An Examination of the Type of Instruction that Facilitates Preservice Teachers' Development of Specialized Content Knowledge of Division with Fractions

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A Thesis

in

The Department

of

Education

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Arts (Child Study) at
Concordia University
Montreal, Quebec, Canada

August 2007

Library and Archives Canada
Published Heritage Branch
395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque et Archives Canada
Direction du Patrimoine de l'édition
395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence
ISBN: 978-0-494-34765-2
Our file Notre référence
ISBN: 978-0-494-34765-2

ABSTRACT

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Ball, Hill, and Bass (2005) proposed that facilitating the development of preservice teachers' specialized content knowledge (SCK) warrants investigation. Specialized content knowledge is mathematical knowledge that is used to teach mathematics and should thus be uniquely understood by teachers. Moreover, the type of mathematics content knowledge (i.e., conceptual and procedural) that would enhance preservice teachers' SCK is not well understood. In line with Ball et al., objectives of the present study were to (a) examine whether content-focused instruction addressing division with fractions would enhance preservice teachers' SCK of this topic and (b) examine the impact of instructional content (i.e., conceptual content, and a combination of conceptual and procedural content) on preservice teachers' gains in SCK of division with fractions (SCK-DF).

Eight preservice teachers were assigned to one of two treatment conditions (i.e., conceptual content, and a combination of conceptual and procedural content) and their development of SCK-DF was measured using semi-structured pre- and post-interviews. The interview data were coded using the Analysis of Specialized Content Knowledge of Division with Fractions rubric, developed by the researcher.

In general, the results supported the hypothesis that (a) participants' overall SCK-DF was enhanced after instruction and (b) the participants who received the combination of conceptual and procedural instructional content demonstrated greater gains in their development in SCK-DF. Taken together, the results from the current study broaden the current understanding of ways to foster preservice teachers' SCK and have implications for future research in the area.

Acknowledgements

I would like to express my appreciation to all whose teaching, guidance, and encouragement made it possible for me to complete this project. I am indebted first and foremost to my research supervisor, Dr. Helena P. Osana for her time and effort in designing the study, as well as revising and editing many drafts. I would also like to thank Dr. Miranda D'Amico and Dr. Guy L. Lacroix, for their willingness to act as my committee members as well as provide me with critical comments and encouraging feedback throughout the year. Finally, I would like to thank the preservice teachers for their participation in this study. Without them, none of this would have been possible.

Dedication

In memory of my mother, Sylvie Rayner (1950 to 2007), and to Brian Rayner, Philippe Mari, Sabrina and David Rayner, thank you for your support and encouragement.

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CHAPTER 1: STATEMENT OF THE PROBLEM

Central to the goal of producing mathematically proficient students is effective mathematics instruction, which is largely dependent on a teacher's deep understanding of the content (Ball, 1990b; Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005; Ma, 1999; Osana, Lacroix, Tucker, & Desrosiers, 2007). What is surprising, however, is that despite the sound logic of this statement, only a paucity of research has explored the depth and type of content knowledge necessary for teaching mathematics (Ball et al., 2005). Indeed, Wu (1997) suggested that efforts to improve mathematics education has primarily focused on pedagogy (i.e., teaching standards and curriculum) while largely ignoring the role of content knowledge. In response to this notion, Schoenfeld (1988) proposed that, "research on the psychology of teaching and learning needs to be expanded both in scope and in breadth…explorations of learning also need to become more focused and detailed as we begin to elaborate on what it means to think mathematically" (p. 165). Therefore, while pedagogical improvements are a necessary aspect of successful mathematics education reform, a teacher's content knowledge is as relevant as his or her pedagogical skills because a teacher's ability to effectively carry out curricular decisions and learning objectives (e.g., task selection) is impacted by his or her content knowledge of the subject (Ball et al., 2005; Osana, Lacroix, et al., 2007; Wu, 1997).

In addressing the role of teachers' content knowledge and how it relates to student performance, Hill et al. (2005) developed a questionnaire to measure the content knowledge implicated in the teaching of mathematics. Specifically, the content knowledge for teaching mathematics questionnaire (CKT-M) measured two aspects of mathematics content knowledge: (a) teachers' *common content knowledge* (CCK) and, (b) teachers' *specialized content knowledge* (SCK). For instance, participants were requested to determine the value of x in $10^x = 1$. This example was considered to be an index of CCK because the question addresses mathematics that is typically taught in school, and therefore it can be considered as something that may commonly be known among adults. Conversely, an item measuring SCK taps into a form of mathematics content knowledge that is used to teach mathematics; and should thus be uniquely understood by teachers. In particular, evidence of having SCK would require the ability to use knowledge of mathematics to do the following: (a) to represent numbers and operations in meaningful ways, (b) to provide justifications that support mathematical rules, and (c) to analyze whether and a student's strategy and solution is, or is not, mathematically reasonable.

Hill et al. (2005) developed and used the CKT-M to empirically investigate whether teachers' CCK and SCK would predict first and third graders' improved performance on a standardized mathematics assessment (i.e., the Basic Battery for first and third graders, and Terra Nova for kindergarteners). In particular, Hill et al. conducted a longitudinal investigation of students from 115 elementary schools, comparing their test scores at the beginning of the school year to their test scores at the end. It was found that the teachers' questionnaire scores positively predicted students' increased test scores from both grades, clearly illustrating the link between the mathematical knowledge teachers display during instruction and students' mathematical understanding. In fact, the teachers who scored in the upper quartile predicted above-average gains in students' test scores, an improvement that is considered to be equivalent to receiving two to three weeks of instruction (Ball et al., 2005). Based on these findings, Ball et al. (2005) put forth the argument that efforts to improve teachers' mathematical knowledge is a key component to the mathematical success of future students, and, that such initiatives should be carried out by providing inservice and preservice teachers with content-focused professional development and coursework.

The proposition of emphasizing mathematical content during preservice teachers' methods courses is further substantiated when one considers that as future teachers, these courses may have the greatest impact on their students' academic growth (Ball & Rowan, 2004). Further, most opportunities for teachers to modify any changes to their instructional practice occur during their university training, before they formally join the teaching profession (National Commission on Teaching and America's Future, 1997). Taken together, instructional content that facilitates the development of SCK, particularly for preservice teachers, warrants investigation. Based on this notion, the first objective of the proposed study was to examine whether content-focused instruction addressing division with fractions will enhance preservice teachers' specialized content knowledge of this topic.

Moreover, the type of mathematics content knowledge (i.e., conceptual and procedural knowledge) that would develop preservice teachers' SCK is not well understood. An ongoing debate within the mathematics education community is whether instruction of mathematics should emphasize a conceptual understanding over a procedural understanding of disciplinary topics (Wu, 1999a). Indeed, Budd et al. (2005) argued that one of the myths regarding mathematics education is the belief that children's meaningful understanding of the subject emerges if they are encouraged to invent strategies to solve problems rather than learn and understand standard algorithms. For instance, Kamii and Dominick (1998), proponents of this view, suggested that promoting students' fluency in procedural knowledge can harm their conceptual understanding of mathematics. This notion, unfortunately, fails to consider the difficulties students will encounter later on during their mathematics career, particularly in algebra, when the understanding of standard algorithms is necessary (Budd et al., 2005). Further, this myth fails to account for the research that has empirically substantiated that conceptual and procedural knowledge develop iteratively (Rittle-Johnson, Siegler, & Alibali, 2001) as well as an alternate conception of procedural knowledge recently put forward by Star (2000, 2002a, 2002b, 2005; Star & Seifert, 2006). Therefore, whether the content-focused coursework suggested by Ball et al. (2005) should exclusively concentrate on understanding mathematics conceptually or should additionally highlight the relevance of understanding mathematics procedurally remains to be seen. For this reason, in this study, I examined the impact of two types of content of instruction (i.e., conceptual content, and a combination of conceptual and procedural content) on preservice teachers' gains in specialized content knowledge of division with fractions (SCK-DF).

CHAPTER 2: REVIEW OF THE LITERATURE

The reform movements of the 1980s and 1990s have fueled the desire to promote students' "mathematical power" which refers to the following mathematical abilities: (a) a capacity to reason mathematically, (b) an ability to engage in problem solving, (c) a demonstration of an understanding of the connections among mathematical concepts, and (d) a facility in communicating mathematics to others (Kilpatrick, Swafford, & Findell, 2001). These different skills have influenced the goals of mathematics instruction and the conception of valuable mathematical knowledge, otherwise referred to as *mathematical proficiency* (Kilpatrick et al., 2001). As a consequence of the numerous mathematical skills reflected in mathematical power, Kilpatrick et al. (2001) expanded on the notion of what is implied when the term "thinking mathematically" is used (Schoenfeld, 1988) and proposed a comprehensive definition of the components or "strands" of mathematical proficiency. That is, mathematical proficiency requires *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning,* and a *productive disposition.* Further, Kilpatrick et al. suggested that these components are interdependent and therefore all strands play an equally vital role in students' development of mathematical proficiency.

The Nature of Conceptual and Procedural Knowledge

Contradicting this view, however, is the dichotomous perception of two types of mathematical knowledge: conceptual knowledge and procedural knowledge. As a consequence of how these two types of understanding are differentiated, their respective roles in the development of mathematical proficiency continues to be a hotly debated issue in the mathematics education community (Hiebert & Lefevre, 1986; Rittle-Johnson et al., 2001; Star, 2005; Wu 1999a). Conceptualizing distinctions among ways of knowing mathematics can be traced back to Scheffler's (1968) theory of knowledge in which he suggested that knowledge can function as a means of either "knowing that" or "knowing how to." In this theory, Scheffler posits that "knowing that" is a condition of propositional knowledge because it is knowledge that is understood in terms of what you know or believe to be true or false. For example, an individual may know that 2 + 2 is the same as 1 + 3 based on evidence that the sum of both pairs of numbers is 4. On the other hand, "knowing how to" is a condition of procedural knowledge because this type of knowledge cannot be understood as being true or false and rather involves knowledge that denotes skill or competence (e.g., knowing how to explain 2 + 2 is the same as 1 + 3).

Parallel to Scheffler's (1968) conditions of knowledge, and in line with other proposed distinctions of mathematical understanding (e.g., Gelman and Gallistel's differentiation between counting skills and principles, 1978; VanLehn's discrepancy between schematic and teleologic, 1983), Hiebert and Lefevre (1986) define two types of mathematics knowledge, (i.e., conceptual knowledge and procedural knowledge) within a similar dichotomous context. Further, it is important to note that Hiebert and Lefevre's definitions have been widely used and accepted within the field of mathematics education and research. Specifically, they defined conceptual knowledge as:

> knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3-4)

Hiebert and Lefevre also defined procedural knowledge as follows:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols. (pp. 7-8)

Star (2005), however, contended that Hiebert and Lefevre's (1986) definition of conceptual knowledge equates the possession of conceptual knowledge with having a connected understanding of concepts, and does not leave room for the possibility that an individual may possess a fragmented understanding of mathematical concepts. That is, in this definition, conceptual knowledge is referred to as being "rich in relationships" and can therefore be considered as a "connected web of knowledge;" the term *concept,* however, is not synonymous with connected ideas, but rather simply refers to a notion or a model. In line with this reasoning, Star (2005) proposed that conceptual knowledge may also include the understanding of concepts in absence of an understanding of the connections among them. For instance, Hiebert and Wearne (1996) found that some students in second grade through fourth grade were capable of accurately performing the standard algorithm for multidigit addition and subtraction problems but were unable to solve the same problems using a non-proportional model (i.e., one red chip is equivalent to 10 and one yellow chip is equivalent to 1). The students whose problems solving performance fit this profile were grouped as the "nonunderstanders" as a result of demonstrating a poorly connected understanding of concepts of place value associated with multidigit addition and subtraction. Based on these findings, I argue that because the students were capable of executing the procedure, they did possess some understanding of the concept of addition and subtraction. Unfortunately, the nonunderstanders' inability to model their solution not only highlights the students' weak connections between concepts of place value and operations, but also illustrates that conceptual knowledge can be disconnected and superficial.[1]

In addition, based on Hiebert and Lefevre's (1986) definition of procedural knowledge, one may assume that, unlike conceptual knowledge, procedural knowledge by nature is disconnected and therefore may not be deeply understood (Star, 2005). Indeed, the notion that knowledge of the links between procedures is largely sequential (i.e., follows a step-by-step procedure) substantiates this proposition. It is important to consider, however, that in the quest to deepen the understanding of conceptual and procedural knowledge, only a paucity of educational research has examined mathematic topics beyond elementary school mathematics (e.g., algebra, geometry, and calculus; Star, 2000). As such, Star (2000) suggested that within the context of elementary mathematic topics, it becomes challenging to conceive of procedural knowledge as comprising more than an understanding of the rules and the steps that are used to solve a problem. When the context of using mathematical procedures is extended, however, the definition of procedural knowledge can be re-conceptualized as a type of knowledge that comprises more than a sequential understanding of algorithms.

Hiebert and Lefevre's (1986) conceptualization of procedural knowledge only encompasses an algorithmic perspective, and, as a consequence, they admit that "...not

[1] Knowledge that is understood deeply refers to the abstract understanding of information such that it may be recalled and applied flexibly in order to maximize the performance of a given task. Conversely, when information is understood superficially, its application becomes limited to the context in which it was learned and is therefore used inflexibly or may be characteristically applied using trial and error (de Jong & Ferguson-Hessler, 1996).

all knowledge fits nicely into one class or the other. Some knowledge lies at the intersection. Heuristic strategies for solving problems, which themselves are objects of thought, are examples" (p. 9). Schoenfeld (1979) defined heuristics strategies as a strategy that relies on knowledge of general mathematical concepts or procedures to derive a solution. The generality of these procedures implies that the focus of these rules extends beyond computation and rather addresses mathematical properties common to all numbers (Wu, 2001). Furthermore, the use of heuristics strategies also demonstrates a capacity to deviate from the typical problem solving pattern in attainting a solution (Star & Seifert, 2006).

Consider, for example, an individual with a less sophisticated procedural understanding of division with fractions as being an individual who would always solve that type of problem by recalling and applying the "inverse and multiply" rule. Star (2005) described this quality of procedural understanding as inflexible or superficial because of the limited ways in which the procedure for solving such problems is known. Further, this conceptualization of procedural knowledge is an example of the algorithmic perspective espoused in Hiebert and Lefevre's (1986) definition of procedural knowledge. Consequently, an individual that possesses Star's *deep procedural* knowledge would demonstrate the capacity to apply various procedures to solve division with fractions problems as a result of knowing general mathematical principles and understanding how they relate to that type of problem.

Deep Procedural Knowledge

The notion of deep procedural knowledge is plausible when one considers de Jong and Ferguson-Hessler's (1996) argument for dimensions of knowledge. That is, rather than confounding type and quality of knowledge, they suggest that type and quality are separate factors of knowledge. In accordance with this theory, Star (2000, 2005) argued that it is possible to regard mathematical knowledge in terms of type (i.e., conceptual and procedural) crossed with quality (i.e., superficial and deep).

In addition to de Jong and Ferguson-Hessler's theory of dimensions of knowledge, Star's (2000, 2005) argument of the existence of deep procedural knowledge is additionally based on the following theories. More specifically, Davis (1983) described the mental processes that a student may experience when faced with a novel problem, one for which they cannot simply recall the strategy that is typically used. More specifically, the planning involved in deciphering an appropriate solution strategy would entail considering procedures based on the goals, or subgoals, that the procedure may attain. That is, rather than selecting a procedure because it matches the problem type, procedures are chosen based on the knowledge of the outcome when a particular procedure is applied, illustrating an abstract understanding of the problem and an ability to understand procedures outside the specific context in which they may have originally been introduced.

Similarly, Ohlsson and Rees (1991) suggested two types of knowledge that may be used to guide problem solving behaviours during the completion of a task. That is, problem solving can be guided by knowledge of mathematical principles (e.g., the laws of numbers) as well as by understanding the purpose of each step in a series of procedures. Therefore, in addition to considering the goals of a procedure, a deep procedural understanding may also enable an individual to consider whether the purpose of using a given rule is in line with the task environment or situation.

Finally, in VanLehn (1990) and VanLehn and Brown (1980), the concept of *teleologic semantics* is suggested as a form of procedural knowledge. Specifically,

> The teleologic semantics of a procedure is knowledge about the purposes of each of its parts and how they fit together. Such knowledge is the province of true masters of the procedure...Teleological semantics is the meaning possessed by one who knows not only the surface structure of a procedure but also the details of its design. (VanLehn & Brown, 1980, p. 95)

More generally, understanding the teleology of a procedure promotes *planning knowledge*, an issue that has been previously addressed by Davis (1983) and Ohlsson and Rees (1991). Indeed, VanLehn (1983) and VanLehn and Brown (1980) extend the concept of planning knowledge to include acknowledging the goals of a procedure (Davis, 1983) and the environment in which a procedure is used (Ohlsson & Rees, 1991), but also incorporate an understanding of the constraints that the situation may impose on procedures as well as any heuristics that are relevant to the situation.

In light of these theories, Star and Seifert (2006) considered the possibility that procedures can be executed with understanding, implying that the goal of procedural understanding is not the automatic execution of a series of steps. Clearly, substantiating this notion would have serious implications for the manner in which mathematics is taught. Specifically, if procedures may only be understood through memorization, the benefits in enhancing students' procedural knowledge would be questionable. Indeed, previous research has demonstrated that rote learning of mathematical procedures tends to make mathematical problem solving less meaningful for students (Hiebert et al., 1997).

Moreover, Hiebert et al. (1997) add that rote learning poses constraints on students' problem solving behaviours, hindering their ability to recall procedures and increasing the likelihood of error. For this reason, the National Council of Teachers of Mathematics (1989) suggested that the memorization of procedures should be de-emphasized during instruction and that conceptual learning should play a key role in fostering students' understanding of the subject. If, however, the development of students' *flexibility* in the execution of mathematical procedures is realizable, then the role of procedural knowledge in mathematics instruction would need to be revised.

In addressing this issue, Star and Seifert (2006; also see Star, 2002a) examined: (a) whether elementary students can demonstrate flexibility in the use of algebra procedures, and (b) what instructional context promotes this type of understanding. Star (2002b) considered that a person who has flexibility possesses an understanding of multiple solution procedures and has the ability to invent new procedures to solve unfamiliar problems or consider the most economical solution for familiar problems. Thirty-six randomly selected sixth-graders who did not possess any prior formal knowledge of solving linear equations participated in five 1-hour experimental instruction sessions. During the first instruction session, and subsequent to the administration of a pretest, all participants were provided with instruction that introduced five basic transformations that are used when solving linear equations (i.e., compute, combine, expand, subtract from both, and divide). Following that session, participants were randomly assigned to either the control group or the treatment group. During the instruction sessions, the participants from both groups were instructed to solve linear equations using the five transformations and the order in which to use the transformations was not specified. The task given to the treatment group differed in that the participants were requested re-solve each linear equation using a different order of transformations; the task for the participants in the control group did not involve resolving problems. Rather, they were required to solve a series of different but isomorphic problems only once.

The results demonstrated that although the participants from the treatment group solved fewer problems and made more transformation errors during the instruction sessions, the posttest revealed no significant differences in the number of problems attempted, the percentage of problems solved without transformation errors, or the percentage of problems with a correct answer between the two groups. The results did, however, indicate differences in the participants' ability to demonstrate flexibility in using transformations. That is, while participants from both groups executed one of three invented equation-solving procedures (i.e., change in variable, cancel terms, or divide before expanding), significant differences between the two groups was supported with respect to the ability to invent novel strategies. Not only were students from the treatment group more likely to invent their own solutions, but the treatment group took advantage of more of the opportunities where inventions could be used. Further, on the posttest, students from the treatment condition more often presented unique transformation sequences, supporting the notion that participants from that group demonstrated a significantly greater knowledge of multiple procedures. Taken together, the findings demonstrate that knowledge of multiple procedures and invention does not necessarily harm a student's ability to derive solutions with accuracy, and that this form of mathematical understanding may effectively be facilitated by incorporating the alternative ordering task during instruction. More generally, these results cut the ties that are commonly perceived between facts and algorithms on the one hand, and drill and practice on the other (Gravemeijer & van Galen, 2003).

Moreover, the implications of the findings in Star and Seifert (2006) suggest that algebra learners can benefit from their understanding of multiple procedures in the same way that elementary learners have an advantage with the ability to invent their own algorithms to solve single digit and multidigit problems. Indeed, the findings from previous research examining children's development of understanding whole number operations recommends that students' use of invented algorithms should also be promoted rather than only encouraging students to rely on standard algorithms (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). It has been suggested that the use of an invented algorithm denotes flexibility in thinking about numbers and an abstract understanding of problems, which enables students to efficiently solve problems as a result of this deepened understanding of number concepts.

Understanding Fractions

The Role of Fractions

Although Star (2000) suggested that the exploration of deep procedural mathematics knowledge may be more suitable at the middle- and secondary-levels, Behr, Lesh, Post, and Silver (1983) and Wu (2001) contended that the topic of fractions may be perceived as a form of "pre-algebra." Indeed, similar to algebra, this topic makes use of abstract and general mathematical procedures in order to solve operations and are therefore expressed using symbolic notation. Based on this notion, Wu (2001) proposed that fractions is a key topic in helping students transition their mathematical perspective from the particular procedures and concepts addressed at the elementary level to the more general manner in which operations are performed in algebra. Moreover, in addition to the value of understanding fractions based on its connection with algebra, a topic that is notoriously known as a gatekeeper course[2], students' understanding of fractions is vital because of its deep connections with other rational number concepts such as percents, decimals, ratios, rates, and quotients (Saxe, Gearhart, & Nasir, 2001).

Unfortunately, although the understanding of fractions plays an important role in elementary mathematics education, it is also a topic that both elementary and preservice students find extremely challenging to grasp (Rayner, Pitsolantis, & Osana, 2007; Saxe et al., 2001). It has been posited that the source of students' difficulties with this topic is largely attributed to the way in which fractions is addressed in students' mathematics textbooks, the instructional techniques used in the classroom, and teachers' difficulties understanding this topic (Askey, 1999; Ma, 1999; Niemi, 1996; Saxe et al., 2001; Wu, 1999b). While it is beyond the scope of this proposed study to address the issue of how fractions are presented in students' textbooks, I will critically review the content of fractions at the elementary level. Following that, I will provide a summary of the research that underscores both inservice and preservice teachers' difficulties with this topic. This review of the literature of fractions will not only address the source of students' difficulties with fractions, but will also serve to provide the reader with a conceptual, procedural, and deep procedural understanding of division with fractions.

Teaching Fractions

In general, Askey (1999) and Wu (1999b) noted that the topic of fractions is presented in a complex manner that lacks mathematical explanations. More specifically,

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[2] Algebra is often referred to as a gatekeeper course because the students that can succeed in this topic often advance to other topics in mathematics while those who do not succeed may be permanently left behind (Wu, 2001).

teachers often introduce students to the concept of fractions with at least five different meanings of its symbolic representation (Wu, 1999b). Wu considered this to be problematic for students because the numerous representations present a dramatic increase of the ways in which the same symbol may be understood as students move from whole numbers to fractions. Indeed, the same fractions symbol can represent different fractions (i.e., parts of a whole, the size of a portion when it is equally distributed, the quotient when the divisor is larger than the dividend, a ratio of two numbers, and an operator[3]). Despite this symbolic commonality, these concepts carry different meanings, making it challenging for students to understand the definition of a fraction without any ambiguity (Wu, 1999b).

Furthermore, the teaching of fractions has been criticized for an over-emphasis on conceptual explanations (Wu, 1999b) despite the simplicity and relevance of mathematically procedural explanations. The prominence of conceptual justifications during instruction on this topic is problematic for a few reasons. First, although it is extremely valuable for students when teachers model the algorithms used to solve operations with fractions using concrete or semi-concrete methods, the numbers that can be used in these examples is limited to simple fractions, such as $\frac{1}{2}$ or $\frac{2}{3}$. As soon as the fraction becomes less common, concretely modeling a solution becomes complex and thus less effective and confusing.

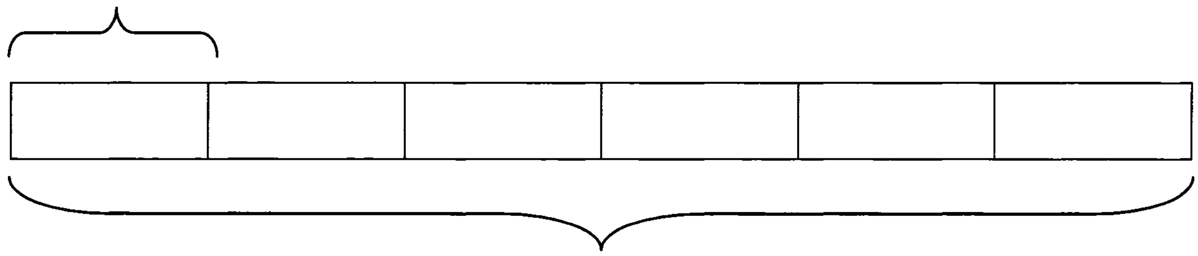
To illustrate, consider the following division with fractions word problem where the goal is to discover how many groups of a certain length are comprised within a total

[3] An operator is an instruction that carries out a process, for example "$\frac{2}{3}$ of".

quantity: Mary has 6 feet of ribbon. She plans on making bracelets with this ribbon. Each bracelet requires $\frac{2}{3}$ of a foot of ribbon. How many bracelets can Mary make? Figure 1 contains a semi-concrete explanation of how to solve this problem (Reys, Lindquist, Lambdin, & Smith, 2007). The first step in solving this problem involves dividing each of the 6-feet pieces into thirds because the problem requires the discovery of how many groups of $\frac{2}{3}$ are in 6 feet. As a result of dividing each foot into thirds, six groups of three have been created; otherwise expressed as 6 multiplied by 3. When the inverse and multiply rule is applied, $\frac{2}{3}$ is inversed and becomes $\frac{3}{2}$ and this fraction is multiplied by 6; in carrying out the initial step of this calculation, 6 is multiplied by 3. The second step requires discerning the number of groups of $\frac{2}{3}$s. In this case, this reflects the action of dividing 18 by 2 because groups of $\frac{2}{3}$s are removed from the total quantity. Following this action, the number of groups that have been formed can be counted, providing a solution to the word problem that reflects the number sentence $6 \div \frac{2}{3}$.
An obstacle, however, arises when more complicated fractions are involved in a problem; it becomes tremendously tedious to solve problems like $\frac{2}{64} \div \frac{5}{17}$ using a concrete or semi- concrete strategy. Therefore, if students are only expected to understand how to solve problems with simple fractions, notwithstanding the fact that complex fractions do exist in the real world, the foundation of their knowledge of fractions becomes weak and unstable (Askey, 1999; Wu, 1999a).

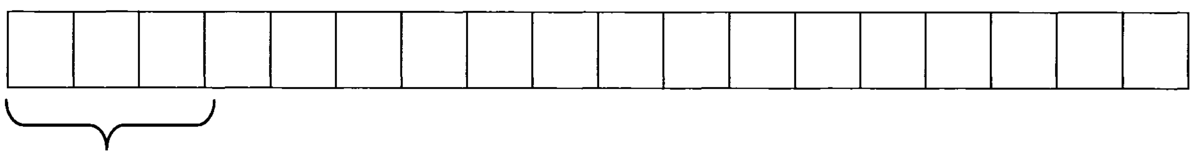
This Mary has 6 feet of ribbon. She plans on making bracelets with this ribbon. Each bracelet requires $\frac{2}{3}$ of a foot of ribbon. How many bracelets can Mary make?

1 foot



6 feet

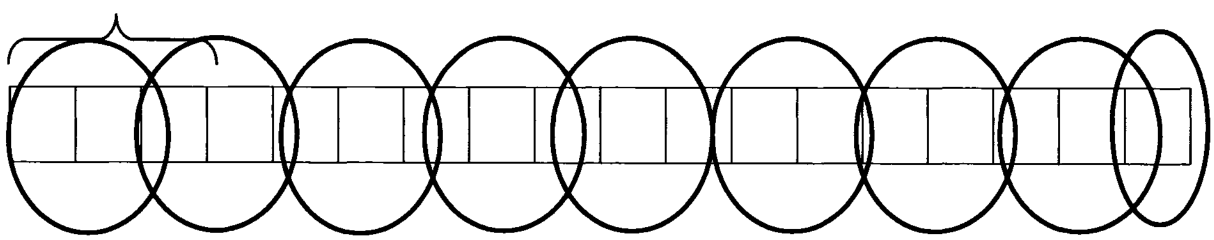
1. Divide each foot into thirds because we are looking for how many groups of $\frac{2}{3}$ go into 6 feet.



1 foot

As a result of dividing each foot into thirds, we have created 6 groups of 3, for a total of 18 sections. Notice that this reflects the first calculation when executing the inverse and multiply rule, $6 \div \frac{2}{3}$ becomes $6 \times \frac{3}{2}$ and the first calculation is 6 x 3.

1 foot



2. We have continued by encircling groups where in each group there is $\frac{2}{3}$ of a section. 9 groups of $\frac{2}{3}$ have been formed. This step reflects the second calculation executed $\frac{18}{2}$ or $18 \div 2 = 9$.

Figure 1. Explanation of a division with fractions word problem using a semi-concrete method.

Similarly, an inadequate understanding of fractions can also be promoted if a student's understanding of operations with fractions is only procedural in nature. The notion that only one type of mathematics knowledge should be used as a vehicle to understand fractions and operations with them contradicts the iterative nature of the relationship between conceptual and procedural knowledge that has been empirically supported (Rittle-Johnson et al., 2001). Specifically, Rittle-Johnson et al. (2001) found that incremental developments in one type of knowledge yields gains in the other, and in turn, these gains in knowledge sustain and extend the initially developed understanding. In addition, relations between the development of conceptual and procedural knowledge indicates a bidirectional association, and, although Rittle-Johnson and Alibali (1999) and Hiebert and Wearne (1996) suggested that conceptual knowledge may have a greater influence on the understanding of procedures than the reverse, the nature of the iterative relationship implies that procedural knowledge may lead to developments in conceptual knowledge and vice versa[4]. Taken together, the second reason why an over-emphasis on conceptual explanations is criticized is because proficiency in mathematics requires an understanding of both mathematical concepts and procedures (Star, 2005; Wu, 1999a), and as such, both types of understanding should play an equally important role in the classroom.

A final reason that supports the notion of teaching both mathematical concepts and procedures is based on research that has demonstrated that instruction that only uses

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[4] Findings from other research (i.e., Byrnes, 1992; Byrnes & Wasik, 1991) contradict this notion and rather propose that the development of conceptual and procedural knowledge is sequential, conceptual knowledge preceding the development of procedural knowledge. Based on studies that have illustrated that under certain circumstances, such as multiplication with fractions, it is possible to correctly execute a procedure without developing a deeper understanding of the underlying concepts (Niemi, 1996), I argue that the iterative and bidirectional theory appears to be a more appropriate explanation addressing the variance in the sequence of development of these two types of knowledge (see Rittle-Johnson & Alibali, 1999).

concrete models to teach mathematics is ineffective. This type of instruction fails to promote mastery of understanding and applying algorithms, limiting students' capacity to use mathematics in applied situations (Gravemeijer & van Galen, 2003). In terms of the topic of division with fractions, not all interpretations of division with fractions lend themselves to justifying the standard algorithm. In fact, aside from the *measurement model* (i.e., the grouping interpretation of division with fractions that requires forming a number of groups of a certain size) illustrated in Figure 1, the *partitive mode*l (i.e., the grouping interpretation of division with fractions that requires finding the size of each group; Ball, 1990b; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Reys et al., 2007) and the *factors and product model* (i.e., an interpretation of division with fractions that requires calculating a missing factor; Askey, 1999; Ma, 1999) are models that do not reflect the actions performed while executing the algorithm (i.e., the denominator in the second number is multiplied by the numerator in the first number, and this product is divided by the product of the denominator in the first number and the numerator in second number). As such, I suggest that only promoting a conceptual understanding of division with fractions may produce a similar outcome as when students are expected to learn the division with fractions algorithm by rote; that is, it limits their understanding of the algorithm.

A final critique of the teaching of fractions is that during instruction on operations with fractions, connections between the algorithms and properties of whole number operations are not made (Askey, 1999; Wu, 1999b). It is a disservice to students to present mathematical topics as isolated curricular subjects in place of highlighting the myriad of connections among mathematical concepts and procedures. Indeed, highlighting the connections among mathematical topics would build students' mathematical understanding rather than promote a disconnected at best, and illogical at worst, progression from one topic to the next. For instance, connected to the division with fractions algorithm is the understanding that division is the inverse of multiplication (Wu, 1999b). The notion that $24 \div 3 = 8$ is the same as $8 \times 3 = 24$ is valid because the former may be interpreted as findings how many groups of 3 are in 24 (i.e., the measurement division model). The latter is interpreted as 8 groups of 3; either way, both interpretations suggest that there are a number of groups of objects (i.e., 8) with a number of objects in each group (i.e., 3) and these values multiplied together make up a total number of objects. Further, the principle that division is the inverse of multiplication is a general rule of mathematics and as such can be represented as: $m \div n = k$ is the same as $m = n \times k$, where k, m, and n are whole numbers (Wu, 1999a).

Figure 2 indicates how this mathematical principle applies to division with fractions, and thus can be used to justify the inverse and multiply rule (Wu, 1999a). As it can be seen in Figure 2, using a mathematical rule (i.e., that division is the inverse of multiplication) to justify a standard algorithm may be considered a powerful tool to add to a student's repertoire of mathematics knowledge. Moreover, the example in Figure 2 substantiates the notion that procedural justifications can be incorporated during instruction in a meaningful manner. As I previously mentioned, Kamii and Dominick (1998) argued that teaching standard algorithms, such as to invert and multiply, to elementary students can harm their understanding of mathematics. Their reasoning is based on the belief that standard algorithms enable students to engage in problem solving without forcing them to consider their own reasoning skills.

1. Because $m \div n = k$ is the same as $m = n \times k$, $\frac{a}{b} \div \frac{c}{d} = \frac{x}{y}$ is the same as

   $\frac{a}{b} = \frac{c}{d} \times \frac{x}{y}$.

2. To solve for $\frac{x}{y}$, multiply both sides of the equation by $\frac{d}{c}$ and the result is

   $\frac{x}{y} = \frac{d}{c} \times \frac{a}{b}$.

3. For this reason $\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$.

For example, $\frac{12}{15} \div \frac{17}{33}$ is the same as $\frac{12}{15} = \frac{17}{33} \times \frac{x}{y}$

Execute step 2 $\left(\frac{33}{17}\right)\frac{12}{15} = \frac{17}{33}\left(\frac{33}{17}\right)\frac{x}{y}$ and so $\frac{33}{17} \times \frac{12}{15} =$ is the same as $\frac{12}{15} \div \frac{17}{33}$

Figure 2. Explanation of the inverse and multiply rule using the principle division is the inverse of multiplication.

Lending partial support to Kamii and Dominick's (1998) reasoning are the results in Carpenter et al. (1998). In particular, Carpenter et al. found that compared to the students who initially used the standard algorithms, the students who used invented strategies first demonstrated a greater understanding of base-ten concepts and a capacity to successfully transfer their knowledge to solve novel tasks. Moreover, these students displayed fewer procedural bugs when performing the subtraction algorithm. Based on the results, and in line with Kamii and Dominick, one may agree that teaching standard algorithms is obsolete because of the benefits of using invented algorithms. This argument, however, may be spurious because the students who used invented algorithms first were also exposed to the standard algorithms during instruction. Therefore, a more appropriate recommendation would be to provide students with the opportunity to construct invented strategies before instruction of standard algorithms provides students with a strong foundation of multidigit operation concepts; from this basis, students can develop a meaningful understanding of procedures used with multidigit addition and subtraction.

Additionally, while it is true that introducing students to algorithmic procedures in an ad hoc fashion may prevent students from developing a more meaningful understanding of mathematics (Wu, 1999b), the appropriate teaching of standard algorithms should nevertheless play a central role in mathematics instruction. That is, because standard algorithms are created using mathematical reasoning, they can play a meaningful role during instruction if the mathematics that supports the algorithm is incorporated in the justification. Moreover, motivating students to use algorithms is in line with an inherent goal of using mathematics, to be able to solve a complex problem in the simplest and most efficient way possible (Carpenter et al., 1998; Wu, 1999a). Taken together, students will only be able to achieve this goal of mathematical proficiency if the teacher's instruction facilitates, among other things, conceptual understanding and procedural fluency (Kilpatrick et al., 2001).

To this point, I have discussed how instruction on division with fractions can promote conceptual understanding (see Figure 1) as well as procedural understanding (see Figure 2). An aspect of Kilpatrick et al.'s (2001) definition of procedural fluency, however, is the capacity to flexibly perform procedures. Based on Star and Seifert's (2006) definition of flexibility, then, developing students' procedural fluency of division with fractions would involve instruction that teaches multiple strategies for solving these types of problems. Two alternate procedural strategies (i.e., complex fraction and common denominator; Tirosh, 2000) that can be used to solve division with fraction problems are shown in Figure 3.

In line with Star and Seifert's (2006) findings, developing students' flexibility enhances their reasoning skills and promotes a student's capacity to understand mathematical problems from an abstract perspective, facilitating the application of efficient strategies with familiar problems. Also, flexibility promotes transfer skills, and therefore allows students to invent new strategies when they encounter novel problems. Taken together, I argue that all of the ways of knowing fractions that I have presented may promote the goal of focusing on "substantive mathematics" (Ma, 1999) and teaching for mathematical proficiency (Kilpatrick et al., 2001), and therefore should be incorporated in fractions instruction.

Alternative 1: Complex Fractions

The product of a number and its reciprocal is 1. $a \times \frac{1}{a} = 1$ or $a \div 1 = a$

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \times \frac{\frac{d}{c}}{\frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$$

This shows how to symbolically represent the division of any two fractions, or a whole number and a fraction, as a fraction. Multiplying the same number (in this case the strategic use of $\frac{d}{c}$) to the denominator (i.e., the divisor) and to the numerator (i.e., the dividend) produces an equivalent fraction whereby the equivalent fraction's denominator is 1. Therefore, based on the principle that $a \div 1 = a$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Alternative 2: Common Denominator

First, the fractions are renamed as like fractions.

$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bd} \div \frac{cb}{db}$

Following that the first numerator is divided by the second numerator.

$ad \div cb$

Apply this to the denominators.

$bd \div db = 1$,

and the result is $\frac{ad \div cb}{1}$ which can be represented as $\frac{ad}{bc}$.

Figure 3. Two alternative procedural explanations of solving division with fractions.

Inservice and Preservice Teachers' Difficulties with Fractions

Tightly integrated with fractions instruction is the quality of mathematical understanding the teacher possesses (Askey, 1999; Ball, 1990b). That is, teachers cannot teach for mathematical proficiency with division with fractions if they do not themselves have a deep mathematical understanding of the topic (Kilpatrick et al., 2001). In a recent study, Ma (1999) interviewed 23 U.S. teachers and 72 Chinese teachers to gain an understanding of the nature of their subject-matter knowledge. Four standard elementary topics (i.e., subtraction with regrouping, multidigit multiplication, division by fractions, and the relationship between perimeter and area) were couched in a classroom scenario and the teachers were requested to discuss how they would handle problems that typically arise during instruction of these topics. For instance, with respect to the topic of division with fractions, the teachers were required to compute $1\frac{3}{4} \div \frac{1}{2}$ as well as to develop a word problem that reflects this number sentence. The fractions interview task was as follows:

> Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$? (p. 55)

Across all four topic areas, Ma (1999) found that the knowledge the Chinese teachers possessed was qualitatively different from that of the U.S. teachers. In particular, the findings suggested that Chinese teachers articulated mathematical ideas that reflected a rich and interconnected understanding of mathematical concepts. The results from the division with fractions topic supports this general conclusion. Of the 21 U.S. teachers who attempted to compute the number sentence, only nine (i.e., 43%) correctly computed the answer to the problem. Of the teachers who could not execute the calculation, four demonstrated an unclear understanding of the algorithm, five vaguely recalled the rule and as consequence could not remember certain details of the procedure (such as which fraction to invert), two correctly applied the algorithm but calculated the wrong solution, and one participant performed the wrong strategy and thus produced an incorrect answer.

With respect to representing the number sentence in a word problem, 6 U.S. teachers could not think of a story and 16 produced stories that conveyed their conceptual misconceptions of division with fractions. One of the U.S. teachers did produce a mathematically reasonable story but, unfortunately, her answer to the problem resulted in a fractional number of children, a notion that is unrealistic. Ma (1999) considered this response as pedagogically problematic. In general, the story problems that the U.S. teachers generated highlighted a few misconceptions of the concept of dividing a fraction by a fraction (i.e., $1\frac{3}{4} \div \frac{1}{2}$). Specifically, the teachers produced stories that conveyed dividing $1\frac{3}{4}$ by 2, or reflected multiplying $1\frac{3}{4}$ by $\frac{1}{2}$. Taken together, the teachers not only displayed a weak understanding of the procedure, but also demonstrated a deficient understanding of the concepts related to this topic.

Based on these results, Ma (1999) concluded that the U.S. teachers' difficulty with this task was the result of their inadequate understanding of the procedure and their misconceptions surrounding the concept of division with fractions. Moreover, another result from Ma's study was that the general pedagogical skills the teachers possessed could not aid them in successfully representing the concept of division with fractions, a conclusion that further supports the importance of teachers' content knowledge.

In contrast, all 72 Chinese teachers carried out a procedure to arrive at the correct answer. Interestingly, some of the Chinese teachers voluntarily offered elaborate justifications of the rule, and as such, Ma began asking the Chinese teachers to describe how they understood the mathematical reasoning behind the algorithm. The answers to this question not only comprised the traditional explanation (i.e., how dividing two numbers is the same as multiplying by its reciprocal), but alternative computational approaches (i.e., using decimals, applying the distributive property, and strategies where only division was used) were also put forward as a result of the teachers' conception that under certain circumstances these alternate approaches can be easier and less complicated than the traditional inverse and multiply strategy.

In addition, while only one U.S. teacher was able to create a conceptually accurate representation of the number sentence (i.e., generate a word problem), 90% of the Chinese teachers were able to complete this task, once again illuminating a gap in the quality of understanding between these two group of teachers. Further, of the 65 Chinese teachers who were successful, 12 teachers presented more than one story to reflect three different interpretations of division with fractions: the measurement model, the partitive model, and the factors and product model. Specifically, a story reflecting the measurement model would revolve around finding out how many halves are represented in $1\frac{3}{4}$. The partitive model presents a slightly different interpretation in which $1\frac{3}{4}$ is half of something. Finally, a story based on the factors and product model would involve $1\frac{3}{4}$

being the product of two factors, one of which is $\frac{1}{2}$. These results demonstrated that, unlike the U.S. teachers, almost all of the Chinese teachers displayed at least one of the three aspects of SCK (i.e., knowledge of how to represent numbers and operations in meaningful ways). Further, Ma (1999) suggested that their comprehensive conceptual and procedural understanding of division of fractions was the vehicle for the development of this knowledge.

Similar to the U.S. teachers in Ma's (1999) study, the preservice teachers in Ball (1990a) also demonstrated an inadequate understanding of division. The prospective teachers were administered three types of questions (i.e., division with fractions, division by zero, and division involving algebraic equations) and their responses were analyzed with respect to their ability to produce correct answers and to justify their solution strategy. The division with fractions number sentence was the same one that was used in Ma (i.e., $1\frac{3}{4} \div \frac{1}{2}$). Compared to the U.S. teachers from the Ma study, however, the preservice teachers demonstrated a stronger grasp of the algorithm, indicated by the higher number of participants that were able to correctly answer the problem (i.e., 17 out of 19, or 89%). Despite the fact that they were more proficient in executing the algorithm, only 5 preservice teachers were able to able to appropriately represent the problem in a real-world situation, a finding that is consistent with Ma's study. The pattern of results that emerged from this problem was similarly found for the division by zero problem and the division involving algebra equation.

Also contributing to the understanding of preservice teachers' knowledge of rational number concepts, Tirosh, Fischbein, Graeber, and Wilson (1998) examined 147 prospective elementary teachers' content knowledge of this topic. Participants' content knowledge was measured using a questionnaire and semi-structured interview, both of which were developed within a framework of three dimensions of mathematics knowledge: (a) an *algorithmic dimension*, (b) an *intuitive dimension*, and (c) a *formal dimension*. Similar to Kilpatrick et al.'s (2001) strands of mathematical proficiency, Tirosh et al. posited that these three dimensions play an interdependent role in facilitating the development of conceptual knowledge and thus inconsistencies in any one of the dimensions may be the source of students' and adults' misconceptions of mathematical concepts.

The first, or the algorithmic dimension, comprises knowledge of the procedures and justifications of the step by step sequence of rules used in standard algorithms. Similar to the findings in Ball (1990a), Tirosh et al. (1998) found that the majority of preservice teachers accurately performed the algorithms when solving addition, subtraction, and multiplication problems with fractions or decimals. A distinction, however, between the findings from both of these studies was that the prospective teachers in Tirosh et al. struggled with executing the standard algorithm used with *division* with fractions and decimal numbers. In general, the preservice teachers demonstrated more difficulties with the division with decimal number problems compared to the division problems with fractions. Those who did display misconceptions of the algorithm used with division with fractions, however, typically inversed the dividend (i.e., the first fraction in the number sentence) instead of the divisor (i.e., the second fraction in the number sentence). Interestingly, even the preservice teachers who were mathematics majors demonstrated difficulties computing accurate answers to these problems (i.e., percentage of correct answers ranged from 57% to 93%). Nonetheless, in general, they were more successful when compared to the other preservice teachers (i.e., percentage of correct answers ranged from 12% to 69%).

Moreover, to assess the participants' knowledge of the mathematical reasoning behind each step in standard algorithms, the preservice teachers were requested to explain each step that was executed in a problem previously solved by a hypothetical elementary student. The results indicated that most of the preservice teachers were unable to justify the steps of the standard algorithms used with operations with fractions and with decimals. For instance, only 8 preservice teachers provided acceptable justifications for the steps involved in a division problem with one decimal number.

Tirosh et al. (1998) proposed that the intuitive dimension involves knowledge of mathematics that is accepted without proof. More simply put, this type of knowledge is the "common sense" understanding that arises from problem solving situations (e.g., multiplying two whole numbers always produces a quantity that is larger than either factor). In the study's questionnaire, this type of understanding was assessed by asking the preservice teachers to produce models (i.e., verbal, graphical, or concrete) of rational numbers and operations with rational numbers. Similar to the results in Ball (1990a), it was found that the preservice teachers showed deficiencies in their abilities to represent operations involving rational numbers, particularly when fractions were included in the number sentence. In addition, the interview questions tapped into the preservice teachers' beliefs of the outcomes of operations with rational numbers. It was found that the preservice teachers incorrectly transferred concepts of multiplication and division with whole numbers to rational numbers. For instance, 60% of the preservice teachers contended that "multiplications always makes bigger" and 51% suggested that "division always makes smaller," two notions that do not hold for all rational numbers.

Finally, Tirosh et al. (1998) proposed that the formal dimension includes knowledge of the definitions of concepts and procedures relevant to the topic in question. In comparing the participants who were mathematics majors to the other preservice teachers, extensive differences in their formal knowledge were found. For example, of the mathematics majors, 92% provided correct definitions of rational and irrational numbers compared to the 23% of the non-mathematics majors.

In summary, the findings from Tirosh et al. (1998) highlight distinctions between the two groups of prospective teachers, demonstrating that the mathematics majors tended to display a stronger understanding of rational numbers. Overall, however, the results depict preservice teachers' content knowledge of this topic as rule-bound and fragile as evidenced by the large percentage who upheld false conceptions of operations with rational numbers, including an inability to justify the steps in algorithms and to provide suitable representations of rational numbers and operations with them. These findings contribute to the understanding of preservice teachers' content knowledge of rational numbers and build on the results that were found in Ball (1990a, 1990b).

In general, the results from several studies (i.e., Ball, 1990a, 1990b; Ma, 1999; Tirosh et al., 1998) paint a bleak picture with respect to inservice and preservice teachers' content knowledge of fractions, and division with fractions in particular. In accordance with this conclusion, opportunities for teachers to teach for mathematical proficiency, at least in this area, appear to be slim. That is, as a consequence of their fragmented understanding of this topic, the ways in which they can address this topic in the classroom will be limited (Wu, 1999b). Furthermore, the findings from Ball (1990a), Ma (1999), and Tirosh et al. (1998) are a cause for concern because it may be implied based on the findings from Hill et al. (2005) that students who have teachers with a similar understanding of division with fractions will perform poorly in this area. As such, the prospect for most of these students to continue understanding mathematics beyond this topic (e.g., algebra) will diminish (Wu, 2001). Taken together, I propose that it is crucial that the content of instruction of division with fractions at the preservice teacher level facilitates the mathematical knowledge that will necessitate effective instruction in the future (Ball et al., 2005).

Theoretical Framework

Based on the results from Tirosh et al. (1998), Tirosh (2000) continued to examine preservice teachers' knowledge of rational numbers by assessing their subject-matter knowledge (i.e., common content and specialized content knowledge; Ball et al., 2005) and their pedagogical content knowledge (PCK) as it relates to division with fractions. Shulman (1986) initiated the research efforts to better understand teachers' PCK and proposed that in addition to subject-matter knowledge, teachers also need to be capable of understanding how students come to comprehend mathematics. In particular, Shulman suggested that teachers with PCK should consider, based on their understanding of students' conceptions of mathematics, which representations of concepts and principles would successfully develop their students' knowledge. Moreover, PCK also comprises knowledge of students' common misconceptions of mathematics. A theoretical model highlighting the relationship between types of teacher knowledge (i.e., SCK, CCK, and PCK) and the types of mathematical content knowledge (i.e., conceptual understanding and procedural fluency) is presented in Figure 4.

In Tirosh (2000), the preservice teachers' subject-matter knowledge was measured as a function of their ability to correctly calculate division with fractions problems and derive number sentences from word problems. The same number sentence problems and word problems were used to gauge the participants' PCK. In line with Shulman's (1986) definition of PCK, the preservice teachers were requested to identify incorrect strategies commonly demonstrated by elementary students. These items were presented at the beginning of a mathematics methods course in which the preservice teachers were enrolled.

The results from this pretest indicated that the majority of the preservice teachers correctly performed the calculations and accurately derived number sentences from the word problems (i.e., 25 out of 30, and 29 out of 30, respectively). In addition, the other items indicated that most of the preservice teachers (i.e., 26 out of 30) attributed the source of students' errors to misapplication of the standard algorithm. Further, the preservice teachers attributed this misapplication to poor memorization of the procedure, underscoring the fundamental belief that mathematics is a discipline of facts and procedures that need to be memorized. Notably, none of the preservice teachers considered that students may use alternate ways of solving these types of problems, and only a few of them were aware of the tendency of many children to falsely assume that concepts associated with operations of whole numbers can always extend to fractions.

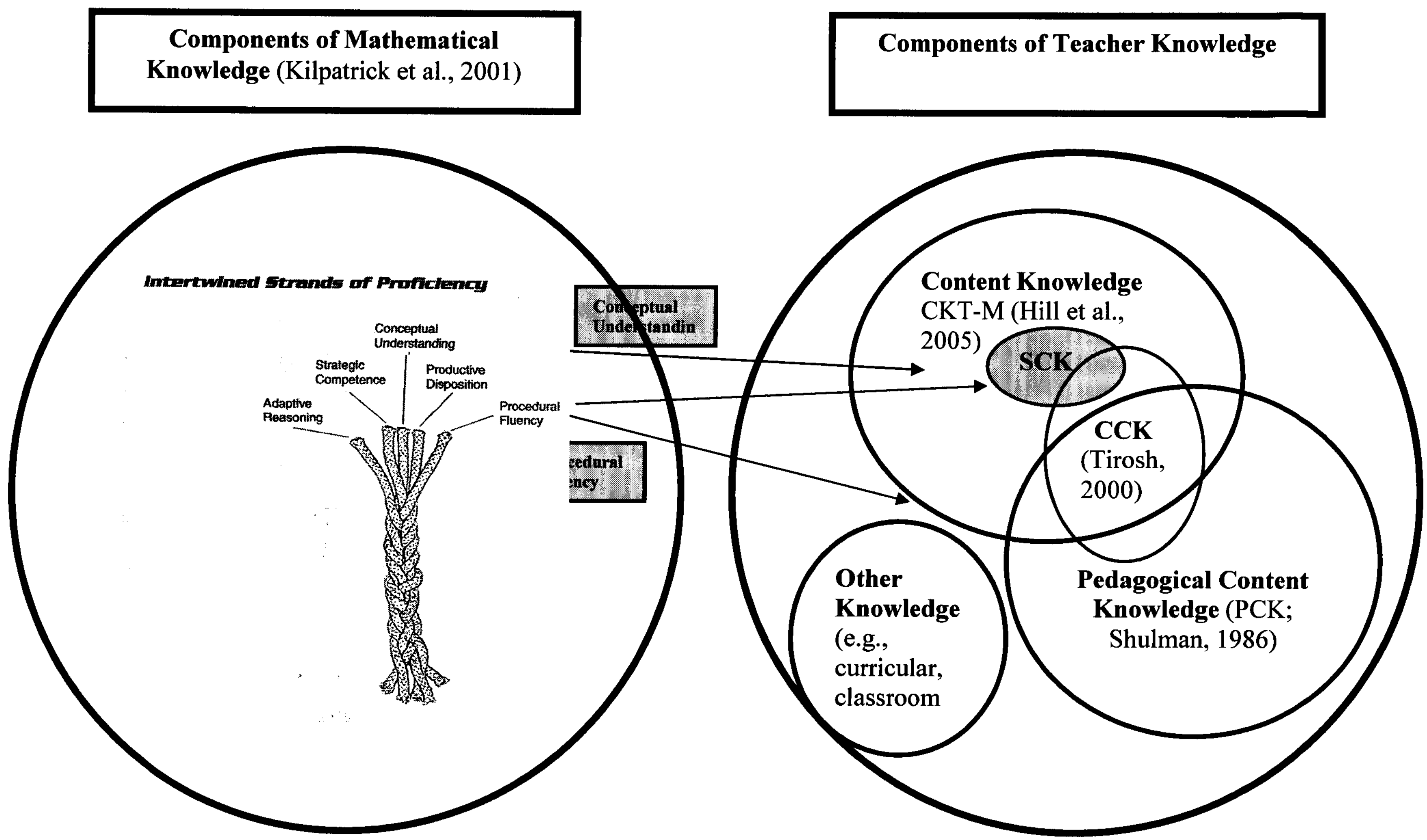

Figure 4. Theoretical framework for proposed investigation.

Based on these findings, the preservice teachers were provided with instruction that emphasized the sources of students' errors in this area as well as multiple ways of solving division with fraction problems. It is important to note that these ways of knowing fractions were not presented concretely or semi-concretely, but rather Tirosh (2000) took the opportunity to expose the preservice teachers to several meaningful algorithmic strategies that are all based on mathematical procedures. The three strategies she used are the same ones that were presented in Figure 2 (i.e., the standard algorithm) and Figure 3 (i.e., an algorithm based on principle the product of a number and its reciprocal is one, and applying the standard algorithm used with multiplication with fractions but using division in place of multiplication).

A posttest assessment, identical to the pretest, was administered at the end of the course to evaluate changes in the preservice teachers' subject-matter knowledge and pedagogical content knowledge in this topic. The results from the posttest demonstrated an inversed pattern of responses pertaining to the questions addressing children's mistakes and sources of misunderstanding. In particular, the vast majority of the preservice teachers demonstrated familiarity with a variety of sources of incorrect methods of performing calculations and 28 preservice teachers were able to display more than one possible algorithmically-based error for each number sentence.

In addition, the results from the questions with the word problem demonstrated that more preservice teachers at the end of the course were aware of the misconceptions of division with fractions commonly upheld by elementary students. The first misunderstanding relates to the tendency of extending concepts of operations with whole numbers to operations with fractions (e.g., "the dividend is greater than the divisor"). The second misunderstanding is based on the application of "multiplication makes bigger, division makes smaller" to all fractions. Finally, the preservice teachers were able to associate students' confusion with an over reliance of the partitive model, which constrains the development of a meaningful understanding of the algorithm; the participants' PCK clearly improved after receiving this instruction. In sum, the findings highlight the notion that instruction that emphasizes several ways of knowing mathematics and children's misconceptions, even in absence of conceptually based instruction, may be an effective pathway to improving preservice teachers' subject-matter and pedagogical content knowledge.

The findings from Tirosh (2000) are crucial given the paucity of research that has examined how content knowledge can help foster preservice teachers' knowledge for effective teaching. That is, it was found that preservice teachers' development of PCK was not only enhanced by the instruction on children's misconceptions, but it was also improved as a result of instruction on the mathematical reasoning that supports the standard algorithm used with division with fractions as well as alternate algorithms that are equally effective in solving these types of problems.

Although Tirosh's (2000) findings are critical, her study did not compare which type of mathematical content (i.e., conceptual, or a combination of conceptual and procedural) has a greater impact on preservice teachers' development of subject-matter and pedagogical content knowledge. In addition, her measures disproportionately focus on preservice teachers' PCK, despite the importance of facilitating their content knowledge (Ball et al., 2005). Finally, Tirosh's method of measuring teachers' subject-matter knowledge was more in line with Ball et al.'s (2005) notion of CCK. Thus, the question of how to enhance preservice teachers' SCK through content based methods of instruction remains open. The development of SCK at the preservice teacher level should be one of the goals of instruction because, in line with the findings from Hill et al. (2005), teachers with adequate SCK, and CCK, may have a positive impact on their students' performance in mathematics.

As I previously mentioned, Ball et al. (2005) proposed that SCK is made up of three primary skills: (a) an ability to analyze students' errors, (b) the capacity to strategically represent numbers and manipulatives that are in line with a given instructional goal, and (c) a facility in explaining the meaning behind an algorithm. I argue that in relation to division with fractions, all three skills require a sound conceptual and procedural understanding of the topic.

For instance, in Tirosh (2000), preservice teachers' knowledge of common student errors improved as a result of forming connections between the multiple algorithms used in division with fractions and children's misconceptions of this topic. This finding illustrates that although a teacher's knowledge of common student errors is an aspect of pedagogical content knowledge (Carpenter, Fennema, & Franke, 1996), this skill is nevertheless connected to the content knowledge used when teachers analyze students' errors, which is an aspect of SCK. Therefore, the results from this study may be used to suggest that procedural content knowledge may indeed play a role in developing preservice teachers' SCK. Moreover, because the students' misconceptions that were addressed during the mathematics methods course implemented by Tirosh comprised both conceptual and procedural errors, it is possible that procedural content knowledge is not the only factor that contributed to this result. Based on this aspect of the study, I suggest that a deep understanding of both concepts and procedures pertaining to division with fractions is necessary in order to comprehensively assess students' misunderstandings.

In addition, multiple ways of understanding division with fractions (i.e., conceptually and procedurally) provides a broader and deeper knowledge base from which numbers and activities can be strategically selected in order to effectively promote learning objectives. For instance, one of the Chinese teachers in Ma (1999) applied the distributive property in her solution strategy because the numbers in the problem (i.e., $1\frac{3}{4} \div \frac{1}{2}$) were amenable to this property. Specifically, because of the invert and multiply rule, $\frac{1}{2}$ becomes 2, which is an easy number to distribute across $1\frac{3}{4}$ (i.e., because $1\frac{3}{4}$ is the same as $1 + \frac{3}{4}$, distributing 2 across $1\frac{3}{4}$ would mean $(1 \text{ x } 2) + (\frac{3}{4} \text{ x } 2)$). Clearly, certain numbers are more conducive to the application of particular properties and strategies than others; the capacity to match strategies with appropriate numbers and manipulatives emerges from a deep understanding of mathematics, implying that the mastery of both conceptual and procedural knowledge is necessary.

Finally, gains in both conceptual and procedural knowledge of division with fractions would provide teachers with the opportunity to discuss with their students the multiple ways of understanding the algorithm. To illustrate, consider the case of a preservice teacher presented in Eisenhart et al. (1993). "Ms. Daniels" experienced many obstacles in her quest for teaching for understanding during her preservice teaching internship. Like many preservice teachers, Ms. Daniels' procedural understanding of mathematics was adequate (Ball, 1990b). Unfortunately, her underdeveloped conceptual knowledge limited the ways in which she could teach certain mathematical topics. For instance, when a student requested that Ms. Daniels justify the algorithm for division with fractions, Ms. Daniels initiated her response with a real world explanation. Unfortunately, she was forced to revert back to focusing on the computational explanation once she realized that her real word example was an example of multiplication with fractions, not division with fractions. Indeed, an implication of Eisenhart et al.'s study is that this teaching moment could have been much more powerful had Ms. Daniels possessed a deeper conceptual and procedural understanding of the topic. In sum, the evidence discussed here highlights the benefits of promoting both a conceptual and a procedural understanding of division with fractions when developing preservice teachers' SCK of this topic.

The Present Study

Given the empirically supported links between teachers' sound mathematics knowledge and student performance in mathematics (Ball et al., 2005), the objectives of the present study were twofold. The first objective was to examine changes in preservice teachers' SCK of division with fractions (SCK-DF) after having received content-focused instruction on the topic. Based on the findings from Tirosh (2000), it was hypothesized that, in general, preservice teachers' SCK of this topic would be enhanced. A second objective was to determine which type of mathematical content (i.e., conceptually-focused instruction or a combination of conceptual and procedurally-based instruction) promotes greater gains in SCK. In line with the theoretical model presented in Figure 4, it was also hypothesized that the preservice teachers receiving the combination of conceptual and procedural content during the instruction sessions would demonstrate greater gains in their development in SCK-DF compared to participants in the conceptual condition only.

CHAPTER 3: METHOD

The present study had two objectives. The first objective was to examine whether content-focused instruction addressing division with fractions would enhance preservice teachers' specialized content knowledge of this topic. In other words, would preservice teachers' SCK-DF develop as a result of having received content-focused instruction on this topic? The second objective was to examine the impact of two types of content of instruction (i.e., conceptual content, and a combination of conceptual and procedural content) on preservice teachers' gains in SCK-DF. That is, would the preservice teachers who receive instructional content that emphasizes a combination of conceptual and procedural explanations experience greater gains in their development of SCK compared to those only receiving conceptual content?

Research Design

To address these research questions, a quasi-experimental design was used. In particular, the participants were assigned to one of two groups, and I provided each group with distinct types of content during instruction that addresses the topic of division with fractions. Type of content was the independent variable. Specifically, the instructional content for the control group's instructional content involved employing mathematical concepts to explain the standard algorithm for division with fractions problems and focused on of the various ways division with fractions can be interpreted. The instructional content for the treatment group involved using mathematical concepts and procedures to meaningfully justify both traditional and alternative algorithms used to solve division with fractions problems.

The measure of preservice teachers' SCK was the dependent variable. The preservice teachers' SCK –DF was measured using a pretest and posttest audio-recorded interview. In addition, at the end of each instruction session, I administered a paper and pencil test to measure participants' understanding of the instruction. This assessment was incorporated as a means of treatment integrity, ensuring that the participants' understanding of the instruction was consolidated and therefore contributed to the outcome measure. A representation of the research design may be viewed in Figure 5.

Participants

I used a convenience sampling technique to recruit 8 undergraduate students in the Early Childhood and Elementary Education program (i.e., preservice teachers) at Concordia University. At the time of testing, the participants were registered in a mathematics methods course (EDUC 388/4, Teaching Mathematics II) for the Winter 2007 session. The majority of the sample was in their 3rd or 4th year of undergraduate study and completed EDUC 387/2 (Teaching Mathematics I) in the Fall 2006 semester. Participants varied in age and previous teaching experience. The instructor of the Teaching Mathematics II course, Dr. Helena Osana, was contacted in order to obtain permission to recruit the students in the course on a voluntary basis. Dr. Osana is an educational psychologist whose research focuses on the relationship between student cognitions and elementary teachers' thinking and practice. One aspect of the mathematics methods course is to enhance teachers' understanding of mathematical concepts and associated classroom practices. Based on this notion, preservice teachers are provided with an instruction of elementary mathematics that emphasizes content knowledge, understanding of student learning, and instructional design (Concordia University, Department of Education, 2006).

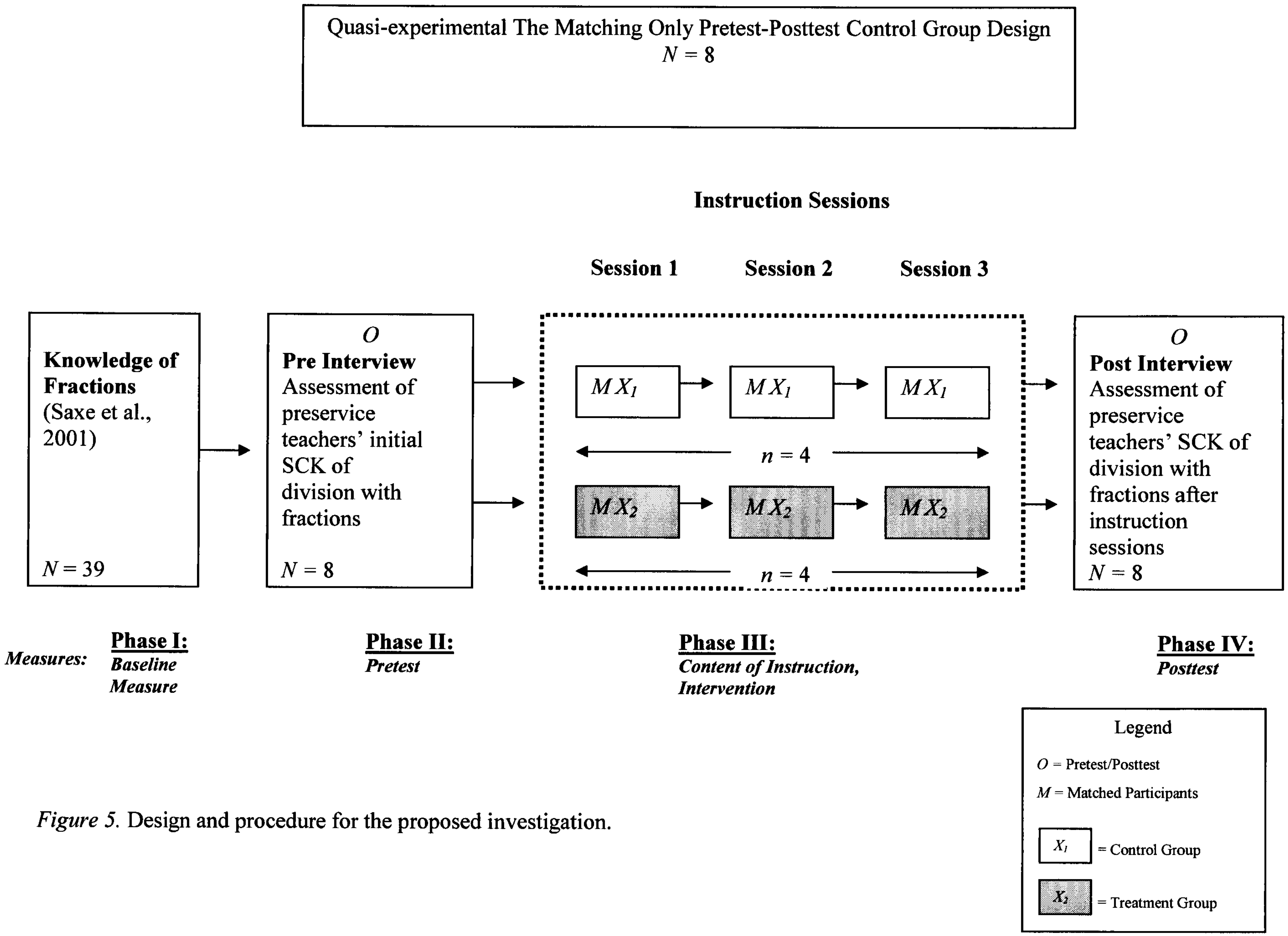

Figure 5. Design and procedure for the proposed investigation.

Measures of Preservice Teachers' Understanding of Fractions

General Knowledge of Fractions

All students enrolled in Dr. Osana's course were required to complete the Knowledge of Fractions Assessment (KFA; see Appendix A). This test was previously developed by Saxe et al. (2001) for elementary students and was used in Rayner et al. (2007) to measure preservice teachers' conceptual and procedural knowledge of fractions. On the KFA, computational accuracy with respect to the following topics of fractions are indices of procedural knowledge: (a) computing the answer to addition and subtraction fraction problems, (b) filling missing numerators of equivalent fractions, and (c) symbolically representing a fraction that reflects a picture of a part-whole representation of a fractional quantity. In addition, Saxe et al.'s measure of students' conceptual understanding of fractions on the KFA uses the accuracy of answers to test items that assessed estimation and problem solving skills.

Saxe et al.'s (2001) KFA was validated using a sample of upper elementary students. Specifically, confirmatory factor analysis was used to distinguish conceptually-orientated and procedural test items. A three-factor model was used: the first was a measure of general knowledge of fractions, the second was a measure of procedural skills, and the third was a measure of conceptual understanding. Results of the confirmatory factor analysis demonstrated strong support that the two sets of items are indices of independent areas of competence. Specifically, the confirmatory factor analysis using the students' responses to the posttest indicated that all fit indices were high (Bentler-Bonett NFI = .984, Bentler-Bonett NNFI = .985, CFI = .994). Similarly, for the pretest, the best fit indices were high (Bentler-Bonett NFI = 0.981, Bentler-Bonett NNFI = .979, CFI = .992).

Cronbach's alpha indicated internal consistency for each scale. For the conceptually-orientated scale, the indices were .73 (pretest) and .83 (posttest); for the computational scale, the indices were .86 (pretest) and .87 (posttest; Saxe et al., 2001).

The participants' scores on the KFA may be viewed in Table 1. Given that this test was created for and validated using upper elementary students, preservice teachers' performance on the test will not be used in the analyses. Rather, this test was used as a means of identifying preservice teachers' initial mathematical abilities with respect to the topic of fractions. An awareness of the preservice teachers' initial understanding of fractions enabled me to incorporate variance in the participants' mathematical abilities within each group, similar to a typical classroom, as well as allowed me to match mathematical abilities across the two groups. Moreover, Tirosh (2000) found that the preservice teachers at the beginning of the mathematics methods course demonstrated misconceptions of division with fractions similar to elementary students, suggesting that this assessment tool has face validity for the purpose of measuring preservice teachers' initial understanding of fractions.

Specialized Content Knowledge

In line with the theory of mathematical knowledge for teaching espoused in Ball et al. (2005; also see Hill et al., 2005), pre and post semi-structured interviews were audio-recorded to assess any changes in the preservice teachers' specialized content knowledge of division with fractions subsequent to their participation in the instruction sessions. As I previously mentioned, Ball et al. proposed that SCK is made up of three primary skills: (a) an ability to analyze students' errors, (b) the capacity to strategically represent numbers and manipulatives that are in line with a given instructional goal, and (c) a facility in explaining

Table 1

The Proportion of the Preservice Teachers' Correct Responses on the KFA (N = 8)

Participant ID	KFA Procedural	KFA Conceptual	KFA Total
CG-1	64.70%	66.67%	65.52%
CG-2	94.12%	100.00%	96.55%
CG-3	100.00%	91.67%	96.55%
CG-4	82.35%	75.00%	79.31%
TG-5	100.00%	83.33%	93.10%
TG-6	100.00%	83.33%	93.10%
TG-7	94.12%	50.00%	75.86%
TG-8	88.26%	75.00%	82.76%

Note. CG-1, CG-2, CG-3, and CG-4 were the participants in the control group and TG-5, TG-6, TG-7, and TG-8 were the participants in the treatment group.

the meaning behind an algorithm. Although the CKT-M (Hill et al., 2005) has been shown to measure teachers' common content knowledge and specialized content knowledge, the questionnaire items extend to topics in elementary mathematics in addition to division with fractions. Given that the content of the instruction sessions in the present study will only address the topic of division with fractions, a general assessment the participants' SCK was not appropriate. Therefore, rather than use Hill et al.'s (2005) CKT-M, I have developed a series of interview items and propose that they measure preservice teachers' SCK of division with fractions. The pre-interview items may be viewed in Appendix B and the post-interview items are presented in Appendix C.

Conducting interviews as oppose to administering a questionnaire was justified because the small sample allows for the collection of rich interview data and these data can be analyzed using both qualitative and quantitative methods. Furthermore, little is known regarding the development of preservice teachers' SCK, and therefore a data collection method that lends itself to qualitative analyses is warranted and may provide specific hypotheses for future research.

As can be seen in Appendix B and Appendix C, the interview items presented fictitious elementary students' incorrect solutions to a series of division with fractions problems (see Tirosh, 2000). In particular, some of the items illustrated that the student incorrectly matched a word problem to the symbolic representation of a division with fractions problem. Other items demonstrated a student incorrectly using the division with fractions algorithm when solving symbolically represented problems. Based on these solutions, participants were asked to identify the source of the student's misunderstanding and propose a follow-up problem that would strategically highlight the misconstrued concepts or procedures in a more meaningful manner.

The last item presented a word problem that reflected the measurement model of division with fractions (e.g., how many groups of $\frac{4}{7}$ are in 12; see Figure 1) and a number sentence. In the context of this specific item, the participants were requested to use the structure and content of the word problem to discuss how they would explain the division with fractions algorithm to their students. The measurement division with fractions model was used because compared to the other two interpretations (i.e., partitive model and factors and products model), only measurement division with fractions word problems comprise actions that mirror the steps carried out when performing the traditional algorithm. The same question was applied to the number sentence. That is, the participants were asked to use the problem $\frac{5}{12} \div \frac{3}{8}$ to justify the standard algorithm.

Assessment of Content

At the end of every instruction session, 20 minutes were dedicated to administering a short written assessment of the participants' understanding of the content (see Appendix D). Prior to discussing the specific problems, it is important to note that the goal of each instruction session was to promote the preservice teachers' content knowledge of division with fractions to examine how improving mathematical content knowledge fosters their SCK. The content covered during instruction, therefore, did not teach to aspects of SCK. I developed the assessments of content to ensure that the preservice teachers consolidated the content of the instruction, and therefore, based on this objective, the problems reflected the content of the instruction and not aspects of SCK. The content of the instruction may be viewed in Appendix E and is based on the literature that has illustrated various ways of understanding division with fractions and the algorithms used to solve these types of problems (Ma, 1999; Star & Seifert, 2006; Tirosh, 2000; Wu, 1999a).

For the control group, the assessments of content entailed presenting a classroom scenario, similar to the one in Ma's study (1999), to assess the participants' conceptual understanding of the three interpretations of division with fractions. That is, participants were given a division with fractions number sentence (e.g., $9 \div \frac{3}{4}$) and were subsequently asked to write a real-world situation to match the symbolic representation. In addition, the control group's assessment of content for all three instruction sessions comprised the same format, but the real-world situation that they were requested to record reflected the interpretation of division with fractions that was covered on that day. That is, at the end of Session 1, the participants were requested to develop a real-world situation that reflected the measurement model. At the end of Session 2, the participants were requested to develop a real-world situation that reflected the partitive model, and at the end of Session 3, the participants were requested to develop a real-world situation that reflected the factors and product model.

Given that the treatment group also received an instruction session that emphasized the conceptual underpinnings of the division with fractions algorithm using the measurement model, the assessment of content for Session 1 mirrored the one that was administered to the control group. The assessment of content administered at the end of Session 2 comprised two division with fractions problems presented in symbolic form, whereby the participants in the treatment group were asked to use those numbers sentences to explain why $\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$ (Wu, 1999a). The assessment of content distributed at the end of Session 3 to the treatment group asked participants to use a different algorithm and re-solve two division with fractions problems (see Star & Seifert, 2006).

Procedure

Dr. Helena Osana was contacted in order to obtain permission to recruit the students in her class. Subsequent to her approval, I requested that she make an announcement in class briefly describing the study and to invite the students who were interested in participating to fill in their contact information on a sign-up sheet. Further, Dr. Osana explained to the students that those who added their name and contact information to the list did not guarantee their participation in the study; that is, when I individually contacted them, they were free to decline their participation. In the remainder of this section, each phase of the research design presented in Figure 5 will be described in detail.

Phase I: Sample Selection

Subsequent to ethical approval, and approximately 2 months into the winter semester, all students enrolled in EDUC 388/4 were requested to complete Saxe et al.'s (2001) knowledge of fractions assessment prior to receiving any instruction on fractions. Given that the test has an important pedagogical function, it was a mandatory part of the course and was therefore administered during the first half of one of their classes. Prior to the administration of the test, I explained to the preservice teachers that they would be given 30 minutes to complete the test. Shortly after the 30 minute period, I made a copy of the tests that belonged to the preservice teachers who completed the sign-up sheet mentioned above and then returned their original test back to them. The preservice teachers who did not write their name on the sign-up sheet also kept their original test, but no copies of their test responses were made. All of the preservice teachers received the answers to the problems, allowing them to use the test as a review for an upcoming examination in the course. For ethical reasons, I administered the test; Dr. Osana was not present during the administration of the test nor when the original tests were returned to the preservice teachers who were interested in participating.

Following the administration and scoring of the Saxe et al. (2001) assessment, I individually contacted the preservice teachers who placed their name on the sign-up sheet. At that time, I reviewed the details of the study to the students (e.g., dates and duration of the instruction sessions and duration of the interviews) as well as the details of compensation[5]. It was also made clear that participation was voluntary and would therefore not influence their grade in the class. In fact, the participating preservice teachers were also informed that Dr. Osana would not be aware of who would be involved in the study. Eight of the students who filled their contact information on the sign-up sheet agreed to participate.

Phase II: Pre-interview

Following this, I conducted individual interviews with the 8 preservice teachers who were able and willing to participate in the study. The pre-interview measured their specialized content knowledge of division with fractions at the onset of the study. During this interview, the participants' responses were recorded using a digital audio-recording device. Further, participants were administered a consent form (see Appendix F) that would clearly stipulate the following information: (a) participation in the research project was completely voluntary, (b) participants would not receive grades on the practice

[5] The compensation was administered in the form of a draw, and the compensation items were two kits of mathematics manipulatives.

problems to be completed during the sessions, (c) whether or not participants decide to participate in the study would not influence the grade they receive in EDUC 388/4, (d) participants should not inform Dr. Osana of their participation until after the their final course grade had been submitted, (e) participants must use their student identification numbers in place of writing their names on the practice problems, and (f) at anytime during or after their participation in the study, they had the right to withdraw from the study. After the interview, participants were asked not to discuss the content addressed during the interview and instruction sessions with other students.

It is important to note that prior to the pre-interview, all of the preservice teachers received instruction of division with fractions during the mathematics methods course. The content of this instruction was also included in Session 1 for both groups.

Phase III: Content-focused Instruction

Following the initial interviews, the participants were randomly assigned to one of the two groups and the test scores from the KFA were used to ensure that the mathematical ability of each student in the control group was matched as closely as possible to a specific student in the treatment group (see Table 1). A summary of the content addressed during the three content-focused instruction sessions for the control and treatment group may be viewed in Figure 6.

I prepared the content of the instruction for all of the sessions prior to conducting the study. I conducted the instruction sessions for three consecutive weeks. The instructional sequence for the control group was as follows. In Session 1, the participants were instructed on the conceptual underpinnings of the traditional algorithm using the measurement model of division. In Session 2, concepts underlying the partitive model of

Session #	Control Group (*n* = 4)	Treatment Group (*n* = 4)
1	Measurement Model	
2	Partitive Model	Mathematical Interpretation
3	Factors & Product Model	Common Denominators Complex Fractions

Figure 6. Content that was addressed during the instruction sessions.

division with fractions were addressed, and in Session 3, concepts tied to the product and factors model of division with fractions were examined. The instructional sequence for the treatment group was as follows. In Session 1, the participants were instructed on the conceptual underpinnings of the traditional algorithm using measurement model of division. In Session 2, the mathematical explanation of the traditional algorithm and certain general properties of mathematics that may also be employed when solving division with fractions problems were explored. Finally, in Session 3, the application of all alternate algorithms was examined. Irrespective of group, each instruction session lasted 1 hour and 15 minutes, whereby the initial 55 minutes was dedicated to instruction and the final 20 minutes was used to complete the appropriate assessment of content measure.

In addition, the assessments of content that were administered to both groups at the end of instruction Session 1 were evaluated and returned to them at the beginning of instruction Session 2. This procedure was repeated once for the assessment of content that was administered at the end of instruction Session 2. With respect to the assessment of content that was administered at the end of instruction Session 3, the original copy and feedback was returned to each of the participants at the beginning of the post-interview. I made copies of their responses and returned the original assessment of content to each of the participants. None of the preservice teachers demonstrated any misconceptions of the content that was addressed during the instruction sessions. All copies of the participants' assessments of content were retained and stored in a locked filing cabinet, along with the signed consent forms.

Phase IV: Post-interview

After the three instruction sessions, I contacted the participants in each group and requested their participation in the final stage of the data collection, the post-interview. While the post-interview items were consistent with the structure and format of the items from the pre-interview, the numbers that were used in the pre-interview items differed from the numbers that were used in the post-interview items. The post-interview was also audio-recorded for subsequent analyses.

Upon completion of the data collection, the PowerPoint instruction slides and practice problems were e-mailed to the participants. Included in this e-mail was a brief description of the results and the identification of the two participants whose names were randomly picked to receive compensation. Finally, because the study terminated after the mathematics methods course, only the PowerPoint instruction for Session 1 was posted on the on-line course conference to allow all students (i.e., those that did not participate or attend the instruction sessions) access to this information. The decision to post the Powerpoint slides and assessments of content after as opposed to during data collection was to prevent participants from one group to view the alternate groups' instructional content.

CHAPTER 4: ANALYSIS AND RESULTS

The current study was designed to address two research questions. The first research question was concerned with the examination of whether content-focused instruction addressing division with fractions enhanced the preservice teachers' specialized content knowledge of division with fractions (SCK-DF). To answer this question, the data were analyzed in two ways. First, each participant's score on the pre-interview was compared to his or her score on the post-interview. Following that, the development of SCK-DF was examined, using descriptive analyses, as a function of area of SCK (i.e., analyzing student solutions, representing numbers and operations in meaningful ways, and justifying mathematical rules and procedures) in order to determine how each of the three areas developed as a consequence of the content-focused instruction.

The second research question addressed whether and how differences in gains in SCK-DF were the result of the type of content of instruction. In order to address this research question, I examined group differences in SCK-DF quantitatively as well as qualitatively. For the quantitative analyses, two Mann-Whitney *U* tests were performed on the participants' ranked pre- and post-interview scores. One Mann-Whitney *U* was used to analyze group differences in participants' final interview score on the pre-interview, and a second Mann-Whitney *U* was used in the same way for participants' post-interview scores.

Added to that, I analyzed the group differences qualitatively whereby the participants' performance on both pre- and post-interviews was described as a function of group. With respect to the qualitative comparison of the groups' development of SCK-

DF, in this chapter I will discuss each interview item[6] and the total interview scores by group on the pre- and post-interview. In addition, I will also report the frequency with which the participants used Division Language and demonstrated Misconceptions of Division with Fractions (MDF) by group. In line with the ethical guidelines concerned with participants' anonymity, the participants will henceforth be referred to using an identification number (i.e., CG-1, CG-2, CG-3, CG-4, TG-5, TG-6, TG-7, and TG-8)[7].

It is important to note that for both research questions, the interview data will be described in this chapter in three sections, each of which accords to one of the three areas of SCK. That is, Section I (i.e., items 1a through 5a) will describe the items that corresponded to analyzing student solutions, Section II (i.e., items 1b through 5b) will describe the items that addressed representing numbers and operations in meaningful ways, and Section III (i.e., items 6 and 7) will describe the items that reflected justifying mathematical rules and procedures.

Data Preparation

Subsequent to conducting the interviews, I developed a coding sheet (see Appendix G), on which the responses were transcribed, to prepare the data for analysis. That is, rather than transcribe the participants' interview responses in a sequential fashion, a list of coding ideas tailored to each item were used as a guide to organize the participants' responses. In general, for the items that corresponded to analyzing the hypothetical student's solution (i.e., Section I, items 1a through 5a), the list of coding ideas comprised: (a) addressing the concept of division, (b) identifying the error

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[6] Please note that for clarity of argument, I will not discuss the interview items in the same order as they appear on the pre- and post-interview.
[7] Note that CG refers to the participants in the control group and TG refers to the participants in the treatment group.

displayed by the student, and (c) identifying an alternate error displayed by the student. With respect to the items that addressed the participants' ability to represent numbers and mathematical ideas in meaningful ways (i.e., Section II, items 1b through 5b), the list of coding ideas comprised: (a) any misconceptions of division with fractions that the participant demonstrated and (b) the participant's follow-up problem that was designed to address the student's misconceptions of division with fractions. Finally, for the last two items that required the participant to justify the invert and multiply algorithm using the context of the problem provided (i.e., Section III, items 6 and 7), the list of coding ideas comprised: (a) addressing the use of multiplication in the standard algorithm, (b) addressing the meaning behind inverting the divisor, and (c) addressing the remainder. Each interview item and the corresponding list of coding ideas are presented in the coding sheet (see Appendix G).

Coding and Scoring

Research Question 1

Subsequent to transcribing the interviews and organizing the transcripts by coding idea, each participant received an SCK-DF numerical score (i.e., SCK-DF Item Score) for each item on the interview. The scores were assigned using a rubric that is presented in Appendix H. In general, an SCK-DF Item Score of 0 indicated that the participant's response to the item in question was inappropriate. In line with this reasoning, an SCK-DF Item Score of 2 indicated an accurate response to the item and a SCK-DF Item Score of 1 was indicative of an answer that was partially correct. Each participant's SCK-DF Item Scores were summed, and the sum of these scores was the participant's Total SCK-DF Score.

Data Reduction

Following the coding and scoring of the data, I calculated several levels of scores from the interviews. A diagram describing these calculations may be viewed in Figure 7. That is, for each of the 8 participants, there is a Total SCK-DF Score (ranging from 0 to 24), or the total sum of the scores received from each of the 12 items on the interviews. I calculated a group score for each item referred to as Group SCK-DF Item Score (ranging from 0 to 8). Finally, the Total Group SCK-DF Score represents the sum of the participants' Total SCK-DF Scores that I summed as a function of group (ranging from 0 to 96).

Research Question 2

Following that, I used the Analysis of Specialized Content Knowledge of Division with Fractions (ASCK-DF), a rubric created to assign codes to the participants' statements. In particular, the ASCK-DF comprised two types of codes: (a) Student Codes, which reflected the hypothetical students' involved in the classroom scenarios, and (b) Participant Codes, which reflected the participants' misconceptions of division with fractions and their use of Division Language. Together, these codes served as a means for qualitatively describing the group differences in SCK-DF on the pre- and post-interview (i.e., addressing the second research question). Please refer to Appendix I for a detailed description of the ASCK-DF.

Student Codes for the Classroom Scenarios

The Student Codes varied as a function of section. That is, the codes for Section I reflected children's common misconceptions of division with fractions identified by previous research (see Tirosh, 2000). In particular, the misconception codes addressed (a)

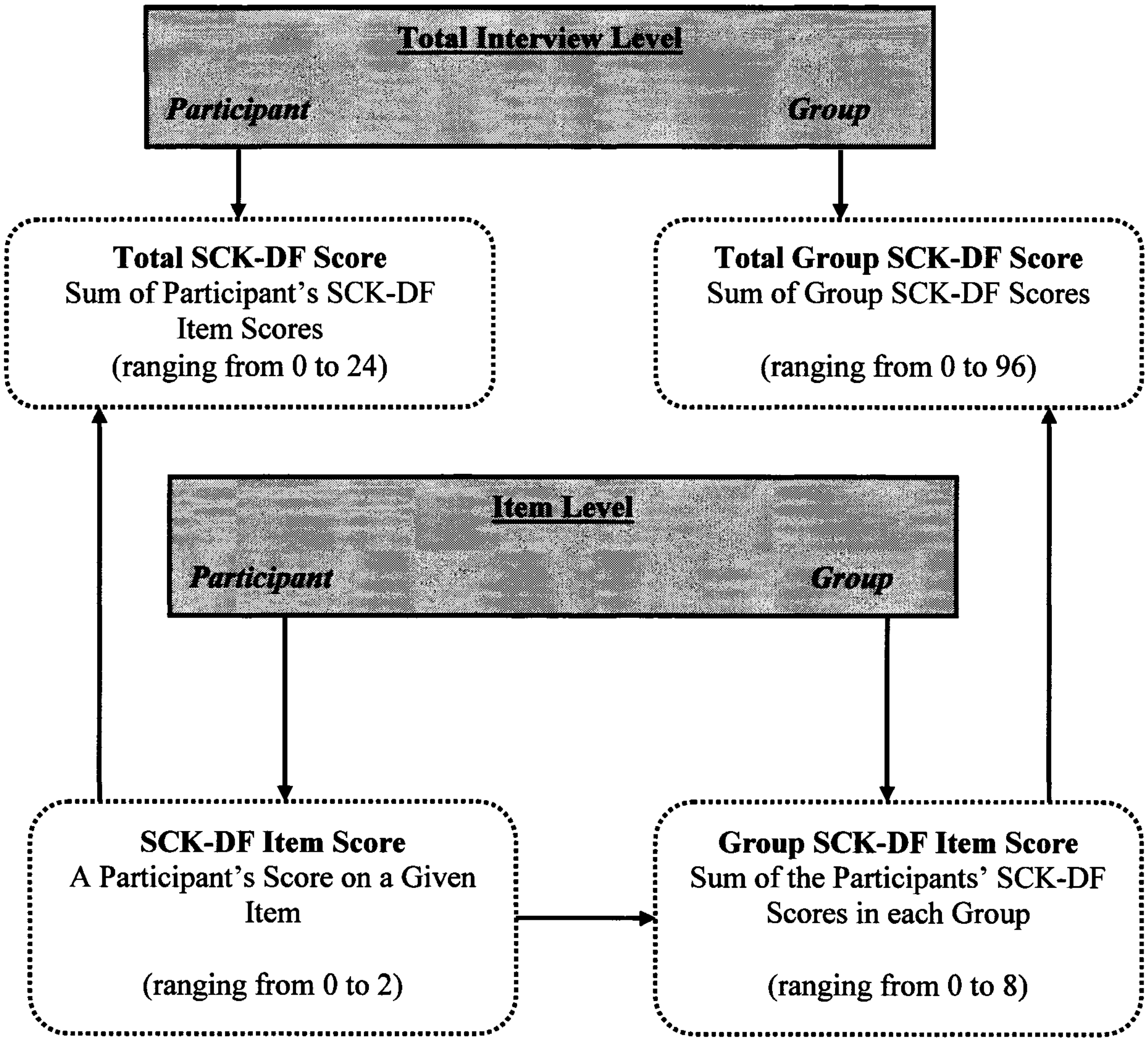

Figure 7. A description of the various interview score calculations.

algorithmically-based errors (e.g., inverting the dividend instead of the divisor); (b) intuitively based mistakes (e.g., the value of the divisor cannot be larger than the value of the dividend); (c) mistakes based on procedural knowledge (e.g., division is commutative); and (d) mistakes based on conceptual knowledge (e.g., confusing the concept of division with the concept of multiplication). Finally, in situations where the participant did not provide any analysis of the student's error, the No Error Analysis code was used. Some of the Student Codes for this section were also generated from the interview data themselves.

The Student Codes for Section II derived from the interview data and primarily addressed whether the participant's goal was to improve the student's conception of division with fractions or provide the student with a more effective solution strategy. In both cases, the pedagogical method that the participant suggested was coded. In other words, I was concerned with the materials that the participant wanted to use (e.g., pictures, manipulatives, or symbols), whether the participant offered more than one method of correcting the misconception or problem, and whether he or she commented on the connection between the student's misconception of division with fractions and other operations or mathematical concepts and procedures.

Similar to Section II, the Student Codes for Section III also emerged from the data. Specifically, in addition to a participant's ability to justify the standard algorithm, I was concerned with: (a) whether the participant volunteered multiple methods of justifying the algorithm, (b) whether his or her method was not addressed during the instruction sessions and thus was invented by the participant, (c) whether the participant used knowledge of concepts of division with fractions or knowledge of procedures associated with this topic, and (d) whether the method used on item 6 was also applied to item 7, thus indicating an ability to flexibly apply the same method to distinct contexts.

Participant Codes Reflecting Participants' MDFs and Use of Division Language

In general, the Participant Codes did not differ as a function of section. More specifically, the same codes were used for all of the items except on items 6 and 7, for which some codes reflecting the participant's misconceptions in his or her justification of the algorithm were added. For example, on item 7, several participants identified an incorrect number of groups in his or her pictorial model of the algorithm, and failed to recognize that as a consequence, the model did not reflect the answer to the problem.

Overall, the MDF codes in all three sections were identical to the Student Codes for Section I. Some additional misconceptions, however, that were not among the Section I Student Codes emerged from the data. As such, some additional Participant Codes were added to the rubric. Some participants, for instance, confused the action of division with the act of partitioning an area. Moreover, a few participants incorrectly matched a number sentence to a word problem that was presented during the interview, situating the divisor as the dividend.

In addition to the participants' MDFs, the use of Division Language was also coded. The Division Language codes reflected the participants' use of formal mathematical vocabulary to communicate concepts and procedures associated with division with fractions (see ASCK-DF). According to Zazkis (2002), the notion of formal vocabulary refers to mathematical language that communicates a specific mathematical concept or procedure. In line with this notion, I coded various types of Division Language to reflect whether Division Language was used as a means of communicating terminology, concepts, or procedures associated with division and division with fractions (e.g., size of one group, invert and multiply, or quotient).

RESULTS

Research Question 1: The General Development of SCK-DF

A comparison of the pre- and post-interview SCK-DF Item Scores may be viewed in Table 2. In general, all but 1 participant demonstrated a higher Total SCK-DF Score on the post-interview compared to the pre-interview. Specifically, CG-3's Total SCK-DF Score was 1 point lower on the post-interview compared to CG-3's pre-interview Total SCK-DF Score. What had a significant impact on her reduced Total SCK-DF Score on the post-interview was the high number of items whose score remained unchanged (i.e., 9). In addition, she was one of the participants with a limited number increased SCK-DF Item Scores on the post-interview (post-interview SCK-DF Item Score was increased for 4 items on the post-interview), suggesting a relatively small improvement in SCK-DF from the pre-interview to the post-interview compared to the other participants. The other participant (i.e., CG-4) who also received a score increase on 4 items demonstrated only a slightly higher Total SCK-DF Score on the post-interview. Indeed, her score pattern indicated that of the 12 items, the scores of 4 items increased, of 3 items decreased, and 5 items remained the same. At the other end of the spectrum, TG-8 displayed a score increase on 9 items on the post-interview. As for the other participants, CG-2 and TG-5 demonstrated a score increase on 6 items and CG-1, TG-6, and TG-7 each experienced a score increase on 5 items. Thus, overall, the preservice teachers' improvement on the post-interview suggests that their SCK-DF was enhanced following the content-focused instruction.

Table 2

The Participants' SCK-DF Item Scores Compared as a Function of Time ($N = 8$)

Participant ID	Interview Section									
	Section I: Analyzing Student Solutions									
Item	1a		2a		3a		4a		5a	
Interview	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
CG-1	1	1	2	2	2	0	2	2	0	1*
CG-2	1	1	2	2	2	2	1	2*	2	2
CG-3	2	1	2	2	2	2	2	1	2	1
CG-4	0	1	1	1	1	2*	2	2	2	2
TG-5	1	2*	1	2*	2	2	1	1	2	2
TG-6	0	2*	2	2	0	2*	2	2	2	2
TG-7	1	2*	2	2	2	2	0	1*	2	1
TG-8	1	0	0	2*	1	2*	1	1	2	1
	Section II: Representing Numbers and Operations in Meaningful Ways									
Item	1b		2b		3b		4b		5b	
Interview	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
CG-1	0	2*	0	0	2	1	1	0	1	2*
CG-2	2	1	1	2*	2	2	0	1*	1	2*
CG-3	1	1	1	2*	2	2	1	1	1	1
CG-4	0	0	2	1	0	1*	2	0	2	1
TG-5	0	2*	2	2	2	2	0	1*	1	2*
TG-6	0	2*	0	2*	1	1	1	1	2	2
TG-7	1	1	2	2	2	2	1	2*	1	2*
TG-8	1	2*	1	2*	1	2*	1	2*	1	2*
	Section III: Justifying Mathematical Rules and Procedures									
Item	6		7							
Interview	Pre	Post	Pre	Post						
CG-1	0	2*	0	1*						
CG-2	1	2*	1	1						
CG-3	1	1	0	1*						
CG-4	0	2*	0	0						
TG-5	0	2*	0	0						
TG-6	2	2	0	1*						
TG-7	2	2	0	2*						
TG-8	1	2*	1	2*						

Note. The values with an * indicates a post-interview SCK-DF Item Score increase.

Based on the data from Table 2, I determined how each of the three areas of SCK improved following the content-focused instruction. In particular, irrespective of group, I calculated the mean number of participants whose item scores increased on the post-interview (please see note in Table 2). Please refer to Figure 8 for the mean number of participants whose SCK-DF Item Scores increased on the post-interview compared as a function of section. The calculations indicated that the highest mean participant score increase was in Section III, whereby an average of 5 of the 8 participants experienced score increases. Following that, in Section II, an average of 4 (specifically, $M = 3.80$) of the participants demonstrated greater scores on the post-interview. The area of SCK that benefited the least from the content-focused instruction was Section I, (analyzing students solutions), whereby an average of 2 participants (specifically, $M = 2.20$) displayed higher scores on the post-interview. In summary, these data indicate that the majority of the participants experienced an overall increase in SCK-DF subsequent to receiving content-focused instruction and the area of SCK that benefited most was the ability to justify the standard algorithm.

Research Question 2: Group Differences in the Development of SCK-DF

Quantitative Analyses

The Total SCK-DF Scores and the Total Group SCK-DF Scores for the pre- and post-interviews may be viewed in Table 3. With respect to the research question pertaining to the differential impact of the two types of content of instruction (i.e., conceptual content, and a combination of conceptual and procedural content) on preservice teachers' gains in SCK of division with fractions, two Mann-Whitney *U* tests

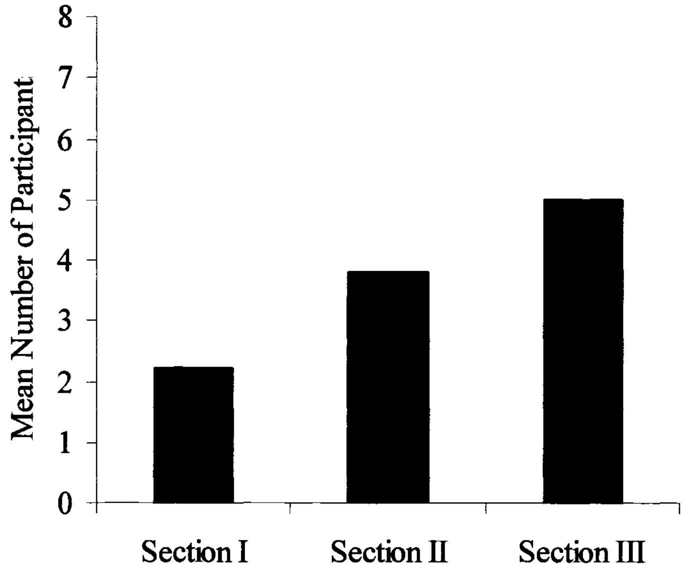

Figure 8. Section comparison of the mean number of participants who improved their SCK-DF Item Score on the post-interview.

Table 3

A Comparison of Total SCK-DF Scores and Total Group SCK-DF Score as a Function of Time (*N* = 8)

	Pre-interview	Post-interview
Participant ID	Total SCK-DF Score (*M*)	Total SCK-DF Score (*M*)
Control Group		
CG-1	11	14
CG-2	16	20
CG-3	17	16
CG-4	12	13
Total Group SCK-DF Score	56 (14.00)	63 (15.75)
Treatment Group		
TG-5	12	20
TG-6	12	21
TG-7	16	21
TG-8	12	20
Total Group SCK-DF Score	52 (13.00)	82 (20.50)
Total	108 (13.50)	145 (18.13)

were performed. The critical alpha level of .05 was used as the criterion for both tests.

Pre-interview Group Differences

The participants' original Total SCK-DF Scores on the pre-interview were ordered, and then ranked. Following that, a Mann-Whitney *U* test was used to compare the ranks for the control group ($n = 4$) to the treatment group ($n = 4$). The results indicated that the median pre-interview score was 12.00 and the interquartile range (IQR) was 4.00. The results from the Mann-Whitney *U* indicated no significant difference between groups on the pre-interview, $U = 7.00$ (corrected for ties $U = 7.50$), $p = .886$.

Post-interview Group Differences

The participants' original Total SCK-DF Scores on the post-interview were ordered, ranked, and a Mann-Whitney *U* test was used to compare the ranks for the control group ($n = 4$) to the treatment group ($n = 4$). The median post-interview score was 20.00 and the $IQR = 6.25$. The results from the Mann-Whitney *U* indicated a significant difference between groups for the post-interview, $U = 1.00$ (corrected for ties $U = 1.00$), $p = .05$, one-tailed.

Qualitative Analysis of Section I: Analyzing Student Solutions

I will now continue the discussion with a description of *how* the participants and groups' SCK-DF changed as a function of the type of content of instruction. Please refer to Figure 9 for a list of the pre- and post-interview items and expected responses for Section I.

To begin, the pattern of change of Group SCK-DF Item Score was different for both groups in this section (see Figure 10). In particular, on the pre-interview, the control group demonstrated a greater Total Group SCK-DF Score for Section I ($M =$

Item	Pre-interview	Post-interview	Expected Response
1a	$\frac{1}{4} \div 4 = 4 \div 4 = 1$	$\frac{1}{6} \div 6 = 6 \div 6 = 1$	1) Inverting the wrong number and 2) dividing versus multiplying
2a	$\frac{1}{4} \div \frac{3}{5}$ "I can't do this because $\frac{3}{5}$ is bigger than $\frac{1}{4}$, so you can't share less among more"	$\frac{1}{3} \div \frac{4}{7}$ "I can't do this because $\frac{4}{7}$ is bigger than $\frac{1}{3}$, so you can't share less among more"	The divisor can be bigger than the dividend. First half of the statement is correct.
3a	$320 \div \frac{1}{3} = \frac{1}{3} \div 320 = \frac{1}{3} \times \frac{1}{320} = \frac{1}{960}$	$410 \div \frac{1}{2} = \frac{1}{2} \div 410 = \frac{1}{2} \times \frac{1}{410} = \frac{1}{820}$	Misuse of the commutative property and correct application of the standard algorithm.
4a	*Kate's mom bought 6 bars of chocolate for her and her friends. If each person had* $\frac{1}{2}$ *of a bar of chocolate, how many people altogether ate chocolate?* $6 \div 2 = 3$	*Harriet has a paper to write for her political science course. She has set aside 10 hours to write the paper. It takes her* $\frac{1}{2}$ *of an hour to write a page. How many pages can she write in 10 hours?* $10 \div 2 = 5$	The student confused finding how many groups of $\frac{1}{2}$ with splitting 10 in half or distributing 10 among 2 groups.
5a	*Jerry bought* $3\frac{1}{3}$ *pounds of flour for $2.00. How much did he pay per pound?* $2 \times 3\frac{1}{3} = \frac{20}{3}$ or $6.66 per pound	*Nigella used* $4\frac{1}{4}$ *cups of flour to bake 2 lemon meringue cakes for a dinner party. Shelly, one of her dinner guests loved the cake so much she decided to bake one lemon meringue cake the next day. How much flour did Shelly use to bake this cake?* $2 \times 4\frac{1}{4} = \frac{34}{4}$ or $8\frac{1}{2}$ cups of flour	Looking for the size of each group versus a multiplication problem where the number of groups and number in each group is known

Figure 9. A description of the interview items in Section I, and the expected responses.

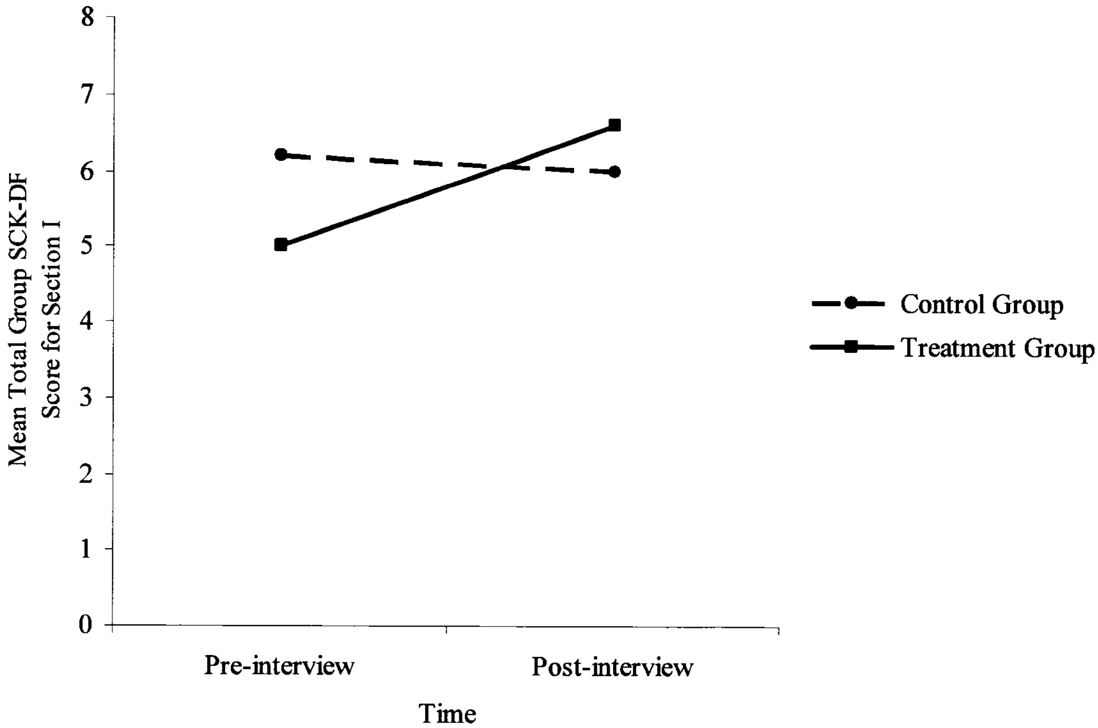

Figure 10. A group comparison of the mean Total Group SCK-DF Score for all the items in Section I.

6.20) compared to that of the treatment group (*M* = 5.00). On the post-interview, however, not only did the treatment group's Total Group SCK-DF Score for Section I improve (i.e., *M* = 6.60), but it also exceeded the control group's Total Group SCK-DF Score for this section (i.e., *M* = 6.00). The mean Total Group SCK-DF Scores for Section I on the post-interview imply that the treatment group experienced an overall increase in SCK-DF while the control group did not. Based on the difference between the groups' mean Total Group SCK-DF Scores for Section I at the onset of the study, however, the treatment group's greater gain in this area of SCK-DF should be interpreted with caution.

A group comparison of the Group SCK-DF Item Scores for the items in this section, as well as the frequencies of Division Language codes and MDFs, are presented in Table 4. In general, both groups demonstrated a relatively low frequency of MDFs for all of the items except item 3a. Further, the treatment group demonstrated a greater decrease in the overall frequency of MDFs on the post-interview compared to the control group.

Despite the groups' difference in the decrease of MDFs on the post-interview, both groups demonstrated an overall increase in the frequency of Division Language that was used in response to the items in this section (i.e., items 1a, 2a, 3a, 4a, and 5a). Indeed, the control group's greatest Division Language frequency increase was demonstrated on items 4a (i.e., frequencies of 2 and 9, respectively) and 5a (i.e., frequencies of 1 and 6, respectively), while the treatment group's greatest Division Language frequency increase was demonstrated on items 1a (i.e., frequencies of 4 and 7, respectively) and 3a (i.e., frequencies of 5 and 10, respectively). It is important to note that items 4a and 5a reflected conceptually-based misconceptions while items 1a and 3a

Table 4

A Group Comparison of Group SCK-DF Item Score, Frequency of Division Language, and Misconceptions of Division with Fraction for Section I at Both Time-points (N = 8)

	Pre-interview		Post-interview	
Variable	Control	Treatment	Control	Treatment
		Item 1a		
Group SCK-DF Item Score	4	3	4	6
Division Language	1	4	1	7
Misconceptions of Division with Fractions (MDF)	1	0	1	1
		Item 2a		
Group SCK-DF Item Score	7	5	7	8
Division Language	1	0	3	4
Misconceptions of Division with Fractions (MDF)	2	1	1	0
		Item 3a		
Group SCK-DF Item Score	7	5	6	8
Division Language	4	5	2	10
Misconceptions of Division with Fractions (MDF)	3	4	4	0
		Item 4a		
Group SCK-DF Item Score	7	4	7	5
Division Language	2	4	9	2
Misconceptions of Division with Fractions (MDF)	0	2	1	0
		Item 5a		
Group SCK-DF Item Score	6	8	6	6
Division Language	1	1	6	2
Misconceptions of Division with Fractions (MDF)	0	0	1	0
Total Group SCK-DF Score for Section I (M)	31 (6.20)	25 (5.00)	30 (6.00)	33 (6.60)

comprised procedurally-based errors, perhaps highlighting a link between the type of content that was addressed during the instruction sessions and the use of Division Language.

Control Group

In general, the control group's Group SCK-DF Item Score for each item in this section did not change from the pre-interview to the post-interview. In fact, the only item where the Group SCK-DF Item Score changed was on item 3a, which decreased by 1 point from the pre-interview to the post-interview. It is important to consider, however, that for the majority of the items on the pre-interview, the Group SCK-DF Item Scores for the control group was high (see Table 4). Specifically, as indicated in Table 4, 3 of 5 the items on the pre-interview received a Group SCK-DF Item Score of 7, and one item received a Group SCK-DF Item Score of 6. Based on the participants' performance on the pre-interview, then, it is logical that more content-focused instruction would only have a minimal positive effect on their final error analysis skills.

Item 1a. One item, however, for which the participants in this group did not collectively perform well on during the pre-interview was item 1a (see Table 4). Moreover, after the instruction sessions, the Group SCK-DF Item Score for this item remained the same. In particular, because CG-3's score decreased on the post-interview and CG-4's increased, there was no change in the overall group score for this item.

Indeed, on the pre-interview, CG-4's statement illustrated no analysis of the student's error. She concluded that:

> CG-4 (1a Pre): Well, first I would have asked her why she came up with this if she had done any little drawings to get a visual of how she actually came up with

> her number sentence to see how she understands it first. That would be the first step that I would do because for me I am very visual so I need to see the actual break down of it. Well she just divided the 4 by the 4 instead of looking at the whole fraction and like drawing it out.

Although it may be argued that her final statement comprised some error analysis, I considered her statement a replication of what was written on the paper. That is, for $\frac{1}{4} \div 4 = 4 \div 4 = 1$, she stated that the student divided 4 by 4 (restating the $4 \div 4$ part of the solution) instead of using the fraction and dividing $\frac{1}{4}$ by 4. The avoidance of analyzing the student's error may reflect her poor understanding of division with fractions.

Added to that, CG-4 confused the action of division (which can either be distributing a total among a number of groups or continuously grouping groups of a certain size from the total) with the act of partitioning, stating that, "Here, for instance, the $\frac{1}{4}$ and then dividing it again by 4, she just divided the 4 directly from the fraction." That is, using a pictorial representation of $\frac{1}{4}$, CG-4 re-partitioned that area into four smaller areas, considering that she was modeling $\frac{1}{4}$ divided by 4. Interestingly, she was not the only participant who displayed this misunderstanding when addressing this item. In fact, in addition to CG-4, 3 participants (i.e., 1 from the treatment group and 2 from the control group) also demonstrated this misconception when addressing how the student's misconception should be corrected.

On the post-interview, however, CG-4 did not display the MDF that was demonstrated in the pre-interview and her SCK-DF Item Score improved because she identified one of the two algorithmically-based errors. Taken together, her performance on the post-interview for this item illustrated some of the progress made in her error analysis skills.

Contrary to CG-4, CG-3's SCK-DF Item Score decreased on the post-interview. Specifically, CG-3 received an SCK-DF Item Score of 2 on the pre-interview, identifying both of the algorithmically-based errors. Unfortunately, CG-3's response to this item on the post-interview only addressed the hypothetical student's general misconception of fractions, resulting in a lower post-interview score for this item.

Although CG-1's SCK-DF Item Score did not change from the pre-interview to the post-interview, her response on the post-interview indicated a slight improvement in her error analysis. That is, on the pre-interview she commented on the fact that the student confused multiplication with division because, according to CG-1, multiplying $\frac{1}{4}$ by 4 closely resembles the solution the hypothetical student represented (i.e., $\frac{1}{4}\times 4=\frac{4}{4}=1$ compared to $\frac{1}{4}\div 4=4\div 4=1$). I considered CG-1's responses to be reasonable, but nonetheless inadequate, given that she did not address the fact that the division symbol remains after the algorithm is applied. On the other hand, in CG-1's response on the post-interview, she stated that the hypothetical student's error was algorithmically-based, noting that the student failed to change the operation from division to multiplication. Her response, however, also demonstrated her own algorithmically-based errors where she stated that correctly performing the algorithm would involve multiplying 6 by 6, illustrating that she inverted the dividend instead of the divisor. She replicated this misunderstanding on item 7 on the pre-interview, where she stated that the standard algorithm involves inverting all fractions.

Item 2a. With respect to items 2a, 4a, and 5a, the Group SCK-DF Item Scores on the pre-interview were high to begin with. As such, there was very little change in the control group participants' responses from the pre-interview to the post-interview for these items (see Table 4). The 3 participants, for instance, who correctly identified the student's misconception of division with fractions on item 2a on the pre-interview were the same 3 participants who did so on the post-interview. Of interest, however, is that in responding to item 2a on the pre-interview, 2 of the 4 participants identified student errors that they themselves demonstrated in their response to the same item. Indeed, CG-4 misunderstood the student's statement, claiming that:

> CG-4 (2a Pre): He doesn't understand the full value of, like, the actual fraction, that even though this might be bigger than, like $\frac{3}{5}$ might be bigger than $\frac{1}{4}$ in actual num[ber] like when you are writing it out you might think that, but when you, but when you divide it into like 5s or whatever, he's not seeing that it could actually be smaller than the first number.

Note that CG-4 believed that the student in 2a misunderstood the concept of a fraction because that student considered that $\frac{1}{4}$ is smaller than $\frac{3}{5}$. Contrary to her analysis, one assumes that the student does indeed understand the concept of a fraction because $\frac{3}{5}$ is in fact bigger than $\frac{1}{4}$. Further, based on her statement, I argue that she displayed the very misconception that she identified in the student, illustrating how a teacher's misconceptions of the content may negatively impact his or her analytical skills of students' solutions, consistent with previous research (van Dooren, Verschaffel, & Onghena, 2002). Moreover, CG-4 continued to display this pattern of identifying a misconception that she herself possessed on the post-interview, on this item and on item 4a. Similarly, on item 2a on the pre-interview, CG-3, stated that:

> CG-3 (2a Pre): I think what he is having trouble seeing is that you are actually taking $\frac{3}{5}$ of $\frac{1}{4}$. You can't share less among more that would be ah, that's more like a subtraction type of thing where you can't take less, you can't take a bigger piece from a smaller piece.

In other words, CG-3 indicated that one of the student's errors was that he or she applied an understanding of subtraction in place of division, and at the same time, she demonstrated an interpretation of multiplication with fractions instead of division with fractions. Interestingly, in addition to identifying this error, CG-3 accurately identified the student's real misconception of division with fractions on the pre-interview. On the post-interview, however, she did not use an interpretation of multiplication with fractions on this item, contributing to the slightly reduced frequency of MDFs on the post-interview for the control group.

Item 4a. With respect to item 4a, the Group SCK-DF Item Score was the same on the pre- and post-interview because CG-2's SCK-DF Item Score increased on the post-interview and CG-3's SCK-DF Item Score decreased by the same amount on the post-interview.

Item 5a. In terms of item 5a, the Group SCK-DF Item Score can be explained by CG-1 and CG-3's post-interview SCK-DF Item Scores. That is, CG-1's response on the pre-interview lacked any analysis of the student's error. Subsequent to participating in the instruction sessions, however, she responded, "He doesn't understand that it is a division problem. But when you break it down with the partitive interpretation, you have another way of seeing it." In contrast, CG-3's score on the post-interview decreased because she considered that the student:

> CG-3 (5a Post): …Knows that when you are supposed to divide, cause sometimes often when we say to children, "well you don't need to divide the fractions, you just need to multiply it by the reverse," sometimes they see fractions that need to be divided and they just multiply it no matter what without reversing the number and without going through the algorithm in their head, without picturing it.

In other words, CG-3 believed that the student failed to write $4\frac{1}{4} \div 2$ and incorrectly wrote $2 \times 4\frac{1}{4}$ because she misapplied the algorithm. This analysis, however, would carry greater weight if the student's incorrect response had been $4\frac{1}{4} \times 2$. Irrespective of this, as indicated in Figure 8, the student demonstrated a misconception of the concept of partitive division, thus explaining why CG-3 did not receive an SCK-DF Item Score of 2 for this response on the post-interview.

Item 3a. Unlike the other items, the Group SCK-DF Item Score on item 3a decreased by 1 point from the pre-interview to the post interview for the control group (see Table 4). On the pre-interview, all of the participants accurately identified that the student's error was the misunderstanding that division is commutative. One of the participants (i.e., CG-4), however, additionally considered that the student demonstrated errors in performing the algorithm. As a matter of fact, CG-4's response not only identified her own misunderstanding of the algorithm, but she revealed on the pre-interview a common misconception of division with fractions that children demonstrate: confusing a whole number divided by $\frac{1}{2}$ with a whole number divided by 2 (Ma, 1999; Tirosh, 2000). Further in this regard, in demonstrating this common misconception, her response illustrated that a possible source of this misunderstanding could be her use of the interpretation of multiplication with fractions as opposed to division with fractions. The following excerpt supports this notion.

> CG-4 (3a Pre): First, I would get him to look at the 320, break that down into $\frac{1}{3}$s to see to show him what they look like, what $\frac{1}{3}$ of 320 is, and then ask him after seeing that if he still thinks that this [the student's solution] is the same. So I would get him first to draw it out into $\frac{1}{3}$s and then look and see if he still thinks it makes sense or not and then we would see from they are depending on what his answer is.

Note that, CG-4 understood $320 \div \frac{1}{3}$ as 320 x $\frac{1}{3}$ because she stated that you are looking for "what $\frac{1}{3}$ *of* 320 is." In addition, her comment regarding breaking down the total into $\frac{1}{3}$s reflects using the partitive model, whereby the number of groups is a whole number (i.e., 3) instead of the fraction that is represented as the divisor (i.e., $\frac{1}{3}$). That is, she distributed the total amount of 320 among 3 groups, as opposed to finding how many groups of $\frac{1}{3}$ make up 320 or that 320 is a $\frac{1}{3}$ of what amount. Added to that, she stated on the post-interview:

> CG-4 (3a Post): If he drew it out, he would see, like, 400 [four flats[8]] and then 1 [one long[9]]. If he has to take $\frac{1}{2}$ of that, he's going to see that he can take half of this [four flats] and this [one long] and then half of this gone [four flats and one long or 410], but he's looking at it like he's taking this [points to the student's solution] and he's just multiplying it across.

Here again, CG-4 distributed the total amount into two groups and was concerned with the size of group; in doing so, she confused $410 \div \frac{1}{2}$ with 410 x $\frac{1}{2}$ (her pictorial solution that matched this response may be viewed in Figure 11). Taken together, CG-4's responses suggested that if the student took a fraction of the whole number, which is the interpretation that is used for multiplication with fractions, the total would be distributed into the correct number of groups, allowing the identification of the size of one group.

Contrary to this notion, the fraction in $320 \div \frac{1}{3}$ and in $410 \div \frac{1}{2}$ represents the size of each group, or a fraction of the total value of one group. CG-4 was not the only participant who demonstrated this particular combination of MDFs. In response to item 3a on the post-interview, CG-1 stated:

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[8] A flat is a square base ten block representing 100 units.
[9] A long is a rectangular base ten block representing 10 units.

$$410 \div \frac{1}{2} = \frac{1}{2} \div 410 = \frac{1}{2} \times \frac{1}{410} = \frac{1}{820}$$

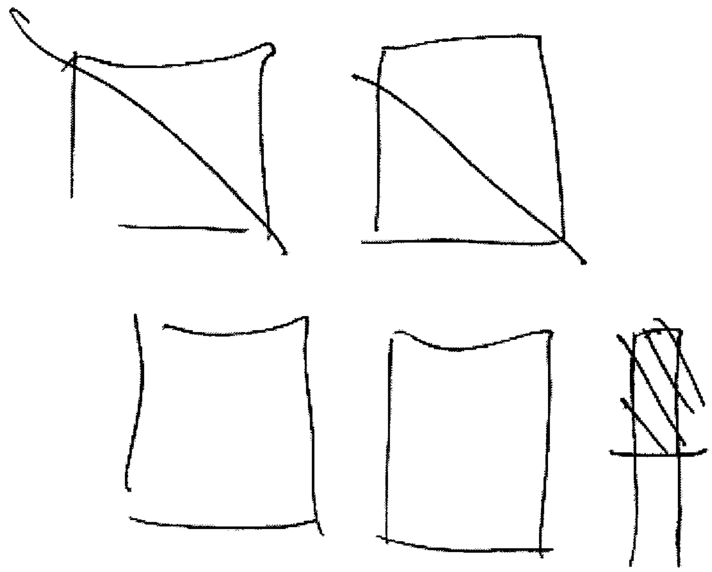


Figure 11. CG-4's pictorial solution on item 3a on the post-interview.

CG-1 (3a Post): When you are given a benchmark like $\frac{1}{2}$, he should know that it is $\frac{1}{2}$ of the whole, so the answer is going to be less [than 410], so 205. So maybe he is not estimating. I know we were taught to look at the numbers, look what they are asking, look at the whole. So his misconception is that he doesn't understand whole and $\frac{1}{2}$.

Note that CG-1 also stated that $410 \div \frac{1}{2}$ is the same as 410 x $\frac{1}{2}$, or knowing what $\frac{1}{2}$ of 410 is. Indeed, on the post-interview, not only did CG-1 confuse a whole number divided by a fraction with a whole number multiplied by a fraction, but she also proposed a different response from the accurate one she offered on the pre-interview, contributing to the control group's lower Group SCK-DF Item Score on the post-interview.

Treatment Group

As I previously mentioned, compared to the control group, the treatment group demonstrated a higher post-interview Total Group SCK-DF Score for Section I (see Table 4). Furthermore, for this group, performance on 4 of the 5 items in this section improved on the post-interview (see Table 4). Interestingly, the 4 items on which the treatment group experienced an improvement in this area of SCK-DF were items that involved procedural knowledge of division with fractions (i.e., items 1a, 2a, and 3a) That is, the error in item 1a addresses two misconceptions of the algorithm: (a) inverting the dividend instead of the divisor, and (b) forgetting to change the operation to multiplication. The error in item 2a highlights a faulty rule of division that elementary students often uphold: the divisor must always be larger than the dividend. On item 3a, the student applied the property of commutativity to division. Although the student error in item 4a was conceptually-based, it addressed the concept of measurement division, a concept of division with fractions that was also emphasized during the treatment group's content-focused instruction. Added to that, the treatment group did not receive any content-focused instruction emphasizing the partitive division model. In line with this, the item where the student error was couched in a partitive division with fractions word problem (i.e., item 5a) was the same item that had a lower Group SCK-DF Item Score on the post-interview.

Item 4a. The score increase from the pre-interview to the post-interview on item 4a was relatively small (i.e., 1 point) compared to the other Group SCK-DF Item Score increases on items 1a, 2a, and 3a. In addition, upon closer examination of the participants' error analysis on item 4a on the pre-interview, it is interesting to note that none of the participants identified the same error, and only 1 participant correctly reported the hypothetical student's error in the problem. The same participant who correctly answered the question on the pre-interview was also the only participant to accurately address this item on the post-interview.

The participant who was responsible for the slight Group SCK-DF Item Score increase was TG-7. In particular, TG-7 was the only participant who did not receive a SCK-DF Item Score of 2 on the pre-interview (her SCK-DF Item Score was 0) and improved her score on the post-interview. Indeed, on the pre-interview she failed to provide any analysis of the student's error. On the post-interview, however, she expressed that the student misunderstands the concept of a fraction, stating, "They are not realizing that fractions again are a world on its own. You can't just take 2 and abandon the 1, because it is not really 2, it's $\frac{1}{2}$." In other words, the hypothetical student applied the denominator in the fraction as a whole number in the problem, changing the place value of the number. The fact that the student was willing to use a number whose place value is different suggested to TG-7 that the student clearly does not possess a relational understanding of the numerator a denominator.

Although TG-8's post-interview response also addressed the student's misconception of fraction concepts, her response suggested that the hypothetical student incorrectly applied one of the three interpretations of a fraction. Based on the fact that TG-8 was more specific when she addressed the student's misconception of fraction concepts, I contend that this was a more sophisticated response compared to the other statements that proposed that the hypothetical student lacked a general understanding of the concept of a fraction. In particular, TG-8 stated:

> TG-8 (4a Post): What is interesting about this word problem, probably done on purpose, is the fact that there is a 1 up here, which probably corresponds to 1 page, and there is a 2 here, which might, to her, correspond to 2 hours. She's saying that for 10 hours she can write 2 pages because for her, I guess she might be taking a guess because for her she sees the 1 page and 1 page every 2 hours. She might be thinking ratio, so she's thinking of fractions as 2 separate numbers. She doesn't see the overall connection between the 2 numbers in a fraction. So that's her major misconception. She sees it as a ratio.

Indeed, TG-8 considered that the student applied a ratio understanding of fractions. In other words, she suspected that rather than consider the fraction as $\frac{1}{2}$ of 1 hour, the student believed that $\frac{1}{2}$ represented the ratio of 1 page to 2 hours, indicating that there are $\frac{1}{2}$ of the number of pages compared to the number of hours. If the student did apply this misunderstanding, then, it is logical that the number sentence would have been $10 \div 2$.

Item 5a. Surprisingly, the treatment group's Group SCK-DF Item Score on item 5a on the pre-interview was 8 and was 6 on the post-interview. Indeed, on the post-interview, 2 of the 4 participants (i.e., TG-7 and TG-8) received a lower score. TG-8's post-interview response, for example, indicated that the student possessed an algorithmically-based misconception of division with fractions. Specifically, she stated:

> TG-8 (5a Post): It is not commutative and I don't know why she is multiplying. You can multiply, but it has to be like this [she writes $4\frac{1}{4}\times\frac{1}{2}$]. I don't know whether she did part of the problem in her head and decided to invert and multiply instead of divide, but she forgot to invert the 2.

Similar to CG-3, although TG-8's response had merit, the student primarily demonstrated a misconception of the concept of partitive division, thus I did not issue an SCK-DF Item Score of 2.

In addition, TG-7 admitted on the post-interview that the student had confused the concept of division with the concept of multiplication. Based on the fact that the student was able to perform the multiplication with fractions algorithm, however, TG-7 concluded that the source of the student's misunderstanding of division with fractions was not mathematical, but rather was a comprehension of the wording used in the problem.

Item 2a. In Section I, the greatest Group SCK-DF Item Score increases were found on items 1a, 2a, and 3a (see Table 4). In addition, the participants in the treatment group displayed fewer MDFs and the majority of the high frequencies of Division Language were displayed on the post-interview by this group and for these three items.

Notably, on items 2a and 3a, the treatment group received the highest possible Group SCK-DF Item Score. On the pre-interview on item 2a, for instance, 2 participants (i.e., TG-5 and TG-8) emphasized the student's misconception of fraction concepts instead of considering the student's specific difficulties with division with fractions. For example, TG-5 reported that although the student falsely believed that the divisor has to be bigger than the dividend, the actual source of the student's inability (to attempt) to solve the problem was a lack of a relational understanding of fractions. TG-5 stated:

> TG-5 (2a Pre): Well, he doesn't know that $\frac{3}{5}$, I think he doesn't know that it can be expressed as a decimal. He doesn't realize it is a number. Yah, I think that is part of the thing. Because he would do what did he say, yah, because he would do 3 divided by 4 probably, right, even though that the one [$\frac{3}{5}$] is bigger than the other [$\frac{1}{4}$]. Yah, so I think that's what he's not getting.

With respect to TG-5's performance on the post-interview on item 2a, he was more concerned with the student's inaccurate notion that the divisor must be larger than the dividend than with the student's relational understanding of fractions. In addition, TG-8 also commented on the student's misunderstanding of the relationship between the numerator and denominator on the pre-interview. More specifically, she stated that the student misunderstood that $\frac{3}{5}$ is bigger than $\frac{1}{4}$ because the value of the numbers in the divisor's numerator and the denominator are larger compared to the value of the numbers in the dividend's numerator and denominator.

> TG-8 (2a Pre): I am wondering whether he is getting confused and whether he is looking at the numbers as separate. The fact that 3 is bigger than 1, and 5 is bigger than 4, which is not how he should be looking at fractions. That the bigger the number doesn't mean that the shaded parts are actually going to be bigger. He needs to look at more the smaller the number, the bigger the part. That's why I don't like this example because the $\frac{3}{5}$ is actually bigger than the $\frac{1}{4}$.

What is awkward about her response is that she stated that the hypothetical student correctly compared the two fractions in the problem (i.e., $\frac{3}{5}$ is greater than $\frac{1}{4}$) but that the rationale supporting this comparison is faulty. In other words, although on the interview item, there is no evidence that the student misunderstood how to compare fractions, TG-8 considered that the hypothetical student lacked a relational understanding of fractions based on his or her correct comparison of two fractions. In contrast, on post-interview, TG-8 accurately compared the fractions in the problem, enabling her to effectively consider the student's misconception of division with fractions.

Item 3a. On the pre-interview on item 3a, TG-8 repeated this pattern of demonstrating a misconception of division with fractions while at the same time highlighting the same misconception in the student's solution. That is, she falsely identified an algorithmically-based error and made the following comment on the second part of the student's solution, "He understands that he has to multiply, but he didn't invert the 2 numbers."

CG-4 and TG-8 were not the only participants who expressed the very misconception that was identified in the student's solution. On the pre-interview on item 3a, for instance, TG-6 reported that in addition to thinking that the commutative property is applicable to division, the rest of the student's solution (i.e., the application of the standard algorithm) was also incorrect. The latter substantiates the notion that the participant misunderstood the standard algorithm because the student did invert the divisor and multiplied the two numbers. Moreover, as it can be seen below in her statement, TG-6 demonstrated the same combination of MDFs as CG-4 and CG-1, using an interpretation of multiplication with fractions and confusing the number of groups in the problem (i.e., that 320 is a $\frac{1}{3}$ of the size of 1 group versus distributing 320 among three groups).

> TG-6 (3a Pre): Well he doesn't understand first how to divide because if you were going to divide it you would reverse this number [the fraction] so it would be 320 divided by 3 over 1 and not $\frac{1}{3}$ divided by 320. He would reverse it, so this would be wrong. He doesn't understand that you would have to reverse this fraction, yah the $\frac{1}{3}$, and also it couldn't be $\frac{1}{960}$ because it [320] is a whole number, so right away you know $\frac{1}{3}$ of that is at least 100 because 100 plus 100 plus 100 is 300, so he doesn't understand that.

Taken together, TG-8 and TG-6's MDFs lowered the treatment group's Group SCK-DF Item Score on item 3a on the pre-interview. Further, the fact that no MDFs were present on the post-interview for this item suggests that correcting their own misconceptions was perhaps the source of their enhanced analytical skills.

Item 1a. Similar to the control group, the treatment group received a low Group SCK-DF Item Score on the pre-interview on item 1a primarily because none of the participants in the treatment group accurately identified the algorithmically-based errors in the student's solution. Also of interest is the fact that none of the participants demonstrated any MDFs either, implying that, in this case, there is no possible link between the participants' misconceptions and their error analysis skills on the pre-interview. Contrary to the control group, however, 3 of the 4 participants in the treatment group received SCK-DF Item Scores of 2 on the post-interview.

The participant whose score did not improve on the post-interview (i.e., TG-8) confused one of the alternate division with fractions algorithms that was addressed during the instruction sessions. Specifically, TG-8 emphasized the student's relational misunderstanding on the pre- and on the post-interview. At the same time, however, she demonstrated her own fragmented understanding of the relationship between the numerator and the denominator on the post-interview.

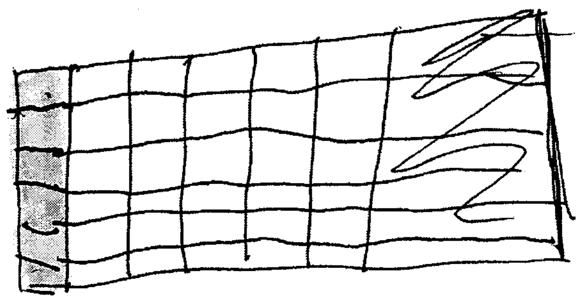
> TG-8 (1a Post): It is obvious that she understands division with whole numbers, 6 divided by 6 she got correctly as 1, but what she is not able to do is to transfer her understanding to fractions to division with fractions. This is a whole number [referring to the fraction], you can't divide it up [split it up]. Whatever you do to the 6, you have to do to the 1, but she's treating $\frac{1}{6}$ as if it is two separate numbers.

She's treating the denominator as a separate number. I am not sure she knows how to handle the $\frac{1}{6}$. The only thing I can think of also is that she is not dividing 6 by both numbers. She's not dividing 1 divided by 6 and 6 divided by 6. If you divide the 6 by 6 then you have to divide the 1 by 6. Six divided by 6 is equal to 1, and that's fine, but 1 divided by 6 equals, you have to do the same thing to the top and bottom for it to be the same ratio.

In accordance with what was written on her interview sheets (see Figure 12), TG-8 perhaps confused two strategies that were addressed during the instruction sessions: the standard algorithm and the common denominators strategy. Indeed, the hypothetical student's answer was intended to suggest that he or she incorrectly applied the standard algorithm. TG-8, however, suggested that the student failed to effectively perform the common denominators strategy, suggesting that the accurate method of solving this problem involved dividing across, or $\frac{1 \div 6}{6 \div 1}$. Although it is possible that the student did indeed incorrectly apply the common denominators strategy (given that the hypothetical student did divide across), the misunderstanding, then, for both the student and TG-8 is that the fractions need to have common denominators prior to dividing the two numerators and dividing the two denominators.

Thus, in general, the results suggest that the treatment group displayed higher post-interview SCK-DF Item Scores in Section I compared to the control group. Although the control group demonstrated relatively high SCK-DF Item Scores in this section on the pre-interview, the treatment group's improved performance on the post-

$$\boxed{\frac{1}{6} \div 6 = 6 \div 6 = 1}$$



$$\frac{1}{6} \div 6 = \frac{\frac{1}{6}}{1}$$

Figure 12. TG-8's pictorial solution on item 1a on the post-interview.

interview indicates a greater gain in this area of SCK following the content-focused instruction.

Qualitative Analysis of Section II: Representing Numbers and Operations in Meaningful Ways

For a list of the interview items in this section and their expected responses, please refer to Figure 13. In analyzing the participants' responses on the post-interview, it becomes apparent that their performance on this section seriously widened the difference in mean Total Group SCK-DF Scores for Section II between the two instructional groups (see Figure 14). As it can be seen in Figure 14, the mean Total Group SCK-DF Score for Section II on the pre-interview was virtually the same for both groups, indicating that on the onset of the study, all participants demonstrated a similar ability to represent numbers and operations in meaningful ways. On the post-interview, however, while the control group's mean Total Group SCK-DF Score for Section II barely increased (on pre-interview M was 4.40 and on the post-interview M was 4.60), the treatment group's mean Total Group SCK-DF Score for Section II on the post-interview ($M =$ 7.20) was close to the highest possible mean value. Indeed, unlike Section I, the fact that the treatment group demonstrated a greater difference on the post-interview compared to the pre-interview cannot be attributed to the treatment group displaying a substantially lower Group SCK-DF Item Scores on the pre-interview (see Table 5). With respect to items 3b and 5b, for example, both groups demonstrated the same Group SCK-DF Item Score on the pre-interview. Along the same lines, on items 1b, 2b, and 4b, the difference in the groups' Group SCK-DF Item Scores on pre-interview was small (i.e., 1 point).

Overall, the frequency of Division Language increased from the pre-interview to

Item	Pre-interview	Post-interview	Expected Response
1b	$\frac{1}{4} \div 4 = 4 \div 4 = 1$	$\frac{1}{6} \div 6 = 6 \div 6 = 1$	Response should address the student's misconception. Responses will vary.
2b	$\frac{1}{4} \div \frac{3}{5}$ "I can't do this because $\frac{3}{5}$ is bigger than $\frac{1}{4}$, so you can't share less among more"	$\frac{1}{3} \div \frac{4}{7}$ "I can't do this because $\frac{4}{7}$ is bigger than $\frac{1}{3}$, so you can't share less among more"	Response should address the student's misconception. Responses will vary.
3b	$320 \div \frac{1}{3} = \frac{1}{3} \div 320 = \frac{1}{3} \times \frac{1}{320} = \frac{1}{960}$	$410 \div \frac{1}{2} = \frac{1}{2} \div 410 = \frac{1}{2} \times \frac{1}{410} = \frac{1}{820}$	Response should address the student's misconception. Responses will vary.
4b	*Kate's mom bought 6 bars of chocolate for her and her friends. If each person had* $\frac{1}{2}$ *of a bar of chocolate, how many people altogether ate chocolate?* $6 \div 2 = 3$	*Harriet has a paper to write for her political science course. She has set aside 10 hours to write the paper. It takes her* $\frac{1}{2}$ *of an hour to write a page. How many pages can she write in 10 hours?* $10 \div 2 = 5$	Response should address the student's misconception. Responses will vary.
5b	*Jerry bought* $3\frac{1}{3}$ *pounds of flour for $2.00. How much did he pay per pound?* $2 \times 3\frac{1}{3} = \frac{20}{3}$ or $6.66 per pound	*Nigella used* $4\frac{1}{4}$ *cups of flour to bake 2 lemon meringue cakes for a dinner party. Shelly, one of her dinner guests loved the cake so much she decided to bake one lemon meringue cake the next day. How much flour did Shelly use to bake this cake?* $2 \times 4\frac{1}{4} = \frac{34}{4}$ or $8\frac{1}{2}$ cups of flour	Response should address the student's misconception. Responses will vary.

Figure 13. A description of the interview items in Section II and the expected responses.

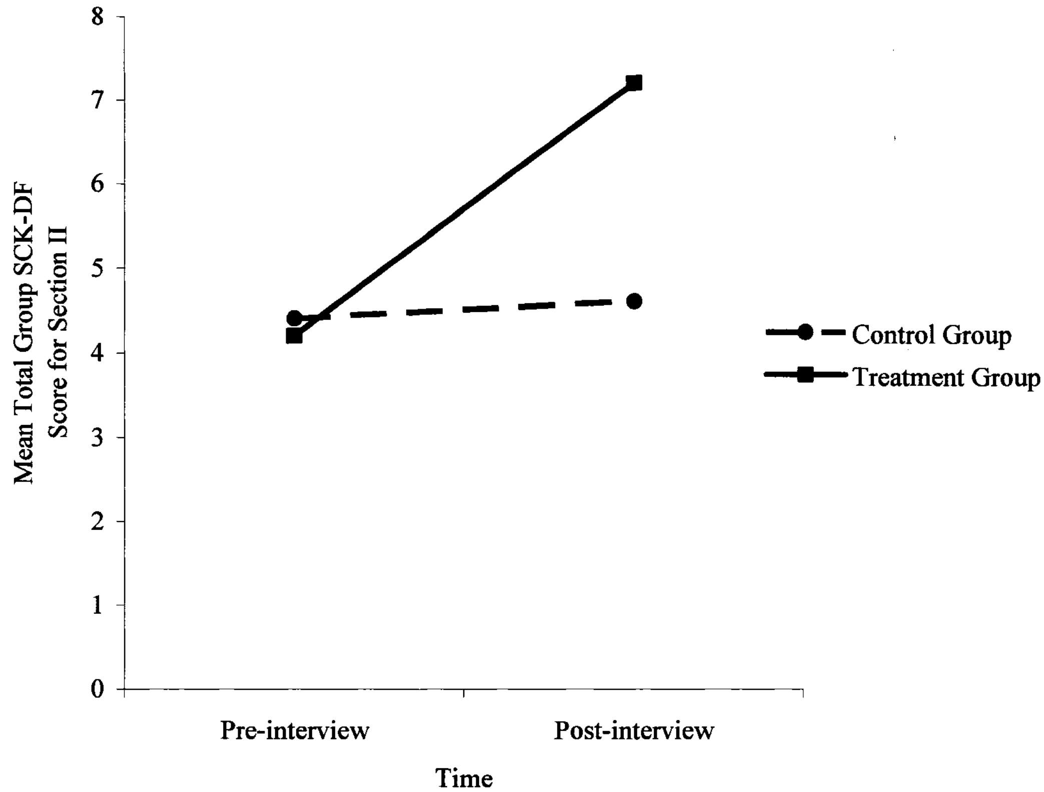


Figure 14. A group comparison of the mean Total Group SCK-DF Score for all the items in Section II.

Table 5

A Group Comparison of Group SCK-DF Item Score, Frequency of Division Language and Misconceptions of Division with Fraction for Section II at Both Time-points ($N = 8$)

	Pre-interview		Post-interview	
Variable	Control	Treatment	Control	Treatment
		Item 1b		
Group SCK-DF Item Score	3	2	4	7
Division Language	1	5	5	8
Misconceptions of Division with Fractions (MDF)	2	4	4	2
		Item 2b		
Group SCK-DF Item Score	4	5	5	8
Division Language	1	3	4	4
Misconceptions of Division with Fractions (MDF)	2	0	2	0
		Item 3b		
Group SCK-DF Item Score	6	6	6	7
Division Language	1	1	2	4
Misconceptions of Division with Fractions (MDF)	3	1	3	0
		Item 4b		
Group SCK-DF Item Score	4	3	2	6
Division Language	0	0	6	0
Misconceptions of Division with Fractions (MDF)	0	1	0	0
		Item 5b		
Group SCK-DF Item Score	5	5	6	8
Division Language	1	3	5	6
Misconceptions of Division with Fractions (MDF)	2	2	0	1
Total Group SCK-DF Score for Section II (*M*)	22 (4.40)	21 (4.20)	23 (4.60)	36 (7.20)

the post-interview for both groups in this section (see Table 5). Interestingly, both groups demonstrated high Division Language frequencies on the same items (i.e., items 1b and 5b) on the post-interview. In addition, the control group also demonstrated a substantial increase in the frequency of Division Language on the post-interview on item 4b. Furthermore similar to the performance on Section I, the highest number of MDFs was displayed on items 1b and 3b. Added to that, the treatment group experienced a decline in the frequency of misconceptions for all items on the post-interview. Contrary to the treatment group's performance, the control group demonstrated a reduction in the frequency of MDFs for 2 of the 5 items only, possibly explaining why the control group's improvements from the pre-interview to the post-interview was not as great as that of the treatment group.

Control Group

Compared to Section I, the control group's performance on the pre-interview for this section was not as strong, indicating that at that time-point, the participants possessed adequate analytical skills but their ability to effectively address the difficulties that were identified was weaker. On average, at both time-points, only 1 participant per item suggested a follow-up problem that would actually address the student's misconceptions as opposed to finding the correct answer. Added to that, a few of the participants proposed problems that were either pedagogically problematic or suggested that the student solve the problem in a manner in which they were not capable.

Despite these issues, all 4 participants in the control group demonstrated an increased SCK-DF Item Score on the post-interview on at least 1 item (see Table 2). Moreover, 2 participants (i.e., CG-2 and CG-3) offered multiple ways of addressing the student's difficulty with the problem, and with the topic of division with fractions, on items 1b, 3b, 4b, and 5b on the post-interview. Further in this regard, CG-2 made links between division with fractions and other mathematical concepts and rules on the post-interview on items 3b and 5b. Taken together, these two positive developments highlight some improvement in the control group's ability to represent mathematical ideas in meaningful ways on items that addressed both procedural and conceptual knowledge of division with fractions.

Item 4b. Surprisingly, despite the fact that the control group did not demonstrate any MDFs on item 4b at either time-point, their Group SCK-DF Item Score for this item on the post-interview was the lowest score in this section (see Table 5). Indeed, although the content-focused instruction for this group emphasized a conceptual understanding of division with fractions, the group demonstrated difficulties in addressing the student's misunderstanding of the concept of measurement division. CG-4, for example, received an SCK-DF Item Score of 2 on the pre-interview because she presented a meaningful way of addressing the student's error, which involved using manipulatives to solve $6 \div 2$ and $6 \div \frac{1}{2}$ in order to provide the student with the opportunity to compare both problems. On the post-interview, however, CG-4 received an SCK-DF Item Score of 0. In particular, the word problem she would have provided to the student was:

> CG-4 (4b Post): She does 6 chores in $\frac{5}{8}$ of an hour. How many chores can she do in one hour? So that she's seeing different, not necessarily the $\frac{1}{2}$, but different fractions in that section where she is actually looking for the whole with the fraction to see.

Not only is her reasoning for suggesting this problem unclear, but the numbers in the partitive division word problem would not correct the misunderstanding that dividing by $\frac{1}{2}$ is the same as dividing by 2.

Unlike CG-4, CG-2's response on the post-interview demonstrated an improved understanding of how to address the student's difficulties with this topic. On the pre-interview, for instance, CG-2 commented on how she would encourage the student to use manipulatives, or act out the problem, to obtain the correct answer. Unfortunately, the misconception that this suggestion addressed was not in line with her error analysis on item 4a.

Conversely, in her post-interview response, CG-2 considered the link between mathematical problems and student learning objectives. That is, given that the word problem I provided in the interview involved the context of time, CG-2 realized that modifying the context of the word problem would be more effective in facilitating the use of manipulatives. The word problem she suggested on the post-interview was, "Her doggy has 10 milk bones for the week. And she decides to take $\frac{1}{2}$ of a milk bone to give him per day. How many days will the dog get a piece of milk bone?" Similar to the problem I presented in the interview, the word problem she suggested required the student to find how many groups of $\frac{1}{2}$ are in the total amount of 10.

Item 1b. Slightly greater than the post-interview Group SCK-DF Item Score on item 4b was the post-interview Group SCK-DF Item Score on item 1b (see Table 5). The increase in Group SCK-DF Item Score from the pre-interview to the post-interview was largely attributed to CG-1's performance on the post-interview. On that interview, she suggested:

> CG-1 (1b Post): I could give her the same problem with different numbers but showing her from the beginning, the grids and stuff like that that we worked with. Let's say, um, like the piece of ribbon and stuff like that because I find that it really helps. So starting with that, showing her how to split it up so that she could see it visually, and then doing it through symbols.

Indeed, using a ribbon, couched in the appropriate measurement division problem, can be an effective method for justifying the invert and multiply algorithm. The reason, however, that this suggestion is problematic is because the number sentence in the problem is $\frac{1}{4} \div 4$, and therefore using this number sentence in a problem where one have a certain length of ribbon, although mathematically feasible, and one would like to know how many pieces fit into that ribbon is pedagogically awkward. That is, it implies finding how many 4 foot pieces fit into a ribbon that is $\frac{1}{4}$ of a foot long. Interestingly, also on the post-interview, CG-1 responded to this problem with the same type of division problem but improved her response by acknowledging that in order to improve the student's understanding of the algorithm, the number sentence presented in the interview should be modified. In particular, to address the student's algorithmically-based misconception, she would use a measurement division word problem whereby 4 is divided by $\frac{1}{4}$. Thus, similar to CG-2's performance on the post-interview on item 4b, CG-1 also demonstrated an improved understanding of the crucial role numbers play in meeting student learning objectives.

Also of interest on the pre-interview is the MDF CG-3 displayed on items 1b and 2b. More specifically, on item 1b, she stated:

> CG-3 (1b Pre): So I would say, so take $\frac{1}{4}$ and draw $\frac{1}{4}$ in. And then you'd be like, if you had to divide that one, if you had to divide the $\frac{1}{4}$ by 4, and then colour, what would be, if you had to divide by 4, what would be left basically? So you divide it by 4 and you get, what's one of those, you get the $\frac{1}{4}$ of the $\frac{1}{4}$.

CG-3's pictorial solution may be viewed in Figure 15. In other words, rather than apply a partitive interpretation when modeling a fraction divided by a whole number, CG-3 represented the fraction using a part-whole model and re-partitioned the shaded area according to the value of the whole number. What she failed to explain is why the answer is a fraction of this re-partitioned area, suggesting that the CG-3 confused the act of partitioning with the act of division.[10] Although it may be argued that CG-3 may not have had this specific misconception because her final statement is mathematically accurate ("you do end up multiplying $\frac{1}{4}$ by $\frac{1}{4}$"), I believe that the only reason this appears to be mathematically reasonable is because of the numbers that were in the problem. This notion is supported by CG-3's response to 2b on the pre-interview. CG-3's pictorial solution is presented in Figure 16. In particular, she suggested that she would encourage

[10] To address why a fraction of $\frac{1}{4}$ is the answer, I will use the same problem, $\frac{1}{4} \div 4$. If an area that represents $\frac{1}{4}$ is re-partitioned into fourths, the answer to this problem is one part of the partitioned area because by re-partitioning the fraction into fourths, one is essentially creating four groups. Therefore the size of one group is the answer.

$$\boxed{\frac{1}{4} \div 4 \rightarrow 4 \div 4 = 1}$$

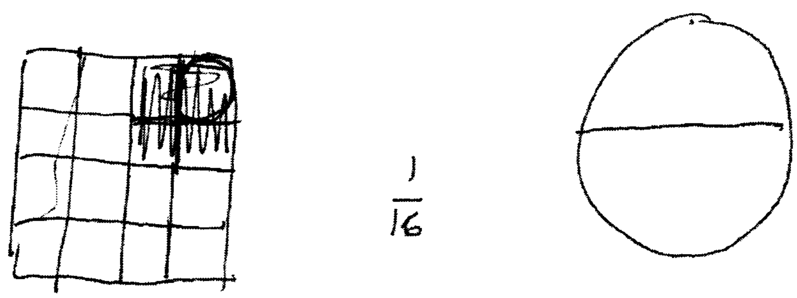


Figure 15. CG-3's pictorial response on item 1b on the pre-interview.

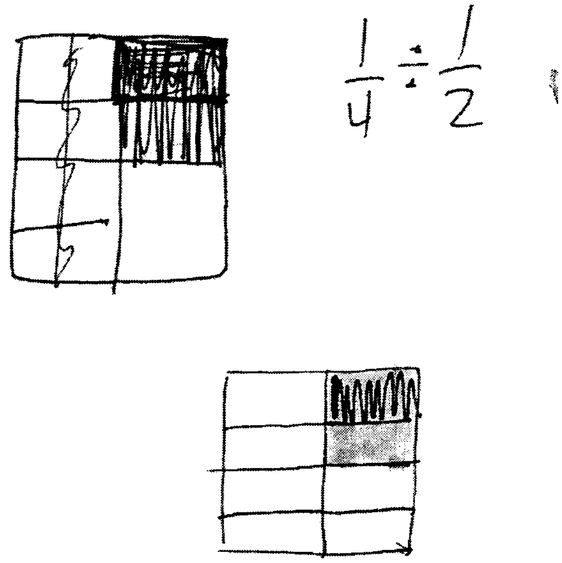


Figure 16. CG-3's pictorial response on item 2b on the pre-interview.

the student to model the solution, and proceeded by attempting to model the solution to $\frac{1}{4} \div \frac{3}{5}$ herself. As she modeled the problem, she explained:

> CG-3 (2b Pre): Again, pictures, so you draw it... $\frac{1}{4}$. So you have $\frac{1}{4}$ and you need to divide the $\frac{1}{4}$ by $\frac{3}{5}$, so um, you take, oh, I forget how to illustrate this. Now I don't remember how to do this. You take, I don't know what I would do. So if I took, ah, if you were dividing $\frac{1}{4}$ by $\frac{1}{2}$ it would be like...can you, can you say that you want to take $\frac{1}{2}$ of the $\frac{1}{4}$ or is that misleading? [This question was directed to me and I did not respond]. Or you want to find what $\frac{1}{2}$ of $\frac{1}{4}$ is? Because if we were dividing $\frac{1}{4}$ by $\frac{1}{2}$ it would be, you would take the $\frac{1}{4}$ and you would divide it into $\frac{1}{2}$, and then you would have this section of the entire square which when you divide it all into equal parts...Oh no what have I done [she redraws another model]. So this is your original $\frac{1}{4}$ and this is your $\frac{1}{2}$, and if you had to divide the entire square in equal parts it gives $\frac{1}{8}$, which isn't the answer I am looking for. I don't know why, I am blanking.

Based on this statement, it is clear that CG-3 was not aware of how to use the partitive interpretation of division to explain the solution to a fraction divided by a whole number or a fraction divided by a fraction. Unfortunately, because this misconception was demonstrated by CG-3 and CG-2 on the post-interview as well, perhaps the content-focused instruction was not effective in correcting this misunderstanding despite the fact that the partitive model was addressed during the instruction sessions.

Item 3b. Of all the items in this section, the highest Group SCK-DF Item Scores on the post-interview were on items 3b and 5b. With respect to 3b, 3 participants (i.e., CG-1, CG-2, and CG-3) received an SCK-DF Item Score of 2 on the pre-interview. CG-2, for instance, stated:

> CG-2 (3b Pre): So just with the picture, drawing out the...actually 320 would be hard to draw out, so then I would definitely not use this problem. I would probably use a smaller...either 32. If we could work with a grid, I think it would be easy to count out 32, like a graph paper, like a big graph paper. I think there wouldn't be a problem working with the number 32, and it would still be similar and familiar to this one in context. And yah, we'd have to draw, like draw it so that he could see where he went wrong because he seems to start to understand the algorithm, just the value of it changes when you place numbers in different spots.

In other words, in order to challenge the student's belief that division is commutative, she would use a smaller number that is easier to represent semi-concretely, and demonstrate that switching the order of each of the numbers changes the role of the number, thus producing a different quotient.

Unlike CG-2's response, CG-4's proposition on the pre-interview was both pedagogically and mathematically problematic, and addressed finding the correct answer to the problem. Moreover, she restated the combination of MDFs that she initially presented on item 3a on the same interview. She stated:

CG-4 (3b Pre): Well, I would get him to start from the beginning again and we could sit down and draw out our 320 circles or squares and then divide that like deal them out into thirds. So that he can see that it is $\frac{1}{3}$ of 320 gives you a certain number and that so we would get the answer for this. And I would get him to do this also to see the difference between the two equations, to show him that it is not the same thing.

The reason why I propose that her suggestion is pedagogically problematic is because drawing 320 objects is long and tedious and an overall ineffective way to represent 320. Further, as a consequence of her MDFs, I also considered that the manner in which she would scaffold the student in finding the correct answer would also be problematic because, as I previously mentioned, she confused $320 \div \frac{1}{3}$ with $320 \times \frac{1}{3}$ and believed that 320 should be distributed among three groups. Although her response on the post-interview remained the same, her SCK-DF Item Score increased by 1 point because her suggestion was in line with the student error that she identified in item 3a.

In contrast, CG-1's SCK-DF Item Score decreased by 1 point, explaining why the Group SCK-DF Item Score for this item remained unchanged from the pre-interview to the post-interview. More specifically, rather than correct the student's misunderstanding of division with fractions, CG-1's follow-up strategy on the post-interview emphasized finding the correct answer.

Item 5b. CG-1, however, was one of the participants who demonstrated a higher SCK-DF Item Score on the post-interview on item 5b. Like to CG-1, CG-2's score also improved on the post-interview. In particular, CG-2 initially attempted to correct the student's misconception on the pre-interview by demonstrating the concept of partitive division with whole numbers (i.e., 2 divided by 4). Following that, she explained that she would apply this reasoning to model the solution to the task at hand (i.e., $2 \div 3\frac{1}{3}$). In doing so, however, the solution strategy she suggested was inaccurate (see first sentence of the excerpt below). She stated:

> CG-2 (5b Pre): So even for myself, I think this is where I would come in, like $2 divided by 3, how much would that give me? And then your last pound, work with only $\frac{1}{3}$ of it as if it was [*sic*] a whole number, but yah, I would go back to working with whole numbers, because with just whole numbers in the problem, maybe he would be able to see division. It was probably just scary to see the $\frac{1}{3}$ and the money. How am I going to divide if this looks bigger than this and all that?

That is, CG-2 explained that because the number of groups was a fraction (i.e., $3\frac{1}{3}$), when the goal is to find out how much 1 pound costs, one cannot distribute the $2 among 3 pounds and then address the fraction afterward; clearly, she did not understand how to model a partitive division problem when the number of groups is a fraction. Rather, the correct method of modeling this problem is to consider how much is in $\frac{1}{3}$ of 1 group, and then multiply that value by 3 to find the size of 1 group. Given that there are 10 thirds in $3\frac{1}{3}$, $2 would be distributed among 10 groups whereby each group represents $\frac{1}{3}$ of 1 group. Subsequent to the content-focused instruction, however, her response on this item improved because she provided the student with a word problem that addressed the student's understanding of the concept of multiplication in addition to the partitive interpretation of division.

In addition, having a fractional number of groups not only impacted CG-1's ability to model the problem; it also affected several participants' understanding of the number sentence reflected in the word problem. Indeed, despite the fact that the word problem on the pre-interview specifies that the goal is to find how much flour costs per pound if $3\frac{1}{3}$ pounds of flour costs \$2, several participants suggested that the number sentence that corresponds to that word problem was $3\frac{1}{3} \div 2$, perhaps indicating that a fractional number of groups was problematic for the participants to consider at the onset of the study. In fact, several participants from both groups displayed this misunderstanding (i.e., TG-5 on item 5a, and CG-3 and CG-4 on item 5b). It is important to note that for this section, this particular MDF would not negatively impact the SCK-DF Item Score because when addressing the student's misunderstanding of partitive division, different numbers can be used, thus a fractional number of groups may be replaced with a whole number.

Item 2b. Slightly lower than the post-interview Group SCK-DF Item Score on items 3b and 5b was the control group's post-interview Group SCK-DF Item Score on item 2b. In addition, the control group's Group SCK-DF Item Score for this item was higher on the post-interview compared to the pre-interview. Contributing to this increase were CG-2 and CG-3's post-interview responses on this item.

As I previously mentioned, CG-3's pre-interview statement on item 2b involved a more complicated method of demonstrating to the student that the two fractions can in fact be divided. More importantly, however, she selected a method that she herself struggled with when carrying it out. In contrast, on the post-interview, CG-3's answer was a simple and effective method of demonstrating to the student that the divisor can be bigger than the dividend. She proposed:

> CG-3 (2b Post): Can you divide one cake, one cake only, can you divide it in two. And yes, if you cut it in half they each get $\frac{1}{2}$. Just emphasize that you can divide smaller numbers by bigger numbers. I would go back to whole numbers [be]cause the problem isn't necessarily fractions and a lot of kids say that.

In other words, her statement highlighted her knowledge of common student misconceptions, a crucial pedagogical tool to have.

Finally, although CG-1's score did not change from the pre-interview to the post-interview, the strategy she proposed on the post-interview illustrated her efforts to incorporate what was discussed during the content-focused instruction. That is, she suggested that she would help the student to use a factors and product word problem, stating that it is that interpretation that helped her consider that it is possible that a division problem can comprise divisor that is larger than the dividend.

Treatment Group

Unlike the control group, the treatment group's performance in this section parallels that from Section I. Indeed, in comparing the pre-interview items from these two sections, the Group SCK-DF Item Scores were virtually identical for all but 1 of the items, highlighting the treatment group's inadequate knowledge regarding both of these areas of SCK-DF at the onset of the study.

After receiving content-focused instruction of division with fractions, however, the treatment group's ability to represent mathematical concepts and procedures in meaningful ways (Section II) was slightly more enhanced than their analytical skills (Section I). This notion is substantiated when the treatment group's post-interview Group SCK-DF Item Scores is compared as a function of section (the treatment group's post-interview mean Total Group SCK-DF Score for Section I was 6.60, and for Section II was $M = 7.20$).

In addition, based on the information from Table 2, it is evident that this pattern of development also emerges at the participant level (i.e., on the post-interview, the treatment group's ability to represent numbers and operations was superior compared to their analytical skills). That is, in examining the number of participants whose score increased from the pre-interview to the post-interview, I note, for instance, that on items 1 and 2, the same number of participants (i.e., 3) demonstrated a higher SCK-DF Item Score on the post-interview in both sections. Moreover, on items 3 and 4, at least 1 participant improved his or her SCK-DF Item Score on the post-interview in both sections (see Table 2). In fact, the primary difference between the treatment group's performance in Section I and the SCK-DF Item Scores in Section II is that the Group SCK-DF Item Score on item 5b was higher compared to that of the Group SCK-DF Item Score on item 5a. It is important to note, with respect to this finding, that irrespective of the fact that the treatment group's content-focused instruction did not address the partitive interpretation of division with fractions, the treatment group demonstrated post-interview improvements in representing this concept in meaningful ways.

Similar to the control group, the participants in the treatment group offered multiple ways of addressing the student's misconception and made links between division with fractions and other mathematical concepts and procedures. Specifically, all 4 participants proposed multiple methods on at least one of items 1b, 2b, 4b, and 5b on the post-interview. Moreover, TG-5 and TG-8 made links between division with fractions and other mathematical topics on items 1b and 3b. Interestingly, it was TG-8 who displayed a SCK-DF Item Score increase on 9 of the 12 interview items on the post-interview. Second to TG-8, was TG-5, who demonstrated a SCK-DF Item Score increase on 6 of the 12 interview items on the post-interview.

Item 2b. As illustrated in Table 5, the treatment group received the highest Group SCK-DF Item Scores on items 2b and 5b. With respect to 2b, 2 participants from the treatment group received an SCK-DF Item Score of 2 on the pre-interview. Therefore, TG-6 and TG-8's improvements on the post-interview were largely responsible for the post-interview Group SCK-DF Item Score of 8.

TG-6, for instance, received the lowest possible SCK-DF Item Score on the pre-interview because she provided an ambiguous response to correcting the hypothetical student's misunderstanding of division with fractions. Specifically, she stated:

> TG-6 (2b Pre): Well, I would give him a problem where he could draw out to see the parts, where he could visually see it, and then maybe go through the model with the words, and then model with the symbols, so he sees the connection between all three. That's what I would do.

Subsequent to the content-focused instruction, however, her response demonstrated a more accurate understanding of the student's misconception, and an enhanced knowledge of how to address the topic of division with fractions in a more meaningful manner. She suggested:

> TG-6 (2b Post): I would make sure he understands what a fraction divided by a whole number is before I would start this, and I would show him this with a model. That you can find a portion of a fraction, that even though it is $\frac{1}{3}$, I would draw it for him and show him that you can still find a portion of a fraction, and that is why we reverse it, to find the portion.

Note that, with her example of a fraction divided by a whole number, not only is the divisor bigger than the dividend, but when one multiplies by the reciprocal of a whole number, one is looking for a fraction of a fraction.

Item 5b. In contrast to item 2b, TG-6 was the only participant who received an SCK-DF Item Score of 2 on the pre-interview. She stated:

> TG-6 (5b Pre): I would give her, maybe, whole numbers. Make sure she understands the basics, like the foundations, the whole number. Like this would be 3 divided by, 2 divided by 3. Just to see like, uh, 3 divided by 2, sorry [participant means 2 divided by 3] to understand that she knows how to divide, she knows how to multiply in this type of context. So, if Jerry bought 3 pounds of flour for $2, how much did he pay per pound? And have her do that to understand that you know how to do it how to solve it. And then maybe I would introduce fractions as a second base [bringing it to another level] but I would really make

sure that she understands how to divide a whole number by a whole number. Yah, even if the answer is a fraction, to make sure she understands the foundation of it.

Notwithstanding the fact that, like several others, she confused the number sentence (i.e., it is $2 \div 3\frac{1}{3}$), her response emphasized enhancing the student's understanding of the partitive interpretation of division with fractions. In addition, although TG-6 received the highest score on the pre- and post-interview, her post-interview response was slightly better because she offered two ways of addressing the student's misconception: using a partitive division word problem and working with numbers in a symbolic context.

Item 1b. Of interest is the improvement in the Group SCK-DF Item Score on item 1b on the post-interview (see Table 5). On the whole, on the pre-interview, the treatment group overlooked the notion that it is crucial to consider whether numbers in the problem effectively meet student learning objectives. More specifically, on the pre-interview, 2 participants (i.e., TG-5 and TG-6) received an SCK-DF Item Score of 0, and the other 2 participants (i.e., TG-7 and TG-8) received an SCK-DF Item Score of 1. In contrast, on the post-interview, all but 1 of the participants received an SCK-DF Item Score of 2. The substantial difference in the Group SCK-DF Item Score on the post-interview, is evidence that, for this item, the participants in the treatment group benefited from the content-focused instruction.

To illustrate, CG-5's response, addressed a misconception that he did not previously identify on item 1a on the pre-interview. He stated:

TG-5 (1b Pre): Well, I don't know, I would give her a whole number of things. One, I guess, just to see if she knows how to divide, and I guess the other thing that is confusing is because the whole number is the divisor, so it is not exactly clear that she knows that this is 4 over 1. One of the points would be to put the whole number over here as the dividend, or whatever, and then I guess the other one would be to make this 4, make it clear that it is a fraction, like, I don't know, make it like $\frac{2}{3}$ or something. I guess I would try those two things.

Although this answer would address the misconception of division with fractions, TG-5 did not identify this misconception earlier during his analysis of the student's solution.

On the post-interview, however, TG-5 offered multiple ways of improving the student's understanding of the algorithm on the post-interview, making links between semi-concrete and symbolic representations of the algorithm. He stated:

> TG-5 (1b Post): Well, anything to show her the idea of division. Show her an operation with the dividend [he pointed to the divisor and I corrected him after he finished this statement] as a fraction, so I guess you got to show her how to do the algorithm. So, I guess you got to draw that out. Initially, we would have a whole number divided by a fraction, draw it for her, and see if she gets that. And I guess we could do one of the proof things we talked about to show her how that works and you could show her how to rewrite it, 6 over $\frac{1}{6}$ as a complex fraction. Just so she understands that it is the divisor that has to be reciprocated, also the multiplication thing because she's not doing that.

Based on this statement, I argue that the content-focused instruction had a positive impact on his post-interview response for this item because I considered that he became more attuned to the critical role numbers play in making mathematical ideas meaningful.

In addition, TG-8's response was also problematic on the pre-interview in that she failed to acknowledge that a fraction of a set of objects can be, and is in her case, a whole number. She stated:

> TG-8 (1b Pre): Maybe I would give it to her in a word problem instead of like this, so I would go, if I had $\frac{1}{2}$ box of cookies and I wanted to divide it into 4, I would go, uh, I have $\frac{1}{2}$ dozen cookies, I have 12 cookies and I give $\frac{1}{2}$ them to my friend, my friend wants to give 3 of her friends including herself an equal amount of cookies, so there is 4 of them, how many cookies does each person get? If I give it to her that way, then she can model it, there's no word problem attached to this, is there? It's symbolic, you can't give her symbolic right off the bat when you don't even know if she understands this [points to $\frac{1}{4}$]...especially if she's a beginner.

Clearly, her word problem is not an example of a meaningful way to address the student's algorithmically-based error. That is, aside from being lengthy and confusing, the type of word problem (i.e., partitive) is not connected to the standard algorithm. Further, because the number sentence in the problem was a fraction divided by a whole number, she overlooked the fact that using numbers may in fact be a more simplistic method of making the algorithm meaningful.

Conversely, on the post-interview, TG-8 considered the possibility that it might be confusing to address the student's misconception using the same numbers in the problem (i.e., $\frac{1}{6} \div 6$), and instead, she suggested that she would use $\frac{1}{4} \div 2$. In doing so, she realized:

> TG-8 (1b Post): I guess what I am also seeing here is that you're doubling the number of parts…it is hard to explain to a child why the answer would work out that way, and first they would have to have number sense. This, to her, should actually make sense if she understood division of fractions, if she modeled it out. So, I guess I would like to do a lot of word problems without her even having to do the symbolic. Then, when she sees patterns such as this one, that when you divide the number of parts doubles, like look that this, 4 multiplied by 2 is 8, maybe that is why the number of parts then. What I might do is explain the wonderful method that you taught. I can teach why the inverse works if she recognizes that pattern, that's why it is good to work with small numbers.

Note that, because she considered different numbers, she became more conscious of the notion that if the correct numbers are chosen, symbols can also be an effective method of promoting understanding.

Item 4b. Another item on which the treatment group demonstrated a substantially low Group SCK-DF Item Score on the pre-interview was item 4b. In general, 3 participants promoted the use of pictures to help the student find the correct answer. TG-6, for example, proposed:

> TG-6 (4b Pre): And to help her understand, I would have her draw out the 6 bars of chocolate and say well, ok, now divide them. If each person had $\frac{1}{2}$ a bar of chocolate, I would divide each bar into halves to show her.

While this suggestion would be effective for the student, it would only provide the student with an alternate solution strategy as opposed to correct the student's misunderstanding.

On the other hand, 3 of the 4 participants' response to this item improved on the post-interview. For instance, TG-5 stated:

> TG-5 (4b Post): I would draw this because if you could do one page per hour, you would multiply it. So if we look to find what hours, "how many do you finish in 1 hour?" 2. So then if you want to do your same thing, if it takes an hour to write a page, how many pages can you write? 10, ok that's obvious. Say so it takes $\frac{1}{2}$ an hour, if you write quicker will you have more pages or less pages? So yah, you could say that 10 divided by 1 is 10, 10 hours divided by 1 hour is 10. 10 divided by $\frac{1}{2}$ an hour, and then you got to show the algorithm I guess. But I would use that counting thing to show that the number should be bigger than 10, 10 divided by 1 is 10, 10 divided by $\frac{1}{2}$ should be bigger because you are writing quicker.

Although TG-5's response to this item focused on finding the correct answer, his proposition was indicative of some progress because he addressed a key concept in division: the smaller the size of each group, the more groups can be made.

Item 3b. Similar to the control group, the treatment group's Group SCK-DF Item Score on item 3b on the pre-interview was 6, and the score only slightly increased on the post-interview (see Table 5). In fact, TG-8 was the only participant who improved her SCK-DF Item Score on the post-interview. On the pre-interview, TG-8 identified two misconceptions: the misuse of the commutative property and an algorithmically-based error. She responded to these misconceptions by suggesting:

> TG-8 (3b Pre): Because, maybe if he is just starting to learn the commutative property, maybe if he is just transferring his knowledge, than 320 is too big for him. Smaller numbers might be easier. Also, I would probably want him to model it because it's the best. I wouldn't want him to directly go to symbols. 320 is harder to model than 12. I would show him why the algorithm, I wouldn't even go to the algorithm, I would have him model first. I would do this after I would explain the commutative property of division, I would have him model how to solve a problem like this, and this is why I would have him use smaller numbers too.

It is important to note that despite having noticed the student's difficulties performing the standard algorithm, she contended that it would be inappropriate to address the algorithm in subsequent problems.

On the post-interview, however, it was evident that her abilities in representing numbers and operations improved. At the beginning of her response to this item on the post-interview, she stated that she would use smaller numbers and address the misuse of commutativity with an equal sharing problem (i.e., "if I have 6 cookies for 2 friends, is that the same as 2 cookies for 6 friends?"). Adding to that, she proposed that she would also discuss whether addition and subtraction is commutative. Although compared to her pre-interview response, these suggestions are clearly indicative of a more effective way of addressing the student's misconception; her final remarks for this item truly highlight her maturation in this area of SCK. She stated:

TG-8 (3b Post): Basically, what I would do is I would have him use way smaller numbers. If I use these numbers [410 and $\frac{1}{2}$] then because he knows the answer...that might help him. If I go 6 divided by 2, 6 times 1 over 2, 6 divided by 2 [represented as a fraction $\frac{6}{2}$] is 3. So these are the steps that I would be showing him. I'd say to him, when your doing multiplication and division like you said in our lessons, they are related to one another, therefore the number sentence can be modified without changing the problem. So it can be 3 times something equals 6. So that's the number sentence. I am going to try to get him to realize it [that division is the inverse of multiplication], but if he doesn't get with multiplication and division, he'll probably get it through addition and subtraction. Now the task is getting the x alone, the question mark alone [which is what she used for x] and the best way to do that would be, you can get rid of the 2 by multiplying, dividing it by itself, because 2 divided by 2 is 1 and 1 times a question mark is the question mark itself. So there is no point in putting the 1 down, so 6 divided by 2, are you proud of me yet, is 3 and then there you go. Here we are just...because if he's doing this [using commutativity] maybe it is just a fluke that he got this right.

In other words, unlike on the pre-interview where she identified an algorithmically-based error but refused to address it with the student afterwards, on the post-interview TG-6 offered a mathematically reasonable justification (i.e., division is the inverse of multiplication) of the algorithm as a "just in case" measure.

Although TG-6 did not positively or negatively impact the post-interview Group SCK-DF Item Score on item 3b, her pre-interview response to this item highlighted the relationship between a teacher's error analysis skills and the ability to represent key mathematical concepts. That is, rather than identify the student's difficulty with properties of division, she considered that the student did not posses the appropriate understanding of the concept of dividing a whole number by a fraction. Unfortunately, her reasoning behind her analysis was faulty (i.e., confusing $320 \div \frac{1}{3}$ with $320 \div 3$), impacting her ability to effectively address the misconception she identified.

> TG-6 (3b Pre): And again, I would do it through pictures and draw it for him and show him. Maybe I would do something like 300 divided by $\frac{1}{3}$ so that way they could see something like it is really...see that it is 100, 100, 100. That is what $\frac{1}{3}$. And then maybe move on to other numbers like 400 and 500...work with whole numbers first and then like add in the 20s and 320, 340.

Her statement, in fact, highlights how the link between error analysis and representing numbers and operations meaningfully can be positively or negatively mediated by the teacher's mathematical knowledge.

In sum, the interview data demonstrates that both groups' performance on the pre-interview in this section was poor. Following the content-focused instruction, however, the treatment group displayed an enhanced ability to represent numbers and operations in meaningful ways. The control group, on the other hand, only demonstrated a slight improvement in this area of SCK-DF.

Qualitative Analysis of Section III: Justifying Mathematical Rules and Procedures

The pre- and post-interview items and the expected responses may be viewed in Figure 17. With respect to the participants' ability to justify the invert and multiply algorithm, both groups demonstrated a parallel pattern of development in SCK-DF (see Figure 18). That is, based on Figure 17, the control group and treatment group demonstrated comparable mean Total Group SCK-DF Scores for Section III on the pre-interview. Moreover, on the post-interview, both groups experienced similar gains in SCK-DF in this section.

A comparison of the Group SCK-DF Item Scores for each group and the frequencies of Division Language and MDFs on items 6 and 7 may be viewed in Table 6. In particular, for both groups, the (a) Group SCK-DF Item Scores on items 6 and 7 were higher on the post-interview, (b) frequency of Division Language in the responses increased on the post- interview, (c) responses to item 6 contained no MDFs at both time points, and (d) responses to item 7 contained some MDFs, but the frequency of these misconceptions decreased on the post-interview.

Control Group

Item 6. As it can be seen in Table 6, the control group's Group SCK-DF Item Score on item 6 increased on the post-interview. Indeed, on the pre-interview, 2 of the 4 participants responded to item 6 using the measurement division interpretation to model a solution and subsequently used the model to justify the steps of the standard algorithm. These 2 participants, however, received an SCK-DF Item Score of 1 because their response did not address the remainder in the answer. That is, when using the measurement interpretation of division to model $7 \div \frac{3}{5}$, 11 groups of three are identified,

Item	Pre-interview	Post-interview	Expected Response
6	*Larry's vegetable garden is 7 feet long. If each vegetable takes up $\frac{3}{5}$ of a foot, how many different types of vegetables can Larry plant?*	*Brian is 5 feet tall. He decides that he wants to measure himself and he notices that he has a bunch of pieces of string on his desk. If every piece of string is $\frac{2}{7}$ of a foot, how many pieces does Brian need to figure out how tall he is?*	Response should address why you multiply by the reciprocal of the divisor. Responses will vary.
7	$\frac{5}{12} \div \frac{3}{8}$	$\frac{7}{15} \div \frac{5}{11}$	Response should address why you multiply by the reciprocal of the divisor. Responses will vary.

Figure 17. A description of the interview items in Section III and the expected responses whereby the participants were asked use the context of the problem to explain why the divisor is inverted and the two values are multiplied.

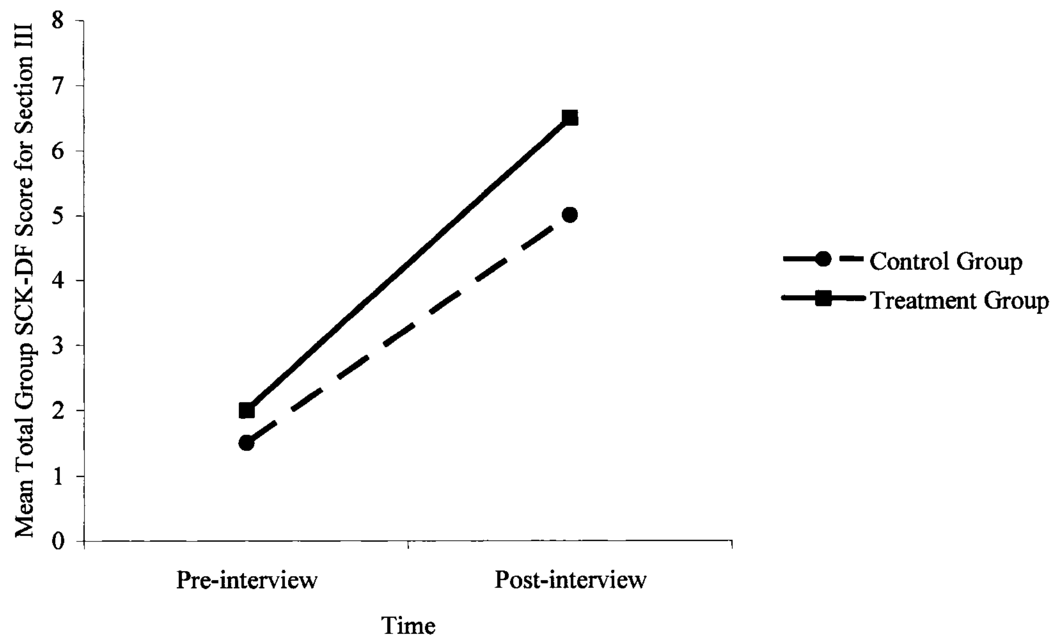


Figure 18. A group comparison of the mean Total Group SCK-DF Score for all the items in Section III.

Table 6

A Group Comparison of Group SCK-DF Item Score, Frequency of Division Language and Misconceptions of Division with Fraction for Section III at Both Time-points (N = 8)

Variable	Pre-interview Control	Pre-interview Treatment	Post-interview Control	Post-interview Treatment
		Item 6		
Group SCK-DF Item Score	2	5	7	8
Division Language	5	6	7	10
Misconceptions of Division with Fractions	0	0	0	0
		Item 7		
Group SCK-DF Item Score	1	1	3	5
Division Language	5	8	8	15
Misconceptions of Division with Fractions (MDF)	12	8	6	1
Total Group SCK-DF Score for Section III (*M*)	3 (1.50)	6 (2.00)	10 (5.00)	13 (6.50)

but there are two remaining parts that do not complete 1 group of 3 parts. According to the model, the two parts appear as though they represent the fraction $\frac{2}{5}$, because each whole was previously partitioned into fifths. Therefore, as part of the justification, the participant must address how the $\frac{2}{5}$ in the model is connected with the remainder, or $\frac{2}{3}$. The other 2 participants in the control group received an SCK-DF Item Score of 0 on the pre-interview.

In contrast to the control group's performance on the pre-interview, all of the participants' post-interview responses to item 6 included an accurate model of the solution and an appropriate justification of the steps in the standard algorithm. Indeed, CG-3 was the only participant whose post-interview score on this item did not increase, primarily because she repeated the same mistake from the pre-interview: she failed to address the remainder.

Item 7. Similar to item 6, the control group demonstrated a higher Group SCK-DF Item Score on item 7 on the post-interview compared to this group's Group SCK-DF Item Score on the pre-interview (see Table 6). In particular, on the pre-interview, only 1 participant (i.e., CG-2) received an SCK-DF Item Score of 1, and the rest received an SCK-DF Item Scores of 0 for this item. CG-2, for instance, applied the same solution method that was used for the previous item, failing, however, to accommodate her strategy to the context of a fraction divided by a fraction, thus resulting in an incomplete justification of the standard algorithm. Specifically, she frequently expressed the misconception that the size of each group corresponds to the numerator of the divisor, as oppose to the *product* of the numerator of the divisor and the denominator of the dividend. Further, she did not comment on the fact that as a result of forming groups of the wrong size, her number of groups did not correspond to the quotient of the problem. For instance, as part of her justification of the standard algorithm using $\frac{5}{12} \div \frac{3}{8}$, she stated:

> CG-2 (7 Pre): Why it would work is because if you have $\frac{5}{12}$ of something and you would divide it by $\frac{3}{8}$, the twelfths, you are making more parts in them and then you are looking technically for how many groups of 8 you'll have in this $\frac{5}{12}$, and you are actually looking for how many 3 groups of 8 you'll have in the $\frac{5}{12}$. So, if you are looking for that then you are going to multiply the number, your whole that you are working with which is $\frac{5}{12}$, you are going to multiply that with your $\frac{8}{3}$. So 5 multiplied by 8 is 40, and then because you are looking for the 3 groups in the 12 you'll divided that by 3 to give you your answer.

Overall, the low pre-interview Group SCK-DF Item Score on item 7 is largely attributed to participants in the control group confusing the concept of division with fractions with the concept of multiplication with fractions (as an example of this misconception, see Figure 19). Indeed, as a consequence of this misconception, all 4 participants modeled the solution to finding a fraction multiplied by a fraction. Although it may be argued that applying a conceptual understanding of multiplication with fractions does not, in this case, indicate a misconception of division with fractions because both numbers are fractions and because the dividend *is* multiplied by the inverted

The number sentence is:

$$\boxed{\frac{5}{12} \div \frac{3}{8}}$$

$$\frac{5}{12} \times \frac{8}{3} = \frac{40}{36} = 1\frac{4}{36}$$

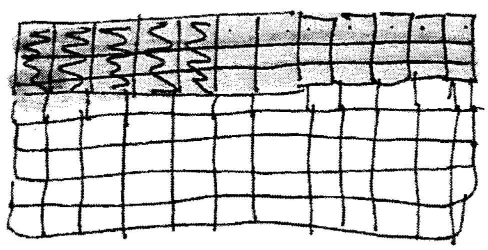


Figure 19. CG-4's pictorial response on item 7 on the pre-interview.

divisor, the participants produced a model of the dividend being multiplied by the divisor, clearly substantiating a misconception in their understanding of how to model a fraction divided by a fraction. Added to that, 1 participant misunderstood the standard algorithm and expressed that all fractions, irrespective of whether the fraction is the dividend, the divisor, or both, are inverted.

Contrary to the substantial improvement the control group demonstrated on item 6, only slight improvements of the control group's Group SCK-DF Item Score was found on item 7 (see Table 6). More specifically, all 4 participants gave responses similar to that of CG-2 on the pre-interview for item 7. As a consequence, more participants demonstrated the misunderstanding of the size of the group and failed to realize that because of their understanding of the size of each group, the total number of groups was inconsistent with the quotient in the problem. Interestingly, however, as a result of using this strategy to respond to the problem, all of the participants used Division Language to communicate their response on the post-interview, whereas CG-2 was the only participant who used Division Language to respond to this item on the pre-interview. In addition, none of the participants repeated the misconceptions displayed on the pre-interview, contributing to a overall decrease in the control group's frequency of MDFs on the post-interview, despite the fact that new misconceptions arose as a consequence of the strategy that was employed to address the item (see Table 6).

Treatment Group

Item 6. As I previously mentioned, the results for the treatment group's responses to items 6 and 7 are similar to that of the control group. To begin with, the treatment group's Group SCK-DF Item Score on item 6 increased from the pre-interview to the post-interview (see Table 6). Specifically, on the pre-interview, 3 of the 4 participants used the measurement division interpretation to model the solution and to highlight how the model may be used as a means of justifying the standard algorithm to students. Different from the control group, however, 2 of the 3 participants addressed the remainder, contributing to the treatment group's higher Group SCK-DF Item Score for this item on the pre-interview. The participant who did not receive any points for this item on the pre-interview was TG-5. As it can be seen in his response, TG-5 verbally explained the model, but failed to understand how this type of division problem can be used to justify the steps in the standard algorithm:

> TG-5 (6 Pre): Because we are supposed to be able to draw this, and I think this is how you draw it... $\frac{3}{5}$...Ok, so we are talking about how many total fifths there are, so there is [*sic*] 35 fifths. I think, 35 over 5, hmmm...if we took three on each we would get, ah $\frac{3}{5}$, $\frac{3}{5}$, I don't know how to do this.

On the post-interview, all of the participants received an SCK-DF Item Score of 2 for this item. Of particular interest, 1 participant in the treatment group did not use conceptual knowledge of division with fractions to answer the question. Indeed, TG-7 explained the algorithm using the mathematical rule that division is the inverse of multiplication. That is, when you change the operation from division to multiplication, the position of the numbers change such that the dividend becomes the product, and the divisor becomes one of the two factors. As a result of solving for the dividend, both sides of the equal sign are multiplied by the reciprocal of the divisor.

Item 7. The treatment group's increase in Group SCK-DF Item Score from the pre-interview to the post-interview on item 7 was greater compared to that of the control group's (see Table 7). Indeed, although both groups' Group SCK-DF Item Score on the pre-interview was 1, 2 participants from the treatment group received an SCK-DF Item Score of 2, and 1 participant received an SCK-DF Item Score of 1 on the post-interview. In fact, one of the 2 participants who received an SCK-DF Item Score of 2 was the same participant who received an SCK-DF Item Score of 1 on the pre-interview; thus, unlike CG-2, this participant improved her score on the post-interview. In particular, TG-8's response to item 7 (i.e., $\frac{5}{12} \div \frac{3}{8}$) on the pre-interview involved applying the same reasoning used on item 6:

> TG-8 (7 Pre): The 12 has to come in where the 3 is, but I don't get why. So it is 5 multiplied by 8, but then I want to divide it in groups of three because it is $\frac{3}{8}$, so 40 divided by 3, which is going to be 36, 40 divided. I don't get why you do 12 multiplied by 3, that's why I am stuck, I don't know what to do with the problem.

She continued to misunderstand the size of the groups and concluded by using interpretations of addition with fractions and multiplication with fraction in place of a concept of division with fractions. She stated:

> TG-8 (7 Pre): So, first what you have to do is make common denominators, because you can't add twelve's and eights [*sic*]. First, I do rows of 5 and then rows of 8, and then what I do…so that gives me 40. After I highlighted the $\frac{5}{12}$, I

only needed $\frac{3}{8}$ of that, so I guess up here you get 5 multiplied by 8 and 12 in each row, because 3 groups of 12, and 12 in each row.

In other words, at first she stated that she wanted to add the two fractions but then she modeled $\frac{5}{12} \times \frac{3}{8}$, illustrating her misuse of the concept of division with fractions. After her participation in the instruction sessions, however, she used procedural knowledge of division with fractions and displayed a mathematically reasonable strategy that was not addressed during any of the instruction sessions. More specifically, she combined various mathematical rules and invented a strategy to explain the steps in the algorithm. She responded:

> TG-8 (7 Post): The way how I would do it is that I want to focus on this number [$\frac{5}{11}$], I want to get rid of the $\frac{5}{11}$ and the way how to do that is by multiplying by its inverse. Yah, I can multiply by its inverse and the reason why is because $\frac{55}{55}$ is equal to 1. When you multiply by its inverse, you end up getting the same product in the numerator and in the denominator and you are making it 1. So $\frac{7}{15}$ multiplied by 1 is $\frac{7}{15}$, $\frac{7}{15}$ times 1 is itself [I mean], divided by 1, sorry, is itself. But the problem is, is because I multiplied this guy [$\frac{5}{11}$] I have to multiply the $\frac{7}{15}$ [by $\frac{11}{5}$].

That is, analogous to the rule that justifies the equal additions algorithm (i.e., adding the same value to the subtrahend and the minuend will not modify the difference), she demonstrated that multiplying the same value to the dividend and the divisor will not change the quotient. Therefore, when you multiply both fractions by the reciprocal of the divisor, the result is the dividend multiplied by the inverted divisor whose product is divided by 1 (because the product of any number multiplied by its reciprocal is 1), and the quotient of any number divided by 1 is itself, substantiating why the divisor is inverted, and why the two numbers are multiplied.

Similar to TG-8, TG-7's response on the post-interview also comprised procedural knowledge of division with fractions as opposed to conceptual knowledge of the topic. Contrary to TG-8, TG-7 spontaneously proposed two methods of justifying the standard algorithm to a student. After using the same justification that was used on item 6 (i.e., $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$) she stated:

> TG-7 (7 Post): But I know you showed us in class with a measurement division problem, but I think I would have to think it out, think it over, because the divisions [*sic*] and you need to know how many parts and it's big and I think maybe that would be too hard for them to understand. Also, if there would be another way to demonstrate it, because of the size of the numbers, because there is 20 parts, 20, so it is hard to break it down. I think this one [the formula], the *a* over *c* [general reference to $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$], we can't use...could we use; we could use the complex function [fraction] in this one right? $\frac{7}{15}$ divided by big, big line $\frac{5}{11}$. Get rid of the denominator and make it over 1, because any number over 1 is itself. So $\frac{11}{5}$, and it becomes a multiplication and that also shows the, um, we

> could have done it even for the previous one because it's got whole numbers. It is a lot easier to do it with the, and the common denominator...can we find a common denominator? Yah, we could use the common denominator, but you don't see the reciprocation [*sic*] on that one.

On item 7 on the pre-interview, however, for which she received an SCK-DF Item Score of 0, she stated, "Well, I guess I would do the same kind of problem, I would show it to them through pictures, because to show it to them only symbolically would be confusing, it is too difficult." Taken together, not only did TG-7 use symbols and algorithms on the post-interview, contradicting her statement during the pre-interview, "to show it to them only symbolically would be confusing," but for both items 6 and 7 on the post-interview, she used procedural knowledge, despite the fact that item 6 is conducive to using the concept of measurement division because of the measurement division word problem that is presented in it.

Summary of Results

In summary, 7 out of the 8 participants' SCK-DF improved as a result of receiving content-focused instruction. Overall, both types of content of instruction benefited participants' ability to justify the standard algorithm used with division with fractions. Moreover, the content-focused instruction promoted the use of Division Language and in several cases reduced participants' MDFs. In addition, the comparison of the pre-interview responses to the post-interview responses indicated that instructional content that combined conceptual and procedural knowledge of division with fractions had a greater impact on the participants' skills in representing mathematics in meaningful and pedagogically-appropriate ways. As a consequence of experiencing gains in this particular area of SCK (i.e., Section III), the participants in the treatment group experienced relatively larger gains in SCK-DF compared to the magnitude of change of SCK-DF experienced by the participants in the control group.

CHAPTER 5: DISCUSSION

In line with the link between a teacher's mathematical content knowledge and student performance in mathematics (Hill et al., 2005), Ball et al. (2005) proposed that the content of preservice teachers' mathematics methods coursework should comprise content-focused instruction, designed to facilitate preservice teachers' development of specialized content knowledge (SCK) of mathematics. In this regard, the first objective of the present study was to examine whether content-focused instruction addressing division with fractions would enhance preservice teachers' SCK of this topic. Moreover, the type of mathematics content knowledge (i.e., conceptual and procedural) that would foster preservice teachers' SCK is not well understood. For this reason, a second objective of this study was to examine the impact of type of instructional content (i.e., conceptual content, or a combination of conceptual and procedural content) on preservice teachers' gains in SCK of division with fractions (SCK-DF).

To address these objectives, I explored the development of eight preservice teachers' SCK-DF. The participants were assigned to one of two treatment conditions (i.e., conceptual content, and a combination of conceptual and procedural content). The preservice teachers' SCK-DF was assessed by conducting pre- and post-interviews, which measured the three areas of SCK as it pertains to the topic of division with fractions. In general, the results support the hypotheses that (a) providing the preservice teachers with content-focused instruction promoted the development of SCK-DF, and that (b) the preservice teachers who received instruction that comprised a combination of conceptual and procedural knowledge of division with fractions demonstrated greater gains in SCK-DF compared to the preservice teachers in the control condition. I will now continue with a discussion of these results, their implications for future research, and concluding remarks.

Effects of Content-Focused Instruction on Developing Preservice Teachers' SCK-DF

With respect to the first research objective, 7 of the 8 preservice teachers' SCK-DF improved after having received content-focused instruction on division with fractions. Moreover, I found evidence that initially, the preservice teachers possessed a fragmented SCK-DF (i.e., 5 of the 8 Total SCK-DF Scores ranged from of 11 to 12, meaning that the majority of the preservice teachers received interview scores that were half of the maximum score). Following the content-focused instruction, however, five of the post-interview scores almost doubled. Taken together, these results suggest that instruction emphasizing either conceptual or a combination of conceptual and procedural knowledge of division with fractions had a positive impact on the preservice teachers' SCK-DF.

Although, the development of all three areas of SCK has not to my knowledge been previously examined, the suggestion that the preservice teachers' overall improvement in SCK-DF was associated with the content-focused instruction is consistent with the Rittle-Johnson et al.'s (2001) theory that mathematical knowledge develops iteratively. That is, because SCK is a type of mathematical knowledge, I propose that the development of SCK parallels the development of conceptual and procedural knowledge in other areas of mathematics. Simply put, Rittle-Johnson et al. found that improvements in one type of knowledge (i.e., conceptual or procedural) promoted the development of the other type of knowledge, as well as sustained or extended already existing knowledge. Thus, I suggest that in the case of the current study, improvements in one area of SCK promoted incremental developments for the other SCK areas, contributing to the participants' overall improvement in their SCK-DF following the content-focused instruction.

Further in this regard, the results from the present study illustrated that both groups' SCK-DF only developed in the same fashion for the items in one section, namely Section III. Moreover, of the three sections within the interview, it was found that the highest mean number of participants demonstrated improved post-interview scores in Section III (i.e., $M = 5.00$ compared to $M = 3.80$ and 2.20), implying that, on average, the preservice teachers' ability to justify the invert and multiply algorithm was the area of SCK that benefited the most from the content-focused instruction. Based on this, I argue that it is the development of this particular area of SCK that contributed to the participants' overall SCK-DF progress on the post-interview.

This particular finding is not surprising given that the content-focused instruction primarily addressed ways of justifying the invert and multiply algorithm. More specifically, both groups received instruction that demonstrated how the measurement model can be used to justify the standard algorithm for division with fractions. Note that, although this method was addressed during the mathematics methods course before I began conducting the pre-interviews, some of the participants on the pre-interview displayed a propensity to confuse how the measurement division model is linked with the steps in the standard algorithm. In addition, none of the participants was capable of transferring his or her understanding of this strategy to a novel context (i.e., a fraction divided by a fraction). As a consequence of the manifestation of this difficulty, on the pre-interview, nearly all of the participants attempted to apply the concept of multiplication with fractions when modeling a fraction divided by a fraction. Taken together, at the beginning of the current study, the preservice teachers demonstrated some conceptual understanding of division with fractions, but their comprehension was inflexible and fragile, highlighting a weak schema of this topic (Baroody, Feil, & Johnson, 2007).

Baroody et al. (2007) used the terms weak and strong schemas to highlight the contrast between deep and superficial mathematical knowledge. That is, *weak schemas* refer to mathematical understanding that entails, "...generalizations local in scope [with] low standards of internal (logical) consistency, precedent-driven comprehension, and no logical basis for a priori reasoning (i.e., predictions are looked up)" (p. 117, Baroody et al., 2007). Baroody et al. proposed that *strong schemas* "...involve generalizations broad in scope, high standards of internal (logical) consistency, principle-driven comprehension, and principled bases for a priori reasoning (i.e., predictions are derived logically)" (p. 117). In line with this reasoning, the preservice teachers on the pre-interview (a) failed to generalize their conceptual understanding of the standard algorithm, (b) switched from applying a conceptual understanding of division with fractions to multiplication with fractions, and (c) were unable to flexibly apply their existing knowledge to derive a solution, illustrating that a weak schema underscored their superficial conceptual understanding of the invert and multiply algorithm.

After the content-focused instruction, however, the preservice teachers displayed a stronger schema for justifying the standard division with fractions algorithm, suggesting that their understanding of this procedure deepened. Evidence that substantiates the presence of a stronger schema for the algorithm lies in the fact that on the post-interview, 7 of the 8 participants accurately applied the measurement division model to explain how to divide a whole number by a fraction. Added to that, although all the participants were unsuccessful in extending a conceptual understanding of division with fractions to a fraction divided by a fraction, none of the participants confused the concept of division with fractions with multiplication.

According to Rittle-Johnson et al.'s (2001) theory of iterative development, this development of understanding should yield developments in the other two areas of SCK. Consistent with this notion, some participants demonstrated improved post-interview scores on the items in Sections I and II as well. The argument that the other two areas improved as a result of enhancing the preservice teachers' knowledge of justifying a mathematical procedure is acceptable because elementary students' common misconceptions of division with fractions, and methods of correcting these misunderstandings, were not addressed during the content-focused instruction. In general, however, fewer of the preservice teachers increased their scores on items that assessed their error analysis skills and abilities to represent numbers and mathematical operations in meaningful ways (i.e., Section I and II). For instance, following Section III was the preservice teachers' performance on the items in Section II, whereby an average of 4 participants improved on the post-interview. Moreover, it was found that on average, only 2 participants improved their scores on the items in Section I.

At first glance, this last finding may be interpreted as being inconsistent with the results from Tirosh (2000), because she found that the treatment in her study improved the preservice teachers' error analysis skills. It is important, however, to consider that her treatment instruction addressed children's misconceptions of division with fractions in addition to multiple ways of justifying the algorithm using procedural mathematics. With respect to the current study, the control group's type of content was conceptual knowledge of division with fractions; thus, I could only present one method of justifying the algorithm to the participants in that group. Moreover, as I previously mentioned, the content-focused instruction was not designed to highlight student misconceptions.

Finally, virtually all of the participants in the control group received the highest item scores on the majority of the items on the pre-interview in Section I, contributing to a relatively low mean number of participants who improved their scores on the items in this section on the post-interview. Taken together, it may be suggested that the low mean number of participants who improved their scores on the items in Section I was the result of a combined effect of the treatment group progress and the control group sustained understanding of this area of SCK. Because these notions begin to delve into the area of group differences, I will elaborate on these interpretations when I discuss the results pertaining to the second research objective.

In addition to improved post-interview scores, the content of the preservice teachers' responses to the items in Section II on the post-interview provide additional support for the notion that this area of SCK (i.e., representing numbers and mathematical operations in meaningful ways) was enhanced. That is, some of the preservice teachers proposed multiple methods of addressing the hypothetical student's misunderstanding of division with fractions, or offered multiple solution methods. More importantly, however, irrespective of the type of content of instruction the preservice teachers received, they considered multiple perspectives of the problem on items that comprised conceptually-based and procedurally-based errors. Furthermore, even though the treatment group's content-focused instruction did not address the partitive model of division with fractions, 2 participants in this group were able to present two methods of correcting the student's misunderstanding of partitive division with fractions on the post-interview. Thus, consistent with the theory of iterative development, it is possible that enhancing the preservice teachers' justification skills using at least one of the two types of mathematical knowledge (i.e., conceptual or procedural) benefited their ability to represent numbers and mathematical operations in relevant ways for problems that addressed *both* types of knowledge. On the whole, despite the fact that the content of instruction only slightly overlapped for both groups, and that only one area of SCK was emphasized, in general, the participants experienced an overall improvement in SCK-DF, which involves both conceptual and procedural knowledge.

Group Differences in SCK-DF

In terms of the second research objective, the quantitative and qualitative analyses supported the hypothesis that the treatment group experienced greater gains in overall SCK-DF compared to the control group following the content-focused instruction sessions. In particular, the Mann-Whitney U tests revealed that the ranked pre-interview scores resulted in a distribution of high and low ranks in both groups. Following the content-focused instruction, the data revealed that the control group comprised the majority of the lower ranks of the post-interview scores. Further, only one post-interview score from the control group was tied with two post-interview scores in the treatment group, thus none of the treatment group's post-interview scores was out-ranked by the control group's post-interview scores. Taken together, the results from the Mann-Whitney U tests imply that the groups' SCK-DF developed differently as a function of the type of content of instruction each group received during the content-focused instruction.

The qualitative analysis was used as a means of determining how the type of content of instruction enhanced the groups' SCK-DF. In general, the analysis yielded several interesting findings and provided some insight into the source of the difference found between the two groups on the post-interview.

To begin, as I previously mentioned, the participants from the control group demonstrated high group item scores in Section I on the pre-interview compared to the participants from the treatment group, who received relatively low group items. Thus, as a result of this difference, in this area of SCK at the onset of the study, the post-interview group item scores for the items in this Section are not easily comparable. That is, although the treatment group displayed higher post-interview scores for the items in Section I, it is difficult to interpret the treatment group's greater gain in error analysis skills. Although I will use the groups' item scores from Section's II and III as a basis for interpreting the groups' differential gains in SCK-DF, I will briefly discuss the group differences found in Section I.

Group Differences for Section I

It is interesting to note that the treatment group's performance on the post-interview for the items in Section I highlights a relationship between the type of content-focused instruction this group received and the development of their error analysis skills. More specifically, on the pre-interview, the treatment group's analysis of the procedurally-based errors was not as strong compared to that of the control group's. Following the content-focused instruction, however, the control group's performance for these items declined slightly while the treatment group clearly demonstrated an improved understanding of these types of errors.

Moreover, while the treatment group's awareness of a hypothetical student's misunderstanding of the concept of measurement division improved, their analysis of the student's understanding of the partitive model declined on the post-interview, perhaps as a consequence of receiving conceptual instruction that only emphasized the measurement model. Taken together, because the treatment group's instruction emphasized both conceptual and procedural content knowledge of division with fractions, it is logical that the treatment group would develop a greater sensitivity to a student's performance on the procedures as well as understanding the measurement model associated with division with fractions.

The notion that the treatment group's development in their error analysis skills was, in part, associated with the content-focused instruction is consistent with what was demonstrated in Tirosh (2000). Indeed, like the preservice teachers in Tirosh, the preservice teachers in the treatment group in this study received multiple ways of using mathematical procedures to understand the division with fractions algorithm. Thus, the treatment group's post-interview performance on the items in Section I suggest that this group's awareness of several procedurally-based methods, in addition to the conceptually-based method for solving division with fraction problems, provided the group with knowledge of more mathematical concepts and procedures associated with this topic, which elementary students often misunderstand.

Group Differences for Section II

The qualitative analyses also revealed that the preservice teachers' ability to represent numbers and operations in meaningful ways was the only area of SCK where both groups' SCK-DF developed differently. Indeed, both groups demonstrated relatively low scores on the pre-interview. Observable differences were found, however, on the items in this section on the post-interview, whereby the treatment group outperformed the control group. In general, the treatment group's responses to the items in Section II were less dependent on the context of the problem, thus addressing the hypothetical student's general misconception. On the whole, these findings suggest that the type of content (i.e., conceptual or a combination of conceptual and procedural knowledge) of instruction created a gap in the group's capacity to meaningfully represent operations and numbers differently, contributing to the differential outcome in the groups' overall development of SCK-DF on the post-interview.

Although the treatment group demonstrated greater gains in their development of SCK-DF in Section II, it should also be noted that participants from both groups demonstrated an improved understanding of the importance in strategically selecting numbers and activities when enhancing students' understanding of mathematics. Similar to the Chinese teacher in Ma (1999) who noticed that the numbers in $1\frac{3}{4} \div \frac{1}{2}$ would easily lend themselves to using the distributive property once the divisor was inverted and the two values were multiplied, more preservice teachers from both groups on the post-interview considered the role of numbers and the context of word problems compared to the pre-interview. In line with what Ball et al. (2005) and Hill et al. (2005) suggested, this development implies that a deepened understanding of either type of mathematical content is a critical competency for effectively promoting learning objectives.

Group Differences for Section III

Based on the case study of Ms. Daniels in Eisenhart et al. (1993), I hypothesized that promoting conceptual and procedural mathematical content during the content-focused instruction sessions would provide the preservice teachers with a greater opportunity to understand the standard algorithm. That is, because Ms. Daniels' conceptual knowledge of the division with fractions algorithm was limited, and her procedural knowledge was superficial, she experienced obstacles in her quest for teaching for understanding. Consistent with my hypothesis, the results from this study indicated that although both groups' growth in justifying the standard algorithm progressed in the same direction, group differences in developmental gains were found. More specifically, 2 participants from the treatment group were capable of accurately responding to item 7 on the post-interview, illustrating that, compared to the other participants, they possessed a stronger grasp of the standard algorithm.

Based on this result, I propose that the treatment group may have developed a deeper understanding of the algorithm as a consequence of having both types of content emphasized during the instruction sessions. Further, in line with the theory of iterative development (Rittle-Johnson et al., 2001), the groups' differential gain in knowledge of the standard algorithm may explain why the ability to represent numbers and operations developed differently for each group. In other words, in accordance with the reasoning that areas of SKC may be enhanced as a consequence of the improvement of one area of SCK, perhaps the quality of this development is an additional factor. For this reason, I suggest that at the source of the treatment group's greater gain in SCK-DF was their deeper understanding of the division with fractions algorithm that developed as a consequence of the type of content of instruction. Thus, this quality of knowledge may have had a greater impact on the development of the other two areas of SCK.

Furthermore, the treatment group's performance on the items in Section III imply that instruction addressing mathematical procedures in a meaningful manner can promote a deep procedural understanding (Star, 2000, 2002a, 2002b, 2005; Star & Seifert, 2006), and that it is possible to possess deep procedural knowledge of this topic. Evidence that some of the preservice teachers developed deep procedural knowledge is based on the fact that 1 of the 2 participants from the treatment group invented her own algorithm to correctly answer item 7 on the post-interview. Indeed, this finding aligns with Carpenter et al.'s (1998) observations of children's invented algorithms, indicating that invented algorithms are not only based on a connected knowledge of mathematical concepts, but algorithmic invention may also emerge from a deep and flexible understanding of mathematical procedures. In line, then, with what Carpenter et al. and Kamii and Dominick (1998) suggested, if students' use of invented algorithms is a goal of effective mathematics instruction because it denotes flexibility in thinking about numbers and operations and can result in more efficient methods of mathematical problem solving, then a student's understanding of mathematical procedures is just as important as his or her knowledge of mathematical concepts.

In addition, the second participant who accurately responded to item 7 on the post-interview, of her own volition, offered multiple ways of justifying the algorithm. The notion that this denotes a deeper understanding of the algorithm is based on Ma's

(1999) suggestion that the Chinese teachers possessed a more comprehensive understanding of division with fractions compared to the U.S. teachers, in part, because they were knowledgeable of multiple methods to compute $1\frac{3}{4} \div \frac{1}{2}$.

Taken together, the performance of these 2 participants for the items in Section III on the post-interview has several implications for the conceptualization of procedural knowledge and the debate regarding its role in the instruction of mathematics. First, understanding the mathematical procedures that underscore the standard algorithm can clearly provide students with the opportunity to understand how to solve several types of division with fraction problems. Indeed, the control group's responses to the items in Section III suggest that having an adequate conceptual understanding of division with fractions does not easily extend to problems where both the dividend and the divisor are fractions, implying that there are situations where procedural knowledge of the standard algorithm is relevant (Budd et al., 2005).

Furthermore, because participants from the treatment group did demonstrate a strong conceptual understanding of the topic after the content-focused instruction, it appears that an appropriate understanding of mathematical procedures does not, as Kamii and Dominick (1998) suggested, harm the development of conceptual knowledge. Rather, because mathematical knowledge develops iteratively (Rittle-Johnson et al., 2001) and the two types of mathematical content knowledge are mutually related (Rittle-Johnson & Alibali, 1999), promoting a meaningful understanding of mathematical procedures can complement students' conceptual knowledge of mathematics.

Overall, the treatment group's gain in SCK-DF suggests that the notion that facts versus higher order thinking accurately characterizes the dichotomy between procedural and conceptual knowledge is false (Wu, 1999a). Rather, and consistent with what Kilpatrick et al. (2001) and Star (2005) espoused, achieving the goal of mathematical proficiency requires fluency in both mathematical concepts and procedures.

Some Positive Effects of the Content-focused Instruction

The Frequency of Division Language

The preservice teachers' use of division language was not a concern when the study and interview measures were designed. Based on Sfard's (2000) premise that mathematical communication is the vehicle for constructing mathematical knowledge, I considered that the participants' use of formal vocabulary of division with fractions warranted some investigation. Indeed, formal vocabulary (see Zazkis, 2002) such as finding the number of groups, the size of groups, invert and multiply, divisor, dividend, and quotient the topic of division with whole numbers and with fractions was not only accessible to the participants in the content-focused instruction sessions, but was also emphasized in their textbooks and in the mathematics methods course content. Irrespective of their exposure to formal mathematical terminology, participants' responses on the pre-interview often comprised informal terms (Zazkis, 2002; e.g., "4 divided by 4" instead of "there is 1 group of 4 in 4). Thus, in these instances, no interpretation of division was communicated when they were conveying the concept of division with fractions. As such, I was concerned with whether the content-focused instruction would increase the use of formal division vocabulary (i.e., Division Language) on the post-interview.

This reasoning was based on the results from Zazkis (2002), where she suggested that the use of formal mathematical vocabulary supported preservice teachers' understanding of mathematical concepts. Zazkis's conclusion transpired from the results of a study that was designed to examine preservice teachers' learning and understanding of fundamental number theory concepts. Although examining preservice teachers' formal and informal vocabulary was not an objective of her study, Zazkis noted themes regarding mathematical communication that emerged from the interviews, particularly with respect to the concept of divisibility. With regard to how the participants interpreted the meaning of the word *divisible,* for example, she noted that some participants would inconsistently use formal vocabulary. More specifically, she suggested that as the participant's confidence in his or her ability to respond to the question declined, so did the use of formal vocabulary, highlighting a positive relationship between their mathematical knowledge and the application of proper terminology to convey that understanding. Similarly, in the current study, the preservice teachers used formal vocabulary more often on the post-interview, irrespective of their prior awareness of the proper terms, perhaps illustrating an improved confidence in their understanding of division with fractions.

Participants' Misconceptions of Division with Fractions

Other benefits of the content-focused instruction included a reduction in the number of misconceptions the preservice teachers demonstrated on the post-interview. The content-focused instruction, for instance, appeared to have improved their accuracy in performing and understanding the invert and multiply algorithm. Indeed, similar to what was found in Ball (1990a), Tirosh (2000), and Tirosh et al. (1998), some of the participants in the current study experienced difficulties applying the invert and multiply algorithm on the pre-interview. Following the content-focused instruction, only 1 participant continued to display an algorithmically-based error on the post-interview. Considering the subject of the content-focused instruction sessions, it is logical that instruction that justifies the steps in the standard algorithm would positively impact future performance on applying the algorithm.

Moreover, perhaps the participants' inadequate understanding of the standard algorithm underscored their general inability to identify the algorithmically-based errors in item 1a on the pre-interview. That is, only 1 participant on the pre-interview considered that the hypothetical student presented in item 1 misunderstood the division with fractions algorithm. In fact, on the pre-interview, the majority of the participants did not comment on the topic of division with fractions when addressing this student's algorithmically-based error. Rather, the preservice teachers were often concerned with the possibility that the student lacked a relational understanding of fractions, or that the student confused the concept of division with the concept of multiplication.

The fact that the preservice teachers were concerned with the student's conceptual understanding when the student demonstrated a procedural misunderstanding contradicts what was found in Osana, Rayner, Desrosiers, and Levesque (2007). More specifically, prior to receiving any instruction in a mathematics methods course, the preservice teachers in Osana, Rayner et al. relied heavily on their procedural knowledge of multidigit subtraction to analyze fictitious student solutions. It was found, that the preservice teachers' justifications of ratings assigned to the students' solutions were primarily concerned with the application of the standard algorithm for multidigit subtraction.

It is possible, however, that the preservice teachers in the present study possessed a stronger conceptual understanding of fractions, division, and division with fractions because they did receive conceptually-based instruction of these topics in the mathematics methods course prior to the study. Most likely, however, because some of the preservice teachers in the current study did hold misconceptions of the steps involved in the standard algorithm used with division with fractions, their impaired understanding could have hampered their ability to detect algorithmically-based errors. In other words, without an accurate understanding of how the algorithm is applied, it becomes increasingly difficult to notice when the algorithm is performed incorrectly. Furthermore, this notion is also supported by that fact that the preservice teachers' error analysis for the other items with procedurally-based student errors (i.e., rules associated with division) emphasized procedural knowledge.

Taken as a whole, these conclusions are consistent with results from van Dooren et al. (2002), who suggested that preservice teachers' mathematical content knowledge influences their evaluations of student solutions. Specifically, it was found that the preservice teachers' scoring decisions of hypothetical student solutions were guided by the way in which they would have solved the problem as well as their understanding of the mathematics involved in it. In line with this reasoning, then, it is plausible to consider that in the current study, the preservice teachers' error analysis skills were influenced by the quality of their understanding of the mathematical content in the problem.

Other Effects of the Content-Focused Instruction

Although the preservice teachers' algorithmically-based misconceptions were virtually absent on the post-interview, some misconceptions of division with fractions still persisted after the content-focused instruction, and some possibly developed as a consequence of it. Specifically, participants from both groups demonstrated two misconceptions: (a) confusing the action of division with the act of partitioning an area and (b) acting out the partitive model using a whole number of groups when the number of groups represented is a fraction.

I contend that the source of these two misconceptions was the preservice teachers' lack of discernment regarding selecting the appropriate model of division on items that were presented in absence of a word problem (see items 1 and 3). In particular, the preservice teachers who demonstrated these misconceptions either did not use a concept of division to interpret the problem or selected a model of division that was more complicated based on the numbers in the problem. With this observation, I suggest that in line with Carpenter et al.'s (1999) notion of direct modeling, the preservice teachers demonstrated the propensity of relying on the context of the word problems to orientate the effective manipulation the of numbers using one of the two division models.

With respect to the former misconception (i.e., confusing the action of division with partitioning), because of the symbolic context, there was no explicit model of division with fractions presented. As such, some of the participants used informal language to model their solution. That is, rather than model the solution to a division with fractions problem using either a measurement (i.e., finding the number groups of a certain size) or a partitive (i.e., finding the size of one group) interpretation, their description of the process of division was ambiguous and thus made it difficult for him or her to conceptualize the answer (i.e., either the size of one group or the number of groups of a certain size; Zazkis, 2002).

On the other hand, the second misconception (i.e., acting out the partitive model and using a whole number of groups when the number of groups is represented as a fraction) had less to do with the use of informal vocabulary and more to do with difficulties modeling an interpretation of division with fractions in the absence of the structure of a word problem. One of the participants who, for example, accurately modeled $4\frac{1}{4} \div 2$ in accordance with the partitive structure of the word problem, incorrectly applied the partitive interpretation to model $320 \div \frac{1}{3}$ and $410 \div \frac{1}{2}$. Note that, when the context of the item was symbolic, she failed to apply a model of division that was consistent with the number sentence (i.e., either finding how many groups are in the total amount when the size of one group is a fraction or finding the size of one group when the number of groups is a fraction). On the other hand, when the interview item provided a word problem, thus clearly outlining the model of division, the participant understood how to manipulate the numbers in the problem in accordance with the model of division.

Further supporting the notion that some preservice teachers experienced difficulties applying models of division in symbolic contexts were data from the post-interview. Despite having effectively applied the measurement model to justify the algorithm on item 6, none of the participants successfully extended this understanding to answer item 7. A primary difference between these items was that the division problem in item 6 was presented within a word problem while the division problem in item 7 was symbolic. As a consequence of this, the preservice teachers who did use the measurement model to respond to item 7 demonstrated misconceptions of justifying the algorithm that did not emerge on the pre-interview.

It is important to consider that, within a numerical context, the problem solver bears the choice of which model to apply, and to effectively carry out this decision, an abstract and flexible understanding of the interpretations of division is a necessity. In particular, one must consider the numbers in the problem and also consider the constraints that are inherent to each interpretation. The partitive model, for instance, is convenient when (a) the divisor is less that the dividend, (b) when the divisor is a whole number, and (c) when the quotient is less that the dividend (Fischbein, Deri, Nello, & Marino, 1985). For this reason, applying a measurement interpretation of division to model $320 \div \frac{1}{3}$ or $410 \div \frac{1}{2}$ is a more logical choice[11]. This is not to say that the partitive model is impossible to use in these cases; considering the partitive model in these situations requires an understanding that will violate the constraints of the partitive model. This is evidently an understanding that certain preservice teachers in the present study did not possess.

Similarly, Tirosh and Graeber (1990) found that of the 15 preservice teachers that contended that the quotient is always less than the dividend, 5 participants maintained this misconception even when they were asked to calculate $4 \div \frac{1}{2}$. In fact, one of the participants incorrectly calculated that 8 is the quotient of $4 \div .5$, and that 2 is the quotient for $4 \div \frac{1}{2}$, even though she had specifically stated that $.5 = \frac{1}{2}$. Based on this

[11] This notion was addressed and justified during the instruction sessions (see Appendix E). In particular, I suggested that for a whole number divided by a fraction, the measurement model is a more appropriate choice, and for a fraction divided by a whole number, the partitive model works best. I also demonstrated how both models are used to solve a problem with a fraction divided by a fraction.

result, and on the fact that several preservice teachers admitted that they were unaware of the measurement model, Tirosh and Graeber suggested that as a consequence of only using the partitive model of division, the preservice teachers lacked a simpler way of interpreting division problems that violated the model's constraints of the partitive model. In terms of the present study, this notion implies that the preservice teachers demonstrated a reliance on the partitive model as a means of solving division with fraction problems, and in certain situations, this involved a more complicated conceptualization of the problem, perhaps confusing their understanding of how to effectively apply the model in accordance with the number sentence.

Although the results from the interviews in Tirosh and Graeber (1990) provide a framework for understanding this misconception, they do not give insight into the resilience of this misconception, particularly with respect to the control group. Indeed, only participants from the control group continued to display this misconception on the post-interview, lending support to the notion that, as a consequence of the treatment group's developed knowledge of procedures associated with division with fractions, they possessed more ways of understanding division with fractions, thus granting them more access, compared to the control group, to alternative interpretations for symbolic division with fractions problems.

Because the notion that the treatment group's content-focused instruction was associated with correcting this misunderstanding, it is possible that aspects of the control group's content-focused instruction contributed to the resilience of this misconception. It is possible, for instance, that the reason the preservice teachers from the control group experienced difficulties applying interpretations of division with fractions within symbolic contexts is because these concepts were only presented within the context of a word problem during the content-focused instruction. Moreover, the preservice teachers in the control group practiced their understanding of the concepts of division with fractions by developing word problems using a specified number sentence and a particular interpretation of division with fractions. Although successfully conceiving pedagogically appropriate word problems is an indication of an adequate understanding of a given concept of division with fractions (Ma, 1999), perhaps the few practice problems did not provide the participants in the control group with enough experience in understanding division with fraction concepts in a more flexible manner.

Some Insight into Other Interesting Findings

On the pre-interview, several participants placed the divisor in the spot for the dividend when addressing a partitive division word problem (i.e., item 5). Because the preservice teachers had only developed an understanding of partitive word problems within the context of whole numbers before the intervention, I suggest that they experienced difficulties intuitively conceptualizing a fractional number of groups.

In support of this, when Graeber, Tirosh, and Glover (1989) examined preservice teachers' misconceptions of multiplication and division, they found that the majority of problems that the preservice teachers answered incorrectly comprised divisors that were larger than the dividend. That is, the preservice teachers would write numerical expressions of these problems whereby the divisor was situated in the number sentence as the dividend. Three of these problems, for instance, involved a decimal number divided by a larger whole number (e.g., $3.25 \div 5$, $.75 \div 5$, and $1.25 \div 5$). This being said, however, these types of problems were easier compared to those in which a whole number dividend was divided by a larger whole number. With further investigation of this pattern, Graeber et al. suggested that the only reason the divisor and dividend were not interchanged when the dividend was a decimal number was because the preservice teachers were fueled by the desire to avoid decimal number divisors. One participant, for example, reported that she initially did write $5 \div 3.25$, but changed her response when she realized that the divisor was a decimal number. Thus, the results from Graeber et al. imply that, among preservice teachers, the misconception that the divisor has to be a whole number appears to be stronger than the notion of the size of the divisor relative to the dividend.

Similarly, in the present study, several preservice teachers on the pre-interview confused the role of the dividend and the divisor when the divisor was a mixed fraction and was larger than the dividend (i.e., $2 \div 3\frac{1}{3}$). In accordance with Graeber et al. (1989), it is possible that the preservice teachers proposed a number sentence that situated the $3\frac{1}{3}$ as the dividend because they may have inadvertently misunderstood that divisors must be whole numbers.

In summary, the results from the current study elucidated four key findings regarding the preservice teachers' development of SCK-DF. Firstly, the content-focused instruction was found to improve the preservice teachers' overall SCK-DF. Moreover, the data addressing the second research objective supported the hypothesis that a combination of conceptual and procedural mathematical content of division with fractions generated greater gains in SCK-DF than the control group. In addition, the frequency with which the preservice teachers used formal vocabulary to communicate notions of division with fractions was greater on the post-interview for both groups. Finally, in general, the content-focused instruction contributed to a reduction in the presence of misconceptions of concepts and procedures associated with division with fractions for both groups.

More generally, these findings underscore the benefit of providing preservice teachers with content-focused instruction, particularly in the development of specialized content knowledge. This undertaking is essentially critical given the impact of a teacher's common and specialized content knowledge on his or her students' performance in mathematics (Hill et al., 2005). With this new information, this study makes a significant contribution to the growing interest in understanding how mathematics methods courses can effectively shape preservice teachers' mathematical knowledge before they join the workforce.

Limitations and Future Research

The findings from the current study, however, should be interpreted cautiously as a result of the small sample. Despite the fact that non-parametric statistical analysis revealed differences in the groups' SCK-DF following the content-focused instruction, this study should be conducted with a larger sample to confirm that the treatment effect can be replicated with a sample that more closely resembles the preservice teacher population. Another potential limitation of the current study is the number of conditions that were used to examine the preservice teachers' development of SCK-DF. In line with this reasoning, future research should consider including a condition where only procedural knowledge is addressed in a deep and meaningful manner to tease out the role of procedural knowledge in absence of conceptual understanding. Finally, preservice teachers' development of SCK was only examined within the context of one topic of elementary mathematics. Although there is a strong argument supporting my decision to examine the development of preservice teachers' SCK within the topic of division with fractions, an ideal extension of this study would be to examine this development longitudinally, and across several topics within elementary mathematics.

Irrespective of these limitations, the results from the current study are promising and suggest that because of the limited body of research that has examined preservice and inservice teachers' SCK, there are a number of pathways that future research could take. For example, it would be worth examining whether and how moderate or high levels of mathematics anxiety interfere when content-focused instruction is used a vehicle for developing preservice teachers' SCK. In addition, this research question would also contribute to the paucity of research that has examined whether experiencing mathematics anxiety compromises performance in mathematics in a variety of anxiety-evoking situations, such as attending a mathematics methods course.

Indeed, research that has examined the relationship between preservice teachers' conceptual and procedural knowledge of fractions and mathematics anxiety concluded that inadequacies in both types of mathematical content knowledge were associated with high levels of mathematics anxiety (Rayner et al., 2007). Further in this regard, and in line with the empirically supported links between working memory, mathematics anxiety, and performance in mathematics (Ashcraft & Kirk, 2001; Ashcraft, Kirk, & Hopko, 1998; Faust, Ashcraft, & Fleck, 1996; Hopko, Ashcraft, & Gute, 1998), Ashcraft and Kirk (2001) suggested that math anxious students may suffer from a lack of working memory resources during instruction; resources that are needed to deeply learn and understand mathematical concepts and procedures. Based on the prevalence of mathematics anxiety within the preservice teacher population (Baloğlu & Koçak, 2006; Bessant, 1995; Hembree, 1990; Kelly & Tomhave, 1985), it is plausible that experiencing mathematics anxiety could moderate any of the associated benefits of content-focused instruction.

Another interesting research question involves comparing the development of preservice teachers' SCK as a function of a variety of instructional methods designed to enhance preservice teachers' mathematical knowledge. With respect to the current study, the instruction sessions followed a model of direct instruction where the mathematical concepts and procedures associated with division with fractions were discussed, and the preservice teachers were required to complete practice problems to demonstrate that the instructional content was consolidated. Conversely, Quinn (2001) examined a mathematics methods course that was premised on the importance of incorporating the appropriate use of manipulatives, technology, and cooperative learning to effectively facilitate preservice teachers' construction of meaningful mathematical knowledge. Quinn found that this strategy of instruction significantly increased the preservice teachers' mathematical content knowledge upon completion of the mathematics methods course. In addition, other research has demonstrated the benefits of complex problem solving on student performance in mathematics (Cognition and Technology Group at Vanderbilt, 1992), particularly in a cooperatively learning environment. Taken together, given that these pedagogical techniques enhance mathematical learning, it is possible that they would equally promote preservice teachers' development of a variety of types of mathematical knowledge, including SCK.

Moreover, although I had not initially intended to examine the frequency of formal vocabulary and the prevalence of the participants' misconceptions of division with fractions, the investigation of these constructs nonetheless contributed to understanding the preservice teachers' knowledge of division with fractions prior to and following the content-focused instruction. The results from this examination could inspire future research to consider the role of mathematical language in preservice teachers' or elementary students' gradual construction of stronger schemas of mathematical concepts and procedures.

Finally, it would be valuable to ascertain whether the association between content-focused professional development and inservice teachers' development of SCK mirrors the relationship I found between content-focused instruction and preservice teachers' development of SCK. As I previously revealed in Chapter 2, a comparison of the findings from Ma (1999) to what was found in Ball (1990a), Tirosh (2000), and Tirsoh et al. (1998) suggest that the mathematical knowledge inservice teachers possess is different to that of preservice teachers. In particular, Ma (1999) indicated that inservice teachers demonstrated inadequacies in his or her conceptual *and* procedural knowledge of division with fractions. As such, the type of content that facilitates preservice teachers' gains in SCK may have a differential impact on inservice teachers' development of this type of mathematical knowledge for teaching.

Conclusions

The results from the present study offer two important contributions to the research on enhancing preservice teachers' specialized content knowledge. The first is that content-focused instruction has the potential of promoting the development of preservice teachers' SCK. The potential effect of content-focused instruction offers a new perspective on the body of literature concerned with the content of mathematics methods courses. Understanding that improvements in preservice teachers' mathematical content knowledge facilitates SCK is particularly important based on the findings from Hill et al. (2005).

The second contribution from the current study concerns the finding that greater gains in SCK-DF was associated with instruction that comprised a combination of conceptual and procedural content regarding division with fractions. This finding outlines the role of procedural knowledge in mathematics learning and builds on the findings from Star and Seifert (2006) by demonstrating that deep procedural knowledge can be developed during instruction on elementary mathematics. Further in this regard, while it is not clear whether deep procedural knowledge can exist for all topics addressed during elementary mathematics, it is evident that preservice teachers' conceptual and procedural competence should be emphasized during their coursework with the goal of developing a deep understanding of both types of content knowledge.

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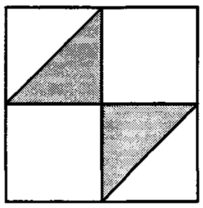
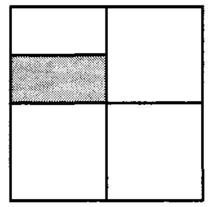
Appendix A

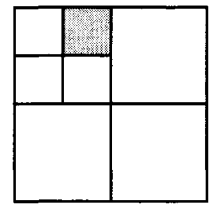
Knowledge of Fractions Assessment

Name______________________________

1) For each picture below, write a fraction to show what part is gray:

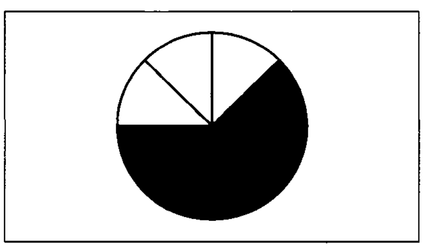
1.1) 

1.2) 

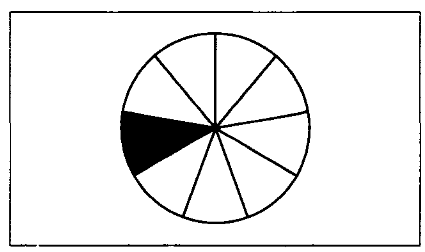
1.3) 

1.1) ______________ 1.2) ________________ 1.3) ____________

2) Circle the fractions that show what part of each circle below is gray:



2.1) $\frac{1}{4}$ $\frac{3}{5}$ $\frac{9}{10}$



2.2) $\frac{1}{9}$ $\frac{1}{3}$ $\frac{2}{5}$

3) Compute the following equations:

3.1)
$$\begin{array}{r} \frac{3}{5} \\ +\frac{1}{5} \\ \hline \end{array}$$

3.2)
$$\begin{array}{r} \frac{2}{10} \\ +\frac{2}{5} \\ \hline \end{array}$$

3.3) $\begin{array}{r} \frac{1}{3} \\ +\frac{1}{2} \\ \hline \end{array}$

3.4) $\begin{array}{r} 7\frac{5}{8} \\ +4\frac{1}{2} \\ \hline \end{array}$

3.5) $\begin{array}{r} \frac{7}{10} \\ -\frac{1}{10} \\ \hline \end{array}$

3.6) $\begin{array}{r} \frac{5}{6} \\ -\frac{1}{3} \\ \hline \end{array}$

3.7) $\begin{array}{r} \frac{2}{3} \\ -\frac{1}{2} \\ \hline \end{array}$

4) Write one fraction that is the same as each fraction below, for example: $\frac{1}{2}=\frac{2}{4}$

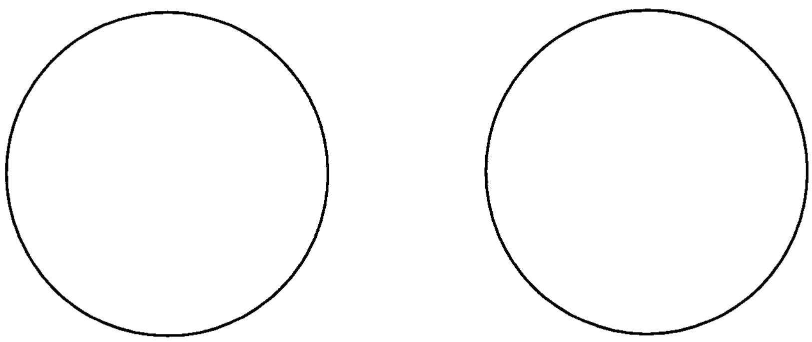
4.1) $\frac{2}{6}=$

4.2) $\frac{1}{5}=$

4.3) $\frac{12}{16}=$

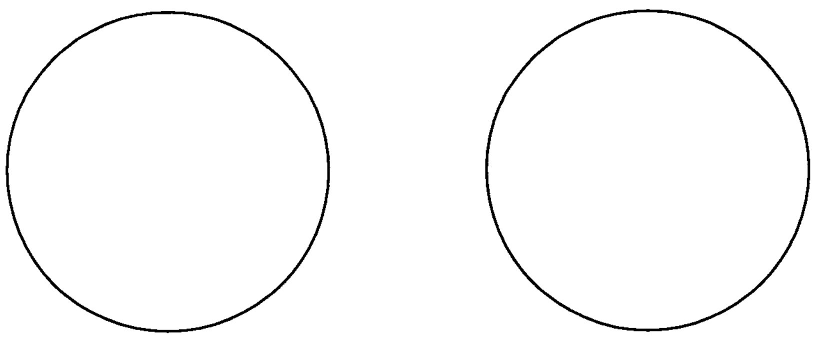
4.4) $\frac{7}{6}=$

5) a. *Four* people are going to share these two pizzas equally. Colour in **one** person's part.



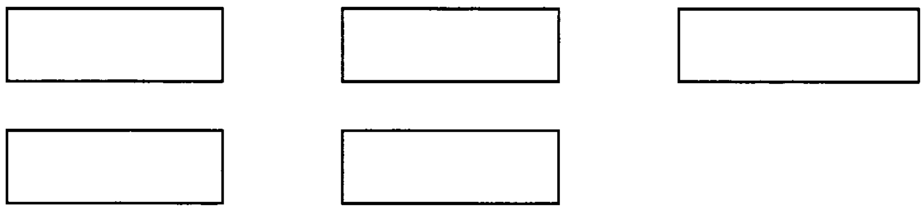
b. Write a fraction that shows how much **one** person gets ______________

6) a. *Three* people are going to share these pizzas equally. Colour in **one** person's part.



b. Write a fraction that shows how much **one** person gets ______________

7) a. *Six* people are going to share these five chocolate bars equally. Colour in **one** person's part.



b. Write a fraction that shows how much **one** person gets ______________

8) Fill in the missing numbers:

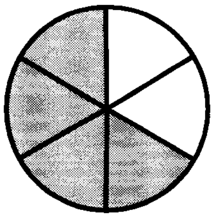
8.1) $\frac{1}{5}=\frac{\square}{10}$

8.2) $\frac{3}{4}=\frac{\square}{8}$

8.3) $2\frac{1}{2}=\frac{\square}{2}$

8.4) $3\frac{1}{4}=\frac{\square}{8}$

9) Circle a, b, c, or d below to show what part of this circle is gray:



a) $\frac{1}{2}+\frac{1}{3}$ b) $\frac{3}{6}+\frac{1}{6}$ c) $1+\frac{1}{3}$ d) 4

10a) John ran $\frac{2}{5}$ of a mile on Thursday and $\frac{3}{5}$ of a mile on Friday. How far did he run altogether on the two days? ______________________________

10 b.) Draw a picture to show your work.

Appendix B

Pre-interview Items

<u>**Pre-interview of Specialized Content Knowledge of Division with Fractions**</u>

Part I and II. The following problems were given to a fifth-grade class to practice division with fractions.

1) $\frac{1}{4} \div 4$

2) $\frac{1}{4} \div \frac{3}{5}$

3) $320 \div \frac{1}{3}$

Below are some students' answers to these problems.

For $\frac{1}{4} \div 4$, Mary's solution looked like this:

$$\boxed{\frac{1}{4} \div 4 = 4 \div 4 = 1}$$

<u>Question:</u>

a) In analyzing her answer, what is Mary's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

For $\frac{1}{4} \div \frac{3}{5}$, Jim's answer was:

"I can't do this because $\frac{3}{5}$ is bigger than $\frac{1}{4}$, so you can't share less among more"

<u>Question:</u>

a) In analyzing his answer, what is Jim's understanding of division with fractions?

b) What problem would you give him to help correct his misconceptions of division with fractions?

For $320 \div \frac{1}{3}$, Billy's answer was:

$$\boxed{320 \div \frac{1}{3} = \frac{1}{3} \div 320 = \frac{1}{3} \times \frac{1}{320} = \frac{1}{960}}$$

Question:

a) In analyzing his answer, what is Billy's understanding of division with fractions?

b) What problem would you give him to help correct his misconceptions of division with fractions?

Part I and II cont. The following word problems were written on the blackboard in a Mr. Heads fifth grade classroom. He then asked each of the students to solve the word problem and write the number sentence reflected in the word problems. Here are some examples of some of the answers two students provided.

Kate's mom gave 6 bars of chocolate to her and her friends. If each person had $\frac{1}{2}$ of a chocolate, how many friends had some chocolate?

Suzy's number sentence for this word problem was

$$\boxed{6 \div 2 = 3}$$

Question:

a) In analyzing her answer, what is Suzy's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

Sabrina's answer for the following word problem was

Jerry bought $3\frac{1}{3}$ *pounds of flour for \$2.00. How much did he pay per pound?*

$$\boxed{2 \times 3\frac{1}{3} = \frac{20}{3} \text{ or \$6.66 per pound}}$$

<u>Question:</u>

a) In analyzing her answer, what is Suzy's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

Part III. During a class on division with fractions, one of your students asks you to explain why the standard algorithm works. You have a word problem and a number sentence written up on the blackboard already and decide to use both of those representations to answer the student's question.

The word problem is the following:

Larry's vegetable garden is 7 feet long. If each vegetable takes up $\frac{3}{5}$ of a foot, how many different types of vegetables can Larry plant?

The number sentence is:

$$\boxed{\frac{5}{12} \div \frac{3}{8}}$$

Appendix C

Post-interview Items

<u>Post-interview of Specialized Content Knowledge of Division with Fractions</u>

Part I and II. The following problems were given to a fifth-grade class to practice division with fractions.

1) $\frac{1}{6} \div 6$

2) $\frac{1}{3} \div \frac{4}{7}$

3) $410 \div \frac{1}{2}$

Below are some students' answers to these problems.

For $\frac{1}{6} \div 6$, Shira's solution looked like this:

$$\boxed{\frac{1}{6} \div 6 = 6 \div 6 = 1}$$

<u>Question:</u>

a) In analyzing her answer, what is Shira's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

For $\frac{1}{3} \div \frac{4}{7}$, Phil's answer was:

"I can't do this because $\frac{4}{7}$ is bigger than $\frac{1}{3}$, so you can't share less among more"

Question:

a) In analyzing his answer, what is Phil's understanding of division with fractions?

b) What problem would you give him to help correct his misconceptions of division with fractions?

For, $410 \div \frac{1}{2}$ David's answer was:

$$\boxed{410 \div \frac{1}{2} = \frac{1}{2} \div 410 = \frac{1}{2} \times \frac{1}{410} = \frac{1}{820}}$$

Question:

a) In analyzing his answer, what is David's understanding of division with fractions?

b) What problem would you give him to help correct his misconceptions of division with fractions?

Part I and II cont. The following word problems were written on the blackboard in a Mr. Heads fifth grade classroom. He then asked each of the students to solve the word problem and write the number sentence reflected in the word problems. Here are some examples of some of the answers two students provided.

Harriet has a paper to write for her political science course. She has set aside 10 hours to write the paper. It takes her $\frac{1}{2}$ of an hour to write a page. How many pages can she write in 10 hours?

Jen's number sentence for this word problem was

$$\boxed{10 \div 2 = 5}$$

<u>Question:</u>

a) In analyzing her answer, what is Jen's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

Jamie's answer for the following word problem was

Nigella used $4\frac{1}{4}$ *cups of flour to bake 2 lemon meringue cakes for a dinner party. Shelly, one of her dinner guests loved the cake so much she decided to bake one lemon meringue cake the next day. How much flour did Shelly use to bake this cake?*

$$\boxed{2\times 4\frac{1}{4}=\frac{34}{4} \text{ or } 8\frac{1}{2} \text{ cups of flour}}$$

<u>Question:</u>

a) In analyzing her answer, what is Jamie's understanding of division with fractions?

b) What problem would you give her to help correct her misconceptions of division with fractions?

Part III. During a class on division with fractions, one of your students asks you to explain why the standard algorithm works. You have a word problem and a number sentence written up on the blackboard already and decide to use both of those representations to answer the student's question.

The word problem is the following:

Brian is 5 feet tall. He decides that he wants to measure himself and he notices that he has a bunch of pieces of string on his desk. If every piece of string is $\frac{2}{7}$ *of a foot, how many pieces does Brian need to figure out how tall he is?*

The number sentence is:

$$\boxed{\frac{7}{15} \div \frac{5}{11}}$$

Appendix D

Assessment of Content

Assessments of Content for the Control Group

Instruction Session 1

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. Using the interpretation of division with fractions that was discussed today, what would you say would be a good story or model for $9 \div \frac{3}{4}$?

What about for $4\frac{2}{5} \div \frac{1}{2}$?

Instruction Session 2 & 3

Assessments of content will be identical to the assessment of content for Instruction Session 1 except the numbers will be different.

Assessments of Content for the Treatment Group

Instruction Session 1

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story problems to show the application of some particular piece of content. Using the interpretation of division with fractions that was discussed today, what would you say would be a good story or model for $9 \div \frac{3}{4}$?

What about for $4\frac{2}{5} \div \frac{1}{2}$?

Instruction Session 2

For the following two division with fractions problems, explain why $\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$.

$$3\frac{8}{11} \div \frac{5}{6}$$

$$\frac{6}{13} \div \frac{3}{7}$$

Instruction Session 3

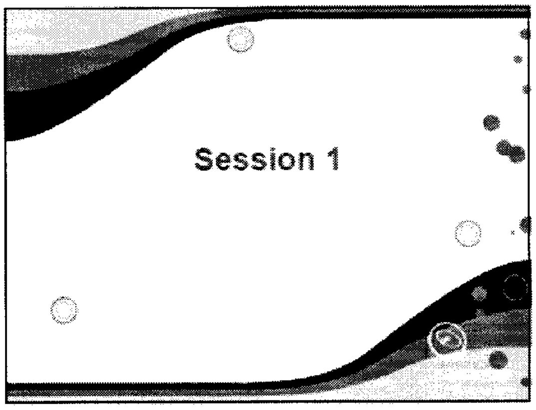
For the following two division with fractions problems, solve each problem twice using a different strategy each time.

$$\frac{8}{9} \div \frac{3}{5}$$

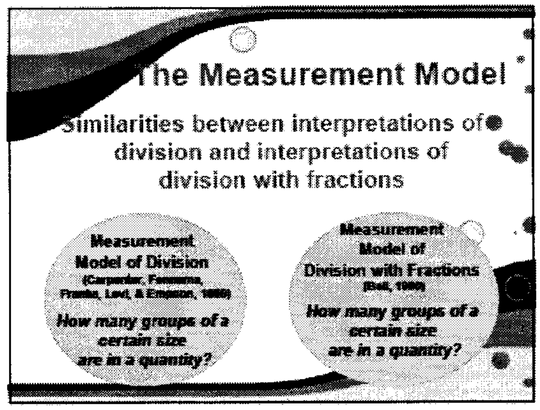
$$\frac{6}{11} \div 5\frac{3}{20}$$

Appendix E

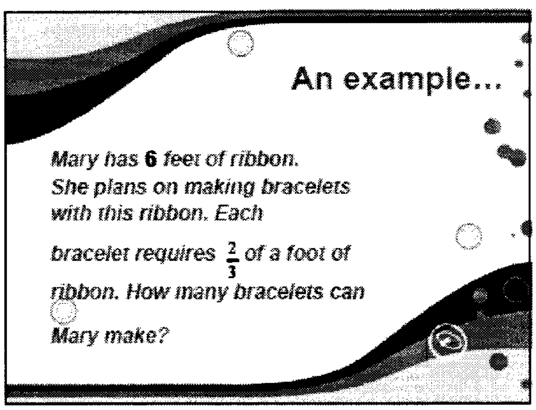
Content of Instruction



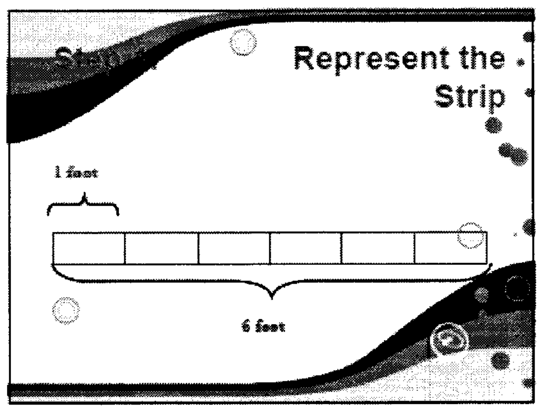
Session 1



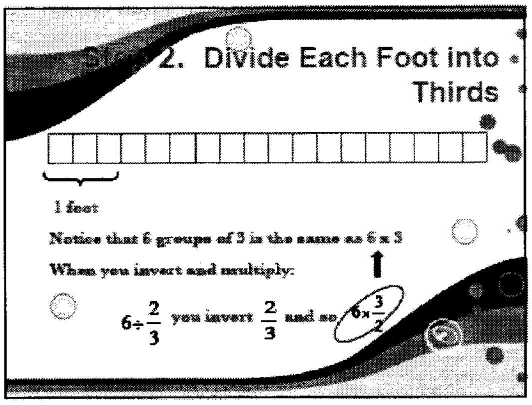
The Measurement Model
Similarities between interpretations of division and interpretations of division with fractions
Measurement Model of Division
How many groups of a certain size are in a quantity?
Measurement Model of Division with Fractions
How many groups of a certain size are in a quantity?

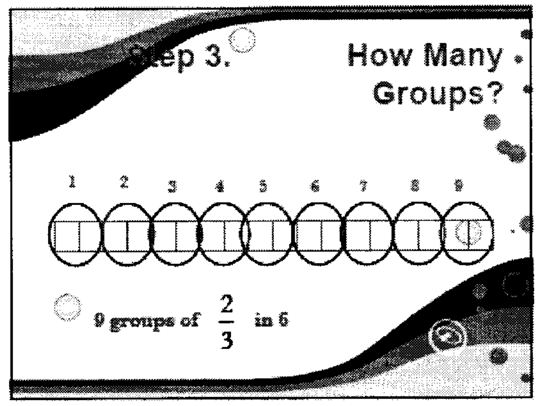


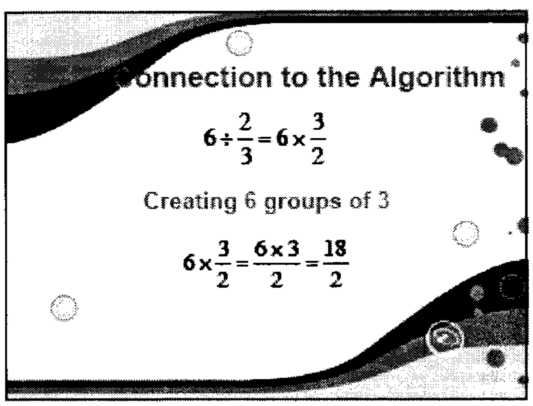
An example...
Mary has **6** feet of ribbon. She plans on making bracelets with this ribbon. Each bracelet requires $\frac{2}{3}$ of a foot of ribbon. How many bracelets can Mary make?

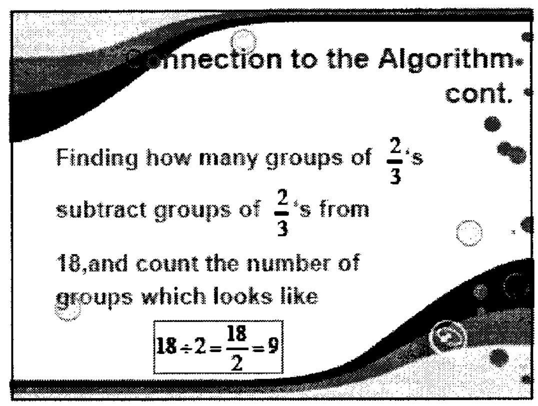


Represent the Strip
1 foot
6 feet


2. Divide Each Foot into Thirds
1 foot
Notice that 6 groups of 3 is the same as 6 x 3
When you invert and multiply:
$6\div\frac{2}{3}$ you invert $\frac{2}{3}$ and so $6\times\frac{3}{2}$


How Many Groups?
1 2 3 4 5 6 7 8 9
9 groups of $\frac{2}{3}$ in 6


onnection to the Algorithm
$6\div\frac{2}{3}=6\times\frac{3}{2}$
Creating 6 groups of 3
$6\times\frac{3}{2}=\frac{6\times3}{2}=\frac{18}{2}$

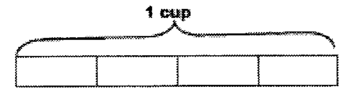

nection to the Algorithm cont.
Finding how many groups of $\frac{2}{3}$'s
subtract groups of $\frac{2}{3}$'s from
18,and count the number of
groups which looks like
$\boxed{18\div2=\frac{18}{2}=9}$

Fraction Divided by a Fraction

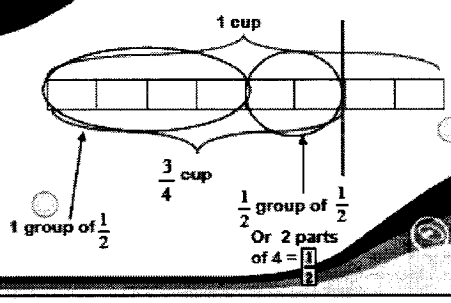
$$\frac{3}{4} \div \frac{1}{2}$$

Stacy has $\frac{3}{4}$ of a cup of flour. For her brownie recipe she needs $\frac{1}{2}$ cup. If she wants to use all of her flour, how many times can she use the recipe?

How can this be solved?

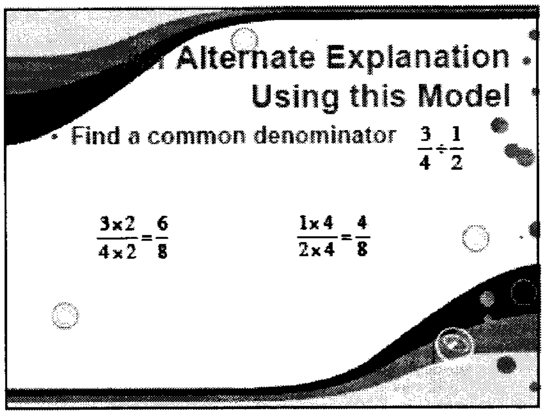


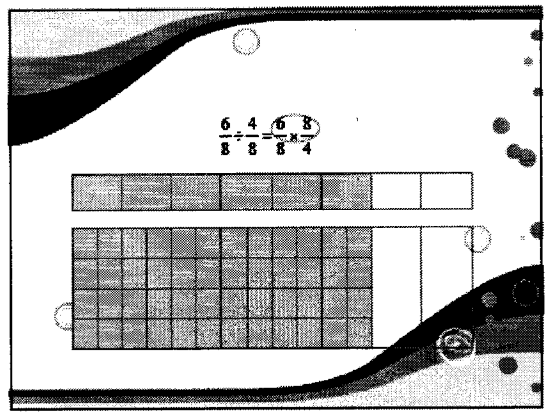
How many groups of $\frac{1}{2}$ are in $\frac{3}{4}$?

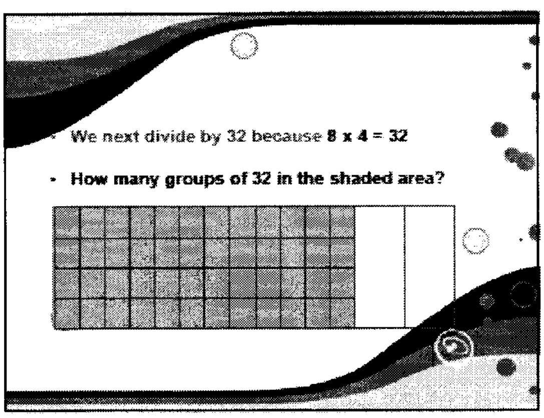


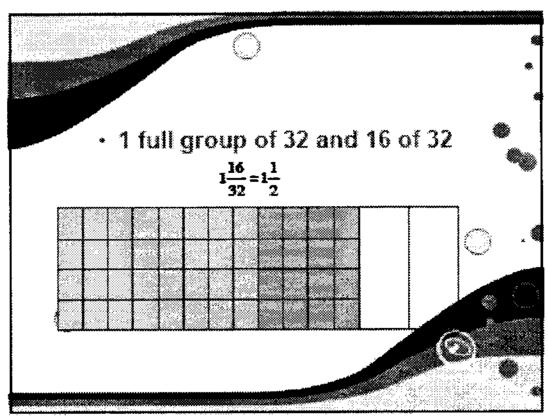
Some Problems to Consider

- When trying to make connections between the model and the algorithm do not illustrate 3 groups of 2, but you do "see" the 6


Alternate Explanation
Using this Model
Find a common denominator $\frac{3}{4} \div \frac{1}{2}$
$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}$
$\frac{1 \times 4}{2 \times 4} = \frac{4}{8}$


$\frac{6}{8} \div \frac{4}{8} = \frac{6}{8} \times \frac{8}{4}$


We next divide by 32 because 8 x 4 = 32
How many groups of 32 in the shaded area?


1 full group of 32 and 16 of 32
$1\frac{16}{32} = 1\frac{1}{2}$

Control Group

Session 2

The Partitive Model

- The number in each group is unknown
(Carpenter, Fennema, Franke, Levi, Empson, 1999)

Gene has 4 tomato plants. There are the same number of tomatoes on each plant. All together there are 20 tomatoes. How many tomatoes are there on each tomato plant?

Division With Fractions

- The interpretation extends itself to division with fractions
- <u>Two situations</u>
  1. Division of a fraction by a whole number
  2. Division of a fraction by a fraction

Whole-Number Divisors

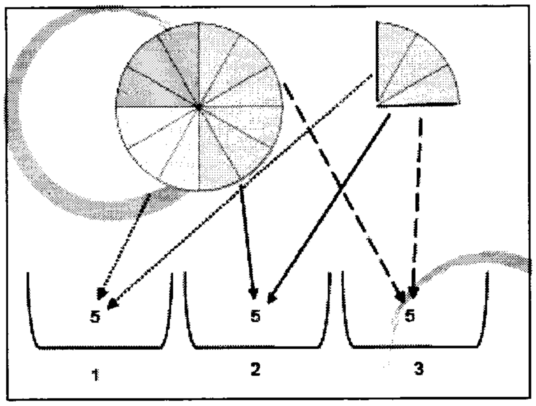
- The total amount is a fraction
- The divisor is a whole number

Fraction ÷ Whole Number (i. e., number of groups) **=**
the size of each group

For example…

It takes Suzy $1\frac{1}{4}$ hours to do 3 of her chores.

How much time does it take to do one chore?



Fractional Divisor

- Total amount is a fraction
- Divisor is a fraction
- Still looking for the size of one group

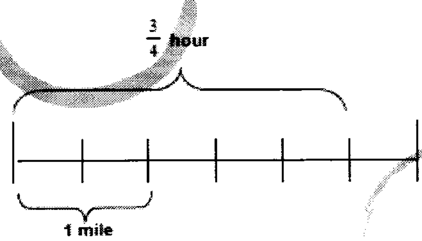
Fraction ÷ Fractional Number of Groups
= the size of each group

For example…

Aidan found out that if she walks really fast during her morning exercise, she
can cover $2\frac{1}{2}$ miles in $\frac{3}{4}$ of an hour.

She wonders how fast she is walking in miles per hour.

Using a Number line



What to do?

- To find out how many miles are walked in 1 hour, find how many miles are walked in $\frac{1}{4}$ of an hour
- How many halves are in each quarter or distributing halves across 3 groups

- In $2\frac{1}{2}$ miles, there are 5 halves
- If we put 1 half in each group, 2 halves remain
- To distribute 2 halves in 3 groups,

  consider that $\frac{1}{2}=\frac{3}{6}$, therefore the 2 remaining

  halves can be evenly distributed into 3 groups,

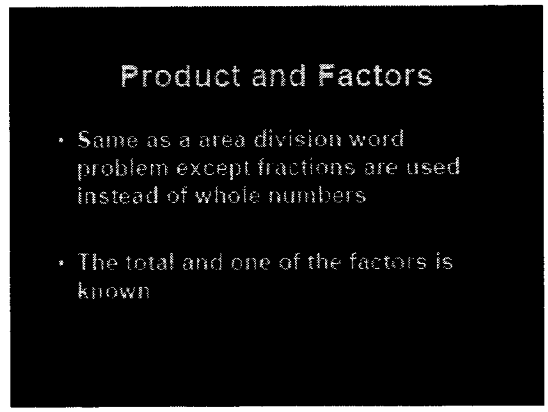
  whereby each group receives an additional $\frac{2}{6}=\frac{1}{3}$

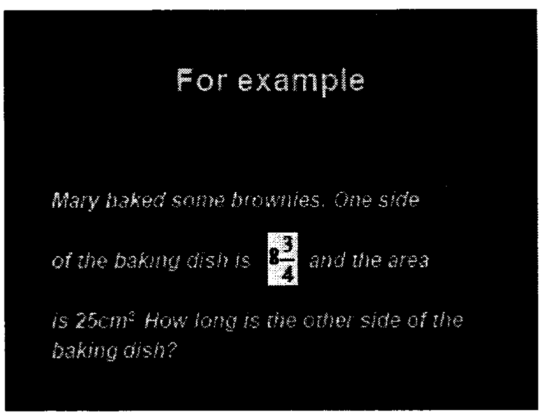
$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

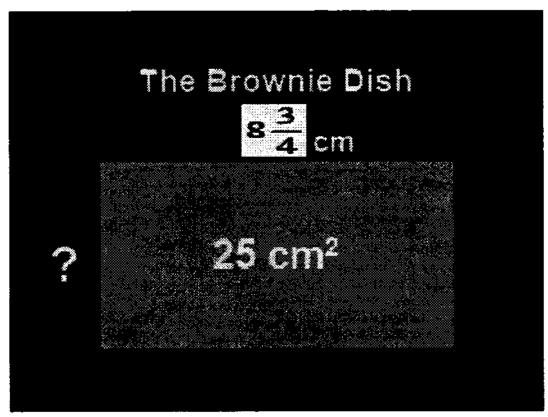
$$\frac{5}{6}\times 4=\frac{20}{6}=3\frac{2}{6}=3\frac{1}{3}$$

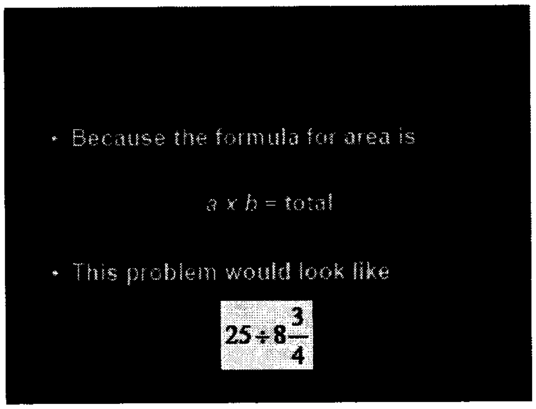
Aidan walks $3\frac{1}{3}$ miles in 1 hour

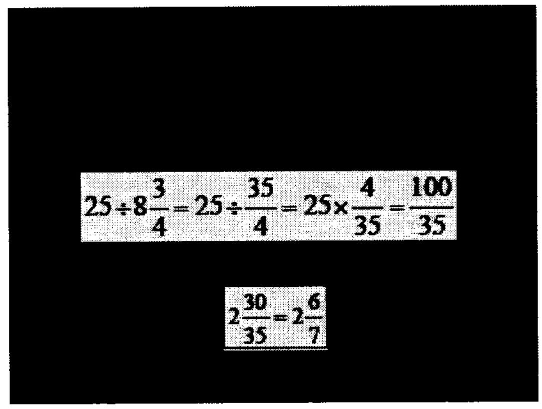

Session 3

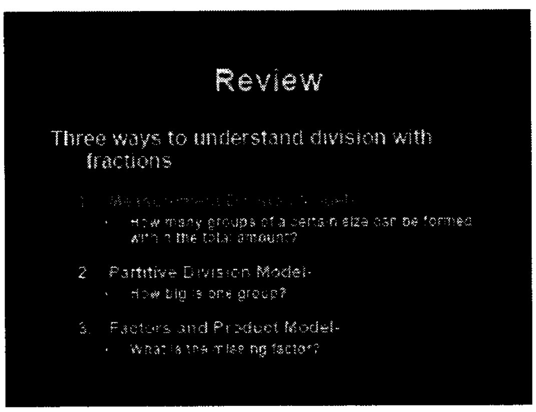

Product and Factors
• Same as a area division word problem except fractions are used instead of whole numbers
• The total and one of the factors is known

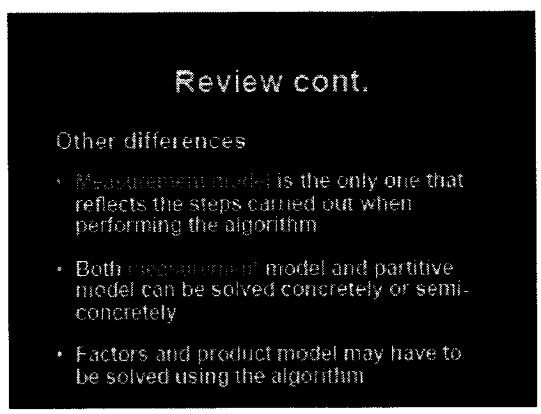

For example
Mary baked some brownies. One side of the baking dish is $8\frac{3}{4}$ and the area is 25cm^2 How long is the other side of the baking dish?


The Brownie Dish
$8\frac{3}{4}$ cm
?
25 cm^2


• Because the formula for area is
a x b = total
• This problem would look like
$25 \div 8\frac{3}{4}$


$25 \div 8\frac{3}{4} = 25 \div \frac{35}{4} = 25 \times \frac{4}{35} = \frac{100}{35}$
$2\frac{30}{35} = 2\frac{6}{7}$


Review
Three ways to understand division with fractions
1.
• How many groups of a certain size can be formed within the total amount?
2. Partitive Division Model-
• How big is one group?
3. Factors and Product Model-
• What is the missing factor?


Review cont.
Other differences
• is the only one that reflects the steps carried out when performing the algorithm
• Both model and partitive model can be solved concretely or semi-concretely
• Factors and product model may have to be solved using the algorithm

Session 2

The Algorithm

- Can the algorithm be taught using procedures and still be meaningful for the students?
- Understanding is important
  1. Multiple strategies
  2. Understanding the mathematical reasoning that the algorithm is based on
  3. Fractions that are not conducive to semi-concrete or concrete methods

An example...

Mary wants to buy a series of desks for her room. The wall that she will place them against measures $10\frac{3}{16}$ *feet. Each desk is* $4\frac{17}{33}$ *feet long. How many desks should Mary buy?*

A Key Factor in Understanding the Algorithm

- The inverse and multiply rule is based on the principle that *division of whole numbers is the inverse of multiplying whole numbers* (Wu, 1999)
- So...

$m \div n = k$ is the same as $m = n \times k$, when m, n, and k are whole numbers

Extending the Rule

$m \div n = k$ is the same as $m = n \times k$; when m, n, and k are whole numbers

$$\frac{a}{b} \div \frac{c}{d} = \frac{x}{y} \quad \text{is the same as} \quad \frac{a}{b} = \frac{c}{d} \times \frac{x}{y}$$

$m \div n = k$ $\qquad$ $m = n \times k$

Solve for "k" or $\frac{x}{y}$

Multiply both sides of $\frac{a}{b} = \frac{c}{d} \times \frac{x}{y}$

by $\frac{d}{c}$ so $\left(\frac{d}{c}\right) \times \frac{a}{b} = \cancel{\left(\frac{d}{c}\right) \times \frac{c}{d}} \times \frac{x}{y}$

and then $\boxed{\frac{x}{y} = \frac{d}{c} \times \frac{a}{b}}$

For this Reason

$$\frac{a}{b} \div \frac{c}{d} = \frac{d}{c} \times \frac{a}{b}$$

Back to the example…

Mary wants to buy a series of desks for her room. The wall that she will place them against measures $10\frac{3}{16}$ feet. Each desk is $4\frac{17}{33}$ feet long. How many desks should Mary buy?

Division is the Inverse of Multiplication

$10\frac{3}{16} \div 4\frac{17}{33} = ?$ $\quad$ $10\frac{3}{16} = 4\frac{17}{33} \times ?$

Solve for "?"

$$4\frac{17}{33} = \frac{149}{33}$$

$$\left(\frac{33}{149}\right) \times 10\frac{3}{16} = \frac{149}{33} \times \left(\frac{33}{149}\right) \times ?$$

$$\left(\frac{33}{149}\right) \times 10\frac{3}{16} = 1 \times ?$$

Finish the Calculation Using the Distributive Property

$$\frac{33}{149} \times 10\frac{3}{16} = \left(\frac{33}{149}\right) \times \left(10 + \frac{3}{16}\right)$$

$$\left(\frac{33}{149} \times 10\right) + \left(\frac{33}{149} \times \frac{3}{16}\right)$$

Distributive Property Continued

$$\frac{330}{149} + \frac{99}{2384}$$

$$\frac{330 \times 16}{149 \times 16} + \frac{99}{2384}$$

$$\frac{5280}{2384} + \frac{99}{2384}$$

$$\frac{5379}{2384} = \boxed{2\frac{611}{2384}}$$

Session 3

Other Algorithms

Do you think it is possible to use an algorithm other than the "inverse and multiply" algorithm?

Two Alternate Strategies

1. Common Denominator

2. Complex Fractions

Key principles

(where *a* can be any number)

1. The product of a number and its reciprocal is 1

$$a \times \frac{1}{a} = 1$$

2. A number divided by 1 is that number

$$a \div 1 = a$$

Key Principles Cont.

3. Equivalent fractions
   - Two fractions are = when the numerator and denominator is multiplied by the same number
   - Because $\frac{a}{a} = 1$, it is the same as multiplying the fraction by 1

Applying the Rule for Common Denominator

Step 1 Find a common denominator for both values

Step 2 Divide the 1st numerator by the 2nd

Step 3 Divided the 1st denominator by the 2nd

Seeing the Common Denominator Rule in Action

For the following problem: $\boxed{4\frac{2}{3} \div 1\frac{1}{6}}$

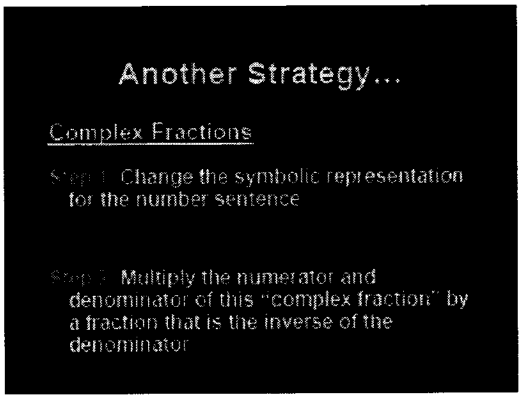
Step 1. $\frac{14}{3} \div \frac{7}{6} = \left(\frac{2 \times 14}{2 \times 3}\right) \div \frac{7}{6}$

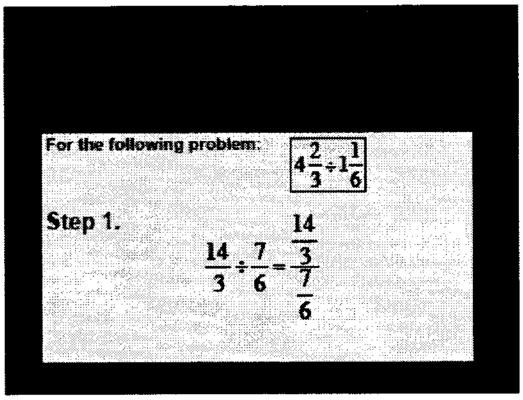
(notice this is where the principle for equivalent fractions comes in)

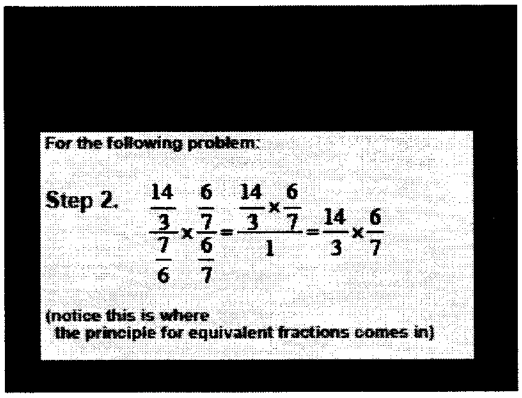
Step 2 & 3. $\frac{28}{6} \div \frac{7}{6} = \frac{28 \div 7}{6 \div 6} = \frac{4}{1}$

The Answer

$\frac{4}{1} = 4$ because $a \div 1 = a$


Another Strategy…
Complex Fractions
Step 1: Change the symbolic representation for the number sentence
Step 2: Multiply the numerator and denominator of this "complex fraction" by a fraction that is the inverse of the denominator


For the following problem: $\boxed{4\frac{2}{3} \div 1\frac{1}{6}}$
Step 1.
$\frac{14}{3} \div \frac{7}{6} = \frac{\frac{14}{3}}{\frac{7}{6}}$


For the following problem:
Step 2.
$\frac{\frac{14}{3}}{\frac{7}{6}} \times \frac{\frac{6}{7}}{\frac{6}{7}} = \frac{\frac{14}{3} \times \frac{6}{7}}{1} = \frac{14}{3} \times \frac{6}{7}$
(notice this is where the principle for equivalent fractions comes in)

Appendix F

Consent Form

Concordia
UNIVERSITY

CONSENT FORM TO PARTICIPATE IN RESEARCH

I agree to participate in a program of research being conducted by Vanessa Rayner, a graduate student in the MA Child Study Program, of the Department of Education at Concordia University, in collaboration with her advisor, Dr. Helena Osana, of the Department of Education at Concordia University.

A. PURPOSE

I have been informed that the purpose of the research is to study the development of specialized mathematics knowledge of division with fractions. This will involve providing preservice teachers with one of two types of instruction on this topic. Prior to and subsequent to receiving instruction on this topic, the preservice teachers' specialized knowledge of division with fractions will be assessed through two individual interviews to detect any changes in their understanding. In addition, another purpose of the study is to determine which type of instruction yields greater gains in their specialized knowledge of this topic. This research is important in part because it will help Dr. Osana best organize and teach future mathematics methods courses for students at Concordia University.

B. PROCEDURES

The research will be conducted at the university. I will be asked to participate in a two 45-minute individual interviews that will be digitally audio-recorded. These interviews will be conducted before and after three 1- hour and 15-minute instruction sessions that will take place once a week for three consecutive weeks. During these sessions, I will be audio-recorded and towards the end of each instruction session, I will be asked to complete some paper-and-pencil practice problems on the topic of fractions for 20 minutes.

C. CONDITIONS OF PARTICIPATION

- My participation in the research project is completely voluntary and confidential.
- Should I agree to participate, only the Principal Investigator, Ms. Vanessa Rayner, will know of my decision, and will not inform the instructor of this course. Only the Principal Investigator will have access to the consent forms and will keep them in a secure location during the project's duration.
- Should I agree to participate, I will not inform Dr. Helena Osana of my participation until after the final grades for EDUC 388/4 have been submitted. Additionally, Dr. Osana will not have access to any information that will be able to identify me as a participant.
- During the project's duration, all data will be stored in a locked box in a locked research office, and will be accessed only by the Principal Investigator. During and after the project , all data will be stored in a locked cabinet in a locked research office, and will be accessible only to the Principal Investigator.
- I understand that the data from this study may be presented and/or published, but no identifying information will be disclosed.
- My participation in the research project will not be graded, nor it will it influence my grade in EDUC 388/4.
- If I decide at any time that I do not wish to continue participating, I have been told to contact Ms. Vanessa Rayner, at v_rayner@education.concordia.ca (or on FirstClass) or 514-240-7134 and all information about and from me will be erased from the research files.
- If I have any questions at any time about this project, I have been informed to contact Ms. Vanessa Rayner, at v_rayner@education.concordia.ca (or on FirstClass).

I HAVE CAREFULLY STUDIED THE ABOVE AND UNDERSTAND THIS AGREEMENT. I FREELY CONSENT AND VOLUNTARILY AGREE TO PARTICIPATE IN THIS STUDY.

FULL NAME (please print): ____

SIGNATURE: ____

DATE: ____

Appendix G

Coding Sheet

Ss Name	Interview	Item		Evidence	S Codes	P Codes	Score (0,1, 2)
Response addresses the concept of division							
Response notices the errors in the algorithm (inverting the wrong number and dividing versus multiplying							
Alternate error identified by Ss							
		1b					
Demonstrates misconceptions of division with fractions while addressing the follow-up problem							
The follow-up problem suggested would address the misconceptions suggested earlier							
		2a					
Response addresses the concept of division							
Response notices the errors in the students statement (the divisor can be bigger)							
Alternate error identified in the students response							
Misunderstands the students statement (i.e., believe that the first half of the students statement is false and/or that the second half is true)							
		2b					
Demonstrates misconceptions of division with fractions while addressing the follow-up problem							
The follow-up problem suggested would address the misconceptions suggested earlier							
		3a					
Response addresses the concept of division							
Response notices the errors in the students answer (misuse of the commutative property)							
Response notices that the algorithm was applied correctly							
Alternate error was identified							
		3b					
Demonstrates misconceptions of division with fractions while addressing the follow-up problem							
The follow-up problem suggested would address the misconceptions suggested earlier							
		4a					
Associates the problem as a multiplication problem with whole numbers							
Associates the problem as a measurement division problem							
Identifies students misunderstanding of the							

problem (that it is a measurement division problem, looking for the number of groups vs. a partitive division problem where the total is being split in 1/2 or divided into 2 groups)				
Believes that the students error is not mathematical				
Explains error from a mathematical perspective. Use of terminology for concepts of division (i.e., measurement or partitive).				
4b				
Use the appropriate model of division as a follow-up problem (i.e., measurement)				
Use alternate model of division or operation to address misconception				
5a				
Associates the problem as a partitive division problem				
Identifies students misunderstanding of the problem (that it is a partitive division problem, looking for the size of each group vs. a multiplication problem where the number of groups and number in each group is known)				
Believes that the students error is not mathematical				
Explains error from a mathematical perspective. Use of terminology for concepts of division (i.e., measurement or partitive).				
5b				
Use the appropriate model of division as a follow-up problem (i.e., partitive)				
Use alternate model of division or operation to address misconception				
6				
Appropriately justifies why the algorithm involves multiplication				
Appropriate justifies why the algorithm involves inverting the second number in the number sentence				
*Explains the remainder (notice whether the model was completed during the explanation or whether the accurate answer was calculated)				
7				
Appropriately justifies why the algorithm involves multiplication				

Appropriate justifies why the algorithm involves inverting the second number in the number sentence					
*Explains the remainder (notice whether the model was completed during the explanation or whether the accurate answer was calculated)					

Appendix H

Interview Scoring Rubric

Item	Score	Scoring Guideline
1a	0	**No error or unreasonable error identified.**
	1	**Reasonable error identified.**
	2	**Correct error identified.**
1b	0	*Follow-up problem does not address misconception.*
	1	*Follow-up problem addresses finding the correct answer and/or not all misconceptions identified were addressed.*
	2	*Follow-up problem addresses all misconceptions identified.*
2a	0	**No error or unreasonable error identified.**
	1	**Reasonable error identified.**
	2	**Correct error identified.**
2b	0	*Follow-up problem does not address misconception.*
	1	*Follow-up problem addresses finding the correct answer and/or not all misconceptions identified were addressed.*
	2	*Follow-up problem addresses all misconceptions identified.*
3a	0	**No error or unreasonable error identified.**
	1	**Reasonable error identified.**
	2	**Correct error identified.**
3b	0	*Follow-up problem does not address misconception.*
	1	*Follow-up problem addresses finding the correct answer and/or not all misconceptions identified were addressed.*
	2	*Follow-up problem addresses all misconceptions identified.*
4a	0	**No error or unreasonable error identified.**
	1	**Reasonable error identified.**
	2	**Correct error identified.**
4b	0	*Follow-up problem does not address misconception.*
	1	*Follow-up problem addresses finding the correct answer and/or not all misconceptions identified were addressed.*
	2	*Follow-up problem addresses all misconceptions identified.*
5a	0	**No error or unreasonable error identified.**
	1	**Reasonable error identified.**
	2	**Correct error identified.**
5b	0	*Follow-up problem does not address misconception.*
	1	*Follow-up problem addresses finding the correct answer and/or not all misconceptions identified were addressed.*
	2	*Follow-up problem addresses all misconceptions identified.*
6	0	<u>Unable to link response to the steps of the algorithm.</u>
	1	<u>Can only justify part of the algorithm. Note if answer was calculated and remainder not explained, score as 1.</u>
	2	<u>Able to justify the algorithm and if the accurate answer was calculated the remainder was explained.</u>
7	0	<u>Unable to link response to the steps of the algorithm.</u>
	1	<u>Can only justify part of the algorithm. Note if answer was calculated and remainder not explained, score as 1 (only if the problem was modeled)</u>
	2	<u>Able to justify the algorithm and if the accurate answer was calculated the remainder was explained (only if the problem was modeled).</u>

Appendix I

Analysis of Specialized Content Knowledge of Division with Fractions (ASCK-DF)

Analysis of Specialized Content Knowledge of Division with Fractions (ASCK-DF)

Student Codes

Section I (items 1a through 5a). Analyzing Student Solutions. The following codes were used only once per item and reflect the participants identification of the student's misconceptions of division with fraction or students' misconceptions in general.

Interview Question: *In analyzing his or her answer, what is his or her understanding of division with fractions?*

Algorithmically based mistake **(AB)** - The student demonstrated a computational "bug" most likely as a result from memorizing the algorithm without understanding it.

1. **ABI-** inverting the dividend instead of the divisor
2. **ABM-** forgetting to multiply the dividend and inverted divisor
3. **ABNI-** both the dividend and the divisor should be inverted
4. **ABF-** inverting all fractions

Intuitively based mistake- **(IB)** – Errors that result from using intuitions held about division as oppose to knowledge about division.

5. **IBW-** the divisor must be a whole number
6. **IBD-** the divisor must be less than the dividend
7. **IBQ-** the quotient must be less than the dividend

Mistakes based on procedural knowledge **(PB)** – Errors that result from a fragmented procedural understanding of division with fractions (i.e., the properties of this operation).

8. **PBC-** division is commutative

Mistakes based on conceptual knowledge **(CB)** – Errors that result from a fragmented conceptual understanding of division with fractions (i.e., the properties of this operation).

9. **CBI-** applying an incorrect interpretation of division (i.e., rather than looking for how many groups of a certain size makes up the total amount, confused the fraction as a number of groups. For example rather than finding how many groups of $\frac{1}{2}$ there are in the total amount, the total amount was equally distributed into 2 groups)
10. **CBO-** using the wrong operation because of misunderstanding that when information regarding the number of groups or size of a group is missing in the problem is indicative of a division problem.

11. **CBF**- the student misunderstands concepts of fractions (i.e., part-whole, quotient and ratio)
12. **NEA**- no analysis of error. Restated what is written on the interview.

Section II (items 1b through 5b). Representing Numbers and Operations in Meaningful Ways. The following codes were used only once per item but can be used in combination with one another. The codes reflect how the participant would address the student's error.

Interview Question: *What problem would you give him or her to help correct his or her misconceptions of division with fractions?*

<u>Focus on How to Obtain the Correct Answer</u> (*CA*) – The participant described a problem that would help show the student how the correct answer can be obtained using the following methods:

1. *CAP*- using pictures to model the problem
2. *CAM*- using manipulatives to model the problem
3. *CAS*- using symbols to model the problem
4. *CAA*- using an algorithm to model the problem
5. *CADP*- used more than one method
6. *CAL*- made links to other mathematical concepts and procedures
7. *CAW-PD OR MD OR FP*- using a word problem, either partitive, measurement or factors and product

<u>Focus on Challenging the Student's Misconception</u> *(CM)*- The participant described a problem that would challenge what the student displayed using the following methods:

8. *CMP*- using pictures to model the problem
9. *CMM*- using manipulatives to model the problem
10. *CMS*- using symbols to model the problem
11. *CMA*- using an algorithm to model the problem
12. *CMDP*- used more than one method
13. *CML*- made links to other mathematical concepts and procedures
14. *CMW-PD OR MD OR FP*- using a word problem, either partitive, measurement or factors and product

Section III (items 6 and 7). Justifying Mathematical Rules and Procedures. The following codes were used only once per item but can be used in combination with one another. The codes reflect how the participant justifies the standard algorithm used with division with fractions.

Interview Question: *During a class on division with fractions, one of your students asks you to explain why the algorithm requires the divisor to be inverted and the two values t* *be multiplied. You have a word problem and a number sentence written up on the blackboard already and decide to use both of those representations to answer the student's question.*

Ability to Answer the Question (JA)

1. JANL- able to model the problem but could not link what they did to the algorithm
2. JAL- able to model the problem and linked what they did to the algorithm
3. JAI- unable to answer the question

Strategy Used to Answer the Question (JA)

4. JAS- used the same method for both items
5. JADPM- offered more than one answer
6. JADPI- used a strategy not addressed during the instruction sessions

Mathematical Knowledge Used to Answer the Question (JA)

7. JAP- used procedural knowledge
8. JAC- used conceptual knowledge

Participant Codes

I (all items). The Participants' Understanding of Division with Fractions. The following codes can be used as often as they appear within every response and are used to describe the participants understanding of this topic.

Division Language (DA)- this includes using division language to answer the question (i.e., making reference to the number of groups, the size of each group, the whole or total amount)

1. **DL-** division terminology (e.g., divisor, dividend, or quotient)
2. **DL-P-** rule associated with division and division with fractions (e.g., invert and multiply)
3. **DL-MD OR PD-** terminology that expresses the concept of partitive or measurement division (e.g., groups, number of groups, and a word problem)

Division with Whole Numbers (**DW**)- this includes using their knowledge of division with whole numbers to respond to the question.

Unclear Statement (US)- the participant's statement was unclear.

Error Statement (ER)- the participant's statement comprised an error (e.g., whatever your number is you have to make it into the full ratio of what they are like the same thing and then so your getting your total number of little squares and your taking)

Misconceptions of Division- this includes the same codes as in the previous error analysis section.
Algorithmically based mistake (**AB**) - The participant demonstrated a computational "bug" most likely as a result from memorizing the algorithm without understanding it.

4. **ABI-** inverting the dividend instead of the divisor
5. **ABM-** forgetting to multiply the dividend and inverted divisor
6. **ABNI-** both the dividend and the divisor should be inverted
7. **ABF-** inverting all fractions

Intuitively based mistake- (**IB**) – Errors that result from using intuitions held about division as oppose to knowledge about division.

8. **IBW-** the divisor must be a whole number
9. **IBD-** the divisor must be less than the dividend
10. **IBQ-** the quotient must be less than the dividend

Mistakes based on procedural knowledge (**PB**) – Errors that result from a fragmented procedural understanding of division with fractions (i.e., the properties of this operation).

11. **PBC-** division is commutative

Mistakes based on conceptual knowledge (**CB**) – Errors that result from a fragmented conceptual understanding of division with fractions (i.e., the properties of this operation).

12. **CBI-** applying an incorrect interpretation of division (i.e., rather than looking for how many groups of a certain size makes up the total amount, confused the fraction as a number of groups. For example rather than finding how many groups of $\frac{1}{2}$ there are in the total amount, the total amount was equally distributed into 2 groups)
13. **CBO-** using the concept of the wrong operation, namely multiplication with fractions, to describe the concept of division with fractions
14. **CBF-** the participant misunderstands concepts of fractions (i.e., part-whole, quotient and ratio)
15. **CBP-** the participant confuses the act of partitioning an area with the act of division
16. **CBN-** the participant confuses the number sentence and situates the total as the divisor instead of the dividend.

Misconceptions of the Division with Fractions Algorithm (MA)- misconceptions participants demonstrated while justifying the standard algorithm.

17. **MA-div-** the participant demonstrated misconceptions of what number acts as the divisor

18. **MA-mult-** the participant demonstrated misconceptions of which two numbers are multiplied
19. **MA-FB-** the participant falsely believes their justification reflects the answer