Frobenius Structures on Orbit Spaces of Coxeter Groups and Hurwitz Spaces

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ABSTRACT

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Here we describe the Frobenius Manifold as a geometric reformulation of the solution space to the WDVV equations. Relations between Frobenius Algebras, Frobenius Manifolds and 2D-Topological Field Theories are shown, and we examine the A_n case from the class of polynomial solutions to WDVV as Topological Landau-Ginzburg Models. The A_n case is also described from the point of view of singularity theory from which it originated, and we show Dubrovin's constructions for Frobenius manifolds on the orbit spaces of Coxeter groups and Hurwitz spaces with the A_n case as the main example.

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Introduction

Here we present a system of differential equations from the papers of physicists concerning 2D-topological field theories from the early 90's. Their problem was to find a quasihomogeneous function $F(t), t = (t^1...t^n)$ such that the third derivatives of it for all t are structure constants of an associative algebra. Solving for the prepotential $F(t)$, we get a complicated system of partial differential equations called the WDVV equations. (Named for physicists E. Witten, R. Dijkgraaf, E. Verline and H. Verlinde.) Dubrovin has given a beautiful geometric re-formulation of the solution space to WDVV into a Frobenius Manifold, which helps to determine interesting solutions [7].

Physically, these solutions to WDVV describe the moduli space of Topological Conformal Field Theories, where the prepotential $F(t)$ encodes all the data of the correspondent theory. The tangent vectors on the moduli space of these theories are the physical operators used to perturb their Lagrangians. There are two large classes of Frobenius manifolds: those that are described by the unfoldings of singularities (polynomial moduli: topological Landau-Ginzburg models, and complex moduli: topological B-models) and those that are described by quantum cohomologies (Kähler moduli: topological A-models). The famous mirror conjecture relates these two families, most often by showing the equivalence of their prepotentials [10].

Frobenius manifolds have been known in singularity theory since K. Saito's paper and Saito's theorem which says the residue form and product on a Jacobian algebra give a flat metric, where the residue form and algebra have a ring structure on the tangent sheaf to the space of parameters of a deformation [14], [2]. Dubrovin's Frobenius structure on a manifold defines such a ring structure on the tangent sheaf with a flat connection, and a flat metric in addition to some compatibility conditions.

To describe physical theories, it is necessary to preserve certain symmetries, so the outline of finding the Frobenius manifolds invariant under the actions of the Coxeter symmetry groups is a good one. Also, it has been proven that certain tensor products of Frobenius Manifolds are also Frobenius Manifolds, so interesting TCFT models might be built from the basic ones on the space of orbits of Coxeter Groups. One of Dubrovin's conjectures is that for a class of solutions to WDVV with good analytic properties, the monodromy group of the resulting Frobenius Manifold is finite. He also conjectures that all polynomial solutions to WDVV are constructed in this way.

These particular Frobenius structures can also be described by a Hurwitz space with certain restrictions [4],[7],[16]. A Hurwitz space is the moduli space of pairs (L, λ) where L is a compact genus g Riemann surface, and λ is a degree N meromorphic function. The critical points of λ give the canonical coordinates of the Frobenius structure, and the ramification points of the covering. The covering is a collection of N copies of CP^1 glued at the branchcuts. Two coverings are called equivalent if they can be obtained from one another by a permutation of sheets. The meromorphic function λ which is invariant under the action of a finite Coxeter group W acting on L, will be called the superpotential of the construction, from which the prepotential of the correspondent Frobenius manifold is found.

Chapter 1

WDVV Equations and Frobenius Structure

We now give the definitions of WDVV equations, Frobenius algebras and Frobenius manifolds, and show that Frobenius manifolds give a coordinate-free geometrization of the solutions to WDVV [7], [9], [2]. We then show the example of main consideration throughout the following chapters, give the physical normalization for the prepotentials, and describe the class of polynomial solutions to WDVV [7].

1.1 WDVV Equations, Frobenius Algebras and Frobenius Manifolds

Definition 1.1.1 *The WDVV system is the following system of nonlinear partial differential equations and 3 conditions, where the third derivatives of the function $F(t)$ (prepotential or free energy) of n variables $t = (t^1, ..., t^n)$ satisfy: (Sum over repeated indices is assumed throughout this paper.)*

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\delta \partial t^\mu} = \frac{\partial^3 F(t)}{\partial t^\gamma \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\delta \partial t^\mu} \tag{1.1.1}$$

The third derivatives of $F(t)$ will be denoted as

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

The Three Conditions of Normalization, Associativity and Quasihomogeneity are

1) Normalization: $\eta_{\alpha\beta}$ *is a constant, symmetric, nondegenerate matrix*

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t) \tag{1.1.2}$$

with inverse

$$\eta^{\alpha\beta} = (\eta_{\alpha\beta})^{-1}$$

2) Associativity: The functions

$$c^{\gamma}_{\alpha\beta}(t) := \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t) \tag{1.1.3}$$

are structure constants of an associative n-dimensional algebra A_t *with generators* $e_1, ..., e_n$ *and commutative multiplication*

$$e_\alpha * e_\beta = c^{\gamma}_{\alpha\beta} e_\gamma = c^{\gamma}_{\beta\alpha} e_\gamma \tag{1.1.4}$$

The basis vector e_1 *is the unit for all the algebras* A_t

$$c^{\beta}_{1\alpha}(t) := \eta^{\beta\gamma}\eta_{\gamma\alpha} = \delta^{\beta}_{\alpha} \tag{1.1.5}$$

3) Quasihomogeneity: $F(t)$ *must be quasihomogeneous in its variables up to a quadratic polynomial. (Since the addition of one does not change the third derivatives.)*

$$F(c^{d_1}t^1, \ldots, c^{d_n}t^n) = c^{d_F} F(t^1, \ldots, t^n) + \textit{quadratic terms} \tag{1.1.6}$$

for any nonzero c and some numbers (weights) $d_1, \ldots, d_n, d_F$*. The quasihomogeneity condition is generalized in terms of the Euler vector field. We assume there exists a vector field E*

$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha \tag{1.1.7}$$

where

$$Lie_E F(t) = E(F) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha F = d_F F + \textit{quadratic terms} \tag{1.1.8}$$

Remark 1.1.1 *The associativity condition is equivalent to the WDVV equations.*

Writing out the associativity condition we have for all α, β, γ,

$$(e_\alpha * e_\beta) * e_\gamma = e_\alpha * (e_\beta * e_\gamma)$$

$$(e_\alpha * e_\beta) * e_\gamma = \left(\eta^{\delta\lambda} c_{\alpha\beta\lambda} e_\delta\right) * e_\gamma = \eta^{\delta\lambda} c_{\alpha\beta\lambda} \eta^{\lambda\mu} c_{\delta\gamma\mu} e_\lambda$$

$$e_\alpha * \left(e_\beta * e_\gamma\right) = e_\alpha * \eta^{\delta\lambda} c_{\beta\gamma\lambda} e_\delta = \eta^{\delta\lambda} c_{\beta\gamma\lambda} \eta^{\lambda\mu} c_{\alpha\delta\mu} e_\lambda$$

Since the generators e_λ are independent and the constant matrix $\eta^{\delta\lambda}$ is invertible, we have

$$c_{\alpha\beta\lambda} \eta^{\lambda\mu} c_{\delta\gamma\mu} = c_{\beta\gamma\lambda} \eta^{\lambda\mu} c_{\alpha\delta\mu}$$

which is equivalent to equation (1.1.1)

A Frobenius Algebra is a finite dimensional vector space with multiplication and bilinear form.

Definition 1.1.2 *An algebra A over $\boldsymbol{C}$ is a Frobenius Algebra if:*

(i) It is a commutative associative $\boldsymbol{C}$-algebra with a unity e

(ii) It admits a $\boldsymbol{C}$-bilinear symmetric nondegenerate inner product

$$A \times A \to \boldsymbol{C}, a, b \mapsto \langle a, b \rangle \tag{1.1.9}$$

being invariant in the following sense:

$$\langle a * b, c \rangle = \langle a, b * c \rangle \tag{1.1.10}$$

We may have a family of Frobenius Algebras depending on the parameters $t = (t^1, \ldots, t^n)$. Denoting the space of parameters by M, we will have a fiber bundle

$$t \in M^{\downarrow A_t} \tag{1.1.11}$$

which will be identified with the tangent bundle TM of the manifold M. We may now define the Frobenius Manifold. Let M be an n-dimensional manifold.

Definition 1.1.3 *M is a Frobenius Manifold if the structure of a Frobenius Algebra is specified on any tangent plane $T_t M$ at any point t in M smoothly depending on the point such that*

(F1) The invariant inner product $\langle , \rangle$ is a flat metric on M.

(F2) The unity vector field e is covariantly constant w.r.t. the Levi-Civita connection ∇ for the metric $\langle , \rangle$

$$\nabla e = 0 \tag{1.1.12}$$

i.e., the unity vector field e is flat.

(F3) (Potentiality) Let

$$c(u, v, w) := \langle u * v, w \rangle \tag{1.1.13}$$

The following 4-tensor is required to be symmetric in the fields u, v, w, z

$$(\nabla_z c)(u, v, w) \tag{1.1.14}$$

(F4) The Euler vector field E must be determined on M such that

$$\nabla(\nabla E) = 0 \tag{1.1.15}$$

and the associated one-parameter group of diffeomorphisms acts by conformal transformation of the metric $\langle , \rangle$ and by rescalings on the Frobenius algebras T_tM. i.e. For arbitrary vector fields u and v, and some constants D and d_1:

$$Lie_E \langle u, v \rangle := E \langle u, v \rangle - \langle [E, u], v \rangle - \langle u, [E, v] \rangle = D \langle u, v \rangle \tag{1.1.16}$$

and

$$Lie_E(u * v) := [E, u * v] - [E, u] * v - u * [E, v] = d_1 u * v \tag{1.1.17}$$

Remark 1.1.2 *Some remarks are in order:*

(1) The metric here denotes a complex non-degenerate symmetric bilinear form.

(2) The Potentiality condition is equivalent to the existence of a closed 1-form $\epsilon := \langle e, \cdot \rangle$ on M, so one may replace (1.1.12) by $Lie_e \langle \cdot, \cdot \rangle = 0$

(3) If the vector fields X, Y, W, Z are flat, then the condition of Potentiality

$$\nabla_X (Y * Z) - Y * \nabla_X (Z) - \nabla_Y (X * Z) + X * \nabla_Y (Z) - [X, Y] * Z = 0$$

is equivalent to the total symmetry of both

$$c(U, V, W) := \langle U * V, W \rangle$$

and

$$\nabla_Z c(X, Y, Z)$$

(4) *We consider only the case where the scaling constant* $d_1 \neq 0$*, and then designate* $d_1 = 1$ *by a rescaling of* E*.*

(5) *Frobenius manifolds are Pseudo-Riemannian manifolds where the bilinear form corresponds to the Riemannian metric. The metric and corresponding Levi-Civita connection must be flat.*

1.2 Coordinate-Free Formulation and the A_n Example

Theorem 1 *Any solution of the WDVV equations with $d_1 \neq 0$ defined in a domain of $t \in M$ determines in this domain the structure of a Frobenius manifold by the formulae:*

$$\partial_\alpha * \partial_\beta := c^\gamma_{\alpha\beta}(t)\partial_\gamma \tag{1.2.1}$$

$$\langle \partial_\alpha, \partial_\beta \rangle := \eta_{\alpha\beta} \tag{1.2.2}$$

Here $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$ and $e := \partial_1$. Conversely, locally any Frobenius manifold with such structure admits a solution of the WDVV equations.

Proof. For a solution F of the WDVV equations, the metric (1.2.2) is constant in the coordinates t^α, so it is flat on M. In the flat coordinates covariant derivatives are partial derivatives, so the unity vector field e is covariantly constant. Also since partial derivatives commute, the expression

$$\nabla_z c\,(u, v, w) = \partial_\delta c_{\alpha\beta\gamma}\,(t) = \frac{\partial^4 F\,(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\delta}$$

is totally symmetric in the four vector fields. The final property is satisfied, since the 1-parameter group of diffeomorphisms for the vector field (1.1.7)

$$Lie_E F(t) = E(F) = \sum_\alpha d_\alpha t^\alpha \partial_\alpha F + \text{quadratic terms}$$

acts by rescalings defined for an algebra A with unit e by:

$$a * b \longmapsto ka * b, e \longmapsto ke$$

for a, b from A and k nonzero constant. And in the flat coordinates, $\nabla\,(\nabla E) = 0$. Conversely, locally on a Frobenius manifold M we can choose flat coordiantes so that the inner product is constant. Since M is a Pseudo-Riemannian manifold, the Levi-Civita connection by definition is compatible with the metric g, and also $\nabla g := 0$. This gives the normalization condition. The covariant constancy of e allows by a linear change of coordinates to set $e := \frac{\partial}{\partial t^1}$. The tensors $\partial\gamma c_{\alpha\beta\gamma}(t)$ and $c_{\alpha\beta\gamma}(t)$ being symmetric in vector fields ∂_m imply the existence of the prepotential function F whose third and fourth derivative tensors to which they correspond, are symmetric. Remark (1.1) shows the structure of the associative algebra, which is equivalent to the WDVV equations for F. This gives the Associativity condition. The generalized Quasihomogeneity condition is satisfied by F. By the fourth property of Frobenius Manifolds, the Euler vector field gives the quasihomogeniety of F. Given (1.1.16), (1.1.17) and that $\nabla(\nabla E) = 0$, we have

$$Lie_E c_{\alpha\beta\gamma} = (1 + D)c_{\alpha\beta\gamma}$$

In terms of the prepotential F,

$$Lie_E \partial_\alpha \partial_\beta \partial\gamma F = (1 + D)\partial_\alpha \partial_\beta \partial\gamma F$$

Since Lie_E commutes with the covariant derivative,

$$\partial_\alpha \partial_\beta \partial\gamma [Lie_E F - (1 + D)F] = 0$$

and the generalized quasihomogeneity condition is obtained:

$$Lie_E F = (1 + D)F + \text{quadratic terms}$$

End of proof.

Example 1.2.1 *We will see in chapter 3 that the following is an example of a Frobenius manifold, and it lends itself to a construction on the orbit space of the Coxeter group A_n. Consider M the affine space of all polynomials*

$$M = \{\lambda(p) = p^{n+1} + a_n p^{n-1} + ... + a_1 | a_1, ..., a_n \in C\} \tag{1.2.3}$$

At any point, its tangent space is a vector space of polynomials with degree less than n. The algebra A_λ on the tangent space (also called a Milnor ring) is endowed with multiplication

$$A_\lambda = C[p]/(\lambda)'(p) \tag{1.2.4}$$

The inner product, unity vector field and Euler vector field are respectively:

$$< f, g >_\lambda = res_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)} \tag{1.2.5}$$

where $\lambda'(p) = \frac{d\lambda}{dp}$

$$e \equiv \frac{\partial}{\partial a_1} \tag{1.2.6}$$

$$E \equiv \frac{1}{n+1}\sum_i (n-i+1)a_i \frac{\partial}{\partial a_i} \tag{1.2.7}$$

1.3 Physical Normalization and Polynomial Solutions to WDVV

We now introduce the normalization for the prepotential F prescribed by the physical literature.

Lemma 1 *The scaling transformations generated by the Euler vector field E (1.1.8) act by linear conformal transformations of the metric $\eta_{\alpha\beta}$*

$$Lie_E\eta_{\alpha\beta} = (d_F - d_1)\eta_{\alpha\beta} \tag{1.3.1}$$

Proof. Differentiating (1.1.8) wrt t^1, t^α and t^β and recalling $Lie_E\partial_1 = -d_1\partial_1$, we obtain the Lie derivative of the metric. End of Proof.

Corollary 1 *If $\eta_{11} = 0$ and all the roots of $E(t)$ are simple then by a linear change of coordinates t^α the matrix $\eta_{\alpha\beta}$ can be reduced to the antidiagonal form*

$$\eta_{\alpha\beta} = \delta_{\alpha\beta,n+1}$$

In these coordinates $F(t)$ has the following form for some function $f(t^2,\ldots,t^n)$

$$F(t) = \frac{1}{2}(t^1)^2t^n + \frac{1}{2}t^1\sum_{\alpha=2}^{n-1}t^\alpha t^{n-\alpha+1} + f(t^2,\ldots,t^n) \tag{1.3.2}$$

The sum

$$d_\alpha + d_{n-\alpha+1}$$

does not depend on α and

$$d_F = 2d_1 + d_n.$$

When the degrees are normalized so that $d_1 = 1$, they have the form

$$d_\alpha = 1 - q_\alpha \quad d_F = 3 - d$$

for numbers $q_1,\ldots,q_n, d_n, d$ given by

$$q_1 = 0,\ q_n = d,\ q_\alpha + q_{n-\alpha+1} = d$$

Proof. If $\langle e_1, e_1\rangle = 0$ then vector e_n may still be chosen to be an eigenvector of the scaling transformations of the Euler vector field (i.e. roots of $E(t)$). Using only such eigenvectors on the orthogonal complement of the span of e_1 and e_n, $\eta_{\alpha\beta}$ can be reduced to the antidiagonal form. Recalling $\eta_{\alpha\beta} := c_{1\alpha\beta}$, the antidiagonal form in these coordinates determines the above form of the prepotential F. Independence of the sum $d_\alpha + d_{n-\alpha+1}$ and $d_F = 2d_1 + d_n$ follow directly from the action of the scaling transformations on the metric, (1.3.1). End of proof.

Example 1.3.1 *Let us look at the* $n = 3$ *case in the algebra* A_t *with basis* $e_1 = 1$, e_2, e_3 *and prepotential function* F *for some function* $f(x, y)$

$$F(t) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + f(t_2, t_3) \tag{1.3.3}$$

The multiplication table (with subscripts of f as partial derivatives) is given by

$$\begin{aligned} e_2^2 &= f_{xxy}e_1 + f_{xxx}e_2 + e_3 \\ e_2 e_3 &= f_{xyy}e_1 + f_{xxy}e_2 \\ e_3^2 &= f_{yyy}e_1 + f_{xyy}e_2 \end{aligned} \tag{1.3.4}$$

The associativity condition

$$(e_2^2)e_3 = e_2(e_2 e_3) \tag{1.3.5}$$

gives the following partial differential equation for $f(x, y)$

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy} \tag{1.3.6}$$

Note that (1.3.5) is the only associativity equation for $n = 3$*, i.e.,*

$$(e_3^2)e_2 = e_3(e_3 e_2)$$

gives nothing new.

Dubrovin has conjectured that any solution of WDVV with good analytic properties has a discrete group for its monodromy group, as we shall investigate in chapter 3. Starting this way, Frobenius manifolds are constructed on the orbit spaces of Coxeter groups, generating a class of solutions that are polynomial in nature. Let us now describe these with examples from dimension $n = 3$.

Polynomial Solutions of WDVV

Consider Frobenius Manifolds whose structure constants are all analytic at the point

$t = 0$. The Frobenius algebra $A_0 := T_{t=0}M$ has at point $t = 0$ structure constants $c^{\gamma}_{\alpha\beta}(0)$ and basis vectors $e_1, \ldots, e_n$. The q_α as defined in Corrolory (1) give the degrees of the basis vectors as

$$deg e_\alpha = q_\alpha$$

The germ of the Frobenius Manifold near $t = 0$ is a deformation of the algebra A_0, and thus the algebras A_t for $t \neq 0$ are deformations of A_0. This analytic deformation is physically relevant as we shall see in chapter 2. The algebra A_0 corresponds to the *primary chiral algebra* of a topological conformal field theory; an operator algebra of the perturbed topological field theory. If in the normalization of (1.3.2) we constrain that the degrees $deg t_\alpha$ be positive real numbers, then $0 < d < 1$. Paired with the quasihomogeneity condition (1.1.6), this amounts to finding the polynomial solutions $F(t)$ of the WDVV equations.

Example 1.3.2 *We consider the case of dimension $n = 3$. The prepotential is, as before (1.3.3)*

$$F(t) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + f(t_2, t_3)$$

The degrees as prescribed by the normalization are:

$$deg t^1 = 1 \tag{1.3.7}$$

$$deg t^2 = 1 - \frac{d}{2}$$

$$deg t^3 = 1 - d$$

$$deg f = 3 - d$$

The function

$$f(x, y) = \sum a_{pq} x^p y^q$$

must satisfy the quasihomogeneity condition, i.e.

$$a_{pq} \neq 0 \quad when \quad p + q - 3 = (\frac{p}{2} + q - 1)d$$

Now the function f has two possible forms.

1) For n odd: $n, m \in \boldsymbol{N}$,

$$f = \sum_k a_k x^{4-2km} y^{kn-1} \qquad d = \frac{n-2m}{n-m}$$

2) For m odd: $n, m \in \boldsymbol{N}$,

$$f = \sum_k a_k x^{4-km} y^{kn-1} \qquad d = \frac{2(n-m)}{2n-m}$$

The powers in f must be nonnegative, and cases for which f is a cubic or lower are uninteresting. This leaves three possibilities:

$$a) \quad f = ax^2y^{n-1} + by^{2n-1}, \quad n \geq 3 \tag{1.3.8}$$

$$b) \quad f = ay^{n-1}, \quad n \geq 5$$

$$c) \quad f = ax^3y^{n-1} + bx^2y^{2n-1} + cxy^{3n-1} + dy^{4n-1}, \quad n \geq 2$$

As in the example preceding, f must satisfy the partial differential equation (1.3.6) from which we may solve for n. In (1.3.6) f with form a) deems n $= 3$; with form b) there is no solution; with form c) deems n $= 2$ or n $= 3$. We thus have as the three remaining polynomial solutions for WDVV with positive degrees of t^α in dimension 3:

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^2 t_3^2}{4} + \frac{t_3^5}{60} \tag{1.3.9}$$

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3}{6} + \frac{t_2^2 t_3^3}{6} + \frac{t_3^7}{210} \tag{1.3.10}$$

$$F = \frac{t_1^2 t_3 + t_1 t_2^2}{2} + \frac{t_2^3 t_3^2}{6} + \frac{t_2^2 t_3^5}{20} + \frac{t_3^{11}}{3960} \tag{1.3.11}$$

The prepotential (1.3.9) has the same form as in Example 1 with n $= 3$, that we will see in chapter 3 may be constructed on the orbit space of the Coxeter group A_n. In the same vein, polynomials (1.3.10) and (1.3.11) are related to the Coxeter groups B_n and H_n respectively.

Chapter 2

Topological Field Theories

Here we describe topological field theories as background independent quantum field theories, describe the matter sector of such theories, and give Atiyah's axioms [1],[7],[10], [11]. We then show that the matter sector for a 2D-topological field theory is always encoded by a Frobenius algebra[1],[7],[9]. The moduli space generated by topological conformal field theories is a Frobenius manifold, and we give the example of Topological Landau-Ginzburg models, which we will see in chapter 3 corresponds to the Frobenius manifold of Example (1.1) [7],[10].

2.1 2D-Topological Field Theories and Atiyah's Axioms

A quantum field field theory (QFT) in its Lagrangian formulation may be specified on a D-dimensional manifold Σ as:

1) A family of *local fields* $\varphi_\alpha(x)$, $x \in \Sigma$. These may be functions or sections of a fiber bundle over Σ. A metric $g_{ij}(x)$ is usually one of the fields (the gravity field).

2) A *Lagrangian* $L = L(\varphi, \varphi_x \ldots)$ and classically, the Euler Lagrange equations:

$$\frac{\delta S}{\delta \varphi_\alpha(x)} = 0 \tag{2.1.1}$$

$$S[\varphi] = \int_\Sigma L(\varphi, \varphi_x \ldots) d\Sigma \tag{2.1.2}$$

3) A *Quantization procedure* via the path integral approach where a path integration measure $[d\varphi]$ is constructed (but almost never well-defined) and the partition

function Z_Σ results from path integration over the space of all fields $\varphi(x)$.

$$Z_\Sigma = \int [d\varphi] e^{-S[\varphi]} \tag{2.1.3}$$

Correlation functions (The output of a QFT: its physical observables) are defined similarly:

$$\langle \varphi_\alpha(x)\varphi_\beta(x)\cdots\rangle_\Sigma = \int [d\varphi]\varphi_\alpha(x)\varphi_\beta(x)\cdots e^{-S[\varphi]} \tag{2.1.4}$$

Here the definition of a QFT involves a choice of a manifold Σ on which the QFT lives and a choice of metric, a background field. Thus, the correlation functions are calculated in a certain background. We now consider a class of 2D-QFTs: 2D-topological field theories (TFT). These are background independent QFTs; those whose correlation functions do not depend upon the choice of metric. TFTs are invariant wrt arbitrary changes of the metric $g_{ij}(x)$ on a 2D surface Σ:

$$\delta g_{ij}(x) = arbitrary, \qquad \delta S = 0$$

As a step towards a rigorous account of TFTs, Atiyah formulated axioms describing them for arbitrary dimension. These axioms describe correlators of fields in the matter sector of a 2D-TFT. In this sector, the local fields $\varphi_1(x), \ldots, \varphi_n(x)$ do not contain a metric on the surface Σ. Atiyah found that in the matter sector, the correlators of the fields obey three simple axioms. Here we describe the matter sector and give Atiyah's axioms for dimension $D = 2$.

Matter Sector for a 2D-TFT:

1) A: the space of local physical states. We assume A is finite-dimensional.

$$dim A = n < \infty$$

2) The assignment $\mathcal{T}$

$$\mathcal{T} : (\Sigma, \partial\Sigma) \longmapsto v_{(\Sigma,\partial\Sigma)} \in A_{(\Sigma,\partial\Sigma)} \tag{2.1.5}$$

which only depends on the topology of the pair $(\Sigma, \partial\Sigma)$ for Σ an oriented 2-surface, and $\partial\Sigma$ its oriented boundary. We are assigning to each local physical state v_i an oriented 2-surface with oriented boundary. (Note that when the surface is closed, it's boundary is null.) The linear space $A_{(\Sigma,\partial\Sigma)}$ is:

$$A_{(\Sigma,\partial\Sigma)} = \begin{cases} \mathbb{C} \text{ if } \partial\Sigma = \emptyset \\ A_1 \otimes \ldots \otimes A_k \text{ if } \partial\Sigma \text{ consists of } k \text{ oriented cycles } C_1 \ldots C_k \end{cases} \tag{2.1.6}$$

$$A_i = \begin{cases} A \text{ if } C_i \text{ is oriented with } \Sigma \\ A^* (\text{dual}) \text{ otherwise} \end{cases} \tag{2.1.7}$$

Atiyah's Axioms for a 2D-TFT:

The 2D-TFT $\mathcal{T}$ satisfies 3 axioms. (Only the orientation of the boundary $\partial\Sigma$ is shown. A cycle C_i will be oriented with the surface Σ if Σ remains to the left traversing in the direction of C_i. Assume the surfaces are oriented wrt the external normal vector.)

1) *Normalization:*

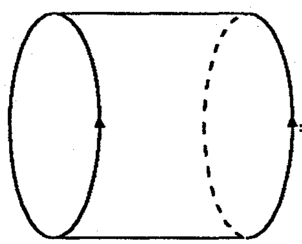


Figure 2.1: Normalization

2) *Multiplicativity:* If

$$(\Sigma, \partial\Sigma) = (\Sigma_1, \partial\Sigma_1) \cup (\Sigma_2, \partial\Sigma_2) \tag{2.1.8}$$

then

$$v_{(\Sigma,\partial\Sigma)} = v_{(\Sigma_1,\partial\Sigma_1)} \otimes v_{(\Sigma_2,\partial\Sigma_2)} \in A_{(\Sigma,\partial\Sigma)} \tag{2.1.9}$$

and

$$A_{(\Sigma,\partial\Sigma)} = A_{(\Sigma_1,\partial\Sigma_1)} \otimes A_{(\Sigma_2,\partial\Sigma_2)} \tag{2.1.10}$$

3) *Factorization:* The operation of contraction for tensor products:

$$A_1 \otimes \ldots \otimes A_k \to A_1 \otimes \ldots \otimes \hat{A}_i \otimes \ldots \otimes \hat{A}_j \otimes \ldots \otimes A_k \tag{2.1.11}$$

is defined when A_i and A_j (the hats denote their omission) are dual to each other and identity on the other factors. Pictorially, we see that if $(\Sigma, \partial\Sigma)$ and $(\Sigma', \partial\Sigma')$ are identical outside of a ball and inside are as in figure 2.2, then

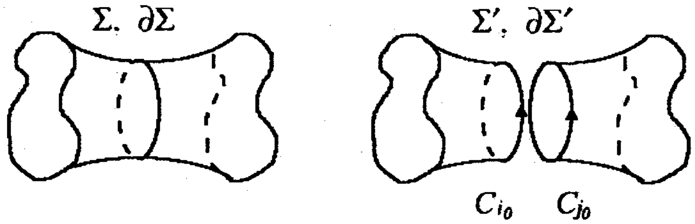


Figure 2.2: Factorization

$$v_{(\Sigma,\partial\Sigma)} = i_0 j_0 \ \ contraction\ of\ v_{(\Sigma',\partial\Sigma')} \tag{2.1.12}$$

is obtained by gluing together the cycles C_{i_0} and C_{j_0}.

Now we present a symmetric polylinear function $v_{g,s}$ on the space of states A; the genus g correlators of the fields $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_s}$ For example:

And in some basis $\varphi_1, \ldots, \varphi_n$in A:

$$v_{g,s}(\varphi_{\alpha_1} \otimes \ldots \otimes \varphi_{\alpha_s}) =: \langle \varphi_{\alpha_1} \ldots \varphi_{\alpha_s} \rangle_g \tag{2.1.13}$$

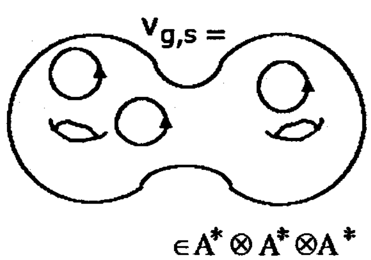


Figure 2.3: Here $g = 2$ and $s = 3$

2.2 2D-Topological Field Theories and Frobenius Algebras

Now we come to the main theorem of the chapter. The space of states A (the matter sector of a 2D-TFT) carries a natural structure of a Frobenius algebra, and all the genus g correlators of the fields can be expressed very simply in terms of this algebra [7].

Theorem 2 *Let (I) The tensors* c, η, *on* A *form a Frobenius algebra structure with*

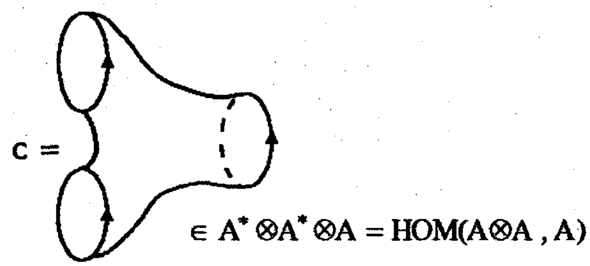


Figure 2.4: Multiplication

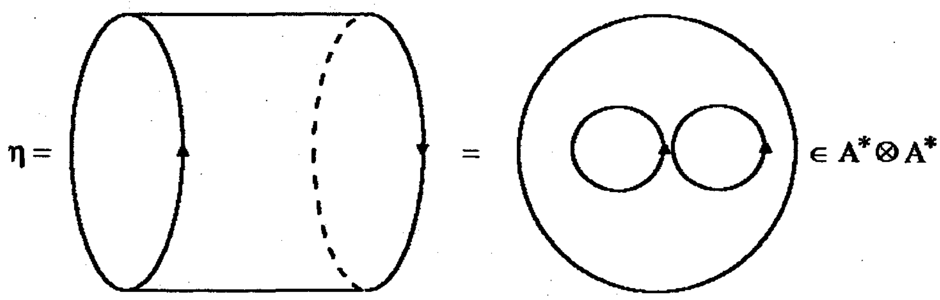


Figure 2.5: Inner Product

unity e *defined as in figure 2.6.*

$e=$  $\in A$

Figure 2.6: Unity

(II) Let the Handle operator H be defined as in figure 2.7.

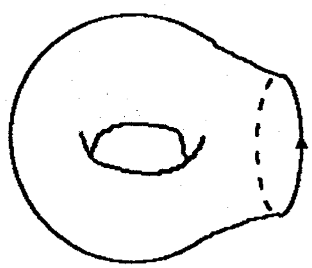
$H=$  $\in A$

Figure 2.7: Handle operator

Then, the genus g correlators may be expressed as the following R.H.S. product in the algebra.

$$\langle \varphi_{\alpha_1} \dots \varphi_{\alpha_k} \rangle_g = \langle \varphi_{\alpha_1} * \dots * \varphi_{\alpha_k}, H_g \rangle \tag{2.2.1}$$

Proof:

(I) The algebra must be commutative. Looking at figure 2.4, the multiplication c is seen to be commutative since we may always exchange pant legs by a homeomorphism. Similarly, by figure 2.5 the inner product η is seen to be symmetric by a homeomorphism. The multiplication must also be associative, and this is demonstrated in figure 2.8:

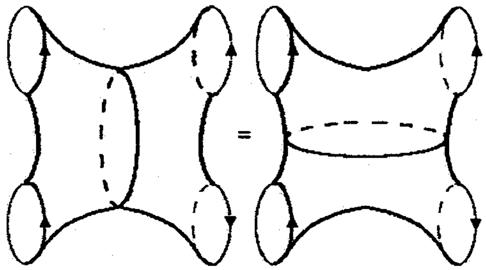


Figure 2.8: Associativity

The more general k-product is realised by the k-leg pants in figure 2.9:

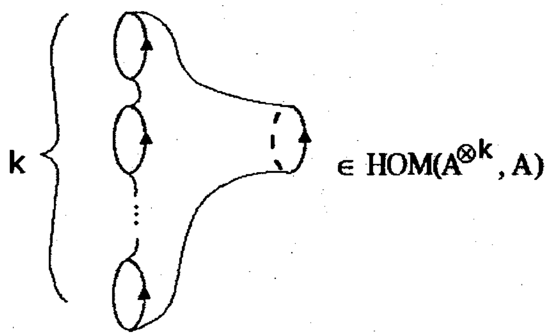


Figure 2.9: k-product

The Identity:

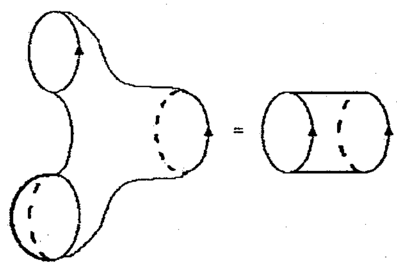


Figure 2.10: Identity

The inner product η must be nondegenerate, so we find its inverse. Let

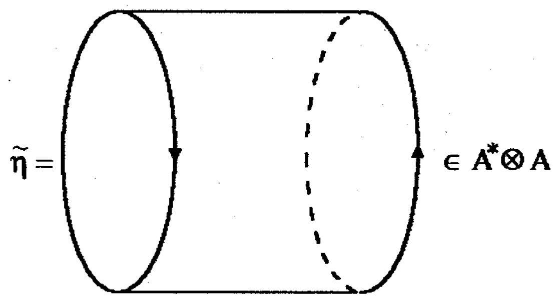


Figure 2.11: Inverse

Then, $\tilde{\eta}\eta$ is easily seen to give the identity cobordism, so $\tilde{\eta} = \eta^{-1}$.

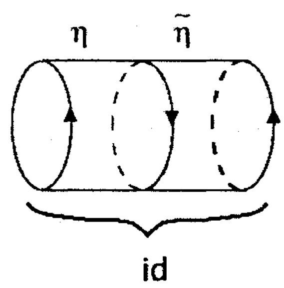


Figure 2.12: Nondegeneracy

The multiplication c (figure 2.4) must be compatible with the inner product η (figure 2.5) as indicated by the following:

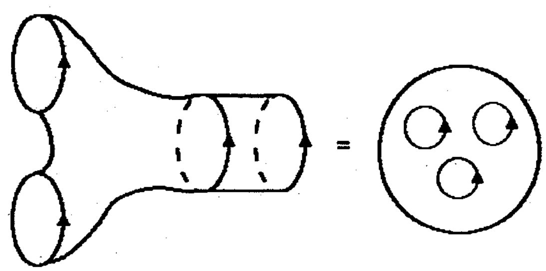


Figure 2.13: Invariance

(II) For any $v_{g,s}$, we may construct these correlation functions using members of the algebra. For example, using the multiplication c, the inner product η, the 3-leg product and 3 handle operators H_g, we have:

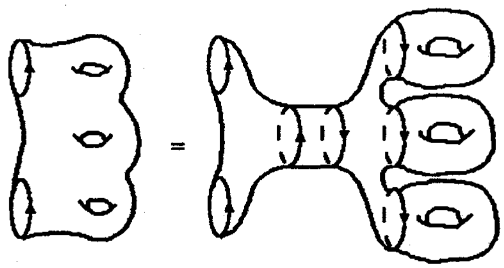


Figure 2.14: Correlation functions

End proof.

For a TFT $\mathcal{T}$ (2.1.5) and unity e (figure 2.6), the image of e under $\mathcal{T}$ gives the vector space $A := \mathcal{T}(e)$. The image of the co-unity θ (e with reversed boundary orientation) under $\mathcal{T}$ gives the dual space $A^* := \mathcal{T}(\theta)$ Note that the pairing η and unity e may be used to define θ:

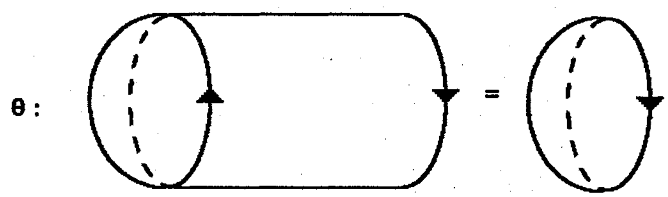


Figure 2.15: Co-unity θ

The nondegeneracy of η given the co-pairing η^{-1} as seen in figure 2.14, yields an isomorphism between A and A^* - the identity cobordism as seen in figure 12. This is the content of Atiyah's axiom [1] $\partial\Sigma \mapsto A \Rightarrow \partial\bar{\Sigma} \mapsto A^*$.

The boundary $\partial\Sigma$ of Σ (if $\partial\Sigma \neq \emptyset$) consists of oriented cycles C_i. C_i will correspond to A if oriented with Σ, and will correspond to A^* if oriented against Σ. The *gluing* along a cycle C_i (the disjoint union of $(\Sigma_1, \partial\Sigma_1)$ and $(\Sigma_2, \partial\Sigma_2)$) corresponds to the tensor product contractions of dual vectors. We now see several examples of this [11], using the notations from chapter 1:

Example 2.2.1 $c^{\gamma}_{\alpha\beta} : A \otimes A \to A \in A^* \otimes A^* \otimes A$, *a* $(2,1)$*-tensor* $e_\alpha \otimes e_\beta = c^{\gamma}_{\alpha\beta} e_\gamma$

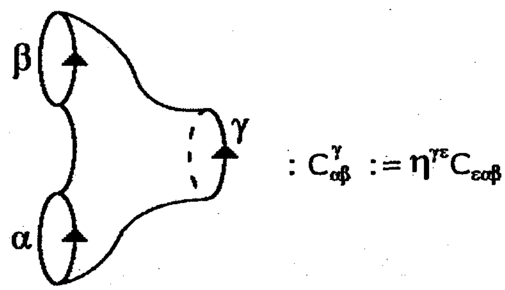


Figure 2.16: Multiplication c

Example 2.2.2 $e : C \to A \in A$, *vectors*

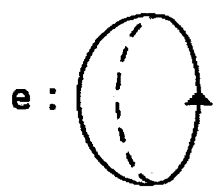


Figure 2.17: Unity e

Example 2.2.3 $\theta : A \to C \in A^*$, *co-vectors*

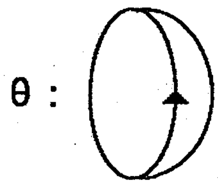


Figure 2.18: Co-Unity θ

Example 2.2.4 $\eta : A \otimes A \to C \in A^* \otimes A^*$, *a* $(2,0)$*-tensor* $\eta_{\gamma\epsilon} =< e_\gamma, e_\epsilon >$

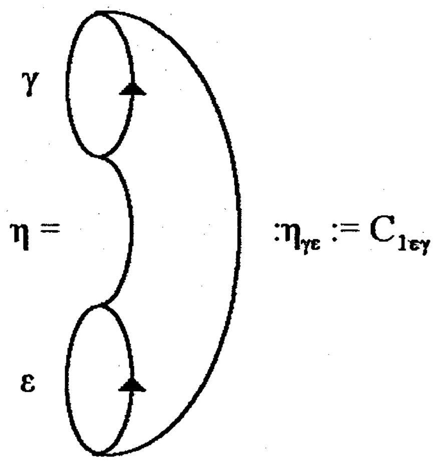


Figure 2.19: Pairing η

Example 2.2.5 $\eta^{-1} : \boldsymbol{C} \rightarrow A \otimes A \in A \otimes A$, *a* $(0, 2)$*-tensor* $\eta^{\gamma\epsilon} =< e_\gamma, e_\epsilon >$

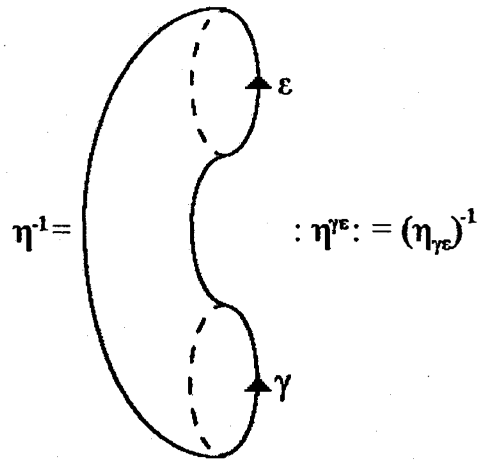


Figure 2.20: Co-pairing η^{-1}

Example 2.2.6 $id : A \rightarrow A \in A^* \otimes A$

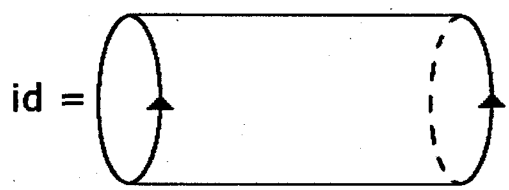


Figure 2.21: Identity

Example 2.2.7 $c_{\epsilon\alpha\beta} : A \otimes A \otimes A \in A^* \otimes A^* \otimes A^*$, *a* $(3,0)$*-tensor* $c_{\epsilon\alpha\beta} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\epsilon}$

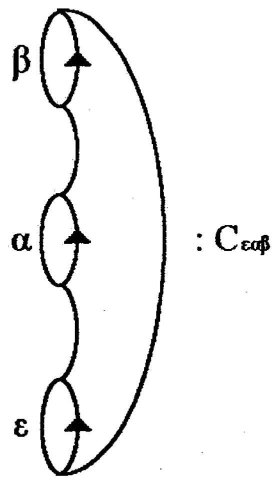


Figure 2.22: 3-point function

Example 2.2.8 *Using the co-pairing* $\eta^{\gamma\epsilon}$ *and 3-point function* $c_{\epsilon\alpha\beta}$*, we can recover the multiplication* c *by gluing along* ϵ*; contracting* ϵ

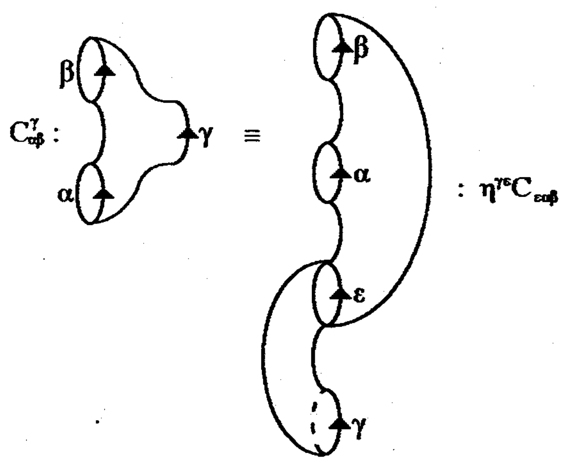


Figure 2.23: $c^\gamma_{\alpha\beta} = \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}$

Example 2.2.9 *Similarly, we can construct the 3-point function $c_{\epsilon\beta\gamma}$ using the pairing $\eta_{\epsilon\alpha}$ and the multiplication $c^{\alpha}_{\beta\gamma}$*

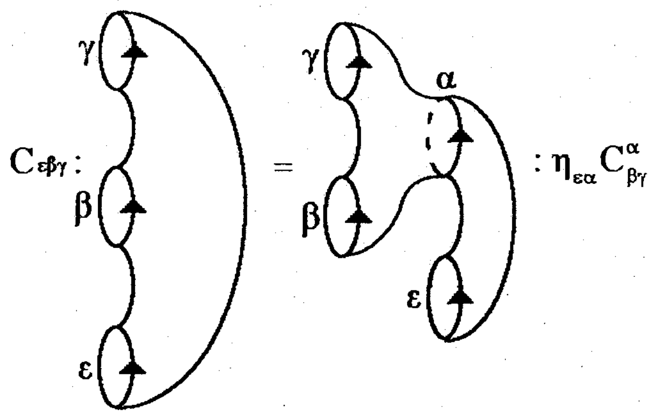


Figure 2.24: $c_{\epsilon\beta\gamma} = \eta_{\epsilon\alpha} c^{\alpha}_{\beta\gamma}$

Remark 2.2.1 *Here it is easy to see that $\eta_{\alpha\beta}$ lowers indices and $\eta^{\alpha\beta}$ raises indices.*

Example 2.2.10 *The Frobenius algebra may also be characterized by equipping A with the multiplication c, unit e, co-unit θ, a co-multiplication μ and the so-called Frobenius relation shown in the following figure.*

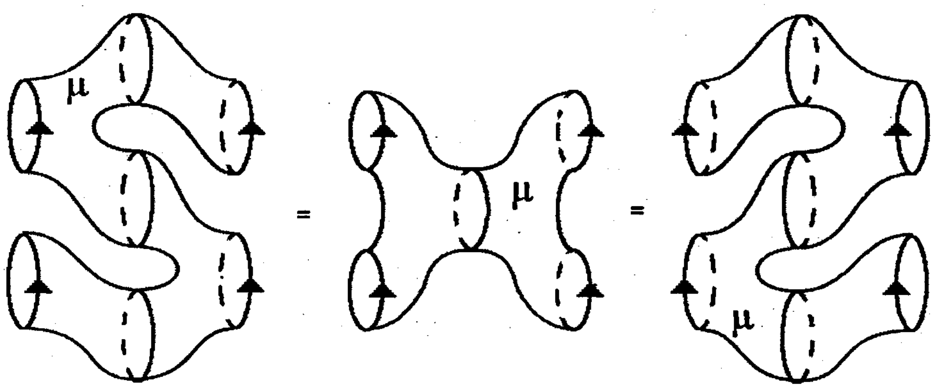


Figure 2.25: Frobenius Relation

From this, the vector space A will be associative and of finite dimension, the co-multiplication will be associative, and (A, θ) will define a Frobenius algebra [11].

2.3 Topological Conformal Field Theories and Topological Landau-Ginzburg Models

Theorem 2 says that the matter sector of 2D-TFT is always encoded by a Frobenius algebra. The Frobenius algebra on the space of states A is called the *primary chiral algebra* of the TFT. To preserve as much information as possible in generating the correlators from the Lagrangian, not only the Lagrangian but also its topologically invariant deformations are considered:

$$L \to L + \sum t^\alpha L_\alpha^{(pert)}$$

where t^α are coupling constants. We now have a moduli space of TFTs. A large, physically relevant class of such moduli spaces of TFTs are *topological conformal field theories* (TCFT). There is a physical theorem which asserts the *canonical moduli space of a TCFT carries the structure of a Frobenius Manifold* [7]. Examples of TCFT include the Topological A-models (Kähler moduli) and Topological B-models (complex moduli) famously related by mirror symmetry and ones we consider next: Topological Landau Ginzburg models (polynomial moduli, which are included in the family of topological B-models).

Example 2.3.1 *Topological Landau-Ginzburg (LG) Models:*
The Bosonic part of the LG action is:

$$S = \int d^2z\left(\left|\frac{\partial p}{\partial z}\right|^2 + |\lambda'(p)|^2\right) \tag{2.3.1}$$

where $\lambda(p)$ is a holomorphic function called the superpotential, and S is a functional of the holomorphic $p(z)$ called the superfield. The Classical states correspond to the critical points of $\lambda(p)$, where

$$p_i := p(z) \text{ for } \lambda'(p_i) = 0, i = 1 \dots n$$

A family of LG models (the moduli space of the LG-theory) is obtained by deforming the superpotential

$$\lambda = \lambda(p; t^1, \dots, t^n)$$

for parameters $t = (t^1, \dots, t^n)$, and the Frobenius structure on the space of parameters *is given by:*

$$\langle \partial, \partial' \rangle_\lambda = \sum_{\lambda'=0} res \frac{\partial(\lambda dp)\partial'(\lambda dp)}{d\lambda(p)} \tag{2.3.2}$$

$$\langle \partial, \partial', \partial'' \rangle_\lambda = \sum_{\lambda'=0} res \frac{\partial(\lambda dp)\partial'(\lambda dp)\partial''(\lambda dp)}{d\lambda(p)dp} \tag{2.3.3}$$

where the vector fields $\partial, \partial', \partial''$ on the space of parameters are taken keeping p constant. For the particular superpotential

$$\lambda(p) = p^{n+1}$$

its deformed superpotential matches the polynomial of Example (1.1). We will see this in chapter 3.

Chapter 3

Unfoldings of Singularities and the Orbit Space of a Coxeter Group

We will describe the unfoldings of singularities of A_n type in section one and verify its Frobenius structure [2],[14]. In section two we describe the Frobenius structure on the space of orbits of a Coxeter group [4],[7], and in the third section we verify that the structures from Examples (1.1) and (2.1) are Frobenius, and coincide with the Frobenius structure on the orbit space of Coxeter group A_n [7], [17].

3.1 Unfoldings of Singularities

Using unfoldings, one may construct a product and flat metric on the space of parameters $M = \mathbf{C}^n$ and establish canonical coordinates that determine the Euler vector field [2],[14]. In this section we will see how our main example of A_n-type exhibits a natural Frobenius Manifold structure in the unfoldings of singularities. The unfolding F_ξ of the polynomial $f(z) = z^{n+1}$ gives the space of parameters a rich structure. Consider only germs, so $M = \mathbf{C}^n$. By choosing a basis for the vector space Q_{F_ξ} (the Jacobian algebra), the tangent space $T_\xi\mathbf{C}^n$ is given the structure of a commutative algebra with unit. Here the product at a point $\xi \in \mathbf{C}^n$ is denoted by $*_\xi$, and the ξ_0-axis gives the identity vector field $\frac{\partial}{\partial \xi_0}$ in all $T_\xi\mathbf{C}^n$. A theorem by K. Saito states there is an isomorphism of vector bundles over $\mathbf{C}^n$, so that at any point ξ the isomorphism $Q_{F_\xi} \to T_\xi\mathbf{C}^n$ transports the bilinear form θ to a flat metric on $\mathbf{C}^n$ [14]. Let us consider the unfolding F_ξ, flat metric and bilinear form θ in our main example.

Example 3.1.1 *A_n-type unfolding: Let us consider the polynomial $f(z) = z^{n+1}$ and* *its unfolding*

$$F_\xi = z^{n+1} + \xi_{n-1}z^{n-1} + \ldots + \xi_0 \quad (\xi \in \mathbf{C}^n) \tag{3.1.1}$$

Its space of parameters is the affine space of all polynomials F_ξ. At any point, the tangent space is the vector space of polynomials with degree $\leq n-1$. The product $\alpha *_\xi \beta$ at this point of $Q_{F_\xi} = \mathbf{C}[z]/ < F_\xi' >$ is the remainder of $\alpha\beta$ in the Euclidean division by F_ξ'.

The one-form θ is

$$\alpha \mapsto \frac{1}{2\pi i}\int_\Gamma \frac{\alpha dz}{F_\xi'} = -Res_\infty \frac{\alpha dz}{F_\xi'} \tag{3.1.2}$$

To see that it defines a flat metric, we look for the flat coordinates in which θ is constant. To do this we invert and solve the equation

$$w^{n+1} = F_\xi(z)$$

$$w = z + \mathcal{O}(z^{-1})$$

expanding the solution for z large, so that

$$z = w + \frac{t_{n-1}}{w} + \ldots + \frac{t_0}{w^n} + \mathcal{O}\left(\frac{1}{w^{n+1}}\right) \tag{3.1.3}$$

where $t_0 \ldots t_{n-1}$ is a basis of the vector space of symmetric polynomials. Then $t_0 \ldots t_{n-1}$ are seen to be the flat coordinates by,

$$\frac{\partial}{\partial t_i}(F_\xi) = F_\xi'(z(w,t))\frac{\partial z}{\partial t_i} = F_\xi'(z(w,t)w^{-n+i} \tag{3.1.4}$$

and using

$$F_\xi'(z)dz = (n+1)w^n dw \tag{3.1.5}$$

we have

$$g(\frac{\partial}{\partial t_i}(F_\xi), \frac{\partial}{\partial t_j}(F_\xi)) = -Res_{z=\infty}F_\xi'(z(w,t))w^{-2n+i+j}dz \tag{3.1.6}$$

$$= -(n+1)Res_{w=\infty}w^{-n+i+j}dw \tag{3.1.7}$$

$$= (n+1)\delta_{i+j,n-1} \tag{3.1.8}$$

$$\tag{3.1.9}$$

the metric flat and nondegenerate everywhere.

Now we would like to establish the canonical coordinates and the Euler vector field.

The canonical coordinates are chosen to be the critical values of $F_\xi : x_i = F_\xi(q_i)$. The vector fields forming a basis for the algebra Q_{F_ξ} are then $\frac{\partial}{\partial x_i}$. For any polynomial $P \in \mathbb{C}[z_1, \dots, z_k]$ with m critical points, the vector field is written in canonical coordinates as:

$$P = \sum_{i=1}^{m} P(q_i) \frac{\partial}{\partial x_i} \tag{3.1.10}$$

The unity vector field is:

$$1 = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \tag{3.1.11}$$

The unfolding F_ξ itself is now an Euler vector field

$$F_\xi = \sum_{i=1}^{m} F_\xi(q_i) \frac{\partial}{\partial x_i} = \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i} =: E \tag{3.1.12}$$

since it rescales the product $*_\xi$ according to (1.1.17). Lastly, we check the Euler vector field E acts by conformal transformations of the metric and that $\nabla(\nabla E) = 0$. Looking at the Euclidean division of F_ξ by F_ξ', we may write the Euler vector field E in coordinates $(\xi_0, \dots, \xi_{n-1})$ as

$$E_\xi = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial}{\partial \xi_j} \tag{3.1.13}$$

In flat coordinates, and recalling from (3.1.3)

$$t_i = -\xi_i + B_i(\xi_{i+1}, \dots, \xi_{n-1}) \quad 0 \le i \le n-1$$

we assume $deg(\xi_j) = n - j + 1$ so that $deg(F_\xi(z)) = n + 1$, $deg(B_i) = n - i + 1$ and both F_ξ and B_i are homogeneous. Then,

$$E \cdot t_i = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial t_i}{\partial \xi_j} \tag{3.1.14}$$

$$= \frac{n-i+1}{n+1} t_i - \left(\frac{n-i+1}{n+1} B_i - \sum_{j=i+1}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial B_i}{\partial \xi_j}\right) \tag{3.1.15}$$

$$= \frac{n-i+1}{n+1} t_i \tag{3.1.16}$$

by the homogeneity of B_i, so that in flat coordinates,

$$E_\xi = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} t_j \frac{\partial}{\partial t_j} \tag{3.1.17}$$

Any space of parameters of a versal unfolding is isomorphic to the space of orbits of a Coxeter group, and we will see their Frobenius structure in the next section. Also Dubrovin conjectured that all polynomial solutions of the WDVV equations are potentials of these structures. This was later proved by Hertling [8].

3.2 Frobenius Structure on the Orbit Space of a Coxeter Group

In this section, we first define the intersection form, flat pencil of metrics and the monodromy group of a Frobenius manifold [4],[7]. We then show the Frobenius structure on the orbit space of a Coxeter group. These manifolds are polynomial in nature and each possesses a finite Coxeter group as its monodromy group [7].

Given a Frobenius manifold M we may use the invariant inner product η (1.1.2), to define another flat metric $(,)^*$ called the *intersection form.*

Definition 3.2.1 *The intersection form is given by*

$$(x,y)^* := i_E(x*y) \tag{3.2.1}$$

for $x, y \in T^*M$ and i_E the inner derivative of a 1-form with the Euler vector field E.

The components of $(,)^*$ in flat coordinates t^α are:

$$g^{\alpha\beta} := (dt^\alpha, dt^\beta)^* = E^\epsilon(t)c_\epsilon^{\alpha\beta}(t) \tag{3.2.2}$$

where

$$c_\epsilon^{\alpha\beta}(t) := \eta^{\alpha\sigma}c_{\sigma\epsilon}^{\beta}(t) \tag{3.2.3}$$

Here η has been used to extend the multiplication and Frobenius structure from the tangent bundle to the cotangent bundle. Having established these metrics as flat, Dubrovin proved further that any linear combination of them is also flat, defining the *flat pencil of metrics.* Consider two non-proportional metrics $(,)_1^*$ and $(,)_2^*$ and their corresponding Levi-Civita connections ∇_1^i and ∇_2^i.

Definition 3.2.2 *Two metrics form a flat pencil if:*

1) The following metric is flat for λ arbitrary

$$g^{ij} = g_1^{ij} + \lambda g_2^{ij} \tag{3.2.4}$$

2) The Levi-Civita connection for this metric has the form

$$\Gamma_k^{ij} = \Gamma_{1k}^{ij} + \lambda\Gamma_{2k}^{ij} \tag{3.2.5}$$

Consider the metrics $(,)^*$ and $\langle,\rangle^*$ on the Frobenius Manifold M, where $\langle,\rangle^*$ is induced on T^*M by $\eta = \langle,\rangle$ and the Euler vector field E is linear in the flat coordinates.

Proposition 3.2.1 *The metrics $(,)^*$ and $\langle,\rangle^*$ form a flat pencil on M.*

Remark 3.2.1 *On M, the difference tensor is defined by:*

$$\Delta^{ijk} = g_2^{is}\Gamma_{1s}^{jk} - g_1^{is}\Gamma_{2s}^{jk} \tag{3.2.6}$$

Dubrovin's geometry of flat pencils of metrics [7], gives the following proposition:

Proposition 3.2.2 *For a flat pencil of metrics a vector field $f = f^i\partial_i$ exists such that the difference tensor (3.2.6) and the metric g_1^{ij} have the form*

$$\Delta^{ijk} = \nabla_2^i\nabla_2^j f^k \tag{3.2.7}$$

$$g_1^{ij} = \nabla_2^i f^j + \nabla_2^j f^i + c g_2^{ij} \tag{3.2.8}$$

for some constant c. The vector field satisfies the equations

$$\Delta_s^{ij}\Delta_l^{sk} = \Delta_s^{ik}\Delta_l^{sj} \tag{3.2.9}$$

where

$$\Delta_k^{ij} := g_{2ks}\Delta^{sij} = \nabla_{2k}\nabla_2^i f^j \tag{3.2.10}$$

$$(g_1^{is}g_2^{jt} - g_2^{is}g_1^{jt})\nabla_{2s}\nabla_{2t}f^k = 0 \tag{3.2.11}$$

Conversely, for a flat metric g_2^{ij} and solution f of (3.2.9),(3.2.11) the metrics g_1^{ij} and g_2^{ij} form a flat pencil.

Later in this section, we will see that from the intersection form, Euler vector field and unity vector field, one can uniquely reconstruct the Frobenius structure. We now describe the monodromy group of a Frobenius Manifold.. The intersection form or *contravariant metric* $(,)^*$ is degenerate on the *discriminat locus* Σ where the *discriminant* $\Delta(t)$ vanishes:

$$\Delta(t) := det(g^{\alpha\beta}(t)) = 0 \tag{3.2.12}$$

$\Sigma \subset M$ where

$$\Sigma := \left\{t\Delta(t) := det(g^{\alpha\beta}(t)) = 0\right\} \tag{3.2.13}$$

Since $(,)^*$ and η are defined outside of the discriminant locus Σ, a Frobenius manifold defined by $(M/\Sigma, (,)^*)$ is not simply connected. Thus at any point there will be a nontrivial holonomy group at any point, being a discrete subgroup of $O(n, \mathbf{C})$ [4]. An isometry Φ can be specified of a domain Ω in n-dimensional complex Euclidean space E^n to the universal cover of M/Σ:

$$\Phi : \Omega \to \widehat{M/\Sigma} \tag{3.2.14}$$

Then the action of the fundamental group $\pi_1(M/\Sigma)$ on the universal cover is lifted to an action by the isometries of E^n. This isometry (3.2.14) is contructed by fixing a point $p_0 \in M$ outside of Σ and expressing $y = (y^1, \ldots, y^n)$ the flat coordinates of $(,)^*$ in terms of the flat coordinates $t = (t^1, \ldots, t^n)$ of η. Germs of functions $y^i(t^1, \ldots, t^n)$ will be multivalued around Σ, and the set of non-contractible loops γ around Σ correspond to linear affine transformations of the y^i's. In this way, the map and group homomorphism μ is obtained:

$$\mu : \pi_1(M/\Sigma) \to Isometries(E^n) \tag{3.2.15}$$

Definition 3.2.3 *The image of the fundamental group under μ defines the monodromy group $W(M)$ of the Frobenius Manifold:*

$$W(M) := \mu(\pi_1(M/\Sigma)) \subset Isometries(E^n) \tag{3.2.16}$$

Remark 3.2.2 *The flat coordiantes $y = (y^1, \ldots, y^n)$ are found by solving the following system, where $\widehat{\nabla}$ denotes the Levi-Civita connection for the intersection form (the Gauss-Manin connection):*

$$\widehat{\nabla}^\alpha \widehat{\nabla}_\beta y := g^{\alpha\epsilon}(t)\partial_\alpha \partial_\beta y + \Gamma_\beta^{\alpha\epsilon}(t)\partial_\epsilon y = 0 \tag{3.2.17}$$

for $\alpha, \beta = 1, \ldots, n$

Inversely, we next describe the construction of polynomial Frobenius manifolds whose monodromy group is a Coxeter group preserving invariant the intersection form $(,)^*$. Let W be a finite Coxeter group; a finite group of linear transformations of an n-dimensional Euclidean space V generated by reflections [5]. The orbit space $M = V/W$ has the structure of an affine variety, where the coordinate ring of M is identified with the coordinate ring of W-invariant polynomials over V. The coordinate ring of M has as a basis invariant homogeneous polynomials y^i. Their degrees d_i are invariants of the group W. The maximal degree h is called the Coxeter number of

W.

$$d_i := deg(y^i) \tag{3.2.18}$$

$$d_1 = h > d_2 \geq \ldots \geq d_{n-1} > d_n = 2 \tag{3.2.19}$$

For example, group A_n has degrees $d_i = n + 2 - i$ and group B_n has degrees $d_i = 2(n - i + 1)$. The action of W is extended to the complexified space

$$M = V \otimes \mathbf{C}/W$$

Coordinates on V will be denoted by x^a.The Euler vector field is:

$$E = \frac{1}{h}(d_1 y^1 \partial_1 + \ldots + d_n y^n \partial_n) = \frac{1}{h} x^a \frac{\partial}{\partial x^a} \tag{3.2.20}$$

The invariant coordinates will be denoted as y^n and normalized as:

$$y^n = \frac{1}{2h}((x^1)^2) + \ldots + (x^n)^2)) \tag{3.2.21}$$

where $(.,.)$ denotes the W-invariant Euclidean metric on V, and is extended onto M as a complex quadratic form. We denote here by $(.,.)^*$ the contravariant metric on the cotangent bundle T^*M induced by the W-invariant Euclidean metric on V.

Lemma 2 *The Euclidean metric of V induces a polynomial contravariant metric (,)* (the intersection form) on the space of orbits*

$$g^{ij}(y) = (dy^i, dy^j)^* := \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^a} \tag{3.2.22}$$

and the corresponding polynomial contravariant Levi-Civita connection (the Gauss-Manin connection)

$$\Gamma^{ij}_k(y) dy^k = \frac{\partial y^i}{\partial x^a}\frac{\partial^2 y^j}{\partial x^a \partial x^b} dx^b \tag{3.2.23}$$

Remark 3.2.3 *The intersection form (3.2.22) and Gauss-Manin connection (3.2.23) are graded homogeneous polynomials that depend linearly on* y^1, *with degrees:*

$$deg\ g^{ij}(y) = d_i + d_j - 2 \tag{3.2.24}$$

$$deg\ \Gamma^{ij}_k(y) = d_i + d_j - d_k - 2 \tag{3.2.25}$$

Theorem 3 *There exists a unique, up to an equivalence, Frobenius Structure on the space of orbits of a finite Coxeter group with the intersection form (3.2.22), the Euler* *vector field (3.2.20) and the unity vector field* $e := \frac{\partial}{\partial y^1}$

To prove this main theorem, we give the Saito metric (3.2.26) in lemma 3, Saito flat coordinates $t^1(x), \dots, t^n(x)$ in lemma 4, and the associated components of the intersection form and Gauss-Manin connection in lemma 5. Then the existence of the Frobenius structure on the orbit space of a Coxeter group is proven in lemma 6 with uniqueness following.

Lemma 3 *The triangular matrix*

$$\eta^{ij}(y) := \partial_1 g^{ij}(y) = 0 \; for \; i + j > n + 1 \tag{3.2.26}$$

has constant nonzero antidiagonal elements

$$c_i := \eta^{i(n-i+1)}, \tag{3.2.27}$$

and

$$c := det(\eta^{ij}) = (-1)^{\frac{n(n-1)}{2}} c_1, \dots, c_n \neq 0 \tag{3.2.28}$$

Lemma 4 *There exist homogeneous polynomials* $t^1(x), \dots, t^n(x)$ *of respective degrees* $d_1, \dots, d_n$ *such that the matrix*

$$\eta^{\alpha\beta} := \partial_1 (dt^\alpha, dt^\beta)^* \tag{3.2.29}$$

is constant

Lemma 5 *For coordinate* t^n *normalized as in (3.2.21), we have the following (with no summation over repeated indices):*

$$g^{n\alpha} = \frac{d_\alpha}{h} t^\alpha \tag{3.2.30}$$

$$\Gamma^{n\alpha}_\beta \frac{(d_\alpha - 1)}{h} \delta^\alpha_\beta \tag{3.2.31}$$

Lemma 6 *Let* $t^1, \dots, t^n$ *be the Saito flat coordinates on the space of orbits of a finite Coxeter group and*

$$\eta^{\alpha\beta} = \partial_1 (dt^\alpha, dt^\beta)^* \tag{3.2.32}$$

be the corresponding constant Saito metric. Then there exists a quasihomogeneous polynomial F(t) of degreee 2h+2 such that

$$(dt^\alpha, dt^\beta)^* = \frac{(d_\alpha + d_\beta - 2)}{h} \eta^{\alpha\lambda} \eta^{\beta\mu} \partial_\lambda \partial_\mu F(t) \tag{3.2.33}$$

The polynomial $F(t)$ determines on the space of orbits a polynomial Frobenius structure with the structure constants

$$c_{\alpha\beta}^{\gamma}(t) = \eta^{\gamma\epsilon}\partial_\alpha\partial_\beta\partial_\epsilon F(t) \tag{3.2.34}$$

the unity

$$e = \partial_1 \tag{3.2.35}$$

the Euler vector field

$$E = \sum(1 - \frac{degt^\alpha}{h})t^\alpha\partial_\alpha \tag{3.2.36}$$

and the invariant inner product η.

Proof. Using lemma 4 and proposition 3.2, we represent for some vector field $f^\beta(t)$ the tensor $\Gamma_\gamma^{\alpha\beta}(t)$ as

$$\Gamma_\gamma^{\alpha\beta}(t) = \eta^{\alpha\epsilon}\partial_\eta\partial_\gamma f^\beta(t) \tag{3.2.37}$$

Also, $\Gamma_\gamma^{\alpha\beta}(t)$ must satisfy the conditions

$$g^{\alpha\sigma}\Gamma_\sigma^{\beta\gamma} = g^{\beta\sigma}\Gamma_\sigma^{\alpha\gamma} \tag{3.2.38}$$

For $\alpha = n$, lemma 4 and the Euler identity it follows that

$$(d_\gamma - 1)g^{\beta\gamma} = \sum_\epsilon \eta^{\beta\epsilon}(d_\gamma - d_\epsilon + h)\partial_\epsilon f^\gamma = (d_\gamma + d_\beta - 2)\eta^{\beta\epsilon}\partial_\epsilon f^\gamma \tag{3.2.39}$$

which gives the symmetry condition (3.2.41) by defining

$$\frac{F^\gamma}{h} := \frac{f^\gamma}{d_\gamma - 1} \tag{3.2.40}$$

$$\eta^{\beta\epsilon}\partial_\epsilon F^\gamma = \eta^{\gamma\epsilon}\partial_\epsilon F^\beta \tag{3.2.41}$$

Hence there exists a quasihomogeneous polynomial function $F(t)$ in $t^1, \ldots, t^n$ with degree $2h + 2$ such that

$$F^\alpha = \eta^{\alpha\epsilon}\partial_\epsilon F \tag{3.2.42}$$

Equation (3.2.33) follows from (3.2.41) and (3.2.39), and writing

$$c_\gamma^{\alpha\beta}(t) = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F \tag{3.2.43}$$

we have the following:

$$\Gamma_\gamma^{\alpha\beta} = \frac{(d_{\beta-1})}{h} c_\gamma^{\alpha\beta} \tag{3.2.44}$$

$$c_\gamma^{\alpha\beta} c_\sigma^{\gamma\epsilon} = c_\gamma^{\alpha\epsilon} c_\sigma^{\gamma\beta} \tag{3.2.45}$$

$$c_\beta^{1\alpha} = \delta_\beta^\alpha \tag{3.2.46}$$

End Proof.

This structure is unique. For a polynomial Frobenius structure on M with Euler vector field (3.2.20) and Saito invariant metric, $F(t)$ must satisfy (3.2.33) up to a quadratic polynomial. We consider in the Saito flat coordinates

$$dt^\alpha \cdot dt^\beta = \eta^{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t) dt^\gamma \tag{3.2.47}$$

and by the definition of the intersection form we have

$$i_E(dt^\alpha, dt^\beta) = \frac{1}{h}\sum_\gamma d_\gamma t^\gamma \eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu\partial_\gamma F(t) \tag{3.2.48}$$

$$= \frac{1}{h}(d_\alpha + d_\beta - 2)\eta_{\alpha\lambda}\eta^{\beta\mu}\partial_\lambda\partial_\mu F(t) = (dt^\alpha, dt^\beta)^* \tag{3.2.49}$$

3.3 Example of Frobenius Structure on the Orbit Space of Coxeter Group A_n

The group $W = A_n$ acts on $\mathbf{R}^{n+1} = (\xi_0, \xi_1, \dots, \xi_n)$ by permutations

$$(\xi_0, \xi_1, \dots, \xi_n) \mapsto (\xi_{\sigma(0)}, \xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$$

restricted to the hyperplane

$$\xi_0 + \xi_1 + \ldots + \xi_n = 0 \tag{3.3.1}$$

The invariant metric is the Euclidean metric on (3.3.1), and the invariant polynomials are the symmetric polynomials. A homogeneous basis in this ring of invariant polynomials is given by the elementary symmetric polynomials:

$$a_k = (-1)^{n-k+1}(\xi_0\xi_1 \ldots \xi_k \ldots + ...), \quad k = 1, \dots, n \tag{3.3.2}$$

The complexified space of orbits $M = \mathbf{C}^n/A_n$ is then identified with the space of polynomials $\lambda(p)$ from example (1.1). We now show that the Frobenius structure on M from lemma 6 coincides with the structures of examples (1.1) and (2.1) [7], [17].

Theorem 4 *1. For example (1.1), the inner product* $\langle , \rangle$ *and tensor* $c(.,.,.) = \langle . * ., . \rangle_\lambda$ *have the form*

$$\langle \partial', \partial'' \rangle_\lambda = -\sum Res_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)}{d\lambda(p)} \tag{3.3.3}$$

$$c(\partial', \partial'', \partial''') = -\sum Res_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)\partial'''(\lambda(p)dp)}{dpd\lambda(p)} \tag{3.3.4}$$

2. Let $q^1, \dots, q^n$ *be the critical points of the polynomial* $\lambda(p)$,

$$\lambda'(q^i) = 0, \quad i = 1, \dots, n$$

and

$$u^i = \lambda(q^i), \quad i = 1, \dots, n \tag{3.3.5}$$

be the critical values. Here $u^1, \dots, u^n$ *are local coordinates on* M *near* λ *where* $\lambda(p)$ *has only simple roots. These local coordinates are the canonical coordinates for multiplication of example (1.1) and in these canonical coordinates the metric* *from example (1.1) has the diagonal form*

$$\langle , \rangle|_\lambda = \sum_{i=1}^{n} \eta_{ii}(u)(du^i)^2, \quad \eta_{ii}(u) = \frac{1}{\lambda''(q^i)} \tag{3.3.6}$$

$\partial', \partial'', \partial'''$ are any tangent vectors on M in a point λ, where derivatives are taken keeping p constant. $\lambda'(p)$ and $\lambda''(p)$ are the first and second derivatives wrt p.

3. *The metric on M induced by the invariant Euclidean metric at a point λ for which $\lambda(p)$ has simple roots may be written as*

$$(\partial', \partial'')_\lambda = -\sum Res_{d\lambda=0} \frac{\partial'(log\lambda(p)dp)\partial''(log\lambda(p)dp)}{dlog\lambda(p)} \tag{3.3.7}$$

Proof. 1. Equation (3.3.3) corresponds to the invariant inner product from example (1.1):

$$\langle f, g\rangle_\lambda = Res_{p=\infty}\frac{f(p)g(p)}{\lambda'(p)}$$

By letting $\partial' = f$, $\partial' = g$ and $\lambda'(p) = \frac{d\lambda(p)}{dp}$ and denoting by ω the meromorphic differential:

$$\omega = \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)}{d\lambda(p)} \tag{3.3.8}$$

we apply residue theorem on ω. The inner products are seen to correspond to each other since the sum of residues of a meromorphic differential on the Riemann p-sphere is zero.

$$Res_{p=\infty}\omega + \sum Res_{|\lambda|<\infty}\omega = 0 \tag{3.3.9}$$

Equation (3.3.4) corresponds to the multiplication from example 1.1. Using equation (3.3.9) and letting $f(p) = \partial'(\lambda(p))$, $g(p) = \partial''(\lambda(p))$, and $h(p) = \partial'''(\lambda(p))$ to re-write it as:

$$c(\partial', \partial'', \partial''') = Res_{p=\infty}\frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)\partial'''(\lambda(p)dp)}{dpd\lambda(p)} \tag{3.3.10}$$

For polynomials $q(p)$ and $r(p)$ where $deg(q) < n$, $f(p)g(p) = q(p) + r(p)\lambda'(p)$, and in the Milnor ring $\mathbf{C}[p]/(\lambda'(p))$, we will have the multiplication $f * g = q$ so that

$$Res_{p=\infty}\frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)\partial'''(\lambda(p)dp)}{dpd\lambda(p)} = Res_{p=\infty}\frac{q(p)h(p)}{d\lambda(p)} + Res_{p=\infty}r(p)h(p)dp \tag{3.3.11}$$

Since the second residue vanishes, the first residue is the inner product:

$$\langle q, h\rangle_\lambda = \langle f * g, h\rangle_\lambda = c(f, g, h) \tag{3.3.12}$$

2. The intersection form in the hyperplane coordinates has the form:

$$g^{ab} = \delta^{ab} - \frac{1}{n+1} \tag{3.3.13}$$

Denote the roots of $\lambda'(p)$ by q^i.

$$u_i = \lambda(q_i) \quad i = 1, \ldots, n \tag{3.3.14}$$

$$\partial_i \lambda(p)|_{p=q_j} = \delta_{ij} \tag{3.3.15}$$

Using (3.3.14),(3.3.15) and the Lagrange interpolation formula, we get

$$\partial_i \lambda(p) = \frac{1}{p-q^i} \frac{\lambda'(p)}{\lambda''(q^i)} \tag{3.3.16}$$

Since $\lambda(p)$ and $\lambda'(p)$ are given by

$$\lambda(p) = (p + \xi_1 + \ldots + \xi_n) \prod_{a=1}^{n} (p - \xi_a)$$

$$\lambda'(p) = \prod_{i=1}^{n} (p - q^i)$$

it follows that

$$(\partial_i \xi_1 = \ldots + \partial_i \xi_n) \prod_{b=1}^{n} (p - \xi_b) - \sum_{a=1}^{n} \frac{\lambda(p)}{p - \xi_a} \partial_i \xi_a = \partial_i \lambda(p) \tag{3.3.17}$$

Substituting $p = \xi_a$ in the previous equation, we get

$$\partial_i \xi_a = -\frac{1}{(\xi_a - q^i)\lambda''(q^i)}, \quad i, a = 1, \ldots, n \tag{3.3.18}$$

From (3.3.3), (3.3.4), and (3.3.15) we obtain

$$\langle \partial_i, \partial_j \rangle = -\delta_{ij} \frac{1}{\lambda''(q^i)} \tag{3.3.19}$$

$$c(\partial_i, \partial_i, \partial_i) = \langle \partial_i * \partial_i, \partial_i \rangle = -\frac{1}{\lambda''(q^i)} \tag{3.3.20}$$

Now we see that $u^1 \dots u^n$ are the canonical coordinates, since in the algebra we have

$$\partial_i * \partial_j = \delta_{ij}\partial_i \tag{3.3.21}$$

3.From (3.3.4) and (3.3.15), we have

$$g_{ii}(u) := (\partial_i, \partial_i) = -\frac{1}{u^i \lambda''(q^i)} \tag{3.3.22}$$

Using this, we obtain

$$(d\xi_a, d\xi_b) = \sum_{i=1}^{n} \frac{1}{g_{ii}} \frac{\partial \xi_a}{\partial u^i} \frac{\partial \xi_b}{\partial u^i} \tag{3.3.23}$$

$$= -\sum_{i=1}^{n} \frac{u^i}{(\xi_a - q^i)(\xi_b - q^i)\lambda''(q^i)} \tag{3.3.24}$$

$$= -\sum_{i=1}^{n} Res_{d\lambda=0} \frac{\lambda(p)}{(p - \xi_a)(p - \xi_b)\lambda'(p)} \tag{3.3.25}$$

$$= \delta_{a,b} - \frac{1}{n+1} \tag{3.3.26}$$

So the intersection form (3.3.7) coincides with the W-invariant metric. End Proof.

Chapter 4

Hurwitz Spaces

Hurwitz spaces are moduli spaces of pairs $(\mathcal{L}, \lambda)$, where $\mathcal{L}$ is a Riemann surface of genus g and λ is a meromorphic function on $\mathcal{L}$ of degree $N = n + 1$. We will see that these spaces with certain restrictions may be given the structure of a Frobenius manifold [3],[4],[7],[13],[16],[15]. Dubrovin builds the Frobenius structure on a covering of the Hurwitz space, which is necessary for the more general cases $g > 0$. There is also the notion of choosing between different *primary differentials* (or primitive forms), that produce different solutions to WDVV but are also related by *Legendre transformations* [7]. The main A_n example may also be described as a Frobenius manifold constructed on a Hurwitz space. The simplest class of such Hurwitz spaces where $g = 0$, $\mathcal{L}$ is the Riemann sphere and λ are rational functions from $\mathcal{L} \to \mathcal{L}$, is where our main A_n example falls [7],[13].

4.1 Hurwitz Spaces and Hurwitz Covers

Specifically, the Hurwitz space $M = H_{g;n_0,\ldots,n_m}$ is the space of equivalence classes $[\lambda : \mathcal{L} \to \mathbf{C}P^1]$ of N-fold branched covers with the following properties:

- n simple ramification points $P_1, \ldots, P_n \in \mathcal{L}$ with distinct finite images $u^1, \ldots, u^n \in \mathbf{C} \subset \mathbf{C}P^1$. These are the critical values of $\lambda : u^j = \lambda(P_j)$, $d\lambda|_{P_j=0}$, $j = 1, \ldots, n$
- The pre-image $\lambda^{-1}(\infty)$ consists of $m + 1$ points: $\lambda^{-1}(\infty) = \infty_0, \ldots, \infty_m$ and the ramification index of the map p at a point ∞_j is n_j $(1 \leq n_j \leq N)$
- The Riemann-Hurwitz formula gives the dimension n of space M as $n = 2g + N + 2m$, (where $N = n_0 + \ldots + n_m$) in terms of the genus g of $\mathcal{L}$, degree N of λ and number of simple finite branch points m.

- The one-dimensional affine group acts on M by:

$$(\mathcal{L}; \infty_0, \ldots, \infty_m; \lambda; \ldots) \longmapsto (\mathcal{L}; \infty_0, \ldots, \infty_m; a\lambda + b; \ldots) \tag{4.1.1}$$

$$u^i \longmapsto au^i + b, \quad i = 1, \ldots, n \tag{4.1.2}$$

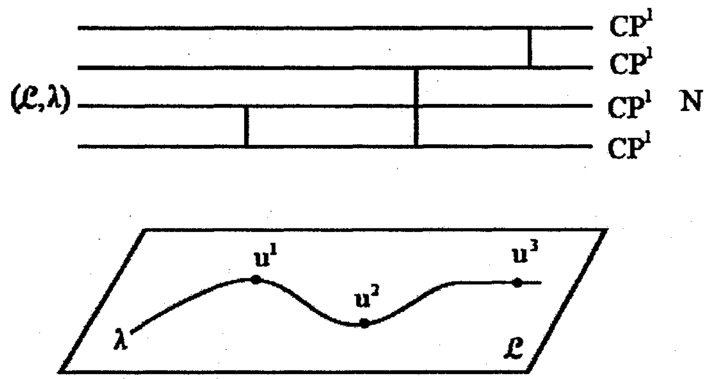


Figure 4.1: $(\mathcal{L}, \lambda)$

Remark 4.1.1 *Some remarks are in order:*

(1) The finite branch (ramification) points being simple expresses that only two sheets are glued at each point.

(2) There are $m+1$ points on the covering projecting to $\lambda(P) = \infty$ on the base. The numbers $(n_i + 1)$ give the number of sheets glued at every point, where n_i are the ramification indices from above.

(3) Two coverings are called equivalent if one can be obtained from the other by a permutation of sheets. $M = H_{g;n_0,\ldots,n_m}$ is the space of equivalence classes of sheets.

Dubrovin constructs a Frobenius structure on a covering of the Hurwitz space. The ramification points $u^1, \ldots, u^n$ will be the canonical coordinates for the multiplication of the tangent vector fields:

$$\partial_i * \partial_j = \delta_{ij}\partial_i, \quad \partial_i := \frac{\partial}{\partial u^i} \tag{4.1.3}$$

The unit vector field e and the Euler vector field E generate the action of the affine group (4.1.2):

$$e = \sum_{i=1}^{n} \partial_i \tag{4.1.4}$$

$$E = \sum_{i=1}^{n} u^i \partial_i \tag{4.1.5}$$

1-forms Ω on a manifold with a Frobenius algebra on the tangent planes are called *admissible* if a Frobenius manifold structure is determined by the invariant inner product:

$$< \partial', \partial'' >_\Omega := \Omega(\partial' * \partial'') \tag{4.1.6}$$

A quadratic differential Q is called $d\lambda$-divisible when it has the form $Q = q d\lambda$ where the differential q has no poles in the branch points of $\mathcal{L}$. The corresponding 1-form Ω_Q on the Hurwitz space M is determined by any Q holomorphic for $|\lambda| < \infty$ on $\mathcal{L}$. Since the 1-form $\Omega_Q = 0$, we may include also multivalued quadratic differentials on the universal covering of $\mathcal{L}$. The monodromy transformation along a cycle γ acts by

$$Q \longmapsto Q + q_\gamma d\lambda \tag{4.1.7}$$

On a suitable covering $\widehat{M}$ of M, the metrics will be defined by the 1-forms corresponding to these multivalued differentials Q. The covering $\widehat{M} = \widehat{M}_{g;n_0,\dots,n_m}$ is the space of sets

$$(\mathcal{L}; \infty_0, \dots, \infty_m; \lambda; k_0, \dots, k_m; a_1, \dots, a_g; b_1, \dots, b_g) \in M_{g;n_0,\dots,n_m} \tag{4.1.8}$$

with the same $\mathcal{L}$,$\infty_0, \dots, \infty_m$ and λ from M, plus a canonical basis of cycles $a_1, \dots, a_g$; $b_1, \dots, b_g$ on $\mathcal{L}$. The branch points $P_1, \dots, P_n$ are the local coordinates on $\widehat{M}$, and in the neighbourhood of P near ∞^i:

$$k_i^{n_i+1}(P) = \lambda(P), \quad P \text{ near } \infty^i \tag{4.1.9}$$

where n_i is the ramification index at ∞^i.

Admissible quadratic differentials on the Hurwitz space are constructed as squares $Q = \phi^2$ of *primary differentials* ϕ on $\mathcal{L}$ (or a covering of $\mathcal{L}$). There are five types of primary differentials. (All differentials have zero a-periods except the holomorphic ϕ_{s^i} below. Also the coefficients $\delta_i, \alpha_i, \beta_i, \gamma_i$ are independent on the point in $\widehat{M}$.) The five types of primary differentials with their characteristic singularities are:

1. A normalized Abelian differential of the second kind:

$$\phi = \phi_{t^{i;\alpha}}(P) := -\frac{1}{\alpha} dk_i^\alpha(P), \quad P \ near \ \infty^i; \quad i = 0, \ldots, m, \quad \alpha = 1, \ldots, n_i \tag{4.1.10}$$

2. A normalized Abelian differential of the second kind:

$$\phi := \delta_i \phi_{v^i} \tag{4.1.11}$$

$$\phi_{v^i}(P) := -d\lambda(P), \quad P \ \ near \ \infty^i; \quad i = 1, \ldots, m \tag{4.1.12}$$

3. A normalized Abelian differential of the third kind:

$$\phi := \alpha_i \phi_{w^i}(P), \quad res_{\infty^i} \phi_{w^i} = 1, \quad res_{\infty^0} \phi_{w^i} = -1; \quad i = 1, \ldots, m \tag{4.1.13}$$

4. A normalized multivalued differential. (The differential undergoing analytic continuation along b_k on $\mathcal{L}$ tranforms as:

$$\phi := \beta_i \phi_{r^i}(P), \quad \phi_{r^i}(P + b_j) - \phi_{r^i}(P) = -\delta_{ij} d\lambda(P); \quad i = 1, \ldots, g \tag{4.1.14}$$

5. A normalized holomorphic differential:

$$\phi := \gamma_i \phi_{s^i}, \quad \oint_{a_j} \phi_{s^i} = \delta_{ij}; \quad i = 1, \ldots, g \tag{4.1.15}$$

For any primary differential ϕ and corresponding multivalued quadratic differential $Q = \phi^2$, Ω_Q will be an admissible 1-form on the Hurwitz space M. The metric corresponding to Ω_{ϕ^2} is defined for two tangent fields ∂', ∂'' on $\widehat{M}$ as:

$$ds_{\phi^2} = < \partial', \partial'' >_{\phi^2} := \Omega_{\phi^2}(\partial' * \partial'') \tag{4.1.16}$$

This gives a Frobenius structure on $\widehat{M}$ for any ϕ. For the function λ, a multivalued function p on $\mathcal{L}$ is introduced:

$$p(P) := p.v. \int_{\infty_0}^{P} \phi \tag{4.1.17}$$

where the principal value is defined by omitting the divergent part of the integral as a function of the local parameter k_0. Now $\phi = dp$ and $\lambda(p)$ on $\mathcal{L}$ is locally a function of the complex variable p.

4.2 Hurwitz Spaces and Frobenius Manifolds

We now come to the main theorem of the chapter [7]:

Theorem 5 *Let $\widehat{M}$ be open in M and specify that $\phi(P_i) \neq 0, \quad i = 1, \ldots, N$. For any primary differential (4.1.10)-(4.1.15), the multiplication (4.1.3), unity (4.1.4), Euler vector field (4.1.5) and 1-form Ω_{ϕ^2} determine on $\widehat{M}$ a structure of a Frobenius manifold. The corresponding flat coordinates $t^A, \quad A = 1, \ldots, N$ consist of the five parts:*

$$t^A = (t^{i;\alpha},\ i = 0, \ldots, m,\ \alpha = 1, \ldots, n_i;\ p^i, q^i,\ i = 1, \ldots, m;\ r^i, s^i,\ i = 1, \ldots, g) \tag{4.2.1}$$

given by:

1.
$$t^{i,\alpha} = res_{\infty_i} k_i^{-\alpha} p d\lambda \tag{4.2.2}$$

2.
$$p^i = p.v. \int_{\infty_0}^{\infty_i} dp \tag{4.2.3}$$

3.
$$q^i = -res_{\infty_i} \lambda dp \tag{4.2.4}$$

4.
$$r^i = \oint_{b_i} dp \tag{4.2.5}$$

5.
$$s^i = -\frac{1}{2\pi i} \oint_{a_i} \lambda dp \tag{4.2.6}$$

The metric (4.1.16) in these coordinates have the (non-zero) forms:

a.
$$\eta_{t^{i;\alpha} t^{i;\beta}} = \frac{1}{n_i + 1} \delta_{ij} \delta_{\alpha+\beta, n_i+1} \tag{4.2.7}$$

b.
$$\eta_{p^i q^j} = \frac{1}{n_i + 1} \delta_{ij} \tag{4.2.8}$$

c.

$$\eta_{r^i s^i} = \frac{1}{2\pi i}\delta_{ij} \tag{4.2.9}$$

And, for any other primary differential ϕ, the 1-form Ω_{ϕ^2} is admissible on the Frobenius manifold.

Proof of the theorem.

For the multiplication (4.1.3), the metric ds^2_ϕ (4.1.16) is diagonal in the coordinates $u^1, \ldots, u^N$:

$$ds^2_\phi = \eta_{ii}(u)(du^i)^2 \quad i = 1, \ldots, N \tag{4.2.10}$$

$$\eta_{ii} = res_{P_i}\frac{\phi^2}{d\lambda} \tag{4.2.11}$$

We now define a Darboux-Egoroff metric, then prove a lemma that for any ϕ, ds^2_ϕ will be Darboux-Egoroff, and will satisfy invariance conditions that give the second and fourth properties of a Frobenius manifold (definition 1.1.3).

Definition 4.2.1 *A Darboux-Egoroff metric is flat, potential and diagonal. A diagonal metric $ds^2 = \eta_{ii}(du^i)^2$ is called potential if $\exists$ a function $V \ni \partial_i V = \eta_{ii}$ for all i. A potential diagonal metric is flat if the rotation coefficients $\gamma_{ij}, \quad i \neq j$*

$$\gamma_{ij}(u) := \frac{\partial_j\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \tag{4.2.12}$$

satisfy the following for i, j, k distinct for all γ_{ij}:

$$\partial_k\gamma_{ij} = \gamma_{ik}\gamma_{kj} \tag{4.2.13}$$

$$\sum_{k=1}^{n}\partial_k\gamma_{ij} = 0 \tag{4.2.14}$$

Remark 4.2.1 *The rotation coefficients of the invariant metric also satisfy;*

$$\sum_{k=1}^{n} u^k\partial_k\gamma_{ij} = -\gamma_{ij} \tag{4.2.15}$$

Lemma 7 *For any primary differential ϕ listed above (4.1.10)-(4.1.15), the metric (4.1.16) is Darboux-Egoroff and also satisfies the invariance conditions:*

$$Lie_e ds^2_\phi = 0 \tag{4.2.16}$$

$$Lie_E ds^2_\phi \textit{ is proportional to } ds^2_\phi \tag{4.2.17}$$

The rotation coefficients of the metric do not depend on the choice of the primary differential ϕ.

Proof of the lemma:

Let z_a be a local parameter near ∞_a: $z_a = k_a^{-1}$. First, a pairing of differentials $\omega^{(1)}$, $\omega^{(2)}$ is defined. $\omega^{(1)}$ and $\omega^{(2)}$ are holomorphic on $\mathcal{L}/(\infty_0 \cup \ldots \cup \infty_m)$, and at the infinite points behave as:
(Where $i = 1, 2$ and $c^{(i)}_{k,a}$, $r^{(i)}_{k,a}$, $A^{(i)}_\alpha$, $p^{(i)}_{s,\alpha}$, $q^{(i)}_{s,\alpha}$ are constants.)

$$\omega^{(i)} = \sum_k c^{(i)}_{k,a} z_a^k dz_a + d \sum_{k>0} r^{(i)}_{k,a} \lambda^k log\lambda, \quad P \to \infty_a \tag{4.2.18}$$

$$\oint_{a_\alpha} \omega^{(i)} = A^{(i)}_\alpha \tag{4.2.19}$$

$$\omega^{(i)}(P + a_\alpha) - \omega^{(i)}(P) = dp^{(i)}_\alpha(\lambda), \quad p^{(i)}_\alpha(\lambda) = \sum_{s>0} p^{(i)}_{s,\alpha} \lambda^s \tag{4.2.20}$$

$$\omega^{(i)}(P + b_\alpha) - \omega^{(i)}(P) = dq^{(i)}_\alpha(\lambda), \quad q^{(i)}_\alpha(\lambda) = \sum_{s>0} q^{(i)}_{s,\alpha} \lambda^s \tag{4.2.21}$$

The bilinear pairing for $\omega^{(1)}$, $\omega^{(2)}$ is defined as:
(Where P_0 is a marked point on $\mathcal{L} \ni \lambda(P_0) = 0$)

$$< \omega^{(1)}\omega^{(2)} > := - \sum_{a=0}^{m} \left[\sum_{k \geq 0} \frac{c^{(1)}_{-k-2,a}}{k+1} c^{(2)}_{k,a} + c_{-1,a} v.p. \int_{P_0}^{\infty_a} \omega^{(2)} + 2\pi i \ v.p. \int_{P_0}^{\infty_a} r^{(1)}_{k,a} \lambda^k \omega^{(2)} \right] + \tag{4.2.22}$$

$$+ \frac{1}{2\pi i} \sum_{\alpha=1}^{g} \left[- \oint_{a_\alpha} q^{(1)}_\alpha(\lambda) \omega^{(2)} + \oint_{b_\alpha} p^{(1)}_\alpha(\lambda) \omega^{(2)} + A^{(1)}_\alpha \oint_{b_\alpha} \omega^{(2)} \right]$$

With these definitions, the following may be proved:

Lemma 8 *The following identity holds*

$$res_{P_j} \frac{\omega^{(1)}\omega^{(2)}}{d\lambda} = \partial_j < \omega^{(1)}\omega^{(2)} > \tag{4.2.23}$$

Corollary 2 *The pairing (4.2.22) of differentials $\omega^{(1)}, \omega^{(2)}$ is symmetric up to an additive constant not depending on the moduli.*

From the previous lemma, we see the rotation coefficients of the metric (4.1.16) are symmetric:

$$\eta_{jj}(u) = \partial_j < \phi\phi >, \quad j = 1, \ldots, N \tag{4.2.24}$$

To prove the rotation coefficients satisfy (4.2.13) consider the differential

$$\partial_i \partial_j \int \partial_k \phi, \quad i, j, k, \text{distinct} \tag{4.2.25}$$

which has poles only in P_i, P_j, P_k. The contour integral will be zero along a domain $\partial\tilde{\mathcal{L}}$ obtained by cutting $\mathcal{L}$ along a canonical basis passing through P_0. Connecting P_0 at a vertex of the resulting $4g$-gon to the points $\infty_0, \ldots, \infty_m$ and cutting along these paths yields $\partial\tilde{\mathcal{L}}$. The sum of the residues vanishes, and by the symmetry of the rotation coefficients (4.2.13) is obtained, from:

$$\partial_j \sqrt{\eta_{ii}} \partial_k \sqrt{\eta_{ii}} + \partial_i \sqrt{\eta_{jj}} \partial_k \sqrt{\eta_{jj}} = \sqrt{\eta_{kk}} \partial_i \partial_j \sqrt{\eta_{kk}} \tag{4.2.26}$$

Similarly, the rotation coefficients are independent of the primary differential ϕ. Consider the differential:

$$\partial_i \phi \int \partial_j \varphi, \quad i \neq j \tag{4.2.27}$$

where φ is another primary differential. Since the sum of the residues vanishes, and the rotation coefficients are symmetric,

$$\sqrt{\eta_{jj}^{\varphi}} \partial_i \sqrt{\eta_{jj}^{\phi}} = \sqrt{\eta_{ii}^{\phi}} \partial_j \sqrt{\eta_{ii}^{\varphi}} \tag{4.2.28}$$

the rotation coefficients are the same for either metric. To prove (4.2.16) an operator D_e on functions $f = f(P, u)$ is defined: (and extended as the Lie derivative by requiring $dD_e = D_e d$)

$$D_e f := \frac{\partial f}{\partial \lambda} + \partial_e f \tag{4.2.29}$$

Then for any ϕ,

$$D_e \phi = 0 \tag{4.2.30}$$

For the metric (4.1.16), by using (4.2.24), the identity (4.2.14) is also obtained:

$$\partial_e \eta_{jj} = 0 \tag{4.2.31}$$

Similarly to prove the identity (4.2.17) an operator D_E is defined:

$$D_E := \lambda \frac{\partial}{\partial \lambda} + \partial_E \tag{4.2.32}$$

Then for any ϕ,

$$D_E \phi = [\phi]\phi \tag{4.2.33}$$

For the primary differentials ϕ listed as in (4.1.10)-(4.1.15), the numbers $[\phi]$ are given by:

$$\begin{aligned} [\phi_{t^{i;\alpha}}] &= \frac{\alpha}{n_i + 1} \\ [\phi_{v^i}] &= 1 \\ [\phi_{w^i}] &= 0 \\ [\phi_{r^i}] &= 1 \\ [\phi_{s^i}] &= 0 \end{aligned} \tag{4.2.34}$$

From this, we may write:

$$\partial_E \eta_{ii}^{\phi} = (2[\phi] - 1)\eta_{ii}^{\phi} \tag{4.2.35}$$

which gives also (4.2.15). End proof of lemma 7.
The identity (4.2.16) is equivalent to condition (F2) from the definition (1.1.3) of the Frobenius manifold. Also, the identity (4.2.17) is equivalent to the third condition from (F4), equation (1.1.17). Given the Euler field (4.1.5) and multiplication (4.1.3), the first and second conditions of (F4) equations (1.1.15), (1.1.17) are satisfied. Condition (F3) is given by the following lemma [7][9]:

Lemma 9 *If g is an Darboux-Egoroff metric, (with respect to the canonical coordinates $u^1, \ldots, u^n$), then the Frobenius structure with canonical basis $\frac{\partial}{\partial u^i}$ has ∇c symmetric.*

Since we have a Darboux-Egoroff metric for any primary differential ϕ invariant with respect to the multiplication (4.1.3), condition (F3) is satisfied. To complete the Frobenius structure, the flat coordinates for the flat metric are established. Denote the coordinates by t^A, and define

$$\phi_A := -\partial_{t^A} \lambda dp \tag{4.2.36}$$

Then by lemma (7),

$$< \partial_{t^A}, \partial_{t^B} >_\phi = \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\phi_A \phi_B}{d\lambda} = \partial_e < \phi_A \phi_B > \tag{4.2.37}$$

The non-zero coefficients $c_{k,a}^{(A,B)}$, $A_\alpha^{(A,B)}$, $q_\alpha^{(A,B)}$ for the differentials $\phi_{A,B}$ are as defined in equations (4.2.18)-(4.2.21). Using $D_e \phi_B = 0$ from (4.2.30):

$$\partial_e c_{k,a}^{(B)} = \frac{k+1}{n_a+1} c_{k-n_a-1,a}^{(B)} \tag{4.2.38}$$

$$\partial_e \int_{P_0}^{\infty_a} \phi_B = \frac{c_{-n_a-2,a}^{(B)}}{n_a+1} + \left(\frac{\phi_B}{d\lambda} \right)_{P_0} \tag{4.2.39}$$

$$\partial_e \oint_{a_\alpha} \lambda \phi_B = \oint_{a_\alpha} \phi_B \tag{4.2.40}$$

$$\partial_e \oint_{b_\alpha} \phi_B = -\frac{\phi_B}{d\lambda}(P + b_\alpha) + \frac{\phi_B}{d\lambda}(P) \tag{4.2.41}$$

Then using the bilinear pairing (4.2.22), the forms of the metrics (4.2.7)-(4.2.9) are obtained:

$$\partial_e < \phi_A \phi_B > = -\sum_{a=0}^{m} \left(\frac{1}{n_a+1} \sum_{k=0}^{n_a-1} c_{-k-2,a}^{(A)} c_{k-n_a-1,a}^{(B)} \right) - \tag{4.2.42}$$

$$-\sum_{a=1}^{m} \frac{1}{n_a-1} \left(c_{-1,a}^{(A)} c_{-n_a-2,a}^{(B)} + c_{-1,a}^{(B)} c_{-n_a-2,a}^{(A)} \right) - \frac{1}{2\pi i} \sum_{\alpha=1}^{g} \left(q_{1\alpha}^{(A)} A_\alpha^{(B)} + A_\alpha^{(A)} q_{1\alpha}^{(B)} \right)$$

End proof of the theorem.

4.3 Hurwitz Space $H_{0;n}$ and the A_n-example

Example 4.3.1 *The Hurwitz space $H_{0;n}$ corresponding to the main A_n-type example consists of all polynomials of the form*

$$\lambda(p) = p^{n+1} + a_n p^{n-1} + \ldots + a_1, \qquad a_1, \ldots, a_n \in \boldsymbol{C} \tag{4.3.1}$$

The affine transformations $\lambda \mapsto a\lambda + b$ act on (4.3.1) by

$$p \mapsto a^{\frac{1}{n+1}} p, \quad a_i \mapsto a_i a^{\frac{n-i+2}{n+1}} \ \ \textit{for} \ \ i > 1, \quad a_1 \mapsto a a_1 + b \tag{4.3.2}$$

To show the Frobenius structure, we show the metric (4.2.10) is Darboux-Egoroff and is flat for the flat coordinates. For primary differential $\phi = dp$, the flat coordinates (4.2.2) correspond to the flat coordinates in example (3.1). Let us denote these flat coordinates as ξ_α. By an affine transformation we can set the sum of the roots to zero and leading coefficient to one. Thus:

$$\lambda(p) = (p + \xi_1 + \ldots + \xi_n) \prod_{\alpha=1}^{n} (p - \xi_\alpha) \tag{4.3.3}$$

$$= \prod_{\alpha=0}^{n} (p - \xi_\alpha) \quad \textit{where} \ \ \xi_0 = -\sum_{\alpha=1}^{n} \xi_\alpha \tag{4.3.4}$$

Since the roots are not independent:

$$\frac{\partial log \lambda(p)}{\partial \xi_i} = \frac{1}{p - \xi_0} - \frac{1}{p - \xi_i}, \quad i = 1, \ldots, n \tag{4.3.5}$$

and for the metric (3.3.7) from chapter 3:

$$(\partial_{\xi_i}, \partial_{\xi_j}) = -\sum res_{d\lambda=0} \left[\left(\frac{1}{p - \xi_o} - \frac{1}{p - \xi_i} \right) \left(\frac{1}{p - \xi_o} - \frac{1}{p - \xi_j} \right) \frac{\lambda}{\lambda'} dp \right] \tag{4.3.6}$$

Then using (3.3.9) as in theorem 4,
we have for $i = j$:

$$(\partial_{\xi_i}, \partial_{\xi_j}) = res_{p=\xi_0} \left[\frac{1}{(p - \xi_0)^2} \frac{\lambda}{\lambda'} dp \right] \tag{4.3.7}$$

$$= 1 \tag{4.3.8}$$

and for $i \neq j$:

$$(\partial_{\xi_i}, \partial_{\xi_j}) = res_{p=\xi_0}\left[\frac{1}{(p-\xi_0)^2}\frac{\lambda}{\lambda'}dp\right] + res_{p=\xi_i}\left[\frac{1}{(p-\xi_i)^2}\frac{\lambda}{\lambda'}dp\right] \tag{4.3.9}$$

$$= 2 \tag{4.3.10}$$

Giving constant coefficients:

$$(\partial_{\xi_i}, \partial_{\xi_j}) = 1 + \partial_{ij} \tag{4.3.11}$$

for the metric we denote here by g:

$$g = \sum_{i,j=1}^{n} (\partial_{\xi_i}, \partial_{\xi_j}) d\xi_i d\xi_j \tag{4.3.12}$$

$$= \sum_{i,j=1}^{n} (1 + \partial_{ij}) d\xi_i d\xi_j \tag{4.3.13}$$

$$= \sum_{i=0}^{n} (d\xi_i)^2 \,\Big|_{\xi_0 = -\sum_{i=1}^{n} \xi_i} \tag{4.3.14}$$

We have seen that g is Darboux-Egoroff since in canonical coordiantes $u_i = \lambda(q_i)$ (using notations from chapter 3, theorem 4) g is diagonal and potential:

$$g = \sum_{i=1}^{n} \frac{(du_i)^2}{\lambda''(q_i)} \tag{4.3.15}$$

$$\frac{1}{\lambda''(q_i)} = \frac{1}{(n+1)}\frac{\partial a_n}{\partial u_i} \tag{4.3.16}$$

with potentiality given by

$$\eta_{ii} = \frac{1}{\lambda''(q_i)} = \frac{\partial}{\partial u_i} V \tag{4.3.17}$$

where

$$V = \frac{a_n}{n+1} \tag{4.3.18}$$

Conclusion and Further Developments

In summary, the solution space of the WDVV system was shown to carry a Frobenius Manifold structure, and Frobenius Algebras were shown to correspond to the matter sector of 2D-Topological Field Theories. The A_n example was given from the class of polynomial solutions to WDVV, and the topological Landau-Ginzburg model was seen to correspond to the A_n case. We saw also how Frobenius structures are built on the orbit spaces of Coxeter groups, and how Hurwitz Frobenius Manifolds are constructed. The A_n case originally described as the unfolding of a versal singularity was seen also to lend itself to the Coxeter orbit space construction and Hurwitz space construction.

Frobenius structures and their prepotentials have been found on the orbit spaces of B_n and D_n [17], in addition to that found by Dubrovin for A_n which we describe here. It would be interesting to construct an equivariant Hurwitz space via its correspondent metric, one which reflects the invariance of λ under the action of W, and reveal further the rather intriguing differential geometry of these Hurwitz Frobenius manifolds. This particular problem has motivated the present write-up.

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