Robust Stabilization of Interconnected Systems by Means of Structurally Constrained Controllers

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ABSTRACT

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This dissertation deals with performance analysis and robust stabilizability verification of large-scale interconnected systems with respect to the class of linear time-invariant (LTI) decentralized controllers. These problems are formulated and tackled in four phases. First, an interconnected system with some unstable decentralized fixed modes (DFM) is considered. It is well-known that there is no stabilizing LTI decentralized controller for such a system; hence, a method is proposed to change the structure of the controller from decentralized to a proper overlapping form, with respect to which the system is stabilizable. This change in the control configuration is carried out by introducing some interactions among the isolated controllers, which leads to the elimination of the undesirable DFMs. The approach utilized in this thesis is based on the graph theory and, in fact, transforms the knowledge of the system into a number of bipartite graphs. A simple combinatorial algorithm is subsequently proposed to address the problem under consideration.

The second problem investigated here is the characterization of all classes of LTI structurally constrained controllers with respect to which a given interconnected system has no fixed modes. Similar to the ideas and notions proposed to handle the preceding problem, an efficient method is presented to tackle the problem in this case. Since establishing a transmission link between a pair of local controllers would certainly incur cost, an implementation expenditure is attributed to each possible link. The proposed approach can also be used to attain the implementation cost associated with any suitable class of controllers obtained. As a by-product of this result, all classes of LTI stabilizing structurally

constrained controllers with the minimum implementation cost can be characterized accordingly.

A LTI structurally constrained control system is considered next, which is subject to parametric uncertainties. Moreover, a region of uncertainty in the form of a semi-algebraic set is envisioned to parametrize the range of variations for uncertain parameters. It is asserted that if the system is stabilizable via a given constrained controller at the nominal point, then it is almost always stabilizable at any operating point in the region of uncertainty. In other words, the points for which the system has some persistent fixed modes lie on an algebraic variety. A method is subsequently proposed to derive this variety.

In the end, it is assumed that a stabilizing decentralized controller isdesigned for a pseudo-hierarchical large-scale system based on its hierarchical approximation. A LQ cost function is defined to evaluate the effectiveness of this indirect controller design for the system. It is shown that a reasonably tight upper bound on this performance index can be straightforwardly obtained by solving a constrained optimization problem with only three variables.

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I would like to express my gratitude to my supervisor, Dr. Amir G. Aghdam, for his insightful comments and ideas for developing the results of this thesis. Without his ongoing encouragement, support, and invaluable suggestions, this work would not be possible.

My sincere gratitude also goes to Javad Lavaei, who not only played an important role in my life as a husband, but also provided me with fruitful ideas and discussions. With no doubt, his advice and comments were essential in the development of a great part of my work.

Last but not least, I would like to thank my parents for their unconditional love and teaching me how to think and learn. I am deeply and forever indebted to them for life.

To my spouse

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Chapter 1

Introduction

The problem of controlling large-scale interconnected systems has been of increasing interest in the past few decades due to its striking applications in many real-world systems such as cooperative robots, communication networks, power systems, to name only a few. The control of this type of systems is often carried out by applying a local controller to each interacting subsystem. In the case when the local controllers are permitted to fully exchange information, the corresponding controller is, in fact, centralized. A centralized controller normally outperforms all other control structures with respect to any given cost function. Nevertheless, due to many practical limitations, it often turns out that the local controllers are restricted to communicate only partially with each other. This may degrade the performance of the system and, more importantly, make the system lose stabilizability. The latter case is sometimes referred to as structurally constrained control. Decentralized control is a special type of constrained controllers, where the local controllers are prohibited from exchanging information and are totally isolated. In this case, each local controller observes only the output of a single subsystem to construct the control command for the same subsystem. In some control applications such as formation flying, the local controllers are neither isolated nor fully interacting with each other. This kind of control configuration is often referred as a decentralized overlapping control structure.

The notion of decentralized fixed modes (DFM) was introduced in the literature to characterize those modes of a LTI system which are fixed with respect to any LTI decentralized controller. Analogously, the notion of a decentralized overlapping fixed mode (DOFM) introduced in prior literature characterizes those modes of the system (if any), which are immovable with respect to all classes of LTI structurally constrained controllers.

This work mainly concentrate on investigating different aspects of *decentralized* and *decentralized overlapping* controller design, and in particular the robustness and stabilization properties. To this end, three chapters are provided in the remainder of the thesis to elaborate on these problems, which will be outlined next.

The problem of eliminating the unwanted DFMs of a system is tackled in Chapter 2, using an efficient graph-theoretic method. The core idea of this part of the dissertation is to establish new links between certain pairs of local controllers in order to change the control configuration from decentralized to decentralized overlapping. This technique leads to characterizing all the decentralized overlapping control structures with respect to which the system has no undesirable fixed modes. It is noteworthy that this result is primarily obtained based on a graph-theoretic approach and different notions for bipartite graphs. Furthermore, a cost is attributed to each communication link in order to take the implementation expenditure for any link into account. Among the obtained classes of control structures, all the ones with the minimum implementation cost have also been systematically identified.

In Chapter 3, a novel technique is proposed to obtain all classes of LTI structurally constrained controllers with respect to which the system has no fixed modes. This problem is in particular important for interconnected systems with several subsystems, where identifying the classes of controllers which fit into the control objectives is a formidable task, in light of the fact that the number of all classes of controllers exponentially depends on the number of subsystems. The classes of LTI stabilizing structurally constrained controllers with the minimum implementation cost are then characterized. It is shown how a noticeable amount of time can be saved in order to obtain these classes, if a certain technique proposed in this chapter is utilized.

In Chapter 4, the robust stabilizability of uncertain LTI systems with respect to any class of structurally constrained controllers is investigated. To this end, it is assumed that the system is polynomially uncertain, and that the corresponding region of uncertainty is a semi-algebraic set. It is proved that if the system has no DOFMs at some point belonging to the uncertainty region, then the points for which the system has a DOFM lie on an algebraic variety. As a result, if a system has no DOFM at a given nominal point, it almost always has no DOFMs at any operating point. Furthermore, since finding the exact algebraic variety can be formidable in general, a simple method is proposed to compute a dominant subset of it, in the sense that the dimension of this subset is greater than that of its complement.

In Chapter 5, it is assumed that a decentralized controller has been designed for the hierarchical model of a pseudo-hierarchical system which meets certain control objectives. Moreover, it is supposed that this controller stabilizes the pseudo-hierarchical system, while it may deteriorate the overall performance. A LQ performance index is defined to assess the discrepancy between the pseudo-hierarchical system and the corresponding reference hierarchical model under this decentralized controller. An optimization problem with only three variables is derived whose solution is indeed an upper bound on this cost function. It is subsequently proved that as the pseudo-hierarchical system approaches the corresponding reference hierarchical model, this bound goes to zero, and in the ideal case, the bound is equal to zero.

It is worth mentioning that the results obtained in this masters thesis are published/will appear for publication in the following journals and conferences:

- **Somayeh Sojoudi** and Amir G. Aghdam, "Elimination of Decentralized Fixed Modes by Employing Optimal Information Exchange," accepted for publication in *Automatica*.

- **Somayeh Sojoudi**, Javad Lavaei and Amir G. Aghdam, "Robust Control of LTI Systems by Means of Structurally Constrained Controllers," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1721-1726, Sep. 2007.

- **Somayeh Sojoudi** and Amir G. Aghdam, "Performance Analysis for a Class of Decentralized Control Systems," in *Proceedings of 2008 American Control Conference*, Seattle, Washington, US, Jun 2008.

- **Somayeh Sojoudi** and Amir G. Aghdam, "Characterizing All Classes of LTI Stabilizing Structurally Constrained Controllers by Means of Combinatorics," in *Proceedings of 46th IEEE Conference on Decision and Control*, New Orleans, USA, pp. 4415-4420, Dec. 2007.

- Javad Lavaei, **Somayeh Sojoudi** and Amir G. Aghdam, "Rational Optimization using Sum-of-Squares Techniques," in *Proceedings of 46th IEEE Conference on Decision and Control*, New Orleans, USA, pp. 4723-4730, Dec. 2007.

- Javad Lavaei, **Somayeh Sojoudi** and Amir G. Aghdam, "Pole Assignment with Improved Control Performance by Means of Periodic Feedback," in *Proceedings of 46th IEEE Conference on Decision and Control*, New Orleans, USA, pp. 1082-1087, Dec. 2007.

- **Somayeh Sojoudi**, Javad Lavaei and Amir G. Aghdam, "Optimal Information Flow

Structure for Control of Interconnected Systems," in *Proceedings of 2007 American Control Conference*, New York City, USA, pp. 1508-1513, July 2007.

- Javad Lavaei, **Somayeh Sojoudi** and Amir G. Aghdam, "Stability Criterion for General Proper Systems with Constrained Control Structure," in *Proceedings of 2007 American Control Conference*, New York City, USA, pp. 1648-1653, July 2007.

Moreover, the following papers are submitted for publication:

- **Somayeh Sojoudi** and Amir G. Aghdam, "Robust Controllability and Observability: A Gramian Approach," *submitted for journal publication.*

- **Somayeh Sojoudi** and Amir G. Aghdam, "Characterization of Optimal Stabilizing Structurally Constrained Controllers By means of Graph-Theoretic Approach," *submitted for journal publication.*

- **Somayeh Sojoudi** and Amir G. Aghdam, "Performance Analysis of Decentralized Controllers Designed Based on Hierarchical Model of the System," *submitted for journal publication.*

- Javad Lavaei, **Somayeh Sojoudi** and Amir G. Aghdam, "Pole placement by Means of Periodic Feedback," *submitted for journal publication.*

- **Somayeh Sojoudi**, Javad Lavaei and Amir G. Aghdam, "Robust Controllability and Observability Degrees of Polynomially Uncertain Systems," *submitted for conference publication.*

Chapter 2

Elimination of Decentralized Fixed Modes of a System by Employing Optimal Information Exchange

2.1 Abstract

This work deals with the stabilizability of interconnected systems via linear time-invariant (LTI) decentralized controllers. Given a controllable and observable system with some distinct decentralized fixed modes (DFM), it is desired to find a desirable control structure (in terms of information flow) for it. Since a decentralized controller consists of a number of non-interacting local controllers, the objective here is to establish certain interactions between the local controllers in order to eliminate the undesirable DFMs. This objective is achieved by translating the knowledge of the system into some bipartite graphs. Then, the notions of minimal sets and maximal subgraphs are introduced, which lead to a simple combinatorial algorithm for solving the underlying problem. Moreover, the proposed technique can be applied to the quotient system corresponding to the strongly connected

subsystems, to displace the quotient fixed modes (QFM). The efficacy of the results obtained is demonstrated in two illustrative examples.

2.2 Introduction

Numerous real-world systems can be modeled as the interconnected systems consisting of a number of subsystems. The control of an interconnected system is often carried out by means of a set of local controllers, corresponding to the interacting subsystems [7; 26; 15]. It is sometimes assumed that the local controllers can fully communicate with each other in order to elevate their effectiveness over the entire system cooperatively. However, this design technique is often problematic as the required data transmission between two particular local controllers (or equivalently, two subsystems) can be unjustifiably expensive or occasionally infeasible. Consequently, it is normally desired that the local controllers either exchange partial information or act independently of each other. The latter case, where the overall controller consists of a set of isolated local controllers, is referred to as decentralized control in the literature [22; 7; 26]. The control structure in a decentralized control system is, in fact, block-diagonal. It is to be noted that the decentralized control theory has found applications in large space structures, power systems, communication networks, etc. [20; 12; 9; 21]. A wide variety of properties of the decentralized control systems are extensively studied in the literature and different design techniques are proposed [16; 8; 17].

One of the important problems in decentralized control design for interconnected systems is the stabilizability verification. The notion of a decentralized fixed mode (DFM) was introduced in [29] to identify those modes of a system which are fixed with respect to any LTI decentralized controller. Since a DFM may be movable with respect to a nonlinear or time-vary decentralized controller, the notion of a quotient fixed mode (QFM) was introduced in [10] to identify those modes of the system which are fixed with respect to any general decentralized controller (not merely LTI ones).

Various methods are introduced in the literature to characterize DFMs and QFMs [7; 2; 3; 6; 18]. For instance, the method given in [7] provides the existence conditions for DFMs in terms of the rank of a set of matrices. As a computationally more efficient technique, the papers [18; 19] propose simple graph-theoretic approaches to verify whether an unrepeated mode of the system is a DFM or a QFM.

Given an interconnected system with at least one unstable DFM, the question arises: Can a stabilizing LTI controller be designed for this system by establishing new information flow channels in the control configuration (which will roughly possess a decentralized structure)? This question has been addressed in a number of papers to some extent by making certain permissible interactions between the local controllers. The work [4] uses this idea to tackle the underlying problem, but it fails to obtain the minimum number of required interactions to achieve stabilizability. This shortcoming limits the effectiveness of the method in practical applications considerably. The paper [28] deals with the pole-assignability problem for interconnected systems by means of partially interacting LTI local controllers. A cost is first attributed to the communication link between any pair of local controllers in order to formulate the implementation expenditure. Nevertheless, the work [28] considers only a particular class of the modes, due to the complexity of the problem in the general case. This particular class is, in fact, the fixed modes which result from the structure of the system, rather than an exact matching of the parameters of the system. This class of fixed modes is referred to as structurally fixed mode (SDFM) [24]. The method proposed in [28] leads to a near-optimal solution by solving two separate optimization problems. A simpler method to handle the same problem (i.e., eliminating the SDFMs of a system) is more recently presented in [5].

The work [30] tackles the problem of eliminating the DFMs by introducing a centralized controller (i.e. interactions between all subsystems). The advantage of this method is that the controller obtained is robust. Although this work introduces several interactions, it attempts to justify the underlying idea by utilizing the notion of low-rank matrices.

When some local controllers are capable of interacting with each other, the overall controller is said to be a decentralized overlapping controller [13]. The stabilizability of an interconnected system by means of LTI decentralized overlapping controllers has been investigated thoroughly in [13; 14; 27] using the new notions of decentralized overlapping fixed mode (DOFM) and quotient overlapping fixed mode (QOFM). It is to be noted that the decentralized overlapping control theory was initially introduced for the systems with some overlapping subsystems. For this type of systems, it is often desired to design decentralized overlapping controllers whose overlapping structure coincides with that of their corresponding subsystems [25]. The analysis and design of this class of decentralized overlapping control systems has been intensively studied in the literature, primarily in the Expansion-Contraction framework [11].

In the present work, it is assumed that the given interconnected system has some distinct undesirable DFMs. A cost is assigned for establishing a link between any pair of local controllers. This can, for instance, reflect the data transmission cost required for a communication link between the control stations. The ultimate goal can be described in two steps. The first step is to characterize all the decentralized overlapping control structures with respect to which the system has no undesirable fixed modes. The second step is to determine the optimal overlapping structure which minimizes the implementation cost (associated with establishing new links between local controllers). To this end, it is first shown that the unrepeated fixed modes of the system with respect to any overlapping control structure can be identified using a graph-theoretic approach. Then, the notions of minimal sets and maximal graphs are introduced to present a simple procedure for solving the problem under study. As a by-product of the proposed development, all the decentralized overlapping control structures whose implementation cost are less than any given value and are capable of eliminating the undesirable DFMs can also be characterized efficiently. Finally, similar ideas are employed to displace the distinct QFMs of the system.

2.3 Preliminaries

Consider a LTI interconnected system $\mathscr{S}$ consisting of ν subsystems $S_1, S_2, ..., S_\nu$ with the following state-space representation for its i-th subsystem ($i \in \bar{\nu} := \{1, 2, ..., \nu\}$):

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + f_i(x(t), u(t)) \\ y_i(t) &= C_{ii}x_i(t) + D_{ii}u_i(t) + g_i(x(t), u(t)) \end{aligned} \tag{2.1}$$

where $x_i(t) \in \Re^{n_i}$, $u_i(t) \in \Re^{m_i}$ and $y_i(t) \in \Re^{r_i}$ are the state, the input and the output of the subsystem S_i, respectively, and $f_i(x(t), u(t))$ and $g_i(x(t), u(t))$ denote the effect of the other subsystems on S_i through its incoming interconnections. Assume that:

$$\begin{aligned} f_i(x(t), u(t)) &= \sum_{j=1,\ j \neq i}^{\nu} A_{ij}x_j(t) + \sum_{j=1,\ j \neq i}^{\nu} B_{ij}u_j(t), \\ g_i(x(t), u(t)) &= \sum_{j=1,\ j \neq i}^{\nu} C_{ij}x_j(t) + \sum_{j=1,\ j \neq i}^{\nu} D_{ij}u_j(t) \end{aligned} \tag{2.2}$$

for any $i \in \bar{\nu}$. The state-space model of the system $\mathscr{S}$ can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{\nu} B_j u_j(t) \\ y_i(t) &= C_i x(t) + \sum_{j=1}^{\nu} D_{ij}u_j(t), \quad i \in \bar{\nu} \end{aligned} \tag{2.3}$$

where A is a matrix with (i, j) block entry A_{ij}, for any $i, j \in \bar{\nu}$, and:

$$\begin{aligned} B_j &= \begin{bmatrix} B_{1j}^T & B_{2j}^T & \cdots & B_{\nu j}^T \end{bmatrix}^T, \\ C_j &= \begin{bmatrix} C_{j1} & C_{j2} & \cdots & C_{j\nu} \end{bmatrix}, \quad j \in \bar{\nu} \end{aligned} \tag{2.4}$$

A structurally constrained controller for the system $\mathscr{S}$ consists of ν local controllers, partially interacting with each other. The following definition will prove convenient in formulating the interaction policy between different subsystems (or equivalently, between the local controllers).

Definition 1 *Given a structurally constrained controller, define the control interaction set* $\mathbf{K}$ *associated with this controller as a set which contains the entry* k_{ij}*,* $i, j \in \bar{\mathrm{v}}$*, if and only if* $y_j(t)$ *can contribute to the construction of* $u_i(t)$ *in the controller.*

Define the set $\mathbf{K}_d := \{k_{11}, k_{22}, ..., k_{\nu\nu}\}$. Any controller whose structure complies with $\mathbf{K}_d$ is composed of ν isolated (non-interacting) local controllers; i.e., there is no data transmission between the local controllers. A controller with this structure will be referred to as a decentralized controller throughout the work [15]. Moreover, any controller whose interaction set $\mathbf{K}$ is not equal to $\mathbf{K}_d$ but includes it (i.e., $\mathbf{K} \neq \mathbf{K}_d$, $\mathbf{K}_d \subset \mathbf{K}$) is called an overlapping controller [13].

The DFMs of $\mathscr{S}$ are indeed the modes of the system which are fixed with respect to all LTI controllers complying with the control interaction set $\mathbf{K}_d$ [7]. Furthermore, in the case of an overlapping controller with the interaction set $\mathbf{K}$, the DOFMs of the system $\mathscr{S}$ w.r.t. $\mathbf{K}$ are the modes of the system which are fixed under any LTI controller whose structure complies with $\mathbf{K}$ [13].

2.4 Main results

Consider the system $\mathscr{S}$ given by (2.3), and assume that it has some distinct undesirable DFMs. It is desired to displace these undesirable fixed modes using a proper control structure, in order to meet the design specifications. Let these undesirable modes be denoted by $\sigma_1, \sigma_2, ..., \sigma_\mu$. By definition, there is no LTI controller complying with $\mathbf{K}_d$ to displace any of these modes. Hence, it is desired to expand the control interaction set $\mathbf{K}_d$ by adding another set $\mathbf{K}_e$ to it such that none of these unwanted modes will be immovable with respect to the new control interaction set $\mathbf{K}_d \cup \mathbf{K}_e$. This problem is investigated in the sequel for a particular case first, and then is extended to the general case.

2.4.1 Displacing a single unrepeated DFM

Assume that σ is an arbitrary unrepeated mode of the system $\mathscr{S}$. One possible state-space realization for this system is given by:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} \sigma & 0 \\ 0 & \mathbf{A} \end{bmatrix} \mathbf{x}(t) + \sum_{j=1}^{\nu} \mathbf{B}_j u_j(t) \\ y_i(t) &= \mathbf{C}_i \mathbf{x}(t) + \sum_{j=1}^{\nu} \mathbf{D}_{ij} u_j(t), \quad i \in \bar{\nu}\end{aligned} \tag{2.5}$$

where the matrices $\mathbf{A}, \mathbf{B}_j, \mathbf{C}_i$ and $\mathbf{D}_{ij},\ i, j \in \bar{\nu}$ can be obtained by using a proper similarity transformation, but their exact form is not essential in the main development (it is to be noted that DFMs are invariant under any similarity transformation). Define the matrix M as follows:

$$M := \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_\nu \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (\mathbf{A} - \sigma I)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_\nu \end{bmatrix} - \begin{bmatrix} \mathbf{D}_{11} & \cdots & \mathbf{D}_{1\nu} \\ \vdots & \ddots & \vdots \\ \mathbf{D}_{\nu 1} & \cdots & \mathbf{D}_{\nu\nu} \end{bmatrix} \tag{2.6}$$

Note that since the multiplicity of σ is assumed to be 1, it is not an eigenvalue of the matrix $\mathbf{A}$. Denote the (i, j) block entry of M with $M_{ij} \in \Re^{r_i \times m_j}$, for any $i, j \in \bar{\nu}$.

A procedure will be introduced next, to construct the graphs required to verify which modes of the system are DFMs.

Procedure 1 *[18]*

Construct a bipartite graph $\mathscr{G}$ with two sets of vertices $\mathscr{V}$ (set 1) and $\bar{\mathscr{V}}$ (set 2) and the tagged vertices $1, 2, \ldots, \nu$ in each of the two sets. For any $i, j \in \bar{\nu}$, carry out the following steps:

1) Connect vertex j of the set $\mathscr{V}$ to vertex i of the set $\bar{\mathscr{V}}$ if $M_{ij} = 0$.

2) Mark vertex i of the set $\mathscr{V}$ if the first column of the matrix $\mathbf{C}_i$ is a zero vector.

3) Mark vertex j of the set $\bar{\mathscr{V}}$ if the first row of the matrix $\mathbf{B}_j$ is a zero vector.

Definition 2 *Consider an arbitrary graph $\hat{\mathscr{G}}$ with ζ vertices in any of its two sets, labeled $1, 2, ..., \zeta$. A subgraph of $\hat{\mathscr{G}}$ is said to span the vertices of $\hat{\mathscr{G}}$, if the labels of its vertices are distinct and form the set $\{1, 2, ..., \zeta\}$.*

Identify every subgraph of $\mathscr{G}$ which satisfies the following criteria:

i) It is a complete bipartite subgraph.

ii) All of its vertices are marked.

iii) It spans the vertices of the graph $\mathscr{G}$.

Denote all such subgraphs with $\mathscr{G}_1, \mathscr{G}_2, ..., \mathscr{G}_w$. Moreover, denote set 1 and set 2 (see Procedure 1) of the graph $\mathscr{G}_j$ with $\mathscr{V}_j$ and $\bar{\mathscr{V}}_j$, respectively, for any $j \in \{1, 2, ..., w\}$.

As an example, assume that the graph $\mathscr{G}$ for the mode σ of a given system which is obtained from the procedure 1, is the one depicted in Figure 2.4. It can be easily observed from this graph that vertices 1 and 2 of the set $\mathscr{V}$ and vertices 3 and 4 of the set $\bar{\mathscr{V}}$ fulfill the three criteria pointed out earlier. Therefore, σ is a DFM of the system (note that marked vertices are denoted by filled circles).

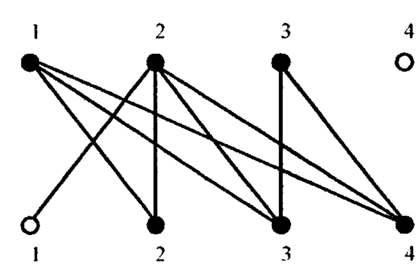

Figure 2.1: The graph $\mathscr{G}$ of a given system.

The following lemma is elicited from [18].

Lemma 1 *The mode σ is a DFM of the system $\mathscr{S}$ if and only if the nonnegative integer w is strictly positive.*

Assume for now that w is strictly positive, and consequently the mode σ is fixed with respect to any LTI controller complying with $\mathbf{K}_d$. It is desired to obtain all overlapping control structures which are able to displace this mode.

Procedure 2 *For any given set $\{k_{i_1 j_1}, k_{i_2 j_2}, ..., k_{i_z j_z}\}$, form a bipartite graph $\mathscr{G}(\{k_{i_1 j_1}, k_{i_2 j_2}, ..., k_{i_z j_z}\})$ as follows:*

- *Put $\nu + z$ vertices in set 1 and set 2 of the graph $\mathscr{G}(\{k_{i_1 j_1}, ..., k_{i_z j_z}\})$.*
- *Assign the labels $1, 2, ..., \nu, j_1, j_2, ..., j_z$ to the vertices of set 1.*
- *Assign the labels $1, 2, ..., \nu, i_1, i_2, ..., i_z$ to the vertices of set 2.*
- *Consider any two arbitrary vertices of the graph $\mathscr{G}(\{k_{i_1 j_1}, ..., k_{i_z j_z}\})$ which do not pertain to the same set of vertices. Let the labels of these two vertices be λ_1 (in set 1) and λ_2 (in set 2). Connect these two vertices to each other in the graph $\mathscr{G}(\{k_{i_1 j_1}, ..., k_{i_z j_z}\})$ if and only if there is an edge between vertex λ_1 of $\mathscr{V}$ and vertex λ_2 of $\bar{\mathscr{V}}$ in the graph $\mathscr{G}$.*

It is notable that some labels in the graph $\mathscr{G}(\{k_{i_1 j_1}, k_{i_2 j_2}, ..., k_{i_z j_z}\})$ are recurrent. The next theorem proposes a simple method to verify whether or not the mode σ is a DOFM of the system $\mathscr{S}$ with respect to a given control interaction set.

Theorem 1 *Given the set $\{k_{p_1 q_1}, k_{p_2 q_2}, ..., k_{p_\alpha q_\alpha}\}$, the mode σ is not a DOFM of the system $\mathscr{S}$ w.r.t. $\mathbf{K}_d \cup \{k_{p_1 q_1}, ..., k_{p_\alpha q_\alpha}\}$ if and only if the graph $\mathscr{G}(\{k_{p_1 q_1}, ..., k_{p_\alpha q_\alpha}\})$ does not contain a complete bipartite subgraph with all vertices marked, which spans its vertices (see Definition 2).*

Sketch of the proof: The proof will be given here for $\alpha = 1$, as its generalization is straightforward. By obtaining two transformation matrices discussed in [13] and pursuing the approach given therein, it can be easily verified that σ is a DOFM of the system $\mathscr{S}$ w.r.t. the control interaction set $\mathbf{K}_d \cup \{k_{p_1 q_1}\}$ if and only if it is a DFM of the following system:

$$
\begin{aligned}
\dot{\tilde{x}}(t) &= A\tilde{x}(t) + \sum_{j=1}^{v} B_j \tilde{u}_j(t) + B_{p_1} \tilde{u}_{v+1}(t) \\
\tilde{y}_i(t) &= C_i \tilde{x}(t) + \sum_{j=1}^{v} D_{ij} \tilde{u}_j(t) + D_{ip_1} \tilde{u}_{v+1}(t), \quad i \in \bar{v} \\
\tilde{y}_{v+1}(t) &= \tilde{C}_{q_1} \tilde{x}(t) + \sum_{j=1}^{v} D_{q_1 j} \tilde{u}_j(t) + D_{q_1 p_1} \tilde{u}_{v+1}(t)
\end{aligned}
\tag{2.7}
$$

Note that this system has one input and one output more than the system $\mathscr{S}$. The proof follows by applying the graph-theoretic approach given in [18] (which was explained in Lemma 1 for the system $\mathscr{S}$) to the system given in (2.7). ■

To clarify the result of Theorem 1, consider again the system whose graph $\mathscr{G}$ is depicted in Figure 2.4. Assume that it is desired to verify if σ remains a fixed mode after adding the controller k_{14} to the decentralized control structure to obtain $\mathbf{K} = \mathbf{K}_d \cup \{k_{14}\}$. For this purpose, consider the graph $\mathscr{G}(\{k_{14}\})$ sketched in Figure 2.2. This graph does not comprise a subgraph with the properties pointed out in Theorem 1. Therefore, the mode σ is not a DOFM of the system w.r.t. $\mathbf{K}_d \cup \{k_{14}\}$.

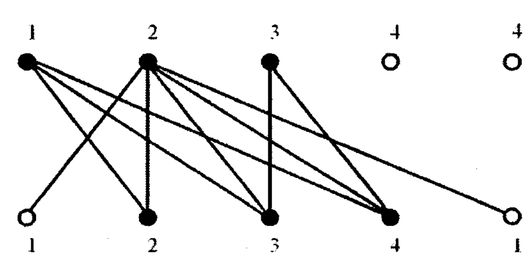

Figure 2.2: The graph $\mathscr{G}(\{k_{14}\})$ derived from the graph $\mathscr{G}$ in Figure 2.4.

So far, it is shown how the existence of a DOFM can be concluded from a bipartite graph. This result will be used next to characterize all desirable overlapping control structures.

Definition 3 *The set* $\{k_{p_1q_1}, k_{p_2q_2}, \ldots, k_{p_\alpha q_\alpha}\}$ *is said to be minimal w.r.t.* σ *if and only if the mode* σ *is not a DOFM of the system* $\mathscr{S}$ *w.r.t.* $\mathbf{K}_d \cup \{k_{p_1q_1}, k_{p_2q_2}, \ldots, k_{p_\alpha q_\alpha}\}$*, while it is a DOFM of* $\mathscr{S}$ *w.r.t.* $\mathbf{K}_d \cup \{k_{p_1q_1}, \ldots, k_{p_{j-1}q_{j-1}}, k_{p_{j+1}q_{j+1}}, \ldots, k_{p_\alpha q_\alpha}\}$ *for any* $j \in \{1, 2, \ldots, \alpha\}$.

Definition 4 *A subgraph of the graph* $\mathscr{G}$ *is said to be maximal if:*

i) It is a complete bipartite subgraph.

ii) All of its vertices are marked.

iii) The set of the labels of its vertices is equal to the set $\overline{\mathrm{V}}$ *(note that this condition is slightly different from spanning the vertices, as the labels can be recurrent here).*

iv) The graph $\mathscr{G}$ *has no other subgraph satisfying criteria (i), (ii) and (iii) given above such that it includes this subgraph.*

Using the combinatorial algorithms, the maximal subgraphs of $\mathscr{G}$ can be easily identified (analogously to the algorithms for finding the complete bipartite graphs with maximum number of edges). Denote such subgraphs with $\tilde{\mathscr{G}}_1, \tilde{\mathscr{G}}_2, \ldots, \tilde{\mathscr{G}}_{\tilde{w}}$. Moreover, denote set 1 and set 2 of vertices of the graph $\tilde{\mathscr{G}}_j$ with $\tilde{\mathscr{V}}_j$ and $\bar{\tilde{\mathscr{V}}}_j$, respectively, for any $j \in \{1, 2, \ldots, \tilde{w}\}$. It is to be noted that the number $\tilde{w}$ is typically small, due to the generic property of the fixed modes. The subsequent remark aims to present a bound on the number $\tilde{w}$.

Remark 1 *From the definition of a maximal graph, the sets of vertices* $\tilde{\mathscr{V}}_1, \ldots, \tilde{\mathscr{V}}_{\tilde{w}}$ *are all distinct. Moreover, it is straightforward to show that one of the sets* $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_w$ *is exactly the same as* $\tilde{\mathscr{V}}_j$*, for any* $j \in \{1, 2, \ldots, \tilde{w}\}$*. These two facts point to the inequality* $\tilde{w} \leq w$.

Consider again the system whose graph $\mathscr{G}$ is depicted in Figure 2.4. The graph $\mathscr{G}(\{k_{13}, k_{24}\})$ derived from $\mathscr{G}$ is depicted in Figure 2.3. It can be easily verified that this graph does not have any subgraph with the properties stated in Theorem 1. Hence, σ is not a DOFM of this system. In contrast, if any of the control components k_{13} or k_{24} is deleted from the control structure, the mode σ will become a DOFM. As a result, the set $\{k_{13}, k_{24}\}$ is minimal w.r.t. σ in this example.

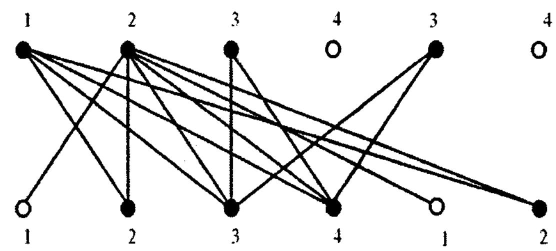

Figure 2.3: The graph $\mathscr{G}(\{k_{13}, k_{24}\})$ derived from the graph $\mathscr{G}$ in Figure 2.4.

Theorem 2 *Assume that the set $\{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\}$ is minimal w.r.t. the mode σ. Then, the number α is less than or equal to $\tilde{w}$.*

Proof: From the definition of a minimal set, the mode σ is a DOFM of the system $\mathscr{S}$ w.r.t. the control interaction set $\mathbf{K}_d \cup \{k_{p_1q_1}, ..., k_{p_{j-1}q_{j-1}}, k_{p_{j+1}q_{j+1}}, ..., k_{p_\alpha q_\alpha}\}$, for any $j \in \{1, 2, ..., \alpha\}$. Hence, it can be concluded from Theorem 1 that the graph $\mathscr{G}(\{k_{p_1q_1}, ..., k_{p_{j-1}q_{j-1}}, k_{p_{j+1}q_{j+1}}, ..., k_{p_\alpha q_\alpha}\})$ has a complete bipartite subgraph with marked vertices, which spans the vertices of the graph. This subgraph should include either the duplicated vertex q_i in its set 1 or the duplicated vertex p_i in its set 2, for all $i \in \{1, 2, ..., j-1, j+1, ..., \alpha\}$. On the other hand, it is straightforward to show that there exists an integer $f_j \in \{1, ..., \tilde{w}\}$ such that this subgraph is included in $\tilde{\mathscr{G}}_{f_j}$ (in light of the definition of a maximal graph). Thus, one comes to the conclusion immediately that the following logic statement is true:

$$\left(q_i \in \tilde{\mathscr{V}}_{(f_j)}\right) \vee \left(p_i \in \bar{\tilde{\mathscr{V}}}_{(f_j)}\right),\ \forall i \in \{1,2,...,j-1,j+1,...,\alpha\} \tag{2.8}$$

where $\vee$ is the logic operation *OR*. Now, to prove Theorem 2 by contradiction, assume that $\tilde{w} < \alpha$. Since all the natural numbers $f_1, f_2, ..., f_\alpha$ belong to the set $\{1,2,...,\tilde{w}\}$ and also the inequality $\tilde{w} < \alpha$ holds, it can be concluded from the Dirichlet's Principle that at least two of the values $f_1, f_2, ..., f_\alpha$ are identical. Without any loss of generality, assume that $f_1 = f_2 = f$ for some positive number f. Consider the relation (2.8) for the values $j = 1$ and $j = 2$. The amalgamation of these two sets of relations will arrive at the following true statement:

$$\left(q_i \in \tilde{\mathscr{V}}_f\right) \vee \left(p_i \in \bar{\tilde{\mathscr{V}}}_f\right),\ \forall i \in \{1,2,...,\alpha\} \tag{2.9}$$

The relation (2.9) yields that the graph $\mathscr{G}(\{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\})$ includes a complete bipartite subgraph with the properties pointed out in Theorem 1. This implies that the mode σ is a DOFM w.r.t. $\mathbf{K}_d \cup \{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\}$, which contradicts the original assumption of minimality. ■

Theorem 2 states that if by adding more than $\tilde{w}$ communication links to the decentralized control structure the mode σ is no longer fixed, then some of the links are redundant and have no contribution in displacing the mode. It is worth mentioning that the result of Theorem 2 significantly diminishes the computational burden of finding all the minimal sets. One can use the following theorem to develop an algorithm for finding the minimal sets systematically.

Theorem 3 *The set $\{k_{p_1q_1}, k_{p_2q_2}, \cdots, k_{p_\alpha q_\alpha}\}$ is minimal w.r.t. σ if and only if the criteria given below both hold:*

- *For any $j \in \{1,2,...,\alpha\}$, there exists an integer $f_j \in \{1,...,\tilde{w}\}$ such that the statements:*

$$\begin{aligned} &(q_i \in \tilde{\mathscr{V}}_{(f_j)}) \vee (p_i \in \bar{\tilde{\mathscr{V}}}_{(f_j)}), \forall i \in \{1,...,j-1,j+1,...,\alpha\} \\ &(q_j \notin \tilde{\mathscr{V}}_{(f_j)}) \wedge (p_j \notin \bar{\tilde{\mathscr{V}}}_{(f_j)}) \end{aligned} \tag{2.10}$$

  are true, where $f_1, f_2, ..., f_{\tilde{w}}$ are all distinct (note that $\wedge$ is the logic operation AND).

- *There exists no integer $f \in \{1,...,\tilde{w}\}$ such that the following logic statement is true:*

$$(q_i \in \tilde{\mathscr{V}}_f) \vee (p_i \in \bar{\tilde{\mathscr{V}}}_f), \ \forall i \in \{1,2,...,\alpha\} \tag{2.11}$$

Proof: The proof of this theorem follows directly from the discussions given in the proof of Theorem 2. The details are omitted here. ■

Theorem 3 implicitly proposes a simple method to compute all the minimal sets w.r.t. to the fixed mode σ.

Remark 2 *Although graph-based problems are computationally intractable in general and hence very difficult to solve, the graph component of the technique proposed here has a very particular form and the existing SOS methods can be employed to efficiently handle it. This can be carried out, for instance, in line with the ideas used in [23] for solving the MAX-CUT problem in graph theory.*

2.4.2 Displacing multiple unrepeated DFMs

The methodology presented in the preceding subsection will be deployed here to characterize all the control interaction sets $\mathbf{K}_e$ such that the DFMs $\sigma_1, \sigma_2, ..., \sigma_\mu$ are all movable w.r.t to $\mathbf{K}_d \cup \mathbf{K}_e$. Although the mode σ_i, $i \in \{1,2,...,\mu\}$, is by assumption an unrepeated DFM of the system $\mathscr{S}$, its multiplicity as a regular mode of the system can be greater than 1. In this case, the aforementioned method cannot be applied to the system directly. As a remedy for this problem, one can consider a generic static decentralized controller and apply it to the system $\mathscr{S}$ so that the multiplicity of the mode σ_i, $i \in \{1,2,...,\mu\}$, will be exactly equal to 1 in the resultant system [7; 1]. Therefore, with no loss of generality, assume henceforth that the mode σ_i, $i \in \{1,2,...,\mu\}$, is not only an unrepeated DFM but also an unrepeated mode of the system $\mathscr{S}$.

Remark 3 *Since the costs attributed to communication links are fixed and time-invariant, they normally depend on some exterior factors such as the distance between subsystems or the capacity of a channel. This implies that although these costs are defined for the original system, the same values can also be considered for the system obtained by applying a generic decentralized controller (used to reduce the multiplicity of the repeated modes of the original system, if any, to 1), and the two systems have the same cost-optimal configuration.*

For any $i \in \{1,2,...,\mu\}$, obtain all minimal sets associated with the mode σ_i using the approach given in the previous subsection, and denote them with $\mathbf{K}_e^{i,1}, \mathbf{K}_e^{i,2}, ..., \mathbf{K}_e^{i,z_i}$. The following corollary states how the underlying problem can be treated.

Corollary 1 *Given the control interaction set $\mathbf{K}_e$, none of the modes $\sigma_1, \sigma_2, ..., \sigma_\mu$ are DOFMs of the system $\mathscr{S}$ w.r.t. $\mathbf{K}_d \cup \mathbf{K}_e$ if and only if there exist integers $g_1, g_2, ..., g_\mu$ with the following property:*

$$\{\mathbf{K}_e^{1,g_1} \cup \mathbf{K}_e^{2,g_2} \cup \cdots \cup \mathbf{K}_e^{\mu,g_\mu}\} \subseteq \mathbf{K}_e \tag{2.12}$$

Proof: The proof follows immediately from the definition of a minimal set. ■

In practice, it is desired that the addition of the set of interconnection links $\mathbf{K}_e$ to the control structure be as inexpensive as possible. In order to take the expenditure of this inclusion into account, it is assumed that the cost of implementing the communication link k_{ij} is prespecified by the designer, and is denoted by $\mathscr{C}_{ij}$, for any $i, j \in \bar{\nu}$. Note that this cost should normally be defined in terms of the exterior factors associated with the system such as the distance between the subsystems or the nature of the outputs to be transmitted to the other subsystems. This restriction results from the fact that different state-space representations are used throughout the work for different modes, which may lead to the inconsistency in the cost evaluation. By virtue of Corollary 1, finding the least costly $\mathbf{K}_e$ with the aforementioned property (displacing certain fixed modes) can be translated into obtaining all the sets $\mathbf{K}_e$ representable as $\mathbf{K}_e^{1,g_1} \cup \mathbf{K}_e^{2,g_2} \cup \cdots \cup \mathbf{K}_e^{\mu,g_\mu}$ for some integers $g_1, g_2, \ldots, g_\mu$, and computing their associated costs accordingly to determine which one is the least expensive.

Remark 4 *It may turn out that certain communication channels between some of the subsystems cannot be established by any means. In order to take this constraint into account, two strategies can be pursued. First, one can allot very large cost values to impermissible links so that they do not appear in the optimal configuration. Alternatively, one can first obtain all the minimal sets, and then rule out the ones containing inadmissible channels. In other words, the optimization must be carried out over the minimal sets with permissible elements.*

2.4.3 Displacing Quotient Fixed Modes (QFM)

The following procedure provides the steps to construct the graphs required to verify which modes of the system are QFMs.

Procedure 3 *[18]*

Construct a bipartite graph $\mathscr{G}$ with two sets of vertices $\mathscr{V}$ (set 1) and $\bar{\mathscr{V}}$ (set 2) and the tagged vertices $1, 2, \ldots, \nu$ in each of the two sets. For any $i, j \in \bar{\nu}$, carry out the following steps:

1) *For any $\mu_1, \mu_2 \in \bar{\nu}$, connect vertex μ_1 of set $\mathscr{V}$ to vertex μ_2 of set $\bar{\mathscr{V}}$ if either the j-th column of C_{μ_1} or the j-th row of B_{μ_2} is a zero vector for all $j \in \{1, 2, \ldots, n\}$.*

2) Mark vertex μ_1 of set $\mathcal{V}$ if the i-th column of the matrix C_{μ_1} is a zero vector, for any $\mu_1 \in \bar{v}$.

3) Mark vertex μ_2 of set $\bar{\mathcal{V}}$ if the i-th row of the matrix B_{μ_2} is a zero vector, for any $\mu_2 \in \bar{v}$.

Although two different procedures are introduced to construct the associated graphs of DFMs and QFMs, it is shown in [18] that the two graphs constructed by following Procedures 1 and 3 should satisfy exactly the same properties in order for a mode of the system to be a DFM or QFM. Therefore, all the results obtained in the previous subsections for displacing DFMs can be extended to QFMs using the same framework (and the same terminology). However, one should note that there are some fundamental differences between the control components in the two cases. More specifically, the elements of the control interaction sets are non-LTI for QFMs, in general, while the corresponding elements are LTI in the case of DFMs.

Note that the necessary and sufficient condition for the displaceability of QFMs (using the proposed method) is that they are distinct DFMs (this comes from the fact that each QFM is also a DFM). Then, the procedure given in the previous subsection to displace multiple unrepeated DFMs can be employed to treat the problem.

Remark 5 *It is worth mentioning that if a DFM is not a QFM, it can be displaced by using a proper non-LTI controller (such as sampled data controller under some mild conditions). Thus, if the cost of a communication link outweighs that of a more sophisticated control law (nonlinear or time-varying), one can employ a controller with a more complex function and a less complex communication structure.*

2.5 Numerical examples

Example 1: (Displacing DFMs)

Let $\mathscr{S}$ be a system consisting of four single-input single-output (SISO) subsystems with the following decoupled state-space matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \; B_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6 \end{bmatrix},$$
$$B_3 = \begin{bmatrix} 0 \\ 7 \\ 9 \\ -5 \end{bmatrix}, \; B_4 = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 7 \end{bmatrix} \tag{2.13}$$

and

$$C_1 = \begin{bmatrix} 0 & 2 & 4 & 3 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & -6 & 0 & 8 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} 0 & 4 & 0 & -9 \end{bmatrix}, \; C_4 = \begin{bmatrix} 5 & 1 & 0 & 7 \end{bmatrix},$$
$$D_{11} = -5, \; D_{12} = 10, \; D_{13} = 27, \; D_{14} = 23,$$
$$D_{21} = 32, \; D_{22} = 60, \; D_{23} = -3, \; D_{24} = 56/3, \tag{2.14}$$
$$D_{31} = -25, \; D_{32} = -62, \; D_{33} = 43, \; D_{34} = -21,$$
$$D_{41} = -4.5, \; D_{42} = 40, \; D_{43} = 16, \; D_{44} = 7,$$

Consider now the mode $\sigma = 1$. The graph $\mathscr{G}$ associated with this mode (by following the steps given in Procedure 1) is depicted in Figure 2.5(a). This graph contains a complete bipartite subgraph with vertex 1 from set 1 of $\mathscr{G}$ and vertices 2, 3 and 4 from set 2 of $\mathscr{G}$ such that its vertices are all marked and also it spans the vertices of $\mathscr{G}$. Therefore, it results from Lemma 1 that the mode 1 is a DFM of the system. Likewise, it can be shown that the mode 3 is also a DFM of the system, while the modes 2 and 4 are not. Note that the control interaction set corresponding to a decentralized controller in this example is equal to $\mathbf{K}_d = \{k_{11}, k_{22}, k_{33}, k_{44}\}$.

It is desired now to expand the structure of the control system from decentralized to overlapping so that both of these unstable DFMs can be displaced. Let the mode $\sigma = 1$ be treated first. It is straightforward to obtain the relation $w = 2$ from the graph $\mathscr{G}$ in Figure 2.5(a). Therefore, the subgraphs $\mathscr{G}_1$ and $\mathscr{G}_2$ will have the following sets of vertices:

$$\mathscr{V}_1 = \{1\}, \quad \bar{\mathscr{V}}_1 = \{2,3,4\}, \quad \mathscr{V}_2 = \{1,2,3\}, \quad \bar{\mathscr{V}}_2 = \{4\} \tag{2.15}$$

On the other hand, the graph $\mathscr{G}$ has two maximal subgraphs which are the same as $\mathscr{G}_1$ and $\mathscr{G}_2$. It can now be concluded from Theorem 3 that the minimal sets w.r.t. $\sigma = 1$ are:

$$\mathbf{K}_e^{1,1} = \{k_{14}\}, \quad \mathbf{K}_e^{1,2} = \{k_{12}, k_{34}\}, \quad \mathbf{K}_e^{1,3} = \{k_{13}, k_{24}\} \tag{2.16}$$

Note that as expected from Theorem 2, these sets have at most 2 elements, due to the relation $\tilde{w} = 2$. Analogously, the minimal sets w.r.t. $\sigma = 3$ can be obtained as:

$$\mathbf{K}_e^{3,1} = \{k_{31}\}, \quad \mathbf{K}_e^{3,2} = \{k_{41}\} \tag{2.17}$$

(note that $\tilde{w}$ for the mode $\sigma = 3$ is equal to 1). It results from Corollary 1 that the modes 1 and 3 are not DOFMs of the system w.r.t. the control interaction $\mathbf{K}_d \cup \mathbf{K}_e$ if and only if the following condition is satisfied for the set $\mathbf{K}_e$:

$$\exists \zeta_1 \in \{1,2,3\}, \exists \zeta_2 \in \{1,2\}: \quad \{\mathbf{K}_e^{1,\zeta_1} \cup \mathbf{K}_e^{3,\zeta_2}\} \subseteq \mathbf{K}_e \tag{2.18}$$

Assume now that all of the communication links have the same cost, i.e., $\mathscr{C}_{ij} = 1, \ i,j \in \{1,2,3,4\}$. In this case, the least costly $\mathbf{K}_e$ will be $\{k_{14}, k_{31}\}$ or $\{k_{14}, k_{41}\}$ with the implementation cost of 2. The graph $\mathscr{G}(\{k_{14}, k_{31}\})$ corresponding to the modes 1 and 3 are depicted in Figures 2.5(b) and 2.5(c), respectively. It can be easily verified that none of these graphs has a subgraph with the properties mentioned in Theorem 1. This validates the obtained result stating that the modes 1 and 3 are not the DOFMs of the system $\mathscr{S}$ w.r.t. $\mathbf{K}_d \cup \{k_{14}, k_{31}\}$.

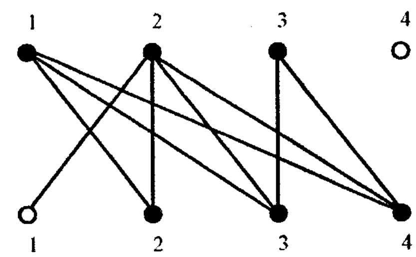

Figure 2.4: The graph $\mathscr{G}$ of a given system.

To show the effect of implementation expenditure on obtaining the set $\mathbf{K}_e$, assume that:

$$\begin{aligned} &\mathscr{C}_{14}=5,\ \mathscr{C}_{12}=2,\ \mathscr{C}_{34}=2,\ \mathscr{C}_{13}=1, \\ &\mathscr{C}_{24}=1,\ \mathscr{C}_{31}=5,\ \mathscr{C}_{41}=4 \end{aligned} \tag{2.19}$$

In this case, the set $\mathbf{K}_e$ will be equal to $\{k_{13}, k_{24}, k_{41}\}$ with the implementation cost of 6.

Example 2: (Displacing QFMs)

Let $\mathscr{S}$ be a strictly proper system consisting of three two-input two-output subsystems with the following state-space matrices:

$$
\begin{aligned}
&\mathbf{A} = \operatorname{diag}([1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ 5]),\\
&\mathbf{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix},\\
&\mathbf{C}_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}^T, \quad \mathbf{C}_2 = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}^T, \quad \mathbf{C}_3 = \begin{bmatrix} 5 & 6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}^T
\end{aligned}
\tag{2.20}
$$

The graphs corresponding to different modes of the system are depicted in Figures 2.6, 2.7 and 2.8. It can be concluded that $\sigma = 2$ is a QFM of the system, as the vertices 2 and 3 from set 1, and vertex 1 from set 2 along with their edges form a complete bipartite subgraph such that its vertices are all marked, and moreover it spans the vertices of $\mathscr{G}$ (see Definition 2). It can be easily verified that the mode $\sigma = 3$ is also a QFM, and that none of the remaining modes is a QFM of the system $\mathscr{S}$. It is to be noted that the control interaction set corresponding to a decentralized controller in this example is equal to $\mathbf{K}_d = \{k_{11}, k_{22}, k_{33}, k_{44}, k_{55}\}$.

Let the mode $\sigma = 2$ be treated first. It can be easily observed that the subgraph $\mathscr{G}_1$ will have the following sets of vertices:

$$
\mathscr{V}_1 = \{2,3\}, \quad \bar{\mathscr{V}}_1 = \{1\} \tag{2.21}
$$

Hence, $\mathscr{G}_1$ is the only maximal subgraph of the graph $\mathscr{G}$. It can be concluded from Theorem

3 that the minimal sets w.r.t. $\sigma = 2$ are:

$$\mathbf{K}_e^{2,1} = \{k_{31}\}, \quad \mathbf{K}_e^{2,2} = \{k_{21}\} \tag{2.22}$$

Analogously, the minimal set w.r.t. $\sigma = 3$ is obtained as:

$$\mathbf{K}_e^{3,1} = \{k_{31}\} \tag{2.23}$$

Since the mode $\sigma = 3$ has merely one minimal set $K_e^{3,1}$, therefore this set must be included in the control interaction set $\mathbf{K}_e$. On the other hand, this set is also minimal for the mode $\sigma = 2$, implying that both QFMs can be displaced simultaneously by employing the control interaction set $\mathbf{K}_d \cup \{k_{31}\}$. It is to be noted that the link $K_e^{2,2}$ is only effective in displacing the mode $\sigma = 2$, and hence implementing it along with $K_e^{3,1}$ to obtain $\mathbf{K}_d \cup \{k_{31}, k_{21}\}$ would incur a redundant cost (as a result of Corollary 1).

If it is practically not possible to establish the communication link k_{31}(as a result of Corollary 4), the minimal sets $K_e^{3,1} = K_e^{2,1} = \{k_{31}\}$ must be passed over, and hence $\sigma = 2$ would be the only displaceable QFM; i.e., in that case $\sigma = 3$ cannot be displaced.

2.6 Conclusions

This work tackles the stabilizability problem for an interconnected system with a number of distinct undesirable decentralized fixed modes (DFM), by means of the structurally constrained controllers. It is well-known that a linear time-invariable (LTI) decentralized controller comprising a set of isolated local controllers cannot displace any DFMs. Thus, the objective of this work is to establish some interactions between the local controllers in order to displace the undesirable DFMs. To this end, the knowledge of the system is transformed into a number of bipartite graphs (corresponding to the unwanted DFMs). Subsequently, the notions of minimal sets of interactions and maximal subgraphs are introduced. A simple procedure is then proposed to characterize all the possible sets of interactions which maintain the mentioned property. It is also shown that the results can be extended to consider the displacement of quotient fixed modes (QFM), which by definition are immovable with respect to any type of decentralized control law (nonlinear or time-varying). The numerical examples provided elucidate the efficacy of the present work.

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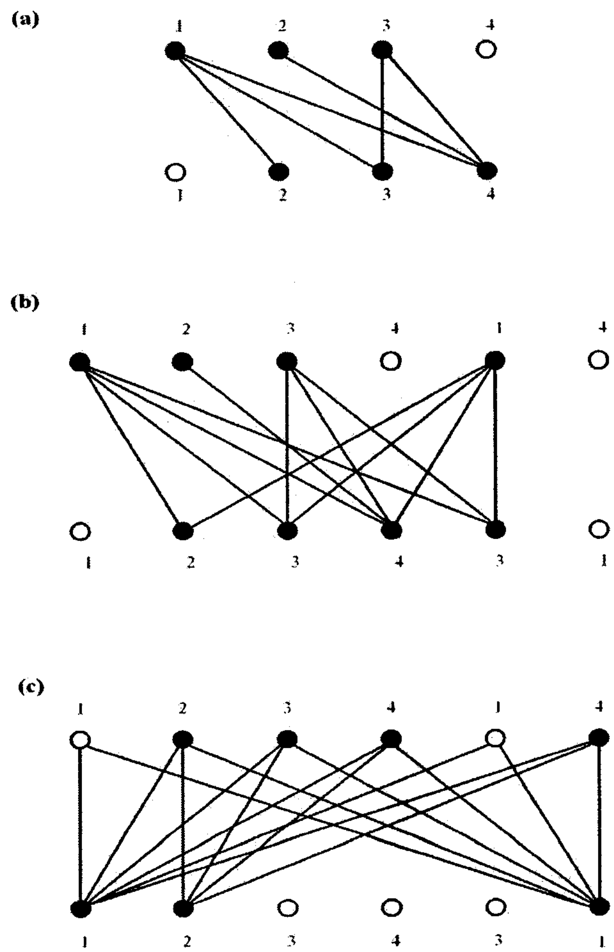

Figure 2.5: a) The graph $\mathscr{G}$ associated with the mode $\sigma = 1$; b) the graph $\mathscr{G}(\{k_{14}, k_{31}\})$ corresponding to the mode $\sigma = 1$; c) the graph $\mathscr{G}(\{k_{14}, k_{31}\})$ corresponding to the mode $\sigma = 3$.

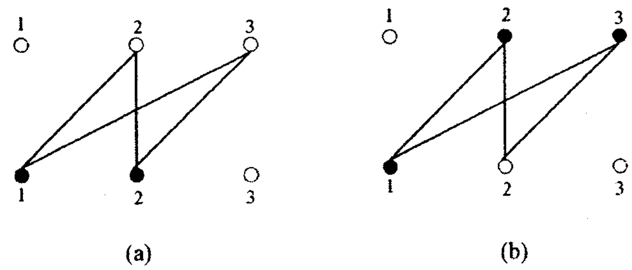

Figure 2.6: The graphs associated with the modes $\sigma = 1$ and $\sigma = 2$ are sketched in (a) and (b), respectively.

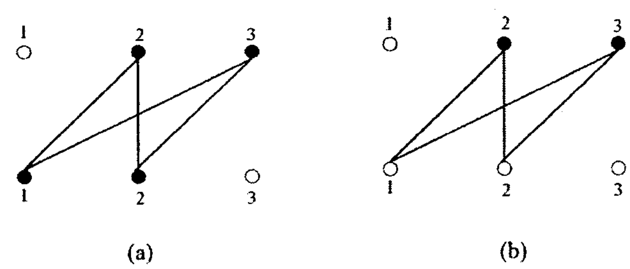

Figure 2.7: The graphs associated with the modes $\sigma = 3$ and $\sigma = 4$ are sketched in (a) and (b), respectively.

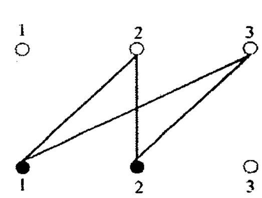

Figure 2.8: The graph associated with the mode $\sigma = 5$

Chapter 3

Characterizing all Classes of LTI Stabilizing Structurally Constrained Controllers by Means of Combinatorics

3.1 Abstract

The focus of this work is directed towards the problem of characterizing the information flow structures of all classes of LTI structurally constrained controllers with respect to which a given interconnected system has no fixed modes. Any class of structurally constrained controllers can be described by a set of communication links, which delineates how the local controllers of any controller in that class interact with each other. To achieve the objective, a cost is first attributed for establishing any communication link in the control structure. These costs are part of deign specifications and represent the expenditure of data transmission between different subsystems. A simple graph-theoretic method is then proposed to characterize all the relevant classes of controllers systematically. As a by-product of this approach, all classes of LTI stabilizing structurally constrained controllers

with the minimum implementation cost are attained using a novel algorithm. The primary advantages of this approach are its simplicity and computational efficiency. The efficacy and importance of this work are thoroughly illustrated in a numerical example.

3.2 Introduction

A great number of real-world plants can be regarded as interconnected systems with several interacting subsystems [1]. A typical controller for an interconnected system is composed of a set of local controllers corresponding to different subsystems. Normally, each local controller should receive some information from all the subsystems in order for the resultant controller to achieve best possible performance. This case is referred to as the centralized control strategy in the literature. Most of the control design techniques spontaneously arrive at centralized controllers, which may have practical problems as far as control implementation is concerned [2]. More precisely, there are some practical issues which may hinder employing a general centralized controller for an interconnected system. The primary reason is that the transmission of information from a subsystem to a local controller of another subsystem may be infeasible or quite costly. Such problems appear, for instance, in formation flying where the shadow phenomenon occurs for a specific time interval [3]. The case when some transmission links are remarkably costly comes about in the systems whose subsystems are geographically remote, e.g., in a power system with several stations in different cities. Furthermore, for a system consisting of several subsystems, the computational complexity associated with the centralized control structure can be quite high. These practical restrictions introduce the motivation for utilizing structurally constrained controllers [4].

Decentralized control is a particular type of structurally constrained controllers which has attracted a considerable amount of interest in the control community. A decentralized controller is the union of a number of local controllers which do not exchange information [5; 6]. Decentralized control theory has found applications in a wide range of real-world systems such as communication networks, power systems and traffic networks. Different decentralized design techniques have thoroughly been investigated and well-documented [7; 8; 9; 10].

In many control applications, there may exist overlapping between certain subsystems of an interconnected system. In such systems, it is often desired to design a structurally constrained controller whose local controllers partially interact with each other with the same overlapping topology as their corresponding subsystems. This conceptual notion is envisaged as decentralized overlapping control strategy in the literature. This class of structurally constrained controllers has been studied intensively, mostly in the framework of Expansion-Inclusion principle [11; 12; 13].

The most important problem in conjunction with the structurally constrained control design is the stabilizability verification. To address this problem, the notion of a decentralized fixed mode (DFM) was introduced in [14] to identify the modes of the system which are fixed with respect to any LTI decentralized controller. Several methods are proposed accordingly to characterize the DFMs of a system efficiently. For instance, the paper [7] proposes a simple graph-theoretic approach to obtain the unrepeated DFMs of a system without having to experience numerical difficulties. Since a DFM may be eliminated by means of a nonlinear or time-varying decentralized controller, the notion of quotient fixed modes (QFM) was introduced in [15] to characterize all the modes of a system which are immovable with respect to any nonlinear and time-varying decentralized controller.

More recently, the notion of decentralized overlapping fixed modes (DOFM) was introduced in [16] to mathematically describe the fixed modes of a system with respect to any given class of structurally constrained LTI controllers. A method is also proposed in [16] to obtain the DOFMs of the system efficiently. One should take note of the fact that DOFMs do not necessarily correspond to the overlapping systems discussed earlier, and is defined for any arbitrary system. In addition, the notion of a quotient decentralized overlapping fixed mode (QOFM) was introduced in [17] as an extension of the notion of a QFM to the overlapping control structure.

Consider an interconnected system with ν subsystems. It can be easily verified that there exist $2^{(\nu^2)} - 1$ classes of structurally constrained controllers for this system. Since this number grows exponentially with ν, choosing the classes which fit into the control objectives may be a formidable task. It is worth mentioning that several approaches have been proposed in the literature for the design of structurally constrained controllers to achieve any objective such as pole-placement or LQ optimality. However, all of these methods require that the structure of the desired controller be known *a priori*.

Given an interconnected system, the focal problem of this work is to find all classes of LTI structurally constrained controllers with respect to which the system has no fixed modes. To handle this problem, the main concept of the graph-theoretic approach of the recent paper [18] has been exploited. Note that the work [18] addresses the problem of eliminating undesirable DFMs of a system, by adding some transmission links between the local controllers. Modified definitions of the notions of *maximal graph* and *minimal set* introduced in [18] are utilized here to address the underlying problem. Moreover, a cost is attributed to establish a communication link between any pair of local controllers. The classes of LTI stabilizing structurally constrained controllers with the minimum implementation cost are then obtained. This is achieved by proposing an efficient method which avoids unnecessary computations. The technique introduced in this work is also applicable to the approach given in [18] for finding the optimal control interaction sets which eliminate the unwanted DFMs.

3.3 Problem formulation

Consider a LTI interconnected system $\mathscr{S}$ consisting of ν subsystems $S_1, S_2, ..., S_\nu$. Let the i^{th} subsystem of $\mathscr{S}$, $i \in \bar{\nu} := \{1, 2, ..., \nu\}$, be modeled as:

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + f_i(x(t), u(t)) \\ y_i(t) &= C_{ii}x_i(t) + D_{ii}u_i(t) + g_i(x(t), u(t)) \end{aligned} \tag{3.1}$$

where $x_i(t) \in \Re^{n_i}$, $u_i(t) \in \Re^{m_i}$ and $y_i(t) \in \Re^{r_i}$ are the state, the input and the output of the subsystem S_i, respectively. Moreover, $f_i(x(t), u(t))$ and $g_i(x(t), u(t))$ in the above state-space representation are the interconnection signals which account for the effect of different subsystems on S_i through its incoming interconnections. Assume that these interconnection signals can be represented by:

$$\begin{aligned} f_i(x(t), u(t)) &= \sum_{j=1,\ j\neq i}^{\nu} A_{ij}x_j(t) + \sum_{j=1,\ j\neq i}^{\nu} B_{ij}u_j(t), \\ g_i(x(t), u(t)) &= \sum_{j=1,\ j\neq i}^{\nu} C_{ij}x_j(t) + \sum_{j=1,\ j\neq i}^{\nu} D_{ij}u_j(t) \end{aligned} \tag{3.2}$$

for any $i \in \bar{\nu}$. Now, the model of the system $\mathscr{S}$ will be as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{\nu} B_j u_j(t) \\ y_i(t) &= C_i x(t) + \sum_{j=1}^{\nu} D_{ij}u_j(t), \quad i \in \bar{\nu} \end{aligned} \tag{3.3}$$

where:

$$\begin{aligned} B_j &= \begin{bmatrix} B_{1j}^T & B_{2j}^T & \cdots & B_{\nu j}^T \end{bmatrix}^T, \\ C_j &= \begin{bmatrix} C_{j1} & C_{j2} & \cdots & C_{j\nu} \end{bmatrix}, \quad j \in \bar{\nu} \end{aligned} \tag{3.4}$$

Denote the modes of the system $\mathscr{S}$ with $\sigma_1, \sigma_2, ..., \sigma_n$, and assume that all the modes are distinct. As a consequence of this assumption, one can suppose with no loss of generality that the matrix A is equal to:

$$A = \text{diag}([\sigma_1\ ,\ \sigma_2\ ,\ ...\ ,\ \sigma_n]) \tag{3.5}$$

(this can be achieved by using a proper similarity transformation, if necessary). Any structurally constrained controller for the system $\mathscr{S}$ comprises ν control agents corresponding to various subsystems as well as a number of transmission links. Each transmission link provides the output of a certain subsystem to a specific control agent which will be used to construct its control signal. The symbol k_{ij} $(i, j \in \bar{\nu})$ is used throughout the work to describe the link which transmits the output of the j^{th} subsystem to the i^{th} control agent (or equivalently, the i^{th} subsystem).

Any class of structurally constrained controllers is formulated in [18] via a set, referred to as *control interaction set*. This concept will be clarified in the following definition.

Definition 1 *Given a class of structurally constrained controllers, define its associated control interaction set* $\mathbf{K}$ *as a set which includes only the symbols* k_{ij}, $i, j \in \bar{\nu}$ *whose corresponding transmission links exist in the control structure.*

Consider any arbitrary control interaction set $\mathbf{K}$. It is well-known that certain modes of the system $\mathscr{S}$ may be fixed under all LTI controllers belonging to the class of structurally constrained controllers defined by $\mathbf{K}$. These fixed modes (if any) are referred to as decentralized overlapping fixed modes (DOFM) w.r.t. (with respect to) $\mathbf{K}$.

Assume henceforth that the system $\mathscr{S}$ is controllable and observable (the results obtained can be easily extended to the case when the system is detectable and stabilizable). Let $\mathscr{C}_{ij}$ denote the pre-specified cost of establishing the transmission link k_{ij}, for any $i, j \in \bar{\nu}$. The implementation cost associated with any set $\mathbf{K}$ is clearly equal to the sum of the costs of its components. Note that in the case when a communication link cannot exist (due to the reasons pointed out earlier such as the shadow phenomenon), its associated cost is infinity. The problems explained below will be addressed in the next section:

- Since any class of structurally constrained controllers is useful only when the system is stabilizable with respect to that class, it is first desired to characterize all classes of LTI structurally constrained controllers with respect to which the system $\mathscr{S}$ has no DOFMs.

- The second objective is to seek the class(es) of LTI structurally constrained controllers (among the ones characterized by addressing the above problem) whose corresponding implementation cost is minimum.

3.4 Main results

The following procedures are essential in developing the main results of this work.

Procedure 1 ([7]) *Corresponding to the mode σ_i, $i \in \bar{n} := \{1,2,...,n\}$, construct a bipartite graph $\mathscr{G}_i$ with two sets $\mathscr{V}$ (set 1) and $\bar{\mathscr{V}}$ (set 2). Put ν vertices in each of these sets and label them as $1,2,\ldots,\nu$. For any $\lambda_1, \lambda_2 \in \bar{\nu}$, carry out the steps given below:*

- *Connect the vertex λ_1 of the set $\mathscr{V}$ to the vertex λ_2 of the set $\bar{\mathscr{V}}$ if and only if the following equation holds:*

$$\begin{aligned} &C_{\lambda_1} \times diag\left(\left[\frac{1}{\sigma_1 - \sigma_i}, \ldots, \frac{1}{\sigma_{i-1} - \sigma_i}, 0,\right.\right. \\ &\left.\left.\frac{1}{\sigma_{i+1} - \sigma_i}, \ldots, \frac{1}{\sigma_n - \sigma_i}\right]\right) \times B_{\lambda_2} - D_{\lambda_1\lambda_2} = 0 \end{aligned} \tag{3.6}$$

- *Mark the vertex λ_1 of the set $\mathscr{V}$ if the i^{th} column of the matrix C_{λ_1} is a zero vector. Likewise, mark the vertex λ_2 of the set $\bar{\mathscr{V}}$ if the i^{th} row of the matrix B_{λ_2} is a zero vector.*

Procedure 2 *For a given control interaction set $\mathbf{K} = \{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\}$ and any $i \in \bar{n}$, construct a bipartite graph $\mathscr{G}_i(\mathbf{K})$ with two sets of α vertices. Label the vertices in set 1 with $q_1, q_2, ..., q_\alpha$ and the vertices in set 2 with $p_1, p_2, ..., p_\alpha$. Mark all vertices in set 1 of $\mathscr{G}_i(\mathbf{K})$ whose corresponding vertices in set 1 of $\mathscr{G}_i$ (the ones with the same labels) are marked. Mark the vertices of set 2 in a similar fashion. Moreover, connect two vertices* *of set 1 and set 2 of $\mathscr{G}_i(\mathbf{K})$ if the vertices with the same labels in $\mathscr{G}_i$ are connected to each other.*

Since $q_1, q_2, \ldots, q_\alpha$ are not necessarily distinct numbers, the labels in set 1 of the graph $\mathscr{G}_i(\mathbf{K})$ formed in Procedure 2 can be recurrent. The same argument holds true for its second set of vertices. It can be observed that $\mathscr{G}_i$ acts as a look-up table for constructing the graph $\mathscr{G}_i(\mathbf{K})$. As an example to clarify this point, assume the graph $\mathscr{G}_1$ to be the one depicted in Figure 3.1. By considering $\mathbf{K}$ as $\{k_{11}, k_{22}, k_{13}\}$, the graph $\mathscr{G}_1(\mathbf{K})$ can be obtained easily in terms of $\mathscr{G}_1$ as shown in Figure 3.2.

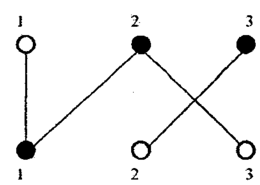

Figure 3.1: The graph $\mathscr{G}_1$ for a certain system.

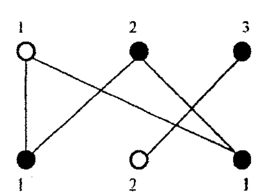

Figure 3.2: The graph $\mathscr{G}_1(\mathbf{K})$ corresponding to the set $\mathbf{K} = \{k_{11}, k_{22}, k_{13}\}$ and the system with the graph $\mathscr{G}_1$ sketched in Figure 3.1.

Lemma 1 *The mode σ_i, $i \in \bar{n}$, is a DOFM of the system $\mathscr{S}$ with respect to the set $\mathbf{K} = \{k_{p_1q_1}, k_{p_2q_2}, \ldots, k_{p_\alpha q_\alpha}\}$ if and only if its corresponding graph $\mathscr{G}_i(\mathbf{K})$ satisfies any of the following properties:*

i) All vertices in set 1 of $\mathscr{G}_i(\mathbf{K})$ are marked.

ii) All vertices in set 2 of $\mathscr{G}_i(\mathbf{K})$ are marked.

iii) $\mathscr{G}_i(\mathbf{K})$ includes a complete bipartite subgraph for which both of the conditions given below hold:

- *All its vertices (in both sets) are marked.*
- *For any $j \in \{1,2,...,\alpha\}$, the j^{th} vertex of either set 1 or set 2 of $\mathscr{G}_i(\mathbf{K})$ is included in the subgraph.*

Proof: The proof can be derived straightforwardly from the results of [7] and [16]. The details are omitted here (see Theorem 1 in [18] for a similar result). ■

Definition 2 *The control interaction set $\mathbf{K} = \{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\}$ is said to be minimal w.r.t. σ_i, $i \in \bar{n}$, if and only if σ_i is not a DOFM of the system $\mathscr{S}$ w.r.t. $\mathbf{K}$, while it is a DOFM w.r.t. $\mathbf{K} - \{k_{p_jq_j}\}$ for any $j \in \{1,2,...,\alpha\}$.*

The important property of a minimal set is that it has no redundant transmission link, in the sense that all transmission links in the set contribute to the stabilizability of the system. As an example, consider again the system whose corresponding graph $\mathscr{G}_1$ is depicted in Figure 3.1. It is easy to verify that $\{k_{11}, k_{22}, k_{13}\}$ is not a minimal set for this system, while $\{k_{11}, k_{23}\}$ is a minimal one.

Definition 3 *A subgraph of the graph $\mathscr{G}_i$, $i \in \bar{n}$, is said to be maximal if:*

i) It is a complete bipartite graph.

ii) All of its vertices in both sets are marked.

iii) The properties (i) and (ii) given above will not hold if any new vertex is added to the subgraph from the graph $\mathscr{G}_i$.

iv) It includes at least one edge.

Using a proper combinatorial algorithm, these subgraphs of the graph $\mathscr{G}_i$ can be easily identified (analogously to the algorithms developed for finding the complete bipartite subgraphs with maximum number of edges). Denote such subgraphs with $\tilde{\mathscr{G}}_1^i, \tilde{\mathscr{G}}_2^i, ..., \tilde{\mathscr{G}}_{w_i}^i$, for any $i \in \bar{n}$. Moreover, denote set 1 and set 2 of the vertices of the graph $\tilde{\mathscr{G}}_j^i$ with $\tilde{\mathscr{V}}_j^i$ and $\bar{\tilde{\mathscr{V}}}_j^i$, respectively, for any $i \in \bar{n}$ and $j \in \{1, 2, ..., w_i\}$.

To clarify Definition 3, consider $\mathscr{G}_1$ as the graph sketched in Figure 3.3. This graph has two maximal subgraphs with the following sets of vertices:

- $\tilde{\mathscr{V}}_1^1 = \{1\}$ and $\bar{\tilde{\mathscr{V}}}_1^1 = \{2, 3\}$.
- $\tilde{\mathscr{V}}_2^1 = \{1, 2\}$ and $\bar{\tilde{\mathscr{V}}}_2^1 = \{2\}$.

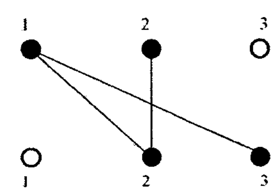

Figure 3.3: A graph $\mathscr{G}_1$ with two maximal subgraphs.

Theorem 1 *Assume that the control interaction set* $\mathbf{K} = \{k_{p_1q_1}, k_{p_2q_2}, ..., k_{p_\alpha q_\alpha}\}$ *is minimal w.r.t.* σ_i, $i \in \bar{n}$. *Then,* α *is less than or equal to* $w_i + 2$.

Proof: From the definition of a minimal set, the mode σ_i is a DOFM of the system $\mathscr{S}$ w.r.t. $\mathbf{K}^j := \mathbf{K} - \{k_{p_jq_j}\}$ for any $j \in \{1, 2, ..., \alpha\}$. Hence, at least one of the properties (i), (ii) or (iii) mentioned in Lemma 1 is satisfied for the graph $\mathscr{G}_i(\mathbf{K}^j)$ (introduced in Procedure 2) for any $j \in \{1, 2, ..., \alpha\}$. Proof of the theorem will be provided now by contradiction. Therefore, assume that $\alpha > w_i + 2$.

Suppose that there are two distinct numbers $j_1, j_2 \in \{1, 2, ..., \alpha\}$ so that property (i) of Lemma 1 is met for both of the graphs $\mathscr{G}_i(\mathbf{K}^{j_1})$ and $\mathscr{G}_i(\mathbf{K}^{j_2})$. Hence, all the vertices in set 1 of the graph $\mathscr{G}_i(\mathbf{K})$ are marked. This implies that the system $\mathscr{S}$ has a DOFM w.r.t the set $\mathbf{K}$ (by virtue of Lemma 1). This contradiction means that condition (i) of Lemma 1 can be true for at most one of the graphs $\mathscr{G}_i(\mathbf{K}^1), \mathscr{G}_i(\mathbf{K}^2), ..., \mathscr{G}_i(\mathbf{K}^\alpha)$. The same argument can be made for condition (ii) of Lemma 1. Therefore, condition (iii) of Lemma 1 is true for more than w_i graphs of $\mathscr{G}_i(\mathbf{K}^1), \mathscr{G}_i(\mathbf{K}^2), ..., \mathscr{G}_i(\mathbf{K}^\alpha)$ (note that $\alpha - 2 > w_i$). With no loss of generality, assume that each of the graphs $\mathscr{G}_i(\mathbf{K}^1), \mathscr{G}_i(\mathbf{K}^2), ..., \mathscr{G}_i(\mathbf{K}^\lambda)$ has a complete bipartite subgraph satisfying property (iii) of Lemma 1, where $\lambda = w_i + 1$. Every of these complete bipartite subgraphs is also a subgraph of one of the graphs $\tilde{\mathscr{G}}_1^i, \tilde{\mathscr{G}}_2^i, ..., \tilde{\mathscr{G}}_{w_i}^i$. Since there are λ complete bipartite subgraphs while the number of these graphs is less than λ (note that $w_i < \lambda$), it can be concluded from Dirichlet's Principle that the subgraphs of two of the graphs $\mathscr{G}_i(\mathbf{K}^1), \mathscr{G}_i(\mathbf{K}^2), ..., \mathscr{G}_i(\mathbf{K}^\lambda)$ correspond to the same maximal graph. Without any loss of generality, assume that $\mathscr{G}_i(\mathbf{K}^1)$ and $\mathscr{G}_i(\mathbf{K}^2)$ both have complete bipartite subgraphs with the properties mentioned in condition (iii) of Lemma 1, which are included in $\tilde{\mathscr{G}}_1^i$. Thus:

$$q_\beta \subseteqq \tilde{\mathscr{V}}_1^i \ \text{ or } \ p_\beta \subseteqq \bar{\tilde{\mathscr{V}}}_1^i, \ \forall \beta \in \{2,3,4,...,\alpha\} \tag{3.7}$$

and:

$$q_\beta \subseteqq \tilde{\mathscr{V}}_1^i \ \text{ or } \ p_\beta \subseteqq \bar{\tilde{\mathscr{V}}}_1^i, \ \forall \beta \in \{1,3,4,...,\alpha\} \tag{3.8}$$

It follows from (3.7) and (3.8) that:

$$q_\beta \subseteqq \tilde{\mathscr{V}}_1^i \ \text{ or } \ p_\beta \subseteqq \bar{\tilde{\mathscr{V}}}_1^i, \ \forall \beta \in \{1,2,3,4,...,\alpha\} \tag{3.9}$$

Using the above relation, one can conclude from Lemma 1 that the mode σ_i is a DOFM of the system $\mathscr{S}$ w.r.t. $\mathbf{K}$, because the graph $\mathscr{G}_i(\mathbf{K})$ encompasses a complete bipartite subgraph satisfying the third property of the lemma. This contradicts the initial assumption, and hence completes the proof. ■

Theorem 1 introduces an important property of maximal sets. Moreover, its proof implicitly proposes a simple method to compute all the minimal sets corresponding to the mode σ_i, denoted by $\mathbf{K}_m^{i,1}, \mathbf{K}_m^{i,2}, \ldots, \mathbf{K}_m^{i,z_i}$, for any $i \in \bar{n}$. Assume for now that all of these sets are attained. The question arises: how can one obtain a set $\mathbf{K}$ such that the system $\mathscr{S}$ has no DOFMs with respect to the control interaction set $\mathbf{K}$? This question will be answered in the next corollary.

Corollary 1 *For any control interaction set* $\mathbf{K}$*, the system* $\mathscr{S}$ *has no DOFMs w.r.t.* $\mathbf{K}$ *if and only if there exist distinct integers* $j_1, j_2, \ldots, j_n$ *with the following property:*

$$\left(\mathbf{K}_m^{1,j_1} \cup \mathbf{K}_m^{2,j_2} \cup \cdots \cup \mathbf{K}_m^{n,j_n}\right) \subseteq \mathbf{K} \tag{3.10}$$

where $1 \leq j_i \leq z_i$*, for any* $i \in \bar{n}$*.*

Proof: The proof is straightforward, and is omitted here. ■

Remark 1 *Since the minimal sets* $\mathbf{K}_m^{i,1}, \mathbf{K}_m^{i,2}, \ldots, \mathbf{K}_m^{i,z_i}$*,* $i \in \bar{n}$*, have already been identified, Corollary 1 can be utilized to attain all the control structures with respect to which the system has no fixed modes. For any control interaction set* $\mathbf{K}$ *obtained, one can take advantage of the method given in [16] to design a LTI structurally constrained controller complying with* $\mathbf{K}$ *so that all the modes of the system are placed at any desirable locations.*

3.4.1 An efficient algorithm to obtain the optimal control interaction set(s)

It is evident that the implementation cost corresponding to a control interaction set $\mathbf{K} = \{k_{p_1q_1}, k_{p_2q_2}, \ldots, k_{p_\alpha q_\alpha}\}$ is equal to $\sum_{i=1}^{\alpha} \mathscr{C}_{p_iq_i}$. The objective here is to obtain all the optimal control interaction set(s), i.e. the ones with the properties that not only does the system $\mathscr{S}$ have no DOFMs with respect to them, but their corresponding cost is also minimum.

Consider a control interaction set $\mathbf{K}$. It can be inferred from Corollary 1 that if this set is optimal, then there exist distinct integers $j_1, j_2, \ldots, j_n$ such that

$$\mathbf{K} = \mathbf{K}_m^{1,j_1} \cup \mathbf{K}_m^{2,j_2} \cup \cdots \cup \mathbf{K}_m^{n,j_n} \tag{3.11}$$

Now, draw a table $\mathscr{T}$ with n columns so that the minimal sets $\mathbf{K}_m^{i,1}, \mathbf{K}_m^{i,2}, ..., \mathbf{K}_m^{i,z_i}$ are placed in the i^{th} column of $\mathscr{T}$ in an arbitrary order, for any $i \in \{1,2,...,n\}$. As a result of the equation (3.11), the most straightforward way to obtain the optimal control interaction set(s) is to consider all the possible ways that n sets can be chosen from different columns of $\mathscr{T}$, and for each of the selections, to compute the cost associated with the union of the chosen sets to observe which one has the least cost. This would result in $z_1 \times z_2 \times \cdots \times z_n$ combinatorial operations. A method will be proposed next to significantly reduce this number, in general.

To cast light on the idea here, let start with a particular case. Assume that $z_1 = z_2 = \cdots = z_n = z$, and that $\mathbf{K}_m^{2,1} \subseteq \mathbf{K}_m^{1,1}$. The latter relation means that not only the mode σ_1, but also the mode σ_2 is movable with respect to the interaction set $\mathbf{K}_m^{1,1}$. An indication of the fact is that if the set $\mathbf{K}_m^{1,1}$ is chosen from the first column of the table $\mathscr{T}$, there is no need to opt a set from the second column, as it just incurs cost. Hence, in this case, the number of ways in which $\mathbf{K}_m^{1,1}$ is selected from column 1 is equal to z^{n-2}, rather than z^{n-1}. Consequently, if it is known that the relation $\mathbf{K}_m^{2,1} \subseteq \mathbf{K}_m^{1,1}$ holds, computing $z^{n-2}(z-1)$ combinations is obviated. Typically, there are several such relations (not just one), which makes the method computationally more efficient. Consequently, it is very important to find out the relations such as $\mathbf{K}_m^{2,1} \subseteq \mathbf{K}_m^{1,1}$. This can be done by carrying out $\binom{nz}{2}$ operations, where every operation considers two sets from different columns of the table $\mathscr{T}$ and verifies whether one of them is a subset of the other one. It is worth mentioning that since $\binom{nz}{2}$ is much smaller than $z^{n-2}(z-1)$ in general, this "so-called" pre-processing which aims at finding the desired relations is quite advantageous.

Attribute a set $\mathbf{T}^{i,j}$ to each minimal set $\mathbf{K}_m^{i,j}$, for any $i \in \bar{n},\ j \in \{1,2,...,z_i\}$, such that the set $\mathbf{T}^{i,j}$ contains a number r if and only if at least one of the sets in the r^{th} column of the table is a subset of $\mathbf{K}_m^{i,j}$. As pointed out earlier in a particular case, the sets $\mathbf{T}^{i,j}$'s can be constructed by carrying out $\binom{z_1+z_2+\cdots+z_n}{2}$ operations. The aforementioned discussion leads to the following corollary.

Corollary 2 *Consider a control interaction set* $\mathbf{K}$. *The system* $\mathscr{S}$ *has no DOFMs w.r.t.* $\mathbf{K}$ *if and only if there exist integers* $i_1, i_2, ..., i_j$ *and* $i'_1, i'_2, ..., i'_j$ $(j \leq n)$ *such that:*

$$\left(\mathbf{K}_m^{i_1,i'_1} \cup \mathbf{K}_m^{i_2,i'_2} \cup \cdots \cup \mathbf{K}_m^{i_j,i'_j}\right) \subseteq \mathbf{K} \tag{3.12}$$

where:

$$\mathbf{T}^{i_1,i'_1} \cup \mathbf{T}^{i_2,i'_2} \cup \cdots \cup \mathbf{T}^{i_j,i'_j} = \{1,2,...,n\} - \{i_1,i_2,...,i_j\} \tag{3.13}$$

Moreover, if the relation (3.12) turns to an equality, i.e.:

$$\mathbf{K}_m^{i_1,i'_1} \cup \mathbf{K}_m^{i_2,i'_2} \cup \cdots \cup \mathbf{K}_m^{i_j,i'_j} = \mathbf{K} \tag{3.14}$$

then $\mathbf{K}$ *is a cost optimal candidate.*

The main advantage of the relation (3.14) over (3.11) to obtain the optimal control interaction set $\mathbf{K}$ is that j can be noticeably less than n. Note that in the case when j is solely one unit less than n, as asserted earlier, there is still a significant saving in the computation time. At this point, a simple algorithm can be devised to obtain all the sets $\mathbf{K}$ representable as (3.14).

Remark 2 *In this work, a mode is considered as being either fixed or movable. In other words, the problem is formulated by giving a binary status to each mode, in terms of being or not being a DOFM. However, a mode which is not a DOFM, can be very close to being a DOFM. In this case, the input energy to displace this mode can be undesirably huge [19]. This can cause important practical problems such as input saturation. In order to take this grave issue into consideration, one can define an inherent cost, aside from the implementation cost. This cost should reflect how flexible the modes of the system are. For instance, the notion of approximate decentralized fixed modes (ADFM) introduced in [20] can be used for this purpose. In that case, in the last stage where a control interaction set* *is to chosen from the possible sets resulted from Corollary 2, both of the implementation cost and the inherent cost should be considered.*

3.5 Numerical example

Consider a system $\mathscr{S}$ consisting of three single-input single-output (SISO) subsystems with the following decoupled state-space matrices:

$$
\begin{aligned}
&A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \\
&B_3 = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 6 & 1 \end{bmatrix} \\
&C_2 = \begin{bmatrix} 0 & 0 & 4 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 10 & 3 & 0 \end{bmatrix} \\
&D_{11} = -1, \;\; D_{12} = 23, \;\; D_{13} = 3, \\
&D_{21} = -3, \;\; D_{22} = 20, \;\; D_{23} = 10, \\
&D_{31} = -15, \;\; D_{32} = 5, \;\; D_{33} = -8,
\end{aligned}
\tag{3.15}
$$

It is desired now to characterize all control interaction sets with respect to which the system $\mathscr{S}$ has no DOFMs. The graphs $\mathscr{G}_1$, $\mathscr{G}_2$ and $\mathscr{G}_3$ corresponding to $\sigma_1 = 1$, $\sigma_2 = 2$ and $\sigma_3 = 3$ are depicted in Figures 3.4, 3.5 and 3.6, respectively. The graph $\mathscr{G}_1$ has two maximal subgraphs with the following sets of vertices:

$$
\begin{aligned}
\bar{\mathscr{V}}_1^1 &= \{1\}, \quad \bar{\bar{\mathscr{V}}}_1^1 = \{2,3\} \\
\bar{\mathscr{V}}_2^1 &= \{1,2\}, \quad \bar{\bar{\mathscr{V}}}_2^1 = \{2\}
\end{aligned}
\tag{3.16}
$$

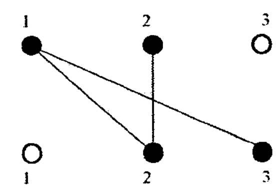

Figure 3.4: The graph $\mathscr{G}_1$ corresponding to the mode $\sigma_1 = 1$ of the system given in (3.15).

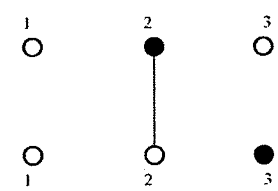

Figure 3.5: The graph $\mathscr{G}_2$ corresponding to the mode $\sigma_2 = 2$ of the system given in (3.15).

Analogously, it can be easily verified that the graphs $\mathscr{G}_2$ and $\mathscr{G}_3$ have no maximal subgraphs. Therefore:

$$w_1 = 2, \quad w_2 = 0, \quad w_3 = 0 \tag{3.17}$$

Using the method proposed in this work, all the minimal sets with respect to different modes of the system $\mathscr{S}$ can be obtained straightforwardly based on the maximal graphs found. These sets are tabulated in Table 3.1.

It results from Corollary 1 that the system $\mathscr{S}$ has no DOFMs with respect to a control interaction set $\mathbf{K}$ if and only if there exit integers $\gamma_1 \in \{1,2\}$ and $\gamma_2, \gamma_3 \in \{1,2,...,8\}$ such that:

$$(\mathbf{K}_m^{1,\gamma_1} \cup \mathbf{K}_m^{2,\gamma_2} \cup \mathbf{K}_m^{3,\gamma_3}) \subseteq \mathbf{K} \tag{3.18}$$

As stated in Subsection A of Section III, in order to characterize the desired sets $\mathbf{K}$, all possible ways of choosing three sets from different columns of Table 3.1 should be considered and for each of them, their corresponding union must be computed. This results in $2 \times 8 \times 8 = 128$ combinations. Nonetheless, Corollary 2 can be exploited to diminish

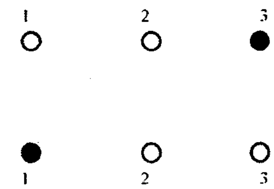

Figure 3.6: The graph $\mathscr{G}_3$ corresponding to the mode $\sigma_3 = 3$ of the system given in (3.15).

the number of possible combinations. To this end, one may take note of the fact that for any choice of $\mathbf{K}_m^{1,1}$ in column 1 of the table, there is no need to choose a set from column 2 (because of the relation $\mathbf{K}_m^{1,1} = \mathbf{K}_m^{2,3}$). Moreover, it can be deduced from the relation $\mathbf{K}_m^{1,2} = \mathbf{K}_m^{2,6} = \mathbf{K}_m^{3,8}$ that if $\mathbf{K}_m^{1,2}$ is chosen from column 1, no sets are needed to be chosen from the remaining columns. This exposition is indeed the interpretation of Corollary 2. Hence, it suffices to merely consider the union of the first set in column 1 with all sets in column 3 as well as the second set of column 1 itself. In other words, 9 combinations are required to be constructed using Corollary 2, while Corollary 1 requires 128. Obtaining

Table 3.1: Minimal sets corresponding to different modes

$\sigma_1 = 1$	$\sigma_2 = 2$	$\sigma_3 = 3$
$\mathbf{k}_m^{1,1} = \{k_{13}\}$	$\mathbf{k}_m^{2,1} = \{k_{11}\}$	$\mathbf{k}_m^{3,1} = \{k_{21}\}$
$\mathbf{k}_m^{1,2} = \{k_{12}, k_{33}\}$	$\mathbf{k}_m^{2,2} = \{k_{21}\}$	$\mathbf{k}_m^{3,2} = \{k_{31}\}$
	$\mathbf{k}_m^{2,3} = \{k_{13}\}$	$\mathbf{k}_m^{3,3} = \{k_{22}\}$
	$\mathbf{k}_m^{2,4} = \{k_{23}\}$	$\mathbf{k}_m^{3,4} = \{k_{32}\}$
	$\mathbf{k}_m^{2,5} = \{k_{22}, k_{33}\}$	$\mathbf{k}_m^{3,5} = \{k_{12}, k_{23}\}$
	$\mathbf{k}_m^{2,6} = \{k_{12}, k_{33}\}$	$\mathbf{k}_m^{3,6} = \{k_{11}, k_{23}\}$
	$\mathbf{k}_m^{2,7} = \{k_{31}, k_{12}\}$	$\mathbf{k}_m^{3,7} = \{k_{11}, k_{33}\}$
	$\mathbf{k}_m^{2,8} = \{k_{22}, k_{31}\}$	$\mathbf{k}_m^{3,8} = \{k_{12}, k_{33}\}$

the 9 resultant sets and eliminating the identical ones will lead to the fact that the system has no DOFM with respect to a control interaction set **K** if and only if **K** includes at least one of the 8 sets given below:

$$
\begin{aligned}
&\{k_{13},k_{21}\},\ \{k_{13},k_{31}\},\ \{k_{13},k_{22}\},\ \{k_{13},k_{32}\},\\
&\{k_{13},k_{12},k_{23}\},\ \{k_{13},k_{11},k_{23}\},\ \{k_{13},k_{11},k_{33}\},\\
&\{k_{12},k_{33}\}
\end{aligned}
\tag{3.19}
$$

It is worth mentioning that there exist $2^{(3^2)}-1=511$ classes of structurally constrained controllers for this example. However, the desired ones are characterized here as only 9 minimal sets.

Now, it is desired to obtain the *cost optimal* control interaction set(s). Assume that $\mathscr{C}_{ij}=1,\ i,j\in\{1,2,3\}$, which implies that establishing any of the transmission links would incur the same cost. As discussed earlier, among the sets given in (3.19), the ones whose corresponding costs are minimum should be identified. In this case, the five sets which have only 2 elements are the cost optimal ones with the minimum cost 2.

3.6 Conclusions

This work aims to obtain all classes of LTI structurally constrained controllers with respect to which a given interconnected system has no fixed modes. To this end, the notions of maximal graph and minimal set are exploited to formulate the problem in the graph theory framework. A cost is then allocated for establishing a communication link between any pair of controllers. An algorithm is subsequently developed to identify the stabilizing control structures with the minimum implementation cost. The significance of this contribution is illustrated in a numerical example. This work takes advantage of the recent results presented in the literature to handle similar problems.

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Chapter 4

Robust Control of LTI Systems by Means of Structurally Constrained Controllers

4.1 Abstract

This work investigates the stabilizability of uncertain LTI systems via structurally constrained controllers. First, a LTI uncertain system is considered whose state-space matrices depend polynomially on the uncertainty vector, defined over some region. It is shown that if the system is stabilizable by a structurally constrained controller in one point belonging to the region, then it is stabilizable by a controller with the same structure in all points belonging to the region, except for the ones located on an algebraic variety. Thus, if a system is stabilizable via a constrained controller at the nominal point, then it is almost always stabilizable at any operating point around the nominal model. It is also shown how this algebraic variety (or the dominant subvariety of it) can be computed efficiently. The results obtained in this work encompass a broad range of the existing results in the literature on

robust stability of the LTI systems.

4.2 INTRODUCTION

Numerous real-world systems can be envisaged as interconnected systems consisting of a number of subsystems [1]. The overall controller for such a system is often composed of a set of local controllers corresponding to the individual subsystems. In an unconstrained control structure, the outputs of all the subsystems are accessible by any local controller. This type of controller is referred to as a centralized controller. However, in many control applications, each local controller can only use the information of a subset of subsystems. This control constraint is due, primarily, to some practical issues discussed below:

i) Interconnected systems often have several subsystems. Hence, a centralized controller for such large-scale systems can potentially be costly, in light of the required computations and transmission of information between the subsystems. In order to reduce the control expenditure for this type of systems, it is desirable to impose certain constraints on the control structure. A manifest example of this case is the traffic control system [2].

ii) For the interconnected systems with geographically distributed subsystems, transmission of information between two specific subsystems can be quite costly and prone to reliability problems. This is the case, for instance, in power systems, where the interacting power stations are located in remote places.

iii) In some interconnected systems, the output of certain subsystems may be inaccessible for some other subsystems in specific time intervals. For example, this can occur frequently in the flight formation problem, due to the shadow phenomenon [3].

It follows from the above discussion that a constrained control structure is often more desirable for the interconnected systems [4]. The structure of such a controller is sometimes represented by a binary matrix, referred to as the information flow matrix [5]. Note that the information flow matrix corresponding to any system is part of the design specifications and is contingent upon the characteristics of the system and the control implementation cost as noted above.

A special case of structurally constrained controllers, often referred to as decentralized control, has been extensively studied in the literature [6; 7]. A decentralized controller comprises a number of non-interacting local controllers, which implies that the corresponding information flow matrix is block diagonal. Another type of structurally constrained control is the one in which some local controllers overlap in accordance with the overlapping structure of their corresponding subsystems [8; 9]. This class of control structure is called decentralized overlapping structure, and has been investigated in the literature in the Expansion-Inclusion framework [10].

The problem of stabilizability of systems (with known parameters) with respect to LTI *decentralized* and *decentralized overlapping* controllers has been investigated intensively, and several methodologies are presented accordingly for controller design [5; 11; 12; 13]. The notion of decentralized fixed modes (DFM) was introduced in [6] to identify those modes of a LTI system (if any) which cannot be shifted by using any LTI decentralized controller. As a generalization of DFM, the notion of decentralized overlapping fixed modes (DOFM) was introduced in [13] to characterize those modes of a LTI system which are immovable with respect to the class of LTI structurally constrained controllers with a given information flow matrix (of any arbitrary structure). A simple graph-theoretic approach is also provided in [13] to obtain the DOFMs of any system efficiently.

The papers surveyed so far have merely considered the problem of structurally constrained stabilization for systems with known parameters. However, the real-world systems are uncertain to some degree. Under this circumstance, a region of uncertainty is usually envisaged to describe the range of uncertainty, along with a set of relations to characterize the uncertain parameters of the system.

In the early works, the region of uncertainty was assumed to be the whole space and besides the uncertain parameters of the system were considered uncorrelated. The notions of structural controllability and structurally fixed modes were then defined based on these assumptions. Structural controllability was introduced in [14] to determine whether the uncontrollability of a LTI system is resulted from its structure or from the exact parameter matching in the system. Structural controllability is studied in several papers, e.g. see [15; 16; 17; 18]. Furthermore, the notion of structurally fixed modes was defined in [19] to characterize those DFMs that are resulted from the structure of the system, and hence remain fixed regardless of the numerical values of the system's nonzero parameters.

Although the notions of structural controllability and structurally fixed modes are very useful in robust control design problems, they fail to address the very important practical issue of correlation between the nonzero parameters of the system [18]. In other words, in many practical problems different parameters of the system are correlated to each other, and belong to known regions in the parameter space. As a simple example, consider a RLC circuit and assume that the numerical values of its elements are known with a maximum error of 10%. In this case, every coefficient of the system transfer function can be written parametrically in terms of three quantities: the resistance, the capacitance and the inductance. This implies that all the coefficients of the transfer function are correlated, and that the uncertainty region is, indeed, a cube.

This work deals with the robust stabilizability of LTI systems via structurally constrained controllers. It is assumed that the state-space matrices of the system are polynomially uncertain, and that the uncertainty variables of the system belong to a known region. It is shown that if the system has no DOFMs at some point belonging to the region, then the points for which the system has a DOFM lie on an algebraic variety. As a result, if a system has no DOFM at its nominal point, it almost always has no DOFMs at any operating point. Furthermore, since finding the exact algebraic variety can be formidable in general, a simple method is proposed to compute a dominant subset of it, in the sense that the dimension of this subset is greater than that of its complement. It is noteworthy that the robust stabilizability problem has been investigated in a number of papers, e.g. see [20; 21; 22; 23; 24; 25; 26]. Nonetheless, these works formulate the problem in some special cases, e.g. SISO systems, centralized controllers or polytopic uncertainties. In contrast, this work tackles the robust stabilizability problem in the most general case. The results provided here encompass the ones presented in the literature for structural controllability and structurally fixed modes.

4.3 ROBUSTNESS PROPERTY OF THE MODES OF A LTI SYSTEM

Consider an uncertain LTI interconnected system $\mathscr{S}(\alpha)$ with unknown, nevertheless fixed (or slowly time-varying) parameters, consisting of v subsystems with the following state-space representation:

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + \sum_{i=1}^{v} B_i(\alpha)u_i(t) \\ y_i(t) &= C_i(\alpha)x(t) + \sum_{j=1}^{v} D_{ij}(\alpha)u_j(t), \quad i \in \bar{v} := \{1,2,...,v\} \end{aligned} \tag{4.1}$$

where:

- $x(t) \in \Re^n$ is the state, and $u_i(t) \in \Re^{m_i}$ and $y_i(t) \in \Re^{r_i}$, $i \in \bar{v}$, are the input and output of the i^{th} subsystem of $\mathscr{S}(\alpha)$, respectively.

- $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_\mu \end{bmatrix}$ represents the vector variable corresponding to the uncertainty in the system, and belongs to a given region $\mathscr{D}$ of dimension μ.

- $A(\alpha), B_i(\alpha), C_i(\alpha)$ and $D_{ij}(\alpha)$, $i, j \in \bar{\mathbf{v}}$, are matrix polynomials of the variable α.

Define now the following matrices:

- $B(\alpha)$ is a block matrix whose i^{th} block column is equal to $B_i(\alpha)$, for any $i \in \bar{\mathbf{v}}$.

- $C(\alpha)$ is a block matrix whose i^{th} block row is equal to $C_i(\alpha)$, for any $i \in \bar{\mathbf{v}}$.

- $D(\alpha)$ is a block matrix whose $(i, j)^{\text{th}}$ block entry is equal to $D_{ij}(\alpha)$, for any $i, j \in \bar{\mathbf{v}}$.

Assume that the system $\mathscr{S}(\alpha)$ is to be controlled by means of a structurally constrained controller. The constraint on the control structure determines which outputs $y_i(t)$ ($i \in \bar{\mathbf{v}}$) are available to construct any specific input $u_j(t)$ ($j \in \bar{\mathbf{v}}$) of the system (as the local control command). In order to simplify the formulation of the control constraint, a block matrix $\mathscr{K}$ with binary entries is defined, where its $(i, j)^{\text{th}}$ block entry, $i, j \in \bar{\mathbf{v}}$, is a $m_i \times r_j$ matrix with all elements equal to 1 if the output of the j^{th} subsystem can contribute to the construction of the input of the i^{th} subsystem, and is a $m_i \times r_j$ zero matrix otherwise. The matrix $\mathscr{K}$ represents the control constraint, and is referred to as the information flow matrix [5]. In the special case, when the entries of the matrix $\mathscr{K}$ are all equal to 1, the corresponding controller is centralized, when $\mathscr{K}$ is block diagonal, the corresponding controller is decentralized.

Consider the system $\mathscr{S}(\alpha)$ for an arbitrary value of α, namely α_0. The notion of decentralized fixed modes (DFM) introduced in [5] for general proper systems corresponds to the modes of the system $\mathscr{S}(\alpha_0)$ which are fixed w.r.t. (with respect to) a *block-diagonal* information flow matrix $\mathscr{K}$. Decentralized overlapping fixed modes (DOFM) were then defined in [13] to identify those modes of the system $\mathscr{S}(\alpha_0)$ which are immovable w.r.t. any LTI controller complying with the given information flow matrix $\mathscr{K}$.

Let $\mathbf{K}$ denote the set of all constant matrices of dimension $(m_1+\cdots+m_\nu)\times(r_1+\cdots+r_\nu)$ with the property that their zero entries coincide with those of the matrix $\mathscr{K}$. In fact, the set $\mathbf{K}$ parameterizes all the structurally constrained *static* controllers complying with the information flow matrix $\mathscr{K}$.

Lemma 1 *Given $\alpha_0 \in \mathscr{D}$, assume that σ is a mode of the system $\mathscr{S}(\alpha_0)$. There exists no static or dynamic LTI structurally constrained controller complying with $\mathscr{K}$ to move the mode σ if and only if the following relation holds:*

$$\sigma \in sp\left\{A(\alpha_0)+B(\alpha_0)K(I-D(\alpha_0)K)^{-1}C(\alpha_0)\right\}, \quad \forall\, K \in \mathbf{K} \tag{4.2}$$

Proof: For the case when $\mathscr{K}$ is block diagonal, the proof follows from the result obtained in [5], which states that if a mode is fixed w.r.t. the *static* LTI decentralized controllers, then it is also fixed w.r.t. the dynamic LTI decentralized controllers. The proof can be extended to the general case of non-block diagonal information flow structure, on noting that there exists a bijective morphism between DFMs and DOFMs (as substantiated in [13]). ■

The following definitions will prove convenient in presenting the main results of this work.

Definition 1 *An algebraic variety refers to the set of common zeros of a number of polynomials. The notation $\mathscr{V}(f_1(\alpha),...,f_\lambda(\alpha))$ will be used throughout this work to refer to the algebraic variety generated by the common roots of the polynomials $f_1(\alpha),...,f_\lambda(\alpha)$.*

Definition 2 *An irreducible algebraic variety is said to be an affine variety.*

In this work, only varieties in the real space (as opposed to the complex space) are considered. Hence, the term "real space" describing the type of said varieties will be omitted hereafter for simplicity. It is worth mentioning that an algebraic variety generated by a set of μ-variate polynomials is of dimension $\mu - 1$. Nonetheless, this variety can be considered as the union of a number of affine varieties such that some of them are of pure dimension $\mu - 1$ and the others have smaller dimensions.

Notation 1 *Given the variables $\alpha_1, \alpha_2, ..., \alpha_\lambda$, all the monomials of maximum degree ρ in the form of $\alpha_1^{\rho_1}\alpha_2^{\rho_2}...\alpha_\lambda^{\rho_\lambda}$ are denoted (in an arbitrary order) by $\Phi_\rho^1(\alpha_1,...,\alpha_\lambda), \Phi_\rho^2(\alpha_1,...,\alpha_\lambda)$, $..., \Phi_\rho^\xi(\alpha_1,...,\alpha_\lambda)$, where $\xi = \sum_{i=0}^{\rho}\binom{i+\lambda-1}{\lambda-1}$.*

The next theorem presents the main result of this chapter.

Theorem 1 *The following statements are true for the robust stability of the system $\mathscr{S}(\alpha)$ in the region $\mathscr{D}$:*

a) Assume that the system $\mathscr{S}(\alpha^*)$ has no DOFM w.r.t. $\mathscr{K}$, for some $\alpha^* \in \mathscr{D}$. There exists an algebraic variety of dimension $\mu - 1$ such that for any arbitrary α_0 belonging to $\mathscr{D}$, the system $\mathscr{S}(\alpha_0)$ does not have any DOFM w.r.t. $\mathscr{K}$ if and only if α_0 does not pertain to this variety.

b) If there exists a point $\alpha^* \in \mathscr{D}$ such that the system $\mathscr{S}(\alpha^*)$ has no DOFMs w.r.t. $\mathscr{K}$, then for almost all values of α_0 belonging to $\mathscr{D}$, the system $\mathscr{S}(\alpha_0)$ also has no DOFMs w.r.t. $\mathscr{K}$.

Proof of part (a): Define the following:

$$q(s,\alpha,K) = \det\left(sI - A(\alpha) - B(\alpha)K(I - D(\alpha)K)^{-1}C(\alpha)\right)\det(I - D(\alpha)K) \tag{4.3}$$

It is straightforward to show that $q(s,\alpha,K)$ is a polynomial in terms of the variables s, α and K, which are a scalar, a vector and a matrix, respectively. It results from Lemma 1 that the system $\mathscr{S}(\alpha_0)$ has no DOFM w.r.t. $\mathscr{D}$ if and only if there exists a matrix $K_0 \in \mathbf{K}$ such that the polynomials $q(s,\alpha_0,K_0)$ and $q(s,\alpha_0,0_{K_0})$ (which are functions of s only) are coprime, where 0_{K_0} denotes a zero matrix with the same size as K_0. Note that the zeros of the polynomial $q(s,\alpha_0,0_{K_0})$ are indeed the modes of the system $\mathscr{S}(\alpha_0)$. The coprimeness of these two polynomials will be formulated next.

One can rewrite the polynomial $q(s,\alpha,K)$ as:

$$q(s,\alpha,K) = \sum_{i=0}^{n} q_i(\alpha,K)s^i \tag{4.4}$$

for some polynomials $q_0(\alpha,K),...,q_n(\alpha,K)$. Construct a $2n \times 2n$ Sylvester matrix by using the following rule:

Consider the first row of this matrix as:

$$\begin{bmatrix} q_n(\alpha,K) & q_{n-1}(\alpha,K) & \cdots & q_0(\alpha,K) & 0\cdots & 0 \end{bmatrix} \tag{4.5}$$

and the $(n+1)^{th}$ *row as:*

$$\begin{bmatrix} q_n(\alpha,0_{K_0}) & q_{n-1}(\alpha,0_{K_0}) & \cdots & q_0(\alpha,0_{K_0}) & 0\cdots & 0 \end{bmatrix} \tag{4.6}$$

Now, for any $i \in \{2,3,...,n,n+2,...,2n\}$, *the* i^{th} *row of the Sylvester matrix is obtained from the* $(i-1)^{th}$ *row by shifting it by one to the right and circularly shifting the rightmost entry to the leftmost position.* Denote the determinant of the resultant Sylvester matrix by the polynomial $r(\alpha,K)$. It can be inferred from Sylvester's theorem that the polynomials $q(s,\alpha_0,K_0)$ and $q(s,\alpha_0,0_{K_0})$ are coprime if and only if $r(\alpha_0,K_0) \neq 0$. One can conclude from this result and the existence condition given earlier for DOFMs of the system $\mathscr{S}(\alpha_0)$, that the system $\mathscr{S}(\alpha_0)$ has no DOFMs w.r.t. $\mathscr{K}$ if and only if there exists a matrix $K_0 \in \mathbf{K}$ such that the polynomial $r(\alpha_0,K_0)$ is nonzero. This condition will be further simplified in the sequel.

Assume that the matrix $\mathscr{K}$ has j nonzero entries. Denote the nonzero scalar variables of the matrix variable $K \in \mathbf{K}$ with $k_1,k_2,...,k_j$, in an arbitrary order. One can decompose the polynomial $r(\alpha,K)$ as follows:

$$r(\alpha,K) = \sum_{i=0}^{\bar{z}} r_i(\alpha)\Phi_z^i(k_1,k_2,...,k_j) \tag{4.7}$$

for some polynomials $r_1(\alpha),...,r_{\bar{z}}(\alpha)$, where $z = n(r_1 + \cdots + r_v)$ and $\bar{z} = \sum_{i=0}^{z} \binom{i+j-1}{j-1}$. It is notable that the polynomials $r_1(\alpha),...,r_{\bar{z}}(\alpha)$ cannot be all identical to zero. This fact results from the assumption that the system $\mathscr{S}(\alpha^*)$ has no DOFMs over the region $\mathscr{D}$ w.r.t. $\mathscr{K}$, which implies that $r(\alpha^*, K^*) \neq 0$ for some $K^* \in \mathbf{K}$. On the other hand, it can be easily verified that for a given scalar α_0, there exists a matrix $K_0 \in \mathbf{K}$ for which $r(\alpha_0, K_0) \neq 0$ if and only if the polynomials $r_1(\alpha),...,r_{\bar{z}}(\alpha)$ are not concurrently equal to zero at $\alpha = \alpha_0$.

The aforementioned results can be summarized as follows: *The system $\mathscr{S}(\alpha_0)$ has no DOFMs w.r.t. $\mathscr{K}$ if and only if α_0 does not pertain to the algebraic variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$.* This completes the proof of part (a).

Proof of part (b): The proof follows from part (a) and on noting that the dimension of the algebraic variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ is equal to $\mu - 1$, while that of the region $\mathscr{D}$ is known to be equal to μ. ■

Remark 1 *As an illustrative example of the above results, assume that the region $\mathscr{D}$ is an oval in a plane. Theorem 1 implies that there are three possible scenarios regarding the DOFMs of the system in this region (DOFMs are represented by black points, while the non-DOFMs are shown by gray points):*

1) All points inside the region are DOFMs (Figure 4.1).

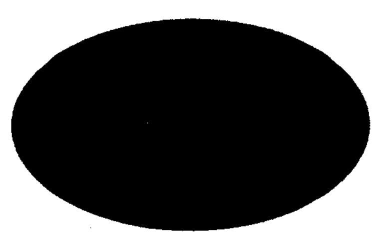

Figure 4.1: Explanation of DOFMs for the example given in Remark 1.

2) None of the points inside the region are DOFMs (Figure 4.2).

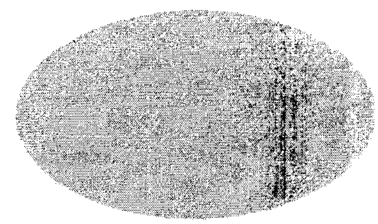

Figure 4.2: Explanation of DOFMs for the example given in Remark 1.

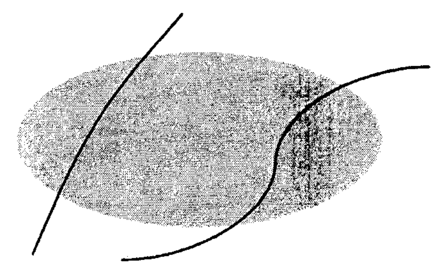

Figure 4.3: Explanation of DOFMs for the example given in Remark 1.

3) Almost all of the points inside the region are not DOFMs (Figure 4.3).

Motivated by the results of Theorem 1, the notion of *structurally robust fixed modes* (SRFM) will now be introduced.

Definition 3 *The uncertain system $\mathscr{S}(\alpha)$ is said to have no SRFM in the region $\mathscr{D}$ w.r.t. $\mathscr{K}$, if there exists an α belonging to $\mathscr{D}$, denoted by α^*, such that the system $\mathscr{S}(\alpha^*)$ does not have any DOFM w.r.t. the information flow matrix $\mathscr{K}$. Note that if $\mathscr{S}(\alpha)$ has some SRFMs, then for any α_0 belonging to the region $\mathscr{D}$, the system $\mathscr{S}(\alpha_0)$ has at least one DOFM w.r.t. $\mathscr{K}$.*

Corollary 1 *If the system $\mathscr{S}(\alpha)$ has at least one SRFM over the region $\mathscr{D}$ w.r.t. $\mathscr{K}$, then it also has some SRFMs over the whole space w.r.t. $\mathscr{K}$.*

Proof: A proof by contradiction will be provided here. Assume that the system $\mathscr{S}(\alpha)$ has no SRFM over the whole space w.r.t. $\mathscr{K}$, while it has some SRFMs over the region $\mathscr{D}$. It can be concluded from part (a) that there exists an algebraic variety such that the system $\mathscr{S}(\alpha)$ has DOFMs only for the uncertainties in this variety. Since the dimension of this variety is $\mu-1$ (and that of the region $\mathscr{D}$ is μ), the region $\mathscr{D}$ is not contained by this variety. As a result, there exist infinitely many points belonging to $\mathscr{D}$, which do not belong to this algebraic variety, and consequently, their corresponding systems do not have any DOFMs. This contradicts the initial assumption that the system $\mathscr{S}(\alpha)$ has some SRFMs over the region $\mathscr{D}$ w.r.t. $\mathscr{K}$. ■

Remark 2 *As a by-product of part (b) of Theorem 1, if the system $\mathscr{S}(\alpha)$ has no DOFMs at the nominal point $\alpha=\alpha^*$, then the system almost always has no DOFMs at any operating point $\alpha=\alpha_0$ either.*

Theorem 1 states that there is an algebraic variety whose intersection with the region $\mathscr{D}$ leads to the points α for which the system $\mathscr{S}(\alpha)$ has some DOFMs w.r.t. $\mathscr{K}$. Hence, identification of this variety can be helpful to provide a precise insight into the robust stabilizability of the system. This algebraic variety is characterized by the polynomials $r_1(\alpha), r_2(\alpha), ..., r_{\bar{z}}(\alpha)$. Since $\bar{z}$ is typically a very large number, finding all these polynomials and the geometric shape of their common zeros is quite cumbersome. It is desired now to present a simple algorithm to obtain a dominant subset of this algebraic variety.

In light of the discussion given earlier, the algebraic variety $\mathscr{V}(r_1(\alpha), ..., r_{\bar{z}}(\alpha))$ is the union of a number of affine varieties. Some of these affine varieties are of pure dimension $\mu-1$ and the remaining ones are of dimensions less than $\mu-1$. More precisely, the dimensions of the latter subvarieties normally do not exceed $\max(0, \mu-\bar{z})$ (rather than $\mu-2$), and since $\bar{z}$ is typically much greater than μ, these subvarieties are likely to be empty in general [27]. Hence, one can come to the conclusion that the affine varieties of pure dimension $\mu-1$ play the primary role in the robust stabilizability of the system $\mathscr{S}$, and the effect of the other affine varieties (if any) is negligible. The following theorem states how the dominant part of $\mathscr{V}(r_1(\alpha), ..., r_{\bar{z}}(\alpha))$ can be identified efficiently.

Theorem 2 *Let two generic matrices K_1 and K_2 be chosen from the set* $\mathbf{K}$. *Compute first the function $q(s,\alpha,K_i)$ and subsequently $r(\alpha,K_i)$ in terms of q, for $i=1,2$. Denote the greatest common divisor (gcd) of $r(\alpha,K_1)$ and $r(\alpha,K_2)$ with $h(\alpha)$. The variety $\mathscr{V}(h(\alpha))$ is included in the variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$, and also contains all the affine varieties of $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ with the pure dimension $\mu-1$.*

Proof: Define the following polynomial:

$$l(\alpha)=\gcd(r_1(\alpha),r_2(\alpha),...,r_{\bar{z}}(\alpha)) \tag{4.8}$$

By virtue of the celebrated results on algebraic sets, one can conclude that not only is the variety $\mathscr{V}(l(\alpha))$ a subset of the variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$, every affine variety in $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ of pure dimension $\mu-1$ is also a subset of $\mathscr{V}(l(\alpha))$ [27]. Therefore, to prove the theorem, it suffices to substantiate that $l(\alpha)$ is identical to $h(\alpha)$ for generic choices of K_1 and K_2. This can be easily shown by commencing from (4.7) and performing some additional manipulations. The details are omitted here for brevity. ∎

Theorem 2 proposes a simple method to obtain the dominant component of said variety, which broadly speaking, causes the uncertain system $\mathscr{S}(\alpha)$ not to be stabilizable by means of LTI structurally constrained controllers. It is to be noted that the varieties $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ and $\mathscr{V}(h(\alpha))$ are the same in the univariate case ($\mu=1$), but not necessarily the same in the multivariate case. This results from the fact that a set of multivariate polynomials can be relatively prime, while they have some common roots.

Remark 3 *The notion of structurally fixed modes introduced in [19] characterizes those modes (if any) of a system which continue to be DFMs after arbitrarily perturbing the nonzero parameters of the system matrices. For a deterministic system $\mathscr{S}$, assume that the system matrices A,B,C and D have accumulatively e nonzero entries. Let the region $\mathscr{D}$ be $\Re^e$. It is desired now to construct the uncertain replica of $\mathscr{S}$, denoted by $S(\alpha)$.*

Define the vector α as $\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_e \end{bmatrix}$, and replace each nonzero entry of the system matrices with one of the variables $\alpha_1, \alpha_2, \ldots, \alpha_e$ so that none of these variables is recurrent. Denote the new matrices with $A(\alpha), B(\alpha), C(\alpha)$ and $D(\alpha)$, and the corresponding system with $S(\alpha)$. It can be easily shown that the notion of a structurally robust fixed mode for the system $\mathscr{S}(\alpha)$ is the same as the notion of a structurally fixed mode for the system $\mathscr{S}$. This implies that the robust stabilizability framework introduced here encompasses one of the relevant well-known results in the literature. In this case, Theorem 1 conforms to the famous result that a system with no structurally fixed modes has generically no DFMs.

Remark 4 *Using a technique similar to the one exploited in Remark 3, it can be easily shown that the formulation presented in this work and the subsequent developments encompass the existing results on structural controllability. In this case, the result of Theorem 1 is in accordance with the celebrated result that if a system is structurally controllable, then almost all systems with the same structure are also controllable.*

4.3.1 Practical applications

A number of immediate applications of this work are encapsulated below:

- The prevailing method for controlling an uncertain interconnected system with unknown, nevertheless fixed (or slowly time-varying) parameters by means of a structurally constrained LTI controller is adaptive control [28; 29; 30]. Nevertheless, an adaptation law is feasible only if the system is stabilizable over the uncertainty region w.r.t. the desirable class of controllers. The present work provides a systematic method to check this feasibility criterion. The results obtained can also be used to develop more effective adaptation laws, compared to the existing ones.

- The problem of robust stability w.r.t. parameter variation is widely studied in the literature. It aims to discover whether a controller designed for a system in the nominal point can stabilize it over the whole region of uncertainty [31; 32]. However, robust stabilizability is a prerequisite for any relevant technique; i.e. the system is required to be stabilizable over the uncertainty region. Consequently, the present work provides a technique to verify structural stabilizability as a necessary condition for the robust stability of the system.

- In some real-world applications involving interconnected systems such as formation flight of spacecraft, the control structure is to be devised first. To attain this primary objective, the matrix $\mathscr{K}$ should undoubtedly be found in such a way that the system will be stabilizable w.r.t. the corresponding class of controllers. Therefore, different forms for the matrix $\mathscr{K}$ can be considered, and the method proposed here can be pursued to explore which information flow matrix leads to an effective controller from the robust stabilizability perspective.

- The robust stabilizability problem tackled here is closely related to the problem of sensitivity analysis w.r.t. the parameters of a system. Thus, the results attained in the present work can be exploited to carry out sensitivity analysis, which manifestly plays an important role in studying how reliable the designed controller is in reality.

4.4 NUMERICAL EXAMPLE

Consider an uncertain third-order system $\mathscr{S}(\alpha)$, $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$, with the following state-space matrices:

$$A(\alpha) = \begin{bmatrix} \alpha_2 & 0 & \alpha_1^3 - \alpha_2^3 \\ 0 & \alpha_1 & \alpha_1 - \alpha_2 \\ 0 & \alpha_1 & 0 \end{bmatrix} \tag{4.9a}$$

$$B(\alpha) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_2^2 & 0 & 0 \\ 0 & \alpha_1 & \alpha_1 + \alpha_2 \end{bmatrix} \tag{4.9b}$$

$$C(\alpha) = \begin{bmatrix} \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_1^3 + \alpha_2^3 \\ \alpha_1 & 0 & 0 \end{bmatrix} \tag{4.9c}$$

$$D(\alpha) = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_1 \\ \alpha_1 & 0 & \alpha_2 \end{bmatrix} \tag{4.9d}$$

(the system consists of three SISO subsystems). Define now two regions of uncertainty as follows:

$$\mathscr{D}_1 = \{\alpha \,:\, 1 \leq \alpha_2^2 - 2\alpha_1^2 \leq 2,\ \alpha_1 > 0,\ \alpha_2 > 0\} \tag{4.10a}$$

$$\mathscr{D}_2 = \{\alpha \,:\, 1 \leq \alpha_1^2 - 2\alpha_2^2 \leq 2,\ \alpha_1 > 0,\ \alpha_2 > 0\} \tag{4.10b}$$

It is desired to check the robust stabilizability of the system $\mathscr{S}$ over the regions $\mathscr{D}_1$ and $\mathscr{D}_2$. To this end, two different control structures are delineated below:

1. Let the information flow matrix $\mathscr{K}$ be:

$$\mathscr{K} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.11}$$

In light of Theorem 2, two generic matrices should be chosen first. Let these matrices be:

$$K_1 = \begin{bmatrix} -1 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.12}$$

The polynomials $r(s,K_1)$ and $r(s,K_2)$ can be simply obtained as:

$$\begin{aligned} r(s,K_1) &= -\alpha_2^4\alpha_1^3(-\alpha_1-\alpha_2+\alpha_1^2\alpha_2^2+\alpha_2^3\alpha_1+\alpha_2^4) \\ &\quad \times (\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2)^2(6\alpha_1-\alpha_2)^3(\alpha_1-\alpha_2)^3, \\ r(s,K_2) &= -8\alpha_2^4\alpha_1^3(-\alpha_1-\alpha_2+\alpha_1^2\alpha_2^2+\alpha_2^3\alpha_1+\alpha_2^4) \\ &\quad \times (\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2)^2(5\alpha_2+\alpha_1)^3(\alpha_1-\alpha_2)^3 \end{aligned} \tag{4.13}$$

Consequently:

$$\begin{aligned} h(\alpha) &= \gcd(r(s,K_1),r(s,K_2)) \\ &= \alpha_2^4\alpha_1^3(-\alpha_1-\alpha_2+\alpha_1^2\alpha_2^2+\alpha_2^3\alpha_1+\alpha_2^4)(\alpha_2^2+\alpha_1\alpha_2+\alpha_1^2)^2(\alpha_1-\alpha_2)^3 \end{aligned} \tag{4.14}$$

Since only the real values of α are of importance, one can consider the variety $\mathscr{V}(p(\alpha))$, instated of $\mathscr{V}(h(\alpha))$, where:

$$p(\alpha) = \alpha_1\alpha_2(-\alpha_1-\alpha_2+\alpha_1^2\alpha_2^2+\alpha_2^3\alpha_1+\alpha_2^4)(\alpha_1-\alpha_2) \tag{4.15}$$

It is worth noting that the curves $\alpha_1-\alpha_2=0$ and $\alpha_1=0$ correspond to the specific perturbations which make the system unobservable or uncontrollable (and hence the corresponding DOFMs are, in fact, centralized fixed modes as well), while the curves $\alpha_2=0$ and $-\alpha_1-\alpha_2+\alpha_1^2\alpha_2^2+\alpha_2^3\alpha_1+\alpha_2^4=0$ represent the perturbations which generally lead to observable and controllable fixed modes. One can verify that for this example, the exact algebraic variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ introduced in Theorem 1 is the same as $\mathscr{V}(p(\alpha))$ over the field of real numbers.

In order to discover whether or not the system $\mathscr{S}(\alpha)$ has some DOFMs over the region $\mathscr{D}_1$ w.r.t. $\mathscr{K}$, it suffices to check if the polynomial $p(\alpha)$ takes zero values in this region. For any point (α_1, α_2) belonging to $\mathscr{D}_1$, the inequalities $\alpha_2 > \sqrt{2}\alpha_1$ and $\alpha_2 > 1$ both hold. The first inequality implies that $\alpha_1 - \alpha_2 \neq 0$. On the other hand, it follows directly from the second inequality that $-\alpha_1 - \alpha_2 + \alpha_1^2\alpha_2^2 + \alpha_2^3\alpha_1 + \alpha_2^4 > 0$. This means that $p(\alpha)$ cannot vanish in the region $\mathscr{D}_1$, or equivalently, that the system $\mathscr{S}(\alpha)$ has no DOFM in the region $\mathscr{D}_1$ w.r.t. $\mathscr{K}$. In contrast, it is easy to show that the polynomial $p(\alpha)$ is equal to zero for $(\alpha_1, \alpha_2) = (1.5312, 0.75)$. Since this point belongs to the region $\mathscr{D}_2$, the system $\mathscr{S}(\alpha)$ has some DOFMs for certain points in the region $\mathscr{D}_2$ w.r.t. $\mathscr{K}$.

2. Assume now that the information flow matrix $\mathscr{K}$ is:

$$\mathscr{K} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{4.16}$$

Pursuing the methodology outlined in the previous case, it can be easily deduced that:

$$h(\alpha) = \alpha_1(\alpha_1 - \alpha_2)^3 \tag{4.17}$$

Hence, the variety $\mathscr{V}(r_1(\alpha),...,r_{\bar{z}}(\alpha))$ includes two lines and some affine varieties of dimensions less than $\mu - 1 = 1$. This implies that any of these affine varieties should be of dimension 0, and hence would comprise a finite number of points. On the other hand, none of the regions $\mathscr{D}_1$ and $\mathscr{D}_2$ has an intersection with the line $\alpha_1 - \alpha_2 = 0$. This means that the system $\mathscr{S}(\alpha)$ has no DOFMs w.r.t. $\mathscr{K}$ for every α pertaining to either $\mathscr{D}_1$ or $\mathscr{D}_2$, except possibly for a finite number of points α.

4.5 CONCLUSIONS

This work investigates the robust control of LTI systems subject to polynomial uncertainties over a given region. It is shown that if the system is stabilizable at some point in the given region by means of a structurally constrained controller, then it is also stabilizable via a controller of the same structure at any point in the region, as long as those points do not lie on an algebraic variety. This result shows that if the nominal model of the system is stabilizable by means of a structurally constrained controller, so is the system at almost all operating points. A numerical example is given to illustrate the importance of the results obtained and their efficacy in dealing with the most general forms of uncertainties. The proposed formulation and the subsequent developments encompass the existing results reported in prior literature on the structural controllability and structurally fixed modes.

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Chapter 5

Performance Evaluation of Decentralized Controllers Designed Based on a Hierarchical Approximation of the System Model

5.1 Abstract

This work is concerned with decentralized controller design for large-scale interconnected systems of pseudo-hierarchal structures. Given such a system, a decentralized controller can be designed for the hierarchical model of the system, as opposed to the original pseudo-hierarchical one, by means of the available techniques. Although this indirect controller design is often fascinating for the sake of computational simplicity, it may not provide the desired performance for the original pseudo-hierarchical system, as the controller has been designed for its hierarchical model. In order to make certain that this approach is appropriate for the system, a LQ cost function is defined to evaluate the discrepancy between the

pseudo-hierarchical system and its hierarchical model under the designed decentralized controller. To compute this performance index, a discrete Lyapunov equation should be solved; due to the large-scale nature of the system, this equation by no means can be handled for many real-world systems. This gives rise to attaining an upper bound on this cost function, as opposed to its exact value. For this purpose, a novel technique is proposed, which only requires solving a simple constrained optimization problem with three variables. It is also proved that as the pseudo-hierarchical system approaches its hierarchical model, this bound goes to zero; and in the ideal case, the bound will become zero.

5.2 Introduction

Large-scale systems often appear in reality and, consequently, they constitute an important class of systems [1]. Since such systems normally comprise many subsystems, their control is intricate. To alleviate the issue, the decentralized theory was developed in the literature and its distinctive aspects have been exhaustively investigated through the last three decades [2; 3]. A decentralized controller is indeed composed of a number of isolated local controllers corresponding to the subsystems (or control channels) of a large-scale system. To further diminish the complexity of the decentralized controller design, it is desired that the large-scale system possesses a hierarchical structure [4; 5]. In this case, the design problem for the system can be broken down into a number of parallel design subproblems corresponding to diverse subsystems. The benefit of this design technique is twofold. Indeed not only is handling everything with the order of subsystem far simpler than with the order of the large-scale system (as the subsystems are likely modest-sized), the parallel computation is also intriguing.

Many real-world systems associated with remarkable applications, such as formation flight, underwater vehicles, automated highway, robotics, satellite constellation, etc., maintain a leader-follower or equivalently hierarchical structure [6; 7; 8]. Furthermore, it is shown in [9] there is a broad class of continuous-time non-hierarchical system whose structures can be converted to hierarchical ones in the discrete-time domain after discretization. For a hierarchical system, designing a decentralized stabilizing controller is tantamount to designing a set of stabilizing local controllers corresponding to different subsystems, after removing all the existing interconnections between the subsystems [5]. This worthwhile result is quite beneficial as it noticeably simplifies the decentralized stabilizability problem. In addition, a technique is provided in [10] to design a near-optimal decentralized controller for hierarchical systems. This idea is developed in [11] to decentralize any given centralized controller without losing its fundamental properties. These points reveal that designing a decentralized controller for a hierarchical system with the aim of achieving certain design specifications has been a focal problem in the past several years and there are some concrete methods to do so. In contrast, there are only a few fledgling controller design techniques for general large-scale systems, which are not satisfactorily efficient.

Although many systems possess a hierarchical structure either originally or after discretization, there exist numerous non-hierarchical systems which tend to be hierarchical. More precisely, such systems either have a few weak interconnections between their subsystems whose removal will make the system maintain a hierarchical structure, or their discrete-time models have this characteristic [12]. This type of systems will be referred to as pseudo-hierarchical systems throughout this work. Given a pseudo-hierarchical large-scale system, decentralized controller design can be performed for its hierarchical model, as opposed to itself, by exploiting the available techniques. Even though this straightforward approach is appealing, the decentralized controller designed may not have the required properties for the pseudo-hierarchical system. As one possible scenario, the controller may even destabilize the original system, whereas it definitely stabilizes its hierarchical model. Apart from this stabilizability issue, which rarely occurs if the removed interconnections are weak enough, the performance of the pseudo-hierarchical system under this controller can be undesirably different from what expected to be. In this regard, it is desired to undergo a performance analysis for the system in order to make certain that this indirect design technique is suitable for the given pseudo-hierarchical large-scale system.

This work deals with the above-mentioned problem. To this end, it is assumed that a decentralized controller has been designed for the hierarchical model of a pseudo-hierarchical system to attain specific control objectives. Moreover, it is supposed that this controller stabilizes the pseudo-hierarchical system, while it may deteriorate the required performance. A LQ cost function is appropriately defined to assess the discrepancy between the pseudo-hierarchical system and its hierarchical model under this decentralized controller. The smaller this performance index is, the closer to each other the two closed-loop system are. Obtaining this cost function involves solving a discrete Lyapunov equation; as a result of the large-scale nature of the system, this equation by no means can be handled. This gives rise to attaining an upper bound on this cost function, as an alternative goal. For this purpose, a novel technique is proposed and it is subsequently shown that an optimization problem with solely three variables needs to be solved in order to compute this bound. The main distinguishing feature of this work is to present a simple optimization problem. To elucidate that the obtained bound is not unnecessarily conservative, it is proved that as the pseudo-hierarchical system approaches its hierarchical model, this bound goes to zero; and in the ideal case, the bound is equal to zero.

5.3 Preliminaries and problem formulation

Consider a large-scale interconnected system $\mathscr{S}$ consisting of v subsystems, where its i^{th} subsystem S_i is represented as:

$$
\begin{aligned}
x_i[k+1] &= \sum_{j=1}^{v} A_{ij}x_j[k] + B_i u_i[k] \\
y_i[k] &= C_i x_i[k], \quad i \in \bar{v} = \{1, 2, \cdots, v\}
\end{aligned}
\tag{5.1}
$$

In the above equation, $x_i[k] \in \mathbb{R}^{n_j}$, $u_i[k] \in \mathbb{R}^{m_i}$ and $y_i[k] \in \mathbb{R}^{r_i}$ stand for the state, the input and the output of S_i, respectively. Sketch now a digraph $\mathscr{G}$ associated with the system $\mathscr{S}$ as follows:

- Put v vertices corresponding to different subsystems of the system $\mathscr{S}$.
- For any $i, j \in \bar{v}$, $i \neq j$, connect vertex i to vertex j with a directed edge if $A_{ij} \neq 0$.
- For any $i, j \in \bar{v}$, if there is an edge between vertex i and vertex j, attribute the weight $\|A_{ij}\|_F$ to that edge, where $\|\cdot\|_F$ represents the Frobenius norm operator.

The graph $\mathscr{G}$ specifies the topology of information transmission between the subsystems. From this perspective, it plays an important role in the stability and stabilizability analyses of the system. Whenever this graph has no directed cycles, the system $\mathscr{S}$ is said to be hierarchal. For any $i \in \bar{v}$, define the isolated subsystem $\bar{S}_i$ as below:

$$
\begin{aligned}
\bar{x}_i[k+1] &= A_{ii}\bar{x}_i[k] + B_i \bar{u}_i[k] \\
\bar{y}_i[k] &= C_i \bar{x}_i[k]
\end{aligned}
\tag{5.2}
$$

This work is concerned with the decentralized control of the system $\mathscr{S}$. For the case when the graph $\mathscr{G}$ is acyclic, designing a stabilizing decentralized controller is tantamount to designing v local controllers separately such that the i^{th} local controller stabilizes the isolated subsystem $\bar{S}_i$, for all $i \in \bar{v}$. This simple, nonetheless noticeable fact implies that when the graph $\mathscr{G}$ is acyclic, the decentralized controller design can be carried out very straightforwardly, compared to the general case. It is to be noted that as discussed earlier, several methods are proposed in the literature to design a LTI decentralized controller for a hierarchical system in order to achieve some pre-defined objectives.

The question arises what happens if the graph $\mathscr{G}$ is not acyclic. In this case, some of the edges of the graphs $\mathscr{G}$ should be removed in order to arrive at an acyclic graph. However, this makes sense once the weights of the removed edges are not comparable with those of the remaining ones as well as the values $\|A_{11}\|_F, \|A_{22}\|_F, \ldots, \|A_{\nu\nu}\|_F$. It is stated in [13] that there exist numerous systems for which all the edges of the graph $\mathscr{G}$ can be removed. Nevertheless, in order convert the graph $\mathscr{G}$ to an acyclic one, only a number of the edges are required to be removed (which may be performed in a non-unique way).

Assume now that some of the edges are removed and for the resultant hierarchical system, a LTI decentralized controller K is designed using the available approaches. This controller is to be applied to the original system $\mathscr{S}$. With no doubt, the closed-loop system $\mathscr{S}$ might be unstable, or if not, its corresponding performance may be poor. Therefore, it is desired in this work to evaluate the performance of the hierarchical system under its associated controller K with respect to its counterpart (i.e. the original closed-loop system). To this end, it is assumed that this closed-loop system is stable, as the more important issue of performance degradation is central to this work. It is worth mentioning that this closed-loop stability can be guaranteed using the available methods when the removed interconnections are weak enough.

In the sequel, represent the hierarchical system under the designed LTI decentralized controller K as:

$$x_h[k+1] = A_h x_h[k] \tag{5.3}$$

and the original system $\mathscr{S}$ under the same controller as:

$$x_c[k+1] = A_c x_c[k] \tag{5.4}$$

Note that $x_c[0] = x_h[0]$. With no loss of generality, the matrix A_h can be assumed to be lower-block diagonal. In order assess the closeness of the systems given in (5.3) and (5.4), it is to be measured how close the states $x_h[k]$ and $x_c[k]$ are. This can be evaluated through the following performance index:

$$J_d = \sum_{k=0}^{\infty} (x_c[k] - x_h[k])^T (x_c[k] - x_h[k]) \tag{5.5}$$

Definition 1: Define the performance indices J_c and J_h as below:

$$J_c = \sum_{k=0}^{\infty} x_c[k]^T Q x_c[k], \quad J_h = \sum_{k=0}^{\infty} x_h[k]^T Q x_h[k] \tag{5.6}$$

Definition 2: The controller K is said to be μ suboptimal, if the inequality $\frac{J_d}{J_h} < \mu$ holds.

Some works, e.g. [13], define the suboptimality on the ratio $\frac{J_c}{J_h}$, as opposed to $\frac{J_d}{J_h}$. However, it is manifest that the smallness of $\frac{J_c}{J_h}$ does not necessarily prove the closeness of $x_c[k]$ and $x_h[k]$. The objective of this work is to obtain a proper and easy-to-compute μ by which the controller K is suboptimal. Nevertheless, the following practical restrictions are also made.

Assumption 1: A discrete Lyapunov equation with the order of any subsystem can be obtained, whereas a discrete Lyapunov equation with the order of the system, i.e. ν, cannot be obtained due to the large-scale nature of the system.

Assumption 2: Although ν is so large that a Lyapunov equation of order ν cannot be computed, lower and upper bounds on the eigenvalues of a matrix of order ν can be obtained.

Regarding Assumption 2, it is quite important to note that solving a Lyapunov equation of order ν is much more difficult than estimating the eigenvalues of a matrix of order ν, as the former one is involved in ν^2 variables but the latter one in $\nu + 1$ (regardless of their linearity or bilinearity). In order to obtain the main results of this work, one more assumption is required to be made. This will be explained next.

It is evident that J_h and J_c satisfy the relations:

$$J_h = x_h[0]^T P_h x_h[0], \quad J_c = x_c[0]^T P_c x_c[0] \tag{5.7}$$

where:

$$A_h^T P_h A_h - P_h + I = 0, \quad A_c^T P_c A_c - P_c + I = 0 \tag{5.8}$$

Assumption 3: The closed-loop system given in (5.4) is stable with the Lyapunov function P_h.

It is to be noted that Assumption 3 is more restrictive than only the stability of the system (5.4), and is met when the removed edges has no noticeable weights. There are several sufficient conditions in the literature, each of which assures the validity of this assumption.

5.4 Main results

In what follows, the performance deviation J_d will be formulated.

Lemma 1 *The performance index J_d can be written as*

$$J_d = \begin{bmatrix} x_h[0]^T & x_c[0]^T \end{bmatrix} P_d \begin{bmatrix} x_h[0] \\ x_c[0] \end{bmatrix} \tag{5.9}$$

where:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} P_d \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T - P_d + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} = 0 \tag{5.10}$$

Proof: Augmenting the closed-loop systems (5.3) and (5.4) results in:

$$\begin{bmatrix} x_h[k+1] \\ x_c[k+1] \end{bmatrix} = \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_c[k] \end{bmatrix} \tag{5.11}$$

On the other hand, the performance index J_d can be rewritten as:

$$J_d = \sum_{k=0}^{\infty} \left[x_h[k]^T \; x_c[k]^T \right] \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_c[k] \end{bmatrix} \tag{5.12}$$

It is well-known that the performance index J_d, which also corresponds to the system , can be written as (5.9) where its component P_d satisfies the equation (5.10). This completes the proof. ■

Due to Assumption 1, the performance deviation P_d cannot be directly computed from Lemma 1 in order to compute the ratio $\frac{J_d}{J_h}$ precisely. Hence, the notion of μ optimality is helpful here in order to obtain a reasonable upper bound on this ratio. This will be carried out in the sequel.

Lemma 2 *Given a matrix H of proper dimension, assume the following inequality is satisfied:*

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T H \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - H + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} < 0 \tag{5.13}$$

Then, the inequality given below holds:

$$J_d < \begin{bmatrix} x_h[0]^T & x_c[0]^T \end{bmatrix} P_d \begin{bmatrix} x_h[0] \\ x_c[0] \end{bmatrix} \tag{5.14}$$

Proof: It can be concluded from the relations (5.10) and (5.13) that:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T (H - P_d) \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - (H - P_d) < 0 \tag{5.15}$$

Since both of the matrices A_c and A_h are assumed to be Schur, it results from the above inequality that $P_d < H$. The proof follows immediately from this result on noting the equation (5.9). ■

Theorem 1 *Consider an optimization problem in three scalar variables k_1, k_2, k_3 which aims to compute the infimum of the objective function $k_1 + 2k_2 + k_3$ subject to the matrix inequalities $k_1 > 1$ and:*

$$(k_1 - 1)(k_3 R_2 + I) + (k_2 R_1 - I)\left(k_2 R_1^T - I\right) < 0 \tag{5.16}$$

where:

$$R_1 = A_c^T P_h A_h - P_h, \quad R_2 = A_c^T P_h A_c - P_h \tag{5.17}$$

Denote the solution of this optimization problem with μ. The controller K is μ suboptimal.

Proof: Consider any k_1, k_2 and k_3 satisfying the inequalities given in Theorem 1. It can be straightforwardly shown by utilizing the Schur complement formula that:

$$\begin{bmatrix} (k_3 R_2 + I) & (k_2 R_1 - I) \\ \left(k_2 R_1^T - I\right) & (1 - k_1) I \end{bmatrix} < 0 \tag{5.18}$$

Obviously, the above inequality can be rearranged as:

$$\begin{bmatrix} (1 - k_1) I & \left(k_2 R_1^T - I\right) \\ (k_2 R_1 - I) & (k_3 R_2 + I) \end{bmatrix} < 0 \tag{5.19}$$

Combining the relations (5.8), (5.17) and (5.19) will lead to:

$$\begin{bmatrix} k_1(A_h^T P_h A_h - P_h) + I & k_2(A_h^T P_h A_c - P_h) - I \\ k_2(A_c^T P_h A_h - P_h) - I & k_3(A_c^T P_h A_c - P_h) + I \end{bmatrix} < 0 \tag{5.20}$$

The above inequality can be rewritten as:

$$\begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix}^T H \begin{bmatrix} A_h & 0 \\ 0 & A_c \end{bmatrix} - H + \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} < 0 \tag{5.21}$$

where:

$$H = \begin{bmatrix} k_1 P_h & k_2 P_h \\ k_2 P_h & k_3 P_h \end{bmatrix} \tag{5.22}$$

Therefore, it can be inferred from Lemma 2 that:

$$
\begin{aligned}
J_d &< \left[\begin{array}{cc} x_h[0]^T & x_c[0]^T \end{array} \right] H \left[\begin{array}{c} x_h[0] \\ x_c[0] \end{array} \right] \\
&= (k_1 + 2k_2 + k_3) x_h[0]^T P_c x_h[0] \\
&= (k_1 + 2k_2 + k_3) J_h
\end{aligned} \tag{5.23}
$$

(note that $x_h[0] = x_c[0]$). Thus,

$$
\frac{J_d}{J_h} < k_1 + 2k_2 + k_3 \tag{5.24}
$$

This inequality illustrates why the objective function $k_1 + 2k_2 + k_3$ is to be minimized, and completes the proof. ■

Theorem 1 proposes a simple optimization problem associated with only three scalar variables which is able to obtain an upper bound on the ratio $\frac{J_d}{J_h}$. In this regard, it is interesting to note that the inequality constraints of this optimization problem are always feasible. To prove this, it suffices to consider $k_1 = 2,\ k_2 = 0$ and k_3 as a very large number; since it is assumed in Assumption 3 that the Lyapunov function P_h detects the stability of the system (5.3), the matrix R_2 is negative define, which causes the inequality (5.16) to hold.

Due to the large-scale nature of the system $\mathscr{S}$ and Assumption 1, it may turn out that the matrix inequality (5.16) not to be handleable. Thus, it is preferred to convert the matrix inequality (5.27) into a scalar one. This goal is achieved by means of the following theorem.

Theorem 2 *Denote with μ the infimum of the objective function $k_1 + 2k_2 + k_3$ subject to the matrix inequalities $k_1 > 1$ and:*

$$
(k_1 - 1)(-1 + k_3 m_1) - 1 - k_2^2 m_2 - k_2 m_3 > 0 \tag{5.25}
$$

where,

$$m_1 = \underline{\lambda}(-R_2) \tag{5.26a}$$

$$m_2 = \bar{\lambda}\left(R_1 R_1^T\right) \tag{5.26b}$$

$$m_3 = \bar{\lambda}\left(-R_1 - R_1^T\right) \tag{5.26c}$$

(the operators $\bar{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ represent the maximum and minimum eigenvalues of a matrix). The controller K is μ suboptimal.

Proof: It is easy to verify that the matrix inequality (5.16) is guaranteed to hold, provided the scalar inequality given below is satisfied:

$$\underline{\lambda}\left((k_1 - 1)(-k_3 R_2 - I)\right) > \bar{\lambda}\left((I - k_2 R_1)\left(I - k_2 R_1^T\right)\right) \tag{5.27}$$

Consider now arbitrary scalar k_1, k_2, k_3 satisfying the inequality (5.25). It can be deuced from the above discussion and Theorem 1 that substantiating the validity of the inequality (5.27) proves this theorem. To show this, one can write:

$$\underline{\lambda}\left((k_1 - 1)(-k_3 R_2 - I)\right) = (k_1 - 1)(-1 + k_3 m_1) \tag{5.28}$$

Moreover, it results from Lemma 2.1 in [14] that:

$$\begin{aligned}
&\bar{\lambda}\left((I - k_2 R_1)\left(I - k_2 R_1^T\right)\right) = 1 \\
&+ \bar{\lambda}\left(k_2\left(-R_1 - R_1^T\right) + k_2^2 R_1 R_1^T\right) \\
&\leq 1 + k_2 \bar{\lambda}\left(-R_1 - R_1^T\right) + k_2^2 \bar{\lambda}\left(R_1 R_1^T\right) \\
&\leq 1 + k_2 m_3 + k_2^2 m_2
\end{aligned} \tag{5.29}$$

The inequalities (5.25), (5.28) and (5.29) all together lead to the relation (5.27). ■

Remark 1 *As before, it can be simply shown that the inequalities given in Theorem 2 are always feasible (by considering $k_1 = 2$, $k_2 = 0$ and k_3 as a large enough number). It is to be noted that m_1 is positive in light of Assumption 3.*

To solve the optimization problem given in either Theorem 1 or Theorem 2, an important quantity is to be obtained first. More specifically, the Lyapunov function P_h is an essential ingredient of these optimizations. As a consequence of Assumption 1, this matrix cannot be computed using the common methods. However, since its associated matrix A_h is assumed to be lower-block diagonal, it can be found by solving several successive Lyapopuv and Sylvester equations of subsystems' orders (as opposed to the system's order). To clarify this issue, assume that there are only two subsystems (i.e. $\nu = 2$). In this case, the equation $A_h^T P_h A_h - P_h + I = 0$ can be equivalently decomposed as:

$$A_{22}^T P_3 A_{22} - P_3 + I = 0 \tag{5.30a}$$

$$A_{11}^T P_2 A_{22} + A_{21}^T P_3 A_{22} - P_2 = 0 \tag{5.30b}$$

$$\begin{aligned} &A_{11}^T P_1 A_{11} + A_{11}^T P_2 A_{21} + A_{21}^T P_2 A_{11} \\ &+ A_{21}^T P_3 A_{21} - P_1 + I = 0 \end{aligned} \tag{5.30c}$$

where:

$$P_h = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \tag{5.31}$$

Since the hierarchical closed-loop system is stable, the matrices A_{11} and A_{22} are both Schur. At this point, the Lyapunov equation (5.30a), which is of subsystem's order, can be solved to find the matrix P_3. Substituting the result in the equation (5.30b) will arrive at a Sylvester equation, which has a unique solution P_2 (because the eigenvalues of A_{11} and A_{22} are all inside the unit circle). At last, the Lyapunov equation (5.30c) can be solved for the matrix variable P_1. This illustrates how Assumption 1 can be bypassed here. It is worth mentioning that in the general case, the corresponding Lyapunov and Sylvester equations can be systematically obtained, which resemble the equations given in (5.30).

The question arises whether the μ obtained in Theorems 1 or 2 is very conservative or not. To answer this, an elegent result on the tightness of this bound will be presented next.

Theorem 3 *The closer to each other the matrices A_h and A_c are, the smaller the μ obtained from Theorems 1 or 2 is. In addition, whenever A_h and A_c are identical, both of these theorems arrive at the same solution $\mu = 0$.*

Sketch of proof: The proof will be provided here by focusing on Theorem 2 and for the case when $A_h = A_c$. For this purpose, one may notice that $R_1 = R_2 = -I$, and consequently, $m_1 = m_2 = 1$ and $m_3 = 2$. Now, Theorem 2 states (after some simplifications) that μ is equal to the infimum of $k_1 + 2k_2 + k_3$ under the inequality constraints $k_1 > 1$ and:

$$k_1 k_3 - k_1 - k_3 - 2k_2 - k_2^2 > 0 \tag{5.32}$$

The later inequality is equivalent to:

$$(k_1 - 1)(k_3 - 1) \geq (k_2 + 1)^2 \tag{5.33}$$

Hence, μ is equal to 0, and is attained once $k_1 = k_3 \to 1^+$ and $k_2 = -1$. This completes this part of the proof. The remaining ones can be proved in a similar line. ■

Remark 2 *The results obtained in this work can be analogously developed to tackle the following problem:*

Assume that the system $\mathscr{S}$ is under perturbation. Design a LTI decentralized controller for the nominal model of the system. Now, the matrices A_h and A_c correspond to the closed-loop nominal and perturbed A-matrices, respectively. In this case, the ratio $\frac{J_d}{J_c}$ describes the closeness of the nominal closed-loop system and its perturbed counterpart. This ration is again desired to be obtained.

5.5 Numerical example

Consider an interconnected system $\mathscr{S}$ with nine SISO subsystems of order 1, and assume that the interconnections from subsystem i to subsystem j, $\forall i, j \in \{1, 2, ..., 9\}$, $i < j$, are weaker than the remaining ones. Hence, to design a decentralized controller for the system with nine local controllers, one can eliminate these weak interconnections and accomplish the design procedure for the obtained hierarchical model. Assume now that a static decentralized controller has been designed for the hierarchical model using this approach. To perform the performance analysis for the pseudo-hierarchical system under the designed controller, two different choices will be considered for the close-loop matrix A_{nh} in the sequel.

Consider first the matrix A_{nh} as follows:

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 1 & 0.3 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0.3 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0.3 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & .04 & 0.5 & 0.6 & 0 & 0.5 & 1 & 1 \\ 0.01 & 0 & 0 & 0.1 & 0.1 & 0 & 0.5 & 1 & 2 \\ 0.4 & 0.9 & 0.04 & 0.03 & 0 & 0.3 & 0.05 & 1 & 0 \end{bmatrix}$$

It can be observed that the lower diagonal entries of this matrix have smaller values than the upper diagonal ones in general, in light of the existing weak interconnections. The hierarchical form of this matrix, denoted by A_h in this work, will be obtained as:

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 0 & 0 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In can be concluded in this example that J_d and J_h are equal to 67.9336 and 38.6145, respectively, which lead to the relation $\frac{J_d}{J_h} = 1.7593$. On the other hand, an upper bound on $\frac{J_d}{J_h}$ can be attained from Theorem 1 by solving an optimization problem arriving at:

$$k_1 = 2.131, \;\; k_2 = -1.7529, \;\; k_3 = 5.7894$$

Due to the relation $\mu = \min(k1 + 2k_2 + k_3)$, the upper bound μ on the ratio $\frac{J_d}{J_h}$ is equal to 4.4145, whereas its exact value has been computed to be 1.7593. Note that although the obtained upper bound is at least twice greater than the exact value, it has been attained through a quite simple procedure, which is helpful for large-scale systems. It is noteworthy that these values point to the fact that the weak interconnections are not so weak that their presence can be ignored. This originates from the existence of some large lower diagonal entries, such as $\frac{0.9}{4.35}$, which are comparable and greater than some of the upper diagonal entries.

Since Theorem 1 proposes an optimization with matrix variables, its handling may be formidable for certain large-scale problems. Thus, let the scalar optimization provided in Theorem 2 be utilized here to obtain an upper bound μ. In this case, the variables k_1, k_2

and k_3 will be obtained to be equal to 1.5206, -0.9144 and 5.2813, respectively, which correspond to the upper bound limit $\mu = 4.973$. Even though the matrix optimization in Theorem 1 seems to be oversimplified in Theorem 2, the reasonable closeness between 4.4145 and 4.973 indicates that Theorem 2 presents a very simple scalar optimization problem without making the result noticeably conservative.

Consider now another scenario for the value of the matrix A_{nh} as follows:

$$\frac{1}{4.35}\begin{bmatrix} 1 & 0.5 & 2 & 0.1 & 0.5 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 1 & 1.5 & 0.5 & 1 & 0 & 1 & 0.2 & 0.25 \\ 0 & 0 & 1 & 0 & 0.2 & 1 & 0.2 & 0.5 & 0.31 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0.05 & 0.1 & 0.01 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0.2 \\ 0.05 & 0 & 0 & 0 & 0 & 1 & 0.8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0.001 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, the previously weak interconnections have further been weakened in order for the pseudo-hierarchical system to approach its hierarchical model. Notice that the hierarchical model of this system is still given by the matrix A_h presented for the previous scenario. It can be observed that the quantities J_d and J_h are equal to 0.0117 and 38.6145, respectively. The smallness of J_d manifestly confirms the closeness of hierarchical and pseudo-hierarchical models to each other. The upper bound limit μ will be obtained as 0.0041 and 0.0671, using Theorems 1 and 2, respectively. These results are in accordance with the statement of the Theorem 3.

5.6 Conclusions

This work tackles the performance analysis for large-scale systems with pseudo-hierarchal structures. It is assumed that a stabilizing decentralized controller has been designed for such a system, based on its hierarchical model, to achieve some control objectives. Since this indirect design technique may not result in a good performance for the real pseudo-hierarchical system, this work aims to investigate the closeness of the system and its hierarchical model under the designed controller. For this purpose, a LQ cost function is properly defined to measure the discrepancy between the closed-loop nominal system and its hierarchical counterpart. Since computing the exact value of this cost function is involved with solving a large-scale Lyapunov equation, it is desired instead to obtain an upper bound on it. To this end, a simple optimization problem with only three variables is proposed here to attain this upper bound. In addition, it is shown that the closer the pseudo-hierarchical system to its hierarchical model is, the smaller this bound is; and in the case when the models are identical, the bound is equal to zero. This demonstrates that the bound obtained through a simple optimization problem is not unreasonably conservative. The ideas developed here are illustrated in a numerical example in the end.

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