SOME NEW RESULTS ON NONLINEAR FILTERING WITH POINT PROCESS OBSERVATIONS

Shu Zhang

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Abstract

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The problem of stochastic filtering is concerned with estimating a signal based upon the partial and noisy observations of the signal. The nonlinear filtering theory has been applied in variety of fields including target detection and tracking, communication networks, mathematical finance, medical sciences, etc. In this thesis, we present some new results on nonlinear filtering with point process observations. These results are motivated by some problems from mathematical finance (cf. Zeng (2003)) and are based upon the novel techniques developed recently by Hu, Ma and Sun (2007).

First, we rigorously derive the filtering equations with point process observations under conditions which are weaker than the usual assumptions. Then, we investigate the uniqueness of solutions to the filtering equations, in particular, we obtain the Poisson expansions for the unnormalized optimal filters. Finally, we introduce a recursive numerical method to approximate the unnormalized optimal filters.

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Contents

Chapter 1

Introduction

In early studies, Brownian motion was one of the most popular continuous-time stochastic processes. This model has many real world applications such as medelling stock market fluctuations and describing evolution of physical characteristics in the fossil record, etc. However, sometimes Brownian motion is just used for the sake of convenience rather than accuracy. In recent studies, people are becoming more and more interested in discrete-valued stochastic processes, e.g. the point processes. The point processes have lots of applications. For example, we can use point processes to model asset prices, stock prices, exchange rates and commodity prices, etc.

In this thesis, we consider the doubly stochastic Poisson process, which is an important class of point processes. Let $(N(t))_{t\geq 0} = (N_1(t), N_2(t), \ldots, N_n(t))_{t\geq 0}$ be n independent standard Poisson processes. We consider

$$Y(t) = \begin{pmatrix} Y_1(t) \\ \vdots \\ Y_n(t) \end{pmatrix} = \begin{pmatrix} N_1(\int_0^t \lambda_1(X(s))ds) \\ \vdots \\ N_n(\int_0^t \lambda_n(X(s))ds) \end{pmatrix}.$$

Herein, $X(t)$ can be used to model the evolution of the intrinsic values of some stock, while $Y(t)$ denotes the cumulative numbers of trades occurred at n price levels up to time t. This is a partially observed model, which has been adopted in many fields of applications, such as economics, engineering, informatics, etc.

Note that although we can construct the Poisson process model for stock trading prices, and the complete information of the prices and trading times can be captured through the observations, it does not mean that the intrinsic values of the stock are known easily. To well distinguish the stock intrinsic values from stock trading prices, we refer the readers to the Macromovement and Micromovement models introduced in [Zeng (2003)]. The macromovement model refers to the closing price behavior while the micromovement model refers to the transactional price behavior. It is also pointed out in [Zeng (2003)] that noise would create the major distinction between these mac- and micromovement models, and they could not be ignored when we estimate the intrinsic values from stock trading prices, especially in the high-frequency data. Discrete noise, clustering noise and nonclustering noise are the three main types of noise, which normally comes from either noised trading or the trading mechanism. Therefore, in the mathematical finance studies, it is more meaningful for the investors to investigate a stock's intrinsic values rather than its trading prices.

Since the stock intrinsic values are not observable, we need to utilize the information of the stock prices and trading times to estimate them. In this thesis, we will apply the theory of nonlinear filtering to this problem. In particular, we will apply the novel techniques recently developed by Hu, Ma and Sun (cf. Hu et al. (2007)). Note that although Poisson process and Brownian motion are qualitatively different, they present striking similarities when the martingale point of view is adopted. The martingale theory and the Ito's differential equations can be parallelly generalized for the Poisson process system. In fact, it has been found that the role of the Stieltjes integration in the Poisson process theory is similar to that of the Ito's stochastic integration in the Wiener system (cf. Protter (1990)). Besides, almost all results for Brownian motion can be found the counterparts in the setting of Poisson process. Therefore, it is quite worthy to review the Wiener system and its corresponding filtering theory, even though it is not the central topic of this thesis.

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with history $(\mathcal{F}_t)_{t\geq 0}$. A $(P, \mathcal{F}_t)$-Wiener process is a continuous process $(B_t)_{t\geq 0}$ such that B_t is adapted to $\mathcal{F}_t$ for all $t \geq 0$ and, for all $0 \leq s \leq t$,

$$B_t - B_s \text{ is } P-\text{independent of } \mathcal{F}_s,$$

$$B_t - B_s \text{ is Gaussian } (0, t-s).$$

If $(B_t)_{t\geq 0}$ is a $(P, \mathcal{F}_t)$-Wiener process, it is easy to verify that B_t and $B_t^2 - t$ are $(P, \mathcal{F}_t)$-martingales. By virtue of Brownian motion, we can consider the stochastic differential equation:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

equivalently, the stochastic integral equation:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

Now, we are ready to state the filtering problem with respect to Wiener process. We refer the readers to [Xiong (2008), pp.1-3] for related examples such as the wireless communication and the environment protection, etc. A filtering model consists of two parts: one is the signal process which we will estimate, and the other one is the observation process which provides the information about the signal.

The signal process X_t is normally a d-dimensional process which is governed by the stochastic differential equation:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t,$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^n$ are continuous maps, and B is an $n-$dimensional Brownian motion. The observation process Y_t is an m-dimensional process governed by the stochastic equation:

$$Y_t = \int_0^t h(X_s)ds + W_t,$$

where $h : \mathbb{R}^d \to \mathbb{R}^m$ is a continuous map and W is an m-dimensional Bronian motion that is independent of B.

Denote $\mathcal{F}_t^Y := \sigma(\{Y_s, 0 \leq s \leq t\}, \mathcal{N})$, where $\mathcal{N}$ is the collection of P-null sets. Then the goal of filtering is to estimate the conditional distributions

$$\pi_t(\cdot) := P(X_t \in \cdot | \mathcal{F}_t^Y), \quad t \geq 0.$$

We define

$$L_t = \exp\left\{\int_0^t \langle h(X_s), dY_s\rangle - \frac{1}{2}\int_0^t |h(X_s)|^2 ds\right\}$$

and

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = L_t.$$

Under reasonable conditions, it can be shown that Q defines a probability measure. Denote by $\hat{E}$ the expectation with respect to Q and define $\sigma_t(f) = \hat{E}[f(X_t)L_t|\mathcal{F}_t^Y]$. Then, we have

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)}.$$

π_t is called the optimal filter and σ_t is called the unnormalized optimal filter.

It can be shown that π_t satisfies the Kushner-FKK (Fujisaki-Kallianpur-Kunita) equation (cf. Fujisaki et al. (1972)) and σ_t satisfies the Zakai equation (cf. Zakai (1969)):

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(Lf)ds + \int_0^t (\pi_s(hf) - \pi_s(h)\pi_s(f))dI_s,$$

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Lf)ds + \int_0^t \sigma_s(hf)dY_s,$$

where L is the generator of X, f is any bounded function belonging to the domain of L, and $I_s = Y_s - \int_0^s \pi_u(f)du$ is the innovation process. There are many different ways to prove the uniqueness of solutions to the above filtering equations (cf. e.g., Kurtz and Ocone (1988), Bhatt et al. (1995) and Kurtz (1998)). In Hu, Sun and Ma (2007), the authors use an interesting Wiener chaos expansion method to establish the uniqueness of solutions of the filtering equations under very weak conditions on the observation function h.

In this thesis, we first derive the Zakai equation with respect to Poisson process in Chapter 2. To this end, we introduce the setting of the thesis at the beginning. Then we derive two equivalent forms of the Zakai equation by replacing Y_t in the Wiener system with $Y_t - t$ in the Poisson system. In Chapter 3, we use the Poisson expansion, which is parallel to the Wiener chaos expansion, to prove the uniqueness of solutions to the Zakai equation. For simplicity, we consider one dimensional case (one stock) only, however, all the results hold for the multi-dimensional case (multi-stocks). In Chapter 4, the objective is to compute the unnormalized optimal filters. Under certain assumptions, we drive a recursive expansion for the unnormalized filter density. Moreover, we introduce a numerical algorithm with bounded error. For the future work, it worthy to do some simulation for the proposed model. It will show whether the stock value process X_t can be well estimated from the stock trading price Y_t.

Chapter 2

Filtering Equations

2.1 Model and Reference Measure

Throughout this chapter, we assume that E is a complete separable metric space with Borel σ-algebra $\mathcal{B}(E)$. Let $(X(t))_{t\geq 0}$ be a càdlàg time-homogeneous Markov process taking values in E and living on a complete probability space $(\Omega, \mathcal{F}, P)$. $(X(t))_{t\geq 0}$ is called the signal process, which cannot be observed directly. Denote by $B_b(E)$ the family of bounded Borel measurable functions on E and denote by $P(t, x, \Gamma)$ $(t \geq 0, x \in E, \Gamma \in \mathcal{B}(E))$ the transition function for X. Then, we have the transition semigroup $(T_t)_{t\geq 0}$ defined by $T_t f(x) = \int_E f(y) P(t, x, dy)$ for $f \in B_b(E)$. Set $\mathcal{C} := \{f \in B_b(E) : \lim_{t\downarrow 0} T_t f(x) = f(x), \forall x \in E\}$. Then $\mathcal{C} \supset C_b(E)$, the family of bounded continuous functions on E. Define

$$\mathcal{D} = \left\{ f \in \mathcal{C} : \exists\ Lf \in \mathcal{C}\ s.t.\ \ T_t f(x) = f(x) + \int_0^t (T_s Lf)(x) ds, \forall x \in E \right\}.$$

L is called the weak generator of X and $\mathcal{D}$ is its domain (cf. Kouritzin and Long (2003)). Note that $\mathcal{D}$ is bounded pointwise dense in $B_b(E)$ and hence measure determining.

Let $(N(t))_{t\geq 0} = (N_1(t), N_2(t), \ldots, N_n(t))_{t\geq 0}$ be n independent standard Poisson processes on $(\Omega, \mathcal{F}, P)$, which are assumed to be independent of $(X(t))_{t\geq 0}$. Suppose that for $1 \leq i \leq n$, $\lambda_i : E \to (0, \infty)$ is Borel measurable and

$$\int_0^t \lambda_i(X(s))ds < \infty, \quad P - a.s., \forall t > 0. \tag{2.1}$$

Let the observation process $(Y(t))_{t\geq 0}$ be a doubly stochastic Poisson process as follows:

$$Y(t) = \begin{pmatrix} Y_1(t) \\ \vdots \\ Y_n(t) \end{pmatrix} = \begin{pmatrix} N_1(\int_0^t \lambda_1(X(s))ds) \\ \vdots \\ N_n(\int_0^t \lambda_n(X(s))ds) \end{pmatrix}.$$

Denote $\mathcal{F}_t^Y := \sigma(\{Y(s), 0 \leq s \leq t\}, \mathcal{N})$, $\mathcal{F}_\infty^X := \sigma(\{X(s), 0 \leq s < \infty\}, \mathcal{N})$ where $\mathcal{N}$ is the collection of P-null sets, and $\mathcal{F}_t := \mathcal{F}_t^Y \vee \mathcal{F}_\infty^X$. Then, we have

$$E[e^{iu^T(Y(t)-Y(s))}|\mathcal{F}_s] = e^{\sum_{j=1}^n (e^{iu_j}-1)\int_s^t \lambda_j(X(v))dv}, \quad \forall u \in \mathbb{R}^n.$$

Note that $X, N_1, N_2, \ldots, N_n$ are independent under P. However, this does not imply that $X, Y_1, Y_2, \ldots, Y_n$ are independent under P. In the following, we will show that there exists a probability measure Q on $(\Omega, \mathcal{F})$ such that $X, Y_1, Y_2, \ldots, Y_n$ are independent and $Y_1, Y_2, \ldots, Y_n$ are standard Poisson processes.

Lemma 2.1.1 For $1 \leq i \leq n$, let $(T_k(i))_{k=1}^\infty$ be the jump times of $(Y_i(t))_{t\geq 0}$. Define the process $(\hat{L}(t))_{t\geq 0}$ by

$$\hat{L}(t) := \prod_{i=1}^n \hat{L}_i(t),$$

where

$$\hat{L}_i(t) := \prod_{k\geq 1} \mu_i(T_k(i))1_{\{T_k(i)\leq t\}} \exp\left\{\int_0^t (1-\mu_i(s))\lambda_i(X(s))ds\right\}$$

and

$$\mu_i(t) = \frac{1}{\lambda_i(X(t))}.$$

Then $(\hat{L}(t))_{t\geq 0}$ is a $(P, \mathcal{F}_t)$-martingale.

Proof. By [Brémaud (1981), pp.165-166], $(\hat{L}(t))_{t\geq 0}$ is a nonnegative $(P, \mathcal{F}_t)$-local martingale and hence a $(P, \mathcal{F}_t)$-supermartingale. It suffices to show that $E[\hat{L}(t)] = 1$ for $t \geq 0$. Let P^X and P^N be the marginal probabilities with respect to X and N, respectively. Note that $\hat{L}$ is pathwise defined. We need to show that $E^N[\hat{L}(t)] = 1$, P^X-a.s., for $t \geq 0$. Since

$$\hat{L}(t) = 1 + \sum_{i=1}^{n} \int_0^t \hat{L}(s-)(\mu_i(s) - 1)d\left(Y_i(s) - \int_0^s \lambda_i(X(u))du\right),$$

$E^N[\hat{L}(t)] \leq 1$, P^X-a.s. By assumption (2.1), we get

$$\int_0^t E^N[\hat{L}(t)] \cdot |(\mu_i(s) - 1)\lambda_i(X(s))|ds < \infty, \quad P^X - a.s.$$

Then $\hat{L}$ is a $(P^N, \mathcal{F}_t)$-martingale, P^X-a.s. and hence $E^N[\hat{L}(t)] = 1$, P^X-a.s., for $t \geq 0$. Therefore, $E[\hat{L}(t)] = 1$ for $t \geq 0$. The proof is complete.

Now we can define $\frac{dQ}{dP}|_{\mathcal{F}_t} = \hat{L}(t)$ for $t \geq 0$. Then Q can be extended to be a probability measure on $(\Omega, \mathcal{F})$ and Y is a standard Poisson process under Q. Define $L(t) = 1/\hat{L}(t)$ and denote by E^Q the expectation with respect to Q. Then $\frac{dP}{dQ}|_{\mathcal{F}_t} = L(t)$ and $E^Q[L(t)] = 1$ for $t \geq 0$. For $f \in B_b(E)$, we define

$$\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y].$$

$(\pi_t)_{t\geq 0}$ is called the *optimal filter.* By Bayes' formula, we get

$$\pi_t(f) = \frac{E^Q[f(X_t)L(t)|\mathcal{F}_t^Y]}{E^Q[L(t)|\mathcal{F}_t^Y]} := \frac{\sigma_t(f)}{\sigma_t(1)}. \tag{2.2}$$

$(\sigma_t)_{t\geq 0}$ is called the *unnormalized optimal filter.*

2.2 Zakai Equation

We will derive the Zakai equation for $(\sigma_t)_{t\geq 0}$. Fix a constant $T > 0$ and denote by $\mathcal{M}_b(E)$ the family of finite signed measures on $(E, \mathcal{B}(E))$.

Lemma 2.2.1 Let $(v_t)_{0\leq t\leq T}$ be an $\mathcal{M}_b(E)$-valued càdlàg process. Suppose that

$$\sum_{i=1}^{n} Q\left[\int_0^T |v_s|(|\lambda_i - 1|)ds < \infty\right] = 1.$$

Then

$$\begin{aligned} v_t(f) &= v_0(f) + \int_0^t v_s(Lf)ds + \sum_{i=1}^{n}\int_0^t v_{s-}[(\lambda_i - 1)f]d(Y_i(s) - s), \\ &\qquad Q - a.s., \quad \forall f \in \mathcal{D} \end{aligned} \tag{2.3}$$

is equivalent to

$$\begin{aligned} v_t(f) &= v_0(T_t f) + \sum_{i=1}^{n}\int_0^t v_{s-}[(\lambda_i - 1)T_{t-s}f]d(Y_i(s) - s), \quad Q - a.s., \\ &\qquad \forall f \in B_b(E). \end{aligned} \tag{2.4}$$

Proof. Without loss of generality we consider only the one-dimensional case, i.e. $n = 1$. For $f \in \mathcal{D}$, define

$$\begin{aligned} v(t,f) &= v_0(T_t f) + \int_0^t v_{s-}[(\lambda - 1)T_{t-s}f]d(Y_s - s) \\ &\quad - \left\{v_0(f) + \int_0^t v_s(Lf)ds + \int_0^t v_{s-}[(\lambda - 1)f]d(Y_s - s)\right\} \\ &= [v_0(T_t f) - v_0(f)] + \\ &\quad \left\{\int_0^t v_{s-}[(\lambda - 1)T_{t-s}f]d(Y_s - s) - \int_0^t v_{s-}[(\lambda - 1)f]d(Y_s - s)\right\} \\ &\quad - \int_0^t v_s(Lf)ds. \end{aligned} \tag{2.5}$$

Note that $T_t f - f = \int_0^t T_s Lf ds$. We obtain by Fubini's Theorem that

$$v(t,f) = \int_0^t v_0(T_s Lf) - \int_0^t v_s(Lf)ds + I(t, Lf), \tag{2.6}$$

where

$$I(t,f) := \int_0^t \int_0^{t-s} v_{s-}[(\lambda-1)T_r f]dr d(Y_s - s).$$

For $n \in \mathbb{N}$, define the stopping time T_n by

$$T_n := \inf\left\{0 \leqslant t \leqslant T : \int_0^t |v_s|(|\lambda - 1|)ds > n\right\}. \tag{2.7}$$

Then $T_n \uparrow T$ as $n \to \infty$, Q-a.s. We also define

$$I(t,f,n) := \int_0^{t\wedge T_n} \int_0^{t-s} v_{s-}[(\lambda-1)T_r f]dr d(Y_s - s),$$

$$v(t,f,n) := \int_0^t v_0(T_s Lf) - \int_0^t v_s(Lf)ds + I(t, Lf, n). \tag{2.8}$$

Part I First, suppose that (2.3) holds. Let $f \in \mathcal{D}$. By the stochastic Fubini's theorem, (2.7) and (2.3), we get

$$\begin{aligned} I(t,f,n) &= \int_0^t \int_0^{t-r} I_{\{s\leqslant T_n\}} v_{s-}((\lambda-1)T_r f)d(Y_s - s)dr \\ &= \int_0^t \left[v_{(t-r)\wedge T_n}(T_r f) - v_0(T_r f) - \int_0^{(t-r)\wedge T_n} v_s(LT_r f)ds \right] dr. \end{aligned} \tag{2.9}$$

The last term of (2.9) is equal to

$$\begin{aligned} \int_0^t \int_0^t I_{\{s\leqslant t-r\}} I_{\{s\leqslant T_n\}} v_s(LT_r f)dsdr &= \int_0^t \int_0^t I_{\{s\leqslant T_n\}} I_{\{r\leqslant t-s\}} v_s(LT_r f)drds \\ &= \int_0^{t\wedge T_n} v_s\left(\int_0^{t-s} LT_r f dr\right) ds \\ &= \int_0^{t\wedge T_n} v_s(T_{t-s} f)ds - \int_0^{t\wedge T_n} v_s(f)ds. \end{aligned}$$

Consequently,

$$\begin{aligned} I(t,f,n) &= \int_0^t v_{s\wedge T_n}(T_{t-s}f)ds - \int_0^t v_0(T_s f)ds \\ &\quad - \int_0^{t\wedge T_n} v_s(T_{t-s}f)ds + \int_0^{t\wedge T_n} v_s(f)ds. \end{aligned}$$

We complete the proof through the following three steps.

(a) Let $f = R_\alpha \varphi$ for some $\alpha > 0$ and $\varphi \in \mathcal{D}$. Hereafter $R_\alpha := \int_0^\infty e^{-\alpha t} T_t dt$, $\alpha > 0$, is the resolvent of $(T_t)_{t \geq 0}$. Then we obtain by (2.8) that

$$v(t, f, n) = \int_0^t I_{\{s > T_n\}} v_{T_n}(T_{t-s} Lf) ds - \int_0^t I_{\{s > T_n\}} v_s(Lf) ds.$$

Letting $n \to \infty$, we get $v(t, f) = 0$.

(b) Let $f \in \mathcal{D}$. Define $f_k := kR_k f$, $k \in \mathbb{N}$. Note that $f_k \to f$ boundedly and pointwise as $k \to \infty$. By the bounded convergence theorem, $v_0(T_t f_k) \to v_0(T_t f)$ and $v_t(f_k) \to v_t(f)$ as $k \to \infty$. Moreover, by (2.7), we get

$$E^Q \left[\left| \int_0^{t \wedge T_n} v_{s-}((\lambda - 1) T_{t-s} f_k) d(Y_s - s) - \int_0^{t \wedge T_n} v_{s-}((\lambda - 1) T_{t-s} f) d(Y_s - s) \right| \right] \to 0$$

as $k \to \infty$ for each n. Then $v(t, f) = 0$, Q-a.s, on $\{t \leq T_n\}$. Thus $v(t, f) = 0$, Q-a.s. Therefore (2.4) holds for any $f \in \mathcal{D}$ by (2.5) and (2.3).

(c) Let $f \in B_b(E)$. Since $\mathcal{D}$ is bounded pointwise dense in $B_b(E)$, there exists a sequence $\{f_n\}_{n \geq 1} \subset \mathcal{D}$ such that $\sup_{n \geq 1} \|f_n\|_\infty < \infty$ and $\lim_{n \to \infty} f_n = f$ boundedly and pointwise. Therefore (2.4) holds for f by (b), the dominated convergence theorem and the stopping time argument (cf. (b)).

Part II Conversely, suppose that (2.4) holds for any $f \in B_b(E)$. Let $f \in \mathcal{D}$. We obtain that

$$\int_0^t v_s(Lf) ds = \int_0^t v_0(T_s Lf) + \int_0^t \int_0^u v_{s-}[(\lambda - 1) T_{u-s} Lf] d(Y_s - s) du.$$

Hence, by (2.6), we get

$$
\begin{aligned}
v(t,f) &= I(t,Lf) - \int_0^t \int_0^u v_{s-}[(\lambda-1)T_{u-s}Lf]d(Y_s - s)du \\
&= \int_0^t \int_s^t v_{s-}[(\lambda-1)T_{u-s}Lf]dud(Y_s - s) \\
&\quad - \int_0^t \int_0^u v_{s-}[(\lambda-1)Lf]d(Y_s - s)du \\
v(t,f,n) &= \int_0^t \int_s^t I_{\{s\leqslant T_n\}} v_{s-}[(\lambda-1)T_{u-s}Lf]dud(Y_s - s) \\
&\quad - \int_0^t \int_0^u v_{s-}[(\lambda-1)T_{u-s}Lf]d(Y_s - s)du \\
&= \int_0^t \int_0^{u\wedge T_n} v_{s-}[(\lambda-1)T_{u-s}Lf]d(Y_s - s)du \\
&\quad - \int_0^t \int_0^u v_{s-}[(\lambda-1)T_{u-s}Lf)d(Y_s - s)du \to 0 \text{ as } n\to\infty, \quad Q-a.s.
\end{aligned}
$$

Therefore (2.3) holds.

Theorem 2.2.2 Suppose that

$$\sum_{i=1}^n \int_0^T E[\lambda_i(X(t))]dt < \infty. \tag{2.10}$$

Then, for $0 \leq t \leq T$, we have

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(Lf)ds + \sum_{i=1}^n \int_0^t \sigma_{s-}[(\lambda_i - 1)f]d(Y_i(s) - s), \quad \forall f \in \mathcal{D}. \tag{2.11}$$

Proof. Note that (2.10) is equivalent to

$$\sum_{i=1}^n \int_0^T E^Q[L(t)|\lambda_i(X(t)) - 1|]dt < \infty.$$

Following the proof of [Brémaud (1981), R9, pp.177], we can show that

$$\sigma_t(f) = \sigma_0(T_t f) + \sum_{i=1}^n \int_0^t \sigma_{s-}[(\lambda_i - 1)T_{t-s}f]d(Y_i(s) - s), \quad Q-a.s., \quad \forall f \in B_b(E).$$

Therefore, the proof follows from Lemma 2.2.1.

Chapter 3

Uniqueness of Solutions to the Zakai Equation via Poisson Expansions

To simplify notation, we consider only the one dimensional case, i.e. $n = 1$, throughout this chapter. But the similar results hold for general n.

Theorem 3.1 Let $(v_t^i)_{0\le t\le T}$, $i = 1, 2$, be two $\mathcal{M}_b(S)$-valued càdlàg processes. Suppose that for all $n \in \mathbb{N}$,

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_n} \nu^2(T_{t_n}(|\lambda-1|T_{t_{n-1}-t_n}(|\lambda-1|\cdots T_{t_1}|\lambda-1|)))dt_{n-1}\cdots dt_2 dt_1 < \infty, \quad (3.1)$$

and the following conditions hold for $i = 1, 2$.

(i) $v_0^i = \nu$.

(ii) For $t \in [0, T]$, v_t^i is $\mathcal{F}_t^Y$-measurable.

(iii) $\int_0^T E^Q[|v_t^i|^2(|\lambda - 1|)]dt < \infty$.

(iv) For any $f \in \mathcal{D}$, $\{v_t^i(f)\}_{0\leq t\leq T}$ is an $\{\mathcal{F}_t^Y\}_{0\leq t\leq T}$ semi-martingale with

$$v_t^i(f) = v_0^i(f) + \int_0^t v_s^i(Lf)ds + \int_0^t v_s^i[(\lambda-1)f]d(Y_s - s), \quad Q-a.s.$$

Then $v_t^1 = v_t^2$ for all $t \in [0,T]$. Moreover, we have the unique Poisson expansion

$$\begin{aligned} v_t^i(f) &= \nu(T_t f) + \int_0^t \nu(T_{t_1}[(\lambda-1)T_{t-t_1}f])d(Y_{t_1}-t_1) \\ &+ \int_0^t\int_0^{t_1} \nu(T_{t_2}[(\lambda-1)T_{t_1-t_2}[(\lambda-1)T_{t-t_1}f]])d(Y_{t_2}-t_2)d(Y_{t_1}-t_1) + \cdots, \\ & \qquad Q-a.s., \quad \forall f \in B_b(E). \end{aligned} \tag{3.2}$$

Proof. Set $v_t = v_t^1$ or $v_t = v_t^2$ for $t \in [0,T]$. Then $\int_0^T E^Q[|v_t|^2(|\lambda-1|)]dt < \infty$ by condition (iii). Let $f \in B_b(E)$. By Lemma 2.2.1, $v_t(f) \in L^2(\Omega, \mathcal{F}_t, Q)$ for $t \in [0,T]$ and

$$v_t(f) = v_0(T_t f) + \int_0^t v_{t_1}[(\lambda-1)T_{t-t_1}f]d(Y_{t_1}-t_1), \quad Q-a.s.$$

Denote $(\lambda-1)_n := ((-n)\vee(\lambda-1))\wedge n$, $n \in \mathbb{N}$. Then, by the dominated convergence theorem, we get

$$v_t(f) = v_0(T_t f) + \lim_{n\to\infty}\int_0^t v_{t_1}[(\lambda-1)_n T_{t-t_1}f]d(Y_{t_1}-t_1), \quad Q-a.s.$$

Hereafter the limit is taken in the L^2-sense.

Apply the above argument to $(\lambda-1)_n T_{t-t_1}f$. By $\int_0^T E^Q[|v_t|^2(|\lambda-1|)]dt < \infty$, the dominated convergence theorem and (3.1), we get

$$\begin{aligned} v_t(f) &= v_0(T_t f) + \lim_{n\to\infty}\int_0^t v_0(T_{t_1}[(\lambda-1)_n T_{t-t_1}f]d(Y_{t_1}-t_1) \\ &+ \lim_{n\to\infty}\lim_{n_1\to\infty}\int_0^t\int_0^{t_1} v_{t_2}[(\lambda-1)_{n_1}T_{t_1-t_2}[(\lambda-1)_n T_{t-t_1}f]d(Y_{t_2}-t_2)]d(Y_{t_1}-t_1) \\ &= v_0(T_t f) + \int_0^t v_0(T_{t_1}[(\lambda-1)T_{t-t_1}f]d(Y_{t_1}-t_1) \\ &+ \lim_{n\to\infty}\lim_{n_1\to\infty}\int_0^t\int_0^{t_1} v_{t_2}[(\lambda-1)_{n_1}T_{t_1-t_2}[(\lambda-1)_n T_{t-t_1}f]]d(Y_{t_2}-t_2)d(Y_{t_1}-t_1), \\ & \qquad Q-a.s. \end{aligned}$$

Then, by orthogonality, the first two terms of the above summation must be the first two terms of the unique Poisson expansion of $v_t(f)$. Repeat this procedure, by induction, we obtain the unique Poisson expansion (3.2) of $v_t(f)$. Therefore, $v_t^1 = v_t^2$ for all $t \in [0, T]$.

Corollary 3.2 Suppose that $\int_0^T E[\lambda(X(t))]dt < \infty$ and

$$\sum_{n=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_n} \nu^2(T_{t_n}(|\lambda - 1|T_{t_{n-1}-t_n}(|\lambda - 1| \cdots T_{t_1}|\lambda - 1|)))dt_{n-1} \cdots dt_2 dt_1 < \infty. \tag{3.3}$$

Then $\{\sigma_t\}_{0 \le t \le T}$ is the unique $\mathcal{M}_b(S)$-valued solution to the Zakai equation (2.11). Moreover, we have the unique Poisson expansion

$$\begin{aligned} \sigma_t(f) &= \nu(T_t f) + \int_0^t \nu(T_{t_1}[(\lambda - 1)T_{t-t_1 f}])d(Y_{t_1} - t_1) \\ &+ \int_0^t \int_0^{t_1} \nu(T_{t_2}[(\lambda - 1)T_{t_1 - t_2}[(\lambda - 1)T_{t-t_1} f]])d(Y_{t_2} - t_2)d(Y_{t_1} - t_1) + \cdots, \\ & \qquad Q - a.s. \ \ \forall f \in B_b(E). \end{aligned} \tag{3.4}$$

Proof. By Theorems 2.2.2 and 3.1, we only need to show that

$$E^Q[\int_0^T \{\sigma_t(|\lambda - 1|)\}^2 \, dt] < \infty.$$

Define $(\lambda - 1)_n := ((-n) \vee (\lambda - 1)) \wedge n$ for $n \in \mathbb{N}$. Let $\{\sigma_t^{(\lambda-1)_n}\}_{0 \le t \le T}$ be the unnormalized filtering process with respect to $(\lambda - 1)_n$, i.e.

$$\sigma_t^{(\lambda-1)_n}(f) := E^{Q,(\lambda-1)_n}[f(X(t))L^{(\lambda-1)_n}(t)|\mathcal{F}_t^{Y,(\lambda-1)_n}], \quad f \in B_b(E).$$

Hereafter, we use $Y_t^{(\lambda-1)_n}$, $\mathcal{F}_t^{Y,(\lambda-1)_n}$, $Q^{(\lambda-1)_n}$ and $E^{Q,(\lambda-1)_n}$ to denote respectively Y_t, $\mathcal{F}_t^Y$, Q and E^Q corresponding to the observation function $(\lambda - 1)_n$. Since X and

Y^* are independent under Q^*, by Fatou's Lemma, Theorem 2.2.2, Theorem 3.1 and dominated convergence theorem, we get

$$
\begin{aligned}
&E^Q\left[\int_0^T \{\sigma_t(|\lambda-1|)\}^2 dt\right] \\
&= \int_0^T \int \left(\int |\lambda-1|(X_t) L_t dQ^X\right)^2 dQ^Y dt \\
&\leq \varliminf_{n\to\infty} \int_0^T \int \left(\int |\lambda-1|(X_t) L_t^{(\lambda-1)_n} dQ^X\right)^2 dQ^Y dt \\
&= \varliminf_{n\to\infty} \int_0^T \int \left(\int |\lambda-1|(X_t) L_t^{(\lambda-1)_n} dQ^{(\lambda-1)_n,X}\right)^2 dQ^{(\lambda-1)_n,Y} dt \\
&= \varliminf_{n\to\infty} E^{Q,(\lambda-1)_n}\left[\int_0^T \{\sigma_t(|\lambda-1|)\}^2 dt\right] \\
&\leq \varliminf_{n\to\infty} \varliminf_{n_1\to\infty} E^{Q,(\lambda-1)_n}\left[\int_0^T \{\sigma_t(|(\lambda-1)_{n_1}|\}^2 dt\right] \\
&\leq \sum_{n=1}^{\infty} \int_0^T \int_0^{t_1} \cdots \int_0^{t_n} \nu^2(T_{t_n}(|\lambda-1| T_{t_{n-1}-t_n}(|\lambda-1| \cdots T_{t_1}|h|))) \\
&\qquad\qquad \cdot dt_{n-1} \cdots dt_2 dt_1 \\
&< \infty,
\end{aligned}
$$

where $Q^{*,X}$ and $Q^{*,Y}$ denote the marginal probabilities of X and Y with respect to Q^*, respectively.

Remark 3.3 The assumptions of Corollary 3.2 are weak and can be verified for a large class of unbounded intensity functions λ. For example, suppose that $d\nu = u_0 dm$ with $u_0 \in L^2(E;m)$, $(T_t)_{t\geq 0}$ is a contraction semigroup on $L^2(E;m)$ and $h := (\lambda-1) \in L^{2^{n+1}}(E;m)$ for any $n \in \mathbb{N}$. Then (3.3) holds. In fact, noting that $(T_t f(x))^n =$

$(E_x[f(X_t)])^n \leq E_x[f^n(X_t)] = T_t f^n(x)$ for any $f \geq 0$ on E and $n \in \mathbb{N}$, we get

$$\begin{aligned}
&\nu(T_{t_{n+1}}(|h|T_{t_n - t_{n+1}}(|h|T_{t_{n-1}-t_n}\cdots)))\\
&= \int_E u_0 T_{t_{n+1}}(|h|T_{t_n - t_{n+1}}(|h|T_{t_{n-1}-t_n}\cdots))dm\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E [T_{t_{n+1}}(|h|T_{t_n - t_{n+1}}(|h|T_{t_{n-1}-t_n}\cdots))]^2 dm\right)^{1/2}\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E |h|^2(T_{t_n - t_{n+1}}(|h|T_{t_{n-1}-t_n}\cdots))^2 dm\right)^{1/2}\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E |h|^4 dm\right)^{1/4}\left(\int_E (T_{t_n - t_{n+1}}(|h|T_{t_{n-1}-t_n}\cdots))^4 dm\right)^{1/4}\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E |h|^4 dm\right)^{1/4}\left(\int_E (T_{t_n - t_{n+1}}(|h|^2(T_{t_{n-1}-t_n}\cdots)^2))^2 dm\right)^{1/4}\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E |h|^4 dm\right)^{1/4}\left(\int_E |h|^4(T_{t_{n-1}-t_n}\cdots)^4 dm\right)^{1/4}\\
&\leq \left(\int_E u_0^2 dm\right)^{1/2}\left(\int_E |h|^4 dm\right)^{1/4}\left(\int_E |h|^8 dm\right)^{1/8}\left(\int_E (T_{t_{n-1}-t_n}\cdots)^8 dm\right)^{1/8}\\
&\cdots\\
&< \infty.
\end{aligned}$$

Another example is as follows. If $d\nu = u_0 dm$ with $u_0 \in L^2(E;m)$, $h := (\lambda - 1) \in L^4(E;m)$ and the semigroup $(T_t)_{t\geq 0}$ satisfies

$$\|T_t f\|_{L^4} \leq c\|T_t f\|_{L^2}, \quad \forall f \in L^2(E;m),\ 0 \leq t \leq T$$

for constant $c > 0$, then we have

$$\sum_{n=1}^{\infty}\int_0^t\int_0^{t_1}\cdots\int_0^{t_n}\nu^2(T_{t_n}(|h|T_{t_{n-1}-t_n}(|h|\cdots T_{t_1}|h|)))dt_{n-1}\cdots dt_2 dt_1 < \infty.$$

Chapter 4

Recursive Poisson Expansions for the Unnormalized Optimal Filters

In order to use the established Poisson expansion (3.4) for practical computations, we must approximate the multiple Poisson integrals by truncation. In this chapter, we will develop a recursive algorithm for computing an approximation of the unnomalized optimal filter. To make the presentation more transparent, we consider a time-homogeneous diffusion signal.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. Suppose $(B(t))_{t\geq 0}$ and $(N(t))_{t\geq 0}$ are standard one-dimensional Brownian motion and Poisson process on $(\Omega, \mathcal{F}, \mathcal{P})$, respectively. The signal $(X(t))_{t\geq 0}$ and observation $(Y(t))_{t\geq 0}$ are given as follows.

$$\begin{aligned} X(t) &= X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dB(s), \\ Y(t) &= N\left(\int_0^t \lambda(X(s))ds\right). \end{aligned}$$

The following conditions are assumed:

(i) $(B(t))_{t\geq 0}$ and $(N(t))_{t\geq 0}$ are independent of each other and of $X(0)$.

(ii) The functions b, σ, and $\lambda \geq 0$ on $\mathbb{R}$ are infinitely differentiable and all the derivatives are bounded.

(iii) $X(0)$ has a density p with respective to the Lebesgue measure dx and p is a smooth rapidly decreasing function on $\mathbb{R}$.

By (2.2), we have

$$\pi_t(f) = \frac{\sigma_t(f)}{\sigma_t(1)} = \frac{E^Q[f(X_t)L(t)|\mathcal{F}_t^Y]}{E^Q[L(t)|\mathcal{F}_t^Y]}, \quad \forall f \in B_b(\mathbb{R}).$$

Similar to [Zakai (1969), pp.232], one finds that under assumptions (i)-(iii) there exists a random field $u(t,x)$, $t \geq 0$, $x \in \mathbb{R}$, such that

$$\sigma_t(f) = \int_{\mathbb{R}} u(t,x)f(x)dx, \quad \forall f \in B_b(\mathbb{R}).$$

$u(t,x)$ is called the unnormalized filtering density function.

Denote by $T_t\varphi$ the solution of the equation

$$\begin{aligned} \frac{\partial v(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2(\sigma^2(x)v(t,x))}{\partial x^2} - \frac{\partial(b(x)v(t,x))}{\partial x}, \quad t > 0, \\ v(0,x) &= \varphi(x). \end{aligned}$$

Let $T > 0$ be a fixed constant and consider a partition $0 = t_0 < t_1 < \ldots < t_M = T$ of $[0,T]$. Denote $\Delta_i = t_i - t_{i-1}$. Then, we obtain the following result by Corollary 3.2.

Theorem 4.1 Under Assumptions (i)-(iii), we have

$$\begin{aligned} u(t_0,x) &= p(x), \\ u(t_i,x) &= T_t u(t_{i-1},\cdot)(x) \\ &\quad + \sum_{k\geq 1} \int_0^{\Delta_i}\int_0^{s_k}\cdots\int_0^{s_2} T_{t-s_k}(\lambda-1)\cdots(\lambda-1)T_{s_1}u(t_{i-1},\cdot)(x) \\ &\qquad d(Y^{(i)}(s_1)-s_1)\cdots d(Y^{(i)}(s_k)-s_k) \end{aligned}$$

for $i = 1,\ldots,M$, where $Y^{(i)}(t) = Y(t+t_{i-1}) - Y(t_{i-1})$, $0 \leq t \leq \Delta_i$.

To simplify notation, for an $\mathcal{F}^Y_{t_{i-1}}$-measurable function $g = g(x, \omega)$ and $0 \le t \le \Delta_i$, we define

$$\begin{aligned} F_0^{(i)}(t,g)(x) &:= T_t g(x) \\ F_k^{(i)}(t,g)(x) &:= \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} T_{t-s_k}(\lambda - 1)\cdots(\lambda-1)T_{s_1} g(x), \\ &\qquad d(Y^{(i)}(s_1) - s_1)\cdots d(Y^{(i)}(s_k) - s_k), \quad k \ge 1. \end{aligned}$$

Then

$$u(t_i, x) = \sum_{k\ge 0} F_k^{(i)}(\Delta_i, u(t_{i-1}, \cdot))(x), \quad i = 1, \ldots, M.$$

Denote by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$ the norm and the inner product of $L^2(\mathbb{R}, dx)$, respectively. Then, there exists a constant $c > 0$ such that (cf. [Rozovskii (1990)])

$$\|T_t \varphi\|_0 \le e^{ct}\|\varphi\|_0.$$

By induction, for every $t \in [0, \Delta_i]$, $i = 1, \ldots, M$ and $k \ge 0$, the operator $g \to F_k^{(i)}(t, g)$ is linear and bounded from $L^2(\Omega, Q; L^2(\mathbb{R}, dx))$ to itself and

$$E^Q \|F_k^{(i)}(t,g)\|_0^2 \le e^{ct}[(ct)^k/k!]E^Q\|g\|_0^2.$$

This implies that $u(t_i, \cdot) \in L^2(\mathbb{R}, dx)$, Q-a.s. By Theorem 4.1 and induction, we get the following result.

Theorem 4.2 If $\{e_n\}$ is an orthonormal basis in $L^2(\mathbb{R}, dx)$ and random variables $\psi_n(i)$, $n \ge 0$, $i = 0, \ldots, M$ are defined recursively by

$$\begin{aligned} \psi_n(0) &= (p, e_n)_0, \\ \psi_n(i) &= \sum_{k\ge 0}\left(\sum_{l\ge 0}(F_k^{(i)}(\Delta_i, e_l), e_n)_0 \psi_l(i-1)\right), \quad i = 1, \ldots, M, \end{aligned}$$

then

$$u(t_i, \cdot) = \sum_{n \geq 0} \psi_n(i) e_n, \quad P-a.s.$$

Now we can use Theorem 4.2 to develop a recursive algorithm for $(\sigma_t)_{t\geq 0}$. For simplicity, we assume that the partition of $[0,T]$ is uniform, i.e. $\Delta_i = \Delta$ for all $i = 1, \dots, M$. Let $\{e_n\}$ be the Hermite basis in $L^2(\mathbb{R}, dx)$:

$$e_n(x) = \frac{1}{\sqrt{2^n \pi^{1/2} n!}} e^{-x^2/2} H_n(x),$$

where $H_n(x)$ is the n-th Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, n \geq 0.$$

Recursive algorithm: Given a positive integer κ, define random variables $\psi_{n,k}(i)$, $n = 0, \dots, \kappa$, $i = 0, \dots, M$, by

$$\begin{aligned} \psi_{n,\kappa}(0) &= (p, e_n)_0, \\ \psi_{n,\kappa}(i) &= \sum_{l=0}^{\kappa} ((T_\Delta e_l, e_n)_0 + (T_\Delta(\lambda - 1)e_l, e_n)_0 (Y(t_i) - Y(t_{i-1}) - \Delta) \\ &+ (1/2)(T_\Delta(\lambda-1)^2 e_l, e_n)_0((Y(t_i) - Y(t_{i-1}) - \Delta)^2 - \Delta))\psi_{n,\kappa}(i-1), \\ & \qquad i = 1, \dots, M. \end{aligned}$$

Then

$$u_\kappa(t_i, x) = \sum_{n=0}^{\kappa} \psi_{n,\kappa}(i) e_n(x).$$

Remark 4.3 Similar to [Lototsky et al. (1997) and Lototsky and Rozovskii (1997)], the following type of error bound can be established.

$$\max_{1 \leq i \leq M} \sqrt{E^Q \| u_\kappa(t_i, \cdot) - u(t_i, \cdot)\|_0^2} \leq c\Delta + \frac{c(\gamma)}{\kappa^{\gamma - 1/2}\Delta}.$$

Then, by appropriate choice of the parameters Δ and κ, we can make the errors to be arbitrary small. The above algorithm looks especially promising if the parameters of the model, i.e. b, σ, λ and p, are known. In this case, the values of $(T_\Delta e_l, e_n)_0$, $(T_\Delta(\lambda - 1)e_l, e_n)_0$ and $(T_\Delta(\lambda - 1)^2 e_l, e_n)$, $n, l = 1, \ldots, \kappa$, can be pre-computed and stored. So only increments of the observations are required at each step of the algorithm, which largely increases the on-line speed of the algorithm.

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