Decentralized Control of Uncertain Interconnected Time-Delay Systems

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ABSTRACT

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In this thesis, novel stability analysis and control synthesis methodologies are proposed for uncertain interconnected time-delay systems. It is known that numerous real-world systems such as multi-vehicle flight formation, automated highway systems, communication networks and power systems can be modeled as the interconnection of a number of subsystems. Due to the complex and distributed structure of this type of systems, they are subject to propagation and processing delays, which cannot be ignored in the modeling process. On the other hand, in a practical environment the parameters of the system are not known exactly, and usually the nominal model is used for controller design. It is important, however, to ensure that robust stability and performance are achieved, that is, the overall closed-loop system remains stable and performs satisfactorily in the presence of uncertainty.

To address the underlying problem, the notion of *decentralized fixed modes* is extended to the class of linear time-invariant (LTI) time-delay systems, and a necessary and sufficient condition is proposed for stabilizability of this type of systems by means of a finite-dimensional decentralized LTI output feedback controller. A near-optimal decentralized servomechanism control design method and a cooperative predictive control scheme are then presented for uncertain LTI hierarchical interconnected systems. A H_∞ decentralized overlapping control design technique is provided consequently which guarantees closed-loop stability and disturbance attenuation in the presence of delay. In particular, for the case of highly uncertain time-delay systems, an adaptive switching control methodology is proposed to achieve output tracking and disturbance rejection. Simulation results are provided throughout the thesis to support the theoretical findings.

To my parents

for their love and support

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List of Abbreviations

UAV	Unmanned Aerial Vehicle
LTI	Linear Time-Invariant
TDS	Time-Delay System
NCS	Networked Control System
DFM	Decentralized Fixed Mode
CFM	Centralized Fixed Mode
DOFM	Decentralized Overlapping Fixed Mode
FDE	Functional Differential Equation
ODE	Ordinary Differential Equation
LQ	Linear Quadratic
LQR	Linear Quadratic Regulator
LMI	Linear Matrix Inequality

Chapter 1

Introduction

1.1 Motivations

There has been a considerable amount of attention in the literature recently towards high-performance control design for interconnected systems [41]. Networked unmanned aerial vehicles (UAV), automated highway systems and automated manufacturing processes all involve multiple, interacting, highly dynamic components [127,132,136]. Elements of such systems, usually called subsystems, are distributed in space and must exchange information with each other using sensing and communication networks. Furthermore, the overall representation of an interconnected system often involves high-order dynamics with several input and output channels [82,135].

For these types of systems, since it is not typically feasible to perform all the control computations at a single location, it is desirable to have a distributed control scheme in order to obtain a more reliable closed-loop system which is less sensitive to failures and has lower computational complexity [123]. On the other hand, distributed implementation of a high-performance centralized controller requires high levels of connectivity among subsystems. Since it is not often realistic to assume that all output measurements can be transmitted to every local controller, there are normally some constraints on the information exchange among different subsystems; i.e., full output observation is typically not possible [130]. A special case of constrained control structure is the one with diagonal (or block-diagonal) information flow structure, which is often referred to as a *decentralized control* system. In this type of systems, each local control station only has access to the measurements of its corresponding subsystem for generating the local control input [83, 149]. All control stations are involved, however, in the overall control operation.

The complexity of control in the above-mentioned problems is considerably increased as the modeling parameters are subject to error and uncertainty, the sensors and measurements are noisy, and the disturbances affect the actuators. Since different agents share their measurements through a communication network, certain problems such as communication noise and delay should also be taken into consideration in control design [41, 80, 123]. Furthermore, interruptions and data loss and node failures may occur in a communication network. The conventional design approaches, where control and communication problems are investigated separately, fail to address these types of problems efficiently.

One of the main challenges in the problem of *network control systems* (NCS), where a communication network with limited bandwidth is utilized to transfer the sensor data and compute the control commands, is control analysis and synthesis in the presence of the undesirable delay in transmission and processing of data. The effect of the network-induced delay on the performance of the NCS is investigated in several papers where time-delay system (TDS) theory is employed to tackle the problem [20, 102, 118].

Time-delay systems are also called systems with aftereffect or dead-time, hereditary systems whose governing equations are referred to as differential-difference equations. Such equations belong to the class of functional differential equations

(FDE) which are infinite dimensional, as opposed to ordinary differential equations (ODE). There are several examples of aftereffect (time-delay) phenomena in biology, chemistry, economics, mechanics, physics, population dynamics, as well as in engineering sciences [15, 49, 63]. Neglecting the effect of delay in the system model can result in the degradation of the system performance or even instability; hence, it is essential to investigate the effect of delay on control design. For instance, the stability margin of the overall system can be highly sensitive to delay and small variation in delay may lead to considerable discrepancy between theoretical and experimental developments.

Many of the classical control design techniques are not effective enough in the presence of time delay. The most naive design approach for a time-delay system is to use a proper finite-dimensional approximation (e.g., Padé approximation) for the delay. However, for typical size of delay in engineering problems, such approximations are known to have major shortcomings in the design of model-based high-performance stabilizing controllers [63]. In the simple case of fixed known delays, such approximations often introduce high-order transfer functions which in turn lead to the same level of complexity as the direct design techniques with no finite-dimensional approximation. In the case of time-varying delays, such approaches can potentially be disastrous in terms of stability and oscillations.

In general, the problem of decentralized control design for a physical interconnected system can be described as follows. Consider an interconnected system with an arbitrary directed graph (digraph). The system is assumed to be subject to noise, disturbance, and parametric perturbation. It is desired to obtain a structurally constrained distributed control scheme with the following properties:

- It has good regulation properties in the sense that it reduces the effect of sensor noise, rejects the effect of disturbances in the system, and follows any given reference trajectory with a "good" precision in steady state.

- It provides a robust control performance, in the sense that the overall closed-loop system performs satisfactorily in the presence of

  (i) uncertainty in the parameters of the system;

  (ii) uncertain and time-varying delay in the communication link between different subsystems

- It is flexible and fault tolerant, in the sense that it can operate in the presence of a wide variety of faults which may occur in a practical environment.

1.1.1 Applications

In what follows, two specific applications for the problem described in the preceding subsection are presented.

- There has been a growing interest in the application of cooperative control theory in a network of coordinated UAVs. These applications include a wide range of civilian and military missions such as surveillance, mapping, patrolling, convoy protection, search and rescue. Such missions can be accomplished in a more efficient manner using vehicles with small size and low cost [132]. Due to the repetitive and dangerous nature of these tasks, they are more suitable to be carried out by autonomous vehicles. As an example, consider a mission scenario where a team of autonomous UAVs need to cooperate in order to monitor the evolution of a forest fire boundary or the dispersion of a pollutant (e.g. an oil spill) in water. A cursory analysis of the problem indicates that the vehicles have to dynamically cooperate in order to optimize the covered area and to adapt to changes of the monitoring patterns. It is to be noted that the monitoring patterns change as functions of external disturbances like wind affecting the forest fire spread and the aerial platforms, or water currents changing the velocity and the dispersion of the oil spill patterns [132].

- There are several advantages in the multiple spacecraft technology compared to traditional monolithic one, including improvement in the resolution of the remote sensing. In addition, spacecraft flying in formation demonstrate increased robustness and reconfigurability features. The Canadian Space Agency, Department of National Defense, NASA, and the US Air force have described spacecraft formation flying as a key technology for the 21st century (e.g., see [9]).

1.2 Background and Literature Review

1.2.1 Decentralized control

Control of large-scale complex interconnected systems has attracted much attention in various engineering disciplines [31,41,107,135]. An interconnected system consists of a number of dynamic components often referred to as subsystems, which interact with each other internally through so called interconnection signals. Physical examples of interconnected systems include energy distribution systems, transportation systems, flight formation, robotics, financial systems. Some important problem in control of interconnected systems will be discussed next.

Consider an interconnected system S consisting of 3 subsystems, and let the ith subsystem be denoted by S_i, $i = 1, 2, 3$. A centralized controller C for the system S uses the measured outputs of all the subsystems to generate the control signal for each subsystem as shown in Figure 1.1. There are two main drawbacks concerning the centralized control structure in real-world applications. First, a centralized control structure requires all local measurements to be transmitted to one single point, which can be very expensive in spatially distributed systems. Second, this type of control structure has a single point of failure and hence may not be reliable, as a fault in the centralized control station can affect the control signals of all

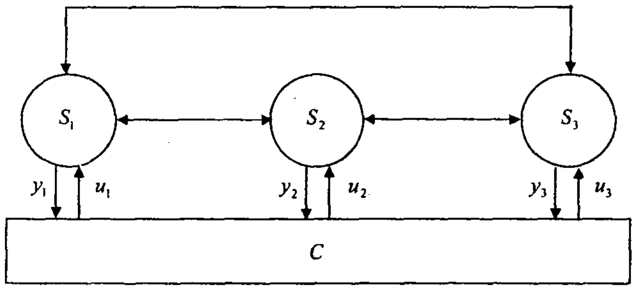


Figure 1.1: A centralized control structure for an interconnected system consisting of three subsystems

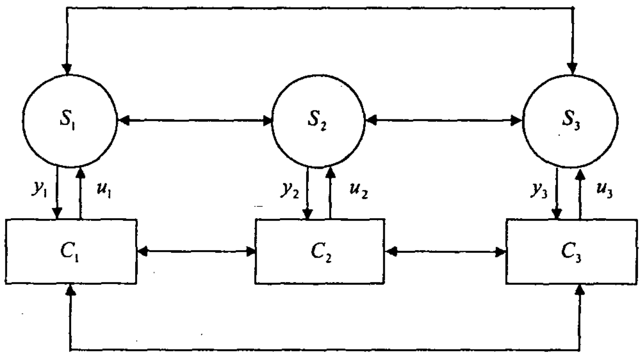


Figure 1.2: A distributed control structure for the interconnected system of Figure 1.1

subsystems.

Distributed control structure was introduced in the literature to address some of the shortcomings of centralized control [123, 138]. A distributed controller for the system of Figure 1.1 is demonstrated in Figure 1.2. In this type of control structure, each subsystem is driven by a local controller which generates the local input signal from the local information as well as the information transmitted from other subsystems. This improves the reliability of control operation significantly, as there is no longer a single point of failure for the overall closed-loop system.

Despite the improved reliability in the distributed control structure, it is important to note that all control agents need to communicate with each other to share their local information. To address this drawback, one can use a decentralized

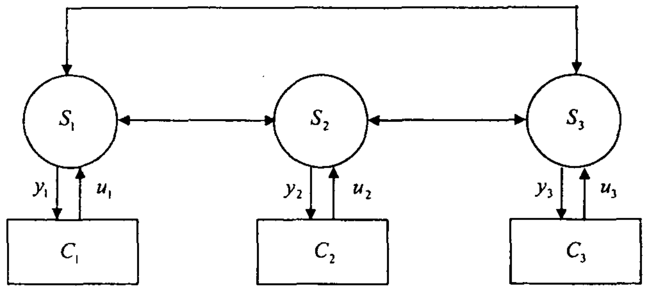


Figure 1.3: A decentralized control structure for the interconnected system of Figure 1.1

control structure, where the local control agents operate independently (without sharing their information) shown in Figure 1.3 for the system of Figure 1.1. In other words, in the decentralized control structure each subsystem can only access its local output to produce the corresponding local control input.

On the other hand, elimination of the communication links in the fully decentralized control structure can lead to the poor performance compared to the centralized or distributed control case. This introduces a tradeoff between the overall performance and the communication cost. As an alternative to a fully decentralized control structure, one can use partial information exchange by maintaining some of the more important communication links. This type of control structure is often referred as *decentralized overlapping* or simply *overlapping* structure. One example of an overlapping control structure for the system of Figure 1.1 is depicted in Figure 1.4.

Design of a high-performance decentralized (overlapping) control for a LTI interconnected system has attracted a considerable amount of attention in recent years. The research in this area is focused on finding the existence conditions for a stabilizing decentralized (overlapping) controller, and developing techniques to find such a controller. For example, the notion of *decentralized fixed modes* (DFM) was introduced in [149] to identify those LTI systems that cannot be stabilized by means

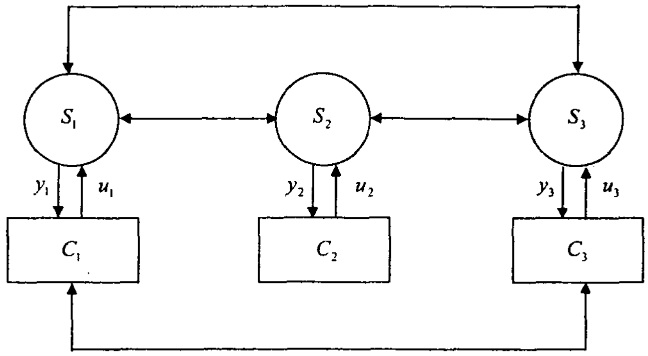


Figure 1.4: An overlapping control structure for the interconnected system of Figure 1.1

of a LTI decentralized controller. This notion was later extended in [7] to address the stabilizing problem with respect to a LTI controller with any overlapping structure.

On the other hand, the problem of optimal output regulation has been investigated in the literature extensively, and different analytical and numerical techniques are proposed to tackle the problem. Given an interconnected system and a performance index, the objective is to find a decentralized feedback law which results in a sufficiently small performance index. The existing approaches for this problem can be categorized as follows:

1. The first approach neglects the effect of interconnections in the control design procedure. Hence, the resultant closed-loop system with the local controllers obtained by this approach may perform poorly, or even be unstable [66].

2. Another approach is to obtain a decentralized static output feedback law by using iterative numerical algorithms [19]. However, it is known that by employing a dynamic feedback law instead of a static one, the overall performance of the system can be improved significantly.

3. The third approach deals with a system with a hierarchical structure [135]. The advantage of this method compared to the second approach discussed above is that it reduces off-line computations. However, this method is inferior to the

second approach, because the static gains are computed one at a time, while in the second approach all of the static gains are determined simultaneously.

There are also some other design techniques which arrive at either some sophisticated differential matrix equations or some non-convex relations, which are very difficult to solve in general [133]. Furthermore, [130] characterizes those optimal decentralized control problems which can be formulated as a convex optimization.

In addition, there are a number of results dealing with decentralized control design with disturbance rejection and attenuation property; e.g., see the decentralized servomechanism controller proposed in [25, 26, 29]. This requires the dynamics behavior of the unmeasurable exogenous disturbances is known. Decentralized H_∞ control design technique are also investigated in the literature to achieve disturbance attenuation (see for example [155]).

1.2.2 Time-delay systems

There is a great number of monographs published in the area of time-delay systems since 1963. The reader can refer, for instance, to the survey papers such as [64, 128] or special issues such as [44, 129]. What can motivate such an increasing interest and ongoing research activities in this field? The following points can address this question, to some extent.

- Many real-world processes include aftereffect phenomenon in their inner dynamics. On the other hand, it is often desirable in engineering problems to model the process as accurately as possible in order to simulate the behavior of the system with a sufficiently high precision. Hence, in the design of high-performance controllers for real-world processes including aftereffect phenomenon, it is crucial to take this phenomenon into consideration in the modeling phase.

- In addition, actuators, sensors, and field networks (that are important components of feedback control systems) often introduce delays in the dynamics. These elements are commonly used in communications, information and control technology [118], high-speed communication networks [20], teleoperated systems [117] and robotics [5].

- Some of the properties of delay are surprising; for instance, it can be shown that injecting delay in some cases can be beneficial from control prospectives [15]. This property of time-delay in control systems has been investigated in a number of case studies in the literature, such as delayed resonators [55], time-delay controllers and observers [126], limit cycle control in nonlinear systems [2].

- In spite of their complexity, time-delay systems often appear as simple infinite-dimensional models representing the systems whose dynamics are governed by partial differential equations (PDE). For instance, hyperbolic PDEs can be locally regarded as neutral delay systems [50, 65].

Modeling of time-delay systems

A classical hypothesis in the modeling of physical processes is to assume that the future behavior of the deterministic system can be summed up in its present state only. In the case of ODEs, the n-dimensional state $x(t)$ evolves in the Euclidean space $\mathbb{R}^n$. Now, in order to take an influence of the past into account, it is required to introduce a deviated time-argument. This in turn means that the state can no longer be a vector $x(t)$ defined at a discrete value of time t. Thus, in *functional differential equations* (FDEs), the state must be a function of $x(t)$ in the past time-interval $[t-h, t]$, where h is a strictly positive constant.

Consider the following general form of a time-delay system [49]

$$\dot{x}(t) = f(x_t, t, u_t) \tag{1.1a}$$

$$y(t) = g(x_t, t, u_t) \tag{1.1b}$$

$$x_t(\theta) = x(t+\theta), \qquad -h \leq \theta \leq 0 \tag{1.1c}$$

$$u_t(\theta) = u(t+\theta), \qquad -h \leq \theta \leq 0 \tag{1.1d}$$

$$x(\theta) = \varphi(\theta), \qquad t_0 - h \leq \theta \leq t_0 \tag{1.1e}$$

where h is the delay and t_0 is the initial time. Let $\gamma([-h,0],\mathbb{R}^n)$ be the space of continuous functions mapping the interval $[-h,0]$ into $\mathbb{R}^n$. The initial condition φ must be prescribed as a function belongs to $\gamma([-h,0],\mathbb{R}^n)$.

By using the step-by-step method initiated by Bellman [11], one can show that the resulting solution $x(t)$ is a succession of some polynomial functions of t, in increasing degree at each interval $[kh,(k+1)h]$. The nature of the corresponding solution (and its corresponding initial value) distinguishes FDEs from ODEs [128].

The systems represented by FDEs introduced above are often referred to as *retarded time-delay systems.* Another type of time-delay systems referred to as *neutral time-delay systems*, which involve the same highest derivation order for some components of $x(t)$ at both time t and past time(s) $t' < t$, resulting in an increased mathematical complexity. Neutral systems are described as [49]

$$\dot{x}(t) = f(x_t, t, \dot{x}_t, u_t)$$

The solutions of retarded systems become more smooth as time increases. This property does not hold for neutral systems due to the implied difference equation involving $\dot{x}(t)$ [10].

Stability analysis of time-delay systems

Time-delay is known to have complex (and sometimes surprising) effects on stability. While its destabilizing effect is investigated intensively in the literature, time-delay can also be helpful in stabilization (e.g., see [122] for the case of retarded systems). For example, the system represented by $\ddot{y}(t) + \dot{y}(t) - y(t-h) = 0$ is unstable for $h = 0$, but asymptotically stable for $h = 1$ (see also other examples in [1]).

The Krasovskii-type approach

For both classes of retarded and neutral time-delay systems, checking eigenvalue conditions for FDEs is much harder than those for ODEs. This explains why numerous stability approaches have been investigated for time-delay systems. The level of difficulty of such approaches depends on different factors; and in particular, stability analysis is more challenging for the case of of neutral time-delay systems. Stability analysis is also more difficult in presence of time-varying delays, nonlinear equations, and parameter uncertainty [49]. A brief description of these methods can be found in the survey paper [128] and a more complete one in the monographs [50,65]. While there are general results for stability independent of delay, one may expect less conservative stability conditions using delay-dependent approaches. In the engineering applications, information on the range of delay is generally available and delay-dependent criteria are likely to result in better performances.

One of the most commonly used generalizations of the Lyapunov direct method for time-delay systems is done by Krasovskii [128], which involves functionals instead of classical positive definite functions [65]. Some other techniques for stability analysis of time-delay systems are:

- The comparison techniques, which are based on differential inequalities [68].
- The method proposed by Razumikin [128], which involves a Lyapunov function whose derivative has to be negative only for special solutions of the system.

The first approach provides a very general framework to the stability study of complex systems, and is capable of estimating the stability domain. Nevertheless, the resultant stability condition may turn out to be conservative since the underlying problem formulation is non-convex.

On the other hand, while the Lyapunov-Razumikhin technique also arrives at conservative results in general, it applies to time-varying delays with only boundedness restriction on the delay itself (i.e. $0 \leq h(t) < \infty$), whereas classical Krasovskii techniques require a bounded derivative (i.e. $\dot{h}(t) \leq \delta$, for some δ) as well.

Linear time-delay systems

A linear time-invariant (LTI) time-delay system can be described by the following state-space representation [49]

$$\begin{aligned} \dot{x} &= \sum_{l=1}^{q} D_l \dot{x}(t - w_l) + \sum_{i=0}^{k} [A_i x(t - h_i) + B_i u(t - h_i)] \\ &\quad + \sum_{j=1}^{r} \int_{t-\tau_j}^{t} [G_j x(\theta) + H_j u(\theta)] d\theta \\ y(t) &= \sum_{i=0}^{k} C_i x(t - h_i) + \sum_{j=1}^{r} \int_{t-\tau_j}^{t} N_j x(\theta) x(\theta) d\theta \end{aligned} \tag{1.2}$$

where A_0 represents instantaneous feedback gains, and $h_0 = 0$. In the above equation, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input of the system. Also, $y(t) \in \mathbb{R}^p$ is the system output with discrete delay gains C_i, $i = 1, \ldots, k$ and distributed delay weights N_j, $j = 1, \ldots, r$. The parameters h_i's, $i = 1, \ldots, k$, represent the discrete-delay phenomena with the corresponding gain matrices A_i's and B_i's, $i = 1, \ldots, k$, for the delayed state and delayed input, respectively. The sum of integral terms corresponds to the distributed delay effects, weighted by G_j's and H_j's, $j = 1, \ldots, r$, over the time intervals $[t - \tau_j, t]$. The matrices D_l, $l = 1, \ldots, q$ introduced the neutral part to the formulation.

On the other hand, it is common in the literature to only assume discrete delays in the input and state as follows

$$\dot{x} = \sum_{i=0}^{k} [A_i x(t - h_i) + B_i u(t - h_i)]$$

Denote the Laplace transform of $u(t)$ and $y(t)$ with $U(s)$ and $Y(s)$, respectively; then one can find the relation between $U(s)$ and $Y(s)$ in equation (1.2) as

$$Y(s) = C(s)(sI - A(s))^{-1}B(s)U(s)$$
$$C(s) = \sum_{i=0}^{k} C_i e^{-sh_i} + \sum_{j=1}^{r} N_j \frac{1-e^{-s\tau_j}}{s}$$
$$A(s) = \sum_{l=0}^{q} D_l s e^{-s\omega_l} + \sum_{i=0}^{k} A_i e^{-sh_i} + \sum_{j=1}^{r} G_j \frac{1-e^{-s\tau_j}}{s}$$
$$B(s) = \sum_{i=0}^{k} B_i e^{-sh_i} + \sum_{j=1}^{r} H_j \frac{1-e^{-s\tau_j}}{s}$$

The solution of (1.2) can be expressed in terms of the roots of the characteristic equation $\Delta(s) = 0$ and the corresponding spectrum $\sigma(A)$, which are defined as follows [63]

$$\Delta(s) = \det(sI - A(s)), \qquad \sigma(A) = \{s \in \mathbb{C}, \Delta(s) = 0\}$$

In addition, for a retarded time-delay system with discrete delays, the stability of the system is completely determined by its characteristic equation, which is given by

$$\Delta(s) = \det(sI - A_0 - \sum_{i=1}^{k} A_i e^{-sh_i})$$

Specifically, the system is stable if and only if $\Delta(s)$ has no roots in the closed right-half complex plane.

Consider the following LTI system with single delay in state

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \tag{1.3}$$

where A_0 and A_1 are given $n \times n$ real matrices. Sufficient conditions for asymptotic stability of the system (1.3) provided by Razumikhin Theorem are given below [49].

Theorem 1.1. *The time-delay system (1.3) with the maximum time delay $\bar{h}$ is asymptotically stable if there exists a bounded quadratic Lyapunov function V such* *that for some $\epsilon > 0$, it satisfies the inequality*

$$V(x) \geq \epsilon \|x\|^2$$

and its derivative along the system trajectory $\dot{V}(x(t))$ satisfies

$$\dot{V}(x(t)) \leq -\epsilon \|x(t)\|^2$$

whenever

$$V(x(t+\xi)) \leq \eta V(x(t)), \qquad -\bar{h} \leq \xi \leq 0$$

for some real constant scalar $\eta > 1$.

A different form of sufficient conditions for asymptotic stability can also be provided in terms of the existence of a specific Lyapunov functional. These conditions are given by the Lyapunov-Krasovskii Theorem, which is presented below [49].

Theorem 1.2. *The time-delay system (1.3) with the maximum time delay $\bar{h}$ is asymptotically stable if there exists a bounded quadratic Lyapunov-Krasovskii functional $V(\phi)$ such that for some $\epsilon > 0$, it satisfies*

$$V(\phi) \geq \epsilon \|\phi(0)\|^2$$

and its derivative along the system trajectory,

$$\dot{V}(\phi) = \dot{V}(x_t)|_{x_t=\phi}$$

satisfies

$$\dot{V}(\phi(t)) \leq -\epsilon \|\phi(0)\|^2$$

Robust control of linear uncertain time-delay systems

To noticeably ease the discussion in this section, consider the following time-delay system

$$\dot{x} = \sum_{i=0}^{k} A_i x(t-h_i) + Bu(t) \tag{1.4}$$

where

$$(A_0, A_1, \dots, A_k, B) \in \Omega \tag{1.5}$$

and Ω is a compact set referred to as the uncertainty region. In the robust control problem, it is desired to find a state feedback law $u = Kx$ which stabilizes the system given by (1.4) for all the admissible uncertainties characterized by (1.5).

Bounded parametric uncertainty

Two major classes of parametric uncertainty models which are often considered in the literature can be categorized as follows:

- The first category is the traditional norm-bounded uncertainty analysis, in which the system matrices $\omega = (A_0, A_1, \dots, A_k, B)$ are decomposed into two components:

  1. The nominal (deterministic) term $\omega^n = ({A_0}^n, {A_1}^n, \dots, {A_k}^n, B^n)$,
  2. The uncertain term $\Delta\omega = (\Delta A_0, \Delta A_1, \dots, \Delta A_k, \Delta B)$.

  Therefore, $\omega = \omega^n + \Delta\omega$ and

  $$\begin{aligned} A_0 &= {A_0}^n + \Delta A_0, \\ A_1 &= {A_1}^n + \Delta A_1, \\ &\vdots \\ A_k &= {A_k}^n + \Delta A_k, \\ B &= B^n + \Delta B \end{aligned}$$

  The uncertain term is written as

  $$[\Delta A_0 \;\; \Delta A_1 \;\; \dots \;\; \Delta A_k \;\; \Delta B] = LF\,[E_0 \;\; E_1 \;\; \dots \;\; E_k]$$

  where $L, E_0, E_1, \dots, E_k$ are known constant matrices, and F is an uncertain matrix satisfying

  $$\|F\| \leq 1$$

In other words, the uncertainty region Ω can be expressed as

$$\Omega = \{(A_0{}^n + LFE_0, A_1{}^n + LFE_1, \dots, A_k{}^n + LFE_k) \mid \; \|F\| \le 1\}$$

- The second category is called polytopic uncertainty. In this case, there exist ν elements of the set Ω (where ν is any arbitrary positive integer) denoted by

  $$\omega^j = \left(A_0{}^j, A_1{}^j, \dots, A_k{}^j, B^j\right), \qquad j = 1, 2, \dots, \nu$$

  known as vertices, such that Ω can be expressed as the convex hull of these vertices represented as [17]

  $$\Omega = \text{co}\left\{\omega^j | j = 1, 2, \dots, \nu\right\}$$

  In other words, the uncertain set Ω consists of all the convex linear combinations of the vertices

  $$\Omega = \left\{\sum_{j=1}^{\nu} a_j \omega^j | a_j \ge 0, j = 1, 2, \dots, \nu; \sum_{j=1}^{\nu} a_j = 1\right\}$$

  Notice that in practice, there are often some uncertain parameters in the system, which may vary between a lower and upper bound. Moreover, these uncertain parameters often appear linearly in the system matrices. In this case, the collection of all the possible system matrices form a polytopic set [17]. The vertices of this region of uncertainty can be calculated by setting the parameters to either lower or upper bound. If there are n_p uncertain parameters, it is easy to see that there will be $\nu = 2^{n_p}$ vertices.

Uncertain delay

In most of the physical applications, the value of the delay in the system is not known perfectly. However, some *a priori* information about the delay is often available. Two types of uncertain delays have been considered in the literature

- *Constant but unknown delay:* It is sometimes supposed that the delay is not known, but its value is fixed (does not change by time). In this case, an upper bound on the delay is assumed to be available. In other words, for the delay h_i in (1.4),

$$0 \le h_i \le \bar{h}_i, \qquad i = 1, 2, \ldots, k$$

(note that by assumption $h_0 = 0$).

- *Time-varying delay:* In some practical problems, the value of delay changes by time. In many of the reported results in this context, it is assumed that upper bounds on the time-varying delay and its derivative are available; i.e.

$$0 \le h_i(t) \le \bar{h}_i, \qquad \frac{d}{dt} h_i(t) \le \alpha_i < 1, \qquad i = 1, 2, \ldots, k$$

for some constant scalars $\bar{h}_i$ and α_i. This is a realistic assumption in most of the real-world systems with time-varying delay.

Robust stability and stabilizability problems for time-delay systems

The robust control problem and robust stability analysis have attracted much attention in control literature, and numerous papers have been published in this area. To recall some of these works, the reader can refer to the monographs [15, 49] and the references cited therein. It is very difficult to survey all these papers and discuss every single approach individually, but the corresponding results can generally be classified in two main categories: frequency-domain analysis and time-domain analysis. In the latter case, the Lyapunov-Krasovskii Theorem is employed, which is proved to be efficient in practice.

In the sequel, some of the relevant results given in [33] are discussed. Consider a system whose dynamics is subject to constant (but unknown) delay. Assume that the uncertainty in the system dynamics is modeled as norm-bounded parameter perturbation. It is desired to find a stabilizing memoryless state feedback controller.

The resultant stability conditions depend on the size of the delay and are expressed in terms of linear matrix inequalities (LMIs), as will be discussed next.

Problem statement

Consider an uncertain linear time-delay system described by the following state-space representation

$$\begin{aligned}\dot{x}(t) =& (A + \Delta A(t))\, x(t) + (B + \Delta B(t))\, u(t) \\ & + (A_d + \Delta A_d(t))\, x(t-\tau)\end{aligned} \tag{1.6a}$$

$$x(t) = \phi(t), \qquad t \in [-\tau, 0] \tag{1.6b}$$

$$z(t) = Cx(t) + Du(t) \tag{1.6c}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $z(t) \in \mathbb{R}^q$ is the controlled output. Furthermore, τ is a constant time delay, and $\phi(.)$ is the initial condition. A, A_d, B, C and D are known real constant matrices of appropriate dimensions which describe the nominal system associated with (1.6), and $\Delta A, \Delta A_d, \Delta B$ are real matrix functions representing time-varying parameter uncertainties. The admissible uncertainties are assumed to be of the form

$$[\Delta A(t) \quad \Delta B(t)] = LF(t)[E_a \quad E_b], \qquad \Delta A_d(t) = L_d F_d(t) E_d \tag{1.7}$$

where $F(t)$ and $F_d(t)$ are unknown real time-varying matrices satisfying

$$\|F(t)\| \leq 1, \qquad \|F_d(t)\| \leq 1, \qquad \forall t \tag{1.8}$$

and L, L_d, E_a, E_b and E_d are known real constant matrices which characterize how the uncertain parameters in $F(t)$ and $F_d(t)$ alter the nominal matrices A, A_d and B. In the sequel, the definitions of robust stability and robust stabilization are given.

Definition 1.1. *The system (1.6) is said to be robustly stable if the equilibrium solution $x(t) \equiv 0$ of the FDE associated to the system with $u(t) \equiv 0$ is globally uniformly asymptotically stable for all admissible $\Delta A(t)$ and $\Delta A_d(t)$.*

Definition 1.2. *Assume that a control law $u(t) = Kx(t)$ is found for system the (1.6) such that the resulting closed-loop system is robustly stable for any constant time delay τ satisfying $0 \leq \tau \leq \bar{\tau}$, for a given strictly positive scalar $\bar{\tau}$. In this case, the system (1.6) is said to be robustly stabilizable.*

The following theorem borrowed from [33], provides robust stability results for time-delay system (1.6).

Theorem 1.3. *Consider the system described by (1.6) with $u(t) \equiv 0$. Then, for any given strictly positive scalar $\bar{\tau}$, this system is robustly stable for any constant time delay τ satisfying $0 \leq \tau \leq \bar{\tau}$ if there exist symmetric positive definite matrices X, X_1 and X_2, and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_5$ satisfying the following LMI*

$$\begin{bmatrix} M(X, X_1, X_2) & XE^T & A_d(X_1 + X_2)E_d^T & \bar{\tau}XN^T \\ * & -J_1 & 0 & 0 \\ * & * & -J_2 & 0 \\ * & * & * & -J_3 \end{bmatrix} < 0$$

where

$$\begin{aligned} M(X, X_1, X_2) &= X(A + A_d)^T + (A + A_d)X + A_d(X1 + X2)A_d^T \\ &\hat{L}J_1\hat{L}^T + \alpha_3 L_d L_d^T \end{aligned}$$

$$\hat{L} = [L \quad L_d], \qquad E = \left[E_a^T \ \ E_d^T\right]^T, \qquad N = \left[A^T \ \ E_a^T \ \ A_d^T \ \ E_d^T\right]^T \tag{1.9a}$$

$$J_1 = \text{diag}\left\{\alpha_1 I, \alpha_2 I\right\}, \qquad J_2 = \alpha_3 I - E_d(X_1 + X_2)E_d^T \tag{1.9b}$$

$$J_3 = \text{diag}\left\{X_1 - \alpha_4 LL^T, \alpha_4 I, X_2 - \alpha_5 L_d L_d^T, \alpha_5 I\right\} \tag{1.9c}$$

Also, the robust stabilizability condition is provided in an LMI form in the following theorem [33].

Theorem 1.4. *Consider the system described by (1.6). Then, for a given strictly positive scalar $\bar{\tau}$, this system is robustly stabilizable for any constant time delay τ satisfying $0 \leq \tau \leq \bar{\tau}$ if there exist symmetric positive definite matrices X, X_1 and X_2, a matrix Y, and positive scalars $\alpha_1, \alpha_2, \ldots, \alpha_5$ satisfying the following LMI*

$$\begin{bmatrix} M_c(X,Y,X_1,X_2) & E_c{}^T(X,Y) & A_d(X_1+X_2)E_d{}^T & \bar{\tau}N_c{}^T(X,Y) \\ * & -J_1 & 0 & 0 \\ * & * & -J_2 & 0 \\ * & * & * & -J_3 \end{bmatrix} < 0$$

where J_1, J_2 and J_3 are given by (1.9b), (1.9c), and

$$\begin{aligned} M_c(X,Y,X_1,X_2) =& X(A+A_d)^T + (A+A_d)X + Y^TB^T + BY \\ & + A_d(X_1+X_2)A_d{}^T + \hat{L}J_1\hat{L}^T + \alpha_3 L_d L_d{}^T \\ E_c =& [XE_a{}^T + Y^TE_b{}^T \quad XE_d{}^T]^T \\ N_c(X,Y) =& [XA^T + Y^TB^T \quad XE_a{}^T + Y^TE_b{}^T \quad XA_d{}^T \quad XE_d{}^T]^T \end{aligned}$$

Moreover, a stabilizing control law is given by $u(t) = YX^{-1}x(t)$.

In the following section, a summary of the main contribution of each chapter of the thesis is provided.

1.3 Contributions of Thesis

The main contributions of this thesis are as follows:

- Chapter 2 investigates the stabilization problem for interconnected linear time-invariant (LTI) time-delay systems by means of a linear time-invariant output feedback decentralized controller. The delays are assumed to be commensurate and can appear in the states, inputs, and outputs of the system. First, the canonical forms for time-delay systems with commensurate delays are introduced and *centralized fixed modes* (CFM) for this type of systems are defined.

A numerically efficient technique is also proposed for characterizing the CFMs for any LTI time-delay system with commensurate delays. *Decentralized fixed modes* (DFM) are defined subsequently, and a necessary and sufficient condition for decentralized stabilizability of the interconnected time-delay systems is obtained. Finally, two numerical examples are given to illustrate the importance of the results.

- Chapter 3 concerns with decentralized output regulation of hierarchical systems subject to input and output disturbances. It is assumed that the disturbance can be represented as the output of a LTI system with unknown initial state. The primary objective is to design a decentralized controller with the property that not only does it reject the degrading effect of the disturbance on the output (for a satisfactory steady-state performance), it also results in a small Linear quadratic (LQ) cost function (implying a good transient behavior). To this end, the underlying problem is treated in two phases. In the first step, a number of modified systems are defined in terms of the original system. The problem of designing a LQ centralized controller which stabilizes all the modified systems and rejects the disturbance in the original system is considered, and it is shown that this centralized controller can be efficiently found by solving a LMI problem. In the second step, a method recently presented in the literature is exploited to decentralize the designed centralized controller. It is shown that the obtained controller satisfies the pre-determined design specifications including disturbance rejection. A numerical example is presented to elucidate the efficacy of the proposed control law.

- Chapter 4 investigates the control problem for a group of cooperative spacecraft with communication constraints. It is assumed that a set of cooperative local controllers with all-to-all communication is given which satisfies the desired objectives of the formation. It is to be noted that due to the information exchange between the local controllers, the overall control structure can be considered centralized, in general. However, communication in flight formation is expensive. Thus, it is desired to have some form of decentralization in control structure, which has a lower communication requirement. A decentralized controller is obtained based on the technique originally proposed in [70, 79], which consists of local estimators such that each local controller estimates the state of the whole formation implicitly. Necessary and sufficient conditions for the stability of the formation under the proposed decentralized controller is attained, and its robustness is studied. It is then shown that the resultant decentralized controller can be converted to a cooperative predictive controller in such a way that most of the features of its centralized counterpart such as the collision avoidance capability are preserved.

- In Chapter 5, a decentralized overlapping static output feedback law is proposed to control a LTI interconnected system. It is assumed that an overlapping information flow structure is given which determines which output measurements are available for any local control agent. Uncertain transmission delay is also considered in communication links among different subsystems. Each subsystem is assumed to be subject to input disturbances with finite energy (or power). A necessary condition for the existence of a stabilizing overlapping controller is obtained which is easy to check. Furthermore, a LMI-based design methodology is proposed to achieve internal stability and H_∞ disturbance attenuation. Simulations are presented to demonstrate the efficacy of the developed results.

- Adaptive switching control schemes are known to be very effective for handling large uncertainty in controlling dynamical systems. Most of the existing switching control techniques are developed specifically for finite-dimensional LTI systems. In many practical applications, however, it is essential to take time delay into consideration in the modeling as the overall closed-loop system can be highly sensitive to delay. In Chapter 6, a multi-model switching control algorithm is proposed for retarded time-delay systems. It is assumed that the plant is represented by a family of known multi-input multi-output, observable, LTI models with multiple delays in the states, and that corresponding to each model in the known family, there exists a high-performance finite-dimensional LTI controller. In addition, it is supposed that a bound on the magnitude of the external inputs and disturbances is available. It is then shown that the proposed switching controller can stabilize the uncertain system, and that under some mild conditions, output tracking can be achieved in the given problem setting.

1.4 Publications

The results of this Ph.D. thesis are published or submitted for publication in the following journals and conferences:

- Chapter 2
  - A. Momeni and A. G. Aghdam, "A necessary and sufficient condition for stabilization of decentralized time-delay systems with commensurate delays," in *Proceedings of the 47^{th} IEEE Conference on Decision and Control*, Cancun, Mexico, pp. 5022–5029, 2008.
  - A. Momeni, A. G. Aghdam, and E. J. Davison, "On the stabilization of decentralized time-delay systems," to be submitted to *IEEE Transactions on Automatic Control*, 2009.
- Chapter 3

- J. Lavaei, A. Momeni, and A. G. Aghdam, "LQ suboptimal decentralized controllers with disturbance rejection property for hierarchical systems," *International Journal of Control*, vol. 81, no. 11, pp. 1720-1732, 2008.
- J. Lavaei, A. Momeni, and A. G. Aghdam, "A near-optimal decentralized servomechanism controller for hierarchical interconnected systems," in *Proceedings of the 46^{th} IEEE Conference on Decision and Control*, New Orleans, LA, pp. 5023-5030, 2007.

- Chapter 4

  - J. Lavaei, A. Momeni, and A. G. Aghdam, "A model predictive decentralized control scheme with reduced communication requirement for spacecraft formation," *IEEE Transactions on Control Systems Technology*, vol. 16, no. 2, pp. 268-278, 2008.
  - J. Lavaei, A. Momeni, and A. G. Aghdam, "A cooperative model predictive control technique for spacecraft formation flying," in *Proceedings of American Control Conference*, New York, NY, pp. 1493-1500, 2007.

- Chapter 5

  - A. Momeni and A. G. Aghdam, "Overlapping control systems with delayed communication channels: stability analysis and controller design," to appear in *Proceedings of American Control Conference*, 2008.
  - A. Momeni and A. G. Aghdam, "Overlapping control systems with delayed communication channels: stability analysis and controller design," submitted to *Automatica*, 2009.

- Chapter 6

– A. Momeni and A. G. Aghdam, "An adaptive tracking problem for a family of retarded time-delay plants," *International Journal of Adaptive Control and Signal Processing*, vol. 21, pp. 885-910, 2007.

– A. Momeni and A. G. Aghdam, "Switching control for time-delay systems," in *Proceedings of American Control Conference*, Minneapolis, MN, pp. 5432-5434, 2006.

Chapter 2

Stabilization of Decentralized Time-Delay Systems

2.1 Introduction

Design of a high-performance controller for interconnected systems is an important challenge in control theory [28, 74, 136]. Networked unmanned aerial vehicles (UAV), automated highway systems and automated manufacturing processes all involve multiple, interacting, and highly dynamic entities [31, 83, 135]. Elements of these systems, usually called subsystems, are distributed in space and must coordinate with each other using sensing and communication networks. Furthermore, the overall representation of an interconnected system often imposes high-order dynamics with several input and output channels. For this type of systems, since it is not typically feasible to perform all the control computations at one single point, it is more desirable to have a distributed control scheme. By means of a distributed implementation, a more reliable control system is obtained which is less sensitive to failures, and has lower computational requirement [136]. In addition, distributed implementation of a high-performance centralized controller requires high levels of connectivity between the subsystems. Therefore, since it is not realistic to assume that all output measurements can be transmitted to every local controller, there are some constraints on information exchange imposed between different subsystems; i.e., full output access is rarely possible. A special case of constrained control structure is the one with diagonal (or block-diagonal) information flow, which is often referred to as a decentralized control system [81, 149]. In this type of systems, each local control station only has access to the measurements of its corresponding subsystem for generating the local control input [28]. All control stations contribute, however, to the overall control operation.

A fundamental question in the analysis and design of decentralized control systems is that under what conditions a set of local feedback control laws exists to achieve stability or arbitrary pole-placement in a given region in the s-plane. The notion of a *decentralized fixed mode* (DFM) was introduced in [149] to address this question for finite-dimensional Linear time-invariant (LTI) systems. It was shown in [28] that a DFM remains fixed in the complex plane, using any LTI decentralized dynamic controller. In other words, there exists no LTI decentralized controller to place a mode of a LTI system freely in the complex plane if and only if that mode is fixed with respect to any constant decentralized controller.

On the other hand, actuators, sensors, and communication networks in feedback control systems often introduce delays in closed-loop dynamics. There are numerous examples in biology, high-speed communication networks, robotics, etc. where the effect of delay cannot be neglected in control design and analysis [15]. Time-delay systems have been studied extensively in the past few decades and several results have been reported in the literature (for example, see [49, 104] and references therein). The dynamics of this type of systems is represented by a class of functional differential equations (FDE) which are infinite-dimensional, as opposed to ordinary differential equations (ODE). The stability margin of a time-delay system can be highly sensitive to delay and small variation in delay may lead to instability [105].

The stability analysis for time-delay systems has been a topic of longstanding interest, and particularly LTI systems with commensurate delays have been investigated intensively (see [49], [57], [153] and the references therein). To study the stability of this class of time-delay systems, a two-variable criterion was introduced in [57]. Further development of this technique led to a variety of stability tests for systems with commensurate delays, such as polynomial elimination and pseudo-delay methods. As an alternative, frequency sweeping tests are also effective tools for analyzing the stability of LTI systems with commensurate delays [49]. It is worth mentioning that most of the existing results on this subject have been developed for systems with a centralized control structure. However, there has been a growing interest recently in the problem of decentralized stabilization of large-scale time-delay interconnected systems (see, e.g. [151]).

This manuscript deals with the problem of stabilizability of the interconnected time-delay systems with commensurate delays in the state variables, inputs and outputs, by means of decentralized controllers. It is shown that by considering delay operators as the elements of a properly defined ring of polynomials, the original delay-differential system representation can be converted into a ring model description. In order to find the stabilizability conditions for the system under the LTI output feedback control, the concepts of controllability and observability are first used to obtain a canonical state-space representation (analogously to the Kalman canonical form in the finite-dimensional case) of this class of time-delay systems. Next, the notion of μ-centralized fixed modes (μ-CFM) is introduced for this class of time-delay systems, and it is shown that a mode of a time-delay system is both controllable and observable if only if it is movable by means of a static output feedback controller. The notion of fixed modes is also extended to decentralized LTI time-delay systems in order to define μ-decentralized fixed modes (μ-DFM) for this type of systems in a manner similar to [149]. A simple necessary and sufficient condition is sought in this chapter to check when a LTI time-delay system can be stabilized using a decentralized LTI dynamic controller. A computational algorithm is then proposed to obtain the set of μ-DFMs of an interconnected time-delay system. Furthermore, some algebraic conditions are provided to determine if a mode of a time-delay system is a μ-DFM.

The remainder of the chapter is organized as follows. In Section 2.2, a convenient notation is given and the problem statement is introduced. The main results of the chapter which are the stabilizability conditions for decentralized LTI time-delay systems are then presented in Section 2.3. Two numerical examples are provided in Section 2.4 to illustrate the importance of the results.

2.2 Problem Formulation

2.2.1 Notations

- h is the delay, and λ is the delay operator; i.e. $\lambda f(t) = f(t-h)$, where f is a function of time t.

- $R[\lambda]$ denotes the ring of polynomials in λ with real coefficients, where λ is the delay operator.

- $A(\lambda) \in R^{m\times n}[\lambda]$ denotes the set of $m \times n$ matrices over $R[\lambda]$.

- For $A(\lambda) \in R^{n\times n}[\lambda]$ with degree k in λ, let $A(\lambda)x(t)$ be defined as follows

$$A(\lambda)x(t) = \sum_{j=0}^{k} A^j x(t-jh)$$

  where $A^j \in \mathbb{R}^{n\times n}$ is a constant matrix for any $j \in \{0, 1, \dots, k\}$.

- Given a matrix $F \in \mathbb{C}^{r\times s}$ with its i-th column represented by f_i, $i = 1, 2, \dots, s$, define vec(F) as

$$\text{vec}(F) = \begin{bmatrix} f_1{}^T & f_2{}^T & \cdots & f_s{}^T \end{bmatrix}^T$$

2.2.2 Preliminaries

Consider the following interconnected LTI time-delay system with ν subsystems subject to commensurate delays [49]

$$\begin{aligned} \dot{x}(t) &= \sum_{j=1}^{k_1} A^j x(t-jh) + \sum_{i=1}^{\nu}\sum_{j=1}^{k_2} B_i^j u_i(t-jh) \\ y_i(t) &= \sum_{j=1}^{k_3} C_i^j x(t-jh), \qquad i \in \bar{\nu} := \{1,2,\ldots,\nu\} \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{p_i}$ are the input and output of the i-th local subsystem, respectively. The matrices $A^j \in \mathbb{R}^{n\times n}$, $B_i^j \in \mathbb{R}^{n\times m_i}$ and $C_i^j \in \mathbb{R}^{p_i\times n}$ are assumed to be real and constant. It is to be noted that in (2.1), commensurate delays can exist in the input, state and output.

Using the λ-operator, the system (2.1) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + \sum_{i=1}^{\nu} B_i(\lambda)u_i(t) \\ y_i(t) &= C_i(\lambda)x(t), \qquad i \in \bar{\nu} \end{aligned} \tag{2.2}$$

where $A(\lambda) \in R^{n\times n}[\lambda]$, $B_i(\lambda) \in R^{n\times m_i}[\lambda]$, and $C_i(\lambda) \in R^{p_i\times n}[\lambda]$. In the problem of decentralized control system design, the primary goal is to find ν local output controllers to stabilize the system. In this work, it is desired to design ν local stabilizing controllers of the following form

$$\begin{aligned} \dot{z}_i(t) &= \Gamma_i z_i(t) + R_i y_i(t) \\ u_i(t) &= Q_i z_i(t) + K_i y_i(t), \qquad i \in \bar{\nu} \end{aligned} \tag{2.3}$$

where $z_i(t) \in \mathbb{R}^{\eta_i}$ is the state of the i-th local controller. Γ_i, R_i, Q_i and K_i are the real constant matrices of appropriate size.

Definition 2.1. *Consider the LTI time-delay interconnected system (2.1). Corresponding to $A(\lambda) \in R^{n\times n}[\lambda]$, the matrix $A(e^{-sh})$ is defined as*

$$A(e^{-sh}) := A(\lambda)|_{\lambda=e^{-sh}} \tag{2.4}$$

It is straightforward to verify that

$$\mathcal{L}\{A(\lambda)x(t)\} = A(e^{-sh})X(s)$$

where $\mathcal{L}\{.\}$ denotes the Laplace transform operator, and $X(s)$ is the Laplace transform of $x(t)$.

Definition 2.2. *Similar to $A(e^{-sh})$ and corresponding to the system (2.1), the matrices $B_i(e^{-sh})$ and $C_i(e^{-sh})$, $i \in \bar{\nu}$, can also be defined as*

$$B_i(e^{-sh}) := B_i(\lambda)|_{\lambda=e^{-sh}}, \quad C_i(e^{-sh}) := C_i(\lambda)|_{\lambda=e^{-sh}}$$

Furthermore, let $B(\lambda)$ and $C(\lambda)$ be constructed as follows

$$B(\lambda) = \begin{bmatrix} B_1(\lambda) & B_2(\lambda) & \dots & B_\nu(\lambda) \end{bmatrix} \tag{2.5}$$

$$C^T(\lambda) = \begin{bmatrix} C_1{}^T(\lambda) & C_2{}^T(\lambda) & \dots & C_\nu{}^T(\lambda) \end{bmatrix} \tag{2.6}$$

and define

$$B(e^{-sh}) := B(\lambda)|_{\lambda=e^{-sh}}, \qquad C(e^{-sh}) := C(\lambda)|_{\lambda=e^{-sh}} \tag{2.7}$$

Remark 2.1. *It is important to recognize that $A(e^{-sh})$, $B(e^{-sh})$ and $C(e^{-sh})$ are matrix quasi-polynomials of s. This property is very important in developing the main results of the chapter.*

Definition 2.3. *Consider ν local controllers given in (2.3). Define the following matrices*

$$\Gamma := \textit{block diagonal}[\Gamma_1, \Gamma_2, \dots, \Gamma_\nu], \qquad R := \textit{block diagonal}[R_1, R_2, \dots, R_\nu]$$

$$Q := \textit{block diagonal}[Q_1, Q_2, \dots, Q_\nu], \qquad K := \textit{block diagonal}[K_1, K_2, \dots, K_\nu]$$

Define also

$$K^e = \begin{bmatrix} K & Q \\ R & \Gamma \end{bmatrix} \tag{2.8}$$

Definition 2.4. *Consider the system (2.2), and assume that there is no delay in the system, i.e. $h = 0$. Then the eigenvalue $\lambda \in \mathrm{sp}(A)$ is called a DFM of the system if it is fixed with respect to any constant decentralized feedback gain matrix K whose i-th entry on the main diagonal is an arbitrary $m_i \times p_i$ matrix. In other words, λ is a DFM of the system if*

$$\lambda \in \bigcap \mathrm{sp}(A + BKC), \qquad \forall K = \text{block diag}[K_1, K_2, \ldots, K_\nu] \tag{2.9}$$

One can easily verify (2.9) numerically, using proper software such as MATLAB with a random number generator to generate the gain matrices. It is to be noted that a similar approach can be used to characterize centralized fixed modes *(CFM) of a system, which are, in fact, the unobservable OR uncontrollable modes of it [30].*

Problem Statement: The objective is to find a necessary and sufficient condition for the stabilizability of the interconnected system (2.1) under the decentralized output feedback of the form (2.3).

2.3 Main Results

It is desired now to investigate the stability of the system (2.2) under the controller of the form (2.3). The following lemma is the key for the proof of Theorem 2.3.

Lemma 2.1. *Consider the decentralized controller (2.1); then, there exist integers η_i, $i = 1, 2, ..., \mu$ and a matrix K^e given by (2.8), such that the closed-loop system obtained by applying (2.3) to (2.1) is asymptotically stable if and only if all roots of the quasi-polynomial*

$$\det(sI - A^e(e^{-sh}) - B^e(e^{-sh})K^eC^e(e^{-sh}))$$

are located in the open left-half complex plane, where

$$A^e(e^{-sh}) = \begin{bmatrix} A(e^{-sh}) & 0 \\ 0 & 0 \end{bmatrix}, \quad B^e(e^{-sh}) = \begin{bmatrix} B(e^{-sh}) & 0 \\ 0 & I \end{bmatrix}, \quad C^e(e^{-sh}) = \begin{bmatrix} C(e^{-sh}) & 0 \\ 0 & I \end{bmatrix} \tag{2.10}$$

and $A(e^{-sh})$, $B(e^{-sh})$, $C(e^{-sh})$ *are all given in Definitions 2.1 and 2.2.*

Proof: By augmenting the states of all ν local controllers in (2.3), the following state-space model is obtained for the decentralized controller

$$\dot{z}(t) = \Gamma z(t) + Ry(t)$$
$$u(t) = Qz(t) + Ky(t)$$

where

$$z^T(t) = \begin{bmatrix} z_1{}^T(t) & z_2{}^T(t) & \dots & z_\nu{}^T(t) \end{bmatrix}$$
$$y^T(t) = \begin{bmatrix} y_1{}^T(t) & y_2{}^T(t) & \dots & y_\nu{}^T(t) \end{bmatrix}$$
$$u^T(t) = \begin{bmatrix} y_1{}^T(t) & y_2{}^T(t) & \dots & y_\nu{}^T(t) \end{bmatrix}$$

On the other hand, the state space equations of the system (2.2) in the Laplace domain can be written as follows

$$sX(s) = A(e^{-sh})X(s) + B(e^{-sh})U(s)$$
$$Y(s) = C(e^{-sh})X(s)$$

where $X(s)$, $U(s)$, $Y(s)$ are Laplace transforms of $x(t)$, $u(t)$ and $y(t)$, respectively.

If the above decentralized feedback law is applied to the system (2.2), the closed-loop system in the Laplace domain can be described by

$$s\begin{bmatrix} X(s) \\ Z(s) \end{bmatrix} = \begin{bmatrix} A(e^{-sh}) + B(e^{-sh})KC(e^{-sh}) & B(e^{-sh})Q \\ RC(e^{-sh}) & \Gamma \end{bmatrix} \begin{bmatrix} X(s) \\ Z(s) \end{bmatrix}$$

Therefore, the system (2.1) under the decentralized controller with the local feedback laws in (2.3) is asymptotically stable if and only if all the zeros of $\pi(s)$, defined below

$$\pi(s) := \det\left(sI - \begin{bmatrix} A(e^{-sh}) + B(e^{-sh})KC(s) & B(e^{-sh})Q \\ RC(e^{-sh}) & \Gamma \end{bmatrix} \right)$$

are in the open left-half complex plane. In addition, it is straightforward to show that the above expression simplifies to

$$\pi(s) = \det(sI - A^e(e^{-sh}) - B^e(e^{-sh})K^eC^e(e^{-sh}))$$

where $A^e(e^{-sh})$, $B^e(e^{-sh})$, $C^e(e^{-sh})$ are defined in (2.10). This completes the proof. ■

The following lemma states that the characteristic equation of a matrix $A(\lambda) \in R^{n\times n}[\lambda]$ cannot have roots at $+\infty + \mathbf{i}b$ (for finite or infinite b) and $a \pm \mathbf{i}\infty$ (for finite a), where $\mathbf{i}^2 = -1$. This lemma is used for developing the main results of the chapter.

Lemma 2.2. *Given $A(\lambda) \in R^{n\times n}[\lambda]$ with degree k in λ, let the characteristic equation be represented by*

$$\phi(s) = \det\left(sI - A(e^{-sh})\right) \tag{2.11}$$

Furthermore, let $s = a + \mathbf{i}b$ be an arbitrary complex root of $\phi(s)$, where a and b are real numbers. Then,

- *a is not an arbitrarily large positive number ($a \neq +\infty$);*
- *if a is finite (i.e., if $a \neq -\infty$), then b is finite as well.*

Proof: The characteristic equation $\phi(s)$ has the following form:

$$\phi(s) = \zeta_0(s) + \sum_{l=1}^{l_f} \zeta_l(s)e^{-lhs} \tag{2.12}$$

where $l_f := k^n$, $\zeta_0(s)$ is a monic polynomial of degree n, and the functions $\zeta_l(s)$, $l = 1, 2, \ldots, l_f$, are polynomials of degree at most $n - 1$ [49]. Since $\phi(s)$ has a principal term, a cannot be an arbitrarily large positive number [49], and hence $a \neq +\infty$. On the other hand, if s in (2.12) is replaced by $a + \mathbf{i}b$, two equations (in terms of a and b) will be obtained, which correspond to the real and imaginary parts of (2.12). Both of these equations can be expressed as a combination of polynomials, exponentials, sinusoidals, and their products. More specifically, one of these two equations (depending on whether n is even or odd) can be written as

$$P_0(a,b) + \sum_{l=1}^{l_f} P_l^1(a,b)e^{-lha}\sin(lhb) + \sum_{l=1}^{l_f} P_l^2(a,b)e^{-lha}\cos(lhb) = 0 \qquad (2.13)$$

where $P_0(a,b)$ is a polynomial of degree n with respect to b, and $P_l^1(a,b)$, $P_l^2(a,b)$, $l = 1,2,\ldots,l_f$, are polynomials of degree at most $n-1$ with respect to b. Now, let a be a fixed finite number and assume that b goes to infinity. In this case, one can verify that the left side of (2.13) will go to infinity as well. Therefore, $a \pm \mathbf{i}\infty$ cannot be a root of $\phi(s)$. This completes the proof. ∎

2.3.1 Kalman canonical representation of LTI time-delay systems with commensurate delays

In this subsection, the following LTI time-delay system with commensurate delays is considered

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + B(\lambda)u(t) \\ y(t) &= C(\lambda)x(t) \end{aligned} \qquad (2.14)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. Moreover, $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are matrices over $R[\lambda]$ with appropriate size. The transfer function of the system (2.14) is given by

$$G(s) = C(e^{-sh})\left(sI - A(e^{-sh})\right)^{-1} B(e^{-sh})$$

where $A(e^{-sh})$, $B(e^{-sh})$ and $C(e^{-sh})$ are given by (2.4)-(2.7). Controllability and observability of the time-delay systems will be defined next.

Definition 2.5. *In this work, the system (2.14) is called controllable if the matrix*

$$\left[\begin{array}{cccc} B(\lambda) & A(\lambda)B(\lambda) & \cdots & (A(\lambda))^{n-1}B(\lambda) \end{array}\right] \qquad (2.15)$$

is full-rank over $R[\lambda]$ [115]. Similarly, the system (2.14) is called observable if the matrix

$$\left[\begin{array}{cccc} C^T(\lambda) & A^T(\lambda)C^T(\lambda) & \cdots & \left(A^T(\lambda)\right)^{n-1}C^T(\lambda) \end{array}\right]^T \qquad (2.16)$$

is full-rank over $R[\lambda]$.

It can be shown that the system (2.14) is controllable if and only if [46]

$$\text{rank}\begin{bmatrix} sI - A(e^{-sh}) & B(e^{-sh}) \end{bmatrix} = n, \qquad \forall s \in \mathbb{C} \tag{2.17}$$

Analogously, the system (2.14) is observable if and only if

$$\text{rank}\begin{bmatrix} sI - A(e^{-sh}) \\ C(e^{-sh}) \end{bmatrix} = n, \qquad \forall s \in \mathbb{C} \tag{2.18}$$

Based on the controllability and observability notions given above, a set of unimodular transformations are defined in [115] over $R[\lambda]$, to separate the uncontrollable and unobservable parts of a time-delay system. One can then arrive at a Kalman canonical form for time-delay systems with commensurate delays. This is pointed out in the following theorem.

Theorem 2.1. *Consider the system (2.14). Assume that*

- *The rank of the controllability matrix corresponding to the pair* $(A(\lambda), B(\lambda))$ *is* $n_c < n$. *Define* $n_{\bar{c}} := n - n_c$.
- *The rank of the observability matrix corresponding to the pair* $\left(\bar{C}_{c_1}(\lambda), \bar{A}_{c_1}(\lambda)\right)$ *is* $n_{oc} < n_c$ *where* $\left(\bar{C}_{c_1}(\lambda), \bar{A}_{c_1}(\lambda), \bar{B}_{c_1}(\lambda)\right)$ *is the controllable part of the system (2.14). Also, define* $n_{\bar{o}c} := n_c - n_{oc}$.

Then,

1. *There exists a unimodular matrix* $\tilde{T}(\lambda)$ *such that the triple* $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ *defined as*

$$\left(C(\lambda)\tilde{T}(\lambda), \tilde{T}^{-1}(\lambda)A(\lambda)\tilde{T}(\lambda), \tilde{T}^{-1}(\lambda)B(\lambda)\right)$$

has the following form

$$\tilde{A}(\lambda) = \begin{bmatrix} \tilde{A}_{11}(\lambda) & 0 & \tilde{A}_{13}(\lambda) \\ \tilde{A}_{21}(\lambda) & \tilde{A}_{22}(\lambda) & \tilde{A}_{23}(\lambda) \\ 0 & 0 & \tilde{A}_{33}(\lambda) \end{bmatrix}$$

$$\tilde{B}(\lambda) = \begin{bmatrix} \tilde{B}_1(\lambda) \\ \tilde{B}_2(\lambda) \\ 0 \end{bmatrix}, \quad \tilde{C}(\lambda) = \begin{bmatrix} \tilde{C}_1(\lambda) & 0 & \tilde{C}_2(\lambda) \end{bmatrix} \tag{2.19}$$

where $\tilde{A}_{11}(\lambda) \in R^{n_{oc}\times n_{oc}}[\lambda]$, $\tilde{A}_{22}(\lambda) \in R^{n_{\bar{o}c}\times n_{\bar{o}c}}[\lambda]$, $\tilde{A}_{33}(\lambda) \in R^{n_{\bar{c}}\times n_{\bar{c}}}[\lambda]$, $\tilde{B}_1(\lambda) \in R^{n_{oc}\times m}[\lambda]$, $\tilde{B}_2(\lambda) \in R^{n_{\bar{o}c}\times m}[\lambda]$, $\tilde{C}_1(\lambda) \in R^{p\times n_{oc}}[\lambda]$, $\tilde{C}_2(\lambda) \in R^{p\times n_{\bar{c}}}[\lambda]$, *and the triple* $\left(\tilde{C}_1(\lambda), \tilde{A}_{11}(\lambda), \tilde{B}_1(\lambda)\right)$ *is both controllable and observable.*

2. *The transfer function matrix is given by*

$$G(s) = \tilde{C}_1(e^{-sh})\left(sI - \tilde{A}_{11}(e^{-sh})\right)^{-1}\tilde{B}_1(e^{-sh})$$

where $\tilde{C}_1(e^{-sh})$, $\tilde{A}_{11}(e^{-sh})$, *and* $\tilde{B}_1(e^{-sh})$ *are obtained from* $\tilde{C}_1(\lambda)$, $\tilde{A}_{11}(\lambda)$, *and* $\tilde{B}_1(\lambda)$, *respectively, by substituting* λ *with* e^{-sh}, *similar to Definitions 2.1 and 2.2.*

Remark 2.2. *The triple* $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ *will be referred to as the Kalman canonical form of the original system* $(C(\lambda), A(\lambda), B(\lambda))$ *(analogously to the finite-dimensional case).*

2.3.2 Centralized fixed modes for LTI time-delay systems with commensurate delays

Definition 2.6. *For* $A(\lambda) \in R^{n\times n}[\lambda]$, *let the set* $\Omega_\mu(A(\lambda))$ *be defined as*

$$\Omega_\mu(A(\lambda)) = \{s | s \in \mathbb{C},\ \mathrm{Re}\{s\} \geq \mu,\ \det\left(sI - A(e^{-sh})\right) = 0\} \tag{2.20}$$

The above set is indeed the set of the modes of $A(\lambda)$ *in the closed right of the line* $\mathrm{Re}\{s\} = \mu$. *It is worth mentioning that* $\Omega_\mu(A(\lambda))$ *is a finite set [121].*

Let K_c denote the set of all $m \times p$ matrices with arbitrary real entries. The following definition is essential in the presentation of the main results of the chapter.

Definition 2.7. *Consider the system (2.14). For a constant $\mu \in \mathbb{R}$, the set of μ-centralized fixed modes (μ-CFM) of the system (2.14), denoted by*

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right),$$

is defined as follows

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0, \forall K \in \mathrm{K}_c\}$$

where

$$\phi(s) = \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)$$

Notice that

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right) \subseteq \Omega_\mu \left(A(\lambda)\right)$$

In what follows, it is shown that a controllable and observable mode (in the sense of (2.17) and (2.18)) of the time delay system (2.14) which lies in the right side of the line $\mathrm{Re}\{s\} = \mu$ cannot be a μ-CFM and vice versa. To this end, the following lemma, a modified version of a result obtained originally in [7], is essential.

Lemma 2.3. *Let matrices $A(e^{-sh}) \in \mathbb{C}^{n\times n}$, $\hat{B}_i(e^{-sh}) \in \mathbb{C}^{n\times \pi_i}$, and $\hat{C}_i(e^{-sh}) \in \mathbb{C}^{\pi_i \times n}$, $i \in \bar{\nu}$, be given. For any $s \in \mathbb{C}$,*

$$sI - A(e^{-sh}) - \sum_{i=1}^{\nu} \hat{B}_i(e^{-sh})\hat{K}_i\hat{C}_i(e^{-sh})$$

is not full-rank for all $\hat{K}_i \in \mathbb{R}^{\pi_i \times \pi_i}$ if and only if

$$\begin{bmatrix} sI - A(e^{-sh}) & \hat{B}_1(e^{-sh}) & \hat{B}_2(e^{-sh}) & \cdots & \hat{B}_\nu(e^{-sh}) \\ \hat{C}_1(e^{-sh}) & L_1 & 0 & \cdots & 0 \\ \hat{C}_2(e^{-sh}) & 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{C}_\nu(e^{-sh}) & 0 & 0 & \cdots & L_\nu \end{bmatrix}$$

is not full-rank for all $\pi_i \times \pi_i$ real matrices L_i, $i \in \bar{\nu}$.

Lemma 2.4. *Consider the system (2.14), and choose an arbitrary* $s_0 \in \Omega_\mu(A(\lambda))$ *and a finite* $\mu \in \mathbb{R}$. *A necessary and sufficient condition for*

$$s_0 \notin \Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c) \tag{2.21}$$

is that the following two statements hold

i) $\operatorname{rank}\begin{bmatrix} s_0 I - A(e^{-s_0 h}) & B(e^{-s_0 h}) \end{bmatrix} = n;$

ii) $\operatorname{rank}\begin{bmatrix} s_0 I - A^T(e^{-s_0 h}) & C^T(e^{-s_0 h}) \end{bmatrix} = n.$

Proof: Define

$$\rho_0(K) = \det\left(s_0 I - A(e^{-s_0 h}) - B(e^{-s_0 h}) K C(e^{-s_0 h})\right)$$

as a $(m \times p)$-variable polynomial in entries of K. It is shown in the following that $\rho_0(K)$ is identically zero if and only if at least one of the statements in this lemma is violated. In the sequel, suppose that for all $K \in \mathrm{K}_c$

$$\rho_0(K) \equiv 0 \tag{2.22}$$

Construct the matrices $\hat{B}(e^{-s_0 h})$ and $\hat{C}(e^{-s_0 h})$ as follows

$$\hat{B}(e^{-s_0 h}) = \begin{cases} \begin{bmatrix} B(e^{-s_0 h}) & 0_{n\times(p-m)} \end{bmatrix}, & p > m \\ B(e^{-s_0 h}), & m \geq p \end{cases}$$

$$\hat{C}(e^{-s_0 h}) = \begin{cases} \begin{bmatrix} C(e^{-s_0 h}), \\ 0_{(m-p)\times n} \end{bmatrix}, & m > p \\ C(e^{-s_0 h}), & p \geq m \end{cases}$$

where $0_{n\times(p-m)}$ and $0_{(m-p)\times n}$ are zero matrices of the specified dimensions. Therefore, it follows from (2.22) that for all $\hat{K} \in \mathbb{R}^{\pi\times\pi}$

$$s_0 I - A(e^{-s_0 h}) - \hat{B}(e^{-s_0 h}) \hat{K} \hat{C}(e^{-s_0 h})$$

is not full-rank, where $\pi := \max\{m, p\}$. From Lemma 2.3, it is concluded that

$$\begin{bmatrix} s_0 I - A(e^{-s_0 h}) & \hat{B}(e^{-s_0 h}) \\ \hat{C}(e^{-s_0 h}) & L \end{bmatrix} \tag{2.23}$$

is not full-rank for all $\pi \times \pi$ constant real matrices L. On the other hand, the above matrix can be written as

$$\begin{bmatrix} M_1(e^{-s_0 h}) & M_2(e^{-s_0 h}) + N_2 Q_2 \end{bmatrix}$$

where

$$M_1(e^{-s_0 h}) = \begin{bmatrix} s_0 I - A(e^{-s_0 h}) \\ \hat{C}(e^{-s_0 h}) \end{bmatrix}, \qquad M_2(e^{-s_0 h}) = \begin{bmatrix} \hat{B}(e^{-s_0 h}) \\ 0 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad Q_2 = L$$

It results from Lemma 1 presented in [7] that at least one of the two statements in the present lemma is violated. Since the above argument is reversible, the proof of necessity follows immediately. ■

The following theorem presents a simple approach for finding μ-CFMs of system (2.14) from the Kalman canonical decomposition.

Theorem 2.2. *Suppose that $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ is the corresponding Kalman canonical form for the triple $(C(\lambda), A(\lambda), B(\lambda))$. Then,*

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0, \forall K \in \mathrm{K}_c\}$$

where

$$\phi(s) = \prod_{i=2}^{3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

and $\tilde{A}_{22}(e^{-sh})$, $\tilde{A}_{33}(e^{-sh})$ are obtained from $\tilde{A}_{22}(\lambda)$, $\tilde{A}_{33}(\lambda)$, respectively, similar to Definitions 2.1 and 2.2.

Proof: Consider $\phi(s)$ in Definition 2.7. It can be shown that

$$\phi(s) = \det\left(sI - \tilde{A}_{11}(e^{-sh}) - \tilde{B}_1(e^{-sh})K\tilde{C}_1(e^{-sh})\right) \times \det\left(sI - \tilde{A}_{22}(e^{-sh})\right) \times \det\left(sI - \tilde{A}_{33}(e^{-sh})\right)$$

From Theorem 2.1, it is known that $\left(\tilde{C}_1(\lambda), \tilde{A}_{11}(\lambda), \tilde{B}_1(\lambda)\right)$ is both controllable and observable. On the other hand, according to Lemma 2.4, there is no finite $s \in \mathbb{C}$ such that

$$\det\left(sI - \tilde{A}_{11}(e^{-sh}) - \tilde{B}_1(e^{-sh})K\tilde{C}_1(e^{-sh})\right) = 0$$

for any $K \in \mathrm{K}_c$. This means that s belongs to $\Lambda_\mu\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right)$ if and only if it is a root of $\prod_{i=2}^{i=3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$. ∎

2.3.3 Decentralized fixed modes for LTI time-delay systems with commensurate delays

Definition 2.8. *Consider the system (2.2), and let K_d denote the set of all block diagonal matrices given below*

$$\mathrm{K}_d = \left\{K | K = \textit{block diagonal}[K_1, K_2, \ldots, K_\nu], K_i \in \mathbb{R}^{m_i \times p_i}, i \in \bar{\nu}\right\} \tag{2.24}$$

For a constant $\mu \in \mathbb{R}$, the set of μ-decentralized fixed modes (μ-DFM) of the system (2.2), denoted by

$$\Lambda_\mu\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right),$$

is defined as follows

$$\Lambda_\mu\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) = \{s | s \in \mathbb{C},\ \mathrm{Re}\{s\} \geq \mu,\ \phi(s) = 0,\ \forall K \in \mathrm{K}_d\}$$

where

$$\phi(s) = \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)$$

Using Definition 2.8, a necessary and sufficient condition for the stabilizability of the system (2.2) under decentralized LTI controllers is obtained later on in Theorem 2.3. However, some initial results need to be drived first. Lemma 2.5 is essential for proving the necessity of the condition, while Lemma 2.7 is required to show the condition obtained is sufficient as well, using the result of Lemma 2.6.

Lemma 2.5. *Consider the system (2.2) and define*

$$A^e(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad B^e(\lambda) = \begin{bmatrix} B(\lambda) & 0 \\ 0 & I \end{bmatrix}, \quad C^e(\lambda) = \begin{bmatrix} C(\lambda) & 0 \\ 0 & I \end{bmatrix} \tag{2.25}$$

Denote with K_d^e *the set of all* $(m+p) \times (m+p)$ *real constant matrices of the form (2.8). Then, for any given set of integers* $\eta_1 \geq 0, \ldots, \eta_\nu \geq 0$ *and any* $\mu \in \mathbb{R}$

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) \subseteq \Lambda_\mu \left(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e\right) \tag{2.26}$$

Proof: The proof is carried out for the special case of $\eta_1 = 1$ and $\eta_i = 0$, $i = 2, \ldots, \nu$; for the general case it can be easily followed from induction. The matrix K_d^e has the same form as the matrix K^e given in (2.8), i.e.

$$K_d^e = \begin{bmatrix} K_1 & & \bigcirc & & q_1 \\ & K_2 & & & 0 \\ & & \ddots & & \vdots \\ & \bigcirc & & K_\nu & 0 \\ r_1 & 0 & \cdots & 0 & \gamma_1 \end{bmatrix}$$

In addition, let K be defined as

$$K = \text{block diagonal}\left[K_1, K_2, \ldots, K_\nu\right] \tag{2.27}$$

It is easy to verify that for any $K \in \mathrm{K}_d$

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) = \Lambda_\mu \left(C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), \mathrm{K}_d\right)$$

Similarly to the case of non-delay case discussed in [149], it can be shown that

$$\Lambda_\mu \left(C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), \mathrm{K}_d\right) \subseteq \Lambda_\mu \left(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1}\right)$$

where K_{c_1} is the set of all $m_1 \times p_1$ matrices (i.e. $\mathrm{K}_{c_1} = \mathbb{R}^{m_1 \times p_1}$) and

$$\Lambda_\mu \left(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1}\right) = \{s | s \in \mathbb{C},\ \mathrm{Re}\{s\} \geq \mu,\ \phi(s) = 0,\ \forall K^* \in \mathrm{K}_{c_1}\} \quad (2.28)$$

$$\phi(s) = \det \left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh}) - B_1(e^{-sh})K^* C_1(e^{-sh})\right) \quad (2.29)$$

Thus, one can conclude that

$$\Lambda_\mu \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) \subseteq \Lambda_\mu \left(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1}\right) \quad (2.30)$$

Choose an arbitrary $K \in \mathrm{K}_d$, and consider the triple

$$\left(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda)\right)$$

From Theorem 2.1, there exists a unimodular matrix $T(\lambda) \in R^{n \times n}[\lambda]$ that transforms the state-space model to the Kalman canonical form given below

$$T^{-1}(\lambda) \left(A(\lambda) + B(\lambda)KC(\lambda)\right) T(\lambda) = \begin{bmatrix} \tilde{A}_{11}(\lambda) & 0 & \tilde{A}_{13}(\lambda) \\ \tilde{A}_{21}(\lambda) & \tilde{A}_{22}(\lambda) & \tilde{A}_{23}(\lambda) \\ 0 & 0 & \tilde{A}_{33}(\lambda) \end{bmatrix} \quad (2.31)$$

and

$$T^{-1}(\lambda) B_1(\lambda) = \begin{bmatrix} \tilde{B}_1(\lambda) \\ \tilde{B}_2(\lambda) \\ 0 \end{bmatrix}, \quad C_1(\lambda) T(\lambda) = \begin{bmatrix} \tilde{C}_1(\lambda) & 0 & \tilde{C}_2(\lambda) \end{bmatrix} \quad (2.32)$$

It results from Theorem 2.2 that $\phi(s)$ defined in (2.29) can be written as

$$\phi(s) = \prod_{i=2}^{i=3} \det \left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

On the other hand,

$$\Lambda_\mu(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e) = \{s | s \in \mathbb{C},\ \mathrm{Re}\{s\} \geq \mu,$$
$$\psi(s) = 0,\ \forall K_d^e \in \mathrm{K}_d^e,\ q_1 \in \mathbb{R},\ r_1 \in \mathbb{R},\ \gamma_1 \in \mathbb{R}\} \quad (2.33)$$

where

$$\psi(s) = \det(sI - A^e(e^{-sh}) - B^e(e^{-sh})K_d^e C^e(e^{-sh}))$$

Consequently, $\psi(s)$ can be written as

$$\psi(s) = \det \begin{bmatrix} sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh}) & -B_1(e^{-sh})q_1 \\ -r_1 C_1(e^{-sh}) & s - \gamma_1 \end{bmatrix}$$

where K is given in (2.27). Equivalently,

$$\psi(s) = \det \left(\begin{bmatrix} T^{-1}(e^{-sh}) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} \times \right.$$
$$\left. \begin{bmatrix} sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh}) & -B_1(e^{-sh})q_1 \\ -r_1 C_1(e^{-sh}) & s - \gamma_1 \end{bmatrix} \times \begin{bmatrix} T(e^{-sh}) & 0_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} \right)$$

where $T(e^{-sh})$ is obtained by substituting λ with e^{-sh} in $T(\lambda)$ (which was introduced in (2.31) and (2.32)). Furthermore, $\psi(s)$ can be expressed as

$$\psi(s) = \det \begin{bmatrix} sI - \tilde{A}_{11}(e^{-sh}) & 0 & -\tilde{A}_{13}(e^{-sh}) & -\tilde{B}_1(e^{-sh})q_1 \\ -\tilde{A}_{21}(e^{-sh}) & sI - \tilde{A}_{22}(e^{-sh}) & -\tilde{A}_{23}(e^{-sh}) & -\tilde{B}_2(e^{-sh})q_1 \\ 0 & 0 & sI - \tilde{A}_{33}(e^{-sh}) & 0 \\ -r_1\tilde{C}_1(e^{-sh}) & 0 & -r_1\tilde{C}_2(e^{-sh}) & s - \gamma_1 \end{bmatrix} \quad (2.34)$$

On the other hand, since

$$\phi(s) = 0 \ \Rightarrow\ \det\left(sI - \tilde{A}_{22}(e^{-hs})\right) = 0 \ \vee\ \det\left(sI - \tilde{A}_{33}(e^{-hs})\right) = 0$$

One can conclude from special structure of (2.34) that for any $K \in \mathrm{K}_d$ and for all $q_1, r_1, \gamma_1 \in \mathbb{R}$, any root of $\phi(s)$ will be a root of $\psi(s)$ as well. Thus, using (2.30), (2.28), and (2.33), one can arrive at (2.26). ∎

Lemma 2.6. *Let the arbitrary positive real scalar σ_0 and complex scalar s_0 be given. Define the disk $\mathcal{D}(s_0, \sigma_0)$ as*

$$\mathcal{D}(s_0, \sigma_0) = \{s | s \in \mathbb{C}, |s - s_0| < \sigma_0\} \tag{2.35}$$

Consider the system (2.2) and the set K_d of block diagonal matrices defined in (2.24). For any $K \in \mathrm{K}_d$, define

$$\phi(s, K) := \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right) \tag{2.36}$$

Define also $\bar{\mathcal{D}}(s_0, \sigma_0)$ as the boundary of the disk $\mathcal{D}(s_0, \sigma_0)$; i.e.

$$\bar{\mathcal{D}}(s_0, \sigma_0) := \{s | s \in \mathbb{C}, |s - s_0| = \sigma_0\}.$$

If $\phi(s, 0)$ is nonzero on $\bar{D}(s_0, \sigma_0)$, then there exists a positive γ such that for all $K \in \mathrm{K}_d$ with $\|K\| < \gamma$, the number of roots of $\phi(s, K)$ and $\phi(s, 0)$ inside $\mathcal{D}(s_0, \sigma_0)$ are the same, where $\|.\|$ denotes any induced norm.

Proof: Since $\phi(s, 0)$ is nonzero on $\bar{\mathcal{D}}(s_0, \sigma_0)$, one can find $\eta > 0$ such that $|\phi(s, 0)| \geq \eta$ for all $s \in \bar{\mathcal{D}}(s_0, \sigma_0)$. On the other hand, $\phi(s, K)$ can be written in the following form

$$\phi(s, K) = \xi_0(s, K) + \sum_{l=1}^{l_f} \xi_l(s, K) e^{-lhs} \tag{2.37}$$

where

$$\begin{aligned} \xi_0(s, K) &= \sum_{\tau=0}^{n} a_\tau(K) s^\tau \\ \xi_l(s, K) &= \sum_{\tau=0}^{n_l} b_{\tau,l}(K) s^\tau \end{aligned} \tag{2.38}$$

In the above equations, $a_\tau(K)$ and $b_{\tau,l}(K)$ are polynomials in $k_i(\alpha, \beta)$; i.e., the (α, β) element of the matrix K_i, $i = 1, 2, \ldots, \nu$, $\alpha = 1, 2, \ldots, m_i$ and $\beta = 1, 2, \ldots, p_i$. One can conclude from (2.37) and (2.38) that

$$|\phi(s, K) - \phi(s, 0)| \leq \sum_{\tau=0}^{n} |a_\tau(K) - a_\tau(0)||s|^\tau + \sum_{l=1}^{lf} |e^{-lhs}| \sum_{\tau=0}^{n_l} |b_{\tau,l}(K) - b_{\tau,l}(0)||s|^\tau \tag{2.39}$$

Furthermore, if $|s - s_0| = \sigma_0$, then $|s| \leq |s_0| + \sigma_0$. Therefore, for $|s - s_0| = \sigma_0$,

$$|\phi(s,K) - \phi(s,0)| \leq \sum_{\tau=0}^{n} |a_\tau(K) - a_\tau(0)|(|s_0| + \sigma_0)^\tau + \sum_{l=1}^{lf} e^{lh(|s_0|+\sigma_0)} \sum_{\tau=0}^{n_l} |b_{\tau,l}(K) - b_{\tau,l}(0)|(|s_0| + \sigma_0)^\tau \quad (2.40)$$

Since $a_\tau(K)$ and $b_{\tau,l}(K)$ are continuous functions of $k_i(\alpha,\beta)$, thus there exists a $\gamma > 0$ such that if $\|K\| < \gamma$, then

$$\begin{aligned} |a_\tau(K) - a_\tau(0)| &< \frac{\eta}{(l_f+1)(n+1)(|s_0|+\sigma_0)^\tau} \\ |b_{\tau,l}(K) - b_{\tau,l}(0)| &< \frac{\eta e^{-lh(|s_0|+\sigma_0)}}{(l_f+1)(n_l+1)(|s_0|+\sigma_0)^\tau} \end{aligned} \quad (2.41)$$

Consequently, from (2.40) and (2.41) it can be deduced that for any K with $\|K\| < \gamma$

$$|\phi(s,K) - \phi(s,0)| < \eta \leq |\phi(s,0)|, \qquad \forall s \in \bar{D}(s_0,\sigma_0) \quad (2.42)$$

The proof follows directly from Rouché's Theorem [131]. ■

Lemma 2.7. *Consider the system (2.2), the set* K_d *of block diagonal matrices defined in (2.24), and the characteristics equation* $\phi(s,K)$, $K \in \mathrm{K}_d$, *defined in (2.36). Let* s_j $(j \in \mathbb{N})$, *denote the roots of* $\phi(s,0)$, *and assume that the set of closed right-half plane roots of* $\phi(s,0)$ *(referred to as unstable roots hereafter) is represented by* $\{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_\beta}\}$. *If*

$$\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) = \emptyset, \quad (2.43)$$

then

1. *There exists a positive* γ *such that for all* $K \in \mathrm{K}_d$ *with* $\|K\| < \gamma$, *the number of unstable roots of* $\phi(s,K)$ *is not greater than the number of unstable roots of* $\phi(s,0)$.

2. *For any* $\xi > 0$, *there exists a* $\hat{K} \in \mathrm{K}_d$ *with* $\|\hat{K}\| < \xi$ *such that* $\phi(s_j,\hat{K}) \neq 0$ *for all* $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\}$.

Proof: Given an arbitrary $\epsilon > 0$, define Θ_ϵ as

$$\Theta_\epsilon = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} > -\epsilon\} \tag{2.44}$$

Since the roots of $\phi(s, 0)$ are separated in the right side of any line parallel to the imaginary axis [115], one can find $\epsilon^* > 0$ such that Θ_{ϵ^*} does not include any stable poles of $\phi(s, 0)$. Furthermore, consider the disk $\mathcal{D}(\rho - \epsilon^*, \rho)$, which is centered at $\rho - \epsilon^*$ in the complex plane and has the radius ρ. It is easy to show that the point $-\epsilon^*$ lies on $\bar{\mathcal{D}}(\rho - \epsilon^*, \rho)$, i.e. the boundary of $\mathcal{D}(\rho - \epsilon^*, \rho)$. In addition,

$$\lim_{\rho \to \infty} \mathcal{D}(\rho - \epsilon^*, \rho) = \Theta_{\epsilon^*} \tag{2.45}$$

From Lemma 2.2, it can be deduced that there exists a ρ^* such that for any $\rho > \rho^*$, all the unstable roots of $\phi(s, 0)$ are placed in $\mathcal{D}(\rho - \epsilon^*, \rho)$. In this case, $\phi(s, 0)$ is nonzero over $\bar{\mathcal{D}}(\rho - \epsilon^*, \rho)$ for any $\rho > \rho^*$. In addition, according to Lemma 2.6, there exists a γ such that the number of roots of $\phi(s, K)$ in the disk $\mathcal{D}(\rho - \epsilon^*, \rho)$, for all $K \in \mathrm{K}_d$ with $\|K\| < \gamma$, is equal to the number of roots of $\phi(s, 0)$ in the same disk if $\rho > \rho^*$. This implies the first statement of the lemma.

In order to prove the second part, define the following set for all $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\}$

$$\Pi_j := \{K | K \in \mathrm{K}_d \text{ and } \phi(s_j, K) = 0\} \tag{2.46}$$

Since s_j, $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\}$ is not a DFM, $\phi(s_j, K)$ is a non-constant polynomial in K. Thus, Π_j is a hyper-surface in the parameter space of K (for the definition of hyper-surface, see [40]). Moreover, in any non-empty open set of the parameter space of K, there exists a $\hat{K}$ such that $\hat{K} \notin \bigcup_j \Pi_j$. This completes the proof of the second statement of the lemma. ■

Theorem 2.3. *A necessary and sufficient condition for the existence of an asymptotically stabilizing LTI decentralized controller for the system (2.2) with the local dynamic control law given by (2.3) is that*

$$\Lambda_0 \left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d \right) = \emptyset \tag{2.47}$$

Proof of necessity: Assume that

$$\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) \neq \emptyset \tag{2.48}$$

Consider an arbitrary s_0 in $\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right)$. From Lemma 2.5, one can conclude that

$$s_0 \in \Lambda_0\left(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e\right) \tag{2.49}$$

According to Definition 2.8, for any $K_d^e \in \mathrm{K}_d^e$

$$\det(s_0 I - A^e(e^{-s_0 h}) - B^e(e^{-s_0 h}) K_d^e C^e(e^{-s_0 h})) = 0 \tag{2.50}$$

Using Lemma 2.1, one can infer that there is no asymptotically stabilizing LTI decentralized controller for the system (2.1) with the local dynamic control law given by (2.3). This completes the proof of necessity.

Proof of sufficiency: Consider $\phi(s, K)$ defined in (2.36). Denote the set of roots of $\phi(s, 0)$ with $\{s_1, s_2, \ldots\}$, and the finite set of unstable roots of $\phi(s, 0)$ with $\{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_\beta}\}$. Based on Lemma 2.7, there exist a positive real scalar γ and a matrix $\hat{K} \in \mathrm{K}_d$ of the form

$$\hat{K} := \text{block diag}[\hat{K}_1, \hat{K}_2, \ldots, \hat{K}_\nu] \tag{2.51}$$

with $\|\hat{K}\| < \gamma$ such that

1. for all $K \in \mathrm{K}_d$ with the property $\|K\| < \gamma$, the number of unstable roots of $\phi(s, K)$ is less than or equal to the number of unstable roots of $\phi(s, 0)$;

2. $\phi(s_j, \hat{K}) \neq 0$, for all $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\}$.

The former statement implies that the number of unstable modes of $A(\lambda)+B(\lambda)KC(\lambda)$ is not greater than the number of unstable modes of $A(\lambda)$ (multiplicities counted) provided the magnitude of the feedback gain is sufficiently small.

Define the following set of matrices

$$\hat{K}(i) := \begin{cases} \text{block diag}[0_{m_1 \times p_1}, \ldots, 0_{m_{i-1} \times p_{i-1}}, \hat{K}_i, \ldots, \hat{K}_\nu], & 1 \le i \le \nu \\ 0_{m \times p}, & i = \nu + 1 \end{cases} \tag{2.52}$$

Let $\pi \in \bar{\nu}$ be such that

$$\phi(s_j, \hat{K}(\pi)) \neq 0, \qquad \text{for all } j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\} \tag{2.53}$$

and

$$\phi(s_j, \hat{K}(\pi + 1)) = 0, \qquad \text{for some } j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\} \tag{2.54}$$

It is to be noted that such a π always exists, because

$$\phi(s_j, \hat{K}(1)) = \phi(s_j, \hat{K}) \neq 0 \tag{2.55}$$

whereas

$$\phi(s_j, \hat{K}(\nu + 1)) = \phi(s_j, 0) = 0 \tag{2.56}$$

for all $j \in \{\alpha_1, \alpha_2, \ldots, \alpha_\beta\}$. Define $\hat{A}(\lambda) := A(\lambda) + B(\lambda)\hat{K}(\pi + 1)C(\lambda)$; then, from the above discussion $\hat{A}(\lambda)$ has an unstable mode denoted by s^* whereas one can find a matrix $\hat{K}_\pi \in \mathbb{R}^{m_\pi \times p_\pi}$ such that s^* is not a mode of $\hat{A}(\lambda) + B_\pi(\lambda)\hat{K}_\pi C_\pi(\lambda)$. Using Lemma 2.4, it is concluded that s^* is not an uncontrollable or unobservable mode. Therefore, one can find a dynamic output feedback controller for the π-th local control station which places this mode in the left-half complex plane [56]. Hence, the number of unstable modes of $\hat{A}$ can be reduced at least by one, via the above local feedback controller.

In addition, note that the number of unstable modes of $\hat{A}(\lambda)$ is not greater than the number of unstable modes of $A(\lambda)$. This follows from the first statement of Lemma 2.7 and the fact that

$$\|\hat{K}(\pi + 1)\| \le \|\hat{K}\| \tag{2.57}$$

Applying the above procedure iteratively, one will arrive at a decentralized output feedback dynamic controller to stabilize the system (2.1) (note that $A(\lambda)$ has a finite number of unstable modes). ■

Remark 2.3. *The iterative procedure presented in the proof of sufficiency part of Theorem 2.3 can be used to develop a stabilization technique for the decentralized time-delay control systems. The only requirement is the existence of an efficient algorithm to move the non-fixed unstable modes of the time-delay system to the open left-half complex plane.*

2.3.4 Characterization of decentralized fixed modes for time-delay systems

A numerical algorithm is now presented to find $\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right)$, i.e. the set of unstable decentralized fixed modes of the system (2.2). Note that this set is required for applying the condition of Theorem 2.3.

Algorithm 1:

1) Compute $\Omega_0\left(A(\lambda)\right)$ using the MATLAB toolbox *DDE-BIFTOOL* [37].

2) Choose a feedback gain $K_d \in \mathrm{K}_d$ by employing a random number generator.

3) Find $\Omega_0\left(A(\lambda) + B(\lambda)K_dC(\lambda)\right)$ using the MATLAB toolbox *DDE-BIFTOOL*.

4) Obtain

$$\Omega^* = \Omega_0\left(A(\lambda)\right) \cap \Omega_0\left(A(\lambda) + B(\lambda)K_dC(\lambda)\right)$$

The set Ω^* resulted from the above algorithm is equal to $\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right)$, for almost all $K_d \in \mathrm{K}_d$ (for a detailed description of "almost all" see [28]). It is to be noted that a similar algorithm can be employed to obtain $\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c\right)$.

Remark 2.4. *Using the toolbox* DDE-BIFTOOL, *the right-most roots of a quasi-polynomial characteristic equation can be found numerically. The roots are first approximated using a linear multi-step (LMS) method. The approximated roots are then adjusted accordingly, using a newton iteration. The convergence is guaranteed under generic conditions.*

A set of algebraic conditions will be provided next (analogously to the finite-dimensional case) to characterize the DFMs of the system (2.2). In a manner similar to the one provided in [7], the following theorem can be obtained using Lemma 2.3.

Theorem 2.4. *Consider the system (2.2). The mode $s \in \Omega_\mu(A(\lambda))$ is a μ-DFM if and only if at least one of the following conditions holds*

i.

$$\operatorname{rank}\begin{bmatrix} sI - A(e^{-sh}) & B_1(e^{-sh}) & B_2(e^{-sh}) & \cdots & B_\nu(e^{-sh}) \end{bmatrix} < n$$

ii.

$$\operatorname{rank}\begin{bmatrix} sI - A(e^{-sh}) \\ C_1(e^{-sh}) \\ C_2(e^{-sh}) \\ \cdots \\ C_\nu(e^{-sh}) \end{bmatrix} < n$$

iii. There exists at least a partition of the set $\bar{\nu}$ into non-empty disjoint subsets $\{i_1, \ldots, i_k\}$ and $\{i_{k+1}, \ldots, i_\nu\}$ such that

$$\operatorname{rank}\begin{bmatrix} sI - A(e^{-sh}) & B_{i_1}(e^{-sh}) & \cdots & B_{i_k}(e^{-sh}) \\ C_{i_{k+1}}(e^{-sh}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{i_\nu}(e^{-sh}) & 0 & \cdots & 0 \end{bmatrix} < n$$

In the sequel, alternative necessary and sufficient conditions are presented for characterizing the μ-DFMs of a time-delay system. Using this characterization, one can define a μ-ADFM (approximate decentralized fixed mode). The following lemma is borrowed from [8], and is used in the proof of Theorem 2.5.

Lemma 2.8. *Consider a singula matrix $M_0 \in \mathbb{C}^n$, and define $M_i := \theta_i {\omega_i}^T$, where θ_i, $\omega_i \in \mathbb{C}^n$ and $i \in \bar{\rho} := \{1, 2, \cdots, \rho\}$. Then,*

$$\det\left(M_0 + \sum_{i=1}^{\rho} \mu_i M_i\right) = 0, \qquad \forall \mu_i \in \mathbb{R}$$

if and only if the following conditions are all satisfied

$$\det\begin{bmatrix} M_0 & \theta_{i_1} & \cdots & \theta_{i_\eta} \\ {\omega_{i_1}}^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ {\omega_{i_\eta}}^T & 0 & \cdots & 0 \end{bmatrix} = 0$$

for any non-empty set $\{i_1, i_2, \cdots, i_\eta\}$ which is a subset of the set $\bar{\rho}$, where $\eta = 1, 2, \ldots, \rho$.

Now, let the state-space equations (2.2) be rewritten as

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + \sum_{i=1}^{\nu^*} b_i^*(\lambda)u_i^*(t) \\ y_i^*(t) &= c_i^*(\lambda)x(t), \qquad i \in \bar{\nu^*} := \{1, 2, \cdots, \nu^*\} \end{aligned} \tag{2.58}$$

where u_i^*, y_i^* are scalar input and output, and $\nu^* = \sum_{i=1}^{\nu} m_i p_i$ ($b_i^*(\lambda)$'s and $c_i^*(\lambda)$'s can be easily obtained from $B_i(\lambda)$'s and $C_i(\lambda)$'s, respectively, using the Kronecker product [28]). Then, it can be verified that the closed-loop system obtained by applying the controller $u_i = K_i y_i$, $i \in \bar{\nu}$ to the system (2.2) is equivalent to the closed-loop system obtained by applying the controller $u_i^* = k_i y_i^*$, $i \in \bar{\nu}^*$ to (2.58), where k_i's are defined by

$$\begin{bmatrix} k_1 & k_2 & \ldots & k_{\nu^*} \end{bmatrix} = \begin{bmatrix} \text{vec}(K_1)^T & \text{vec}(K_2)^T & \ldots & \text{vec}(K_\nu)^T \end{bmatrix}$$

Now, using Lemma 2.8 and the above discussion the following theorem result is obtained.

Theorem 2.5. *Given the system (2.2), the mode $s \in \Omega_\mu(A(\lambda))$ is a μ-DFM with respect to* K_d *if and only if all of the following conditions hold*

$$\mathrm{rank}\begin{bmatrix} sI - A(e^{-sh}) & b^*_{i_1}(e^{-sh}) & \cdots & b^*_{i_\eta}(e^{-sh}) \\ c^*_{i_1}(e^{-sh}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c^*_{i_\eta}(e^{-sh}) & 0 & \cdots & 0 \end{bmatrix} < n + \eta$$

for any non-empty set $\{i_1, i_2, \cdots, i_\eta\}$ which is a subset of $\bar{\nu^*}$, where $\eta = 1, 2, \ldots, \nu^*$.

This paves the way for defining a μ-ADFM of a time-delay system, which provides a measure of how close a mode is to being a μ-DFM. Suppose that $\mathrm{cond}_i(s)$ denotes the condition measure of the i-th matrix introduced in the above theorem [27]. Furthermore, Let

$$\kappa = \min_{i \in \bar{\nu^*}} \ \mathrm{cond}_i(s) \tag{2.59}$$

Following an argument similar to the one presented in [27], the mode $s \in \Omega_\mu(A(\lambda))$ is called a μ-ADFM of magnitude κ; in the particular case, s is a μ-DFM when $\kappa = \infty$.

2.4 Numerical Examples

Example 2.1. *Consider an interconnected system S consisting of two subsystems S_1 and S_2 with the respective state-space representations given by*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -4 & 7 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-2h) \\ x_2(t-h) \end{bmatrix} \\ &\quad + \begin{bmatrix} 3 \\ 3 \end{bmatrix} z_1(t) + \begin{bmatrix} 6 \\ 1 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t-h) \end{aligned} \tag{2.60a}$$

$$y_1(t) = 8x_1(t) - 6x_2(t) - 2x_2(t-h) + w_1(t)$$

$$\begin{bmatrix} \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} x_4(t-2h)$$

$$+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} z_2(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_2(t) + \begin{bmatrix} 0 \\ -4 \end{bmatrix} u_2(t-2h) \tag{2.60b}$$

$$y_2(t) = -2x_3(t) + x_4(t) - 2x_4(t-2h) + w_2(t)$$

where $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}$ are the local input and output corresponding to S_i, for $i = 1, 2$. In addition, $[x_1^T \ x_2^T]^T$ and $[x_3^T \ x_4^T]^T$ are the state vectors of the subsystems S_1 and S_2, respectively. Furthermore, $\chi_1(t)$ and $\chi_2(t)$ are the incoming interconnection signals of the subsystems S_1 and S_2, respectively, and are assumed to be as follows

$$\begin{aligned} \chi_1(t) &= \frac{1}{3}x_4(t) - x_4(t-h) \\ \chi_2(t) &= -4x_1(t) + 3x_2(t) + x_2(t-h) \end{aligned} \tag{2.61}$$

The signals $w_1(t)$ and $w_2(t)$ represent the direct effect of the state of one subsystem on the output of the other subsystem, and are considered to be

$$\begin{aligned} w_1(t) &= x_3(t) - e^2 x_3(t-2h) \\ w_2(t) &= x_2(t) - e x_2(t-h) \end{aligned} \tag{2.62}$$

Using the λ-operator, the state-space model for the interconnected system S can be written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -4-\lambda^2 & 7+\lambda & 0 & -1-3\lambda \\ -1-\lambda^2 & 5 & 0 & -1-3\lambda \\ -4 & 3+\lambda & 1 & 0 \\ 0 & 0 & 0 & -2+2\lambda \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 1+\lambda & 0 \\ 0 & 0 \\ 0 & -1-4\lambda^2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{2.63}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 8 & -6-2\lambda & 1-e^2\lambda^2 & 0 \\ 0 & 1-e\lambda & -2 & 1-2\lambda^2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

One can easily verify that $s = 1$ is a mode of the system S for all $h \geq 0$.

Assume initially that $h = 0$ (finite-dimensional case). In this case, denote the controllability and observability matrices of the system S with M_{c0} and M_{o0}, respectively. It is easy to show that

$$\text{rank } M_{c0} = 4, \qquad \text{rank } M_{o0} = 4 \tag{2.64}$$

Hence, for $h = 0$, the system S is both controllable and observable, which implies that it does not have any CFM. Furthermore, it can be verified that in this case the system S does not have any DFM either [28]. Thus, the modes of the system S, including $s = 1$, can be placed arbitrarily in the complex plane using both centralized and decentralized output feedback controllers.

Now, assume that $h = 1$. It can be verified in this case that $s = 1$ is a controllable and observable mode using the criteria given in (2.17) and (2.18). Therefore, *according to Lemma 2.4 this mode of the system is not a μ-centralized fixed mode, for any finite $\mu \in \mathbb{R}$, and a static output feedback $u(t) = Ky(t)$, $K \in \mathbb{R}^{2\times 2}$ can displace this mode of the system.*

Next, it is aimed to investigate if there exists a decentralized LTI finite-dimensional output feedback controller to stabilize the system. Consider the following static decentralized output feedback

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

It can be shown using Symbolic Math Toolbox *that for $s = 1$,*

$$\det\left(sI - A(e^{-hs}) - B(e^{-hs})KC(e^{-hs})\right)$$

is zero for any 2×2 diagonal matrix K. Thus, it can be concluded that $s = 1$ is an unstable DFM for the underlying system, and as a result (from Theorem 2.3) there is no LTI finite-dimensional decentralized output feedback controller to stabilize the system.

Example 2.2. *Consider the following 2-input 2-output interconnected system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h) + \sum_{i=1}^{2} b_i u(t) + \sum_{i=1}^{2} b_{d_i} u(t-h) \\ y_1(t) &= c_1 x(t), \qquad y_2(t) = c_2 x(t) \end{aligned} \tag{2.65}$$

Let the delay h be equal to 1, and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 5 & 3 & 1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}, \quad b_{d1} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad b_{d2} = \begin{bmatrix} 1 \\ -6 \\ -4 \end{bmatrix},$$

$$c_1 = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, \qquad c_2 = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}$$

Using MATLAB toolbox DDE-BIFTOOL [37], one can obtain the unstable open-loop modes of the above system, which are

$$\{1, 1.555\}$$

Now, consider a diagonal static output feedback controller $u = Ky$ for the system. The state-space model of the closed-loop system can be written as

$$\begin{aligned} \dot{x}(t) &= (A + BKC)x(t) + (A_d + B_dKC)x(t-h) \\ y(t) &= Cx(t) \end{aligned} \tag{2.66}$$

One can use a diagonal random gain matrix to check the stabilizability of the system with respect to the decentralized LTI finite-dimensional output feedback controllers. For example, using the following gain matrix

$$K_0 = \begin{bmatrix} 0.769 & 0 \\ 0 & 0.232 \end{bmatrix} \tag{2.67}$$

the set of the unstable modes of the closed-loop system (2.66) will be

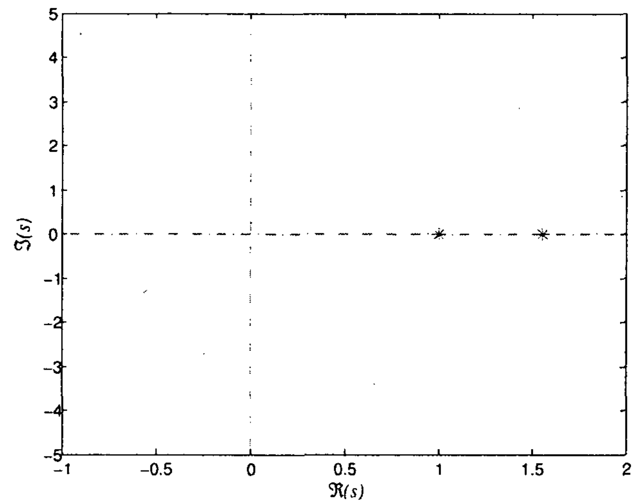
$$\{0.13, 0.329 \pm \jmath 0.752, 4.344\}$$

Therefore, it is concluded that

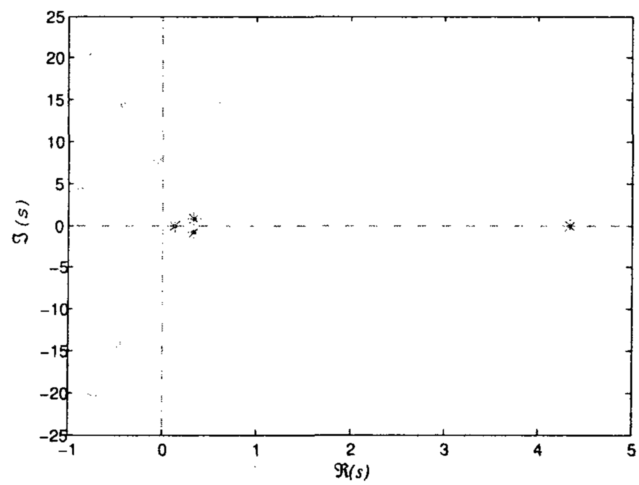
$$\Lambda_0 = \emptyset \tag{2.68}$$

This means that the system does not have any unstable DFM (i.e. 0-DFM, according to Definition 2.8), and hence it can be stabilized using a proper decentralized output feedback controller. The location of the open-loop and closed-loop modes of the system in a part of the complex plane is sketched in Figure 2.1.

Example 2.3. *Consider the system given in Example 2.2, with the only difference that A_d is zero here. The mode $s = 1$ is an unstable ADFM for this system with the magnitude $\kappa(h)$ which is defined in (2.59) and is a function of the delay h. The value of $\kappa(h)$ is obtained for $h \in (0, 5]$ and plotted in Figure 2.2(a). For $h = 0$, the*



(a)

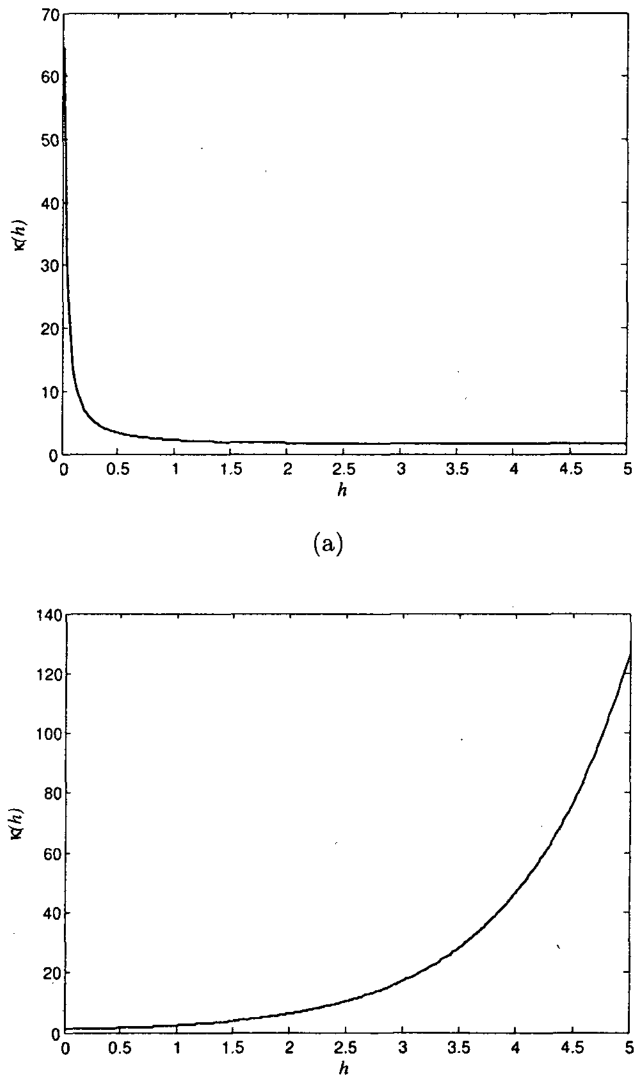


(b)

Figure 2.1: (a) The open-loop modes of system (2.65); (b) the closed-loop modes of system (2.65) under the gain K_0 given by (2.67).

system has an unstable DFM at $s = 1$, *i.e.* $\kappa(0) = \infty$. *Figure 2.2(a) shows that* $\kappa(h)$ *decreases with* h, *implying that the presence of delay in the dynamics of the system would make the system stabilizable with respect to the decentralized dynamic output feedback controllers (this interesting observation can be regarded as one of the many surprising results reported in the literature for time-delay control systems).*

Assume now that $b_i = 0$, $i = 1, 2$, *as well. In this case,* $\kappa(h)$ *increases with* h *as depicted in Figure 2.2(b). This means that the larger the delay in the input is, the more difficult it is to stabilize the system using a decentralized dynamic output feedback controller. It is to be noted that the condition measure used in this example is defined as* $cond(.) = 1/\underline{\sigma}(.)$, *where* $\underline{\sigma}(.)$ *is the smallest singular value of a matrix [27].*



(a)

(b)

Figure 2.2: The ADFM measure $\kappa(h)$ *corresponding to the mode* $s = 1$ *for the system of Example 2.3 when: (a)* $A_d = 0$, *(b)* $A_d = 0$ *and* $b_i = 0$, $i = 1, 2$.

Chapter 3

LQ Suboptimal Decentralized Controllers with Disturbance Rejection Property for Hierarchical Interconnected Systems

3.1 Introduction

In the control literature, an interconnected system is often referred to a system with a set of interacting subsystems [141]. Systems with different types of interaction topologies have been investigated in the literature, among which the class of *hierarchical* interconnected systems has drawn special attention in recent publications due to its broad applications, e.g. in formation flying, underwater vehicles, automated highways, robotics, satellite constellations, etc., which have leader-follower structures or structures with virtual leaders [38, 54, 83, 143, 144]. An interconnected system is said to be hierarchical if its structural graph is acyclic; i.e., if it does not have any directed cycles. It is shown in [4] that given a continuous-time interconnected system which does not have a hierarchical structure, under certain conditions its discrete-time equivalent model can have a hierarchical form. To stabilize such a system, one can design a set of stabilizing local controllers for the individual subsystems. In some cases, it is permissible that these local controllers partially exchange information [73, 144]. In general, the need for this type of structurally constrained controllers can be originated from some practical limitations concerning, for instance, the geographical distribution of the subsystems or the computational complexity associated with the centralized controller design for large-scale systems [75]. The case when these local controllers operate independently (i.e., they do not interact with each other), is referred to as decentralized feedback control [28, 83, 135].

Various aspects of the decentralized control theory have been extensively investigated in the past few decades. The papers [28,47,71,74] study the decentralized stabilizability of a system by using the notions of decentralized fixed modes and quotient fixed modes. Furthermore, different approaches are proposed in the literature to solve the pole-placement problem by means of decentralized controllers [59, 86]. High-performance decentralized control design techniques, on the other hand, have been investigated in [76, 130].

Since the real-world systems are usually vulnerable to external disturbances, the controller for a hierarchical interconnected system is desired to satisfy the following properties:

i) The disturbances must be rejected in the steady state.

ii) A predefined H_2 performance index should be minimized to achieve a fast transient response with a satisfactorily small control energy.

iii) The structure of the controller to be designed should be decentralized.

There exist a number of works which have addressed the problem of designing a controller satisfying the properties (i) and (iii) given above, and the controller obtained is regarded as decentralized servomechanism controller [25, 26, 29]. The paper [25] parameterizes all those decentralized controllers which reject the unmeasurable exogenous disturbances with known dynamics.

Moreover, the design of a controller which meets the criteria (ii) and (iii) has been studied intensively in several papers. In contrast to the H_2 optimal centralized controller which can be simply obtained from the Riccati equation, the H_2 optimal decentralized control problem involves sophisticated differential/nonconvex matrix equations [133, 142]. As a result, the available techniques often seek a near-optimal solution, rather than a globally optimal one. For instance, a method is proposed in [66], which cuts off all the interconnections between the subsystems and designs local optimal controllers for the isolated subsystems accordingly. The main shortcoming of this approach is that the controller obtained may destabilize the system, as a result of neglecting the system's interconnection parameters in the design procedure. Another approach for handling the underlying problem is to consider only static decentralized controllers [19, 94]. This imposes a stringent constraint on the controller, which can lead to a poor closed-loop performance. More recently, a method is provided in [75] for decentralization of any given centralized controller of desired performance. As a by-product of the results in [75], it is shown that the decentralized controller obtained from a H_2 optimal centralized controller using the above-mentioned technique is H_2 near-optimal. The only requirement of this approach is that the nominal model of the system is known by all control agents; i.e., every local controller must have a belief about the model of the entire system. This idea is further developed in [83] for the flight formation problem in a model predictive control framework. The paper [35], on the other hand, aims to design a controller for which all the aforementioned criteria (i), (ii) and (iii) hold. Since a set of nonlinear equations is derived in [35] for control design, this work cannot solve the problem for general large-scale systems efficiently.

This chapter presents a novel design strategy to obtain a high-performance decentralized control law for hierarchical interconnected systems, which satisfactorily eliminates the effect of unmeasurable external disturbances with known dynamics. It is assumed that the state of each subsystem is available in its corresponding local output (this is not an unrealistic assumption in many applications such as vehicle formation problems [137]), and that the modeling parameters of the whole system are available (with some error) in any local station. It is to be noted that once a centralized controller is designed to achieve the properties (i) and (ii), its decentralized duplicate (which can be obtained by using the method in [75]) does not necessarily maintain the same properties. To bypass this hurdle, a centralized controller is designed first, which should satisfy certain constraints (inspired by the conditions given in [83]). This control design is formulated in terms of LMI (which can be carried out straightforwardly). The centralized controller elicited from the LMI problem is then converted to a decentralized one via the approach presented in [75]. Two important issues concerning the designed controller are investigated subsequently. First, since the knowledge of each local controller about the whole system is inexact in practice, a procedure is proposed to measure the closeness of the designed controller performance to that of the optimal controller in terms of the statistical information on the parameter deviation. This enables the designer to assess the performance of the proposed controller in the physical environment. It will be demonstrated later in a formation flying example that the proposed control law outperforms the methods surveyed earlier. Furthermore, the robustness analysis results are presented for the case when the system is subject to perturbation and input delay.

The chapter is organized as follows. In Section 3.2, the problem is formulated and some important assumptions are given. The decentralization procedure introduced in [75] is briefly explained in Section 3.3, and a reference servomechanism controller is then designed in Section 3.4 by using the LMI framework. The structure of the controller obtained is converted to the decentralized form in Section 3.5, and its performance is evaluated in terms of the statistical information on the degree of accuracy of the modeling parameters. In Section 3.6, the proposed control law is fine-tuned to account for delay and also to reduce the sensitivity of the overall system to parameter variation. Simulation results are given in Section 3.7 to demonstrate the effectiveness of the proposed method.

3.2 Problem Formulation

Consider a hierarchical interconnected system $\mathcal{S}$, whose i-th subsystem $\mathcal{S}_i$, $i \in \bar{\nu} := \{1, 2, ..., \nu\}$, is represented by

$$\begin{aligned} \dot{x}_i(t) &= \sum_{j=1}^{i} A_{ij} x_j(t) + B_i u_i(t) + E_i \omega(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{3.1}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and the input of the subsystem $\mathcal{S}_i$, respectively. Furthermore, $y_i \in \mathbb{R}^{r_i}$ is the output of $\mathcal{S}_i$ to be regulated, and $\omega \in \mathbb{R}^q$ is the disturbance vector. Assume that the state $x_i(t)$ of the subsystem $\mathcal{S}_i$ is locally available, and that there is no measurement noise; i.e., the measured output of the i-th subsystem is equal to $x_i(t)$. Suppose that the disturbance $\omega(t)$ can be expressed as

$$\begin{aligned} \dot{z}(t) &= \Lambda z(t) \\ \omega(t) &= \mathcal{C} z(t) \end{aligned} \tag{3.2}$$

where the pair $(\mathcal{C}, \Lambda)$ is observable. Furthermore, the vector $z(0)$ in (3.2) and the elements of the matrix E_i in (3.1) are arbitrary, nevertheless unknown.

The system $\mathcal{S}$ can be represented as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E\omega(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.3}$$

where

$$\begin{gathered} x(t) = \begin{bmatrix} x_1(t)^T & x_2(t)^T & \cdots & x_\nu(t)^T \end{bmatrix}^T, \quad u(t) = \begin{bmatrix} u_1(t)^T & u_2(t)^T & \cdots & x_\nu(t)^T \end{bmatrix}^T \\ y(t) = \begin{bmatrix} y_1(t)^T & y_2(t)^T & \cdots & y_\nu(t)^T \end{bmatrix}^T, \quad E = \begin{bmatrix} E_1^T & E_2^T & \cdots & E_\nu^T \end{bmatrix}^T \\ B = \operatorname{diag}\left(\begin{bmatrix} B_1 & B_2 & \cdots & B_\nu \end{bmatrix}\right), \quad C = \operatorname{diag}\left(\begin{bmatrix} C_1 & C_2 & \cdots & C_\nu \end{bmatrix}\right) \end{gathered} \tag{3.4}$$

and A is a lower block triangular matrix whose (i, j) block entry is equal to A_{ij}, for any $i, j \in \bar{\nu}, \ j \leq i$. Define now

$$n := \sum_{i=1}^{\nu} n_i, \quad m := \sum_{i=1}^{\nu} m_i, \quad r := \sum_{i=1}^{\nu} r_i \tag{3.5}$$

Suppose that the initial state $x(0)$ is a random variable with a given mean X_μ and variance X_σ. Define X_0 as

$$X_0 := \mathcal{E}\left\{x(0)x(0)^T\right\} = X_\sigma + X_\mu X_\mu^T \tag{3.6}$$

where $\mathcal{E}\{\cdot\}$ represents the expectation operator. Furthermore, assume that the elements of the matrix E given in (3.3) are arbitrary and unknown. The objective of this chapter is introduced in the problem statement given below.

Problem 1: Design a decentralized LTI controller K_d (with block diagonal information flow structure [28]), such that the following conditions hold

i) The state $x(t)$ goes to zero as $t \to \infty$, provided $z(0) = 0$.

ii) The output $y(t)$ approaches zero as $t \to \infty$, regardless of the initial state $z(0)$.

iii) When $z(0)$ is a zero vector, the following performance index

$$J := \mathcal{E}\left\{\int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt\right\} \tag{3.7}$$

is satisfactorily small (note that the above cost function reflects the closed-loop performance). In (3.7), $R \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are positive definite and positive semi-definite matrices, respectively.

Since E is an unknown matrix and can take any arbitrary value, it is desired that the controller K_d be independent of E. The following assumption is made without loss of generality.

Assumption 3.1. *The matrices B_i, C_i and $\mathcal{C}$ satisfy the following rank conditions*

$$rank(\mathcal{C}) = q, \quad rank(B_i) = m_i, \quad rank(C_i) = r_i, \quad i \in \bar{\nu} \tag{3.8}$$

Let the rank conditions given above hold. One can easily deduce from the results of [24] that the following two assumptions are required for the existence of the desired controller K_d.

Assumption 3.2. *The matrices given below are all full-rank*

$$\begin{bmatrix} A_{ii} - \lambda_j I & B_i \\ C_i & 0 \end{bmatrix}, \quad \forall\, i \in \bar{\nu} \quad and \quad \forall\, j \in \{1, 2, ..., p\} \tag{3.9}$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ denote the eigenvalues of Λ. Furthermore, the inequality $m_i \geq r_i$ holds for all $i \in \bar{\nu}$.

Assumption 3.3. *The pair (A_{ii}, B_i) is stabilizable for all $i \in \bar{\nu}$.*

3.3 Preliminaries

In this section, it is desired to present the gist of the decentralization procedure given in [75]. Assume for now that $z(0) = 0$, i.e., the system $\mathcal{S}$ is disturbance free. Define the following vectors

$$\begin{aligned} x^i(t) &= \begin{bmatrix} x_1(t)^T & \dots & x_{i-1}(t)^T & x_{i+1}(t)^T & \dots & x_\nu(t)^T \end{bmatrix}^T, \\ u^i(t) &= \begin{bmatrix} u_1(t)^T & \dots & u_{i-1}(t)^T & u_{i+1}(t)^T & \dots & u_\nu(t)^T \end{bmatrix}^T, \end{aligned} \tag{3.10}$$

for any $i \in \bar{\nu}$. Consider an arbitrary centralized LTI controller K_c with the following state-space representation

$$\begin{aligned} \dot{\eta}_c(t) &= \Gamma\eta_c(t) + \Omega x(t) \\ u(t) &= M\eta_c(t) + Nx(t) \end{aligned} \tag{3.11}$$

where $\eta_c \in \mathbb{R}^\mu$. There exist constant matrices $\mathbf{\Omega}^i, \mathbf{\Omega}_i, \mathbf{M}^i, \mathbf{M}_i, \mathbf{N}^i, \mathbf{N}_i, \mathbf{N}^{\bar{i}}$ and $\mathbf{N}_{\bar{i}}$, such that the above controller can be expressed by a decomposed representation as follows

$$\begin{aligned} \dot{\eta}_c(t) &= \Gamma\eta_c(t) + \mathbf{\Omega}^i x^i(t) + \mathbf{\Omega}_i x_i(t) \\ u^i(t) &= \mathbf{M}^i\eta_c(t) + \mathbf{N}^i x^i(t) + \mathbf{N}_i x_i(t) \\ u_i(t) &= \mathbf{M}_i\eta_c(t) + \mathbf{N}^{\bar{i}} x^i(t) + \mathbf{N}_{\bar{i}} x_i(t) \end{aligned} \tag{3.12}$$

for any $i \in \bar{\nu}$. Similarly, there exist matrices $\mathbf{A}^i, \mathbf{A}^{\bar{i}}, \mathbf{A}_i$ and $\mathbf{B}^i$ (derived from A and B) such that the system $\mathcal{S}$ given in (3.1) can be decomposed as follows

$$\begin{aligned} \dot{x}^i(t) &= \mathbf{A}^i x^i(t) + \mathbf{A}_i x_i(t) + \mathbf{B}^i u^i(t) \\ \dot{x}_i(t) &= \mathbf{A}^{\bar{i}} x^i(t) + A_{ii} x_i(t) + B_i u_i(t) \end{aligned} \tag{3.13}$$

for any $i \in \bar{\nu}$. Define K_{d_i} as a local controller for the subsystem $\mathcal{S}_i$, $i \in \bar{\nu}$, with the following state-space representation:

$$\begin{aligned} \dot{\eta}_{d_i}(t) &= \begin{bmatrix} \mathbf{A}^i + \mathbf{B}^i\mathbf{N}^i & \mathbf{B}^i\mathbf{M}^i \\ \mathbf{\Omega}^i & \Gamma \end{bmatrix} \eta_{d_i}(t) + \begin{bmatrix} \mathbf{A}_i + \mathbf{B}^i\mathbf{N}_i \\ \mathbf{\Omega}_i \end{bmatrix} x_i(t) \\ u_i(t) &= \begin{bmatrix} \mathbf{N}^{\bar{i}} & \mathbf{M}_i \end{bmatrix} \eta_{d_i}(t) + \mathbf{N}_{\bar{i}} x_i(t) \end{aligned} \tag{3.14}$$

Define also K_d as a decentralized controller consisting of the local controllers K_{d_1}, K_{d_2},...,K_{d_ν}. The following theorem is borrowed from [75].

Theorem 3.1. *Assume that $x(0)$ is a known vector. The state and the input of the system $\mathcal{S}$ under the centralized controller K_c are the same as those of the system $\mathcal{S}$ under the decentralized controller K_d, if the initial state of the local controller K_{d_i} is chosen as*

$$\eta_{d_i}(0) = \begin{bmatrix} x^i(0) \\ 0 \end{bmatrix}, \quad i \in \bar{\nu} \tag{3.15}$$

Theorem 3.1 states that the centralized controller K_c can be transformed to an equivalent decentralized controller K_d, if the initial state $x(0)$ is a known vector and any local controller K_{d_i}, $i \in \bar{\nu}$, knows the exact initial states of the other subsystems. It is to be noted that these are not realistic assumptions in practice, and thus the result of Theorem 3.1 cannot be applied to real-world problems. However, this result will be used later for the subsequent developments of the chapter, when the practical limitations are taken into account. As the first step, assume that $x(0)$ is only statistically known, and hence let the following initial state be deployed

$$\eta_{d_i}(0) = \begin{bmatrix} X_\mu^i \\ 0 \end{bmatrix}, \quad i \in \bar{\nu} \tag{3.16}$$

instead of the one in (3.15). In the sequel, the internal stability of the system $\mathcal{S}$ under the decentralized controller K_d will be investigated.

Definition 3.1. *Consider the system $\mathcal{S}$ given by (3.1). The modified system $\mathbf{S}^i$, $i \in \bar{\nu}$, is defined to be a system obtained by removing all the interconnections going to the i-th subsystem in $\mathcal{S}$. The state-space representation of the modified system $\mathbf{S}^i$ is as follows*

$$\begin{aligned} \dot{x}(t) &= \tilde{A}^i x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{3.17}$$

where $\tilde{A}^i$ is derived from A by replacing the first $i-1$ block entries of its i-th block row with zeros. It is to be noted that based on this definition $\mathbf{S}^1 = \mathcal{S}$.

Definition 3.2. *Define the isolated subsystem $\mathcal{S}_i$, $i \in \bar{\nu}$, as a system obtained from the subsystem $\mathcal{S}_i$ by eliminating all of its incoming interconnections.*

In the following a necessary and sufficient condition for stability of the interconnected system $\mathcal{S}$ under the proposed decentralized controller K_d is given [83].

Theorem 3.2. *The system $\mathcal{S}$ is internally stable under the controller K_d if and only if the system $\mathbf{S}^i$ is stable under the controller K_c, for all $i \in \bar{\nu}$.*

A centralized servomechanism controller will be given in the next section, which will be used later as a reference to obtain the desired decentralized controller.

3.4 A Reference Centralized Servomechanism Controller

To avoid trivial cases, assume with no loss of generality that all of the eigenvalues of Λ lie in the closed right-half plane. It can be concluded from Assumptions 3.1, 3.2 and 3.3, and the results of [24], that there exist three nonunique matrices $\mathcal{B}, \mathcal{M}$ and $\mathcal{N}$, such that any minimum order *centralized* controller satisfying the requirements (i) and (ii) of Problem 1 can be represented by

$$\dot{\eta}_c(t) = \mathcal{A}\eta_c(t) + \mathcal{B}y(t) \tag{3.18a}$$

$$u(t) = \mathcal{M}\eta_c(t) + \mathcal{N}x(t) \tag{3.18b}$$

where

$$\mathcal{A} := \text{diag}([\underbrace{\Lambda \quad \Lambda \quad \cdots \quad \Lambda}_{r \text{ times}}]) \tag{3.19}$$

and where $(\mathcal{A}, \mathcal{B})$ is controllable. The objective of this section is to solve the problem introduced below.

Problem 2: Find the matrices $\mathcal{B}, \mathcal{M}$, and $\mathcal{N}$, so that the centralized controller given by (3.18) has the following properties

i) It satisfies the criteria (i) and (ii) of Problem 1.

ii) It stabilizes all of the systems $\mathbf{S}^2, ..., \mathbf{S}^{\nu}$.

iii) $\mathcal{B}$ is a block diagonal matrix, where the dimension of its i-th block entry is $r_i p \times r_i$, for any $i \in \bar{\nu}$.

The centralized controller solving Problem 2 will be converted to a decentralized form in the next section. It is to be noted that the conditions (ii) and (iii) given above are required in the decentralization procedure, as will be shown in the following Lemma.

Lemma 3.1. *Problem 2 has a solution, if and only if there exist a block diagonal matrix $\mathcal{B}$, matrices $\mathcal{M}$ and $\mathcal{N}$, and positive definite matrices $P_1, P_2, ..., P_\nu$ with the following properties*

$$\begin{bmatrix} \tilde{A}^i & 0 \\ \mathcal{B}C & \mathcal{A} \end{bmatrix}^T P_i + P_i \begin{bmatrix} \tilde{A}^i & 0 \\ \mathcal{B}C & \mathcal{A} \end{bmatrix} - P_i \begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T P_i +$$
$$\left(R^{-\frac{1}{2}} \begin{bmatrix} B \\ 0 \end{bmatrix}^T P_i + R^{\frac{1}{2}} \begin{bmatrix} \mathcal{N} & \mathcal{M} \end{bmatrix} \right)^T \times \left(R^{-\frac{1}{2}} \begin{bmatrix} B \\ 0 \end{bmatrix}^T P_i + R^{\frac{1}{2}} \begin{bmatrix} \mathcal{N} & \mathcal{M} \end{bmatrix} \right) +$$
$$\begin{bmatrix} \mathcal{Q} & 0 \\ 0 & 0 \end{bmatrix} < 0, \qquad i \in \bar{\nu} \tag{3.20}$$

Proof: Combining (3.18a) and (3.17) results in the augmented system given below

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}_c(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}^i & 0 \\ \mathcal{B}C & \mathcal{A} \end{bmatrix} \begin{bmatrix} x(t) \\ \eta_c(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad i \in \bar{\nu} \tag{3.21}$$

It is inferred from [24] that the desired controller exists, if and only if there are matrices $\mathcal{M}$ and $\mathcal{N}$, and a block diagonal matrix $\mathcal{B}$ such that the static controller $u(t) = \begin{bmatrix} \mathcal{N} & \mathcal{M} \end{bmatrix} \begin{bmatrix} x(t) \\ \eta_c(t) \end{bmatrix}$ stabilizes all of the augmented systems given by (3.21). Moreover, it follows from [69] that this stabilizability problem is equivalent to the solvability of the matrix inequality problem given in (3.20). ∎

Theorem 3.3. *Problem 2 has a solution if and only if there exist block diagonal matrices $\mathcal{B}$ and W, matrices $\mathcal{M}$ and $\mathcal{N}$, and positive definite matrices P_1, P_2,...,*

P_ν,V_1, V_2,..., V_ν, such that the following matrix inequality problem

$$\begin{bmatrix} \Phi_i & \bar{\Phi}_i \\ \bar{\Phi}_i^T & -I \end{bmatrix} < 0, \quad \forall i \in \bar{\nu} \tag{3.22}$$

is feasible, where

$$\begin{aligned}
\Phi_i = {} & \begin{bmatrix} \tilde{A}^i & 0 \\ 0 & \mathcal{A} \end{bmatrix}^T P_i + P_i \begin{bmatrix} \tilde{A}^i & 0 \\ 0 & \mathcal{A} \end{bmatrix} + V_i \left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I \right) V_i \\
& - P_i \left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I \right) V_i - V_i \left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I \right) P_i \\
& + \begin{bmatrix} I \\ 0 \end{bmatrix} C^T \left(W^T W - \mathcal{B}^T W - W^T \mathcal{B} \right) C \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \\
\bar{\Phi}_i = {} & \begin{bmatrix} \left(R^{-\frac{1}{2}} \begin{bmatrix} B \\ 0 \end{bmatrix}^T P_i + R^{\frac{1}{2}} \begin{bmatrix} \mathcal{N} & \mathcal{M} \end{bmatrix} \right)^T & \begin{bmatrix} 0 & 0 \\ \mathcal{B}C & 0 \end{bmatrix}^T + P_i \end{bmatrix}, \quad i \in \bar{\nu}
\end{aligned} \tag{3.23}$$

Proof of necessity: Assume that Problem 2 has a solution. It can be concluded from Lemma 3.1 that there exist a block diagonal matrix $\mathcal{B}$, matrices $\mathcal{M}$ and $\mathcal{N}$, and positive definite matrices $P_1, P_2, ..., P_\nu$, such that the matrix inequality problem given in (3.20) is feasible. One can easily verify that the matrix inequality problem (3.22) for the matrix variables $\mathcal{B}, \mathcal{M}, \mathcal{N}, W, P_1, ..., P_\nu, V_1, ..., V_\nu$, is the same as the one expressed by (3.20), with $W = \mathcal{B}$ and $V_i = P_i$, $\forall i \in \bar{\nu}$.

Proof of sufficiency: Suppose that there exist block diagonal matrices $\mathcal{B}$ and W, matrices $\mathcal{M}$ and $\mathcal{N}$, and positive definite matrices $P_1, ..., P_\nu, V_1, ..., V_\nu$, such that the matrix inequality problem (3.22) is feasible. Applying the Schur complement's formula to (3.22), one can conclude that

$$\bar{\Phi}_i \bar{\Phi}_i^T + \Phi_i < 0, \quad i \in \bar{\nu} \tag{3.24}$$

On the other hand, it is known that

$$(P_i - V_i)\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)(P_i - V_i) \geq 0, \quad C^T(B - W)^T(B - W)C \geq 0 \tag{3.25}$$

The above inequalities are equivalent to the following ones

$$\begin{aligned} &V_i\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)V_i - V_i\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)P_i \\ &- P_i\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)V_i \geq -P_i\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)P_i, \end{aligned} \tag{3.26a}$$

$$C^T W^T W C - C^T \mathcal{B}^T W C - C^T W^T \mathcal{B} C \geq -C^T \mathcal{B}^T \mathcal{B} C \tag{3.26b}$$

The inequalities (3.24), (3.26a) and (3.26b) lead to the following

$$\begin{aligned} &\bar{\Phi}_i\bar{\Phi}_i^T + \begin{bmatrix} \tilde{A}^i & 0 \\ 0 & \mathcal{A} \end{bmatrix}^T P_i + P_i \begin{bmatrix} \tilde{A}^i & 0 \\ 0 & \mathcal{A} \end{bmatrix} - P_i\left(\begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}^T + I\right)P_i \\ &- \begin{bmatrix} I \\ 0 \end{bmatrix} C^T \mathcal{B}^T \mathcal{B} C \begin{bmatrix} I & 0 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} < 0, \quad i \in \bar{\nu} \end{aligned} \tag{3.27}$$

The proof follows from Lemma 3.1 and from the observation that the expressions in the left sides of the inequalities (3.20) and (3.27) are identical. ■

Remark 3.1. *It can be easily verified that the matrix inequalities (3.22) turn to be LMIs in the case when V and W are constant matrices.*

Consider now the i-th isolated subsystem $\mathcal{S}_i$

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_i u_i(t) + E_i\omega(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{3.28}$$

Pursuing the method proposed in [24] and using Assumptions 3.1, 3.2 and 3.3, one can obtain the matrices $\mathcal{B}_i, \mathcal{M}_i$ and $\mathcal{N}_i$, for any $i \in \bar{\nu}$, such that the controller

$$\begin{aligned} \dot{\eta}_{c_i}(t) &= \mathcal{A}_i \eta_{c_i}(t) + \mathcal{B}_i y_i(t) \\ u_i(t) &= \mathcal{M}_i \eta_{c_i}(t) + \mathcal{N}_i x_i(t) \end{aligned} \tag{3.29}$$

attenuates the state $x_i(0)$ of the system given in (3.28) to zero provided $z(0) = 0$, and regulates $y_i(t)$ to zero for any arbitrary $z(0)$, where

$$\mathcal{A}_i := \text{diag}([\underbrace{\Lambda \;\; \Lambda \;\; \cdots \;\; \Lambda}_{r_i \text{ times}}]) \tag{3.30}$$

Define now the following matrices

$$\mathcal{B}_o = \text{diag}\left(\left[\begin{array}{cccc} \mathcal{B}_1 & \mathcal{B}_2 & \cdots & \mathcal{B}_\nu \end{array}\right]\right), \quad \mathcal{M}_o = \text{diag}\left(\left[\begin{array}{cccc} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_\nu \end{array}\right]\right),$$
$$\mathcal{N}_o = \text{diag}\left(\left[\begin{array}{cccc} \mathcal{N}_1 & \mathcal{N}_2 & \cdots & \mathcal{N}_\nu \end{array}\right]\right) \tag{3.31}$$

By considering $\mathcal{B} = \mathcal{B}_0$, $\mathcal{M} = \mathcal{M}_0$, and $\mathcal{N} = \mathcal{N}_0$, it can be shown that the controller (3.18) is a solution to Problem 2 (due to the hierarchical structure of the system). Therefore, from Lemma 3.1 there exist positive definite matrices $P_1^0, \ldots, P_\nu^0$, such that the matrix inequalities (3.20) hold for $\mathcal{B} = \mathcal{B}_0$, $\mathcal{M} = \mathcal{M}_0$, $\mathcal{N} = \mathcal{N}_0$ and $P_i = P_i^0$, $\forall i \in \bar{\nu}$. It is to be noted that the quadratic terms with respect to P_i in (3.20) are eliminated, which implies that (3.20) is a LMI with respect to P_i, and thus can be solved using the available LMI solvers.

An algorithm is introduced next, which aims at designing a centralized controller solving Problem 2, while it meets the condition (iii) of Problem 1 as well.

Algorithm 3.1.

i) Set $W = \mathcal{B}_0$ and $V_i = P_i^0$ for all $i \in \bar{\nu}$.

ii) Minimize the objective function trace$(P_1 X_0)$ with respect to the variables $\mathcal{B}$, $\mathcal{M}$, $\mathcal{N}$ and $P_1, \ldots, P_\nu \geq 0$, subject to the inequality constraints (3.22) (which *are LMIs (according to Remark 3.1), and the constraint that $\mathcal{B}$ is block diagonal (note that X_0 is defined in (3.6)).*

iii) *If $\sum_{i=1}^{\nu} \|V_i - P_i\| + \|W - \mathcal{B}\| \leq \delta$, where δ is a prescribed permissible deviation, then stop. Otherwise, set $V_i = P_i$, $i \in \bar{\nu}$, and $W = \mathcal{B}$, and go to (ii).*

Let the matrices $\mathcal{B}$, $\mathcal{M}$ and $\mathcal{N}$ obtained in Algorithm 3.1 be denoted by $\mathcal{B}_{opt}$, $\mathcal{M}_{opt}$ and $\mathcal{N}_{opt}$, respectively. It can be easily verified that the controller (3.18) with the parameters $\mathcal{B}_{opt}$, $\mathcal{M}_{opt}$ and $\mathcal{N}_{opt}$ satisfies the requirements of Problem 2 and the condition (iii) of Problem 1.

Remark 3.2. *The objective function trace(P_1X_0) introduced in Step 2 of Algorithm 3.1 is, in fact, equivalent to the performance index J given by (3.7). More details on this can be found in [67].*

Remark 3.3. *As pointed out earlier, the matrix inequalities given by (3.20) are satisfied for $\mathcal{B} = \mathcal{B}_0$, $\mathcal{M} = \mathcal{M}_0$, $\mathcal{N} = \mathcal{N}_0$, and $P_i = P_i^0$, $i \in \bar{\nu}$. On the other hand, by setting $V_i = P_i^0$ and $W = \mathcal{B}_0$, the LMIs (3.22) will be equivalent to the matrix inequalities (3.20). This implies that the LMI problem given in Step 2 of Algorithm 3.1 is feasible. In addition, it is evident that this algorithm is monotone decreasing and convergent, and should ideally stop when $W = \mathcal{B}$ and $V_i = P_i$ for all $i \in \bar{\nu}$. This results from the conditions under which the inequalities (3.26a) and (3.26b) turn to the equalities. However, Step 3 is required in order for the algorithm to halt in a finite number of iterations.*

The centralized servomechanism controller obtained here will be used in the next section to find a high-performance decentralized servomechanism controller.

3.5 Optimal Decentralized Servomechanism Controller

Consider the centralized controller $\tilde{K}_c$ of the form (3.11) with the following parameters

$$\begin{aligned} \dot{\eta}_c(t) &= \mathcal{A}\eta_c(t) + \mathcal{B}_{opt}Cx(t) \\ u(t) &= \mathcal{M}_{opt}\eta_c(t) + \mathcal{N}_{opt}x(t) \end{aligned} \tag{3.32}$$

The methodology proposed in Section 3.3 can now be applied to the centralized controller $\tilde{K}_c$ in order to obtain a decentralized controller denoted by $\tilde{K}_d$. For this purpose, let the above controller be decomposed as

$$\begin{aligned} \dot{\eta}_c(t) &= \mathcal{A}\eta_c(t) + \mathbf{B}^i_{opt}\mathbf{C}^i x^i(t) + \mathbf{B}^{opt}_i C_i x_i(t) \\ u^i(t) &= \mathbf{M}^i_{opt}\eta_c(t) + \mathbf{N}^i_{opt}x^i(t) + \mathbf{N}^{opt}_i x_i(t) \\ u_i(t) &= \mathbf{M}^{opt}_i\eta_c(t) + \mathbf{N}^{\bar{i}}_{opt}x^i(t) + \mathbf{N}^{opt}_{\bar{i}} x_i(t) \end{aligned} \tag{3.33}$$

where the matrices $\mathbf{C}^i, \mathbf{B}^i_{opt}, \mathbf{B}^{opt}_i, \mathbf{M}^i_{opt}, \mathbf{M}^{opt}_i, \mathbf{N}^i_{opt}, \mathbf{N}^{opt}_i, \mathbf{N}^{\bar{i}}_{opt}$ and $\mathbf{N}^{opt}_{\bar{i}}$ are derived from $C, \mathcal{B}_{opt}, \mathcal{M}_{opt}$ and $\mathcal{N}_{opt}$ [75]. Therefore, the state-space representation of the local controller $\tilde{K}_{d_i}$, $i \in \bar{\nu}$, will be obtained as follows

$$\begin{aligned} \dot{\eta}_{d_i}(t) &= \begin{bmatrix} \mathbf{A}^i + \mathbf{B}^i\mathbf{N}^i_{opt} & \mathbf{B}^i\mathbf{M}^i_{opt} \\ \mathbf{B}^i_{opt}\mathbf{C}^i & \mathcal{A} \end{bmatrix} \eta_{d_i}(t) + \begin{bmatrix} \mathbf{A}_i + \mathbf{B}^i\mathbf{N}^{opt}_i \\ \mathbf{B}^{opt}_i C_i \end{bmatrix} x_i(t) \\ u_i(t) &= \begin{bmatrix} \mathbf{N}^{\bar{i}}_{opt} & \mathbf{M}^{opt}_i \end{bmatrix} \eta_{d_i}(t) + \mathbf{N}^{opt}_{\bar{i}} x_i(t) \end{aligned} \tag{3.34}$$

Suppose that the initial state of the controller $\tilde{K}_{d_i}$ is equal to $\eta_{d_i}(0) = \begin{bmatrix} X^{i\,T}_{\mu} & 0_{1\times rp} \end{bmatrix}^T$, for all $i \in \bar{\nu}$, where $0_{1\times rp}$ denotes the $1 \times rp$ zero matrix. It is desired to prove that $\tilde{K}_d$ is a solution of Problem 1.

Theorem 3.4. *The decentralized controller $\tilde{K}_d$ satisfies the requirements (i) and (ii) of Problem 1 for the system $\mathcal{S}$.*

Proof: Since $\tilde{K}_c$ given by (3.32) is designed in Section 3.4 in such a way that it stabilizes the modified system S^i for any $i \in \bar{\nu}$, it can be concluded from Theorem 3.2 that the state $x(t)$ of the system $\mathcal{S}$ under the decentralized controller $\tilde{K}_d$ goes to zero as $t \to \infty$, provided $z(0) = 0$. Thus, the requirement (i) of Problem 1 is satisfied. Denote the block diagonal matrix $\mathcal{B}_{opt}$ as

$$\mathcal{B}_{opt} = \text{diag}\left(\left[\begin{array}{cccc} \mathcal{B}_{11}^{opt} & \mathcal{B}_{22}^{opt} & \cdots & \mathcal{B}_{\nu\nu}^{opt} \end{array} \right]\right) \tag{3.35}$$

It can be easily verified that $\mathbf{B}_i^{opt}$ introduced in (3.33) is equal to

$$\mathbf{B}_i^{opt} = \left[\begin{array}{ccccccc} 0_{r_i \times r_1 p} & \cdots & 0_{r_i \times r_{i-1} p} & \mathcal{B}_{ii}^{opt\,T} & 0_{r_i \times r_{i+1} p} & \cdots & 0_{r_i \times r_\nu p} \end{array} \right]^T, \quad i \in \bar{\nu} \tag{3.36}$$

Furthermore, $\mathbf{B}_{opt}^i \mathbf{C}^i$ is derived from $\mathcal{B}_{opt} C$ by removing its i-th block column (which is equal to $\mathbf{B}_i^{opt} C_i$). This observation along with the fact that $\mathcal{B}_{opt}$ and C are block diagonal, yields that the i-th block row of $\mathbf{B}_{opt}^i \mathbf{C}^i$ is a zero matrix. Using this result and substituting (3.36) into (3.34), one can rearrange the entries of the state vector $\eta_{d_i}(t)$ in order to come up with the following state-space representation for the local controller $\tilde{K}_{d_i}$

$$\begin{aligned} \dot{\tilde{\eta}}_{d_i}(t) &= \left[\begin{array}{cc} \mathcal{A}_i & 0 \\ L_{i_1} & L_{i_2} \end{array} \right] \tilde{\eta}_{d_i}(t) + \left[\begin{array}{cc} \mathcal{B}_{ii}^{opt} & 0 \\ 0 & L_{i_3} \end{array} \right] \left[\begin{array}{cc} y_i(t) & 0 \\ 0 & x_i(t) \end{array} \right] \\ u_i(t) &= L_{i_4} \tilde{\eta}_{d_i}(t) + \mathbf{N}_{i_2}^{opt} x_i(t) \end{aligned} \tag{3.37}$$

where $\mathcal{A}_i$ is defined in (3.30). Apply now the decentralized controller $\tilde{K}_d$ to the system $\mathcal{S}$. Each interconnection signal coming into the subsystem $\mathcal{S}_i$ from the other subsystems is composed of two main components: one is exponentially decaying (because the requirement (i) of Problem 1 is fulfilled) and hence does not affect the regulation of y_i, and the other one is an unbounded component whose effect is similar to $\omega(t)$ in (3.2). This unbounded component together with the disturbance term $E_i\omega(t)$ can be modeled in the state-space representation of the subsystem $\mathcal{S}_i$ as an embedded term $\tilde{E}_i\omega(t)$, where $\omega(t)$ is obtained from (3.2) with a proper initial condition $z(0)$. As a result, the i-th subsystem can be modeled as

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_i u_i(t) + G_i r_i(t) + \tilde{E}_i \omega(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{3.38}$$

where $r_i(t)$ represents the exponentially decaying component of the incoming interconnections. Since the form of $\tilde{K}_{d_i}$ in (3.37) complies with that of the controller proposed in [24], $y_i(t)$ approaches zero as $t \to \infty$, when the local controller $\tilde{K}_{d_i}$ (given by (3.37)) is applied to the system given by (3.38). This completes the proof. ■

So far, it is shown that the decentralized controller $\tilde{K}_d$ satisfies the requirements (i) and (ii) of Problem 1. The performance of the resultant decentralized controller will be investigated next (note that the performance index was defined by (3.7) in the requirement (iii) of Problem 1).

Assume that the centralized controller $\tilde{K}_c$ is applied to the system $\mathcal{S}$. Denote the corresponding performance index (3.7) with J_{opt}. Note that J_{opt} is achieved by solving a constrained optimization problem, and ideally it is desired to have the same performance for the decentralized control system. However, in reality there will be a deviation between the decentralized performance index and J_{opt}. A method will be given next to measure this deviation.

The performance index J associated with the system $\mathcal{S}$ under the decentralized controller $\tilde{K}_d$ can be written as trace$(P_d X_0^d)$, where P_d is derived from a Lyapunov equation [67], and

$$X_0^d = E\left\{ \begin{bmatrix} x(0) \\ X_\mu^1 \\ X_\mu^2 \\ \vdots \\ X_\mu^\nu \end{bmatrix} \begin{bmatrix} x(0)^T & X_\mu^{1T} & \dots & X_\mu^{\nu T} \end{bmatrix} \right\} = \begin{bmatrix} X_0 & X_\mu X_\mu^{1T} & \cdots & X_\mu X_\mu^{\nu T} \\ X_\mu^1 X_\mu^T & X_\mu^1 X_\mu^{1T} & \cdots & X_\mu^1 X_\mu^{\nu T} \\ \vdots & \vdots & \ddots & \vdots \\ X_\mu^\nu X_\mu^T & X_\mu^\nu X_\mu^{1T} & \cdots & X_\mu^\nu X_\mu^{\nu T} \end{bmatrix} \tag{3.39}$$

According to Theorem 3.1, if X_μ^i is equal to $x^i(0)$ for all $i \in \bar{\nu}$, then the state and the input of the centralized closed-loop system are the same as those of the corresponding decentralized closed-loop system. Hence, J_{opt} can alternatively be written as trace$(P_d X_0^c)$, where

$$X_0^c = E\left\{ \begin{bmatrix} x(0) \\ x^1(0) \\ x^2(0) \\ \vdots \\ x^{\nu}(0) \end{bmatrix} \begin{bmatrix} x(0)^T & x^1(0)^T & \dots & x^{\nu}(0)^T \end{bmatrix} \right\} \tag{3.40}$$

Therefore, the error between J and J_{opt} can be obtained as follows

$$J_{opt} - J = \text{trace}\left(P_d \begin{bmatrix} 0 & \text{cov}(X_\mu, X_\mu^1) & \cdots & \text{cov}(X_\mu, X_\mu^\nu) \\ \text{cov}(X_\mu^1, X_\mu) & \text{cov}(X_\mu^1, X_\mu^1) & \cdots & \text{cov}(X_\mu^1, X_\mu^\nu) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_\mu^\nu, X_\mu) & \text{cov}(X_\mu^\nu, X_\mu^1) & \cdots & \text{cov}(X_\mu^\nu, X_\mu^\nu) \end{bmatrix} \right) \tag{3.41}$$

where $\text{cov}(\pi_1, \pi_2) = E\{\pi_1 \pi_2^T\} - E\{\pi_1\} E\{\pi_2^T\}$ for any arbitrary column vectors π_1 and π_2. One can use the equations (3.40) and (3.41) to obtain the relative error between J and J_{opt}.

Remark 3.4. *One can use the equation (3.41) to assess the closeness of the decentralized performance index J to its optimal centralized counterpart J_{opt}. In addition, it can be deduced from (3.41) that the more the initial state $x(0)$ tends to be deterministic, the closer J becomes to J_{opt}, and in the case of a deterministic initial state, J is equal to J_{opt}. This observation along with the result of Theorem 3.4 confirms that $\tilde{K}_d$ is a solution of Problem 1.*

Remark 3.5. *The centralized servomechanism controller $\tilde{K}_c$ is obtained in Section 3.4 by solving an optimization problem subject to the constraint that all of the systems $\mathbf{S}^2, \mathbf{S}^3, ..., \mathbf{S}^\nu$ are stable under the controller $\tilde{K}_c$. This constraint is essential in deriving the decentralized controller. Nevertheless, when the interconnections* *between the subsystems of $\mathcal{S}$ are relatively weak, the aforementioned constraint is not expected to have much influence on the solution of the optimization problem. In other words, in the presence of sufficiently weak interconnections, $\tilde{K}_c$ obtained by solving the optimization problem automatically stabilizes the systems $\mathbf{S}^2, ..., \mathbf{S}^\nu$. This implies that the above constraint is unlikely to degrade the global optimality of the solution. This point is further clarified later in Example 3.1.*

Remark 3.6. *The results obtained can easily be extended to the tracking problem in the presence of a nonzero reference input which can be expressed similarly by (3.2), with all of its poles located in the closed left-half plane. This can be carried out by defining an augmented system and converting the tracking problem for the original system to a regulation one for the augmented system [48].*

3.6 Practical Considerations in Control Design

The decentralized servomechanism controller given in the previous section was designed based on some rather unrealistic assumptions. For instance, the exact model of the overall system has been assumed to be known by all local subsystems. In order to modify the control design procedure to make it more suitable in a practical framework, the following issues will now be considered in the control design

- The knowledge of the system parameters is not identical from the viewpoints of different subsystems.

- The system $\mathcal{S}$ is subject to perturbation in the sense that its parameters A and B are uncertain.

- There exist delay in the interconnection and input signals.

The following definitions will prove convenient in the development of the remaining results.

Definition 3.3. *The system* $\tilde{\mathbf{S}}^i$*,* $i \in \bar{\nu}$*, is defined to be a system obtained from* $\mathcal{S}$ *by carrying out the following steps*

- *All of the interconnections coming into the i-th subsystem are removed.*
- *All of the parameters of the system, except for those of the subsystem* $\mathcal{S}_i$ *and its outgoing interconnections, are replaced by the values representing the belief of the subsystem* $\mathcal{S}_i$ *about those parameters.*

Definition 3.4. *Let the perturbed model of the system* $\mathcal{S}$ *be denoted by* $\bar{\mathcal{S}}$*. The system* $\bar{\mathbf{S}}^i$*,* $i \in \bar{\nu}$*, is defined to be a system obtained from* $\tilde{\mathbf{S}}^i$ *by replacing the parameters of the i-th subsystem as well as its outgoing interconnections, with the corresponding perturbed values of the system* $\bar{\mathcal{S}}$*. Let the state-space representation of* $\bar{\mathbf{S}}^i$ *be represented as*

$$\begin{aligned} \dot{x}(t) &= \bar{A}^i x(t) + \bar{B}^i u(t) \\ y(t) &= \bar{C}^i x(t) \end{aligned} \tag{3.42}$$

In order to proceed with the control design, all assumptions made earlier for the system $\mathcal{S}$ (Assumptions 3.1, 3.2 and 3.3) are also required to hold for the systems $\tilde{\mathbf{S}}^i$, $i \in \bar{\nu}$ (because $\tilde{\mathbf{S}}^i$ is the i-th subsystem's $\mathcal{S}_i$ belief of the system $\mathcal{S}$). Suppose that all of those assumptions are satisfied. For any $i \in \bar{\nu}$, design a centralized servomechanism controller $\tilde{K}_c^i$ for the system $\tilde{\mathbf{S}}^i$ using the methodology explained in Section 3.3. Then, convert the centralized controller $\tilde{K}_c^i$ to a decentralized servomechanism controller $\tilde{K}_d^i$ as pointed out in Section 3.5, and denote its local controllers with $\tilde{K}_{d_1}^i, \tilde{K}_{d_2}^i, ..., \tilde{K}_{d_\nu}^i$. Define now the decentralized servomechanism controller $\hat{K}_d$ as a controller consisting of the local controllers $\tilde{K}_{d_1}^1, \tilde{K}_{d_2}^2, ..., \tilde{K}_{d_\nu}^\nu$. It is to be noted that $\hat{K}_d$ is a modified version of the controller $\tilde{K}_d$ obtained in Section 3.5, as it is attained through a procedure which takes into account the practical issues such as uncertainties and non-identical beliefs of the subsystems about the system parameters. It is interesting to observe that in the case when any two different subsystems have exactly the same belief of the system $\mathcal{S}$, then $\tilde{K}^i_{d_l}$ will be identical to $\tilde{K}^j_{d_l}$, for any $i, j, l \in \bar{\nu}$.

Definition 3.5. *Let $\hat{\mathcal{S}}$ represent the system obtained from $\bar{\mathcal{S}}$ by considering the delay d_{i_j} for the j-th input of its i-th subsystem, for any $i \in \bar{\nu}$ and $j \in \{1, 2, ..., m_i\}$. Accordingly, $\hat{\mathbf{S}}^i$, $i \in \bar{\nu}$, is defined to be a system obtained from $\bar{\mathbf{S}}^i$ by considering the delay d_{i_j} for the j-th input of its i-th subsystem, for any $i \in \bar{\nu}$ and $j \in \{1, 2, ..., m_i\}$.*

It is to be noted that $\hat{\mathcal{S}}$ is the perturbed version of the ideal system $\mathcal{S}$ which is also affected by delay. Moreover, $\hat{\mathbf{S}}^i$, $i \in \bar{\nu}$, is derived from the system $\tilde{\mathbf{S}}^i$ by imposing the delay and perturbations on its i-th subsystem only. It is also worth noting that $\hat{\mathcal{S}}$ is, in fact, a more accurate model for the system represented by $\mathcal{S}$, in a practical environment. Following an approach similar to the one used in Theorem 3.2, one can arrive at the next theorem which reveals an important characteristic of the above model.

Theorem 3.5. *The system $\hat{\mathcal{S}}$ is stable under the decentralized controller $\hat{K}_d$ if and only if the system $\hat{\mathbf{S}}^i$ is stable under the centralized controller $\tilde{K}^i_c$ for any $i \in \bar{\nu}$.*

Remark 3.7. *Theorem 3.5 translates the stability of the decentralized control system into that of a set of centralized control systems. This approach is very useful for obtaining some permissible bounds on the uncertain parameters of the system $\hat{\mathcal{S}}$ and the delay. For instance, in the case of delay-free systems, the problem can be reduced to finding the sensitivity of the eigenvalues of a number of matrices to the variations in their entries. This has been addressed in the literature using different mathematical approaches [36], [34], and it is known that this sensitivity depends on several factors such as the norm of the perturbation matrix, condition number or eigenvalue condition number [36], and whether or not there are any repeated eigenvalues (and the corresponding multiplicities), in general.*

Analogously to [25], the following result on the asymptotic regulation property of the proposed decentralized control system can be deduced.

Theorem 3.6. *Assume that the system $\hat{S}$ is stable under the decentralized controller $\hat{K}_d$. The output $y(t)$ is regulated to zero for any $z(0)$.*

Theorem 3.6 states that as long as the system $\hat{S}$ is stable under the decentralized servomechanism controller $\hat{K}_d$, the desired output regulation is achieved. Hence, the sole concern regarding the controller $\hat{K}_d$ is that it should maintain the stability of the closed-loop system, which can be ensured by obtaining a number of conditions using [36] and the references therein.

Remark 3.8. *The deviation in the performance index for the case when all individual subsystems assume the same modeling parameters for the system $\mathcal{S}$ was obtained in Section 3.5, as expressed by the equation (3.41). One can pursue the same methodology here in order to attain a similar result under the assumptions made in this section, which describe a more pragmatic case.*

3.7 Numerical Example

Example 3.1. *Consider a system $\mathcal{S}$ consisting of two interconnected subsystems with the following state-space representation for its first subsystem S_1*

$$\begin{aligned} \dot{x}_1(t) &= \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} x_1(t) + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u_1(t) + E_1\omega(t), \\ y_1(t) &= \begin{bmatrix} -1 & 2 \end{bmatrix} x_1(t) \end{aligned} \tag{3.43}$$

and the following representation for its second subsystem S_2

$$\begin{aligned} \dot{x}_2(t) &= \begin{bmatrix} -1 & 2 \end{bmatrix} x_1(t) - 3x_2(t) + 5u_2(t) + E_2\omega(t), \\ y_2(t) &= 3x_2(t) \end{aligned} \tag{3.44}$$

where

- *the disturbance input $\omega(t)$ is assumed to be the scalar exponential function e^t.*

- *E_1 and E_2 are unknown matrices of proper dimensions, which account for the unmeasurable nature of the disturbance in the system.*

Assume that the initial state of the system is a random variable with X_0 (defined in (3.6)) equal to I. It is desired to design a decentralized controller K_d to solve Problem 1 with $Q = R = I$. To this end, an initial stabilizing centralized controller which can reject the disturbance $\omega(t)$ is to be designed first. This controller is obtained using the method proposed earlier, and is given below

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{M}_o = \begin{bmatrix} -3.3182 & 0 \\ 0 & -1.0393 \end{bmatrix},$$
$$\mathcal{N}_o = \begin{bmatrix} 0.9231 & -4.4856 & 0 \\ 0 & 0 & -1.0147 \end{bmatrix} \tag{3.45}$$

Using Algorithm 3.1 for optimizing the performance of the initial controller iteratively (starting from the initial controller provided above), one will arrive at a centralized controller $\tilde{K}_c$ described in (3.32) with the state-space matrices

$$\mathcal{B}_{opt} = \begin{bmatrix} 3.3253 & 0 \\ 0 & 1.6471 \end{bmatrix}, \quad \mathcal{M}_{opt} = \begin{bmatrix} -0.9348 & -0.0207 \\ 0.0988 & -0.5580 \end{bmatrix},$$
$$\mathcal{N}_{opt} = \begin{bmatrix} 0.8214 & -4.2823 & -0.0513 \\ 0.0480 & -0.1015 & -0.9764 \end{bmatrix} \tag{3.46}$$

The resultant quadratic performance index J corresponding to the initial controller (3.1) given to Algorithm 3.1 and the optimal controller $\tilde{K}_c$ are given by 8.6425 and 3.9422, respectively. This sizable reduction in the cost function points to the effectiveness of Algorithm 3.1. Now, decentralize the controller $\tilde{K}_c$ using the procedure *proposed in Section 3.5, to obtain the local controllers* $\tilde{K}_{d_1}$ *and* $\tilde{K}_{d_2}$ *described by*

$$\dot{\eta}_{d_1}(t) = \begin{bmatrix} -7.8820 & 0.4940 & -2.7898 \\ 0 & 1.0000 & 0 \\ 4.9412 & 0 & 1.0000 \end{bmatrix} \eta_{d_1}(t) + \begin{bmatrix} -0.7602 & 1.4925 \\ -3.3253 & 6.6506 \\ 0 & 0 \end{bmatrix} x_1(t)$$

$$u_1(t) = \begin{bmatrix} -0.0513 & -0.9348 & -0.0207 \end{bmatrix} \eta_{d_1}(t) + \begin{bmatrix} 0.8214 & -4.2823 \end{bmatrix} x_1(t) \tag{3.47}$$

and

$$\dot{\eta}_{d_2}(t) = \begin{bmatrix} 1.8214 & -6.2823 & -0.9348 & -0.0207 \\ 4.4642 & -9.8468 & -2.8043 & -0.0621 \\ -3.3253 & 6.6506 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix} \eta_{d_2}(t) + \begin{bmatrix} -0.0513 \\ -0.1539 \\ 0 \\ 4.9412 \end{bmatrix} x_2(t)$$

$$u_2(t) = \begin{bmatrix} 0.0480 & -0.1015 & 0.0988 & -0.5580 \end{bmatrix} \eta_{d_2}(t) - 0.9764 x_2(t) \tag{3.48}$$

respectively. It is worth mentioning that these local controllers are attained based upon the assumption that every subsystem knows precisely the parameters of the other subsystem, but not necessarily its initial state. To evaluate the performance of the controller $\tilde{K}_d$*, suppose that the real initial state* $x(0)$ *is equal to* $\begin{bmatrix} 1.5 & 1.5 & 1.5 \end{bmatrix}^T$. *This represents an inferior scenario in light of the relation* $X_0 = I$ *(in fact, it can be easily verified that the initial state given above is noticeably far from its mean). Now, consider two cases as follows*

1. *Assume that*

$$E_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 5 \end{bmatrix} \tag{3.49}$$

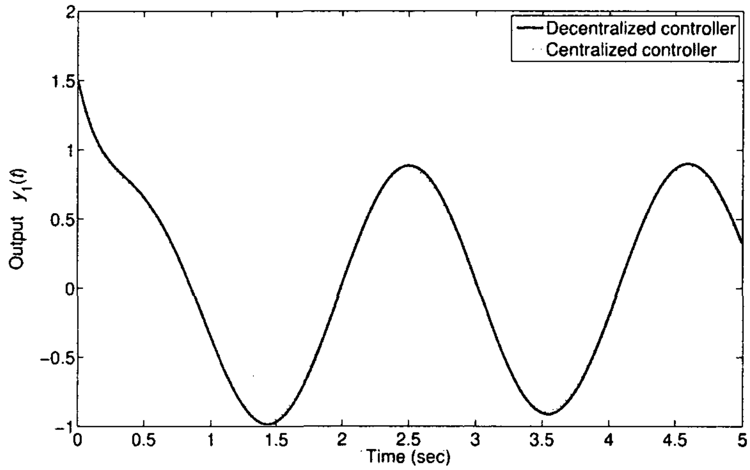
and that each local controller knows the initial state of the other subsystem with -100% *error. This means that the initial states* η_{d_1} *and* η_{d_2} *are zero vectors. Let the external input* $\sin(3t)$ *(in addition to the disturbance input* $\omega(t)$*) be applied to the system* $\mathcal{S}$. *The outputs of the first and the second subsystems of* *the system $\mathcal{S}$ under the controllers $\tilde{K}_c$ and $\tilde{K}_d$ are depicted in Figure 3.1. As can be observed, these two controllers perform almost identically such that the discrepancy in their corresponding signals is barely visible (in particular in the output $y_1(t)$). This figure also illustrates that the disturbance is rejected very quickly and that the steady-state trajectory is reached rather shortly, although the error in the initial state estimate was significantly large.*

2. *Assume that E_1 and E_2 are the same as the ones given in (3.49), and that each local controller knows the initial state of the other subsystem with* 5000% *error (i.e., a quite substantial error in the initial state estimation). Hence,*

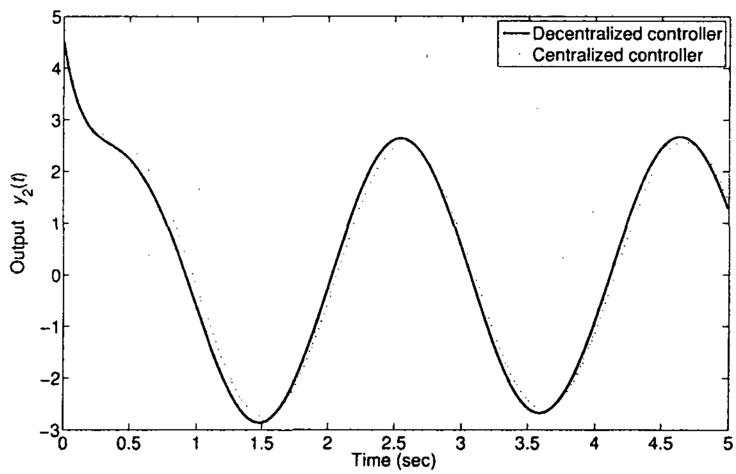
$$\eta_{d_1} = \begin{bmatrix} 75 & 0 & 0 \end{bmatrix}, \quad \eta_{d_2} = \begin{bmatrix} 75 & 75 & 0 & 0 \end{bmatrix} \tag{3.50}$$

Let the external unbounded input $t \times \sin(t)$ be applied to the system $\mathcal{S}$. The outputs of the first and the second subsystems of the system $\mathcal{S}$ under the controllers $\tilde{K}_c$ and $\tilde{K}_d$ are depicted in Figure 3.2. This figure substantiates how insensitive the decentralized controller $\tilde{K}_d$ is to the initial state's prediction error.

Remark 3.9. *The method proposed in this chapter can be applied to various systems such as flight formation, power systems, etc. Nevertheless, for the sake of simplicity and in order to show the design details, a low-order system was examined in the above example.*

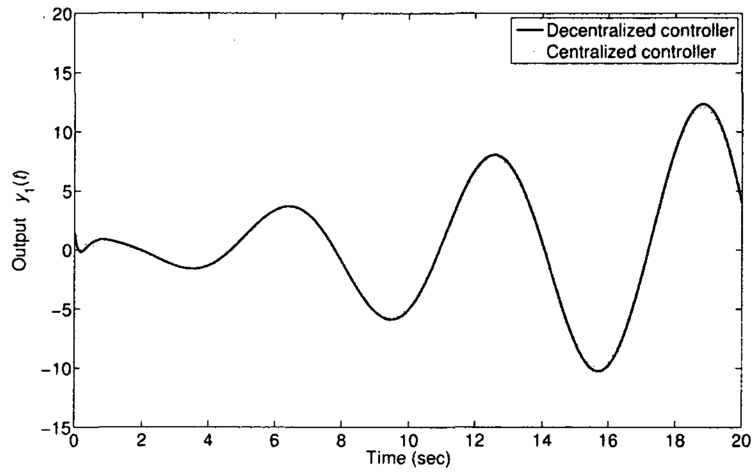


(a)

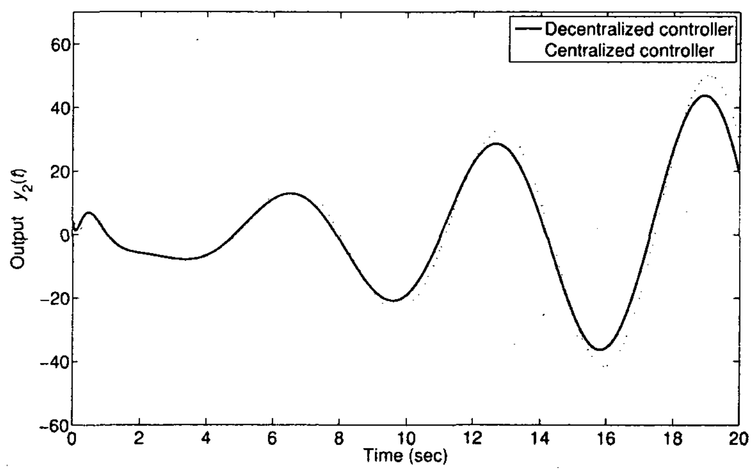


(b)

Figure 3.1: The outputs $y_1(t)$ and $y_2(t)$ of the centralized and decentralized control systems in the presence of -100% prediction error for the initial state.



(a)



(b)

Figure 3.2: The outputs $y_1(t)$ and $y_2(t)$ of the centralized and decentralized control systems in the presence of 5000% prediction error for the initial state.

Chapter 4

A Cooperative Predictive Control Technique for Spacecraft Formation Flying

4.1 Introduction

Formation flying control involving a number of spacecraft in order to accomplish a mission cooperatively has been of special interest in the recent years [145], [38]. The manner of cooperation between the spacecraft determines the architecture of the formation, which has been classified in the literature as five main categories: leader-follower, behavioral, virtual, cyclic, and multi-input multi-output. This classification is, in fact, based on the topology of communication between the spacecraft controllers. In practice, it is desired to minimize the number of communication links, as the data transmission is expensive in deep space applications. Lack of sufficient number of communication links, on the other hand, may cause several problems such as deterioration of the overall control performance, inability to avoid collision and/or to detect obstacles, inefficient formation reconfiguration, etc.

In [137], a formation consisting of a number of physically decoupled spacecraft in deep space is defined in terms of the relative positions between the spacecraft as well as the spacecraft attitudes. Basically, the approach introduced in [137] enables each spacecraft to systematically calculate the distance between any pair of spacecraft using its locally measurable information, with no communication requirement.

The method proposed in [139] considers a static controller for any spacecraft formation. It is assumed that this controller is designed to satisfy desired specifications. Since this controller takes advantage of all the communication links, it is very difficult to implement it in practice. Hence, a method is introduced consequently which aims to eliminate some of the communication links from the control structure, and to estimate the lost information by means of local observers. The resulting controller behaves closely to the original one, in general, after an initial transient. The decentralized controller obtained is much more complex than the centralized one at the price of simpler structure, i.e. fewer communication links. Although [138] presents a novel idea, it suffers from two practical drawbacks:

- In the control design procedure, certain conditions are required to be satisfied (as will be discussed later). First of all, these requirements do not hold in many practical cases. Furthermore, there is no systematic pole-placement technique via a decentralized *static* controller (which is also required in the corresponding control design procedure).

- Any crucial occurrence, such as collision, may happen during the transient time due to the overshoots. As a remedy to avoid these unwanted incidents, one may reduce the transient time by deploying high gains in the local observers. Nevertheless, this may cause saturation in the actuators.

In many formation control techniques the model of the entire formation is copied in all the local controllers. This can be envisaged as an open-loop strategy for the control design, which is known to be sensitive to parameter variations. The methods in [139] and [138] are further developed to address this issue by introducing fewer communication links.

On the other hand, a near-optimal control law with no communication link is proposed in [79] for flight formations with hierarchical LTI models in relative coordinates, and its key features such as stability and robustness are investigated thoroughly. In this chapter, the results of [70] will be extended to surmount a more general formation control problem. It is assumed that a centralized controller consisting of a set of interacting local controllers for the formation is designed to achieve the desired specifications such as optimal performance, collision avoidance, etc. The objective here is to design another controller which performs almost the same as the original controller, while its communication requirement is significantly lower. Throughout this chapter, the terms *centralized* and *decentralized* controller are referred to the original multivariable controller (consisting of the interacting local controllers), and the proposed controller with reduced number of communication links, respectively.

To this end, the formation is first described by a hierarchical linear time-invariant (LTI) model. A decentralized controller is then derived from any given centralized controller. The idea concealed behind this approach is that each local controller estimates the unavailable states of other spacecraft according to its belief about the model of the formation. Robust stability of the designed decentralized controller and its closeness to the reference centralized one are investigated in this work, analogously to the results obtained in [79] and [70].

The proposed control law may still suffer from the following drawbacks:

- Most of the time, a linearized model cannot describe the formation accurately for a long period of time.
- Since the local controller of any spacecraft uses the nominal models of other

spacecraft, the discrepancy between this model and the real one may cause problem.

- Unlike the centralized controller, the proposed controller is incapable of avoiding collision and obstacles, detecting certain faults, efficient reconfiguration, etc.

In order to ameliorate the above-mentioned limitations, the proposed controller has been reformulated in the predictive-control framework. More precisely, the communication links required for the implementation of the centralized controller which were eliminated in the proposed decentralized controller are replaced with weak communication links which transmit and receive information in certain time instants only. This implies that instead of removing the communication links perpetually, the communication rate is reduced as a compromise in the trade-off between the performance and communication cost. It will be shown later how the new model predictive controller takes the above issues into consideration.

4.2 Decentralized Implementation of a Centralized Controller

Consider a formation $\mathcal{F}$ consisting of ν spacecraft. Assume that the model of the formation expressed either in the relative coordinates or in the absolute coordinates has a hierarchical structure with the following state-space model for the i^{th} spacecraft:

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + \sum_{j=1}^{i-1} H_{ij}z_{ij}(t) + B_i u_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{4.1}$$

where $x_i(t) \in \Re^{n_i}$, $u_i(t) \in \Re^{m_i}$, $y_i(t) \in \Re^{r_i}$, $i \in \bar{\nu} := \{1, 2, ..., \nu\}$, are the state, the input and the measurable output of the i^{th} spacecraft, respectively, and $z_{ij}(t)$

is a signal representing the effect of the j^{th} spacecraft on the dynamics of the i^{th} spacecraft. The signal $z_{ij}(t)$, $i, j \in \bar{\nu}$, $j < i$, can be regarded as an input for the model of the i^{th} spacecraft coming out of the j^{th} spacecraft as an output, which can be modeled as $z_{ij}(t) = L_{ij}x_j(t)$. Assume now that $z_{ij}(t)$ is measurable for the j^{th} spacecraft, i.e., it can be computed from $y_j(t)$. Define:

$$A_{ij} := H_{ij}L_{ij}, \quad i, j \in \bar{\nu}, \ j < i \tag{4.2}$$

The formation $\mathcal{F}$ consists of all of the spacecraft in (4.1), and is represented by the following state-space model:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4.3}$$

where:

$$u(t) := \begin{bmatrix} u_1(t)^T & \dots & u_\nu(t)^T \end{bmatrix}^T, \tag{4.4a}$$

$$x(t) := \begin{bmatrix} x_1(t)^T & \dots & x_\nu(t)^T \end{bmatrix}^T, \tag{4.4b}$$

$$y(t) := \begin{bmatrix} y_1(t)^T & \dots & y_\nu(t)^T \end{bmatrix}^T \tag{4.4c}$$

and

- A is a lower block-diagonal matrix whose (i, j) block entry is A_{ij}, for any $i, j \in \bar{\nu}$, $i < j$.
- B and C are block diagonal matrices with the respective i^{th} block entries B_i and C_i, for any $i \in \bar{\nu}$.

Consider a centralized LTI controller K_c with the following state-space representation:

$$\begin{aligned} \dot{\eta}_c(t) &= \Gamma\eta_c(t) + \Omega y(t) \\ u(t) &= M\eta_c(t) + Ny(t) \end{aligned} \tag{4.5}$$

where $\eta_c \in \Re^{\mu}$ is the controller state. It is assumed that the controller K_c has been designed by using any proper technique to achieve any prespecified control objectives such as optimal energy, collision avoidance, etc. The implementation of the centralized controller K_c requires several communication links in general, so that all of the spacecraft can share their outputs with each other. Since this is not pragmatic, it is desired now to implement the centralized control K_c in an equivalent decentralized fashion. To this end, define the following vectors:

$$\begin{aligned} x^i &:= \begin{bmatrix} x_1^T & \dots & x_{i-1}^T & x_{i+1}^T & \dots & x_\nu^T \end{bmatrix}^T, \\ u^i &:= \begin{bmatrix} u_1^T & \dots & u_{i-1}^T & u_{i+1}^T & \dots & u_\nu^T \end{bmatrix}^T, \\ y^i &:= \begin{bmatrix} y_1^T & \dots & y_{i-1}^T & y_{i+1}^T & \dots & y_\nu^T \end{bmatrix}^T \end{aligned} \tag{4.6}$$

for any $i \in \bar{\nu}$.

Notation 4.1. *The following notations will prove to be convenient in the development of the main results:*

- *Consider a $\nu \times \nu$ block diagonal matrix T with block entries $I_{r_1 \times r_1}, I_{r_2 \times r_2}, \dots, I_{r_\nu \times r_\nu}$. Denote the i^{th} block column of T with T_i and the matrix obtained from T by removing T_i with T^i, for any $i \in \bar{\nu}$.*
- *Similarly, define $\bar{T}$ as a $\nu \times \nu$ block diagonal matrix with the block entries $I_{m_1 \times m_1}, I_{m_2 \times m_2}, \dots, I_{m_\nu \times m_\nu}$. Denote the i^{th} block column of $\bar{T}$ with $\bar{T}_i$ and the matrix obtained from $\bar{T}$ by removing $\bar{T}_i$ with $\bar{T}^i$, for any $i \in \bar{\nu}$.*
- *Consider a $\nu \times \nu$ block diagonal matrix $\tilde{T}$ with block entries $I_{n_1 \times n_1}, I_{n_2 \times n_2}, \dots, I_{n_\nu \times n_\nu}$. Denote the i^{th} block column of $\tilde{T}$ with $\tilde{T}_i$ and the matrix obtained from $\tilde{T}$ by removing $\tilde{T}_i$ with $\tilde{T}^i$, for any $i \in \bar{\nu}$.*

One can easily conclude that:

$$
\begin{aligned}
&y(t) = \begin{bmatrix} T^i & T_i \end{bmatrix} \begin{bmatrix} y^i(t) \\ y_i(t) \end{bmatrix}, \quad u(t) = \begin{bmatrix} \bar{T}^i & \bar{T}_i \end{bmatrix} \begin{bmatrix} u^i(t) \\ u_i(t) \end{bmatrix} \\
&x(t) = \begin{bmatrix} \tilde{T}^i & \tilde{T}_i \end{bmatrix} \begin{bmatrix} x^i(t) \\ x_i(t) \end{bmatrix}, \quad i \in \bar{\nu}
\end{aligned}
\tag{4.7}
$$

Substituting the equation (4.7) into (4.5), the controller K_c can be written as follows:

$$
\begin{aligned}
\dot{\eta}_c(t) &= \Gamma \eta_c(t) + \mathbf{\Omega}^i y^i(t) + \mathbf{\Omega}_i y_i(t) \\
u^i(t) &= \mathbf{M}^i \eta_c(t) + \mathbf{N}^i y^i(t) + \mathbf{N}_i y_i(t) \\
u_i(t) &= \mathbf{M}_i \eta_c(t) + \mathbf{N}^{\bar{i}} y^i(t) + \mathbf{N}_{\bar{i}} y_i(t)
\end{aligned}
\tag{4.8}
$$

for any $i \in \bar{\nu}$, where:

$$
\begin{aligned}
\begin{bmatrix} \mathbf{\Omega}^i & \mathbf{\Omega}_i \end{bmatrix} &:= \mathbf{\Omega} \begin{bmatrix} T^i & T_i \end{bmatrix}, \\
\begin{bmatrix} \mathbf{M}^i \\ \mathbf{M}_i \end{bmatrix} &:= \begin{bmatrix} \bar{T}^i & \bar{T}_i \end{bmatrix}^T \mathbf{M}, \\
\begin{bmatrix} \mathbf{N}^i & \mathbf{N}_i \\ \mathbf{N}^{\bar{i}} & \mathbf{N}_{\bar{i}} \end{bmatrix} &:= \begin{bmatrix} \bar{T}^i & \bar{T}_i \end{bmatrix}^T \mathbf{N} \begin{bmatrix} T^i & T_i \end{bmatrix}
\end{aligned}
\tag{4.9}
$$

Likewise, the system $\mathcal{S}$ given in (4.3) can be decomposed as follows:

$$
\begin{aligned}
\dot{x}^i(t) &= \mathbf{A}^i x^i(t) + \mathbf{A}_i x_i(t) + \mathbf{B}^i u^i(t) \\
\dot{x}_i(t) &= \mathbf{A}^{\bar{i}} x^i(t) + A_{ii} x_i(t) + B_i u_i(t) \\
y^i(t) &= \mathbf{C}^i x^i(t)
\end{aligned}
\tag{4.10}
$$

for any $i \in \bar{\nu}$, where:

$$
\begin{aligned}
\begin{bmatrix} \mathbf{A}^i & \mathbf{A}_i \\ \mathbf{A}^{\bar{i}} & A_{ii} \end{bmatrix} &:= \begin{bmatrix} \tilde{T}^i & \tilde{T}_i \end{bmatrix}^T A \begin{bmatrix} \tilde{T}^i & \tilde{T}_i \end{bmatrix}, \\
\begin{bmatrix} \mathbf{B}^i & 0 \\ 0 & B_i \end{bmatrix} &:= \begin{bmatrix} \tilde{T}^i & \tilde{T}_i \end{bmatrix}^T B \begin{bmatrix} \bar{T}^i & \bar{T}_i \end{bmatrix}, \\
\begin{bmatrix} \mathbf{C}^i & 0 \\ 0 & C_i \end{bmatrix} &:= \begin{bmatrix} T^i & T_i \end{bmatrix}^T C \begin{bmatrix} \tilde{T}^i & \tilde{T}_i \end{bmatrix}
\end{aligned}
\tag{4.11}
$$

Using the equations (4.8) and (4.10), one can find the following equation relating $x_i(t)$ and $y_i(t)$ to $u_i(t)$:

$$\begin{bmatrix} \dot{x}^i(t) \\ \dot{\eta}_c(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A}^i + \mathbf{B}^i\mathbf{N}^i\mathbf{C}^i & \mathbf{B}^i\mathbf{M}^i \\ \boldsymbol{\Omega}^i & \Gamma \end{bmatrix} \begin{bmatrix} x^i(t) \\ \eta_c(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}^i\mathbf{N}_i \\ \boldsymbol{\Omega}_i \end{bmatrix} y_i(t) + \begin{bmatrix} \mathbf{A}_i \\ 0 \end{bmatrix} x_i(t) \tag{4.12}$$

$$u_i(t) = \begin{bmatrix} \mathbf{N}^{\bar{i}}\mathbf{C}^i & \mathbf{M}_i \end{bmatrix} \begin{bmatrix} x^i(t) \\ \eta_c(t) \end{bmatrix} + \mathbf{N}_{\bar{i}} y_i(t)$$

Combining the relations $z_{ij}(t) = L_{ij}x_j(t)$ and $A_{ij} = H_{ij}L_{ij}$, $i, j \in \bar{\nu}$, $j < i$, leads to the equation $\mathbf{A}_i x_i(t) = \mathbf{H}^i z^i(t)$, where:

$$z^i(t) = \begin{bmatrix} 0^T & \cdots & 0^T & z_{(i+1)i}(t)^T & \cdots & z_{\nu i}(t)^T \end{bmatrix}^T \tag{4.13}$$

and $\mathbf{H}^i$ is a block diagonal matrix whose (j, j) block entry is equal to H_{ji} for any $j \in \bar{\nu}$, $i < j$, and 0 otherwise. Define now K_{d_i} as a controller for the i^{th} spacecraft whose state-space representation is given by:

$$\dot{\eta}_{d_i}(t) = \begin{bmatrix} \mathbf{A}^i + \mathbf{B}^i\mathbf{N}^i\mathbf{C}^i & \mathbf{B}^i\mathbf{M}^i \\ \boldsymbol{\Omega}^i & \Gamma \end{bmatrix} \eta_{d_i}(t) + \begin{bmatrix} \mathbf{B}^i\mathbf{N}_i \\ \boldsymbol{\Omega}_i \end{bmatrix} y_i(t) + \begin{bmatrix} \mathbf{H}^i \\ 0 \end{bmatrix} z^i(t) \tag{4.14}$$

$$u_i(t) = \begin{bmatrix} \mathbf{N}^{\bar{i}}\mathbf{C}^i & \mathbf{M}_i \end{bmatrix} \eta_{d_i}(t) + \mathbf{N}_{\bar{i}} y_i(t)$$

It is to be noted that by assumption $z^i(t)$ is measurable for the i^{th} spacecraft, i.e., it can be computed from $y_i(t)$. Define K_d as a decentralized controller consisting of the local controllers $K_{d_1}, K_{d_2}, \ldots., K_{d_\nu}$.

Theorem 4.1. *The formation $\mathcal{F}$ under the centralized controller K_c and the decentralized controller K_d behaves identically in the sense that it has the same state* *under both constraints, provided the following conditions hold:*

$$\eta_{d_i}(0) = \begin{bmatrix} x^i(0) \\ 0 \end{bmatrix}, \quad i \in \bar{\nu} \tag{4.15}$$

Proof: As pointed out earlier, the decomposed model of the formation given in (4.10) under controller K_c given in (4.8) results in the controller (4.12) for the i^{th} spacecraft. The proof follows on noting that the controller (4.14) is the same as (4.12) due to the relation $\mathbf{A}_i x_i(t) = \mathbf{H}^i z^i(t)$ and the equation (4.15). ∎

Theorem 4.1 states that the centralized controller K_c for the whole formation can be transformed into an equivalent decentralized controller K_d if the controller K_{d_i} for the i^{th} spacecraft, $i \in \bar{\nu}$, knows exactly the initial states and the modeling parameters of all other spacecraft. It is to be noted that this is not a realistic assumption in practice. To remedy the drawback of the inaccurate knowledge of the initial states, the following initial state will be deployed:

$$\eta_{d_i}(0) = \begin{bmatrix} \hat{x}^i(0) \\ 0 \end{bmatrix}, \quad i \in \bar{\nu} \tag{4.16}$$

instead of the one in (4.15), where $\hat{x}^i(0)$ is the estimate of $x^i(0)$ which is available to the i^{th} spacecraft. Choosing this new initial state can induce some nonzero residues for the unstable modes of the decentralized control system, and consequently make the formation unstable [70]. Hence, the internal stability of the formation $\mathcal{F}$ under the decentralized controller K_d will be investigated in the sequel.

Definition 4.1. *Consider the formation $\mathcal{F}$ given by (4.3). The modified formation $\mathcal{F}_i$, $i \in \bar{\nu}$, is defined to be a formation obtained from $\mathcal{F}$ by neutralizing the effect of spacecraft $1, 2, ..., i-1$ on the i^{th} spacecraft model. The state-space representation of the modified formation $\mathcal{F}_i$ is as follows:*

$$\begin{aligned} \dot{x}(t) &= \tilde{A}^i x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{4.17}$$

where $\tilde{A}^i$ is derived from A by replacing the first $i-1$ block entries of its i^{th} block row with zeros. It is to be noted that $\mathcal{F}^1 = \mathcal{F}$.

Definition 4.2. *Define the decoupled model of the i^{th} spacecraft, $i \in \bar{\nu}$, as:*

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + B_i u_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{4.18}$$

Note that in the decoupled model the effect of the other spacecraft is vanished.

Theorem 4.2. *The formation $\mathcal{F}$ is internally stable under the decentralized controller K_d if and only if the modified formation $\mathcal{F}_i$ is stable under the centralized controller K_c, for all $i \in \bar{\nu}$.*

Proof: For the special case when the matrix C is an identity matrix, the proof is provided in [79]. Following a similar technique, the proof can be accomplished in the general case. ■

Given the centralized controller K_c, its decentralized counterpart K_d obtained earlier can be applied to the formation $\mathcal{F}$ if and only if the easy-to-check conditions given in Theorem 4.2 hold. To compare these two controllers, it is known that the centralized controller K_c suffers from the following communication difficulties:

- The number of communication links grows with the square of ν.
- The communication links should be synchronized.
- The controller is vulnerable to the communication links failure in the sense that if one of them fails, the overall controller will not operate normally.

The main advantage of the decentralized controller K_d is that it does not have the above-mentioned difficulties. However, there are a few practical issues regarding the controller K_d which will be addressed in the following subsections.

4.2.1 Performance evaluation

Since the i^{th} spacecraft exploits the initial state $\hat{x}^i(0)$ instead of $x^i(0)$ due to its unavailability, the performance of the formation $\mathcal{F}$ under the decentralized controller K_d will not be identical to that of $\mathcal{F}$ under the centralized controller K_c. In order to evaluate the discrepancy between the performances in the centralized and the decentralized cases, consider the following cost function:

$$J = \int_0^\infty \Delta x(t)^T Q \Delta x(t) dt \tag{4.19}$$

where Q is a given positive definite matrix and $\Delta x(t)$ denotes the state of the formation given by (4.3) under the controller K_d minus that under the controller K_c. Define the vector ΔX_0 as:

$$\begin{aligned}&\Big[0_{1\times n}\ ,\ \left(\hat{x}^1(0) - x^1(0)\right)^T\ ,\ 0_{1\times\mu}\ ,\ \left(\hat{x}^2(0) - x^2(0)\right)^T,\\ &0_{1\times\mu}\ ,\ \dots\ ,\ \left(\hat{x}^\nu(0) - x^\nu(0)\right)^T\ ,\ 0_{1\times\mu}\Big]^T\end{aligned} \tag{4.20}$$

where $n := n_1 + n_2 + \cdots + n_\nu$ and $0_{i\times j}$ is a $i \times j$ zero matrix, for any positive integers i and j.

Define also Γ_d, Ω_d, M_d and N_d as block diagonal matrices whose i^{th} block entries, denoted by $\Gamma_{d_i}, \Omega_{d_i}, M_{d_i}$ and N_{d_i}, respectively, are as follows:

$$\begin{aligned}\Gamma_{d_i} &:= \begin{bmatrix} \mathbf{A}^i + \mathbf{B}^i\mathbf{N}^i\mathbf{C}^i & \mathbf{B}^i\mathbf{M}^i \\ \mathbf{\Omega}^i & \Gamma \end{bmatrix},\\ \Omega_{d_i} &:= \begin{bmatrix} \mathbf{B}^i\mathbf{N}_{\bar{i}}C_i + \mathbf{A}_i \\ \mathbf{\Omega}_{\bar{i}}C_i \end{bmatrix},\\ M_{d_i} &:= \begin{bmatrix} \mathbf{N}^{\bar{i}}\mathbf{C}^i & \mathbf{M}_{\bar{i}} \end{bmatrix},\quad N_{d_i} := \mathbf{N}_{\bar{i}}C_i\end{aligned} \tag{4.21}$$

for any $i \in \bar{\nu}$. It can be concluded from (4.14) that the closed-loop model of the formation $\mathcal{F}$ under the decentralized controller K_d will be given by:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\eta}_{d_1}(t) \\ \vdots \\ \dot{\eta}_{d_\nu}(t) \end{bmatrix} = A_d \begin{bmatrix} x(t) \\ \eta_{d_1}(t) \\ \vdots \\ \eta_{d_\nu}(t) \end{bmatrix} \tag{4.22}$$

where

$$A_d := \begin{bmatrix} A + BN_d & BM_d \\ \Omega_d & \Gamma_d \end{bmatrix} \tag{4.23}$$

Similarly to Theorem 4 in [70], one can obtain the following theorem which presents a simple methodology to calculate the cost function J.

Theorem 4.3. *Assume that the formation $\mathcal{F}$ given by (4.3) is stable under the decentralized controller K_d. The cost function J is equal to $\Delta X_0^T P_d \Delta X_0$, where the matrix P_d is the solution of the following Lyapunov equation:*

$$A_d{}^T P_d + P_d A_d + \Phi^T Q \Phi = 0 \tag{4.24}$$

with $\Phi := \begin{bmatrix} I_{n\times n} & 0_{n\times(\nu\mu+(\nu-1)n)} \end{bmatrix}$.

4.2.2 Distributed model of the formation

So far, it has been assumed in constructing K_d from the centralized controller K_c that any two different spacecraft consider the same model for the formation $\mathcal{F}$. This assumption is not pragmatic in general, and hence the controller K_d should be modified to account for discrepancy in the corresponding models. This modification in the proposed decentralized controller can be carried out in line with the methodology presented in [79]. The obtained controller possesses similar properties as the decentralized controller K_d in terms of robustness to parameter variation and stability. The details are omitted here for brevity.

4.2.3 Robust stability

In practice, the LTI model (4.3) cannot precisely describe the formation $\mathcal{F}$. Let the exact model of the formation $\mathcal{F}$, which is a perturbed form of the nominal model, be described by:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A)x(t) + (B + \Delta B)u(t) \\ y(t) &= (C + \Delta C)x(t) \end{aligned} \tag{4.25}$$

where $\Delta A \in \mathcal{L}_1, \Delta B \in \mathcal{L}_2$ and $\Delta C \in \mathcal{L}_3$ represent the parametric uncertainties accounting for nonlinearity, error in system identification, etc. Note that $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ are the regions describing the permissible variations of the uncertain parameters.

Regarding the uncertain model of the formation, it is straightforward to assert the following result (by using an approach analogous to the one given in [70]):

The formation with the perturbed model (4.25) is more likely to be robustly stable under the designed controller K_d rather than under the reference centralized controller K_c.

This reveals the superiority of the decentralized controller K_d over its centralized counterpart, in terms of robustness.

To verify whether or not the formation $\mathcal{F}$ with the uncertain model (4.25) is stable under the proposed decentralized controller, some information about the regions $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ is required to be available. For instance, the special case when these regions establish semi-algebraic sets (i.e., each of them is constructed by the positivity of a number of polynomials) has been investigated intensively in the literature, and several efficient methods have been proposed [22, 72].

4.3 Predictive-Control Based Approach

Concerning the decentralized controller K_d, there are some practical issues as follows:

i) The model of each spacecraft is nonlinear and time-varying, while it is assumed here to be LTI. However, as pointed out in [62], this assumption is valid for sufficiently short period of time.

ii) The controller K_d is unable to account for the effect of time-varying perturbation in the model of the formation.

iii) The structure of the local controller for each spacecraft is contingent upon the modeling matrices and initial states of other spacecraft in an open-loop manner. Although this will not affect the stability of the formation provided the aforementioned conditions hold, it can degrade the performance of the formation.

iv) The centralized controller K_c can be designed in such a way that it is capable of avoiding a possible collision, detecting a fault, or passing a barrier without hitting it. In contrast, the decentralized controller K_d does not necessarily have these capabilities.

In order to ameliorate the applicability of the controller and address the above issues to some extent, a pseudo decentralized controller $\tilde{K}_d$ will be proposed now based on the decentralized controller K_d.

Procedure 1: *Consider a sampling period h, and assume for now that any spacecraft can measure the states of the other spacecraft at the sampling instants $0, h, 2h, \ldots$. For any $i \in \bar{\nu}$, apply the controller given by (4.14) to the i^{th} spacecraft in the time interval $[0, h)$, with the initial state $\eta_{d_i}(0) = \begin{bmatrix} \hat{x}^i(0)^T & 0 \end{bmatrix}^T$, where $\hat{x}^i(0)$ denotes the states of the other spacecraft measured at time $t = 0$ (as discussed earlier). At the instant $t = h$, measure the states of the other spacecraft to obtain $\hat{x}^i(h)$. For the time interval $[h, 2h)$, apply the controller given by (4.14) (as before) to the i^{th} spacecraft, with the new initial state $\eta_{d_i}(h) = \begin{bmatrix} \hat{x}^i(h)^T & 0 \end{bmatrix}^T$. Following the same strategy, the state of the controller (4.14) at the time instants $2h, 3h, \ldots$ should be updated. The union of these local controllers will be referred to as the* *pseudo decentralized controller $\tilde{K}_d$.*

The controller $\tilde{K}_d$ has the following advantages:

- The linear model considered in (4.3) can describe the formation in the intervals of duration h with a high precision.

- Any controller observes the states of the other spacecraft with a low rate, to compensate for the negative effects discussed in (ii) and (iii) above.

- For any positive integer τ, the controller of the i^{th} spacecraft, $i \in \bar{\nu}$, observes the states of the other spacecraft at $t = \tau h$. Then, it can predict the trajectory of the whole formation in the interval $[\tau h, (\tau + 1)h)$ from the state of its controller (see the equations (4.12) and (4.14)). If it is known that no collision for the i^{th} spacecraft will occur in the interval $[\tau h, (\tau + 1)h)$, the i^{th} local controller of $\tilde{K}_d$ proposed above will be applied to the i^{th} spacecraft in this interval; otherwise an emergency local controller should be applied to the i^{th} spacecraft in this interval. This emergency controller can be designed by using the existing techniques [13, 14], and can have any general form, i.e. nonlinear and time-varying.

- For any positive integer τ, the controller of the i^{th} spacecraft measures the states of other spacecraft at $t = (\tau + 1)h$, and compares it with their predictions obtained in terms of the measurements at $t = \tau h$. If there is a sizable discrepancy between them, it implies that a fault has occurred in the formation, and a proper action (e.g. reconfiguration) should be taken.

There is an important issue which needs to be considered in the design of $\tilde{K}_d$. More specifically, the formation under the controller $\tilde{K}_d$ is envisaged as a closed-loop system, but there are jumps in some of its states at the instants $h, 2h, \ldots$ (note that there is no jump in the state of the formation). If the closed-loop system does not satisfy a number of conditions, these jumps might destabilize the closed-loop system for sufficiently small values of h. It is desired to find a reliable lower bound on h, which guarantees the stability of the closed-loop formation control system.

4.3.1 Reliable sampling periods

Assume that the formation $\mathcal{F}$ with the uncertain model (4.25) is robustly stable over the regions $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ with respect to the decentralized controller K_d, as discussed earlier. The objective here is to determine the admissible values of h, which do not violate the stability of the formation under the predictive-based controller $\tilde{K}_d$.

Obtain the A-matrix of the closed-loop system composed of the model (4.25) and the feedback controller K_d, and denote it with $A_d + \Delta A_d$. Clearly, when there is no uncertainty in the model, ΔA_d will be a zero matrix. Due to the existence of $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$, the matrix ΔA_d belongs to a region, denoted by $\mathcal{L}$. Note that $\mathcal{L}$ can simply be obtained in terms of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and the nominal parameters of the model.

Suppose that the volumes of the regions $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are so small that there exists a common Lyapunov function P and a *strictly negative* number β such that the inequality:

$$(A_d + \Delta A_d)^T P + P(A_d + \Delta A_d) - 2\beta P < 0 \tag{4.26}$$

holds for any ΔA_d belonging to the region $\mathcal{L}$. It is worth mentioning that the problem of obtaining the matrix P and the scalar β has intensively been investigated in the literature, and several methods are proposed accordingly. For instance, if the regions $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are characterized by some matrix polynomials, the LMI approaches given in [22,72] can be employed to treat the problem in question. The interpretation of equation (4.26) is that not only is the formation $\mathcal{F}$ robustly stable with a common Lyapunov function, but the modes of the corresponding closed-loop system are also not allowed in a certain neighborhood of the $j\omega$ axis in the s-plane in oder to attain a desirable stability margin.

Define now the following:

$$\Pi := \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ I_n^1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_\mu & 0 & 0 & \cdots & 0 & 0 \\ I_n^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & I_\mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ I_n^\nu & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_\mu \end{bmatrix}, \tag{4.27}$$

$$\mathbf{x}_l(t) := \begin{bmatrix} x(t)^T & \eta_{d_1}(t)^T & \eta_{d_2}(t)^T & \cdots & \eta_{d_\nu}(t)^T \end{bmatrix}^T$$

where I_n^i is derived from the $n \times n$ identity matrix by removing its i^{th} row, for any positive integer i. Note that each zero term in the above expression represents a zero matrix with proper dimension.

It is straightforward to verify that the state of the formation $\mathcal{F}$ given by the model (4.25) under the decentralized controller $\tilde{K}_d$ satisfies the relations:

$$\dot{\mathbf{x}}_l(t) = (A_d + \Delta A_d)\mathbf{x}_l(t), \quad t \neq 0, h, 2h, \ldots \tag{4.28a}$$

$$\mathbf{x}_l(\kappa h) = \Pi \mathbf{x}_l(\kappa h^-), \quad \kappa = 0, 1, 2, \ldots \tag{4.28b}$$

where $\mathbf{x}_l(\kappa h^-)$ denotes the left limit of $\mathbf{x}_l(t)$ at $t = \kappa h$. The equation (4.28b) is the mathematical representation of the fact that the local controllers measure the state of the whole formation at $t = \kappa h$, and adapt themselves accordingly.

The following theorem presents a condition on h which guarantees that the formation $\mathcal{F}$ with the uncertain model under the decentralized predictive controller $\tilde{K}_d$ will remain stable.

Theorem 4.4. *The stability of the formation $\mathcal{F}$ under the decentralized controller $\tilde{K}_d$ is guaranteed if the sampling period h satisfies the following equation:*

$$h \geq -\frac{1}{\beta} \ln \left(\|\Pi\| \sqrt{\frac{n\alpha_2}{\alpha_1}} \right) \tag{4.29}$$

where $\|\cdot\|$ is the Frobenius-norm operator, and α_1 and α_2 are the minimum and the maximum eigenvalues of the matrix P (note that all eigenvalues of P are real and positive).

Proof: It follows from the equation (4.28a) that $\mathbf{x}_l(\kappa h^-) = e^{(A_d+\Delta A_d)h}\mathbf{x}_l((\kappa - 1)h)$, for $\kappa = 1, 2, 3, \ldots$. This relation along with the equation (4.28b) leads to:

$$\mathbf{x}_l((\kappa+1)h^-) = e^{(A_d+\Delta A_d)h}\left(\Pi e^{(A_d+\Delta A_d)h}\right)^{\kappa}\mathbf{x}_l(0) \tag{4.30}$$

for any nonnegative integer κ. On the other hand, it can be concluded from (4.28a) that $\mathbf{x}_l(t)$ converges to zero as the continuous argument t approaches infinity, if and only if $\mathbf{x}_l(\kappa h^-)$ decays to zero as the discrete argument κ approaches infinity. Hence, the equation (4.30) yields that the formation $\mathcal{F}$ is stable under the controller $\tilde{K}_d$ if and only if all of the eigenvalues of $\Pi e^{(A_d+\Delta A_d)h}$ lie inside the unit circle. It is well known that the latter condition holds if the Frobenius norm of $\Pi e^{(A_d+\Delta A_d)h}$ is less than 1. Now, one can write:

$$\begin{aligned}\|\Pi e^{(A_d+\Delta A_d)h}\|^2 &= \sum_{i=1}^{n}\|\Pi e^{(A_d+\Delta A_d)h}\mathbf{e}_i\|^2 \\ &\leq \|\Pi\|^2\sum_{i=1}^{n}\|e^{(A_d+\Delta A_d)h}\mathbf{e}_i\|^2\end{aligned} \tag{4.31}$$

where e_i is the i^{th} standard basis for the n-dimensional coordinate space. Moreover, it results from the equation (4.26) and the work [17] that:

$$\|e^{(A_d+\Delta A_d)h}\mathbf{e}_i\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{\beta h}\|\mathbf{e}_i\| = \sqrt{\frac{\alpha_2}{\alpha_1}}e^{\beta h}, \quad i = 1, \ldots, n \tag{4.32}$$

The proof follows from the equations (4.31) and (4.32). ■

So far, it is assumed that any spacecraft can measure the states of the other spacecraft at the sampling instants $0, h, 2h, \ldots$. However, if some of the states cannot be measured at either all instants or even some instants (due to the shadow phenomena [137]), the update in the controller for these specific states should inevitably be ignored at those instants. In this case, a sufficient condition for the stability of the formation in terms of h can be attained by pursuing an approach similar to the proof of Theorem 4.4.

4.4 Simulation Results

Example 4.1. *Consider a leader-follower formation $\mathcal{F}$ consisting of three spacecraft with the exact linearized model given in [79]. Label the leader as spacecraft 1, and the followers as spacecraft 2 and spacecraft 3. The aim is to design a controller which satisfies the following properties:*

- *All spacecraft fly at the same desired speed.*
- *The desired Euclidean distances between spacecraft are achieved.*
- *The communication requirements are reasonably low.*

To this end, the LTI model of the formation $\mathcal{F}$ in the relative coordinates is obtained in [79] to be:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \tag{4.33}$$

where A and B are given in [79], and

$$\begin{aligned} x_1(t) &= \begin{bmatrix} x_{11}(t) & x_{12}(t) \end{bmatrix}^T, \\ x_2(t) &= \begin{bmatrix} x_{21}(t) & x_{22}(t) & x_{23}(t) & x_{24}(t) \end{bmatrix}^T, \\ x_3(t) &= \begin{bmatrix} x_{31}(t) & x_{32}(t) & x_{33}(t) & x_{34}(t) \end{bmatrix}^T, \\ u_i(t) &= \begin{bmatrix} u_{i1}(t) & u_{i2}(t) \end{bmatrix}^T, \quad i = 1, 2, 3 \end{aligned} \tag{4.34}$$

Here, $x_1(t)$ denotes the state of the leader, and $x_2(t)$ and $x_3(t)$ represent the states of spacecraft 2 and 3 (i.e. the followers), respectively. More specifically:

1. $x_{11}(t)$ *and* $x_{12}(t)$ *are the speed error of the leader (speed of the leader minus its desired speed) along the* x *and* y *axes, respectively.*

2. $x_{i1}(t)$ *and* $x_{i2}(t)$, $i = 2, 3$, *are the distance error (distance between spacecraft* i *and* $i - 1$ *minus their desired distance) along the* $\mathbf{x}$ *and* $\mathbf{y}$ *axes, respectively.*

3. $x_{i3}(t)$ *and* $x_{i4}(t)$, $i = 2, 3$, *are the speed error (speed of spacecraft* i *minus its desired speed) along the* $\mathbf{x}$ *and* $\mathbf{y}$ *axes, respectively.*

4. $u_{i1}(t)$ *and* $u_{i2}(t)$, $i = 1, 2, 3$, *are the acceleration of spacecraft* i *along the* $\mathbf{x}$ *and* $\mathbf{y}$ *axes, respectively.*

Since the given LTI model for $\mathcal{F}$ *is rather simplified, the unmodeled dynamics of the formation will be considered as perturbations here. Let the corresponding perturbed model of the formation be as follows:*

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = 1.1A \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + 0.9B \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} \tag{4.35}$$

Assume that the centralized controller K_c *with the control law* $u(t) = 2Nx(t)$ *satisfies the design specifications, where* N *is the LQR gain obtained for the same system in [79] (note that* N *is derived from the Riccati equation). Consider now the initial states* $x_1 = [400, 1200]$, $x_2 = [2000, 2400, 1200, 1600]$, $x_3 = [2800, 3200, 800, 1200]$. *Decentralize the controller* K_c *to arrive at the controller* K_d, *by employing the method pointed out earlier. Moreover, construct the decentralized predictive controller* $\tilde{K}_d$ *by assuming that each spacecraft knows the states of the other spacecraft with* 10% *error at* $t = 0$, *and can measure them accurately at the subsequent sampling instants* $h, 2h, ...,$ *where* h *is the sampling time. It is desired to show that using the controller* $\tilde{K}_d$ *instead of* K_d *is vital.*

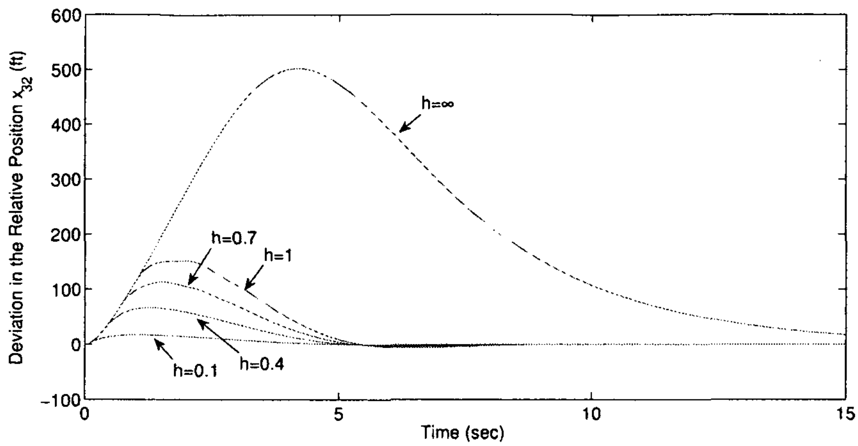


Figure 4.1: The difference between the relative position of spacecraft 2 and 3 in both cases of centralized and decentralized controllers for different values of h.

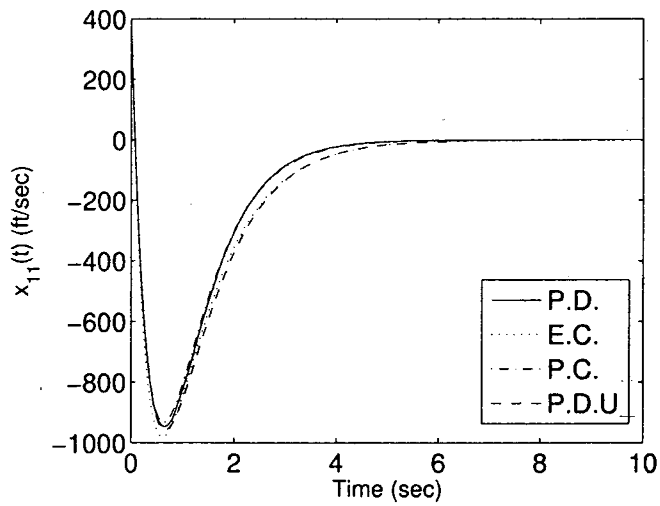
On considering the perturbed model (4.35) for the formation, the difference between the relative position of spacecraft 2 with respect to spacecraft 3 along the **x**-*axis under the centralized and the decentralized controllers is depicted in Figure 4.1 for different values of h. It can be easily observed from this figure that the difference between the relative position under centralized and decentralized controllers vanishes as h becomes smaller. This implies that for small values of h, the formation with the perturbed model behaves almost identically under the centralized controller K_c and the decentralized controller K_d. Moreover, in the case of a large h, i.e., when the decentralized controller measures the states of the formation less frequently, there can be a huge difference between the formation under centralized and decentralized controllers. This, in turn, may lead to a collision in the formation under the decentralized control law. It is to be noted that between the sampling instants the decentralized controller operates in an open-loop fashion when it comes to processing the non-identical information, and hence this time interval should ideally be short.*

Suppose now that any spacecraft can measure the states of the other spacecraft accurately at the sampling instants $0, h, 2h, \ldots$ (with a possible exclusion of 0), *while the model of the formation is perturbed. Some of the states of the formation corresponding to four different cases are depicted in Figures 4.2, 4.3 and 4.4. The abbreviations P.D., P.D.U, P.C., and E.C. in these figures represent the decentralized controller applied to the perturbed model of the formation with* $h = 0.4$*, the decentralized controller applied to the perturbed model of the formation with* $h = \infty$*, the centralized controller applied to the perturbed model of the formation, and the centralized controller applied to the exact model of the formation, respectively.*

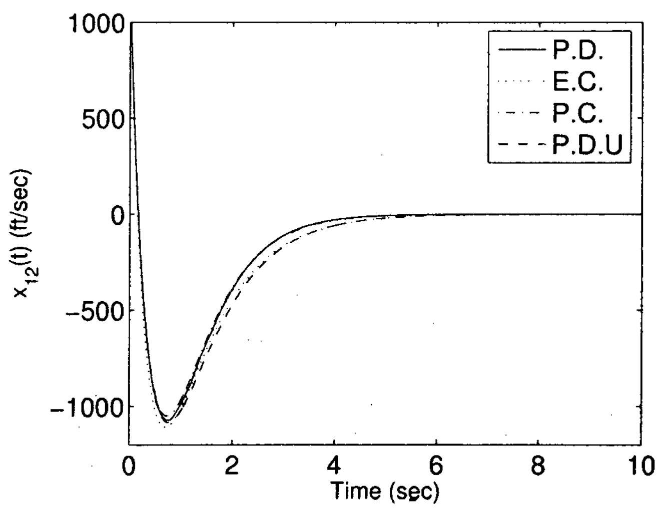
Construct a new decentralized controller $\tilde{K}_d$ *by assuming that each spacecraft considers zero initial states for the other two spacecraft at* $t = 0$ *(because they may not be measurable initially), but it can measure the states at the sampling instants* $t = h, 2h, \ldots$ *precisely. Consider now the following specifications for the formation:*

- *The initial positions of spacecraft 1, 2 and 3 are* $(0, 0), (210, 150)$ *and* $(460, 340)$*, respectively.*
- *The initial velocity vectors of spacecraft 1, 2 and 3 are* $(500, 500), (500, 580)$ *and* $(660, 500)$*, respectively.*
- *The desired velocity vector for all spacecraft is* $(0, 100)$*.*
- *The desired relative distance of the first spacecraft with respect to the second one is* $(50, -50)$*.*
- *The desired relative distance of the second spacecraft with respect to the third one is* $(50, -50)$*.*

The trajectory of the formation under the decentralized controller $\tilde{K}_d$ *is sketched in Figure 4.5 for* $h = \infty$ *(left plot) and* $h = 2$ *(right plot). Note that spacecraft 1, 2 and 3 in the plots are shown by the symbols* $+, \diamond$ *and* $*$*, as indicated in the legends. It can be observed from these plots that the formation converges to its desired trajectory faster for* $h = 2$ *(in general, the transient response is longer for a slower* h*).*



(a)



(b)

Figure 4.2: The state variables x_{11} and x_{12} resulted from four different control setups.

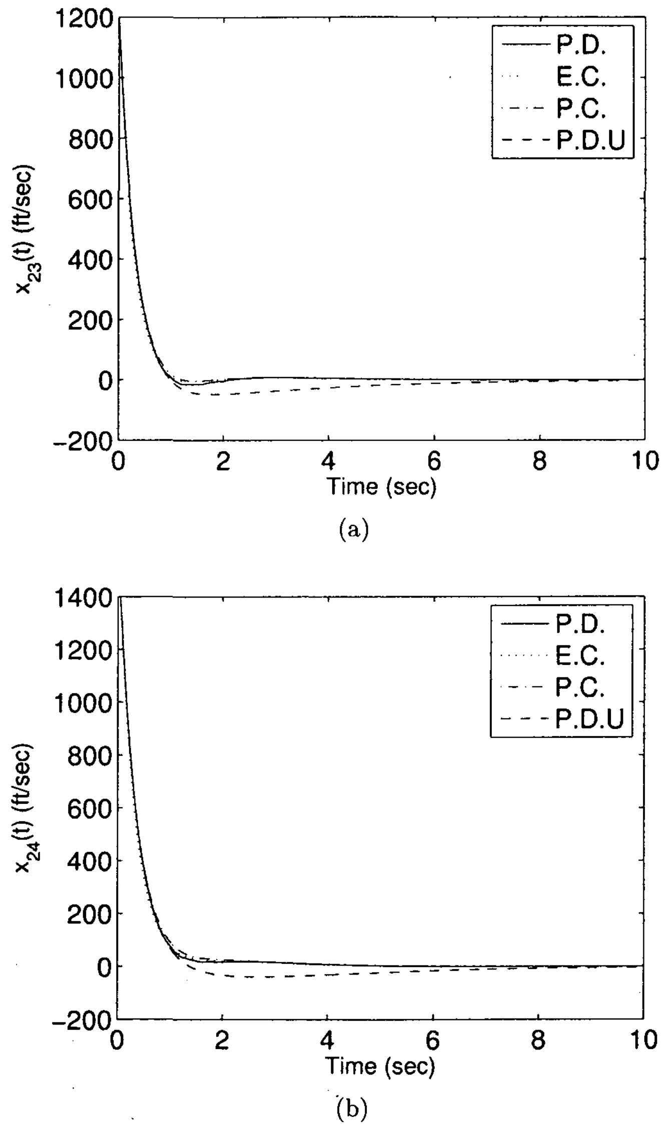
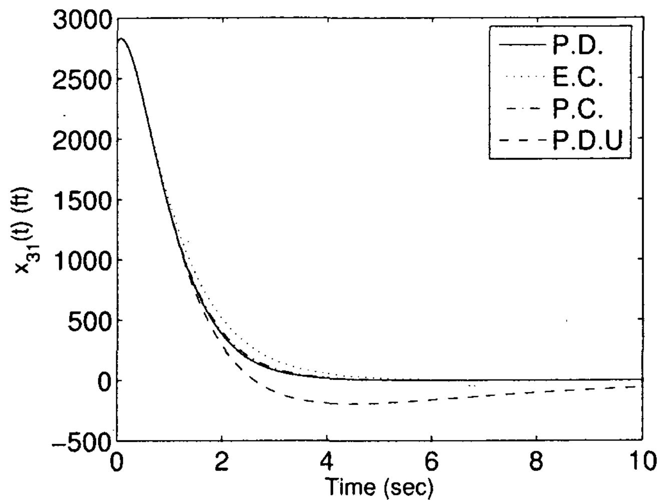
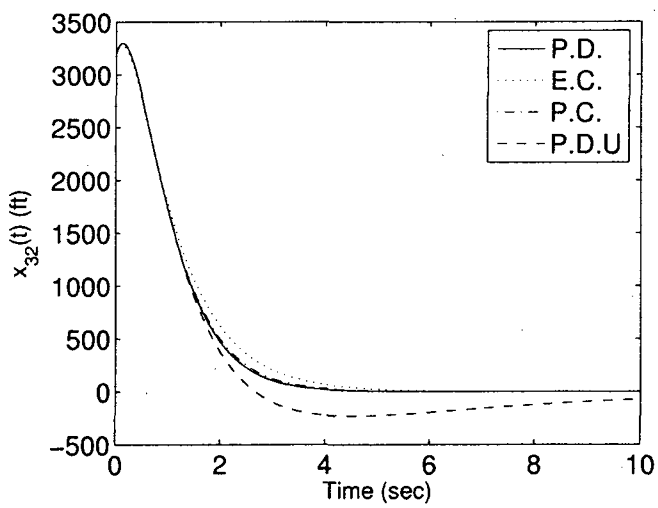


Figure 4.3: The state variables x_{23} and x_{24} resulted from four different control setups.

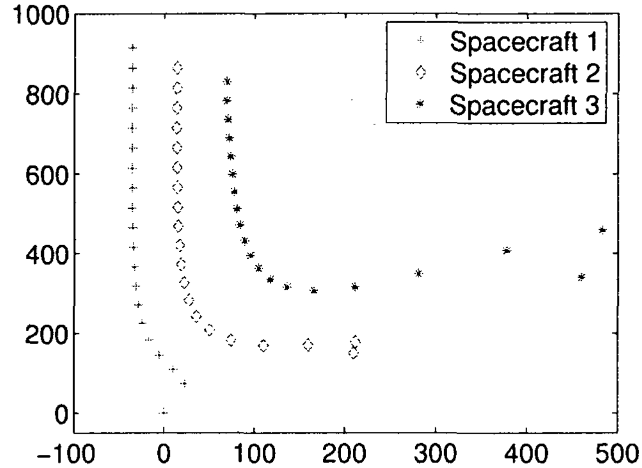


(a)

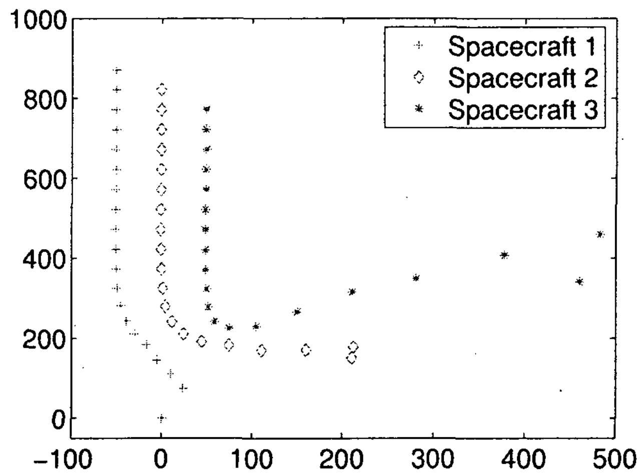


(b)

Figure 4.4: The state variables x_{31} and x_{32} resulted from four different control setups.



(a)



(b)

Figure 4.5: (a) The trajectory of the formation under the decentralized controller K_d *for* $h = \infty$ *(b) the trajectory of the formation under the decentralized controller* K_d *for* $h = 2$.

Chapter 5

Overlapping Control Systems with Delayed Communication Channels: Stability Analysis and Controller Design

5.1 Introduction

Design of structurally constrained control systems has been of special interest recently and various aspects of it have been vastly studied in the literature [83,84,149]. This type of systems have a wide range of real-world applications, e.g., in multi-agent systems [38]. The cooperative nature of control paradigm in such systems is characterized based on the topology of communication between control agents. Typically, it is not realistic to assume each control agent can use all the measurement signals of the system to generate its local control input. In other words, some kind of constraint on the information flow between different control agents is inevitable.

In a geographically distributed large-scale system such as coordinated vehicles, a decentralized structure is often more desirable in control [136]. Decentralized control theory has attracted several researchers in the past three decades [77, 143, 149]. Particularly, overlapping control has been studied more recently and has found applications in various areas [38, 140, 143]. In [143], an *expansion transformation* is used to convert the original overlapping control problem into a decentralized one. The contraction procedure is applied consequently to provide an appropriate controller for the original system. It is shown that such an approach is more efficient if the system structure itself is overlapping too. The work [77] introduces the notion of a decentralized overlapping fixed mode (DOFM) to characterize the fixed modes of an interconnected system with respect to the class of linear time-invariant (LTI) structurally constrained controllers. The results of [77] are further developed in [78] to identify those modes of the system which are fixed with respect to an overlapping control structure of any general type (nonlinear and time-varying).

In a physical large-scale control system, on the other hand, communication delays inherently exist in information exchange between different control agents. Time-delay in system dynamics has a significant impact on the stability and performance of the system, and needs to be taken into account in controller design. This problem has been investigated intensively in the control literature, e.g. see [15, 103, 124, 148].

Some of the recent developments in delay-dependent stability analysis have been reported in [15,49,90]. Different approaches are proposed for designing a proper feedback controller which satisfies prescribed performance requirements, such as H_∞ disturbance attenuation [42].

In this manuscript, an overlapping control strategy is proposed for interconnected systems consisting of a number of interacting subsystems. Each local controller is assumed to share its local measurements with some of the others local controllers (which are known *a priori*). The signal transmission between different control agents is assumed to be subject to uncertain delay. Furthermore, all actuators are exposed to disturbances, affecting the resultant control signals. The main contributions of this chapter are as follows. A necessary condition for the stabilizability of interconnected systems by means of overlapping output feedback controllers is derived first. A methodology is then proposed using linear matrix inequalities (LMI) to design an overlapping static output feedback controller which stabilizes the system and attenuates the effect of disturbances on the regulated signal. It is assumed in this chapter that the interconnected system possesses a LTI state space representation. The control gain is then decomposed into diagonal and off-diagonal components. A description of the resultant closed-loop system dynamics is presented through the above gain decomposition procedure. This results in a LTI system with an uncertain state-delay. A graph-based algorithm is utilized subsequently to transform the overlapping gain matrix into a block-diagonal form.

This chapter is organized as follows. The problem is formulated in Section 5.2, and the main objectives of the work are presented. In Section 5.3, the closed-loop dynamics of the system under overlapping static output feedback control law is obtained and the matrix block diagonalization procedure is reviewed. Then in Section 5.4, the stability analysis and H_∞ control synthesis are addressed. Section 5.5 presents some simulations which support the theoretical results of the chapter.

5.2 Problem Formulation

5.2.1 Problem statement

Consider a LTI interconnected system $\mathbf{S}$ consisting of ν subsystems. Assume that the state-space model for the i-th subsystem is described by

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{\substack{j=1,\\ j\neq i}}^{\nu} A_{ij}x_j(t) + B_i u_i(t) + E_i w_i(t), \quad i \in \bar{\nu} := \{1, 2, \ldots, \nu\} \quad (5.1)$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state and input for the i-th subsystem, respectively. In (5.1), the term $A_{ij}x_j$, $j \in \bar{\nu}$, represents the effect of the j-th subsystem on the dynamics of subsystem i. The system matrices A_i, B_i, E_i and A_{ij}, $i, j \in \bar{\nu}$ are constant and have appropriate dimensions. Furthermore, $w_i \in \mathbb{R}^{p_i}$ is the disturbance affecting the input of subsystem i, with the property $w_i(t) \in \mathcal{L}_2[0, \infty)$.

By putting together the state-space representations of all ν subsystems, the overall dynamics of the interconnected system **S** can be expressed as

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{5.2}$$

where

$$x(t) = \begin{bmatrix} x_1(t)^T & x_2(t)^T & \dots & x_\nu(t)^T \end{bmatrix}^T$$
$$u(t) = \begin{bmatrix} u_1(t)^T & u_2(t)^T & \dots & u_\nu(t)^T \end{bmatrix}^T$$
$$w(t) = \begin{bmatrix} w_1(t)^T & w_2(t)^T & \dots & w_\nu(t)^T \end{bmatrix}^T$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\nu} \\ A_{21} & A_{22} & \cdots & A_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\nu 1} & A_{\nu 2} & \cdots & A_{\nu\nu} \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_\nu \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & & & 0 \\ & E_2 & & \\ & & \ddots & \\ 0 & & & E_\nu \end{bmatrix}$$

The local measurement signal for the i-th local controller is represented by

$$y_i(t) = C_i x_i(t) \tag{5.3}$$

where $y_i \in \mathbb{R}^{r_i}$, and C_i is a given constant matrix with appropriate dimension.

Assumption 5.1. *For the sake of non-triviality (i.e., to avoid the exact decentralized structure with no overlapping), it is assumed that at least one of the local controllers has access to at least one of the other subsystems' measurement signal through a communication link.*

Let the measurement signal y_j of the j-th subsystem be transmitted to the control agent of subsystem i to construct the local control input u_i, $i, j \in \bar{\nu}, i \neq j$. Denote the received signal with s_j, which can be represented by

$$s_j(t) = y_j(t-h) = C_j x_j(t-h) \tag{5.4}$$

In the above equation, h is the communication delay which is uncertain, but is known to be strictly positive with finite magnitude. For simplicity and without loss of generality, it is assumed here that the communication delay is identical for all channels.

5.2.2 Control objectives

To control the system **S**, let the following local static output feedback controller be considered for the i-th subsystem

$$u_i(t) = K_i s^i(t) \tag{5.5}$$

where $K_i \in \mathbb{R}^{m_i \times r}$, $r := \sum_{i=1}^{\nu} r_i$ and

$$s^i(t) = \left[\begin{array}{ccccccc} s_1(t)^T & \dots & s_{i-1}(t)^T & y_i(t)^T & s_{i+1}(t)^T & \dots & s_\nu(t)^T \end{array} \right]^T$$

In other words, $s^i(t)$ is obtained by replacing $s_i(t)$ with $y_i(t)$ in the vector $s(t)$. Let K_i be written as

$$K_i = \left[\begin{array}{cccc} K_{i1} & K_{i2} & \cdots & K_{i\nu} \end{array} \right] \tag{5.6}$$

where $K_{ij} \in \mathbb{R}^{m_i \times r_j}$ for $i, j \in \bar{\nu}$. Assumption 5.1 implies that there exist distinct integers $i, j \in \bar{\nu}$, for which the gain matrix K_{ij} is nonzero. Note that the local controller for the i-th subsystem is characterized by the set of given K_{ij}'s, where K_{ii} is the control coefficient for the instantaneous local output, and K_{ij}'s, $j \neq i$, are the coefficients of the non-local output signals which are subject to the communication delay. Define $\mathbf{K}$ as an overlapping static output controller whose (i, j) block entry is K_{ij}.

Let the regulated signal be represented by

$$z(t) = \Gamma x(t)$$

where $z \in \mathbb{R}^{\xi}$ and $\Gamma \in \mathbb{R}^{\xi \times n}$ ($n := \sum_{i=1}^{\nu} n_i$). In this chapter:

i) It is desired to find a necessary condition for the existence of a stabilizing overlapping controller $\mathbf{K}$ for the interconnected system $\mathbf{S}$.

ii) A set of distributed overlapping output feedback gains K_i, $i \in \bar{\nu}$, is sought such that for any delay h with a known upper bound,

   – the internal stability of the closed-loop system is achieved.

   – the ∞-norm of the closed-loop gain from $w(t)$ to $z(t)$ is less than a prescribed value γ, i.e.

$$\|T_{zw}\|_{\infty} := \frac{\|z(t)\|_2}{\|w(t)\|_2} < \gamma$$

5.3 Preliminaries

5.3.1 Closed-loop dynamics under the controller K

Consider a distributed overlapping control gain $\mathbf{K}$ with the i-th local output feedback gain denoted by K_i, $i \in \bar{\nu}$, as given in (5.5) and (5.6).

Definition 5.1. *$\mathbb{K}_D$ is the set of all block-diagonal matrices which have ν diagonal entries, where the i-th block entry on the main diagonal is a $m_i \times r_i$ matrix, for all $i \in \bar{\nu}$.*

Definition 5.2. *The decentralized gain matrix $\bar{K}$ is defined as*

1) $\bar{K} \in \mathbb{K}_D$.

2) The (i,i) block entry of $\bar{K}$, is equal to K_{ii}.

Definition 5.3. *Define the overlapping gain matrix $\tilde{K}$ as a matrix of the following form:*

1) Its (i,j) block entry, $i \neq j$, is K_{ij} if the output of subsystem j is available to local controller i, and is a $m_i \times r_j$ zero matrix otherwise.

2) Its (i,i) block entry is a $m_i \times r_i$ zero matrix.

Consider the interconnected system **S** given by (5.2), and let the overlapping static output feedback control law **K** be applied to **S**. The input u_i in (5.5) can then be rewritten as

$$u_i(t) = \sum_{j=1, j\neq i}^{\nu} K_{ij} s_j + K_{ii} y_i$$

From (5.3) and (5.4), it follows that

$$u_i(t) = \sum_{j=1, j\neq i}^{\nu} K_{ij} C_j x_j(t-h) + K_{ii} C_i x_i(t)$$

This leads to the following expression for the input

$$u(t) = \tilde{K} C x(t-h) + \bar{K} C x(t) \qquad (5.7)$$

where

$$C = \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_\nu \end{bmatrix}$$

and $\bar{K}$ and $\tilde{K}$ are introduced in Definitions 5.2 and 5.3. By Substituting (5.7) into (5.2), the closed-loop dynamics of the system **S** under the overlapping static output feedback **K** is obtained as follows

$$\dot{x}(t) = (A + B\bar{K}C)x(t) + B\tilde{K}Cx(t-h) + Ew(t)$$

5.3.2 Matrix block diagonalization procedure

Inspired by [77], the following graph-theoretic algorithm is presented to convert $\tilde{K}$ to a block diagonal matrix H using a single transformation matrix. This diagonalization procedure is used in developing the main results of the chapter.

Algorithm 5.1.

Step 1- *Construct the overlapping graph* $\mathbf{G}$ *as follows:*

a. *Consider two sets of* ν *vertices denoted by* $\mathbf{I}$ *and* $\mathbf{J}$. *Label the vertices in* $\mathbf{I}$ *and* $\mathbf{J}$ *as vertex* 1 *to vertex* ν.

b. *For any* $i, j \in \bar{\nu}$, $i \neq j$, *if there exists a communication link from local controller* j *to local controller* i, *connect vertex* $i \in \mathbf{I}$ *to vertex* $j \in \mathbf{J}$ *with an edge. The gain of this edge is* K_{ij}.

Step 2- *Consider the* i*-th vertex in* $\mathbf{I}$ *and define a new graph* $\mathbf{G}_i$ *which includes all the edges connected to this vertex. Thus, the graph* $\mathbf{G}$ *is partitioned into* ν *subgraphs* $\mathbf{G}_1$, $\mathbf{G}_2$, ..., $\mathbf{G}_\nu$.

Step 3- *Consider the subgraph* $\mathbf{G}_i$, $i \in \bar{\nu}$, *and denote the set of all vertices of* $\mathbf{I}$ *which appear in* $\mathbf{G}_i$ *with* $\mathbf{I}_i$. *Note that* $|\mathbf{I}_i| = 1$, *where* $|.|$ *is the cardinality of a set. In addition, let the set of all vertices of* $\mathbf{J}$ *which appear in* $\mathbf{G}_i$ *be denoted by* $\mathbf{J}_i$. *Suppose that* $|\mathbf{J}_i| = \delta_i$, $i \in \bar{\nu}$; *define* H *as a block-diagonal matrix where its* (i,i) *block entry,* $i \in \bar{\nu}$, *is a block row whose* j*-th block entry,* $j = 1, \ldots, \delta_i$, *is the gain of the edge connecting the only vertex in* $\mathbf{I}_i$ *to the* j*-th vertex in* $\mathbf{J}_i$.

Remark 5.1. *In step 2 of Algorithm 5.1, some vertices of* $\mathbf{J}$ *might appear in more than one subgraph* $\mathbf{G}_i$, $i \in \bar{\nu}$. *In other words, for some distinct* $i, j \in \bar{\nu}$, $\mathbf{J}_i \bigcap \mathbf{J}_j$ *might be non-empty; however,* $\mathbf{I}_i \bigcap \mathbf{I}_j = \emptyset$, $\forall i, j \in \bar{\nu}$, $i \neq j$.

Definition 5.4. *Let $\mathcal{C}_i = \mathbf{J}_i \bigcup \{i\}$, for any $i \in \bar{\nu}$. Define $\mathbb{H}_D$ as the set of all block-diagonal matrices which have ν diagonal block entries, where the i-th block entry, $i \in \bar{\nu}$, is a $m_i \times \mu_i$ matrix itself, and*

$$\mu_i = \sum_{j \in \mathcal{C}_i} r_j \tag{5.8}$$

The following lemma relates the matrix H, obtained from step 3 of Algorithm 5.1, to $\tilde{K}$.

Lemma 5.1. *Assume the block-diagonal matrix $H \in \mathbb{H}_D$ is obtained from $\tilde{K}$ using Algorithm 5.1. One can find a matrix T such that*

$$\tilde{K} = HT \tag{5.9}$$

Proof: Following an approach similar to [73], it is straightforward to show that the matrix H can be derived through only a finite sequence of operations on the columns of $\tilde{K}$, and therefore a unique transformation matrix T can be obtained such that (5.9) holds. ■

As an illustrative example, consider a vehicle formation system $\mathbf{F}$ consisting of 3 vehicles with the i-th input and output ($i = 1, 2, 3$) denoted by $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}^2$, respectively. Suppose that vehicle 2 has access to the local measurements of the other 2 vehicles while vehicles 1 and 3 receive the measurement of vehicle 2 only, and there is no communication link between them (this formation topology is referred to as leader-follower in the literature, where vehicle 1 is the leader and vehicles 2 and 3 are followers [83]). In this case, the structure of the gain matrix $\tilde{K}$ is as follows

$$\tilde{K} = \begin{bmatrix} 0_{1\times 2} & K_{12} & 0_{1\times 2} \\ K_{21} & 0_{1\times 2} & K_{23} \\ 0_{1\times 2} & K_{32} & 0_{1\times 2} \end{bmatrix}$$

where $K_{12}, K_{21}, K_{23}, K_{32} \in \mathbb{R}^{1\times 2}$. Following the procedure given in Algorithm 5.1,

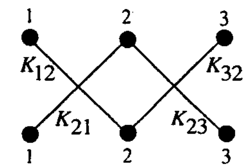


Figure 5.1: The overlapping graph **G** for the formation **F**

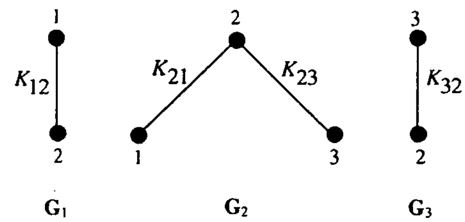


Figure 5.2: The subgraphs obtained from the graph **G** using step 2 of Algorithm 5.1

the overlapping graph **G** corresponding to the matrix $\tilde{K}$ is obtained as shown in Figure 5.1. Furthermore, following step 2 of the algorithm, one can find the subgraphs $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{G}_3$ depicted in Figure 5.2.

Using step 3 of the algorithm, the block diagonal matrix H is obtained as

$$H = \begin{bmatrix} K_{12} & 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} \\ 0_{1\times 2} & K_{21} & K_{23} & 0_{1\times 2} \\ 0_{1\times 2} & 0_{1\times 2} & 0_{1\times 2} & K_{32} \end{bmatrix}$$

Moreover, the transformation matrix T (defined in (5.9)) for this example is

$$T = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ 0_2 & I_2 & 0_2 \end{bmatrix}$$

where I_2 and 0_2 denote the 2×2 identity matrix and the 2×2 zero matrix, respectively.

5.4 Main Results

5.4.1 Stabilizability conditions for overlapping control systems

Definition 5.5. *$\mathbb{G}_D$ is defined as the set of all block-diagonal matrices with ν diagonal entries, where the i-th block entry on the main diagonal is a $m_i \times (r_i + \mu_i)$ matrix itself (μ_i is given by (5.8)).*

It is easy to verify that the block matrix

$$\begin{bmatrix} \bar{K} & H \end{bmatrix}$$

can be converted to a proper block-diagonal matrix of the form given in the above definition by a finite number of elementary column operations. Therefore, let

$$\begin{bmatrix} \bar{K} & H \end{bmatrix} = \hat{K}\Omega \tag{5.10}$$

where $\hat{K} \in \mathbb{G}_D$, and Ω is a proper transformation matrix (associated with the above-mentioned elementary operations).

Definition 5.6. *The system $\hat{\mathbf{S}}$ is defined by the following state-space equations*

$$\dot{x}(t) = Ax(t) + \hat{B}u(t)$$
$$q(t) = \hat{C}^0 x(t) + \hat{C}^1 x(t-h)$$

where

$$\hat{B} = B, \quad \hat{C}^0 = \Omega \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix}, \quad \hat{C}^1 = \Omega \begin{bmatrix} 0_{r\times n} \\ TC \end{bmatrix}$$

(note that $q(t) \in \mathbb{R}^{2r}$, and Ω is given in (5.10)).

Now, let the matrices $\hat{B}$, $\hat{C}^0$ and $\hat{C}^1$ be partitioned as

$$\hat{B} = \begin{bmatrix} \hat{B}_1 & \hat{B}_2 & \cdots & \hat{B}_\nu \end{bmatrix}$$

$$\hat{C}^0 = \begin{bmatrix} \hat{C}_1^0 \\ \hat{C}_2^0 \\ \vdots \\ \hat{C}_\nu^0 \end{bmatrix}, \quad \hat{C}^1 = \begin{bmatrix} \hat{C}_1^1 \\ \hat{C}_2^1 \\ \vdots \\ \hat{C}_\nu^1 \end{bmatrix}$$

where $\hat{B}_i \in \mathbb{R}^{n \times m_i}$ and $\hat{C}_i^\sigma \in \mathbb{R}^{(r_i+\mu_i)\times n}$, for $i \in \bar{\nu}$ and $\sigma = 0, 1$.

The following lemma along with Lemma 2.3 play key roles in obtaining a necessary condition for the stabilizability of the system **S** with respect to the overlapping controller **K** (the following lemma is borrowed from [7]).

Lemma 5.2. *Consider the matrices M_i and N_i, $i = 1, 2, \ldots, \eta$, where $M_i \in \mathbb{C}^{\rho \times \gamma_i}$ and $N_i \in \mathbb{C}^{\rho \times \nu_i}$. A necessary and sufficient condition for the following inequality*

$$\operatorname{rank} \begin{bmatrix} M_1 + N_1 K_1 & M_2 + N_2 K_2 & \cdots & M_\eta + N_\eta K_\eta \end{bmatrix} < \min \left\{ \rho, \sum_{i=1}^{\eta} \gamma_i \right\}$$

to hold for all $K_i \in \mathbb{C}^{\nu_i \times \gamma_i}$, $i = 1, 2, \ldots, \eta$, is that there exists a non-empty subset $\Phi = \{i_1, i_2, \cdots, i_j\}$ of the index set $\{1, 2, \cdots, \eta\}$ for which

$$\operatorname{rank} \begin{bmatrix} M_{i_1} & N_{i_1} & \cdots & M_{i_j} & N_{i_j} \end{bmatrix} < \min \left\{ \rho - \sum_{i \notin \Phi} \gamma_i, \sum_{i \in \Phi} \gamma_i \right\}$$

Theorem 5.1. *A necessary condition for the existence of a stabilizing overlapping controller **K** for the system **S** is that for any $s \in \operatorname{sp}(A)$, $\operatorname{Re}\{s\} \geq 0$, all of the following matrices are full-rank*

$$\begin{bmatrix} sI - A & \hat{B}_{i_1} & \cdots & \hat{B}_{i_l} \\ \hat{C}_{i_{l+1}}^0 + \hat{C}_{i_{l+1}}^1 e^{-sh} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}_{i_\nu}^0 + \hat{C}_{i_\nu}^1 e^{-sh} & 0 & \cdots & 0 \end{bmatrix}$$

where i_1, i_2, ..., i_ν are distinct integers representing any permutation of the set $\bar{\nu}$. Furthermore, $l = 0, \ldots, \nu + 1$ and $\hat{B}_{i_0} = \hat{C}_{i_{\nu+1}}^\sigma = \emptyset$, $\sigma = 0, 1$.

Proof: Equation (5.7) can be expressed as

$$u(t) = \begin{bmatrix} \bar{K} & \tilde{K} \end{bmatrix} \begin{bmatrix} Cx(t) \\ Cx(t-h) \end{bmatrix}$$

It follows from Lemma 5.1 and equation (5.9) that

$$u = \begin{bmatrix} \bar{K} & H \end{bmatrix} \begin{bmatrix} Cx(t) \\ TCx(t-h) \end{bmatrix}$$

Using (5.10), one will obtain

$$u = \hat{K} \left(\Omega \begin{bmatrix} C \\ 0 \end{bmatrix} x(t) + \Omega \begin{bmatrix} 0 \\ TC \end{bmatrix} x(t-h) \right)$$

From the above equation, it is inferred that the system **S** is stabilizable by an overlapping static controller of the form **K** if and only if $\hat{\mathbf{S}}$ is stabilizable by a decentralized static output feedback controller with the output feedback gain $\hat{K} \in \mathbb{G}_D$. A necessary condition for the latter statement to hold is that for any $s \in \text{sp}(A)$ with $\text{Re}\{s\} \geq 0$, there exists a $\hat{K}^* \in \mathbb{G}_D$ such that [106]

$$\det \left(sI - A - \sum_{i=1}^{\nu} \hat{B}_i \hat{K}_i^* \left(\hat{C}_i^0 + \hat{C}_i^1 e^{-sh} \right) \right) \neq 0$$

$\hat{K}_i^* \in \mathbb{R}^{m_i \times (r_i + \mu_i)}$, and

$$\hat{K}^* = \text{block diagonal} \left[\hat{K}_1^*, \hat{K}_2^*, \ldots, \hat{K}_\nu^* \right]$$

Using Lemmas 5.2 and 2.3, it can be shown in a manner similar to the techniques used in [7] that all the rank conditions provided in this theorem must hold for the system **S** to be stabilizable with respect to an overlapping controller of the form **K**. This completes the proof. ■

5.4.2 H$_\infty$ decentralized overlapping control synthesis

It is desired now to find LMI conditions to design a static controller for the system **S** which guarantees the stability and H$_\infty$ performance.

Definition 5.7. *$\mathbb{Q}_D$ is the set of all block diagonal matrices which have ν block-diagonal entries, where the i-th block entry of the main diagonal, $i \in \bar{\nu}$, is a $m_i \times m_i$ matrix itself.*

Theorem 5.2. *Consider the system* **S** *and let the delay h be an arbitrary positive value with a known upper bound $\bar{h}$. Assume that for a given $\gamma > 0$, there exist matrices $Q_1 > 0$, $0 < Q_2 \in \mathbb{Q}_D$, $Y_1 \in \mathbb{K}_D$, $Y_2 \in \mathbb{H}_D$, $R_{11} > 0$, R_{12} and $R_{22} > 0$ which satisfy the LMIs given below*

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & 0 & 0 & 0 \\ * & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & Z_{27} & Z_{28} \\ * & * & Z_{33} & Z_{34} & Z_{35} & 0 & 0 & 0 \\ * & * & * & Z_{44} & Z_{45} & Z_{46} & Z_{47} & Z_{48} \\ * & * & * & * & -0.5\gamma^2 I & 0 & 0 & 0 \\ * & * & * & * & * & -0.5\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -\bar{h}R_{11} & -\bar{h}R_{12} \\ * & * & * & * & * & * & * & -\bar{h}R_{22} \end{bmatrix} < 0 \tag{5.11}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0 \tag{5.12}$$

where

$$
\begin{aligned}
Z_{11} &= A^T Q_1 + Q_1 A + \Gamma^T \Gamma \\
Z_{12} &= Q_1 B + A^T C^T {Y_1}^T + A^T C^T T^T {Y_2}^T \\
Z_{13} &= A^T Q_1 \\
Z_{14} &= A^T C^T {Y_1}^T + A^T C^T T^T {Y_2}^T \\
Z_{15} &= Z_{35} = Q_1 E \\
Z_{22} &= B^T C^T {Y_1}^T + B^T C^T T^T {Y_2}^T + Y_1 C B + Y_2 T C B \\
Z_{23} &= B^T Q_1 \\
Z_{24} &= B^T C^T {Y_1}^T + B^T C^T T^T {Y_2}^T \\
Z_{25} &= Z_{45} = Y_1 C E \\
Z_{26} &= Z_{46} = Y_2 T C E \\
Z_{27} &= Z_{47} = \bar{h} Y_2 T C A \\
Z_{28} &= Z_{48} = \bar{h} Y_2 T C B \\
Z_{33} &= -2Q_1 + \bar{h} R_{11} \\
Z_{34} &= \bar{h} R_{12} \\
Z_{44} &= -2Q_2 + \bar{h} R_{22}
\end{aligned}
\tag{5.13}
$$

Set

$$
\bar{K} = {Q_2}^{-1} Y_1, \qquad \tilde{K} = {Q_2}^{-1} Y_2 T \tag{5.14}
$$

and let the overlapping controller with the above parameters be denoted by $\mathbf{K}^*$. *Then,*

i) the system $\mathbf{S}$ *under the controller* $\mathbf{K}^*$ *is internally stable; and*

ii) the ∞*-norm of the closed-loop transfer function from disturbance input* $w(t)$ *to regulated variable* $z(t)$*, denoted by* $\|T_{zw}\|_\infty$*, is less than* γ*, i.e.*

$$
\|T_{zw}\|_\infty = \|z(t)\|_2 / \|w(t)\|_2 < \gamma \tag{5.15}
$$

Proof: Define

$$\theta(t) = \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix}^T, \qquad v(t) = \begin{bmatrix} w(t)^T & w(t-h)^T \end{bmatrix}^T$$

It is straightforward to show that

$$\begin{aligned} \dot{\theta}(t) &= \begin{bmatrix} A & B \\ \bar{K}CA & \bar{K}CB \end{bmatrix} \theta(t) + \begin{bmatrix} 0 & 0 \\ \tilde{K}CA & \tilde{K}CB \end{bmatrix} \theta(t-h) \\ &+ \begin{bmatrix} E & 0 \\ \bar{K}CE & \bar{K}CE \end{bmatrix} v(t) \\ z(t) &= \begin{bmatrix} \Gamma & 0 \end{bmatrix} \theta(t) \end{aligned} \tag{5.16}$$

Consider the performance index given below

$$\begin{aligned} J(v) &= \int_0^\infty \left[z(t)^T z(t) - \gamma^2 w(t)^T w(t) \right] dt \\ &= \int_0^\infty \left[z(t)^T z(t) - \frac{\gamma^2}{2} v(t)^T v(t) \right] dt \end{aligned}$$

and let the system (5.16) be represented in the following descriptor form

$$\begin{aligned} \dot{\theta}(t) =& \zeta(t) \\ \zeta(t) =& \begin{bmatrix} A & B \\ \bar{K}CA & \bar{K}CB \end{bmatrix} \theta(t) + \begin{bmatrix} 0 & 0 \\ \tilde{K}CA & \tilde{K}CB \end{bmatrix} \theta(t-h) \\ &+ \begin{bmatrix} E & 0 \\ \bar{K}CE & \bar{K}CE \end{bmatrix} v(t) \end{aligned}$$

Define the following Lyapunov-Krasovskii functional for the above system

$$\begin{aligned} V(t) &= \begin{bmatrix} \theta(t) & \zeta(t) \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \zeta(t) \end{bmatrix} \\ &+ \int_{-h}^0 \int_{t+\beta}^t \zeta(\alpha)^T R \zeta(\alpha) d\alpha \, d\beta \end{aligned}$$

where

$$P = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} > 0, \qquad R = \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix} > 0$$

and $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{Q}_D$. Thus, it can be concluded that if the inequality

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ * & Z_{22} & Z_{23} & Z_{24} \\ * & * & -0.5\gamma^2 I & 0 \\ * & * & * & -\bar{h}R \end{bmatrix} < 0$$

holds for

$$\begin{aligned} Z_{11} &= Z_{12} + Z_{12}{}^T + \Phi_1 \\ \Phi_1 &= \begin{bmatrix} \Gamma^T\Gamma & 0 \\ 0 & 0 \end{bmatrix} \\ Z_{12} &= \begin{bmatrix} A^T Q_1 & \Phi_2 \\ B^T Q_1 & \Phi_3 \end{bmatrix} \\ \Phi_2 &= A^T C^T \bar{K}^T Q_2 + A^T C^T \tilde{K}^T Q_2 \\ \Phi_3 &= B^T C^T \bar{K}^T Q_2 + B^T C^T \tilde{K}^T Q_2 \\ Z_{13} &= \begin{bmatrix} Q_1 E & 0 \\ Q_2 \bar{K} C E & Q_2 \tilde{K} C E \end{bmatrix} \\ Z_{14} &= \bar{h} \begin{bmatrix} 0 & 0 \\ Q_2 \tilde{K} C A & Q_2 \tilde{K} C B \end{bmatrix} \\ Z_{22} &= \begin{bmatrix} -2Q_1 + \bar{h}R_{11} & \bar{h}R_{12} \\ * & -2Q_2 + \bar{h}R_{22} \end{bmatrix} \\ Z_{23} &= Z_{13} = Z_{24} = Z_{14} \end{aligned} \tag{5.17}$$

then $\dot{V}(t) < 0$ and $J(v) < 0$ for all nonzero $v(t) \in \mathcal{L}_2[0, \infty)$. According to Theorem 1.2, the above result implies that both statements in this theorem are satisfied. Substitute $\tilde{K}$ in (5.17) with HT as noted in Lemma 5.1, and define

$$Y_1 = Q_2 \bar{K}, \qquad Y_2 = Q_2 H$$

Using the above relations, the LMIs introduced in (5.11)-(5.13) are obtained. On the other hand, since $Q_2 \in \mathbb{Q}_D$ and $\bar{K} \in \mathbb{K}_D$, this implies that Y_1 also belongs to $\mathbb{K}_D$. Similarly, it can be concluded that $Y_2 \in \mathbb{H}_D$. This completes the proof. ∎

Remark 5.2. *Unlike Theorem 5.1, it is required in Theorem 5.2 to use the transformation T to find $\tilde{K}$. One can simply choose Y_2 as $Q_2\tilde{K}$, and consider a structure similar to $\tilde{K}$ for Y_2.*

5.5 Simulation Results

Example 5.1. *Consider a formation flight consisting of 3 unmanned aerial vehicles (UAV) with leader-follower structure. Let UAV 1 be the leader, and UAVs 2, 3 the followers. The objective here is to control the planar motion of the formation. Assume that all UAVs are desired to fly at the same velocity (v_x, v_y) with the distance vector (d_{x_i}, d_{x_i}) between UAVs i and $i+1$, $i = 1, 2$. The model of the formation in the relative coordinate frame is obtained as follows [144]*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ I_2 & 0_2 & -I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 & 0_2 & -I_2 \\ 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} I_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & I_2 \end{bmatrix} \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \\ u_3 + w_3 \end{bmatrix}$$

where

$$x_1 = \begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T, \qquad x_1 = \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \end{bmatrix}^T$$
$$x_3 = \begin{bmatrix} x_{31} & x_{32} & x_{33} & x_{34} \end{bmatrix}^T$$

Assume that the i-th vehicle can measure its state in the relative coordinates (i.e. x_i, $i = 1, 2, 3$) using GPS-based sensors. Thus, $C_1 = I_2$ and $C_2 = C_3 = I_4$. Consider the same communication topology as the one in the illustrative example *of Subsection 5.3.2, and suppose that* $\Gamma = I_{10}$. *The transformation matrix in this case is*

$$T = \begin{bmatrix} 0_{4\times 2} & I_{4\times 4} & 0_{4\times 4} \\ I_{2\times 2} & 0_{2\times 4} & 0_{2\times 4} \\ 0_{4\times 2} & 0_{4\times 4} & I_{4\times 4} \\ 0_{4\times 2} & I_{4\times 4} & 0_{4\times 4} \end{bmatrix}$$

Using the above transformation, it is straightforward to show that the rank conditions in Theorem 5.1 hold for $h < 1$. *A proper control design technique will be employed next to achieve stability.*

Consider the H_∞ *control synthesis provided in Theorem 5.2 with* $\bar{h} = 0.1$ *and* $\gamma = 0.15$, *and assume that*

$$w_1(t) = 0, \quad w_2(t) = w_3(t) = 160 \times \sin(20\pi t) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using the LMIs given by (5.11)-(5.13), the following overlapping static feedback control parameters are obtained

$$\bar{K} = \begin{bmatrix} \bar{K}_{11} & 0_{2\times 4} & 0_{2\times 4} \\ 0_{2\times 2} & \bar{K}_{22} & 0_{2\times 4} \\ 0_{2\times 2} & 0_{2\times 4} & \bar{K}_{33} \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} 0 & 0 & -23.51 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -23.51 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7.83 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$\bar{K}_{11} = \begin{bmatrix} -11.76 & 0 \\ 0 & -11.76 \end{bmatrix}$$

$$\bar{K}_{22} = \begin{bmatrix} 7.83 & 0 & -11.76 & 0 \\ 0 & 7.83 & 0 & -11.76 \end{bmatrix}$$

$$\bar{K}_{33} = \begin{bmatrix} 23.51 & 0 & -11.76 & 0 \\ 0 & 23.51 & 0 & -11.76 \end{bmatrix}$$

For $h = 0.1$*, the state variables of the system under the controller given above are depicted in Figures 5.3, 5.4 and 5.5. It can be verified that the formation remains stable for all* $h < 0.85$*. However, the performance of the closed-loop system obtained by applying the proposed overlapping controller to the formation deteriorates as* h *increases. Suppose that UAVs 1, 2 and 3 are initially located in* $(0, 0)$, $(-450, 100)$, $(-200, 850)$*, respectively. Let also*

$$d_{x_1} = \begin{bmatrix} 50 & 100 \end{bmatrix}^T, \quad d_{x_2} = \begin{bmatrix} 50 & -150 \end{bmatrix}^T$$

and assume that the leader is moving in the $x - y$ *plane with the constant velocity vector* $[200 \quad 100]^T$*. The trajectory of the formation under the proposed overlapping controller for* $h = 0.1$ *is sketched in Figure 5.6.*

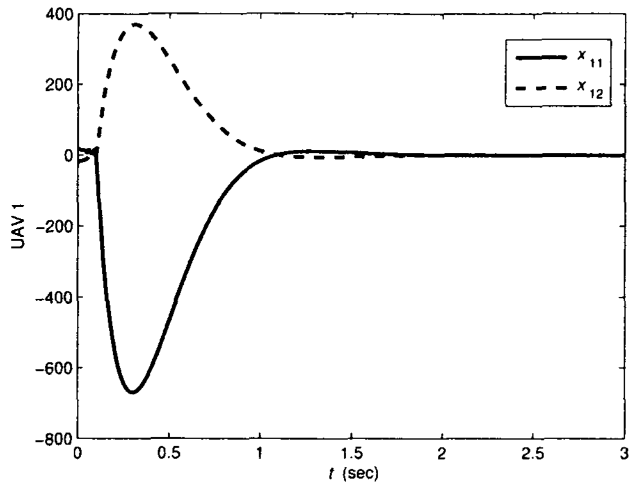


Figure 5.3: The state response of vehicle 1 for $h = 0.1$ *in Example 5.1*

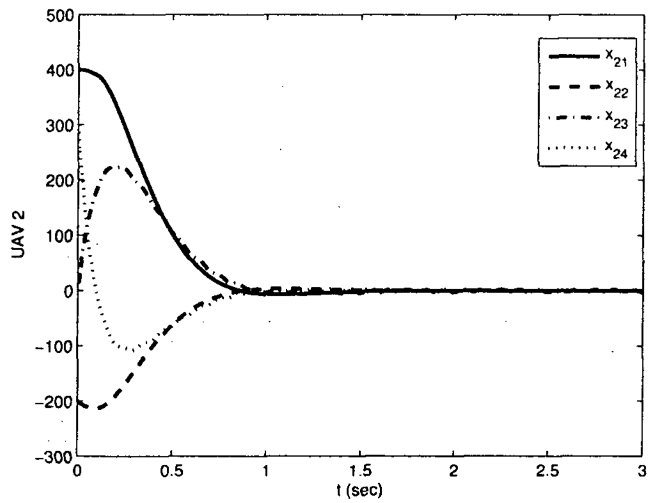


Figure 5.4: The state response of vehicle 2 for $h = 0.1$ *in Example 5.1*

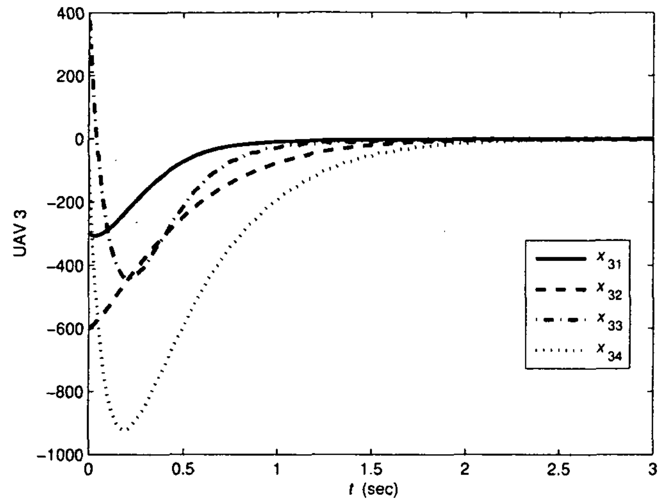


Figure 5.5: The state response of vehicle 3 for $h = 0.1$ *in Example 5.1*

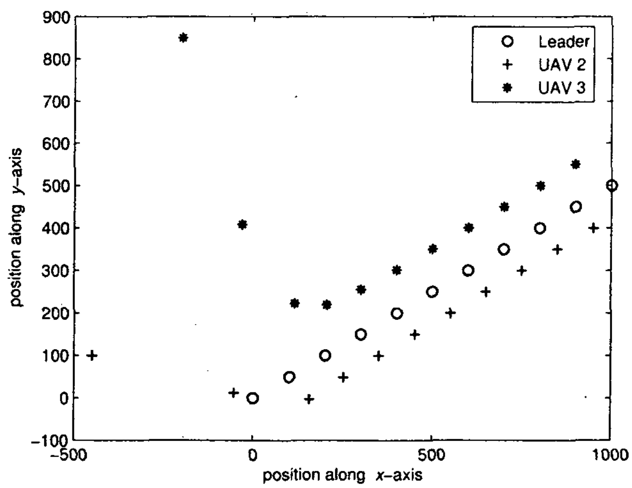


Figure 5.6: Planar motion of the formation for $h = 0.1$ *in Example 5.1*

Chapter 6

An Adaptive Switching Control Scheme for Uncertain LTI Time-Delay Systems

6.1 Introduction

In conventional adaptive control design, it is typically assumed that the actual plant is fixed, and can be described by a linear time-invariant (LTI) model which is unknown, but a good deal of *a priori* information on the plant is known; this information typically includes a knowledge of the upper bound on plant's order, the relative degree, the sign of the high-frequency gain, and minimum phase property. There have been some developments made to relax some of the classical assumptions adopted in conventional adaptive control. For example, some improvements have been made to remove the required information on the sign of the high-frequency gain [85,113,119,150], and to weaken the other requirements [96,111,146]. However, certain assumptions on the right-half plane zeros are required [100].

The adaptive control of systems via switching methods is a relatively new line of research which was motivated to weaken the classical *a priori* information, and can be traced back to [110], in which a number of questions about the classical assumptions in conventional adaptive control were raised. Switching controllers are nonlinear controllers, which can be used to stabilize and regulate systems with highly uncertain plant models. This is accomplished by using a dictionary of controllers, and by switching from one controller to another at appropriate time instants. There has been a considerable amount of interest towards switching control methods and its applications in the literature recently; e.g., see [3,21,51,98,101,101,114].

In the adaptive switching control approach using a family of plants, it is typically assumed that the plant is not necessarily fixed, i.e., the plant representation may change from one plant model to another; in this case, it is assumed that the plant model belongs to a known set of models [92,93]. Thus, to implement the adaptive controller, the first step required is to design a finite set of controllers (using either a model based, or an experimental approach) which provides the required performance for the family of plant models [21,45,98,99,112,114]. Then, on applying a so called "switching scheme", each controller is applied to the plant sequentially, and eventually, switching stops in finite time. This implies that as long as the plant remains unchanged, the switching controller will remain locked on one of the appropriate controllers which fulfills the closed-loop performance requirements.

Fu and Barmish [45] considered a compact set of LTI models to represent a plant and imposed an *a priori* upper bound on the order of plants in this set. They showed that Lyapunov stability can be achieved in this case, by applying a finite set of controllers. Miller and Davison [98] reduced this *a priori* information, to the knowledge required about the order of a LTI stabilizing compensator. They then simplified the compactness assumption required on the set of possible plant models to just a finite set of models. As a result of this, one can design a high-performance LTI controller, e.g. an optimal controller, for each plant model in the known set.

In [21], a class of multi-variable switching control algorithms was introduced which does not require a knowledge of the actual family of plant models. Using this procedure, the only information which is required to be known, is a set of controllers corresponding to the set of plant models, which contains a stabilizing controller for each plant model. A comprehensive survey of switching control systems is presented in [87]. These methods can be very effective when wide-band tracking or disturbance rejection of a physical plant, which can be described by a family of plant models, is required.

All methods described above, assume that the model of the LTI plant to be controlled is finite dimensional, which is unrealistic in many "real world" applications. As it is well studied in monographs [49, 63, 116], there are several examples of aftereffect phenomenon (which is represented by time-delay systems) in biology, chemistry, economics, mechanics, physics, population dynamics, as well as in engineering sciences. Since neglecting the effect of delay in the model of the system can result in the degradation of the system performance, it should be taken into account in control design. For instance, the stability margin of the overall system can be highly sensitive to delay and small variation in delay may lead to instability. This gives motivation to the present work, which studies the switching control of time-delay LTI plants with uncertain parameters.

Controller design for fixed model time-delay systems has been investigated extensively in the literature recently [23, 32, 64]. The problems of stability and stabilizability of discrete-time switched linear systems whose subsystems are subject to state delays are also investigated in [109], using the LMI approach. In [61], the uniform asymptotic stability of a class of linear switched system with time-delay is studied and the notion of common Lyapunov functional method is introduced. Furthermore, the stabilization problem for switched linear systems with unknown time-varying delays and arbitrary switching signals is addressed in [52]. Time-delay systems with Markovian jump, on the other hand, has been investigated in [16, 89, 91]. However, in all the existing works, it is assumed that the switching signal can be available instantaneously. In other words, when a switch from one subsystem to another occurs, it can be observed immediately. Furthermore, no online supervision and adaptation is required in the above cited papers.

In this chapter, it is assumed that the plant is described by a continuous-time retarded time-delay LTI model, which belongs to a known family of plant models. It is also assumed that a set of controllers exists to satisfactorily control the models in the known set. A switching control scheme is then proposed that uses the input-output information of the system to achieve online supervision and adaption, and to compute the switching instants. The present work is an extension of the switching control scheme proposed by Miller and Davison in [98] for finite dimensional LTI systems.

The remainder of the chapter is organized as follows. The problem of controlling a family of time-delay systems is formulated in Section 6.2. A method is then proposed in Section 6.3 to obtain an upper-bound signal for the error in two phases. This upper-bound signal is essential in finding the switching instants which are later used to develop the switching scheme. Two illustrative example are presented in Section 6.4, which demonstrate the effectiveness of the proposed switching technique.

6.2 Problem Formulation

Consider an uncertain plant whose model at any point in time belongs to a given finite set of models $\mathbf{P} := \{\mathrm{P}_1, \ldots, \mathrm{P}_p\}$. This can represent a plant or process which is subject to parameter jump (or rapid change of parameters) at distinct time instants due, for instance, to sudden change of operating point. Assume that any model P_i, $i \in \bar{\mathrm{p}} := \{1, 2, \ldots, p\}$, in the set $\mathbf{P}$ can be represented by a linear time-invariant (LTI) time-delay system, whose dynamics is represented by a retarded differential equation of the following form

$$\begin{aligned}
\dot{x}(t) &= A_i^0 x(t) + \sum_{j=1}^{m} A_i^j x(t - h_i^j) + B_i u(t) + E_i \omega(t) \\
y(t) &= C_i x(t) + F_i \omega(t) \\
x(r) &= \phi(r), \qquad -\bar{h}_i \le r \le 0
\end{aligned} \tag{6.1}$$

where $x(t) \in \mathbb{R}^{n_i}$ is the state, $u(t) \in \mathbb{R}^{v}$ is the control input, $y(t) \in \mathbb{R}^{r}$ is the output, and $\omega(t) \in \mathbb{R}^{\varsigma}$ is the exogenous disturbance. Furthermore, h_i^j's are the delays in the state of the plant P_i, which are assumed to be constant and satisfy the inequality $0 < h_i^1 < \ldots < h_i^m$. Let h_i^1, the smallest delay and h_i^m, the biggest delay in the states of the plant P_i be denoted by $\underline{h}_i$ and $\bar{h}_i$, respectively. In addition, the initial function of (6.1), denoted by $\phi(r)$, is assumed to be piecewise continuous. It is also supposed that $\omega(t)$ is a bounded piecewise continuous disturbance signal. The system matrices $A_i^j \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times \nu}$, $E_i \in \mathbb{R}^{n_i \times \varsigma}$, $C_i \in \mathbb{R}^{r \times n_i}$, and $F_i \in \mathbb{R}^{r \times \varsigma}$ are matrices with constant entries, for all $i \in \bar{\mathrm{p}}$ and $j \in \{0, 1, \ldots, m\}$.

Assumption 6.1. *The system (6.1) is assumed to be observable. (for the definition of observability for time-delay retarded systems, see [95]).*

Assumption 6.2. *For each plant model P_i, $i \in \bar{p}$, a high-performance LTI controller K_i is available with the following state-space representation*

$$\begin{aligned}
\dot{z}(t) &= G_i z(t) + H_i y(t) + J_i y_{ref}(t) \\
u(t) &= K_i z(t) + L_i y(t) + M_i y_{ref}(t)
\end{aligned} \tag{6.2}$$

where $z(t) \in \mathbb{R}^{l}$ is the state of the controller, and $y_{ref} \in \mathbb{R}^{r}$ is the reference signal which is assumed to be bounded.

Remark 6.1. *It will be assumed with no loss of generality throughout the chapter that all the controllers K_i, $i \in \bar{p}$ have the same order. It is to be noted that this condition can always be met by adding unobservable stable modes to the controllers, if necessary [98].*

The objective of this chapter is to propose a switching mechanism so that output tracking is achieved in the presence of external disturbances. In other words, it is desired to switch between the feedback gains $\tilde{\mathrm{K}}_i$ at appropriate time instants so that the tracking error approaches zero as $t \to \infty$.

Remark 6.2. *Throughout this chapter, the main requirement for control design is that each controller K_i stabilizes the corresponding plant P_i (for the stability analysis of time-delay systems, see [18, 43, 49]). However, when it is desired to achieve exact tracking for a certain class of reference inputs and disturbances, additional conditions on the control structure needs to be imposed. This will be discussed later in Corollary 6.2.*

Define the following augmented vectors

$$\tilde{x} := \begin{bmatrix} x \\ z \end{bmatrix}, \quad \tilde{u} := \begin{bmatrix} u \\ \dot{z} \end{bmatrix}, \quad \tilde{y} := \begin{bmatrix} y \\ z \\ y_{ref} \end{bmatrix}$$

The dynamic feedback control problem corresponding to the pair (P_i, K_i) can now be expressed as the static feedback control problem corresponding to the pair ($\tilde{\mathrm{P}}_i$, $\tilde{\mathrm{K}}_i$), where the augmented controller $\tilde{\mathrm{K}}_i$ and the augmented plant $\tilde{\mathrm{P}}_i$ are given by equations (6.3) and (6.4) below, respectively

$$\tilde{u} = \tilde{K}_i \tilde{y} \tag{6.3}$$

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}_i^0 \tilde{x}(t) + \sum_{j=1}^{m} \tilde{A}_i^j \tilde{x}(t - h_i^j) + \tilde{B}_i \tilde{u}(t) + \tilde{E}_i \omega(t) \\ \tilde{y}(t) &= \tilde{C}_i \tilde{x}(t) + \tilde{D}_i y_{ref}(t) + \tilde{F}_i \omega(t) \end{aligned} \tag{6.4}$$

and where

$$\tilde{A}_i^0 = \begin{bmatrix} A_i^0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_i^j = \begin{bmatrix} A_i^j & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{C}_i = \begin{bmatrix} C_i & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}, \quad \tilde{D}_i = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \quad \tilde{F}_i = \begin{bmatrix} F_i \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \quad \tilde{K}_i = \begin{bmatrix} L_i & K_i & M_i \\ H_i & G_i & J_i \end{bmatrix}$$

In addition, since the controller $\tilde{\mathrm{K}}_i$ stabilizes the model $\tilde{\mathrm{P}}_i$ according to Remark 6.2, all of the roots of the following characteristic equation

$$\tilde{\Delta}_i(s) = \det(sI - \tilde{A}_i^0 - \tilde{B}_i\tilde{K}_i\tilde{C}_i - \sum_{j=1}^{m} \tilde{A}_i^j e^{-h_i^j s}) = 0 \tag{6.5}$$

lie in the open left-half of the complex plane.

6.3 Main Results

In this section, the switching control scheme proposed in [98] is modified to account for the delay in the state of the system. The main objective is to introduce a switching control scheme to stabilize an uncertain plant which can be described by the family of models (6.1).

6.3.1 Preliminaries

Consider the following retarded time-delay state equation

$$\begin{aligned} \dot{x}(t) &= A_i^0 x(t) + \sum_{j=1}^{m} A_i^j x(t - h_i^j) + f(t) \\ \phi(r) &= x(r), \qquad -\bar{h}_i \le r \le 0 \end{aligned} \tag{6.6}$$

where $x(t) \in \mathbb{R}^{n_i}$ is the state and $f(t) \in \mathbb{R}^{n_i}$ is the piecewise continuous bounded input function. Furthermore, the system parameters A_i^j, h_i^j and $\bar{h}_i$ are the same as the ones in (6.1), and $\phi(r)$ is the piecewise continuous initial function. From the functional differential equations theory [50], it is known that the solution of (6.6) can be written as

$$x(t) = x(\phi, 0) + \int_0^t X_i(t-\tau) f(\tau) d\tau \tag{6.7}$$

where $x(\phi, 0)$, the homogeneous part of the solution in (6.7), is given by

$$x(\phi, 0) = X_i(t)\phi(0) + \sum_{j=1}^{m} \int_{-h_i^j}^{0} X_i(t - r - h_i^j) A_i^j \phi(r) dr \tag{6.8}$$

$X_i(t)$ in (6.7) and (6.8) is the fundamental matrix for the corresponding retarded state equation which satisfies the following matrix functional differential equation [60]

$$\dot{X}_i(t) = A_i^0 X_i(t) + \sum_{j=1}^{m} A_i^j X_i(t - h_i^j)$$

with the initial condition given by

$$X_i(r) = \begin{cases} I_{q_i}, & r = 0 \\ 0_{q_i}, & r \in [-\bar{h}_i, 0) \end{cases}, \quad q_i := n_i + l, \quad i \in \bar{\text{p}}$$

where I_{q_i} denotes the $q_i \times q_i$ identity matrix, and 0_{q_i} is the $q_i \times q_i$ zero matrix. Furthermore, it is known that there exist constants α_i and λ_i, so that [50]

$$\|X_i(t)\| \leq \alpha_i e^{\lambda_i t}, \quad \forall t > 0, \quad i \in \bar{\text{p}} \tag{6.9}$$

where $\|.\|$ represents the 2-norm of a vector, or the corresponding induced 2-norm of a matrix. As a result, it can be easily concluded that there exists a constant σ_i, such that

$$\|x(\phi, 0)\| \leq \sigma_i e^{\lambda_i t} \times \max_{-\bar{h}_i \leq r \leq 0} \|\phi(r)\|, \quad \forall t > 0, \quad i \in \bar{\text{p}}$$

Moreover, consider the characteristic equation corresponding to the retarded state equation (6.6) as follows

$$\Delta_i(s) = \det(sI - A_i^0 - \sum_{j=1}^{m} A_i^j e^{-h_i^j s}) = 0$$

and define λ_{0_i} as

$$\lambda_{0_i} = \max\{\text{Re}\{s\} : \Delta_i(s) = 0\}$$

Then, it can be easily verified that λ_i in (6.9) is greater than or equal to λ_{0_i} [50]. Consequently, If the system given by (6.6) is asymptotically stable, then one can choose λ_i in (6.9) as a strictly negative value.

6.3.2 Finding an upper bound on the initial function

Lemma 6.1. *Consider the system (6.1). Let the initial function be denoted by $\phi(r)$, where $r \in [-\bar{h}_i, 0]$. Then, for every arbitrary $T > 0$ and $i \in \bar{p}$, the matrix $Q_i(r,T)$ defined by*

$$Q_i(r,T) := \int_0^T \int_{-\bar{h}_i}^{0^+} \Theta_i{}'(t,\tau) C_i{}' C_i \Theta_i(t,r) d\tau \, dt \tag{6.10}$$

is invertible for all $r \in [-\bar{h}_i, 0]$, where

$$\Theta_i(t,r) = X_i(t-r)\delta(r) + \sum_{j=1}^{m} X_i(t-r-h_i^j) A_i^j u_{-1}(r+h_i^j) \tag{6.11}$$

and $X_i(t)$ is the fundamental matrix for the corresponding retarded differential equation of plant P_i ($\delta(.)$ and $u_{-1}(.)$ are Dirac delta and unit step functions, respectively).

Proof: If $u(t)$ and $\omega(t)$ are identically zero in the interval $[0,T]$, the output of the system (6.1) can be obtained as follows

$$y(t) = C_i(X_i(t)\phi(0) + \sum_{j=1}^{m} \int_{-h_i^j}^{0} X_i(t-r-h_i^j) A_i^j \phi(r) dr)$$

Using (6.11) and the sifting property of Dirac delta, $y(t)$ can be rewritten as

$$y(t) = C_i \int_{-\bar{h}_i}^{0^+} \Theta_i(t,r)\phi(r) dr \tag{6.12}$$

Multiplying both sides of (6.12) by $\Theta_i{}'(t,\tau)C_i{}'$ and integrating over t and τ result in

$$\int_{-\bar{h}_i}^{0^+} \int_0^T \Theta_i{}'(t,\tau) C_i{}' y(t) dt \, d\tau = \int_{-\bar{h}_i}^{0^+} \int_0^T \Theta_i{}'(t,\tau) C_i{}' C_i \int_{-\bar{h}_i}^{0^+} \Theta_i(t,r)\phi(r) dr \, dt \, d\tau$$

$$= \int_{-\bar{h}_i}^{0^+} [\int_0^T \int_{-\bar{h}_i}^{0^+} \Theta_i{}'(t,\tau) C_i{}' C_i \Theta_i(t,r) d\tau \, dt] \phi(r) dr$$

From the definition given by (6.10), the following can be obtained

$$\int_{-\bar{h}_i}^{0^+}\int_0^T \Theta_i'(t,r)C_i'y(t)dt\,dr = \int_{-\bar{h}_i}^{0^+} Q_i(r,T)\phi(r)dr \tag{6.13}$$

Suppose now, that $Q_i(r,T)$ is not full-rank for some $r_0 \in [-\bar{h}_i, 0]$. Then a nonzero vector φ_0 exists, such that $Q_i(r_0,T)\varphi_0 = 0$. Define $\nu(r)$ as

$$\nu(r) = \begin{cases} \varphi_0, & r = r_0 \\ 0_{n_i}, & r \neq r_0 \end{cases}$$

where 0_{n_i} is the zero vector in $\mathbb{R}^{n_i}$. Therefore, if $y(t)$ is identically zero for all $t \in [0,T]$, then $\phi(r) = \nu(r)$ and $\phi(r) = 0$ will be two possible solutions for (6.13). On the other hand, since the system (6.1) is observable, the equation (6.13) must have a unique solution for $\phi(r)$. This means that the observability assumption is violated and thus, it can be concluded that $Q_i(r,T)$ should be invertible for all $r \in [-\bar{h}_i, 0]$. ■

Corollary 6.1. *If $u(t)$ and $\omega(t)$ are identically zero in the interval $[0,T]$, it follows that the vector $\phi(r)$ given by*

$$\phi(r) = Q_i^{-1}(r,T)\int_0^T \Theta_i'(t,r)C_i'y(t)dt \tag{6.14}$$

is the unique solution of (6.13).

Proof: It follows from Lemma 6.1 that the inverse of $Q_i(r,T)$ exists and thus, (6.14) gives a solution for $\phi(r)$ in (6.13). In addition, it can be concluded from the observability assumption that this solution is unique. ■

It is to be noted that Lemma 6.1 provides only a sufficient condition for non-singularity of the matrix (6.10). The matrix $Q_i(r,T)$ is known as the observability gramian for the time-delay system (6.1). One can use the methods given in [12, 95, 120] to check the observability of time-delay systems.

Remark 6.3. *By substituting (6.11) into (6.10), the matrix $Q_i(r,T)$, for $r \in [-\bar{h}_i, 0)$, can be rewritten as*

$$Q_i(r,T) := \sum_{l=1}^{m} \int_0^T X_i'(t) C_i' C_i X_i(t - r - h_i^l) A_i^l u_{-1}(r + h_i^l) dt$$
$$+ \sum_{j=1}^{m} \sum_{l=1}^{m} \int_0^T \int_{-\bar{h}_i}^0 u_{-1}(\tau + h_i^j) A_i^{j\prime} X_i'(t - \tau - h_i^j) \qquad (6.15)$$
$$C_i' C_i X_i(t - r - h_i^l) A_i^l u_{-1}(r + h_i^l) d\tau\, dt$$

The following additional notations (derived from the above expression) are useful for the development of the the further result

$$\psi_1{}^i(r) := \sum_{l=1}^{m} \int_0^T X_i'(t) C_i' C_i X_i(t - r - h_i^l) A_i^l u_{-1}(r + h_i^l) dt$$
$$\psi_2{}^i(r) := \sum_{j=1}^{m} \sum_{l=1}^{m} \int_0^T \int_{-\bar{h}_i}^0 u_{-1}(\tau + h_i^j) A_i^{j\prime} X_i'(t - \tau - h_i^j) \qquad (6.16)$$
$$C_i' C_i X_i(t - r - h_i^l) A_i^l u_{-1}(r + h_i^l) d\tau\, dt$$

Lemma 6.2. *Consider the system (6.1). Assume that $u(t) = 0$ for all $t \in [0, T]$, where T is any arbitrary positive nonzero value. Then, there exists a constant β_i, so that for any arbitrary continuous initial condition $\phi(r)$ and every disturbance $\omega(t)$*

$$\max_{-\bar{h}_i \le r \le 0} \|\phi(r)\| \le \beta_i \sup_{t \ge 0} \|\omega(t)\| + \sup_{-\bar{h}_i \le r < 0} \frac{1}{\eta_i(r)} \Upsilon_i(y)$$

where $\eta_i(r)$ is the smallest singular value of $Q_i(r,T)$, and

$$\beta_i = \sup_{-\bar{h}_i \le r < 0} \frac{1}{\eta_i(r)} \int_0^T \|(\int_t^T \Theta_i'(\tau, r) C_i' C_i X_i(\tau - t) E_i d\tau) +$$
$$\Theta_i'(t, r) C_i' F_i \| dt \qquad (6.17a)$$
$$\Upsilon_i(y) = \| \int_0^T \Theta_i'(t, r) C_i' y(t) dt \| \qquad (6.17b)$$

Proof: Using an approach similar to the proof of Lemma 6.1, it can be shown that

$$\int_{-\bar{h}_i}^{0^+} \int_0^T \Theta_i'(t, r) C_i' (y(t) - F_i \omega(t)) dt\, dr = \int_{-\bar{h}_i}^{0^+} Q_i(r, T) \phi(r) dr$$
$$+ \int_{-\bar{h}_i}^{0^+} \int_0^T \int_0^t \Theta_i'(t, r) C_i' C_i X_i(t - \tau) E_i \omega(\tau) d\tau\, dt\, dr$$

Since the system (6.1) is observable, it follows from Lemma 6.1 that

$$\begin{aligned}\phi(r) =& Q_i^{-1}(r,T)[\int_0^T \Theta_i'(t,r)C_i'y(t)dt \\ & - \int_0^T \Theta_i'(t,r)C_i'F_i\omega(t)dt - \xi(r)]\end{aligned}$$

where

$$\xi(r) = \int_0^T \int_0^t \Theta_i'(t,r)C_i'C_iX_i(t-\tau)E_i\omega(\tau)d\tau\, dt$$

It is concluded from Fubini's Theorem that $\xi(r)$ can be rewritten as

$$\begin{aligned}\xi(r) &= \int_0^T \int_\tau^T \Theta_i'(t,r)C_i'C_iX_i(t-\tau)E_i\omega(\tau)dt\, d\tau \\ &= \int_0^T [\int_\tau^T \Theta_i'(t,r)C_i'C_iX_i(t-\tau)E_idt]\omega(\tau)d\tau \\ &= \int_0^T [\int_t^T \Theta_i'(\tau,r)C_i'C_iX_i(\tau-t)E_id\tau]\omega(t)dt\end{aligned}$$

Consequently, $\phi(r)$ can be obtained as follows

$$\begin{aligned}\phi(r) =& \int_0^T Q_i^{-1}(r,T)\Theta_i'(t,r)C_i'y(t)dt \\ & - \int_0^T Q_i^{-1}(r,T)[\Theta_i'(t,r)C_i'F_i \\ & + \int_t^T \Theta_i'(\tau,r)C_i'C_iX_i(\tau-t)E_id\tau]\omega(t)dt\end{aligned}$$

where $r \in [-\bar{h}_i, 0]$. By taking the norm of both sides of the above equation, using the related inequalities, and noting that $\|Q_i^{-1}(r,T)\| = 1/\eta_i(r)$, the upper bound for $\|\phi(r)\|$ given in Lemma 6.2 is obtained. Note that since the function $\phi(r)$ is continuous, and the interval for r is finite, hence

$$\max_{-\bar{h}_i \le r \le 0} \phi(r) = \sup_{-\bar{h}_i \le r < 0} \phi(r)$$

■

Remark 6.4. *To find the upper bound function given in Lemma 6.2, it is not required to obtain the inverse of the observability gramian matrix $Q_i(r,T)$. This* *reduces the computational complexity of the proposed switching algorithm. Nevertheless, integration of matrix exponentials is numerically difficult, in general. It is shown in the following two remarks that the upper bounds on β_i and $\Upsilon_i(y)$ in (6.17) can be found without matrix integration.*

Remark 6.5. *Applying triangle inequality to (6.17a) yields*

$$\beta_i \leq \sup_{-\bar{h}_i \leq r < 0} \frac{1}{\eta_i(r)} [\int_0^T \| \int_t^T \Theta_i'(\tau, r) C_i' C_i X_i(\tau - t) E_i d\tau \| dt + \int_0^T \|\Theta_i'(t, r) C_i' F_i \| dt]$$

Thus

$$\beta_i \leq \sup_{-\bar{h}_i \leq r < 0} \frac{1}{\eta_i(r)} [\int_0^T \int_t^T \|\Theta_i'(\tau, r)\| \|C_i'\| \|C_i\| \|X_i(\tau - t)\| \|E_i\| d\tau \, dt$$
$$+ \int_0^T \|\Theta_i'(t, r)\| \|C_i'\| \|F_i\| dt]$$

It follows from (6.11) that

$$\beta_i \leq \sup_{-\bar{h}_i \leq r < 0} \frac{1}{\eta_i(r)} [\int_0^T \int_t^T \sum_{j=1}^m u_{-1}(r - h_i^j) \|A_i^{j'}\| \|X_i'(\tau - r - h_i^j)\| \|C_i'\| \|C_i\|$$
$$\|X_i(\tau - t)\| \|E_i\| d\tau \, dt + \int_0^T \sum_{j=1}^m u_{-1}(r - h_i^j) \|A_i^{j'}\| \|X_i'(t - r - h_i^j)\| \|C_i'\| \|F_i\| dt]$$

The following inequality is then resulted from (6.9)

$$\beta_i \leq \sup_{-\bar{h}_i \leq r < 0} \frac{1}{\eta_i(r)} \left[\int_0^T \sum_{j=1}^m u_{-1}(r - h_i^j) \|A_i^{j'}\| \alpha_i \left\{ \int_t^T e^{\lambda_i(\tau - r - h_i^j)} u_{-1}(\tau - r - h_i^j) \|C_i'\| \|C_i\| \right.\right.$$
$$\left.\left. \alpha_i e^{\lambda_i(\tau - t)} \|E_i\| d\tau + e^{\lambda_i(t - r - h_i^j)} u_{-1}(t - r - h_i^j) \|C_i'\| \|F_i\| \right\} dt \right]$$

Remark 6.6. *Applying triangle inequality to (6.17b) yields*

$$\Upsilon_i(y) \leq \int_0^T \|\Theta_i'(t, r)\| \|C_i'\| \|y(t)\| dt$$

It follows from (6.11) that

$$\Upsilon_i(y) \leq \int_0^T \sum_{j=1}^m u_{-1}(r - h_i^j) \|A_i^{j'}\| \|X_i'(t - r - h_i^j)\| \|C_i'\| \|y(t)\| dt$$

The following inequality can then be obtained from (6.9),

$$\Upsilon_i(y) \leq \int_0^T \sum_{j=1}^m u_{-1}(r - h_i^j) \|A_i^{j'}\| \alpha_i e^{\lambda_i(t - r - h_i^j)} u_{-1}(t - r - h_i^j) \|C_i'\| \|y(t)\| dt$$

In the next Proposition, it is shown that one can choose T such that $Q_i(r,T)$ is computed more efficiently.

Proposition 6.1. *If the time interval $T > 0$ is chosen smaller than the smallest delay in the states of each of the models, i.e. $T < \min\{\underline{h}_1, \underline{h}_2, \ldots, \underline{h}_p\}$, and if the matrices A_i^0, $i \in \bar{p}$ are invertible, then the matrices ${\psi_1}^i(r)$ and ${\psi_2}^i(r)$ in (6.16) can be written as follows*

$$\begin{aligned} {\psi_1}^i(r) =& \sum_{l=1}^{m} u_{-1}(r+h_i^l)u_{-1}(T-h_i^l-r)\left[\int_{r+h_i^l}^{T} e^{A_i^{0'}t}C_i'C_ie^{A_i^0t}dt\right] \\ & e^{-A_i^0r}e^{-A_i^0h_i^l}A_i^l \end{aligned} \quad (6.18a)$$

$$\begin{aligned} {\psi_2}^i(r) =& \sum_{j=1}^{m}\sum_{l=1}^{m} u_{-1}(r+h_i^l)u_{-1}(T-h_i^l-r)A_i^{j'}\left(A_i^{0'}\right)^{-1} \\ & \times\left[\int_{r+h_i^l}^{T} e^{A_i^{0'}t}C_i'C_ie^{A_i^0t}dt - \int_{r+h_i^l}^{T} C_i'C_ie^{A_i^0t}dt\right]e^{-A_i^0(r+h_i^l)}A_i^l \end{aligned} \quad (6.18b)$$

Proof: It is known that for $t < h_i^1$

$$X_i(t) = e^{A_i^0t}u_{-1}(t) \quad (6.19)$$

Substituting $X_i(t)$ given by (6.19) into (6.16), one can easily verify the expression given for ${\psi_1}^i(r)$ in (6.18a). In addition, ${\psi_2}^i(r)$ can be simplified as

$$\begin{aligned} {\psi_2}^i(r) =& \sum_{j=1}^{m}\sum_{l=1}^{m} A_i^{j'}\int_0^T\left(\int_{-h_i^j}^{t-h_i^j} e^{-A_i^{0'}(\tau+h_i^j)}d\tau\right)e^{A_i^{0'}t} \\ & C_i'C_ie^{A_i^0(t-r-h_i^l)}u_{-1}(t-r-h_i^l)A_i^lu_{-1}(r+h_i^l)dt \end{aligned}$$

It follows then by integrating with respect to τ that

$${\psi_2}^i(r) = \sum_{j=1}^{m}\sum_{l=1}^{m} A_i^{j'}\int_0^T [A_i^{0'}]^{-1}(e^{A_i^{0'}t}-I)C_i'C_ie^{A_i^0(t-r-h_i^l)}u_{-1}(t-r-h_i^l)A_i^lu_{-1}(r+h_i^l)dt$$

It can be concluded that

$$\begin{aligned} {\psi_2}^i(r) =& \sum_{j=1}^{m}\sum_{l=1}^{m} u_{-1}(r+h_i^l)A_i^{j'}\left(A_i^{0'}\right)^{-1}\Big[\int_0^T e^{A_i^{0'}t}C_i'C_ie^{A_i^0t}u_{-1}(t-r-h_i^l)dt \\ & - \int_0^T C_i'C_ie^{A_i^0t}u_{-1}(t-r-h_i^l)dt\Big]e^{-A_i^0(r+h_i^l)}A_i^l \end{aligned}$$

Therefore, the expression given for ${\psi_2}^i(r)$ in (6.18b) is obtained. ■

Remark 6.7. *The expressions obtained for* ${\psi_1}^i(r)$ *and* ${\psi_2}^i(r)$ *in Proposition 6.1 involve the standard matrix exponential integrals, for which a computationally efficient method is introduced in [88]. This substantially reduces the computational complexity required for finding the matrix* $Q_i(r,T)$.

6.3.3 Finding an upper bound for the state

Lemma 6.3. *There exist strictly positive constants* γ_{i_1}, γ_{i_2}, γ_{i_3}, *and* $\lambda_i < 0$, *so that the solution of (6.4) satisfies the following inequality*

$$\begin{aligned}\|\tilde{x}(t)\| \leq & \gamma_{i_1} e^{\lambda_i t} \max_{-\bar{h}_i \leq r \leq 0} \|\tilde{\phi}(r)\| + \int_0^t e^{\lambda_i(t-\tau)} \gamma_{i_3} \|\tilde{\omega}(\tau)\| d\tau \\ & + \int_0^t e^{\lambda_i(t-\tau)} [\gamma_{i_2} \|\tilde{u}(\tau) - \tilde{K}_i(\tilde{y}(\tau) - \tilde{D}_i y_{ref}(\tau))\|] d\tau\end{aligned}$$

Proof: The retarded differential equation for $\tilde{x}(t)$ given by (6.4) can be rewritten as

$$\begin{aligned}\dot{\tilde{x}}(t) =& (\tilde{A}_i + \tilde{B}_i\tilde{K}_i\tilde{C}_i)\tilde{x}(t) + (\tilde{E}_i + \tilde{B}_i\tilde{K}_i\tilde{F}_i)\tilde{\omega}(t) \\ & + \tilde{B}_i[\tilde{u}(t) - \tilde{K}_i(\tilde{y}(t) - \tilde{D}_i y_{ref}(t))] + \sum_{j=1}^{m} \tilde{A}_i^j \tilde{x}(t - h_i^j)\end{aligned} \tag{6.20}$$

One can express $\tilde{x}(t)$ using an equation similar to (6.7), as follows

$$\tilde{x}(t) = \tilde{x}(\tilde{\phi},0) + \int_0^t \tilde{X}_i(t-\tau)\{(\tilde{E}_i + \tilde{B}_i\tilde{K}_i\tilde{F}_i)\tilde{\omega}(\tau) + \tilde{B}_i[\tilde{u}(\tau) - \tilde{K}_i(\tilde{y}(\tau) - \tilde{D}_i y_{ref}(\tau))]\} d\tau$$

where $\tilde{X}_i(t)$, $i \in \bar{\text{p}}$ is the fundamental matrix for the retarded state equation in (6.20). Consequently, it can be concluded that $\|\tilde{x}(t)\|$ satisfies the following inequality

$$\begin{aligned}\|\tilde{x}(t)\| \leq & \|\tilde{x}(\tilde{\phi},0)\| + \int_0^t \|\tilde{X}_i(t-\tau)\| \|\tilde{E}_i + \tilde{B}_i\tilde{K}_i\tilde{F}_i\| \|\tilde{\omega}(\tau)\| d\tau \\ & + \int_0^t \|\tilde{X}_i(t-\tau)\| \|\tilde{B}_i\| \|\tilde{u}(\tau) - \tilde{K}_i(\tilde{y}(\tau) - \tilde{D}_i y_{ref}(\tau))\| d\tau\end{aligned}$$

It is known that there exist constants α_i, λ_i, and σ_i, such that the following inequalities hold

$$\|\tilde{X}_i(t)\| \leq \alpha_i e^{\lambda_i t} \tag{6.21}$$

$$\|\tilde{x}(\phi, 0)\| \leq \sigma_i e^{\lambda_i t} \times \max_{-\bar{h}_i \leq r \leq 0} \|\tilde{\phi(r)}\| \tag{6.22}$$

Since the closed-loop system in (6.20) corresponds to the pair ($\tilde{\mathrm{P}}_i$, $\tilde{\mathrm{K}}_i$), it is asymptotically stable, and hence all of its poles given by the roots of the characteristics equation (6.5) lie in the open right-half plane. Therefore, λ_i can be chosen strictly negative. The upper bound for $\|\tilde{x}(t)\|$ is then obtained from (6.21) and (6.22), as follows

$$\|\tilde{x}(t)\| \leq \sigma_i e^{\lambda_i t} \max_{-\bar{h}_i \leq r \leq 0} \|\tilde{\phi}(r)\| + \int_0^t \alpha_i e^{\lambda_i (t-\tau)} \|\tilde{E}_i + \tilde{B}_i \tilde{K}_i \tilde{F}_i\| \|\tilde{\omega}(\tau)\| d\tau$$
$$+ \int_0^t \alpha_i e^{\lambda_i (t-\tau)} \|\tilde{B}_i\| \|\tilde{u}(\tau) - \tilde{K}_i(\tilde{y}(\tau) - \tilde{D}_i y_{ref}(\tau))\| d\tau$$

The proof follows by choosing

$$\gamma_{i_1} = \sigma_i, \quad \gamma_{i_2} = \alpha_i \|\tilde{B}_i\|, \quad \gamma_{i_3} = \alpha_i \|\tilde{E}_i + \tilde{B}_i \tilde{K}_i \tilde{F}_i\|$$

■

Lemma 6.4. *Assume that σ_i and λ_i satisfy (6.22) and let α_i be equal to $\sqrt{q_i}\sigma_i$. Then the following inequality holds:*

$$\|\tilde{X}_i(t)\| \leq \alpha_i e^{\lambda_i t}$$

Proof: It is known that

$$\|\tilde{X}_i(t)\| \leq \|\tilde{X}_i(t)\|_F \tag{6.23}$$

where $\|.\|_F$ denotes the Frobenius norm. On the other hand,

$$\|\tilde{X}_i(t)\|_F^2 = \sum_{s=1}^{q_i} \|\tilde{x}(\tilde{\phi}_i^s, 0)\|^2 \tag{6.24}$$

where $\tilde{\phi}_i^s$, $s = 1, 2, \ldots, q_i$, $i \in \bar{\mathrm{p}}$, is defined as follows

$$\tilde{\phi}_i^s(r) = \begin{cases} e_s, & r = 0 \\ 0_{q_i}, & r \in [-\bar{h}_i, 0) \end{cases}$$

The vector e_s in the above definition is the s^{th} column of the $q_i \times q_i$ identity matrix. In addition, from (6.22)

$$\|\tilde{x}(\tilde{\phi}_s, 0)\|^2 \leq \sigma_i^2 e^{2\lambda_i t} \tag{6.25}$$

for all $s = 1, \ldots, q_i$. The following inequality is directly obtained from (6.24) and (6.25)

$$\|\tilde{X}_i(t)\|_F^2 \leq q_i \sigma_i^2 e^{2\lambda_i t} \tag{6.26}$$

The proof follows immediately from (6.23) and (6.26). ■

The following procedure can be used to obtain the constants λ_i, σ_i, and α_i such that the inequalities (6.21) and (6.22) in Lemma 6.3 hold.

Step 1) Use the Mikhailov diagram [108] to find the smallest negative value for λ_i.

Step 2) Find the constant σ_i, based on the value obtained for λ_i in Step 1 and by using the following relations [108]

$$\sigma_i = \sqrt{\frac{\alpha_{i_2}}{\alpha_{i_1}}}, \qquad \alpha_{i_1} = \lambda_{\min}(R_i),$$
$$\alpha_{i_2} = \lambda_{\max}(R_i) + \sum_{j=1}^{m} h_i^j \lambda_{\max}(S_i^j)$$

where R_i and S_i^j, $j = 1, \ldots, m$ are $q_i \times q_i$ positive-definite matrices with real entries, which satisfy the LMI conditions given below

$$M(R_i, S_i^1, \ldots, S_i^m) - 2\lambda_i N(R_i) < 0, \quad i \in \bar{\mathrm{p}}$$

The matrices M and N in the left side of the above inequality are given by

$$M(R_i, S_i^1, \ldots, S_i^m) := \mathbf{A}_i{}' R_i \mathbf{E}_i + \mathbf{E}_i{}' R_i \mathbf{A}_i + \text{diag}\left\{ \sum_{j=1}^{m} S_i^j, -e^{-2\lambda_i h_i^1} S_i^1, \ldots, -e^{-2\lambda_i h_i^m} S_i^m \right\}$$

and

$$N(R_i) := \operatorname{diag}\{R_i, 0_{q_i}, \ldots, 0_{q_i}\}$$

where

$$\mathbf{A}_i = [\tilde{A}_i^0 + \tilde{B}_i\tilde{K}_i\tilde{C}_i \ \tilde{A}_i^1 \ \cdots \ \tilde{A}_i^m], \quad \mathbf{E}_i = [I_{q_i} \ 0_{q_i} \ \cdots \ 0_{q_i}]$$

Step 3) Use the results of Lemma 6.4 and the value obtained for σ_i in Step 2, to find α_i.

6.3.4 Switching algorithm

It is desired now to develop a switching control strategy. Suppose that the constants α_i, β_i, σ_i, λ_i, γ_{i_1}, γ_{i_2} and γ_{i_3} are all chosen such that Lemmas 6.2 and 6.3 both hold, and define $\bar{h} := \max\{\bar{h}_1, \ldots, \bar{h}_p\}$. The proposed switching scheme consists of two phases.

Phase 1: Finding the bound on the initial function. It is assumed that $\tilde{u}(t) = 0$ for $t \in [0, T]$, where T is any arbitrary positive (nonzero) constant, and $z(r) = 0$ for $r \in [-\bar{h}, 0]$. Define

$$\rho_j := \sup_{-\bar{h}_j \le r < 0} \frac{1}{\eta_j(r)} \| \int_0^T \Theta_j{}'(t,r) C_j{}' y(t) dt \|$$

Suppose that $\|\omega(t)\| \le \bar{b}$, and let the following auxiliary signals for $j \in \bar{\mathrm{p}}$ and $t \in [0, T]$ be defined

$$\dot{r}_j = \lambda_j r_j(t) + \gamma_{j2} \|\tilde{K}_j(\tilde{y}(t) - \tilde{D}_j y_{ref}(t))\| + \gamma_{j3}\bar{b}$$

with the initial condition $r_j(0) = 0$. Define also

$$\mu_j := \rho_j + \beta_j \bar{b} \tag{6.27}$$

If the plant is P_j, then it follows from Lemma 6.2 that

$$\max_{-\bar{h}_j \le r \le 0} \|\tilde{\phi}(r)\| \le \mu_j$$

Phase 2: Searching the gains. Let the control input be

$$\tilde{u}(t) = \tilde{K}_i\tilde{y}(t), \quad t \in (t_i, t_{i+1}]$$

Consider the following p auxiliary signals

$$\dot{r}_j(t) = \lambda_j r_j(t) + \gamma_{j_2}\|\tilde{u}(t) - \tilde{K}_j(\tilde{y}(t) - \tilde{D}_j y_{ref}(t))\| + \gamma_{j_3}\bar{b}$$

with $r_j(T^+) = r_j(T) + \gamma_{j_1}e^{\lambda_j T}\mu_j$, $j \in \bar{\text{p}}$, and let the filtered signal be given by

$$\dot{\tilde{r}}(t) = \tilde{\lambda}\tilde{r}(t) + (\lambda - \tilde{\lambda})\|\tilde{y}(t) - \tilde{D}y_{ref}(t)\|, \quad \tilde{r}(T) = 0$$

where $\lambda := \min\{\lambda_i : i \in \bar{\text{p}}\}$ and $\tilde{\lambda} < \lambda$. It is to be noted that since the matrix D_j is considered the same for all plant models, it is simply denoted by $\tilde{D}$ in the above equation. $r_j(t)$ gives an upper bound on the norm of the state for $t \geq T$, when the plant is P_j. Moreover, $\tilde{r}$ filters $\tilde{y} - \tilde{D}y_{ref}$ to obtain a smooth error signal. The switching instants are now recursively defined as follow: set $t_1 := T$, and for any $i \in \{2, \ldots, p+1\}$ define t_i as

$$\min\{t \geq t_{i-1} | \text{ there exists a time } \tilde{t} \in [T, t], \text{ for which } \tilde{r}(\tilde{t}) = \|\tilde{C}_{i-1}\|r_{i-1}(\tilde{t}) + \|\tilde{F}_{i-1}\|\tilde{b} + \epsilon\}$$

where ϵ is any arbitrary (small) positive number.

6.3.5 Properties of the proposed switching controller

The properties of the switching algorithm proposed in the previous subsection is now presented through Theorem 6.1 and Corollary 6.2. However, Lemma 6.5 needs to be derived first. This lemma is essential for the proof of Theorem 6.1.

Lemma 6.5. *Suppose that* $\tilde{\lambda} < \lambda \leq \lambda_j < 0$, $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\phi_1(.) \geq 0$, $\phi_2(.) \geq 0$,

$$\dot{v}_1(t) = \lambda_j v_1(t) + \phi_1(t), \quad v_1(0) \geq 0$$

$$\dot{v}_2(t) = \tilde{\lambda} v_2(t) + (\lambda - \tilde{\lambda})\phi_2(t), \quad v_2(0) = 0.$$

If $\phi_2(t) \leq \gamma_1 v_1(t) + \gamma_2$ *for* $t \geq 0$, *then* $v_2(t) \leq \gamma_1 v_1(t) + \gamma_2$ *for* $t \geq 0$ *as well.*

Proof: Set

$$\hat{e}(t) := \gamma_1 v_1(t) + \gamma_2 - v_2(t)$$

then

$$\dot{\hat{e}}(t) = \tilde{\lambda}\hat{e} + \underline{(\lambda - \tilde{\lambda})[-\phi_2(t) + \gamma_1 v_1(t) + \gamma_2]} + \underline{(\lambda_j - \lambda)\gamma_1 v_1(t)} + \underline{\gamma_1 \phi_1(t) - \lambda\gamma_2}$$

with $\hat{e}(0) = \gamma_1 v_1(0) + \gamma_2$. However, all of the underlined terms on the right side of the above equation are non-negative. Since $\hat{e}(0)$ is non-negative as well, it follows that $\hat{e}(t) \geq 0$ for $t \geq 0$. ■

Theorem 6.1. *Suppose that $y_{ref}(t)$ and $\omega(t)$ are bounded piecewise continuous signals, and that $\|\omega(t)\| \leq \bar{b}$ for $t \geq 0$. For any continuous initial function $\phi(r)$, $r \in [-h, 0]$, the closed-loop system under the proposed switching algorithm has the following properties:*

(i) the gain will eventually locks onto a controller in the set of $\{\tilde{K}_i : i \in \bar{p}\}$.

(ii) the state $\tilde{x}(t)$ will be bounded for all $t \geq 0$.

Proof: Let $y_{ref}(t)$, $\omega(t)$ be piecewise continuous signals. Assume that $\|\omega(t)\| \leq \bar{b}$ for $t \geq 0$. Let $\phi(r)$ be any arbitrary initial function. Suppose now that the real plant is P_j, $j \in \bar{p}$. It follows from Lemma 6.2 and the definition of μ_j in (6.27), that $\max_{-\bar{h}_j \leq r \leq 0} \|\tilde{\phi}(r)\| \leq \mu_j$. Now, using the result of Lemma 6.3

$$\|\tilde{x}(t)\| \leq \gamma_{j_1} e^{\lambda_j t} \max_{-\bar{h}_j \leq r \leq 0} \|\tilde{\phi}(r)\| + \int_0^t e^{\lambda_j (t-\tau)} [\gamma_{j_2} \|\tilde{u}(\tau) - \tilde{K}_j(\tilde{y}(\tau) - \tilde{D}_j y_{ref}(\tau))\| + \gamma_{j_3} \|\tilde{\omega}(\tau)\|] d\tau$$

for $t > T$. Consequently,

$$\|\tilde{x}(t)\| \leq \gamma_{j_1} \mu_j e^{\lambda_j t} + \int_0^t e^{\lambda_j (t-\tau)} [\gamma_{j_2} \|\tilde{u}(\tau) - \tilde{K}_j(\tilde{y}(\tau) - \tilde{D}_j y_{ref}(\tau))\| + \gamma_{j_3} \bar{b}] d\tau$$

Thus, for $t > T$

$$\|\tilde{x}(t)\| \leq r_j(t) \tag{6.28}$$

and hence

$$\|\tilde{y}(t) - \tilde{D}_j y_{ref}(t)\| \le \|\tilde{C}_j\| r_j(t) + \|\tilde{F}_j\|\tilde{b}$$

It follows from Lemma 6.5 that for $t > T$,

$$\tilde{r}(t) \le \|\tilde{C}_j\| r_j(t) + \|\tilde{F}_j\|\tilde{b}$$

Therefore, it can be concluded that the first property holds. It remains to show that $\tilde{x}(t)$ is bounded. From (6.28) it suffices to show that r_j is bounded. Let the final gain be $\tilde{K}_i$. Since $y_{ref}(t)$ and $\omega(t)$ are bounded piecewise continuous signals, $r_i(t)$ is also bounded. This leads to the boundedness of $\tilde{r}(t)$. On the other hand, for $t \ge T$

$$\int_T^t e^{\lambda_j(t-\tau)}(\lambda - \tilde{\lambda})\|\tilde{y}(\tau) - \tilde{D}y_{ref}(\tau)\|d\tau = \tilde{r}(t) + (\lambda_j - \tilde{\lambda})\int_T^t e^{\lambda_j(t-\tau)}\tilde{r}(\tau)d\tau$$

Thus,

$$\sup_{t>T}\int_T^t e^{\lambda_j(t-\tau)}\|\tilde{y}(\tau)\|d\tau < \infty$$

This results in the boundedness of $r_j(t)$. ■

The following three assumptions will be used to achieve zero tracking error for a pre-specified set of reference inputs and disturbances.

Assumption 6.3. *The entries of $y_{ref}(t)$ and $\omega(t)$ are assumed to be described by the following differential equation*

$$(.)^{(\nu)} + \xi_{\nu-1}(.)^{(\nu-1)} + \ldots + \xi_1(.)^{(1)} + \xi_0(.) = 0 \tag{6.29}$$

with independent initial conditions. Moreover, the roots of

$$s^\nu + \xi_{\nu-1}s^{\nu-1} + \ldots + \xi_1 s + \xi_0 = 0 \tag{6.30}$$

are assumed to be distinct and purely imaginary.

Assumption 6.4. *For every $i, l \in \bar{p}$, the matrix $\tilde{K}_l$ has the property that the roots of the following equation do not lie on the imaginary axis*

$$\Delta_i(s) = \det(sI - \tilde{A}_i^0 - \tilde{B}_i\tilde{K}_l\tilde{C}_i - \sum_{j=1}^m \tilde{A}_i^j e^{-h_i^j s}) = 0$$

Condition of Assumption 6.2 implies that $\tilde{K}_l$ is designed such that the error signal, when the plant is $\tilde{P}_i$, can be written in terms of strictly growing or strictly decaying exponentials and not sinusoids or constants.

Assumption 6.5. *Suppose that controllers K_i, $i \in \bar{p}$, are designed such that each one of them solves the servomechanism problem for the corresponding plant P_i and a certain class of reference inputs and disturbance signals (e.g. constant signals). In other words, assume that K_i is designed in such a way that exact tracking of y_{ref} is achieved in the presence of external disturbances in the system, for a certain class of reference inputs and disturbance signals. In this case, it is required first to design a servo-compensator for each plant model (like the finite-dimensional case [24]). This is, in fact, accomplished by introducing open-loop poles equal to the roots of (6.30). One should then find a stabilizing feedback controller for the overall system (consisting of the plant and the corresponding servo-compensator). It can be shown for both finite and infinite-dimensional cases that such a controller configuration has the property that the zeros of each entry in the transfer function matrix from the reference and the disturbance to the error includes all the roots of (6.30).*

Corollary 6.2. *Consider the system (6.1) and suppose that the conditions of Assumptions 6.3, 6.4 and 6.5 hold. In addition, assume that $\|\omega(t)\| \leq \bar{b}$ for $t \geq 0$, where $\bar{b}$ is, in fact, a known bound on the norm of disturbance. Then, for any continuous initial function $\phi(r)$, $r \in [-h, 0]$, the error signal $e(t)$ resulted by applying the proposed switching scheme approaches zero as $t \to \infty$.*

Proof: Let y_{ref} and ω be such that any of its elements satisfies (6.29). It is known from Theorem 6.1 that the gain will eventually becomes fixed and $\tilde{x}$ and $\tilde{y}$ will always remain bounded. By Assumption 6.4, the final control gain has the property that the corresponding LTI closed-loop system has no poles on the imaginary axis. In addition, it follows from Assumption 6.5 that $E(s)$ (the Laplace transform of

$e(t)$) does not have any pole at the roots of (6.30). Therefore, the error is a sum of weighted exponentials corresponding to the poles of the closed-loop system, none of which lie on the imaginary axis. Then, it follows from the boundedness of e (which follows from the boundedness of $\tilde{y}$) that only the decaying exponential terms have non-zero weights, and hence $e(t) \to 0$ as $t \to \infty$. ■

Remark 6.8. *It is to be noted that the system may lock onto a stabilizing controller, which is not necessarily the high-performance controller designed for the corresponding plant model. However, to avoid this problem, one may use an approach similar to [39] to design the controllers such that each one stabilizes only one of the plant models.*

6.4 Numerical Examples

Example 6.1. *Consider the following single-input single-output system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + x_2(t-h_0) + u(t) + \omega(t) \\ \dot{x}_2(t) &= -2x_1(t) - 3x_2(t) \\ y(t) &= cx_1(t) \end{aligned} \tag{6.31}$$

A family of three plant models is considered as follows

$$P_1: \quad h_0 = 0.1, \ c = 1$$

$$P_2: \quad h_0 = 0.6, \ c = 1$$

$$P_3: \quad h_0 = 0.1, \ c = -1$$

y_{ref} *is assumed to be a square wave of magnitude* 1 *and period* 4 *sec. Let the following PI controllers be used to achieve reference tracking and disturbance rejection for piecewise constant input signals*

$$K_1: \quad \dot{z} = e, \ u = 12z + 6e$$

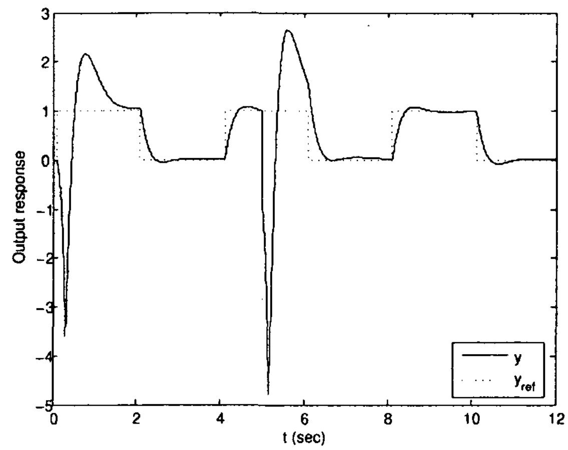
$$K_2: \quad \dot{z} = e, \ u = 16z + 8e$$

$$K_3: \quad \dot{z} = e, \ u = -12z - 6e$$

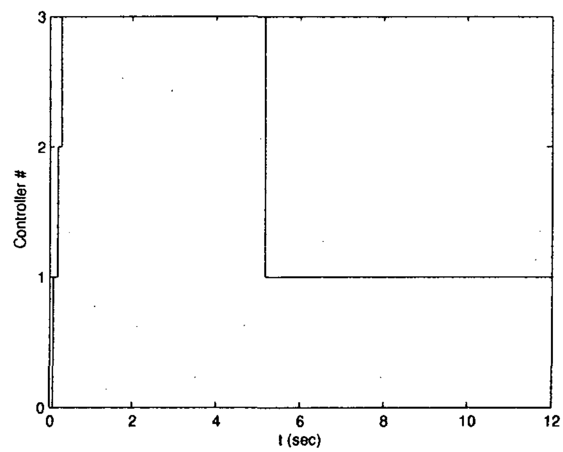
Suppose that $\omega(t) = 0$, and choose T in Phase 1 equal to 0.1 sec. This implies that for $t \in [0, 0.1]$ (i.e., during Phase 1), the control signal applied to the system is identical to zero, and immediately after that different controllers are examined. The system will first switch to the controller K_1. Since this is not the stabilizing controller for P_3, the error will hit the upper bound signal, and thus the system will switch to the controller K_2. This will also destabilize the plant and eventually at $t = 0.29$ sec, the system will switch to the stabilizing controller K_3. Figure 6.1(a) depicts the output response of the system and Figure 6.1(b) gives the switching sequence. It is to be noted that in the above switching sequence, the plant will examine two destabilizing controllers K_1 and K_2 but as it can be seen from Figure 6.1(a), the resultant transient magnitude is about 3.5 at approximately $t = 0.29$ sec which is good.

Assume now that at $t = 5$ sec, the plant changes from P_3 to P_1. As a result, the error will hit its corresponding upper bound signal in about 0.15 sec and the system will then switch to K_1, which is the stabilizing controller for P_1. It is to be noted that one of the shortcomings of most switching control schemes is the large magnitude of the transient response. One can use the multi-layer switching mechanism introduced in [58, 154] to improve the transient response.

Example 6.2. *In metal cutting processes, cutting tool vibrations and chatter might lead to poor quality of the final products. It is well-known that among various sources of chatter, regenerative chatter is the most detrimental [97]. Hence, it is very important to suppress and control the adverse effects of regenerative chatter. Various techniques are proposed in the literature for regenerative chatter suppression including passive and active methods [147, 152]. Active chatter absorbers result in a better performance and robustness, in general. In this example, it is shown how the proposed switching supervisory control can be utilized to actively control regenerative vibrations in metal cutting tools which might run in different operating points.*



(a)



(b)

Figure 6.1: (a) Output response for the system (6.31), using the proposed switching scheme (b) Switching control sequence for the numerical example, using the proposed scheme.

Consider a lumped single degree of freedom model for the metal cutting process. This model can be described by the following retarded time-delay differential equation [49]

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k+f}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{f}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{f}{m} \end{bmatrix} (u(t) + \xi(t)) \tag{6.32}$$

$$y(t) = x_1(t)$$

where ξ is the input disturbance, and $u(t)$ is the input feed rate which should be determined such that the vibration of $y(t)$, the tool end-point position, is suppressed with acceptable transient response. Furthermore, τ denotes the state delay and is equal to $\tau = 2\pi/\Omega$, where Ω is the rotational speed of the workpiece. The system parameters m, c, k and f in (6.32) are given by

$$m = 20\ kg, \qquad c = 10^3\ Ns/m, \qquad k = 5\times10^6\ N/kg, \qquad f = 5\times10^6 b\ N/kg \tag{6.33}$$

where b is the cutting depth. All parameters of the system except Ω and b are assumed to be fixed. The operating point of the system, on the other hand, depends on the values of Ω and b. Four typical operating points for the system are given by

$$1 : b = 0.02\ m, \quad \Omega = 2000\ rpm \tag{6.34a}$$

$$2 : b = 0.02\ m, \quad \Omega = 200\ rpm \tag{6.34b}$$

$$3 : b = 0.03\ m, \quad \Omega = 2000\ rpm \tag{6.34c}$$

$$4 : b = 0.04\ m, \quad \Omega = 2000\ rpm \tag{6.34d}$$

(It is to be noted that the above set of parameters (6.33) and operating points (6.34) are within the typical range of values given in [6].) The above set of operating points constitute a family of four models $\{P_1, P_2, P_3, P_4\}$. It is to be noted that all four models are highly underdamped and oscillatory. Corresponding to each P_i, $i = 1, 2, 3, 4$,

a controller K_i is designed which tracks constant reference inputs, rejects constant disturbances and meets certain performance specifications. For this example, the following controllers are obtained using the robust servomechanism design method provided in [24], and the control design technique proposed by Theorem 1.4

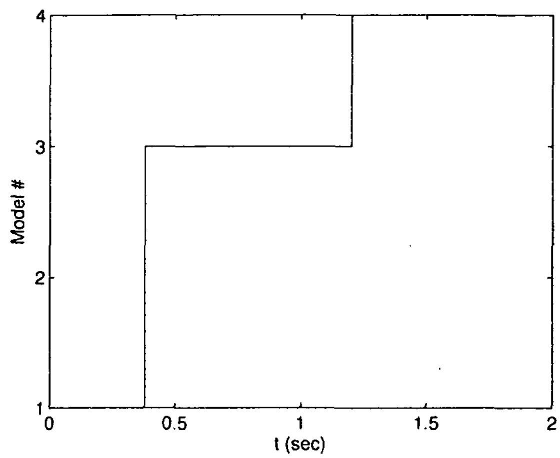
$$K_1 : \begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -8000 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 2.048 \\ -4877 \end{bmatrix} e(t) \\ u(t) = 7832 z_1(t) + 4877 z_2(t) + 1280 e(t) \end{cases}$$

$$K_2 : \begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -8000 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 2.048 \\ -5373 \end{bmatrix} e(t) \\ u(t) = 7856 z_1(t) + 5373 z_2(t) + 1920 e(t) \end{cases}$$

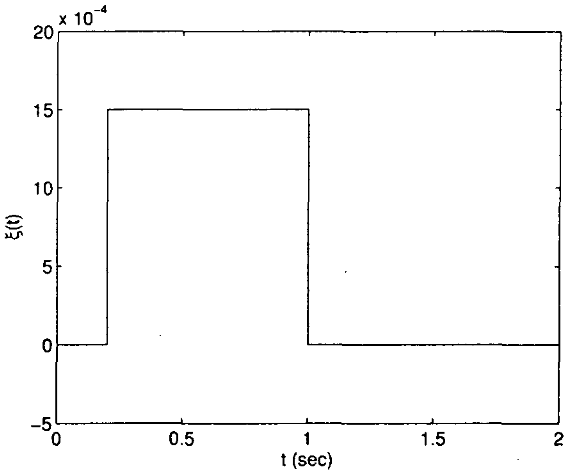
$$K_3 : \begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -8000 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1.024 \\ -3810 \end{bmatrix} e(t) \\ u(t) = 7856 z_1(t) + 3810 z_2(t) + 960 e(t) \end{cases}$$

$$K_4 : \begin{cases} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -8000 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 1.024 \\ -3379 \end{bmatrix} e(t) \\ u(t) = 5260 z_1(t) + 3379 z_2(t) + 853.3 e(t) \end{cases}$$

Let T (in the Phase 1 of the switching algorithm) be equal to 0.1 *sec, and the jump in the parameters occurs as depicted in Figure 6.2(a). Moreover, suppose that the disturbance signal ξ is the square wave shown in Figure 6.2(b). The results obtained for the proposed switching control method are demonstrated in Figures 6.3(a) and 6.3(b). The output response obtained for initial state $x(0) = [10^{-5} \;\; 0]'$ is depicted in Figure 6.3(a), and the switching instants are given in Figure 6.3(b). It can be seen from these figures that the proposed adaptive switching controller can find the correct* *controller and falsify the other candidates rapidly, in the presence of disturbance and sudden change of system parameters. Thus, the proposed switching supervisory controller succeeds in attenuating the vibration and maintaining the output chattering in an acceptable range.*

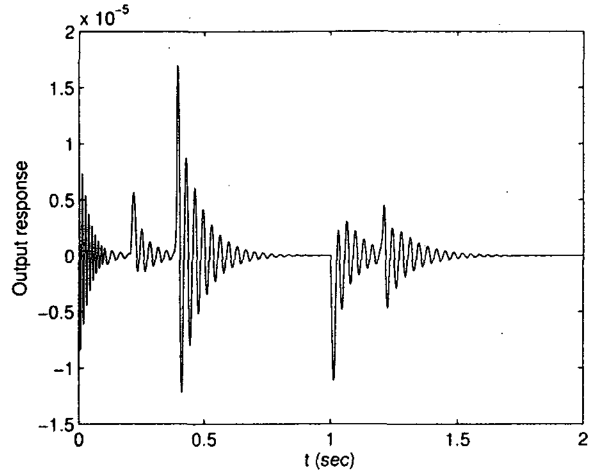


(a)

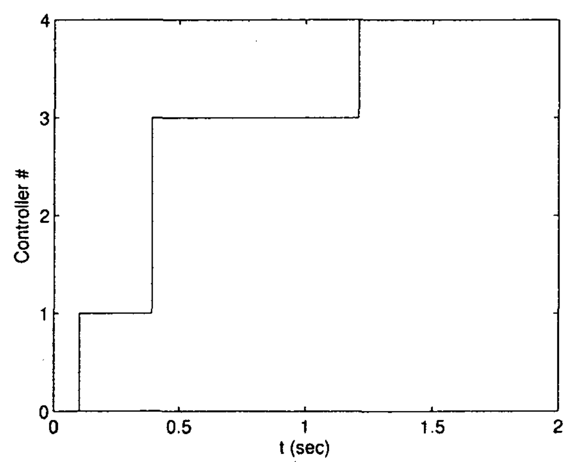


(b)

Figure 6.2: (a) Jump in the model parameters for the numerical example (b) Disturbance signal $\xi(t)$ for the numerical example.



(a)



(b)

Figure 6.3: (a) Output response for the system (6.32) with the operating parameters (6.34), using the proposed scheme (b) Switching control sequence for the system of Example 6.2, using the proposed scheme.

Chapter 7

Conclusions

7.1 Summary

The developed results in this thesis can be summarized as follows:

The problem of stabilization of linear time-invariant (LTI) time-delay interconnected systems using decentralized output feedback control is investigated in Chapter 2. It is assumed that the system is subject to the input/output and state commensurate delays. The notion of decentralized fixed modes (DFM) introduced for finite-dimensional LTI systems in [149] is extended to the class of time-delay systems with known fixed delays, and a necessary and sufficient condition is obtained consequently for the stabilizability of this type of systems under decentralized finite-dimensional LTI output feedback controllers. The existing results on decentralized stabilization of LTI time-delay systems provide sufficient conditions only [53, 125]; this substantiates the importance of the results presented in this chapter.

In Chapter 3, a near-optimal decentralized servomechanism controller is designed for a LTI hierarchical interconnected system. The controller obtained performs satisfactorily with respect to a prescribed LQ cost function, and is capable of rejecting unmeasurable external disturbances of known dynamics. The case when the system is subject to perturbation and input delay is also investigated, and necessary and sufficient conditions to achieve the stability and disturbance rejection for the closed-loop system are obtained. The controller obtained relies on the information of every individual subsystem about the overall system, and since this information is inexact in practice, a procedure is presented to assess the degradation of the performance of the decentralized control system as a result of the erroneous information.

In Chapter 4, the decentralized implementation of a given centralized controller for an interconnected LTI system with a particular focus on spacecraft formation is investigated. The objective is to meet the design specifications with a reduced communication cost. A decentralized control law is first derived from a given centralized controller based on a recently proposed technique in the literature. Then, stability and robust stability of the formation under the proposed control law are studied, and the closeness of the resultant decentralized controller to the reference centralized controller is evaluated. The main advantage of the proposed decentralized controller is the elimination of the communication links between the local controllers of different spacecraft. However, this can potentially have a negative impact on the output performance in the presence of uncertainties, mismatch of the beliefs of different local controllers about the system model, etc. To address this trade-off between the communication cost and the robust performance, a predictive control scheme is proposed as the main contribution of this chapter, to implement the controller obtained. The resultant decentralized model predictive control strategy constitutes rather weak communication links between the local controllers as the information exchange can be carried out periodically with a low rate. The results obtained can be extended to the case of LTI time-delay systems to account for delay in the transmission and processing of data.

Chapter 5 deals with the stability analysis and the control design problem for finite-dimensional LTI interconnected systems with a given communication topology using decentralized overlapping controllers. The subsystems are assumed to be subject to input disturbances with finite energy (or power). Furthermore, the information flow among different agents (local controllers) is subject to transmission delay. First, some rank conditions are given which are necessary for the existence of an overlapping output feedback controller. Then, a LMI-based design method is proposed for solving H_∞ control synthesis problem to attenuate the effect of disturbance in the regulated output.

Chapter 6 introduced an adaptive switching control approach for highly uncertain retarded time-delay LTI systems. A switching control scheme is proposed to stabilize and regulate the system, as an extension of the method introduced in [98] for finite dimensional LTI systems. In this switching control scheme, it is assumed that the plant can be described by a family of retarded time-delay LTI models. It is also assumed that a set of high-performance controllers is available, so that the actual plant model can be stabilized and regulated by at least one controller in this set.

7.2 Suggestions for Future Work

In what follows, some of the possible extensions to the results obtained in this thesis as well as some relevant problems for future study are presented.

- *Structurally decentralized fixed modes:* Regarding the problem investigated in Chapter 2, one may consider perturbation in the modeling parameters and time-delay to extend the notion of *structurally fixed modes* introduced in [134] to this class of systems.

- *Mixed H_2 and H_∞ decentralized overlapping output feedback control design for interconnected time-delay systems:* As discussed earlier in the thesis, it is often desirable to find a distributed control structure for a LTI interconnected system in which local control agents can communicate with each other according to a given communication topology while the information flow among different components in the system is subject to transmission delay. Now, it would be very helpful to design local dynamic output feedback controllers for the subsystems in order to satisfy combined H_2 and H_∞ criteria for the overall system simultaneously.

- *Adaptive switching control for interconnected time-delay systems:* As a natural extension of the results in Chapter 6, the jump in both time-delay and modeling parameters of an interconnected system can be considered. A decentralized switching controller can be designed accordingly to handle these types of large uncertainties.

The author hopes that this manuscript could contribute to the area of distributed cooperative control systems. To the author's knowledge, the advances in this field help develop new technologies which can provide a safer and more comfortable life for mankind.

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