On Fontaine Sheaves

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A Thesis
In the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of Doctor of Philosophy (Mathematics) at
Concordia University
Montréal, Québec, Canada

September, 2009

Your file Votre référence
ISBN: 978-0-494-63432-5
Our file Notre référence
ISBN: 978-0-494-63432-5

ABSTRACT

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In this thesis we focus our research on constructing two new types of Fontaine sheaves, $\mathbb{A}^{\nabla}_{max}$ and $\mathbb{A}_{max}$ in the third chapter and the fourth one respectively and in proving some of their main properties, most important the localization over small affines. This pair of new sheaves plays a crucial role in generalizing a comparison isomorphism theorem of Faltings for the ramified case.

In the first chapter we introduce the concept of p-adic Galois representation and provide and analyze some examples.

The second chapter is an overview of the Fontaine Theory. We define the concept of semi-linear representation and study the period rings introduced by Fontaine while understanding their importance in classifying the p-adic Galois representations.

ACKNOWLEDGEMENTS

I dedicate this thesis to my beloved teachers.

Contents

List of Symbols

$\mathbb{Z}_p$	the ring of p-adic integers
$\mathbb{Q}_p$	the field of p-adic numbers
$\bar{\mathbb{Q}}_p$	a fixed algebraic closure of $\mathbb{Q}_p$
$\mathbb{C}_p$	the p-adic completion of $\bar{\mathbb{Q}}_p$
$\mathbb{F}_p$	the finite field with p elements
$Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$	the Galois group of $\bar{\mathbb{Q}}_p$ over $\mathbb{Q}_p$
GL_d	the general linear group
χ	the cyclotomic character
μ_p	the group of p-th roots of unity
$\mathbb{G}_m$	the multiplicative group scheme
$T_p\mathbb{G}_m$	the p-adic Tate module of $\mathbb{G}_m$
T_pE	the p-adic Tate module of the elliptic curve E
$\mathcal{O}_K$	the ring of integers of the p-adic field K
k	the the residue field of K
$\overline{K}$	an algebraic closure of K
$\mathbb{C}_K$	the completion of $\overline{K}$
G_K	the Galois group of $\overline{K}$ over K

$\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G}_K)$	the category of p-adic representations of G_K
$\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(\mathrm{G}_K)$	the category of Hodge-Tate representations of G_K
$\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\mathrm{G}_K)$	the category of de Rham representations of G_K
$\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\mathrm{G}_K)$	the category of semi-stable representations of G_K
$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\mathrm{G}_K)$	the category of crystalline representations of G_K
$\mathbb{W}(R)$	the ring of Witt vectors with coeficients in R
Vec_K	the category of finite dimensional K-vector spaces
Gr_K	the category of graded K-vector spaces
$Gr_{K,f}$	the category of graded K-vector spaces of finite dimension over K
Fil_K	the category of finite dimensional filtered K-vector spaces
MF^{φ}_K	the category of filtered φ-modules over K
$MF^{\varphi,N}_K$	the category of filtered φ-modules over K endowed with a monodromy operator N
X	smooth proper scheme over $\mathcal{O}_K$
X_K	the generic fiber of X
X_k	the special fiber of X
$X_{\overline{K}}$	the geometric generic fiber of X
X^{et}	the small étale site on X
$\mathrm{Sh}(X^{et})$	the category of sheaves of abelian groups on X^{et}
H^i_{et}	the i-th étale cohomology group
H^i_{dR}	the i-th de Rham cohomology group
H^i_{cris}	the i-th crystalline cohomology group

$\mathfrak{X}$ Falting's Grothendieck topology on X

$\overline{\eta}$ a geometric generic point of X

Introduction

The general aim of this thesis is to study p-adic local Galois representations. More precisely let us fix a prime integer $p > 0$, a finite extension K of $\mathbb{Q}_p$, an algebraic closure of K, $\overline{K}$ and let us denote by G_K the Galois group of $\overline{K}$ over K. Then a p-adic representation of G_K is a finite dimensional $\mathbb{Q}_p$-vector space V on which G_K acts linearly and continuously. In chapter 1 we give an alternative way of thinking about these objects as well as many examples of such representations coming from algebraic geometry.

The category of p-adic representations of G_K which we denote $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ has a filtration by sub-categories as follows:

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_K) \subset \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \subset \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \subset \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K) \subset \mathrm{Rep}_{\mathbb{Q}_p}(G_K),$$

where the upper-scripts cris, st, dR, HT refer to crystalline, semi-stable, de Rham and Hodge-Tate representations. These are defined using Fontaine's rings $B_{\mathrm{cris}}, B_{\mathrm{st}}, B_{\mathrm{dR}}, B_{\mathrm{HT}}$ and the respective functors: $D_{\mathrm{cris}}, D_{\mathrm{st}}, D_{\mathrm{dR}}, D_{\mathrm{HT}}$.

The Fontaine rings and functors are described in chapter 2, where many examples of representations and their images under Fontaine's functors are given. We should point out that these examples are known and we only worked out some of the details of the respective calculations. In fact "the comparison isomorphisms", i.e. theorems comparing p-adic étale cohomology of the geometric generic fiber of a smooth, proper, connected scheme X over K to other cohomology theories associated to X allows one to decide the nature of the G_K representations $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$. The cohomology theories we refer to are: the Hodge cohomology of X, the de Rham cohomology of X, the log crystalline cohomology of the special fiber of a semi-stable, proper model of X over $\mathcal{O}_K$ (if X has semi-stable reduction) or the crystalline cohomology of the special fiber of a smooth proper model of X over $\mathcal{O}_K$ (if X has good reduction). For example a consequence of the crystalline comparison isomorphism is that if X has good reduction over $\mathcal{O}_K$ then $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation for all $i \geq 0$. The comparison isomorphisms (for trivial coefficients) are stated in chapter 2.

Recently, in [AI1], a new proof of the crystalline comparison isomorphism (with non-trivial coefficients) for smooth, proper connected schemes X over K with good reduction was given in the case K is unramified over $\mathbb{Q}_p$. Our work is an attempt to generalize these results for the case when the ramification degree of K is larger then 1. For this we use Faltings's topology $\mathfrak{X}_{\overline{K}}$ associated to X and a smooth, proper model of it and construct new Fontaine sheaves of rings on this topology. The definition of Faltings's topology, which is a Grothendieck topology, is recalled in chapter 3. Moreover, for all $n \geq 1$ we construct in chapter 3 a family of sheaves on $\mathfrak{X}_{\overline{K}}$, $(\mathbb{A}^{\nabla}_{\text{max,n}})_{n\geq 1}$ and in chapter 4 the family of sheaves $(\mathbb{A}_{\text{max,n}})_{n\geq 1}$. We also study the properties of these sheaves of rings in these chapters. For the moment we have only constructed these sheaves in the case K unramified over $\mathbb{Q}_p$ but it is possible to construct them even in the case when K is ramified. These rings will be used in sequel work to define a Riemann-Hilbert correspondence between p-adic locally constant sheaves on X and F-isocrystals on the special fiber of the fixed smooth model of X over $\mathcal{O}_K$.

Chapter 1

p-Adic Galois representations

Let $\mathbb{Q}_p \subset \bar{\mathbb{Q}}_p \subset \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ and put $G := Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Definition 1.0.1. A p-adic representation of G is a finite dimensional $\mathbb{Q}_p$-vector space V, with a continuous linear action $\rho : G \to Aut(V)$. By continuity one understands that the action map:

$$G \times V \to V \text{ sending } (\sigma, v) \to \sigma v$$

is continuous. The category of such representations is denoted $\mathrm{Rep}_{\mathbb{Q}_p}(G)$.

To better understand the notion of continuity of ρ choose a basis $\mathrm{e} := \{e_1, e_2, ..., e_d\}$ of V. For any $\sigma \in G$ we have that:

$$\sigma e_i = \sum_{1 \leq j \leq d} a_{ji}(\sigma) e_j.$$

Consider now the matrix $A(\sigma) := (a_{ij}(\sigma)) \in GL_d(\mathbb{Q}_p)$ ($A(\sigma)$ is invertible since $\sigma \in G$). We then have a non-canonical isomorphism of groups: $Aut(V) \cong GL_d(\mathbb{Q}_p)$ via the map $\sigma \to A(\sigma)$. Via the above isomorphism one extends the action

$G \to Aut(V) \to GL_d(\mathbb{Q}_p)$ and we still denote it ρ. One obtains:

$$\rho : G \to GL_d(\mathbb{Q}_p);\ \rho(\sigma) = A(\sigma).$$

On one hand note that $GL_d(\mathbb{Q}_p) \subset \bar{\mathbb{Q}}_p^{d^2}$ and since the latest is a topological space with the product topology induced by the p-adic metric on $\bar{\mathbb{Q}}_p$, one can endow $GL_d(\mathbb{Q}_p)$ with the subspace topology.

On the other hand, $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a profinite topological group. We obtain that ρ is a map between topological groups and so the notion of continuity is clear. ρ is a continuous homomorphism. We have that for $\sigma, \tau \in G$, $\rho(\tau\sigma) = \rho(\tau \circ \sigma) = \rho(\tau) \cdot \rho(\sigma)$ where the latest product is the multiplication of matrices in $GL_d(\mathbb{Q}_p)$.

If V is a finite dimensional $\mathbb{Q}_p$-vector space of basis $\{e\}$ then define $\rho_e(\sigma) \in GL_d(\mathbb{Q}_p)$ such that $\sigma e = \rho_e(\sigma)e$. Remark that the map $\rho = \rho_e$ depends on the basis e of V: if e' is another basis, $e = M \cdot e'$ for some $M \in GL_d(\mathbb{Q}_p)$ (the change of basis matrix), then

$$\rho_{e'}(\sigma) = M\rho_e(\sigma)M^{-1} \qquad (*)$$

since $\sigma e = \sigma(Me') = M\sigma e' = M\rho_{e'}(\sigma)e' = M\rho_{e'}(\sigma)M^{-1}$ (for the second equality one uses the fact that $M \in GL_d(\mathbb{Q}_p)$ and that $\sigma_{|\mathbb{Q}_p} = id$).

We say that ρ_e and $\rho_{e'}$ are conjugate. More precisely, two continuous homomorphisms of topological groups $\rho, \rho' : G \to GL_d(\mathbb{Q}_p)$ are equivalent $\rho \sim \rho'$ if there exists an invertible matrix M such that for every $\sigma \in G$ the equation $(*)$ holds. One can easily see that " $\sim$ " is an equivalence relation.

We have an equivalence between the following sets:

$\{V \mid V$ is a p-adic representation$\}/\,iso \rightleftarrows \{\rho : G \to GL_d(\mathbb{Q}_p) \mid \rho$ is a continuous homomorphism$\}/\sim \qquad (**)$

We've just seen the implication from left to right while vice-versa, we can associate to every continuous homomorphism ρ the vector space $V' = \mathbb{Q}_p^d$ and define the G-action on V' as: $\sigma x = \rho(\sigma)(x_1, x_2, ..., x_d)^t$ for $\sigma \in G$ and $x = (x_1, x_2, ..., x_d)^t \in V'$. Now, if we start with a p-adic representation V with the continuous action ρ, we get a new p-adic representation V' according to the above construction. We need to prove that $V \cong V'$ as G-representations in other words that there exists an isomorphism $f : V \to V'$, $\mathbb{Q}_p$-linear such that $f(\sigma v) = \sigma f(v)$ for every $\sigma \in G$ and every $v \in V$ (i.e. f is G-equivariant). For this, we choose f to be the application sending the basis $e = \{e_1, ..., e_d\}$ into the canonical basis of $\mathbb{Q}_p^d$ i.e. $f(e_1) = (1, 0, ..., 0)$, $f(e_2) = (0, 1, ..., 0)$, etc.

In this way we obtain an equivalent definition of the p-adic representations.

Remark 1.0.2. If K is a finite field extension of $\mathbb{Q}_p$ one works similarly for $Gal(\bar{\mathbb{Q}}_p/K)$.

Also note that if $\rho : G \to GL_d(\mathbb{Q}_p)$ is continuous then $Im(\rho)$ is compact since G is compact. It is known that $GL_d(\mathbb{Z}_p)$ is a maximal compact subgroup of $GL_d(\mathbb{Q}_p)$ and that any other maximal compact subgroup of $GL_d(\mathbb{Q}_p)$ is conjugate to $GL_d(\mathbb{Z}_p)$. It follows that up to conjugation one can factor ρ as:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & GL_d(\mathbb{Q}_p) \\ & \searrow & \uparrow {\scriptstyle i} \\ & & GL_d(\mathbb{Z}_p) \end{array}$$

where $i : GL_d(\mathbb{Z}_p) \to GL_d(\mathbb{Q}_p)$ is the inclusion map.

Consider now the $\mathbb{Q}_p$-vector space V, of finite dimension d, with its continuous linear G-action and denote by $\{e\}$ a basis of it. Let L be the free $\mathbb{Z}_p$-submodule of V generated by e, so we have that $L \subset V$ and $\sigma L \subset L$ possibly after conjugating ρ for all $\sigma \in G$. Since $L \cong \mathbb{Z}_p^d$ we have that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V$ and $L/pL \cong \mathbb{F}_p^d$ so one gets a representation of G on a $\mathbb{F}_p$-vector space, namely L/pL.

We analyze now some examples of p-adic representations.

1.1 The Tate module of $\mathbb{G}_m$

Definition 1.1.1. $\mathbb{G}_m$ is the algebraic group defined by the set $\mathbb{A}^1 - \{0\}$ with the multiplication map $m : (\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^1 - \{0\}) \to \mathbb{A}^1 - \{0\}$ and inverse $i : \mathbb{A}^1 - \{0\} \to$

$\mathbb{A}^1 - \{0\}$ defined by $m(x,y) = xy$ and $i(x) = x^{-1}$ respectively. (Recall that a variety A is an algebraic group if one has morphisms $m : A \times A \to A$ and $i : A \to A$ which make the points of A into an abelian group).

Denote by $\mathbb{G}_m[p^n](\bar{\mathbb{Q}}_p)$ the subgroup of p^n-torsion points over $\bar{\mathbb{Q}}_p$. We have that $\mathbb{G}_m[p^n](\bar{\mathbb{Q}}_p) = \mu_{p^n}(\bar{\mathbb{Q}}_p)$, where $\mu_{p^n}(\bar{\mathbb{Q}}_p) := \mu_{p^n} = \{x \in \bar{\mathbb{Q}}_p \mid x^{p^n} = 1\}$ is the group of p^n-th roots of unity in $\bar{\mathbb{Q}}_p$.

Via this remark, $\mathbb{G}_m[p^n](\bar{\mathbb{Q}}_p)$ is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank 1. In order to prove this, fix a primitive p^n-th root of unity, say ζ. Then every element $\alpha \in \mu_{p^n}$ can be uniquely written as $\alpha = \zeta^j$ for some $j \in \mathbb{Z}/p^n\mathbb{Z}$ and $\{\zeta\}$ is a basis of μ_{p^n}. Then, since $\mathbb{Z}/p^n\mathbb{Z} \cong \mu_{p^n}$ as abelian groups via the map sending $j \to \zeta^j$, one defines the $\mathbb{Z}/p^n\mathbb{Z}$-module structure on μ_{p^n} via the action $j * \zeta := \zeta^j$.

Now, G acts on μ_{p^n} as follows: for every $\sigma \in G$ and $\varepsilon \in \mu_{p^n}$ one has that $\sigma(\varepsilon) \in \mu_{p^n}$ since

$$(\sigma(\varepsilon))^{p^n} = \sigma(\varepsilon^{p^n}) = \sigma(1) = 1$$

and μ_{p^n} becomes a G-representation. Since σ is an automorphism and ζ is primitive then also $\sigma(\zeta)$ is primitive so $\sigma(\zeta) \in \mu_{p^n} - \mu_{p^{n-1}}$, $\sigma(\zeta) = \zeta^{a_\sigma}$ with $a_\sigma \in (\mathbb{Z}/p^n\mathbb{Z})^*$. Hence we get a (continuous) homomorphism of groups:

$$\chi_n : G \to (\mathbb{Z}/p^n\mathbb{Z})^*$$

defined by $\chi_n(\sigma) = a_\sigma$ such that $\sigma(\zeta) = \zeta^{a_\sigma}$.

In order to prove that χ_n is continuous, note that since the topology of $(\mathbb{Z}/p^n\mathbb{Z})^*$ is discrete, it is enough to check that $ker(\chi_n)$ is open. We have that $ker(\chi_n) = \{\sigma \in G \mid \chi_n(\sigma) = 1\} = \{\sigma \in G \mid \sigma(\zeta) = \zeta\} = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta))$ and since

$$Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)/Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta)) \cong Gal(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p)$$

we obtain that $ker(\chi_n)$ is of finite index. Since it is also closed, it follows that it is open ([Ro], 3.3).

One defines now ***the Tate module*** of $\mathbb{G}_m$, $T_p\mathbb{G}_m$:

$$T_p\mathbb{G}_m = T_p\mu := \varprojlim \mu_{p^n} = \{(\alpha_0, \alpha_1, ...) \mid \alpha_i \in \mu_{p^i} \text{ and } \alpha_{i+1}^p = \alpha_i, \forall i \geq 0\}$$

where the projective limit is taking with respect to the Frobenius morphism: $\mu_{p^{i+1}} \to \mu_{p^i}$ sending $\alpha \to \alpha^p$.

Since μ_{p^n} is a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank 1 we have that $T_p\mu$ is a free $\mathbb{Z}_p$-module of rank 1 and consequently $T_p\mathbb{G}_m$ is a free $\mathbb{Z}_p$-module of rank 1, a generator for example being $\epsilon = (1, \zeta_1, \zeta_2, ...)$ where ζ_i is a primitive p^i-th root of unity and $\zeta_{i+1}^p = \zeta_i$. One obtains that $T_p\mu = \mathbb{Z}_p\epsilon$ and we have an action of G on $T_p\mu$ given by:

$$\sigma(\alpha_0, \alpha_1, ...) = (\sigma\alpha_0, \sigma\alpha_1, ...).$$

In particular, $\sigma\epsilon = (\sigma 1, \sigma\zeta_1, \sigma\zeta_2, ...) = (1, \zeta_1^{\chi_1(\sigma)}, \zeta_2^{\chi_2(\sigma)}, ...)$.

Recall that $\chi_i : G \to (\mathbb{Z}/p^i\mathbb{Z})^*$. These maps are compatible i.e. the diagram:

$$\begin{array}{ccc} G & \xrightarrow{\chi_i} & (\mathbb{Z}/p^i\mathbb{Z})^* \\ & \searrow^{\chi_{i+1}} & \downarrow \cdot p \\ & & (\mathbb{Z}/p^{i+1}\mathbb{Z})^* \end{array}$$

is commutative, hence we get a (continuous) homomorphism:

$$\chi = \varprojlim \chi_i : G \to \varprojlim(\mathbb{Z}/p^i\mathbb{Z})^* = \mathbb{Z}_p^*$$

called the ***Cyclotomic character***. Note that χ is continuous since it is a projective limit of continuous maps.

Then $\sigma(\epsilon) = \epsilon^{\chi(\sigma)} := \chi(\sigma) \cdot \epsilon$ (we write the action additively, it is a convention) and we have that:

$$T_p\mu = \mathbb{Z}_p\epsilon = \mathbb{Z}_p(1) = \mathbb{Z}_p$$

where by $\mathbb{Z}_p(1)$ we mean $\mathbb{Z}_p$ with G-action given by $\sigma x = \chi(\sigma)x$ for $\sigma \in G$ and $x \in \mathbb{Z}_p$.

1.2 The Tate module of an Elliptic Curve

Let $E/\mathbb{F}_p$ be an ordinary elliptic curve (i.e. $p \nmid a_p := 1+p-\#E(\mathbb{F}_p)$) and consider the subgroup of p^n-torsion points over $\bar{\mathbb{F}}_p$:

$$E[p^n](\bar{\mathbb{F}}_p) \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$$

which is a free $\mathbb{Z}_p/p^n\mathbb{Z}_p$-module of rank 1 (see [Sil, Corollary 6.4]).

Denote by $G_{\mathbb{F}_p}$ the absolute Galois group of $\mathbb{F}_p$ i.e. $Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. Let's look at the action of $G_{\mathbb{F}_p}$ on $E[p^n](\bar{\mathbb{F}}_p)$: if $P \in E[p^n](\bar{\mathbb{F}}_p)$, then since $[p^n]P = 0$, we have that

$$[p^n](\sigma P) = \sigma([p^n]P) = \sigma(0) = 0$$

for every $\sigma \in G_{\mathbb{F}_p}$.

As in the previous section, one defines the ***Tate module*** of E as:

$$T_pE := \varprojlim E[p^n](\bar{\mathbb{F}}_p) \cong \varprojlim \mathbb{Z}_p/p^n\mathbb{Z}_p = \mathbb{Z}_p$$

which is a free $\mathbb{Z}_p$-module of rank 1. And as before, one has a continuous action of $G_{\mathbb{F}_p}$, call it $\bar{\varphi}_E : G_{\mathbb{F}_p} \to GL_1(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$.

Note that we have a continuous surjection $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to G_{\mathbb{F}_p}$ and so, by composing it with $\bar{\varphi}_E$ we get a continuous homomorphism, φ_E:

$$\varphi_E : G \to \mathbb{Z}_p^* \text{ given by } \sigma x = \varphi_E(\sigma)x.$$

Further denote by $\mathbb{Z}_p(\varphi_E) := T_pE \cong \mathbb{Z}_p$ together with its G action.

Consider now the following exact sequence:

$$0 \to I \to G \to G_{\mathbb{F}_p} \to 0.$$

where $I = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{unr})$ is the inertia group. Remark that φ_E is unramified since for any $\gamma \in I$ we have that $\varphi_E(\gamma) = 1$ (by definition a character is unramified if it is trivial on the inertia group).

Note that for the cyclotomic character $\chi : G \to \mathbb{Z}_p^*$, which is totally ramified we have a factorization:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) & \longrightarrow & G_{\mathbb{F}_p} & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & & & \\ & & Gal(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) & & & & & & \end{array}$$

We have that for all $\sigma \in G$, $\sigma\epsilon = \epsilon^{\chi(\sigma)}$ and therefore $\chi(\sigma) = 1 \Longleftrightarrow \sigma(\epsilon) = \epsilon$. From this it follows that $ker(\chi) = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p(\zeta_{p^\infty}))$. So we get an isomorphism:

$$\chi : Gal(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^*.$$

In general, if $\varphi : G \to \mathbb{Z}_p^*$ is a continuous character, let's denote by $T := \mathbb{Z}_p(\varphi)$ the G-representation defined as previously by $\mathbb{Z}_p$ with the G-action $\sigma x = \varphi(\sigma)x$. If we want continuous unramified representations then we use the fact that $G_{\mathbb{F}_p} \cong \widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ which is a pro-cyclic group generated by the Frobenius automorphism $Fr : \bar{\mathbb{F}}_p \to \bar{\mathbb{F}}_p$ sending $x \to x^p$ hence φ is determined by $\varphi(Fr)$.

Finally, define $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p =: \mathbb{Q}_p(\varphi) \cong \mathbb{Q}_p$ with the G-action given by $\sigma x = \varphi(\sigma)x$, with $x \in \mathbb{Q}_p$.

1.3 Further examples

As previously denote by $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and remark that if K is a finite field extension of $\mathbb{Q}_p$ one works similarly for $Gal(\bar{\mathbb{Q}}_p/K)$.

1.3.1 Dimension one representations.

For this case the p-adic representations correspond to characters. We've seen in the section 1.1 the cyclotomic character, $\chi : G \to \mathbb{Z}_p^*$. This corresponds to the 1-dimensional representation:

$$\mathbb{Q}_p(1) = \mathbb{Q}_p(\chi) = \mathbb{Q}_p \text{ as vector space, with action given by } \sigma * x = \chi(\sigma)x,\ x \in \mathbb{Q}_p.$$

We have that $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \varprojlim \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

For $n \in \mathbb{Z}$ one defines the 1-dimensional representation:

$$\mathbb{Q}_p(n) = \mathbb{Q}_p(\chi^n) = \mathbb{Q}_p \text{ as vector space, with action given by } \sigma * x = \chi^n(\sigma)x.$$

where $\chi^n : G \to \mathbb{Z}_p^*$ is also a cyclotomic character and $\chi^n(\sigma) = (\chi(\sigma))^n$.

Remark now that if $\psi : G \to \mathbb{Z}_p^*$ is any continuous character then one defines similarly:

$$\mathbb{Q}_p(\psi) = \mathbb{Q}_p \text{ as vector space, with G-action: } \sigma * x = \psi(\sigma)x,\ x \in \mathbb{Q}_p.$$

Recall (from section 1.2) that ψ is unramified if for every $\sigma \in I \Longrightarrow \psi(\sigma) = 1$. We have that ψ factors as $\psi : G/I \to G_{\mathbb{F}_p}$ and moreover $G/I \cong G_{\mathbb{F}_p} \cong \widehat{\mathbb{Z}} = \langle Fr \rangle$. So, if one wants an unramified character it is enough to determine its value on the Frobenius automorphism Fr, $\psi(Fr) = a \in \mathbb{Z}_p^*$. We will then have $\psi(Fr^\alpha) = a^\alpha$ so ψ will be completely determined.

1.3.2 Dimension two representations.

Let $\rho : G \to GL_2(\mathbb{Q}_p)$ be a continuous homomorphism. Further, let $E/\mathbb{Q}_p$ be an elliptic curve and consider its Tate module:

$$T_pE := \varprojlim E[p^n](\bar{\mathbb{Q}}_p).$$

Since char$(\mathbb{Q}_p) = 0$, following [Sil, Prop. 7.1, Chapter 3] we have that T_pE is a free $\mathbb{Z}_p$-module of rank 2.

Consider now $V_E := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ which is a 2-dimensional representation over $\mathbb{Q}_p$. From the equivalence of sets $(**)$ from the first paragraph, V_E corresponds to a continuous homomorphism $\rho_E : G \to GL_2(\mathbb{Q}_p)$. We have that the determinant of this map is the cyclotomic character, $\det(\rho_E) = \chi$. To see this, take the composition of the following maps:

$$G \xrightarrow{\rho_E} GL_2(\mathbb{Q}_p) \xrightarrow{det} \mathbb{Q}_p^*$$

which one denotes $\det\rho_E : G \to \mathbb{Q}_p^*$. Clearly $\det\rho_E$ is a continuous character. It follows that $\mathbb{Q}_p(\det\rho_{\mathrm{E}})$ is 1-dimensional.

Note that we have the Weil pairing (bilinear, alternating, non-degenerate, Galois invariant (see [Si1, §8, Chapter 3])):

$$<,>: E[p^n] \times E[p^n] \to \mu_{p^n}$$

so we get a map (by using the universal property of the exterior product):

$$V_E \wedge V_E \to \mathbb{Q}_p(1)$$

sending $x \wedge y \to \langle x, y\rangle$ and obtain that $\mathbb{Q}_p(\det\rho_{\mathrm{E}}) = \mathrm{V_E} \wedge \mathrm{V_E} \cong \mathbb{Q}_{\mathrm{p}}(1) = \mathbb{Q}_{\mathrm{p}}(\chi)$ since $\dim_{\mathbb{Q}_{\mathrm{p}}}\mathbb{Q}_{\mathrm{p}}(\det\rho_{\mathrm{E}}) = \dim_{\mathbb{Q}_{\mathrm{p}}}\mathbb{Q}_{\mathrm{p}}(1)$.

Case 1. Suppose that $E/\mathbb{Q}_p$ is a Tate curve i.e. that the valuation of its j-invariant is negative, $v(j(E)) < 0$. Following [Si2, Theorem 5.3 (Tate), Chapter 5], this is equivalent to the case when E has split multiplicative reduction. Moreover, via [Si2, Theorem 5.3(a) (Tate), Chapter 5] there exists a unique $q_E \in \mathbb{Q}_p$ with $|q_E| < 1$ such that E is isomorphic over $\bar{\mathbb{Q}}_p$ with $E_{q_E} := \bar{\mathbb{Q}}_p^*/q_E^{\mathbb{Z}}$, where $q_E^{\mathbb{Z}} := \{q_E^n \mid n \in \mathbb{Z}\}$ is a discrete subgroup of $\bar{\mathbb{Q}}_p^*$. q_E is called the **Tate period**. The quotient $E_{q_E} := \bar{\mathbb{Q}}_p^*/q_E^{\mathbb{Z}}$ is an abelian group which admits a natural structure of G-module via the action on $\bar{\mathbb{Q}}_p^*$. So one has the isomorphism of G-modules:

$$E(\bar{\mathbb{Q}}_p) \cong \bar{\mathbb{Q}}_p^*/q_E^{\mathbb{Z}}$$

and one further obtains that:

$$E[p^n](\bar{\mathbb{Q}}_p) = \{[x] \in \bar{\mathbb{Q}}_p^*/q_E^{\mathbb{Z}} \mid [x]^{p^n} = [1]\}.$$

Note that if $\zeta^{(n)} \in \mu_{p^n} - \mu_{p^{n-1}}$ (i.e. $\zeta^{(n)}$ is primitive p^n-th root of unity) then so is $(\zeta^{(n)})^i$, $0 \le i \le p^n$. Moreover, put $q_E^{(n)} := q_E^{1/p^n} = \sqrt[p^n]{q_E}$ so $(q_E^{(n)})^{p^n} = q_E \in q_E^{\mathbb{Z}}$ and one obtains

$$E[p^n](\bar{\mathbb{Q}}_p) = \{(\zeta^{(n)})^i (q_E^{(n)})^j, 0 \le i < p^n, 0 \le j < p^n\}$$

which is isomorphic to a free $\mathbb{Z}/p^n\mathbb{Z}$-module of rank 2 with basis $\{\zeta^{(n)}, q_E^{(n)}\}$.

Remark that for every $\sigma \in G$, we have $\sigma(\zeta^{(n)}) = (\zeta^{(n)})^{\chi(\sigma)}$ and $\sigma(q_E^{(n)}) = q_E^{(n)}(\zeta^{(n)})^{a_\sigma}$ for some $a_\sigma \in \mathbb{Z}/p^n\mathbb{Z}$.

Fix now the basis $\{q_E^{(n)}, \zeta^{(n)}\}$. We then get a map:

$$\rho_{E,n} : G \to GL_2(\mathbb{Z}/p^n\mathbb{Z}) \text{ sending } \sigma \to \begin{pmatrix} 1 & 0 \\ a_\sigma & \chi_n(\sigma) \end{pmatrix}.$$

Recall that $T_pE := \varprojlim E[p^n](\bar{\mathbb{Q}}_p)$ and that $V_E := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. By passing now to the limit we obtain a map:

$$\rho_E : G \to GL_2(\mathbb{Z}_p) \subset GL_2(\mathbb{Q}_p) \text{ sending } \sigma \to \begin{pmatrix} 1 & 0 \\ a_\sigma & \chi(\sigma) \end{pmatrix}.$$

We also have that a_σ determines a map $a : G \to \mathbb{Z}_p$ sending $\sigma \to a_\sigma$.

Proposition 1.3.1. *a_σ is a 1-cocycle (i.e. $a_{\sigma\tau} = a_\sigma + \sigma * a_\tau$ for $\sigma, \tau \in G$).*

Proof. On one hand, by using the definition we have that $(\sigma\tau)(q_E^{(n)}) = q_E^{(n)}(\zeta^{(n)})^{a_{\sigma\tau}}$.

On the other hand,

$$\begin{aligned}(\sigma\tau)(q_E^{(n)}) &= \sigma(\tau(q_E^{(n)})) \\ &= \sigma(q_E^{(n)}(\zeta^{(n)})^{a_\tau}) \\ &= \sigma(q_E^{(n)})\sigma((\zeta^{(n)})^{a_\tau}) \\ &= q_E^{(n)}(\zeta^{(n)})^{a_\sigma}(\zeta^{(n)})^{a_\tau\chi(\sigma)} \\ &= q_E^{(n)}(\zeta^{(n)})^{a_\sigma + a_\tau\chi(\sigma)}.\end{aligned}$$

We obtain that $a_{\sigma\tau} = a_\sigma + a_\tau\chi(\sigma) = a_\sigma + \sigma * a_\tau$ (where $*$ is the action of G on $\mathbb{Q}_p(1)$).

□

Remark 1.3.2. One can also show that a_σ is a 1-cocycle by using the fact that ρ_E is a group homomorphism. For $\sigma, \tau \in G$, $\rho_E(\sigma\tau) = \rho_E(\sigma)\rho_E(\tau)$ is equivalent to

$$\begin{pmatrix} 1 & 0 \\ a_\sigma & \chi(\sigma) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ a_\tau & \chi(\tau) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{\sigma\tau} & \chi(\sigma\tau) \end{pmatrix}$$

and hence $a_{\sigma\tau} = a_\sigma + \chi(\sigma)a_\tau$.

One can easily prove that the following sequence:

$$0 \longrightarrow \frac{\mathbb{Z}}{p^n\mathbb{Z}}(\chi) \xrightarrow{\varphi} E[p^n] \xrightarrow{\psi} \frac{\mathbb{Z}}{p^n\mathbb{Z}} \longrightarrow 0$$

is an exact sequence of G-modules where $\varphi(1) := \zeta^{(n)}$, $\psi(\zeta^{(n)}) = 0$ and $\psi(q_E^{(n)}) = 1$. By taking now projective limit and after tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$ one obtains the exact sequence of G-modules:

$$0 \longrightarrow \mathbb{Q}_p(1) \xrightarrow{\varphi} V_E \xrightarrow{\psi} \mathbb{Q}_p \longrightarrow 0 \quad (*).$$

This further induce a long exact sequence of group cohomology:

$$0 \longrightarrow \mathbb{Q}_p(1)^G \longrightarrow V_E^G \longrightarrow \mathbb{Q}_p^G \xrightarrow{\delta} H^1(G, \mathbb{Q}_p(1)) \longrightarrow \cdots \qquad (**)$$

where $\delta(1) = [a]$, a being our Kummer cocycle determined by the fact that $\sigma(q_E^{(n)}) = q_E^{(n)}(\zeta^{(n)})^{a_\sigma}$.

Moreover, $\mathbb{Q}_p^G = \mathbb{Q}_p^{Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} = \mathbb{Q}_p$ and we claim that $\mathbb{Q}_p(1)^G = 0$ and also $V_E^G = 0$.

Firstly, take an element $x \in \mathbb{Q}_p(1)^G = \mathbb{Q}_p(\chi)^G$ hence $\sigma x = x$ and $\sigma x = \chi(\sigma)x$ for any $\sigma \in G$. By choosing now $\sigma \in G$ such that $\chi(\sigma) \neq 1$ we obtain that $\mathbb{Q}_p(1)^G = 0$.

We've seen that the elements $q_E^{(n)}$ and $\zeta^{(n)}$ form a basis of $E[p^n](\bar{\mathbb{Q}}_p)$ so that a basis of T_pE is given by $e := \varprojlim_n \zeta^{(n)}$ and $f := \varprojlim_n q_E^{(n)}$. This allows us to compute explicitly the Galois action on T_pE. For $\sigma \in G$ we then have:

$$\sigma e = \varprojlim_n \sigma(\zeta^{(n)}) = \chi(\sigma)e$$

and

$$\sigma f = \varprojlim_n \sigma(q_E^{(n)}) = \varprojlim_n q_E^{(n)}(\zeta^{(n)})^{a_\sigma} = f + a_\sigma e.$$

We also obtain that $\{e', f'\}$ is a basis of V_E where $e' := e \otimes 1$ and $f' := f \otimes 1$ and moreover that $\sigma f' = f' + a_\sigma e'$ and $\sigma e' = \chi(\sigma)e'$. By using a similar type of argument as in the proof of $\mathbb{Q}_p(1)^G = 0$ one also has that $V_E^G = 0$.

One obtains that the sequence (**) becomes:

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\delta} H^1(G, \mathbb{Q}_p(1)) \longrightarrow \cdots$$

and that further (*) is non-split as an extension of representations of G. Note that $V_E^G = 0$ is equivalent to a non-splitting of (*) since if (*) would have split then we would have had that $\mathbb{Q}_p \hookrightarrow V_E^G$.

Moreover, if K is a p-adic field and E/K is an elliptic curve with split multiplicative reduction then following [BC, Example 2.2.4] one can show that (*) is non-split as a sequence of $\mathbb{Q}_p$-representations of $G_{K'}$ for all finite extensions K'/K inside $\overline{K}$.

Case 2. Assume that $E/\mathbb{Q}_p$ is an elliptic curve with good ordinary reduction at p i.e. $\tilde{E}/\mathbb{F}_p$ is an elliptic curve and $p \nmid a_p := 1 + p - \#\tilde{E}/\mathbb{F}_p$ (where $\tilde{E}$ is the reduction curve). Following [Sil, Theorem 7.4, Chapter 4] and [Sil, Theorem 3.5, Chapter 5], this is equivalent to saying that the formal group of E, $\hat{E}$ has height 1. Via [Sil, Proposition 2.1, Chapter 7] we have an exact sequence:

$$0 \longrightarrow \hat{E}(\bar{\mathbb{Q}}_p) \longrightarrow E(\bar{\mathbb{Q}}_p) \longrightarrow \tilde{E}(\bar{\mathbb{F}}_p) \longrightarrow 0$$

hence:

$$0 \longrightarrow \hat{E}[p^n](\bar{\mathbb{Q}}_p) \longrightarrow E[p^n](\bar{\mathbb{Q}}_p) \longrightarrow \tilde{E}[p^n](\bar{\mathbb{F}}_p) \longrightarrow 0\,.$$

Let's remark now that $\tilde{E}[p^n](\bar{\mathbb{F}}_p) \cong \frac{\mathbb{Z}}{p^n\mathbb{Z}}(\psi)$ for $\psi : G \to \mathbb{Z}_p^*$ unramified character $(\mathrm{char}(\bar{\mathbb{F}}_p) = p$ and I acts trivially on $\bar{\mathbb{F}}_p)$.

By taking now projective limits we obtain:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_p\hat{E} & \longrightarrow & T_pE & \xrightarrow{f} & T_p\tilde{E} & \longrightarrow & 0 \quad (**) \\ & & \downarrow \cong & & & & \downarrow \cong & & \\ & & \mathbb{Z}_p(\varphi) & & & & \mathbb{Z}_p(\psi) & & \end{array}$$

Denote by $\{e_1\}$ a basis of $T_p\hat{E}$ over $\mathbb{Z}_p$ and complete it to a basis $\{e_1, e_2\}$ of T_pE over $\mathbb{Z}_p$ such that $f(e_2) = 1$ where $\{1\}$ is the basis of $\mathbb{Z}_p(\psi)$ (remark that $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{T_p\hat{E}} = \mathrm{rank}_{\mathbb{Z}_p}\mathrm{T_p\tilde{E}} = 1$ and that $\mathrm{rank}_{\mathbb{Z}_p}\mathrm{T_pE} = 2$). For $\sigma \in G$ we clearly have that:

$$\sigma e_1 = \varphi(\sigma)e_1. \quad (1)$$

We want to compute now σe_2.

Apply f and on one hand we obtain: $f(\sigma e_2) = \sigma f(e_2) = \sigma 1 = \psi(\sigma) \cdot 1 = \psi(\sigma)$.

On the other hand, $f(\psi(\sigma)e_2) = \psi(\sigma)f(e_2) = \psi(\sigma)$.

One obtains that $f(\sigma e_2 - \psi(\sigma)e_2) = 0$ hence $\sigma e_2 - \psi(\sigma)e_2 \in ker(f)$ which is a subgroup of T_pE and so $\sigma e_2 - \psi(\sigma)e_2 = a_\sigma e_1$ for some $a_\sigma \in \mathbb{Z}_p$.

It follows that: $\sigma e_2 = a_\sigma e_1 + \psi(\sigma)e_2$. (2)

From (1) and (2) we obtain that the matrix of σ in the basis $\{e_1, e_2\}$ is :

$$\rho_E(\sigma) = \begin{pmatrix} \varphi(\sigma) & a_\sigma \\ 0 & \psi(\sigma) \end{pmatrix}$$

hence $\det(\rho_{\mathrm{E}}(\sigma)) = \chi(\sigma) = \psi(\sigma)\varphi(\sigma)$. Consequently one obtains that: $\varphi(\sigma) = \chi(\sigma)\psi^{-1}(\sigma)$ and further we can write:

$$\rho_E(\sigma) = \begin{pmatrix} \chi(\sigma)\psi^{-1}(\sigma) & a_\sigma \\ 0 & \psi(\sigma) \end{pmatrix}.$$

After tensoring (**) with $\mathbb{Q}_p$ over $\mathbb{Z}_p$ (same procedure as in the *Case 1*) one obtains the exact sequence of G-modules:

$$0 \longrightarrow \mathbb{Q}_p(\chi\psi^{-1}) \xrightarrow{1 \to e_1'} V_E \xrightarrow[e_2' \to 1]{e_1' \to 0} \mathbb{Q}_p(\psi) \longrightarrow 0\,,$$

$\{e_1', e_2'\}$ being a $\mathbb{Q}_p$-basis of V_E, where $e_1' := e_1 \otimes 1$ and $e_2' := e_2 \otimes 1$.

Case 3. Suppose that $E/\mathbb{Q}_p$ is an elliptic curve with good supersingular reduction at p i.e. $p \mid a_p := 1 + p - \#\tilde{E}/\mathbb{F}_p$.

In this case we have no general formula for ρ_E but as in the previous cases, $\det(\rho_{\mathrm{E}}(\sigma)) = \chi(\sigma)$ for any $\sigma \in G$.

In this case, since there are no p-power points of the reduction curve, we have that $T_p\tilde{E} = 0$ and so, from the exact sequence

$$0 \longrightarrow T_p\hat{E} \longrightarrow T_pE \longrightarrow T_p\tilde{E} = 0$$

we obtain that $T_pE \cong T_p\hat{E}$. Following [Sil, Theorem 3.1(v), Chapter 5], the height of the formal group $\hat{E}$ associated to E is 2. Since $T_p\hat{E}$ is irreducible we get that T_pE is also irreducible.

We have that $V_E = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an irreducible representation i.e. for $\sigma \in G$:

$$\rho_E(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

where $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathbb{Z}_p$.

Also note that the exact sequence of group-cohomology doesn't give us any information.

However, an important result is obtained by using Tate's Theorem (see [Ta, Theorem 3, Corollary 2] or [I1, Theorem 2.2.15]), namely that $T_pE \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$. We also have this isomorphism for elliptic curves in the *Case 1* or *Case 2* and we will prove this in the next chapter, Proposition 2.1.14.

Remark 1.3.3. If $E/\mathbb{Q}_p$ is an elliptic curve with additive reduction (i.e. $\tilde{E}/\mathbb{F}_p$ has a cusp), after a change of basis the reduction type becomes good (i.e. $\tilde{E}/\mathbb{F}_p$ is an elliptic curve) or semi-stable (multiplicative reduction (i.e. $\tilde{E}/\mathbb{F}_p$ has a node))) (see [Sil, Proposition 5.4]).

Remark 1.3.4. A more general example than the ones analyzed in subsections 1.3.1 and 1.3.2 is the étale cohomology.

Suppose that K is a finite extension of $\mathbb{Q}_p$. If X is a proper and smooth variety over K, then the étale cohomology $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic representation, where $X_{\overline{K}} = X \times_{\text{Spec}K} \text{Spec}\overline{K}$. The étale cohomology was the motivation for the study of the p-adic representations and Fontaine was the one who succeeded in constructing a functor relating the étale and the crystalline cohomologies of a p-divisible group. The existence of this functor was conjectured by Grothendieck in 1970.

Since $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = (\varprojlim_n H^i_{ét}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, one needs first to understand $H^i_{ét}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$ for $i \geq 0$.

If X is a curve of genus g over K (i.e. a smooth, projective, irreducible algebraic variety of dimension 1), then following [I2, Theorem 2.10.5] and [Mil, Proposition 14.2], one obtains that:

$$\varprojlim H^i_{ét}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z}) = \begin{cases} \varprojlim \mu_{p^n}(K) = T_p(\mu_{p^\infty}) \cong \mathbb{Z}_p, & \text{i=0;} \\ \varprojlim Jac(X)_{\overline{K}}[p^n] = T_p(Jac(X_{\overline{K}})), & \text{i=1;} \\ \mathbb{Z}_p, & \text{i=2;} \\ 0, & \text{i} \geq 3. \end{cases}$$

and so, after tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$, one further obtains that:

$$H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = \begin{cases} \mathbb{Q}_p, & \text{i=0;} \\ V_p(Jac(X_{\overline{K}})), & \text{i=1;} \\ \mathbb{Q}_p, & \text{i=2;} \\ 0, & \text{i} \geq 3. \end{cases}$$

Remark that $H^1_{ét}(X_{\overline{K}}, \mathbb{Q}_p) = V_p(Jac(X_{\overline{K}}))$ and hence a $\mathbb{Q}_p$- representation of dimension $2g$. Consequently, if $g = 1$ (i.e. if X is an elliptic curve and hence the Jacobian $Jac(X) \cong X$ following [Si2, Proposition 2.6, Chapter 2]) we recover the example 1.3.2. Moreover, from the above description it is clear that the examples analyzed in subsections 1.3.1 and 1.3.2 are special cases of the étale cohomology.

Chapter 2

Fontaine Theory

We've seen in the previous chapter that Algebraic Geometry provides interesting p-adic representations of $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ i.e. continuous representations of G on finite dimensional $\mathbb{Q}_p$-vector spaces V.

Fontaine constructed period rings $\mathbb{C}_p, B_{\text{HT}}, B_{\text{cris}}, B_{\text{st}}, B_{\text{dr}}$ in [Fo1] and [Fo2], which are topological $\mathbb{Q}_p$-algebras with an action of G and some additional structures compatible with this action (for example: Frobenius φ, a filtration Fil, a monodromy operator N and a differential operator ∂) and using them was able to describe p-adic G-representations in terms of semi-linear data.

2.1 Hodge-Tate theory.

1) We will first analyse what happens when we tensor a p-adic representation of G_K with $\mathbb{C}_p$.

Definition 2.1.1. Let $V \in \text{Rep}_{\mathbb{Q}_p}(\text{G})$. Then V is Hodge-Tate (HT) if we have an isomorphism as $\mathbb{C}_p$-modules with (semi-linear) G_K-action

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i=1}^{d} \mathbb{C}_p(n_i)$$

where $d = \dim_{\mathbb{Q}_p} V$. The numbers n_i, $1 \leq i \leq d$ are called Hodge-Tate numbers (and are not necessarily distinct).

For example $\mathbb{Q}_p(n) \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$ is HT since $\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p(n)$.

We have the following central result which is known under the name "Hodge-Tate comparison isomorphism"

Theorem 2.1.2 (Fa2, Chapter 3, Theorem 4.1). *Let X the a smooth, proper, geometrically connected scheme over K. Then for all $i \geq 0$ we have a canonical isomorphism as $\mathbb{C}_p$-modules with (semi-linear) G_K-action*

$H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \oplus_{a+b=i}(H^a(X, \Omega^b_{X/K}) \otimes_K \mathbb{C}_p(-b))$

The theorem 2.1.2 has the following consequence:

Corollary 2.1.3. *If X is a smooth, proper geometrically connected scheme over K, then for every $i \geq 0$, the p-adic G_K-representations $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ are Hodge-Tate, with Hodge-Tate numbers given by the Betti numbers of the base change of X to the complex numbers.*

The theorem 2.1.2 is a deep result but corollary 2.1.3 can in some examples be deduced using elementary methods. We will examine such examples in the next section.

2.1.1 Elementary examples

Firstly we will focus on classifying the representations $\rho_E : G \to GL_n(\mathbb{Q}_p)$. In order to do this, it is easier to consider representations over $\mathbb{C}_p$ which is complete and algebraically closed:

$$\mathbb{Q}_p \subset \bar{\mathbb{Q}}_p \subset \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$$

Definition 2.1.4. A $\mathbb{C}_p$-representation of G is a finite dimensional $\mathbb{C}_p$-vector space W equipped with a continuous *semilinear* G-action $G \times W \to W$ (i.e. $\sigma(ax) = \sigma(a)\sigma(x)$ for all $a \in \mathbb{C}_p$, $x \in W$, $\sigma \in G$).

We denote by $\mathrm{Rep}_{\mathbb{C}_p}(G)$ the category whose objects are $\mathbb{C}_p$-representations of G and if V, W are two such, a morphism $f : V \to W$ is a $\mathbb{C}_p$-linear map which satisfies $f(\sigma v) = \sigma f(v)$ for any $\sigma \in G$ and $v \in V$.
If V is a $\mathbb{Q}_p$-representation of G then $W = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ is an object of $\mathrm{Rep}_{\mathbb{C}_p}(G)$. We will mostly work with representations arising in this way.

Let now $e := \{e_1, e_2, ..., e_n\}$ be a $\mathbb{C}_p$-basis of W. For any $\sigma \in G$ we can uniquely write:

$$\sigma e_i = \textstyle\sum_{1 \le j \le n} a_{ji}(\sigma) e_j \text{ for all } 0 \le i \le n.$$

Consider now $A(\sigma) := (a_{ij}(\sigma)) \in GL_n(\mathbb{C}_p)$ ($A(\sigma)$ is invertible since $\sigma \in G$). Then we get a continuous map $A : G \to GL_n(\mathbb{C}_p)$ defined by $\sigma \to A(\sigma)$.

Remark 2.1.5. One works similarly if one replaces $\mathbb{Q}_p$ with a p-adic field K (i.e. a field of characteristic zero, complete with respect to a fixed discrete valuation, having a perfect residue field k of characteristic $p > 0$). Then one has

$$\mathbb{Q}_p \subset K \subset \overline{K} \subset \mathbb{C}_K = \widehat{\overline{K}}$$

and one denotes by $G_K := Gal(\overline{K}/K)$.

Proposition 2.1.6. *Suppose $\{e\}$ is a basis of W. Then:*

a) A_e defined as above is a 1-cocycle;

b) If one further choose another basis $\{f\}$ of W then A_e and A_f are cohomologous i.e. there exists a matrix $M \in GL_n(\mathbb{C}_p)$ such that $A_e(\sigma) = \sigma(M)A_f(\sigma)M^{-1}$.

Proof. a) Let $\sigma, \tau \in G$. For every $x \in W$ one obtains:

$$\begin{aligned} A_e(\sigma\tau)x &= (\sigma\tau)x \\ &= \sigma(\tau x) \\ &= \sigma(A_e(\tau)x) \\ &= \sigma(A_e(\tau))\sigma x \\ &= \sigma(A_e(\tau))A_e(\sigma)x \end{aligned}$$

and hence $A_e(\sigma\tau) = \sigma(A_e(\tau))A_e(\sigma)$ i.e. A_e is a 1-cocycle;

b) If $\{f\}$ is another basis of W, by letting M to be the change of basis matrix, we have that $e = M \cdot f$ and hence:

$$\sigma e = \sigma(Mf) = \sigma(M)\sigma f = \sigma(M)A_f(\sigma)f = (\sigma(M)A_f(\sigma)M^{-1})e.$$

□

On the other hand, $\sigma e = A_e(\sigma)e$ and one obtains $A_e(\sigma) = \sigma(M)A_f(\sigma)M^{-1}$ (twisted conjugation).

Definition 2.1.7. Two cocycles A, B are cohomologous if $A(\sigma) = \sigma(M)B(\sigma)M^{-1}$.

Note that being cohomologous is an equivalence relation; denote it "$\sim$".

Definition 2.1.8. $H^1_{cont}(G, GL_n(\mathbb{C}_p)) = \{cocycles\}/\sim$.

Also remark that if $n > 1$ then $GL_n(\mathbb{C}_p)$ is not abelian hence $H^1_{cont}(G, GL_n(\mathbb{C}_p))$ is not a group, just a pointed set. However, $H^1_{cont}(G, GL_n(\mathbb{C}_p))$ classifies the n-dimensional semilinear continuous representations of G up to isomorphism. We have a bijection between the following sets:

$$\{W \mid W \text{ is a } \mathbb{C}_p\text{-representation of } G\}/ \longleftrightarrow H^1_{cont}(G, GL_n(\mathbb{C}_p))$$

$$\text{given by:} \qquad (W, e) \to A_e$$

$$\text{and} \qquad W_A \leftarrow A$$

where $W_A = \mathbb{C}_p^n$ as a $\mathbb{C}_p$-vector space, with semilinear action of G given by the multiplication of A. More concretely, we have the following:

Proposition 2.1.9. *$[A] = [B] \in H^1_{cont}(G, GL_n(\mathbb{C}_p)) \Leftrightarrow W_A \cong W_B$ as semilinear G-representations.*

Proof. ($\Leftarrow$) Take M the matrix of the isomorphism in the canonical basis. The claim follows.

($\Rightarrow$) If $A \sim B$ then let $M \in GL_n(\mathbb{C}_p)$ such that $A(\sigma) = \sigma(M)B(\sigma)M^{-1}$ for every $\sigma \in G$. Let $\{e\}$ be a basis of W_A such that $\sigma e = A(\sigma)e$ and $\{f\}$ be a basis of W_B such that $\sigma e = B(\sigma)e$.

Define $\psi : W_A \to W_B$ such that $\psi(e) = Mf$.

Obviously ψ is an isomorphism of $\mathbb{C}_p$-vector spaces. We need to show that it commutes with the action of G.

Indeed, we have that:

$$\begin{aligned}\psi(\sigma e) &= \psi((A(\sigma)e)) \\ &= A(\sigma)\psi(e) \\ &= A(\sigma)Mf \\ &= \sigma(M)B(\sigma)M^{-1}Mf \\ &= \sigma(M)B(\sigma)f \\ &= \sigma(M)\sigma f \\ &= \sigma(Mf) \\ &= \sigma\psi(e).\end{aligned}$$

□

Let us now examine some easy applications of the above.

Suppose that $\varphi : G \to \mathbb{Z}_p^*$ is a continuous character. We take $V := \mathbb{Q}_p(\varphi)$ and extend scalars to $\mathbb{C}_p$ by defining:

$$W := V \otimes_{\mathbb{Q}_p} \mathbb{C}_p := \mathbb{C}_p(\varphi).$$

Note that $\mathbb{C}_p(\varphi) = \mathbb{C}_p$ as a vector space with a continuous semilinear action: $\sigma x = \varphi(\sigma)\sigma(x)$. Note also that since $\mathbb{Z}_p^* \subset \mathbb{C}_p^* = GL_1(\mathbb{C}_p)$, we can think of φ as $\varphi : G \to GL_1(\mathbb{C}_p)$. In this way, φ is a 1-cocycle.

One question arises, namely, what does it mean that $\mathbb{C}_p(\varphi) \cong \mathbb{C}_p$ as semilinear G-representations?

Following Proposition 2.1.9, $\mathbb{C}_p(\varphi) \cong \mathbb{C}_p$ as semilinear G-representations if and only if $[\varphi] = [1] \in H^1_{cont}(G, GL_1(\mathbb{C}_p))$ (note that the cocycle corresponding to $\mathbb{C}_p$ is $1 : G \to \mathbb{C}_p^*$ defined by $1(\sigma) = 1$ for any $\sigma \in G$).

Moreover, $[\varphi] = [1] \in H^1_{cont}(G, GL_1(\mathbb{C}_p)) \Leftrightarrow$ there exists $\gamma \in \mathbb{C}_p^*$ such that $\varphi(\sigma) = \sigma(\gamma)1(\sigma)\gamma^{-1}$. We've obtained:

$$\mathbb{C}_p(\varphi) \cong \mathbb{C}_p \text{ in } \mathrm{Rep}_{\mathbb{C}_p}(G) \Leftrightarrow \exists \gamma \in \mathbb{C}_p^* \text{ such that } \varphi(\sigma) = \sigma(\gamma)\gamma^{-1}\ \forall \sigma \in G.$$

In other words, $\mathbb{C}_p(\varphi) \cong \mathbb{C}_p$ in $\mathrm{Rep}_{\mathbb{C}_p}(G) \Leftrightarrow \exists \gamma \in \mathbb{C}_p^*$ such that $\varphi(\sigma) = \frac{\sigma(\gamma)}{\gamma}$ for any $\sigma \in G$.

Of crucial importance in the Fontaine Theory is the Ax-Sen-Tate Theorem (see [BC, Theorem 2.2.7]):

Theorem 2.1.10 (Ax-Sen-Tate). *For any p-adic field K we have that $K = \mathbb{C}_K^{G_K} = \widehat{\overline{K}}^{G_K}$ (i.e. there are no transcendental invariants) and $\mathbb{C}_K(r)^{G_K} = 0$ for $r \neq 0$ (i.e., if $x \in \mathbb{C}_K$ and $\sigma(x) = \chi(\sigma)^{-r}x$, for all $\sigma \in G_K$ and some $r \neq 0$ then $x = 0$). Also $H^1_{cont}(G_K, \mathbb{C}_K(r)) = 0$ if $r \neq 0$ and $H^1_{cont}(G_K, \mathbb{C}_K)$ is 1-dimensional over K.*

More generally, if $\eta : G_K \to \mathbb{Z}_p^*$ is a continuous character and $\mathbb{C}_K(\eta)$ denotes $\mathbb{C}_K$ with the twisted G_K-action $\sigma x = \eta(\sigma)\sigma(x)$ then $\mathbb{C}_K(\eta)^{G_K} = 0$ if $\eta(I_K)$ is infinite and $\mathbb{C}_K(\eta)^{G_K}$ is 1-dimensional over K if $\eta(I_K)$ is finite. Also, $H^1_{cont}(G_K, \mathbb{C}_K(\eta)) = 0$ if $\eta(I_K)$ is infinite.

Proposition 2.1.11. *a) $\mathbb{C}_p(1) \not\cong \mathbb{C}_p$ as G-representations;*

b) If $m \neq n \in \mathbb{Z}$, then $\mathbb{C}_p(m) \not\cong \mathbb{C}_p(n)$ as G-representations.

Proof. a) Suppose that $\mathbb{C}_p(1) \cong \mathbb{C}_p$ as G-representations over $\mathbb{C}_p$. Then also their G-invariants are isomorphic i.e. $\mathbb{C}_p^G(1) \cong \mathbb{C}_p^G$. Following Ax-Sen-Tate theorem for $K = \mathbb{Q}_p$ we obtain $0 = \mathbb{C}_p^G(1) \cong \mathbb{C}_p^G = \mathbb{Q}_p$ which is absurd.

b) Suppose that $\mathbb{C}_p(m) \cong \mathbb{C}_p(n)$ as G-representations over $\mathbb{C}_p$. Also, suppose that $m > n$. We then have:

$$\mathbb{C}_p(m)(-n) \cong \mathbb{C}_p(n)(-n)$$

hence $\mathbb{C}_p(m-n) \cong \mathbb{C}_p$. Again, by taking G-invariants, Ax-Sen-Tate theorem leads us to $0 = \mathbb{C}_p^G(m-n) \cong \mathbb{C}_p^G = \mathbb{Q}_p$ which is absurd.

□

Proposition 2.1.12. *Let $\psi : G \to \mathbb{Z}_p$ be an unramified character. Then $\mathbb{C}_p(\psi) \cong \mathbb{C}_p$ as G-representations.*

Remark 2.1.13. Firstly, note that $\mathbb{Q}_p(\psi) \ncong \mathbb{Q}_p$ as G-representations.

In order to prove this, suppose that $f : \mathbb{Q}_p \to \mathbb{Q}_p(\psi)$ is an isomorphism. Then $f(1) = e \neq 0$ where $\{e\}$ is a basis of $\mathbb{Q}_p(\psi)$ and for any $\sigma \in G$ we have that

$$e = f(1) = f(\sigma 1) = \sigma f(1) = \sigma e = \psi(\sigma)\sigma(e) = \psi(\sigma)e$$

since σ acts trivially on $\mathbb{Q}_p$.

It follows that $\psi(\sigma) = 1$ for any $\sigma \in G$ and hence ψ is the trivial character. So except for the trivial character $\mathbb{Q}_p(\psi) \ncong \mathbb{Q}_p$ as G-representations.

Proof of Proposition 2.1.12 We want to prove that $\mathbb{C}_p(\psi) \cong \mathbb{C}_p$ as G-representations, in other words, that $\mathbb{Q}_p(\psi)$ is HT of HT-number zero for unramified characters (since $\mathbb{Q}_p(\psi) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p(\psi) \cong \mathbb{C}_p$).

We will construct an isomorphism $f : \mathbb{C}_p \to \mathbb{C}_p(\psi)$. By putting $f(1) = e \in \mathbb{C}_p^*$, we have:

$$e = f(1) = f(\sigma 1) = \sigma f(1) = \sigma e = \psi(\sigma)\sigma(e).$$

In other words, we need $e \in \mathbb{C}_p^*$ satisfying $e = \psi(\sigma)\sigma(e)$ for any $\sigma \in G$.

We claim that one can consider $e \in \mathcal{O}^*$, where $\mathcal{O} := \mathcal{O}_{\mathbb{C}_p}$ is the ring of integers of $\mathbb{C}_p$. For this, remark that for every $\sigma \in I$, where $I = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p^{ur}) \subset G$ is the inertia group, we have that $\sigma e = e = \sigma(e)$ since $\psi(\sigma) = 1$, ψ being unramified and hence trivial on inertia. It follows that $e \in \mathbb{C}_p^I = \widehat{\mathbb{Q}_p^{ur}} \supseteq \mathcal{O}$.

Assume now that $v(p) = 1$ (otherwise one normalizes the valuation) and remark that $\mathcal{O}/p\mathcal{O} = \bar{\mathbb{F}}_p$. As in Chapter 1, we have that ψ factors through G/I:

$$\psi : G/I \cong Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong Gal(\widehat{\mathbb{Q}_p^{ur}}/\mathbb{Q}_p) \to \mathbb{Z}_p^*$$

Also recall that $Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \langle Fr \rangle$ and put $\psi(Fr) = \alpha \in \mathbb{Z}_p^*$.

Let $v(e) = n \in \mathbb{Z}$ and define $e' := e/p^n$. For every $\sigma \in G$ we have that $\sigma(e') = \sigma(e)/p^n$ and hence:

$$\sigma(e')\psi(\sigma) = \tfrac{\sigma(e)}{p^n}\psi(\sigma) = \tfrac{e}{p^n} = e'.$$

So $\sigma(e')\psi(\sigma) = e'$ and moreover $v(e') = 0$. It follows that we can assume that $e \in \mathcal{O}^*$.

Following [Iw, Section 2.3], if $\sigma \in G/I = Gal(\widehat{\mathbb{Q}_p^{ur}}/\mathbb{Q}_p)$ it is enough to find $e' \in \mathcal{O}^*$ such that $Fr(e')\psi(Fr) = e'$.

Now, $Fr - id : \mathcal{O} \to \mathcal{O}$ is surjective and since $\psi(Fr) = \alpha$ we obtain that it is enough to find $e' \in \mathcal{O}^*$ such that $Fr(e') = e'\alpha^{-1}(\text{mod} p\mathcal{O})$. For this, note that $X^p - \bar{\alpha}^{-1}X$ is separable in $\bar{\mathbb{F}}_p[X]$(since $D(X^p - \bar{\alpha}^{-1}X) = -\bar{\alpha}^{-1} \neq 0$). We apply now Hensel's Lemma and get:

$$e' \in \varprojlim_n \mathcal{O}/p^n\mathcal{O} = \hat{\mathcal{O}} = \mathcal{O}.$$

In this way, we've proved that there exists an element $e \in \widehat{\mathbb{Q}_p^{ur}}$, $e \neq 0$, such that $Fr(e)\psi(Fr) = e$. We obtain that $\sigma(e)\psi(\sigma) = e$ for any $\sigma \in G$ and consequently that we have an isomorphism of G-representations $f : \mathbb{C}_p \to \mathbb{C}_p(\psi)$.

□

As a consequence, we have that:

$$\mathbb{C}_p(\chi^n\psi) \cong (\mathbb{C}_p(\psi))(\chi^n) \cong \mathbb{C}_p(\chi^n) = \mathbb{C}_p(n).$$

where $\psi : G \to \mathbb{Z}_p$ is unramified and $\chi : G \to \mathbb{Z}_p^*$ is the cyclotomic character.

This implies that if ψ is unramified, then $V := \mathbb{Q}_p(\chi^n\psi)$ is HT.

We move now to the elliptic curves and prove the following

Proposition 2.1.14. *Let $E/\mathbb{Q}_p$ be an elliptic curve as in the* Case 1 *or* Case 2 *of 1.3.2 (i.e. with good ordinary reduction or a Tate curve respectively). Then V_E is HT. More exactly,*

$$V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$$

as G-representations.

Proof. We consider the exact sequence:

$$0 \to \mathbb{Q}_p(\chi\psi^{-1}) \to V_pE \to \mathbb{Q}_p(\psi) \to 0$$

where ψ is unramified if E has good ordinary reduction or trivial if E is a Tate curve.

Since any $\mathbb{Q}_p$-algebra is flat, by tensoring with $\mathbb{C}_p$ over $\mathbb{Q}_p$, we get:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}_p(\chi\psi^{-1}) & \longrightarrow & V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p & \longrightarrow & \mathbb{C}_p(\psi) & \longrightarrow & 0 \\ & & \downarrow \cong & & & & \downarrow \cong & & \\ & & \mathbb{C}_p(1) & & & & \mathbb{C}_p & & \end{array}$$

where for the upper left isomorphism, one uses the fact that ψ^{-1} is also unramified since ψ is and Proposition 2.1.12 and for the right upper only the Proposition 2.1.12.

And so we've obtained:

$$0 \longrightarrow \mathbb{C}_p(1) \xrightarrow{f} V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{g} \mathbb{C}_p \longrightarrow 0 \quad (*)$$

We want $V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$ as G-representations. This is equivalent to proving that the sequence $(*)$ is split as a sequence of G-modules. However, $\mathbb{C}_p$ being projective (since it is a vector space), $(*)$ is split as a sequence of vector spaces and $V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$. So firstly we find a splitting of $(*)$ just as $\mathbb{C}_p$-vector spaces. Define:

$$s : \mathbb{C}_p \to V_pE \otimes_{\mathbb{Q}_p} \mathbb{C}_p \text{ by } s(1) := \alpha \text{ such that } g(\alpha) = 1.$$

Remark that for every $a \in \mathbb{C}_p$ we then have $s(a) = as(1) = a\alpha$ and so $g(s(a)) = a$.

Consider now the element $\sigma\alpha - \alpha \in V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, $\sigma \in G$. Since $g(\sigma\alpha - \alpha) = g(\sigma\alpha) - g(\alpha) = \sigma g(\alpha) - g(\alpha) = \sigma 1 - 1 = 1 - 1 = 0$ it follows that $\sigma\alpha - \alpha \in ker(g)$ (note that $g(\sigma\alpha) = \sigma g(\alpha)$ since g is a homomorphism of G-modules).

Now, since $(*)$ is exact one obtains that $\sigma\alpha - \alpha \in Im(f)$ hence $\sigma\alpha - \alpha = f(a_\sigma)$ for some $a_\sigma \in \mathbb{C}_p(1)$.

Define now:

$$\beta : G \to \mathbb{C}_p(1) \text{ by } \beta(\sigma) = a_\sigma.$$

Then β is a 1-cocycle. In order to prove this, let $\sigma, \tau \in G$, apply f to $\beta(\sigma\tau)$ and use the fact that f is injective. Concretely, we have:

$$\begin{aligned} f(a_{\sigma\tau}) &= f(\beta(\sigma\tau)) = \sigma\tau\alpha - \alpha \\ &= \sigma(\tau\alpha - \alpha) + \sigma\alpha - \alpha \\ &= \sigma f(a_\tau) + f(a_\sigma) \\ &= f(\sigma a_\tau) + f(a_\sigma) = f(\sigma a_\tau + a_\sigma) \end{aligned}$$

and since f is injective we obtain that: $a_{\sigma\tau} = \sigma a_\tau + a_\sigma$ i.e. $\beta(\sigma\tau) = \sigma\beta(\tau) + \beta(\sigma)$.

Now, since $\chi(I)$ is infinite, following Ax-Sen-Tate's Theorem (Theorem 2.1.10) we have that $H^1_{cont}(G, \mathbb{C}_p(1)) = 0$ and consequently the class $[\beta] = 0 \in H^1_{cont}(G, \mathbb{C}_p(1))$. In other words, β is a coboundary and so there exists an element $\gamma \in \mathbb{C}_p(1)$ such that $\beta(\sigma) = a_\sigma = \sigma\gamma - \gamma = \chi(\sigma)\sigma(\gamma) - \gamma$ for any $\sigma \in G$.

Since s may not be G-equivariant, we modify now this section by letting:

$$t : \mathbb{C}_p \to V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p;$$
$$t(1) := \alpha - f(\gamma)$$

such that t is G-equivariant.

Remark that $g \circ t = 1_{\mathbb{C}_p}$ since $g(t(1)) = g(\alpha - f(\gamma)) = g(\alpha) = 1$.

Recall that $\{e \otimes 1, f \otimes 1\}$ is a basis of $V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and since $f(1) = e \otimes 1$ we have that $f(\gamma) = \gamma(e \otimes 1)$. Moreover, one can take $\alpha := f \otimes 1$ and so $t(1) = f \otimes 1 - \gamma(e \otimes 1)$. Also recall from the previous chapter that the action of G on the basis is given by $\sigma(f \otimes 1) = f \otimes 1 + a_\sigma e \otimes 1$ and $\sigma(e \otimes 1) = \chi(\sigma) e \otimes 1$, $\sigma \in G$.

Consequently,

$$\sigma t(1) = \sigma(f \otimes 1 - \gamma(e \otimes 1)) = f \otimes 1 + a_\sigma e \otimes 1 - \sigma(\gamma)\chi(\sigma) e \otimes 1$$
$$= f \otimes 1 + (\chi(\sigma)\sigma(\gamma) - \gamma) e \otimes 1 - \sigma(\gamma)\chi(\sigma) e \otimes 1 = f \otimes 1 - \gamma(e \otimes 1) = t(1)$$

for all $\sigma \in G$ and so t is G-equivariant.

We obtain that $V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$ as G-representations.

$\square$

Remark 2.1.15. In general, if $E/\mathbb{Q}_p$ is an elliptic curve, then since $V_E \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p \oplus \mathbb{C}_p(1)$ (by using Tate's Theorem (see [Ta, Theorem 3, Corollary 2] or [I1, Theorem 2.2.15])) it follows that V_E is HT (see also Corollary 2.1.3).

2) B_{HT}.

The theory of Hodge-Tate p-adic representations can be better expressed in a slightly different language, as follows.

We first define the category of graded vector spaces over a field. Following [BC], we have:

Definition 2.1.16. A $\mathbb{Z}$-graded vector space over a field F is an F-vector space V equipped with direct sum decomposition $\oplus_{q\in\mathbb{Z}}V_q$ where V_q are F-subspaces of V. One also defines the $q-th$ graded piece of D to be $gr^q(V) = V_q$. The morphisms $T : V \to V'$ between graded F-vector spaces are F-linear maps that respect the grading, in other words $T(V_q) \subseteq V'_q$ for all $q \in \mathbb{Z}$. The category of the graded vector spaces over the field F is denoted Gr_F and if $\dim_F V < \infty$ one denotes by $Gr_{F,f}$ the corresponding subcategory.

Definition 2.1.17. We have a covariant functor $D_K : \mathrm{Rep}_{\mathbb{C}_K}(G_K) \to Gr_K$ defined by:

$$D_K(W) = \oplus_{q\in Z}(W \otimes_{\mathbb{C}_K} \mathbb{C}_K(q))^{G_K} = (W \otimes_{\mathbb{C}_K} (\oplus_{q\in Z} C_K(q)))^{G_K}$$

which is left-exact.

Following Serre-Tate Lemma ([BC, Lemma 2.3.1]) we have that D_K takes values in $Gr_{F,f}$ and that $\dim_K D_K(W) \le \dim_{\mathbb{C}_K} W$ with equality if and only if W is HT.

An easy application of Ax-Sen-Tate's theorem is the computation:

$D_K(\mathbb{C}_K(r)) = \oplus_{q\in Z}(\mathbb{C}_K(r) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q))^{G_K} = \oplus_{q\in Z}(\mathbb{C}_K(q+r))^{G_K} = K\langle -r\rangle$ where by $F\langle r\rangle$ one denotes the F-vector space F endowed with the grading such that the unique non-vanishing graded component is the one in degree r, $r \in \mathbb{Z}$.

Definition 2.1.18. The Hodge-Tate ring of K is the $\mathbb{C}_K$-algebra $B_{\mathrm{HT}} = \oplus_{q\in\mathbb{Z}}\mathbb{C}_K(q)$, the multiplication being defined via the natural maps $\mathbb{C}_K(q) \otimes_{\mathbb{C}_K} \mathbb{C}_K(q') \cong \mathbb{C}_K(q+q')$.

Remark 2.1.19. B_{HT} is a $\mathbb{C}_K$-graded vector space with a $\mathbb{C}_K$-semi-linear G_K-action (which respects the ring structure and the grading).

If one chooses a basis of $\mathbb{C}_K(1)$, one has that:

$$B_{\mathrm{HT}} = \mathbb{C}_K[t, \tfrac{1}{t}] = \oplus_{q\in\mathbb{Z}}\mathbb{C}_K t^q = \oplus_{q\in\mathbb{Z}}\mathbb{C}_K(q)$$

and the G_K-action is given by $\sigma \cdot t^q = \chi(\sigma)^q t^q$.

We've used the fact that there is an isomorphism: $f : \mathbb{C}_K(q) \to t^q\mathbb{C}_K$ given by $f(a) = t^q a$. Note that f is G_K equivariant since for $x \in \mathbb{C}_K(q)$ and $\sigma \in G_K$:

$$\begin{aligned} f(\sigma * x) &= f(\chi^q(\sigma)\sigma(x)) = \chi^q(\sigma)f(\sigma(x)) \\ &= \chi^q(\sigma)t^q\sigma(x) = \sigma \cdot t^q\sigma(x) = \sigma \cdot f(x). \end{aligned}$$

Remark 2.1.20. A very important result is obtained by using Ax-Sen-Tate's theorem, namely that:

$$(B_{\mathrm{HT}})^{G_K} = (\oplus_{q\in\mathbb{Z}}\mathbb{C}_K(q))^{G_K} = K.$$

Moreover, for any $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ one has that $D_K(W) = \oplus_{q\in Z}(W\otimes_{\mathbb{C}_K}\mathbb{C}_K(q))^{G_K} = (W \otimes_{\mathbb{C}_K} B_{\mathrm{HT}})^{G_K}$ in Gr_K, the grading being induced from B_{HT}.

We introduce now the functor $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to Gr_{K,f}$ defined by:

$$D_{\mathrm{HT}}(V) := D_K(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K},\ V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{G_K}),$$

with grading induced by the one on B_{HT}.

Definition 2.1.21. Let $\mathrm{Rep}_{\mathrm{HT}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ be the full subcategory of p-adic representations of G_K which are HT.

Remark 2.1.22. The functor $D_{\mathrm{HT}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to Gr_{K,f}$ defined above is faithful functor (see [BC, Lemma 2.4.10]) but not full. For this, let $\eta : G_K \to \mathbb{Z}_p^*$ be any finite order character, $\eta \neq 1$. We then have:

$$D_{\mathrm{HT}}(\mathbb{Q}_p(\eta)) = (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K}$$
$$= (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} (\oplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)))^{G_K} = \oplus_{q\in\mathbb{Z}} \mathbb{C}_K(\chi^q \eta)^{G_K} = K\langle 0\rangle$$

where for the last equality one uses the Ax-Sen-Tate theorem. By using the same theorem, we also obtain that:

$$D_{\mathrm{HT}}(\mathbb{Q}_p) = (\mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K}$$
$$= (\mathbb{Q}_p \otimes_{\mathbb{Q}_p} (\oplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)))^{G_K} = \oplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)^{G_K} = K\langle 0\rangle.$$

It follows that $D_{\mathrm{HT}}(\mathbb{Q}_p(\eta)) = D_{\mathrm{HT}}(\mathbb{Q}_p)$ but note that there is no non-zero homomorphism from $\mathbb{Q}_p \to \mathbb{Q}_p(\eta)$ in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

In order to prove this, let $f : \mathbb{Q}_p \to \mathbb{Q}_p(\eta)$ be a homomorphism in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and put $f(1) := x$. Then, for any $\sigma \in G_K$, one has that:

$$x = f(1) = f(\sigma 1) = \sigma \cdot f(1) = \sigma \cdot x = \eta(\sigma)x$$

so $x = \eta(\sigma)x$. Choose now $\sigma \in G_K$ such that $\eta(\sigma) \neq 1$. It follows that $x = 0$ so $f = 0$.

□

One proceeds further in refining the category $\mathrm{Rep}_{\mathrm{HT}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ to a category that includes all representations coming from geometry. One also needs to refine the target semi-linear algebra category $Gr_{K,f}$ to a richer one. For this, one introduces the filtered modules:

Definition 2.1.23. A filtered module over a commutative ring R is an R-module M equipped with a collection $\{Fil^i M\}_{i\in\mathbb{Z}}$ of R-submodules which is decreasing i.e. $Fil^{i+1}M \subseteq Fil^i M$ for all $i \in \mathbb{Z}$. We say that the filtration is exhaustive if $\cup Fil^i M = M$ and the filtration is separated if $\cap Fil^i M = 0$.

For a filtered R-module M, one defines the associated graded module:

$$gr^{\bullet}(M) = \oplus_i (Fil^i M / Fil^{i+1} M).$$

Similarly, if k is a field, a filtered k-algebra is a k-algebra A equipped with an exhaustive and separated filtration $\{A^i\}$ of k-subspaces (k-vector spaces) such that

$A^i \cdot A^j \subseteq A^{i+j}$ for all $i, j \in \mathbb{Z}$ and $1 \in A^0$. The associated graded algebra is $gr^\bullet(A) = \oplus_i(Fil^i A / Fil^{i+1} A)$.

Remark 2.1.24. Following Definition 2.1.16, if $(V, \{Fil^i(V)\})$ is a filtered vector space over F and $\dim_F V < \infty$ then the filtration is exhaustive if and only if $Fil^i(V) = V$ for $i \ll 0$ and separated if and only if $Fil^i(V) = 0$ for $i \gg 0$. We denote by Fil_F the category of finite dimensional filtered vector spaces $(V, \{Fil^i(V)\})$ over F with exhaustive and separated filtration. Note that a morphism between two such objects is a linear map $T : V' \to V$ which is filtration compatible i.e. $T(Fil^i(V')) \subseteq Fil^i(V)$ for all $i \in \mathbb{Z}$.

The reason for introducing a new type of period ring is the following: for a smooth proper variety X over $\mathbb{C}$, Faltings' comparison isomorphism theorem (Theorem 2.1.2) leads to:

$$H^n_{Hodge}(X) = \oplus_q H^{n-q}(X, \Omega^q_{X/K}) \cong D_{HT}(H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)) = (H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{HT})^{G_K}$$

and so, in order to improve the comparison between the the étale and the graded Hodge cohomology via B_{HT} (note that $H^n_{Hodge}(X)$ is a graded K-vector space), one needs to replace the graded K-algebra with a filtered one, which will be called B_{dR}, such that $gr^\bullet(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$.

Also one hopes that the new functor D_{dR} defined on $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ with values in the category of filtered K-vector spaces is finer then D_{HT}. We will see in Proposition 2.2.14 that this is the situation and that one has an isomorphism of graded K- vector spaces: $gr^\bullet(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$.

2.2 de Rham theory

We briefly review now the construction of B_{dR}. For the notion of Witt vectors and their properties see [Se, Chapter 2, §2].

Firstly, for any $\mathbb{F}_p$-algebra A, one can construct an associated perfect $\mathbb{F}_p$-algebra $R(A)$ (see [BC, Proposition 4.2.3]):

$$R(A) = \varprojlim A = \{(x_0, x_1, ...) \in \prod_{n\geq 0} A \mid x_{i+1}^p = x_i \text{ for all } i \geq 0\}$$

the inverse limit being taken with respect to the Frobenius map: $Fr : A \to A$ defined by $Fr(a) = a^p$.

Note that $R(A)$ is perfect. For this, observe that the $p-th$ power map on $R(A)$ is surjective because if $(y_n)_{n\geq 0} \in R(A)$ then by letting $x_0 := y_0^{1/p}$ one constructs a compatible sequence $(x_n)_{n\geq 0} \in R(A)$ which maps to $(y_n)_{n\geq 0}$.

It is also injective since if $x = (x_n)_{n\geq 0} \in R(A)$ such that $x_n^p = 0$ for all $n \geq 0$ then the compatibility condition ($x_n^p = x_{n-1}$ for any $n \geq 1$) leads to $x_{n-1} = 0$ for all $n \geq 1$ hence $x = 0$.

Definition 2.2.1. Let S be a commutative $\mathbb{F}_p$-algebra and let $\varphi : S \to S$ be defined by $\varphi(x) = x^p$, $x \in S$. φ is an $\mathbb{F}_p$-algebra homomorphism called Frobenius. We say that S is perfect if φ is an isomorphism.

We will be interested in the following $\mathbb{F}_p$-algebra:

Definition 2.2.2. $\underline{R}(\mathcal{O}_{\overline{K}}) := R(\mathcal{O}_{\overline{K}}/(p)) = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$,

where the inverse limit is taken with respect to Frobenius.

Remark 2.2.3. $\underline{R}(\mathcal{O}_{\overline{K}})$ is a perfect $\mathbb{F}_p$-algebra in view of the above discussion though $\mathcal{O}_{\overline{K}}/(p)$ is not a perfect ring (for example $(p^{1/p})^p = 0$ while $p^{1/p} \neq 0$ in $\mathcal{O}_{\overline{K}}/(p)$).

Note also that since $\mathcal{O}_{\overline{K}}/(p) = \mathcal{O}_{\mathbb{C}_K}/(p)$, sometimes it is more convenient to work with $R(\mathcal{O}_{\mathbb{C}_K}/(p)) = \underline{R}(\mathcal{O}_{\overline{K}})$ since $\mathcal{O}_{\mathbb{C}_K}$ is p-adically separated and complete. For example, we have the following:

Lemma 2.2.4. *The multiplicative map of sets:*

$\varprojlim \mathcal{O}_{\mathbb{C}_K} \mapsto \varprojlim \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} = R(\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K})$, *defined by:*

$(x^{(n)})_{n\geq 0} \mapsto (x^{(n)} \text{mod} p)$, *with inverse given by:*

$(x_n)_{n\geq 0} \mapsto (x^{(n)})_{n\geq 0}$, where $x^{(n)} = lim_{m\to\infty}\widehat{x_{n+m}}^{p^m}$, for arbitrary lifts $\hat{x}_i \in \mathcal{O}_{\mathbb{C}_K}$ of $x_i \in \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ for all $i \geq 0$, is bijective.

Proof. We have to show that the inverse map makes sense (the direct one makes sense and is multiplicative clearly). For this, for each $n \geq 0$ and $m' \geq m \geq 0$, one has that:

$$\widehat{x_{n+m'}}^{p^{m'-m}} \equiv \widehat{x_{n+m}}(\mathrm{mod} p)$$

hence $\widehat{x_{n+m'}}^{p^{m'}} \equiv \widehat{x_{n+m}}^{p^m}(\mathrm{mod} p^{m+1})$ so the sequence $(\widehat{x_{n+m}}^{p^m})_m$ is Cauchy and so the limit $x^{(n)} = lim_{m\to\infty}\widehat{x_{n+m}}^{p^m}$ makes sense for any $n \geq 0$ (since $\mathcal{O}_{\mathbb{C}_K}$ is complete, the sequence $(\widehat{x_{n+m}}^{p^m})_m$ is convergent).

We still have to prove that the limit $x^{(n)}$ is independent of the choice of liftings. So, for any $n \in \mathbb{N}$ let $\tilde{x}_n$ and $\hat{x}_n$ be two liftings of x_n and put $\tilde{x}_n = \hat{x}_n + \nu_n$, with $\nu_n \in p\mathcal{O}_{\mathbb{C}_K}$. Then $\widetilde{x_{n+m}}^{p^m} - \widehat{x_{n+m}}^{p^m} = \sum_{k=1}^{p^m} C_{p^m}^k \widehat{x_{n+m}}^{p^m-k}\nu_n^k$. Since the p-adic valuation $v_p(C_{p^m}^k) = m - v_p(k)$ we obtain that $v_p(\widetilde{x_{n+m}}^{p^m} - \widehat{x_{n+m}}^{p^m}) \geq m$ and further that the limit is unique.

□

Remark 2.2.5. Via Lemma 2.2.4 one can identify $R := \underline{R}(\mathcal{O}_{\overline{K}})$ with:

$$\varprojlim \mathcal{O}_{\mathbb{C}_K} = \{(x^{(n)})_n \mid x^{(n)} \in \mathcal{O}_{\mathbb{C}_K}, x^{(n+1)^p} = x^{(n)} \text{ for all } n \geq 0\}$$

The laws of multiplication and addition are given by the following formulae: for any $x, y \in R$ and $n \in \mathbb{N}$,

$$(xy)^{(n)} = x^{(n)}y^{(n)}$$
$$(x+y)^{(n)} = \lim_{m\to\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$$

Moreover, R is a domain. Now, one gives R a valuation by defining $v_R(x) = v_p(x^{(0)})$ for all $x \in R$. One proves that v_R is a valuation on R and that R is v_R-adically separated and complete of residue field $\overline{k}$ (see [BC, Lemma 4.3.3]).

Now, for any natural number $n \geq 1$ we have a ring homomorphism:

$\theta_n : W_n := \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ given by $(s_0, ..., s_{n-1}) \longrightarrow \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}$, where $\tilde{s}_i \in \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ is a lift of s_i for every i, where W_n is the ring of Witt vectors of length n (on $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ valued points).

Denote by $u_n : W_{n+1} \to W_n$ the homomorphism defined by Frobenius composed with the truncation map. Also let $v_n : \mathcal{O}_{\overline{K}}/p^{n+1}\mathcal{O}_{\overline{K}} \to \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ be the truncation map.

We have that for every $n \in \mathbb{N}$, $n \geq 1$ the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{W}_{n+1}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) & \xrightarrow{\theta_{n+1}} & \mathcal{O}_{\overline{K}}/p^{n+1}\mathcal{O}_{\overline{K}} \\ \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle v_n} \\ \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) & \xrightarrow{\theta_n} & \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} \end{array}$$

This follows easily since:

$$\begin{array}{ccc} (s_0, s_1, ..., s_n) & \xrightarrow{\theta_{n+1}} & \sum_{i=0}^{n} p^i \tilde{s}_i^{p^{n-i}} \\ \downarrow{\scriptstyle u_n} & \equiv & \downarrow{\scriptstyle v_n} \\ (s_0^p, s_1^p, ..., s_{n-1}^p) & \xrightarrow{\theta_n} & \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-i}} \end{array}$$

By taking now the inverse limit one obtains a continuous G_K-equivariant morphism:

$$\theta : \varprojlim_{u_n} \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \to \varprojlim_{v_n} \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}} = \mathcal{O}_{\mathbb{C}_K}$$

Remark 2.2.6. The inverse limit of the projective system $(\mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}), u_n)_{n\in\mathbb{N}}$ is identified with the ring of Witt vectors $\mathbb{W}(R)$.

In order to prove this, we have that the truncation maps $\mathbb{W}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \to \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ are defining a morphism between the projective systems $(\mathbb{W}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}), \mathbb{W}(Fr))_n$ and $(\mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}), u_n)_n$ where $Fr : \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \to \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ is the p-power map. Since the Witt functor $\mathbb{W}(.)$ commutes with the projective limits and via Definition 2.2.2 we get that the first system is $\mathbb{W}(R)$ and consequently we have a ring homomorphism:

$$\mathbb{W}(R) \to \varprojlim_{u_n} \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$$

given by $(s_0, s_1, ...) \to ((s_0^{(n)}(p), s_1^{(n)}(p), ..., s_{n-1}^{(n)}(p)))_{n\in\mathbb{N}}$. This is bijective with inverse given by $((s_0^{(n)}, s_1^{(n)}, ..., s_{n-1}^{(n)}))_{n\in\mathbb{N}} \to ((s_m^{(n+m)})_{n\in\mathbb{N}})_{m\in\mathbb{N}}$. It is also continuous with respect to the p-adic topology on $\mathcal{O}_{\mathbb{C}_K}$.

Note also that the map $\psi_n : \mathbb{W}(R) \to \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$ defined by:

$$(s_0, s_1, ...) \to (s_0^{(n)}(p), s_1^{(n)}(p), ..., s_{n-1}^{(n)}(p))$$

verifies the relation: $\psi_n = u_n \circ \psi_{n+1}$.

We want an explicit formula for $\theta : \mathbb{W}(R) \to \mathcal{O}_{\mathbb{C}_K}$ so let us compute it on the Teichmueller lifts. For $r = (r^{(n)})_{n\geq 0} \in R$ (which is sent to $[r] = (r, 0, 0, ...)$ via the map $R \to \mathbb{W}(R)$), we have that:

$$\begin{aligned}\theta([r]) &= \varprojlim \theta_n(\psi_n([r])) = \varprojlim \theta_n([r^{(n)}(\text{mod}p)]) \\ &= \varprojlim (r^{(n)})^{p^n}(\text{mod}p^n) = \varprojlim r^{(0)}(\text{mod}p^n) = r^{(0)}.\end{aligned}$$

and hence $\theta(\sum [c_n]p^n) = \sum c_n^{(0)} p^n$.

It follows that for a general Witt vector $(r_0, r_1, ...) = \sum [r_n^{p^{-n}}]p^n$,

$$\theta((r_0, r_1, ...)) = \sum \theta([r_n^{p^{-n}}])p^n = \sum (r_n^{p^{-n}})^{(0)} p^n = \sum r_n^{(n)} p^n.$$

Moreover, θ is surjective since the map $r \to r^{(0)}$ from $R \to \mathcal{O}_{\mathbb{C}_K}$ is surjective.

Choose now $\tilde{p} \in R$ such that $\tilde{p}^{(0)} = p$ (in other words $\tilde{p} = (p, p^{1/p}, p^{1/p^2}, ...) \in R = \varprojlim \mathcal{O}_{\mathbb{C}_K}$ so $v_R(\tilde{p}) = v_p(p) = 1$. Let also $\xi := [\tilde{p}] - p = (\tilde{p}, -1, ...) \in \mathbb{W}(R)$ and remark that $\theta(\xi) = 0$. Moreover, following [BC, Proposition 4.4.3], $ker(\theta)$ is a principal ideal generated by ξ.

The ring of Witt vectors $\mathbb{W}(R)$ is a subring of $\mathbb{W}(R)[\frac{1}{p}]$ and θ induces a G_K- equivariant surjection $\theta_K : \mathbb{W}(R)[\frac{1}{p}] \to \mathcal{O}_{\mathbb{C}_K}[\frac{1}{p}] = \mathbb{C}_K$ and since $\mathbb{W}(R)[\frac{1}{p}]$ is not complete one replaces it with its $ker(\theta_K)$-adic completion, namely:

$$B_{\text{dR}}^{+} := \varprojlim_n \mathbb{W}(R)[\tfrac{1}{p}]/(ker\theta_K)^n.$$

θ_K induces a natural G_K-equivariant surjection $\theta^+_{\mathrm{dR}} : B^+_{\mathrm{dR}} \to \mathbb{C}_K$. Since $\mathbb{W}(R)[\frac{1}{p}]$ is an integral domain and $ker(\theta_K) = ker(\theta)[\frac{1}{p}]$ a principal maximal ideal, the localization ring $\mathbb{W}(R)[\frac{1}{p}]_{ker(\theta_K)}$ is an integral domain (being the localization of one), with maximal ideal that is principal (call it $I := ker(\theta_K)\mathbb{W}(R)[\frac{1}{p}]_{ker(\theta_K)}$) and moreover $\mathbb{W}(R)[\frac{1}{p}]_{ker(\theta_K)}$ is separated for the I-adic topology (see [BC, Corollary 4.4.5]) hence it is noetherian. Consequently (see [Al-Io, Theorem 2.3.15]) it is a discrete valuation ring hence its completion B^+_{dR} is a discrete valuation ring and moreover of residue field $\mathbb{C}_K$ (for further details see [BC, Proposition 4.4.6]).

One defines now the field of p-adic periods B_{dR}:

Definition 2.2.7. B_{dR}:=FracB^+_{dR}.

Remark 2.2.8. B_{dR} is equipped with its natural G_K-action and G_K-stable filtration via the powers of the maximal ideal of B^+_{dR}, in other words, $\mathrm{Fil}^1 B_{\mathrm{dR}}$ is the maximal ideal of B^+_{dR} hence generated by ξ and for all $i \in \mathbb{Z}$, $\mathrm{Fil}^i B_{\mathrm{dR}}$ is the fractional ideal $(Fil^1 B_{\mathrm{dR}})^i$.

Choose now an element $\varepsilon \in R$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ (hence $\varepsilon^{(n)}$ is a primitive p^n-th root of 1) and consider the Teichmueller representant $[\varepsilon] \in \mathbb{W}(R)$. We have that $\theta([\varepsilon] - 1) = \varepsilon^{(0)} - 1 = 0$ hence $[\varepsilon] - 1 \in ker(\theta) \subseteq ker(\theta^+_{dR})$.

We have that $(\varepsilon - 1)^{(0)} = \lim_n(\varepsilon^{(n)} + (-1)^{(n)})^{p^n} = \lim_n(\zeta_{p^n} - 1)^{p^n}$ (for $p \neq 2$) and hence:

$$v_R(\varepsilon - 1) = v_p((\varepsilon - 1)^{(0)}) = \lim_n(p^n v_p(\zeta_{p^n} - 1)) = \lim_n \frac{p^n}{p^{n-1}(p-1)} = \frac{p}{p-1} > 1.$$

Since $[\varepsilon] - 1 \in Fil^1 B_{\mathrm{dR}}$ we get that $[\varepsilon] = 1 + ([\varepsilon] - 1)$ is a 1-unit in B^+_{dR}. Moreover, one obtains a well defined element of B^+_{dR} namely the logarithm

$$t := log([\varepsilon]) = log(1 + ([\varepsilon] - 1)) = \sum_{n \geq 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n}.$$

Concretely, by defining $s_n := \sum_{k=1}^{n}(-1)^{k+1}\frac{([\varepsilon]-1)^k}{k}$, then for $m > n$ we have that

$$s_m - s_n = \sum_{k=n+1}^{m}(-1)^{k+1}\frac{([\varepsilon]-1)^k}{k} \in (ker(\theta_K))^{n+1}$$

hence $|s_m - s_n| < \frac{1}{n+1} \to 0$ in the $ker(\theta_K)$-adic topology. It follows that the sequence $(s_n)_n$ is Cauchy and since B_{dR}^+ is complete with respect to the $ker(\theta_K)$-adic topology, we get that $(s_n)_n$ is convergent.

Following [Fo4, Proposition 3.1] or [BC, Proposition 4.4.8], the element $t = log([\varepsilon])$ is a uniformizer of B_{dR}^+. We have that $\mathrm{Fil}^i B_{\mathrm{dR}} = B_{\mathrm{dR}}^+ t^i$ and moreover note that the action of G_K on $t = log([\varepsilon])$ is given by:

$$\sigma t = \sigma log([\varepsilon]) = log(\sigma[\varepsilon]) = log([\varepsilon^{\chi(\sigma)}]) = log([\varepsilon]^{\chi(\sigma)}) = \chi(\sigma) log[\varepsilon] = \chi(\sigma) t.$$

Consequently,

$$Gr(B_{\mathrm{dR}}) = \oplus_{\imath \in Z} Gr^i(B_{\mathrm{dR}}) = \oplus_{\imath \in Z} \left(\frac{Fil^i B_{\mathrm{dR}}}{Fil^{i+1} B_{\mathrm{dR}}}\right) \cong \oplus_{\imath \in Z} \mathbb{C}_K(\imath) = B_{\mathrm{HT}}.$$

Remark 2.2.9. We also have the important relation:

$$(B_{\mathrm{dR}}^+)^{G_K} = B_{\mathrm{dR}}^{G_K} = K$$

which follows by means of Ax-Sen-Tate's theorem. Concretely, we have a canonical G_K-equivariant embedding $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ and by taking G_K-invariants one obtains a natural map $K \hookrightarrow B_{\mathrm{dR}}^{G_K}$. Since the G_K action on B_{dR} respects the filtration we get an injection $Gr(B_{\mathrm{dR}}^{G_K}) \hookrightarrow (Gr(B_{\mathrm{dR}}))^{G_K} = B_{\mathrm{HT}}^{G_K} = K$ hence $Gr(B_{\mathrm{dR}}^{G_K})$ is 1-dimensional over K which further implies that $B_{\mathrm{dR}}^{G_K}$ is 1-dimensional over K.

One further introduces the covariant functor D_{dR} valued in the category of finite dimensional K-vector spaces Vec_K:

Definition 2.2.10. $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to Vec_K$ given by $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$.

Following [Fo3, Proposition 1.4.2 and Proposition 1.5.2] or [BC, Theorem 5.2.1] we have that $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$. In case of equality one says that V is a de Rham representation.

Let also $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ be the full subcategory of the de Rham representations.

Remark 2.2.11. Note that if $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ then $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$ has a natural structure of object in Fil_K, the category of finite dimensional filtered K-vector spaces with exhaustive and separated filtration. For this, recall that B_{dR} has an exhaustive, separated and G_K-stable K-linear filtration $Fil^i(B_{\mathrm{dR}}) = t^i B_{\mathrm{dR}}^+$ and hence one obtains a G_K-stable K-linear filtration $\{V \otimes_{\mathbb{Q}_p} Fil^i(B_{\mathrm{dR}})\}$ on $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ which further induces (after taking G_K-invariants) an exhaustive and separated filtration on $D_{\mathrm{dR}}(V)$, namely:

$$Fil^i D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K}.$$

The main result in the theory of de Rham representations is the following "de Rham comparison isomorphism theorem":

Theorem 2.2.12 (T. Tsuji (T, Theorem 4.10.2), G. Faltings (Fa3, Theorem 8.1)). *Let X be a smooth, proper geometrically connected scheme over K. Then, for every $i \geq 0$ we have a canonical isomorphism as B_{dr}-vector spaces, respecting the G_K-action and the filtrations*

$$H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{dR}(X/K) \otimes_K B_{\mathrm{dR}}.$$

The theorem has the following

Corollary 2.2.13. *If X is a smooth, proper geometrically connected scheme over K then the p-adic G_K-representations $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ are de Rham and moreover the filtration on $D_{\mathrm{dR}}(H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^i_{\mathrm{dR}}(X/K)$ is the Hodge filtration.*

In some simple examples the corollary above can be actually deduced using only elementary methods, which we'll examine in the next section.

2.2.1 Examples of de Rham representations

Example. $V = \mathbb{Q}_p(n)$ is de Rham for all $n \in \mathbb{Z}$. Viewing $\mathbb{Q}_p(n)$ as $\mathbb{Q}_p$ with G_K-action given by χ^n we have that $D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = Kt^{-n}$ so $\dim_K D_{\mathrm{dR}}(V)$=$\dim_{\mathbb{Q}_p} V = 1$.

Moreover,

$$Fil^i(D_{\mathrm{dR}}(\mathbb{Q}_p(n))) = \begin{cases} 0, & i > -n \\ D_{dR}(\mathbb{Q}_p(n)), & i \leq -n \end{cases}$$

We prove now that the de Rham representations are always Hodge-Tate while the equivalence holds only for the 1-dimensional case:

Proposition 2.2.14. *If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is a de Rham representation then V is Hodge-Tate. Moreover, if $dim_{\mathbb{Q}_p} V = 1$ then V is Hodge-Tate if and only if V is de Rham.*

Proof. We prove firstly that if V is de Rham then V is Hodge-Tate.

Note that $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$ and that we have the i^{th} filtration $Fil^i D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K} \subseteq D_{\mathrm{dR}}(V)$.

One can show by induction that:

$$(1) \qquad \dim_K Gr(D_{\mathrm{dR}}(V)) = \dim_K D_{\mathrm{dR}}(V),$$

where $Gr(D_{\mathrm{dR}}(V)) = \oplus_{\imath \in Z} Gr^i(D_{\mathrm{dR}}(V)) = \oplus_{\imath \in Z}(\frac{Fil^i D_{\mathrm{dR}}(V)}{Fil^{i+1} D_{\mathrm{dR}}(V)})$.

In order to prove this, consider the finite filtration:

$$D_{\mathrm{dR}}(V) = Fil^{io} D_{\mathrm{dR}}(V) \supseteq Fil^{io+1} D_{\mathrm{dR}}(V) \supseteq \ldots \supseteq Fil^{jo} D_{\mathrm{dR}}(V) = 0$$

We have the exact sequence:

$0 \longrightarrow Fil^{io+1} D_{\mathrm{dR}}(V) \subseteq D_{\mathrm{dR}}(V) \longrightarrow Gr^{io} D_{\mathrm{dR}}(V) \longrightarrow 0$ hence:

$$(2) \qquad \dim_K D_{\mathrm{dR}}(V) = \dim_K Gr^{io} D_{\mathrm{dR}}(V) + \dim_K Fil^{io+1} D_{\mathrm{dR}}(V).$$

Similarly, from $0 \longrightarrow Fil^{io+2} D_{\mathrm{dR}}(V) \subseteq Fil^{io+1} D_{\mathrm{dR}}(V) \longrightarrow Gr^{io+1} D_{\mathrm{dR}}(V) \longrightarrow 0$,

(2) becomes:

$\dim_K D_{\mathrm{dR}}(V) = \dim_K Gr^{io} D_{\mathrm{dR}}(V) + \dim_K Gr^{io+1} D_{\mathrm{dR}}(V) + \dim_K Fil^{io+2} D_{\mathrm{dR}}(V)$.

We continue the procedure and since

$0 = Fil^{jo} D_{\mathrm{dR}}(V) \subseteq Fil^{jo+1} D_{\mathrm{dR}}(V) \cong Gr^{jo+1} D_{\mathrm{dR}}(V)$, we obtain that:

$\dim_K D_{\mathrm{dR}}(V) = \oplus_{i=j_0+1}^{i_0} \dim_K Gr^i(D_{\mathrm{dR}}(V))$.

Now, recall that $D_{\mathrm{HT}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K} = (V \otimes_{\mathbb{Q}_p} (\oplus_{i \in Z} C_p(i)))^{G_K} =$

$= \oplus_{i \in Z}(V \otimes_{\mathbb{Q}_p} C_p(i))^{G_K}$

and that V is Hodge-Tate if and only if $\dim_K D_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p}(V)$.

We want to prove that if V is de Rham then V is Hodge-Tate. Consider the exact sequence of G_K modules:

$$(3) \qquad 0 \longrightarrow t^{i+1} B_{\mathrm{dR}}^+ \longrightarrow t^i B_{\mathrm{dR}}^+ \longrightarrow C_p(i) \longrightarrow 0.$$

We've used the fact that since tB_{dR}^+ is maximal ideal of B_{dR}^+, one has:

$$\frac{t^i B_{\mathrm{dR}}^+}{t^{i+1} B_{\mathrm{dR}}^+} = t^i\left(\frac{B_{\mathrm{dR}}^+}{tB_{\mathrm{dR}}^+}\right) = t^i C_p \cong C_p(i).$$

By tensoring (3) with V and taking G_K invariants we obtain:

$$0 \longrightarrow (V \otimes_{\mathbb{Q}_p} t^{i+1} B_{\mathrm{dR}}^+)^{G_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} t^i B_{\mathrm{dR}}^+)^{G_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} C_p(i))^{G_K} \longrightarrow H^1(G_K, V \otimes_{\mathbb{Q}_p} t^{i+1} B_{\mathrm{dR}}^+) \longrightarrow \cdots$$

i.e. $0 \longrightarrow Fil^{i+1} D_{\mathrm{dR}}(V) \longrightarrow Fil^i D_{\mathrm{dR}}(V) \longrightarrow Gr^i(D_{\mathrm{HT}}(V)) \longrightarrow \ldots$.

Hence we obtain an injection:

$Gr^i(D_{\mathrm{dR}}(V)) = \frac{Fil^i D_{\mathrm{dR}}(V)}{Fil^{i+1} D_{\mathrm{dR}}(V)} \hookrightarrow Gr^i(D_{\mathrm{dR}}(V))$ and consequently:

$$(4) \qquad Gr(D_{\mathrm{dR}}(V) \subseteq D_{\mathrm{HT}}(V).$$

Recall now that $\dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$ and that since V is de Rham, $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$. By using now (1), (4) gives us:

$$\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR}}(V) = \dim_K Gr(D_{\mathrm{dR}}(V)) \leq \dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$$

so we have equality everywhere and hence V is Hodge-Tate. This completes the proof of the first implication.

We prove now that if V is HT and $\dim_{\mathbb{Q}_p} V$=1 then V is de Rham.

Firstly, since $\dim_{\mathbb{Q}_p} V$=1, via the subsection 1.3.1, we have that the dimension 1 representations correspond to characters hence $V = \mathbb{Q}_p(\varphi)$ where $\varphi : G_K \to \mathbb{Z}_p^*$ is a continuous character.

V is HT and of dimension 1 so $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p(i)$ as G_K-modules for some $i \in \mathbb{Z}$.

In other words, we get that $\mathbb{Q}_p(\varphi) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathbb{C}_p(i)$ and consequently the isomorphism: $\mathbb{C}_p(\varphi) \cong \mathbb{C}_p(i)$ as G_K-modules.

We twist now by χ^{-i} where $\chi : G_K \to \mathbb{Z}_p^*$ is the cyclotomic character and by letting $\psi := \varphi\chi^{-i}$ we further obtain:

$$\mathbb{C}_p(\psi) = \mathbb{C}_p(\varphi\chi^{-i}) \cong \mathbb{C}_p.$$

We claim that it is enough to show that $D_{\mathrm{dR}}(V) \neq \phi$.

Then, since $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V = 1$ we obtain the equality of dimensions:

$$\dim_K D_{\mathrm{dR}}(V) = 1 = \dim_{\mathbb{Q}_p} V$$

hence V is de Rham.

Proof of the Claim: We have a G_K-equivariant map: $f : \mathbb{C}_p \cong \mathbb{C}_p(\psi)$ and let $f(1) := \gamma$.

Take $\sigma \in G_K$. We then have:

$$\sigma * \gamma = \sigma * f(1) = f(\sigma \cdot 1) = f(1) = \gamma.$$

On the other hand, $\sigma * f(1) = \psi(\sigma)\sigma(f(1)) = \psi(\sigma)\sigma(\gamma)$

and so we get that $\gamma = \psi(\sigma)\sigma(\gamma)$ or equivalently $\psi^{-1}(\sigma)\gamma = \sigma(\gamma)$ (5).

Consider now $x \in \mathcal{O}^*_{\widehat{K^{ur}}} \subseteq B^+_{\mathrm{dR}}$. Following (5) we have that $\sigma x = \psi^{-1}(\sigma)x$ for $\sigma \in G_K$.

Let now $e := 1 \otimes xt^{-i} \in V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$. Note that $\sigma(1) = \varphi(\sigma)$ since $1 \in V = \mathbb{Q}_p(\varphi)$.

We have that:

$$\begin{aligned}\sigma e &= \sigma(1) \otimes \sigma(x)\sigma(t^{-i}) = \varphi(\sigma) \otimes \psi^{-1}(\sigma)x\chi^{-i}(\sigma)t^{-i} \\ &= \varphi(\sigma) \otimes \varphi^{-1}(\sigma)xt^{-i} = 1 \otimes xt^{-i} = e\end{aligned}$$

and hence that $e \in (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} = D_{\mathrm{dR}}(V)$. Moreover, $D_{\mathrm{dR}}(V) = K \cdot e$.

The claim follows.

□

Remark 2.2.15. The functor $D_{dR} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to Fil_K$ is faithful, exact and compatible with the tensor product and duality ([BC, Proposition 6.3.3]) but not full.

We prove that D_{dR} is not full. Firstly write $D_{\mathrm{dR,K}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$ for accuracy. Following [BC, Proposition 6.3.8], for any complete discretely-valued extension K'/K inside $\mathbb{C}_K$, the natural map $D_{\mathrm{dR,K}}(V) \otimes_K K' \to D_{\mathrm{dR,K'}}(V)$ is an isomorphism in $Fil_{K'}$. In particular, V is de Rham as a G_K-representation if and only if V is de Rham as a $G_{K'}$- representation.

As consequence, we claim that if $\rho : G_K \to Aut(V)$ is a p-adic representation with finite image on I_K, then V is de Rham and $D_{\mathrm{dR,K}}(V) = (K\langle 0\rangle)^{\oplus \dim_{\mathbb{Q}_p}(V)}$.

It is then clear that $D_{\mathrm{dR,K}}$ is not full since $D_{\mathrm{dR,K}}(V) \in Fil_K$ has lost all information about V.

Now, for the proof of the above claim, choose L/K finite extension with $\rho(I_L) = 1$ and let $K' := \widehat{L^{ur}}$ so in particular $G_{K'} = I_L$. Since $V^{G_{K'}} = V$ we have that $D_{\mathrm{dR,K'}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{K'}} = V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}{}^{G_{K'}} = V \otimes_{\mathbb{Q}_p} K' = (K'\langle 0\rangle)^{\oplus \dim_{\mathbb{Q}_p}(V)}$ and hence $\dim_{K'} D_{\mathrm{dR,K'}}(V) = \dim_{\mathbb{Q}_p} V$.

It follows that V is de Rham as a $G_{K'}$-representation and by the above remark as a G_K-representation. Since $D_{\mathrm{dR,K}}(V) \otimes_K K' \cong (K'\langle 0\rangle)^{\oplus \dim_{\mathbb{Q}_p}(V)}$ the result follows.

Remark 2.2.16. We claim that the Frobenius automorphism $\varphi : \mathbb{W}(R)[\frac{1}{p}] \to \mathbb{W}(R)[\frac{1}{p}]$ does not preserve $ker(\theta_K)$.

Recall that $ker(\theta_K) = ker(\theta)[\frac{1}{p}]$ is principal ideal generated by $\xi := [\tilde{p}] - p = (\tilde{p}, -1, ...) \in \mathbb{W}(R)$ so it is enough to show that $\theta_K(\varphi([\tilde{p}] - p)) \neq 0$.

We have that:

$$\varphi([\tilde{p}] - p) = \varphi([\tilde{p}]) - p\varphi(1) = [\tilde{p}]^p - p = [\tilde{p}^p] - p$$

hence $\theta_K(\varphi([\tilde{p}]-p)) = \theta_K([\tilde{p}^p]) - p = p^p - p \neq 0$ in $\mathbb{C}_K = \mathcal{O}_{\mathbb{C}_K}[\frac{1}{p}]$.

It follows that φ does not naturally extend to $B^+_{\mathrm{dR}} := \varprojlim_n \mathbb{W}(R)[\frac{1}{p}]/(ker\theta_K)^n$. (One can also see this by taking the element $[\tilde{p}^{1/p}] - p$ which is invertible in B^+_{dR} but if $\varphi : B^+_{\mathrm{dR}} \to B^+_{\mathrm{dR}}$ would be a natural extension of $\varphi : \mathbb{W}(R)[\frac{1}{p}] \to \mathbb{W}(R)[\frac{1}{p}]$ then we would have that $\varphi(1/([\tilde{p}^{1/p}] - p)) = 1/([\tilde{p}] - p) \notin B^+_{\mathrm{dR}}$ since $\theta_K([\tilde{p}] - p) = 0$.)

So one would like to complete $\mathbb{W}(R)[\frac{1}{p}]$ such that the completion is still endowed with a Frobenius map. For this one defines a subring of B^+_{dR}, namely:

2.3 Crystalline theory

In this paragraph, K_0 will be the maximal unramified extension of $\mathbb{Q}_p$ in K.

We have the following definition:

Definition 2.3.1. $B^+_{\mathrm{cris}} = \{x \in B^+_{\mathrm{dR}} \mid x = \sum_{n=0}^{\infty} x_n \frac{\xi^n}{n!}$ such that $x_n \to 0$ in $\mathbb{W}(R)[\frac{1}{p}]\}$.

Also, let $B_{\mathrm{cris}} := B^+_{\mathrm{cris}}[\frac{1}{t}]$.

Definition 2.3.2. Let K be a p-adic field.

1) A filtered φ-module over K is a triple $(D, \varphi, Fil^\bullet)$ where D is a finite dimensional K_0-vector space, φ is a Fr-semilinear (i.e. $\varphi(a \cdot d) = Fr(a) \cdot \varphi(d)$, for any $a \in K_0$ and $d \in D$) and bijective endomorphism of D, where Fr is the Frobenius automorphism of $K_0 = \mathbb{W}(k)[1/p]$ and $\{Fil^i\}$ is a decreasing exhaustive and separated filtration on $D_K = D \otimes_{K_0} K$.

A morphism $D' \to D$ between two filtered φ-modules is a K_0-linear map compatible with $\varphi' : D' \to D'$ and also $\varphi : D \to D$ and has scalar extension $D'_K \to D_K$ that is a morphism in Fil_K. One denotes by MF^{φ}_K the category of filtered φ-modules over K.

2) A (φ, N)-module over K_0 is a finite dimensional K_0-vector space equipped with a bijective Frobenius semilinear endomorphism $\varphi : D \to D$ (i.e. an isocrystal over K_0) equipped with a K_0-linear endomorphism $N : D \to D$ (called monodromy operator)

such that $N \circ \varphi = p\varphi \circ N$. The notion of morphism between such objects is the obvious one. One denotes by $Mod_{K_0}^{\varphi,N}$ the category of (φ, N)-modules over K_0.

A filtered (φ, N)-module over K is a (φ, N)-module D over K_0 for which $D_K = D \otimes_{K_0} K$ is endowed with a structure of object in Fil_K. The notion of morphism between such objects is the obvious one. One denotes by $MF_K^{\varphi,N}$ the category of filtered (φ, N)-modules over K.

Further, let A^0_{cris} be the $\mathbb{W}(R)$-subalgebra of $\mathbb{W}(R)[1/p]$ generated by the elements $\{\frac{\xi^n}{n!}\}_{n\in\mathbb{N}}$, in other words A^0_{cris} is the divided power envelope of $\mathbb{W}(R)$ with respect to the ideal $\xi\mathbb{W}(R)$ where $\xi = [\tilde{p}] - p$. Also let A_{cris} be the p-adic completion of A^0_{cris}:

$$A_{\text{cris}} = \{\textstyle\sum_{n\geq 0} a_n \frac{\xi^n}{n!} \mid a_n \in \mathbb{W}(R), a_n \to 0 \text{ in the } p\text{-adic topology}\}.$$

We have the following:

Proposition 2.3.3. *$t = log[\varepsilon] \in A_{cris}$.*

Proof. Since $[\varepsilon] - 1 \in ker(\theta) = \xi\mathbb{W}(R)$, it follows that $[\varepsilon] - 1 = v\xi$ for some $v \in \mathbb{W}(R)$. Moreover,

$$t = log([\varepsilon]) = \textstyle\sum_{n\geq 1}(-1)^{n+1}\frac{v^n\xi^n}{n} = \sum_{n\geq 1}(-1)^{n+1}(n-1)!v^n\frac{\xi^n}{n!}$$

and so, since $a_n := (-1)^{n+1}(n-1)!v^n \to 0$ in $\mathbb{W}(R)$ relative to the p-adic topology (remark that $v_p((n-1)!) \to \infty$ when $n \to \infty$), we get $t = \sum_{n\geq 1} a_n \frac{\xi^n}{n!} \in A_{\text{cris}}$.

□

Proposition 2.3.4. *([T, Lemma A3.1]) We have that $([\varepsilon] - 1)^{p-1} \in pA_{\text{cris}}$.*

Proof. Denote by $e := (\varepsilon_{n+1})_{n\geq 0}$ (so $e^p = \varepsilon$) and let also $s := ([e]-1)^{p-1} + \sum_{k=1}^{p-1} C_p^k([e]-1)^{p-k-1}$. We then have:

$$\begin{aligned} s \cdot ([e] - 1) &= \textstyle\sum_{k=0}^{p-1} C_p^k([e] - 1)^{p-k} \\ &= ([e] - 1 + 1)^p - 1 = [\varepsilon] - 1 \end{aligned}$$

and consequently one obtains:

$$
\begin{aligned}
([\varepsilon]-1)^{p-1} &= s^{p-1}\cdot([e]-1)^{p-1} \\
&= s^{p-1}(s-\sum_{k=1}^{p-1} C_p^k([e]-1)^{p-k-1}) \\
&= p!\tfrac{s^p}{p!} - s^{p-1}p\sum_{k=1}^{p-1}\tfrac{1}{p}C_p^k([e]-1)^{p-k-1} \in pA_{\text{cris}}.
\end{aligned}
$$

□

Corollary 2.3.5. *([T, Corollary A3.2])* $t^{p-1} \in pA_{\text{cris}}$.

Proof. For any $n \geq p+1$, since $(n-1)!$ is divisible by p, we have that $\frac{([\varepsilon]-1)^n}{n} = (n-1)!\frac{([\varepsilon]-1)^n}{n!} \in pA_{\text{cris}}$ hence:

$$t \equiv \sum_{n=1}^{p}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \mod pA_{cris},$$

in other words it is enough to consider the above finite truncation of the sum.

By the previous Proposition we have that $p^{-1}([\varepsilon]-1)^{p-1} \in A_{\text{cris}}$ and hence $t \equiv r([\varepsilon]-1) \mod pA_{\text{cris}}$, $r \in A_{\text{cris}}$, since the terms for $1 \leq n < p$ are A_{cris}-multiples of $[\varepsilon]-1$. We apply again the previous Proposition and get that $t^{p-1} \in pA_{\text{cris}}$.

□

Denote by φ the Frobenius endomorphism of $\mathbb{W}(R)[\frac{1}{p}]$. The answer to the question of how does φ act on the subring A^0_{cris} is provided by the following important:

Lemma 2.3.6. *The $\mathbb{W}(R)$-subalgebra A^0_{cris} is φ-stable and also G_K-stable.*

Proof. We have that $\varphi(\xi) = [\tilde{p}]^p - p = (\xi+p)^p - p = \xi^p + p\alpha$ for some $\alpha \in \mathbb{W}(R)$ hence:

$$\varphi(\xi) = p\cdot((p-1)!\tfrac{\xi^p}{p!}+\alpha)$$

and so $\varphi(\xi^n) = p^n\cdot((p-1)!\frac{\xi^p}{p!}+\alpha)^n$ for all $n \geq 1$. Since $\frac{p^n}{n!} \in \mathbb{Z}_p$ for all $n \geq 1$ we obtain that $\varphi(\xi^n/n!) \in A^0_{\text{cris}}$ for all $n \geq 1$ and since A^0_{cris} is generated by the elements $\{\xi^n/n!\}_n$, the first claim follows.

Now, since θ is G_K-equivariant and $\theta(\xi) = 0$ we have that $\theta(\sigma(\xi)) = 0$ for any $\sigma \in G_K$ so $\sigma(\xi) \in ker(\theta) = \xi\mathbb{W}(R)$. Consequently, $\sigma(\xi) = c(\sigma)\xi$, with $c(\sigma) \in \mathbb{W}(R)$, for any $\sigma \in G_K$.

□

Remark 2.3.7. One extends by continuity φ and the action of G_K to A_{cris}.

Following the Definition 2.3.1, we have that $B^+_{\text{cris}} = A_{\text{cris}}[\frac{1}{p}]$ and that $B_{\text{cris}} = B^+_{\text{cris}}[\frac{1}{t}] = A_{\text{cris}}[\frac{1}{p}, \frac{1}{t}] = A_{\text{cris}}[\frac{1}{t}]$ (inverting t makes p become a unit since $t^{p-1} \in pA_{\text{cris}}$ via Corollary 2.3.5).

The rings B_{cris} and B^+_{cris} are G_K-stable $\mathbb{W}(R)[\frac{1}{p}]$-subalgebras of B_{dR} and B^+_{dR} respectively.

We compute now:

$$\varphi(t) = \varphi(log[\varepsilon]) = log(\varphi([\varepsilon])) = log([\varepsilon^p]) = log([\varepsilon]^p) = plog[\varepsilon] = pt$$

and further extend φ to B_{cris} by putting $\varphi(t^{-1}) = p^{-1}t^{-1}$.

One further defines the following functors:

Definition 2.3.8. 1) $D_{\text{cris}} : \text{Rep}_{\mathbb{Q}_p}(G_K) \to \text{Vec}_{K_0}$ given by $D_{\text{cris}} := (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K}$;

2) $D_{\text{st}} : \text{Rep}_{\mathbb{Q}_p}(G_K) \to \text{Vec}_{K_0}$ given by $D_{\text{cris}} := (V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{G_K}$.

We have that $\dim_{K_0} D_{\text{cris}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ ([Fo3, Proposition 1.4.2 and Proposition 1.5.2] or [BC, Theorem 5.2.1]) and we say that V is crystalline if the equality holds. Denote by $\text{Rep}^{\text{cris}}_{\mathbb{Q}_p}$ the full subcategory of crystalline p-adic representations of G_K.

Also, $\dim_{K_0} D_{\text{st}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ and we say that V is semi-stable if the equality holds. Similarly, denote by $\text{Rep}^{\text{st}}_{\mathbb{Q}_p}$ the full subcategory of semi-stable p-adic representations of G_K.

Remark 2.3.9. Note that there is a natural exhaustive and separated descending filtration on $D_{\text{cris}}(V) \otimes_{K_0} K$ via the natural injection on $D_{\text{dR}}(V)$. Recall that we've extended the action of the Frobenius endomorphism φ to A_{cris} and B_{cris}. Following [BC, Theorem 9.1.8], φ is injective on A_{cris} and in particular, the induced Frobenius on $B_{\text{cris}} = A_{\text{cris}}[1/t]$ is also injective.

One obtains that D_{cris} is valued in MF_K^{φ}. As D_{dR}, the covariant functor

$D_{\text{cris}} : \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K) \to MF_K^{\varphi}$ is exact and commutes with tensor products and duals.

Moreover, one can prove that if V is a crystalline Galois representation then we have an isomorphism as B_{cris}-modules which respects Galois actions, Frobenius and filtrations

$$D_{\text{cris}}(V) \otimes_{K_0} B_{\text{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\text{cris}}$$

([BC, Proposition 9.1.9]) and by using [Fo2, Theorem 5.3.7] that D_{cris} is fully faithful ([BC, Proposition 9.1.11]). This is a non-trivial result and recall that D_{dR} and D_{HT} are not full.

The central result in the crystalline theory is the following "crystalline comparison isomorphism theorem":

Theorem 2.3.10 (Fa3, Theorem 5.6). *Let X be a smooth proper scheme, geometrically connected over K with good reduction. Then for every $i \geq 0$ we have canonical isomorphisms as B_{cris}-modules, which respects the G_K-actions, the Frobenii and the filtrations.*

$$H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H^i_{\text{cris}}(\overline{X}/W) \otimes_W B_{\text{cris}},$$

where we have denoted $\overline{X}$ the special fiber of a smooth model of X over $\mathcal{O}_K$ and $W := W(k)$ and $K_0 := W[1/p]$.

The above theorem has the immediate consequence

Corollary 2.3.11. *Let X be as in the theorem 2.3.10, then for every $i \geq 0$ the G_K-representation $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline and moreover*

$D_{\text{cris}}\big(H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \cong H^i_{\text{cris}}(\overline{X}/W) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as Frobenius modules. The filtration on $D_{\text{cris}}\big(H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)\big) \otimes_{K_0} K \cong H^i_{\text{dR}}(X/K)$ is the Hodge filtration.

We arrive now to the main characters of the thesis, namely the rings $A_{\max}$, $B^+_{\max}$ and $B_{\max}$. These were first defined by P. Colmez in [Col, Chapter 3, Section 2].

Definition 2.3.12. Let $A_{\max}$ denote the p-adic completion of the ring $A_{\mathrm{dil}} := A_{\mathrm{inf}}[Y_0]/(pY_0 - \xi)$, where $A^+_{\mathrm{inf}} := \mathrm{W}(R)$ and recall that $\xi = [\tilde{p}] - p$ is a generator of the ideal $\mathrm{Ker}(\theta : A_{\mathrm{inf}} \longrightarrow \mathcal{O}_{\mathbb{C}_p})$ and Y_0 is a variable. One observes that $A_{\max}$ is p-torsion free and denote by $B^+_{\max} := A_{\max} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Moreover the series $t = \sum_{n=1}^{\infty}(-1)^{n-1}([\varepsilon]-1)^n/n$ converges in $A_{\max}$ and we denote by $B_{\max} := A_{\max}[1/t]$.

The group G_K acts naturally on $A_{\max}, B^+_{\max}, B_{\max}$ and the natural Frobenius on A^+_{inf} extends to a Frobenius on all three rings. We have natural inclusions of rings $A_{\mathrm{cris}} \subset A_{\max}$ and $B_{\mathrm{cris}} \subset B_{\max} \subset B_{\mathrm{dR}}$ which are G_K-equivariant.

The main usefulness of $B_{\max}$ is that it allows to the calculation of the functor D_{cris}. More precisely, Colmez proved in [Col, Chapter 3, Section 4] the following

Theorem 2.3.13 (Colmez). *Let V be a p-adic representation of G_K. Then the inclusion $B_{\mathrm{cris}} \subset B_{\max}$ induces an isomorphism as filtered, Frobenius modules:*

$$D_{\mathrm{cris}}(V) \cong \left(V \otimes_{\mathbb{Q}_p} B_{\max}\right)^{G_K}.$$

Geometric interpretation of the ring $A_{\max}$

We claim that $A_{\max}$ is a **formal dilation**, i.e. we are claiming that the ring A_{dil}, it is a **dilation** in the sense of Bosch, Luetkebohmert, Raynaud, Néron models. More precisely we have an isomorphism of rings $A^+_{\mathrm{inf}}/pA^+_{\mathrm{inf}} \cong R$, moreover the natural projection on the first component gives a ring homomorphism $R \longrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ whose kernel is generated by $\tilde{p} \in R$ (see the beginning of Chapter 3). In other words we have ring homomorphisms $A^+_{\mathrm{inf}} \longrightarrow R \longrightarrow \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and the kernel of the composition is the ideal of A^+_{inf} generated by (p, ξ). Let $X := \mathrm{Spec}(A^+_{\mathrm{inf}})$ and denote by $\overline{X} := \mathrm{Spec}(R)$

its special fiber. We have closed immersions of affine schemes:

$$Y := \mathrm{Spec}(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \hookrightarrow \overline{X} \hookrightarrow X$$

and it follows that $\mathrm{Spec}(A_{\mathrm{dil}})$ is the dilation of Y in X, in other words it is the affine ring of a certain open of the blowing-up of X at the ideal $I = (p, \xi)$. Therefore A_{max} can be seen as the affine ring of the formal completion along its special fiber of the above mentioned dilation.

As such A_{max} has a natural universal property (see [BLR, Proposition 3.2.1(b)]).

In the next two chapters we will discuss modp^n versions of A_{max} and sheafified versions of these constructions.

2.4 Semi-stable theory

Definition 2.4.1. Denote by $B_{\mathrm{st}} := B_{\mathrm{cris}}[log[\tilde{p}]]$ the polynomial algebra with coefficients in B_{cris} and also let $u := log[\tilde{p}]$.

One extends the action of φ and also of G_K to B_{st} by putting:

$\varphi(log[\tilde{p}]) := p \cdot log[\tilde{p}]$

and $\sigma(log[\tilde{p}]) := log[\tilde{p}] + \alpha(\sigma)t$, for any $\sigma \in G_K$, where $\alpha(\sigma) \in \mathbb{Z}_p$ such that $\sigma(\tilde{p}) = \varepsilon^{\alpha(\sigma)}\tilde{p}$.

N is called the B_{cris}-derivation of B_{st} normalized by $N(u) = -1$.

One verifies that $N\varphi = p\varphi N$ (note that $N\varphi(u) = -p = p\varphi N(u)$) and that the action of G_K commutes with φ and also with N since:

$$\sigma(Nu^n) = \sigma(nu^{n-1}) = n(u + \alpha(\sigma)t)^{n-1}$$
$$\text{and } N(\sigma u^n) = N((u + \alpha(\sigma)t)^n) = n((u + \alpha(\sigma)t)^{n-1})$$

for any $\sigma \in G_K$, $n \in \mathbb{N}$.

Remark 2.4.2. We have the following important result:

$$B_{\text{cris}}^{G_K} = B_{\text{st}}^{G_K} = K_0.$$

Note that since $\mathbb{W}(k) \subseteq \mathbb{W}(R) \subseteq A_{\text{cris}}$ one obtains that $K_0 = \mathbb{W}(k)[1/p] \subseteq B_{\text{cris}}$ and further since $B_{\text{cris}} \subseteq B_{\text{st}} \subseteq B_{\text{dR}}$ we get: $K_0 \subseteq B_{\text{cris}}^{G_K} \subseteq B_{\text{st}}^{G_K} \subseteq B_{\text{dR}}^{G_K} = K$.

Following [Fo1, Proposition 4.7], the natural G_K-equivariant map

$B_{\text{cris}} \otimes_{K_0} K \to B_{\text{dR}}$ is injective. Moreover, following [Fo2, Theorem 4.2.4] or [BC, Theorem 9.2.10], the homomorphism of $B_{\text{cris}} \otimes_{K_0} K$-algebras $B_{\text{st}} \otimes_{K_0} K \to B_{\text{dR}}$ (which sends $u \in B_{\text{st}}$ to $u \in B_{\text{dR}}$) is injective. The result follows by using the injectivity of the second map (the injectivity of the first one only leads to $B_{\text{cris}}^{G_K} = K_0$).

Remark 2.4.3. We have that $B_{\text{st}}^{N=0} = B_{\text{cris}}$. This follows easily since by taking an element $f = \sum_{n=0}^{m} a_n u^n \in B_{\text{st}}^{N=0}$, where $a_n \in B_{\text{cris}}$ for all $0 \leq n \leq m$, then $N(f) = 0$ is equivalent to: $\sum_{n=1}^{m} n \cdot a_n u^{n-1} = 0$ and consequently $a_n = 0$ for all $1 \leq n \leq m$ hence $f = a_0 \in B_{\text{cris}}$.

We use now this remark and the fact that G_K commutes with N (previously proved) and obtain that:

$$D_{\text{st}}^{N=0}(V) = ((V \otimes_{\mathbb{Q}_p} B_{\text{st}})^{G_K})^{N=0} = (V \otimes_{\mathbb{Q}_p} B_{\text{st}}^{N=0})^{G_K} = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K} = D_{\text{cris}}(V)$$

in MF_K^{φ} for all $V \in \text{Rep}_{\mathbb{Q}_p}(G_K)$.

Consequently, if V is semi-stable and the monodromy operator N vanishes on $D_{\text{st}}(V)$, then $D_{\text{cris}}(V) = D_{\text{st}}(V)$ and so $\dim_{K_0} D_{\text{cris}}(V) = \dim_{K_0} D_{\text{st}}(V) = \dim_{\mathbb{Q}_p}(V)$ hence V is crystalline.

Also, if V is crystalline then:

$\dim_{\mathbb{Q}_p}(V) = \dim_{K_0} D_{\text{cris}}(V) = \dim_{K_0} D_{\text{st}}^{N=0}(V) \leq \dim_{K_0} D_{\text{st}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ and so V is semi-stable hence the crystalline representations are semi-stable.

We conclude that if one works with semi-stable representations, by observing if N vanishes or not one keeps track of the crystalline representations.

As for the crystalline case, by using now the additional structure on B_{st}, we have that D_{st} is valued in $MF_K^{\varphi,N}$ and same as D_{cris}, the functor $D_{\text{st}} : \text{Rep}_{\mathbb{Q}_p}^{\text{st}}(G_K) \to MF_K^{\varphi,N}$

is exact, commutes with tensor products and duals and is fully faithful.

Here are some examples of semi-stable and crystalline representations:

1. $\mathbb{Q}_p(n)$ is crystalline for all $n \in \mathbb{Z}$.

Since $\overline{K} \hookrightarrow R$ and since $\mathbb{W}(R)[1/p] \subseteq B_{\text{cris}}$ (see the remark 2.4.2) one also has the inclusion $\widehat{K_0^{ur}} = \mathbb{W}(\overline{K})[1/p] \subseteq B_{\text{cris}}$. Note that $t^n \in B_{\text{cris}}$ for all $n \in \mathbb{Z}$, where $t = log[\varepsilon]$ and since $\varphi : B_{\text{cris}} \to B_{\text{cris}}$ is compatible with the Frobenius automorphism of $\widehat{K_0^{ur}}$ it follows that $\varphi(t^n) = p^n \cdot t^n$(recall that $\varphi(t) = p \cdot t$).

Thus $D_{\text{cris}}(\mathbb{Q}_p(n)) := (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K}$ on which G_K acts through χ^{-n} has basis $\{e \otimes t^{-n}\}$ where $\{e\}$ is the basis of $\mathbb{Q}_p(n)$ over $\mathbb{Q}_p$. Since G_K acts on $\mathbb{Q}_p(n)$ through χ^n we have that $\sigma \cdot (e \otimes t^{-n}) = \chi^n(\sigma)e \otimes \chi^{-n}(\sigma)t^{-n} = e \otimes t^{-n}$ for any $\sigma \in G_K$, in other words that $\{e \otimes t^{-n}\}$ is G_K-equivariant.

We obtain that $\dim_{K_0} D_{\text{cris}}(\mathbb{Q}_p(n)) = 1 = \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ hence $\mathbb{Q}_p(n)$ is crystalline.

2. Let A be an abelian variety over K. Then following [CI1, Theorem 4.7], $V_A := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline if and only if A has good reduction.

As a consequence, if E/K is an elliptic curve with good reduction over $\mathcal{O}_K$ then V_E is crystalline.

3. If E/K is an elliptic curve with semi-stable and bad reduction over $\mathcal{O}_K$ then following [Br, Theorem 5.3.2], V_E is semi-stable and not crystalline.

4. Suppose that $[k : k^p] = p^d < \infty$, where k is the residue field of K, K being a finite extension of $\mathbb{Q}_p$ and let K_0 be a closed subfield of K, of the same residue field k and absolutely unramified. Let $\{\bar{t}_1, ..., \bar{t}_d\}$ be a p-basis of k and $t_1, ..., t_d$ be the liftings of $\bar{t}_1, ..., \bar{t}_d$ in $\mathcal{O}_{K_0}$. Let $i \in \{1, ..., d\}$ and $X_i := \mathbb{G}_m/(t_i^{\mathbb{Z}})$. Moreover, denote by:

$$T_p(X_i) := \varprojlim_n (\overline{K}^{\times}/t_i^{\mathbb{Z}})_{p^n-tors}$$

its Tate module. Let $\varepsilon = (\varepsilon^{(0)}, \varepsilon^{(1)}, ...) \in R = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ (so $\varepsilon^{(n)}$ is a primitive p^n-th root of 1). We then have:

$$T_p(X_i) := \varprojlim_n \{(\varepsilon^{(n)})^i (t_i^{(n)})^j, 0 \leq i, j < p^n\}$$

and hence $T_p(X_i) = \mathbb{Z}_p e \oplus \mathbb{Z}_p f$ where $e = \varprojlim_n \varepsilon^{(n)}$ and $f = \varprojlim_n t_i^{(n)}$. As computed before (see subsection 1.3.2, *Case 1*), the action of G_K on $V_p(X_i)$ is given by $\sigma e = \chi(\sigma) e$ and $\sigma f = f + a_i(\sigma) e$ where $a_i : G_K \to \mathbb{Z}_p$ is the 1-cocycle describing the action of G_K on $(t_i^{(n)})$, namely $\sigma t_i^{(n)} = (\varepsilon^{(n)})^{a_i(\sigma)} t_i^{(n)}$ and so the matrix of σ in the basis (e, f) is:

$$\begin{pmatrix} \chi(\sigma) & a_i(\sigma) \\ 0 & 1 \end{pmatrix}.$$

Recall that the action of G_K on $t = log([\varepsilon])$ is given by:

$$\sigma t = \sigma log([\varepsilon]) = log(\sigma[\varepsilon]) = log([\varepsilon^{\chi(\sigma)}]) = log([\varepsilon]^{\chi(\sigma)}) = \chi(\sigma) log[\varepsilon] = \chi(\sigma) t.$$

Let now $e' := t^{-1} e$. We then have that $\sigma e = \chi^{-1}(\sigma) t^{-1} \chi(\sigma) e = t^{-1} e = e'$ for all $\sigma \in G_K$ and so e' is G_K-invariant.

Also let $\alpha_i := log(t_i^{-1}[\tilde{t}_i])$ where $\tilde{t}_i \in R$ such that $\tilde{t}_i^{(0)} = t_i$ and $u_i = [\tilde{t}_i] - t_i$, $1 \leq i \leq d$. Via [Bri2, Proposition 2.3.7], we have that:

$$\alpha_i := log(1 + t_i^{-1} u_i) = \textstyle\sum_{n=1}^{\infty} (-1)^{n-1} t_i^{-n} (n-1)! \frac{u_i^n}{n!} \in A_{\text{cris}}.$$

The action of G_K on α_i is given by:

$$\sigma \alpha_i = log(\sigma t_i^{-1} \sigma[\tilde{t}_i]) = log(t_i^{-1} [\varepsilon^{(n)}]^{a_i(\sigma)} [\tilde{t}_i])$$

since $\sigma t_i = \sigma \tilde{t}_i^{(0)} = (\varepsilon^{(0)})^{a_i(\sigma)} \tilde{t}_i^{(0)} = \tilde{t}_i^{(0)} = t_i$. We obtain that $\sigma \alpha_i = \alpha_i + a_i(\sigma) t$.

Define now $f' := -t^{-1} \alpha_i e + f$ and for $\sigma \in G_K$ we have that:

$$\begin{aligned} \sigma f' &= -\chi^{-1}(\sigma) t^{-1} (a_i(\sigma) t + \alpha_i) \chi(\sigma) e + f + a_i(\sigma) e = \\ &= -a_i(\sigma) e - t^{-1} \alpha_i e + f + a_i(\sigma) e \\ &= -t^{-1} \alpha_i e + f = f'. \end{aligned}$$

and consequently f' is G_K-invariant.

Let $v := \lambda e + \mu f \in (V_p(X_i) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$, $\lambda, \mu \in B_{\mathrm{dR}}$. Then

$$\sigma v = v \Leftrightarrow (\sigma(\lambda)\chi(\sigma) + \sigma(\mu)a_i(\sigma))e + \sigma(\mu)f = \lambda e + \mu f$$

in other words $\sigma(\lambda)\chi(\sigma) + \sigma(\mu)a_i(\sigma) = \lambda$ and $\sigma\mu = \mu$ in the basis (e, f). It follows that $\mu \in B_{\mathrm{dR}}^{G_K} = K$ and by letting $\lambda' := \lambda + \mu t^{-1}\alpha_i \in B_{\mathrm{dR}}$ one has $v = \lambda' e + \mu f'$. For $\sigma \in G_K$, we have that:

$$\sigma(\lambda') = \sigma(\lambda) + \mu\chi^{-1}(\sigma)t^{-1}(\alpha_i + a_i(\sigma)t) = \sigma(\lambda) + \mu\chi^{-1}(\sigma)t^{-1}\alpha_i + \mu\chi^{-1}(\sigma)a_i(\sigma)$$

and since $\sigma(\lambda)\chi(\sigma) + \mu a_i(\sigma) = \lambda$, multiplying the above relation by $\chi(\sigma)$ leads to: $\sigma(\lambda')\chi(\sigma) = \lambda'$. Consequently $\sigma(t\lambda') = \chi(\sigma)t\sigma(\lambda') = t\lambda'$ i.e. $t\lambda' \in B_{\mathrm{dR}}^{G_K} = K$.
One obtains that (e', f') is a G_K-equivariant basis of the K-vector space $D_{\mathrm{dR}}(V_p(X_i))$ hence $\dim_K D_{\mathrm{dR}}(V_p(X_i)) = 2 = \dim_{\mathbb{Q}_p} V_p(X_i)$ so $V_p(X_i)$ is a de Rham representation. Moreover, $V_p(X_i)$ is also crystalline since $\alpha_i, t^{-1} \in B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$.

Remark 2.4.4. In the classical case of the Tate curve, $V_p(\mathbb{G}_m/q^{\mathbb{Z}})$ is only de Rham. Concretely, let K be a p-adic field, fix $q \in K$ with $|q| < 1$ and set $E_q := \overline{K}^\times/q^{\mathbb{Z}}$ as a G_K-module through the action on $\overline{K}^\times$. Then $E_q(\overline{K})[p^n] = \{(\varepsilon^{(n)})^i(q^{(n)})^j, 0 \leq i, j < p^n\}$ where $\varepsilon^{(n)}$ are the p^n-th roots of 1 chosen as in the previous example and $q^{(n)}$-th are the elements defined by $q^{(0)} = q$ and $(q^{(n+1)})^p = q^{(n)}$. Consequently, a basis of $T_p(E_q)$ is (e, f) where $e = \varprojlim_n \varepsilon^{(n)}$ and $f = \varprojlim_n q^{(n)}$. As in subsection 1.3.2, *Case 1*, the action of G_K on $T_p(E_q)$ is given by $\sigma e = \chi(\sigma)e$ and $\sigma f = f + a(\sigma)e$ where $a(\sigma)$ is as before the 1-cocycle describing the action of G_K on $(q^{(n)})$, namely $\sigma q^{(n)} = (\varepsilon^{(n)})^{a(\sigma)}q^{(n)}$.

Define by $\tilde{q} := (q^{(0)}, q^{(1)}, ...) \in R$ and note that $\sigma(\tilde{q}) = (\sigma q^{(0)}, \sigma q^{(1)}, ...) = \tilde{q}\varepsilon^{a(\sigma)}$, $\sigma \in G_K$ and that $\theta_K([\tilde{q}]/q^{(0)} - 1) = \theta_K(q/q - 1) = 0$.

Consider now the series

$$log([\tilde{q}]/q) = log(1 + ([\tilde{q}]/q - 1)) = \sum_{n\geq 1}(-1)^{n+1}\frac{([\tilde{q}]/q-1)^n}{n}.$$

This element makes sense and converges in B_{dR}^+. Concretely, by defining $s_n := \sum_{k=1}^n (-1)^{k+1}\frac{([\tilde{q}]/q-1)^k}{k}$, then for $m > n$ we have that

$$s_m - s_n = \sum_{k=n+1}^{m}(-1)^{k+1}\frac{([\tilde{q}]/q-1)^k}{k} \in (ker(\theta_K))^{n+1}$$

hence $|s_m - s_n| < \frac{1}{n+1} \to 0$ in the $ker(\theta_K)$-adic topology. It follows that the sequence $(s_n)_n$ is Cauchy and since B_{dR}^+ is complete with respect to the $ker(\theta_K)$-adic topology, we get that $(s_n)_n$ is convergent.

One can define now the element $u := log_p(q) + log([\tilde{q}]/q)" = "log([\tilde{q}])$. This plays the role of the α_i from the previous example.

Observe that the action of G_K on u is given by

$$\sigma u = log([\sigma\tilde{q}]) = log([\tilde{q}] \cdot [\varepsilon^{a(\sigma)}]) = log([\tilde{q}]) + log([\varepsilon^{a(\sigma)}]) = log([\tilde{q}]) + a(\sigma)log([\varepsilon]) = u + a(\sigma)t.$$

We show now that $V_p(E_q)$ is de Rham. A $\mathbb{Q}_p$-basis of $V_p(E_q)$ being (e, f), we need to find a G_K-equivariant basis of $D_{\mathrm{dR}}(V_p(E_q))$ in terms of $e \otimes 1$ and $f \otimes 1$. As in the previous example, finding a G_K-invariant vector is easy: consider $e' := e \otimes 1/t$ and note that:

$$\sigma e' = \chi(\sigma)e \otimes \chi^{-1}(\sigma)t^{-1} = e \otimes 1/t = e'.$$

Now, the second vector is linearly independent to e' so it has to have nonzero $f \otimes 1$ component. Since $\sigma f = f + a(\sigma)e$ one can search for f' of the form $f' = e \otimes x + f \otimes 1$, for some $x \in B_{dR}$.

Then $\sigma f' = f'$ is equivalent to $\chi(\sigma)e \otimes \sigma x + f \otimes 1 + a(\sigma)e \otimes 1 = e \otimes x + f \otimes 1$, in other words to $e \otimes \chi(\sigma)\sigma x + f \otimes 1 + e \otimes a(\sigma) = e \otimes x + f \otimes 1$ hence $\chi(\sigma)\sigma x + a(\sigma) = x$. Multiplying this relation by t we get:
$(\chi(\sigma)t)(\sigma x) + a(\sigma)t = xt$. Further one can write it $\sigma t \sigma x + a(\sigma)t = xt$ i.e.
$\sigma(xt) + a(\sigma)t = xt$. This is equivalent to $\sigma(xt) - xt = -a(\sigma)t$ and observe now that $\sigma(-u) = -\sigma u = -u - a(\sigma)t$ and so we can take $xt := -u$ hence $x = -u/t$. Consequently, $f' = -e \otimes u/t + f \otimes 1$ is G_K-invariant and we obtain that:
$\dim_K D_{\mathrm{dR}}(V_p(E_q)) = 2 = \dim_{\mathbb{Q}_p} V_p(E_q)$ so $V_p(E_q)$ is a de Rham representation.

Chapter 3

The sheaf $\mathbb{A}^{\nabla}_{max}$

In this chapter we define a new type of Fontaine sheaf, $\mathbb{A}^{\nabla}_{\max}$, we prove some properties of it and we study its localization over small affines, the main result being that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$, where $A^{\nabla}_{\max}$ is the ring defined by O. Brinon in [Bri 2].

Let $p > 0$ be a prime integer, K a finite, unramified extension of $\mathbb{Q}_p$ with residue field k and $\mathcal{O}_K$ the ring of integers of K.

3.1 The rings $A_{\max,n}$

Recall from the previous chapter that we have a ring homomorphism for every $n \in \mathbb{N}$, $n \geq 1$:

$\theta_n : W_n := \mathbb{W}_n(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}) \longrightarrow \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ given by $(s_0, ..., s_{n-1}) \longrightarrow \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}$, where $\tilde{s}_i \in \mathcal{O}_{\overline{K}}/p^n\mathcal{O}_{\overline{K}}$ is a lift of s_i for every i. Also note that $\mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \cong A^+_{\inf}/p^n A^+_{\inf}$, where $A^+_{\inf} = \mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ and $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$, the inverse limit being taken with respect to Frobenius. In order to prove this, we use the projection on the first n components:

$$\pi_n : A^+_{\text{inf}} \mapsto \mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$$

$$(s_0, s_1, ..., s_n, ...) \mapsto (s_0, s_1, ..., s_{n-1}),$$

with $ker(\pi_n) = \{(s_0, s_1, ...s_n, ...) \in A^+_{\text{inf}} \mid s_0 = s_1 = ... = s_{n-1} = 0\} = p^n A^+_{\text{inf}}$. (Recall that $p^n = (\underbrace{0, 0, ..., 0}_{n}, 1, 0, ...)$ and that $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ is perfect.)

Moreover, we have the following:

Proposition 3.1.1. *The kernel of the projection $\bar{q}_n : \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}) = \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} \mapsto \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ on the $n+1$-th factor of the limit is generated by $\tilde{p}^{p^n}$.*

Proof. To simplify the notations, put $R := \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$. Let $x = (x_m)_{m \geq 0} \in R$. Then our map sends $(x_m)_{m\geq 0} \xrightarrow{\bar{q}_n} x_n$. Recall that we have a bijective map:

$\varprojlim \mathcal{O}_{\overline{K}} \mapsto \varprojlim \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}} = R$, defined by:

$(x^{(n)})_{n\geq 0} \mapsto (x^{(n)} \text{modp})$, with inverse given by:

$(x_n)_{n\geq 0} \mapsto (x^{(n)})_{n\geq 0}$, where $x^{(n)} = lim_{m\to\infty} \widehat{x_{n+m}}^{p^m}$, for arbitrary lifts $\hat{x}_i \in \mathcal{O}_{\overline{K}}$ of $x_i \in \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ for all $i \geq 0$. Remark that, since

$v_R(x) = v(x^{(0)}) = v((x^{(n)})^{p^n}) = p^n v(x^{(n)})$ for $n \geq 0$, then

$v_R(x) \geq p^n \Leftrightarrow v(x^{(n)}) \geq 1 \Leftrightarrow x^{(n)}(\text{modp}) = 0.$

One obtains in this way a better description of $ker(\bar{q}_n) = \{x \in R/v_R(x) \geq p^n\} = \{x \in R/x^{(n)}(\text{modp}) = 0\}$.

Now, since $v_R(\tilde{p}^{p^n}) = v(p^{p^n}) = p^n$, it's clear that $(\tilde{p}^{p^n}) \subseteq ker(\bar{q}_n)$. For the other inclusion, let $x \in ker(\bar{q}_n)$. Subsequently, $v(x^{(0)}) \geq p^n$ hence $x^{(0)} = p^{p^n} y^{(0)}$, for some $y^{(0)} \in \mathcal{O}_{\overline{K}}$. Since $(x^{(n)})_n$ is compatible we have that $(x^{(1)})^p = x^{(0)} = p^{p^n} y^{(0)}$ and one obtains $x^{(1)} = p^{p^{n-1}} y^{(1)}$, $y^{(1)} \in \mathcal{O}_{\overline{K}}$ and moreover $(y^{(1)})^p = y^{(0)}$ (recall that the multiplication in R (through the above mentioned bijection) is $(st)^{(n)} = (s)^{(n)}(t)^{(n)}$ and that $\mathcal{O}_{\overline{K}}$ is normal). We construct in this way a compatible sequence $y = (y^{(n)})_n \in R$ such that $x = \tilde{p}^{p^n} y$.

□

The projection $\bar{q}_n$ induces a ring homomorphism:

$q_n : \mathbb{W}_n(\underline{R}(O_{\overline{K}})) \mapsto \mathbb{W}_n(O_{\overline{K}}/pO_{\overline{K}})$, given by:

$(s_0, s_1, ..., s_{n-1}) \mapsto (s_0^{(n)}(\mathrm{mod}p), s_1^{(n)}(\mathrm{mod}p), ..., s_{n-1}^{(n)}(\mathrm{mod}p))$

Note that since q_n is surjective we have the isomorphism:

$\mathbb{W}_n(\underline{R}(O_{\overline{K}}))/ker(q_n) \cong \mathbb{W}_n(O_{\overline{K}}/pO_{\overline{K}}) = W_n$.

Remark 3.1.2. The above map is denoted by q_n in order to simplify the notations (it should be called $q_{n,n}$ where the first n indicates the length of the Witt vector while the second indicates the component (in this case the $n+1$-th)). Note also that $q_n(\xi(\mathrm{mod}p^n)) = pr_n(\xi_{n+1})$ (i.e. the first n components of ξ_{n+1}) while $q_{n,n-1}(\xi(\mathrm{mod}p^n)) = \xi_n$ (recall that $\xi = [\tilde{p}]-p = (\tilde{p}, 0, 0, ...)-(0, 1, 0, 0, ...) \in A^+_{\mathrm{inf}}$ where $\tilde{p} = (p, p^{1/p}, p^{1/p^2}, ..) \in \underline{R}(O_{\overline{K}})$, $\tilde{p}^{(n)} = p^{1/p^n}$, so $q_{n,n-1}(\xi(\mathrm{mod}p^n)) = (\tilde{p}^{(n-1)}(\mathrm{mod}p), 0, ..., 0) - (0, 1, 0, .., 0) = (p^{1/p^{n-1}}, 0, ..., 0) - (0, 1, 0, ..., 0) = \tilde{p}_n - p = \xi_n)$. Recall also that $\tilde{p}_n = [p^{1/p^{n-1}}] \in W_n$ is the Teichmueller lift of $p^{1/p^{n-1}} \in O_{\overline{K}}/pO_{\overline{K}}$.

Proposition 3.1.3. *The kernel of the ring homomorphism q_n is the ideal generated by $\{[\tilde{p}]^{p^n}, V([\tilde{p}]^{p^n}), V^2([\tilde{p}]^{p^n}), ..., V^{n-1}([\tilde{p}]^{p^n})\}$.*

Proof. For $n = 1$ the statement is obvious by using Proposition 3.1.1. For $n \geq 2$ we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{W}_{n-1}(\underline{R}(O_{\overline{K}})) & \xrightarrow{V\circ(*)^p} & \mathbb{W}_n(\underline{R}(O_{\overline{K}})) & \xrightarrow{pr_1\circ(*)^{1/p^n}} & \mathbb{W}_1(\underline{R}(O_{\overline{K}})) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_1} & & \\ 0 & \longrightarrow & \mathbb{W}_{n-1}(O_{\overline{K}}/pO_{\overline{K}}) & \xrightarrow{V} & \mathbb{W}_n(O_{\overline{K}}/pO_{\overline{K}}) & \xrightarrow{pr_1} & \mathbb{W}_1(O_{\overline{K}}/pO_{\overline{K}}) & \longrightarrow & 0 \end{array}$$

One can easily check the exactness of the second row so we omit it. For the first one, remark that $(V\circ(*)^p)((s_0, s_1, ..., s_{n-2})) = (0, s_0^p, s_1^p, ..., s_{n-2}^p)$, $s_i \in \underline{R}(O_{\overline{K}})$, $0 \leq i \leq n-2$, and that $(pr_1 \circ (*)^{1/p^n})((0, s_0^p, s_1^p, ..., s_{n-2}^p)) = pr_1((0, s_0^{1/p^{n-1}}, s_1^{1/p^{n-1}}, ..., s_{n-2}^{1/p^{n-1}})) = 0$.

On the other hand, $V \circ (*)^p$ is injective since Verschiebung is injective and $(*)^p$ is bijective due to the fact that $\underline{R}(O_{\overline{K}}))$ is perfect. Similarly, $pr_1 \circ (*)^{1/p^n}$ remains surjective (for $s_0 \in \mathbb{W}_1(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$, we have that $(pr_1 \circ (*)^{1/p^n})((s_0^{p^n}, s_1, ..., s_{n-1})) = s_0$, where $(s_0^{p^n}, s_1, ..., s_{n-1}) \in \mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})))$.

Take now $(s_0, s_1, ..., s_{n-1}) \in ker(pr_1 \circ (*)^{1/p^n})$ so $s_0^{1/p^n} = 0$. Since $\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$ is perfect it follows that $s_0 = 0$ and consequently $(s_0, s_1, ..., s_{n-1}) = (V \circ (*)^p)((s_1^{1/p}, s_2^{1/p}, ..., s_{n-1}^{1/p}))$ hence $ker(pr_1 \circ (*)^{1/p^n}) \subseteq Im(V \circ (*)^p)$.

One obtains that the first row is exact.

Note that the first square diagram is exact since, for a choice of $s_i \in \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$, $0 \leq i \leq n-2$, we have:

$$\begin{array}{ccc} (s_0, s_1, ..., s_{n-2}) & \xrightarrow{V\circ(*)^p} & (0, s_0^p, s_1^p, ..., s_{n-2}^p) \\ \downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} \\ (s_0^{(n-1)}(p), s_1^{(n-1)}(p), ..., s_{n-2}^{(n-1)}(p)) & \xrightarrow{V} & (0, s_0^{(n-1)}(p), s_1^{(n-1)}(p), ..., s_{n-2}^{(n-1)}(p)) \end{array}$$

Also the second square diagram commutes since, for a choice of $s_i \in \underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})$, $0 \leq i \leq n-1$, we have:

$$\begin{array}{ccc} (s_0, s_1, ..., s_{n-1}) & \xrightarrow{pr_1\circ(*)^{1/p^n}} & (s_0^{1/p^n}) \\ \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle q_1} \\ (s_0^{(n)}(p), s_1^{(n)}(p), ..., s_{n-1}^{(n)}(p)) & \xrightarrow{pr_1} & (s_0^{(n)}(p)) \end{array}$$

One applies further the induction hypothesis at the level of kernels in the main diagram.

□

Definition 3.1.4. Let A be a p-adically complete $\mathcal{O}_K$-algebra and T a variable. Then we denote by $A\{T\} := \varprojlim A[T]/p^n A[T]$.

We define now the rings $A_{\max,n} = W_n[\delta]/(p\delta - \xi_n)$ and let $A_{\max} := \varprojlim_n A_{\max,n}$. We then have:

$A_{\max} = A^+_{\inf}\{[\frac{\xi}{p}]\} = A^+_{\inf}\{\delta\}/(p\delta - \xi) = \{\sum_{i\geq 0} a_i \delta^i$ such that $a_i \in A^+_{\inf}$ and $a_i \to 0$ when $i \to \infty\}$. Let $A'_{\max,n} := W_n[\delta]/(p\delta - \xi_{n+1})$. (By ξ_{n+1} we mean here the projection on the first n components of this vector namely $pr_n(\xi_{n+1}) = \underbrace{(p^{1/p^n}, -1, 0, ..., 0)}_{n}$.)

Note that we also have that:

$V^i([\tilde{p}]^{p^n}) = p^i([\tilde{p}]^{p^n})^{p^{-i}} = p^i[\tilde{p}]^{p^{n-i}} = p^i(\xi+p)^{p^{n-i}} = p^i(p(\delta+1))^{p^{n-i}} \equiv p^{i+p^{n-i}}\delta^{p^{n-i}} \equiv$ $0 \bmod p^n A_{\max}$, where for the first equality one uses the Witt coordinatization ($(r_0, r_1, ...) = \sum p^n[r_n^{p^{-n}}]$ (or one computes it directly)).

By using Proposition 3.1.3 one obtains that $ker(q_n) \subseteq p^n A_{\max}$.

We will also use this remark in order to prove the following important:

Proposition 3.1.5. $A_{\max}/p^n A_{\max} \cong A'_{\max,n}$.

Proof. For $n = 1$, $A'_{\max,1} = W_1[\delta]/(p\delta - \xi_2) = \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(p\delta - \xi_2) \cong \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(\xi_2) = \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(p^{1/p})$, since $p\delta = 0$ (reduction modulo p).

On the other hand, since $ker(q_1) \subseteq pA_{\max}$ we have that:

$$\frac{A_{\max}}{pA_{\max}} = A_{\max}/(p, ker(q_1))A_{\max} = \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p,ker(q_1))(A^+_{\inf}\{\delta\}/(p\delta-\xi))}$$
$$= \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p,ker(q_1),p\delta-\xi)A^+_{\inf}\{\delta\}/(p\delta-\xi)} \cong A^+_{\inf}[\delta]/(p, ker(q_1), p\delta - \xi)A^+_{\inf}[\delta]$$
$$\cong \frac{A^+_{\inf}[\delta]/pA^+_{\inf}[\delta]}{(p,ker(q_1),p\delta-\xi)A^+_{\inf}[\delta]/pA^+_{\inf}[\delta]} \cong \frac{(A^+_{\inf}/pA^+_{\inf})[\delta]}{(ker(q_1),\xi(\mathrm{mod} p))(A^+_{\inf}[\delta]/pA^+_{\inf}[\delta])}$$
$$\cong \mathbb{W}_1(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))[\delta]/(ker(q_1), \xi(\mathrm{mod} p)) \cong W_1[\delta]/(\xi_2) = \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(\xi_2) = A'_{\max,1}.$$

Note that, since $\mathbb{W}_1(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))/ker(q_1) \cong W_1$ and $q_1(\xi(\mathrm{mod} p)) = \xi_2$, q_1 induces the isomorphism: $\mathbb{W}_1(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))[\delta]/(ker(q_1), \xi(\mathrm{mod} p)) \cong W_1[\delta]/(\xi_2)$.

Similarly, for the general case, since $ker(q_n) \subseteq p^n A_{\max}$, we obtain that:

$$\frac{A_{\max}}{p^n A_{\max}} = A_{\max}/(p^n, ker(q_n))A_{\max} = \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p^n,ker(q_n))(A^+_{\inf}\{\delta\}/(p\delta-\xi))}$$
$$= \frac{A^+_{\inf}\{\delta\}/(p\delta-\xi)}{(p^n,ker(q_n),p\delta-\xi)A^+_{\inf}\{\delta\}/(p\delta-\xi)} \cong A^+_{\inf}[\delta]/(p^n, ker(q_n), p\delta - \xi)A^+_{\inf}[\delta]$$
$$\cong \frac{A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta]}{(p^n,ker(q_n),p\delta-\xi)A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta]} \cong \frac{(A^+_{\inf}/p^n A^+_{\inf})[\delta]}{(ker(q_n),p\delta-\xi(\mathrm{mod} p^n))(A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta])}$$
$$\cong \mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))[\delta]/(ker(q_n), p\delta - \xi(\mathrm{mod} p^n)) \cong W_n[\delta]/(p\delta - \xi_{n+1}) = A'_{max,n}.$$

Remark that, since $\mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))/ker(q_n) \cong W_n$ and $q_n(\xi(\mathrm{mod} p^n)) = \xi_{n+1}$, q_n induces the isomorphism: $\mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))[\delta]/(ker(q_n), p\delta - \xi(\mathrm{mod} p^n)) \cong W_n[\delta]/(p\delta - \xi_{n+1})$.

Above we've also used the isomorphisms of rings $A^+_{\inf}/p^n A^+_{\inf} \cong \mathbb{W}_n(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ and $A^+_{\inf}[\delta]/p^n A^+_{\inf}[\delta] \cong (A^+_{\inf}/p^n A^+_{\inf})[\delta]$.

The result follows.

□

Remark 3.1.6. One can also prove the previous Proposition by showing that there is a surjective map $A_{\max} \twoheadrightarrow A'_{\max,n}$ whose kernel is $p^n A_{\max}$. We will see later (Lemma 3.2.5) that for any positive integers $m > n$ we also have an isomorphism of rings $A_{\max}/p^n A_{\max} \cong A_{\max,m}/p^n A_{\max,m}$.

Note that, via the isomorphism $A_{\max}/p^n A_{\max} \cong A'_{\max,n}$, we have a surjective map of rings:

$$q'_n : A_{\max}/p^n A_{\max} \to A_{\max,n}$$

sending $pr_n(\xi_{n+1}) \to \xi_n$, induced by Frobenius on W_n and that we also have a map:

$$u_n : A_{\max,n+1} \to A_{\max}/p^n A_{\max}$$

sending $\xi_{n+1} \to pr_n(\xi_{n+1})$, induced by the natural projection $W_{n+1} \to W_n$.

3.2 Definition of the sheaf $\mathbb{A}^{\nabla}_{\max}$

Let now X be a scheme of finite type over $\mathcal{O}_K$ and also let M be an algebraic extension of K. One denotes by X^{et} the small étale site on X and by X_M^{fet} the finite étale site of $X_M = X \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(M)$. Further, one denotes by $\mathrm{Sh}(X^{et})$ and $\mathrm{Sh}(X_M^{fet})$ the categories of sheaves of abelian groups of these two sites, respectively. Following [AI1] we will construct the site $\mathfrak{X}_M$. Firstly, one has the following:

Definition 3.2.1. ([AI1, Definition 2.1]) Let E_{X_M} be the category defined as follows:
1) the objects consist of pairs $(g : U \to X, f : W \to U_M)$ such that g is an étale morphism and f is a finite étale morphism. One further denotes by (U, W) this object to simplify the notations;
2) a morphism $(U', W') \to (U, W)$ in E_{X_M} is a pair (α, β), where $\alpha : U' \to U$ is a morphism over X and $\beta : W' \to W$ is a morphism commuting with $\alpha \otimes_{\mathcal{O}_K} Id_M$.

Definition 3.2.2. ([AI1, Definition 2.3]) We say that a family of morphisms $\{(U_i, W_i) \to$

$(U, W)\}_{i\in I}$ has the property $(*)$ if either:

i)$\{U_i \to U\}_{i\in I}$ is a covering in X^{et} and $W_i \cong W \times_U U_i$ for every $i \in I$, the morphism $W \to U$ used in the fibre product being the composition $W \to U_M \to U$,

or

ii) $U_i \cong U$ for all $i \in I$ and $\{W_i \to W\}_{i\in I}$ is a covering in X_M^{fet}.

One further endows E_{X_M} with the topology generated by the families having the property $(*)$ and one denotes by $\mathfrak{X}_M$ the associated site. One calls $\mathfrak{X}_M$ the locally Galois site associated to (X, M).

Definition 3.2.3. ([AI1, Definition 2.4]) A family $\{(U_{ij}, W_{ij}) \to (U, W)\}_{i\in I, j\in J}$ is called a strict covering family if:

i) For each $i \in I$ there exists an étale morphism $U_i \to X$ such that one has $U_i \cong U_{ij}$ over X for all $j \in J$;

ii)$\{U_i \to U\}_{i\in I}$ is a covering in X^{et};

iii) For each $i \in I$ the family $\{W_{ij} \to W \times_U U_i\}_{j\in J}$ is a covering in X_M^{fet}.

Each strict covering family is a covering family (see [AI1, Remark 2.5]).

Let now (U, W) be an object of E_{X_M}. A. Iovita and F. Andreatta defined in [AI1] (Definition 2.10) the presheaf $\mathcal{O}_{\mathfrak{X}_M}$ on E_{X_M}, by requiring that $\mathcal{O}_{\mathfrak{X}_M}(U, W)$ consists of the normalization of $\Gamma(U, \mathcal{O}_U)$ in $\Gamma(W, \mathcal{O}_W)$. They also proved ([AI1, Proposition 2.11]) that the presheaf $\mathcal{O}_{\mathfrak{X}_M}$ is a sheaf.

Now, if X is a scheme of finite type over $\mathcal{O}_K$, $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ is the sheaf of rings on $\mathfrak{X}_{\overline{K}}$ defined by requiring that for every object (U, W) in $\mathfrak{X}_{\overline{K}}$, the ring $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(U, W)$ is the normalization of $\Gamma(U, \mathcal{O}_U)$ in $\Gamma(W, \mathcal{O}_W)$. Note that $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ is a sheaf of $\mathcal{O}_{\overline{K}}$-algebras.

Let $\hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}} := \varprojlim_n \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}} \in Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$.

Also, let $\underline{\mathcal{R}}(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ be the sheaf of rings in $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the inverse system $\{\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}\}$, the transition maps being given by Frobenius.

For every $s \in \mathbb{N}$ we define now the sheaf of rings $\mathbb{A}^+_{\mathrm{inf},s,\overline{K}} := \varprojlim \mathrm{W}_{s,\overline{K}}$ where

$\mathbb{W}_{s,\overline{K}} := \mathbb{W}_s(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ is the sheaf $(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})^s$ with ring operations defined by Witt polynomials and the transition maps in the projective limit are defined by Frobenius.

We further define the sheaf of rings $\mathbb{A}^+_{inf,\overline{K}} := \varprojlim \mathbb{W}_{n,\overline{K}}$ in $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$, where the transition maps in the projective limit are defined as the composite of the projection $\mathbb{W}_{n+1,\overline{K}} \to \mathbb{W}_{n,\overline{K}}$ and the Frobenius on $\mathbb{W}_{n,\overline{K}}$ and $\mathbb{W}_{n,\overline{K}} := \mathbb{W}_n(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$ is the sheaf $(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})^n$ with ring operations defined by Witt polynomials.

We also have a morphism $\theta_{\overline{K}} : \mathbb{A}^+_{\mathrm{inf},\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}}$ of objects of $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$; we construct it at the beginning of the *4-th* chapter.

$\mathbb{A}^+_{\mathrm{inf},\overline{K}}$ and $\mathbb{A}^+_{\mathrm{inf,s},\overline{K}}$ are endowed with an operator, φ, which is the canonical Frobenius associated to the Witt vector construction and are sheaves of $\mathcal{O}_K$-algebras.

We are able now to construct the first sheaf mentioned at the beginning, $\mathbb{A}^{\nabla}_{\mathrm{max},\overline{K}}$.

Firstly, let $\mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}} := A_{\mathrm{max},n} \otimes_{W_n} \mathbb{W}_{n,\overline{K}} = W_n[\delta]/(p\delta - \xi_n) \otimes_{W_n} \mathbb{W}_{n,\overline{K}}$ i.e. $\mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}}$ is the sheaf on $\mathfrak{X}_{\overline{K}}$ associated to the pre-sheaf given by

$(\mathcal{U}, \mathcal{W}) \mapsto A_{\mathrm{max,n}} \otimes_{W_n} \mathbb{W}_{n,\overline{K}}(\mathcal{U}, \mathcal{W})$ for $(\mathcal{U}, \mathcal{W}) \in \mathfrak{X}_{\overline{K}}$.

Consider the map $r_{n+1} : \mathbb{W}_{n+1,\overline{K}} \mapsto \mathbb{W}_{n,\overline{K}}$ given by the natural projection composed with Frobenius. This induces a natural map $r_{n+1,\overline{K}} : \mathbb{A}^{\nabla}_{\mathrm{max,n+1},\overline{K}} \mapsto \mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}}$.

Let $\mathbb{A}^{\nabla}_{\mathrm{max},\overline{K}}$ be the sheaf in $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the family $\{\mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}}\}_n$ with the transition maps $\{r_{n+1,\overline{K}}\}_n$.

Secondly, let $\mathbb{A}'^{\nabla}_{\mathrm{max,n},\overline{K}}$ be the sheaf on $\mathfrak{X}_{\overline{K}}$ associated to the pre-sheaf given by

$(\mathcal{U}, \mathcal{W}) \mapsto A_{\mathrm{max}}/p^n A_{\mathrm{max}} \otimes_{W_n} \mathbb{W}_{n,\overline{K}}(\mathcal{U}, \mathcal{W})$ for $(\mathcal{U}, \mathcal{W}) \in \mathfrak{X}_{\overline{K}}$.

As for $\mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}}$, r_{n+1} induces a natural map $r'_{n+1,\overline{K}} : \mathbb{A}'^{\nabla}_{\mathrm{max,n+1},\overline{K}} \mapsto \mathbb{A}'^{\nabla}_{\mathrm{max,n},\overline{K}}$.

Similarly, let $\mathbb{A}'^{\nabla}_{\mathrm{max},\overline{K}}$ be the sheaf in $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ defined by the family $\{\mathbb{A}'^{\nabla}_{\mathrm{max,n},\overline{K}}\}_n$ with the transition maps $\{r'_{n+1,\overline{K}}\}_n$.

Also, note that $\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ is the sheaf associated to the pre-sheaf

$(\mathcal{U}, \mathcal{W}) \mapsto \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U}, \mathcal{W})/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U}, \mathcal{W})$.

In order to simplify the notations denote by $\mathbb{A}^{\nabla}_{\mathrm{max}} := \mathbb{A}^{\nabla}_{\mathrm{max},\overline{K}}$, $\mathbb{A}^{\nabla}_{\mathrm{max,n}} := \mathbb{A}^{\nabla}_{\mathrm{max,n},\overline{K}}$,

$\mathbb{A}'^{\nabla}_{\max} := \mathbb{A}'^{\nabla}_{\max,\overline{K}}$, $\mathbb{A}'^{\nabla}_{\max,n} := \mathbb{A}'^{\nabla}_{\max,n,\overline{K}}$, $\bar{\mathcal{O}}_{\mathfrak{X}} := \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$, $\mathbb{W}_n := \mathbb{W}_{n,\overline{K}}$ and $\mathbb{A}^+_{\inf} := \mathbb{A}^+_{\inf,\overline{K}}$.

Further let $r''_{n+1} : W_{n+1}[\delta]/(p\delta - \xi_{n+1}) \mapsto W_n[\alpha]/(p\alpha - \xi_n)$ i.e. $r''_{n+1} : A_{\max,n+1} \mapsto A_{\max,n}$ be the map of rings defined by the natural projection composed with Frobenius.

Since $r''_{n+1}(\widetilde{p}_{n+1}) = \widetilde{p}_n$, we have that $r''_{n+1}(\delta) = r''_{n+1}(\frac{\xi_{n+1}}{p}) = \frac{r''_{n+1}(\xi_{n+1})}{p} = \frac{r''_{n+1}(\widetilde{p}_{n+1}-p)}{p} = \frac{\widetilde{p}_n - p}{p} = \frac{\xi_n}{p} = \alpha$, hence $r''_{n+1}(p\delta - \xi_{n+1}) = p\alpha - \xi_n$. It follows that r''_{n+1} is well defined.

Let us remark now that, since $A'_{\max,1} = \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(\xi_2)$, we have a nice description of $\mathbb{A}'^{\nabla}_{\max,1}$, namely $\mathbb{A}'^{\nabla}_{\max,1} = A'_{\max,1} \otimes_{W_1} \mathbb{W}_1 = \frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}[\delta]/(\xi_2) \otimes_{\frac{\mathcal{O}_{\overline{K}}}{p\mathcal{O}_{\overline{K}}}} (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}) = (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(\xi_2) = (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(p^{1/p})$. We'll use this fact in the proof of the following:

Lemma 3.2.4. *For every n we have an exact sequence of sheaves:*

$$0 \longrightarrow \mathbb{A}'^{\nabla}_{\max,n} \xrightarrow{f} \mathbb{A}'^{\nabla}_{\max,n+1} \xrightarrow{g} \mathbb{A}'^{\nabla}_{\max,1} \longrightarrow 0,$$

where f is the map of sheaves associated to the Verschiebung $\mathbb{V} : \mathbb{W}_n \mapsto \mathbb{W}_{n+1}$ and $g = r'_{2,\overline{K}} \circ r'_{3,\overline{K}} \circ ... \circ r'_{n+1,\overline{K}}$.

Proof. Firstly, let us fix an object $(\mathcal{U}, \mathcal{W})$ of $\mathfrak{X}$ and denote by $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$.

For $(s_0, s_1, ..., s_{n-1}) \in \mathbb{W}_n(S/pS)$, since $(r_2 \circ r_3 \circ ... r_{n+1})(0, s_0, ..., s_{n-1}) = (r_2 \circ r_3 \circ ... r_n)(0, s_0^p, ..., s_{n-2}^p) = ... = (r_2 \circ r_3)(0, s_0^{p^{n-2}}, s_1^{p^{n-2}}) = r_2(0, s_0^{p^{n-1}}) = 0$, one obtains that $g \circ f = 0$.

Let's prove now the surjectivity of g. Denote by $s : \mathbb{W}_{n+1} \mapsto \mathbb{W}_1 = \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ the natural projection and by s' the induced map of sets $\mathbb{W}_{n+1}(S/pS) \xrightarrow{s'} \mathbb{W}_1(S/pS)$ sending $(s_0, s_1, ..., s_n)$ to s_0. Since $ker(s') = \{(s_0, s_1, ..., s_n) \in (S/pS)^{n+1}/s_0 = 0\} \cong \mathbb{W}_n(S/pS) = (S/pS)^n$, it's clear that $ker(s)$ is identified with $\mathbb{W}_n$ via Verschiebung. Note that $ker(s)$ is a W_{n+1}-module via the projection map $W_{n+1} \mapsto W_n$ composed with Frobenius on W_n and since $\mathbb{W}_n$ is a W_n-module. We obtain that:

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} ker(s) \cong A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_n.$$

Since $s'(\xi_{n+2}) = s'(\tilde{p}_{n+2} - p) \equiv p^{1/p^{n+1}}(\text{mod}p)$, it follows that

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1 \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p^{n+1}}\bar{\mathcal{O}}_{\mathfrak{X}}[\delta]. \qquad (1)$$

Now, since $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$ is a normal ring, Frobenius to the n-th power $\varphi^n : S/p^{1/p^n}S \to S/pS$ is injective (for this, let $x \in S$ such that $\varphi^n(\bar{x}) = \bar{\bar{0}}$, so $x^{p^n} = p \cdot y$ for some $y \in S$. Since S is normal it follows that $x = p^{1/p^n} \cdot y'$, $y' \in S$ i.e. $x \in S/p^{1/p^n}S$, in other words $\bar{x} = 0$). So we have an injection $\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p^n}\bar{\mathcal{O}}_{\mathfrak{X}} \mapsto \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$

On the other hand, by [AI2], Lemma 4.4.1, (v), Frobenius on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective with kernel $p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ hence we have an isomorphism $\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}} \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$. Consequently, Frobenius to the n-th power on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective with kernel $p^{1/p^n}\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ hence we have an isomorphism $\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p^n}\bar{\mathcal{O}}_{\mathfrak{X}} \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$. (2)

From (1) and (2), one obtains that

$$A_{\max}/p^{n+1}A_{\max} \otimes_{W_{n+1}} \mathbb{W}_1 \cong \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}[\delta],$$

Since $\varphi^n \circ s = r_2 \circ r_3 \circ \ldots \circ r_{n+1} : \mathbb{W}_{n+1} \mapsto \mathbb{W}_1 = \bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ is surjective, after tensoring with $A_{\max}/p^{n+1}A_{\max}$ over W_{n+1}, and since tensoring is right exact, we obtain a surjective map $\mathbb{A}'^{\nabla}_{\mathrm{max,n+1}} \xrightarrow{g} (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta] \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$ where the last isomorphism follows from (2).

Also by (2) it follows that $(\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]/(p^{1/p}) \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$, in other words $\mathbb{A}'^{\nabla}_{\mathrm{max,1}} \cong (\bar{\mathcal{O}}_{\mathfrak{X}}/p^{1/p}\bar{\mathcal{O}}_{\mathfrak{X}})[\delta]$ and so the right exactness of the displayed sequence is proved.

Now we need to prove the left exactness of our sequence. We will show that it is left exact on stalks. For this, let x be a point of X. Recall that $A'_{\mathrm{max,n}} = W_n[\delta]/(p\delta - \xi_{n+1})$. Since $\frac{\xi_{n+1}}{p} = \frac{\tilde{p}_{n+1} - p}{p} = \frac{\tilde{p}_{n+1}}{p} - 1$, we have that $A'_{\mathrm{max,n}} \cong W_n[\delta]/(p\delta - \tilde{p}_{n+1})$.

Define $B := W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$, and similarly, denote by $C := W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$ and by $D := (\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})[\delta]$.

Let's remark that $B/(p\delta - \tilde{p}_{n+1})B$ is the stalk $\mathbb{A}'^{\nabla}_{\mathrm{max,n,x}}$ of $\mathbb{A}'^{\nabla}_{\mathrm{max,n}}$ at x, that $C/(p\delta - \tilde{p}_{n+2})C$ is the stalk $\mathbb{A}'^{\nabla}_{\mathrm{max,n+1,x}}$ of $\mathbb{A}'^{\nabla}_{\mathrm{max,n+1}}$ at x and that $D/\tilde{p}_{n+2}D$ is the stalk $\mathbb{A}'^{\nabla}_{\mathrm{max,1,x}}$ of $\mathbb{A}'^{\nabla}_{\mathrm{max,1}}$ at x ($\mathbb{A}'^{\nabla}_{\mathrm{max,1,x}} = D/p^{1/p}D \cong D/\tilde{p}_{n+2}D$ by using the isomorphism from (2)).

The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{f_x} & C & \xrightarrow{s_x} & D & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle p\delta-\tilde{p}_{n+1}} & & \Big\downarrow{\scriptstyle p\delta-\tilde{p}_{n+2}} & & \Big\downarrow{\scriptstyle -\tilde{p}_{n+2}} & & \\
0 & \longrightarrow & B & \xrightarrow{f_x} & C & \xrightarrow{s_x} & D & \longrightarrow & 0
\end{array}$$

where f_x is the map sending $\delta \mapsto \delta$ and inducing the Verschiebung $W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}) \mapsto W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$ and s_x is the natural projection.

Since the Verschiebung is injective and since B (respectively C) is a free $W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$-module (respectively $W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$-module), one obtains that the map f_x is injective. Also D is a free $\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}$-module and the rows in the above diagram are exact.

Let's check now the exactness of the two square diagrams of the main one.

For the first square diagram, since $\delta \mapsto \delta$ it's enough to verify the exactness on coefficients. Let $s \in W_n(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$, $s = (s_0, s_1, ..., s_{n-1})$. We have that $\tilde{p}_{n+1} \cdot s = (p^{1/p^n}, 0, ..., 0) \cdot (s_0, s_1, ..., s_{n-1}) = (p^{1/p^n} s_0, p^{1/p^{n-1}} s_1, ..., p^{1/p} s_{n-1})$ and since $\tilde{p}_{n+2} \cdot V(s) = (p^{1/p^{n+1}}, 0, ..., 0) \cdot (0, s_0, ..., s_{n-1}) = (0, p^{1/p^n} s_0, ..., p^{1/p} s_{n-1})$, one obtains that $V(\tilde{p}_{n+1} \cdot s) = \tilde{p}_{n+2} \cdot V(s)$. The composition of the maps on the left lower side of the first square diagram will then be $V(p\delta s - \tilde{p}_{n+1} s) = p\delta V(s) - V(\tilde{p}_{n+1} \cdot s) = p\delta V(s) - \tilde{p}_{n+2} \cdot V(s) = (p\delta - \tilde{p}_{n+2}) V(s)$, which is exactly what the composition of the maps on the right upper side gives us. We obtain that the first square diagram is commutative, i.e.:

$$\begin{array}{ccc}
B & \xrightarrow{f_x} & C \\
{\scriptstyle p\delta-\tilde{p}_{n+1}}\Big\downarrow & \equiv & \Big\downarrow{\scriptstyle p\delta-\tilde{p}_{n+2}} \\
B & \xrightarrow{f_x} & C
\end{array}.$$

Similarly, for the second one, if $t \in W_{n+1}(\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x})$, $t = (t_0, t_1, ..., t_n)$, then:

$$\begin{array}{ccc}
(t_0, t_1, ..., t_n) & \xrightarrow{\qquad s_x \qquad} & t_0 \\
{\scriptstyle p\delta-\tilde{p}_{n+2}}\Big\downarrow & \equiv & \Big\downarrow{\scriptstyle -\tilde{p}_{n+2}} \\
(p\delta - \tilde{p}_{n+2}) \cdot t & \xrightarrow{s_x} & -\tilde{p}_{n+2} t_0 = -p^{1/p^{n+1}} t_0
\end{array}$$

With the same type of argument as for the first square diagram we conclude that the second one is commutative i.e.:

$$\begin{array}{ccc} C & \xrightarrow{s_x} & D \\ {\scriptstyle p\delta-\tilde{p}_{n+2}}\downarrow & \equiv & \downarrow{\scriptstyle -\tilde{p}_{n+2}} \\ C & \xrightarrow{s_x} & D \end{array}$$

Note that the sequence of cokernels $B/(p\delta - \tilde{p}_{n+1})B \mapsto C/(p\delta - \tilde{p}_{n+2})C$ is the map on stalks associated to f. We want to prove its injectivity. By the Snake Lemma in the main diagram this is equivalent to showing that the kernel of the multiplication by $p\delta - \tilde{p}_{n+2}$ on C surjects into the kernel of the multiplication by $-\tilde{p}_{n+2}$ on D. Let's remark that $\tilde{p}_{n+2} = p^{1/p^{n+1}}$ in $\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$ and that, since $\cdot p$ itself kills D, the kernel of the multiplication by $p^{1/p^{n+1}}$ on D is $p \cdot p^{-1/p^{n+1}}D = p^{\frac{p^{n+1}-1}{p^{n+1}}}D = \tilde{p}_{n+2}^{p^{n+1}-1}D$. Take now $v \in D$(so in particular $p^{\frac{p^{n+1}-1}{p^{n+1}}} \cdot v \in ker(\cdot p^{1/p^{n+1}})$) and let $x \in C$ be the lift of v under s_x defined by taking the Teichmueller lifts of the coefficients of x with respect to the $\bar{\mathcal{O}}_{\mathfrak{X}_x}/p\bar{\mathcal{O}}_{\mathfrak{X}_x}$-basis of D. Define $u := \sum_{i=0}^{p^{n+1}-1} p^i\delta^i\tilde{p}_{n+2}^{p^{n+1}-i-1}v$. We have that:

$$\begin{aligned}(p\delta - \tilde{p}_{n+2})u &= \sum_{i=0}^{p^{n+1}-1} p^{i+1}\delta^{i+1}\tilde{p}_{n+2}^{p^{n+1}-i-1}v - \sum_{i=0}^{p^{n+1}-1} p^i\delta^i\tilde{p}_{n+2}^{p^{n+1}-i}v \\ &= \delta^{p^{n+1}}p^{p^{n+1}}v - \tilde{p}_{n+2}^{p^{n+1}}v = 0\end{aligned}$$

since $\delta^{p^{n+1}}p^{p^{n+1}}v \equiv 0(\mathrm{mod} p)$ and $\tilde{p}_{n+2}^{p^{n+1}}v = p \cdot v = 0$ on D.

On the other hand, $s_x(u) = p^0\delta^0\tilde{p}_{n+2}^{p^{n+1}-1} \cdot v = \tilde{p}_{n+2}^{p^{n+1}-1} \cdot v = p^{\frac{p^{n+1}-1}{p^{n+1}}} \cdot v$ hence the kernel of the multiplication by $p\delta - \tilde{p}_{n+2}$ on C surjects into the kernel of the multiplication by $-\tilde{p}_{n+2}$ on D which is what we wanted. The left exactness of the diagram of sheaves follows and with this, one completes the proof.

□

Consider now the map of sheaves

$$u_{n,\overline{K}} : \mathbb{A}^{\nabla}_{\mathrm{max,n+1}} \to \mathbb{A}'^{\nabla}_{\mathrm{max,n}}$$

associated to the map of pre-sheaves induced by $u_n : A_{\mathrm{max,n+1}} \to A_{\mathrm{max}}/p^n A_{\mathrm{max}}$ (defined before Lemma 3.2.4) and by the natural projection $\mathbb{W}_{n+1}(\mathcal{U}, \mathcal{W}) \to \mathbb{W}_n(\mathcal{U}, \mathcal{W})$.

Also consider the map of sheaves

$$q'_{n,\overline{K}} : \mathbb{A}'^{\nabla}_{\text{max,n}} \to \mathbb{A}^{\nabla}_{\text{max,n}}$$

associated to the map of pre-sheaves induced by $q'_n : A_{\max}/p^n A_{\max} \to A_{\text{max,n}}$ (defined as well before Lemma 3.2.4) and by Frobenius $\mathbb{W}_n(\mathcal{U}, \mathcal{W}) \to \mathbb{W}_n(\mathcal{U}, \mathcal{W})$.

Write $q'_{\overline{K}} := \{q'_{n,\overline{K}}\}_n : \mathbb{A}'^{\nabla}_{\max} \to \mathbb{A}^{\nabla}_{\max}$ and $u_{\overline{K}} := \{u_{n,\overline{K}}\}_n : \mathbb{A}^{\nabla}_{\max} \to \mathbb{A}'^{\nabla}_{\max}$.

In order to conclude the comparison between $\mathbb{A}'^{\nabla}_{\text{max,n}}$ and $\mathbb{A}^{\nabla}_{\text{max,n}}$ let's prove the following:

Lemma 3.2.5. *For any positive integers $m \geq n+2$ we have an isomorphism of rings $A_{\max}/p^n A_{\max} \cong A_{\text{max,m}}/p^n A_{\text{max,m}}$ and the map $u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ \ldots \circ r_{m,\overline{K}} : \mathbb{A}^{\nabla}_{\text{max,m}} \to \mathbb{A}'^{\nabla}_{\text{max,n}}$ induces an isomorphism $\mathbb{A}^{\nabla}_{\text{max,m}}/p^n \mathbb{A}^{\nabla}_{\text{max,m}} \cong \mathbb{A}'^{\nabla}_{\text{max,n}}$.*

Proof. We defined at the beginning of the chapter the surjective maps q_m and the reduction π_m. Their composition is the surjective map

$$q_{m,m-1} \circ \pi_m : \mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \twoheadrightarrow \mathbb{W}_m(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}})) \twoheadrightarrow \mathbb{W}_m(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})$$

sending $(s_0, s_1, \ldots) \mapsto (s_0^{(m-1)}(\text{mod}p), \ldots, s_{m-1}^{(m-1)}(\text{mod}p))$, which induces the surjection:

$$\mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))\{\delta\} \twoheadrightarrow \mathbb{W}_m(\mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}})[\delta] = W_m[\delta]$$

defined by $\sum_{i\geq 0} a_i\delta^i \to \sum_{i\geq 0} \bar{a}_i\delta^i$, where $\bar{a}_i = (q_{m,m-1}\circ\pi_m)(a_i) = q_{m,m-1}(a_i\text{mod}p^m)$. Further we get a surjective map $\psi_m : A_{\max} \to A_{\text{max,m}}$ and for any integers $m \geq n+2$, $\psi_m(p^n A_{\max}) = p^n A_{\text{max,m}}$ since $\psi_m(p^n \sum_{i\geq 0} a_i\delta^i) = \bar{p}^n \sum'_{i\geq 0} \bar{a}_i\delta^i = p^n \sum'_{i\geq 0} \bar{a}_i\delta^i$ where by $\sum'$ we mean finite sum (for the latest equality remark that $q_{m,m-1}(p^n\text{mod}p^m) = (0, \ldots, 0, 1, 0, \ldots, 0) \in W_m$ for $m \geq n+2$). The second isomorphism theorem for rings gives us now: $A_{\max}/p^n A_{\max} \cong A_{\text{max,m}}/p^n A_{\text{max,m}}$. More explicit, let μ be the surjective map obtained by composing ψ_m with the reduction modulo p^n map, so

$$\mu : A_{\max} \to A_{\text{max,m}} \to A_{\text{max,m}}/p^n A_{\text{max,m}}, \text{ sending}$$

$$\sum_{i\geq 0} a_i(\tfrac{\xi}{p})^i \to \sum'_{i\geq 0} \bar{a}_i(\tfrac{\xi_m}{p})^i \to \sum'_{i\geq 0} \bar{a}_i(\text{mod}p^n)(\tfrac{\xi_m(\text{mod}p^n)}{p})^i.$$

Then $ker(\mu) = \psi_m^{-1}(p^n A_{\max,m}) = p^n \psi_m^{-1}(A_{\max,m}) = p^n A_{\max}$ and so one obtains $A_{\max}/p^n A_{\max} \cong A_{\max,m}/p^n A_{\max,m}$.

Remark that the finiteness of the sum appears since $a_i \to 0$ in the strong topology of $\mathbb{W}(\underline{\mathcal{R}}(\mathcal{O}_{\overline{K}}))$ (the p-adic topology) i.e. there exists a natural number $N > 0$ such that $a_j \equiv 0 (\text{mod } p^j)$ for all $j \geq N$ (we can take $N > m$ so one has $p^m \mid p^N$ hence $a_j \equiv 0(\text{mod } p^m)$ for all $j \geq N$).

One can write $\cdot p^n$ on $\mathbb{W}_m$ as $\mathbb{V} \circ \varphi$ where $\mathbb{V}$ is the Verschiebung and φ Frobenius. Recall that φ is surjective on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$ by [AI2], Lemma 4.4.1(v). As in Lemma 3.2.4 we get an isomorphism $\mathbb{W}_m/p^n\mathbb{W}_m \cong \mathbb{W}_n$ induced by the natural projection on the first n components. One obtains that, via this identification, the map $u_n \circ r_{n+2} \circ ... \circ r_m$: $\mathbb{W}_m \to \mathbb{W}_n$ is φ^{m-n-1} and that at the level of rings sends $\xi_m \in W_m$ to $pr_n(\xi_{n+1}) \in W_n$.

We have that $(V^s(\tilde{p}_m))^{p^n} = (p^s \cdot \tilde{p}_m)^{p^{n-s}} = p^{sp^{n-s}} \cdot \frac{\tilde{p}_m^{p^{n-s}}}{p^{p^{n-s}}} \cdot p^{p^{n-s}} = p^{(1+s)p^{n-s}} \cdot \frac{\tilde{p}_m^{p^{n-s}}}{p^{p^{n-s}}} = 0$ in $A_{\max,m}/p^n A_{\max,m}$ since $(1+s)p^{n-s} \geq n$, $0 \leq s \leq n$ (the inequality follows easily by induction over $n-s$: for $n = s$ the inequality reads $s+1 \geq s$ and for $n = s+1$: $(1+s) \cdot p \geq 1+s$; suppose that for $n = s+k$, $k > 0$, $(1+s)p^k \geq s+k$ holds, then for $n = s+k+1$ we get $(1+s) \cdot p^{k+1} \geq (s+k) \cdot p \geq s+k+1$).

Now, $\tilde{p}_m^{p^n}$ generates the kernel of φ^{m-n-1} on $\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}}$. On one hand, $\varphi^{m-n-1}(\tilde{p}_m^{p^n}) = \varphi^{m-n-1}((p^{n-m+1})) = (p) = 0$ on S/pS(recall that $S = \bar{\mathcal{O}}_{\mathfrak{X}}(\mathcal{U}, \mathcal{W})$). For the other inclusion let $x \in ker(\varphi^{m-n-1})$ so $x^{p^{m-n-1}} = p \cdot y$ for some $y \in S$. Since S is normal it follows that $x = p^{1/p^{m-n-1}} \cdot y'$, $y' \in S$, hence $x \in (\tilde{p}_m^{p^n})$.

We obtain that $\{\mathbb{V}^s(\tilde{p}_m^{p^n})\}_{0 \leq s \leq n}$ generates the kernel of φ^{m-n-1} on $\mathbb{W}_n$.

Similarly it follows that $W_m/p^n W_m \cong W_n$ and that $\{V^s(\tilde{p}_m^{p^n})\}_{0 \leq s \leq n}$ generates the kernel of φ^{m-n-1} on W_n.

Let's prove now that $p^n \mathbb{A}^{\nabla}_{\max,m} = ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$.

Firstly, let $x \otimes_{W_m} y \in A_{\max,m} \otimes_{W_m} \mathbb{W}_m(\mathcal{U}, \mathcal{W})$. Since $p^n \in W_m$ we have $p^n(x \otimes_{W_m} y) = p^n x \otimes_{W_m} y = x \otimes_{W_m} p^n y \in ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$ clearly.

Secondly, let $\sum_i x_i \otimes_{W_m} y_i \in ker(u_{n,\overline{K}} \circ r_{n+2,\overline{K}} \circ ... \circ r_{m,\overline{K}})$. The element $\sum_i x_i \otimes_{W_m} y_i$ is mapped to $\sum_i \bar{x}_i \otimes_{W_n} pr_n(y_i) = 0 \in A_{\max,m}/p^n A_{\max,m} \otimes_{W_n} \mathbb{W}_n(\mathcal{U}, \mathcal{W})$ (here we use the isomorphism $A_{\max}/p^n A_{\max} \cong A_{\max,m}/p^n A_{\max,m}$). We conclude that $\sum_i x_i \otimes_{W_m} y_i \in p^n(A_{\max,m} \otimes_{W_m} \mathbb{W}_m(\mathcal{U}, \mathcal{W}))$ and so the second inclusion also holds. The second claim of the Lemma follows.

□

We study now the localization of $\mathbb{A}^{\nabla}_{\max}$ over small affines.

Let $\mathcal{U}$=Spf($R_{\mathcal{U}}$) be a small affine open of the étale site on X, X^{et}. This is an object such that $R_{\mathcal{U}} \otimes_{\mathcal{O}_K} k$ is geometrically irreducible over k and there are parameters $T_1, T_2, ..., T_d \in R_{\mathcal{U}}^{\times}$ such that the map $R_0 := \mathcal{O}_K\{T_1^{\pm 1}, T_2^{\pm 1}, ..., T_d^{\pm 1}\} \subset R_{\mathcal{U}}$ is formally étale.

We define $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ to be the p-adic completion of the sub-$\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$-algebra of $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\frac{1}{p}]$ generated by $p^{-1}ker(\vartheta)$ where the map ϑ is defined as follows (we keep the notations of [AI1]):

For every n, let ϑ_n be the composition of the projection (reduction modulo p^n map): $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$, of the map $\mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})$ induced by the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the n-th component (see Proposition 3.1.1) and of $\theta_n : \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}) \to \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}}$ (defined at the beginning of the chapter).

Then define $\vartheta : \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \widehat{\overline{R}}_{\mathcal{U}} = \varprojlim \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}}$ to be the map $x \to \varprojlim \vartheta_n(x)$.

In [*Bri*1, §6] it is proved that $ker(\vartheta)$ is a principal ideal generated by ξ. We also have a Frobenius φ on $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ induced by the Frobenius on $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$. Remark that if $x \in \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ belongs to $ker(\vartheta)$ and if $n \in \mathbb{N}_{>0}$, one can write $x^{[n]} = p^{[n]}(x/p)^n \in A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ ($x^{[n]}$ is the n-th divided power of x i.e. $x^n/n!$) and hence there exists a natural homomorphism $A^{\nabla}_{\mathrm{cris}}(\overline{R}_{\mathcal{U}}) \to A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ (which is injective according to [Bri2, Proposition 2.3.2]). $A^{\nabla}_{\mathrm{cris}}(\overline{R}_{\mathcal{U}})$ is the p-adic completion of the $\mathbb{W}(k)$-DP envelope of $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ with respect to the kernel of the map ϑ defined above (see [AI1, §2.3] or [Bri1, §6] for details).

Note that ϑ makes sense since the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) & \xrightarrow{\vartheta_{n+1}} & \overline{R}_{\mathcal{U}}/p^{n+1}\overline{R}_{\mathcal{U}} \\ & \searrow^{\vartheta_n} & \downarrow{\scriptstyle mod(p^n)} \\ & & \overline{R}_{\mathcal{U}}/p^n\overline{R}_{\mathcal{U}} \end{array}$$

Let g_n be the composite of the projection (reduction modulo p^n map) $\mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$ and of the map $v_n : \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})$ induced by the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the $n+1$-th component (defined similar to q_n). As in the proof of Proposition 3.1.5, since $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) = \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\delta]/(p\delta - \xi)$ we have that (denote by $R := \mathcal{R}(\overline{R}_{\mathcal{U}})$):

$$\begin{aligned} A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) &= \tfrac{\mathbb{W}(R)[\delta]/(p\delta-\xi)}{p^n(\mathbb{W}(R)[\delta]/(p\delta-\xi))} \cong \tfrac{\mathbb{W}(R)[\delta]/(p\delta-\xi)}{(p^n, p\delta-\xi)\mathbb{W}(R)[\delta]/(p\delta-\xi)} \\ &\cong \tfrac{\mathbb{W}(R)[\delta]}{(p^n, p\delta-\xi)} \cong \tfrac{\mathbb{W}(R)[\delta]/p^n\mathbb{W}(R)[\delta]}{(p^n, p\delta-\xi)\mathbb{W}(R)[\delta]/p^n\mathbb{W}(R)[\delta]} \cong \mathbb{W}_n(R)[\delta]/(p\delta - \xi(\mathrm{mod} p^n)) \quad (1) \end{aligned}$$

so $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))[\delta]/(p\delta - \xi(\mathrm{mod} p^n))$ and since $g_n(\xi) = \xi_{n+1}$, we get a map $g'_n : A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}}) = A_{\max}/p^n A_{\max} \otimes_{W_n} (\mathbb{W}_n(\overline{R}_{\mathcal{U}}))$.

We have the following important result:

Proposition 3.2.6. *The ring $A^{\nabla}_{\max}$ is p-torsion free.*

Proof. By [Bri2, Proposition 2.3.7] and [Bri2, Remark 2.3.8], $A^{\nabla}_{\max}$ can be identified with a sub-ring of $A_{\max}$. According to [Bri2, Proposition 3.5.3] we have that $\varphi(A_{\max}) \subset A_{\mathrm{cris}} \subset A_{\max}$ where φ is the Frobenius and $A_{\mathrm{cris}} = \{\sum_{i\geq 0} a_i \xi^{[i]} \mid a_i \in A^{+}_{\inf}$ and $a_i \to 0$ when $i \to \infty\}$.

Now, let $x \in A_{\max}$ such that $p^n \cdot x = 0$ for some $n > 0$. Then $\varphi(p^n \cdot x) = p^n \cdot \varphi(x) = 0$ in A_{cris}. Since A_{cris} has no p-torsion by [Bri1, Proposition 6.1.10], it follows that $\varphi(x) = 0$. Moreover, since Frobenius is injective on $A_{\max}$, we obtain that $x = 0$ and so $A_{\max}$ is p-torsion free and consequently $A^{\nabla}_{\max}$ is free of p-torsion.

□

We will use this result in the proof of the following:

Theorem 3.2.7. *a) For every $n \in \mathbb{N}^*$ the map $g'_n : A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ is injective;*

b) The map $\mathbb{A}'^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ defined by $q'_{\overline{K}}$ is an isomorphism. (note that $q'_{\overline{K}} := \{q'_{n,\overline{K}}\}_n : \mathbb{A}'^{\nabla}_{\max} \to \mathbb{A}^{\nabla}_{max}$ is defined before Lemma 3.2.5).

Proof. a) We have that $\overline{R}_{\mathcal{U}}$ is a normal ring and that Frobenius is surjective on $\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ by [Bri1, Proposition. 2.0.1] and as in the proof of Proposition 3.1.1 we get that the kernel of the projection $\mathcal{R}(\overline{R}_{\mathcal{U}}) = \varprojlim \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} \to \overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}}$ on the $n+1$-th component is generated by $\tilde{p}^{p^n}$.

As in the proof of Lemma 3.2.5 we have that $(V^s([\tilde{p}]))^{p^n} = (p^s \cdot [\tilde{p}])^{p^{n-s}} = p^{sp^{n-s}} \cdot \frac{[\tilde{p}]^{p^{n-s}}}{p^{p^{n-s}}} \cdot p^{p^{n-s}} = p^{(1+s)p^{n-s}} \cdot \frac{[\tilde{p}]^{p^{n-s}}}{p^{p^{n-s}}} = 0$ in $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$, $0 \le s \le n$. Now, via Proposition 3.1.3, we obtain that $\{\mathrm{V}^s([\tilde{p}])\}^{p^n}_{0 \le s \le n}$ generate the kernel of v_n. As in the proof of Proposition 3.1.5, it follows that:

$$A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{W}_n(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p\delta - \xi_{n+1}), \quad (2)$$

where the isomorphism is induced by the map $g_n : \mathbb{W}(\mathcal{R}(\overline{R}_{\mathcal{U}})) \to \mathbb{W}_n(\mathcal{R}(\overline{R}_{\mathcal{U}}))$.

We prove a) by induction on n. For $n = 1$ the map

$A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/pA^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,1}(\overline{R}_{\mathcal{U}})$ becomes

$(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p\delta - \xi_2) \to ((\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}))[\delta]/(\xi_2)$

via the above isomorphism and the remark before Lemma 3.2.4. By using now [AI1, Proposition 2.13] and [AI1, Proposition 2.14] we have an injective map

$\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}} = \bar{\mathcal{O}}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}})/p\bar{\mathcal{O}}_{\mathfrak{X}}(\overline{R}_{\mathcal{U}}) \to (\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}})$ hence

$(\overline{R}_{\mathcal{U}}/p\overline{R}_{\mathcal{U}})[\delta]/(p^{1/p}) \to ((\bar{\mathcal{O}}_{\mathfrak{X}}/p\bar{\mathcal{O}}_{\mathfrak{X}})(\overline{R}_{\mathcal{U}}))[\delta]/(p^{1/p})$ is injective and so the case $n = 1$ is proved (recall that $\xi_2(\mathrm{mod} p) = p^{1/p}$).

By Proposition 3.2.6, $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ has no p-torsion hence we have the exact sequence:

$$0 \longrightarrow \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} \xrightarrow{\cdot p} \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{p^{n+1} A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} \longrightarrow \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{pA^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} \longrightarrow 0$$

This is compatible with the exact sequence obtained by taking the localizations in the exact sequence of Lemma 3.2.4 i.e. we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} & \xrightarrow{\cdot p} & \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{p^{n+1} A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} & \longrightarrow & \frac{A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})}{p A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})} & \longrightarrow & 0 \quad (*) \\ & & \downarrow g'_n & & \downarrow g'_{n+1} & & \downarrow g'_1 & & \\ 0 & \longrightarrow & \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}}) & \xrightarrow{f'} & \mathbb{A}'^{\nabla}_{\max,n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{g'} & \mathbb{A}'^{\nabla}_{\max,1}(\overline{R}_{\mathcal{U}}) & \longrightarrow & 0 \end{array}$$

where the maps $f' = f_{\overline{R}_{\mathcal{U}}}$ and $g' = g_{\overline{R}_{\mathcal{U}}}$ are induced by f and g respectively (see Lemma 3.2.4).

The second square diagram of the main one is commutative since:

$$\begin{array}{ccc} \sum b_i(\frac{\xi}{p})^i(\mathrm{mod}p^{n+1}) & \longrightarrow & \sum b_i(\frac{\xi}{p})^i(\mathrm{mod}p) \\ \downarrow g'_{n+1} & \equiv & \downarrow g'_1 \\ \sum b_i(\frac{\xi_{n+2}}{p})^i(\mathrm{mod}p^{n+1}) & \xrightarrow{g'} & \sum b_i(\frac{\xi_2}{p})^i(\mathrm{mod}p) \end{array}$$

where the bottom map is induced by Frobenius to the n-th power φ^n composed with the projection and we have that $(proj \circ \varphi^n)(\xi_{n+2}) = \xi_2$ and for the vertical maps we use the fact that $g'_n(\xi(\mathrm{mod}p^n)) = \xi_{n+1}$. Moreover, $b_i \in A^+_{\mathrm{inf}}(\overline{R}_{\mathcal{U}})$ such that $b_i \to 0$ in the p-adic topology and so the above sums are finite.

The first square diagram of the main one is also commutative since:

$$\begin{array}{ccc} \sum b_i(\frac{\xi}{p})^i(\mathrm{mod}p^n) & \xrightarrow{\cdot p} & \sum p \cdot b_i(\frac{\xi}{p})^i(\mathrm{mod}p^{n+1}) \\ \downarrow g'_n & \equiv & \downarrow g'_{n+1} \\ \sum b_i(\frac{\xi_{n+1}}{p})^i(\mathrm{mod}p^n) & \xrightarrow{f'} & \sum p \cdot b_i(\frac{\xi_{n+2}}{p})^i(\mathrm{mod}p^{n+1}) \end{array}$$

For the commutativity of the above diagram one uses the fact that f' induces the Verschiebung at the level of the Witt vectors so that we have:

$$f'(\xi_{n+1}) = V(\xi_{n+1}) = (0, p^{1/p^n}, 0, ..., 0) - V(p)$$

and since $V(p) = V(FV(1)) = (VF)(V(1)) = pV(1) = p^2$ we get that

$$f'(\xi_{n+1}) = (0, p^{1/p^n}, 0, ..., 0) - p^2.$$

On the other hand, $p \cdot \xi_{n+2} = VF([p^{1/p^{n+1}}]) - p^2 = (0, p^{1/p^n}, 0, ..., 0) - p^2$ and consequently $f'(\xi_{n+1}) = p \cdot \xi_{n+2}$.

Now we apply the inductive hypothesis (g'_n injective) and use the Snake Lemma in the main diagram, $(*)$, so at the level of kernels we get:

$0 \longrightarrow ker(g'_{n+1}) \longrightarrow 0$ hence g'_{n+1} is injective (one can also see this directly by diagram chase). Claim a) follows.

b) We prove that for every $n \in \mathbb{N}^*$ we have $q'_{n,\overline{K}} \circ u_{n,\overline{K}} = r_{n+1,\overline{K}}$ and $u_{n,\overline{K}} \circ q'_{n+1,\overline{K}} = r'_{n+1,\overline{K}}$.

For the first relation, let's remark that the following diagram is commutative:

$$\begin{array}{ccc} A_{\max,n+1} \otimes_{W_{n+1}} \mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{u_{n,\overline{K}}} & A_{\max,n} \otimes_{W_n} \mathbb{W}_n(\overline{R}_{\mathcal{U}}) \\ & \searrow^{r_{n+1,\overline{K}}} & \downarrow q'_{n,\overline{K}} \\ & & A_{\max}/p^n A_{\max} \otimes_{W_n} \mathbb{W}_n(\overline{R}_{\mathcal{U}}) \end{array}$$

since
$$\begin{array}{ccc} \xi_{n+1} \otimes_{W_{n+1}} 1 & \xrightarrow{u_{n,\overline{K}}} & pr_n(\xi_{n+1}) \otimes_{W_n} 1 \\ & \searrow^{r_{n+1,\overline{K}}} & \downarrow q'_{n,\overline{K}} \\ & & \xi_n \otimes_{W_n} 1 \end{array}$$

and also
$$\begin{array}{ccc} (s_0, s_1, ..., s_n) & \xrightarrow{u_n} & (s_0, s_1, ..., s_{n-1}) \\ & \searrow^{r_{n+1}} & \downarrow q'_n \\ & & (s_0^p, s_1^p, ..., s_{n-1}^p) \end{array}$$

For the second relation, we obtain similarly that the following diagram is commutative:

$$\begin{array}{ccc} A_{\max}/p^{n+1} A_{\max} \otimes_{W_{n+1}} \mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) & \xrightarrow{q'_{n+1,\overline{K}}} & A_{\max,n+1} \otimes_{W_{n+1}} \mathbb{W}_{n+1}(\overline{R}_{\mathcal{U}}) \\ & \searrow^{r'_{n+1,\overline{K}}} & \downarrow u_{n,\overline{K}} \\ & & A_{\max}/p^n A_{\max} \otimes_{W_n} \mathbb{W}_n(\overline{R}_{\mathcal{U}}) \end{array}$$

$$\text{since } pr_{n+1}(\xi_{n+2}) \otimes_{W_{n+1}} 1 \xrightarrow{q'_{n+1,\overline{K}}} \xi_{n+1} \otimes_{W_{n+1}} 1, \quad \xi_{n+1} \otimes_{W_{n+1}} 1 \xrightarrow{u_{n,\overline{K}}} pr_n(\xi_{n+1}) \otimes_{W_n} 1, \quad pr_{n+1}(\xi_{n+2}) \otimes_{W_{n+1}} 1 \xrightarrow{r'_{n+1,\overline{K}}} pr_n(\xi_{n+1}) \otimes_{W_n} 1$$

$$\text{and also } (s_0, s_1, ..., s_n) \xrightarrow{q'_{n+1}} (s_0^p, s_1^p, ..., s_n^p), \quad (s_0^p, s_1^p, ..., s_n^p) \xrightarrow{u_n} (s_0^p, s_1^p, ..., s_{n-1}^p), \quad (s_0, s_1, ..., s_n) \xrightarrow{r'_{n+1}} (s_0^p, s_1^p, ..., s_{n-1}^p)$$

By taking now $\varprojlim$, the two above mentioned relations give us: $q'_{\overline{K}} \circ u_{\overline{K}} = id$ and $u_{\overline{K}} \circ q'_{\overline{K}} = id$ respectively. Claim b) follows; $u_{\overline{K}}$ defines the inverse of $q'_{\overline{K}}$.

□

Corollary 3.2.8. *The induced map* $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \longrightarrow \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) = \varprojlim \mathbb{A}^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ *is an isomorphism.*

Proof. One shows that the transition maps $\mathbb{A}'^{\nabla}_{\max,n+1}(\overline{R}_{\mathcal{U}}) \to \mathbb{A}'^{\nabla}_{\max,n}(\overline{R}_{\mathcal{U}})$ factor via $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})/p^n A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ for all $n \geq 1$ and by taking projective limit and further using the fact that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ is complete, one obtains that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}'^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$. By Theorem 3.2.7.b) we have an isomorphism $\mathbb{A}'^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$ and consequently we obtain that $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}}) \cong \mathbb{A}^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$.

□

Chapter 4

The sheaf $\mathbb{A}_{max}$

Let $p > 0$ be a prime integer, K a finite, unramified extension of $\mathbb{Q}_p$ with residue field k, $\mathcal{O}_K$ the ring of integers of K and denote by K_0 the maximal unramified subfield of $\overline{K}$ and by $\mathcal{O}_{K_0}$ its ring of integers.

Recall that we have a morphism $\theta_{\overline{K}} : \mathbb{A}^+_{\mathrm{inf},\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}}$ of objects of $Sh(\mathfrak{X}_{\overline{K}})^{\mathbb{N}}$ constructed as follows: let $(\mathcal{U}, \mathcal{W})$ be an object of $\mathfrak{X}_{\overline{K}}$. Denote by $S = \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U}, \mathcal{W})$ and for fixed $n \in \mathbb{N}$, consider the diagram of sets:

$$\begin{array}{ccc} (S/p^nS)^n & \xrightarrow{a_n} & S/p^nS \\ & \searrow^{b_n} & \uparrow \exists! c_n \\ & & (S/pS)^n \end{array}$$

where b_n is the natural projection and $a_n(s_0, s_1, ..., s_{n-1}) := \sum_{i=0}^{n-1} p^i s_i^{p^{n-1-i}}$.

There exists a unique map of sets, call it $c_n : (S/pS)^n \to S/p^nS$ making the diagram commutative i.e. $c_n \circ b_n = a_n$.

We have that $c_n(s_0, s_1, ..., s_{n-1}) := \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}$, where $\tilde{s}_i \in S/p^nS$ is a lift of $s_i \in S/pS$ for all $0 \leq i \leq n-1$ and let's remark that c_n is well defined:

For this, let $(c_0, c_1, ..., c_n) \in (S/pS)^n$ such that $c_i \equiv s_i(p)$ for all $0 \leq i \leq n-1$. Then $c_i^{p^{n-1-i}} \equiv s_i^{p^{n-1-i}}(p^{n-i})$ and by multiplying the latest relation by p^i we obtain that $p^i c_i^{p^{n-1-i}} \equiv p^i s_i^{p^{n-1-i}}(p^n)$ for all $0 \le i \le n-1$. It follows that $\sum_{i=0}^{n-1} p^i \tilde{c}_i^{p^{n-1-i}} \equiv \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}(p^n)$, which is equivalent to $c_n(c_0, c_1, ..., c_{n-1}) \equiv c_n(s_0, s_1, ..., s_{n-1})(p^n)$, in other words c_n is well defined.

The map c_n induces a ring homomorphism $c_{n,(\mathcal{U},\mathcal{W})} : \mathbb{W}_n(S/pS) \to S/p^n S$, which is functorial in $(\mathcal{U}, \mathcal{W})$, in other words a morphism of presheaves $\mathbb{W}_{n,\overline{K}} \xrightarrow{c_n} \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$. One denotes by $\theta_{n,\overline{K}}$ the induced morphism on the associated sheaves and let:

$$\theta_{\overline{K}} := \{\theta_{n,\overline{K}}\} : \mathbb{A}^+_{\mathrm{inf},\overline{K}} = \varprojlim \mathbb{W}_{n,\overline{K}} \to \hat{\bar{\mathcal{O}}}_{\mathfrak{X}_{\overline{K}}} = \varprojlim(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}})$$

Assume that X is a smooth scheme over $\mathcal{O}_K$ and that $\mathcal{O}_K = \mathbb{W}(k)$ is absolutely unramified.

Let $\mathcal{O}_X$ be the sheaf on the site $\mathfrak{X}_{\overline{K}}$ defined by $\mathcal{O}_X(\mathcal{U}, \mathcal{W}) := \mathcal{O}_X(\mathcal{U})$.

For every $n \ge 1$ one defines the sheaf $\mathbb{W}_{X,n,\overline{K}} := \mathbb{W}_n(\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}) \otimes_{\mathcal{O}_K} \mathcal{O}_X$ of $\mathcal{O}_{K_0}$-algebras and also the morphism of sheaves of $\mathcal{O}_{K_0} \otimes_{\mathcal{O}_K} \mathcal{O}_X$-algebras $\theta_{X,n,\overline{K}}$: $\mathbb{W}_{X,n,\overline{K}} \to \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}/p^n\bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$ associated to the following map of presheaves: firstly take an object $(\mathcal{U}, \mathcal{W})$ of $\mathfrak{X}_{\overline{K}}$ such that $\mathcal{U}$=Spf$(R_\mathcal{U})$ is affine (i.e. $R_\mathcal{U} = \mathcal{O}_X(\mathcal{U}, \mathcal{W})$). Clearly $S = \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{U}, \mathcal{W})$ has a natural $R_\mathcal{U}$-algebra structure. Define now:

$$\theta_{n,(\mathcal{U},\mathcal{W})} : \mathbb{W}_n(S/pS) \otimes_{\mathcal{O}_K} R_\mathcal{U} \to S/p^n S \text{ by } (x \otimes r) \to c_n(x)r.$$

Also denote by $\tau_{X,n,\overline{K}}$ the sheaf of ideals $\mathrm{Ker}(\theta_{n,(\mathcal{U},\mathcal{W})})$.

Let now $\mathcal{U}$=Spf$(R_\mathcal{U})$ be a small affine open of the étale site on X, X^{et}, with parameters $T_1, T_2, ..., T_d \in R_\mathcal{U}^\times$ (recall the definition of small affines from the previous chapter). Further, for $n \ge 0$, let $R_{\mathcal{U},n} := R_\mathcal{U}[\zeta_n, T_1^{1/p^n}, ..., T_d^{1/p^n}]$, where $R_{\mathcal{U},0} = R_\mathcal{U}$, ζ_n is a primitive p^n-th root of unity with $\zeta_{n+1}^p = \zeta_n$ and such that T_i^{1/p^n} is a fixed p^n-th root of T_i in $\overline{R}_\mathcal{U}$ with $(T_i^{1/p^{n+1}})^p = T_i^{1/p^n}$ for any $1 \le i \le d$. Moreover, consider the category $\mathfrak{U}_{n,\overline{K}}$ consisting of objects $(\mathcal{V}, \mathcal{W})$ and a morphism to $(\mathcal{U}, \mathrm{Spf}(R_{\mathcal{U},n}) \otimes_{\mathcal{O}_K} \overline{K})$. The morphisms of this category are the morphisms of objects over $(\mathcal{U}, \mathrm{Spf}(R_{\mathcal{U},n}) \otimes_{\mathcal{O}_K} \overline{K})$ and the covering families of an object $(\mathcal{V}, \mathcal{W})$ are the covering families of $(\mathcal{V}, \mathcal{W})$ regarded as object of $\mathfrak{X}_{\overline{K}}$. Given a sheaf $\mathcal{F}$ on $\mathfrak{X}_{\overline{K}}$, one writes $\mathcal{F}|_{\mathfrak{U}_{n,\overline{K}}}$ for $u_*(\mathcal{F})$ where $u : \mathfrak{U}_{n,\overline{K}} \to \mathfrak{X}_{\overline{K}}$ is the forgetful functor.

Let now $(\mathcal{V},\mathcal{W}) \in \mathfrak{U}_{n,\overline{K}}$ with $\mathcal{V} = Spf(R_{\mathcal{V}})$ affine and let $S := \bar{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}(\mathcal{V},\mathcal{W})$. Remark that $T_i^{1/p^n} \in R_{\mathcal{U},n} \subset S$ for all $1 \leq i \leq d$ since S is the normalization of $\Gamma(\mathcal{V},\mathcal{O}_{\mathcal{V}}) = R_{\mathcal{V}}$ in $\Gamma(\mathcal{W},\mathcal{O}_{\mathcal{W}})$. Also denote by:

$$\widetilde{T}_i := ([T_i],[T_i^{1/p}],...,[T_i^{1/p^n}],...) \in \varprojlim \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$$

the inverse limit being taken with respect to the map $\mathbb{W}_{n+1}(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1}) \to \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$ defined as the composition between the natural projection

$\mathbb{W}_{n+1}(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1}) \to \mathbb{W}_n(R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1})$ and the map induced by the Frobenius: $R_{\mathcal{U},n+1}/pR_{\mathcal{U},n+1} \to R_{\mathcal{U},n}/pR_{\mathcal{U},n}$. Note that the image of $\tilde{T}_i$ in $\mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n})$ is $(T_i^{1/p^n},0,...,0)$ i.e. the Teichmueller lift of T_i^{1/p^n}. For all $1 \leq i \leq d$, define now:

$$X_i := 1 \otimes T_i - \widetilde{T}_i \otimes 1 \in \mathbb{W}_n(R_{\mathcal{U},n}/pR_{\mathcal{U},n}) \otimes_{\mathcal{O}_K} R_{\mathcal{U}}$$

and remark that these elements also live in $\mathbb{W}_n(S/pS) \otimes_{\mathcal{O}_K} R_{\mathcal{V}}$.

We prove now the following:

Lemma 4.0.9. *([AI1, Lemma 2.28]) The kernel of the map* $\theta_{n,(\mathcal{V},\mathcal{W})} : \mathbb{W}_n(S/pS) \otimes_{\mathcal{O}_K} R_{\mathcal{V}} \to S/p^nS$ *is the ideal generated by* $(\xi_n, X_1, ..., X_d)$.

Proof. We claim that $\xi_n = \tilde{p}_n - p = [p^{1/p^{n-1}}] - p$, which is a well defined element of $\mathbb{W}_n(S/pS)$ (note that $p^{1/p^{n-1}} \in S$), generates $\ker(c_n)$, $c_n : \mathbb{W}_n(S/pS) \to S/p^nS$.

Recall now that $R_0 = \mathcal{O}_K\{T_1^{\pm1},...,T_d^{\pm1}\}$ and so $R_0/p^nR_0 = (\mathcal{O}_K/p^n\mathcal{O}_K)[T_1^{\pm1},...,T_d^{\pm1}]$. We have that the kernel of the ring homomorphism $R_0/p^nR_0 \otimes R_0/p^nR_0 \to R_0/p^nR_0$ defined by $x \otimes y \to xy$ is the ideal $I = (T_1 \otimes 1 - 1 \otimes T_1, .., T_d \otimes 1 - 1 \otimes T_d)$. Note that $R_0/p^nR_0 \hookrightarrow R_{\mathcal{V}}/p^nR_{\mathcal{V}}$ is étale hence I also generates the kernel of the map $R_{\mathcal{V}}/p^nR_{\mathcal{V}} \otimes R_{\mathcal{V}}/p^nR_{\mathcal{V}} \to R_{\mathcal{V}}/p^nR_{\mathcal{V}}$ $(*)$, defined by $x \otimes y \to xy$. We tensor now $(*)$ with S/p^nS over $R_{\mathcal{V}}/p^nR_{\mathcal{V}}$ and since base changing of an étale morphism is étale ([Mi, Proposition 2.11, (c)]) we obtain that I generates the kernel of $S/p^nS \otimes R_{\mathcal{V}}/p^nR_{\mathcal{V}} \to S/p^nS$.

Proof of the Claim ([AI1, Lemma 2.17]): Firstly recall that $c_n(s_0, s_1, ..., s_{n-1}) := \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}}$, where $\tilde{s}_i \in S/p^nS$ is a lift of $s_i \in S/pS$ for all $0 \leq i \leq n-1$.

One computes now $c_n(\xi_n) = (p^{1/p^{n-1}})^{p^{n-1}} - p = p - p = 0$ hence $\xi_n \in ker(c_n)$.

We will prove that if $x \in ker(c_n)$ then $x \in \xi_n \mathbb{W}_n(S/pS)$ and we show this statement by induction on n: for $n = 1$, $c_1 = id$ and $\xi_1 = \tilde{p} = 0 \in \mathbb{W}_1(S/pS) = S/pS$.

Let $n > 1$ and assume that the statement is true for $n-1$. Further, let $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in ker(c_n)$ and recall that $\mathbb{V} : \mathbb{W}_{n-1}(S/pS) \to \mathbb{W}_n(S/pS)$ is the Verschiebung i.e. $\mathbb{V}(s_0, s_1, ..., s_{n-2}) = (0, s_0, s_1, ..., s_{n-2})$, for $(s_0, s_1, ..., s_{n-2}) \in \mathbb{W}_{n-1}(S/pS)$.

We will prove that there exist elements $\beta \in \mathbb{W}_n(S/pS)$ and $\gamma \in \mathbb{W}_{n-1}(S/pS)$ such that $\alpha = \xi_n\beta + \mathbb{V}(\gamma)$.

We have that $c_n(\alpha) = c_n((\alpha_0, \alpha_1, ..., \alpha_{n-1})) = \sum_{i=0}^{n-1} p^i \tilde{s}_i^{p^{n-1-i}} = 0$ and hence:

$p \mid \tilde{\alpha_0}^{p^{n-1}}$. Put $\tilde{\alpha}_0^{p^{n-1}} = pc$, $c \in S$.

Let now $R_\mathcal{V} \subset S'(\subset S)$ be a finite and normal extension containing $\tilde{\alpha}_0$ and $p^{1/p^{n-1}}$. In particular S' is noetherian and integrally closed.

Now, for every height one prime ideal $\wp$ of S', since $S'_\wp$ is noetherian, integrally closed (because S' is noetherian and integrally closed respectively), and since dim$S'_\wp$=ht$\wp$=1, it follows ([Al-Io, Theorem 2.3.15]) that $S'_\wp$ is a DVR.

Remark that for every height one prime ideal $\wp$, $p \in S' - \wp$. We have that $\tilde{\alpha}_0^{p^{n-1}}/p = c \in S'_\wp$ and moreover, since $S'_\wp$ is a DVR, we obtain that $\tilde{\alpha}_0/p^{1/p^{n-1}} = c' \in S'_\wp$ (note that $\tilde{\alpha}_0^{p^{n-1}} = p \cdot c$ leads to $v(\alpha_0) \geq v(p^{1/p^{n-1}})$).

It follows that $\tilde{\alpha}_0/p^{1/p^{n-1}}$ lives in the intersection of the localizations of S' at every height one prime ideal. Since S' is an integral closed noetherian domain we have that $\bigcap_{\wp, ht(\wp)=1} S'_\wp = S'$ ([Ma, 2, Theorem 38]) or [Ha, Proposition 6.3 A]). Consequently, $\tilde{\alpha}_0/p^{1/p^{n-1}} \in S'$.

Let β_0 be the image of $\tilde{\alpha}_0/p^{1/p^{n-1}}$ in S/pS so $\alpha_0 = p^{1/p^{n-1}}\beta_0$ in S/pS. Moreover, define $\beta := (\beta_0, 0, ..., 0) \in \mathbb{W}_n(S/pS)$.

Note that $\tilde{p}_n \cdot \beta = (p^{1/p^{n-1}}, 0, ..., 0) \cdot (\beta_0, ..., 0) = (\alpha_0, 0, ..., 0)$ and that $p \cdot \beta = (0, 1, 0, ..., 0) \cdot (\beta_0, 0, ..., 0) = F\mathbb{V}((\beta_0, 0, ..., 0)) = (0, \beta_0^p, 0, ..., 0)$, where F is the Frobenius map (see [Se, Chapter 2, § 6]). We have that:

$$\alpha - \xi_n\beta = \alpha - (\tilde{p}_n - p)\beta = \alpha - \tilde{p}_n\beta + p\beta =$$
$$= (\alpha_0, \alpha_1, ..., \alpha_{n-1}) - (\alpha_0, 0, ..., 0) + (0, \beta_0^p, 0, ..., 0) \in \mathbb{V}(\mathbb{W}_n(S/pS)).$$

That is, there exists $\gamma \in \mathbb{W}_{n-1}(S/pS)$ such that $\alpha - \xi_n\beta = \mathbb{V}(\gamma)$. We then have $c_n(\mathbb{V}(\gamma)) = c_n(\alpha - \xi_n\beta) = 0$ (for this, recall that $c_n(\alpha) = c_n(\xi_n) = 0$). Moreover, since $c_n(\mathbb{V}(\gamma)) = \psi(c_{n-1}(\gamma))$ where by ψ_n one denotes the isomorphism $\psi_n : S/p^{n-1}S \cong pS/p^nS$, we obtain that $c_{n-1}(\gamma) = 0$. By using now the induction hypothesis, there is a $\delta \in \mathbb{W}_{n-1}(S/pS)$ such that $\gamma = \xi_{n-1}\delta$.

Write now $\delta = (\delta_0, \delta_1, ..., \delta_{n-2})$. We use now the following property of the multiplication of Witt vectors: $(r, 0, ..., 0, ...) \cdot (a_0, a_1, ..., a_n, ...) = (ra_0, r^p a_1, ..., r^{p^n} a_n, ...)$ (see [Se, Chapter 2, § 6]) and obtain:

$$\xi_{n-1}\delta = (p^{1/p^{n-2}}, 0, ..., 0) \cdot (\delta_0, \delta_1, ..., \delta_{n-2}) - p \cdot \delta =$$
$$= (p^{1/p^{n-2}}\delta_0, p^{1/p^{n-3}}\delta_1, ..., p\delta_{n-2}) - p \cdot \delta$$

hence $\mathbb{V}(\xi_{n-1}\delta) = (0, p^{1/p^{n-2}}\delta_0, p^{1/p^{n-3}}\delta_1, ..., p\delta_{n-2}) - \mathbb{V}(p\delta)$ (1)

Moreover,

$$\xi_n \cdot \mathbb{V}(\delta) = \xi_n \cdot (0, \delta_0, \delta_1, ..., \delta_{n-2}) = \tilde{p}_n \cdot (0, \delta_0, \delta_1, ..., \delta_{n-2}) - p\mathbb{V}(\delta) =$$
$$= (p^{1/p^{n-1}}, 0, ..., 0) \cdot (0, \delta_0, \delta_1, ..., \delta_{n-2}) - p\mathbb{V}(\delta) =$$
$$= (0, p^{1/p^{n-2}}\delta_0, p^{1/p^{n-3}}\delta_1, ..., p\delta_{n-2}) - p\mathbb{V}(\delta) \quad (2)$$

Now, since $\mathbb{V}$ is additive, (1) and (2) lead to: $\mathbb{V}(\xi_{n-1}\delta) = \xi_n\mathbb{V}(\delta)$ and one further obtains that:

$$\alpha = \xi_n\beta + \mathbb{V}(\gamma) = \xi_n\beta + \mathbb{V}(\xi_{n-1}\delta) = \xi_n\beta + \xi_n\mathbb{V}(\delta) = \xi_n(\beta + \mathbb{V}(\delta))$$

and so ξ_n generates the kernel of c_n, the claim being proved.

□

Theorem 4.0.10. *There exists a unique continuous sheaf $\mathbb{A}_{\max}$ on $\mathfrak{X}_{\overline{K}}$ of $\mathbb{A}^{\nabla}_{\max}$-algebras such that for every small affine $\mathcal{U} = Spec(R_{\mathcal{U}})$ of X^{et} we have a canonical isomorphism as $A^{\nabla}_{\max}(\overline{R}_{\mathcal{U}})$-algebras: $\mathbb{A}_{\max}(\overline{R}_{\mathcal{U}}) \cong A_{\max}(\overline{R}_{\mathcal{U}})$. Here the algebra $A_{\max}(\overline{R}_{\mathcal{U}})$ is the one defined in [Bri 2, Definition 2.3.3].*

Proof. Let us fix a small affine $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$ and a choice of $\overline{R}_{\mathcal{U}}$. Let us now fix $n \geq 0$ and let us recall that we defined at the beginning of this section a certain category $\mathfrak{U}_{\overline{K},n}$. Fix $T_1, T_2, ..., T_d$ parameters of $R_{\mathcal{U}}$ let us recall that we have chosen for every $1 \leq i \leq d$ a compatible family of p-power roots $(T_i^{1/p^n})_{n=0}^{\infty}$ and also a compatible family of p-power roots on 1, $\varepsilon := (\varepsilon^{(n)})_{n=0}^{\infty}$. With these choices let us recall that we have defined the elements $X_i := 1 \otimes T_i - \widetilde{T}_i \otimes 1 \in \mathbb{W}_{X,n,\overline{K}}(\mathcal{V}, \mathcal{W})$ for any $(\mathcal{V}, \mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$. We define the presheaf $\mathcal{A}_{\mathcal{U},n}$ on $\mathfrak{U}_{\overline{K},n}$ by

$$(\mathcal{V}, \mathcal{W}) \longrightarrow \mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W}) := \mathbb{W}_{X,n,\overline{K}}(\mathcal{V}, \mathcal{W})[Y_0, Y_1, Y_2, ..., Y_d]/(pY_0 - \xi_n, pY_i - X_i)_{1 \leq i \leq d},$$

for $(\mathcal{V}, \mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$. If we denote by $y_1^{(n)}, y_2^{(n)}, ..., y_d^{(n)}$ the images of $Y_1, Y_2, ..., Y_d$ in $\mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})$, let us remark that $\mathbb{A}^{\nabla}_{\max,n}(\mathcal{V}, \mathcal{W}) \subset \mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})$ and moreover we have $\mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W}) = \mathbb{A}^{\nabla}_{\max,n}(\mathcal{V}, \mathcal{W})[y_1^{(n)}, ..., y_d^{(n)}]$. In fact $\mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})$ is a free $\mathbb{A}^{\nabla}_{\max,n}(\mathcal{V}, \mathcal{W})$-module with basis the monomials in $y_1^{(n)}, y_2^{(n)}, ..., y_d^{(n)}$, therefore the presheaf $\mathcal{A}_{\mathcal{U},n}$ is in fact a sheaf on $\mathfrak{U}_{\overline{K},n}$.

Let us first remark that we have a natural morphism of $\mathcal{O}_K$-algebras: $R_0 := \mathcal{O}_K[T_1^{\pm 1}, T_2^{\pm 1}, ..., T_d^{\pm 1}] \longrightarrow \mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})$ given by $T_i \longrightarrow \widetilde{T}_i \otimes 1 + X_i$, for $1 \leq i \leq d$. We remark that as $\widetilde{T}_i$ is a unit in $\mathbb{W}_n(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}})(\mathcal{V}, \mathcal{W})$ and as $X_i = py_i$ in $\mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})$ and therefore nilpotent in that ring, it follows that $\widetilde{T}_i \otimes 1 + X_i \in \mathcal{A}_{\mathcal{U},n}(\mathcal{V}, \mathcal{W})^{\times}$ and so the definition makes sense.

We extend the morphism $\theta_n : \mathbb{A}^{\nabla}_{\max,n}|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow (\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}})|_{\mathfrak{U}_{\overline{K},n}}$ to a morphism $\theta_{\mathcal{U},n} : \mathcal{A}_{\mathcal{U},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}}/p^n\mathcal{O}_{\mathfrak{X}})|_{\mathfrak{U}_{\overline{K},n}}$ by sending $y_i^{(n)}$ to 0, for all $1 \leq i \leq d$.

For each $(\mathcal{V},\mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$ we have a diagram of rings and ring homomorphisms

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}) & \xrightarrow{f_{n,1}} & \mathcal{A}_{\mathcal{U},1}(\mathcal{V},\mathcal{W}) \\ \uparrow & & \uparrow \\ R_0 & \longrightarrow & R_{\mathcal{V}} \end{array}$$

Let us recall that $\mathcal{A}_{\mathcal{U},1}(\mathcal{V},\mathcal{W}) = \mathbb{A}^{\nabla}_{\max,1}(\mathcal{V},\mathcal{W})[y_1^{(1)},...,y_d^{(1)}] = (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})(\mathcal{V},\mathcal{W})[y_1^{(1)},...,y_d^{(1)}]$ and so the morphism $R_{\mathcal{V}} \longrightarrow \mathcal{A}_{\mathcal{U},1}$ in the diagram is the natural one. With this definition the diagram is commutative and moreover $\mathrm{Ker}(f_{n,1})$ is a torsion ideal of $\mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W})$. As $R_{\mathcal{V}}$ is étale over R_0, there is a unique R_0-morphism

$$R_{\mathcal{V}} \longrightarrow \mathcal{A}_{\mathcal{U},n}(\mathcal{V},\mathcal{W}),$$

making the two triangles commute and so we obtain a morphism of sheaves on $\mathfrak{U}_{\overline{K},n}$, $h_{\mathcal{U},n} : \mathbb{W}_{X,n,\overline{K}}|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow \mathcal{A}_{\mathcal{U},n}$.

Now let us denote by $\mathfrak{U}_{\overline{K}}$ the full subcategory of $\mathfrak{X}_{\overline{K}}$ consisting of pairs $(\mathcal{V},\mathcal{W})$ such that the map $\mathcal{V} \longrightarrow X$ factors through $\mathcal{U}$. We endow $\mathfrak{U}_{\overline{K}}$ with the topology induced from $\mathfrak{X}$ and consider $\mathfrak{U}_{\overline{K},n}$ as a sub-topology of it. Our construction proceeds in several steps, as follows:

Step 1 *The sheaf $\mathcal{A}_{\mathcal{U},n}$ on $\mathfrak{U}_{\overline{K},n}$ extends uniquely to a sheaf which we denote $\mathbb{A}_{\max,\mathfrak{U},n}$ on the whole of $\mathfrak{U}_{\overline{K}}$.*

For this let us fix an étale open $\mathcal{V}$ of X^{et} such that the structure map $\mathcal{V} \longrightarrow X$ factors through $\mathcal{U}$ and let $\mathcal{V}^{\mathrm{fet}}$ (respectively $\mathcal{V}^{\mathrm{fet}}_n$) denote the sub-site of $\mathfrak{U}_{\overline{K}}$ consisting of pairs $(\mathcal{V},\mathcal{W})$ (respectively consisting of pairs $(\mathcal{V},\mathcal{W})$ such that the structure map $\mathcal{W} \longrightarrow \mathcal{V}$ factors through $\mathrm{Spf}(R_{\mathcal{V},n})\otimes_{\mathcal{O}_K} K$. We recall that $R_{\mathcal{V},n} = R_{\mathcal{V}}[\zeta_n, T_1^{1/p^n},...,T_d^{1/p^n}]$.)

To prove the claim it would be enough to prove that the restriction of $\mathcal{A}_{\mathcal{U},n}$ to $\mathcal{V}^{\mathrm{fet}}_n$ extends uniquely to $\mathcal{V}^{\mathrm{fet}}$, for all $\mathcal{V}$ as above. Let $\Delta_{\mathcal{V}} := \pi_1^{\mathrm{alg}}(\mathcal{V}_{\overline{K}},\overline{\eta})$, and by Δ_n its open subgroup of elements which fix $R_{\mathcal{V},n}$.

We have the following natural diagram of categories and functors:

$$\begin{array}{ccc} \mathrm{Sh}(\mathcal{V}^{\mathrm{fet}}) & \xrightarrow{\mathrm{Res}} & \mathrm{Sh}(\mathcal{V}_n^{\mathrm{fet}}) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L}_n \\ \mathrm{Rep}(\Delta_{\mathcal{V}}) & \xrightarrow{\mathrm{Res}} & \mathrm{Rep}(\Delta_n) \end{array}$$

where $\mathcal{L}$ and $\mathcal{L}_n$ are the localization functors: if $\mathcal{F}$ is a sheaf on $\mathcal{V}^{\mathrm{fet}}$, respectively on $\mathcal{V}_n^{\mathrm{fet}}$, then $\mathcal{L}(\mathcal{F}) := \mathcal{F}(\overline{R}_{\mathcal{V}})$, respectively $\mathcal{L}_n(\mathcal{F}) := \mathcal{F}(\overline{R}_{\mathcal{V}})$. Therefore we have $\mathcal{L}_n(\mathrm{Res}(\mathcal{F})) \cong \mathrm{Res}(\mathcal{L}(\mathcal{F}))$ and so the diagram is commutative. Both $\mathcal{L}$ and $\mathcal{L}_n$ are equivalences of categories, therefore in order to prove that $\mathcal{A}_{\mathcal{U},n}$ (seen as sheaf on $\mathcal{V}_n^{\mathrm{fet}}$) extends uniquely to a sheaf on $\mathcal{V}^{\mathrm{fet}}$ it is enough to show that the Δ_n-action on $A_{\mathcal{V},n} := \mathcal{L}_n(\mathcal{A}_{\mathcal{U},n})$ extends uniquely to a $\Delta_{\mathcal{V}}$-action.

Let us remark that $\mathbb{A}^{\nabla}_{\mathrm{max,n}}(\overline{R}_{\mathcal{V}})[y_1, ..., y_d] = A^{\nabla}_{\mathrm{max,n}}(\overline{R}_{\mathcal{V}})[y_1, ..., y_d]$, where until the end of this chapter we denoted $y_i := y_i^{(n)}, 1 \le i \le d$. As $A^{\nabla}_{\mathrm{max,n}}(\overline{R}_{\mathcal{V}})$ has a canonical action of $\Delta_{\mathcal{V}}$, we only need to define the action on y_i, $1 \le i \le d$. For this let us denote by $c_i : \Delta_{\mathcal{V}} \longrightarrow \mathbb{Z}_p$ the cocycle defined by: if $\sigma \in \Delta_{\mathcal{V}}$

$$\sigma\big((T_i^{1/p^m})_{m=0}^{\infty}\big) = (T_i^{1/p^m})_{m=0}^{\infty}\varepsilon^{c_i(\sigma)}.$$

Let us remark that after we fixed the choices of p-power roots of T_i and of 1, the cocycles c_i are uniquely determined for every $1 \le i \le d$. Let us denote for every such i and every $\sigma \in \Delta_{\mathcal{V}}$ by $e_i(\sigma) \in A_{\mathrm{max,n}}$ the image under the natural map $A_{\mathrm{max}} \longrightarrow A_{\mathrm{max,n}}$ of the element

$$(1 - [\varepsilon]^{c_i(\sigma)})/p \in A_{\mathrm{max}}.$$

Then, for every $\sigma \in \Delta_{\mathcal{V}}$, we define

$$\sigma(y_i) := y_i + e_i(\sigma)\widetilde{T}_i \otimes 1 \in A_{\mathcal{V},n}.$$

By the definition above, $A_{\mathcal{V},n}$ is now a representation of $\Delta_{\mathcal{V}}$ and so let us denote by $\mathbb{A}_{\max,\mathfrak{U},n}$ the unique sheaf on $\mathfrak{U}_{\overline{K}}$ such that for every $\mathcal{V}$ as above we have natural isomorphisms as $\Delta_{\mathcal{V}}$-representations $\mathbb{A}_{\max,\mathfrak{U},n}(\overline{R}_{\mathcal{V}}) \cong A_{\mathcal{V},n}$. It follows that $\mathbb{A}_{\max,\mathfrak{U},n}|_{\mathfrak{U}_{\overline{K},n}} = \mathcal{A}_{\mathcal{U},n}$.

Step 2 *extension of the morphisms $h_{\mathcal{U},n}$ and $\theta_{\mathcal{U},n}$*

We'd like to show that $h_{\mathcal{U},n} : \mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow \mathcal{A}_{\mathcal{U},n}$ and $\theta_{\mathcal{U},n} : \mathcal{A}_{\mathcal{U},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K},n}}$ extend uniquely to morphisms of sheaves $h_{\mathcal{U}} : \mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}} \longrightarrow \mathbb{A}_{\max,\mathfrak{U},n}$ and respectively $\theta_{\mathcal{U},n} : \mathbb{A}_{\max,\mathfrak{U},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}}$.

a) *The extension of $h_{\mathcal{U},n}$.* As the natural inclusion $\mathbb{W}_n(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p\mathcal{O}_{\mathfrak{X}_{\overline{K}}}) \longrightarrow \mathbb{A}^{\nabla}_{\max,n}$ is in fact defined over all $\mathfrak{X}_{\overline{K}}$, it is enough to show that the natural morphism induced by $h_{\mathcal{U},n}$, $\mathcal{O}_X|_{\mathfrak{U}_{\overline{K},n}} \longrightarrow \mathcal{A}_{\mathcal{U},n}$ extends to the whole of $\mathfrak{U}_{\overline{K}}$. Let us fix $\mathcal{V}$ as above, then it is enough to show that the map induced by $h_{\mathcal{U},n}$, $R_{\mathcal{V}} \longrightarrow A_{\mathcal{V},n}$ is $\Delta_{\mathcal{V}}$-invariant. But this map is completely determined by the map $R_0 \longrightarrow A_{\mathcal{V},n}$. In the end we have to prove that the images of T_i, $1 \le i \le d$, are $\Delta_{\mathcal{V}}$-invariant. Let us recall, $h_{\mathcal{U},n}(T_i) = \widetilde{T}_i \otimes 1 + X_i = \widetilde{T}_i \otimes 1 + py_i$. Therefore,

$$\sigma(h_{\mathcal{U},n}(T_i)) = \sigma(\widetilde{T}_i) \otimes 1 + p\sigma(y_i) = [\varepsilon]^{c_i(\sigma)}\widetilde{T}_i \otimes 1 + p(e_i(\sigma)\widetilde{T}_i \otimes 1 + y_i) =$$

$$= [\varepsilon]^{c_i(\sigma)}\widetilde{T}_i \otimes 1 + (1 - [\varepsilon]^{c_i(\sigma)})\widetilde{T}_i \otimes 1 + X_i = h_{\mathcal{U},n}(T_i).$$

b) *The extension of $\theta_{\mathcal{U},n}$.*

Following the same line of arguments as above, after fixing a small affine $\mathcal{V}$, we need to prove that the map induced by $\theta_{\mathcal{U},n}$, $A_{\mathcal{V},n} \longrightarrow (\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})(\overline{R}_{\mathcal{V}})$ is $\Delta_{\mathcal{V}}$-equivariant. It is then enough to look at the images of $y_i, 1 \le i \le d$. Let us choose such an i and let $\sigma \in \Delta_{\mathcal{V}}$. We have

$$\theta_{\mathcal{U},n}(\sigma(y_i)) = \theta_{\mathcal{U},n}(y_i + e_i(\sigma)\widetilde{T}_i \otimes 1) = \theta_{\mathcal{U},n}(y_i) + \theta_{\mathcal{U},n}(e_i(\sigma))\theta_{\mathcal{U},n}(\widetilde{T}_i \otimes 1) = T_i\theta_n(e_i(\sigma)).$$

Now $e_i(\sigma) \in A_{\max,n}$ and we have $(1 - [\varepsilon]^{c_i(\sigma)})/p = a_i(\sigma)(\xi/p)$ in $A_{\max}$, with $a_i(\sigma) \in A^+_{\inf}$, we have that $e_i(\sigma) = b_i(\sigma)\delta_n$, where $b_i(\sigma) \in W_n$ is the image of $a_i(\sigma)$ and $\delta_n \in A_{\max,n}$

is the image of Y_0. Therefore $\theta_n(e_i(\sigma)) = \theta_n(b_i(\sigma))\theta_n(\delta_n) = 0$ and so $\theta_{\mathcal{U},n}(\sigma(y_i)) = 0 = \sigma(\theta_{\mathcal{U},n}(y_i))$.

Now let us remark that for every $n \geq 0$, we have natural morphisms of sheaves $\mathbb{A}_{\max,\mathfrak{U},n+1} \longrightarrow \mathbb{A}_{\max,\mathfrak{U},n}$ induced by the natural morphism $\mathbb{A}^{\nabla}_{\max,n+1}|_{\mathfrak{U}} \longrightarrow \mathbb{A}^{\nabla}_{\max,n}|_{\mathfrak{U}}$, which make the family $\mathbb{A}_{\max,\mathfrak{U}} := \{\mathbb{A}_{\max,\mathfrak{U},n}\}_{n\geq 0}$ into a projective system of torsion sheaves, i.e. a continuous sheaf. Moreover, the family of maps $\{h_{\mathcal{U},n}\}_{n\geq 0}$ induces a morphism of continuous sheaves $h_{\mathcal{U}} : \mathcal{O}_{\hat{\mathcal{U}}} \longrightarrow \mathbb{A}_{\max,\mathfrak{U}}$ and the family $\{\theta_{\mathcal{U},n}\}_{n\geq 0}$ induces a morphism of continuous sheaves $\theta_{\mathcal{U}} : \mathbb{A}_{\max,\mathfrak{U}} \longrightarrow \hat{\mathcal{O}}_{\mathfrak{U}_{\overline{K}}}$. Here we have denoted by $\mathcal{O}_{\hat{\mathcal{U}}}$ the continuous sheaf $\{\mathcal{O}_{\mathcal{U}}/p^n\mathcal{O}_{\mathcal{U}}\}_{n\geq 0}$ and $\hat{\mathcal{O}}_{\mathfrak{U}_{\overline{K}}}$ is the continuous sheaf $\{(\mathcal{O}_{\mathfrak{X}_{\overline{K}}}/p^n\mathcal{O}_{\mathfrak{X}_{\overline{K}}})|_{\mathfrak{U}_{\overline{K}}}\}_{n\geq 0}$.

Step 3. *Gluing of* $\mathbb{A}_{\max,\mathfrak{U}_{\overline{K}},n}$.

We choose a covering $\{\mathcal{U}_j\}_j$ of X by small affines. For each j, we have defined unique continuous sheaves $\mathbb{A}_{\max,\mathfrak{U}_j}$ on $\mathfrak{U}_{j,\overline{K}}$. By the uniqueness, these sheaves glue to give a unique continuous sheaf $\mathbb{A}_{\max}$ on $\mathfrak{X}_{\overline{K}}$, together with morphisms of sheaves $h : \mathbb{A}^{+}_{\inf} \longrightarrow \mathbb{A}_{\max}$, $\mathbb{A}^{\nabla}_{\max} \longrightarrow \mathbb{A}_{\max}$ and $\theta : \mathbb{A}_{\max} \longrightarrow \hat{\mathcal{O}}_{\mathfrak{X}_{\overline{K}}}$, such for every j, their restrictions to $\mathfrak{U}_{i,\overline{K}}$ are the ones defined above.

□

The continuous sheaf $\mathbb{A}_{\max}$ constructed above have nice properties which we summarize in the following

Theorem 4.0.11. *Let us fix $n \geq 1$.*

1) The sheaf $\mathbb{A}_{\max}$ has a decreasing filtration by sheaves of ideals $\mathrm{Fil}^r\mathbb{A}_{\max} := (Ker(\theta))^n$, for all $r \geq 0$.

2) There is a unique connection $\nabla := \{\nabla_n\}_{n\geq 0} : \mathbb{A}_{\max} \longrightarrow \mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ such that

a) $\nabla|_{\mathbb{A}^{\nabla}_{\max}} = 0$

b) for every $n \geq 0$ and every small affine $\mathcal{U}$ of X with parameters $T_1, T_2, ..., T_d$ and for every pair $(\mathcal{V}, \mathcal{W})$ in $\mathfrak{U}_{\overline{K},n}$, if we denote as before the elements $y_1, y_2, ..., y_d \in$

$\mathbb{A}_{\max,n}(\mathcal{V},\mathcal{W})$, *then* $\nabla_n(y_i) = 1 \otimes dT_i \in \mathbb{A}_{\max,n}(\mathcal{V},\mathcal{W}) \otimes_{R_\mathcal{V}} \Omega^1_{R_\mathcal{V}/\mathcal{O}_K}$.

3) The connection described at 2) has the property that it is integrable and $\mathbb{A}_{\max}^\nabla = (\mathbb{A}_{\max})^\nabla$.

4) We have $\nabla(\mathrm{Fil}^r\mathbb{A}_{\max}) \subset \mathrm{Fil}^{r-1}\mathbb{A}_{\max} \otimes_{\mathcal{O}_{\hat{X}}} \Omega^1_{\hat{X}/\mathcal{O}_K}$ *for every* $r \geq 1$, *i.e.* ∇ *satisfies the Griffith transversality property with respect to the respective filtration.*

Proof. Let us first remark that the properties 2) a) and b) define a unique connection on the restrictions of the sheaf $\mathbb{A}_{\max,n}$ to $\mathfrak{U}_{\overline{K},n}$. We'd like to show that it extends uniquely to a connection on the whole of $\mathfrak{U}_{\overline{K}}$. For this it would be enough to show that if we fix an affine open $\mathcal{V}$ of X^{et} such that the structure map $\mathcal{V} \longrightarrow X$ factors through $\mathcal{U}$, the connection $\nabla_n : A_{\mathcal{U},n} \longrightarrow A_{\mathcal{U},n} \otimes_{R_\mathcal{V}} \Omega^1_{R_\mathcal{V}/\mathcal{O}_K}$ induced by ∇_n is $\Delta_\mathcal{V}$-equivariant. It is enough to check the elements y_i, $1 \leq i \leq d$. Let $\sigma \in \Delta_\mathcal{V}$. Then on the one hand we have $\sigma(\nabla_n(y_i)) = \sigma(1 \otimes dT_i) = 1 \otimes dT_i$. On the other hand $\nabla_n(\sigma(y_i)) = \nabla\big(y_i + e_i(\sigma)\widetilde{T}_i \otimes 1\big) = \nabla(y_i) = 1 \otimes dT_i$, which shows that indeed ∇_n is $\Delta_\mathcal{V}$-equivariant.

Properties 3), 4) are local therefore it is enough to verify them on the restriction $\mathcal{A}_{\mathcal{U},n}$ of $\mathbb{A}_{\max,n}$ to $\mathfrak{U}_{\overline{K},n}$, and in that case $\mathcal{A}_{\mathcal{U},n}$ is a free $\mathbb{A}_{\max}^\nabla|_{\mathfrak{U}_{\overline{K},n}}$-module with basis the monomials in $y_1, y_2, ..., y_d$. Therefore everything follows from the local definition of ∇_n.

□

Chapter 5

Concluding remarks

We suspect that the sheaves $\mathbb{A}_{\max}$ and $\mathbb{A}_{\max}^{\nabla}$ can be defined for the case when K is ramified over $\mathbb{Q}_p$ and we would like to extend our theory from chapters 3 & 4 and to prove "localization over small affines"-equivalent theorems for this general case. Concretely, we expect that the localizations $\mathbb{A}_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})$ and $\mathbb{A}_{\max}(\overline{R}_{\mathcal{U}})$ are respectively isomorphic to the rings $A_{\max}^{\nabla}(\overline{R}_{\mathcal{U}})$ and $A_{\max}(\overline{R}_{\mathcal{U}})$ for a "small" affine $\mathcal{U} = \mathrm{Spec}(R_{\mathcal{U}})$.

Let X be a smooth proper scheme over $\mathcal{O}_K$ with geometrically connected fibers. Let us now introduce the natural functors $u : \mathfrak{X} \to X_{\overline{K}}^{et}$ and $v : X^{et} \to \mathfrak{X}$ defined as follows:

$u(\mathcal{U}, \mathcal{W}) = \mathcal{W}$ and $v(\mathcal{U}) = (\mathcal{U}, \mathcal{U}_{\overline{K}})$ respectively.

One further defines the morphisms

$u_* : \mathrm{Sh}(X_{\overline{K}}^{et}) \to \mathrm{Sh}(\mathfrak{X})$ and $v_* : \mathrm{Sh}(\mathfrak{X}) \to \mathrm{Sh}(X^{et})$ analogous to the push-forward in the following way: $u_*(\mathbb{L})(\mathcal{U}, \mathcal{W}) = \mathbb{L}(\mathcal{W})$ and $v_*(\mathcal{F})(\mathcal{U}) = \mathcal{F}(\mathcal{U}, \mathcal{U}_{\overline{K}})$ respectively, where $\mathbb{L}$ is a sheaf on $X_{\overline{K}}^{et}$ and $\mathcal{F}$ a sheaf on $\mathfrak{X}$.

Denote now by $\mathbb{L}$ a locally constant $\mathbb{Q}_p$-sheaf on X_K^{et} which we view via base change to $X_{\overline{K}}^{et}$ and then applying u_* as a sheaf on $\mathfrak{X}$. We would like to construct a functor named $\mathbb{D}_{\text{max}}^{\text{ar}}$ which makes a (Riemann-Hilbert) correspondence between the category of locally constant sheaves on X_K^{et} and the category of sheaves of $\mathcal{O}_{X_K}$-modules endowed with an integrable connection, a filtration and a Frobenius endomorphism on $\hat{X}$, where by $\hat{X}$ we mean the completion of X along the special fiber X_k. We define this functor by:

$$\mathbb{D}_{\text{max}}^{\text{ar}}(\mathbb{L}) = v_*(\mathbb{L} \otimes \mathbb{A}_{\text{max}})^{G_K}.$$

We then make the following:

Conjecture: $\mathbb{D}_{\text{max}}^{\text{ar}}(\mathbb{L}) \cong \mathbb{D}_{\text{cris}}^{\text{ar}}(\mathbb{L})$ *as sheaves of* $\mathcal{O}_{X_K}$*-modules on* X_K^{et},

where the sheaf $\mathbb{D}_{\text{cris}}^{\text{ar}}(\mathbb{L})$ was defined by F. Andreatta and A. Iovita in [AI1] by setting $\mathbb{D}_{\text{cris}}^{\text{ar}}(\mathbb{L}) = v_*(\mathbb{L} \otimes \mathbb{A}_{\text{cris}})^{G_K}$ and $\mathbb{A}_{\text{cris}}$ is a sheaf on $\mathfrak{X}$ also constructed in [AI1].

We hope that this conjecture is true since the functor D_{cris} defined in the second chapter (see Definition 2.3.8) doesn't loose any information if one replaces the ring B_{max} with B_{cris}. Concretely, if V is a p-adic representation of G_K then

$$(V \otimes_{\mathbb{Q}_p} B_{\text{max}})^{G_K} \cong (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{G_K} = D_{\text{cris}}(V)$$

as filtered modules (see Theorem 2.3.13 (Colmez)).

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