Compact polyhedral surfaces of an arbitrary genus and determinants of Laplacians

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Abstract. Compact polyhedral surfaces (or, equivalently, compact Riemann surfaces with conformal flat conical metrics) of an arbitrary genus are considered. After giving a short self-contained survey of their basic spectral properties, we study the zeta-regularized determinant of the Laplacian as a functional on the moduli space of these surfaces. An explicit formula for this determinant is obtained.

1 Introduction

There are several well-known ways to introduce a compact Riemann surface, e.g., via algebraic equations or by means of some uniformization theorem, where the surface is introduced as the quotient of the upper half-plane over the action of a Fuchsian group. In this paper we consider a less popular approach which is at the same time, perhaps, the most elementary: one can simply consider the boundary of a connected (but, generally, not simply connected) polyhedron in three dimensional Euclidean space. This is a polyhedral surface which carries the structure of a complex manifold (the corresponding system of holomorphic local parameters is obvious for all points except the vertices; near a vertex one should introduce the local parameter $\zeta = z^{2\pi/\alpha}$, where α is the sum of the angles adjacent to the vertex). In this way the Riemann surface arises together with a conformal metric; this metric is flat and has conical singularities at the vertices. Instead of a polyhedron one can also start from some abstract simplicial complex, thinking of a polyhedral surface as glued from plane triangles.

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The present paper is devoted to the spectral theory of the Laplacian on such surfaces. The main goal is to study the determinant of the Laplacian (acting in the trivial line bundle over the surface) as a functional on the space of Riemann surfaces with conformal flat conical metrics (polyhedral surfaces). The similar question for *smooth* conformal metrics and arbitrary holomorphic bundles was very popular in the eighties and early nineties being motivated by string theory. The determinants of Laplacians in flat singular metrics are much less studied: among the very few appropriate references we mention [DP89], where the determinant of the Laplacian in a conical metric was defined via some special regularization of the diverging Liouville integral and the question about the relation of such a definition with the spectrum of the Laplacian remained open, and two papers [K93], [AS94] dealing with flat conical metrics on the Riemann sphere.

In [KK09] (see also [KK04]) the determinant of the Laplacian was studied as a functional

$$\mathcal{H}_g(k_1, \dots, k_M) \ni (\mathcal{X}, \omega) \mapsto \det \Delta^{|\omega|^2}$$

on the space $\mathcal{H}_g(k_1, \dots, k_M)$ of equivalence classes of pairs $(\mathcal{X}, \omega)$, where $\mathcal{X}$ is a compact Riemann surface of genus g and ω is a holomorphic one-form (an Abelian differential) with M zeros of multiplicities $k_1, \dots, k_M$. Here $\det \Delta^{|\omega|^2}$ stands for the determinant of the Laplacian in the flat metric $|\omega|^2$ having conical singularities at the zeros of ω. The flat conical metric $|\omega|^2$ considered in [KK09] is very special: the divisor of the conical points of this metric is not arbitrary (it should be the canonical one, i. e. coincide with the divisor of a holomorphic one-form) and the conical angles at the conical points are integer multiples of 2π. Later in [KK07] this restrictive condition has been eliminated in the case of polyhedral surfaces of genus one.

In the present paper we generalize the results of [KK09] and [KK07] to the case of polyhedral surfaces of an arbitrary genus. Moreover, we give a short and self-contained survey of some basic facts from the spectral theory of the Laplacian on flat surfaces with conical points. In particular, we discuss the theory of self-adjoint extensions of this Laplacian and study the asymptotics of the corresponding heat kernel.

2 Flat conical metrics on surfaces

Following [T86] and [KK07], we discuss here flat conical metrics on compact Riemann surfaces of an arbitrary genus.

2.1 Troyanov's theorem

Let $\sum_{k=1}^N b_k P_k$ be a (generalized, i.e., the coefficients b_k are not necessary integers) divisor on a compact Riemann surface $\mathcal{X}$ of genus g. Let also $\sum_{k=1}^N b_k = 2g-2$. Then, according to Troyanov's theorem (see [T86]), there exists a (unique up to a rescaling) conformal (i. e. giving rise to a complex structure which coincides with that of $\mathcal{X}$) flat metric $\mathbf{m}$ on $\mathcal{X}$ which is smooth in $\mathcal{X} \setminus \{P_1, \dots, P_N\}$ and has simple singularities of order b_k at P_k. The latter means that in a vicinity of P_k the metric $\mathbf{m}$ can be represented in the form

$$\mathbf{m} = e^{u(z,\bar{z})}|z|^{2b_k}|dz|^2, \tag{1}$$

where z is a conformal coordinate and u is a smooth real-valued function. In particular, if $\beta_k > -1$ the point P_k is conical with conical angle $\beta_k = 2\pi(b_k+1)$. Here we construct the metric $\mathbf{m}$ explicitly, giving an effective proof of Troyanov's theorem (cf. [KK07]).

Fix a canonical basis of cycles on $\mathcal{X}$ (we assume that $g \geq 1$, the case $g=0$ is trivial) and let $E(P,Q)$ be the prime-form (see [F73]). Then for any divisor $\mathcal{D} = r_1 Q_1 + \dots r_m Q_M - s_1 R_1 - \dots - s_N R_N$ of degree zero on $\mathcal{X}$ (here the coefficients r_k, s_k are positive integers) the meromorphic differential

$$\omega_{\mathcal{D}} = d_z \log \frac{\prod_{k=1}^M E^{r_k}(z, Q_k)}{\prod_{k=1}^N E^{s_k}(z, R_k)}$$

is holomorphic outside $\mathcal{D}$ and has first order poles at the points of $\mathcal{D}$ with residues r_k at Q_k and $-s_k$ at R_k. Since the prime-form is single-valued along the $\mathbf{a}$-cycles, all $\mathbf{a}$-periods of the differential $\omega_{\mathcal{D}}$ vanish.

Let $\{v_\alpha\}_{\alpha=1}^g$ be the basis of holomorphic normalized differentials and $\mathbb{B}$ the corresponding matrix of $\mathbf{b}$-periods. Then all $\mathbf{a}$- and $\mathbf{b}$-periods of the meromorphic differential

$$\Omega_{\mathcal{D}} = \omega_{\mathcal{D}} - 2\pi i \sum_{\alpha,\beta=1}^g ((\Im \mathbb{B})^{-1})_{\alpha\beta} \Im \left(\int_{s_1 R_1 + \dots s_N R_N}^{r_1 Q_1 + \dots r_M Q_M} v_\beta \right) v_\alpha$$

are purely imaginary (see [F73], p. 4).

Obviously, the differentials $\omega_{\mathcal{D}}$ and $\Omega_{\mathcal{D}}$ have the same structure of poles: their difference is a holomorphic 1-form.

Choose a base-point P_0 on $\mathcal{X}$ and introduce the following quantity

$$\mathcal{F}_{\mathcal{D}}(P) = \exp \int_{P_0}^P \Omega_{\mathcal{D}}.$$

Clearly, $\mathcal{F}_{\mathcal{D}}$ is a meromorphic section of some *unitary* flat line bundle over $\mathcal{X}$, the divisor of this section coincides with $\mathcal{D}$.

Now we are ready to construct the metric $\mathbf{m}$. Choose any holomorphic differential w on $\mathcal{X}$ with, say, only simple zeros $S_1, \dots, S_{2g-2}$. Then one can set $\mathbf{m} = |u|^2$, where

$$u(P) = w(P)\mathcal{F}_{(2g-2)S_0 - S_1 - \dots S_{2g-2}}(P) \prod_{k=1}^{N} [\mathcal{F}_{P_k - S_0}(P)]^{b_k} \tag{2}$$

and S_0 is an arbitrary point.

Notice that in the case $g = 1$ the second factor in (2) is absent and the remaining part is nonsingular at the point S_0.

2.2 Distinguished local parameter

In a vicinity of a conical point the flat metric (1) takes the form

$$\mathbf{m} = |g(z)|^2 |z|^{2b} |dz|^2$$

with some holomorphic function g such that $g(0) \neq 0$. It is easy to show (see, e. g., [T86], Proposition 2) that there exists a holomorphic change of variable $z = z(x)$ such that in the local parameter x

$$\mathbf{m} = |x|^{2b} |dx|^2 \,.$$

We shall call the parameter x (unique up to a constant factor c, $|c| = 1$) *distinguished*. In case $b > -1$ the existence of the distinguished parameter means that in a vicinity of a conical point the surface $\mathcal{X}$ is isometric to the standard cone with conical angle $\beta = 2\pi(b + 1)$.

2.3 Euclidean polyhedral surfaces.

In [T86] it is proved that any compact Riemann surface with flat conformal conical metric admits a proper triangulation (i. e. each conical point is a vertex of some triangle of the triangulation). This means that any compact Riemann surface with a flat conical metric is a *Euclidean polyhedral surface* (see [B07]) i. e. can be glued from Euclidean triangles. On the other hand as it is explained in [B07] any compact Euclidean oriented polyhedral surface gives rise to a Riemann surface with a flat conical metric. Therefore, from now on we do not discern compact Euclidean polyhedral surfaces and Riemann surfaces with flat conical metrics.

3 Laplacians on polyhedral surfaces. Basic facts

Without claiming originality we give here a short self-contained survey of some basic facts from the spectral theory of Laplacian on compact polyhedral surfaces. We start with recalling the (slightly modified) Carslaw construction (1909) of the heat kernel on a cone, then we describe the set of self-adjoint extensions of a conical Laplacian (these results are complementary to Kondratjev's study ([K67]) of elliptic equations on conical manifolds and are well-known, being in the folklore since the sixties of the last century; their generalization to the case of Laplacians acting on p-forms can be found in [M99]). Finally, we establish the precise heat asymptotics for the Friedrichs extension of the Laplacian on a compact polyhedral surface. It should be noted that more general results on the heat asymptotics for Laplacians acting on p-forms on piecewise flat pseudomanifolds can be found in [C83].

3.1 The heat kernel on the infinite cone

We start from the standard heat kernel

$$H_{2\pi}(x,y;t) = \frac{1}{4\pi t}\exp\{-(x-y)\cdot(x-y)/4t\} \tag{3}$$

in the space $\mathbb{R}^2$ which we consider as the cone with conical angle 2π. Introducing the polar coordinates (r,θ) and (ρ,ψ) in the x and y-planes, one can rewrite (3) as the contour integral

$$H_{2\pi}(x,y;t) =$$

$$\frac{1}{16\pi^2 it}\exp\{-(r^2+\rho^2)/4t\}\int_{C_{\theta,\psi}}\exp\{r\rho\cos(\alpha-\theta)/2t\}\cot\frac{\alpha-\psi}{2}\,d\alpha, \tag{4}$$

where $C_{\theta,\psi}$ denotes the union of a small positively oriented circle centered at $\alpha=\psi$ and the two vertical lines, $l_1=(\theta-\pi-i\infty, \theta-\pi+i\infty)$ and $l_2=(\theta+\pi+i\infty,\theta+\pi-i\infty)$, having mutually opposite orientations.

To prove (4) one has to notice that

1) $\Re\cos(\alpha-\theta)<0$ in vicinities of the lines l_1 and l_2 and, therefore, the integrals over these lines converge.

2) The integrals over the lines cancel due to the 2π-periodicity of the integrand and the remaining integral over the circle coincides with (3) due to the Cauchy Theorem.

Observe that one can deform the contour $C_{\theta,\psi}$ into the union, A_θ, of two contours lying in the open domains $\{\theta-\pi<\Re\alpha<\theta+\pi\,,\,\Im\alpha>0\}$ and

$\{\theta - \pi < \Re\alpha < \theta + \pi\,, \Im\alpha < 0\}$ respectively, the first contour goes from $\theta+\pi+i\infty$ to $\theta-\pi+i\infty$, the second one goes from $\theta-\pi-i\infty$ to $\theta+\pi-i\infty$. This leads to the following representation for the heat kernel $H_{2\pi}$:

$$H_{2\pi}(x, y; t) =$$
$$\frac{1}{16\pi^2 it}\exp\{-(r^2+\rho^2)/4t\}\int_{A_\theta}\exp\{r\rho\cos(\alpha-\theta)/2t\}\cot\frac{\alpha-\psi}{2}\,d\alpha. \quad (5)$$

The latter representation admits a natural generalization to the case of the cone C_β with conical angle β, $0 < \beta < +\infty$. Notice here that in case $0 < \beta \le 2\pi$ the cone C_β is isometric to the surface $z_3 = \sqrt{(\frac{4\pi^2}{\beta^2} - 1)(z_1^2 + z_2^2)}$.

Namely, introducing the polar coordinates on C_β, we see that the following expression represents the heat kernel on C_β:

$$H_\beta(r, \theta, \rho, \psi; t) =$$
$$\frac{1}{8\pi\beta it}\exp\{-(r^2+\rho^2)/4t\}\int_{A_\theta}\exp\{r\rho\cos(\alpha-\theta)/2t\}\cot\frac{\pi(\alpha-\psi)}{\beta}\,d\alpha\,. \quad (6)$$

Clearly, expression (6) is symmetric with respect to (r, θ) and (ρ, ψ) and is β-periodic with respect to the angle variables θ, ψ. Moreover, it satisfies the heat equation on C_β. Therefore, to verify that H_β is in fact the heat kernel on C_β it remains to show that $H_\beta(\cdot, y, t) \longrightarrow \delta(\cdot - y)$ as $t \to 0+$. To this end deform the contour A_ψ into the union of the lines l_1 and l_2 and (possibly many) small circles centered at the poles of $\cot\frac{\pi(\cdot-\psi)}{\beta}$ in the strip $\theta - \pi < \Re\alpha < \theta + \pi$. The integrals over all the components of this union except the circle centered at $\alpha = \psi$ vanish in the limit as $t \to 0+$, whereas the integral over the latter circle coincides with $H_{2\pi}$.

3.1.1 The heat asymptotics near the vertex

Proposition 1 *Let $R > 0$ and $C_\beta(R) = \{x \in C_\beta : \mathrm{dist}(x, \mathcal{O}) < R\}$. Let also dx denote the area element on C_β. Then for some $\epsilon > 0$*

$$\int_{C_\beta(R)} H_\beta(x, x; t)\,dx = \frac{1}{4\pi t}\mathrm{Area}(C_\beta(R)) + \frac{1}{12}\left(\frac{2\pi}{\beta} - \frac{\beta}{2\pi}\right) + O(e^{-\epsilon/t}) \quad (7)$$

as $t \to 0+$.

Proof (cf. [F94], p. 1433). Make in (6) the change of variable $\gamma = \alpha - \psi$ and deform the contour $A_{\theta-\psi}$ into the contour $\Gamma^-_{\theta-\psi} \cup \Gamma^+_{\theta-\psi} \cup \{|\gamma| = \delta\}$, where the oriented curve $\Gamma^-_{\theta-\psi}$ goes from $\theta-\psi-\pi-i\infty$ to $\theta-\psi-\pi+i\infty$ and intersects the real axis at $\gamma=-\delta$, the oriented curve $\Gamma^+_{\theta-\psi}$ goes from $\theta-\psi+\pi+i\infty$ to $\theta-\psi+\pi-i\infty$ and intersects the real axis at $\gamma=\delta$, the circle $\{|\gamma|=\delta\}$ is positively oriented and δ is a small positive number. Calculating the integral over the circle $\{|\gamma|=\delta\}$ via the Cauchy Theorem, we get

$$H_\beta(x,y;t)-H_{2\pi}(x,y;t)=$$
$$\frac{1}{8\pi\beta it}\exp\{-(r^2+\rho^2)/4t\}\int_{\Gamma^-_{\theta-\psi}\cup\Gamma^+_{\theta-\psi}}\exp\{r\rho\cos(\gamma+\psi-\theta)/2t\}\cot\frac{\pi\gamma}{\beta}\,d\gamma \tag{8}$$

and

$$\int_{C_\beta(R)}\left(H_\beta(x,x;t)-\frac{1}{4\pi t}\right)dx=$$
$$\frac{1}{8\pi it}\int_0^R dr\,r\int_{\Gamma_0^-\cup\Gamma_0^+}\exp\{-\frac{r^2\sin^2(\gamma/2)}{t}\}\cot\frac{\pi\gamma}{\beta}\,d\gamma\,. \tag{9}$$

The integration over r can be done explicitly and the right hand side of (9) reduces to

$$\frac{1}{16\pi i}\int_{\Gamma_0^-\cup\Gamma_0^+}\frac{\cot(\frac{\pi\gamma}{\beta})}{\sin^2(\gamma/2)}\,d\gamma+O(e^{-\epsilon/t}). \tag{10}$$

(One can assume that $\Re\sin^2(\gamma/2)$ is positive and separated from zero when $\gamma\in\Gamma_0^-\cup\Gamma_0^+$.) The contour of integration in (10) can be changed for a negatively oriented circle centered at $\gamma=0$. Since $\mathrm{Res}(\frac{\cot(\frac{\pi\gamma}{\beta})}{\sin^2(\gamma/2)},\gamma=0)=\frac{2}{3}(\frac{\beta}{2\pi}-\frac{2\pi}{\beta})$, we arrive at (7).

Remark 1 The Laplacian Δ corresponding to the flat conical metric $(d\rho)^2+r^2(d\theta)^2, 0\le\theta\le\beta$ on C_β with domain $C_0^\infty(C_\beta\setminus\mathcal{O})$ has infinitely many self-adjoint extensions. Analyzing the asymptotics of (6) near the vertex $\mathcal{O}$, one can show that for any $y\in C_\beta, t>0$ the function $H_\beta(\cdot,y;t)$ belongs to the domain of the Friedrichs extension Δ_F of Δ and does not belong to the domain of any other extension. Moreover, using a Hankel transform, it is possible to get an explicit spectral representation of Δ_F (this operator has an absolutely continuous spectrum of infinite multiplicity) and to show that the Schwartz kernel of the operator $e^{t\Delta_F}$ coincides with $H_\beta(\cdot,\cdot;t)$ (see, e. g., [T97] formula (8.8.30) together with [C10], p. 370.)

3.2 Heat asymptotics for compact polyhedral surfaces

3.2.1 Self-adjoint extensions of a conical Laplacian

Let $\mathcal{X}$ be a compact polyhedral surface with vertices (conical points) $P_1, \dots, P_N$. The Laplacian Δ corresponding to the natural flat conical metric on $\mathcal{X}$ with domain $C_0^\infty(\mathcal{X} \setminus \{P_1, \dots, P_N\})$ (we remind the reader that the Riemannian manifold $\mathcal{X}$ is smooth everywhere except the vertices) is not essentially self-adjoint and one has to fix one of its self-adjoint extensions. We are to discuss now the choice of a self-adjoint extension.

This choice is defined by the prescription of some particular asymptotical behavior near the conical points to functions from the domain of the Laplacian; it is sufficient to consider a surface with only one conical point P of the conical angle β. More precisely, assume that $\mathcal{X}$ is smooth everywhere except the point P and that some vicinity of P is isometric to a vicinity of the vertex $\mathcal{O}$ of the standard cone C_β (of course, now the metric on $\mathcal{X}$ no more can be flat everywhere in $\mathcal{X} \setminus P$ unless the genus g of $\mathcal{X}$ is greater than one and $\beta = 2\pi(2g-1)$).

For $k \in \mathbb{N}_0$ introduce the functions $V_\pm^k$ on C_β by

$$V_\pm^k(r, \theta) = r^{\pm\frac{2\pi k}{\beta}} \exp\{i\frac{2\pi k\theta}{\beta}\}; \quad k > 0\,,$$

$$V_+^0 = 1, \quad V_-^0 = \log r\,.$$

Clearly, these functions are formal solutions to the homogeneous problem $\Delta u = 0$ on C_β. Notice that the functions V_-^k grow near the vertex but are still square integrable in its vicinity if $k < \frac{\beta}{2\pi}$.

Let $\mathcal{D}_{\min}$ denote the graph closure of $C_0^\infty(\mathcal{X} \setminus P)$, i.e.,

$$U \in \mathcal{D}_{\min} \Leftrightarrow \exists u_m \in C_0^\infty(\mathcal{X} \backslash P), W \in L_2(\mathcal{X}) : u_m \to U \text{ and } \Delta u_m \to W \text{ in } L_2(\mathcal{X}).$$

Define the space $H_\delta^2(C_\beta)$ as the closure of $C_0^\infty(C_\beta \setminus \mathcal{O})$ with respect to the norm

$$||u; H_\delta^2(C_\beta)||^2 = \sum_{|\alpha| \le 2} \int_{C_\beta} r^{2(\delta - 2 + |\alpha|)} |D_x^\alpha u(x)|^2 dx.$$

Then for any $\delta \in \mathbb{R}$ such that $\delta - 1 \ne \frac{2\pi k}{\beta}, k \in \mathbb{Z}$ one has the a priori estimate

$$||u; H_\delta^2(C_\beta)|| \le c||\Delta u; H_\delta^0(C_\beta)|| \tag{11}$$

for any $u \in C_0^\infty(C_\beta \setminus \mathcal{O})$ and some constant c being independent of u (see, e.g., [NP92], Chapter 2).

It follows from Sobolev's imbedding theorem that for functions from $u \in H^2_\delta(C_\beta)$ one has the point-wise estimate

$$r^{\delta-1}|u(r,\theta)| \leq c||u; H^2_\delta(C_\beta)||. \tag{12}$$

Applying estimates (11) and (12) with $\delta = 0$, we see that functions u from $\mathcal{D}_{\min}$ must obey the asymptotics $u(r,\theta) = O(r)$ as $r \to 0$.

Now the description of the set of all self-adjoint extensions of Δ looks as follows. Let χ be a smooth function on $\mathcal{X}$ which is equal to 1 near the vertex P and such that in a vicinity of the support of χ $\mathcal{X}$ is isometric to C_β. Denote by $\mathfrak{M}$ the linear subspace of $L_2(\mathcal{X})$ spanned by the functions $\chi V^k_\pm$ with $0 \leq k < \frac{\beta}{2\pi}$. The dimension, $2d$, of $\mathfrak{M}$ is even. To get a self-adjoint extension of Δ one chooses a subspace $\mathfrak{N}$ of $\mathfrak{M}$ of dimension d such that

$$(\Delta u, v)_{L_2(\mathcal{X})} - (u, \Delta v)_{L_2(\mathcal{X})} = \lim_{\epsilon \to 0+} \oint_{r=\epsilon} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r} \right) = 0$$

for any $u, v \in \mathfrak{N}$. To any such subspace $\mathfrak{N}$ there corresponds a self-adjoint extension $\Delta_{\mathfrak{N}}$ of Δ with domain $\mathfrak{N} + \mathcal{D}_{\min}$.

The extension corresponding to the subspace $\mathfrak{N}$ spanned by the functions χV^k_+, $0 \leq k < \frac{\beta}{2\pi}$ coincides with the Friedrichs extension of Δ. The functions from the domain of the Friedrichs extension are bounded near the vertex.

From now on we denote by Δ the Friedrichs extension of the Laplacian on the polyhedral surface $\mathcal{X}$; other extensions will not be considered here.

3.2.2 Heat asymptotics

Theorem 1 *Let $\mathcal{X}$ be a compact polyhedral surface with vertices $P_1, \dots, P_N$ of conical angles $\beta_1, \dots, \beta_N$. Let Δ be the Friedrichs extension of the Laplacian defined on functions from $C_0^\infty(\mathcal{X} \setminus \{P_1, \dots, P_N\})$. Then*

1. *The spectrum of the operator Δ is discrete, all the eigenvalues of Δ have finite multiplicity.*

2. *Let $\mathcal{H}(x, y; t)$ be the heat kernel for Δ. Then for some $\epsilon > 0$*

$$\operatorname{Tr} e^{t\Delta} = \int_{\mathcal{X}} \mathcal{H}(x, x; t)\, dx = \frac{\text{Area}(\mathcal{X})}{4\pi t} + \frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right\} + O(e^{-\epsilon/t}), \tag{13}$$

   as $t \to 0+$.

3. *The counting function, $N(\lambda)$, of the spectrum of Δ obeys the asymptotics $N(\lambda) = O(\lambda)$ as $\lambda \to +\infty$.*

Proof. 1) The proof of the first statement is a standard exercise (cf. [K93]). We indicate only the main idea leaving the details to the reader. Introduce the closure, $H^1(\mathcal{X})$, of $C_0^\infty(\mathcal{X}\setminus\{P_1,\dots,P_N\}$ with respect to the norm $|||u|||=||u;L_2||+||\nabla u;L_2||$. It is sufficient to prove that any bounded set S in $H^1(\mathcal{X})$ is precompact in the L_2-topology (this will imply the compactness of the self-adjoint operator $(I-\Delta)^{-1}$). Moreover, one can assume that the supports of functions from S belong to a small ball B centered at a conical point P. Now to prove the precompactness of S it is sufficient to make use of the expansion with respect to eigenfunctions of the Dirichlet problem in B and the diagonal process.

2)Let $\mathcal{X}=\cup_{j=0}^N K_j$, where $K_j,\ j=1,\dots,N$ is a neighborhood of the conical point P_j which is isometric to $C_{\beta_j}(R)$ with some $R>0$, and $K_0=\mathcal{X}\setminus\cup_{j=1}^N K_j$.

Let also $K_j^{\epsilon_1}\supset K_j$ and $K_j^{\epsilon_1}$ be isometric to $C_{\beta_j}(R+\epsilon_1)$ with some $\epsilon_1>0$ and $j=1,\dots,N$.

Fixing $t>0$ and $x,y\in K_j$ with $j>0$, one has

$$\int_0^t ds\int_{K_j^{\epsilon_1}}(\psi\{\Delta_z-\partial_s\}\phi-\phi\{\Delta_z+\partial_s\}\psi)\,dz= \tag{14}$$

$$\int_0^t ds\int_{\partial K_j^{\epsilon_1}}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dl(z)-\int_{K_j^{\epsilon_1}}(\phi(z,t)\psi(z,t)-\phi(z,0)\psi(z,0))\,dz \tag{15}$$

with $\phi(z,t)=\mathcal{H}(z,y;t)-H_{\beta_j}(z,y;t)$ and $\psi(z,t)=H_{\beta_j}(z,x;t-s)$. (Here it is important that we are working with the heat kernel of the Friedrichs extension of the Laplacian, for other extensions the heat kernel has growing terms in the asymptotics near the vertex and the right hand side of (14) gets extra terms.) Therefore,

$$H_{\beta_j}(x,y;t)-\mathcal{H}(x,y;t)=$$

$$\int_0^t ds\int_{\partial K_j^{\epsilon_1}}\left(\mathcal{H}(y,z;s)\frac{\partial H_{\beta_j}(x,z;t-s)}{\partial n(z)}-H_{\beta_j}(z,x;t-s)\frac{\partial\mathcal{H}(z,y;s)}{\partial n(z)}\right)dl(z)$$

$$=O(e^{-\epsilon_2/t})$$

with some $\epsilon_2>0$ as $t\to0+$ uniformly with respect to $x,y\in K_j$. This implies that

$$\int_{K_j}\mathcal{H}(x,x;t)dx=\int_{K_j}H_{\beta_j}(x,x;t)dx+O(e^{-\epsilon_2/t}). \tag{16}$$

Since the metric on $\mathcal{X}$ is flat in a vicinity of K_0, one has the asymptotics

$$\int_{K_0} \mathcal{H}(x,x;t)dx = \frac{\text{Area}(K_0)}{4\pi t} + O(e^{-\epsilon_3/t})$$

with some $\epsilon_3 > 0$ (cf. [MS67]). Now (13) follows from (7).

3) The third statement of the theorem follows from the second one due to the standard Tauberian arguments.

4 Determinant of the Laplacian: Analytic surgery and Polyakov-type formulas

Theorem 1 opens a way to define the determinant, $\det\Delta$, of the Laplacian on a compact polyhedral surface via the standard Ray-Singer regularization. Namely introduce the operator ζ-function

$$\zeta_\Delta(s) = \sum_{\lambda_k>0} \frac{1}{\lambda_k^s}, \tag{17}$$

where the summation goes over all strictly positive eigenvalues λ_k of the operator $-\Delta$ (counting multiplicities). Due to the third statement of Theorem 1, the function ζ_Δ is holomorphic in the half-plane $\{\Re s > 1\}$. Moreover, due to the equality

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left\{\text{Tr}\, e^{t\Delta} - 1\right\} t^{s-1}\, dt \tag{18}$$

and the asymptotics (13), one has the equality

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \left\{ \frac{\text{Area}\,(\mathcal{X})}{4\pi(s-1)} + \left[\frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right\} - 1 \right] \frac{1}{s} + e(s) \right\}, \tag{19}$$

where $e(s)$ is an entire function. Thus, ζ_Δ is regular at $s = 0$ and one can define the ζ-regularized determinant of the Laplacian via usual ζ-regularization (cf. [R73]):

$$\det\Delta := \exp\{-\zeta'_\Delta(0)\}\,. \tag{20}$$

Moreover, (19) and the relation $\sum_{k=1}^N b_k = 2g-2$; $b_k = \frac{\beta_k}{2\pi} - 1$ yield

$$\zeta_\Delta(0) = \frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right\} - 1 = \left(\frac{\chi(\mathcal{X})}{6} - 1 \right) + \frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} + \frac{\beta_k}{2\pi} - 2 \right\}, \tag{21}$$

where $\chi(\mathcal{X}) = 2 - 2g$ is the Euler characteristics of $\mathcal{X}$.

It should be noted that the term $\frac{\chi(\mathcal{X})}{6} - 1$ at the right hand side of (21) coincides with the value at zero of the operator ζ-function of the Laplacian corresponding to an arbitrary *smooth* metric on $\mathcal{X}$ (see, e. g., [S88], p. 155).

Let $\mathbf{m}$ and $\tilde{\mathbf{m}} = \kappa\mathbf{m}$, $\kappa > 0$ be two homothetic flat metrics with the same conical points with conical angles $\beta_1, \dots, \beta_N$. Then (17), (20) and (21) imply the following *rescaling property* of the conical Laplacian:

$$\det\Delta^{\tilde{\mathbf{m}}} = \kappa^{-\left(\frac{\chi(\mathcal{X})}{6}-1\right)-\frac{1}{12}\sum_{k=1}^{N}\left\{\frac{2\pi}{\beta_k}+\frac{\beta_k}{2\pi}-2\right\}}\det\Delta^{\mathbf{m}} \tag{22}$$

4.1 Analytic surgery

Let $\mathbf{m}$ be an arbitrary smooth metric on $\mathcal{X}$ and denote by $\Delta^{\mathbf{m}}$ the corresponding Laplacian. Consider N nonoverlapping connected and simply connected domains $D_1, \dots, D_N \subset \mathcal{X}$ bounded by closed curves $\gamma_1, \dots, \gamma_N$ and introduce also the domain $\Sigma = \mathcal{X} \setminus \cup_{k=1}^N D_k$ and the contour $\Gamma = \cup_{k=1}^N \gamma_k$.

Define *the Neumann jump operator* $R : C^\infty(\Gamma) \to C^\infty(\Gamma)$ by

$$R(f)|_{\gamma_k} = \partial_\nu(V_k^- - V_k^+),$$

where ν is the outward normal to $\gamma_k = \partial D_k$, the functions V_k^- and V^+ are the solutions of the boundary value problems $\Delta^{\mathbf{m}} V_k^- = 0$ in D_k, $V^-|_{\partial D_k} = f$ and $\Delta^{\mathbf{m}} V^+ = 0$ in Σ, $V^+|_\Gamma = f$. The Neumann jump operator is an elliptic pseudodifferential operator of order 1, and it is known that one can define its determinant via the standard ζ-regularization.

In what follows it is crucial that the Neumann jump operator does not change if we vary the metric within the same conformal class.

Let $(\Delta^{\mathbf{m}}|D_k)$ and $(\Delta^{\mathbf{m}}|\Sigma)$ be the operators of the Dirichlet boundary problem for $\Delta^{\mathbf{m}}$ in domains D_k and Σ respectively, the determinants of these operators also can be defined via ζ-regularization.

Due to Theorem B^* from [BFK92], we have

$$\det\Delta^{\mathbf{m}} = \left\{\prod_{k=1}^N \det(\Delta^{\mathbf{m}}|D_k)\right\} \det(\Delta^{\mathbf{m}}|\Sigma)\,\det R\,\{\mathrm{Area}(\mathcal{X}, \mathbf{m})\}\,\{l(\Gamma)\}^{-1}, \tag{23}$$

where $l(\Gamma)$ is the length of the contour Γ in the metric $\mathbf{m}$

Remark 2 We have excluded the zero modes of an operator from the definition of its determinant, so we are using the same notation $\det A$ for the determinants of operators A with and without zero modes. In [BFK92] the determinant of an operator A with zero modes is always equal to zero, and what we call here $\det A$ is called the modified determinant in [BFK92] and denoted there by $\det^* A$. .

An analogous statement holds for the flat conical metric. Namely let $\mathcal{X}$ be a compact polyhedral surface with vertices $P_1, \dots, P_N$ and g be a corresponding flat metric with conical singularities. Choose the domains D_k, $k = 1, \dots, N$ being (open) nonoverlapping disks centered at P_k and let $(\Delta|D_k)$ be the Friedrichs extension of the Laplacian with domain $C_0^\infty(D_k \setminus P_k)$ in $L_2(D_k)$. Then formula (23) is still valid with $\Delta^{\mathbf{m}} = \Delta$ (cf. [KK04] or the recent paper [LMP07] for a more general result).

4.2 Polyakov's formula

We state this result in the form given in ([F92], p. 62). Let $\mathbf{m}_1 = \rho_1^{-2}(z, \bar{z})\widehat{dz}$ and $\mathbf{m}_2 = \rho_2^{-2}(z, \bar{z})\widehat{dz}$ be two *smooth* conformal metrics on $\mathcal{X}$ and let $\det\Delta^{\mathbf{m}_1}$ and $\det\Delta^{\mathbf{m}_2}$ be the determinants of the corresponding Laplacians (defined via the standard Ray-Singer regularization). Then

$$\frac{\det\Delta^{\mathbf{m}_2}}{\det\Delta^{\mathbf{m}_1}} = \frac{\mathrm{Area}(\mathcal{X}, \mathbf{m}_2)}{\mathrm{Area}(\mathcal{X}, \mathbf{m}_1)} \exp\left\{ \frac{1}{3\pi} \int_{\mathcal{X}} \log\frac{\rho_2}{\rho_1} \partial^2_{z\bar{z}} \log(\rho_2\rho_1)\widehat{dz} \right\}. \tag{24}$$

4.3 Analog of Polyakov's formula for a pair of flat conical metrics

Proposition 2 *Let $a_1, \dots, a_N$ and $b_1, \dots, b_M$ be real numbers which are greater than -1 and satisfy $a_1 + \dots + a_N = b_1 + \dots + b_M = 2g - 2$. Let also T be a connected C^1-manifold and let*

$$T \ni t \mapsto \mathbf{m}_1(t), \quad T \ni t \mapsto \mathbf{m}_2(t)$$

be two C^1-families of flat conical metrics on $\mathcal{X}$ such that

1. *For any $t \in T$ the metrics $\mathbf{m}_1(t)$ and $\mathbf{m}_2(t)$ define the same conformal structure on $\mathcal{X}$,*

2. *$\mathbf{m}_1(t)$ has conical singularities at $P_1(t), \dots, P_N(t) \in \mathcal{X}$ with conical angles $2\pi(a_1 + 1)$, $\dots$, $2\pi(a_N + 1)$.*

3. *$\mathbf{m}_2(t)$ has conical singularities at $Q_1(t), \dots, Q_M(t) \in L$ with conical angles $2\pi(b_1 + 1)$, $\dots$, $2\pi(b_M + 1)$,*

4. *For any $t \in T$ the sets $\{P_1(t), \dots, P_N(t)\}$ and $\{Q_1(t), \dots, Q_M(t)\}$ do not intersect.*

Let x_k be distinguished local parameter for $\mathbf{m}_1$ near P_k and y_l be distinguished local parameter for $\mathbf{m}_2$ near Q_l (we omit the argument t).

Introduce the functions f_k, g_l and the complex numbers $\mathbf{f_k}$, $\mathbf{g_l}$ by

$$\mathbf{m}_2 = |f_k(x_k)|^2|dx_k|^2 \quad near \quad P_k; \qquad \mathbf{f_k} := f_k(0),$$

$$\mathbf{m}_1 = |g_l(y_l)|^2|dy_l|^2 \quad near \quad Q_l; \qquad \mathbf{g_l} := g_l(0).$$

Then the following equality holds

$$\frac{\det\Delta^{\mathbf{m}_1}}{\det\Delta^{\mathbf{m}_2}} = C\ \frac{\mathrm{Area}\,(\mathcal{X}, \mathbf{m}_1)}{\mathrm{Area}\,(\mathcal{X}, \mathbf{m}_2)}\ \frac{\prod_{l=1}^M |\mathbf{g_l}|^{b_l/6}}{\prod_{k=1}^N |\mathbf{f_k}|^{a_k/6}}, \tag{25}$$

where the constant C is independent of $t \in T$.

Proof. Take $\epsilon > 0$ and introduce the disks $D_k(\epsilon)$, $k = 1, \dots, M+N$ centered at the points $P_1, \dots, P_N$, $Q_1, \dots, Q_M$; $D_k(\epsilon) = \{|x_k| \le \epsilon\}$ for $k = 1, \dots, N$ and $D_{N+l} = \{|y_l| \le \epsilon\}$ for $l = 1, \dots, M$. Let $h_k : \overline{\mathbb{R}}_+ \to \mathbb{R}$, $k = 1, \dots, N+M$ be smooth positive functions such that

1.
$$\int_0^1 h_k^2(r) r dr = \begin{cases} \int_0^1 r^{2a_k+1} dr = \frac{1}{2a_k+2}, & \text{if } \ k = 1, \dots, N \\ \int_0^1 r^{2b_l+1} dr = \frac{1}{2b_l+2}, & \text{if } \ k = N+l, \ l = 1, \dots, M \end{cases}$$

2.
$$h_k(r) = \begin{cases} r^{a_k} \ \text{ for } \ r \ge 1 \ \text{ if } \ k = 1, \dots, N \\ r^{b_l} \ \text{ for } \ r \ge 1 \ \text{ if } \ k = N+l, \ l = 1, \dots, M \end{cases}$$

Define two families of *smooth* metrics $\mathbf{m}_1^\epsilon$, $\mathbf{m}_2^\epsilon$ on $\mathcal{X}$ via

$$\mathbf{m}_1^\epsilon(z) = \begin{cases} \epsilon^{2a_k} h_k^2(|x_k|/\epsilon)|dx_k|^2, & z \in D_k(\epsilon), \quad k = 1, \dots, N \\ \mathbf{m}(z), & z \in \mathcal{X} \setminus \cup_{k=1}^N D_k(\epsilon)\,, \end{cases}$$

$$\mathbf{m}_2^\epsilon(z) = \begin{cases} \epsilon^{2b_l} h_{N+l}^2(|y_l|/\epsilon)|dy_l|^2, & z \in D_{N+l}(\epsilon), \quad l = 1, \dots, M \\ \mathbf{m}(z), & z \in \mathcal{X} \setminus \cup_{l=1}^M D_{N+l}(\epsilon)\,. \end{cases}$$

The metrics $\mathbf{m}_{1,2}^\epsilon$ converge to $\mathbf{m}_{1,2}$ as $\epsilon \to 0$ and

$$\mathrm{Area}(\mathcal{X}, \mathbf{m}_{1,2}^\epsilon) = \mathrm{Area}(\mathcal{X}, \mathbf{m}_{1,2}).$$

Lemma 1 *Let ∂_t be the differentiation with respect to one of the coordinates on T and let* $\det\Delta^{\mathbf{m}^\epsilon_{1,2}}$ *be the standard ζ-regularized determinant of the Laplacian corresponding to the smooth metric $\mathbf{m}^\epsilon_{1,2}$. Then*

$$\partial_t \log \det\Delta^{\mathbf{m}_{1,2}} = \partial_t \log \det\Delta^{\mathbf{m}^\epsilon_{1,2}}. \tag{26}$$

To establish the lemma consider for definiteness the pair $\mathbf{m}_1$ and $\mathbf{m}_1(\epsilon)$. Due to the analytic surgery formulas from section 4.1 one has

$$\det\Delta^{\mathbf{m}_1} = \left\{\prod_{k=1}^N \det(\Delta^{\mathbf{m}_1}|D_k(\epsilon))\right\} \det(\Delta^{\mathbf{m}_1}|\Sigma)\, \det R\,\{\text{Area}(\mathcal{X}, \mathbf{m}_1)\}\,\{l(\Gamma)\}^{-1}, \tag{27}$$

$$\det\Delta^{\mathbf{m}^\epsilon_1} = \left\{\prod_{k=1}^N \det(\Delta^{\mathbf{m}^\epsilon_1}|D_k(\epsilon))\right\} \det(\Delta^{\mathbf{m}^\epsilon_1}|\Sigma)\, \det R\,\{\text{Area}(\mathcal{X}, \mathbf{m}^\epsilon_1)\}\,\{l(\Gamma)\}^{-1}, \tag{28}$$

with $\Sigma = \mathcal{X} \setminus \cup_{k=1}^N D_k(\epsilon)$.

Notice that the variations of the logarithms of the first factors in the right hand sides of (27) and (28) vanish (these factors are independent of t) whereas the variations of the logarithms of all the remaining factors coincide. This leads to (26).

By virtue of Lemma 1 one has the relation

$$\partial_t \left\{\log \frac{\det\Delta^{\mathbf{m}_1}}{\text{Area}(\mathcal{X}, \mathbf{m}_1)} - \log \frac{\det\Delta^{\mathbf{m}_2}}{\text{Area}(\mathcal{X}, \mathbf{m}_2)}\right\} =$$

$$\partial_t \left\{\log \frac{\det\Delta^{\mathbf{m}^\epsilon_1}}{\text{Area}(\mathcal{X}, \mathbf{m}^\epsilon_1)} - \log \frac{\det\Delta^{\mathbf{m}^\epsilon_2}}{\text{Area}(\mathcal{X}, \mathbf{m}^\epsilon_2)}\right\}. \tag{29}$$

By virtue of Polyakov's formula the r. h. s. of (29) can be rewritten as

$$\sum_{k=1}^N \frac{1}{3\pi}\partial_t \int_{D_k(\epsilon)} (\log H_k)_{x_k\bar{x}_k} \log|f_k| \widehat{dx_k} -$$

$$\sum_{l=1}^M \frac{1}{3\pi}\partial_t \int_{D_{N+l}(\epsilon)} (\log H_{N+l})_{y_l,\bar{y}_l} \log|g_l| \widehat{dy_l}, \tag{30}$$

where $H_k(x_k) = \epsilon^{-a_k} h_k^{-1}(|x_k|/\epsilon)$, $k = 1, \dots, N$ and $H_{N+l}(y_l) = \epsilon^{-b_l} h_{N+l}^{-1}(|y_l|/\epsilon)$, $l = 1, \dots, M$. Notice that for $k = 1, \dots, N$ the function H_k coincides with

$|x_k|^{-a_k}$ in a vicinity of the circle $\{|x_k|=\epsilon\}$ and the Green formula implies that

$$\int_{D_k(\epsilon)}(\log H_k)_{x_k\bar{x}_k}\log|f_k|\widehat{dw_k}=\frac{i}{2}\left\{\oint_{|x_k|=\epsilon}(\log|x_k|^{-a_k})_{\bar{x}_k}\log|f_k|d\bar{x}_k+\right.$$

$$\left.+\oint_{|x_k|=\epsilon}\log|x_k|^{-a_k}(\log|f_k|)_{x_k}dx_k+\int_{D_k(\epsilon)}(\log|f_k|)_{x_k\bar{x}_k}\log H_k dx_k\wedge d\bar{x}_k\right\}$$

and, therefore,

$$\partial_t\int_{D_k(\epsilon)}(\log H_k)_{x_k\bar{x}_k}\log|f_k|\widehat{dx_k}=-\frac{a_k\pi}{2}\partial_t\log|\mathbf{f}_k|+o(1) \tag{31}$$

as $\epsilon\to 0$. Analogously

$$\partial_t\int_{D_{N+l}(\epsilon)}(\log H_{N+l})_{y_l\bar{y}_l}\log|g_l|\widehat{dy_l}=-\frac{b_l\pi}{2}\partial_t\log|\mathbf{g}_l|+o(1) \tag{32}$$

as $\epsilon\to 0$.

Formula (25) follows from (29), (31) and (32). $\square$

4.4 Lemma on three polyhedra

For any metric $\mathbf{m}$ on $\mathcal{X}$ denote by $Q(\mathbf{m})$ the ratio $\det\Delta^{\mathbf{m}}/\text{Area}(\mathcal{X},\mathbf{m})$.

Consider three families of flat conical metrics $\mathbf{l}(t)\sim\mathbf{m}(t)\sim\mathbf{n}(t)$ on $\mathcal{X}$ (here $\sim$ means conformal equivalence), where the metric $\mathbf{l}(t)$ has conical points $P_1(t),\dots,P_L(t)$ with conical angles $2\pi(a_1+1),\dots,2\pi(a_L+1)$, the metric $\mathbf{m}(t)$ has conical points $Q_1(t),\dots,Q_M(t)$ with conical angles $2\pi(b_1+1),\dots,2\pi(b_M+1)$ and the metric $\mathbf{n}(t)$ has conical points $R_1(t),\dots,R_N(t)$ with conical angles $2\pi(c_1+1),\dots,2\pi(c_N+1)$.

Let x_k be the distinguished local parameter for $\mathbf{l}(t)$ near $P_k(t)$ and let $\mathbf{m}(t)=|f_k(x_k)|^2|dx_k|^2$ and $\mathbf{n}(t)=|g_k(x_k)|^2|dx_k|^2$ near $P_k(t)$. Let ξ be an arbitrary conformal local coordinate in a vicinity of the point $P_k(t)$. Then one has $\mathbf{m}=|f(\xi)|^2|d\xi|^2$ and $\mathbf{n}=|g(\xi)|^2|d\xi|^2$ with some holomorphic functions f and g and the ratio

$$\frac{\mathbf{m}(t)}{\mathbf{n}(t)}(P_k(t)):=\frac{|f(0)|^2}{|g(0)|^2}$$

is independent of the choice of the conformal local coordinate. In particular it coincides with the ratio $|f_k(0)|^2/|g_k(0)|^2$.

From Proposition 2, one gets the relation

$$1 = \left\{ \frac{Q(\mathbf{l}(t))}{Q(\mathbf{m}(t))} \frac{Q(\mathbf{m}(t))}{Q(\mathbf{n}(t))} \frac{Q(\mathbf{n}(t))}{Q(\mathbf{l}(t))} \right\}^{-12} =$$

$$\mathrm{C} \prod_{i=1}^{N} \left[\frac{\mathbf{l}(t)}{\mathbf{m}(t)}(R_i(t)) \right]^{c_i} \prod_{j=1}^{L} \left[\frac{\mathbf{m}(t)}{\mathbf{n}(t)}(P_j(t)) \right]^{a_j} \prod_{k=1}^{M} \left[\frac{\mathbf{n}(t)}{\mathbf{l}(t)}(Q_k(t)) \right]^{b_k}, \qquad (33)$$

where the constant C is independent of t.

From the following statement (which we call *the lemma on three polyhedra*) one can see that the constant C in (33) is equal to 1.

Lemma 2 *Let $\mathcal{X}$ be a compact Riemann surface of an arbitrary genus g and let $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ be three conformal flat conical metrics on $\mathcal{X}$. Suppose that the metric $\mathbf{l}$ has conical points $P_1, \dots, P_L$ with conical angles $2\pi(a_1+1), \dots, 2\pi(a_L+1)$, the metric $\mathbf{m}$ has conical points $Q_1, \dots, Q_M$ with conical angles $2\pi(b_1+1), \dots, 2\pi(b_M+1)$ and the metric $\mathbf{n}$ has conical points $R_1, \dots, R_N$ with conical angles $2\pi(c_1+1), \dots, 2\pi(c_N+1)$. (All the points P_l, Q_m, R_n are supposed to be distinct.) Then one has the relation*

$$\prod_{i=1}^{N} \left[\frac{\mathbf{l}}{\mathbf{m}}(R_i) \right]^{c_i} \prod_{j=1}^{L} \left[\frac{\mathbf{m}}{\mathbf{n}}(P_j) \right]^{a_j} \prod_{k=1}^{M} \left[\frac{\mathbf{n}}{\mathbf{l}}(Q_k) \right]^{b_k} = 1\,. \qquad (34)$$

Proof. When $g > 0$ and all three metrics $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ have trivial holonomy, i. e. one has $\mathbf{l} = |\omega_1|^2$, $\mathbf{m} = |\omega_2|^2$ and $\mathbf{n} = |\omega_3|^2$ with some holomorphic one-forms ω_1, ω_2 and ω_3, relation (34) is an immediate consequence of the Weil reciprocity law (see [GH78], §2.3). In general case the statement reduces to an analog of the Weil reciprocity law for harmonic functions with isolated singularities.

5 Polyhedral tori

Here we establish a formula for the determinant of the Laplacian on a polyhedral torus, i.e., a Riemann surface of genus one with flat conical metric. We do this by comparing this determinant with the determinant of the Laplacian corresponding to the smooth flat metric on the same torus. For the latter Laplacian the spectrum is easy to find and the determinant is explicitly known (it is given by the Ray-Singer formula stated below).

In this section $\mathcal{X}$ is an elliptic ($g = 1$) curve and it is assumed that $\mathcal{X}$ is the quotient of the complex plane $\mathbb{C}$ by the lattice generated by 1 and σ, where $\Im\sigma > 0$. The differential dz on $\mathbb{C}$ gives rise to a holomorphic differential v_0 on $\mathcal{X}$ with periods 1 and σ.

5.0.1 Ray-Singer formula

Let Δ be the Laplacian on $\mathcal{X}$ corresponding to the flat smooth metric $|v_0|^2$. The following formula for $\det\Delta$ was proved in [R73]:

$$\det\Delta = C|\Im\sigma|^2|\eta(\sigma)|^4, \tag{35}$$

where C is a σ-independent constant and η is the Dedekind eta-function.

5.1 Determinant of the Laplacian on a polyhedral torus

Let $\sum_{k=1}^N b_k P_k$ be a generalized divisor on $\mathcal{X}$ with $\sum_{k=1}^N b_k = 0$ and assume that $b_k > -1$ for all k. Let $\mathbf{m}$ be a flat conical metric corresponding to this divisor via Troyanov's theorem. Clearly, it has a finite area and is defined uniquely when this area is fixed. Fixing numbers $b_1, \dots, b_N > -1$ such that $\sum_{k=1}^N b_k = 0$, we define the space $\mathcal{M}(b_1, \dots, b_N)$ as the moduli space of pairs $(\mathcal{X}, \mathbf{m})$, where $\mathcal{X}$ is an elliptic curve and $\mathbf{m}$ is a flat conformal metric on $\mathcal{X}$ having N conical singularities with conical angles $2\pi(b_k+1)$, $k = 1, \dots, N$. The space $\mathcal{M}(b_1, \dots, b_N)$ is a connected orbifold of real dimension $2N+3$.

We are going to give an explicit formula for the function

$$\mathcal{M}(\beta_1, \dots, \beta_N) \ni (\mathcal{X}, \mathbf{m}) \mapsto \det\Delta^{\mathbf{m}}.$$

Write the normalized holomorphic differential v_0 on the elliptic curve $\mathcal{X}$ in the distinguished local parameter x_k near the conical point P_k ($k = 1, \dots, N$) as

$$v_0 = f_k(x_k)dx_k$$

and define

$$\mathbf{f}_k := f_k(x_k)|_{x_k=0},\ k = 1, \dots, N. \tag{36}$$

Theorem 2 *The following formula holds true*

$$\det\Delta^{\mathbf{m}} = C|\Im\sigma|\,\mathrm{Area}(\mathcal{X}, \mathbf{m})\,|\eta(\sigma)|^4 \prod_{k=1}^N |\mathbf{f}_k|^{-b_k/6}, \tag{37}$$

where C is a constant depending only on $b_1, \dots, b_N$.

Proof. The theorem immediately follows from (35) and (25).

6 Polyhedral surfaces of higher genus

Here we generalize the results of the previous section to the case of polyhedral surfaces of an arbitrary genus. Among all polyhedral surfaces of genus $g \geq 1$ we distinguish *flat surfaces with trivial holonomy.* In our calculation of the determinant of the Laplacian, it is this class of surfaces which plays the role of the smooth flat tori in genus one. For flat surfaces with trivial holonomy we find an explicit expression for the determinant of the Laplacian which generalizes the Ray-Singer formula (35) for smooth flat tori. As we did in genus one, comparing two determinants of the Laplacians by means of Proposition 2, we derive a formula for the determinant of the Laplacian on a general polyhedral surface.

6.1 Flat surfaces with trivial holonomy and moduli spaces of holomorphic differentials on Riemann surfaces

We follow [KZ03] and Zorich's survey [Z06]. Outside the vertices a Euclidean polyhedral surface $\mathcal{X}$ is locally isometric to a Euclidean plane and one can define the parallel transport along paths on the punctured surface $\mathcal{X} \setminus \{P_1, \dots, P_N\}$. The parallel transport along a homotopically nontrivial loop in $\mathcal{X} \setminus \{P_1, \dots, P_N\}$ is generally nontrivial. If, e.g., a small loop encircles a conical point P_k with conical angle β_k, then a tangent vector to $\mathcal{X}$ turns by β_k after the parallel transport along this loop.

A Euclidean polyhedral surface $\mathcal{X}$ is called *a surface with trivial holonomy* if the parallel transport along any loop in $\mathcal{X} \setminus \{P_1, \dots, P_N\}$ does not change tangent vectors to $\mathcal{X}$.

All conical points of a surface with trivial holonomy must have conical angles which are integer multiples of 2π.

A flat conical metric g on a compact real oriented two-dimensional manifold $\mathcal{X}$ equips $\mathcal{X}$ with the structure of a compact Riemann surface, if this metric has trivial holonomy then it necessarily has the form $g = |w|^2$, where w is a holomorphic differential on the Riemann surface $\mathcal{X}$ (see [Z06]). The holomorphic differential w has zeros at the conical points of the metric g. The multiplicity of the zero at the point P_m with the conical angle $2\pi(k_m+1)$ is equal to k_m [1].

[1] There exist polyhedral surfaces with nontrivial holonomy whose conical angles are all integer multiples of 2π. To construct an example take a compact Riemann surface $\mathcal{X}$ of genus $g > 1$ and choose $2g-2$ points $P_1, \dots, P_{2g-2}$ on $\mathcal{X}$ in such a way that the divisor $P_1 + \dots + P_{2g-2}$ is not in the canonical class. Consider the flat conical conformal metric $\mathbf{m}$ corresponding to the divisor $P_1 + \dots + P_{2g-2}$ according to the Troyanov theorem. This

The holomorphic differential w is defined up to a unitary complex factor. This ambiguity can be avoided if the surface $\mathcal{X}$ is provided with a distinguished direction (see [Z06]), and it is assumed that w is real along this distinguished direction. In what follows we always assume that surfaces with trivial holonomy are provided with such a direction.

Thus, to a Euclidean polyhedral surface of genus g with trivial holonomy we put into correspondence a pair $(\mathcal{X}, w)$, where $\mathcal{X}$ is a compact Riemann surface and ω is a holomorphic differential on this surface. This means that we get an element of *the moduli space*, $\mathcal{H}_g$, *of holomorphic differentials over Riemann surfaces of genus* g (see [KZ03]).

The space $\mathcal{H}_g$ is stratified according to the multiplicities of zeros of w.

Denote by $\mathcal{H}_g(k_1, \ldots, k_M)$ the stratum of $\mathcal{H}_g$, consisting of differentials w which have M zeros on $\mathcal{X}$ of multiplicities $(k_1, \ldots, k_M)$. Denote the zeros of w by $P_1, \ldots, P_M$; then the divisor of the differential w is given by $(w) = \sum_{m=1}^{M} k_m P_m$. Let us choose a canonical basis of cycles (a_α, b_α) on the Riemann surface $\mathcal{X}$ and cut $\mathcal{X}$ along these cycles starting at the same point to get the fundamental polygon $\hat{\mathcal{X}}$. Inside $\hat{\mathcal{X}}$ we choose $M-1$ (homology classes of) paths l_m on $\mathcal{X} \setminus (w)$ connecting the zero P_1 with other zeros P_m of w, $m = 2, \ldots, M$. Then the local coordinates on $\mathcal{H}_g(k_1, \ldots, k_M)$ can be chosen as follows [KZ97]:

$$A_\alpha := \oint_{a_\alpha} w \ , \quad B_\alpha := \oint_{b_\alpha} w \ , \quad z_m := \int_{l_m} w \ , \quad \alpha = 1, \ldots, g; \ m = 2, \ldots, M \ . \tag{38}$$

The area of the surface $\mathcal{X}$ in the metric $|w|^2$ can be expressed in terms of these coordinates as follows:

$$\mathrm{Area}(\mathcal{X}, |w|^2) = \Im \sum_{\alpha=1}^{g} A_\alpha \bar{B_\alpha} \ .$$

If all zeros of w are simple, we have $M = 2g-2$; therefore, the dimension of the highest stratum $\mathcal{H}_g(1, \ldots, 1)$ equals $4g-3$.

The Abelian integral $z(P) = \int_{P_1}^{P} w$ provides a local coordinate in a neighborhood of any point $P \in \mathcal{X}$ except the zeros $P_1, \ldots, P_M$. In a neighborhood of P_m the local coordinate can be chosen to be $(z(P) - z_m)^{1/(k_m+1)}$.

Remark 3 The following construction helps to visualize these coordinates in the case of the highest stratum $H_g(1, \ldots, 1)$.

metric must have nontrivial holonomy and all its conical angles are equal to 4π.

Consider g parallelograms $\Pi_1, \dots, \Pi_g$ in the complex plane with coordinate z having the sides (A_1, B_1), ..., (A_g, B_g). Provide these parallelograms with a system of cuts

$$[0, z_2], \quad [z_3, z_4], \quad \dots, \quad [z_{2g-3}, z_{2g-2}]$$

(each cut should be repeated on two different parallelograms). Identifying opposite sides of the parallelograms and gluing the obtained g tori along the cuts, we get a compact Riemann surface $\mathcal{X}$ of genus g. Moreover, the differential dz on the complex plane gives rise to a holomorphic differential w on $\mathcal{X}$ which has $2g-2$ zeros at the ends of the cuts. Thus, we get a point $(\mathcal{X}, w)$ from $\mathcal{H}_g(1, \dots, 1)$. It can be shown that any generic point of $\mathcal{H}_g(1, \dots, 1)$ can be obtained via this construction; more sophisticated gluing is required to represent points of other strata, or non generic points of the stratum $\mathcal{H}_g(1, \dots, 1)$.

To shorten the notations it is convenient to consider the coordinates A_α,B_α, z_m altogether. Namely, in the sequel we shall denote them by ζ_k, $k = 1, \dots, 2g+M-1$, where

$$\zeta_\alpha := A_\alpha \ , \ \ \zeta_{g+\alpha} := B_\alpha \ , \ \ \alpha = 1, \dots, g \ , \ \ \zeta_{2g+m} := z_{m+1} \ , \ m = 1, \dots, M-1 \tag{39}$$

Let us also introduce corresponding cycles s_k, $k = 1, \dots, 2g+M-1$, as follows:

$$s_\alpha = -b_\alpha \ , \ \ s_{g+\alpha} = a_\alpha \ , \ \ \alpha = 1, \dots, g \ ; \tag{40}$$

the cycle s_{2g+m}, $m = 1, \dots, M-1$ is defined to be the small circle with positive orientation around the point P_{m+1}.

6.1.1 Variational formulas on the spaces of holomorphic differentials

In the previous section we introduced the coordinates on the space of surfaces with trivial holonomy and fixed type of conical singularities. Here we study the behavior of basic objects on these surfaces under the change of the coordinates. In particular, we derive variational formulas of Rauch type for the matrix of **b**-periods of the underlying Riemann surfaces. We also give variational formulas for the Green function, individual eigenvalues, and the determinant of the Laplacian on these surfaces.

Rauch formulas on the spaces of holomorphic differentials. For any compact Riemann surface $\mathcal{X}$ we introduce the prime-form $E(P, Q)$ and the canonical meromorphic bidifferential

$$\mathbf{w}(P,Q) = d_P d_Q \log E(P,Q) \tag{41}$$

(see [F92]). The bidifferential $\mathbf{w}(P,Q)$ has the following local behavior as $P \to Q$:

$$\mathbf{w}(P,Q) = \left(\frac{1}{(x(P)-x(Q))^2} + \frac{1}{6} S_B(x(P)) + o(1) \right) dx(P)dx(Q), \tag{42}$$

where $x(P)$ is a local parameter. The term $S_B(x(P))$ is a projective connection which is called *the Bergman projective connection* (see [F92]).

Denote by $v_\alpha(P)$ the basis of holomorphic 1-forms on $\mathcal{X}$ normalized by $\int_{a_\alpha} v_\beta = \delta_{\alpha\beta}$.

The matrix of $\mathbf{b}$-periods of the surface $\mathcal{X}$ is given by $\mathbf{B}_{\alpha\beta} := \oint_{b_\alpha} v_\beta$.

Proposition 3 *(see [KK09]) Let a pair $(\mathcal{X}, w)$ belong to the space $\mathcal{H}_g(k_1, \dots, k_M)$. Under variations of the coordinates on $\mathcal{H}_g(k_1, \dots, k_M)$ the normalized holomorphic differentials and the matrix of* $\mathbf{b}$*-periods of the surface $\mathcal{X}$ behaves as follows:*

$$\left. \frac{\partial v_\alpha(P)}{\partial \zeta_k} \right|_{z(P)} = \frac{1}{2\pi i} \oint_{s_k} \frac{v_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\,, \tag{43}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial \zeta_k} = \oint_{s_k} \frac{v_\alpha v_\beta}{w} \tag{44}$$

where $k = 1, \dots, 2g + M - 1$; we assume that the local coordinate $z(P) = \int_{P_1}^P w$ is kept constant under differentiation.

Variation of the resolvent kernel and eigenvalues. For a pair $(\mathcal{X}, w)$ from $\mathcal{H}_g(k_1, \dots, k_M)$ introduce the Laplacian $\Delta := \Delta^{|w|^2}$ in the flat conical metric $|w|^2$ on $\mathcal{X}$ (recall that we always deal with the Friedrichs extensions). The corresponding resolvent kernel $G(P,Q;\lambda)$, $\lambda \in \mathbb{C} \setminus \mathrm{sp}\,(\Delta)$

- satisfies $(\Delta_P - \lambda)G(P,Q;\lambda) = (\Delta_Q - \lambda)G(P,Q;\lambda) = 0$ outside the diagonal $\{P = Q\}$,
- is bounded near the conical points i. e. for any $P \in \mathcal{X} \setminus \{P_1, \dots, P_M\}$

  $$G(P,Q;\lambda) = O(1)$$

  as $Q \to P_k$, $k = 1, \dots, M$,

- obeys the asymptotics

$$G(P,Q;\lambda) = \frac{1}{2\pi}\log|x(P)-x(Q)| + O(1)$$

as $P \to Q$, where $x(\cdot)$ is an arbitrary (holomorphic) local parameter near P.

The following proposition is an analog of the classical Hadamard formula for the variation of the Green function of the Dirichlet problem in a plane domain.

Proposition 4 *The following variational formulas for the resolvent kernel $G(P,Q;\lambda)$ hold:*

$$\frac{\partial G(P,Q;\lambda)}{\partial A_\alpha} = 2i\int_{\mathbf{b}_\alpha} \omega(P,Q;\lambda)\,, \tag{45}$$

$$\frac{\partial G(P,Q;\lambda)}{\partial B_\alpha} = -2i\int_{\mathbf{a}_\alpha} \omega(P,Q;\lambda)\,, \tag{46}$$

where

$$\omega(P,Q;\lambda) = G(P,z;\lambda)G_{z\bar{z}}(Q,z;\lambda)\overline{dz} + G_z(P,z;\lambda)G_z(Q,z;\lambda)dz$$

is a closed 1*-form and* $\alpha = 1,\dots,g$*;*

$$\frac{\partial G(P,Q;\lambda)}{\partial z_m} = -2i\lim_{\epsilon\to 0}\oint_{|z-z_m|=\epsilon} G_z(z,P;\lambda)G_z(z,Q;\lambda)dz\,, \tag{47}$$

where $m = 2,\dots,M$. It is assumed that the coordinates $z(P)$ and $z(Q)$ are kept constant under variation of the moduli A_α, B_α, z_m.

Remark 4 One can unite the formulas (45-47) in a single formula:

$$\frac{\partial G(P,Q;\lambda)}{\partial \zeta_k} =$$

$$-2i\left\{\int_{s_k} \frac{G(R,P;\lambda)\partial_R\overline{\partial_R}G(R,Q;\lambda) + \partial_R G(R,P;\lambda)\partial_R G(R,Q;\lambda)}{w(R)}\right\}, \tag{48}$$

where k=1, ..., 2g+M-1.

Proof. We start with the following integral representation of a solution u to the homogeneous equation $\Delta u - \lambda u = 0$ inside the fundamental polygon $\hat{\mathcal{X}}$:

$$u(\xi,\bar{\xi}) = -2i \int_{\partial\hat{\mathcal{X}}} G(z,\bar{z},\xi,\bar{\xi};\lambda)u_{\bar{z}}(z,\bar{z})d\bar{z} + G_z(z,\bar{z},\xi,\bar{\xi};\lambda)u(z,\bar{z})dz\,. \quad (49)$$

Cutting the surface $\mathcal{X}$ along the basic cycles, we notice that the function $\dot{G}(P,\ \cdot\ ;\lambda) = \frac{\partial G(P,\ \cdot\ ;\lambda)}{\partial B_\beta}$ is a solution to the homogeneous equation $\Delta u - \lambda u = 0$ inside the fundamental polygon (the singularity of $G(P,Q;\lambda)$ at $Q = P$ disappears after differentiation) and that the functions $\dot{G}(P,\ \cdot\ ;\lambda)$ and $\dot{G}_{\bar{z}}(P,\ \cdot\ ;\lambda)$ have the jumps $G_z(P,\ \cdot\ ;\lambda)$ and $G_{z\bar{z}}(P,\ \cdot\ ;\lambda)$ on the cycle $\mathbf{a}_\beta$. Applying (49) with $u = \dot{G}(P,\ \cdot\ ;\lambda)$, we get (46). Formula (45) can be proved in the same manner.

The relation $d\omega(P,Q;\lambda) = 0$ immediately follows from the equality $G_{z\bar{z}}(z,\bar{z},P;\lambda) = \frac{\lambda}{4}G(z,\bar{z},P;\lambda)$.

Let us prove (47). From now on we assume for simplicity that $k_m = 1$, where k_m is the multiplicity of the zero P_m of the holomorphic differential w.

Applying Green's formula (49) to the domain $\hat{\mathcal{X}} \setminus \{|z - z_m| < \epsilon\}$ and $u = \dot{G} = \frac{\partial G}{\partial z_m}$, one gets

$$\dot{G}(P,Q;\lambda) =$$

$$2i \lim_{\epsilon\to 0} \oint_{|z-z_m|=\epsilon} \dot{G}_{\bar{z}}(z,\bar{z},Q;\lambda)G(z,\bar{z},P;\lambda)d\bar{z} + \dot{G}(z,\bar{z},Q;\lambda)G_z(z,\bar{z},P;\lambda)dz\,. \quad (50)$$

Observe that the function $x_m \mapsto G(x_m,\bar{x}_m,P;\lambda)$ (defined in a small neighborhood of the point $x_m = 0$) is a bounded solution to the elliptic equation

$$\frac{\partial^2 G(x_m,\bar{x}_m,P;\lambda)}{\partial x_m \partial \bar{x}_m} - \lambda |x_m|^2 G(x_m,\bar{x}_m,P;\lambda) = 0$$

with real analytic coefficients and, therefore, is real analytic near $x_m = 0$.

From now on we write x instead of $x_m = \sqrt{z - z_m}$. Differentiating the expansion

$$G(x,\bar{x},P;\lambda) = a_0(P,\lambda) + a_1(P,\lambda)x + a_2(P,\lambda)\bar{x} + a_3(P,\lambda)x\bar{x} + \dots \quad (51)$$

with respect to z_m, z and $\bar{z}$, one gets the asymptotics

$$\dot{G}(z,\bar{z},Q;\lambda) = -\frac{a_1(Q,\lambda)}{2x} + O(1), \quad (52)$$

$$\dot{G}_{\bar{z}}(z,\bar{z},Q;\lambda) = \frac{\dot{a}_2(Q,\lambda)}{2\bar{x}} - \frac{a_3(Q,\lambda)}{4x\bar{x}} + O(1), \tag{53}$$

$$G_z(z,\bar{z},P;\lambda) = \frac{a_1(P,\lambda)}{2x} + O(1), \tag{54}$$

Substituting (52), (53) and (54) into (50), we get the relation

$$\dot{G}(P,Q,\lambda) = 2\pi a_1(P,\lambda)a_1(Q,\lambda).$$

On the other hand, calculation of the right hand side of formula (47) via (54) leads to the same result. □

Now we give a variation formula for an eigenvalue of the Laplacian on a flat surface with trivial holonomy.

Proposition 5 *Let λ be an eigenvalue of Δ (for simplicity we assume it to have multiplicity one) and let ϕ be the corresponding normalized eigenfunction. Then*

$$\frac{\partial\lambda}{\partial\zeta_k} = 2i\int_{s_k}\left(\frac{(\partial\phi)^2}{w} + \frac{1}{4}\lambda\phi^2\bar{w}\right), \tag{55}$$

where $k = 1, \dots, 2g+M-1$.

Proof. For brevity we give the proof only for the case $k = g+1, \dots, 2g$. One has

$$\iint_{\hat{L}}\phi\dot{\phi} = \frac{1}{\lambda}\iint_{\hat{\mathcal{X}}}\Delta\phi\,\dot{\phi} = \frac{1}{\lambda}\left\{2i\int_{\partial\hat{\mathcal{X}}}(\phi_{\bar{z}}\dot{\phi}d\bar{z} + \phi\dot{\phi}_z\,dz) + \iint_{\hat{\mathcal{X}}}\phi(\lambda\phi)^{\cdot}\right\} =$$

$$\frac{1}{\lambda}\left\{2i\int_{\mathbf{a}_\beta}(\phi_{\bar{z}}\phi_z\,d\bar{z} + \phi\phi_{zz}\,dz) + \dot{\lambda} + \lambda\iint_{\hat{\mathcal{X}}}\phi\dot{\phi}\right\}.$$

This implies (55) after integration by parts (one has to make use of the relation $d(\phi\phi_z) = \phi_z^2dz + \phi\phi_{zz}dz + \phi_{\bar{z}}\phi_z d\bar{z} + \frac{1}{4}\lambda\phi^2 d\bar{z}$). □

Variation of the determinant of the Laplacian. For simplicity we consider only flat surfaces with trivial holonomy having $2g-2$ conical points with conical angles 4π. The proof of the following proposition can be found in [KK09].

Proposition 6 *Let $(\mathcal{X}, w) \in \mathcal{H}_g(1,\dots,1)$. Introduce the notation*

$$\mathbb{Q}(\mathcal{X},|w|^2) := \left\{\frac{\det\Delta^{|w|^2}}{\mathrm{Area}(\mathcal{X},|w|^2)\,\det\Im\mathbf{B}}\right\} \tag{56}$$

where $\mathbf{B}$ *is the matrix of* $\mathbf{b}$*-periods of the surface* $\mathcal{X}$ *and* $\mathrm{Area}(\mathcal{X}, |w|^2)$ *denotes the area of* $\mathcal{X}$ *in the metric* $|w|^2$*.*

The following variational formulas hold

$$\frac{\partial \log \mathbb{Q}(\mathcal{X}, |w|^2)}{\partial \zeta_k} = -\frac{1}{12\pi i} \oint_{s_k} \frac{S_B - S_w}{w} \ , \tag{57}$$

where $k = 1, \dots, 4g-3$*;* S_B *is the Bergman projective connection,* S_w *is the projective connection given by the Schwarzian derivative* $\left\{ \int^P w, x(P) \right\}$*;* $S_B - S_w$ *is a meromorphic quadratic differential with poles of the second order at the zeroes* P_m *of* w*.*

6.1.2 An explicit formula for the determinant of the Laplacian on a flat surface with trivial holonomy

We start with recalling the properties of the prime form $E(P, Q)$ (see [F73, F92], some of these properties were already used in our proof of the Troyanov theorem above).

- The prime form $E(P, Q)$ is an antisymmetric $-1/2$-differential with respect to both P and Q,
- Under tracing of Q along the cycle $\mathbf{a}_\alpha$ the prime-form remains invariant; under the tracing along $\mathbf{b}_\alpha$ it gains the factor

$$\exp\left(-\pi i \mathbf{B}_{\alpha\alpha} - 2\pi i \int_P^Q v_\alpha\right) \ . \tag{58}$$

- On the diagonal $Q \to P$ the prime-form has first order zero with the following asymptotics:

$$E(x(P), x(Q))\sqrt{dx(P)}\sqrt{dx(Q)} =$$
$$(x(Q)-x(P))\left(1 - \frac{1}{12} S_B(x(P))(x(Q) - x(P))^2 + O((x(Q) - x(P))^3\right), \tag{59}$$

  where S_B is the Bergman projective connection and $x(P)$ is an arbitrary local parameter.

The next object we shall need is the vector of Riemann constants:

$$K_\alpha^P = \frac{1}{2} + \frac{1}{2}\mathbf{B}_{\alpha\alpha} - \sum_{\beta=1, \beta\neq\alpha}^{g} \oint_{\mathbf{a}_\beta} \left(v_\beta \int_P^x v_\alpha\right) \tag{60}$$

where the interior integral is taken along a path which does not intersect $\partial\widehat{\mathcal{X}}$.

In what follows the pivotal role is played by the following holomorphic multivalued $g(1-g)/2$-differential on $\mathcal{X}$

$$\mathcal{C}(P) = \frac{1}{\mathcal{W}[v_1,\dots,v_g](P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}} v_{\alpha_1}\dots v_{\alpha_g}(P)\,, \tag{61}$$

where Θ is the theta-function of the Riemann surface $\mathcal{X}$,

$$\mathcal{W}(P) := \det{}_{1\le\alpha,\beta\le g}||v_\beta^{(\alpha-1)}(P)|| \tag{62}$$

is the Wronskian determinant of holomorphic differentials at the point P.

This differential has multipliers 1 and $\exp\{-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha} - 2\pi i(g-1)K_\alpha^P\}$ along basic cycles $\mathbf{a}_\alpha$ and $\mathbf{b}_\alpha$, respectively.

In what follows we shall often treat tensor objects like $E(P,Q)$, $\mathcal{C}(P)$, etc. as scalar functions of one of the arguments (or both). This makes sense after fixing the local system of coordinates, which is usually taken to be $z(Q) = \int^Q w$. In particular, the expression "the value of the tensor T at the point Q in local parameter $z(Q)$" denotes the value of the scalar $Tw^{-\alpha}$ at the point Q, where α is the tensor weight of $T(Q)$.

The following proposition was proved in [KK09].

Proposition 7 *Consider the highest stratum $\mathcal{H}_g(1,\dots,1)$ of the space $\mathcal{H}_g$ containing Abelian differentials w with simple zeros.*

Let us choose the fundamental polygon $\hat{\mathcal{X}}$ such that $\mathcal{A}_P((w))+2K^P=0$, where $\mathcal{A}_P$ is the Abel map with the initial point P. Consider the following expression

$$\tau(\mathcal{X},w) = \mathcal{F}^{2/3} \prod_{m,l=1\;\; m<l}^{2g-2} [E(Q_m,Q_l)]^{1/6}\,, \tag{63}$$

where the quantity

$$\mathcal{F} := [w(P)]^{\frac{g-1}{2}} \mathcal{C}(P) \prod_{m=1}^{2g-2} [E(P,Q_m)]^{\frac{(1-g)}{2}} \tag{64}$$

does not depend on P; all prime-forms are evaluated at the zeroes Q_m of the differential w in the distinguished local parameters $x_m(P) = \left(\int_{Q_m}^P w\right)^{1/2}$. Then

$$\frac{\partial \log\tau}{\partial \zeta_k} = -\frac{1}{12\pi i}\oint_{s_k} \frac{S_B - S_w}{w}\,, \tag{65}$$

where $k=1,\dots,4g-3$.

The following Theorem immediately follows from Propositions 6 and 7. It can be considered as a natural generalization of the Ray-Singer formula (35) to the higher genus case.

Theorem 3 *Let a pair $(\mathcal{X}, w)$ be a point of the space $\mathcal{H}_g(1, \dots, 1)$. Then the determinant of the Laplacian $\Delta^{|w|^2}$ is given by the following expression*

$$\det \Delta^{|w|^2} = C\ \text{Area}(\mathcal{X}, |w|^2)\ \det \Im \mathbf{B}\ |\tau(\mathcal{X}, w)|^2, \tag{66}$$

where the constant C is independent of a point of $\mathcal{H}_g(1, \dots, 1)$. Here $\tau(\mathcal{X}, w)$ is given by (63).

6.2 Determinant of the Laplacian on an arbitrary polyhedral surface of genus $g > 1$

Let $b_1, \dots, b_N$ be real numbers such that $b_k > -1$ and $b_1 + \dots + b_N = 2g - 2$. Denote by $\mathcal{M}_g(b_1, \dots, b_N)$ the moduli space of pairs $(\mathcal{X}, \mathbf{m})$, where $\mathcal{X}$ is a compact Riemann surface of genus $g > 1$ and $\mathbf{m}$ is a flat conformal conical metric on $\mathcal{X}$ having N conical points with conical angles $2\pi(b_1 + 1), \dots, 2\pi(b_N + 1)$. The space $\mathcal{M}_g(b_1, \dots, b_N)$ is a (real) orbifold of (real) dimension $6g + 2N - 5$. Let w be a holomorphic differential with $2g - 2$ simple zeroes on $\mathcal{X}$. Assume also that the set of conical points of the metric $\mathbf{m}$ and the set of zeros of the differential w do not intersect.

Let $P_1, \dots, P_N$ be the conical points of $\mathbf{m}$ and let $Q_1, \dots, Q_{2g-2}$ be the zeroes of w. Let x_k be a distinguished local parameter for $\mathbf{m}$ near P_k and y_l be a distinguished local parameter for w near Q_l. Introduce the functions f_k, g_l and the complex numbers $\mathbf{f_k}$, $\mathbf{g_l}$ by

$$|w|^2 = |f_k(x_k)|^2 |dx_k|^2 \ \text{ near } \ P_k; \qquad \mathbf{f_k} := f_k(0),$$

$$\mathbf{m} = |g_l(y_l)|^2 |dy_l|^2 \ \text{ near } \ Q_l; \qquad \mathbf{g_l} := g_l(0).$$

Then from (25) and (66) and the lemma on three polyhedra from §4.4 it follows the relation

$$\det\Delta^{\mathbf{m}} = C \text{Area}\,(\mathcal{X}, \mathbf{m}) \det \Im \mathbf{B}\ |\tau(\mathcal{X}, w)|^2 \frac{\prod_{l=1}^{2g-2} |\mathbf{g_l}|^{1/6}}{\prod_{k=1}^{N} |\mathbf{f_k}|^{b_k/6}}, \tag{67}$$

where the constant C depends only on $b_1, \dots, b_N$ (and neither the differential w nor the point $(\mathcal{X}, \mathbf{m}) \in \mathcal{M}_g(b_1, \dots, b_N)$) and $\tau(\mathcal{X}, w)$ is given by (63).

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References

[AS94] Aurell, E., Salomonson, P., Further results on Functional Determinants of Laplacians in Simplicial Complexes, hep-th/9405140

[B07] Bobenko A., Lectures on Riemann surfaces, to appear in LNM

[BFK92] Burghelea, D., Friedlander, L., and Kappeler, T., Meyer-Vietoris type formula for determinants of elliptic differential operators, J. of Funct. Anal., **107** 34-65 (1992)

[C10] Carslaw, H. S., The Green's function for a wedge of any angle, and other problems in the conduction of heat, Proc. London Math. Soc., vol. 8 (1910), 365-374

[C83] Cheeger, J., Spectral Geometry of singular Riemannian spaces, J. Diff. Geometry, 18 (1983), 575-657

[DP89] D'Hoker E., Phong, D.H., Functional determinants on Mandelstam diagrams, Comm. Math. Phys. **124** 629–645 (1989)

[F73] Fay, John D., Theta-functions on Riemann surfaces, Lect.Notes in Math., **352** Springer (1973)

[F92] Fay, John D., Kernel functions, analytic torsion, and moduli spaces, Memoirs of the AMS **464** (1992)

[F94] Fursaev D. V., The heat-kernel expansion on a cone and quantum fields near cosmic strings, Class. Quantum Grav., 11 (1994) 1431-1443

[GH78] Griffiths P., Harris J., Principles of Algebraic Geometry; John Wiley and Son, 1978

[K93] Hala Khuri King, Determinants of Laplacians on the space of conical metrics on the sphere, Transactions of AMS, **339**, 525-536 (1993)

[KK07] Klochko Yu, Kokotov A., Genus one polyhedral surfaces, spaces of quadratic differentials on tori and determinants of Laplacians, Manuscripta Mathematica, 122, 195-216 (2007)

[KK04] Kokotov A., Korotkin D., Tau-functions on the spaces of Abelian and quadratic differentials and determinants of Laplacians in Strebel metrics of finite volume, preprint of Max-Planck Institute for Mathematics in the Science, Leipzig, 46/2004; math.SP/0405042

[K07] Kokotov A., Preprint of Max-Planck-Institut für Mathematik in Bonn, 2007(127)

[KK09] A.Kokotov, D.Korotkin, "Tau-functions on spaces of Abelian differentials and higher genus generalization of Ray-Singer formula", Journal of Differential Geometry, 82(2009), 35–100

[K67] Kondratjev, V., Boundary value problems for elliptic equations in domains with conical and angle points, Proc. Moscow Math. Soc., 16(1967), 219-292

[KZ03] Kontsevich, M., Zorich, A., Connected components of the moduli spaces of holomorphic differentials with prescribed singularities, Invent. Math. **153** 631-678 (2003)

[KZ97] Kontsevitch, M., Zorich A., Lyapunov exponents and Hodge theory, hep-th/9701164

[LMP07] Loya P., McDonald P., Park J., Zeta regularized determinants for conic manifolds, Journal of Functional Analysis (2007), 242, N1, 195–229

[MS67] McKean, H. P., Singer, I. M., Curvature and the eigenvalues of the laplacian, J. Diff. Geometry, 1(1967), 43-69

[M99] Mooers, E., Heat kernel asymptotics on manifolds with conic singularities, Journal D'Analyse Mathématique, 78(1999), 1-36

[NP92] Nazarov S., Plamenevskii B., Elliptic boundary value problems in domains with piece-wise smooth boundary, 1992, Moscow, "Nauka"

[S88] Osgood,B., Phillips,R., Sarnak,P., Extremals of determinants of laplacian, Journal of Functional Analysis, Vol. 80, N1, 148-211 (1988)

[R73] Ray D. B., Singer I. M., Analytic torsion for complex manifolds. Ann. of Math., Vol 98 (1973), N1, 154-177

[T97] Taylor M., Partial Differential Equations, vol 2., Springer (Appl. Math. Sc., Vol. 116)

[T86] Troyanov M., Les surfaces euclidiennes à singularités coniques, L'Enseignement Mathématique, **32** (1986), 79-94

[Z06] Zorich A., Flat Surfaces, in collection "Frontiers in Number theory, Physics and Geometry. Vol. 1: On random matrices, zeta functions, and dynamical systems", P. Cartier, B. Julia, P. Moussa, P. Vanhove (Editors), Springer-Verlag, Berlin, 2006, 439–586

[Z71] Zverovich, E.I., Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces, *Russ. Math. Surveys* **26** 117-192 (1971)