Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

STOCHASTIC PERTURBATIONS AND ULAM'S METHOD FOR W-SHAPED MAPS

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(Communicated by the associate editor name)

Abstract. For a discrete dynamical system given by a map $\tau : I \to I$, the long term behavior is described by the probability density function (pdf) of an absolutely continuous invariant measure. This pdf is the fixed point of the Frobenius-Perron operator on $L^1(I)$ induced by τ. Ulam suggested a numerical procedure for approximating a pdf by using matrix approximations to the Frobenius-Perron operator. In [12] Li proved the convergence for maps which are piecewise C^2 and satisfy $|\tau'| > 2$. In this paper we will consider a larger class of maps with weaker smoothness conditions and a harmonic slope condition which permits slopes equal to ± 2. Using a generalized Lasota-Yorke inequality [4], we establish convergence for the Ulam approximation method for this larger class of maps. Ulam's method is a special case of small stochastic perturbations. We obtain stability of the pdf under such perturbations. Although our conditions apply to many maps, there are important examples which do not satisfy these conditions, for example the W-map [7]. The W-map is highly unstable in the sense that it is possible to construct perturbations W_a with absolutely continuous invariant measures (acim) μ_a such that μ_a converge to a singular measure although W_a converge to W. We prove the convergence of Ulam's method for the W-map by direct calculations.

1. **Introduction.** The long term behavior of a discrete time chaotic dynamical system $\tau : I \to I$ is described by the probability density function (pdf) of an absolutely continuous invariant measure (acim). The pdf is the fixed point of the Frobenius-Perron operator on $L^1(I)$ induced by τ, but only in very simple cases can it be computed analytically. A number of methods have, therefore, been developed to approximate the pdf. The most used and best understood is Ulam's method [15] which approximates the invariant pdf by invariant vectors of finite dimensional matrices which are restrictions of the Frobenius-Perron operator to subspaces of $L^1(I)$ generated by finite partitions of I. The convergence of Ulam's method has been proved in [12], for piecewise C^2, piecewise expanding maps of interval satisfying

2000 *Mathematics Subject Classification.* Primary: 37A10; Secondary: 37A05, 37E05.

Key words and phrases. piecewise expanding maps of an interval, absolutely continuous invariant measures, Frobenius-Perron operator, Markov maps, W-shaped maps, Ulam's method, harmonic average of slopes.

The research of the authors was supported by NSERC grants.

$|\tau'| \geq \alpha > 2$. Since then, many generalizations, both one and higher-dimensional, have appeared. We point out articles by R. Murray (e.g. [1, 14], and recently [13]) and a monograph [3]. We also mention results of [7, 8] where more general stability is established for quasi-compact linear operators and, in particular, the Frobenius-Perron operators. Another related result is the compactness theorem [2] which can be used for families of piecewise linear Markov maps τ_n that converge to τ.

However, except for the very special case where the map is piecewise convex with an indifferent fixed point [13], all the foregoing approximation methods, when applied to piecewise expanding maps and their associated Frobenius-Perron operators, require the condition that $|\tau'| \geq \alpha > 2$ in order to establish convergence of the approximating pdfs.

In principle, when trying to approximate the invariant pdf of a map τ satisfying $|\tau'| > 1$, one can consider an iterate τ^k such that $|(\tau^k)'| > 2$ and apply Ulam's method. But, the map τ^k is much more complicated than τ itself, which causes complications in the implementation of the method. Also, this trick does not establish the convergence of Ulam's method for τ.

It is not always true that maps close to each other transfer their statistical behavior. For example, maps close to Keller's W map [7] may behave dynamically very differently than the W map itself. In [10] we show that acims of maps close to the W map actually converge to a singular measure rather than the pdf for the W map. Thus, the lack of $|\text{slope}| > 2$ condition can deny compactness for the pdfs associated with families of approximating maps.

In this paper we consider a larger class of maps with weaker smoothness conditions and a harmonic average slope condition which permits slopes to equal ± 2 on certain intervals. Using a generalized Lasota-Yorke inequality [4], we establish convergence for the Ulam approximation method for this larger class of maps. Since Ulam's method is a special case of small stochastic perturbations, we also obtain stability of the pdf under small stochastic perturbations (Theorem 2). This is important in practical situations, where computer errors play a role.

In Section 2 we review the Ulam method and describe the connection between Ulam's method and small stochastic perturbations. The harmonic average slope condition is described in section 3. For τ that is C^{1+1}, (the derivative of τ satisfies a Lipschitz condition) and satisfies the harmonic average slope condition, we prove that the pdfs derived from the Ulam method form a weakly compact set in $L^1(I)$. In Section 4 we apply Ulam's method to the standard W map which does not satisfy the assumption of Theorem 3.3. This leaves open the possibility that Ulam's method may work even for maps with slope ≤ 2 in magnitude.

2. **Ulam's method and small stochastic perturbations.**

Let $I = [0, 1]$. We consider piecewise expanding piecewise C^1 maps from I into itself. The precise definitions are given in Section 3. Now we review Ulam's method. Let N be a positive integer. We divide the interval $[0, 1]$ into N equal subintervals $I_i = [(i-1)/N, i/N]$, $i = 1, 2, \ldots, N$ and create the matrix $M_N = \left(m_{i,j}^{(N)}\right)$, where

$$m_{i,j}^{(N)} = \frac{m(\tau^{-1}(I_j))}{m(I_i)} \ ,$$

and m is Lebesgue measure on I. Let $h^{(N)}$ be the left invariant vector of M_N normalized to satisfy $\sum_{i=1}^N h_i^{(N)} = N$. Let $f^{(N)} = \sum_{i=1}^N h_i^{(N)} \chi_{I_i}$ be the corresponding normalized density on $[0,1]$. In [12] it was first proved that the densities $f^{(N)}$ converge to the τ-invariant density f if $|\tau'| > 2$.

It has been shown that Ulam's method is a special case of a small stochastic perturbation of the map τ (for example, see [6]). We describe the most relevant form of stochastic perturbation. Let $\mathcal{Q}^{(N)} = \{J_1^{(N)}, J_2^{(N)}, \dots, J_{q(N)}^{(N)}\}$ be a sequence of partitions of $[0,1]$ such that $\max_{1\le i\le q(N)} m(J_i^{(N)}) \to 0$ as $N \to \infty$. A small stochastic perturbation of τ is a Markov stochastic process T_N process with the transition density

$$p_N(x,y) = \frac{1}{m(I_{j(\tau(x))})} \chi_{I_{j(\tau(x))}}(y) \ ,$$

where $j(t)$ is the index $j \in \{1,2,\dots,q(N)\}$ such that $t \in I_j$. The process sends x to $\tau(x)$ and then randomly to a point of $I_{j(\tau(x))}$ according to the uniform probability density. The process T_N acts on densities through the operator $P_N : P_N(g)(y) = \int_{[0,1]} g(x)p_N(x,y)dm(x)$. We have

$$\begin{aligned} P_N(g)(y) &= \int_{[0,1]} g(x)p_N(x,y)dm(x) = \int_{[0,1]} g(x)\frac{1}{m(I_{j(\tau(x))})}\chi_{I_{j(\tau(x))}}(y)dm(x) \\ &= \int_{[0,1]} P_\tau g(x)\frac{1}{m(I_{j(x)})}\chi_{I_{j(x)}}(y)dm(x) \ . \end{aligned}$$

Thus,

$$P_N = Q_N \circ P_\tau \ ,$$

where Q_N is the operator of averaging (conditional expectation) with respect to the partition $\mathcal{Q}^{(N)}$. The process T_N has an L^1 invariant density f_N satisfying

$$\int_{[0,1]} f_N(x)p_N(x,y)dm(x) = f_N(y) \ .$$

The main problem related to small stochastic perturbations is whether the invariant densities f_N converge to the τ-invariant density f, as $N \to \infty$. When this holds we say that τ is acim stable under small stochastic perturbations.

Ulam's matrix with respect to the partition $\mathcal{Q}^{(N)}$ is

$$M_{\mathcal{Q}^{(N)}} = \left(\frac{m(\tau^{-1}(J_i))}{m(I_j)}\right) = \left(\frac{1}{m(I_j)}\int_{I_i}\chi_{I_j}(\tau(x))dm(x)\right) \ ,$$

so $M_{\mathcal{Q}^{(N)}}$ is just the restriction of P_N to the space of densities constant on the elements of the partition $\mathcal{Q}^{(N)}$. If τ has acim stable under small stochastic perturbations, then Ulam's method for τ converges.

A more general model of small stochastic perturbations starts with a family of transition densities $q^{(N)}(x,y)$ with the property

$$\sup_x \int_{(x-r_N,x+r_N)} q^{(N)}(x,y)dm(y) \to 1 \ , \quad N \to \infty \ , \tag{1}$$

for a sequence of positive $r_N \to 0$. Then, the perturbation process is defined by transition densities $p^{(N)}(x,y) = q^{(N)}(\tau(x),y)$. Similarly as above, we have $P_N = Q_N \circ P_\tau$ where the operators Q_N are defined by $Q_N(g) = \int_{[0,1]} g(x)q^{(N)}(x,\cdot)dm(x)$.

3. **Convergence of Ulam's method for maps satisfying harmonic average of slopes condition.** In this section we prove that Ulam's method converges for maps satisfying the harmonic average of slopes condition.

Definition 3.1. Suppose there exists a partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, \dots, q\}$ of I such that $\tau : I \to I$ satisfies the following conditions:
(A) $\tau_i := \tau|_{I_i}$ is C^{1+1}, i.e.,

$$|\tau_i'(x) - \tau_i'(y)| \le M_i |x-y| \ , \ x, y \in I_i$$

for some constant M_i, $i = 1, 2, \dots, q$;
(B) $|\tau_i'(x)| \ge s_i > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.
Then, we say $\tau \in \mathcal{T}(I)$, the class of piecewise expanding transformations.

Let $s := \min_{1\le i\le q} s_i$. We assume that $\tau \in \mathcal{T}(I)$ satisfies the following additional condition:

$$s_H = \max_{i=1,\dots,q-1} \left\{ \frac{1}{s_i} + \frac{1}{s_{i+1}} \right\} < 1 \ . \tag{2}$$

The number $H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}$ is called the harmonic average of a and b. Condition $H(a,b) > 2$ is equivalent to the condition $\frac{1}{a} + \frac{1}{b} < 1$. If τ satisfies $s_H < 1$ we say that τ satisfies the harmonic average of slopes condition.

The main tool in studying acims of piecewise expanding maps is the Perron-Frobenius operator which, for our class of maps, can be defined as

$$P_\tau f(x) = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau I_i}(x) \ .$$

We will consider it on $BV(I)$, the space of functions of bounded variation. For more details about piecewise expanding maps, the Perron-Frobenius operator and the space of functions of bounded variation, $BV(I)$, we refer the reader to [2].

In the following Theorem 3.2, we state a generalized Lasota-Yorke inequality, proved in [4]. We call it "generalized" since it applies to a wider class of maps than the original Lasota-Yorke inequality [9], which requires $|\tau'| > 2$ to be useful. This inequality is always stated under assumption $|\tau'| > 1$, with understanding that one can apply it to an iterate τ^k satisfying the stronger condition. The generalized Lasota-Yorke inequality allows for some slopes to be equal to 2 in magnitude.

Theorem 3.2. *Let $\tau \in \mathcal{T}(I)$. Then, for every $f \in BV([0,1])$,*

$$\bigvee_I P_\tau f \le \eta \bigvee_I f + \gamma \int_I |f| \, dm, \tag{3}$$

where $\gamma = \frac{M}{s^2} + \frac{2}{s \min\limits_{1\le i\le q} m(I_i)}$.

As proved in Theorem 3.2 of [4], if $\tau(0), \tau(1) \in \{0,1\}$, then $\eta \le s_H < 1$. If the condition $\tau(0), \tau(1) \in \{0,1\}$ is not satisfied one uses an extension method to arrive at a similar conclusion, as done in Theorem 3.3 of [4].

Now we can prove the main result of this note, the stability of acims under small stochastic perturbations for maps satisfying the harmonic average of slopes condition.

Theorem 3.3. *Let $\tau \in \mathcal{T}(I)$ satisfy the harmonic average of slopes condition $s_H < 1$ and have a unique acim. Then, the τ-invariant absolutely continuous invariant measure is stable under all small stochastic perturbations which do not increase* *the variation, i.e,* $\bigvee_I Q^{(N)} f \le \bigvee_I f$. *In particular, Ulam's method for such maps converges.*

Proof. Once the inequality (3) with $\eta < 1$ is established the stability of the acim of τ follows by standard methods (e.g. [7]). We provide a short proof for completeness.

For any $f \in BV(I)$, we have

$$\bigvee_I Q^{(N)} P_\tau f \le \bigvee_I P_\tau f \le s_H \bigvee_I f + \gamma \int_I |f| \, dm \, . \tag{4}$$

First, this inequality implies the existence of the invariant density $f_N \in BV(I)$ for the process T_N, for any N. Substituting f_N into the inequality, we obtain that for each N the $P^{(N)}$-invariant density f_N satisfies

$$\bigvee_I f_N \le \frac{\gamma}{1 - s_H} \, .$$

Thus, the family $\{f_N\}$ is weakly compact in L^1. It is known that condition (1) implies that any $*$-weak limit of measures $f_N \cdot m$ is a τ-invariant measure. Since $\{f_N\}$ are weakly compact in L^1, it must be an acim.

In general, we assume that (1) is satisfied. To see that it holds for Ulam's method it is enough to set $r_N = 1/N$ for the N-th approximation. Since, in this case, the density $q^{(N)}(x, y)$ is of the form $\frac{1}{N}\chi_{[k/N,(k+1)/N]}(y)$, where $x \in [k/N, (k+1)/N]$, the left hand side of (1) is always equal to 1. $\square$

Example 1: Let us consider the map

$$W_{3,2}(x) = \begin{cases} 1 - 3x, & 0 \le x < \frac{1}{3}, \\ 3(x - \frac{1}{4}), & \frac{1}{3} \le x < \frac{1}{2}, \\ \frac{1}{2} - 2(x - \frac{1}{2}), & \frac{1}{2} \le x < \frac{3}{4}, \\ 4(x - \frac{3}{4}), & \frac{3}{4} \le x \le 1. \end{cases}$$

It does not satisfy $|\tau'| > 2$ but it has $s_H = 1/3 + 1/2 < 1$. Thus, Theorem 3.3 proves the convergence of Ulam's method for this map.

Since the slope of the third branch of τ is 2 the standard Lasota-Yorke method gives an inequality similar to (3) with constant $\eta = 1$, which is not useful. The only method to improve this, known before [4], was to use an iterate τ^k instead of τ itself. This can be used to prove stability of the acim of τ^k but says nothing about the stability of the acim of τ.

Example 2: This example shows how unstable the behavior of W-map can be. Let us consider a family of maps W_a, $a \ge 0$,

$$W_a(x) = \begin{cases} 1 - 4x, & 0 \le x < \frac{1}{4}, \\ (2 + a)(x - \frac{1}{4}), & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{1}{2} + \frac{a}{4} - (2 + a)(x - \frac{1}{2}), & \frac{1}{2} \le x < \frac{3}{4}, \\ 2(x - \frac{3}{4}), & \frac{3}{4} \le x \le 1. \end{cases}$$

It has been proved in [10] that the W_a-invariant densities f_a do not converge to the W_0-invariant density f_0 as $a \to 0$. On the other hand, for any $a > 0$, map W_a satisfies the assumptions of Theorem 3.3, so Ulam's approximations $f_a^{(N)}$ converge to f_a as $N \to \infty$. Thus, it is possible to find a sequence $N(a) \to \infty$ such that $f_a^{(N(a))}$ do not converge to f_0 as $a \to 0$. In Section 4 we will show that Ulam's method converges for W_0, so $f_0^{(N)} \to f_0$ as $a \to 0$. This is another manifestation of the unstable behavior of the map $W_0 = W$ is.

4. **Convergence of Ulam's method for standard W map.** The standard W map is defined as follows [7]:

$$W(x) = \begin{cases} 1 - 4x, & 0 \le x < \frac{1}{4}, \\ 2(x - \frac{1}{4}), & \frac{1}{4} \le x < \frac{1}{2}, \\ \frac{1}{2} - 2(x - \frac{1}{2}), & \frac{1}{2} \le x < \frac{3}{4}, \\ 2(x - \frac{3}{4}), & \frac{3}{4} \le x \le 1. \end{cases}$$

This map does not satisfy the assumptions of Theorem 3.3, since $s_H = 1$. However, using straightforward calculations, we will prove the convergence of Ulam's method. Depending on the form of N, the matrix M_N can have one of four forms.
(1) $N = 2n$ is even and n is also even. Then, the points $1/4$, $1/2$ and $3/4$ are partition points. The rows of the matrix M_N consist of groups of consecutive $1/4$ s for $1 \le i \le N/4$ and $3N/4 \le i \le N$ and of groups of consecutive $1/2$ s for the remaining i. See M_{12} below for the illustration.

$$M_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

For $i = 1, 2, \ldots, N/2$, in every column the sum of the elements $\sum_{j=1}^{N/2} m_{i,j} = \sum_{j=N/2+1}^{N} m_{i,j} = 3/4$. For $i = N/2+1, N/2+2, \ldots, N$, the similar sums are equal to $1/4$. This shows that the vector

$$h^{(N)} = (3/2, 3/2, \ldots, 3/2, 1/2, \ldots, 1/2, 1/2) \ ,$$

with $h_i^{(N)} = 3/2$ for $1 \le i \le N/2$ and $h_i^{(N)} = 1/2$ for $N/2 + 1 \le i \le N$, is the left invariant vector of M_N.
(2) $N = 2n$ is even with n odd. Then, the point $1/2$ is the partition point as before. The points $1/4$ and $3/4$ are not. They fall exactly in the middle of subintervals $i_1 = (N/2 - 1)/2 + 1$ and $i_2 = N/2 + (N/2 - 1)/2 + 1$. The rows of matrix M_N are similar as in case (1) except for these two indices, for which they are $[3/4, 1/4, 0, \ldots, 0]$. See M_{14} below for the illustration.

$$M_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

The column half sums are the same as in case (1) so the left invariant vector of M_N has the same form as in case (1).

(3) $N = 4k+1$ is odd. Then, none of the points $1/4$, $1/2$ and $3/4$ is a partition point. The point $1/2$ falls exactly in the middle of the central subinterval $i_1 = (N-1)/2+1$. The points $1/4$ and $3/4$ fall at $1/4$ th and $3/4$ th of subintervals $i_2 = (N-1)/4+1$ and $i_3 = N-(N-1)/4-1$. The rows of matrix M_N are similar as in case (1) or (2) for $1 \le i \le (N-1)/4$ and $N-(N-1)/4 \le i \le N$. For i_2 and i_3 they are $[3/4, 1/4, 0, \ldots, 0]$. For i_1 the row has elements $m_{i_1,(N-1)/2} = m_{i_1,(N-1)/2+1} = 1/2$ and all other 0. The remaining rows have form $[0, \ldots, 0, 1/4, 1/2, 1/4, 0, \ldots, 0]$ See M_{13} below for the illustration.

$$M_{13} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4
\end{bmatrix}.$$

The column half sums are the same as before except columns $j_1 = (N-1)/2$ and $j_2 = (N-1)/2+1$. It is still easy to check that the left invariant vector of M_N is of the form

$$h^{(N)} = (3/2, 3/2, \ldots, 3/2, 1, 1/2, \ldots, 1/2, 1/2) ,$$

where 1 is at the central position $(N-1)/2+1$.

(4) $N = 4k+3$ is odd. Again, none of the points $1/4$, $1/2$ and $3/4$ is a partition point. The point $1/2$ falls as before in the middle of central subinterval $i_1 = (N-1)/2+1$. The points $1/4$ and $3/4$ fall at $3/4$ th and $1/4$ th of the subintervals $i_2 = (N-3)/4+1$ and $i_3 = N-(N-3)/4-1$. The rows of matrix M_N are similar as in case (3) except for rows i_2 and i_3 which are $[1/2, 1/4, 1/4, 0, \ldots, 0]$. See M_{15} below for the illustration.

$$M_{15} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4
\end{bmatrix}.$$

Again, the left invariant vector of M_N is of the form

$$h^{(N)} = (3/2, 3/2, \ldots, 3/2, 1, 1/2, \ldots, 1/2, 1/2) ,$$

where 1 is at the central position $(N-1)/2+1$.

For all forms of N, we have $f^{(N)} \to f = 3/2\chi_{[0,1/2]} + 1/2\chi_{[1/2,1]}$, as $N \to \infty$. Thus, the Ulam method is convergent.

Acknowledgments. The authors are grateful to an anonymous reviewer for detailed comments which improved the presentation.

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Received xxxx 20xx; revised xxxx 20xx.

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