# An Analysis of Students’ Difficulties in Learning Group Theory

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ABSTRACT. Research in mathematics education and anecdotal data suggest that undergraduate students often find their introductory courses to group theory particularly difficult. Research in this area, however, is scarce. In this thesis, I consider students' difficulties in their first group theory course and conjecture that they have two distinct sources. The first source of difficulties would pertain to a conceptual understanding of what group theory is and what it studies. The second would relate to the modern abstract formulation of the topics learned in a group theory course and the need to interpret and write meaningful statements in modern algebra. To support this hypothesis, the group concept is explored through a historical perspective which examines the motivations behind developing group theory and its practical uses. Modern algebra is also viewed in a historical context in terms of three defining characteristics of algebra; namely, symbolism, justifications and the study of objects versus relationships. Finally, a pilot study was conducted with 4 students who had recently completed a group theory course and their responses are analysed in terms of their conceptual understanding of group theory and modern algebra. The analysis supports the hypothesis of the two sources. Based on the results, I propose a remediation strategy and point in the direction of future research.

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# Chapter 1

# Introduction

## 1.1 Motivation and goal of the study

Typically, group theory is taught at the undergraduate level in a first Abstract Algebra course. This course is structured around learning group theory through abstract definitions. The emphasis is on solving problems by using abstract definitions, symbolic manipulation and the main theorems of group theory. In this paper, I will argue that this approach might be a source of students' difficulties in learning group theory.

The ultimate goal of this thesis is to support the conjecture that students' difficulties in these courses have (at least) two different sources of clearly different natures. The first are conceptual difficulties that pertain specifically to learning and understanding group theory concepts. The second are difficulties due to having to cope with the abstract manner in which these concepts are typically presented. I surmise that when teaching group theory in a first Abstract Algebra course, the ideas that pertain to group concepts are not distinguished in any way from their abstract representation. Therefore, students do not learn how to connect the ideas and concepts of group theory to their abstract definitions. I propose that students often confuse these two aspects of group theory and do not have a firm grasp of either one.

Little research has been conducted on the teaching and learning of abstract algebra; the majority of research in university level mathematics deals with pre-calculus, calculus, linear algebra and discrete mathematics. (Fukawa & Connelly, 2012; Hazzan, 1999) While many studies report on students' difficulties with abstraction, it is not clear that the notion of abstraction they refer to can be used to discuss students' difficulties in a group theory course. A few studies do report on students' difficulties with abstract algebra concepts but do not suggest how to improve students' theoretical understanding of concepts. (Dubinsky et al., 1994, Hazzan, 1999; Leron et al. 1995; Weber and Larsen, 2004) The study presented in this thesis is a contribution to the scarce research in the teaching and learning of abstract algebra in general, and group theory in particular.

The question that motivated my research is: to what extent is a student's understanding of group theory inextricably linked to abstract symbolism. Do students who have completed a group theory course understand abstract formulations of mathematical concepts such as the abstract definition of a group? How can a student answer the question "what is a group" beyond "it is a set provided with a binary operation that satisfies four axioms"? In particular, the goal of this study is to discuss the questions

- Where does group theory come from?
- What does group theory study?
- What are the defining characteristics of modern algebra?
- What are students' conceptual understandings of the group concept? and
- How do students interpret and create statements in modern algebra?

These are posed in the context of the teaching and learning of a first Abstract Algebra course at the undergraduate level. They are explained through the historical development of the group concept and modern algebra and through the analysis of a pilot study.

## 1.2 Overview of the thesis

In the second chapter of the thesis, I present the historical development of the group concept. I give a sketch of the main developments leading up to the modern abstract formulation of group theory which emerged in the $19^{th}$ century and took hold in the $20^{th}$ century. In this way I hope to portray group theory as a method for solving concrete problems in mathematics and demonstrate the central ideas, motivations and applicability of group theory. I explain how these ideas were originally formulated conceptually and without the use of modern algebra. By providing insight into group theory in its early stages I hope to demonstrate that it is not simply "abstract". In this way I aim to provide a context to what group theory is and what it studies and how various fields in mathematics are explained through group theoretic thought. I will then contrast students' understandings of group theory concepts with the historical development of these concepts.

I trace the origins of group theory to a method used by Lagrange to solve rational polynomial equations. The solution to rational polynomial equations was essentially all there was to algebra prior to the advent of abstract algebra. The methods used to find these solutions revolved around directly manipulating specific polynomials usually through substituting variables, reducing equations and then working backwards. This was what Lagrange was engaged in when he realised that there was a new way to gain insight into the solvability of an equation without the need to work with it directly. This method involved examining the number of values taken on by the roots under all possible permutations. He discovered that this was in direct correlation to the solvability of the polynomial equation. It is often the case in mathematics and the sciences that we can work with objects indirectly such as developing an atomic theory before seeing atoms. Mathematicians developed a method for gaining insight into the solvability of polynomials without working with them directly.

The connection between permutations and solvability of equations sparked an interest in the study of general properties of permutations and answering questions like "How many values can be taken on by a rational function of $n$ variables?" The answers to these questions concerning the general properties of permutations could then be applied to specific polynomials to determine their solvability.

In this way the independent theory of permutations and theories of solvability expanded culminating in Galois' group theoretic proof of the Abel-Ruffini theorem. In section 2.2.2, I present a sketch of how Galois created a method of attaching a unique permutation *group* to every polynomial. This allowed him to determine which polynomials have rational solutions and to demonstrate that there is no general formula for solving polynomials of degree five and higher like Pythagoras or Cardano's formula.

These revelations did not go unnoticed and mathematicians realised that the group theoretic methods of abstracting the important features of a system could be applied to a wide variety of mathematical disciplines. Once a system is put into the context of a group, knowledge can be transferred. For example, the rotations of a polygon have the same structure as permuting the roots of a polynomial. Therefore the knowledge accumulated by algebraists on theories of permutations and polynomials could then be applied to geometry (e.g., to the rotations of regular polygons). Geometers such as Jordan and Klein applied the ideas of group theory to reformulate geometry and saw group theory as the grand unifier of mathematics.

Group theory flourished in a variety of mathematical fields before developing into the more precise and general abstract theory that we see today. In order to understand this abstract formulation of group theory, I will examine modern algebraic thought.

In the $20^{th}$ century, algebra reached soaring heights and great complexity. It achieved the much needed rigour that had been sorely lacking in algebraic proofs. In the third chapter I will examine the way abstract statements in group theory are interpreted and worked with in relation to modern algebra. This will later be used to assess whether students are trained to view algebra in this way following an introductory group theory course.

In Chapter 3, I contrast the mode of thought associated with historical classical algebra and modern algebra in terms of three distinguishing characteristics; namely, the use of symbolism, the nature of justifications in algebra and the study of objects versus the study of generalised relationships. Our current take on a group theory is rife with *symbolic algebra* and I begin by examining Nesselman's commonly used distinction for periods of algebraic symbolism; rhetorical, syncopated and symbolic algebra. (Heeffer, 2008) I discuss the way the type of modern symbols encountered in a group theory course such as $a * b = c$ are to be interpreted in contrast to syncopated symbols and rhetorical definitions. These types of definitions and examples are seen in the way group theory is described in Chapter 2.

Students are often required to give rigorous justifications of symbolic statements and must be at ease working in axiomatic systems. I contrast the nature of justifications in algebra by examining how algebra was historically an intuitive science dominated by Platonic philosophy and physical justifications, and how this ontology was disposed of and replaced by rigour and consistency in axiomatic systems.

Finally I discuss how work on general equations emerged in Viète's analytic art and how this created a new way of asking and answering questions in algebra. For example, the difference between examining a specific expression $x^2 - 3x + 2 = 0$ where the goal is to find specific values, and examining a general equation such as $ax^2 + bx + \mathrm{c} = 0$ where the goal is to explore general relationships, not specific values.

In this way I hope to describe how the modern algebraic statements in group theory are to be interpreted and how this differs from previous classical forms of algebra that students may be more accustomed to from other courses. I postulate that these differences are a source of "abstraction" in a group theory course and I use this to provide a basis to analyze students' understanding of the algebraic statements found in a group theory course. By outlining both the history of the group concept and modern algebra, I hope to demonstrate that group theory and modern abstract thinking *are not* inextricably linked. They can be examined separately and in terms of different, albeit overlapping, developments in mathematics and mathematical philosophy.

In Chapter 4, I review mathematics education research on students' learning of and difficulties with group theory concepts. This should give a general sense of the work being done in this field. The reviewed literature often refers to the fact that group theory is "abstract" and that this is a source of difficulty for students. I present the definitions of abstraction found in the literature which I postulate are not specific to a group theory course and fall short in their goal of describing why group theory in particular is so difficult for undergraduate students.

I then describe a second source of student difficulty described in the literature; the use of canonical procedures in answering questions. This addresses how students often use techniques and formulae they do not understand and are often observed to prefer flawed memorized procedures over theoretical answers. This demonstrates a chronic problem; students feel they should imitate processes rather than understand ideas and are hesitant about using theoretical knowledge to answer questions.

In Chapter 5, I analyze and discuss data collected in the context of a pilot study on students' understanding of groups and their representations. I present a framework for this study which was conducted with four students who had recently completed an introductory undergraduate group theory course. The students answered six questions during audio recorded one-on-one interviews. These questions were categorised in terms of (a) the *familiarity* of the group in the question, i.e., whether or not it was a group they had previously seen in class, and (b) the *familiarity* of the representation of the question, i.e., whether or not it contained the type of abstract formulations they are accustomed to using in their group theory course.

This pilot study and the familiarity/unfamiliarity framework were designed to assess students' understanding of group theory in contrast to the historical development of group concepts and modern algebraic thinking. I attempt to distinguish difficulties that arise from a lack of understanding of group concepts from those that are caused by the modern abstract nature of the course. I hope to gain better understanding of a student's perception of a group and their ability to do mathematics as characterised by modern abstraction.

In Chapter 6, I present a discussion, conclusions and recommendations. In the discussion section, based on the pilot study, I characterise students' difficulties in terms of three factors which I surmise affect a student's ability to understand and answer questions in group theory. The first factor concerns students' basic level of knowledge of group theory concepts as well as their ability to carry out basic calculations and procedures. Secondly, I discuss their theoretical understanding of group concepts that goes beyond the elementary definitions. I contrast this to the historical context of group theory given in Chapter 2. The third factor concerns their ability to interpret the modern abstract formulation of group theory and their skill in writing statements in modern algebra. I base my argument on the three characteristics of modern algebra described in Chapter 3; symbolism, justifications and object vs. relationships.

Next, I conclude from the literature review and pilot study that students tend to view group theory as purely abstract and do not have an understanding of group theory concepts that goes beyond the definitions. They have difficulties interpreting and writing statements in modern abstraction and in correlating and distinguishing modern abstract definitions and concepts before and after entering an introductory group theory course. As a result they cannot evolve beyond an abstract characterization of group theory and they answer questions by imitating formal statements they do not understand. I believe that it is vital for future mathematicians to be trained in symbolic systems and abstraction and to be able to connect this to the concepts they are studying. Students in undergraduate group theory courses, however, might not be getting a deeper sense of what they are studying in these abstract structures or why.

I hypothesize that the source of the difficulties mentioned in the above paragraph, is that, in a group theory course, students are being introduced to two new topics at the same time: the concepts of group theory and modern algebra. I propose that students were not adequately prepared by earlier courses to understand the modern abstract formulations of group theory concepts or to interpret and write statements in modern algebra. Students who enter into an introductory group theory course need to be skilled at formulating statements involving symbolic algebra, rigorously justifying these statements in axiomatic systems and in observing relationships, symmetry and structure. They need to be able to understand group concepts independently of abstract definitions and relate them to theoretical ideas and practical methods for solving concrete problems. Therefore, students should learn to decouple group concepts and modern algebra. Based on the discussion presented in Chapters 2 and 3, I argue that the central ideas and uses of group theory can be examined independently of modern algebra and that the use of symbolic algebra, axiomatic systems and the study of general relationships are not restricted to group theory concepts.

Based on the discussion and conclusion, the thesis ends with pedagogical recommendations that I surmise could help improve students' understanding of group theory concepts and of modern algebra. I also point out a direction for future research.

# Chapter 2

## A history of the development of the group concept

A group theory course generally begins with the abstract definition of group similar to the following one:

> A group is a non-empty set $G$ on which there is a binary operation * such that
>
> 1. if $a$ and $b$ belong to $G$ then $a * b$ is also in $G$ (closure),
> 2. $a(bc) = (ab)c$ for all $a, b, c$ in $G$ (associativity),
> 3. there is an element $1_G \in G$ such that $a * 1_G = 1_G * a = a$ for all $a \in \mathrm{G}$ (identity),
> 4. if $a \in \mathrm{G}$, then there is an element $a^{-1} \in \mathrm{G}$ such that $a * a^{-1} = a^{-1}a = 1_G$ (inverse).

The concept of group is based on this definition as a student goes on into developing the theory. In this chapter I explore how and why group theory was developed and why it is so much more than a set of axioms – the goal is to highlight the *dangers* of presenting group theory "simply" as the theory that develops from these four axioms. (Shay, 2010)

Group theory development started in the mid-18$^{th}$ century with the search for rational solutions to polynomial equations. It evolved out of insights into the inner workings of polynomials by applying concepts such as invariance under permutations. These insights evolved into the creation of a novel way of thinking about algebra that proved extremely useful in problem solving and created a structure which is capable of determining which polynomials are solvable. For over a hundred years, group theory was almost inseparable from permutations of polynomial equations. Then, through the adaptation of group theoretic ideas into a variety of disciplines (notably geometry) it developed into a universal language of symmetry that was abstracted in the 19$^{th}$ century into its current form. By the beginning of the 20$^{th}$ century, notable mathematicians such as Felix Klein (1849-1925) were deeply concerned about what the dominance of an abstracted version of group theory would mean for future work. In his 1926 book *Vorlesungen über höhere Geometrie* (*Lectures on Higher Geometry*, in English) Klein described the dangers he foresaw with the dominance of this abstract system. "This abstract formulation is excellent for the working out of proofs but it does not help one find new ideas and methods." (Wussing, 1984)

## 2.1 The Origins of Group theory

### *2.1.1 Joseph-Louis Lagrange*

The origin of group theory lies in Joseph-Louis Lagrange's (1736-1813) 1770 work *Réflexions sur la théorie algébrique des equations* (Reflections on the solution of algebraic equations, in English) and his search for rational solutions to polynomial equations. Algebra was essentially the study of solving polynomial equations until the early 19$^{th}$ century and the focus at the time of Lagrange was primarily on finding the roots of polynomials that could be solved and also on finding classes of polynomials that could not be solved. At this time there was no clear pattern behind which polynomials were solvable or not solvable by radicals or how to go about discerning whether or not a polynomial was solvable. The methods used revolved around the direct manipulation of polynomials. Since the goal was to find a rational solution, they could only employ algebraic operations i.e.addition, subtraction, multiplication, division and the application of radicals. (Kleiner, 2007; Logan, 2010)

Lagrange examined the work of his contemporaries, such as Alexandre-Théophile Vandermonde(1735 –1796) and Edward Waring (1736 –1798), attempting to find similarities in their methods for finding roots. (Kleiner, 2007) These methods revolved around creating auxiliary rational equations known as *resolvents* that were linked to the original equation but easier to solve.

If this auxiliary equation exists and its roots can be determined algebraically, then the roots of the auxiliary equation can be used to find the roots of the original equation which are rational in terms of the coefficients of the original equation and the roots of the auxiliary equation.

Lagrange began his work on the general cubic $f(x) = x^3 + a_2x^2 + a_1x + a_3 = 0$ with a commonly used reduction; setting $y = x + \frac{a}{3}$ he reduced $x^3 + a_2x^2 + a_1x + a_3 = 0$ to $y^3 + py + q = 0$. He then set $y = z - \frac{p}{z}$ to obtain the resolvent $\varphi = z^6 + qz^3 - \frac{p^3}{27} = 0$.

Here, Lagrange realised that $z^3$ was the root of a quadratic equation and so he had a polynomial of degree six which was composed of quadratic and a pure cubic. So by using the substitution $r = z^3$ he attained the quadratic equation $r^2 + qr - \frac{p^3}{27} = 0$ which has two roots $r_1$ and $r_2$. The cube roots of $r_1$ and $r_2$ are solutions of the resolvent and when combined with the third root of unity $\zeta$, they provide all six solutions to the resolvent $\varphi$ as if $r_1$ and $r_2$ are plugged back into $z^3 - r = 0$ we find that $z = \zeta^k (r_1)^{1/3}$ and $\zeta^k (r_2)^{1/3}$, for $k = 0, 1, 2$ are the solutions to $\varphi$. These solutions can then be backtracked to the original cubic function to find the roots $a, b, c$ of $f(x)$ (the general cubic) which are:

$$a = (r_1)^{1/3} + (r_2)^{1/3},$$

$$b = \zeta (r_1)^{1/3} + \zeta^2 (r_2)^{1/3}, \text{ and}$$

$$c = \zeta^2 (r_1)^{1/3} + \zeta (r_2)^{1/3}.$$

Lagrange then examined the roots of the quadratic, $r_1$ and $r_2$, in terms of the roots of the general cubic, to get $(r_2)^{1/3} = \frac{1}{3}(a + \zeta b + \zeta^2 c)$ and $(r_1)^{1/3} = \frac{1}{3}(a + \zeta^2 b + \zeta c)$. (Waerden, 1985) Here, Lagrange realised that if the roots $a, b, c$ are permuted then this would give him all six values of $z$ corresponding to his equation of degree six. Lagrange recognized that the reason this particular resolvent was capable of solving a cubic in terms of quadratic was because the six permutations only take on two values when $a, b$ and $c$ are permuted. Consequently the solvability of an equation by resolvents is directly correlated to the number of values taken on by the roots of the resolvent when the roots of the original equation are permuted.

While the use of resolvents was commonplace, there was no clear methodology behind them. (Kleiner, 2007) Lagrange developed a new rationale behind his choice of resolvents by relating them to their invariance under permutations. He generally used resolvents of the form $(x_1 + \zeta x_2 + \zeta^2 x_3 + ... + \zeta^{n-1} x_n)^n$ (where $\zeta$ is the n$^{th}$ root of unity), due to their invariance under circular permutations. These are still mathematical objects of particular interest now referred to as *Lagrange resolvents*. (Friedelmeyer, 1986)

Lagrange turned his attention towards examining symmetry and congruence in relation to the roots of unity. The driving force behind Lagrange's work was the ability to find a resolvent expression $\varphi$ that only takes on $r$ values when all permutations are carried out and leaves another expression $\rho$ unaltered. This would later evolve into the concept of *invariance* which would be crucial in discovering which polynomials are solvable and which polynomials are not. If this resolvent expression $\varphi$ exists, then it can be solved through an equation of degree $r$ whose coefficients depend rationally on $\rho$ and on the coefficients of the original equation. (Wussing, 1984) The solutions of a cubic could now be reduced to investigating his resolvent $R = (a + \zeta b + \zeta^2 c)^3$. (Wussing, 1984) Lagrange had in effect reduced the solutions of equations to a "combinatorial calculus" or the "calculus of permutations" as solvability was directly linked to the invariance of resolvents. (Kleiner, 2007, Wussing, 1984) The existence of rational solutions for a given polynomial equation of degree $n$ was the equivalent of finding a resolvent that took on fewer than $n$ values under all $n!$ permutations of the roots.

Lagrange's method for solving polynomial equations was novel and created the foundation for a new and general theory that examined the direct relation between the solvability of an equation and the permutations of the roots or arguments.

Lagrange set in motion the ascent of permutations out of their relative obscurity in combinatorics to the forefront of algebra. (Wussing, 1984) Lagrange never proved or suggested any theorems on solutions by radicals for higher degree polynomials; he believed that they may be solvable and that nothing in his work proved their insolvability. (Wussing, 1984) Lagrange attempted to find a suitable resolvent for the quintic which would have to assume at most four values under all $5! = 120$ permutations of the roots if a reducible resolvent was to be created. He succeeded in constructing a resolvent that takes on six values but of course could not complete the impossible task of finding any resolvents that take on fewer values. (Kiernan, 1971)

His inability to find a resolvent that took on four values led Lagrange to consider the more general question: how many values can be taken on by a rational function of $n$ variables? (Wussing, 1984) The answer to this question would show that it is indeed impossible to find a resolvent for the general quintic that took on fewer than five values under all permutations but this was again not suggested by Lagrange.

His main efforts were entirely pragmatic and focused on specific examples until his fourth section *Conclusion des réflexions* (Conclusion of the reflections, in English) where he began to explore this generalised property. Here he described a general function of $n$ variables $f(x_1,\ldots,x_n)$ and stated that if $f_i$ is a root of the Lagrange resolvent for this function and if $f_i$

takes on $m$ values under all $n!$ permutations of the $n$ roots then $m$ always divides $n!$ (Wussing, 1984)

While Lagrange stated that the value of $m$ must always be a divisor of $n!$ he only proved this for two variables (typical of the lack of rigour provided in proofs during this time period). The modern form of this work is now known as Lagrange's theorem which is commonly taught in an introductory group theory course and states that for any finite group *G*, the order of every subgroup *H* of *G* divides the order of *G* or that $|G|=\frac{[G:H]}{|H|}$.

This concept behind Lagrange's *Conclusions* would lead to an independent theory of permutations and the proof that polynomials of degree five and higher are not solvable by radicals. It showcased the value of exploring general properties of permutations as they are in direct correlation to the solvability of equations. Seeing as this was the bulk of algebra at that time it was a strong motivator. This theory of permutations established the basic ideas, formulae and theorems in group theory which was synonymous with and almost inseparable from the permutations of roots of equations for almost a hundred years.

### *2.1.2 Paolo Ruffini*

Lagrange's *Réflexions* went unmentioned for 27 years and was only referenced in a minor capacity in the 1797 fifth edition of Alexis Claude de Clairaut's (1713-1765) *elements d'algebra (Elements of Algebra,* in English). Two years later, it would be finally taken on by Paolo Ruffini (1765-1822) in his 1799 work *Teoria generale delle equazioni (On the general theory of equations*, in English). He was the first to controversially attempt a proof that equations of degree five or higher cannot be solved by radicals now known as the Abel–Ruffini theorem. Ruffini angered many members of the community as his work flew in the face of the old guard like Gianfrancesco Malfatti (1731-1807) who firmly believed in the existence of an algebraic equation for the general quintic and who had managed to achieve fame for his resolvent. (Wussing, 1984) Ruffini's concepts and methods were harshly criticized. His *Teoria* was long, confusing and disorganised and fell short in its goal to change the minds of mathematicians who could not understand his work. Despite the fact that his proof of the Abel-Ruffini theorem was almost correct and contained only one gap, the confusing and inarticulate presentation along with mathematical community's strongly held belief in the solvability of higher degree polynomials led to its rejection. (Kiernan, 1971) Even Niels Henrik Abel (1802 – 1829) who proved the Abel-Ruffini theorem said of Ruffini that "his memoir is so complicated that it is difficult to judge the validity of his reasoning. It seems to me that his reasoning is not always satisfying." (Ayoub, 1980)

Ruffini cultivated strong connections between permutations and the solvability of algebraic equations and through his work group theory took shape. Ruffini used Lagrange's calculus of permutations and created the concept of the set of all possible permutations of the arguments or roots of a function which he called *permutazione*. He noticed that any permutation could be composed in any way, any number of times and still give back an element of his original *permutazione*; in modern terminology this would be known as the closure property in group theory. (Wussing, 1984) He also explored general types of subsets that could occur from these permutazione by examining how they were embedded in one another. (Shay, 2010)

For example, given a function of four arguments $f(x_1, x_2, x_3, x_4)$ the *permutazione* would contain all $4!$ permutations of the four arguments. An example of such a subset would be to take the two permutations $P = (x_1, x_2)$ and $Q = (x_3, x_4)$ (i.e. $P$ permutes the first and second variables and $Q$ permutes the third and fourth variables), then create a subset from all possible permutations that can be derived through composition of $P$ and $Q$ in any order, any number of times. This generates the infinite set of permutations $\{PQ, QP, P^2, Q^2, P^2Q, PQP, QPQ, QP^2Q, P^3, ...\}$. In this particular example $P^2 = 1$, $Q^2 = 1$ and $PQ = QP$ so this set can be reduced to the finite set containing four permutations $\{1, P, Q, PQ\}$ which are all part of the original *permutazione*. Ruffini stated that any infinite set of this kind will always be a closed subset (in modern terminology) of the original *permutazione* of the function and saw the great importance of this closure property.

Ruffini defined sets of generators for his *permutazione* which he could use to classify the permutations of a function. He introduced four types of groups based on this concept of *generators*. For example, his *permutazione semplice* were cyclic groups where one single permutation could be repeatedly composed with itself to give every possible permutation. The still unnamed "groups" of permutations slowly developed structure through the Ruffini manipulations. He introduced other defining characteristics such as order or as he called it *degree of equality*. (Wussing, 1984)

Ruffini generalised his classifications of functions by probing into the number of possible permutations that could exist for a given polynomial, an idea equivalent to the number of possible subgroups. Our knowledge of subgroups directly evolved from this work on permutations of roots and was entirely motivated by a search for the solvability of equations. (Ayoub, 1980) Ruffini was able to connect the structure of his *permutatzione* to the structure of the polynomials which in turn determined their solvability. (Shay, 2010)

Through an exhaustive search, Ruffini attempted classify all possible *permutazione* that were rational in term of the coefficients of the general quintic. This is equivalent to listing all possible subgroups of the symmetric group on five elements $S_5$ (although he did miss some). He demonstrated that this expression could not take on certain values and concluded as a result that the general quintic could not be solved by radicals. (Ayoub, 1980; Wussing, 1984)

Ruffini repeatedly tried to gain Lagrange's approval of his book *Teoria* but to no avail. Lagrange, along with Legendre and Lacroix were appointed to a committee that examined the validity of Ruffini's claim. Lagrange himself reported that he "found little in it worthy of attention" and "he had understood nothing." This was presumably untrue according to noted author and scientist Henri Gaultier de Claubry (1792-1878), who said that Lagrange had found *Teoria* to be quite good but did not want to cause unrest in the community by giving his approval to this contentious book he had in fact inspired. (Ayoub, 1980)

## 2.2 Independent Theory of Permutations

### *2.2.1 Augustin-Louis Cauchy*

One of Ruffini's greatest advocates was Baron Augustin-Louis Cauchy (1789-1857) who recognized that Ruffini's *Teoria* contained concepts for permutations that could potentially be manipulated and examined independently of equations. (Ayoub, 1980) The systematic development of a theory of permutations was largely attributed to Cauchy. His crucial role in developing permutation theory can be divided into two periods. The first of which began in 1812 with *Sur les fonctions symétriques* (*On Symmetric functions*, in English*)* that was published in two papers in 1815. (Barnett, 2010; Wussing, 1984) In this early period, he was entirely motivated by algebraic solvability and made it clear that his work was to be an extension of Lagrange, Vandermonde, Waring and Ruffini's work on the permutation of roots (Barnett, 2010; Wussing, 1984).

> Messieurs Lagrange and Vandermonde were, I believe, the first to have considered functions of several variables relative to the number of forms they can assume when one substitutes these variables in place of each other [...] Since then, several Italian mathematicians have productively occupied themselves with this matter, and particularly Monsieur Ruffini [...] One of the most remarkable consequences of the work of these various mathematicians is that, for a given number of letters, it is not always possible to form a function which has a specified number of forms. (Wussing, 1984)

While Lagrange and Ruffini investigated the effect of permutations on algebraic expressions, Cauchy examined the permutations themselves. (Kiernan, 1971) Cauchy proceeded to generalise earlier works through a study on the behaviour of arbitrary functions of $n$ variables. He was interested in rational functions where either the coefficients or roots, $x_1, x_2, \ldots, x_n$, were independent variables. Cauchy, in this early period, still worked with auxiliary equations or resolvents. He would take functions $f(x_1, x_2, \ldots, x_n)$ and search for the number of possible values taken on by these functions under the permutation of its roots. If $f_1, \ldots, f_s$ were the values taken on by $f$ under permutation, then $(t - f_1)(t - f_2)\ldots(t - f_s) = 0$ was his auxiliary equation of degree $s$. He then tried to determine possible values of $s$ and proved two vital theorems. (Waerden, 1985)

Theorem 1: The number of values a non-symmetric function of $n$ quantities cannot be less than the largest odd prime $p$ which divides $n$.

Theorem 2: The order of a system of conjugate substitutions on $n$ variables always divides the number of $N$ arrangements that can be formed from these variables.

(Wussing, 1984)

Cauchy's manipulations essentially defined the symmetric group of $n!$ and its subgroups. He spawned an abstracted perspective and methodology in this field. While Lagrange and Ruffini focused on manipulating specific functions, particularly the quintic, Cauchy saw the importance of pondering properties of arbitrary functions of degree $n$ in an effort to discover general properties of permutations. (Ayoub, 1980)

Cauchy began to think of and treat permutations as independent algebraic objects. His increasingly abstracted work on permutation theory provided insight into the subgroups of permutations and provided calculus for Évariste Galois (1811-1832) that allowed him to carry out his calculations on the Abel–Ruffini theorem; calculations previously described by Lagrange as “so long and complicated that they can discourage the most intrepid calculators.” (Wussing, 1984)

### *2.2.2 Évariste Galois*

The incomparable Évariste Galois, a rebel in every sense of the word, would forever change the face of algebra and create what is now known as Galois Theory. Galois could feel the onset of structural changes in mathematics and employed entirely new methodologies. While in Saint-Pélagie prison in 1831, he wrote down his thoughts on calculi in mathematics:

> “From the beginning of this century, the algorithm has attained such levels of complexity that all progress by this means had become impossible.” (Wussing, 1984)

Galois was concerned with “insight into principles” and found the current trend towards elegance and simplicity to be limiting. (Wussing, 1984) He felt that mathematics was falling prey to style over substance.

The significance of Galois' work lies in his direct associations between groups and equations. (Wussing, 1984) He identified the interesting properties of equations as being reflected in a unique permutation *group* tied to that equation which he proved always exists. In essence, the equation itself could be ignored while the properties of the group were investigated.

It is interesting to note that Abel proved the Abel-Ruffini theorem independently of and prior to Galois yet the solution to Abel-Ruffini theorem is almost always referred to in the context of Galois Theory. (Ayoub, 1980) Fringe mathematician Abel self-published a proof that the general quintic is not solvable by radicals in 1824 and more generalised work was released posthumously as he died of tuberculosis in 1829 while attempting to make his proofs known. (Garding, 1994) However, once Galois' work was understood following his death in 1832, the theory of solvable equations became entirely tied up with the theory of solvable groups. Galois surpassed Abel by developing a system based on the structure of the polynomial that determined which equations could be solved. He established an approach for testing the solvability of a polynomial in direct accordance with the solvability of the corresponding group. While Abel had group theoretic aspects to his proof, he never properly used the concept of solvable groups. (Wussing, 1984; Ayoub, 1980)

I will give a sketch of the basic ideas behind Galois' proof of the Abel-Ruffini theorem. Firstly, Galois was the first to focus on which classes of equations could be solved as determined by the structure of the equation. (Shay, 2010) The important feature was the existence of a chain of subgroups of the Galois group. It was not necessary to consider the permutations as acting on the equation as Ruffini and Abel had, Galois focused instead on the permutation groups themselves. (Ayoub, 1980)

Galois' proof of the Abel-Ruffini theorem is based on the construction of a tower of rational extensions. A finite algebraic extension $E/k$ is called a radical tower over $k$ if there is a series of intermediate fields

$$k = E_0 \subset E_1 \subset \dots E_{m-1} \subset E_m = E$$

such that for each $0 < i < m$, $E_{i+1} = E_i(\sqrt[p_i]{\alpha_i})$ where $p_i$ is a prime and $\alpha_i \in E_i^*$. The equation $f(x) = O$ is solvable by radicals if there is a radical tower $E/k$ such that the splitting field $F \subset E$. (Rosen, 1995)

Here, each field extension in the tower is a radical extension of the previous field so if there is a radical tower $E/k$ then the roots of $f(x) = 0$ can be obtained from the coefficients by the successive use of rational operations and the extraction of roots which is the definition of a solution by radicals in group theoretic terms.

Galois defined the group $G_f$ tied to an equation $f$ "as all permutations that leave invariant all relations among the roots over the field of coefficients of the equation." (Wussing, 1984) By deciding to work with the corresponding group instead of performing exceedingly complex analysis on the equations themselves, he completely bypassed the resolvents of Lagrange, Ruffini, and Abel. (Kleiner, 2007)

Galois key insight which allowed him to succeed in developing a complete theory of equation that employed solely group theoretic techniques was that Galois developed the idea of a normal subgroup. The normal subgroup group leaves any polynomial and all of its conjugates invariant. It is a special kind of subgroup of the symmetric group. (Birkhoff, 1937)

The basis of the Galois Theory proof for the Abel-Ruffini theorem is to define the Galois group corresponding to an algebraic equation of degree $n$, $f(\boldsymbol{x}) = x^n + a_1x^{n-1} + \ldots + a_n = 0$. The first step is to examine the *ground field* of the coefficients: this contains all linear combinations of the coefficients with the rational numbers, or $k = \{b_0 + b_1a_1 + \ldots + b_na_n : b_i \in \mathbb{Q}\, i = 1..n\}$ , so $f(x) \in k[x]$ (Aleksandrov et. al., 1963) (assume $k$ to be a field of characteristic zero and, as Galois showed, there is no loss of generality to assume that $f(x)$ is irreducible see (Jacobson, 1974) for proof)

The next step is to find the *splitting field* $F$ of $f(\boldsymbol{x})$ over $k$. The splitting field is the extension of minimal degree over $k$ such that $f(\boldsymbol{x})$ factors completely in $F$. It is composed of all linear combinations of the roots over the ground field. (Rosen, 1995) The splitting field $F$ of $f(\boldsymbol{x})$ over $k$ is obtained from $k$ by adjoining all the roots of $f(x) = 0$ to the ground field $k$.

i.e. if $f(x) = (x - y_1)(x - y_2)(x - y_3)(x - y_4)(x - y_5)$ then the splitting field of $f(x)$ over $k$ is $F = k(y_1, y_2, \ldots, y_5)$ .

To build up the splitting field of $f(x)$ (without loss of generality,) take $y_1$ to be transcendental over the field $k$ and let $y_2$ be transcendental over $k(y_1)$, ($k$ adjoin $y_1$ or all linear combinations of $k$ and $y_1$) and so on to $y_5$ which is transcendental over $k(y_1, y_2, \ldots, y_4)$.

The first step in showing that equations of degree five and higher are not (in general) solvable by radicals is to compute the *Galois group* for the general equation of degree $n$. The

Galois group is the set of all automorphisms of the splitting field $F$ that maps $F$ onto itself relative to $k$. (Aleksandrov et. al., 1963)

Then it must be shown that the Galois group is isomorphic to $S_n$: the symmetric group on $n$ elements. To show that the symmetric group $S_5$ is isomorphic to the Galois group $G(F/k)$ for $f(x) = (x - y_1)(x - y_2)(x - y_3)(x - y_4)(x - y_5)$ (this can easily be generalised for a general polynomial of degree $n$), take any permutation $\sigma$ from the symmetric group $S_5$. This permutation creates an automorphism $\sigma'$ on $F$ that leaves $k$ fixed and permutes the elements $y_n$. For example, if $\sigma$ is the element of $S_5$ which carries out the permutation

$$y_1 \to y_2,\ y_2 \to y_3,\ \ y_3 \to y_1, y_4 \to y_5, y_5 \to y_4$$

$((23154)$ in standard $n$ −cycle notation) this induces the automorphism $\sigma'\left(f(\boldsymbol{x})\right)$ which carries out σ on $f(x)$ and so it transforms

$$(x - y_1)(x - y_2)(x - y_3)(x - y_4)(x - y_5) \to (x - y_3)(x - y_1)(x - y_2)(x - y_5)(x - y_4)$$

which is clearly the same polynomial so σ′ is an element of the Galois group $G(F/k)$ by definition of the Galois group. As $\sigma$ is an arbitrary element, all elements of $S_5$ are in $G(F/k)$

Now, since $|S_5| = 5!$, we know that $|G(F/k)\ | \geq 5!$ since all elements of $S_5$ are contained in the Galois group but there could also be automorphisms in $G(F/k)$ that are not in $S_5$. However, we can see from the way we built up the splitting field that the number of automorphisms contained in $F$ relative to $k$ for a quintic polynomial is at most 5! so $|G(F/k)| = 5!$ and so $G(F/k)$ must be isomorphic to $S_5$ as it contains all elements $S_5$ and has the same cardinality. (Rosen, 1995)

To show that $S_n$ is not a solvable group when $n \geq 5$ (For proof see (Rosen, 1995))

This gives a basic sketch of how Galois solved the Abel-Ruffini theorem and the ability to examine the roots of polynomials by examining the symmetric group and using concepts of congruence and invariance. His use of group theoretic thinking gave great insight into our general knowledge of polynomials.

Galois' life was tragically cut short by a duel at age 20 on May 31$^{st}$ 1832. Much the work he wished to accomplish was incomplete. He threw down the gauntlet to those such as Jacobi and Gauss to pick up where he left off: to finish his proofs and unearth their substance. (Wussing, 1984)

It took some time before Galois' work was understood and assimilated. (Barnett, 2010) While dated January 16th 1831, his *Memoire* was not published until 1846 by mathematician Joseph Liouville (1809-1882). (Wussing, 1984) Galois rejected that he was part of a mathematical tradition and tried to create his own work on the fringes of the society. His writings were highly aphoristic and he is now acknowledged as being a man ahead of his time. Combined with the fact that Galois is notorious for his confusing scribbled manuscripts which were exceedingly difficult to read meant that few could understand his work and so the dissemination of the Galois Theory into the mainstream was a slow process.

A page from Galois' manuscript

### *2.2.3 Cauchy after Galois*

Cauchy's familiarity with Galois' work was apparent in his second stage of developing group theory, beginning in 1844 with *Memoire on the arrangements.* (Wussing, 1984) This work portrayed a post-Galois permutation theory. From 1815 to 1844, Cauchy started to look at permutations with increasing abstraction and his *Memoire* made no mention of polynomial equations. He was now focused on the independent development of the algebraic properties of permutations. (Barnett, 2010; Wussing, 1984)

He defined permutations (or substitutions) as independent operations and created notations that would allow them to be thought of and worked with independently of equations.

He took $n$ independent variables $x_1, x_2, \dots, x_n$ which did not necessarily have to be roots or coefficients of a polynomial, and ordered the place of each variable. He then wrote the variables in the assigned order to create an arrangement $x_1\, x_2 x_3 \dots$ . If the order of the variables changes then the arrangement was replaced by another which could be compared to the first. Cauchy extracted the concept of permuting the roots of a polynomial and generalised it to represent the order of any arrangement of variables. He introduced much of the notation we use today, including the *n*-cycle notation for the symmetric groups, the product of permutations, and the identity permutation. (Wussing, 1984)

With his notation the concept of permutations could now be transferred to other disciplines such as Jordan's work in geometry and physics. (Barnett, 2010; Wussing, 1984)

### *2.2.3 Joseph Alfred Serret*

The works of Galois and Cauchy also had a great impact on the independent theory of permutations. During the early 19th century the rapidly growing theory of permutations and the theory of equation were diverging. Galois and Abel's successful group theoretic proofs which classified the equations solvable by radicals, highlighted the significance of permutations and reunified them with the solvability of algebraic equations. (Wussing, 1984; Barnett, 2010)

The abstraction of permutation groups ran in parallel to an independent theory of permutations that began in the 1840s and acquired great international support. This was reflected in Serret's *Cours d'Algèbre supérieure* (3rd edition) of 1866, which became the definitive text on algebra at that time. (Wussing, 1984) *Cours* was framed in a Cauchian permutation theory and employed his notations, definitions and language. Serret had attended Liouville's seminars on Galois Theory and included a full account of this theory along with proofs of several of Galois' main theorems. (Waerden, 1985)

Serret organised, clarified and amended Galois' fundamental ideas. The language was not strictly group theoretic and amalgamated analysis, number theory and permutation theory.

(Wussing, 1984) Serret was instrumental in pushing permutation theory outside of the realm of equations and created influential notations such as operators and the conjugate subgroup. (Waerden, 1985; Wussing, 1984)

By extending the concept of the permutation group, Serret developed the abstract equivalent of the finite group of linear substitutions which prove extremely useful for geometers. Groups are excellent for solving problems relating to linear equations and they can even be thought of as the minimal structures needed for the solution of linear equations. (Shay, 2010)

Serret began his work with linear function substitutions of the form $\frac{axb}{a'x+b'}$ and then defined $\theta x = ax + \frac{b}{a'}x + b$ and defined it again by the recursion $\theta^2 x = \theta\theta x$, ..., $\theta^m x = \theta\theta^{m-1} x$.... In doing so he proved that the set of all linear functions form a group. (Waerden, 1985; Wussing, 1984)

## 2.3 The adaptation of group theory in Geometry

### *2.3.1 Marie Ennemond Camille Jordan*

Jordan took full advantage of group theory now being applicable outside of algebra. He experimented with a geometric group theory that was removed from its raw, strictly permutation based state. (Wussing, 1984) He had a grand vision for group theory which he shared with a number of mathematicians of that time such asKlein. He saw the group concept as the grand unifier of mathematics and applied it to disciplines such as algebraic geometry, transcendental functions, and theoretical mechanics. His approach enabled him to give a unified presentation of the works of Galois and Cauchy. (Wussing, 1984; Waerden, 1985) For example, he expanded Cauchy's proof of theorem 1 (Section 2.2.1) to state that the order of any subgroup of a finite group divides the order of the group. (Kiernan, 1971) Jordan stated in the preface to *Traité des substitutions et des équations algébriques* that "the aim of the work is to develop Galois' method and to make it a proper field of study, by showing with what facility it can solve all principal problems of the theory of equations." (Wussing, 1984)

In 1870, Jordan accrued all applications of permutations he could find such as algebraic geometry, number theory and function theory under the banner of group theory (Barnett, 2010; Aleksandrov et. al., 1963) along with all that was known of group theory at that time. (Kiernan, 1971) His aim was to survey all areas of mathematics where the theory of permutation groups had been applied or seemed likely to be applicable along with the revolutionary work he himself had achieved.

Jordan's mentality was reflective of the times. The industrial revolution in Europe had led to an increased need for connections between mathematics and the sciences. Physics, particularly mechanics, were of great importance. (Wussing, 1984) The notion of *congruence*, an idea already seen in Cauchy's work on permutations and highlighted in Galois' work, (Kleiner, 2007) helped simplify motion and therefore all geometry, to four groups of axioms and this in turn eased its practical use. (Wussing, 1984) Those such as Georg Friedrich Bernhard

Riemann (1826 –1866) and Hermann Ludwig Ferdinand von Helmholtz (1821 –1894), who published several papers on the subject between 1866 and 1870, brought a great deal of attention to the axiomatization of geometry. There were deep structural changes on how geometry should be perceived. It was slowly being redefined as the "possible motions of physical bodies". (Wussing, 1984)

Jordan and Klein felt that this new motion-geometric conception could best be described by groups representing motions. (Wussing, 1984) In his *Traité* of 1870, Jordan claimed that all motions of a solid in space can be reduced to twists which are completely determined by

a. The position in space of the axis $A$ of the twist
b. The angle r of rotation about $A$
c. The magnitude of the displacement $t$ along $A$ (Wussing, 1984)

Jordan denoted all motion, or twists, in space by $A_{r,t}$. He described, what would now be known as, the generators of a group by any given set of motions $A_{r_1,t_1}, A_{r_2,t_2},\ldots$ which he called "motions that serve as a point of departure." A group would then be what was formed by executing any number of these motions any number of times in any order. He defined groups as having the characteristic property: If $M_1$ and $M_2$ are two motions in this group then $M_1M_2$, the two motions done successively, also belong to the group. (Wussing, 1984) Jordan's motion group is similar to our modern abstract definition but only requires elements to be closed under the group's operator; however his group is strictly meant to represent actual motions of physical bodies through space.

*Traite* embodied most of Jordan's publications on groups up to that time. It directed attention to a large number of difficult problems and introduced many fundamental concepts. He explicitly detailed the notions of isomorphism and homomorphism for substitution groups. He undertook a very thorough study of what he believed were the fundamental concepts of group theory: Cauchy's transitivity, Gauss and Abel's primitivity and Galois' simple and composite groups. (Wussing, 1984) Many of his results have not since been superseded.

Much of Jordan's work is seen in classrooms today and is based on simple visual concepts. For example, he attempted to find all closed groups of motion, $G$, by combining rotations, $R$ with translations, $T$. Here we find the concept of the kernel in the isomorphisms of motion groups. For example $R = G/T$ (Waerden, 1985) states that translations don't rotate hence they do not cause any change in the group of rotation.

Equivalence classes were depicted as well, for example, examining all of the translations of a given rotation or vice versa. Geometry is rarely seen in group theory courses, yet much of the motivation behind developing a sophisticated abstract group theory was geometric.

Jordan, for his part, rejected and feared the path to abstraction and axiomatisation which he felt divests groups of their concrete nature. In the introduction to *Traité,* he dismissed this methodology, particularly that of Leopold Kronecker (1823-1891) of whom he said:

> "We would have liked to present a larger part than we have of the work on the equations of this illustrious author. Various causes have prevented us from so

doing: the totally arithmetic nature of his methods, so different from our own; the difficulty of having completely to reconstitute a number of proofs most often barely indicated; and finally the hope of seeing one day these beautiful theorems, which are now the envy and despair of geometers, grouped in one body of coherent and complete doctrine." (Wussing, 1984)

Kronecker upheld the constructionist viewpoint and insisted that all mathematics should consist of constructive proofs consisting of a finite number of steps. (Kiernan, 1971) This was his answer to the much needed rigor in mathematics which previously relied on general statements and half proofs.

### *2.3.2 Felix Klein and the Erlangen Program*

Jordan made a distinct impression on Klein who went to join Jordan along with good friend Sophus Lie (1842-1899) in Paris following the publication of *Traité*. (Waerden, 1985) Lie and Klein attempted to expand the significance of group theory in the vein of Jordan's hope to unify mathematics. They correlated Jordan's closed systems of motions, or transformation groups, to groups of substitutions.

In 1872, Klein created the *Erlangen Program* in an attempt to classify geometry through the conception of transformation groups as he defined them. "The combination of any number of transformations of space is always equivalent to a single transformation. If now a given system of transformations has the property that any transformation obtained by combining any transformations of the system belongs to that system, it shall be called a group of transformation.” (Klein, 1893)

While Klein enjoyed the elegance of framing geometry in the group theoretic framework of symmetry, he rejected and feared the outcome of the abstract viewpoint in group theory, even though he was instrumental in its formation. Much of his work was lacking in elegant abstraction and resembled the definition given above.

He was highly motivated by projective geometry, manifolds and worked with a myriad of now sophisticated elements of group theory such as the projective group, the group of rigid motions, the group of similarities, the hyperbolic group, the elliptic groups, as well as the geometries associated with them. (Kleiner, 2007) Invariance was a major contributor to the importance and development of this stage of group theory. (Klein, 1893) The Erlangen program claimed that any geometry was dependent on invariant properties under a certain group action. (Kisil, 2007) He showed that geometry could also be characterized by the groups of transformations that leave their fundamental relations invariant and this was true of topology, differential geometry, projective geometry, affine geometry, geometry of inversion. (Birkhoff, 1937) While the Erlangen program itself was largely conceptual, it evolved into a highly praised framework under which geometry could be understood in group theoretic terms and vice versa. For example if we take the group $SL_2(\boldsymbol{R})$ of all $2x2$ matrices of real entries this can be perceived geometrically as a map of complex numbers $z = x + iy$. (Kisil, 2007)

Klein’s explicit use of groups in geometry spawned a conceptual shift in group theoretic thought away from permutations. (Wussing, 1984) Klein wanted to develop a parallel body that defined a theory of transformations based on transformation groups akin to the system developed for permutations. (Kleiner, 2007)

Jordan and Klein, along with others such as Sir William Rowan Hamilton (1805 –1865), were hugely successful with their group theoretic methods of reducing mathematics to describing motion groups. (Wussing, 198) Motion geometry, where motions or transformations of geometric objects are the elements of a group, became its own rewarding domain. (Kleiner, 2007) A general climate of returning to axiomatisation and classification would aid their success in extending the permutation group concept to a theory of transformation groups in the 1870's and 1880's.

Klein's work considerably broadened the conception of a group and its applicability in other fields of mathematics. He did much to promote the viewpoint that group theoretic ideas are fundamental in mathematics. (Kleiner, 2007) One of Klein's great interests was examining the *isometries* of regular polyhedral. He created the set of allowable rotations and explicated all details on the isometries of the cube, octahedron and pentagonal dodecahedron. He found research in this area so rewarding that he exclaimed he "struck a vein of ore." (Wussing, 1984) Klein "solved" the quintic equation with the symmetry group of the icosahedrons in his *Lectures on the Icosahedron* in1884. He discovered deep connections between the groups of rotations of the regular solids and polynomial equations. (Kleiner, 2007) He discovered that the group of rotation of the regular icosahedron is isomorphic to the alternating group of permutations on five elements. Klein's observations of isometries went far beyond previous concepts of congruence. They can be perceived as a true reflection of the nature and difficulties of modern group theory where we examine relationships rather than the elements themselves. (Wussing, 1984)

## 2.4 The abstraction of group theory

The efforts of these geometers had led to an intricate and widespread useful body of group theoretic thought that was derived from and representative of specific mathematical objects. This approach was so useful, a drive developed to generalise these ideas and results so they could be applied in any discipline of mathematics. This notion of an abstracted and generalised group theory was heavily resisted in the 19th century. This type of abstract mathematics was still quite novel (to be discussed in Chapter 3) but would achieve enormous success due to its utility and general changes in certain mathematical circles. (Aleksandrov et. al., 1963; Wussing, 1984)

The abstraction of the group concept is attributed to British mathematician Arthur Cayley (1821 - 1895) (although some sources will credit Walther Franz Anton von Dyck (1856 - 1934) who was instrumental in its development). In 1854, Cayley presented a paper on abstract group theory that has since been recognized as the foundational paper in abstract group theory but the climate was not yet ready and it was generally ignored. It did not contain “concrete representations accepted in mathematical practice.” (Wussing, 1984) As Kline put it “[p]remature abstraction falls on deaf ears, whether they belong to mathematicians or to students.”

Cayley had carefully studied Cauchy's work, and was exceedingly familiar with his ideas. He was also well versed in the symbolic algebra that dominated British mathematics at the time. By drawing on his understanding of both areas, Cayley became the first to construct a definition of what is the modern form of an abstract group.

> "A set of symbols all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a group. These symbols are not in general convertible [commutative], but are associative." (Kleiner, 2007)

Changes in English society resulting from the Industrial Revolution, and the role of universities in promoting research and science ignited the English mathematical community which had been somewhat stagnant ever since the controversy concerning the invention of calculus by Leibniz and Newton. The Analytic Society, founded at Cambridge in 1812, began to implement symbolic manipulation in their university curriculum. There were deep philosophical concerns about the foundations of algebra and symbolic algebra was their response to these concerns. (Barnett, 2010)

Symbolic algebra begins with formal laws on a set of symbols and performs operations on them. These symbols may then be used as a representation of any number of concepts. Until this time, doing algebra consisted in first taking mathematical objects and then use them to derive laws, while symbolic algebra does just the opposite.(as detailed in Chapter 3) (Barnett, 2010)

Many of the properties created by Cayley were familiar to Cauchy's theory of permutations. Though his process is far more general, Cayley's idea of the group fit very well with Cauchy's. For example, Cayley was deeply concerned with the general problem of finding all the groups of any given order $n$. He proved what is now known as Cayley's theorem: every group $G$ is isomorphic to a subgroup of the symmetric group acting on $G$. His search was in fact the same as constructing all subgroups of permutations. (Wussing, 1984)

Cayley's definition of a group was not different or contradictory to any of the work that preceded him. This can be seen in the different definitions of groups. They are abstractly the same but, at the time, they represented different concrete extrapolations specific to a given discipline in mathematics. Cayley's groups, like symbolic algebra, did not represent anything, and could be applied to almost anything.

Eventually the abstract group concept began to take hold. In 1878, Cayley returned to his abstracted theory of 1854. He was firm is his belief that the best approach to group theory was to consider a general problem in abstract symbolism and then deduce theories of specific groups from it. (Kleiner, 2007) Many mathematicians agreed with Cayley's vision of a reformulated and abstractly defined group theory. For example, von Dyck saw a group as "discrete operations that are applied to a certain object, while ignoring the particular form of representation of the individual operations, regarding them as given only in terms of the properties essential for group building."(Wussing, 1984)

The abstract group concept spread rapidly during the 1880s and 1890s. There was still extensive work being done on permutation and transformation groups but the abstract viewpoint was creeping into existing bodies of mathematics. New ideas and results were now being formulated in an abstracted form and new studies emerged that were based on abstraction. (Barnett, 2010, Kleiner, 2007)

By the 1880s, the evolution of the abstraction of group theory was complete but its acceptance was not and it was not noticeable in papers, textbooks, monographs and lectures. (Wussing, 1984) This would not occur until the beginning of the 20$^{th}$ century. In 1904 , J.-A Séguire published his *Eléments*, the first monograph on abstract group theory, in an attempt to remove himself from the tradition of permutation and transformation group that was still dominant. He characterised his new group theory as "many past investigations from different areas have been combined into a more general theory that has not ceased to develop." By the 1920s this abstraction had taken hold and would guide the future axiomatisation of mathematics for those such as David Hilbert (1862-1943). (Wussing, 1984)

After over a hundred year of examining the practicality of using group theoretic thought, a complete abstract transformation took place and now would describe the basics of a group and the axioms defining them. These four axioms largely came from the works of Klein and Lie. They "elaborated the significance of group theory for different areas of mathematics." They had generalised how a "group is a class of unique operations *A*, *B*, *C*, ... such that the combination of any two operations *A*, *B* again yields an operation *C* of the class $A*B = C$." (Wussing, 1984) Lie then found it necessary to include inverses. However Klein and Lie had great trepidations over what they felt was a watered down, “far paler but more precise definition” of group theory which perhaps does not delve into the crux of group theoretic thought. (Wussing, 1984) Klein also addressed how abstract group theory would be learned by students:

> “It makes matters far more difficult for the mind of the student, for he confronts something closed, does not know how one arrives at these definitions, and can imagine absolutely nothing. In general the disadvantage of the method is that it fails to encourage thought. All one must beware of is that one does not violate the four commandments” (Wussing, 1984)

# Chapter 3

# The Development of Modern Algebra

The 20th century saw the acceptance of an entirely novel notion of abstraction in algebra which has since been incorporated into group theory based on the work of Cayley and von Dyck. In this chapter I aim to describe how and why modern algebra emerged and how it differs from earlier forms of algebra. In this way I hope to set a context for discussing how the concepts of group theory detailed in Chapter 2 are thought of today in their abstract form and create a framework for analysing students understanding of modern algebra in their group theory courses. In particular, in Chapter 4 I contrast this notion of abstract with the definitions for abstraction seen in the literature reviewed for the study. In Chapter 5 I use modern abstraction in detailing the representation of question and testing students' understanding of how to interpret group theory questions

Historically, the term "modern algebra" came from Bartel Leendert van der Waerden (1903-1996) who penned three volumes under that name. (Pratt, 2013) While modern algebra is generally interchangeable with the abstract algebra, I will contrast modern algebra with classical algebra in term of three defining characteristics of the "algebraic mode of thought" (Mahoney, 1980)

I will highlight the differences between the modern algebra and classical algebra in terms of three areas:

1) The types of **symbolism** employed in algebra, namely: *rhetorical, syncopated* and *symbolic* algebras
2) What constitutes an appropriate **justification** in algebra, contrasting the *physical intuitive* mathematics to abstract *axiomatic* systems
3) How algebra changed from being a discipline that studies **objects** to a domain that analyzes generalised **relationships and structure**.

While developments in all three areas are all intertwined I will do my best to distinguish them.

## 3.1 Symbolism

The most common categorisations for algebraic symbolism stem from Georg Heinrich Ferdinand Nesselmann's (1811 – 1881) 1842 study on *Greek Algebra.* These are:

1. *Rhetorical algebra*: **no abbreviations or symbols**, all equations are written out in full sentences. It is process for determining an unknown quantity through **logical reasoning.**

2. *Syncopated algebra*: symbols are merely **abbreviations** and are substituted in for common operations and quantities. Solutions for unknown quantities are still **reasoned out logically** in words. Symbols should have a context and represent a geometric or numerical quantity that can be represented in the physical universe.

3. *Symbolic algebra*: Almost all steps are written out in mathematical **symbols**. Solutions are found by manipulating and performing **operations on symbols**. These symbols and operators do not require context and they do not represent anything in particular but are

defined entirely by laws of combination. This is the modern form of symbolism. (Heeffer, 2008;Heeffer, 2009; Sfard and Linchevski, 1994)

### *3.1.1 Rhetorical algebra*

As pointed out in Chapter 2, algebra was historically the study of solving equations. Derived from the Arabic science of *al-jabr,* its evolutionary path has been traced to ancient Babylonian, Egyptian, Indian, Arabic and Greek traditions as a discipline that poses a question and provides a method for determining an unknown. (Heeffer, 2009; Goddijn, 2011)

The period of rhetoric algebra stretches from antiquity to the 16th century and includes traditions such as Arabic algebra, Italian abacus algebra, Iamblichus, and Regiomontanus. Rhetorical algebra consisted of full sentences that expressed ideas and calculations and completely lacked any form of symbolism except for the representations of numbers. (Heeffer, 2009) Without symbols, all there was to algebra was mental arithmetic and all processes were carried out logically. There were no operators, only word problems involving numbers. In many ways, algebra tended to be more of companion for geometry than a discipline on its own and it is difficult to discern what discovered writings truly are rhetorical algebra or merely descriptions of geometric constructions. (Heeffer, 2009)

Example of Rhetorical Algebra

Problem: (Al-Khwarizm circa 825 C.E.) What is the square which combined with ten of its roots will give a sum total of 39?

Solution: … take one-half of roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course, is 9. (Sfard and Linchevski, 1994)

### *3.1.2 Syncopated Algebra*

The first appearance of algebraic symbolism, and the onset of Nesselman's so-called syncopated algebra, is generally attributed to the *father of algebra*, Diophantus of Alexandria, in his *Aritmetic* (circa 250 C.E.), although it is believed that he may have been predated in this process. (Heeffer, 2008; Mahoney, 2003)

Diophantus created individual symbols to represent an unknown quantity; $\varsigma$ symbolised an unknown, $\Delta\upsilon$ was for squares of unknowns and $K^{\upsilon}$ for cubes of unknowns (the equivalent of $x, x^2, x^3$ in modern notation). These representations allowed Diophantus to treat unknowns as known quantities in order to proceed with his calculations through assumptions on the type of unknown. (Mahoney, 1980; Heeffer, 2008) Diophantus employed symbols simply as abbreviations and reasoned out his results by combining them with words. His symbolic system only allowed for one unknown in a given statement and there could not be equations with two or more unknowns. The symbols of that time had nowhere near the sophistication and abstraction of modern algebra. They were essentially time saving devices and did not carry the significance that our symbols do today. (Heeffer, 2009)

Radford suggests that syncopated symbolism can be compared to a "'transitional' language prior to the standard alpha-numeric-based algebraic language."(Radford, 2000) A Diophantus type expression such as *"for what ς does ς and ς combined with 2 give 8?"* is the equivalent to asking a student to answer a question such as *find '*' in the equations: * + * + 2 = 8* which is common practice according to the new Ontario Curriculum of Mathematics. (Radford, 2000) It represents an intermediate step that acclimatises students to symbolism. In these types of problem a student is supposed to rationalise in their mind what possible values of '*' would satisfy the equation. In this example, the symbols are meant to represent natural numbers and the answer is found by making use of knowledge pertaining to the natural numbers. In this type of syncopated algebra, there is no manipulation of symbols; they are essentially a placeholder for an unknown quantity, the value of which is discerned mentally.

Algebraists struggled with the meaning of symbolism and its place in algebra from the end of the $14^{th}$ century to the end of the $16^{th}$ century. The mathematical symbols we are familiar with today only began to appear towards the end of antiquity. Even the use of one of our most common and simplistic symbols only began to surface in print in the $15^{th}$ century; the plus sign made its first appearance in an arithmetic context in Johannes Widmann's Mercantile Arithmetic of 1489.

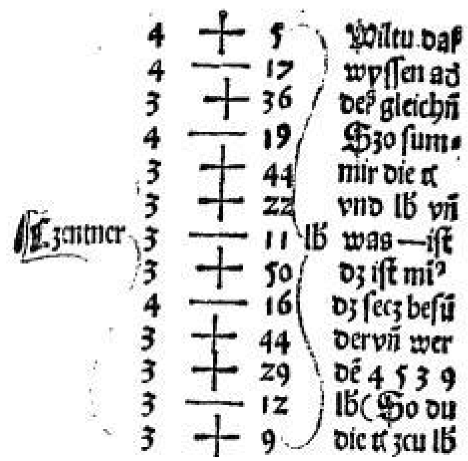

**Figure 1.** The plus sign as it appeared in J. Widmann's Mercantile Arithmetic in 1489. (Heeffer, 2009)

The symbol for the plus sign comes from the Latin word *et* meaning "and"; the symbols might be an abbreviation supposedly used in the Latin based German cossic tradition. (Heeffer, 2009) This likely implies that it was not a symbolic notation but merely a shorthand notation. The first 15$^{th}$ century arithmetic + sign appeared in the context of "3 + 5 makes 8." It predated the equal sign which made its first appearance in 1557.

As speculated by Heeffer (2009), this plus sign represents a mental or physical action and is therefore not a modern symbol. In an action, there is a temporal component that precludes it from being symbolic or abstract in the modern sense. Heeffer writes that the manner this addition is described should be interpreted as "First you have three; after adding five, you find out that you have eight." (Heeffer, 2009) In a syncopated symbolism the expression "3 + 5 = 8" is simply an un-interpreted calculus, – a set of symbols not yet attached to physical quantities. Widmann's syncopated "3 + 5 makes 8" refers to the fact that there are 8 total physical entities present as discerned by a temporal action which is presupposed to combine quantities so it is not a modern symbol. Syncopated algebra still abides by a philosophy of physical quantities and values. The modern symbols in *symbolic algebra* are dependent on a very different philosophy of symbolic reasoning and we shall see how the modern "3 + 5 = 8" is to be interpreted.

### *3.1.3 Symbolic Algebra*

Symbolism was not readily embraced by the mathematical community. The symbolic style of mathematics in general, and algebra in particular, was considered "lower" mathematics simply meeting the demands set forth by the growth of sedentary mercantilism and educational systems. Widmann did not use the plus sign in his arithmetic chapters on the basic operations of addition and subtraction but only in reference to mercantile problems. (Heeffer, 2008) This kind of algebra was considered a more direct and simplistic method than the rigours of Greek synthesis; well suited for the pedagogy of the expanding 16th and 17th centuries school and university system. (Mahoney, 1980) The 16th century symbolic accomplishments, however, were generally not adopted in 17th century mathematical textbooks. (Heeffer, 2008)

As mentioned in Chapter 2, *symbolic algebra* was largely a product of 19th century British mathematics in an attempt to resolve philosophical crises in the mathematical community. (While symbolic work was also coming out of continental Europe, particularly German, at the time the *symbolic algebra* movement came from Britain.) A great rift was caused by the independent creation of calculus by Sir Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) which became matters of national pride and crippled British mathematical advancement until the 19th century. (Ball, 1908)

Both methods of calculus established a critical inverse relationship between algebra and geometry and aided in the development of symbolic algebra but these aspects were exceedingly more predominant in Leibniz's work. (Mahoney, 2003) There was also great philosophical divide at this time between the continental rationalists and British empiricists (see section 3.2). Perhaps in line with the British philosophical tradition at that time and likely because Newton was more a scientist than mathematician, he developed his calculus primarily with regards to physics while continental philosopher Leibniz turned his head towards the works of the new emerging symbolic mathematics. (Struik, 1986) Leibniz thoroughly outlined the importance of symbolism in all aspects of his work and there was a distinct advantage in Leibniz's differentials over Newton's fluxions and motions. (Mahoney, 2003) Leibniz saw the vital importance of developing a meticulous notation system, most of which is still in use; such as the stylized I or $\int$ for integration and $d$ for differentiation. (O'Connor, 1996)

In 1715 the Royal Society decreed that Newton had invented calculus and Leibniz had plagiarised him. England refused to acknowledge Leibniz's methods or any mathematics from other country until 1820. (Ball, 1908) This was of course devastating to the British mathematical community that suffered along with Newton's cumbersome fluxions but would come back with a vengeance in the 19th century embracing the Leibnizian calculus and the heuristic power of a symbolic algebra.

The symbolic algebra movement of the 19th and 20th centuries was marked by a desire to develop a streamlined and universal mathematics. Through the calculus schism, the lesson had been learned on the importance of notation and this had become a new focus for British mathematicians who fought hard for the use of Leibniz's calculus in England and sought to develop a well-organized mathematics of pure symbolism and logic.

This effort was rooted in the so-called Cambridge Network which strived to reform British mathematics and was linked to the Analytical Society founded in 1812 by Charles Babbage (1791–1871), George Peacock (1791–1858) and John Herschel (1792–1871). (Peckhaus, 2003) One of the first accomplishments of the Analytical Society was to adopt Leibniz's notation for calculus. The great success of this change in notation is likely what prompted a newfound interest in symbolism and the development of symbolical algebra. (Guicciardini, 2004)

The works of the Analytical Society were generalised by Duncan Gregory's (1813–1844) "calculus of operations" which aimed to define symbolic algebra as "the science which treats of the combination of operations defined not by their nature, that is by what they are or what they do, but by the laws of combinations to which they are subject." (Peckhaus, 2003) This would become one of the defining principles of modern algebra where modern symbols are defined purely by their "laws of combination." While a syncopated symbol is known to a be a quantity and its value is found by rationalising about the nature of the object the symbol represents, a modern symbol has no preconceptions attached to it; it can represent nothing in particular or absolutely anything. What it represents or if it represents anything at all is irrelevant, the only relevant information is how it is combined and manipulated in accordance with the laws that govern the symbol. The new symbolism being developed was a universal language of analysis which "identified with a concept-less calculational technique." (Hopkins, 2008)

Gregory set forth to develop a symbolic system that was in no way representational of or tied to a specific system. He gave the example of the symbols $a$ and + $a$ which are isomorphic in arithmetic but not in geometry. In a Cartesian system the symbol $a$ may refer to a point on a line while the symbol + $a$ could refer to say the direction of a line. (Peckhaus, 2003) The desired symbolic algebra would not have these types of contextual symbols which take on different meanings in different fields of study.

Gregory believed that mathematics should be purged of all general science symbols as these symbols already have meaning ascribed to them by particular sciences. (Peckhaus, 2003) These symbols lend themselves to preconceived notions relating to the nature of the object the symbol represents and a modern symbol should create absolutely no bias in the mind of the interpreter as to whether it represents an entity in the sciences such as friction, velocity or pressure.

Through the work of Gregory, George Boole (1815-1864) and their successors, algebra entered a paradigm shift based on a pure symbolism that held no connections or connotations to any other discipline. These modern symbols are now commonly seen in group theory. For example, if we examine a set $G = \{a, b, c\}$, these elements should create no bias in the mind as to whether they represent a particular object. This set can be renamed to contain elements $\{x_1, x_2, x_3\}$, {⬡, □, ⌓} or {🕮, 😐, ✈} and still be the same set. There should be absolutely no preconceptions or context inherent to these symbols. This is in stark contrast to previous forms of algebra in which the symbol was always known to represent a certain type of object and then the solution based on knowledge of the type of object the symbol represents.

In line with Gregory's "calculus of operations", these symbols represent "the workings of the combinatory operations." (Mahoney, 1980) The set $\{a, b, c\}$ can be defined by an operation* which states laws of combination, i.e.;

| * | a | b | c |
|---|---|---|---|
| a | a | b | c |
| b | b | c | a |
| c | c | a | b |

The elements of the set $\{a, b, c\}$ are now manipulated independent of context, according to the laws of combination of the binary operation *. All statements concerning these elements are true as long as they are consistent with the laws. This is representative of a modern form of *justification*.

## 3.2 Justification

A vital aspect of mathematics is the need to justify all statements. The manner in which we justify what is allowable and “true” in algebra has changed drastically in the lead up into modern mathematics.

In this section, I will explore the following three stages in the development of justifications of algebra:

1) A strict physical ontology justified by geometric constructions and Platonism,
2) The introduction of non-intuitive entities and the abandoning of Platonism, and
3) Modern logical justifications based on consistency in axiomatic systems.

### ***3.2.1 Physical Ontology and Platonism***

In ancient Greece, philosophy dictated what was allowable in mathematics and how a mathematical statement could be justified. In antiquity, philosophies on abstraction were of supreme importance and abstractions or *universals* were taken to be existent objects. Mathematics uncovered the inherent or *universal* properties which were the *essence* of what existed in the tangible universe. It was a commonly held belief that mathematics is what formed the concrete, that “nature is made up of numbers,” and that mathematics was the science of discovering what was *real*. (Avigad, 2007; Pap, 1957)

The philosophical abstraction of mathematics is what made it true, real and applicable. (Risteski, 2008) As mathematics was a discipline that described real entities, it had to be restricted accordingly in order to be justified. The philosophers of the time dictated what was allowable in the applied component of mathematics. (Klein, 1967; Mahoney, 1980)

Mathematics was contained in a universe of *quantity* with two classes of objects. The first was that of continuous magnitudes; physical shapes such as lines, areas and solids that were restricted to what was constructible in Euclidean geometry (by means of straight-edge and compasses), and no more than three dimensions could exist. (Klein, 1967; Mahoney, 1980; Heeffer, 2009)

The second class contained discrete numbers or collections of units corresponding to the natural numbers which represented quantities. Numbers were imagined to be pure units, "presupposed irreducible quantities" and therefore could not operate with fractional parts. (Klein, 1967; Mahoney, 1980)

This view of mathematics is that of an intuitive science dominated by a physical ontology that could only examine quantities and geometrical forms. All procedures and entities must have bearing and relevance to the "real" world. (Avigad, 2007)

While many traditional philosophers, such as William of Ockham (c. 1285–1349) and his followers in medieval philosophy and British empiricists since Thomas Hobbes (1588–1679), have tried to discredit Platonism as a belief that has "no better rational foundation than a belief in ghosts and fairies" (Pap, 1957), overcoming Platonism was exceptionally difficult. Platonism was the unquestionably dominant philosophical position in the ancient world for a period of more than 800 years. (Gerson, 2004) It is difficult to consider any other philosophy which had such a dominating influence and there will perhaps never again be such unanimity on the ontology of mathematics as when it fell under the realm of Platonic philosophy.

The need to abandon a strictly physical ontology for mathematical constructions and solutions became evident in the $16^{th}$ century. This type of thinking was somewhat forced upon the mathematical community when they encountered entities that were disastrous philosophically but invaluable mathematically, notably negative and imaginary numbers.

### *3.2.2 Non-intuitive objects and the abandoning of Platonism*

Imaginary roots, or algebra which contains $\sqrt{-1}$, surfaced in the algebraic work on quadratics and cubics. The complex numbers did not abide by the standards on allowable mathematical objects of the time. They could not be properly represented geometrically and they were not considered to be quantities in Platonic philosophy.

Rafael Bombelli (1526–1572) was the first to state explicit rules for the multiplication of imaginary numbers. In his 1572 work *Algebra* he refers to *più di meno* (plus than minus, in English) or $\sqrt{-1}$ in modern terminology (note that his work predated the common use of symbolism so his expressions are all rhetorical). Bombelli tried desperately to construct these objects geometrically for if the root of a particular cubic equation that could only be found by using imaginary numbers could be constructed geometrically then imaginary numbers could be tied into the mathematics of quantities and be legitimated. (Bagni, 2009) He attempted geometric constructions of these roots (see Figure 2) drawing on the Arabic tradition of Al-Khwarizm but never produced a proper geometric proof. He could not manage to fit imaginary numbers into the mathematics of quantities and so he stated that his expression for $\sqrt{-1}$ was "in reference to quantities whose meaning was still unclear." (Bagni, 2009)

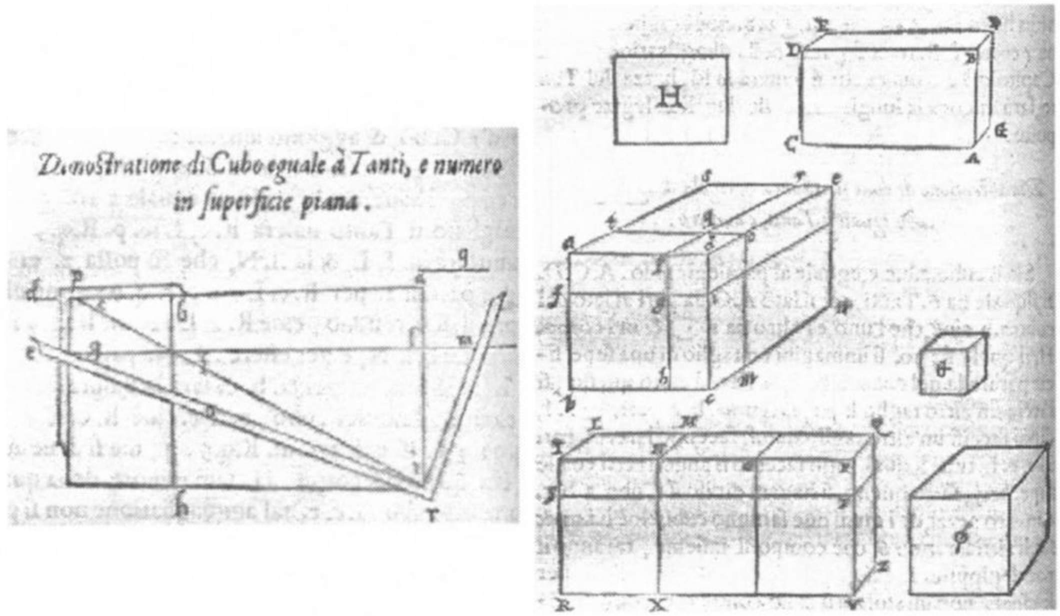


**Figure 2.** Bombelli's attempts at a geometric construction of imaginary roots. (Bagni, 2009)

Despite the fact that geometric constructability was a necessary qualification for acceptable objects in algebra, numerous 16th century mathematicians such as Girolamo Cardano (1501–1576) felt that imaginary numbers were essential to their work on cubics and would not allow them to be ignored. (Heeffer, 2009; Bagni, 2009)

The only way Bombelli and his peers could include imaginary numbers in their work was to define them through the 8 possible multiplications that can be performed:

| | |
|---|---|
| Più uia più di meno, fà più di meno. | $(+1).(+i) = +i$ |
| Meno uia più di meno, fà meno di meno. | $(-1).(+i) = -i$ |
| Più uia meno di meno, fà meno di meno. | $(+1).(-i) = -i$ |
| Meno uia meno di meno, fà più di meno. | $(-1).(-i) = +i$ |
| Più di meno uia più di meno, fà meno. | $(+i).(+i) = -1$ |
| Più di meno uia men di meno, fà più. | $(+i).(-i) = +1$ |
| Meno di meno uia più di meno, fà più. | $(-i).(+i) = +1$ |
| Meno di meno uia men di meno fà meno. | $(-i).(-i) = -1$ |

**Figure 3.** Bombelli's 8 multiplications for imaginary numbers (left) and their modern equivalents (right). (Heeffer, 2009)

Here we can see an example of the philosophy associated with an algebraic justification in a *modern* sense. The expressions for imaginary numbers could not be defined by conventional means as "real" but they were correct as long as they abided by their set of rules.

These ancestors to the axiomatisation of arithmetic and algebra set a precedent for this type of object in algebra. They opened the door for justifications that could not be defended by a philosophical "reality" and the resultant erosion of the intuitive, physical foundations of algebra.

The approach taken to deal with imaginary numbers was well suited for other unallowable and counterintuitive concepts such as negatives. Negative numbers were harshly criticised or ignored and debates on the paradoxes they created were quite heated during the 17$^{th}$ century. (Heeffer, 2009)

The controversy was often rooted in the pure traditional mathematics of quantities. The havoc they wreaked to the revered Greek logic of proportions was described by Antoine Arnauld (1612–1694) in his major philosophical work *The logic of Port-Royal*. (Heeffer, 2008) Here he described an outcome of the arithmetic rules of negative numbers which he deemed counterintuitive to reasoning on proportions. His argument was that if we examine the proportions of two numbers where one number is known to be smaller than the other number, then the proportion of the larger number to the smaller one should be greater than the proportion of the smaller number to the larger one. This rationale is utter nonsense if negative numbers are allowed as $\frac{1}{-1}$ would be considered greater than $-\frac{1}{1}$ as 1 is greater than -1, which contradicts the rules of algebra. (Heeffer, 2008, 2009)

In an effort to maintain the balance, most mathematicians believed that these numbers did not actually represent quantities less than zero but conceived of a negative number as the negation of a real (positive) quantity. (Peckhaus, 2003)

Others such as Leibniz found that the only proper resolution to the incongruity between proportion theory and an algebra that allowed negative numbers was to perform purely symbolic calculations in the vein of the calculations performed on imaginary numbers. If one is only observing rules of signs then there are no philosophical conundrums; i.e., "$\frac{1}{-1} = -\frac{1}{1}$, *pointe finale*." (Heeffer, 2009) The only justification required to check the validity of a statement rests on whether or not the laws of combination are being employed correctly.

In many ways, the conception of modern abstraction was incidental. A genuine desire to push the boundaries of traditional philosophies in an effort to modestly expand the capabilities of algebraists had devastatingly profound consequences. The allowance of symbolic definitions based on laws of combination took on a vastly greater meaning than initially intended. In a desire to not dismiss unconstructible quantities, a new system of mathematics was created and symbolic tables containing definitions based on laws of combination began to spread. (Heeffer, 2008)

In this transformation symbols were imbued with an increased and unforeseen meaning. From the 16th century on, mathematicians began to treat old problems with new methods. The strict physical nature of mathematics had begun to be relaxed and negative and imaginary numbers were in the lexicon of mathematicians who gradually liberated themselves from philosophising on the nature and existence of numbers.

### *3.2.3 Axiomatisation*

It became increasingly clear that abandoning Platonism was required if mathematicians were to continue exploring exciting and fruitful new realms and that a new philosophy was needed to take its place. There was a foundational crisis in the mathematics and many struggled to put mathematical assertions back on solid ground. To this there were many propositions but little consensus. These propositions generally stemmed from the two rival camps of 17th century philosophy: the rationalists and the empiricists.

The three main rationalists were René Descartes (1596–1650), Baruch Spinoza (1632–1677), and Leibniz. They espoused the belief that all knowledge can be gained through reason alone and took mathematics as their model for knowledge. Rationalists claimed that there are significant ways in which our concepts and knowledge are gained independently of sense experience. For Rationalists, what is provided by experience is lesser than what can be conceived of through theoretical concepts and knowledge. (Mahoney, 1980; Horsten, 2012)

Empiricists rejected this claim believing that all concepts and knowledge must come through the senses from experience and took the physical sciences as their model for knowledge. Empiricists are sometimes seen as sceptics who assert that what cannot be shown through experience is neither a concept nor knowledge. (Mahoney, 1980; Horsten, 2012)

In the 19th century, the general philosophical and scientific viewpoint veered toward the empirical in an effort to avoid the Platonist aspects of abstraction still present in rationalistic theories of mathematics. (Horsten, 2012) The rationalists had to reconstruct their fundamentals without any Platonism to make their position defensible.

One framework developed with the objective of showing that mathematics can be reduced to logic. Leibniz had originated the philosophy that mathematics is logic and reason in disguise and "anticipated a time when matters of controversy could be resolved by sitting down and calculating." (Mahoney, 2003) The realm of logic required no empirical justification; it was a rationalist paradise where "truth" was defined through computation. The new building blocks of this brand of mathematics were *intelligibilia*, or "logical objects which have their origin in reason alone." (Peckhaus, 2003)

Algebra took premises and then carried out logical analysis to achieve a conclusion based on calculation. (Mahoney, 2003) Logic proved a tempting choice for mathematicians attempting to establish a foundation for the justification of algebra in anti-Platonist climate; reducing mathematics to logic created a new foundation that was ontologically neutral.

(Peckhaus, 2003, Horsten, 2012) Through Boole, Friedrich Ludwig Gottlob Frege (1848–1925) and Bertrand Russell (1872–1970), algebra slowly formed the link between mathematics and logic. (Mahoney, 2003) Other mathematicians agreed with the idea of truth through calculation but did not agree that algebra could be reduced to logic.

Those such as Giuseppe Peano (1858–1932) and Richard Dedekind (1831–1916) felt that increased rigour would rid mathematics of intuition. They felt they needed to re-establish the certainty that had been lost in the foundations of mathematic by making proofs entirely rigorous. (Segre, 1994)

Peano used logic as an instrument in mathematics. He asserted that even the natural numbers could not truly be defined and instead characterized them as general axiomatic entities and championed the idea of definition by abstraction. (Segre, 1994) He employed logical symbolism and drafted axioms and basic theorems to define arithmetic axiomatically and without intuition. Peano's five axioms are defined as follows:

1. To a particular entity of the system shall be given the name 1.

2. Define an operation by which to each entity $a$ of the system there corresponds another, $a\ +$, also of the system.

3. And two entities whose correspondents are equal, are equal.

4. The entity called 1 is not the correspondent of any [-entity].

5. And finally it shall be the class common to all the classes $s$ which contain the individual 1, and which, if they contain an individual, contain its correspondent. (Segre, 1994)

In this way mathematicians such as Peano and Dedekind rebuilt the foundations of mathematics with axioms and rigour and many mathematicians and philosophers took the nine Dedekind-Peano axioms to be the new foundation for arithmetic.

While there were a multitude of philosophies and disagreements on the foundations of mathematics in the 19th and 20th centuries, the importance of truth through calculation, rigour and axiomatization are consistent with a general modern take on justifications.

Modern arithmetic and algebraic statements consist of logical statements that are rigorously justified in axiomatic systems. For example, if we return to our statement "3 + 5 = 8" in the context of an axiomatize arithmetic system, we have a very different picture than Widmann's syncopated statement. If we take an example of a rhetorical statement such as "I have 3 sons and 5 daughters implies I have 8 children" or our syncopated statement "3 + 5 makes 8" they are justified empirically by a physical reality. These statements are "true" because "if I take 3 objects and 5 objects then I have 8 objects."

For a modern symbolic expression such "3 + 5 = 8" the "truth" of this statement is derived from the fact that the calculation performed is a correct application of laws of combination; the calculation is consistent within the axioms of arithmetic for natural numbers.

These laws designate an arithmetical equivalence of the sum of 3 and 5 with 8, and the partitioning of 8 into the numbers 3 and 5. This is a justification in the modern sense. (Heefer, 2009) In terms of modern algebra, the statement "I have 3 sons and 5 daughters implies I have 8 children" is true because it is a special case of the arithmetic statement "3 + 5 = 8" which can be verified rigorously and axiomatically in a way an empirical statement cannot. (Pap, 1957)

The same can be said about the abstract group $(G,*)$ introduced in section 3.1.3. Correct statements made concerning the set $\{a, b, c\}$ are to be interpreted in much the same way; as calculations abiding by the laws of combination set forth by the operation $*$. Every calculation can be rigorously tested axiomatically. If the symbols are taken to represent something specific, then statements concerning the objects the symbols represent are true because they are special cases of a general rule and are justified as long as they abide by the rule.

This type of algebra is not concerned with what the symbols represent or with specific objects and values. This brings us to the final piece of the modern algebra puzzle which describes what exactly these abstract groups are studying.

### *3.4 Objects vs. Relationships*

The development of modern abstraction was forcibly tied to the rise of algebra; expanding the contents of this discipline and challenging the geometric reign. We have seen that algebra originated as a process for determining an unknown quantity which represented a physical object or geometric construction. At this point algebra was essentially an extension of geometry, merely an "auxiliary technique for solving arithmetical problems." (Mahoney, 1980)

As algebra progressed and developed its own mathematical language, it no longer needed to study geometric objects. In the 16th and 17th centuries, the focus of algebra slowly shifted from finding the value of a particular quantity to observing relationships between general classes of quantities, regardless of their particular value.

Algebra blossomed in the 14th and 15th centuries when the world of mathematics was still emerging from the dark ages and there was a resurgence of readings from antiquity. (Mahoney, 1980) The reintroduction of Diophantus into the stream of intellectual thought brought forth an interest in the treatment of unknowns, symbolically representing discrete quantities. Among those attempting to follow in the footsteps of Diophantus, was French lawyer and mathematician François Viète (1540–1603) with his *ars analytica* or “the analytic art.” (Mahoney, 1980) This “art”, so named because it was not considered to be “proper” mathematics, was a revolutionary symbolic system consisting of three stages (Macbeth, 2004):

1. Take an arithmetic or geometric problem and translate it into the symbolic system;
2. Transform symbolic equations according to rules into canonical forms; and
3. Translate the problem back into a geometric construction or numerical solution based on the symbolic equation

Viète stressed that his art was closely connected to the work of the Greeks but there are two vital distinctions. First of all, Greek symbols for unknowns always represented a specific quantity being calculated, but for Viète the realm of what an unknown could represent was everything for which it made sense to add, subtract, multiply, and divide. (Mahoney, 1980) He practiced a generalised form of mathematics that did not depend on the type of quantity he was working with. Viète often referred to algebra as *logistice speciosa* the "logistic of species" whose species are abstract, general quantities. (Shapiro, 2005; Macbeth, 2004)

Viète's species predated the logical objects of modern algebra and still represented concrete objects in arithmetic and geometry. The importance of his species stems from the fact that he ignored what they represented while he was working with them in his second step. As a result, his algebraic work was not predicated on the nature of the unknown being a natural number or the length of a line. This is in stark contrast to previous forms of algebra whose rationalisations always stemmed from the nature of the object in question. Viète's algebra was not a study of objects.

The second divergence is that Viète created the first mathematical system that allowed the of use symbolic representations for both knowns and unknowns. (Macbeth, 2004) This made Viète the first mathematician to ever symbolise a general equation such as $ax^2 + bx + \mathrm{c} = 0$ (in modern notation) where both the unknown quantity $x$ and known parameters $a, b, c$ are represented symbolically. General equations created an entirely new form of algebra. The types of questions and answers relating to a general equation are vastly different than those pertaining to a specific equation. For example, an equation such as $x^2 - 3x + 2 = 0$ , whether presented in rhetorical, syncopated or symbolic algebra, generally represents a process for determining the value of an unknown. The goal is to find the specific values which satisfies the equation, in this case the values $1$ and $2$. This form of algebra studies unknown mathematical quantities in an effort to determine what these quantities are.

Viète's new general equations did not have this goal, the aim was to describe relationships and find properties that certain quantities share in common with other quantities. This can be seen in Viète's fascination with general relationships that exist between the coefficients and roots of an equation. For example, if you examine the special case of the general quadratic equation $ax^2 + bx + \mathrm{c} = 0$ where the coefficient of the second term $b$ is the negative of two numbers whose product is c, then these two numbers are roots of the equation. (Shapiro, 2005) This holds true regardless of the specific values of $b$ and c, all that matters is that they have that particular relationship to one another. Polynomials that share this property form a species of interest. The goal here is not to find the values of particular polynomials in the species but to describe a class of polynomial that possesses an interesting and potentially useful relationship between the roots and coefficients.

This technique developed a new method of posing problems based on general structure. There developed a quickly rising interest in the study of structure and relationships between abstract species. Manipulating symbols under a set of rules in Viète's general equations showed immediate efficacy in solving old and new problems. (Macbeth, 2004)

Pierre de Fermat (1601–1665) was a great advocate of Viète and saw the enormous power of posing problems in the framework created by general equations. For example, he showed that if $P(a)$ is an extreme value of an algebraic polynomial $P(x)$, then $P(x)$ must be of the form $(x - a)^2R(x)$ and then went on to develop a general method for ascertaining extreme values. (Mahoney, 1980) Here the goal is not to find the function's maximum point $(a, P(a))$ but to describe a generalised structure of a special type of polynomial equation in an effort to engage in the interesting and useful activity of finding maximums. (Fermat's work predated the invention of calculus so finding maximums was not a trivial activity.)

It is curious to note that Fermat still maintained that his work was in the tradition of ancient mathematics even though his statements on the properties of general equations was a concept the ancient mathematicians had not conceived of. (Macbeth, 2004) That was not the opinion of noted philosopher, mathematician, and writer René Descartes who made similar advancements in Geometry. Descartes searched for the most general laws of quantity and believed that generalised mathematics was in fact the approached used by ancient Greek geometers but they "hid their methods to keep others from seeing how easy and unremarkable many of their discoveries really were." (Shapiro, 2005) Klein was of a similar opinion and believed that "the Arithmetic of Diophantus is but a remnant of a more general theory of equations, of a true and more general algebra." (Hopkins, 2011) He felt that Viete's ability to develop this insight made him the "inventor of modern mathematics."(*ibid.*)

In any case, a realisation that algebraists had quickly surpassed the greatest achievements of the ancient Greeks led them to notice that they had greatly extended the heuristic power of algebra with a new focus on relationships rather than specific values and objects.

In modern algebra, a particular value or object is not the topic of interest; it is a generalised study of interesting relationships or a particular structure that may prove useful in problem solving. In the abstract group $(G,*)$, the elements $a, b$ and $c$ are not the objects of interest and a group is not simply a set of symbols governed by laws. The laws of combination define generalised relationships which do not depend on the nature of the elements and can be applied to any number of "species." An abstract group depicts a structure which is interesting to mathematicians and proves highly useful in problem solving.

## 3.5 What is modern algebra?

Now that all the pieces are in place, we can summarise the modern abstraction which took hold in the 19$^{th}$ century and create a framework to analyze group theory in the context of three characteristics described in the three sections.

1) **Symbolism**: A modern symbol should create no bias in the mind. It is an *operative symbolism* that represents the workings of combinatory operations and can be operated upon independently of context and manipulated in accordance with a given set of general rules.

2) **Justification:** The modern algebraic mode of thought is liberated from all ontological commitments. Justifications require intense rigour and rest solely on an ability to be consistent within a given axiom system.

3) **Relationships vs. Objects**: Modern algebra examines generalised relationships and structures, not particular objects or values.

# Chapter 4

# Literature Review on Difficulties in Group Theory

In this chapter, I review several empirical studies that examine students' understanding of and difficulties with group theory. This is meant to illuminate common difficulties students experience in their first group theory course and the mechanisms they use to cope with these difficulties. I also present the frameworks researchers used to describe and analyze student difficulties. This should give a general sense of the research being done in this field.

The literature surveyed for this study describes students' reactions to a variety of introductory group theory questions related to the general abstract concept of a group, subgroups, binary operations, cosets and isomorphism as well as examples of specific groups such as modular arithmetic and symmetric groups. From these studies we can see that students often possess deep rooted misconceptions relating to group concepts and display a general lack of understanding of what constitutes a proper proof in group theory. These problems arise early on and tend to worsen as the course progresses.

Mathematics faculty and students generally consider abstract algebra to be one of the most troublesome courses for undergraduate students. (Dubinsky et. al., 1994) Abstract algebra, however, is of vital and growing importance in the domains of theoretical mathematics, computer science, physics, chemistry and data communications (Hazzan, 1999) and students' lack of understanding of basic concepts often becomes an obstacle for advanced studies in mathematics. Nevertheless, research on the teaching and learning of abstract algebra, in general, and group theory, in particular, appears to be scarce.

The literature I encountered in my research frequently explains student difficulties within the APOS framework where APOS stands for action-process-object-schema. (Dubinsky, 1991; Dubinsky et al., 1992; Brown et. al., 1997). APOS is a framework that describes four different levels students go through when learning mathematical concepts. Within this framework, the learning of a mathematical concept is progression through the action-process-object-schema hierarchy.

The first level in APOS is that of *actions*, or a stage in the development of a concept where a student can perform repeatable manipulations (mental or physical) to transform an object into another object. If one does not evolve beyond this level of understanding, one will view concepts *only* in an algorithmic manner where each step taken by the student must be dictated by a set of step by step instructions. These *actions* need to be interiorized into *processes*; this stage is attained when a student can mentally link multiple inputs and outputs and they can think about the process without performing calculations. These processes crystallize into *objects* when the student can combine and manipulate the processes and transform a process by some action. The objects become encapsulated into *schema* when they represent a coherent set of processes, objects and other schema that are thematized and can be invoked to deal with mathematical situations. (Dubinsky et al., 1994)

The claim derived from the literature, within the APOS framework, is that introducing students to group theory concepts through packaged abstract definitions inverts the way they are accustomed to learning mathematical concepts. (Brown et al., 1997) Students generally learn concepts through a steady growth of knowledge from actions to schema but the definition of a group immediately presents students with a combination of three schema; sets, axioms and binary operations that *encapsulate* the underlying actions, processes and objects. In order to work with a given group, a student should understand these three schema at a "high" level and be able to *de-encapsulate* these schema back into actions. (Brown et al., 1997)

By this theory, students are often incapable of de-encapsulating these schema and resultantly cannot attain a general understanding of group concepts. A student whose goal is to go beyond actions and processes and plugging symbols into definitions must learn how to break down schema and get a general sense of what group theory is and what is studies.

I will describe two main types of difficulties diagnosed in the reviewed literature: difficulties that stem from the level of abstraction in a group theory course and those that arise from a reliance on canonical procedures. Often, students understand concepts only through their abstract definitions and cannot evolve beyond this level of understanding. They cannot cope with abstraction and so they make comparisons to the *familiar* in ways that do not accurately reflect the situation they are working with. Due to a lack a conceptual understanding or mistrust of theoretical answers, students rely on flawed canonical procedures and symbolic manipulations to answer questions.

## 4.1 Reducing Abstraction

When students are presented with the group concept, they are given the abstract definition of a group (G,*) which they must unpack and apply to abstract problem solving.

However, most of the work done by students is at a lower level of abstraction than what they are taught in class. The literature gives several different interpretations of abstraction. Leron et. al (1995) refers to groups being abstract when they do not refer to specific cases and Dubinsky et. al (1994) references abstract groups as those with elements that are undefined for a student. Hazzan (1999) describes abstraction in three ways;

1. Abstraction level as the quality of the relationships between the object of thought and the thinking person
2. Abstraction level as a reflection of the process-object duality
3. Abstraction level as the degree of complexity of the concept of thought (Hazzan, 1999)

The literature suggests that students often reduce the level of abstraction as a coping mechanism for solving problems and in creating their understanding of group concepts. These techniques often end in falsehoods as they do not accurately reflect the entire abstract concept. This coping mechanism is referred to by Hazzan (1999) as *reducing abstraction*. In terms of three descriptions of abstraction given above, Hazzan (1999) suggests that at early stages of understanding groups, students may well construct their own conception of what a group is through objects they are familiar with. This helps them create a closer relation with this concept which makes it less abstract. (Hazzan, 1999) These difficulties with abstraction are often intertwined. In terms of the APOS structure, they can then create processes that associate these familiar objects with each other or simply view objects as processes. If students use these formulated processes, and encapsulate them into an object, then they will mistakenly characterize groups as simpler objects and portray a basic level of misunderstanding of what a group is based on procedural methods. This also makes problems and concepts less complex than they really are as they are now based on simpler object which again makes them less abstract by the above definition. (Dubinsky et al., 1994)

An example of such behavior is that students can be inclined to view a group in terms of its elements or simply as a set. (Dubinsky et al., 1994) In such situations students will, for example, not be able to differentiate between groups of the same cardinality.

***Example 1:*** Dubinsky's et al. (1994) interviewed 24 mathematics teachers taking a summer course on group theory. In their written and verbal assessments, numerous participants stated that they understood $\mathbb{Z}_3$ as the set {0, 1, 2}, {1, 2, 3}, {0, 2, 3}, or {0, 2, 4} or any closed set with three elements. Here the novel object of the group $\mathbb{Z}_3$ is replaced by the familiar and less complex object of a set with three elements.

The methodology of dealing with difficulties by assimilating novel situations into existent schema is common, as is ignoring important characteristics of a group. In the above example, the participants who mistakenly characterized $\mathbb{Z}_3$ as any set with three elements ignored the main characteristics of $\mathbb{Z}_3$, namely its associated binary operation and ability to satisfy the four axioms. (Dubinsky et al., 1994)

The practice of using familiar objects to solve problems may be based on the fact that students expect new knowledge to be based on old knowledge. During the process of de-encapsulation students feel they can reduce abstraction through familiar knowledge. (Hazzan, 1999) In group theory, existing knowledge that seems relevant may not necessarily be applicable. For example, the familiar rules and properties of numbers can change when examined in a group context. This can be particularly confusing when the group elements are numbers and the group operation does not correspond with multiplication of the integers even though the standard notation for a binary operation in group theory * is the same as numerical multiplication. (Weber and Larsen, 2004) Students have a tendency to base their arguments on previous knowledge of how numbers work. They use general properties of numbers and familiar operations and do not contextualize how numbers abide by laws of combination within a particular group. (Hazzan, 1999)

***Example 2:*** In the nine interviews Hazzan (1999) conducted with undergraduate students, Tamara stated that $\mathbb{Z}_3$ was not a group as the element 2 does not have an inverse in $\mathbb{Z}_3$. Her rationale was based on incorporating the known mathematical properties of an identity being the number 1 and that the inverse of an element $x$ is an element $y$ such that $x * y = 1$. She did not incorporate these facts into a new context where the identity mod 3 is the number 0 and * is addition mod 3. As result she failed to appropriately test the axioms under the given binary operation due to her reliance on familiar knowledge. The rules of combination can change for a given set of elements depending on the binary operation and this requires students to understand binary operations as objects. (Brown et al., 1997)

These difficulties increase as a student progresses into more advanced concepts such as subgroups and quotient groups. Even if a student understands what constitutes a group they may have difficulty transferring that knowledge to gain a general understanding of what makes a group a subset of another group. For example, if a student grasps why $\mathbb{Z}_3$ is a group and why $\mathbb{Z}_6$ is group, how does that transfer into an ability to answer the question "is $\mathbb{Z}_3$ a subgroup of $\mathbb{Z}_6$?"

***Example 3:*** In a written questionnaire, 73 out of 113 students incorrectly answered that $\mathbb{Z}_3$ is subgroup of $\mathbb{Z}_6$. (Hazzan, 1999) When discerning if $\mathbb{Z}_3$ is subgroup of $\mathbb{Z}_6$, the operation changes from addition mod 3 to addition mod 6. Students, however, upon seeing $\mathbb{Z}_3$ only think of addition mod 3 regardless of the context. Most students were unable to understand why the operation on a subgroup must be the same as in the larger group. (Dubinsky et al., 1994) Even this does not fully represent the theoretical concept that the operation used on a subset G of a group (H,*) is not just where G happens to agree with * but requires a mental action of restricting * to elements of H that correspond with the elements of G or $*|_G$. (Dubinsky et al., 1994) This level of sophistication in understanding the generalized property of restricting to a set when examining a subset was not exhibited.

Students answers were commonly based on using the more familiar and less complex $\mathbb{Z}_3$ and performed calculations on $\mathbb{Z}_3$ in addition mod 3 than establish how to view $\mathbb{Z}_3$ as a subgroup of $\mathbb{Z}_6$.

## 4.2 Canonical procedures

When faced with a mathematical problem, students often use canonical procedures or automatic responses (Hazzan, 1999). Students are accustomed to being given a familiar situation where they employ a given procedure or algorithm to find the solution. Students often inaccurately represent group concepts due to a reliance on repetitive behaviour patterns. Sometimes students prefer to rely on procedures that they do not understand rather than use theorems and ideas they do or could potentially understand.

***Example 4:*** when a group of students were asked to produce an operation table for a group of order 4, some students used an algorithm which they admittedly did not understand. (Weber and Larsen, 2004)

*1. Choose an identity element and fill in the first row and first column.*

*2. Choose a result for one of the remaining boxes that is not the identity.*

*3. Fill in all boxes without repeating any element in any row or column.*

This Sudoku style method for creating tables represents the cancelation law which is not the equivalent of the definition of a group and does not work in general. The main aspect of group theory used here is that groups have a unique identity element. Some students stated that they felt more at ease with this procedure than using something more reflective of group concepts such as stating that a group with 4 elements must be isomorphic to $\mathbb{Z}_4$ or the Klein-4 group and creating the table for one of these groups which they have worked with before.

Students often have difficulties relating activities such as creating tables and bijections (Weber and Larsen, 2004), performing calculations (Hazzan, 1999) and finding the order of elements to the group concepts pertaining to these activities. (Leron et al., 1995) Students will commonly use algorithms to solve problems without correlating them with their theoretical knowledge.

***Example 5***: in the Hazzan study (1999), the student named Guy was asked to calculate the cosets of the subgroup {1, 2, 4} of $\mathbb{Z}_7$ \{0} with multiplication modulo 7. He began by correctly describing in detail what a coset is:

*Guy: OK, it is known that equivalent classes divide the set into disjoint sets, the whole set. And now, if a coset is like an equivalent class then the cosets divide the ... divide the group into disjoint sets. We know that in each coset the number of elements is equal.*

> Then, to solve the task at hand, he calculated all 6 cosets by multiplying out the subgroup by each element in the group. Only then did he notice that there were two cosets. (Hazzan, 1999) Upon calculating two disjoint cosets each containing three elements, he did not notice that there could be no additional disjoint cosets of three elements in a group containing six elements. He carried out needless calculations rather instead of using his theoretical knowledge of cosets. Perhaps this displays a lack of trust in using a theoretical thought as part of an answer. Students are likely accustomed to using numerical calculations as answers and do not feel safe with providing anything else as a solution.

Students are thought to fear unrestricting or uncertain procedures or procedures with any degree of freedom. This can be viewed in terms of a general "fear of freedom," a phenomenon where many people do not like making choices since, often, choosing one option means losing others. (Leron et al., 1995)

A student's mindset often changes when faced with an open or vague problem versus a practical one. In the above example, Guy displayed an ability to theoretically describe the properties of cosets but chose not to apply this knowledge when faced with a practical problem.

***Example 6:*** Leron et al. (1995) noted that students prioritize differently when they think about a problem generally or practically. In a study conducted on 51 students, they were asked the conceptual question "what properties are preserved under isomorphism?" The students listed properties in more or less the following order: commutativity, cyclicity, order of the group, orders of elements.

When showing whether two given groups are isomorphic they used almost the reverse ordering: the order of the group, the identity element, the orders of the various elements, the "order type" of the groups, is the group cyclic, is it commutative? (Leron et al., 1995)

Calculating order and creating tables are relatively simply and are more along the lines of canonical procedures than proving that all the elements of a group commute or can all be formed by powers of one element.

Students are often unable to develop an intuitive sense of group theoretic concepts and must rely on procedures and definitions. This tendency to rely on canonical procedures instead of factual knowledge proves extremely detrimental when students check for isomorphisms and construct proofs.

Weber and Alcock (2004) gave four undergraduate students, who had just completed an abstract algebra course, five pairs of groups each and asked them to rationalize aloud whether or not they were isomorphic. Collectively, they were only able to prove two out of the twenty correctly. They generally began by checking the number of elements and then tried to set up a bijection. (Weber and Larsen, 2004) The formal definition of an isomorphism states that there must exist a homomorphic bijection between the two groups but this is rarely the best way of checking to see if groups are isomorphic.

> **Example 7:** when asked if ($\mathbb{Z}$, +) is isomorphic to ($\mathbb{Q}$, +) one student attempted a complicated bijection (Weber and Larsen, 2004) rather than testing the properties of these groups to notice that one was cyclic and the other was not. This was consistent with Leron et al.'s study (1995) who typified student responses as "these two groups are isomorphic because I can find a one-to-one function from each element in G to each element in G'."

Most students could not escape the formal definition of an isomorphism and look beyond it to see the general conceptions of what makes a group abstractly the same or different. They insisted on canonical thought; if asked whether two groups are isomorphic they used the definition and created a bijection.

Weber and Alcock (2004) also interviewed mathematicians as part of this study. The majority of the mathematicians they interviewed stated that they could "see" the groups and so they knew what to use to prove whether or not two groups are isomorphic. (Weber and Larsen, 2004) In general, the students could not grasp groups in terms of relationships and structure therefore could not conceive of the congruencies in relationships and structure between two groups. Most students did not see isomorphic groups as being mathematically the same or as reordering and renaming and generally could not understand isomorphism beyond the given definition. One student remarked “my intuition and formal understanding of isomorphic groups are the same.” (Weber and Larsen, 2004)

Often, students can possess intuition, knowledge and insight into a given concept or problem and simply not know how to apply it. This is often apparent when students are faced with the prospect of a theoretical proof. For example, Weber (2001) interviewed undergraduate students who had just completed an abstract algebra course. (Weber and Larsen, 2004) They were asked to think aloud as they proved non-trivial group theoretic propositions and were subsequently given a written test. He noted that out of the students who possessed the factual knowledge to prove given propositions, as shown by a written test, few of them could construct a proof. He concluded that these students cannot apply their knowledge productively and lack the skill and strategies to write proofs.

These students would often use random facts and try to play around with symbols and break down definitions rather than rationalize out an answer theoretically. (Weber and Larsen, 2004) Students generally preferred stating and using definitions or trying to make use of formal sounding statements that they did not necessarily understand. The mathematicians interviewed in this study generally spent time reflecting on what they felt would be useful to prove a proposition and often employed powerful theorems.

Many of the theorems in abstract algebra disguise their meaning and provide general statements students cannot contextualize and so, even when students attempt to use theorems, they often fail. For example, 20 out of the 73 students who believed that $\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_6$ in the Hazzan study used Lagrange’s theorem to justify their answer: 3 divides 6 so $\mathbb{Z}_3$ is a subgroup of $\mathbb{Z}_6$. This is the converse of Lagrange's theorem which is not true in general. Other's used Cauchy's theorem based on the fact that 3 is a prime number.

## 4.3 Conclusions

The literature addresses numerous difficulties relating to the "abstraction" encountered in a group theory course and the coping mechanisms students use to reduce abstraction. While the definitions for abstraction given in the reviewed literature certainly apply to abstract algebra, they are not specific to abstract algebra. The literature indicates that abstract algebra is perhaps the most difficult undergraduate subject. Dubinsky et. al. (1994) states that "many students report that, after taking this course (introductory abstract algebra), they tended to turn off from abstract mathematics." So what makes abstract algebra in particular so diffi cult?

In an effort to answer this question I will examine two areas that are specific to an introductory abstract algebra course; conceptual aspects of group theory and its modern abstract representation. I this way I aim to decouple these two components of group theory and demonstrate how they are both vital to succeeding in a group theory course. This should address common difficulties discovered in the literature such as how students understand group theory concepts as their abstract definitions.

In order to address the second major cause of difficulty raised in the reviewed literature I will examine students' tendencies to use canonical procedures, definitions and theorems. I will examine how familiarity and the ability to use canonical procedures affect the areas of conceptual understanding and abstract representations. Hopefully this will provide a deeper insight into why students experience the difficulties seen in the literature review.

# Chapter 5

# An empirical study

From the literature review in Chapter 4, we can see that students often encounter great difficulties with the concepts typically introduced in undergraduate abstract algebra courses. As stated in Chapter 1, the ultimate goal of this thesis is to support the conjecture that students' common difficulties in these courses have two different sources of clearly different natures. The first are conceptual difficulties that pertain specifically to learning and understanding group theory concepts. This requires a deep knowledge of what group theory is and what it studies as described in Chapter 2. The second are difficulties due to having to cope with the abstract manner in which these concepts are typically presented nowadays. This pertains to the *modern abstraction* found in the presentation of concepts, examples and exercises. This requires students to be able to properly work with operational symbolism, justify statements in an axiomatic system and work with generalised relationships and not specific objects as described in Chapter 3.

In this chapter, I present an empirical study carried out with 4 participants aimed at examining and differentiating the difficulties experienced when working with group concepts from those that might originate from the way in which these concepts are introduced to students (the *modern abstract* representation of group theory).

Based on the literature reviewed in Chapter 4, in the next section I propose a framework for analysing students' difficulties in such a way that the two types of difficulties can be clearly distinguished. In section 5.2, I present the methodology. The results are presented in section 5.3 and analyzed in section 5.4.

## 5.1 A framework for analysing students' difficulties in abstract algebra

In the literature I encountered numerous references, definitions and interpretations of abstraction in group theory. As stated in Chapter 4, Leron et. al. (1995) consistently referred to the abstract as something that is not specific. Hazzan's study (1999) defines abstraction in terms of how close a student is to given problem, whether they see it as an object or a process and how complex the problem is. These descriptions do not apply specifically to group theory and modern algebra. While abstract algebra is certainly "abstract" in a general sense, I believe that examining abstraction that is specific to modern algebra (as defined in Chapter 3) would help in distinguishing the two sources of difficulties mentioned above and isolate difficulties that are specific to abstract algebra courses. The goal is to explain why abstract algebra, in particular, causes such great difficulties for students.

The second goal is to determine students' understanding of the general concepts and ideas of group theory. As seen in the literature review, students often feel uncomfortable giving theoretical answers to question and when they have freedom in answering a question. I aim to test students' conceptual understanding of group theory by contrasting questions that should require a general knowledge of group theory to those that can be answered with remembered knowledge of a specific group. The other main difficulty addressed in the literature review is that students use canonical procedures to solve problems. I postulate that the use of canonical procedures is related to a students' previous exposure to a given type of problem or group.

Therefore I will contrast the familiar and the unfamiliar to gauge students' respective ability in the two sources of difficulties. In an attempt to isolate these two aspects of what might make an abstract algebra course difficult for undergraduate students, I created a set of 6 problems that I used in task-based interviews with 4 participants who had recently passed their first abstract algebra course. The problems were designed to expose whether students' difficulties were arising from group theory concepts or from coping with the modern abstraction that is typically used in presenting concepts and problems in abstract algebra courses.

For the design and also for the posterior analysis, problems were classified into four categories:

1. Familiar group – Familiar Representation: In this category, the group the participants have to work with and its representation are familiar to them.
2. Familiar group – Unfamiliar Representation: In this case, the groups are familiar, but they are represented in an unfamiliar manner.
3. Unfamiliar group – Familiar Representation: In this category, the groups are unfamiliar but they are represented using notation the participants are accustomed to.
4. Unfamiliar group – Unfamiliar Representation: Finally, this category corresponds to situations in which the groups and their representations are unfamiliar to the participants.

The criteria for establishing familiarity were based on (a) the text used in the abstract algebra course that the participants passed prior to the study: *Abstract Algebra* by Dummit and Foote (2004), (b) questions used in a variety of studies on difficulties in abstract algebra, and (c) my experience as an undergraduate student in that same course a few years prior. All problems had to satisfy all criteria to be labeled as familiar or unfamiliar.

### *5.1.1 Familiar and Unfamiliar Groups*

Familiar groups are groups participants of this study have worked with regularly in their abstract algebra course. Examples of these are modular groups $(Z/nZ)$, symmetric groups $(S_n)$, dihedral groups $(D_{2n})$ and cyclic groups $(< x >)$. Having worked with these groups, participants should know the elements and binary operation in a group theoretic context and be accustomed to the procedures and proofs commonly used when working with these groups. Ideally they will be familiar with the structure and main ideas behind each of these groups. These are all things they should have been exposed to in their abstract algebra course.

When working with familiar groups, participants may rely on their prior experiences working with these particular groups. In contrast to this, when working with unfamiliar groups participants cannot rely on past experiences and must apply their general knowledge of group theory. This is meant to illuminate how participants view group theory concepts when they cannot employ familiar knowledge or processes pertaining to a particular group and how dependent they are on canonical procedures.

The two main groups used in this study were the motion (unfamiliar) and symmetric (familiar) groups which have great historical significance. As seen in Chapter 2, the symmetric and motion groups were the first two established groups whose properties were abstracted and generalised to give groups their modern form we know today. The main ideas behind what group theory studies are encapsulated in these two groups and so I surmised they should be useful in discerning whether or not students have a sense of general group theoretic concepts.

They were also selected due to the fact that they can be represented both geometrically and algebraically and are easily reflected in real world examples that can be pictured and rationalised upon by undergraduate students. It was expected that this should allow the study participants to easily work with these groups without the use of abstract symbolic manipulation.

### *5.1.2 Familiar and Unfamiliar Representations*

Familiar representations involve notation that is similar in style to that used in problems and examples given in class, for homework and in the textbook. The familiar representations used in the task-based interviews were derived from the textbook. In these representations, the parameters of a problem (e.g., the elements and operator of the group) are presented symbolically, in a modern abstract state, as described in Chapter 3.

A representation that is familiar to the student allows him or her to perform symbolic manipulations in order to solve the problem; e.g., directly plugging symbols into the group axioms. Furthermore, familiar representations involve an operative symbolism and known laws of combination that can be used independently of a specific context. In contrast to this, unfamiliar representations do not provide, in the phrasing of the question, a familiar symbolic system to work with. In such cases, participants must either create their own meaningful symbolic system to represent the problem or discuss and rationalise out the ideas behind the problem. This should determine if students are capable of differentiating between symbolic algebra and syncopated algebra. This is also meant to elucidate the depth of their understanding of the abstract, symbolic representations they are accustomed to using and whether they can distinguish group theory concepts from symbolic representations or if their understanding of group theory is tied into or limited by abstract definitions and symbolic manipulations.

## 5.2 Methodology

The participants in this study were 3 undergraduate mathematics students and 1 graduate student in the Masters in the Teaching of Mathematics. All 4 participants had completed the same group theory course three months prior to the study and were currently enrolled in the same ring theory course from which they were recruited.

Participants were met individually in one-on-one interviews which lasted approximately 90 minutes. At the beginning of the interview, participants were told that the interest of the study was *how* they went about solving problems, not whether they arrived at the correct answer. This was repeated during the interview if the participant was overly concerned with whether his or her answer was correct. The interview was task-based; there were 6 mathematics problems (tasks) and participants were asked to think aloud while working on them. Interviews were audio recorded and later transcribed for analysis.

Participants were given each of the 6 written problems one at a time and were not allowed to look ahead in the set but could refer back to what they had done. For each question, participants were permitted to work on the provided paper and ask questions to the interviewer. They were not allowed any resource materials such as class notes or textbooks. Participants were given time to work on each problem and to present their attempts, solutions, etc. in written or oral form. Then they were given some prompts, examples or explanations if they experienced difficulties and time to re-think and re-work the problems.

### *5.2.1 The tasks*

The tasks were designed in accordance with the familiarity framework described above. In relation to group concepts, there were four basic problems in the set; one for each combination of familiar/unfamiliar group/representation (problems 1, 3, 4 & 6) which asked the participant to determine whether or not a given set and binary operation formed a group and two problems (2 & 5) on more advanced topics such isomorphisms, quotient groups and subgroups.

The goal was to compare and contrast how participants dealt with group concepts in relation to their familiarity with the group and the presentation of the problem. I will now explain why each question satisfies the familiar/unfamiliar group type and representation designation. I will also analyze how the familiarity/unfamiliarity should affect participants' responses.

*Question 1 &2 - Unfamiliar group and unfamiliar representation*

**Question 1** (U/U)

> Let $M$ = {all possible motions or space-transformations of a 3-dimentional object in 3-dimentional space}.
>
> Does $(M,*)$ form a group, where $*$ is the composition of transformations?
>
> (i.e. if $t_1, t_2 \in M, t_1 * t_2$ is transformation $t_1$ followed by transformation $t_2$)

The group $(M,*)$ is unfamiliar in the context of the typical problems and examples given in an undergraduate abstract algebra course but the underlying concept of an object undergoing a transformation in 3-dimensional space should be familiar to a student who has successfully completed such a course. I expected that the concept of *transformation* was known in both an algebraic and a geometric sense. Linear transformations are studied in mandatory linear algebra course that participants have taken prior to their group theory course. The rotations and translations of an object in 2-dimensional space are commonly studied in late elementary school or early secondary school mathematics. I also expected participants at that stage in a mathematics program to be able to picture and describe all possible motions of an object moving in a 3-dimensional space.

The representation of this group fell under the category of unfamiliar as the phrasing was mainly rhetorical and symbolism was minimal and syncopated – as opposed to the modern symbolic representations students are used to (familiar representations). The verbal description of the problem does not give the participants the elements of the group or the binary operation in a modern symbolic manner and only refers to two arbitrary elements in the group $t_1$ and $t_2$ and operation $*$. The symbols $t_1$ and $t_2$ are simply abbreviations for arbitrary transformations and there are no inherent laws of combination for these elements under this operator. These symbols cannot be easily plugged into formulas or canonical procedures and participants must first deduce the nature of a space-transformation, such as the fact that they are rotations and translations, and get a general sense of the group to answer the question.

The reason behind giving both an unfamiliar group with an unfamiliar representation was so that participants could not rely on past experiences or learned procedures related to the group in addition to not being able to use symbolic manipulation. Therefore, they would be required to rely entirely on their knowledge of group concepts to determine if $(M,*)$ is a group or create their own contextually meaningful symbolism to perform calculations and proofs. I expected that participants who do not have an understanding of the group concept and/or tend to use canonical procedures and heedless symbolic manipulation would experience extensive difficulties with Question 1.

**Question 2 (U/U)**

> There are two proper non-trivial subgroups of $M$. Let $S$ and $T$ be these subgroups. Show that the quotient group $M/S$ is isomorphic to $T$.

Question 2, a continuation of Question 1, is still based on the unfamiliar group $(M,*)$ but concerns slightly more advanced topics from an introductory group theory course, namely subgroups, quotient groups and isomorphisms. As seen in example 3 of the literature review, students who understand the basic concept of group have difficulties integrating this understanding into more advanced group theory topics (i.e. students who display a clear understanding of the groups $\mathbb{Z}_3$ and $\mathbb{Z}_6$ often falsely believe that $\mathbb{Z}_3$ is as subgroup of $\mathbb{Z}_6$).

Question 2 also had an unfamiliar representation which consisted mainly of verbal descriptions and a syncopated symbolism. As in Question 1, the problem was designed in such a way that it could not be solved by pure symbolic manipulation and canonical procedures.

I expected that participants' reactions to proving an isomorphism would be similar to the Leron et al. (1995) and Weber and Larsen (2004) studies (described in the literature review) which inferred that students tend to understand the isomorphism concept simply as a bijection and not as two groups with identical structures or a renaming of elements. Participants should require an understanding of the group $(M *)$, subgroups $S$ and $T$, quotient groups, kernels and isomorphism to answer Question 2.

This question was designed to test participants' abilities to work with and understand slightly more advanced group concepts without relying on past experiences or learned procedures relating to the groups or their elements.

*Question 3 & 5 - Familiar group & unfamiliar representation*

**Question 3 (F/U)**

Let $n \in Z^{+}$ and let $M_n$ = {all linearly independent $nxn$ matrices $|$ all rows contain exactly one 1 and $n-1$ zeroes }. For example,

$$\mathrm{M}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\mathrm{M}_3 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ldots \right\}$$

A matrix $\mathrm{M}_\sigma \in M_n$ acts on an *n*x*n* matrix $\boldsymbol{A}$ from a non-empty set $\boldsymbol{X}$ through matrix multiplication i.e,

$$\mathrm{M}_\sigma(\mathrm{A}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Does $M_n$ form a group for all $n \in Z^{+}$ under the operation of matrix multiplication?

Question 3 and Question 5 (which compares Question 3 to a special case of Question 4 – see below) contain the structure of the permutation group which should be familiar to participants not only from group theory but also from two mandatory statistics courses which require in depth knowledge and application of permutations. Because of their familiarity with permutations, I expected participants to be aware that any group that reorders a set in all possible ways or represents choosing without replacement has the structure of the permutation group and thus satisfies the group axioms. A familiarity with permutations would mean that participants do not need to display a general understanding of the general group concept but only of a particular group learned in class.

The representation of the group is unfamiliar as the elements of the group given in Question 3 are matrices which are not commonly used in a group context or to represent permutations. I expected participants to be familiar with matrices as they had taken mandatory courses in linear algebra prior to their group theory course. The matrix elements cannot be easily plugged into the group axioms to solve Question 3 especially since participants had to show that $M_n$ is a group for all $n \in Z^+$. Question 3 could not be solved by performing calculations (i.e. matrix operations for all $n \in Z^+$).

The solution was contingent on an ability to discover the general structure of the group and ideally identify it as a permutation group with a cardinality of $n!$. The unfamiliar representation should test what a participant sees as important when examining a group when they cannot perform a canonical procedure such as plugging symbols into the group axioms. In contrast to Question 1, this should test whether a familiarity with a group aids in conceptual answers.

**Question 5 (F/U)**

5) For a given set $X = \{x_1, x_2, \dots, x_n\}$ Let $G_n$ be all bijections $\psi: X \to X$

Is the group of functions $G_n$, under the operation composition, isomorphic to the group of matrices $M_n$?

Question 5 is a restatement of the textbook definition of the symmetric group: *"Let Ω be any nonempty set and let* $S_\Omega$ *be the set of all bijections from Ω to itself (i.e., the set of all permutations of Ω) The set* $S_\Omega$ *is a group under function composition:* ∘.*"* (Dummit, 2004) This definition should have been more familiar than matrix representation but less familiar than $S_n$, $n$ −cycles and $n$ −gons. I expected participants to be familiar with the concept of a bijective function acting on a discrete finite set from previous courses as well as their group theory course.

Through these unfamiliar descriptions of the permutation group, Questions 3 and 5 aim at testing participants understanding of the concept of the symmetric group by checking to see if they can recognise the defining properties of a permutation group when presented as a group of matrices or functions. If participants can see that the elements of $M_n$ and $G_n$ represent choosing without replacement or a reordering of a set or that any element of $M_n$ would permute the rows of an arbitrary $nxn$ matrix through matrix multiplication (a hint given in the question,) then they can draw on what they have learnt about permutation groups to check the four axioms but could not rely on symbolic manipulation and canonical procedures due to the unfamiliar representation and difficulty in setting up a bijection.

*<u>Question 4 - Unfamiliar group & familiar representation</u>*

**Question 4 (U/F)**

Let $G = \{ f \mid f$ is a bijective function on a set X$\}$. Is $G$ a group under the operation $f * g = f \circ g$ (the composition of $f$ with $g$)? Prove why.

The group of all bijective functions is not a familiar one but participants have worked with functions and composition of functions in numerous courses. As this group is of an unfamiliar type, participants could not rely on knowledge of this group or familiar procedures to check the group axioms.

The presentation of Question 4 is familiar as the phrasing of the question directly gives the elements and binary operation of this group in a way that allows for symbolic manipulation to be used to find a solution to this problem. Functions under composition abide by laws of combination known to participants and can be applied outside of the context of this group. The solution is straight forward and can be achieved through processes that I expect are familiar to participants who have carried out basic proofs with functions in other courses such as the introductory analysis courses taken prior or concurrently to the their group theory course.

Verifying that the given set and binary operation constitute a group does not require a deep understanding of the group concept but a basic knowledge of bijective functions, symbolic proofs and recalling the abstract definition of group as this question can be answered by plugging functions which have an operative symbolism directly into the group axioms.

*Question 6 – Familiar group & familiar representation*

**Question 6 (F/F)**

| |
|---|
| Show that $S_5$ is a group under the binary operation of composition |

In Question 6, the symmetric group $S_5$ (or all permutations on 5 elements) is directly referred to as $S_5$. The elements of this group under the given binary operation should be very familiar to participants under this name. I expect that participants have seen proofs and explanations as to why $S_n$ is a group and have worked with symmetric groups on their own on numerous occasions in contexts such as classroom examples, assignments, the textbook and tests for their group theory course. Therefore, they should be able to draw on their past experiences and relate their knowledge of the permutation group to a familiar geometric or algebraic representation to execute a familiar process in order to correctly demonstrate Question 6 through a symbolic or geometric proof or a well-defended rationalisation.

This type of problem does not necessarily test an understanding of group concepts as it can be solved by mimicking familiar techniques and is meant to serve as a control question to see how participants approach this type of question in comparison to the less familiar or procedural questions.

## 5.3 Analysis of results

As stated above, 4 students who have successfully completed their first abstract algebra course were interviewed. In what follows, I analyze the results of these interviews, question by question, in comparison to the expectations outlined above. I will describe the effect that the familiar/unfamiliar representation and group type had on participants' responses and the challenges they created. In this way I hope to isolate how these two sources of difficulty affect a group theory student's ability to answer a question.

A discussion of this analysis is presented in the next chapter.

### *5.3.1 Analysis of results question by question*

**Unfamiliar groups with unfamiliar representations: Question 1**

The hypothesis was correct for both categories of unfamiliarity for Questions 1 and 2. In terms of the unfamiliarity of the group; none of the participants were familiar with the concept of space transformations in either 2 or 3 dimensions in the context of group theory. Participants attempted to resolve their unfamiliarity with this group by comparing it to a familiar group. Three of them compared $(M,*)$ to the symmetric or dihedral groups, namely $S_8$, $S_3$ and a non-specific variation of a 3-dimentional dihedral group. For example, after being told that a cube moving in $R^3$ was an example of $(M,*)$, Participant 3 counted off the 8 corners of a cube and then asked if $S_8$ would be the appropriate group for this problem.

The most common "familiar" aspect that participants took from this group was that it had a geometric representation. Their statements on $(M,*)$ were generally based on relating this group to other known groups that had geometric representations. In general they tried to fit this group into the context of the symmetric group because it involved isometries and rotations of an object. They did not examine the general structure of this group.

In terms of representation, the fact that the problem did not explicitly state the laws of combination for the elements and the binary operation, and thus the elements could not be directly plugged into the group axioms, was a great challenge for all participants. Three participants attempted to resolve this difficulty by a comparison to familiar objects; they described the group using a familiar notational system on which they could perform calculations and use it to establish what certain elements were. These notional systems were also mainly based on the geometric aspect of this group. For example, Participant 4 tried to establish a system based on $R^3$ stating that the elements would be of the form $(a, b, c)$ and explained that the identity was the point $(1,1,1)$. He performed mathematical manipulations that were more in line with viewing the elements of $(M,*)$ as scalars rather than vectors. For example, he multiplied the arbitrary element $t_1$ by the point $(1,1,1,)$ to get back $t_1$. Working along similar lines, Participant 3 found the point $(0,0,0)$ to be the identity and defined group inverses as the ability to return to the origin.

Participant 1 completely bypassed her unfamiliarity with the group by plugging symbols $t_1, t_2$ into the definition of a group and taking the integer $1$ as the identity to prove the required axioms. She did not take the rhetoric aspects of the group into much consideration working almost exclusively with these arbitrary elements, even after prompting.

Interviewer: Do you think there's an identity in there?

Participant 1: Well if it's $t_1 t_2$ in M. Okay so $t_1 * t_2$ if you take $1$ times $t_1 * t_2$. If you multiple $1$ by $t_1, t_2$ in M, you're still going to get $t_1, t_2$.

These relations to the familiar for the group and notational system also do not accurately reflect the structure of the group which clearly was not grasped by the participants of this study. Participants did not appear at ease using conceptual answers and preferred to use mathematical comparisons to the familiar.

**Unfamiliar groups with unfamiliar representations: Question 2**

Question 2 proved even more difficult and required an understanding of more advanced group theory concepts of subgroups, quotient groups and isomorphism. Participants were walked through the nature of $(M,*)$, why it was a group and the types of motions it contained so they could apply knowledge of this still fairly unfamiliar group to answer Question 2. All four participants admitted particular weakness in the area of quotient groups and their understanding of the isomorphism concept appeared to be equivalent to the ability to create a bijection between two groups. There were almost no signs of a more evolved understanding of isomorphism. None of the participants considered applying knowledge of $(M,*)$ to discern what the subgroups $S$ and $T$ were. Instead they attempted to apply general concepts and procedures relating to subgroups, quotient groups, isomorphisms and homomorphisms. Participants hardly took the elements or structure of the group into consideration when answering these two questions.

For all four participants, their first instinct was to assume that the way to go about showing the isomorphism in Question 2 was to apply the first isomorphism theorem. This is most likely a result of the similarity between the representation of the question (which asked participants to show that $M/S \cong T$) and the first isomorphism theorem (as stated in the textbook: *"If $\varphi : G \to H$ is a homomorphism of groups, the* $\ker(\varphi) \trianglelefteq G$ *and* $G/\ker(\varphi) \cong \varphi(G)$".

In general, their explanations related to the fact that a homomorphism was required to answer Question 2. However, they either could not figure out what it would be or they attempted to create a purely symbolic homomorphism, such as Participant 1 who again wrote a modern symbolic proof based on arbitrary elements with a syncopated symbolism.

> Participant 1: So $S$and $T$ are already in $M$ because they're subgroups and you have $M/S$. So $M/S$ is isomorphic to $T$. How I would go about it? Say you have an element $x$ is in $S$ and let's say $y$ is in $T$ and both of these are in $M$ so let's say $x, y$ is in M So basically you can just show $(x, y)/x$ is isomorphic to $y$.

Only Participant 3, after being prompted to abandon finding and defining a homomorphism on the groups, thought of establishing what the subgroups $S$ and $T$ were in reference to the question and then managed to rationalise out the proposed isomorphism. This however was based on the false assumption that $M$ had only two elements; rotations and translations (group $(M,*)$ has an infinite number of elements). All other participants continued to use $S$ and $T$ or various other symbols in an attempt to prove general properties about the group and quotient groups and set up a bijection even after the interviewer's intervention.

Participants were not capable of applying a general knowledge of group theory concepts to solve these two questions nor could they create a symbolic system to accurately reflect the system. They often confused the syncopated algebra with symbolic algebra.

**Familiar group unfamiliar representation: Question 3**

The group in Question 3 was not fully recognised as an unfamiliar representation of the familiar symmetric group by any of the participants. Three of them did not perform enough meaningful calculations on the matrices or examine into the properties of these matrices to acknowledge that they represent permutations. They examined individual elements and performed a few calculations but did not pay attention to the structure of the group – something that, considering their level of mathematics, I expected they would identify as a central component of group theory.

The first reaction for all participants was to perform operational procedures with matrices such as multiplying them together or finding their inverses and to consider particular matrices displayed on the page. Their attempts to verify the group axioms were generally based on properties of matrices, not groups, and were too vague and overly generalised. In particular, their approaches for verifying closure and inverses were oversimplifications and insufficient. For example, three participants worked on the assumption that a non-zero determinant was an if and only if condition for checking group inverses and did not see the difference between mathematical inverses existing in general versus inverses existing inside the group $M_n$ as a condition for satisfying the group axiom pertaining to inverses.

For closure, participants generally checked a few matrices and then extrapolated closure for the group. When asked how they knew they would never get a 2 or a row of 0s, Participants 1 and 4 insisted that induction was the only method they could think of to go about answering the question for all $n$. Participant 2 stated that she was *guessing* about closure and asked for a specific example she should look at. Participant 3 tried to recall the double sum formula for matrix multiplication.

Overall, three participants did not get any sense of this group and displayed a general weakness in how to show that a given set and binary operation is a group or how to identify characteristics of a familiar group. The only exception was Participant 3 who thought of examining general properties of the groups $M_2$ and $M_3$ (not just general properties of matrices.) He eventually determined that there were a limited number of choices for each row. Applying knowledge from statistics to group theory, he determined that the choices for each row were "choosing without replacement" and the group had a cardinality of $n!$ . When asked if these characteristics gave him a hint as to what group this was he responded, "No, why would that give me a hint?" and did not identify these characteristics as being clear indicators of the symmetric group. Participant 3's answer was based more on knowledge of matrices and permutations than group theory and he was not able to show that this was in fact a group.

The unfamiliarity of the representation meant that participants could not plug symbols which abide by laws of combination into the group axioms. In a similar manner to their reaction to the unfamiliar representation for $(M,*)$ in Question 1, participants attempted to answer Question 3 by relating the checking of axioms to the familiar. In this case they examined properties of matrices such as inverses and identity; once again without stepping back to get general a sense of the group.

Similar to the desire to use the first isomorphism theorem in Question 2, three out of the four participants assumed (either eventually – Participants 1 and 2, or immediately –

Participant 4) that a proof by induction was the only way to solve Question 3 without being able to explain why knowing that $M_n$ is a group would impact their knowledge on whether or not $M_{n+1}$ is a group. Participants felt that the general and powerful proof by induction was the only way they could solve Question 3 even though they could not explain why it was applicable in this case and did not think of examining the general structure of the group.

**Familiar group unfamiliar representation: Question 5**

For Question 5, all participants attempted to prove that $G_n$ and $M_n$ are isomorphic before examining $G_n$ in any capacity (in a similar fashion to their attempts to construct an isomorphism in Question 2 without analysing the involved subgroups $S$ and $T$.) This shows a continuing trend of answering questions without any reflection on the nature or properties of the groups they are working with. It also reinforces the conjecture that their understanding of isomorphism is equivalent to the existence of a bijection: once more, in Question 5, Participants 1, 3 and 4 tried to directly compare the two groups to show that they were isomorphic by creating a bijection between the matrices in $M_n$ and the set $X = \{x_1, x_2, \dots, x_n\}$.

After being prompted to compare $G_n$ (not $X$) to $M_n$, Participant 3 recalled "I remember from linear algebra that any transformation can work as a matrix," and was again the only one to establish that $G_n$ was a group of permutations in order to develop some sense of isomorphism.

Participant 3: " … (after rereading Question 5) which means things that map elements of $X$ and basically shuffles the order. Okay. $M_n$ on the other hand does the same thing, it takes those three lines and shuffles their order so that, that's the answer."

Participant 2 insisted that the comparison of these unfamiliar groups required unknown information (mainly what a group action is) and that there was "a nice relationship" that she could not figure out or remember.

Since three of the participants did not recognise the permutation or choosing without replacement aspects to both the set of all bijections on a finite set and the matrices $M_n$, it would have proved extraordinarily difficult to answer Question 5; the general feeling was that there was a need to establish an unknown homomorphism to show that the isomorphism exists.

The hypothesis was correct in terms of the unfamiliar representation for Questions 3 and 5. The familiarity of the permutation group was not of great assistance for answering these questions.

**Unfamiliar group familiar representation: Question 4**

The hypothesis was correct for Question 4 as no participant acknowledged previously seeing the given group but the familiar elements and symbols allowed them to prove the group axioms. All participants immediately plugged functions $f$ and $g$ into the group axioms and the mitigating factors were their varying abilities to perform symbolic proofs and their knowledge on the behaviour of functions. Only Participants 3 and 4 rigorously proved all axioms and expressed affinities for symbolic proofs. Participant 2 who showed an appreciation for rationalising out the solution to Question 1, tried to rationalise out the axioms of Question 4. "If $f$ is onto $R$, from $R$ to $R$ and $g$ is onto $R$ then I'm going to get $R$. Yeah if $f$ is 1-1, I can get, then g is 1-1 then $f$ of $g$ is 1-1." While these are correct assumptions, they are incredibly difficult assertions to establish rhetorically, but can be simply accomplished by a symbolic proof which was never done by Participant 2. (For any $x_1, x_2 \in X$ such that $x_1 \neq x_2$, then $g(x_1) \neq g(x_2)$ as $g$ is injective → $f\big(g(x_1)\big) \neq f(g(x_2))$ as $f$ is injective, so $f(g(x))$ is injective by definition of injective. Here we are abiding by the rules of functions possessing the property of injectivity based on the standard modern symbolic definition of injectivity.)

Participant 1 struggled with properties of functions and with checking group axioms. She failed to properly check all the axioms due to misconceptions such as stating that the integer 1 was the identity element of this group; however, she made far more progress on this question than any other question.

The abstract symbolic representation of Question 4 and the familiarity with the elements of the group, allowed the participants to go about answering the question in a correct manner, even though they were not familiar with this particular group or its structure. They did not make vague and general assumptions about functions in Question 4 like they did with the matrices in Question 3. All participants checked (or attempted to check) all axioms and did not attempt to extrapolate or generalise in their justifications – something they did in all other questions. None the participants attempted to get a general sense of this group or why it was a group, and were generally contented by the fact that they were capable of testing the group axioms with symbolic proofs.

**Familiar group familiar representation: Question 6**

All participants stated that they were familiar with $S_n$ and referred to it by some means as being the permutation group on $n$ elements. The hypothesis was incorrect as they were not capable of conveying much else about this group and did not recall any of the processes I expected them to be familiar with; process they had used to work with this group while taking the abstract algebra course such as rotating an $n$ −gon.. No participant was capable of fully checking all of the group axioms.

In general, they felt frustrated by an inability to remember exactly what this group was and decided they were incapable of answering this question since they did not remember how to go about. No participant appeared to have understanding of the structure of the permutation group or why it was in fact a group. This question appeared to be even more difficult than preceding questions as participant appeared dejected by the fact that they could not remember how to prove that $S_5$ is a group.

The only real attempts made to solve this question involved building it up from scratch and writing out all of the elements. For example, Participant 4 felt incapable of working with $S_5$ and decided to work with $S_3$ instead.

Participant 4: "We know $S_3$, $S_3$ is {e, (12), (13), (23), (123), (132)} it has 6 elements and it has the identity so e into (12) is equal to (12), into e is equal to (12).

Interviewer: Does it have inverses?

Participant 4: It does have inverses, (12) into (12) is equal to e, also (13) into (13).

We see once more an inability to work with a group unless it is through working with its elements. Furthermore, Participant 4 only tested a few elements before deciding that this was a group. The way Participant 4 stated that "$S_3$ is {e, (12), (13), (23), (123), (132)}" would indicate that he sees this group as a collection of elements. Participant 2 also felt the need to construct the group and write out all of the elements. "$S_5$ so you're talking about (starts writing $n$ −cycles) I can't write out the group but it's permutations on 5 things right?" and after prompting, she vaguely generalised properties of $S_5$ often fixating on conditions that do not need to be satisfied such as being finite and not being commutative.

Participant 3 described how he would go about solving this problem if he could remember the correct procedure.

Interviewer: So let's say you had this on a test, how would you go about it?

Participant 3: I'd just look at the definition of the set, because they always say $S_5$ equals like a set, there's a rule to that set, I'd just analyze the rule of the set and then play with it to get to find the inverses, to find each I'd take an arbitrary element and I would just find the inverse basically. Which is not that hard because once you have the rule in front of you, you can basically visualise, you can see where things go. It's harder for me to visualise really abstract $S_5$, you know what I mean."

### *5.3.2 General overview of results*

Participants did not display a conceptual understanding of group concepts and generally seemed to understand concepts as their definitions. Their ability to check if a given set and binary operation was a group was usually dependent of an ability to plug symbols into the four axioms to verify them. With little exception, they did not see the structure or general behaviour of the group as important even though group theory is commonly acknowledged to be the study of structure and symmetry.

The intended contrast between the unfamiliar groups (the groups of motions and bijective functions) and the familiar group (the symmetric group) could not be properly established. Participants did not thoroughly investigate group properties and relationships in the representations of the symmetric group in Questions 3 & 5 and did not demonstrate extensive knowledge of this group in Question 6 beyond stating that it is the symmetric group, nor did they remember any familiar processes used to show that this is a group. This would indicate that familiar groups are not of great assistance in problem solving and that presenting general properties or the structure of a group is not of great importance in helping participants answer questions. They seem to neither recall nor examine general group properties or structure.

The contrast between unfamiliar and familiar symbolic representations was clear. Participants needed familiar symbols and rules; without them, they felt incapable of answering questions. Without a familiar symbolic system to work with, participants generally resorted to using symbolic systems that did not accurately reflect the situation being described, extrapolated meaning to syncopated symbolism and simply plugged arbitrary symbols into axioms. Participants put greater emphasis on the specific elements and symbols given in the question than the nature of the group. They did not take a lack of symbols as a sign that they should consider the situation and structure being described and try to establish a sense of the group; instead they often worked with flawed symbolic systems and canonical procedures. They did not appear at ease with conceptual or theoretical answers.

Over all, unfamiliarity was met with flawed comparisons to the familiar both in terms of groups and symbolic representation. When faced with an unfamiliar group, the most common reaction was to try to fit the given problem into the context of a familiar type of group or symbolic representation. This often led to a loose and flawed checking of axioms and properties.

Participants constantly reverted to the familiar and to flawed canonical procedures in answering these questions. When there were similarities between given questions and familiar theorems or processes, they would immediately try to use these theorems without being able to explain why they were useful; e.g., the desire to use the first isomorphism theorem in Question 2 and a proof by induction in Question 3. This would support the hypothesis that students believe that in order to find the solution for a given question, they are supposed to use theorems and procedures that “work” – although they don’t understand them.

# Chapter 6

## Discussion, conclusions and recommendations

In this section, I will analyze the difficulties encountered by the participants of this study in terms of three factors that impact a student's ability to understand group theory and to properly answer questions. These factors are categorized as follows:

1. Difficulties stemming from a lack of knowledge of basic group theory concepts and concepts covered in previous courses;
2. Conceptual difficulties relating to misconceptions of what group theory is and what it studies; and
3. Difficulties that arise due to a lack of knowledge of modern abstraction, which dictates the manner in which they are excepted to think about and to complete proper proofs in group theory.

I postulate that a group theory student should have a reasonable understanding of group theory in terms of these three areas, and be able to combine their knowledge in these three areas, to succeed in their group theory course. First of all, a student needs to know the basics such as the standard definitions of group concepts and particular groups and be able to carry out basic calculations and procedures. This also refers to topics covered in previous courses and an ability to work with entities such as matrices, functions and permutations.

Furthermore, a student should have an understanding of group concepts that goes beyond these standard definitions. They should have a theoretical sense of what they learning and not feel that it is simply abstract. They should espouse a group theoretic mentality when describing mathematical systems. Finally a student should know how to interpret the modern abstract formulation of group theory and be skilled in proofs in modern algebra. This requires them to see modern abstract statements as something deeper that a collection of symbols and be skilled in rigorous symbolic proofs in axiomatic systems.

Overall, participants had large gaps in their basic knowledge, in particular, of matrices, group axioms, and specific groups. They did not seem to have a conceptual grasp of what a group is, what an isomorphism is, the systems described by specific groups or what group theory studies in general. They experienced great difficulties in using modern algebraic proofs to describe systems and as seeing modern algebraic definitions as something meaningful.

## 6.1 Difficulties stemming from lack of knowledge of basic group theory concepts and concepts covered in previous courses

Some of the difficulties encountered were due to a lack of knowledge of topics covered in previous courses. This was apparent in Question 3 where participants demonstrated a weak knowledge of linear algebra which made it difficult for them to work with the matrices in $M_n$. Many of the difficulties experienced in Question 3 can be attributed to poor knowledge of linear algebra. Participants did not want carry out matrix multiplications or were incapable of doing so. As a result, they performed mainly simple or trivial calculations which they then took to infer broader assumptions. For example, Participant 2 carried out *all* possible multiplications on the elements of $M_2$, including multiplying out by the identity, to show that $M_2$ was closed and had inverses. She was then unwilling to work with larger matrices as she did not properly recall how to write products for larger matrices and did not feel comfortable working with larger matrices. She then attempted to extrapolate the results she obtained for $M_2$, for lager $n$ without being able to justify why the results for $M_2$ would apply for larger matrices.

In Question 3, it was given as hint that a matrix in $M_3$ could be multiplied by an arbitrary matrix $A$ (see Chapter 5). This would have shown that any matrix in $M_3$ would permute the rows of an arbitrary matrix, hopefully revealing its connection to the permutation group. The participants were told that this could be used as hint for Questions 3 and 5 but was not strictly required to solve the problem. None of the participants, however, appropriately used this hint. I surmise the reason to be a weakness concerning basic properties of matrix multiplication. Further support to this conjecture was brought by Participant 1 who after multiplying a general matrix $A$ by the identity matrix $I$ obtained a diagonal matrix (not $A$). She did not find this striking or go back to fix it:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \text{ x } \begin{bmatrix}\text{a} & \text{b} & \text{c}\\\text{d} & \text{e} & \text{f}\\\text{g} & \text{h} & \text{i}\end{bmatrix} = \begin{bmatrix}\text{a} & 0 & 0\\0 & \text{e} & 0\\0 & 0 & \text{i}\end{bmatrix}.$$

Considering that participants of this study had successfully passed advanced linear algebra courses, I expected them to be proficient in matrix multiplication; such proficiency would have allowed the participants to work comfortably with $3x3$ matrices and to progress further in this question. However, in numerous instances, participants engaged in trivial activities such as multiplying by an identity matrix, or acknowledged they were not able to perform generalised calculations with matrices of arbitrary size.

Furthermore, they experienced a number of difficulties in remembering basic properties of matrices. For example, all participants were unable to properly use the condition of linear independence in $M_n$; they seemed unclear on the definition and implications of linear independence. This hindered their abilities to get a proper sense of the matrices in this group. For example, Participant 2 seemed to either ignore or not understand the meaning of linear independent:

> Interviewer: So each row contains exactly one 1 and the rest are all 0's and they're linearly independent. (Restating the definition of $M_n$, given in Question 3.)
>
> Participant 2: This row only has one, this row only has one, this row only has one. We're not saying they have to be in separate columns, right?

Similarly, Participant 3, the only participant to correctly answer Question 3, did not initially grasp linear independence and stated that each matrix in $M_3$ had three options for every row which represents "choosing with replacement".

In general, the participants did feel at ease working with matrices in terms of performing calculations and their basic knowledge of matrices. They performed calculations and made use of theorems they did not understand (proof by induction) to answer Question 3.

Their work with matrices can be seen in contrast to their abilities to work with functions in Question 4 which did not pose many technical difficulties, likely because participants have worked with functions regularly since high school. Participants were capable of recalling the relevant definitions for injective and subjective functions and were mainly successful in using these definitions to write out symbolic proofs. In this sense, familiarity with the objects may reduce difficulties. As I discuss later, this is a *genuine* familiarity – as opposed to a familiarity due to exposure but not to understanding.

Participants also appeared to show great weakness in recalling basic knowledge of group concepts. Questions 1 and 2 were particularly useful in identifying apparent weaknesses relating to the testing of the four axioms. In answering Question 1, Participant 4 replaced *closure* with merely having a binary operation. Participant 1 appeared to believe that the closure property meant that you can multiply but not divide: "If you have $t_1, t_2$, you can't get $t_1/\ t_2$ if you do a certain operation."

All participants often checked and referenced commutativity or confused commutativity with a group axiom even though commutativity is not one of the group axioms. For example, in Question 1 Participant 3 used commutativity to show associativity and Participants 2 and 4 stated that the inverse of $t_1 t_2$ was $t_2 t_1$ which is the same element in the abelian group $(M,*)$ and not the inverse.

All participants showed severe difficulties in working with more evolved concepts namely subgroups, quotient groups, homomorphism and isomorphism. They often stated falsehoods; for example, Participant 4 attempted to answer Question 2 by using the "fact" that all proper subgroups are abelian and all abelian subgroups are cyclic.

Interviewer: I'm just saying that the number of proper subgroups is 2 and that's really all I'm telling you. That you have 2 proper subgroups

Participant 4: (Pause) so it's abelian … if it's abelian, the subgroup is cyclic.

In this way, difficulties can be attributed to the participants' abilities to recall definitions and carry out calculations. Due to the nature of this study, the interviewer could remind participants of definitions they had forgotten or recalled erroneously. While a basic knowledge of group theory is important, a larger goal of this study was to view how students when about solving problems. It was disconcerting to note that students generally felt that they should remember concepts rather than understand them.

Participants had a tendency to blame their difficulties on not remembering or not knowing concepts and examples from group theory. They often stated that what was required to solve a given problem was something they could not remember but should have learned in either their group theory course or another course. For example, when trying to show closure for Question 3, Participant 3 stated

Participant 3: "if I just had the formulas for matrix multiplication, if I just looked it up in a linear algebra textbook … there's some double sum thing. A nice neat compact way of writing it right there, it will prove to me closure."

When discussing Question 1, Participant 4 repeatedly related his difficulties to a lack of knowledge from other courses even though the topics he was referencing were not required to solve this problem:

Participant 4: I never thought of 3-d before. I forgot calculus II. [...] This is vector spaces right, I haven't studied vector spaces.

Interviewer: for this you don't need vector spaces.

Participant4: It's a block in my brain.

Participants seemed to experience complete mental blocks as reactions to problems they felt they should remember how to do and would then refuse to go any further. For example, when Participant 2 was trying to define a homomorphism in Question 2, she gave up and said, "It's not occurring to me, I can't remember what we did and it probably had something to do with um, uh permutations on the set and I'm not sure how to apply it." (Question 2 is not related to permutations.)

Participants would commonly feel blocked due to an ability to remember procedures or definitions. In this sense, familiarity acted as an obstacle; but it was in fact *familiarity due to exposure* – participants remembered having been exposed to concepts but they hadn't in fact *learned* the concepts. As participants commonly felt that concepts should be remembered rather than understood, their approaches and knowledge were scarcely conceptual.

## **6.2** Conceptual difficulties relating to basic misconceptions of what group theory is and what it studies

In this section, I examine participants' conceptual understanding of what they learned in their group theory course and their ability to use general group theory concepts in answering questions. This does not refer to their ability to remember basic procedures and definitions. I will examine their understanding of group theory in general as well as the motion group, symmetric group, and the concept of isomorphism.

Overall, participants did not appear to have a conceptual view of what group theory is or what it studies and did not feel at ease giving theoretical answers. They were not aware of where group properties came from or why they are important, interesting and useful. Participant 3 said of group theory that "It's like some deep dark cosmic secret. This is why I like analysis better." The participants felt that what they were studying was simply and entirely "abstract" and their goal was simply to prove statements. This sentiment was indicated by Participant 1 who said in reference to her group theory courses:

> Participant 1: When you do this kind of stuff and you have to prove it's a subgroup or something, I think it's ... I never actually think too much about it. I guess I don't have time to think what it is in terms of … I just kind of do it.

As seen in Chapter 2, group theoretic thinking was historically conceptual and pragmatic; it is a way of examining a system in terms of its structure and symmetry that allows for the use of powerful heuristics. None of the participants appeared to have this impression of group theory or display an ability to understand what a group is beyond listing the four axioms. I would postulate that a conceptual understanding of groups is vital in learning group theory and would have greatly aided participants in answering the questions in this study as well as give them a sense of meaning and purpose in studying abstract algebra. Klein's fear that a student of group theory "confronts something closed, does not know how one arrives at these definitions, and can imagine absolutely nothing" would appear warranted.

The reason for giving unfamiliar groups was so that participants could not employ familiar procedures and would be forced to rely on general theoretical knowledge. This was meant to test their ability to examine a system with a group theoretic mindset. As seen in Chapter 2, group theory is a way of thinking about and classifying mathematical systems. When Jordan was defining the motion group he examined motions in terms of their position, rotation and displacement so he could fit them into group theory. I wanted to see how the participants would go about interpreting the motion group which they had not seen in class.

As seen in Chapter 5, participants tried to understand the unfamiliar group $(M,*)$ by relating it to the symmetric group. The initial important characteristic they identified in this group was that it related to a shape undergoing transformations. They took this to mean that it should be in the family of the symmetric and dihedral groups (possible the only groups with geometric representations they had seen in their abstract algebra course.) The participants of this study did not seem to be aware of which significant properties they should be abstracting or how to relate a mathematical system to group. They used flawed representations they had seen before which pertained to other systems and did not feel at ease giving theoretical answers. There was a certain laxness in the manner participants associated groups with the system being described. They mainly felt that this should relate to some group learned in class and had great difficulty building up this group and describing elements in the group. This group is meant to describe all possible motions but the participants did not get a sense of what this meant in a group theoretic context.

This trend continued in Question 3 where participants did not try to get a general sense of the matrices in $M_n$. As stated in the Weber Alock (2004) study, mathematicians can "see" groups and know what properties are important and what formulae will work. (Weber and Larsen, 2004) With the exception of Participant 3, they did not see the importance of stepping back to get a general sense of the properties of a group but were mainly focused on pushing through the testing of axioms. In this way they did not see a group as a whole structure but only in terms of satisfying conditions.

Participants did not identify the structure of the symmetric group in Questions 3 and 5. Even Participant 3 who acknowledged that the matrices in Question 3 were in direct correspondence to choosing without replacement did not associate this group with the symmetric group. As seen in Chapter 2, the main properties of group theory were extracted directly from the properties of permutations on the roots or arguments of polynomial equations, and group theory was inseparable from permutations for over 100 years. It is disconcerting to observe the lack of knowledge of the symmetric group considering its place at the foundation of all group theory. All participants acknowledged they had seen the symmetric group $S_n$ in class and that it was connected to permutations. They appeared to have simply been exposed to the symmetric group and did not have genuine familiarity with it and could not answer Question 6. Responses to Question 6 were centered on seeing the group as $n$ −cycles and building the group up from its elements. This question was mainly greeted with mental blocks and participants stating they could not remember how to prove that $S_5$ is a group. This would seem to display a great deficiency in understanding of what group theory is and how and why it was developed.

One of the greatest conceptual difficulties in group theory, observed in the reviewed literature and during the course of this study, is that students commonly understand concepts by their abstract definition. In particular, students generally understand isomorphisms as a bijections and not as two groups being structurally the same. The concept of isomorphism was firmly established by Jordan when working with the motion group with the example given in Question 2. This was not grasped at all by the participants who interpreted this isomorphism as the possibility of matching up elements. Participant 1 attempted to construct a proof of $M/S \cong T$ for Question 2 by showing elements were isomorphic:

> Participant 1: Say you have an element $x$ in $S$ and let's say $y$ is in $T$ and both of these are in $M$. So let's say $(x, y)$ is in $M$. So basically you can just show $(x, y)/x$ is isomorphic to $y$.

Stating that two elements are isomorphic to each other is a clear misconception of what isomorphism is and a further support to the conjecture that students tend to understand concepts as their definitions.

In general, the definitions, formulas, and theorems used by the participants did not indicate much conceptual knowledge as to why they were using them or what they represented. All participants seemed to believe that the way to answer questions was by symbolic manipulation or using a powerful formula rather than getting a sense of what the group is or does. They did not appear to place much importance on the conceptual components of group theory. This may be a result of the fact that the ideas and concepts in group theory tend to be condensed or hidden in their abstract formulations. It appears difficult to gain a balance between a clear understanding of the concepts of group theory and the modern abstract formulations and manipulations typically used to teach these concepts.

## 6.3 Difficulties related to abstraction

Here I will analyze familiarity with modern algebra in terms of the three characteristics described in Chapter 3; symbolism, observing relationships (not objects) and axiomatic systems. Abstraction is certainly one of the root causes of difficulties experienced by students of group theory. This difficulty is in essence two fold as students must be capable of completing symbolic proofs but they must also be able to interpret and give meaning to modern symbolic statements and definitions. I will demonstrate the confusion observed in this study between syncopated algebra and symbolic algebra, the apparent laxness in axiomatic justifications and physical justifications and how participants generally saw groups as a collection of object and not in terms of generalised relationships.

### *6.3.1 Symbolism*

As seen in Chapter 3, algebraic thought is commonly characterised according to the role played by symbolism. There is a distinction drawn between syncopated symbols which are abbreviations that can only be used in the context of the problem and modern symbols which are known to be governed by general, pre-stated rules. The participants were fairly adept at symbolic proofs in Question 4, but struggled greatly with proving statements using syncopated symbols, matrices and the symmetric group. The participants did not appear capable of making a clear distinction between modern symbolism and syncopated symbolism when working with these entities in the study.

Modern symbolism was present in the familiar representations, such as $f$ representing a generic bijective function that can be manipulated independently of the specific context of Question 4. Operative symbolism naturally lends itself to being plugged into axioms and equations. There are rules and laws of combination known to be applicable to $f$ by virtue of it being a bijective function such as "for any $x_1, x_2 \in X$ such that $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$". These types of modern symbolic statements represent properties that are specific to certain entities such as bijective functions. A true understanding of symbolism would dictate a direct correlation between a symbolic statement and a behaviour which is always true. Symbolic statements that are used in mathematics have meaning but the participants experienced great difficulties interpreting symbols and using symbolism to accurately define a system.

I postulate that an understanding of symbolism would indicate that a student would not write out a symbolic statement unless it was connected to something that is true. The syncopated symbolisms present in the unfamiliar representations such as the symbols $t_1, t_2$ in Question 1 could not be operated upon independently of the context of the problem. These are simply abbreviations and require additional analysis to discern which rules to apply to these elements.

As seen in the question by question review in Chapter 5, the connection between symbolic statements and true statements was often not grasped by participants: all attempted on several occasions to govern syncopated symbols by rules that did not apply. Participants would often plug syncopated symbols into equations and formulas without establishing what these symbols represented such as identifying them as rotations and translations.

In Questions 1 and 2, all participants, at one point or another, worked directly with the elements $t_1, t_2$, $t_1 * t_2$ or created elements such as $t_1^{-1}, t_2^{-1}$, $t_1'$, $t_2'$ and plugged them into the group axioms without questioning which rules do in fact govern these symbols. For example, Participant 1 wrote $(t_1 t_2)^{-1} = t_1^{-1} t_2^{-1}$ to “prove” that inverses exist for all elements in $M$, without establishing first that $t_1$, $t_2$ or $t_1 t_2$ have inverses and without knowing why the statement $(t_1 t_2)^{-1} = t_1^{-1} t_2^{-1}$ would be true for these elements. Participants also attempted to solve Question 2 by matching up arbitrary elements in a bijection without establishing what $S$ and $T$ were or identifying any of their elements in a meaningful way. For example when Participant 4 was establishing the isomorphism in Question 2 he wrote “Let $t_1, t_2$ be in $M$ such that $\varphi(t_1)$, $\varphi(t_2)$, be in $T$, okay” and proceeded to set up a homomorphism based on this statement without examining what these elements might be. This behaviour was repeated in

Question 5 when participants attempted to create a bijection without establishing the elements of $G_n$.

Participants did not appear to acknowledge that the syncopated symbols $t_1, t_2$ cannot be easily plugged into axioms. They needed to establish behaviour and rules of combination. Participants would commonly write out $t_1^{-1}$ and use it in statements without being able to identify why it existed or what it was. Writing $t_1^{-1}$ is meaningless unless it is stated that this is the opposite or the undoing of transformation $t_1$ and inverses exist because every action can be undone while still remaining in the group $(M,*)$. As is the case for syncopated symbols, the answer is found by rationalising on the nature of the object the symbol represents; in this case rotations and translations. If the elements would have been renamed $(r, t)$ as rotations in radians and translations as represented by vectors, then they could have been operated upon symbolically, but no rules are known for $t_1, t_2$ without further investigation into their properties.

Participants would commonly place great importance on symbols that were written down. For example Participant 2 attributed meaning to the notation of the arbitrary symbols chosen to represent transformations in $(M,*)$. In answering Question 1, she began using $a$ and $b$ as arbitrary elements instead of $t_1$ and $t_2$.

Interviewer: What does it mean to be closed?

Participant 2: It means that if $a$ is an element of $M$ and $b$ is an element of $M$ then $a$ of $b$ is an element of $M$ (pause rereads question) ah no $t_1, t_2$…"

Clearly there is no difference in referring to elements as $a$ and $b$ instead of $t_1,\ t_2$ , this is the nature of algebra, elements can renamed. She felt that working with $a$ and $b$ instead of $t_1,\ t_2$ was relevant and stopped herself from proceeding with her work. She had difficulty regaining her train of thought and thought that what she said was somehow wrong because it was in reference to $a$ and $b$ and not $t_1,\ t_2$.

Symbols were commonly related to ideas and often once a symbolic or numerical language was written down, participants would proceed to work symbolically and essentially ignore the question. Overall, there was a lack of correlation between symbolism and meaning. Symbolic statements were often seen as entirely abstract. They often used symbolic statements as a random mechanism and not as the representation of rules with meaning behind them.

### *6.3.2 Axioms*

In this section I will analyze difficulties with axiomatic justifications and physical justifications. The relative nature of justification in modern mathematics did not resonate with participants who did not appear at ease working in axiomatic systems, (with the exception of Question 4,) or with physical justifications. The fact that truth in modern mathematics must be in direct correspondence to consistency within an axiomatic system was not displayed in the answers given. Participants seemed to generally agree with the concept of "truth through calculation" but were not particularly successful at executing this.

As seen in the question by question review, participants would often not check all of the axioms or elements before stating that a given set and binary operation was group and did not seem to have a deep sense of the strictness of conditions. The repeated occurrences of participants justifying statements by plugging symbols into formulae and axioms without knowing their laws of combination displays an inherent weakness in modern justifications. There were numerous occurrences of unfounded assertions involving the use of numbers to justify group axioms. For example, Participant 4 stated in Question 1 that inverses are obtained by multiplying out by the number -1 and Participant 1 was under the impression that the number 1 was the identity for all groups even though they could not justify these statements in any way. For example Participant 1 stated "If you multiple 1 by $t_1,\ t_2$ in $M$, you're still going to get $t_1,\ t_2$." None of the groups in this study contained numerical elements so the integers 1 and -1 could not be contained in any of the groups. The number 1 is not, for example, the same as the identity transformation in $(M,*)$ or the identity function on a set $X$ in Question 4 and cannot be used in justifying the group axioms. The participants needed to identify these elements and show they were in the group.

Participants often proposed and stuck to unrealistic ways of justifying statements such as using theorems they did not understand or did not apply at all, such as the first isomorphism theorem for Question 2 and proof by induction in Question 3. For the latter question, Participant 1 even proposed performing an infinite number of calculations as a viable method for showing $M_n$ is a group for all $n$.

With the exception of Question 4, participants were generally unable to justify or were apprehensive about appropriately justifying mathematical statements even if they appeared to possess a reasonable level of knowledge on the topic. For Question 6, after clearly displaying an understanding of the concept behind permutations in Questions 3 and 5, Participant 3 could not begin to show that $S_5$ is a group; he said: "You see I don't remember the fancy way of writing it all out." Participants felt that justifications were something they should remember and not understand. Modern justifications are not just a way of writing out something but showing beyond a doubt that a given statement is true in accordance with a given system.

Often participants would state that they were guessing or acting on "instinct" when asked if a certain property was being satisfied or if something was a group. After discerning that $M_2$ was closed in Question 3, Participant 2 then took this to infer that $M_3$ was closed as well. When asked why she responded "I'm just guessing, shh don't tell anyone." After reading Question 1 and following an intervention of the interviewer;

> Interviewer: You don't have to write a formal proof. If that's the way you want to solve it, that's great, but whatever (I just want to know what) your thought process is to get you (from) one place or another.
>
> […]
>
> Participant 3: I think, well I'm using intuition now, intuitively I think it forms a group because I've seen other groups with composition of functions and I know that follows ..., associative, all those things you have to check.

Due to the nature of this study, justifications could be rationalised out or proved symbolically. In particular, Questions 1 and 2 required either the creation of a representative symbolic system or a physical justification. Participants experienced difficulties accepting and constructing proofs justified by conceptual descriptions, logical reasoning and physical "reality".

The responses to Questions 1 and 2 were a bit of a hybrid combining physical justifications with symbolic proofs.

Participants had enormous difficulty grasping the group $(M,*)$ in a purely mathematical sense and generally referred to flawed physical interpretations to prove the axioms. This was demonstrated in Cartesian representations used to describe the group $(M,*)$. It was difficult for the participants to use real world or geometric justifications in their answers to Question 1 while properly representing the group. They also had difficulty abstracting the important features needed to describe the group. Participant 3 stated that left, right and up-down were three categories of motion and then used the aerial terms *roll*, *pitch* and *yaw* to describe $(M,*)$. He had trouble seeing that the group $(M,*)$ contained an infinite number of elements and could not shake the idea that it only contained two or three elements.

In general, participants had great difficulty with conceptual and physical justifications as well as with the mentality and the rigour required for justifications. They usually felt that they should mimic a standard procedure or used flawed symbolic or geometric constructions which did not reflect the reality of the group.

### *6.3.3 Objects vs. relationships*

Here I will examine difficulties incurred when participants view groups as collections of objects. As modern algebra commonly employs symbolism that exists outside of the context of a given problem, the properties of specific elements or objects in a given problem are not necessarily the interest of modern algebra. Group theory does not necessitate the correlating of groups to a particular set of objects. Modern group theory studies $a * b = c$ regardless of what $a, b$ and $c$ are as long as there are rules to govern them under the given binary operation. As detailed previously, participants almost never examined, discussed or referred to the idea of structure or relationships when answering questions. They tended to solely examine, and in fact placed great emphasis on the specific elements in the set. They did not grasp that their goal was to describe how they related to one another.

This can be seen in the answers to the three questions relating to the symmetric group. The fact that only one participant identified the aspects of the symmetric groups in Question 3 and in the definition in Question 5 would indicate that they do not see the symmetric group as a description of permutations nor the need to get a general sense of relationships in a group. The participants almost never stepped back to try and "see" a group. Their responses to Question 6 which essentially described $S_n$ as a collection of $n-$cycles would futher support this.

The work done on isomorphisms also indicated that participants viewed groups as collections of objects. The most common method used to show an isomorphism to match up two groups element by element; only Participant 3 mentioned that two groups are isomorphic if they "do the same thing." The other participants were generally of the mindset an isomorphism is a 1-1 matching between the domain and range of some homomorphism. For example, after Participant 2 read Question 5 she said "so is it isomorphic, probably yes, I would need to define a nice relationship and line them up," and then focused on how matrices could be matched up to functions. This could also be seen in section 6.2 when Participant 1 referred to elements being isomorphic to each other. The view of isomorphisms as the matching up elements bypasses the main idea, which is that two groups have identical structure.

Participants generally felt that groups are simply a collection of objects. For Question 1, they had trouble conceiving of the fact that the elements of group could motions and tried to define them as more static objects. They often felt the need to contextualize elements and create reference points. For example, in Question 1 Participants 3 and 4 felt that the identity should be a point on a grid and that inverses were an ability to go to a specific point. This is the mathematics of *values*; not in line with Viète's general equations which do not need to assign specific values but study generalised relationships.

While Participant 1 was trying to establish whether or not the system in Question 1 was closed, the interviewer made the comparison to the possible transformations of a pencil case. Participant 1 described certain motions of the pencil case which were established as space-transformations but then in reference to closure she stated that it was closed because "it's still going to be a pencil case" and that staying in the group was the equivalent of the object not changing. Similarly Participant 2 saw the group $(M,*)$ as an object remaining in $R^3$ and checked the closure and inverse properties by seeing if the object was still in $R^3$ and not in reference to these elements being space transformations that were part of the original set. Overall, they had great difficulty viewing the elements as relative transformations.

The participants' focus on objects over relationships was apparent in Question 5 not just in terms of setting up isomorphisms element by element but in the way participants viewed the groups they were working with. For example, Participant 1 stated "So for me $G_n$ is just the set $X$," and was not able to see the difference between examining the set of all permutations on $X = \{x_1, x_2, \dots, x_n\}$ and the set $X$.

Matrices were also seen as static elements. Participant 3 also felt that the matrices were isomorphic to the set $X$ rather than to the functions in set $G_n$ and thought of matrices in $M_n$ as the solutions to linear equations. While he could see that the matrices in $M_n$ had their rows permuted, he did not see them as entities capable of permuting. At one point he was even confused as to how the various matrices in $M_{n,}$ were different:

> Participant 3: It doesn't really change anything if you switch the row does it? Because these are 3 linear equations you're solving all at once so it doesn't matter which one you solve first.

Overall, participants did not view structure and relationships as important aspects but continually focused on objects and elements, seeing groups as collections of objects.

## 6.4 Recommendations

The abstract algebra course that the participants passed before engaging in this study is labeled as an introductory course in group theory. It requires students to understand concepts in group theory through definitions and procedures in modern algebra. Students do not appear capable of giving meaning to statements in modern algebra or gaining a concrete understanding of the ideas specific to group theory. The concepts of operative symbolism, studying generalised relationships and structure instead of objects, and working in purely axiomatic systems is relatively new to students whose previous courses did not prepare them to view mathematics with this perspective.

Repeated actions such as plugging meaningless symbols into equations, viewing mathematical entities like matrices and functions as objects and not feeling the need to check all axioms before deciding something is a group, show that students do not approach group theory concepts with the mentality associated with modern abstraction and are not aware of the nature of this type of abstraction. The fact that all participants appeared to view the concept of a group as the ability to satisfy the group axioms would indicate that there is indeed confusion between the conceptual aspects of group theory and their abstract representations.

As argued in the reviewed literature, a great number of difficulties stem from participants dealing with novel situations by assimilating them into existent schemas. This can be alleviated by not combining two sets of novel concepts at the same time and allowing students to use familiar objects when learning unfamiliar schema.

In terms of APOS theory, I would argue that the group concept is introduced to students in their introductory abstract algebra course as a complex schema made of two components: set, binary operation and the group axioms, and operative symbolisms, mathematical structures and axiomatic systems. This second component is not explicitly addressed as something novel to the student. I would argue that (a) the underlying ideas behind group theory concepts (as described in Chapter 2) may be learned without modern abstraction, and that (b) adding more conceptual aspects pertaining specifically to group theoretic thought into a group theory course would alleviate a great number of conceptual difficulties experienced by students being introduced to group theory through an approach intrinsically entangled with modern abstraction.

The concepts of set, binary operation and the group axioms taken outside of the context of modern abstraction are not entirely novel for students starting a group theory course. A set is a simple concept defined as "a collection of well-defined and distinct objects" and students have worked with basic sets such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ for many years and have studied sequences in their integral calculus course. Students have also had experiences with binary operations such as basic addition, subtraction, multiplication, division and composition, and modular arithmetic could be introduced as a novel operation well anchored in their previous experiences. Students have also been required, in their linear algebra and calculus courses, to check axioms or conditions that must be satisfied. Furthermore, the four axioms themselves address basic mathematical properties (inverses, identity, closure and associativity) that are common to courses such as linear algebra.

As demonstrated in Chapter 2, the development of group theoretic concepts was based on work with entities that should be familiar to group theory students, notably polynomial equations, permutations and motions of objects. A more practical and conceptual side of group theory should not prove extraordinarily difficult for students in a group theory course. These concepts were developed in an effort to solve problems and so students can learn group theory in a more familiar manner involving problem solving and using familiar objects. This would allow students to have an understanding of groups that goes beyond their definition and allow them to get a sense of what groups are used for and why they are useful and interesting.

Israel Kleiner proposed a historically focused course in abstract algebra which imparts on students the major ideas of abstract algebra. His rationale is that "history points to the sources of abstract algebra, hence to some of its central ideas, it provides motivation and it makes the subject come to life." (Kleiner, 2007) The results of this study would strongly indicate that students are not aware of the main ideas of group theory. Pedagogical efforts should be made to foster a better understanding of the concepts and ideas central to group theory.

The study carried out in this thesis shows that many of the difficulties that students encounter in their first abstract algebra course may originate from their lack of knowledge of modern abstraction and inability to interpret and create statements in modern algebra. These difficulties could be un-entangled from those specifically pertaining to group theory concepts, and appropriately tackled if students were introduced to modern algebra – not at the same time as they are introduced to group theory[1]. This should allow students to separate modern abstraction from group theory concepts and learn how to differentiate the classical view point of mathematics from the approach they should now be taking to ask and answer questions in accordance with abstract modern mathematics.

As seen in Chapter 3, modern abstraction developed from a variety of disciplines, which are in fact known to students before starting a group theory course. Furthermore, modern abstraction is not unique to group theory. The three defining characteristics of modern

[1] In their introductory course to analysis, students learn to perform analytic proofs while working with familiar objects and concepts from calculus courses. An analysis course focuses on the basics of analysis without introducing a host of new concepts at the same time. In contrast, when students begin modern algebra proofs they do so with unfamiliar objects from group theory. I would say this is analogous to a student taking a course in real analysis without taking an introductory course in analysis. In this situation students would be introduced to analysis while working with novel objects and concepts from topology.

abstraction: operative symbolism, axiomatic systems and studying relationships and structure, represent a way of doing mathematics that can be applied to a variety of concepts and objects that are far more familiar to undergraduate students than group theory concepts. The results of this study would indicate that students do not understand entities such as functions, matrices and symbolic systems in the vein of modern abstract mathematics; perhaps a course focused on understanding those objects from the perspective of modern abstraction would ease and facilitate, later on, their understanding of group theory concepts.

I would argue the in order for a student to truly succeed in a group theory course they should be able to combine a conceptual knowledge of group theory with the ability to construct proofs in modern algebra. As seen in the study, genuine familiarity removes the mentality that a student should replicate things they do not understand. By distinguishing and focusing on the two components in a group theory course, students can gain a proper understanding of group theory concepts, how to apply them and how to combine the two and perhaps not be turned off by abstract mathematics as a result of taking this course.

## 6.5 Directions for future research

Further research is needed to test students' understanding of group theory concepts in relation to modern algebra. I would recommend a more direct comparison between questions that require conceptual knowledge of group theory and questions that require abstract proofs. This could be accomplished by having an equal number of questions from these two categories that are of equivalent difficulty. The only question in this study where students successfully carried out symbolic proofs was Question 4 and this question appeared too easy in relation to the other questions in the study. Clearly the sample size would also need to be larger for more meaningful results.

It would be interesting to put forth a curriculum based on the suggestions made in section 6.4 that would teach students conceptual aspects and practical uses of group theory as well as how to properly construct proofs in modern algebra prior to taking one of these studies. This could be taken in contrast to students who take standard courses in group theory which do not distinguish these two aspects of group theory.

The main goal of this study was to test the conceptual understanding of group theory topics and its connection with modern algebraic statements. The framework used was to give both familiar and unfamiliar type of problems. As noted in the review, there were two distinct kinds of familiarity. There was a *genuine* familiarity which was connected to repeated long term experience, intimate understanding of concepts, and knowledge of related processes. This can be taken in contrast to familiarity through exposure where participants admitted to seeing a topic in class and knew some basics definitions and related procedures but did not have a conceptual understanding. It would be interesting to conduct testing relating students' responses to these two types of familiarities.

A final note on the shortcomings of my empirical study: An intimate knowledge of what the students had learned in their course and the questions on their assignments and tests would have helped in designing the questionnaire and interpreting students' responses. Future research could take this into account.

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