Providing College Level Calculus Students with Opportunities to Engage in Theoretical Thinking

Dalia Challita

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By: Dalia Challita

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Signed by the final Examining Committee:

Helena Osana	Chair
Anna Sierpinska	Examiner
Helena Osana	Examiner
Nadia Hardy	Supervisor

Approved by ______________________________
Chair of Department or Graduate Program Director

____
Dean of Faculty

Date ______________________________

ABSTRACT

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Previous research has reported a procedural, rather than conceptual, approach in college level Calculus courses. In particular, previous studies have shown that theoretical thinking is not a necessary condition for success. This can be gleaned from the exercises on assignments and assessments which constitute all, or most of, students' course grades. This approach has been linked with institutional constraints that are often imposed on these courses. Our belief is that theoretical thinking is necessary for learning Calculus, and that students should be provided with opportunities to engage in this type of thinking. In this thesis, we provide empirical evidence that students can be engaged in theoretical thinking in a college level Calculus course, despite the existing institutional constraints. Students enrolled in a Calculus course were presented with optional tasks intended to engage them in theoretical thinking. We analyze collected data from the perspective of Sierpinska, Nnadozie, and Oktac's (2002) model of theoretical thinking; all students attending class engaged in these optional tasks and our analysis shows that on average, more than half of them engaged in theoretical thinking. We place our study in the context of previous research in the teaching and learning of university introductory (and remedial) level mathematics and of the role that Calculus courses play in the mathematics education of undergraduate students.

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Introduction

Every constraint is an opportunity – John Mason, CMESG 2013

While higher level mathematics courses deeply engage students in activities that can stimulate theoretical thinking, previous studies report that this is hardly the case in college level Calculus courses (e.g., Boesen, Lithner, & Palm, 2010; Hardy, 2010; Lithner, 2000; Selden, Selden, Hauk, & Mason, 1999). We believe that while learning mathematics at any level consists of developing procedural skills, it cannot take place without theoretical thinking, and that thinking about mathematical concepts and relations is fundamental to learning mathematics. In particular, if mathematics is to be used as a tool, there is likely only so much one can do with this tool if one does not understand its fundamental structure and components; for a given problem, should one be faced with a different set of conditions, or new constraints, it may be difficult to proceed without a deep understanding of the involved concepts. Furthermore, as will be discussed later, a lack of understanding has been reported to cause difficulties for students in solving exercises. For at least these reasons, we believe that theoretical thinking is an important part of learning mathematics.

The current work was motivated by the reported procedural approach to teaching college level Calculus courses, and our belief that opportunities to engage in theoretical thinking should be provided to students in the mathematics classroom. Our goal in this research was to find a way to provide such opportunities.

In an ideal setting in which a teacher has a class of a dozen or two students at most, an abundance of time for class activities, and the flexibility in choosing the types of tasks and assessments in which his or her students will engage, plenty of thought-provoking activities might be proposed by the enthusiastic teacher who is seeking to engage his students in theoretical thinking. Unfortunately, actual teaching does not take place in ideal settings. Constraints imposed on the teacher by the institution often limit the teacher's flexibility in his practices. These institutional constraints might take the form of a large class size, or a set of lesson objectives that need to be met in a given time – for example, before the pre-set weekly assignment. These constraints make it somewhat challenging to incorporate non-procedural activities which promote theoretical thinking.

Educational context

The study in this research project took place in one of the six sections of a college level Calculus course, which in North America is often offered at colleges and universities to students applying to programs such as Business, Engineering, and applied sciences. It is perhaps worth emphasizing that Calculus means many different things in different countries and institutions, or even in different contexts within the same institution. The context of this study was a college level Calculus II course (or *Differential and Integral Calculus II* as it is named) at Concordia University in Montreal, Canada. The main topics that are covered are *Riemann Integrals* (including the Fundamental Theorem of Calculus, areas, and techniques of integration), and *Series* (including tests for convergence or divergence of series, and Taylor series expansions). While the titles of the topics might be similar to those included in a Calculus course designed for students in Mathematics programs, they are usually presented in different, perhaps less rigorous ways in these college level mathematics courses.

College level mathematics courses are usually multi-sectioned at Concordia University and in many institutions across North America. The sections are usually administered by a single course examiner who prepares the course outlines, assignments, and exams, which are common to all the sections. In addition to specifying the topics to be taught each week and the order in which they should be taught, the course outline includes a list of "recommended" exercises from a common-assigned textbook which are indicative of the types of problems that will appear on the weekly assignments, and the midterm and final exams. These courses have been characterized as not including activities that stimulate theoretical thinking; the tasks proposed to students are routine and familiar tasks (Hardy, 2009a; Lithner, 2000) that can usually be solved using an algorithm that has previously been presented in class. In particular, one does not need to consider the mathematical properties that are inherent to the concepts in the problems in order to solve them (Lithner, 2004). Furthermore, any theory that is presented to students is disconnected from the subsequent exercises. Students find themselves seeking non-mathematical strategies, such as memorizing worked examples, to most efficiently solve exam questions (Boesen et al., 2010).

It is important to note that the 'non-routine'-ness of a problem does not lie in its general difficulty. For instance, a problem could be generally categorized as 'difficult' because resolving it requires the use of several concepts and techniques; but it might be one that was discussed in class several times and for which students have a solution which could simply be memorized and reproduced. At the university level, this is easily seen with proving exercises; students are sometimes asked to prove a theorem on an exam which has already been proven in class. The proof might be a difficult one, consisting of several levels and involving various ideas; but having memorized the steps of the proof word-for-word, one can merely reproduce it on the test. The 'non-routine'-ness of a task lies, rather, in its unfamiliarity to students. This was pointed to by Boesen et al. (2010) who used tasks that were not necessarily difficult but for which students did not already have a pre-rehearsed algorithm to employ. Hardy (2009a) also points to this idea in suggesting that among the institutional practices are ones that *condition* students to *expect* certain types of tasks on the Calculus examinations, and that this is what makes them 'routine' tasks.

In the following table, we display questions taken from the textbook and exams given in the course in which this study took place. It is easy to see that the textbook example, textbook exercise, and exam question shown are very similar. The three questions involve computing a common characteristic of a mathematical object. The objects, however, are almost identical in every question with only slight modifications. As a result, solving each of these problems can easily become a question of identifying the class of problems to which they each belong, and applying the memorized technique to solve problems in this class. Furthermore, since the modifications are extremely superficial, identifying the class of problems and technique to be used is likely not cognitively demanding and is reduced to simply recalling memorized procedures. In this sense, these exercises become routine tasks. In these particular questions the technique is even prescribed by the question, further facilitating the solving process.

Textbook example	**Textbook exercise**	**Exam question**
Evaluate $\int_0^1 \sqrt{1-x^2}dx$ by interpreting it in terms of areas	Evaluate $\int_{-3}^0 1+\sqrt{9-x^2}dx$ by interpreting it in terms of areas	Evaluate $\int_{-2}^4 f(x)dx$ by interpreting it in terms of areas, where $f(x) = \begin{cases} 1+\sqrt{4-x^2} & \lvert x\rvert \leq 2 \\ 3-x & x>2 \end{cases}$

Table 0.1 - Example of similarity between textbook examples, textbook exercises, and exam questions

Previous research seems to depict this situation quite clearly, but does not offer many practical ways for the teacher to provide opportunities for thinking theoretically in spite of these conditions.

Tool to prompt theoretical thinking

Our tool to engage students in theoretical thinking was a set of questions. We designed questions which were intended to engage students in thinking about concepts and relations rather than procedures, and which could not be entirely solved using a known algorithm or a technique previously used in class. These were presented to students in the form of weekly quizzes which were optional and only counted for bonus marks. The questions will be presented and discussed in detail in chapter 3, but we display one of the questions in the following table to give the reader an idea of the difference between the quiz questions and the so-called routine and procedural questions normally encountered in the course in which the study took place.

Exam question	Q11 (Quiz question)
Evaluate the integral $\int_e^\infty \frac{dx}{xln(x)}$ or show that it diverges.	Consider a function $g(x)$ that is continuous on the interval $(-\infty, \infty)$. Suppose that the integral $\int_1^\infty g(x)dx$ is convergent. Is it true that $\int_{10}^\infty g(x)dx$ is also convergent? Justify your answer.

Table 0.2 – A question typically encountered in the course (left) and one of our quiz questions (right)

Research question

We phrase our research question as *can we provide college level Calculus students with opportunities to engage in theoretical thinking, despite the institutional constraints that are often imposed on these courses?* but also propose splitting this question into smaller parts which perhaps reveals its multiple dimensions. The question can be rephrased as:

– Given the existing institutional constraints,

– can we engage a reasonable number of students in theoretical thinking

– with only reasonable compromises?

– If so, what is one possible way to do so?

We propose that in order to answer this research question positively, i.e., be able to say *Yes*, *it is possible*, we would first have to find a way to engage students in theoretical thinking (we chose this to be the quiz questions), and second, we would have to confirm a) that the instructor managed the course demands (i.e., completed the course outline in a timely manner as intended by the course examiner, while preparing students for the course assessments); b) that a reasonable number of students were engaged in theoretical thinking; and c) that only reasonable compromises were made (e.g., we would have to justify how we created the time for the quizzes, and argue that it did not negatively affect students' readiness for the assessments).

Our aim was not to engage as many students as possible, rather to *provide opportunities* for students to engage in theoretical thinking. However, we mention a 'reasonable number' and 'reasonable compromises' because realistically, it would perhaps be inefficient and not worthwhile if the instructor was to take time away from the usual class-time every week, and a great deal of his own time marking the quizzes, if only 2%, for example, of the quiz participants are engaged in theoretical thinking through the quizzes. What counts as a reasonable number is subjective of course; for us, this meant roughly a third of the quiz participants.

In order to identify instances of theoretical thinking in student responses, we used Sierpinska, Nnadozie, and Oktac's (2002) model of theoretical thinking. We also used the model to analyze the quiz questions and uncover the types of thinking they might invite. Finally, we proposed a method for modeling the questions. We chose four questions for this purposed, and created generalized questions that model each of these questions. Such a question-model uncovers the fundamental structure of the question, and can be used by researchers and practitioners to generate TT-engaging questions.

We found that while some quiz questions were more engaging than others, students were indeed engaged in theoretical thinking through them. Overall, we found that one can incorporate activities that stimulate theoretical thinking in a Calculus course, despite the described constraints.

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This work was carried out for the completion of my Master's thesis, and was conceived after I was appointed by the Department as instructor of one section of the course *Differential and Integral Calculus II*. The design and implementation of the study involved a number of people; namely, my supervisor, Dr. Nadia Hardy, my program director, Dr. Anna Sierpinska, and I. My supervisor and program director recommended that the course I was to teach be the context of the study, and helped me determine the goal and general outline of the study. Throughout the study, I assumed the role of instructor and researcher and throughout this thesis refer to these roles as though they were assumed by separate individuals. The thesis is written in the plural, in the voice of the researchers, although I was the principal researcher in this study while coordinating regularly with my supervisor.

In the first chapter, we review literature that is pertinent to our study. In particular, we draw from previous research that describes the context of our study, and put forth characterizations of theoretical thinking. In this chapter, we introduce the model of theoretical thinking by Sierpinska et al. (2002) which we present in depth in the following *theoretical framework* chapter. In Chapter 3 we present the instruments which we used to engage students in theoretical thinking (the quiz questions) and describe the setting and quiz-taking protocols. We explain how we operationalized the model of theoretical thinking so that it could be used as a working tool to identify instances of theoretical thinking, and give examples of how we analyzed student responses. Finally, we describe the two types of analyses which were carried out and which are elaborated in Chapters 4 and 5. In Chapter 6, we present some conclusions of the study and pose questions which might be interesting to investigate in a future research.

1 Literature review

In this literature review we report works that will help situate our study in the general context of the North American college level mathematics courses institution, and we highlight several characterizations of individuals' thinking in mathematics which we could draw from or strive to complement in order to identify instances of theoretical thinking in our study. We begin by summarizing the accounts of various researchers about the relevance and role of theoretical thinking in college level mathematics courses; next, we highlight some of the institutional constraints which are reportedly imposed on these courses and affect their members (students and instructors in particular); finally, we discuss some of the characterizations of thinking from mathematics education literature. We conclude by discussing the pertinence of these works to our study.

1.1 Mathematical thinking in college level mathematics courses

A large and growing body of literature has investigated the degree to which students are involved in mathematical thinking[1] in college level mathematics courses; often indicating that they are hardly so – if at all (e.g., Hardy & Challita, 2012; Lithner, 2003; Selden et al., 1999). In cases where theory is incorporated in course materials (whether the textbook or class notes), it has been reported to be disconnected from problems and their solutions (Barbé, Bosch, Espinoza, & Gascón, 2005) making it difficult for students to perceive it as useful. Moreover, as Barbé et al. point out, theory is often absent from the

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[1] Throughout the chapter, we use terms such as 'mathematical thinking', 'theoretical thinking', and 'mathematical reasoning' as they appear in the reviewed literature; these terms are often used in the literature in a self-explanatory way. However, when provided by the researchers, we will state their given "definitions" here; the exception will be the complete definition (or *model* as they refer to it) of 'theoretical thinking' provided by Sierpinska et al. (2002), which we will introduce briefly here and elaborate in the *theoretical framework* chapter.

course materials altogether in the sense that if it is incorporated into the curriculum, it is there for the teacher's reference and not to be used by students.

Students in these assessment-driven courses are competent at solving problems which have been described as routine and familiar tasks (Boesen et al., 2010; Lithner, 2003; Selden et al., 1999). Familiar tasks are those which share significant properties with the ones appearing in the textbook, and unfamiliar tasks are those for which students cannot recall a well-rehearsed algorithm or a previously established procedure to solve them (Boesen et al., 2010; Hardy, 2009b). When confronted with familiar tasks, students approach them by trying to recall algorithms from memory (Hardy, 2009a; Lithner, 2000); neither a conceptual understanding of the components involved in the tasks, nor considering their intrinsic mathematical properties, are required in order to solve the tasks (Boesen et al., 2010). On the other hand, Boesen et al. found that solving unfamiliar tasks stimulated creative reasoning in students.

Several researchers have discussed how textbooks typically used in North American Calculus courses seem to reinforce the routinization of tasks (or the needlessness of mathematical thinking); the majority of exercises in these textbooks can be solved without considering the core mathematical properties underlying the concepts involved in the exercises and their solutions (Lithner, 2004) and only require the application of some routinized procedure (Lithner, 2000). A consequence of this seems to be that students use solved examples as templates to solve exercises (Hardy, 2009b); i.e., they copy solutions to solved examples, making only the necessary, minor adjustments. Hardy points out students' reliance on "algorithmic" techniques whereby they recall a set of "instructions" or "steps" (p. 232) which were provided by the instructor or textbook to solve these types of tasks. Not only is this technique not conducive to thinking theoretically, but it can cause difficulties for students because learning these instructions with a lack of a theoretical context can result in students forming an arbitrary list of what needs to be done, and forgetting the order of the steps. Resorting to solved examples, or known solution-algorithms, is a common strategy used to solve familiar tasks, versus, for example, attempting a new way of solving.

However, if mathematical thinking is considered necessary condition for learning mathematics, then this is indeed a concern since students in these college level mathematics courses spend most of their study-time practicing textbook exercises (Lithner, 2001). In fact, the book is still by far “the most pervasive technology to be found in use in mathematics classrooms […]; the textbook has dominated both the perceptions and the practices of school mathematics” (Love & Pimm, as cited in Lithner, 2004, p. 406).

This routinization of tasks is not limited to textbook exercises. In their study, Sierpinska et al. (2002) report that the tasks on one of the investigated examinations were very similar to ones given during the previous years (all of which students have access to) making most of the questions quite predictable. In fact, not only can exam questions often be solved by applying algorithms previously rehearsed in class and which mainly require *procedural* skills, but the time allowed to solve an exam is often quite tight; taking the time to *think* about and explore the questions will leave no time to solve them. It is rather apparent that succeeding on these courses has little to do with thinking mathematically. For example, Sierpinska et al. (2002) discuss how theoretical thinking is essential for understanding Linear Algebra, but not necessary for achieving high scores on the course.

The types of tasks, the restricted time for assessment and the teaching of curriculum material, together with other institutional constraints, have been linked to the absence of activities that stimulate mathematical thinking in these courses (e.g., Barbé et al., 2005; Boesen et al., 2010; Hardy, 2009a).

The way out of these routine tasks is not evident. There are institutional practices which more or less define the way that these courses are run; this will be discussed further in the next section. Among these institutional practices are ones that *condition* students to expect certain types of tasks on the Calculus examinations, making them routine tasks (Hardy, 2009a). Tall (1990) suggests that students "learn to respond to standard questions in a *predictable* manner, but if their understanding is probed in unusual ways, subtle difficulties arise" (p. 49). Further investigations into students' understandings of the involved concepts can reveal fundamental inadequacies (Selden et al., 1999).

1.2 Institutional constraints

Various researchers have described institutional constraints, in particular those that exist in educational institutions such as schools, colleges, and universities. While the natures of the mentioned constraints perhaps differ from one institution to another, one reported effect of these constraints always seems to be the same: less freedom in teaching practices.

Before reflecting on some of these works, we present a characterization of 'institution' as it is adopted by Hardy (2009b) in following Ostrom's (2005) description of this notion. In this characterization, an institution is a structure which organizes a set of *repeated interactions* between its participants, whose aim is to achieve certain outcomes. Institutional practices are different from practices in general, in particular due to the existence of explicit *rules* that regulate institutional practices. These rules are established by authorities who also design *sanctions* against individuals who violate the rules. *Norms* are another important regulatory mechanism of institutions. Unlike rules, norms are not explicit; newcomers learn the norms through repetition and practice, and by having experienced participants point out that "that's not how things are done here" when they violate the norms. Institutions can exist within other institutions; for example, college level mathematics courses form an institution within the larger Departmental institution, which is yet part of the University institution. Participants of an institution have different *positions* and thus assume different roles and carry out different *actions*; for example, students and instructors occupy different positions in college level mathematics courses. In some cases, instructors and course examiners also occupy different positions.

In the context of college level mathematics courses at a large, urban, North American institution, Sierpinska, Bobos, and Knipping (2008) draw attention to several constraints that are normative and which directly affect instructors, and in turn affect students. An example of these constraints is the course outline, which establishes the order in which content has to be delivered and the time that the instructor must invest in delivering it. Other constraints are the assessment tools which are prepared by a course examiner. The instructors then must prepare students to succeed in the course examinations; failing to do so may result in student discontent, negative course evaluations, and possibly in non-rehiring. In particular, Sierpinska et al. suggest that instructors fear that presenting theory or engaging students in reasoning and proving activities will take away from the already-little time available for solving procedural problems (that are central to such courses, as discussed in the previous section) thus increasing failure rates and students' complaints.

Barbé et al. (2005) argue that teachers' practices are highly conditioned by several constraints, but in particular by the "limited scope for action traditionally assigned to the teacher" (p. 235). In the context of a Spanish high school, Barbé et al. point out that while it is up to the teacher to decode the information provided in curricular documentation in order to develop, together with students, mathematical concepts to be learned, the teacher is more or less guided on what to teach (and how) through textbooks, assessment tasks, national examinations, and so on. Hardy (2009a) reports a similar situation in college level Calculus courses; in a review of the assessment materials (such as assignments and exams), one can notice certain "constants" that mark these materials from year to year (such as the type of function whose limit students are asked to calculate). These constants constitute *norms* of these courses which students expect, and thus become constraints which the instructor is bound to.

Institutional constraints are reported in other disciplines as well. For example, Sweet (1998) reports that although the teaching materials in a *Sociology of Education* course at a particular North American university address radical social theory, professors are not entirely free to implement radical pedagogical techniques that correspond with this theory, due to various institutional restrictions that bound the extent of their autonomy in teaching. Sweet argues that instructors of such courses find themselves unable to practice the very essence of what they preach.

1.3 Models of theoretical thinking

1.3.1 General characterizations of thinking in mathematics

In their book *Thinking Mathematically* designed for use by senior high school students, in teacher preparation courses, and in courses for undergraduates in mathematics, Mason, Burton, and Stacey (2010) describe processes that underlie mathematical thinking, and provide a range of questions to be explored using (at least) these processes. The authors describe mathematical thinking as “a dynamic process which, by enabling us to increase the complexity of ideas we can handle, expands our understanding” (p. 144). The processes they refer to are *Specializing and generalizing* (testing the problem using particular cases, and then moving from a few instances to making generalizations about a class of cases), *Conjecturing and convincing* (proposing a supposition after recognizing a growing generalization, and then providing a justification that will convince the most critical reader), *Imagining and expressing* (anticipating, recalling relationships and properties, and then expressing in different ways what is being imagined, e.g., graphically, algebraically), *Stressing and ignoring* (recognizing features or properties and ignoring ones that are not recurrent or important; ultimately to make generalizations), *Extending and restricting* (extending or restricting the context under consideration to simplify or extend the problem), *Classifying* (perceiving something as an instance of a property), and *Characterizing* (describing the distinctive features of something). The authors claim that these are “natural powers and processes” (p. 231) that are inherent to human intelligence and possessed by every child, and that knowing how to use these in mathematical ways is what thinking mathematically is about.

The more specific and well-known Van Hiele model was developed in 1957 by Dina van Hiele-Geldof and Pierre van Hiele to describe the way an individual's thinking develops as he learns geometry. In particular, the model proposes five levels of geometric thought; *visualization* (identifying, naming, comparing, and operating on geometric figures), *analysis* (analyzing components and properties of figures and relationships between figures), *abstraction* (providing informal arguments to interrelate previously discovered properties), *deduction* (deductive reasoning and construction of simple proofs), and *rigor* (comprehending different axiomatic systems; understanding that definitions and axioms are arbitrary). The transition from one level to the next is related to instruction and experience rather than to age, making this model appropriate to describe the learning of geometry at any scholastic level Guberman (2008).

Guberman (2008) proposes a framework for characterizing the development of arithmetical thinking mainly based on the Van Hiele model. The framework consists of defining four groups of features of mathematical behavior, each group indicating a level of arithmetical thinking. The levels are defined first by an a priori formulation based on the researcher's experience in teaching mathematics, and then refined based on students' responses to questionnaire tasks. In fact, the process of refining the initial characterization of thinking reduced the levels from five levels (as in Van Hiele's model) to four, since, the researcher explains, the research population did not include representatives of the fifth level. Guberman suggests that this framework can be used as a tool for teachers to determine the level at which a student stands with respect to arithmetical thinking.

Sierpinska et al. (2002) propose a model of theoretical thinking in which they postulate three main categories of theoretical thinking; *Reflective*, *Systemic*, and *Analytic thinking*. *Reflective thinking* is concerned with reflecting on, investigating, and extending ideas. The aim of *Reflective thinking* is not to immediately solve a task; rather to build and attend to curiosities and mental challenges. *Systemic thinking* is thinking about a system of concepts rather than treating concepts as isolated objects. *Systemic thinking* distinguishes properties from definitions and is aware of the conditional character of statements. *Analytic thinking* is sensitive to the symbolism, structure, and logic of mathematical language. The model also includes features of theoretical thinking that are associated to each of the main categories; we will provide these and a more detailed description of Sierpinska et al.'s model in the next chapter. The model is different from Van Hiele's and Guberman's frameworks at least in that it does not propose *levels* of thinking, rather it lists *features* of thinking corresponding to each of the three main categories – which is more akin to Mason et al.'s (2010) description of mathematical thinking. Sierpinska et al.'s model, however, appears to be more thorough, accounting for the various types of thinking which might be observed. Another important feature of Sierpinska et al.'s model is its *operationalization* (details of how this can be done will be given in chapter 3) making it a *tool* that can be used in empirical studies dealing with theoretical thinking.

1.3.2 Thinking related to problem-solving

While observing the problem-solving approaches of several individuals, Schoenfeld (1987) characterizes their approaches as *novice* and *expert* and describes each approach in terms of the amount of time spent on each of the stages *Read*, *Analyze*, *Explore*, *Plan*, *Implement*, and *Verify*. The terms are rather self-explanatory except for the subtle difference between *explore* and *implement*; although both involve proceeding with solving, the *exploring* phase is carried out more haphazardly, while the *implement* phase is carried out after devising a plan. Schoenfeld finds that an expert problem-solving approach comprises an ample amount of time of each of the stages, especially *analyzing*, and, moreover, alternates between these stages. On the other hand, the novice problem-solver spends all his time (besides reading the problem) *exploring*, often persistently pursuing 'wild mathematical geese' without ever reflecting on his work. Schoenfeld suggests that making students aware of these stages, and of their thinking while problem-solving, could help them develop better problem-solving habits.

In discussing the types of thinking involved in tackling a problem in mathematics, Mason et al. (2010) put forth three phases of thinking that might occur in the problem-solving process that are somewhat similar to those proposed by Schoenfeld (1987). The phases are called *Entry*, *Attack*, and *Review*. They describe the *Entry* phase as becoming acquainted with the problem: noting the information it provides us with, what it requires us to do, and what possible strategies might we be able to use to solve it. The *Attack* phase involves a cyclic process between conjecturing statements which lead to solving the problem and justifying these conjectures. Finally, the *Review* phase is based on checking the validity of the solution (whether the arguments used are appropriate, pertinent to answering the question, clear, and reasonable), reflecting on the key ideas involved in the solution, and exploring whether the result can be extended to a wider context. Similarly to Schoenfeld's (1987) observation, Mason et al. describe the outlined problem solving process as a back-and-forth process where one might refer back to a previous phase in reconsidering their solution.

1.3.3 Conceptual thinking

In discussing individuals' approaches to dealing (mentally) with concepts, Tall and Vinner (1981) differentiate between 'concept definition' and 'concept image' (two terms which were originally coined by Vinner and Hershkowitz (1980)). They consider the former to be a definition of a concept through which the concept is formally introduced – "a form of words used to specify that concept", and the latter to be "the total cognitive structure [in the individual's mind] that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). The researchers explain that the concept definition may be part of the concept image, but that for some individuals it might be a part of the concept image that is not recalled often, or that is even virtually non-existent. They add that concept images of the same concept may vary over time in a contradictory sense, but that this will only actually cause conflict or confusion in the individual's mind in case inconsistent concept images are evoked simultaneously. A more serious type of conflict, the researchers describe, occurs when a concept image is not consistent with the concept definition, as this could give rise to serious misconceptions which can impede the learning of formal theory.

In a later paper, Tall (1995) describes 'advanced mathematical thinking' about a concept as a particular reformulation of the concept's image in relation to its definition (p. 1):

> The cognitive structure in elementary mathematical thinking becomes advanced mathematical thinking when the concept images in the cognitive structure are reformulated as concept definitions and used to construct

> formal concepts that are part of a systematic body of shared mathematical knowledge.

Sfard (1991) suggests that mathematical concepts (such as *limit*) have a dual nature and can be conceived in both ways; *operationally* (as processes) and *structurally* (as objects). With an operational approach, one focuses more on the procedures, algorithms, and actions associated with the concept; while a structural approach treats a mathematical concept as though it refers to some abstract object. Sfard argues that while these two approaches might seem contradictory, they are in fact complementary, and that ultimately each approach is evoked as appropriate in problem-solving. According to Sfard, the process of concept acquisition usually begins with an operational conception and then a structural one, and the transition is achieved through three main stages; *interiorization*, *condensation*, and *reification*. In the interiorization phase, the individual "gets acquainted with the processes which will eventually give rise to a new concept" (p. 18); the condensation phase consists of "'squeezing' lengthy sequences of operations into more manageable units" (p. 19). By this stage, the individual is able to think about the process as a whole without feeling the need to go into details. Finally, and only once the individual is able to completely detach from the procedural aspects of the concept, the reification stage is realized – the individual experiences "an ontological shift… the sudden ability to see something familiar in a whole new light" (p. 19). Sfard suggests that the reification is so difficult that at certain levels it may remain out of reach for some students.

While Tall and Vinner (1981) describe the mental *images* an individual might form of a concept, Sfard (1991) describes two fundamentally different natures of concepts and the *process* through which the transition from one perception to the other might occur for an individual. An ultimate level of thinking is described by each of the authors; for Tall and Vinner, this is achieved when the concept images are tightly based on the concept definition; for Sfard it is attained after the reification stage.

1.4 Conclusions

The works we reviewed reveal that features such as mathematical reasoning and conceptual thinking are very often not required to successfully complete a college level mathematics course. Furthermore, researchers conjecture that several institutional constraints often impede the development of these and other (deemed) important aspects of mathematics in students either by the way in which the material that is to be taught is defined, or by allowing very little time for open-ended tasks that could develop those features, or a combination of these and other constraints. While various studies propose that tasks that stimulate mathematical reasoning and thinking be incorporated in teaching, none of the ones we reviewed propose these in light of the often extremely-rigid institutional constraints, with suggestions of how to incorporate such tasks despite these constraints.

The third section of this chapter was deliberately divided into three sub-sections to emphasize some of the approaches that have been undertaken by mathematics education researchers to characterize types of thinking. In the modest number of works we reviewed, there are various descriptions of the *processes* of acquisition/construction/development of mathematical concepts. There are also classifications of *levels* of thinking about concepts, and characterizations of the types of behaviors or ways of thinking that contribute to efficient problem-solving.

For the purpose of this study, we ultimately required a comprehensive characterization of theoretical thinking in and of itself which we believe is best portrayed by Sierpinska et al.'s (2002) model. Furthermore, Sierpinska et al.'s model has a structure that 'easily' allows the researchers to operationalize the model for the purpose of analyzing data. While Tall's (1995) description of advanced mathematical thinking seems to thoroughly describe conceptual thinking (which is closest to Sierpinska et al.'s *Systemic thinking*), it does not account for the features of *Reflective thinking* which resemble Schoenfeld's (1987) *Analyze* and *Plan* stages, nor for an individual's mathematical linguistic skills (which is accounted for in Sierpinska et al.'s *Analytic thinking*). Mason et al.'s (2010) description of the processes that underlie mathematical thinking also lacks a reference to features described by *Analytic thinking*, but more importantly, although it seems rather comprehensive, its structure did not seem 'easily' operationalizable to us.

2 Theoretical framework

As was explained in the introduction, this research was created for the completion of a Master's degree, and involved researchers and an instructor. In fact, the study was conceived when the instructor was appointed by the Department as the instructor of one section of the course Calculus II (*Differential and Integral Calculus II*). The study was motivated by the researchers who were curious to see whether it was possible to provide students in such a course with opportunities to engage in theoretical thinking, despite the institutional constraints, and viewed the situation as an opportunity for collecting data. The instrument for engaging students in theoretical thinking (and thus for collecting data– the quiz questions), however, was designed almost entirely by the instructor. The instructor designed them using her intuitive understanding of what theoretical thinking is and what types of questions might invite this type of thinking. The researchers assisted the instructor in this design, but at the time, they had not yet a model for theoretical thinking and therefore, were also drawing from their intuition and from their experiences as instructors of the same course.

In order to speak of theoretical thinking later in the study, we, the researchers, first needed to establish what we would refer to as theoretical thinking. For this purpose we used the *model of theoretical thinking* proposed by Sierpinska et al. (2002). Inspired by Vygotsky's (1987) distinction between scientific and everyday (or 'spontaneous') concepts, the authors describe theoretical thinking as developing in opposition to practical thinking. For them, theoretical thinking and the objects about which one thinks belong to different planes of action while practical thinking operates on the same level as its aims and objects; furthermore, the objects of theoretical thinking are systems of concepts, versus isolated objects that may be involved in the practical thinking processes. They understand that theoretical thinking involves reasoning (and reflecting on this reasoning) about concepts while extending beyond the symbolic form or any particular image these concepts may take.

Sierpinska et al. (2002) postulate several features of theoretical thinking (TT) which they group under three main categories: *Reflective*, *Systemic*, and *Analytic thinking*. *Reflective thinking* is concerned with posing and reflecting on curiosities and mental challenges; its aim is not to seek a means to an end, rather, to investigate and extend ideas. An individual engaging in *Reflective thinking* may reconsider his/her solution to a problem and be open to exploring different approaches.

Systemic thinking is described as thinking in terms of a system of concepts rather than isolated objects. *Systemic thinking* is said to be *definitional*, where meanings of concepts are established and recalled based on their definitions and not on a particular event, example, or image; furthermore it is *hypothetical*; it is understood that statements do not exist in the absolute and are only true under a set of conditions. An individual exhibiting *hypothetical thinking* might check whether the conditions of a theorem are satisfied before stating its conclusion. Finally, *Systemic thinking* appreciates mathematical reasoning and is concerned with the consistency of ideas within a conceptual system.

The third category of TT is *Analytic thinking* which involves an "analytical approach to signs" (Sierpinska et al., 2002, p. 35); in particular, thinking analytically involves being aware of the distance between concepts and their symbolic representations while being sensitive to specialized terminology. In Calculus, this might be exemplified in understanding the meaning of the summation notation and being aware of its components; for instance that $\sum_{i=4}^{20} a_i$ represents the sum of the terms a_i from i = 4 till 20, i.e., $a_4 + a_5 + a_6 + \cdots + a_{20}$, or recognizing the difference in symbolic notations between a definite and indefinite integral. Moreover, *Analytic thinking* appreciates the language of mathematics (its general structure, logic, notations, and conventions).

Sierpinska et al.'s model of TT is displayed in the following table with a brief description of the features and categories.

Category of TT • Feature of TT ○ Sub-feature of TT	General description
TT1 Reflective	Theoretical thinking is aimed at reflecting on, investigating, and extending ideas. Its aim is not merely to accomplish tasks, rather to reflect on curiosities and mental challenges
TT2 Systemic	Theoretical thinking is thinking about systems of concepts, where the meaning of a concept is established based on its relations with other concepts and not with things or events
• TT21 Definitional	• The meanings of concepts are stabilized by means of definitions
• TT22 Proving	• Theoretical thinking is concerned with the internal coherence of conceptual systems
• TT23 Hypothetical	• Theoretical thinking is aware of the conditional character of its statements; it seeks to uncover implicit assumptions and study all logically conceivable cases
TT3 Analytic	Theoretical thinking has an analytical approach to signs
• TT31 Linguistic sensitivity ○ TT311 Sensitivity to formal symbolic notations ○ TT312 Sensitivity to specialized terminology • TT32 Meta-linguistic sensitivity ○ TT321 Awareness of the symbolic distance between sign and object ○ TT322 Sensitivity to the structure and logic of mathematical language	

Table 2.1 - Sierpinska et al.'s (2002) model of TT

As a standalone, the model serves as a characterization of TT. However, in order to make it a working tool which can be used to identify occurrences of TT in empirical research, Sierpinska et al. (2002) propose that the model be *operationalized* with *theoretical behaviors* (TB); an individual's display of a TB is taken as a sign that the individual is thinking theoretically – each TB is indicative of a particular feature of TT. In the following table, by listing only one of the TBs for each feature of TT, we exemplify the authors' operationalization of the model for a study of TT in the context of an introductory course in Linear Algebra.

Category of TT • Feature of TT ○ Sub-feature of TT	Sample TB
TT1 Reflective	Displaying an investigative ("researcher's") attitude towards mathematical problems
TT2 Systemic	
• TT21 Definitional	• Referring to definitions in algebraic contexts when deciding upon meanings
• TT22 Proving	• Engaging in proving activity
• TT23 Hypothetical	• Being aware of the conditional character of mathematical statements and engaging in discussions about the possible consequences of adopting different sets of assumptions
TT3 Analytic	
• TT31 Linguistic sensitivity	
○ TT311 Sensitivity to formal symbolic notations	○ Interpreting algebraic expressions in a rigorous way
○ TT312 Sensitivity to specialized terminology	○ Being articulate and using correct terminology
• TT32 Meta-linguistic sensitivity	
○ TT321 Symbolic distance between sign and object	○ Interpreting letters in algebraic expressions as variables
○ TT322 Sensitivity to the structure and logic of mathematical language	○ Being aware of the role and meaning of expressions such as "for all", "for some"; having a sense of the implicit universal quantification of variables in conditional statements; negating the universal quantifier by the existential one and vice versa

Table 2.2 - Model with sample TBs from Sierpinska et al.'s (2002) operationalization for their study involving Linear Algebra

2.1 Sierpinska et al.'s model of TT in the context of a Calculus course

The authors in Sierpinska et al. (2002) characterize TT through listing features of TT, and they name the main categories of these features *Reflective*, *Systemic* and *Analytic thinking*. These features do not constitute an exhaustive list, however, and they emerge as one reflects on what TT is in general and in particular contexts. As pointed out above, the features that they proposed emerged in the context of a study of TT in a Linear Algebra course. When we tried to use the model in the context of a Calculus course, different features of TT emerged. These *new* features do not modify the model; they rather further *clarify* its meaning in relation to the particular context under consideration. In different contexts, different features of TT might fall under the main categories of TT. These features, as well as pertinent-to-context TBs, emerge in the operationalization of the model.

The process through which we identified the relevant features of TT and TBs began with a pilot analysis in which we attempted to analyze 7 students' responses using the model and its operationalization as presented by Sierpinska et al. (2002). It was during this pilot analysis that we realized that the operationalization done for the Linear Algebra study was not entirely representative of the behaviors we might observe in our study, and that operationalizations of the model need to be specific to the context under study. As we carried out the analysis, we were attentive to new TBs and features of TT that emerged from the student responses and which seemed to be representative of the type of thinking which we might encounter in our study. We understood that this would re-occur as we analyzed the entire set of responses.

In the actual study, we began the operationalization through analyzing the research instruments (quiz questions) by listing behaviors which we believed might be provoked by the quiz questions. The operationalization is discussed in detail in the *methodology* chapter. This operationalization is not necessarily an exhaustive one since it was conceived ad hoc for the purpose of analyzing data collected via a specific research tool (the quizzes).

Developing theory from data and constantly comparing the two, as we did in our operationalization, is part of a type of methodology known as *Grounded Theory Methodology*, developed by the sociologists Barney Glaser and Anselm Strauss in 1967. In Grounded theory methodology, theory evolves while conducting research through a continuous interplay between analysis and data collection. In this methodology, theory is initially generated from data, or, if an existing theory seems somewhat appropriate to perform analysis, then it can be used and further developed as collected data are compared to it. This is illustrated by the following figure below. In the figure, the double-sided arrows illustrate the interplay between the objects they refer to.

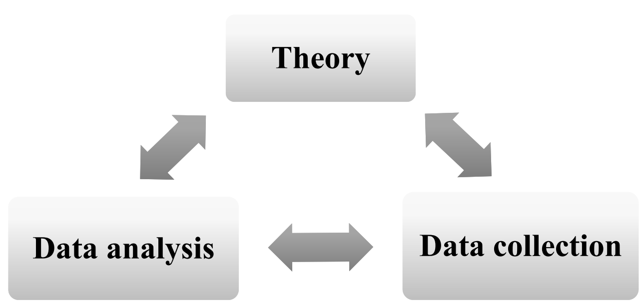


Figure 2.1 - Process involved in Grounded Theory Methodology

In our operationalization, we had already collected the data for our study; data collection was not dependant on the theory to be used, as it can be in Grounded theory methodology. We compare our methodology to Grounded theory methodology since our operationalization was completed through a continuous interplay between analyzing our data, and modifying the TBs and features of TT. We illustrate this with the figure below. The double-sided arrow illustrates the interplay between the operationalization and data analysis in our study.

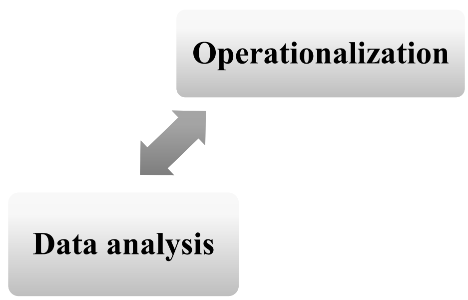


Figure 2.2 - Contribution of our operationalization and analysis to each other

The following table shows the model and its operationalization for the context of this study. As mentioned above, details of how we arrived to this operationalization are discussed later (Chapter 3) – there, we also discuss and contrast some of the different features and TBs.

Category of TT • Feature of TT	TB
TT1 Reflective	$TB1_1$ Displaying an investigative (“researcher’s”) attitude towards mathematical problems ○ $TB1_{11}$ Considering particular cases of a problem ○ $TB1_{12}$ Exploring solution paths ○ $TB1_{13}$ Defining objects in a problem ○ $TB1_{14}$ Connecting components of a problem together ○ $TB1_{15}$ Reflecting on the relationships between concepts in a problem and previously learned concepts ○ $TB1_{16}$ Seeking the requirements of the problem at hand $TB1_2$ Generalizing a solution $TB1_3$ Verifying a solution
TT2 Systemic	
• TT21 Definitional	$TB21_1$ Referring to definitions when deciding upon meaning
• TT22 Proving	$TB22_1$ Engaging in a proving or reasoning activity $TB22_2$ Refuting a general statement by drawing a contradiction $TB22_3$ Referring to a theorem or property $TB22_4$ Referring to previously learned concepts
• TT23 Hypothetical	$TB23_1$ Being aware of the conditional character of a mathematical statement $TB23_2$ Considering particular cases to negate a statement or to state its conditional truth
TT3 Analytic	$TB3_1$ Being sensitive to logical connectives $TB3_2$ Interpreting symbolic expressions in a rigorous way $TB3_3$ Representing a given problem in a different mathematical register

Table 2.3 - The model, operationalized with the features of TT and TBs pertinent to our study

3 Methodology for data collection and operationalization of the model

The question which motivated this study is to determine whether, and how, we can provide students with opportunities to engage in TT in a Calculus II course, despite the institutional constraints that are imposed on this course. The goal of this research then became to answer the question, and to analyze the tool which was used to engage students in TT. In her M.T.M. thesis, Bobos (2004) studied the impact of weekly quizzes administered by the instructor (not her) in an undergraduate Linear Algebra course (based on the vector space theory) on the development of theoretical thinking in the students.. We decided to use the same kind of instructional means (weekly quizzes) to provide students with opportunities to engage in TT in the context of a Calculus II course.

From the perspective of the researchers, two sets of data would be collected: students' responses to the quiz questions, and the questions themselves. Analyzing the former would tell us whether we were able to engage a reasonable number of quiz participants in TT, thus answering the research question which motivated the study; analyzing the latter would characterize the questions and help us evaluate the effectiveness of this tool in light of the model of TT.

However, in order to perform the analyses, we first needed to operationalize the model of TT; this was done by examining the questions which were created for the study, and, later, examining student responses helped refine the operationalization; this will be discussed later. In this section, we first describe how the quiz questions (or research instruments) were created, and then show how they were used to operationalize the model. Next, we explain how the quizzes were implemented, including a description of the setting and participants, and the quiz-taking (and instructor feedback) protocols.

3.1 Research instruments

The questions were not devised prior to the beginning of the course and the study; in fact, as mentioned in the *introduction* chapter, the study was devised with the start of the semester and there became an immediate need for the questions that would be used to engage students in TT. The first few questions were thus designed immediately, and the rest on a weekly basis as needed.

The quiz questions were designed by the instructor of the course and reviewed by the researchers before distributing them in class. Of course, the instructor is one of the researchers in this study, but in designing the quizzes this researcher assumed the role of the instructor since she had to take into account her course outline, and tailor the quizzes around the covered objectives; furthermore, she had no prior research experience in the field, and designed the questions according to her own perception of what 'theoretical thinking' is and the types of questions which might inspire this type of thinking. The quiz-marking (discussed in detail later in this chapter) was also assumed by the instructor who was not marking according to a TT model, but rather, based on whether the responses contained arguments that were mathematically sound. The researchers were mainly involved later in the study, when both the student responses and the quiz questions that had been designed were being analyzed according to the TT model.

We mention the above to clarify the circumstances under which the questions were created, and to differentiate between the works resulting from the ‘instructor’ and the ‘researchers’ in this study.

The questions were designed in a way that they a) relate to material that was covered in class; and b) stimulate TT. While “relating to material that was covered” is fairly unambiguous, “stimulating TT” depended on our perception, as mathematicians, researchers, and instructors of what “theoretical thinking” in the context of an introductory Calculus course means. At this point we had not yet a model for TT, and the design of the questions was not based on a particular, previously conceived model of TT. For us, TT generally meant thinking about concepts rather than recalling known procedures; more precisely, it meant (but was not limited to) referring to underlying meanings in a problem and identifying connections between objects that are not obviously related (for instance, referring to a definite Riemann integral as a real number). It involved using creative, perhaps intuitive, problem-solving techniques, and being able to integrate several ideas to solve tasks for which one’s existing ‘tools’ may not suffice. It is with these notions of TT that we designed the questions.

The hope was that the questions would prompt in students a type of behavior which we could characterize as a display of TT. There was typically not only one correct way to answer the questions, but students were asked to clearly justify their responses so that their thinking is (in a sense) reflected in their response.

In total there were 12 questions. In what follows, we will present each question together with a discussion including the responses we expected and the type of engagement we hoped the question would stimulate – which are not always the same.

3.1.1 Quiz questions

Question 1

Recall the definition of a Riemann integral given in class:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Is it essential that *n* approaches infinity? Give a detailed justification of your answer.

Figure 3.1 - Question 1

We expected that the most common answer would be a version of "yes, for accuracy"[2]. We anticipated, however, that while students would recognize that the above is the definition of a Riemann integral and confirm that '*n*' should approach infinity, they might not readily write about the components which make up the right-hand side of the equality. In asking this question, we hoped that students would reflect on the definition of the Riemann integral and the different concepts involved. In particular, we intended for students to think about the role of '*n*' in the definition, making the connection with the

[2] It is worth noting that students in an introductory Calculus course might not be familiar with the definition of area. In secondary school, the idea of the area of a plane region is usually explained as the limit (in an intuitive sense) of "coverings" of the region with squares with smaller and smaller sides. It is assumed that the area of a square with side a is a^2; sometimes the formula for the area of a rectangle is derived from this assumption; sometimes it is just given. The Riemann integral is based on the assumption (as known, or as an axiom) that the area of a rectangle with sides of length a and b is $a*b$. It also assumed as obvious that the area of a union of two non-overlapping regions is the sum of the areas of the component regions. These intuitive notions are unpacked and questioned only in Measure Theory. Therefore, their acceptance that a Riemann integral serves to calculate an area is based, perhaps, on a naive idea of what area means. Furthermore, the notion of "accuracy", in this context, is supported by visual examples, not by formal calculations.

number of rectangles and the accuracy generally involved with considering rectangles with a smaller base. Perhaps even an example of a function for which it is *not* important that n approaches infinity (e.g., a constant function). We felt that students might not readily justify their answer because writing about the concepts involved on the right-hand side (number of rectangles, the effect of 'n' on the area of the rectangles, etc…) is not a usual exercise in this class, and students might not readily find the language or approach to think about these ideas.

Question 2

Is it true that $\int_a^b f(x)dx + \int_c^d g(x)dx = \int_c^d g(x)dx + \int_a^b f(x)dx$ where a, b, c, and d are real numbers? Clearly justify your answer.

Figure 3.2 - Question 2

We expected that most students would intuitively respond "yes" to this question but perhaps without a sound justification of their answer. Furthermore, we expected that most students would not consider the conditions for which each of these integrals exists in the first place. We hoped that this question would encourage students to think about the transferability of the commutative property of addition of *real numbers* to other objects – in this case *definite integrals*, and consider whether this is possible and why. We anticipated that students might relate the definite integral to a real number or else refer to the definition and relate to limits of sums, realizing that the commutative property of addition holds when these identifications can be made.

Question 3

Recall the method of integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Suppose you are given an integral to evaluate using the method of 'integration by parts'. Explain how you would choose which part of the integrand will take the role of 'u' and which part of the integrand will take the role of 'dv'.

Figure 3.3 - Question 3

Our intention through this question was that students reflect on the roles of 'u' and 'dv' in the formula for 'integrating by parts' [$\int udv = uv - \int vdu$], and how they are each treated in this process. We also hoped that students would follow through and consider the integral resulting on the right-hand side of the formula, which might also affect their choice of 'u' and 'dv' originally. Our expectation was that most students would express that their choice would depend on which part of the integrand is more easily differentiable, and which part is more easily integrable, and that these would take the roles of 'u' and 'dv' respectively. We did not expect most students to immediately consider the integral resulting after applying the rule ($\int vdu$) as it involves thinking about the effect of the first choice on the resulting integral, and perhaps involves a level of abstraction. We hoped, however, that the need to consider it while making their original decision would be noticed after some reflection, or perhaps after writing down the formula and realizing that it needs to be considered.

Question 4

Is it possible to compute the area between the curve representing the function $f(x)$, the x-axis, $x = a$, and $x = b$ using a Riemann sum? Justify your answer.

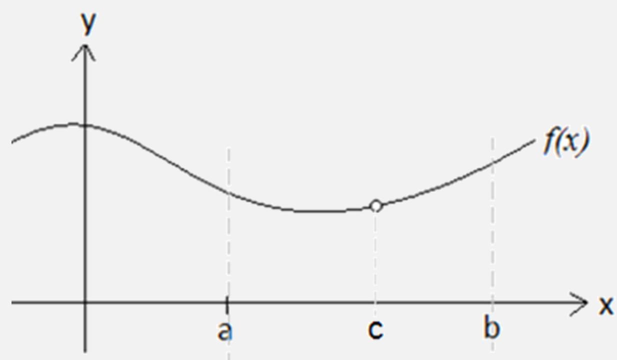


Figure 3.4 - Question 4

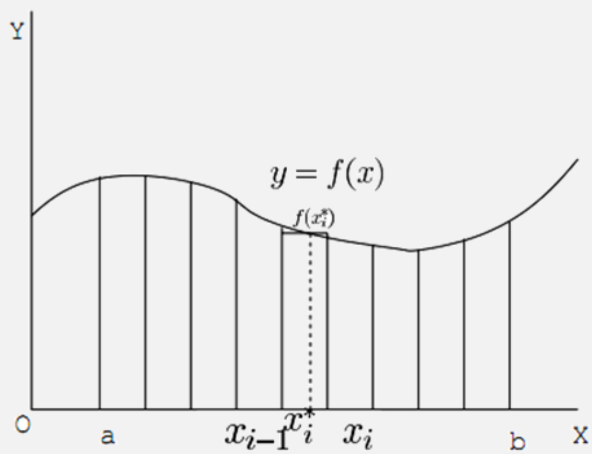
We anticipated that the most common answer would be equivalent to "No, since f is not continuous" because theorems presented in class regarding this matter only consider continuous functions (although continuity is not a necessary condition). Still, we expected that students would engage in discussions about the area under this curve which could require them to think theoretically. We hoped that this question would raise some discussion about 'missing area' that lies above the point $x = c$ and below the 'hole' in the curve; the *size* of this area, and whether it is negligible. Or perhaps a discussion about whether we can view this area as the union of two chunks of area – before and after $x = c$. Furthermore, since students in this course have the notion of *limits*, we anticipated a reference to the limit of the function as x approaches c from the left and right of c.

Question 5

Recall the definition of a Riemann integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i^*)$, where x_i^* is a sample point in the interval $[x_{i-1}, x_i]$.



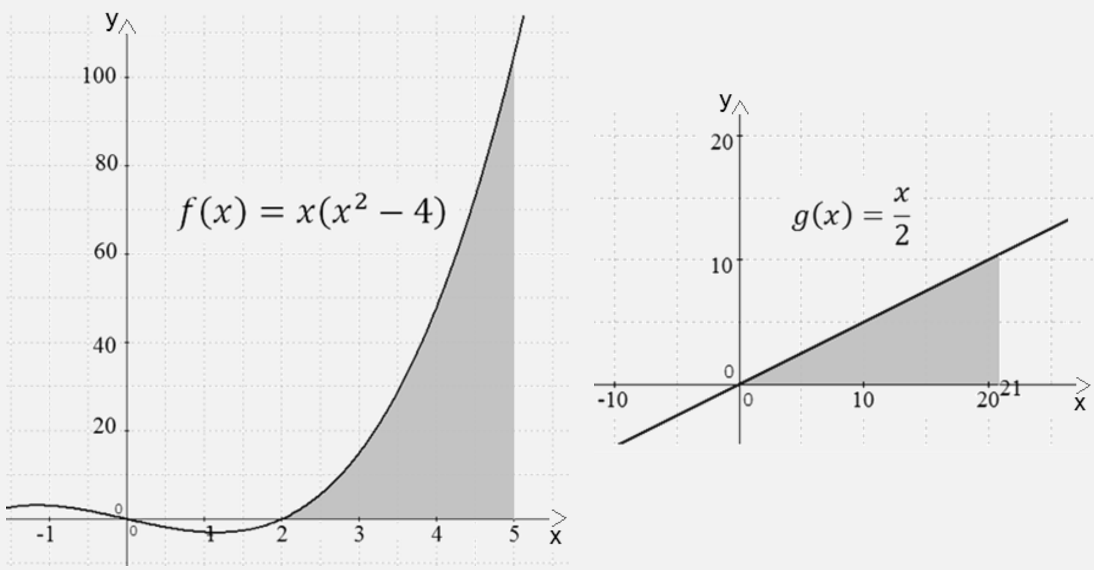
Must the sample point x_i^* be chosen at the same position in each interval, or can it be the right end point in an interval, the left end point in another interval, and any random position in another interval (for example)? Justify your answer.

Figure 3.5 - Question 5

While the sample point could actually be chosen anywhere (since as '*n*' approaches infinity, the position of the sample point becomes irrelevant), we were not quite sure what the most common response to this question would be. We suspected most students would respond "No, the sample point cannot be chosen randomly. It must be chosen in the same position in each interval" for the reason that it was not a familiar situation – every question solved in class involved choosing a fixed position for x_i^* (in fact always the left, right, or middle of the intervals). As for a justification of their response, we thought students might explain that “the Riemann sum will no longer be accurate if x_i^* was chosen randomly”, without a careful consideration of how different the entire sum would actually be once the limit is taken. Through this question, we hoped that students would think about the role of ‘n’ in the formula and consider possibilities other than the ones they are accustomed to.

Question 6

It was explained that to calculate the value of the shaded area below one could instead calculate the value of this shaded area



Your friend does not understand why. Can you explain to him or her why this is true?

Figure 3.6 - Question 6

This question involves noticing that one function is a continuous transformation of the other. In the Calculus course, students are used to representing the value of the shaded area under the first curve as a definite integral, and then using the ‘substitution method for integration’ to arrive to an equivalent integral which represents the value of the shaded area under the second curve. There are probably not many ways to solve this problem, and we intended that students recall the connection between an area and a definite Riemann integral and use this to solve the problem. We expected that some students would not make the connection we intended because of the novelty of this problem in their experience, and not because of a conceptual difficulty. We expected that some students might set up and compute integrals which represent the value of each of the areas and falsely believe that this responds to the problem. While this approach confirms that the two areas are equal in value, it does not explain how we could have known this without actually *computing* the integrals (or the first integral, in particular), which is precisely what the problem demands.

Question 7

Explain, in your own words, why this theorem is true:
If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Figure 3.7 - Question 7

The contrapositive of this theorem is normally presented in the course as a test for the divergence of a series (if $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent). We did not intend for a formal proof as a response to this question, but perhaps some informal discussion or even illustrations to reason why the statement must be true. Also, since the contrapositive of this statement is known to students, we hoped that they might realize this and argue for the truth of the statement by ‘contradiction’ – arguing that if $\lim_{n\to\infty} a_n \neq 0$ then necessarily the series is divergent, contradicting the hypothesis of this statement. Overall, we hoped that this question would provide students with a chance to reason about a statement using their own wording and ideas. We did expect students to engage in a form of reasoning (whether through words or a diagram, etc…).

Question 8

If a sequence $\{a_n\}_{n\geq 1}$ is such that $a_n > 0$ and $a_{n+1} < a_n$ for all n, then can we be sure that the sequence $\{a_n\}_{n\geq 1}$ converges? Justify your answer.

Figure 3.8 - Question 8

The hypothesis of this problem indirectly states that the sequence is lower bounded (by 0, since all terms are positive) and decreasing (since $a_{n+1} < a_n$ for all n), which are sufficient conditions for the convergence of a sequence. The theorem is not a novel one to students but is stated here in a slightly indirect way. We thus hoped that this question would prompt students to reflect about and interpret the given hypothesis, concluding that the sequence must indeed converge, but did not expect that most students would accurately achieve this since it involves various extractions from the hypothesis and then connecting these extractions together. As the previous problem, we felt that this problem invites students to discuss and argue using their own words and reasoning, and we did expect that students would do so. A curiosity we had was whether those students who do realize it is a convergent sequence will presume that it converges to 0, which of course is not necessarily true but may seem so intuitively before careful consideration.

Question 9

If $\sum a_n$ and $\sum b_n$ are series with strictly positive terms, then one of the statements of the Direct Comparison Test theorem is:

[if $a_n < b_n$ for all n, and if $\sum b_n$ converges, then $\sum a_n$ also converges]

Can we write a similar statement if $\sum a_n$ and $\sum b_n$ are series with strictly negative terms and if $\sum b_n$converges? If so, what would be the statement? Explain your choice of this statement, specifying the relationship between a_n and b_n. If not, explain why not.

Figure 3.9 - Question 9

We expected that students would propose a correct statement (equivalent to [if $b_n < a_n$ for all n, and if $\sum b_n$ converges, then $\sum a_n$ also converges]) and some reasoning as to why the statement is true. We hoped that students would consider modeling this problem using a graph, which could help them reason about their response, or else engage in a discussion regarding the partial sums $A_{n=} \sum_{i=1}^{n} a_i$ and $B_{n=} \sum_{i=1}^{n} b_i$ with $b_n < a_n$, and perhaps arrive to a sound reasoning through this discussion.

Question 10

In the paradox of "Achilles and the Tortoise", Achilles is in a footrace with the tortoise.

Achilles allows the tortoise a head start of 100 meters.
If we suppose that Achilles and the tortoise each start running at some constant speed (one very fast and one very slow), then after some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point.

During this time, the tortoise has run a much shorter distance, say, 10 meters. It will then take Achilles some further time to run that distance, during which the tortoise will have advanced farther; and then more time still for Achilles to reach this third point, while the tortoise moves ahead.

Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Therefore, because there are an infinite number of points Achilles must reach where the tortoise has already been, he can never overtake the tortoise!

*   *   *

This is a paradox because there seems to be something contradictory about Achilles being faster, but not ever being able to pass the tortoise due to the infinite number of 'distances' he needs to cross first.

Use graphs, algebra, verbal explanations, or any means to justify how Achilles will come closer, infinitely many times, to the tortoise without passing it AND will pass the tortoise (showing that these two events do <u>not</u> have to lie in contradiction).

Figure 3.10 - Question 10

We posed this problem with a desire to situate *limits* and *convergence* in a so-called real life situation. We hoped that students would realize that the two events indeed do not lie in contradiction; that although Achilles is getting closer and closer to the tortoise and one can talk about this happening indefinitely so long as smaller and smaller time intervals are considered, Achilles actually *passes* the tortoise after some time. We expected that students would provide illustrations to model the situation and perhaps use mathematical tools to engage in a discussion about this problem which is otherwise a non-traditional problem in a Calculus class.

Question 11

Consider a function $g(x)$ that is continuous on the interval $(-\infty, \infty)$.
Suppose that the integral $\int_1^\infty g(x)dx$ is convergent.
Is it true that $\int_{10}^\infty g(x)dx$ is also convergent? Justify your answer.

Figure 3.11 - Question 11

While this statement seems trivially true at first sight, and indeed we expected that most students would state that it is true, such a problem was not previously posed in class and we hoped and expected that the problem would invite students to use graphical or algebraic techniques to justify why the statement must be true. Perhaps students would draw a graph of the function $g(x)$ indicating the area bounded by the curve, the x-axis, and $x = 1$, and showing that the latter integral represents the same area but without the (finite) 'chunk' of area between $x = 1$ and $x = 10$; or else would write the first integral as the sum of two integrals: one over the interval [1,10], and the other over [10, ∞) (which is in fact the second integral); using these types of arguments to justify their response. We also expected that some students might associate the integrals with series to argue that the statement must be true; such problems involving infinite series that begin with $n = 3$, for instance, instead of $n = 1$ were considered in class and it was pointed out that the addition or subtraction of a finite number of terms would not affect that convergence or divergence of an infinite series. We wondered whether students would be able to make the analogy with integrals.

Question 12

Consider a function $g(x)$ that is continuous on $(-\infty, \infty)$. Suppose that the integral $\int_1^\infty g(x)dx$ is convergent.

Is it true that $\int_1^\infty [g(x) + 2]dx$ is also convergent? Justify your answer.

Figure 3.12 - Question 12

We expected students to misinterpret the second integral as being 2 units more than the first – instead of *the integral over [1,∞) of 2* more; and therefore falsely concluding that it is a convergent integral. What we intended, however, is that students use either algebraic or graphical techniques to show that the integral in question is in fact divergent.

3.2 Operationalization of the model

The operationalization took place after the questions were designed. As explained in the *theoretical framework* chapter, we realized that the TBs needed to be specific to the context of our study. Also, from our pilot analysis, we realized that TBs would emerge from student responses. However, in order to have an *a priori* set of behaviors which we could expect to observe in student responses, we decided to list, per question, behaviors which might be displayed while engaging with the question, and which we count as displays of TT (as it is generally described by the model). These lists consisted of behaviors that were more specific to the corresponding question than the previous TBs; for instance they could refer to mathematical objects that appear solely in that question, yet fall under a more general TB. These behaviors took the form of action phrases (as did the TBs) and would act as a rubric while analyzing student responses. Of course, it would not be an entirely rigid rubric since students' responses could contain behaviors which we did not anticipate. In a sense, these behaviors would constitute a more finely grained tool to analyze the responses in search for displays of TT.

Thus, our operationalization process began by listing, for each question, behaviors in the form action phrases which could describe an individual's behavior while answering the question. We called these action phrases '*features of discourse*' and interpret their display as an indication of the occurrence of TT. We listed a set of features of discourse for each question. These are displayed in the tables below. The columns on the right indicate the TBs under which we grouped the adjacent features of discourse.

Question 1

Recall the definition of a Riemann integral given in class:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Is it essential that *n* approaches infinity? Give a detailed justification of your answer

Features of discourse	**Corresponding TB**
Considering particular cases for which it is not important (e.g. *f* constant)	$TB23_2$: Considering particular cases to negate a statement or to state its conditional truth
Defining the variable *n*/ the Riemann sum in this context	$TB1_{13}$: Defining objects in a problem
Explaining the relationship between *n* and Δx/ area/ length of intervals	$TB21_1$: Referring to definitions when deciding upon meaning
Relating the area of a rectangle to its height	$TB1_{14}$: Connecting components of a problem together
Explaining the relationship between the left-hand side and the right-hand side of the equality in the definition	$TB1_{14}$: Connecting components of a problem together

Explaining why area obtained from rectangles with a more narrow base has less error than from those with a wider base	$TB22_1$: Engaging in a proving or reasoning activity
Referring to the effect of increasing *n* on the accuracy of the approximation of the area	$TB22_3$: Referring to a theorem or property

Table 3.1 - Features of discourse for Q1

Question 2

Is it true that $\int_a^b f(x)dx + \int_c^d g(x)dx = \int_c^d g(x)dx + \int_a^b f(x)dx$ where a, b, c, and d are real numbers? Clearly justify your answer.	
Features of discourse	**Corresponding TB**
Posing a question about the characteristics of the functions or on whether the integrals are defined on the given intervals	$TB23_1$: Being aware of the conditional character of a mathematical statement
Relating the definite integral to the limit of a Riemann sum (thus relating concepts)	$TB1_{15}$: Reflecting on the relationships between concepts in a problem and previously learned concepts
Discussing the commutative property of limits (in case the subject identifies the integral with the limit of a Riemann sum)	$TB22_3$: Referring to a theorem or property
Relating the definite integral to a real number (assuming/ arguing that the integral is convergent. Note: Convergence of an integral not yet discussed at this point in course)	$TB1_{15}$: Reflecting on the relationships between concepts in a problem and previously learned concepts
Discussing the commutative property of addition of real numbers (in case the subject identifies the integral with a real number)	$TB22_3$: Referring to a theorem or property
Considering a particular case of the integrand and limits of integration	$TB1_{11}$: Considering particular cases of a problem
Using the Fundamental Theorem of Calculus to write an equivalent expression of the integrals	$TB22_3$: Referring to a theorem or property

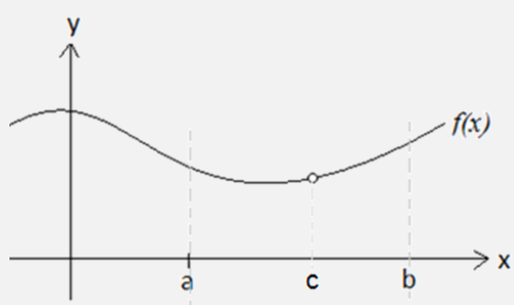
Table 3.2 - Features of discourse for Q2

Question 3

Recall the method of integration by parts: $$\int u\,dv = uv - \int v\,du$$ Suppose you are given an integral to evaluate using the method of 'integration by parts'. Explain how you would choose which part of the integrand will take the role of '*u*' and which part of the integrand will take the role of '*dv*'.	
Features of discourse	**Corresponding TB**
Displaying anticipation for the remainder of the "integration by parts" procedure, i.e., integrability of *vdu* or obtaining an identical integrand to *udv* etc…	$TB1_{12}$: Exploring solution paths
Referring to the formula for *integration by parts* to support argument	$TB22_{3}$: Referring to a theorem or property
Reasoning about the choice of *u* and *dv*	$TB22_{1}$: Engaging in a proving or reasoning activity

Table 3.3 - Features of discourse for Q3

Question 4

Is it possible to compute the area between the curve representing the function $f(x)$, the *x*-axis, $x = a$, and $x = b$ using a Riemann sum? Justify your answer. 	
Features of discourse	**Corresponding TB**
Considering the limit of *f* as *x* approaches the point, *c*, of discontinuity	$TB1_{12}$: Exploring solution paths
Discussing the negligibility of the area of a "segment" under the point of discontinuity,	$TB22_{4}$: Referring to previously learned concepts

or *measure* of a segment	
Dividing the interval into [a, c) U (c, b]	$TB23_2$: Considering particular cases to negate a statement or to state its conditional truth
Referring to the hypothesis of the Fundamental Theorem of Calculus (given such that *f* is continuous)	$TB23_1$: Being aware of the conditional character of a mathematical statement
Considering intervals such that *c* is not a sample point	$TB23_2$: Considering particular cases to negate a statement or to state its conditional truth

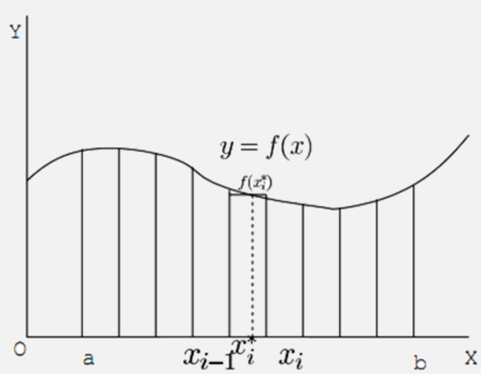
Table 3.4 - Features of discourse for Q4

Question 5

Recall the definition of a Riemann integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i^*)$, where x_i^* is a sample point in the interval $[x_{i-1}, x_i]$.



Must the sample point x_i^* be chosen at the same position in each interval, or can it be the right end point in an interval, the left end point in another interval, and any random position in another interval (for example)? Justify your answer.

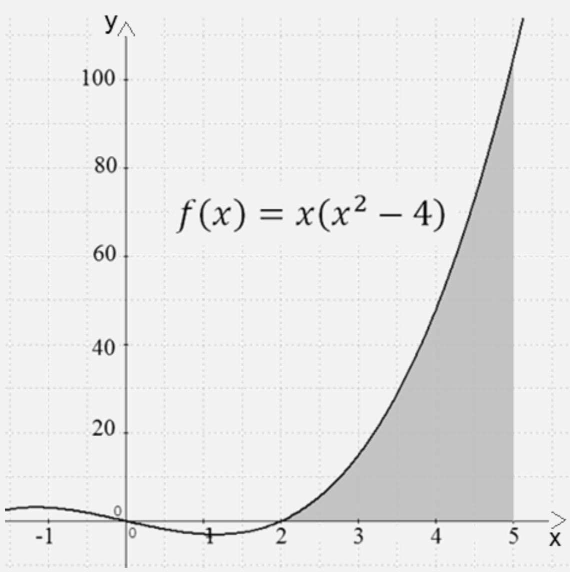
Features of discourse	**Corresponding TB**
Discussing the irrelevance of the position of x_i once the limit of the Riemann sum is	$TB22_4$: Referring to previously learned concepts

taken	
Discussing continuity of f at x_i	TB23$_1$: Being aware of the conditional character of a mathematical statement
Discussing the variance of the Riemann sum (not the limit of-) as the position of x_i varies	TB1$_{11}$: Considering particular cases of a problem
Discussing the practicality in choosing x_i at the same position; x_i is then equal to $a + i\Delta x$, for $1 \leq i \leq n$, for example, if x_i is chosen to be the right end point	TB1$_{14}$: Connecting components of a problem together
Discussing the *necessity* in choosing x_i at the same position if the formula $a + i\Delta x$, for $1 \leq i \leq n$ is to be used	TB21$_1$: Referring to definitions when deciding upon meaning

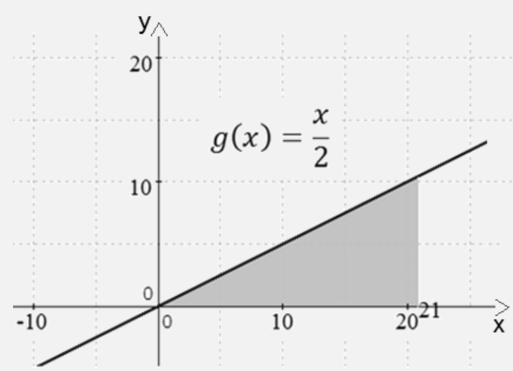
Table 3.5 - Features of discourse for Q5

Question 6

It was explained that to calculate the value of the shaded area below



one could instead calculate the value of this shaded area



Your friend does not understand why. Can you explain to him or her why this is true?

Features of discourse	Corresponding TB
Setting up definite integrals representing each area	TB3$_3$: Representing a given problem in a different mathematical register

Showing how one can obtain the second integral from the first	$TB22_1$: Engaging in a proving or reasoning activity
Exploring the problem; verifying that the two areas are equal in magnitude	$TB1_3$: Verifying a solution
Displaying awareness that the area need not be computed	$TB1_{16}$: Seeking the requirements of the problem at hand
Regarding the variable in the integrand as a dummy variable	$TB3_2$: Interpreting symbolic expressions in a rigorous way

Table 3.6 - Features of discourse for Q6

Question 7

Explain, in your own words, why this theorem is true: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$	
Features of discourse	**Corresponding TB**
Discussing the necessary nature of terms a_n for large values of n if the series is convergent	$TB22_3$: Referring to a theorem or property
Arguing by contradiction	$TB22_2$: Refuting a general statement by drawing a contradiction/ $TB3_1$: Being sensitive to logical connectives
Modeling the behavior of the partial sums of a convergent series or of the terms of a convergent series graphically	$TB3_3$: Representing a given problem in a different mathematical register
Stating the contra-positive of the statement	$TB3_1$: Being sensitive to logical connectives
Explaining why the contra-positive of the statement is true	$TB22_1$: Engaging in a proving or reasoning activity
Stating the definition of an infinite series/ a convergent infinite series	$TB1_{13}$: Defining objects in a problem
Referring to the definition of an infinite series/ a convergent infinite series	$TB21_1$: Referring to definitions when deciding upon meaning

Table 3.7 - Features of discourse for Q7

Question 8

If a sequence $\{a_n\}_{n\geq1}$ is such that $a_n > 0$ and $a_{n+1} < a_n$ for all n, then can we be sure that the sequence $\{a_n\}_{n\geq1}$ converges? Justify your answer.	
Features of discourse	**Corresponding TB**
Concluding from the hypothesis that the sequence is lower bounded and decreasing	$TB22_3$: Referring to a theorem or property
Referring to the theorem for monotone and bounded sequences to conclude the convergence of this sequence	$TB22_3$: Referring to a theorem or property
Referring to the definition of a convergent sequence	$TB21_1$: Referring to definitions when deciding upon meaning
Considering particular examples; particularizing	$TB1_{11}$: Considering particular cases of a problem
Arguing by contradiction	$TB22_2$: Refuting a general statement by drawing a contradiction/ $TB3_1$: Being sensitive to logical connectives
Explaining why the sequence cannot diverge	$TB22_1$: Engaging in a proving or reasoning activity
Interpreting the behavior of the given type of sequence	$TB3_2$ Interpreting symbolic expressions in a rigorous way

Table 3.8 - Features of discourse for Q8

Question 9

If $\sum a_n$ and $\sum b_n$ are series with strictly <u>positive</u> terms, then one of the statements of the Direct Comparison Test theorem is: [if $a_n < b_n$ for all n, and if $\sum b_n$ converges, then $\sum a_n$ also converges] Can we write a similar statement if $\sum a_n$ and $\sum b_n$ are series with strictly <u>negative</u> terms and if $\sum b_n$converges? If so, what would be the statement? Explain your choice of this statement, specifying the relationship between a_n and b_n. If not, explain why not.	
Features of discourse	**Corresponding TB**
Writing a single general statement for positive and negative series by considering	$TB1_2$: Generalizing a solution

the absolute value of terms	
Representing the terms of the sequence S_n graphically	$TB3_3$: Representing a given problem in a different mathematical register
Discussing boundedness and monotony of S_n (where S_n is the sequence of partial sums of the series whose convergence is in question)	$TB1_{15}$: Reflecting on the relationships between concepts in a problem and previously learned concepts
Describing an analogy between the two statements	$TB22_1$: Engaging in a proving or reasoning activity

Table 3.9 - Features of discourse for Q9

Question 10

In the paradox of "Achilles and the Tortoise", Achilles is in a footrace with the tortoise.

Achilles allows the tortoise a head start of 100 meters.
If we suppose that Achilles and the tortoise each start running at some constant speed (one very fast and one very slow), then after some finite time, Achilles will have run 100 meters, bringing him to the tortoise's starting point.

During this time, the tortoise has run a much shorter distance, say, 10 meters. It will then take Achilles some further time to run that distance, during which the tortoise will have advanced farther; and then more time still for Achilles to reach this third point, while the tortoise moves ahead.

Thus, whenever Achilles reaches somewhere the tortoise has been, he still has farther to go. Therefore, because there are an infinite number of points Achilles must reach where the tortoise has already been, he can never overtake the tortoise!

*   *   *

This is a paradox because there seems to be something contradictory about Achilles being faster, but not ever being able to pass the tortoise due to the infinite number of 'distances' he needs to cross first.

Use graphs, algebra, verbal explanations, or any means to justify how Achilles will come closer, infinitely many times, to the tortoise without passing it AND will pass the tortoise (showing that these two events do <u>not</u> have to lie in contradiction).	
Features of discourse	**Corresponding TB**
Indicating that as smaller distances are considered, smaller time intervals are simultaneously considered (not explicit in problem)	$TB1_{14}$: Connecting components of a problem together
Marking the positions of Achilles and the tortoise over equal time intervals; indicating that Achilles *does* pass the tortoise	$TB3_3$: Representing a given problem in a different mathematical register
Deriving equations of motion for Achilles and the tortoise and using them for explanation (for example, to express the time at which Achilles passes the tortoise)	$TB3_3$: Representing a given problem in a different mathematical register
Deriving speed/ time graphs for Achilles and the tortoise and using them for explanation (for example, to express the time at which Achilles passes the tortoise)	$TB3_3$: Representing a given problem in a different mathematical register
Using the concept of a limit to describe the distance between Achilles and the tortoise before Achilles passes the tortoise	$TB22_4$: Referring to previously learned concepts
Explaining why the two situations do not lie in conflict	$TB22_1$: Engaging in a proving or reasoning activity

Table 3.10 - Features of discourse for Q10

Question 11

Consider a function $g(x)$ that is continuous on the interval $(-\infty, \infty)$. Suppose that the integral $\int_1^\infty g(x)dx$ is convergent. Is it true that $\int_{10}^\infty g(x)dx$ is also convergent? Justify your answer.	
Features of discourse	**Corresponding TB**
Expressing the integral $\int_{10}^\infty g(x)dx$ as $\int_1^\infty g(x)dx - \int_1^{10} g(x)dx$	$TB22_3$: Referring to a theorem or property
Explaining why the integral $\int_{10}^\infty g(x)dx$ is convergent	$TB22_1$: Engaging in a proving

after expressing it as $\int_1^\infty g(x)dx - \int_1^{10} g(x)dx$	or reasoning activity
Indicating that and explaining why the integral $\int_1^{10} g(x)dx$ is convergent	$TB22_1$: Engaging in a proving or reasoning activity
Indicating that the integral $\int_a^\infty g(x)dx$ is convergent for all $a \geq 1$	$TB1_2$: Generalizing a solution
Indicating that the integral $\int_a^b g(x)dx$ is convergent over any subinterval [a, b] of [1,∞)	$TB1_2$: Generalizing a solution
Indicating that (10, ∞) is a subinterval of (1, ∞) and explaining that the integral $\int_{10}^\infty g(x)dx$ is thus convergent	$TB22_1$: Engaging in a proving or reasoning activity
Assuming $g(x) > 0$: Drawing an arbitrary graph representing $g(x)$, then indicating the areas represented by the integrals $\int_1^\infty g(x)dx$ and $\int_{10}^\infty g(x)dx$, showing that the latter is included in the former	$TB3_3$: Representing a given problem in a different mathematical register

Table 3.11 - Features of discourse for Q11

Question 12

Consider a function $g(x)$ that is continuous on $(-\infty, \infty)$. Suppose that the integral $\int_1^\infty g(x)dx$ is convergent. Is it true that $\int_1^\infty [g(x) + 2]dx$ is also convergent? Justify your answer.	
Features of discourse	**Corresponding TB**
Splitting the integral $\int_1^\infty [g(x) + 2]dx$ into $\int_1^\infty g(x)dx$ + $\int_1^\infty 2dx$ and arguing that: $\int_1^\infty 2dx$ is divergent; thus the whole integral is divergent	$TB22_1$: Engaging in a proving or reasoning activity
Drawing a graph representing $g(x)$ and then $g(x) +$ 2; indicating that the latter represents an infinite area	$TB3_3$: Representing a given problem in a different mathematical register
Noting the general statement "adding any non-zero constant to the integrand would result in a divergent integral"	$TB1_2$: Generalizing a solution

Table 3.12 - Features of discourse for Q12

The operationalization continued as we analyzed student responses and found behaviors which we had not anticipated. Although this was rare, it did occur, and we simply expanded our lists of behaviors and features of TT as required. Once we compiled the lists of features of discourse (per question, as in the tables above), we grouped similar features of discourse under TBs from Sierpinska et al.'s operationalization. When we found that a group of features of discourse was not accurately represented by any of the existing TBs, we listed a new behavior which accurately described the group and was at the same time indicative of one of the three main categories of TT (*Reflective, Systemic,* and *Analytic*). After grouping all the features of discourse under appropriate TBs, we eliminated TBs which belonged to the original operationalization but which were shown to be irrelevant to our study (i.e., which were not associated with any feature of discourse), and added the new ones that emerged.

In a similar manner, we grouped similar TBs under features of TT, and finally grouped these features of TT under the three main categories of TT. The result of this work was an operationalization of the model that suits the context of our study. In the following table is an extract from the grouping of features of discourse under TBs, which are in turn grouped under features and categories of TT. The complete list can be found in the Appendix.

Category of TT	TT2 Systemic
Feature of TT	TT23 Hypothetical
TB	$TB23_1$ Being aware of the conditional character of a mathematical statement
Features of discourse	• Posing a question about the characteristics of the functions, or about whether the integrals exist on the given intervals (from Q2) • Referring to the hypothesis of the Fundamental Theorem of Calculus (from Q4) • Discussing continuity of f at x_i (from Q5)

Table 3.13 - Grouping similar features of discourse under a TB and category of TT

It is easy to see that features of discourse are similar in form to TBs (they are both action phrases that describe a behavior); however, while features of discourse are specific to the response to a particular question, TBs are more general and describe an entire group of features of discourse. While they are similar in form, they are each meaningful in their own way; for example, features of discourse are question-specific, so that one can use them as a rubric to analyze a student's response. TBs, on the other hand, are more general and link the features of discourse to the features of TT. The more general and encompassing terminology used in TBs makes it easier for one to see why a particular feature of discourse is indeed a display of a corresponding feature of TT.

For the remainder of the study, we could use our operationalization of the model to analyze our collected data. The process of operationalizing the model was ongoing, however; as we analyzed student responses, features of discourse which we had not anticipated sometimes appeared and were added to the list. Finally, we operationalized the model (Table 2.3, displayed in section 2.1) with a total of thirteen TBs with one of the behaviors broken down into six sub-behaviors.

As explained previously, TBs that operationalize the model vary with different mathematical contexts. In what follows, we aim to clarify why the behaviors presented in Table 2.3 are indicative of the features of TT which they fall under in our operationalization (in cases where it is not clear from the phrasing of the behavior). *Reflective thinking*, as we understand it, occurs while an individual is taking a step back from the procedural aspect of the problem, and is taking the time to probe the problem or investigate it further. The purpose of *Reflective thinking* is not merely to accomplish tasks. It takes place precisely when the individual is reflecting. For this reason, we felt that the three TBs *Displaying an investigative ("researcher's") attitude towards mathematical problems*, *Generalizing a solution*, and *Verifying a solution* are indicative of the occurrence of *Reflective thinking* since they are both not directly required to solve the problem. Some of the sub-behaviors, such as "$TB1_{13}$: Defining objects in a problem" and "$TB1_{15}$ Exploring relationships between concepts in a problem and previously learned concepts", seem pertinent to *Systemic thinking* since they regard concepts and definitions. We maintain that such behaviors are indicative of *Reflective thinking*, however, when they are displayed with the intention of exploring the problem and becoming acquainted with it before attempting to resolve it.

Analytic thinking is sensitive to the symbolism, structure, and logic of mathematical language. While activities involving proofs or proving are not very common in the college level Calculus context, students are often required to shift between different ways of modeling problems. For example, in computing the integral of a particular square root function, drawing the graphical representation of the function might save one from lengthy calculations. This is accounted for by the behavior "*Representing a given* *problem in a different mathematical register*" which we added to the operationalization and identified with *Analytic thinking*.

3.3 Setting and participants

As explained in the *education context* section of the *introduction*, the context of this study (a college level Calculus course) is not an uncommon one in North America. Many programs, such as Engineering, Business, and pure and applied sciences, offered by post-secondary institutions often list Calculus I, and some, Calculus II, as prerequisites to joining the programs. These prerequisite courses are often provided by the institution itself. At Concordia University, these courses are stretched over thirteen weeks (one term) and consist of two sessions per week, each 1h15 in length. Assessment (weekly online assignments, one midterm exam and one final exam) is common to all sections and is prepared entirely by the course examiner. However, although the course outline and assessments are determined by the course examiner, it is up to the instructor to manage the class time in a way that the weekly objectives are covered effectively.

This study took place in one section of the Calculus II course at Concordia University. Typically, two main topics are covered in this course: *Riemann Integrals* (including the Fundamental Theorem of Calculus, areas, and techniques of integration), and *Series* (including tests for convergence or divergence of series, and Taylor series expansions). A pre-requisite for this course is Calculus I which mainly covers *Limits* and *Derivatives*.

Students registered in Calculus II vary greatly in age as well as in the programs they are registered in, with the majority enrolled in an engineering program (mainly civil or mechanical). Other programs that students are enrolled in include political science, anthropology, and computer-related programs. Attendance is neither compulsory nor graded. There are between 60 and 70 students registered in each section.

In the Calculus II section considered in this study, there were 64 registered students. We will only consider the 55 students, out of those 64, who contributed at least once to the collected data (since we do not have any data from the remaining 9 students).

3.4 Quiz-taking and instructor-feedback protocols

The quizzes required 10 to 15 minutes of class time each week (depending on how demanding the questions were considered to be). While designing this study the researchers and instructor made an ethical decision: they agreed that taking 10-15 minutes away of class every week would be worthwhile to run the quizzes – assuming that the quizzes indeed provide opportunities for TT and that a reasonable number of participants would be engaged in TT through them; assuming also that the instructor would be able to implement the course outline as expected. To save on some class time for running the quizzes, the instructor used the overhead projector to project some of the notes (which were emailed to the students at least a day before class), but still wrote many items on the board. Usually, statements of theorems and questions (without solutions) were projected and emailed to students, whereas explanations or proofs of theorems, examples, graphs, and solutions to problems were not projected, rather they were written on the board so that students could create their own notes of these items. Of course, the entire set of notes could have been projected with hardly any board-writing, thus saving on even more class time, but the instructor believed that certain items would be clearer when they were written word-for-word (versus projected) on the board since students would be more involved in the creation of the notes (even if they did not take notes). Accordingly, she believed it would be unethical to exclusively use the projector for the purpose of saving time for the sake of the study, and proceeded in using the chalkboard as she felt needed. Furthermore, the instructor prepared the lesson plans as she would have without the quizzes, and was determined to implement the study without sacrificing any part of her lesson plans (except for time-adjustment).

The instructor explained to students that taking the quiz was optional, but that points received on a quiz would be counted as *bonus marks* on their course grade (up to 5%, as agreed with the course examiner) for complete responses or for incomplete responses containing valid arguments. Students were informed that the quiz questions were not typical test questions, and while they may not serve directly as practice for the mid-term and final exams, they will aim to improve their understanding of concepts in the course. Students were asked to justify their answers, providing as much detail as possible; this was important since our sole source of data, and thus our potential window to their ideas and thinking, was the set of responses we would receive from them. Finally, students were asked to work individually but were permitted to refer to their notes.

Copies of students' responses were kept for analysis, but, once corrected by the instructor, were returned to students with a grade and written feedback consisting of minor corrections and suggestions for improving the quality of their responses when they were inadequate. For instance, if a response contained incorrect reasoning, the instructor provided counter-examples or suggestions as to why the reasoning may not be sound; or, if a response was correct but could be structured better or written using clearer terminology, suggestions of how to do so were given. We hoped that such feedback would help students in writing future responses. A response was awarded 1 point if it was valid or 0.5 if it was incomplete but contained elements of sound reasoning. In some cases, even if the response was incorrect but displayed evidence of an awareness of the requirement of the task, 0.5 was awarded as well. Otherwise, if the response was incorrect and did not contain any valid arguments, or if no response was given, no points were awarded. The instructor did not provide students with complete answers to the questions to allow them the freedom of developing their own style in answering the questions, especially considering that there was usually not a single correct answer to a question. Instead, the instructor encouraged students to review the feedback they received and refer to her to further explore questions and their responses.

Examples of responses that received partial or full credit are shown below to illustrate the way in which points were awarded and the type of feedback that was given by the instructor.

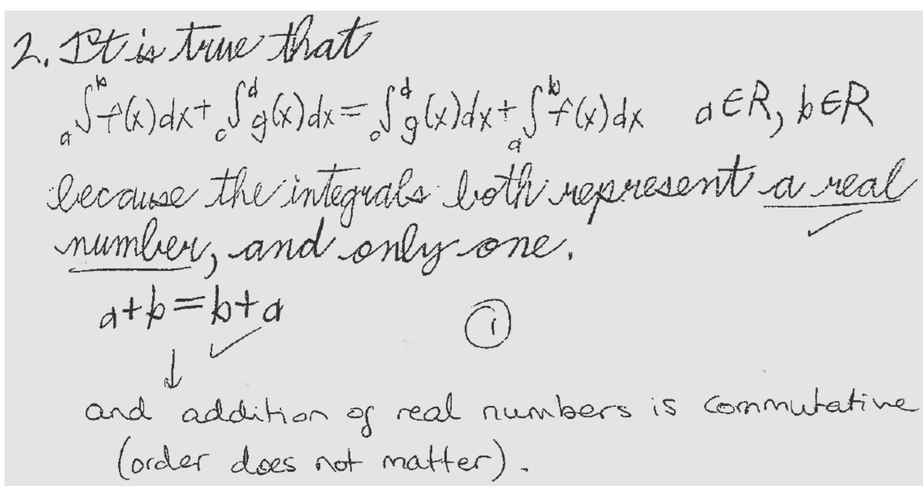


Figure 3.13 - Student response to Q2 which received full credit, and instructor feedback

The student whose response is displayed in Figure 3.13 received full credit but also a suggestion from the instructor (the last two lines in the previous figure) to improve the quality of his answer; in particular, the use of mathematical terminology to describe a property he referred to.

The response to Q11 in Figure 3.14 contains elements of sound reasoning; the student states that "$1 < 10$" and is perhaps referring to the (finite) value of each integral. The reasoning is somewhat unclear, however, and received partial credit. The instructor's feedback consists of underlining (3 of the phrases) and remarks on the top and bottom right of the response in the image.

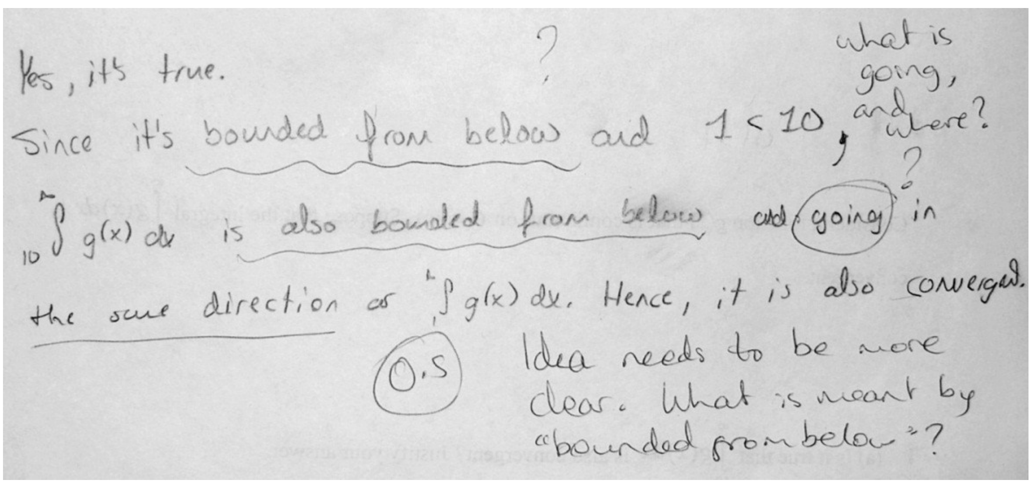


Figure 3.14 - Student response to Q11 which received partial credit, and instructor feedback

3.5 Collected data and devised analyses

In our study, the quiz questions acted as a *tool* to engage students in TT. This tool had neither been tested nor analyzed prior to our study and became an object of analysis. Thus from the perspective of the researchers, two sets of data were collected: the quiz questions and students' responses. We devised two types of analyses, one corresponding to each of these two sets of data. We label them *Question analysis* and *Class analysis* and describe them in detail in the next chapter. The results of the *Question analysis*, presented in the next chapter, would help us characterize the questions by uncovering the types of TT that they each invite. What we had so far – the lists of features of discourse per question – only informed us of question-specific behaviors which the questions might invite, but not of the general types of TT that they might prompt in participants. The *Class analysis,* presented in chapter 5, would uncover the actual engagement of the quiz participants in TT while answering each question.

4 Question analysis

In our study, the quiz questions served as a tool to engage students in TT. We devised the *Question analysis* to uncover the types of TBs that each question invited, as well as the frequency of opportunities that each question provided to engage in TT. In what follows, we present the methodology for the *Question analysis*. We then present the results followed by a brief discussion.

4.1 Methodology of Question analysis

We analyzed the questions in two ways. The first consisted of listing features of discourse (which was presented in chapter 3) based on our examination of the quiz questions. This analysis is entangled with the operationalization of the model of TT: we used the TBs in the model that was operationalized for the study in the Linear Algebra course, as well as newly emerging ones in our study, as guidelines for listing features of theoretical discourse, but also adjusted the operationalization after examining the questions so that it is pertinent to our study; the two thus contributed to each other. We consider this work part of the *Question analysis* because the aim was to uncover the *types* of TBs that each question invited. The second part of the *Question analysis* consisted of recording, for each question, the *number of times* each TB was invited. We extracted this information from the tables of features of discourse that we created for each question, in which we had associated a TB to each feature of discourse. For instance, the tables show that Q1 could prompt $TB1_1$ three times, $TB21_1$ once, $TB22_1$ once, $TB22_3$ once, and $TB23_2$ once. The entire set of results is displayed below.

The *Question analysis* had both a qualitative and quantitative nature. The qualitative aspect lays in this first part of the *Question analysis* (listing the features of discourse) since it involved an in-depth analysis of one question at a time, with no specific rubric against which we listed the features of discourse. The results of this analysis were not numerical, rather a qualitative description of the types of actions each question might invite (indicated by the 'features of discourse'). The second part of the *Question analysis* took on a more quantitative approach in summarizing (by counting) the frequency of opportunities that each question provides to engage in TT.

4.2 Results of Question analysis

As mentioned above, the result of the first part of the *Question analysis* (the listings of the features of discourse) contributed to the operationalization of the model. These results were fully presented in chapter 3.

The results of the second part of the *Question analysis* are presented in Table 4.1 below. These provide a global view of the types of TBs and TT that our questions invite. The table indicates the number of times that each TB was invited by each question (indicated in each row), as well as the number of TBs that each question invited (Count TB per Q). When counting the number of times a TB is invited by a question, we in fact accounted for each feature of discourse that is indicative of a TB; that is, a particular TB is counted more than once if we listed more than one feature of discourse corresponding to this TB. The last 2 rows of the table indicate the total number of times that each TB and category of TT were invited by the questions (Count TB and Count TT respectively).

	REFLECTIVE			SYSTEMIC							ANALYTIC			Count TB per Q
	$TB1_1$ invest	$TB1_2$ gener	$TB1_3$ verif	$TB21_1$ defin	$TB22_1$ reason	$TB22_2$ refut	$TB22_3$ th.prp	$TB22_4$ prev	$TB23_1$ cond	$TB23_2$ partic	$TB3_1$ logic	$TB3_2$ symb	$TB3_3$ repres	
Q1	3			1	1		1			1				7
Q2	3						3		1					7
Q3	1				1		1							3
Q4	1							1	1	2				5
Q5	2			1				1	1					5
Q6	1		1		1							1	1	5
Q7	1			1	1	1	1				2		1	8
Q8	1			1	1	1	2				1	1		8
Q9	1	1			1								1	4
Q10	1				1			1					3	6
Q11		2			3		1						1	7
Q12		1			1								1	3
Count TB	15	4	1	4	11	2	9	3	3	3	3	2	8	
Count TT	20			35							13			

Table 4.1 - Count of TB and TT invited per question and overall

4.3 Discussion of Question analysis

While creating the questions, neither the instructor nor ourselves, the researchers, referred to a particular model of TT. We created the questions with an aim to prompt students in TT as we perceived it. Later, we decided to use Sierpinska et al.'s (2002) model of TT for characterizing TT, and the results indicated by Table 4.1 inform us of the nature of thinking that a particular question might stimulate, according to this model of TT. For instance, the results indicate that Q7 provides a student with one opportunity to engage in *Reflective thinking*, four opportunities to engage in *Systemic thinking*, and three opportunities to engage in *Analytic thinking*.

Of course, a question which invites a particular type of TT will might not engage a student in this type of TT. A question might invite TT but simply be difficult or inaccessible to a student for various reasons such as insufficient mathematical background, or language barriers. However, while designing questions in the future, one might take these factors into consideration in order to create questions which are challenging enough to indeed engage students in TT (not merely in a recollection of skills and procedures) but still be within students' grasps. For us, this means that the *Question analysis* alone does not inform us how effective a question is at engaging students in TT; in fact, this is a characteristic which will be further clarified through the *Class analysis*.

The last column of Table 4.1 reveals the number of TBs each question invites. A higher count means that the question provides the participants with a larger variety of opportunities to engage in TT. Whether or not questions with a higher TB count were indeed more effective at engaging students in TT is revealed in the *Class analysis* where it is clear how many students actually engaged

5 Class analysis

The *Class analysis* is concerned with uncovering a) the level of engagement of the quiz participants, i.e., the portion of participants who were engaged in TT, and to what extent; and b) the type of engagement of the group of quiz participants in TT on every question, i.e., the types of behaviors and thinking they displayed. While the *Question analysis* revealed the types of TB and TT that each question invited, the *Class analysis* would uncover the actual engagement of students in TT through the questions. In particular, these results would highlight the questions which engaged a high (and similarly, low) number of participants in TT, giving us an idea of how effective each question was at engaging students in TT. Furthermore, the results of the *Class analysis* would inform us whether we succeeded at engaging a "reasonable" number of students in TT.

5.1 Methodology of Class analysis

We performed the *Class analysis* by examining each student response to each question. We searched for phrases in each response which could be described by the features of discourse we had listed for the question. Each time such a phrase was found, we noted that the corresponding TB had been displayed by the student. For instance, the response to Q11 (Figure 5.1)

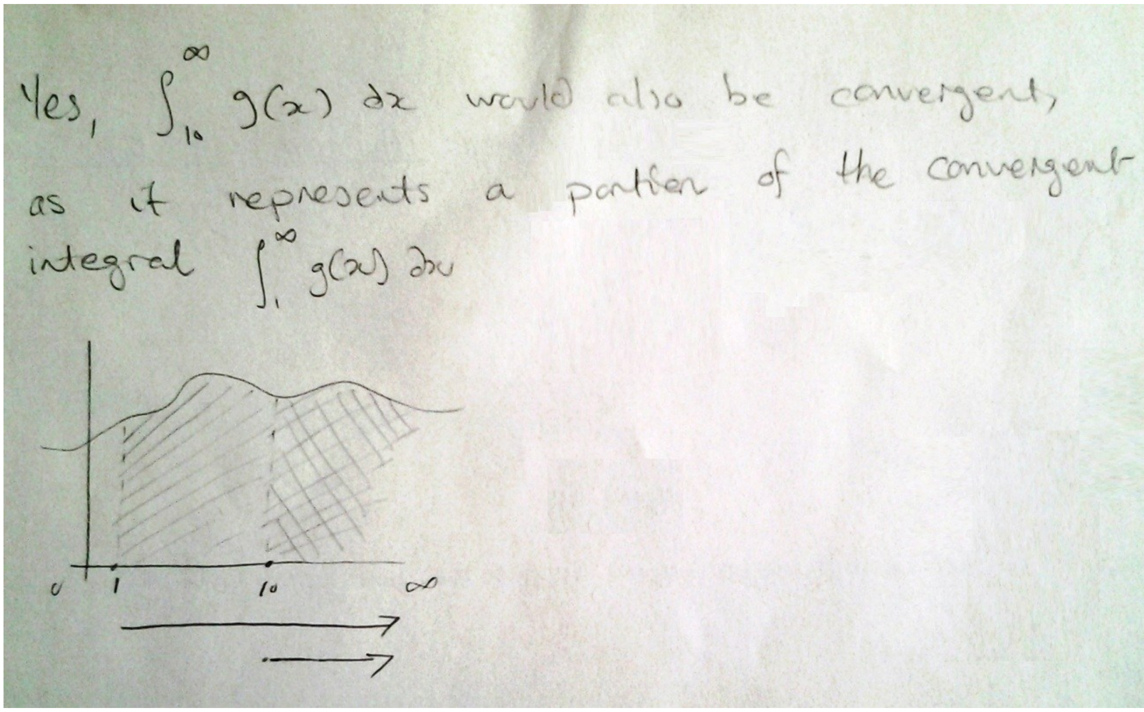


Figure 5.1 - Response with which we associated a feature of discourse

is described by the feature of discourse "*Drawing an arbitrary graph representing* $g(x)$*, then indicating the areas represented by the integrals* $\int_1^\infty g(x)dx$ *and* $\int_{10}^\infty g(x)dx$*, showing that the latter is included in the former*" and is a display of the behavior $TB3_3$ "*Representing a given problem in a different mathematical register*" corresponding to *Analytic thinking*. As explained previously, when a feature of discourse was indicated by a response and was not accounted for in our list, we added it to the list (we previously explained how this analysis was also entangled with the operationalization of the model as it contributed to the list of features of discourse).

We then noted the total number of times each TB had been displayed by the group of participants per question (these results will be shown in Table 5.2). In the next section, we give examples of student responses, explaining how we identified features of discourse, and thus displays of TT, in them. We will not present results indicating the engagement of individual quiz participants since we are concerned with the engagement of the *group* of participants. We will, however, indicate the number (and percentage) of participants who engaged in TT (some students participated but were not engaged in TT).

The *Class analysis* is mainly characterized as quantitative since we were mainly 'counting' the number of times TBs were indicated by student responses. However, there was still a slightly qualitative aspect to this analysis since although we now had a rubric (the list of features of discourse) against which we 'measured' student responses, we were still attentive to phrases which were not already represented by our lists of features of discourse; thus analyzing student responses was not limited to counting, rather also to expanding the list of features of discourse when appropriate.

5.2 Results of Class analysis

While quiz participants varied from one week to another, most of them were regular participants and averaged 36 participants per week. In presenting the results, we do not differentiate between groups of participants that differ in members and consider the group as one entity consisting of those who participated in the quiz on any given day. In fact, the study is concerned with engaging *participating* students in TT, whether they are a part of- or the whole of the class, and whether they are the same or different members each time.

5.2.1 Examples of student responses

We begin this section by providing examples of students' responses in which we could identify occurrences of TT; general results of the *Class analysis* follow.

Example of a display of *Reflective thinking*

Figure 5.2 displays a student's response to Q2. The statement of this problem does not mention the existence of the integral nor the continuity of the functions over the intervals [a, b] and [c, d] respectively; in fact, in the way that the problem is posed, the emphasis is

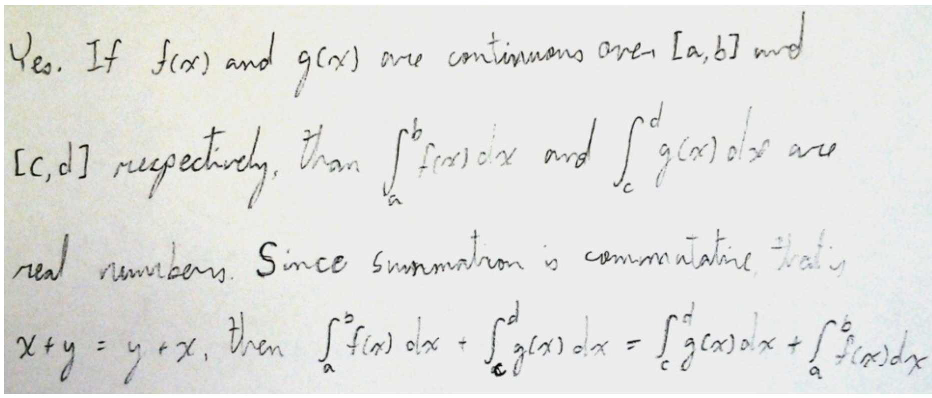

Yes. If $f(x)$ and $g(x)$ are continuous over $[a,b]$ and $[c,d]$ respectively, then $\int_a^b f(x)\,dx$ and $\int_c^d g(x)\,dx$ are real numbers. Since summation is commutative, that is $x+y = y+x$, then $\int_a^b f(x)\,dx + \int_c^d g(x)\,dx = \int_c^d g(x)\,dx + \int_a^b f(x)\,dx$

Figure 5.2 - Response to Q2 indicating *Reflective thinking*

on the re-ordering of the integrals, and it seems quite clear that the question does not expect the reader to state these hypotheses or even consider them. Yet, this student took the initiative to recall the conditions which would allow him to then proceed with the identification of the integral with a real number; in the context of the question, we consider that the student provided additional detail and displayed what we characterize as "an investigative ("researcher's") attitude towards mathematical problems" – a type of behavior which we associate with *Reflective thinking*.

Examples of a display of *Systemic thinking*

Figure 5.3 displays a student's response to Q2. We identified two TBs in this response: The student first associated the definite Riemann integral with a real number, thus relating two different concepts within a system – a behavior associated to *Systemic thinking*.

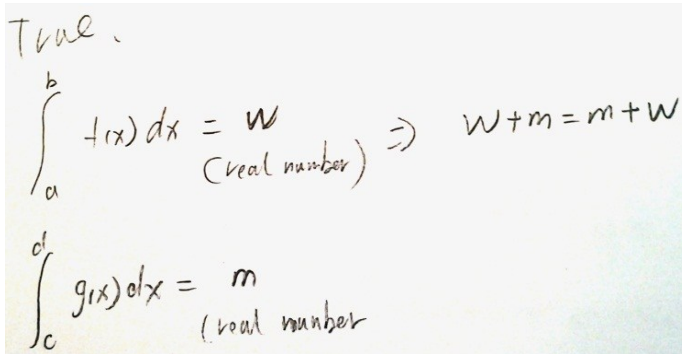


Figure 5.3 - Response to Q2 indicating *Systemic thinking*

The student then used the commutative property of addition of real numbers to establish the sought equality; again a behavior associated with *Systemic thinking* as it involves referring to properties of an operation (addition) on objects in a particular system (the set of real numbers).

Example of a display of *Analytic thinking*

Q6 requires one to explain why the value of the area under the second curve is equivalent to the value of the area under the first. In her response, one student used a 'substitution' with $u = x^2 - 4$ to transform $\int_2^5 x(x^2-4)\mathrm{dx}$ into $\int_0^{21} \frac{u}{2}\mathrm{du}$. She then correctly remarked that this integral is actually equivalent to the integral $\int_0^{21} \frac{x}{2}\mathrm{dx}$ given in the problem since "u is just a variable". This student was able to distance the symbol 'u' from its meaning – a TB which is indicative of *Analytic thinking*.

Example of a display of a behavior indicating two types of thinking

In his response to Q7 a student argued about the truth of a statement by reasoning by contradiction (Figure 5.4):

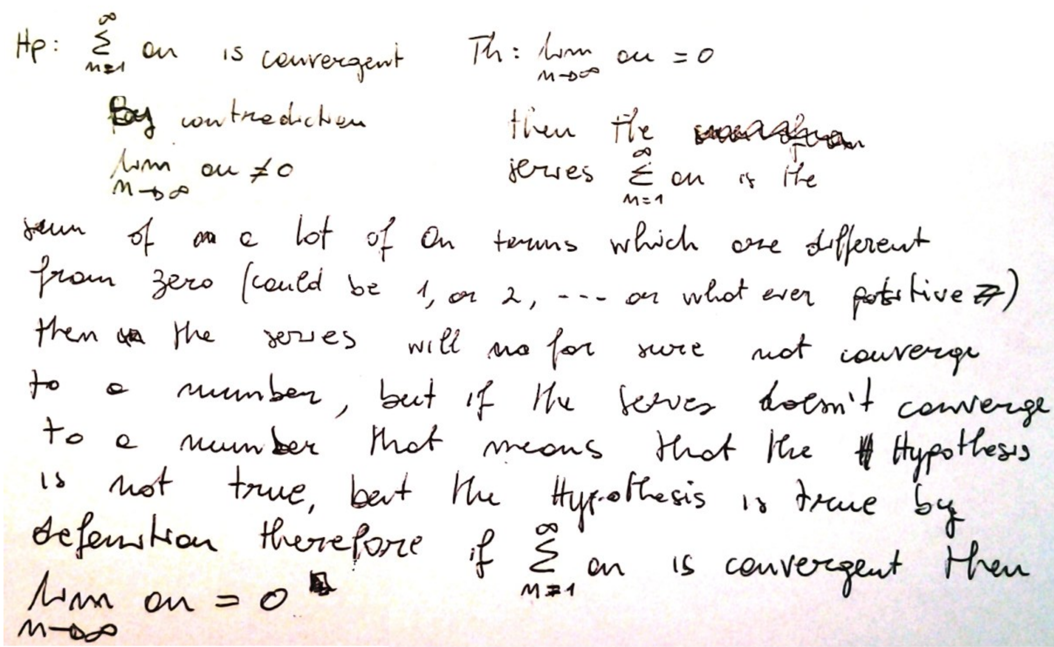


Figure 5.4 - Response 1 to Q7 indicating *Systemic thinking* and *Analytic thinking*

In another response to this question, which we will call, for reference, *Response 2 to Q7*, a student wrote "If $\sum_{n=1}^{\infty} a_n$ converges to a number, it means that it has to be adding smaller and smaller numbers for it to be able to converge. If we said $\sum_{n=1}^{\infty} a_n$ converges and $\lim_{n\to\infty} a_n \neq 0$, that would not be true. It would be diverging." While Response 1 is written more rigorously, both Response 1 and Response 2 contain a conscious use of reasoning by contradiction, and so, an awareness of the logical structure of the argument, which are symptoms of both *Systemic thinking* and *Analytic thinking*.

5.2.2 Engagement in TT

Table 5.1 provides a global view of participation in the quizzes, indicating the number of students who participated in each quiz, and how many of these students engaged in TT, as well as the number of occurrences of TT per question in student responses. The number of occurrences of TT was typically higher than the number of students who engaged in TT since one student response could display more than one TB.

Question	# Participated	# Engaged in TT	% Engaged in TT	# Occurrences of TT	Count TB
Q1	47	33	70 %	59	7
Q2	47	28	60 %	43	7
Q3	41	24	59 %	25	3
Q4	41	34	83 %	47	5
Q5	40	21	53 %	23	5
Q6	39	27	69 %	60	5
Q7	33	23	70 %	36	8
Q8	35	12	34 %	20	8
Q9	35	12	34 %	14	4
Q10	29	23	79 %	32	6
Q11	27	22	81 %	35	7
Q12	27	10	37 %	10	3

Table 5.1 - A global, quantitative view of participation and engagement

Table 5.2 displays the number of times each TB was displayed in student responses for every question (under each TB and across from each question), as well as the total number of times each TB and category of TT was displayed (Count TB and Count TT respectively).

	REFLECTIVE			SYSTEMIC							ANALYTIC		
	$TB1_1$ invest	$TB1_2$ gener	$TB1_3$ verif	$TB21_1$ defin	$TB22_1$ reason	$TB22_2$ refut	$TB22_3$ th.prp	$TB22_4$ prev	$TB23_1$ cond	$TB23_2$ partic	$TB3_1$ logic	$TB3_2$ symb	$TB3_3$ repres
Q1	19			6	5		28			1			
Q2	14						22		7				
Q3	6				19								
Q4	2							2	22	21			
Q5	12			6				5	0				
Q6	19		7		4							4	26
Q7	8			8	4	3	9				3		1
Q8	1			0	6	0	9				0	4	
Q9	0	2			3								9
Q10	0				7			9					16
Q11		2			15		11						7
Q12		0			7								3
Count TB	81	4	7	20	70	3	79	16	29	22	3	8	62
Count TT	92			239							73		

Table 5.2 - Count of TB and TT displayed by participating groups per question and overall

5.3 Discussion of Class analysis

Table 5.1 indicates that there were fewer and fewer quiz participants throughout the semester. This is not surprising since although every student who attended the class participated in the quiz, class attendance is not compulsory and typically fewer students attend class as the semester progresses. However, the fourth column indicates that at least 34% of quiz participants engaged in TT through each quiz and up to 83% engaged on one of the quizzes (Q4), with an average engagement of 61%. For us, this not only meant that the quizzes did provide students with opportunities to engage in TT, but that each question engaged at least a reasonable portion of the quiz participants in TT.

Although our aim was not to engage as many students in TT as possible, rather to simply provide the *opportunity* to engage in TT activities, it was also important for us to engage a "reasonable" number of students in TT (which for us was generally no less than a third of the class, on average) since providing these opportunities did consume class-time and required a significant amount of work by the instructor (preparing the quizzes, correcting them, etc…). Engaging only a very small portion of the class could raise questions as to whether the whole intervention is worthwhile.

Table 5.2 shows that behaviors pertaining to *Systemic thinking* were most prevalent with a total of 239 occurrences, followed by TBs corresponding to *Reflective thinking* with 92 occurrences, and finally *Analytic thinking* with 73 occurrences. These results are consistent with the results obtained in the *Question analysis*: *Systemic thinking* was most invited by the questions, followed by *Reflective*, and then *Analytic thinking*. However, by comparing the number of opportunities provided by the quizzes to engage in each of the three types of thinking to the number of times that students actually engaged in each type of thinking (Table 5.3), we see that students in this class were more inclined to engage in *Systemic thinking* – at least while engaging with the quizzes – since the ratio of 'engaged' to 'invited' is largest for *Systemic thinking*.

	Reflective	Systemic	Analytic
Count TT invited by questions	20	35	13
Count overall engagement in TT	92	239	73

Table 5.3 - Comparison between features of TT invited by questions and those displayed by participants

We also noticed that most of the observed behaviors (62 out of 73) pertaining to *Analytic thinking* were related to modeling a problem using algebraic expressions or graphs. We hardly observed behaviors displaying an awareness of the logical structure of a statement, nor of a strong interpretation of symbolic expressions (together 11 out of 73 of behaviors indicative of *Analytic thinking*). The latter observation is perhaps not very surprising considering the little emphasis on the *meaning* of notation used in Calculus. It often suffices to be able to 'plug into' the variables, without necessarily understanding fully their meaning. Also, proofs are rarely (if ever) a part of exercises; as a result there is rarely a reference to the logical structure of proofs in mathematics in the Calculus context.

From Table 5.2 we can see that some TBs were seldom displayed; these are $TB1_2$ "*Generalizing a solution*" with a total of 4 occurrences, $TB22_2$ "*Refuting a general statement by drawing a contradiction*" with a total of 3 occurrences, and $TB3_1$ "*Being sensitive to logical connectives*" with no occurrences. These behaviors were also among the ones that were shown to be least invited (in the results of the *Question analysis*). Again, this is perhaps not very surprising considering the context of this course and the lack of proving activities or activities involving abstractions and forming generalizations. The '0' entries in Table 5.2 indicate the TBs which we expected a question could invite, but which were not displayed by any student.

Comparing results shown in the second, third, and fifth columns of Table 5.1 (participation, engagement, and occurrences of TT) provides more global information about the questions. For instance, 41 students participated in both Q3 and Q4; however, 34 out of these engaged in TT in Q4, but only 24 in Q3. Furthermore, the number of occurrences of TT is 47 for Q4, compared with only 25 for Q3, likely indicating that several students engaged in more than one type of thinking in Q4 but not in Q3. Similar conclusions can be drawn when comparing the results of Q11 and Q12: despite the same number of participants, many more students were actually engaged in TT in Q11, and many were likely engaged in more than one type of TT as well since 22 participants were engaged and 35 occurrences of TT were observed. The results indicated by Q8 and Q9 are slightly different; these two questions were answered by the same number of participants and also *engaged* the same number of participants in TT. However, the number of *occurrences of TT* for Q9 was 14, and was 20 for Q8. Thus although these two questions engaged the same percentage of participants, the engagement was more fruitful in Q8, in the sense that more participants were engaged in more than one type of TT.

The last column of Table 5.1 reveals the number of TBs each question invites; with the exception of Q8 (which provided at least 8 opportunities to engage in TT but only engaged 34% of the class), the questions which invited a higher number of TBs generally engaged a larger portion of quiz participants in TT. The number of TBs that a question invites, however, does not alone determine how TT-engaging the question will be. This can be seen from the results of analyzing the class engagement through Q4 and Q5, for example; while both invite 5 TBs, 83% of participants engaged in TT through Q4, but only 53% through Q5. Not only did the quiz questions differ in the types of TT in which they engaged students, but some questions were overall much more effective at engaging students in TT than others.

5.4 A further analysis of four questions

As stated in the previous section, the TB count of a question does not alone determine how TT-engaging a question is; there are other factors which play a role in making a question more (or less) TT-engaging in a given context. We conjecture that some of these factors are didactic factors that are intrinsic to the structure of the question; and, since a question is not engag*ing* until it engag*es* an individual, we assume that some factors are context-dependent.

To uncover the intrinsic features of a particular TT-engaging question, one could model the question with a generalized one. On one hand, the model would reveal the underlying structure of the question, independently of the particular mathematical concepts involved; on the other hand, the model would help practitioners and researchers create other questions of a similar type, or refine the ones used in this study.

In the following section, we propose an approach to constructing a model of TT-engaging questions; we sow the first seeds by analyzing and modeling four of the twelve questions that were used in this research. We also discuss the type(s) of thinking that each question might prompt, and conjecture factors (besides the structure of the question) that might make the question more (or less) TT-engaging by referring to the four questions.

5.4.1 Modeling four questions

One way to model a question is by identifying the didactic components of the question that are fixed, and those that are variable. Fixed components are ones which cannot be changed without changing the structure of the question; variable components, on the other hand, can be replaced without modifying the demands of the question. We give the following example: In a Calculus course in which students who were taught the concept of *derivative* and how to compute the derivative of a polynomial function, a question asking whether the derivative of the function $f(x) = 3x^2 - 4x + 2$ (defined on R) exists, and why, will not be modified (except aesthetically) if the function $g(x) = -7x^3 + x^2 + 9$ (defined on R) is considered instead. However, asking to *find* the derivative of the function $f(x) = 3x^2 - 4x + 2$ (defined on R), is a different question.

The former question can be generalized as: "Compute the characteristic X of the object o which belongs to the class O".

The didactic constants in this type of questions are:

- Characteristic X and class O are both familiar to students.
- In class, computing characteristic X of a representative of class O has been taught.
- The question asks to compute the characteristic X of a representative of class O.

The didactic variables in this type of questions are: X and O.

Once the didactic components of a question are identified, a generalized question that models this question can be formulated. Then, representatives of this type of question differ by the values of the didactic variables, and share the constants.

*   *   *

We analyzed four questions – Q4, Q5, Q6, and Q11 – and display the results below. We decided to analyze Q4 and Q5 since they invite the same number of TBs and were attempted by almost the same number of students in our study, but curiously engaged a very different portion of participants in TT (83% and 53% respectively). We decided to analyze Q6 since although the percentage of participants that engaged in TT through Q6 (69%) is not strikingly different from Q4 or Q5, Q6 has the same TB count as Q4 and Q5, and was attempted by roughly the same number of students. Finally, we felt that analyzing Q11 might add to our insight into the factors which contribute to the *high potentiality* of a question to engage individuals in TT, since a high percentage of participants in Q11 engaged in TT (81%). In the table below we remind the reader of the results of quiz participants' engagements with these four questions and the count of TBs we had determined each question could invite.

Question	# Participated	# Engaged in TT	% Engaged in TT	Count TB
Q4	41	34	83 %	5
Q5	40	21	53 %	5
Q6	39	27	69 %	5
Q11	27	22	81 %	7

Table 5.4 - Engagement of participants in TT in Q4, Q5, Q6, and Q11

Question 4

In class, students were given the definition of a Riemann sum and shown how to use it to compute the area bounded by the curve of a function (and other curves) that is continuous (at least on the domain on which the area is to be computed). Q4 asks whether it is possible to compute the area, using a Riemann sum, bounded by the curve of a function that is undefined at one point in the interval over which the area is to be computed. The question thus asks whether this concept (Riemann sum) can be defined for a larger class of functions than that addressed in class. In general terms, this question asks:

> "Is it possible to calculate/ define the characteristic X for a class of objects O?"

The didactic constants in this type of questions are:

- A definition of a characteristic X of a class of objects O' has been given in class.
- The question gives an example of an object o that belongs to a **slightly larger** class O than the class O' assumed in the definition and asks if one can still (or whether it is possible to) calculate or define the characteristic X for this object.

The didactic variables in this type of questions are: O', X, O, and o.

Questions of this type are theory-generating, because they provoke the extension of concepts. Using the terms of Sierpinska et al.'s (2002) model, questions of this type encourage *Reflective thinking* in extending ideas, but also *Systemic thinking* since the question asks one to consider definitions and relationships within a system of concepts. Depending on the values of the didactic variables chosen for a particular question of this type, the question could engage one in *Analytic thinking* as well.

Another core feature of this question is its open-ended approach which encourages exploration; a behavior pertaining to *Reflective thinking*. Also, asking "is it possible", or "is it true that", or "under which conditions is…" and so on, leaves the reader to consider and verify the truth or falsity of a statement, perhaps even depending on conditions imposed by the reader. Questions driven by these types of phrases promote *Hypothetical thinking* – a feature of *Systemic thinking*.

The particular representative of this type of questions in our study, Q4, engaged 83% of participants in TT. We conjecture that aside from the strong potential of this type of questions to invite TT, features particular to this question made it more approachable to students, while remaining sufficiently challenging. For instance, part of the problem was represented visually, which could act as an aid to those who would not have been able to visualize the problem but who find visual representations helpful. Furthermore, some of the mathematical concepts and techniques needed to think about this problem are perhaps within students' reach; for instance, re-writing a sum as a sum of two sums.

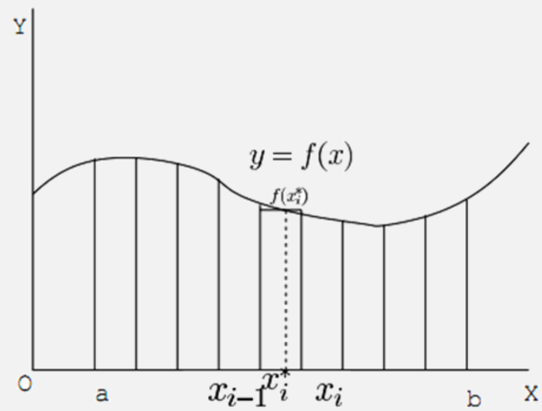
Question 5

In class, the Riemann integral was defined as the limit of a Riemann sum (which was previously defined). In computing the Riemann sum in class, however, the *sample point* was only ever considered (by the teacher and the exercises) as the left endpoint, the right endpoint, or the midpoint of the intervals; furthermore, in every particular question the sample point was chosen in the same position for each sub-interval. The teacher posed Q5 to students knowing that they had only applied the algorithm under these conditions. Q5 presents the definition of the Riemann integral as it was given in class, and asks whether the sample point must be chosen in the same position for each sub-interval and whether this position can be a random position. In other words, the quiz question asks whether the algorithm can be applied under a different condition (arbitrary sample points, in non-consistent positions across the intervals) while generating the same result. We noticed, however, a flaw in the phrasing of the question: In the question, x_i^* is (correctly) referred to as a *sample point*, and thus by definition can assume any position in the interval. Yet, the question (displayed below) interrogates exactly this. However, this flaw does not disable us from modeling the question.

Recall the definition of a Riemann integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

The area S_i of the strip between x_{i-1} and x_i can be approximated as the area of the rectangle of width Δx and height $f(x_i^*)$, where x_i^* is a sample point in the interval $[x_{i-1}, x_i]$.



Must the sample point x_i^* be chosen at the same position in each interval, or can it be the right end point in an interval, the left end point in another interval, and any random position in another interval (for example)? Justify your answer.

Figure 5.5 - Question 5

The phrasing of the question could be changed to

> "Recall that the Riemann integral can be calculated by computing the following limit: $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ where x_i^* is the left endpoint, right endpoint, or midpoint of the i^{th} interval, and is at the same position in each interval. Must the points x_i^* be at the same relative position in each interval, or can it be the right end point in an interval, the left end point in another interval, and a random position in another interval (for example)? Justify your answer.

Then problems of this type can be generally phrased as:

"Must condition C be verified in order to apply algorithm $\mathcal{A}$ and obtain result r(Y), or can it be applied under a more general condition C' while still generating result r(Y)?"

The didactic constants in this type of questions are:

- Result r($\mathcal{A}$) has been defined in class as resulting from algorithm $\mathcal{A}$ subject to condition C.
- The question asks whether condition C is necessary to apply algorithm $\mathcal{A}$ and obtain result r(Y), or whether a more general condition C' can suffice.

The didactic variables in this type of questions are: $\mathcal{A}$, r($\mathcal{A}$), C, and C'.

This type of questions requires one to distinguish between the epistemologically necessary assumptions about the range of the input in an algorithm, and the arbitrary – but traditional – choices one makes about picking values from that range. Therefore, it requires *Hypothetical thinking*. According to Hewitt (1999), students usually cannot distinguish between the arbitrary and necessary because they do not have the necessary historical and theoretical knowledge. This might explain why fewer students engaged in this question.

While problems of this type might be very mathematically involved, Q5 did not prove to be very TT-engaging in the class in which it was posed (as compared with Q4), with a 53% engagement in TT. We partially attribute this to the required level of abstractness in thinking about and solving the problem. Students not only had to imagine different possible positions of the sample point, but conceptualize how or whether these varying positions would affect the Riemann sum and then the limit of the Riemann sum.

In this sense, we felt that the concepts and skills required were slightly out of reach of many students in this course. This does not mean that the general question type is not conducive to TT, but that the values of the variables and the structure of the question might be chosen and designed in a way that is more approachable to students.

Question 6

This quiz question is asking to show the equivalence of two problems. The problems are ones which were practiced a lot throughout the course. One way of showing the equivalence of the two problems involves a process with which students were very familiar as well. Showing the equivalence between two such problems, however, was an entirely new problem for students. The question can be generally described as:

"Show that problem P is equivalent to problem Q".

The didactic constants in this type of questions are:

- Problems P and Q are problems which had been practiced in class.
- The question requires one to show the equivalence of the two problems P and Q.
- One way to show the equivalence of the two problems involves a process that is familiar to students.

The didactic variables in this type of questions are: P and Q.

The exercise of showing the equivalence between two problems is often encountered in mathematics. It is useful for simplifying problems, or turning a problem into one for the solving of which we have more tools. However, showing this equivalence can vary in difficulty relative to a particular context and based on the values chosen for the variables P and Q. In our study, the question of this type, Q6, did not open a wide range of possibilities to show the equivalence while using tools which were accessible to most students in the class. In fact, only 4 out of 39 participants were able to show (using procedural techniques) the equivalence, and only 1 student out of 39 hinted to a conceptual reasoning of why the two problems are equivalent. Some students showed that both problems have the same solution and expressed that although they realize that the equivalence of the two problems must be shown, they are not sure how to do so.

We conjecture that the process of showing the equivalence in Q6 was a very particular one, and that discussing the conceptual aspect of the process involved (transformation of areas) was slightly advanced relative to the Calculus context. Another factor which we felt might have diverted students' attention from showing the equivalence is the low level of generality of the two problems P and Q. The two problems were presented in particular (versus general) terms. We conjecture that this might have prompted many students to *solve* the two problems P and Q (as was the case), sufficing to show that the two have the same solution. Of course, having the same solution is not sufficient to show that two problems are equivalent; two problems could have the same solution without being equivalent. In this sense, perhaps the phrasing of Q6 is poor and the low level of generality of the question was not conducive to engaging students in TT. Perhaps asking "Show that the problem of finding the value of the shaded area in the first figure is equivalent to the problem of finding the value of the shaded area in the second figure", together with presenting problems P and Q in general terms (for example, having generalized constants instead of numerical values at the boundaries of the area regions), could make the question more TT-engaging. Still, the narrow range of possibilities and the complexity of the ideas involved might not make Q6 a successful (as far as TT-engagement is concerned) question.

Question 11

This quiz question asks whether the integral of an arbitrary function $g(x)$ over a domain S' is convergent, knowing that the integral of $g(x)$ over a domain S (a superset of S') is convergent. The mathematical concepts involved – improper integrals and convergence of integrals – were familiar to students in the class. This type of problem, however, was not familiar. The problem type can be described as:

> "Suppose that object O has property P on the domain S. Is it true that object O has property P on a domain S' which is a subset of S?"

The didactic constants in this type of questions are:

- Property P is familiar to students.
- Property P is said to hold for object O on a domain S.
- The object O and property P are fixed throughout the question, while the domain is changed; the question then asks whether the property still holds for the new domain S'.

The didactic variables in this type of questions are: O, P, S, and S'.

More generally, this question type is of the form "Is it true that [if X then Y]" or "Is it true that X implies Y". Questions of this type address the sufficient conditions to obtain a result, and thus promote, perhaps among others, *Systemic thinking*.

The representative of this question type in our study, Q11, addressed the convergence property of improper integrals. Although the problem itself had not been encountered before, the concepts were discussed in depth in class. Furthermore, techniques that could be used to solve the problem were referred to often in class (e.g., expressing an integral as the sum of two integrals, or representing a convergent integral as an area by imposing conditions on the function in the integrand). Thus while requiring reflection, engaging in the problem was likely possible for a large portion of the participants.

5.4.2 Discussion

The questions used in this research all proved to be TT-engaging in the study in which they were used. Some questions appeared to be more engaging than others, however, and our analysis showed that this difference is not fully accounted for by the differences in the TB counts of the questions. Uncovering the fundamental structure of the questions could reveal the particular features of the questions which made them more (or less) TT-engaging than others, and can also serve researchers and practitioners as a tool to design TT-engaging questions.

For four out of the twelve questions, we created a 'generalized' question which models the original question. We highlighted the didactic variables and constants characteristic of each questions, and described the types of thinking or behaviors that these question-types could prompt. We propose, however, that the success of a question at engaging one in TT thinking is not only dependent on the intrinsic characteristics of the question. Although the question models can be used to design new TT-engaging questions, other factors must be taken into consideration when designing the questions.

Some of these factors are inherent to the question, while others are related to the context in which the questions will be posed. Following this idea, we conjecture that there is not an 'absolutely poor' or 'absolutely rich' question, in the sense of how engaging it is in TT.

Factors inherent to the question include the *phrasing* of the question and the *values* which the variables assume. We noticed that questions asking the equivalent of "Is it true that… and why?" are more TT-engaging than questions asking to "Show that…", as they offer the reader more autonomy in thinking about the question. "Justify your answer" or else "Explain why…" was a part of the phrasing of every question in our study. We wanted to ensure that students would justify their answers, as this was our only window to their reasoning about the question, and we made a choice of the wording we would consistently use. This is not a fundamental part of the question however, rather a didactic addition we made, and is thus not included in the model of the questions.

The question models are stated in terms of variables which are replaced by mathematical objects when designing a question and essentially dictate the mathematical content of the question. The values which the variables take may have a strong impact on the complexity of the question. Depending on the context, particular values might make the question challenging enough to prompt TT, yet accessible enough for students to engage. For example, in the model of Q11, the relative relation between the set S and its subset S' might have a significant impact on the actual TT-engagement. If, for example, S' was defined as the union of finite or infinite countable disjoint intervals, the question might fail to engage Calculus students in TT-thinking, while it might be very successful in the context of an Analysis course. Another example of the effect that the values of the variables may have on the actual TT-engagement can be seen in the model constructed for Q4. One of the didactic constants of the model requires that the extension is done over a "slightly larger" class of objects. In Q4, the 'slightly larger' class of objects are functions with one point of discontinuity. We surmise that the question would not have been more complex if there were two, three, or ten points of discontinuity. However, if the larger class of objects was described as "functions with a finite number of points of discontinuity", the question might have attained a significantly higher level of complexity in the given context, perhaps making it too challenging to engage with. While "slightly larger" is a rather vague description of *how much* larger, we keep this phrasing and maintain that its magnitude depends on the context and values of the variables in a particular question.

Besides the values of the variables, the *level of generality* of the values of the variables can affect the way that the question is perceived. In some cases, such as Q6, the particularization of the values of the variables might have inhibited students' engagement in TT; this might not have been the case in Q11 where having particular limits of integration helped students represent the problem graphically and use algebraic techniques more readily. We conjecture that in the case of Q11, the particular values did not inhibit students' TT, but rather facilitated it. While there does not seem to be a general rule (or it might be difficult to formulate such a rule precisely) regarding the level of generality of the values of the variables, one might keep this factor in mind while designing a question and imagine the effect that changing the level of generality might have on the TT-engaging potential of the question. The same can be said about including visual representations in the question. In some cases, a visual representation might facilitate one's thinking; in others it might inhibit creativity. Again, there is perhaps no general rule regarding this factor, but it might be taken into consideration while designing a question.

6 Discussion and conclusions

Previous studies report that in the prerequisite (or college level) Calculus course institution the typical assessments and activities do not engage students in TT; rather, they encourage a procedural and algorithmic mode of thinking. Furthermore, research indicates that college-level Calculus instructors are often compelled to certain institutional constraints that are often imposed on these courses making it somewhat difficult for instructors to introduce activities different than the typical ones. In this study, we did not deny this, nor did we seek alternative situations in which these constraints are loosened or eliminated; rather we took on a different perspective. Through this study we sought to determine whether, and how, class participants in a college level Calculus course could be provided with opportunities to actively engage in TT, despite these constraints, i.e., while abiding by the course outline and requirements. Such a result would suggest a way for an instructor of these courses (who does not necessarily have control over the course content and assessments) to pursue such a method for incorporating activities that stimulate TT.

In pursuit of our goal, we designed questions which we expected would require one to think theoretically in answering them, and presented them to students in the form of weekly quizzes. In adhering to terms used in the literature, the questions can be classified as ‘non-routine’ since they could not be entirely solved using an algorithm or technique that was familiar to students; rather, in answering the quiz questions students would need to rely on expressing and explaining their ideas while using strategies that are not typically used throughout the course tasks. Students could choose whether or not to attempt the quizzes, and would be rewarded with bonus points on their course grade for a meaningful response or part of response.

We analyzed student responses using Sierpinska et al.'s (2002) model of TT in search of displays of TT (*Class analysis*). To do so, we needed to operationalize the model with TBs that are relevant to the context of our study. The set of questions were designed without reference to a model of TT; thus they also constituted data, and analyzing them informed us of the types of TBs and thus TT they invite (*Question analysis*). We noticed that the two analyses complemented each other as far as the TBs were concerned because while we could 'guess' the types of TBs that the questions could invite in order to complete the *Question analysis*, the *Class analysis* helped refine and add to our predictions since it revealed the types of TB that students actually displayed while engaging with the questions. On the other hand, the *Question analysis* allowed a starting point for analyzing student responses. The discrepancy in TBs between the two analyses can be attributed to our over- or under-expectations in analyzing the questions but also to the sample of subjects and the quiz conditions. Perhaps a different student sample, or even the same student sample but working under different quiz conditions (e.g., longer duration), would display a different engagement in TT.

Conversely, the operationalization of the model contributed to both the *Question* and *Class analyses* since it provided a general idea of the types of TBs we might expect in both analyses. The following figure captures the triangulation process which took place throughout the analyses and operationalization.

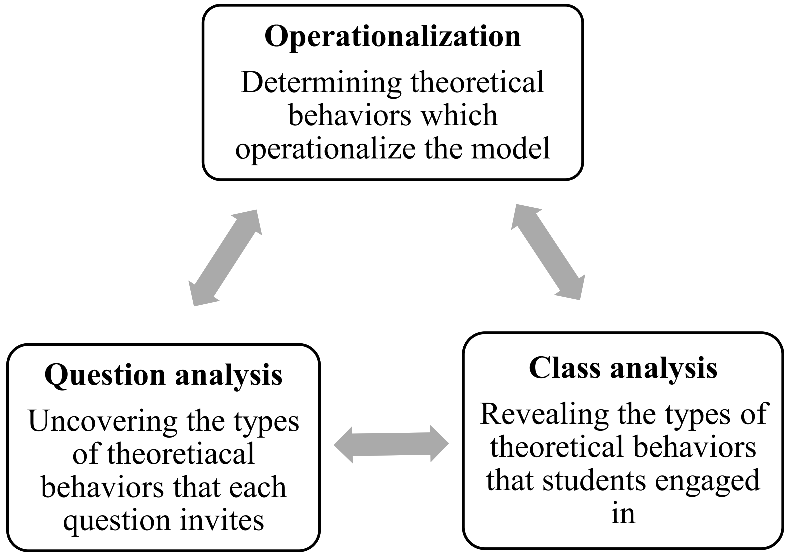


Figure 6.1 - The triangulation process between the analyses and operationalization

The success of our study depended on the occurrence of two events: The first is that the quiz questions indeed provide students with opportunities to engage in TT, and even engage a reasonable number (which for us meant around a third) of participants in TT. This was determined through the *Question* and *Class analyses* which indicated that each question engaged no less than 34% (Q8 and Q9) and up to 83% (Q4), and an average of 61% of the quiz participants in TT. The second is that the quizzes are incorporated without 'disturbing' the structure of the course; mainly, this meant implementing the course outline as required, and preparing students for the common assignments and mid-term and final exams. Indeed, although it was a challenge, the instructor implemented the course outline as set by the course examiner, and the class average for the course was similar to those of the remaining sections (among the highest 3 out of 6).

An event that we believe deserves attention is that every student who was present in class on a day on which a quiz was given chose to take the quiz. For us, this was remarkable, especially considering that the quizzes were carried out at the end of the class and students were aware that they could leave (early) and had no obligation to take the quiz; furthermore, while some students often received points for the work, some consistently did not and still continued taking the quizzes anyway. Moreover, many students were curious to know the correct responses to the questions and often stayed after class to discuss these with the instructor. Some students even praised the quizzes in the course evaluations (run by the university) at the end of the term: “The quizzes provide excellent feedback on our understanding of the theory in class” and “I love the quizzes because they test your knowledge without consequences”. Such events are often unexpected, and for us these, together with the results discussed above, constitute a significant outcome of the study and a strong indication of the success of the tool in engaging students in TT.

6.1 Choosing a model of theoretical thinking to analyze the data

As mentioned previously, we did not have a model of TT in mind prior to designing the questions; the instructor designed the questions based on her intuitive perception of what TT is. Later, the researchers operationalized and refined Sierpinska et al.’s (2002) model based on their understanding of TT and on the collected data. This way of ‘generating’ a tool for analysis from data is not uncommon. As mentioned in the *literature review* chapter, Guberman (2008) follows almost an identical procedure in using the Van Hiele model to characterize arithmetical thinking. In fact, as discussed previously, this methodology falls under ‘grounded theory methodology’ developed initially by the sociologists Barney Glaser and Anselm Strauss.

We analyzed our data using solely Sierpinska et al.’s (2002) characterization of TT. We found Sierpinska et al.’s model particularly appropriate for our needs since it provides primarily a thorough yet general characterization of TT. Some of the other models seem to characterize an ongoing *process* of thinking theoretically; for example, one’s progressive behavior while problem solving. Furthermore, these models often consisted of levels, so that one could assess how advanced an individual’s TT is. This is not exactly what we intended to do; rather, we intended to use the model to identify occurrences of TT in the group of participants without identifying a level of thinking or any progress of any of the individual subjects. The former characterizations are perhaps more useful when building a profile of a single individual’s disposition to TT.

While we would not expect to find results that are contradictory to the ones we found, perhaps analyzing the data through a different lens might provide insight to issues or phenomena that we could not see. At the same time, our results are consistent with other characterizations of TT. For instance, in our operationalization the behaviors corresponding to *Reflective thinking* are similar to stages that both Schoenfeld (1987) and Mason et al. (2010) describe (*Explore* and *Verify*; *Entry phase*, respectively). Similarly, Tall and Vinner’s (1981) description of concept image and concept definition, and Tall’s later description of advanced mathematical thinking as the “[construction of] formal concepts that are part of a systematic body of shared mathematical knowledge” (Tall, 1995, p.1), coincides with ideas related to *Systemic thinking*. Among the characterizations of TT which we came across, we did not find an explicit reference to ideas related to what Sierpinska et al. call *Analytic thinking*.

In retrospect, we realize that the quiz questions and the nature of the quiz-taking sessions did not constitute the ideal conditions for prompting *Reflective thinking* in participants. The short time-intervals provided for answering the quiz questions perhaps did not give students a chance to display behaviors such as *Generalizing a solution*, *Verifying a solution*, or *Investigating various solution paths*. We believe that *Reflective thinking* might be displayed in conditions where one is not bound by any constraints. In investigating different problem-solving approaches, Schoenfeld (1987) provided subjects with an ample amount of time to solve a problem, allowing them the opportunity to explore the problem for an extended amount of time, and thus, perhaps, engage in a broader spectrum of thinking. Having said this, we remind that there was not a wealth of choices in setting the conditions for posing the questions to students and collecting the data we desired while adhering to the course requirements, and in part it is precisely these rigid conditions, stemming from institutional constraints, which gave urgency to our study.

6.2 Limitations of our study

In the way our study was set up, students' only ways of expressing their thinking was through writing. Several issues could stand in the way of this mode of expression, perhaps influencing our findings. Some of these are language difficulties (many students are international students whose first language is not English), a lack of mathematical language which can facilitate the expression of one's thinking, and not being accustomed to expressing ideas in writing in mathematics – thus lacking coherency and structure in a response. These issues might stand in the way of knowing whether the questions indeed engaged a student in TT, but are difficult (if not impossible) to avoid using our methodology. In striving to eliminate or minimize these issues, the instructor asked students to explain their answers in cases where they were not clear.

Many of the TBs that operationalized the model were not in the original operationalization and were ones that we extrapolated from our data (questions and responses). The final list of TBs was thus dependent on our understanding of what TT is. Perhaps a different researcher would infer a different set of TBs. Triangulating the analyses might have thus been useful to increase the validity and credibility of our conclusions.

6.3 Avenues for further research

The analyses showed that not only was *Systemic thinking* the most invited type of thinking by the questions, but also that students were more *likely* to engage in *Systemic thinking* than in a different type of thinking. It might be interesting to investigate why this type of thinking was most likely to occur, and, since we are not assuming that one type of thinking is ‘better’ than the other, whether it would be important to engage students in activities which would promote the two other types of thinking. For instance, in his study and discussion about ‘novice’ and ‘expert’ problem solving approaches, Schoenfeld (1987) finds that an expert problem solving approach devotes an ample amount of time to analyzing, planning, and verifying. These are actions which are akin to behaviors displayed by *Reflective thinking* as we understand it. In this light, perhaps promoting *Reflective thinking* could enhance students’ problem solving approaches. Similarly, promoting *Analytic thinking* might be desirable for specific purposes.

As mentioned previously, the results we obtained are representative of a fairly small and specific group of subjects; it might be interesting to run the quizzes again with a larger group, and perhaps even with a group of students who *previously* completed Calculus II. We wondered whether students' creativity in answering the questions might have been hindered by their attempts to 'mimic' ideas or examples from class, even though the questions were different from the usual exercises they had been exposed to; running the quizzes in a class different from Calculus II might eliminate this possibility, perhaps engaging students in a slightly different way.

In the study, some questions proved to engage students in TT more than others. In modeling four of the quiz questions, we proposed a way to uncover the fundamental structure of a question. The question-models helped identify features of a question which make it TT-engaging, and can also be used to design questions of the same type in the future. We found that some factors that contribute to how TT-engaging a question is in a given context are related to the structure and inherent characteristics of the question (such as the phrasing of the question and the values that the didactic variables take), while others are dependent on the particular context (such as the level of complexity of the designed question relative to the context).

Students in this study engaged in some categories and features of TT more than in others; will Calculus students typically engage more in those categories and features? If so, why would this be the case? A closer look at this question might contribute to our understanding of the TT involved in the learning of different mathematics concepts. When refining and modifying the questions for an iteration of the study, one can place emphasis on engaging students in a *variety* of categories and features of TT or on some *particular* ones. These approaches might give different results, and shed light on different aspects of TT and students engagement in it – which we believe are worth exploring.

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APPENDIX

In what follows, we display the operationalized model including the features of discourse (marked by filled bullets) corresponding to each theoretical behavior. At the end of each feature of discourse phrase is a reference to the question (in parenthesis) for which we had listed the feature of discourse.

TT1 REFLECTIVE

- $TB1_1$ Displaying an investigative ("researcher's") attitude towards mathematical problems
  - $TB1_{11}$ Considering particular cases of a problem
    - Considering a particular case of the integrand and limits of integration (Q2)
    - Discussing the variance of the Riemann sum (not the limit of-) as the position of x_i varies (Q5)
    - Considering particular examples; particularizing (Q8)
  - $TB1_{12}$ Exploring solution paths
    - Displaying anticipation for the remainder of the "integration by parts" procedure, i.e., integrability of *vdu* or obtaining an identical integrand to *udv* etc… (Q3)
    - Considering the limit of f as x approaches the point, c, of discontinuity (Q4)
  - $TB1_{13}$ Defining objects in a problem
    - Defining the variable n/ the Riemann sum in this context (Q1)

- Stating the definition of an infinite series/ a convergent infinite series (Q7)

○ $TB1_{14}$ Connecting components of a problem together

- Relating rectangles' area to height (Q1)
- Explaining the relationship between the left-hand side and the right-hand side of the equality in the definition (Q1)
- Discussing the practicality in choosing x_i at the same position; x_i is then equal to $a + i\Delta x,$ for $1 \leq i \leq n$, for example, if x_i is chosen to be the right end point (Q5)
- Indicating that as smaller distances are considered, smaller time intervals are simultaneously considered (not explicit in problem) (Q10)

○ $TB1_{15}$ Reflecting on the relationships between concepts in a problem and previously learned concepts

- Relating the definite integral to a real number (assuming/arguing that the integral is convergent. Note: Convergence of an integral not yet discussed at this point in course) (Q2)
- Relating the definite integral to the limit of a Riemann sum (thus relating concepts) (Q2)
- Discussing boundedness and monotony of S_n (where S_n is the sequence of partial sums of the series whose convergence is in question) (Q9)

○ $TB1_{16}$ Seeking the requirements of the problem at hand

- Displaying awareness that the area need not be computed (Q6)

○ $TB1_2$ Generalizing a solution

- Writing a single general statement for positive and negative series by considering the absolute value of terms (Q9)
- Indicating that the integral $\int_a^\infty g(x)dx$ is convergent for all $a \geq 1$ (Q11)
- Indicating that the integral $\int_a^b g(x)dx$ is convergent over any subinterval [a, b] of [1,∞) (Q11)
- Remarking that the addition of any non-zero constant to the integrand would result in a diverging integral (Q12)

○ $TB1_3$ Verifying a solution

- Exploring the problem; verifying that two areas are equal in magnitude (Q6)

TT2 SYSTEMIC

TT21 DEFINITIONAL

$TB21_1$ Referring to definitions when deciding upon meaning

- Explaining the relationship between n and delta Δx / area/ length of intervals (Q1)
- Discussing the *necessity* in choosing x_i at the same position if the formula $a + i\Delta x,$ for $1 \leq i \leq n$ is to be used (Q5)

- Referring to the definition of an infinite series/ a convergent infinite series (Q7)
- Referring to the definition of a convergent sequence (Q8)

TT22 PROVING

$TB22_1$ Engaging in a proving or reasoning activity

- Explaining why area obtained from rectangles with a more narrow base has less error than those with a wider base (Q1)
- Reasoning about the choice of u and dv (Q3)
- Showing how one can obtain the second integral from the first (Q6)
- Explaining why the contra-positive of the statement is true (Q7)
- Explaining why the sequence cannot diverge (Q8)
- Describing an analogy between the two statements (Q9)
- Explaining why the two situations do not lie in conflict (Q10)
- Explaining why the integral $\int_{10}^{\infty} g(x)dx$ is convergent after expressing it as $\int_{1}^{\infty} g(x)dx - \int_{1}^{10} g(x)dx$ (Q11)
- Indicating that $(10, \infty)$ is a subinterval of $(1, \infty)$ and explaining that the integral $\int_{10}^{\infty} g(x)dx$ is thus convergent (Q11)
- Indicating that and explaining why the integral $\int_{1}^{10} g(x)dx$ is convergent (Q11)
- Splitting the integral $\int_{1}^{\infty}[g(x) + 2]dx$ into $\int_{1}^{\infty} g(x)dx + \int_{1}^{\infty} 2dx$ and arguing that: $\int_{1}^{\infty} 2dx$ is divergent; thus the whole integral is divergent (Q12)

$TB22_2$ Refuting a general statement by drawing a contradiction

- Arguing by contradiction (Q7, Q8)

$TB22_3$ Referring to a theorem or property

- Referring to the effect of increasing *n* on the accuracy of the area (Q1)
- Discussing the commutative property of limits (in case the subject identifies the integral with the limit of a Riemann sum) (Q2)
- Discussing the commutative property of addition of real numbers (in case the subject identifies the integral with a real number) (Q2)
- Using the Fundamental Theorem of Calculus to write an equivalent expression of the integrals (Q2)
- Referring to the formula for *integration by parts* to support argument (Q3)
- Discussing the necessary nature of terms a_n for large values of n if the series is convergent (Q7)
- Extracting from the hypothesis that the sequence is lower bounded and decreasing (Q8)
- Referring to the theorem for monotone and bounded sequences to conclude the convergence of this sequence (Q8)
- Expressing the integral $\int_{10}^{\infty} g(x)dx$ as $\int_{1}^{\infty} g(x)dx - \int_{1}^{10} g(x)dx$ (Q11)

$TB22_4$ Referring to previously learned concepts

- Discussing the negligibility of the area of a "segment" under the point of discontinuity, or *measure* of a segment (Q4)

- Discussing the irrelevance of the position of x_i once the limit of the Riemann sum is taken (Q5)
- Using the concept of a limit to describe the distance between Achilles and the tortoise before Achilles passes the tortoise (Q10)

TT23 HYPOTHETICAL

TB23$_1$ Being aware of the conditional character of a mathematical statement

- Posing a question about the characteristics of the functions or on whether the integrals are defined on the given intervals (Q2)
- Referring to the hypothesis of the Fundamental Theorem of Calculus (given such that f is continuous) (Q4)
- Discussing continuity of f at x_i (Q5)

TB23$_2$ Considering particular cases to negate a statement or to state its conditional truth

- Considering particular cases for which it is not important (e.g. f constant) (Q1)
- Dividing the interval into [a, c) U (c, b] (Q4)
- Considering intervals such that c is not a sample point (Q4)

TT3 ANALYTIC

TB3$_1$ Being sensitive to logical connectives

- Regarding the variable in the integrand as a dummy variable (Q6)
- Stating the contra-positive of the statement (Q7)
- Arguing by contradiction (Q7)

$TB3_2$ Interpreting symbolic expressions in a rigorous way

- Regarding the variable in the integrand as a dummy variable (Q6)
- Interpreting the behavior of the given type of sequence (Q8)

$TB3_3$ Representing a given problem in a different mathematical register

- Setting up definite integrals representing each area (Q6)
- Modeling the behavior of the partial sums of a convergent series or of the terms of a convergent series (Q7)
- Representing the terms of the sequence S_n graphically (Q9)
- Marking the positions of Achilles and the tortoise over equal time intervals; indicating that Achilles *does* pass the tortoise (Q10)
- Deriving equations of motion for Achilles and the tortoise and using them for explanation (for example, to express the time at which Achilles passes the tortoise) (Q10)
- Deriving speed/time graphs for Achilles and the tortoise and using them for explanation (for example, to express the time at which Achilles passes the tortoise) (Q10)
- Assuming $g(x) > 0$: Drawing an arbitrary graph representing $g(x)$, then indicating the areas represented by the integrals $\int_1^\infty g(x)dx$ and $\int_{10}^\infty g(x)dx$, showing that the latter is included in the former (Q11)
- Drawing a graph representing $g(x)$ and then $g(x) + 2$; indicating that the latter represents an infinite area (Q12)