Concordia University

A Numerical Application to *Optimal Reciprocal Reinsurance Treaties Under the Joint Survival Probability and Joint Profitable Probability* Using a Compound Poisson Model with Exponential Severity

Research project

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1 Introduction

In this section, we summarize the research paper *Optimal Reciprocal Reinsurance Treaties Under the Joint Survival Probability and Joint Profitable Probability* by Cai and Li (2012), on which we base our numerical applications and new results. The main focus of the authors is to design a reinsurance treaty that would be optimal not only to the insurer, but to the reinsurer as well. This goal is established after considering that the stop-loss reinsurance treaties that have been previously designed were proven to be optimal in the eyes of the insurer, but there is no evidence suggesting that they are also in the best interests of the reinsurer.

The issue of possible conflicting interests is tackled by defining and maximizing the joint survival and joint profitable probabilities of the two parties in question. The authors begin by looking at the optimal reinsurance retention under the expected value principle and find the necessary values under the two types of treaties—quota share and stop-loss reinsurance—that maximize the joint survival probability. From a mathematical perspective, the joint profitable probability can be perceived as a special case of the joint survival probability when the initial wealth of both parties are set to zero. However, for interpretational purposes, it is important to consider the two as separate entities. First, it is established that for the quota share reinsurance, the optimal retention is dependent on how the insurer's initial wealth measures up to what can be interpreted as the excess of the pure risk premium including safety loadings over the premium received by the insurer from the insured. For this same treaty type, Cai and Li (2012) find that interestingly enough, when evaluated at the optimal retention value, the survival probability functions of both the insurer and reinsurer are equivalent. In fact, they are identical to the cumulative distribution of the loss when it is equal to the sum of the initial wealth of both parties and the insurance premium received by the insurer from the insured. As a result, when this optimal retention value is used, the contract is optimal and thus fair for both parties. For the stop-loss reinsurance contract studied in Theorems 2 to 4, the optimal retention that maximizes the joint survival probability is also equivalent to this value. In these three theorems, the sufficient and necessary conditions leading to the existence of an optimal stop-loss retention are examined.

Theorems 5 and 6 serve as the foundation that one could use to design general optimal reinsurance contracts by maximizing the joint survival and joint profitable probability functions under general premium principles and among a wide class of reinsurance policies. This is eventually demonstrated through the design of a quota share contract under the variance premium principle in Theorems 7 and 8, and finally a limited stop-loss contract under the expected value principle in Theorems 9 to 12. Treaties that could be considered outside the contents of this paper are proportional surplus and non-proportional excess-of-loss reinsurance contracts. This demonstrates that their findings could serve as tools in the future development of a desired form of reinsurance under a preferred reinsurance premium, that can be made fair in the perspective of both parties involved.

2 Assumptions

An aggregate loss random variable will be considered to model the loss incurred by the insured. Numerical applications for the theorems in Cai and Li (2012) will be executed and analyzed, focusing on the quota share and limited stop-loss reinsurance contracts. We shall consider :

- *The claim severity, B_i, as independent Exponential random variables with rate* $\beta = 0.01$
- *The claim frequency, N, as a Poisson random variable with* $\lambda = 2$.

The Compound Poisson random variable is therefore defined as $X = \sum_{i=1}^{N} B_i$. Consequently, its cumulative distribution function is :

$$F(x) = \sum_{i=1}^{N} \Pr(N = i) \Pr(X \leqslant x \mid N = i) + P(N = 0) = \sum_{i=1}^{N} \frac{2^i e^{-2}}{i!} \Gamma(i; x) + e^{-2},$$

where $\Gamma(i; x)$ is the incomplete gamma function, defined as $\frac{1}{\Gamma(i)} \int_0^x t^{i-1} e^{-t} \, dt$.

3 Numerical Application of Theorem 1

3.1 A quota share reinsurance under the expected value principle

A quota share reinsurance contract will first be studied, with a fixed relative safety loadings of $\theta_R = 0.15$ for the reinsurer. Although not mentioned in the original paper, we will also consider the safety loadings for the insurer, θ_I, at values below, equal to and above $\theta_R = 0.15$. It is important to consider relative safety loadings for both parties involved in the treaty. We can assume that the insurance company is also interested in benefiting from θ_I's advantages, such as covering the expenses of securing and maintaining the business. To study the effect that this variable has on the contract, we shall consider $\theta_I = 0$, 0.15 and 0.4.

Under the expected value principle, we take into account :

$$P_R^f = 1.15 E(f(X))$$

$$P_I^f = (1+\theta_I)E(I_f(x))$$

– $f(x) = (1-b)X$, *the ceded loss covered by the reinsurer*
– $I_f(x) = bX$, *the retained loss covered by the insurer*
– $\mu = E(N)E(B) = \lambda \frac{1}{\beta} = 2 \times 100 = 200$, *the expected aggregate loss of the contract*
– $P_R(b) = (1.15)(1-b)\mu = 230(1-b)$, *the reinsurance premium*
– $P_I(b) = (1+\theta_I)200b$, *the net insurance premium*
– $P_0 = 230 - b(30 - 200\theta_I)$, *the insurance premium paid by the insured to the insurer*
– $p = (1+\theta_R)\mu - P_0 = 230 - P_0$, *the excess of the pure risk premium including safety loadings over* P_0

Numerical results for Theorem 1 are found in the following tables, conditioned on insurer safety loadings. Reinsurer initial wealth is set at values both below and above the average aggregate loss of 200. Insurer initial wealth is set within each specific case to respect the constraints set by the first theorem. Although the authors define the initial wealth of both parties as being strictly positive, we consider cases below zero as well. This leads to more thorough results by providing solutions for each point of Theorem 1, when the insurer safety loading is set to 0.15. This particular value of θ_I leads to $p = 0$, and thus without considering negative values of u_I, the case $u_I < p$ would be ignored. It is of importance to consider the meaning of this negative value. First, since we are dealing with reinsurance treaties, it is logical to assume that we are dealing with fairly large risks. Companies that assume these risks are large and tend to be made up of multiple business lines. We can therefore assume that with sufficient funds, it is possible for them to run a line which possesses a negative initial wealth, with hope of it turning around in the near future.

As mentioned earlier, when the safety loadings for both parties are equal and set to 0.15, p is always 0 and thus optimal results are automatically obtained without excessive calculations. However, for values of insurer safety loadings that differ from 0.15, one must first calculate p^* and P_0^*, the optimal values of these variables, by evaluating each one at b^*, the optimal retention for each case. For $u_I \leq p$, this evaluation simply consists of replacing b in the equations for p and P_0 by the quota retention b^* provided by Theorem 1. For the case $u_I > p$ though, $b_0 = b^*$ is dependent on p, which is also expressed in terms of the optimal quota retention. A quadratic equation must therefore be solved for these particular cases, resulting in the following solutions :

$$b^* = \begin{cases} 0, & \text{for all } \theta_I \text{ and } u_I = p; \\ 1, & \text{for all } \theta_I \text{ and } u_I < p; \\ \frac{(u_I+u_R+30)-\sqrt{(u_I+u_R+30)^2-120u_I}}{60}, & \text{if } \theta_I = 0 \text{ and } u_I > p\,; \\ \frac{u_I}{u_I+u_R}, & \text{if } \theta_I = 0.15 \text{ and } u_I > p; \\ \frac{(u_I+u_R+110)-\sqrt{(u_I+u_R+110)^2-120(80+u_I)}}{60}, & \text{if } \theta_I = 0.4 \text{ and } u_I > p. \end{cases}$$

For each combination of initial wealths, the numerical results for the first theorem can be seen in the following tables, with optimal results showcased in Tables 2, 3 and 4.

θ_I	$P_0 = 230 - b(30 - 200\theta_I)$	$p = 230 - P_0$	Range of p
0	230-30b	30b	[0,30]
0.15	230	0	0
0.4	310-30b	30b-80	[-80,-50]

TABLE 1 – Expressions of important variables in terms of b

u_I	u_R	b^*	$J_S(b^*)$	P_0^*	p^*
0	100	0.0000	0.7874	230	0
0	300	0.0000	0.9272	230	0
15	100	1.0000	0.6296	200	30
15	300	1.0000	0.6296	200	30
35	100	0.2210	0.8160	223.3699545	6.630045473
35	300	0.0967	0.9394	227.1002506	2.899749443

TABLE 2 – Optimal results with $\theta_I = 0$

u_I	u_R	b^*	$J_S(b^*)$
-10	100	1.0000	0.6380
-10	300	1.0000	0.6380
0	100	0.0000	0.7874
0	300	0.0000	0.9272
35	100	0.2593	0.8222
35	300	0.1045	0.9403

TABLE 3 – Optimal results with $\theta_I = 0.15$

u_I	u_R	b^*	$J_S(b^*)$	P_0^*	p^*
-80	100	0.0000	0.787389	310	-80
-80	300	0.0000	0.9272392	310	-80
-60	100	1.0000	0.6380359	280	-50
-60	300	1.0000	0.6380359	280	-50
-20	100	0.4535	0.8324218	296.394103	-66.39410298
-20	300	0.2085	0.946431	303.7458609	-73.74586088

TABLE 4 – Optimal results with $\theta_I = 0.4$

u_I	u_R	b^*	$J_S(b^*)$
5	100	1.0000	0.6380
5	300	1.0000	0.6380
15	100	0.0000	0.7874
15	300	0.0000	0.9272
35	100	0.221	0.8160
35	300	0.09666	0.9394

TABLE 5 – Results for $\theta_I = 0$ taking $p = 15$

u_I	u_R	b^*	$J_S(b^*)$
-95	100	1.0000	0.6035
-95	300	1.0000	0.6035
-65	100	0.0000	0.7874
-65	300	0.0000	0.9272
-35	100	0.2308	0.8176
-35	300	0.0909	0.9386

TABLE 6 – Results for $\theta_I = 0.4$ taking $p = 65$

As one would expect to see, the tables demonstrate the highest values for the joint survival probability when the conditions to obtain b^* are considered. To highlight this, the function is first evaluated at an arbitrary retention ratio of 0.5. In Table 1, we see that each value of θ_I is accompanied by its respective range of possible values for p. An arbitrary value for p is therefore chosen by taking the midpoint of each range, which in fact results in $b = 0.5$. Besides the case $\theta_I = 0.15$ which results in a constant value of zero for p, it is observed that when the quota retention b differs from b^*, the joint survival probability is not maximized.

When $u_I < p$, the initial wealth of the reinsurer is irrelevant in the maximization of the joint survival probability, since $J_S(b^*) = J_S(1) = F(u_I + P_0)$. It is of interest to look at the result when the initial wealth of the insurer is greater than $p = 0$, due to the certain b^* that is obtained. Looking at the table values when $u_I = 35$, we observe that the optimal quota share retention is actually the proportion of the insurer's initial wealth out of the total initial wealth of both parties. Therefore, when $p = 0$, we get that $\frac{I_f(x)}{X} = \frac{u_I}{(u_I+u_R)}$ and $\frac{f(x)}{X} = \frac{u_R}{(u_I+u_R)}$. This makes sense since p, as defined in the paper, can be interpreted as the difference between the pure risk premium with reinsurer safety loadings and the insurance premium actually paid by the insured to the insurer of the contract. If this value is 0, then both the reinsurer and insurer have the same expectations and thus neither party possesses an unfair advantage. As a result, a logical value to accept for the proportion of loss that is to be assumed by each of them, b, is their respective proportion of initial total wealth.

4 Numerical Results of Proposition 1

This section provides results relating to the maximization of the joint profitable probability. We will first consider the case $p = 0$. For our model, Proposition 1 implies that when $\theta_R = \theta_I = 0.15$, $b^* = b$ for any $0 < b < 1$ and thus $J_p(b^*) = F(P_0) = F(230) = 0.6544017$. This means that when the safety loadings for both parties are equivalent and set to 0.15, both the insurer and reinsurer stand to make profit from the contract 65.44 % of the time for any given quota retention value. The choice of b is irrelevant, since in this scenario the premium paid by the insured to the insurer, P_0, is independent of the variable p. When $p \neq 0$, the optimal quota retention is $b^* = 1$ and the optimal values of P_0 must be found in order to calculate $J_p(1) = F(P_0)$ for each case of θ_I that differs from 0.15. These results are summarized below in Table 7. It is observed that the maximum joint profitable probability increases as the insurer safety loadings increases. This can be explained by the proportional relationship between θ_I and P_0 and the fact that $J_p(b)$ depends solely on P_0. The insurance company increases their safety loadings when they assume a higher level of risk. To take on this riskier contract, they would also charger a higher amount to the insured, which is in turn represented by the increase in P_0.

θ_I	b^*	P_0^*	$J_p(b^*)$
0.00	1.0000	200	0.603501
0.15	b $\in$ [0,1]	230	0.654401
0.40	1.0000	280	0.7275728

Table 7 – Numerical results of Proposition 1

5 Summary of Theorems 2, 3 and 4

In the following three theorems, the optimal retention level d^* of a stop-loss reinsurance is studied. Theorem 2 describes the desired value d in the domain of $[0,\infty)$ that maximizes the joint survival probability as the solution to an equation, while Theorem 3 states the necessary and sufficient conditions for its existence. The optimal stop-loss retention that maximizes the joint profitable probability is eventually examined in Theorem 4. It is interesting to note that two conflicts of interests amongst the insurer and reinsurer arise after calculation of the optimal stop-loss retention level. The first issue occurrs with the maximization of the joint survival probability, when the insurer is found to benefit due to survival certainty, meanwhile the reinsurer suffers from bankruptcy risk. Secondly, through maximization of the joint profitable probability, it is seen that the insurer could make risk-free profits whereas the reinsurer is at risk of not only a zero-gain situation, but of losing money as well. These two unfair situations are later resolved with the introduction of a limited stop-loss reinsurance contract and a new optimal limited stop-loss retention level.

6 Summary of Theorems 5 and 6

In Theorems 5 and 6, the necessary conditions that f must comply with to be considered an optimal ceded loss function over the class of all admissible reinsurance policies under a certain premium principle are given. These two theorems form the foundation for all later findings, and they require that both the optimal ceded loss and optimal retained loss functions be non-decreasing in $x \geq 0$. As Theorem 5 states a condition to maximize the joint survival probability, Theorem 6 does the same for the joint profitable probability function. Mathematically, Theorems 5 and 6 provide one with the following two equations, respectively :

$$\beta\sigma^2(1-b^*)^2 - (u_I + u_R + P_0 - \mu)(1-b^*) + u_R = 0 \tag{1}$$

and

$$b^* = \frac{\beta\sigma^2 + \mu - P_0}{\beta\sigma^2}. \tag{2}$$

These two theorems are then used throughout the remainder of the paper to construct two new optimal reinsurance contracts ; a quota share one under the variance principle and a limited stop-loss one under the expected value principle.

6.1 Optimal reinsurance under the variance principle

Beginning with the first of the two previously mentioned contract types, equations (1) and (2) are broken down and re-expressed in terms of a new set of variables, as to design a hypothetical optimal quota share reinsurance under the variance reinsurance premium principle. The following are these expressions, relative to our aggregate loss model :

- $Var(X) = \sigma^2 = E(N)V(B) + E(B)^2V(N) = \lambda(2(\frac{1}{\beta})^2) = 40,000$;
- $P_R(b) = 200(1-b) + \beta(1-b)^2 40,000$;
- $P_I(b) = P_0 - P_R(b)$;
- $q = 200 + 40,000\beta - 200\theta_A b$;
- $\Delta = (u_I + u_R + 200\theta_A b - 200)^2 - 160,000 u_R \beta$.

For numerical purposes, we shall consider the insurance premium received by the insurer from the insured as a variable depending on a safety loadings, θ_A and the expected value of the retained loss. Variations of the safety loadings will represent the insurance company's focus on different age groups of the insureds. We shall therefore consider

$$P_0 = \theta_A E(I_f(x)) = 200\theta_A b = \begin{cases} 100b, & \text{if } \theta_A = 0.5 \text{ (young insureds)}; \\ 300b, & \text{if } \theta_A = 1.5 \text{ (old insureds)}, \end{cases}$$

leading to the above expressions for q and Δ. Note that Δ is in fact the discriminant of the quadratic equation formed by solving the equation in Theorem 5, whereas q is the numerator of b^* in regards to Theorem 6. The results of the latter lead to the study of the maximization of the joint profitable probability and are examined with greater detail in Theorem 8.

7 Numerical Application of Theorem 7

Three different cases arise when searching for the optimal quota share retention under the variance principle, built on the foundation of Δ being less than, equal to, or greater than zero. Each of these three cases are then expanded in order to incorporate fact that although solutions for the roots b_1 and b_2 could very well exist outside of the range [0,1], such solutions cannot be considered as retention ratios and should therefore be discarded. As a result, we obtain the six different cases below. Since our P_0 depends on retention b, the following expressions for the optimal retention b^* in points four and five are found by optimizing P_0 in the equations of Theorem 7 and solving for the variable in question once again.

1. If $\Delta > 0$ and $b_2 = 0$, then $b^* = 0$
2. If $\Delta > 0$ and $b_1 < 0 < b_2 < 1$, then $b^* = b_2$
3. If $\Delta > 0$ and $0 \leq b_1 < b_2 < 1$, then both $b^* = b_1$ and $b^* = b_2$ are optimal solutions
4. If $\Delta = 0$ and $0 < u_I + u_R + 200(\theta_A b - 1) < 80000\beta$, then $b^* = \frac{80000\beta - u_I - u_R - 200}{80000\beta + 200\theta_A}$
5. If $\Delta < 0$ and $0 < q - u_I < 40000\beta$, then $b^* = \frac{-200\theta_A + \sqrt{(200\theta_A)^2 + 4\sigma^2\beta(200 + \sigma^2\beta - u_I)}}{2\sigma^2\beta}$
6. If $\Delta < 0$ and $q - u_I \geq 40000\beta$, then $b^* = 1$

We consider β severity levels of 0.1, 0.5, 1 and 3. The following results are obtained by following the calculations according to the authors' findings :

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-1596400	4260	1	0.2427328
-60	300	-4780400	4260	1	0.2427328
0	100	-1600000	4200	1	0.3942969
0	300	-4760000	4200	1	0.3942969
35	100	-1598775	4165	1	0.4743334
35	300	-4744775	4165	1	0.4743334
150	150	-2360000	4050	1	0.6853708
200	150	-2337500	4000	1	0.7530113
300	150	-2277500	3900	0.975	0.8519364
500	150	-2097500	3700	0.94935	0.9526792

TABLE 8 – Optimal results : Young insureds with $\beta = 0.1$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-7996400	20260	1	0.2427328
-60	300	-23980400	20260	1	0.2427328
0	100	-8000000	20200	1	0.3942969
0	300	-23960000	20200	1	0.3942969
35	100	-7998775	20165	1	0.4743334
35	300	-23944775	20165	1	0.4743334
150	150	-11960000	20050	1	0.6853708
200	150	-11937500	20000	1	0.7530113
300	150	-11877500	19900	0.995	0.8519364
500	150	-11697500	19700	0.98997	0.9515152

TABLE 9 – Optimal results : Young insureds with $\beta = 0.5$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-15996400	40260	1	0.2427328
-60	300	-47980400	40260	1	0.2427328
0	100	-16000000	40200	1	0.3942969
0	300	-47960000	40200	1	0.3942969
35	100	-15998775	40165	1	0.4743334
35	300	-47944775	40165	1	0.4743334
150	150	-23960000	40050	1	0.6853708
200	150	-23937500	40000	1	0.7530113
300	150	-23877500	39900	0.9975	0.8519364
500	150	-23697500	39700	0.995	0.9513732

TABLE 10 – Optimal results : Young insureds with $\beta = 1$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-47996400	120260	1	0.2427328
-60	300	-143980400	120200	1	0.2427328
0	100	-48000000	120200	1	0.3942969
0	300	-143960000	120165	1	0.3942969
35	100	-47998775	120165	1	0.4743334
35	300	-143944775	120050	1	0.4743334
150	150	-71960000	120200	1	0.6853708
200	150	-71937500	120000	1	0.7530113
300	150	-71877500	119900	0.99917	0.8519366
500	150	-71697500	119700	0.99833	0.9512785

TABLE 11 – Optimal results : Young insureds with $\beta = 3$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-1580400	4260	1	0.6701769
-60	300	-4684400	4260	1	0.6701769
0	100	-1560000	4200	1	0.7530113
0	300	-4640000	4200	1	0.7530113
35	100	-1544775	4165	1	0.7926999
35	300	-4610775	4165	1	0.7926999
150	150	-2240000	4050	1	0.8867208
200	150	-2197500	4000	1	0.9139345
300	150	-2097500	3900	0.98788	0.953091
500	150	-1837500	3700	0.925	0.9852765

TABLE 12 – Optimal results : Old insureds with $\beta = 0.1$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-7980400	20260	1	0.6701769
-60	300	-23884400	20260	1	0.6701769
0	100	-7960000	20200	1	0.7530113
0	300	-23840000	20200	1	0.7530113
35	100	-7944775	20165	1	0.7926999
35	300	-23810775	20165	1	0.7926999
150	150	-11840000	20050	1	0.8867208
200	150	-11797500	20000	1	0.9139345
300	150	-11697500	19900	0.99	0.9509453
500	150	-11437500	19700	0.985	0.9852765

TABLE 13 – Optimal results : Old insureds with $\beta = 0.5$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-15980400	40260	1	0.6701769
-60	300	-47884400	40260	1	0.6701769
0	100	-15960000	40200	1	0.7530113
0	300	-47840000	40200	1	0.7530113
35	100	-15944775	40165	1	0.7926999
35	300	-47810775	40165	1	0.7926999
150	150	-23840000	40050	1	0.8867208
200	150	-23797500	40000	1	0.9139345
300	150	-23697500	39900	0.995	0.9510893
500	150	-23437500	39700	0.9925	0.9852765

Table 14 – Optimal results : Old insureds with $\beta = 1$

u_I	u_R	$max\Delta$	$max(q-u_I)$	b^*	$J_S(b^*)$
-60	100	-47980400	120260	1	0.6701769
-60	300	-143884400	120200	1	0.6701769
0	100	-47960000	120200	1	0.7530113
0	300	-143840000	120165	1	0.7530113
35	100	-47944775	120165	1	0.7926999
35	300	-143810775	120050	1	0.7926999
150	150	-71840000	120200	1	0.8867208
200	150	-71797500	120000	1	0.9139345
300	150	-71697500	119900	0.998334	0.9511843
500	150	-71437500	119700	0.9975	0.9852765

Table 15 – Optimal results : Old insureds with $\beta = 3$

To obtain the correct b^* and hence the optimal quota share reinsurance contract for each case, Δ first has to be observed to establish whether it is less than, greater than, or equal to zero. It is important to note that this variable itself depends on b through its inclusion of P_0. Therefore, using the fact that Δ is an increasing function in respect to b, its maximum is considered by replacing b by 1 in order to narrow down which of the six above stated restrictions each specific case belongs to. Looking at Tables 8 through 15, it can be observed that for all of the reconsidered pairs of initial wealths from part one, a maximum delta value of less than zero is observed. Using our model, a positive delta value will only be obtained if the initial wealth of the reinsurer is very low relative to the initial wealth of the insurer.

At first, only the initial wealth pairings that are considered in previous sections of this paper were chosen to be further studied under this new principle, restricting results to cases where $u_I < u_R$. This lead to an optimal retention of $b^* = 1$ for all cases that fell into this category. A possible explanation for these results can be contemplated by examining the significance of q. In the paper, we are only given a mathematical representation of this value in terms of μ, β, σ^2 and P_0, but it is interesting to interpret it as $P_R(0) - P_0$ or the excess of the risk premium under the variance principle over the premium paid by the insured to the insurer in the contract. Defining this variable as so, it can be looked at as the counterpart of $p = (1 + \theta_R)\mu - P_0$, which is similarly defined under the expected value principle. We note that under the variance principle, this value depends on the degree of fluctuation of the aggregate loss. The only way of achieving an optimal quota share retention of 1 is by having $\Delta < 0$ and $q - u_I > 40,0000\beta$. Therefore, ceteris paribus, as q increases, so does the likelihood of the optimal retention ratio being 1 for this specific reinsurance contract. Consequently, we can say that as the expected profit of the contract under this premium principle increases, the insurer has incentive to retain all of the loss, leading to an optimal retention ratio of 1.

Now, referring to only our specific results in the preceding eight tables, the maximized joint survival probabilities for each case were found by applying Theorem 7 of Cai and Li (2012) to our model as follows :

$$J_S(b^*) = \begin{cases} F(u_I + 200\theta_A), & \text{if } b^* = 1\,; \\ F(200 + 80,000\beta - 2\sqrt{40,000\beta(200 + 40,000\beta - 200\theta_A b^* - u_I}), & \text{if } b^* \neq 1. \end{cases}$$

As a result of the joint survival probability depending solely on the insurer's initial wealth and P_0 when $b^* = 1$, it does not vary with changes to the security level β. It can be observed that the optimized value of the function is higher for the group of older insureds, which makes sense since they are in fact charged a higher premium. We must also consider that pairings of initial wealths are predetermined from the work dealing with previous theories, and this leads to strictly focusing on cases where $u_I < u_R$. After this is taken into account, we consider a few cases where the reinsurer's initial wealth exceeds that of its counterpart, and these results can be seen in the last three rows of Tables 8 to 15. We can see that although Δ remains negative, the optimal retention is no longer strictly 1. From this point onwards, let us refer to the term $40,0000\beta$ in points 5 and 6 above as the threshold for each case (θ_A, β, u_I, u_R). This value is of importance because it can be observed in each of the preceding nine tables that when $u_I = \mu = 200$, the threshold is obtained. Consequently, we can say that when the initial wealth of the insured is at most the expected value of the aggregate loss, then the optimal reinsurance contract has a quota share retention of 1. As the initial wealth of the insurer increases passed the average, the optimal retention decreases. Furthermore, for each case, looking at the results as u_R stays constant at 150 and u_I varies, we see that as the insurer's initial wealth increases, b^* decreases and the maximized joint survival probability increases.

8 Numerical Application of Theorem 8

With Theorem 6 serving as its foundation, Theorem 8 focuses on the joint profitable probability function. Cai and Li (2012) state that a quota share reinsurance under the variance principle with $b^* = 1$ or $b^* = \frac{q}{\beta\sigma^2}$ forms an optimal contract within the class of all admissible policies, for $0 \leq q \leq \beta\sigma^2$. Looking at Tables 8 to 15, we remark that only the former optimal quota share retention is observed, as the latter is not present in our data. We can therefore establish that the joint profitable probability function is only maximized for combinations of (θ_A, β, u_I, u_R) such that $u_I < u_R$.

9 Numerical Application of Theorem 9

Theorem 5 is once again used to design an optimal limited stop-loss reinsurance contract under the expected value principle. Theorem 9 focuses on optimality through maximization of the joint survival probability.

Under the expected value principle, the net reinsurance and insurance premiums are defined as follows :

$$P_R(d_1, d_2) = (1 + \theta_R)E(f(x)) = 1.15 \int_{d_1}^{d_1+d_2} S(x)\, dx,$$

$$P_I(d_1, d_2) = (1 + \theta_I)E(X - f(x)) = (1 + \theta_I) \left(200 - \frac{P_R(d_1, d_2)}{1.15}\right).$$

Simplification of the sum of these two equations leads us to a representation of the premium paid by the insured, P_0. Under this specific reinsurance contract and principle, we obtain

$$P_0 = 200(1 + \theta_I) + S(x)(0.15 - \theta_I)d_2.$$

As in Theorem 1, insurer safety loadings values of 0, 0.15 and 0.4 are considered. Calculations are executed using R to generate values of $F(x)$ and hence $S(x)$ for various loss amounts, both above and below the average of 200. Relative to our aggregate loss model, Theorem 9 can therefore be restated as follows :

If $d_1+S(x)(1+\theta_I)d_2 = u_I+200(1+\theta_I)$ *has solutions in* Γ_1 *or* $d_2 = \frac{u_R}{1-1.15S(x)}$ *has solutions in* Γ_2*, then a limited stop-loss reinsurance with retention* $(d_1^*, d_2^*) \in \Gamma_1^* \bigcup \Gamma_2^*$ *is an optimal reinsurance in* F^π*. Here,* Γ_1^* *and* Γ_2^* *are the respective solution sets to these two equations.*

We consider :
- Γ_1 consisting of all retention vectors (d_1, d_2) such that $0 \leq d_1 < u_I + u_R + P_0$, $d_2 \geq 0$ and $d_1 + d_2 > u_I + u_R + P_0$;
- Γ_2 consisting of all retention vectors (d_1, d_2) such that $0 \leq d_1 < u_I + u_R + P_0$, $d_2 \geq 0$ and $d_1 + d_2 \leq u_I + u_R + P_0$.

Table 16 contains important values that will be used in further calculations. We consider various loss amounts and the corresponding premium paid by the insured for the three cases of θ_I that were previously studied.

Loss Amount (x)	S(x)	P_0 ($\theta_I = 0$)	P_0 ($\theta_I = 0.15$)	P_0 ($\theta_I = 0.4$)
50	0.7309879	$200 + 0.10964d_2$	230	$280 - 0.18275d_2$
100	0.6057031	$200 + 0.09086d_2$	230	$280 - 0.15143d_2$
200	0.396499	$200 + 0.05947d_2$	230	$280 - 0.09912d_2$
300	0.2469887	$200 + 0.03705d_2$	230	$280 - 0.06175d_2$
1000	0.0041651	$200 + 0.00062d_2$	230	$280 - 0.00104d_2$

TABLE 16 – Loss amount, survival probability and premium paid by insured

Looking at possible results in Γ_2, we can see that the second equation of this theorem, $d_2 = \frac{u_R}{1-1.15S(x)}$, is independent of θ_I. Accordingly, the results of the second component of the retention vector are as follows for all possible insurer safety loadings values :

Loss Amount (x)	u_R	d_2
50	100	627.4946245
	200	1254.989249
	300	1,882
100	100	329.5528839
	200	659.1057678
	300	989
200	100	183.8146935
	200	367.6293869
	300	551
300	100	139.6720231
	200	279.3440463
	300	419
1000	100	100.4812918
	200	200.9625836
	300	301

TABLE 17 – Results for $d_2 = \frac{u_R}{1-1.15S(x)}$

After consideration of each θ_I value, there are a few observations that can be made. The simplest case arises when dealing with $\theta_I = \theta_R = 0.15$. Here, P_0 is a constant and is therefore independent of both of the retentions, d_1 and d_2. Since Theorem 10 focuses on conditions that lead to the existence of the solutions that we are looking for in Theorem 9, further calculations that result in optimal solutions are present in the next section.

10 Numerical Application of Theorem 10

To study the necessary and sufficient conditions for the existence of solutions to either of the two principle equations in Theorem 9 for our Compound Poisson aggregate model, we must consider :

- $\alpha_R = \frac{1}{1+\theta_R} = \frac{20}{23} \approx 0.869565$
- $d_R = S^{-1}(\alpha_R) = 0$
- $S(0) = 0.8646647$

Correspondingly, it is observed that $S(0) \leq \alpha_R$. By applying Cai and Li (2012) Theorem 10 to our model, solutions to the two showcased equations respectively exist in Γ_1 or Γ_2 if and only if :

$$1.15 \int_0^{u_I+u_R+P_0} S(x)\,dx \leq u_I + P_0.$$

Once again considering $S(x)$ as a constant having been evaluated at a particular loss amount, this results in the following inequality :

$$1.15S(x)(u_I + u_R + P_0) \leq u_I + P_0. \qquad (3)$$

As long as the above inequality holds, an optimal limited stop-loss reinsurance contract exists under the expected value principle. As the authors observe, it is important to note that the optimal retentions (d_1^*,d_2^*) in Γ_2 lead to a contract that is unfair to the insurer in terms of the joint survival probability, while the optimal retentions in Γ_1 provide fairness for both parties. For this reason, further results will be calculated strictly based on Γ_1. Note that values for P_0 regarding all studied values of d_2 can be observed in Tables 25 and 37 for $\theta_I = 0, 0.4$, and it is already known that P_0 is constant at 230 for $\theta_I = 0.15$. Isolating u_I in

(3), we obtain

$$u_I \leq \frac{P_0 - 1.15S(x)(u_R + P_0)}{1.15S(x) - 1}$$

and as a result can find the maximum insurer initial wealth needed for a possible optimal contract under each case. We shall call this maximum value u_I^*. As a result, we can now plug in values of $S(x)$, P_0, u_R and d_2 into the above inequality for each of the cases considered and observe which pairings of (u_I, u_R) lead to a possible optimal contract under our model. For d_2, values of 70 and 300 are used to calculate results for this retention vector component both above and below the average of 200. The following three tables contain these results for each case of θ_I.

Loss Amount (x)	d_2	$u_I{}^*$ for $u_R = 100$	$u_I{}^*$ for $u_R = 200$	$u_I{}^*$ for $u_R = 300$
50	70	319.8352402	847.3452803	1374.85532
	300	294.6180402	822.1280803	1349.63812
100	70	23.23769057	252.8355811	482.4334717
	300	2.339890574	231.9377811	461.5356717
200	70	-120.3495074	-36.53611471	47.27727793
	300	-134.0276074	-50.21421471	33.59917793
300	70	-162.9214778	-123.2494557	-83.57743353
	300	-171.4429778	-131.7709557	-92.09893353
1000	70	-199.5620945	-199.0807891	-198.5994836
	300	-199.7046945	-199.2233891	-198.7420836

TABLE 18 – Possible optimal contract combinations for $\theta_I = 0$

Loss Amount (x)	$u_I{}^*$ for $u_R = 100$	$u_I{}^*$ for $u_R = 200$	$u_I{}^*$ for $u_R = 300$
50	297.5100402	825.0200803	1352.53012
100	329.5528839	229.943175	987.3423907
200	-146.1866074	-62.37321471	21.44017793
300	-190.3279778	-150.6559557	-110.9839335
1000	-229.5186945	-229.0373891	-228.5560836

TABLE 19 – Possible optimal contract combinations for $\theta_I = 0.15$

Loss Amount (x)	d_2	$u_I{}^*$ for $u_R = 100$	$u_I{}^*$ for $u_R = 200$	$u_I{}^*$ for $u_R = 300$
50	70	260.3025402	787.8125803	1315.32262
	300	302.3350402	829.8450803	1357.35512
100	70	-39.80200943	189.7958811	419.3937717
	300	-4.973109426	224.6247811	454.2226717
200	70	-189.2482074	-105.4348147	-21.62142207
	300	-166.4506074	-82.63721471	1.176177931
300	70	-236.0054778	-196.3334557	-156.6614335
	300	-221.8029778	-182.1309557	-142.4589335
1000	70	-279.4458945	-278.9645891	-278.4832836
	300	-279.2066945	-278.7253891	-278.2440836

TABLE 20 – Possible optimal contract combinations for $\theta_I = 0.4$

In Table 19, it is important to note that when $\theta_I = 0.15$, u_I^* is unaltered by a change in d_2. This is due to the fact that for an insurer's safety loadings of this amount, P_0 is independent of d_2. In the previous three tables, we also observe that as the insurer safety loadings increases, the range of u_I^* also increases for each combination of x and d_2. This is representative of the riskiness of each contract in the eyes of the insurer, since the assumed risk is proportional to the safety loadings value they would choose.

We shall now bind Theorems 9 and 10 together. Consequently, it can be said that for each respective value of θ_I taking d_2, x, u_R and a maximum insurer initial wealth of u_I^* as observed in Tables 18 to 20, one can form an optimal reinsurance contract under the expected value principle that maximizes the joint survival probability. For each case of insurer safety loadings, the u_I used for calculation purposes is the least integer of $\min\{u_I^*\}$ for each respective combination of loss amount and d_2. This is done in order to respect the necessary and sufficient conditions for a solution to exists in Γ_1 and also for simplicity of further calculations. For example, with $\theta_I = 0$, for a loss amount of 50 and $d_2 = 70$, we take u_I to be 319. Let these values be hereby known as u_I^C. Taking the minimum value for each combination assures that $u_I^C < u_I^*$ for each pairing of (x,d_2) and the three different values of u_R. The following tables demonstrate these results in the form of potential optimal retention vectors after solving for $d_1 = u_I^C + (1+\theta_I)(200 - S(x)d_2)$ (by Theorem 9's equation for solutions in Γ_1) for each case :

Loss Amount (x)	d_2^*	u_I^C	d_1^*	$J_S(d_1^*, d_2^*)$
50	70	319	467.830847	0.930385
	300	294	274.70363	0.943573
100	70	23	180.600783	0.6862653
	300	2	20.28907	0.7767392
200	70	-121	51.24507	0.4437157
	300	-135	-53.9497	—
300	70	-163	19.710791	0.3694823
	300	-172	-46.09661	—
1000	70	-200	-0.291557	—
	300	-200	-1.24953	—

TABLE 21 – Optimal results for $\theta_I = 0$

Loss Amount (x)	d_2^*	u_I^C	d_1^*	$J_S(d_1^*, d_2^*)$
50	70	297	468.1554741	0.9305126
	300	297	274.8091745	0.9436072
100	70	-1	180.2409005	0.6857298
	300	-1	20.0324305	0.7764517
200	70	-147	51.0818305	0.4433457
	300	-147	-53.792155	—
300	70	-191	19.11740965	0.3680346
	300	-191	-46.2111015	—
1000	70	-230	-0.33529055	—
	300	-230	-1.4369595	—

TABLE 22 – Optimal results for $\theta_I = 0.15$

Loss Amount (x)	d_2^*	u_I^C	d_1^*	$J_S(d_1^*, d_2^*)$
50	70	261	468.363186	0.9309856
	300	302	274.985082	0.9436641
100	70	-40	180.641096	0.6863253
	300	-4	20.604698	0.7782083
200	70	-190	51.143098	0.4434846
	300	-167	-53.52958	—
300	70	-237	18.7951074	0.3672475
	300	-222	-45.735254	—
1000	70	-280	-0.4081798	—
	300	-280	-1.749342	—

TABLE 23 – Optimal results for $\theta_I = 0.4$

Looking at the above tables, one can now form an optimal limited stop-loss reinsurance contract under the expected value principle that maximizes the joint survival probability. Note that the first four columns of each table represent results for initial reinsurer wealth of 100, 200 and 300, due to our choice of u_I^C. In reality though, the definition of the joint survival probability function for a limited stop-loss reinsurance depends on u_R. Analyzing the values prompted us to realize that for all optimal combinations, $J_S(d_1^*, d_2^*) = F(d_2+u_I^C+P_I(d_1, d_2))$. This is due to each studied combination resulting in $d_1 \leq u_I + P_I(d_1, d_2)$ and $d_2 \leq u_R + P_R(d_1, d_2)$. Consequently, the maximized joint survival probability no longer depends on u_R. As a result, we can say that if $d_2 \leq min\{u_R + P_R(d_1, d_2)\}$ for $u_R = 100, 200, 300$, then the evaluated joint survival function is accurate for all considered values of initial reinsurer wealth. In reality, this is true for all cases in Tables 21 to 23.

It is important to note that negative values for d_1 imply an invalid solution and thus no joint survival probabilities are calculated for these retention vectors. Furthermore, this implies that no optimal solution exists for these particular values of d_2. Values of the net reinsurance and insurance premiums are therefore retracted from their respective tables, which are located in the section regarding Theorem 12.

11 Numerical Application of Theorem 11

As a counterpart to Theorem 9 and its focus on maximizing the joint survival probability, Theorem 11 concentrates on maximizing the joint profitable probability. Relative to our aggregate model, it states that :

A limited stop-loss reinsurance with retentions (d_1^*,d_2^*) is an optimal reinsurance in F^π and thus maximizes the joint profitable probability if any of the following three cases hold :

1. *If (d_1^*,d_2^*) $\in \bar{\Gamma}_1$ and satisfies $d_1 + 1.15S(x)d_2 = P_0$ (4)*
2. *If (d_1^*,d_2^*) $\in \bar{\Gamma}_2$ and satisfies $S(x) = \frac{1}{1.15} = \alpha_R = \frac{20}{23}$*
3. *If (d_1^*,d_2^*) $\in \bar{\Gamma}_3$.*

We consider :
- $\bar{\Gamma}_1$ consisting of all retention vectors (d_1, d_2) such that $0 \leq d_1 < P_0$, $d_2 > 0$ and $d_1 + d_2 > P_0$;
- $\bar{\Gamma}_3$ consisting of all retention vectors (d_1, d_2) such that $d_1 \geq 0$ and $d_2 = 0$.

We see in point two of three above that the equation of interest is independent of the retentions (d_1, d_2). As a matter of fact though, $S(x)$ never takes on the specific value of $\frac{20}{23}$. In our model, a maximum joint survival probability of approximately 0.8647 is attained when an aggregate loss of zero is observed. Consequently, $\bar{\Gamma}_2$ is disregarded and focus is turned to optimal retentions in $\bar{\Gamma}_1$ or $\bar{\Gamma}_3$ only.

We shall first focus on solutions in $\bar{\Gamma}_3$, as their existence is straightforward. As mentioned above, we will be studying retention vectors (d_1, d_2) where $d_1 \geq 0$ and $d_2 = 0$. Taking our equations for the net reinsurance and insurance premiums that were established in Theorem 9, we obtain :

$$P_R(d_1, 0) = 0$$

and

$$P_I(d_1, 0) = 200(1 + \theta_I).$$

Furthermore, $J_P(d_1, 0) = F(200(1 + \theta_I))$ for all solutions in $\bar{\Gamma}_3$. The following table displays the maximized values of the joint profitable probability function for each of the three insurer safety loadings considered, for all optimal retention vectors in $\bar{\Gamma}_3$:

θ_I	$J_P(d_1^*, 0)$
0	0.603501
0.15	0.6544017
0.4	0.7275728

Table 24 – Optimal results of joint profitable probability for retention vector in $\bar{\Gamma}_3$

As we can see, θ_I and $J_P(d_1^*, 0)$ are positively correlated and the highest joint profitable probability is thus obtained at the largest value of insurer safety loadings.

12 Numerical Application of Theorem 12

As a result of the observations made in Theorem 11, the concluding theorem will focus on only the first point established by Cai and Li (2012), which is the following :

The equation $d_1 + 1.15S(x)d_2 = P_0$ (4) of Theorem 11 has solutions in $\bar{\Gamma}_1$ if and only if,

$$S(P_0) = S\left(200(1+\theta_I) + S(x)(0.15 - \theta_I)d_2\right) < \alpha_R = \frac{20}{23}.$$

Even if we had not previously chosen to exclude solutions in $\bar{\Gamma}_2$ in Theorem's 11 analysis, Theorem 12 would have lead to the same conclusion. Since it has already been established that our model implies $S(0) < \alpha_R$ and the survival function is obviously decreasing in x, we see that this contradicts the second point of the concluding theorem in Cai and Li (2012) and once again we disregard solutions in the domain of $\bar{\Gamma}_2$. To find which retentions are optimal, we once again consider the cases of P_0 in Table 16 to see which vectors (d_1,d_2) are solutions to the above equation. By inspection, we see that as long as $P_0 > 0$ then $S(P_0) < \frac{20}{23}$. As a result, for each case of θ_I, we can find the minimum value of d_2 that satisfies the above equation.

For $\theta_I = 0$, $P_0 > 0$ implies $d_2 >$ *value less than 0* for all loss amounts, as observed in Table 16. Since $d_2 \in \Gamma$ implies it is at least zero, a solution (d_1, d_2) in $\bar{\Gamma}_1$ exists as long as $d_2 \geq 0$ and hence for all possible retentions $(d_1,d_2) \in \Gamma$. The following table demonstrates values of P_0 for different loss amounts and values of the second component of the retention vector, with an insurer safety loadings of 0. These calculations are needed to find solutions of d_1 in equation (4) for each respective case.

d_2	P_0 for $x=50$	P_0 for $x=100$	P_0 for $x=200$	P_0 for $x=300$	P_0 for $x=1000$
70	207.6748	206.3602	204.1629	202.5935	200.0434
150	216.446	213.629	208.9205	205.5575	200.093
250	227.41	222.715	214.8675	209.2625	200.155
300	232.892	227.258	217.841	211.115	200.186

TABLE 25 – P_0 corresponding to different values of d_2 for each loss amount with $\theta_I = 0$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	148.83	58.8448	148.830847	0.4607698
150	90.35	126.096	90.351815	0.7223607
250	17.25	210.16	17.253025	0.8928136
300	-19.3	252.192	-19.29637	—

TABLE 26 – Optimal results for $\theta_I = 0$ and $x = 50$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	157.601	48.75909955	157.600783	0.6147463
150	109.145	104.4837848	109.144535	0.6272971
250	48.575	174.1396413	48.574225	0.6425375
300	18.29	208.9675695	18.28907	0.6499726

TABLE 27 – Optimal results for $\theta_I = 0$ and $x = 100$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	172.24472	31.9181695	172.24507	0.6108893
150	140.5244	8.3960775	140.52515	0.5041726
250	100.874	113.9934625	100.87525	0.629404
300	81.0488	136.792155	81.0503	0.6344241

TABLE 28 – Optimal results for $\theta_I = 0$ and $x = 200$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	182.7107	19.88259	182.710791	0.6081159
150	162.9515	42.60555	162.951695	0.6133400
250	138.2525	71.00925	138.252825	0.6197949
300	125.903	85.2111	125.90339	0.6229912

TABLE 29 – Optimal results for $\theta_I = 0$ and $x = 300$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	199.7081	0.33529055	199.708443	0.6035785
150	199.3745	0.71847975	199.375235	0.6036671
250	198.9575	1.19746625	198.958725	0.6037779
300	198.749	1.4369595	198.75047	0.6038333

TABLE 30 – Optimal results for $\theta_I = 0$ and $x = 1000$

Once more, the value of d_1 for a loss amount of 50 and $d_2 = 300$ is discarded on the basis of not belonging to Γ. It can also be observed that when $\theta_I = 0$, as the amount of the loss rises, P_0 converges to the expected value of the loss.

As previously seen in Table 16, for $\theta_I = 0.15$, $P_0 = 230 > 0$ for all X values and thus a solution exists for all loss amounts and retention vectors $(d_1,d_2) \in \Gamma$. Therefore, combining the restrictions of Γ and $\bar{\Gamma}_1$, any combination of (d_1,d_2) such that $0 \leq d_1 < 230$, $d_2 > 0$ and $d_1 + d_2 > 230$ forms an optimal reinsurance in F^π that maximizes the joint profitable probability. Here, the previous set of inequalities make up what we can refer to as $\bar{\Gamma}_1^*$.

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	171.1552	58.8448	171.1554741	0.6544017
150	103.904	126.096	103.9045873	0.6544017
250	19.84	210.16	19.84097875	0.6544017
300	-22.192	252.192	-22.1908255	—

TABLE 31 – Optimal results for $\theta_I = 0.15$ and $x = 50$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	181.2408	48.7590995	181.2409005	0.6466404
150	125.516	104.4837848	125.5162153	0.6376034
250	55.86	174.1396414	55.86035875	0.6260547
300	21.032	208.9675692	21.0324305	0.6201746

TABLE 32 – Optimal results for $\theta_I = 0.15$ and $x = 100$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	193.2941	31.9181695	198.0818305	0.6467443
150	151.3445	68.3960775	161.6039225	0.6376034
250	98.9075	113.9934625	116.0065375	0.6260547
300	72.689	136.792155	93.207845	0.6201746

TABLE 33 – Optimal results for $\theta_I = 0.15$ and $x = 200$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	210.11741	19.88259	210.1174097	0.6544017
150	187.39445	42.60555	187.3944493	0.6544017
250	158.99075	71.00925	158.9907488	0.6544017
300	144.7889	85.2111	144.7888985	0.6544017

TABLE 34 – Optimal results for $\theta_I = 0.15$ and $x = 300$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	229.6647095	0.33529055	229.6647095	0.6544017
150	229.2815203	0.71847975	229.2815203	0.6544017
250	228.8025338	1.19746625	228.8025338	0.6544017
300	228.5630405	1.4369595	228.5630405	0.6544017

TABLE 35 – Optimal results for $\theta = 0.15$ and $x = 1000$

The first observation to be made is the negative value of d_1 when $d_2 = 300$ for a loss amount of 50. Since (-22.192,300) does not belong in Γ, this solution is discarded. The others form examples of possible optimal retention vectors that maximize the joint profitable probability of a limited stop-loss reinsurance under the expected value principle. In general, by taking any $d_2 \geq 0$, one can find $d_1 = P_0 - 1.15S(x)d_2$. If the resulting retention vector is an element of Γ, then an optimal reinsurance can be designed when the insurer and reinsurer safety loadings are of equal value. It is important to note that for Table 34, d_1^* and $P_I(d_1^*, d_2^*)$ are equivalent to four decimal places and thus are considered as equal. Consequently, these combinations are assigned the joint profitable probability definition of $F(d_1^* + P_R(d_1^*, d_2^*))$.

Moreover, for $\theta_I = 0.4$, results proving existence of solutions in $\bar{\Gamma}_1$ to the equation in question are not as straightforward. For each loss amount, P_0 can be represented as $280 - cd_2$ (with $c > 0$) and thus a solution in $\bar{\Gamma}_1$ exists as long as $d_2 < \frac{280}{c}$, assuring that $P_0 > 0$. Let us call this maximum value d_2', and the following table demonstrates this respective retention vector component for different loss amounts under an insurer's safety loadings of 0.4 :

Loss amount (x)	d_2'
50	1532.148
100	1849.039
200	2824.859
300	4534.413
1000	269230.769

TABLE 36 – Necessary and sufficient conditions for existence of solutions to (4) in $\bar{\Gamma}_1$ with $\theta_I = 0.4$

Next, we find values of P_0 for each case that will eventually be used in the calculations for d_1. The following are the results :

d_2	P_0 for $x = 50$	P_0 for $x = 100$	P_0 for $x = 200$	P_0 for $x = 300$	P_0 for $x = 1000$
70	267.2075	269.3999	273.0616	275.6775	279.9272
150	252.5875	257.2855	265.132	270.7375	279.844
250	234.3125	242.1425	255.22	264.5625	279.74
300	225.175	234.571	250.264	261.475	279.688
1500	5.875	52.855	131.32	187.375	278.44
1600	-12.4	37.712	121.408	181.2	278.336
1800	-48.95	7.426	101.584	168.85	278.128
1900	-67.225	-7.717	91.672	162.675	278.024
2800	-231.7	-144.004	2.464	107.1	277.088
2900	-249.975	-159.147	-7.448	100.925	276.984
4500	-542.375	-401.435	-166.04	2.125	275.32
4600	-560.65	-416.578	-175.952	-4.05	275.216
269000	-48879.75	-40454.67	-26383.28	-16330.75	0.24
269500	-48971.125	-40530.385	-26432.84	-16361.625	-0.28

TABLE 37 – P_0 corresponding to different values of d_2 for each loss amount with $\theta_I = 0.4$

It can be observed that Table 37 thoroughly supports Table 36. We consider values of d_2 that were studied for the two previous cases, along with ones slightly below and above the respective d_2' for each loss amount. For each x value, we can see that as $d_2 \longrightarrow d_2'$, the value of P_0 remains strictly positive. However, we observe that when $d_2 > d_2'$, the premium paid by the insured takes on a negative value. Since it is established that P_0 must be strictly positive as a consequence of Theorem 12, a negative value represents non-existence of a solution to equation (4) in $\bar{\Gamma}_1$ for $\theta_I = 0.4$. Consequently, we can now say that with an insurer's safety loadings of 0.4, as long as $d_1 \geq 0$ and $d_2 < d_2'$, solutions to equation (4) in $\bar{\Gamma}_1$ exist and lead to the maximization of the joint profitable probability. In Table 38 that follows, whenever (d_1,d_2) respect these restrictions, the retention vector is optimal. In fact, this implies that all pairings in this table such that $d_1 > 0$ are valid optimal solutions. From Table 38, we can thus retract optimal retention vectors and evaluate the maximized joint profitable probability function for all results.

d_2	d_1 for $x = 50$	d_1 for $x = 100$	d_1 for $x = 200$	d_1 for $x = 300$	d_1 for $x = 1000$
70	208.3627	220.6379	241.14342	255.79491	279.591907
150	126.4915	152.7955	196.7359	228.13195	279.125515
250	24.1525	67.9925	141.2265	193.55325	278.542525
300	-27.017	25.591	113.4718	176.2639	278.25103

TABLE 38 – (d_1^*,d_2^*) that maximize joint profitable probability for $\theta_I = 0.4$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	208.3627	58.8448	208.3631858	0.7101857
150	126.4915	126.096	126.492541	0.6892088
250	24.1525	210.16	24.154235	0.6612766
300	-27.017	252.192	-27.014918	—

TABLE 39 – Optimal results for $\theta_I = 0.4$ and $x = 50$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	220.6379	48.75909955	220.6410962	0.7132248
150	152.7955	104.4837848	152.802349	0.6960713
250	67.9925	174.1396413	68.003915	0.6734649
300	25.591	208.9675695	25.604698	0.6616656

TABLE 40 – Optimal results for $\theta_I = 0.4$ and $x = 100$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	241.14342	31.9181695	241.143098	0.6719429
150	196.7359	68.3960775	196.73521	0.5976325
250	141.2265	113.9934625	141.22535	0.4878221
300	113.4718	136.792155	113.47042	0.4259241

TABLE 41 – Optimal results for $\theta_I = 0.4$ and $x = 200$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	255.79491	19.88259	255.7951074	0.7217971
150	228.13195	42.60555	228.132373	0.7150725
250	193.55325	71.00925	193.553955	0.706479
300	176.2639	85.2111	176.264746	0.7021031

TABLE 42 – Optimal results for $\theta_I = 0.4$ and $x = 300$

d_2^*	d_1^*	$P_R(d_1^*, d_2^*)$	$P_I(d_1^*, d_2^*)$	$J_P(d_1^*, d_2^*)$
70	279.591907	0.33529055	279.5918202	0.7270317
150	279.125515	0.71847975	279.125329	0.7264122
250	278.542525	1.19746625	278.542215	0.7256361
300	278.25103	1.4369595	278.250658	0.7252474

TABLE 43 – Optimal results for $\theta = 0.4$ and $x = 1000$

13 Concluding Remarks

The procedures established in order to find optimal solutions in the last few sections are undoubtedly not the only possibilities. For example, in Theorem 10, results were based on a maximum value of initial insurer wealth found using equation (3). By changing this step and focusing on a different variable—say, P_0—we would have preceded differently and obtained a different set of optimal results under this new criteria. Aspects like this make a paper like Cai and Li (2012) interesting to work with, by providing us with ideas and theorems that are extremely well thought out but also flexible and open to interpretation on behalf of the reader.