Comparison of big Heegner points at intersections of Hida families

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Abstract

Given a Hida family $\mathcal{F}$ of tame level W, for a quadratic imaginary field K that satisfies the Heegner hypothesis for W, one can construct some classes in the Galois cohomology of a self-dual twist of Hida's big Galois representation associated to $\mathcal{F}$, which are called big Heegner points. When two families intersect, a natural question is to compare the big Heegner points at the intersection. We show that the specializations at intersections agree up to multiplication by some Euler factor that arise from the difference in the tame levels.

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Contents

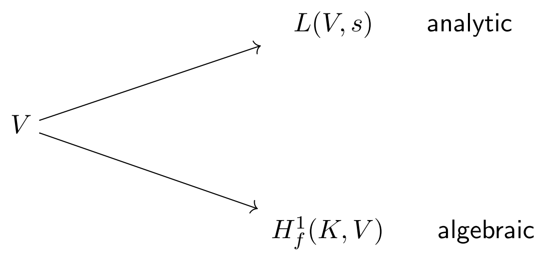
Chapter 1

Introduction

1.1 Motivation

1.1.1 Bloch-Kato conjecture

Let K be a number field, p a rational prime and and fix an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$. Let V be a p-adic geometric Galois representation of G_K [1], i.e. V is a finite dimensional $\mathbb{Q}_p$-vector space, as a G_K-module is unramified almost everywhere and de Rham at all places dividing p. To such a representation one can associate two different objects: a complex L-function $L(V, s)$ and the Bloch-Kato Selmer group $H^1_f(K, V)$.



The L-function is defined by an Euler product over the finite places of K

$$L(V, s) = \prod_{v \nmid p} \det\left(\mathsf{Id} - (\mathsf{Frob}_v^{-1} q_v^{-s})_{|V^{I_v}}\right)^{-1} \prod_{v|p} \det\left(\mathsf{Id} - (\varphi^{-1} q_v^{-s})_{|D_{\mathsf{crys}}(V_{|G_v})}\right)^{-1}$$

where q_v is the cardinality of the residue field at v, $\varphi = \phi^{f_v}$ with ϕ the crystalline Frobenius and f_v is the integer such that $q_v = p^{f_v}$. It is absolutely convergent for $\Re(s) \gg 0$ and conjecturally it has a moremorphic continuation to the whole complex plane; moreover if V is irreducible and $V \not\cong \mathbb{Q}_p(n)$ then $L(V, s)$ should have no poles.

The Bloch-Kato Selmer group is defined as

$$H^1_f(K, V) = \ker\left(H^1(K, V) \longrightarrow \prod_{v \nmid p} H^1(I_v, V) \prod_{v|p} H^1(D_v, V \otimes_{\mathbb{Q}_p} B_{\mathsf{crys}})\right)$$

where D_v is the decomposition group at v and I_v the inertia. It could be thought as a generalization of the Mordell-Weil group of an elliptic curve. A more concrete description, an element of $H^1(K, V)$ that corresponds to an extension

$$0 \to V \to W \to \mathbb{Q}_p \to 0$$

is in $H^1_f(K, V)$ if and only if the sequences

$$\begin{cases} 0 \to (V_{|D_v})^{I_v} \to (W_{|D_v})^{I_v} \to \mathbb{Q}_p \to 0 & v \nmid p \\ 0 \to D_{\mathsf{crys}}(V_{|D_v}) \to D_{\mathsf{crys}}(W_{|D_v}) \to D_{\mathsf{crys}}(\mathbb{Q}_p) \to 0 & v \mid p \end{cases}$$

[1] Fontaine and Mazur conjectured that every p-adic geometric Galois representation comes from geometry, i.e. it can be realized as a subquotient of a Tate twist of an étale cohomology group of a proper and smooth variety over K.

are all exact, where $D_{\mathsf{crys}}(V) := \big(V \otimes B_{\mathsf{crys}}\big)^{G_K}$.

The Bloch-Kato conjecture predicts a strong relationship between the two different objects attached to V. For the purposes of this introduction we will consider the following version

$$\text{(BK)} \qquad \operatorname{ord}_{s=0} L(V^*(1), s) = \dim_{\mathbb{Q}_p} H^1_f(K, V)$$

where $V^*(1)$ is the Kummer dual of V and we assume that V does not contain $\mathbb{Q}_p(n)$ as a subrepresentation. Notice that the conjecture becomes more interesting when V is self-dual, $V \cong V^*(1)$, because in that case only one representation is in the game. Furthermore, a priori $L(V, s)$ is not even defined at $s = 0$ and if it is, it is only through the process of analytic continuation. It amazes me how the process of analytic continuation can put together all the local information coming from the definition of $L(V, s)$ as an Euler product to produce a global invariant of V.

Examples 1.1.1. *Let E be an elliptic curve over $\mathbb{Q}$ and $V = V_p(E)$ the p-adic $G_{\mathbb{Q}}$-representation associated to it. Here $L(V, s) = L(E, s+1)$ where $L(E, s)$ is the usual L-function associated to an elliptic curve. By Weil pairing V is self-dual and if the p-torsion part of the Tate-Shafarevich group of E is finite, the Kummer map $k : E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to H^1_f(\mathbb{Q}, V_p(E))$ is an isomorphism of $\mathbb{Q}_p[G_{\mathbb{Q}}]$-modules. Thus* (BK) *becomes*

$$\text{(BSD)} \qquad \operatorname{ord}_{s=1} L(E, s) = \dim_{\mathbb{Q}_p} E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p.$$

More generally[2] *we can consider an eigenform f of level $\Gamma_0(N)$ and even weight k and $V = V_p(f)\big(\frac{k}{2}\big)$, where $V_p(f)$ is the Galois representation associated to f by Deligne.*

1.1.2 Heegner points and the Gross-Zagier formula

A way to state the modularity of an elliptic curve $E_{/\mathbb{Q}}$ is to say that there exists a modular parametrization defined over $\mathbb{Q}$

$$\Phi_N : X_0(N)_{/\mathbb{Q}} \longrightarrow E_{/\mathbb{Q}}$$

where $X_0(N)$ is the compact modular curve of level $\Gamma_0(N)$. The arithmetic applications of such a map come from the theory of complex multiplication. CM theory provides an analytic construction of class fields of quadratic imaginary fields and combined with the modular parametrization allows the construction of special points on E, called Heegner points, that are defined over such class fields.

Points on $X_0(N)(\mathbb{C})$ are equivalence classes of couples $\big(E(\mathbb{C}), \mathfrak{n}(\mathbb{C})\big)$ where $E_{/\mathbb{C}}$ is an elliptic curve and $\mathfrak{n}$ is cyclic subgroup of order N. To construct Heegner points, let $K/\mathbb{Q}$ be a quadratic imaginary that satisfies the Heegner hypothesis for N, i.e. there exists $\mathfrak{N} \subset \mathcal{O}_K$ ideal such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. For any positive integer c coprime with N, consider

$$Q_c = [(\mathbb{C}/\mathcal{O}_c, \mathfrak{N}_c^{-1}/\mathcal{O}_c))]$$

where $\mathcal{O}_c$ is the order of conductor c of K and $\mathfrak{N}_c = \mathfrak{N} \cap \mathcal{O}_c$. Q_c is called a Heegner point of conductor c on the modular curve. Furthermore

$$Q_c \in X_0(N)\big(H_c\big)$$

where H_c is the ring class field of K corresponding to $\mathcal{O}_c$, i.e. if $\widehat{K}^\times$ are the finite idels of K and if $\widehat{\mathcal{O}}_c$ is the closure of $\mathcal{O}_c$ in $\widehat{K}$, then H_c is the abelian extension of K corresponding to the group of norms $K^\times \widehat{\mathcal{O}}_c^\times$.

The great merit of the Gross-Zagier formula is to show the relation

$$\text{Heegner points} \quad \longleftrightarrow \quad \text{Derivatives of } L-\text{functions.}$$

It states that if we define $P_1 = \Phi_N(Q_1) \in E(H)$ and $P_K = \mathsf{Tr}_{H/K}(P_1) \in E(K)$ then the height of P_K is equal to the derivative of the L-function associated to E at 1 up to a non-zero constant .

$$\langle P_K, P_K \rangle \doteq L'(E/K, 1).$$

[2]After the work of Taylor, Wiles and others, all elliptic curves over $\mathbb{Q}$ are modular.

Recall that $L(E/K,s) = L(E,s)L(E',s)$ where E' is the quadratic twist of E over K.

To generalize the construction of Heegner points it is necessary to see them as cohomology classes. The modular parametrization induces a morphism of abelian varieties $\Phi_N : J_0(N) \to E$ from the Jacobian of $X_0(N)$ to the elliptic curve, as it is isomorphic to its own Jacobian; combining it with the Kummer morphism we get the following commutative diagram

$$\begin{array}{ccc} J_0(N)(K) & \xrightarrow{\Phi_N} & E(K) \\ \downarrow{\scriptstyle \mathsf{Kum}} & & \downarrow{\scriptstyle \mathsf{Kum}} \\ H^1\big(K, \mathsf{Ta}_p(J_0(N))\big) & \xrightarrow{\Phi_N^*} & H^1\big(K, \mathsf{Ta}_p(E)\big) \end{array}$$

that explains why big Heegner points are constructed as cohomology classes of some big Galois representation.

1.1.3 p-adic variation

The theme of p-adic variation is very important in modern number theory. The aim is to use the more arithmetic notion of p-adic topology to put objects of arithmetic interest in families and to be allowed to use limit arguments.

To be a little more precise, let M be an arithmetic object and suppose there exists a p-adic family $\{\mathcal{M}_k\}_{k\in\mathbb{C}_p}$ that varies in a well-behaved manner and for which there exists $k_0 \in \mathbb{C}_p$ such that $\mathcal{M}_{k_0} = M^*$ [3]. If we are able to show that a property $\mathcal{P}$ is true for $\mathcal{M}_k$ for k in a neighborhood of k_0, we may "take the limit for $k \to k_0$" and obtain that also M^* satisfies $\mathcal{P}$.

In this paper, the theme of p-adic variation is applied to the following situations.

$$p - \mathsf{adic\ variation} + \begin{cases} f \ \mathsf{eigenform} \\ V_p(f)\ \mathsf{Galois\ representation} \\ \mathsf{Heegner\ points} \end{cases} = \begin{cases} \mathsf{Hida\ family} \\ \mathsf{big\ Galois\ representation} \\ \mathsf{big\ Heegner\ points} \end{cases}$$

1.2 Setting

1.2.1 Hida families

Fix a prime $p \geq 5$ and embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $\mathcal{O}$ be the ring of integers of F a finite extension of $\mathbb{Q}_p$ and $\Lambda = \mathcal{O}[[\Gamma]]$ the Iwasawa algebra where $\Gamma = 1 + p\mathbb{Z}_p$.

Definition 1.2.1. *If A is a finite and flat, commutative Λ-algebra, a continuous $\mathcal{O}$-algebra map $A \to \overline{\mathbb{Q}}_p$ is arithmetic if the composition*

$$\Gamma \overset{\gamma\mapsto[\gamma]}{\longrightarrow} A^\times \longrightarrow \overline{\mathbb{Q}}_p^\times$$

has the form $\gamma \mapsto \psi(\gamma)\gamma^{r-2}$ for some integer $r \geq 2$ and some finite order character ψ of Γ. We denote the set of arithmetic maps of A by $\mathfrak{X}_{\mathsf{arith}}(A)$ and we may call it also the set of arithmetic points of A.

Λ-adic modular form

A Λ-adic modular form is a formal q-expansion

$$\mathcal{H} = \sum_{n\geq 0} A_n q^n \in \overline{\mathsf{Frac}\Lambda}[[q]]$$

and we let $R_h = \Lambda[\{A_n\}_{n\in\mathbb{N}}]$ be the Λ-algebra generated by the coefficients of $\mathcal{H}$, which we suppose to be finite and flat over Λ. For every arithmetic point $\nu \in \mathfrak{X}_{\mathsf{arith}}(R)$ we denote by

$$\nu(\mathcal{H}) = \sum_{n\geq 0} \nu(A_n) q^n \in \overline{\mathbb{Q}}_p[[q]]$$

[3] Usually we do not get back exactly M but a M^* which is M minus "the Euler factor at p"

the specialization of $\mathcal{H}$ at ν.

Definition 1.2.2. *Let W be a positive integer prime to p. A Hida family of tame level W is a Λ-adic modular form whose specializations at arithmetic points are p-ordinary, p-stabilized newforms of tame level W.*

Recall that a newform is p-ordinary if the pth coefficient of its q-expansion is a p-adic unit. Moreover, with p-stabilized newform of tame level W we mean either a newform for $\Gamma_1(Wp^r)$ with $r \geq 1$ or the p-stabilization of a p-ordinary newform on $\Gamma_1(W)$.

The utility of the theory of Hida families comes from the following theorem (Theorem 1.2 [4]).

Theorem 1.2.1. *Let $h \in S_t(\Gamma_0(Wp), \psi; \mathcal{O})$ be a p-ordinary, p-stabilized newform of tame level W, then there exists a Hida family $\mathcal{H}$ of tame level W whose specialization at an arithmetic point is h.*

For a given h, we can construct the Hida family passing through it considering an irreducible component of the universal ordinary Hecke algebra. Let's recall the construction for $h \in S_t(\Gamma_0(Wp), \omega^u; \mathcal{O})$ p-ordinary and p-stabilized.

Hecke algebra

We keep the normalizations considered in ([5]). Let $X_s(W)_{/\mathbb{Q}}$ be the complete modular curve classifying triples

$$(E, w, \pi) \tag{1.1}$$

where $E_{/\mathbb{Q}}$ is an elliptic curve, w is a cyclic subgroup of order W and $\pi \in E[p^s]$ is a point of exact order p^s. For $a \in \left(\mathbb{Z}/p^s\mathbb{Z}\right)^\times$, the diamond operator $\langle a\rangle$ acts on $X_s(W)_{/\mathbb{Q}}$ by

$$\langle a\rangle(E, w, \pi) = (E, w, a \cdot \pi).$$

For a prime ℓ, let $X_s(W;\ell)_{/\mathbb{Q}}$ be the modular curve classifying quadruples

$$(E, w, \pi, C)$$

where (E, w, π) is as in (1.1) and C is a cyclic subgroup of order ℓ such that $C \cap w = \{0\}$ and $C \cap \langle\pi\rangle = \{0\}$. There are finite morphisms $\alpha, \beta : X_s(W;\ell) \longrightarrow X_s(W)$ given by

$$(E, w, \pi) \xleftarrow{\alpha} (E, w, \pi, C) \xrightarrow{\beta} (E/C, w + C/C, \pi + C).$$

They define a correspondence

$$\begin{array}{ccc} & X_s(W;\ell) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ X_s(W) & \xrightarrow{\quad T_\ell \quad} & X_s(W) \end{array}$$

which acts on various cohomology groups, both covariantly: $T_{\ell,\mathsf{Alb}} = \beta_* \circ \alpha^*$ and contravariantly: $T_{\ell,\mathsf{Pic}} = \alpha_* \circ \beta^*$. As the operators T_ℓ behave differently from the others when $l \mid Wp$, we denote them by U_ℓ.

The diamond operators $\langle a\rangle : X_s(W) \longrightarrow X_s(W)$, $a \in \left(\mathbb{Z}/p^s\mathbb{Z}\right)^\times$, act on cohomology by

$$\langle a\rangle_{\mathsf{Alb}} = \langle a\rangle_*, \qquad \langle a\rangle_{\mathsf{Pic}} = \langle a\rangle^* = \langle a^{-1}\rangle_* = \langle a^{-1}\rangle_{\mathsf{Alb}}.$$

If $f \in S_k(\Gamma_0(W) \cap \Gamma_1(p^s), \psi, \overline{\mathbb{Q}}_p)$ is a p-adic modular form, where $\psi : \left(\mathbb{Z}/p^s\mathbb{Z}\right)^\times \longrightarrow \overline{\mathbb{Q}}_p$ is a Dirichlet character, the action of the diamond operators is given by

$$\langle a\rangle_{\mathsf{Alb}} f = \psi(a) f.$$

If $f = \sum_{n\geq 0} a_n q^n \in \overline{\mathbb{Q}}_p[[q]]$ is viewed as q-expansion, then

$$T_{\ell,\mathsf{Alb}} f = \sum_{n\geq 0} a_{n\ell} q^n + \ell^{k-1}\psi(\ell) \sum_{n\geq 0} a_n q^{n\ell}$$

and

$$U_{\ell,\mathsf{Alb}} f = \sum_{n\geq 0} a_{n\ell} q^n.$$

Let $\mathfrak{h}_{W,s}$ be the $\mathcal{O}$-algebra generated by the Hecke operators $T_{\ell,\mathsf{Alb}}$ $(\ell \nmid Wp)$, $U_{\ell,\mathsf{Alb}}$ $(\ell \mid Wp)$ and the diamond operators $\langle a\rangle_{\mathsf{Alb}}$, $a \in (\mathbb{Z}/p^s\mathbb{Z})^\times$, acting on the space of p-adic cusp forms $S_2(\Gamma_0(W)\cap\Gamma_1(p^s), \overline{\mathbb{Q}}_p)$. Hida's ordinary projector $e^{\mathsf{ord}} = \lim\ U_{p,\mathsf{Alb}}^{m!}$ defines an idempotent in each $\mathfrak{h}_{W,s}$, and these are compatible with the natural surjections $\mathfrak{h}_{W,s} \to \mathfrak{h}_{W,s-1}$ induced by restricting the action to the subspace $S_2(\Gamma_0(W)\cap \Gamma_1(p^{s-1}), \overline{\mathbb{Q}}_p)$. If we define $\mathfrak{h}^{\mathsf{ord}}_{W,s} = e^{\mathsf{ord}}\mathfrak{h}_{W,s}$, then

$$\mathfrak{h}^{\mathsf{ord}}_{W} = \lim_{\leftarrow, s} \mathfrak{h}^{\mathsf{ord}}_{W,s}$$

is the universal ordinary Hecke algebra. We make it into a $\mathcal{O}[[\mathbb{Z}_p^\times]]$-algebra by $[z] \mapsto \langle z\rangle_{\mathsf{Alb}}$ for $z \in \mathbb{Z}_p^\times$, and we find that it is finite and flat over Λ (Theorem 3.1 [3]). We are also interested in the quotient of $\mathfrak{h}^{\mathsf{ord}}_{W}$ that acts faithfully on the space of newforms and we denote it $\mathfrak{h}^{\mathsf{new}}_{W}$.

Recall that $\mathcal{O}[[\mathbb{Z}_p^\times]]$ has a decomposition $\prod_{i\in(\mathbb{Z}/(p-1)\mathbb{Z})^\times} \Lambda e_i$ where

$$e_i = \frac{1}{p-1} \sum_{\delta\in\mu_{p-1}} \omega^{-i}(\delta)[\delta]$$

is an idempotent.

A p-ordinary, p-stabilized newform $h = \sum_{n>0} a_n q^n \in S_t(\Gamma_0(Wp), \omega^u; \mathcal{O})$ determines an arithmetic map

$$h : \mathfrak{h}^{\mathsf{new}}_{W} \longrightarrow \mathcal{O}$$

characterized by $T_{l,\mathsf{Alb}} \mapsto a_l$ for $l \nmid Wp$, $U_{l,\mathsf{Alb}} \mapsto a_l$ for $l \mid Wp$ and

$$[\delta] \mapsto \omega^{t+u-2}(\delta) \qquad\qquad [\gamma] \mapsto \gamma^{t-2}$$

for $\delta \in \mu_{p-1}$, $\gamma \in \Gamma$. There is a decomposition of $\mathfrak{h}^{\mathsf{new}}_{W}$ as a direct sum of its localizations at maximal ideals, and we let $(\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}}$ be the unique local summand through which h factors. As $h(e_i) = 0$ for $i \neq t+u-2$ we must have

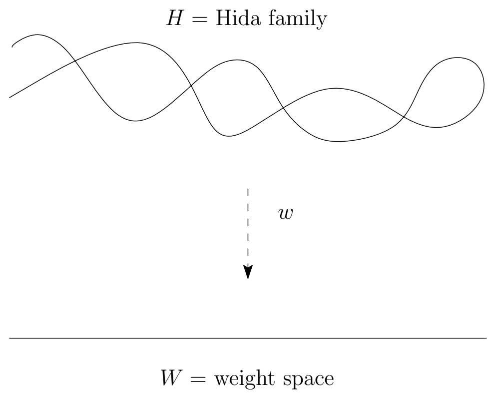
$$(\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}} = e_{t+u-2}(\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}}.$$

The localization of $(\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}}$ at the kernel of h is a discrete valuation ring. Therefore there is a unique minimal prime $\mathfrak{a}' \subset (\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}}$ such that h factors through the integral domain

$$R_h := (\mathfrak{h}^{\mathsf{new}}_{W})_{\mathfrak{w}}/\mathfrak{a}'.$$

The Hida family passing through h is defined by $\mathcal{H} = \sum_{n>0} T_n q^n \in R_h[[q]]$.

Geometrically, we can think of a Hida family as a branched cover of the weight space.



For a Hida family $\mathcal{F}$ of tame level W, we consider $H = \mathsf{Spec}((\mathfrak{h}_W^{\mathsf{new}})_{\mathfrak{w}})$ and $W = \mathsf{Spec}(\Lambda)$; the weight map $w : H \to W$ is induced by the structure morphism $\Lambda \to R$. Moreover, the arithmetic points cannot be branch points because the structure morphism is étale around them (Corollary 1.4 [4]).

1.2.2 Big Galois representation

We recall the construction of the big Galois representation that interpolates the contragredient cohomological representations attached to the eigenforms of the Hida family passing through h. We assume that the residual representation attached to h is absolutely irreducible and p-distinguished.

For $s, W > 0$ integers consider the tower

$$\ldots \longrightarrow X_s(W) \stackrel{\alpha}{\longrightarrow} X_{s-1}(W) \longrightarrow \ldots$$

with respect to the degeneracy maps described on the open modular curve by

$$(E, \mathfrak{n}, \pi) \mapsto (E, \mathfrak{n}, p \cdot \pi).$$

Letting J_s^W be the Jacobian variety of $X_s(W)$, the inverse limit of the system induced by Albanese functoriality,

$$\ldots \longrightarrow \mathsf{Ta}_p(J_s^W) \underset{\mathbb{Z}_p}{\otimes} \mathcal{O} \stackrel{\alpha_*}{\longrightarrow} \mathsf{Ta}_p(J_{s-1}^W) \underset{\mathbb{Z}_p}{\otimes} \mathcal{O} \longrightarrow \ldots$$

is equipped with an action of the algebra $\mathcal{O}[[\mathbb{Z}_p^\times]]$.

Let

$$\mathsf{Ta}_p^{\mathsf{ord}}(J_s^W) = e^{\mathsf{ord}}\big(\mathsf{Ta}_p(J_s^W) \underset{\mathbb{Z}_p}{\otimes} \mathcal{O}\big)$$

and define

$$\mathbf{Ta}_W^{\mathsf{ord}} = \varprojlim \big(\mathsf{Ta}_p^{\mathsf{ord}}(J_s^W), \alpha_*\big)$$

$$\mathbf{T}_h = \mathbf{Ta}_W^{\mathsf{ord}} \underset{\mathfrak{h}_W^{\mathsf{ord}}}{\otimes} R_h.$$

Proposition 1.2.1. *The R_h-module $\mathbf{T}_h$ is free of rank two. As a Galois representation it is unramified outside Wp and the arithmetic Frobenius of a prime $\ell \nmid Wp$ acts with characteristic polynomial $X^2 - T_{\ell,\mathsf{Alb}}X + \ell[\ell]$. Furthermore,*

$$R_h \cong \mathsf{Hom}_\Lambda(R_h, \Lambda)$$

as R_h-modules.

Proof. As we assume that the residual representation attached to h is absolutely irreducible and p-distinguished this follows from Theorem 2.1 [4] and for the Gorenstein property from Theorem 2.1 [7]. □

For later use let's compute the determinant of the residual representation attached to $\mathbf{T}_h$. First reduce modulo the prime ideal $\mathfrak{p}_h$ corresponding to h to get

$$\ell[\ell] \equiv \ell\omega^{t+u-2}(\delta_\ell)\gamma_\ell^{t-2} \pmod{\mathfrak{p}_h} \qquad \forall \ell \nmid Wp$$

where $\ell = \delta_\ell \cdot \gamma_\ell$ under the decomposition $\mathbb{Z}_p^\times \cong \mu_{p-1} \times 1 + p\mathbb{Z}_p$. The reduction modulo the maximal ideal $(p) + \mathfrak{p}_h$ is then equal to $\omega^{t+u-1}(\delta_\ell)$. As the Teichmuller character $\omega : G_\mathbb{Q} \to \mu_{p-1}$ is ramified at p, the determinant of the residual representation is ramified at p if and only if $t + u \not\equiv 1 \pmod{p-1}$.

Definition 1.2.3. *Factor the p-adic cyclotomic character $\epsilon_{\mathsf{cyc}} = \epsilon_{\mathsf{tame}} \cdot \epsilon_{\mathsf{wild}}$ according to the decomposition of $\mathbb{Z}_p^\times$ as $\mu_{p-1} \times 1 + p\mathbb{Z}_p$. Let $h \in S_t(\Gamma_0(Wp), \omega^u; \mathcal{O})$ and define the critical character $\Theta_h : G_\mathbb{Q} \longrightarrow \Lambda^\times$ associated to h by*

$$\Theta_h = \epsilon_{\mathsf{tame}}^{\frac{t+u-2}{2}} \cdot [\epsilon_{\mathsf{wild}}^{1/2}]$$

where $\epsilon_{\mathsf{wild}}^{1/2}$ is the unique square root of ϵ_{wild} taking values in $1 + p\mathbb{Z}_p$.

Remark 1.2.1. *As noted by Howard, the critical character is defined up to multiplication by the quadratic character of conductor p. Therefore, if $f \in S_k(\Gamma_0(Np), \omega^j)$ and $g \in S_r(\Gamma_0(Mp), \omega^s)$ are p-ordinary p-stabilized newforms such that $k + j \equiv r + s \pmod{p-1}$, we can choose $\Theta_f = \Theta_g$.*

1.2.3 Big Heegner points

As before, let W be a positive integer and $h \in S_u(\Gamma_0(Wp), \omega^u; \mathcal{O})$ a p-ordinary, p-stabilized newform of tame conductor W whose residual representation is absolutely irreducible and p-distinguished. The $R_h[G_\mathbb{Q}]$-module $\mathbf{T}_h$ attached to the Hida family passing through h is not self-dual, but if we twist it by Θ^{-1}, the inverse of the critical character associated to h, $\Theta : G_\mathbb{Q} \to \Lambda^\times$, then the twist $\mathbf{T}_h^\dagger$ is self-dual, in fact its determinant is given by the character

$$[\epsilon_{\mathsf{cyc}}]\epsilon_{\mathsf{cyc}} \cdot \Theta^{-2} = [\epsilon_{\mathsf{cyc}}]\epsilon_{\mathsf{cyc}} \cdot [\epsilon_{\mathsf{cyc}}]^{-1} = \epsilon_{\mathsf{cyc}}$$

as $\Theta^2 = [\epsilon_{\mathsf{cyc}}]$ in $R_h^\times$ (as noted in (2) page 5 [2]). Now we are in the favourable setting to talk about p-adic variation of Heegner points. Let $K/\mathbb{Q}$ be a quadratic imaginary field that satisfies the Heegner hypothesis for W and such that its discriminant is prime to W. Let $\mathcal{O}_c$ be the order of conductor c of K and H_c the ring class field of K corresponding to $\mathcal{O}_c$, i.e. if $\widehat{K}^\times$ are the finite idels of K and if $\widehat{\mathcal{O}}_c$ is the closure of $\mathcal{O}_c$ in $\widehat{K}$, then H_c is the abelian extension of K corresponding to the group of norms $K^\times\widehat{\mathcal{O}}_c^\times$.

Then for every positive integer c prime to W, B. Howard constructed in [2] a cohomology class

$$\mathfrak{X}_{h,c} \in \tilde{H}_f^1(H_c, \mathbf{T}_h^\dagger)$$

that lives in J. Nekovar's extended Selmer group of $\mathbf{T}_h^\dagger$ [4] and has the property that when it is specialized at an arithmetic prime of weight 2 and trivial character it essentially recovers the Kummer image of a classical Heegner point. The Bloch-Kato Selmer group is a quotient of J. Nekovar's extended Selmer group, the reason is that it is modelled on a complex L-function rather than on a p-adic one, and sometimes "trivial zeros" arise from the process of p-adic interpolation. Anyway, if we specialize $\mathbf{T}_h^\dagger$ at an arithmetic prime of even weight and not exceptional (Definition 2.4.3 [2]), then the extended Selmer group and the Bloch-Kato Selmer group coincide ((22), (23) above Proposition 2.4.5 [2]).

Construction

Fix a positive integer c prime to W. Let K be an imaginary quadratic field with an ideal $\mathfrak{W}$ of $\mathcal{O}_K = \mathbb{Z} + \varpi\mathbb{Z}$ such that $\mathcal{O}_K/\mathfrak{W} \cong \mathbb{Z}/W\mathbb{Z}$. For each $s \geq 0$ let $\mathcal{O}_{cp^s} = \mathbb{Z} + cp^s\varpi\mathbb{Z}$ be the order of conductor cp^s and consider

$$Q_{c,s} = [(\mathbb{C}/\mathcal{O}_{cp^s}, \mathfrak{W}_{c,s}^{-1}/\mathcal{O}_{cp^s}, [c\varpi])] \in X_s(W)$$

[4] B. Howard actually proves that they live in the Greenberg Selmer group and that in this case Nekovar's and Greenberg's coincide. Proposition 2.4.5 and (21) page 18 [2].

where $\mathfrak{W}_{c,s} = \mathfrak{W} \cap \mathcal{O}_{cp^s}$.

By the theory of complex multiplication $Q_{c,s} \in X_s(W)(L_{c,s})$, where $L_{c,s} = H_{cp^s}(\mu_{p^s})$ (Corollary 2.2.2 [H]).

The natural short exact sequence

$$0 \to J_s^W(L_{c,s}) \otimes \mathcal{O} \to \mathsf{Pic}(X_s(W)_{/L_{c,s}}) \otimes \mathcal{O} \stackrel{\deg}{\to} \mathcal{O} \to 0$$

is Hecke equivariant, and as the action of $U_{p,\mathsf{Alb}}$ on $\mathcal{O}$ is by $p = \deg(U_p)$ there is an induced isomorphism taking the ordinary parts, that we abbreviate as

$$J_s^W(L_{c,s})^{\mathsf{ord}} \cong \mathsf{Pic}(X_s(W)_{/L_{c,s}})^{\mathsf{ord}}.$$

Viewing $Q_{c,s}$ as a divisor on $X_s(W)_{/L_{c,s}}$, we obtain an element

$$e^{\mathsf{ord}} Q_{c,s} \in J_s^W(L_{c,s})^{\mathsf{ord}}.$$

Moreover, if we project to the component where μ_{p-1} acts through ω^{t+u-2} through the idempotent e_{t+u-2} we get a point

$$y_{c,s} = e_{t+u-2} e^{\mathsf{ord}} Q_{c,s} \in J_s^W(L_{c,s})^{\mathsf{ord}}$$

such that

$$y_{c,s}^\sigma = \Theta(\sigma) \cdot y_{c,s} \tag{1.2}$$

for all $\sigma \in \mathsf{Gal}(\overline{\mathbb{Q}}/H_{cp^s})$. Let $\mathfrak{h}_{2,s}^{\mathsf{ord},\dagger}$ denote $\mathfrak{h}_{2,s}^{\mathsf{ord}}$ as a module over itself but with $G_{\mathbb{Q}}$ acting through the character Θ^{-1}, and let $\zeta_s \in \mathfrak{h}_{2,s}^{\mathsf{ord},\dagger}$ be the element corresponding to $1 \in \mathfrak{h}_{2,s}^{\mathsf{ord}}$ under the identification of underlying $\mathfrak{h}_{2,s}^{\mathsf{ord}}$-modules. For any $\mathfrak{h}_{2,s}^{\mathsf{ord}}$-module M we abbreviate

$$M \otimes \zeta_s = M \otimes_{\mathfrak{h}_{2,s}^{\mathsf{ord}}} \mathfrak{h}_{2,s}^{\mathsf{ord},\dagger}.$$

The equality (1.2) implies that

$$y_{c,s} \otimes \zeta_s \in H^0\big(H_{cp^s}, J_s^W(L_{c,s})^{\mathsf{ord}} \otimes \zeta_s\big)$$

and we define the corestriction

$$x_{c,s} = \mathsf{Cor}_{H_{cp^s}/H_c}\big(y_{c,s} \otimes \zeta_s\big) \in H^0\big(H_c, J_s^W(L_{c,s})^{\mathsf{ord}} \otimes \zeta_s\big).$$

We now construct a twisted Kummer map

$$\mathsf{Kum}_s : H^0\big(H_c, J_s^W(L_{c,s})^{\mathsf{ord}} \otimes \zeta_s\big) \longrightarrow H^1\big(\mathfrak{G}_c, \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W) \otimes \zeta_s\big)$$

where $\mathfrak{G}_c = \mathsf{Gal}\big(H_c^{(Wp)}/H_c\big)$ is the Galois group of the maximal extension of H_c unramified outside Wp.

Suppose $A \otimes \zeta_s \in J_s^W(L_{c,s})^{\mathsf{ord}} \otimes \zeta_s$ is fixed by the action of $\mathsf{Gal}(\overline{\mathbb{Q}}/H_c)$. For each $n > 0$ choose a finite extension $L/L_{c,s}$ contained in $H_c^{(Wp)}$ large enough so that there is a point $A_n \in J_s^W(L)^{\mathsf{ord}}$ with $[p^n]A_n = A$. Abbreviating

$$J_s^W[p^n]^{\mathsf{ord}} = e^{\mathsf{ord}}(J_s^W[p^n] \otimes \mathcal{O}),$$

for $\sigma \in \mathfrak{G}_c$

$$b_n(\sigma) = (A_n \otimes \zeta_s)^\sigma - A_n \otimes \zeta_s$$

defines a 1-cocycle with values in

$$J_s^W[p^n]^{\mathsf{ord}} \otimes \zeta_s \cong \big(\mathsf{Ta}_p^{\mathsf{ord}}(J_s^W)/p^n \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W)\big) \otimes \zeta_s$$

whose image in cohomology does not depend on the choice of L or A_n. Taking the inverse limit over n yelds an element

$$\mathsf{Kum}_s(A \otimes \zeta_s) = \lim_{\leftarrow,\ n} b_n \in H^1\big(\mathfrak{G}_c, \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W) \otimes \zeta_s\big).$$

The twisted Kummer map is both $\mathfrak{h}_{2,s}^{\mathsf{ord}}$ and $G_{\mathbb{Q}}$ equivariant. Define

$$\mathfrak{X}_{c,s} = \mathsf{Kum}_s(x_{c,s}) \in H^1\big(\mathfrak{G}_c, \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W) \otimes \zeta_s\big) \tag{1.3}$$

Lemma 2.2.4 [2] says that the map

$$\alpha_* : \mathsf{Ta}_p^{\mathsf{ord}}(J_{s+1}^W) \otimes \zeta_{s+1} \longrightarrow \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W)$$

acts on the cohomology classes (1.3) by $\alpha_*(\mathfrak{X}_{c,s+1}) = U_p \cdot \mathfrak{X}_{c,s}$ therefore we can consider the limit

$$\lim_{\leftarrow,\ s} U_p^{-s}\mathfrak{X}_{c,s} \in H^1\big(\mathfrak{G}_c, \mathbf{Ta}_W^{\mathsf{ord}} \otimes \zeta\big) \tag{1.4}$$

which belongs to that cohomology group because cohomology and inverse limit commute as the group $\mathfrak{G}_c$ satisfies a property of p-finitness, i.e. for all open subgroups $U \subset \mathfrak{G}_c$ and $\forall i \geq 0$ the set $H^i(U, \mathbb{Z}/p\mathbb{Z})$ is finite ([6]).

Define the big Heegner point of conductor c

$$\mathfrak{X}_{h,c} \in H^1\big(H_c, \mathbf{T}_h^{\dagger}\big)$$

to be the image of (1.4) after inflation to H_c-cohomology and projection $\mathbf{Ta}_W^{\mathsf{ord}} \otimes \zeta \longrightarrow \mathbf{T}_h^{\dagger}$.

1.3 Overview of the result

We denote by $\omega : \big(\mathbb{Z}/p\mathbb{Z}\big)^{\times} \to \mu_{p-1}$ the Teichmuller character. Let N, M be positive integers and $p \nmid NM$. Let

$$f = \sum_{n>0} a_n q^n \in S_k(\Gamma_0(Np), \omega^j), \qquad g = \sum_{n>0} b_n q^n \in S_r(\Gamma_0(Mp), \omega^s)$$

be normalized eigenforms of weight greater or equal to 2. Fix a finite extension $F/\mathbb{Q}_p$ which contains all Fourier coefficients of f and g and let $\mathcal{O}$ be its ring of integers. We assume that they are p-ordinary p-stabilized newforms of conductors divisible by N and M respectively. Moreover, we suppose the residual representations attached to them to be absolutely irreducible and p-distinguished. Fix a quadratic imaginary field K where all the prime divisors of NM are split.

Let

$$\mathcal{F} = \sum_{n>0} A_n q^n \in R_f[[q]], \qquad \mathcal{G} = \sum_{n>0} B_n q^n \in R_g[[q]]$$

be the Hida families passing through f and g. We suppose that the two families intersect, i.e. there are continuous $\mathcal{O}$-algebra homomorphisms $\nu : R_f \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p}$, $\nu' : R_g \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p}$ such that $\nu(\mathcal{F}) = \nu'(\mathcal{G})$. We recall that under our hypothesis there are big Galois representations $\mathbf{T}_f, \mathbf{T}_g$ free of rank two over R_f, R_g respectively. The hypothesis that the two branches intersect implies that the residual representations are isomorphic. To assure that the residual representation has a unique p-stabilization we ask that $k + j \equiv r + s \not\equiv 1 \pmod{p-1}$. It implies that the determinant of the residual representation is ramified at p.

We consider the twists $\mathbf{T}_f^{\dagger}, \mathbf{T}_g^{\dagger}$ by the same critical character Θ, which makes sense as the definition of the critical character essentially depends only on the congruence classes of $k + j$ and $r + s \pmod{p-1}$. For every positive integer c prime to NM B.Howard's construction, recalled in the previous section, provides cohomology classes

$$\mathfrak{X}_{f,c} \in H^1\big(H_c, \mathbf{T}_f^{\dagger}\big), \qquad \mathfrak{X}_{g,c} \in H^1\big(H_c, \mathbf{T}_g^{\dagger}\big).$$

Our objective is to compare them when they are specialized at the intersection of the families $\mathcal{F}$ and $\mathcal{G}$. For an $\mathcal{O}_F$-algebra homomorphism $\nu : R \to \mathcal{O}_{\overline{\mathbb{Q}}_p}$ we set $T_\nu^{\dagger} := \mathbf{T}^{\dagger} \underset{R,\nu}{\otimes} \mathcal{O}_{\overline{\mathbb{Q}}_p}$ the specialization of $\mathbf{T}^{\dagger}$ at ν and we denote again by ν the map induced in cohomology $\nu : H^1\big(H_c, \mathbf{T}^{\dagger}\big) \to H^1\big(H_c, T_\nu^{\dagger}\big)$. When ν, ν' correspond to the intersection point of the families, we show that there exists an isomorphism of $G_{\mathbb{Q}}$-modules (Corollary 3.2.1)

$$\Upsilon : T_{f,\nu}^{\dagger} \xrightarrow{\sim} T_{g,\nu'}^{\dagger}$$

and that (Corollary 4.1.1) for all positive integer c prime to NM the induced isomorphism in cohomology

$$\Upsilon_c : H^1\big(H_c, T^{\dagger}_{f,\nu}\big) \xrightarrow{\sim} H^1\big(H_c, T^{\dagger}_{g,\nu'}\big)$$

is such that

$$\nu\left(\prod_{\text{some prime } \ell} E_\ell(f)\mathfrak{X}_{f,c}\right) \mapsto \nu'\left(\prod_{\text{some prime } \ell'} E_{\ell'}(g)\mathfrak{X}_{g,c}\right)$$

where $E_\ell(f), E_{\ell'}(g)$ are some Euler factor that depend on the Hida families passing through f and g and the respective tame levels.

1.3.1 Future perspectives

Howard's construction of big Heegner points was generalized by M. Longo and S. Vigni to a general quaternionic setting over $\mathbb{Q}$ in

[*] M. Longo, S. Vigni, *Quaternion algebras, Heegner points and the arithmetic of Hida families,* Manuscripta Math. 135, (2011), 273-328.

It would be interesting to extend the result of this memoire to the same setting. On one hand, the extension to the indefinite Shimura curve case over $\mathbb{Q}$, where we allow an even number of prime factors of the tame level to be inert in the quadratic imaginary field K, might be useful to prove the analogue of B. Howard's "Horizontal non-vanishing conjecture" (Conjecture 10.3 [*]).

On the other hand, in the definite case over $\mathbb{Q}$, where we allow an odd number of prime factors of the tame level to be inert in K, M. Longo and S. Vigni's construction provides theta elements instead of cohomology classes which can be shown to give rise to anticyclotomic p-adic L-functions and the extension of our result could be used to control the variation of anticyclotomic Iwasawa invariants in Hida families.

Chapter 2

Modular deformation theory

We fix a representation $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$, where $\mathbb{F}$ is a finite field of characteristic p and $\overline{V}$ is the two dimensional $\mathbb{F}$-vector space on which $\bar{\rho}$ acts. Let us recall some definitions.

Definition 2.0.1. .

- *$\bar{\rho}$ is p-ordinary if $\bar{\rho}$ restricted to $G_{\mathbb{Q}_p}$ has an unramified quotient of dimension one over $\mathbb{F}$.*
- *A p-stabilization of $\bar{\rho}$ is a choice of a one-dimensional quotient of $\overline{V}$ on which the $G_{\mathbb{Q}_p}$-action is unramified.*
- *If $\bar{\rho}_{|G_{\mathbb{Q}_p}}$ is reducible, then we say that $\bar{\rho}$ is p-distinguished if the semi-semplification of $\bar{\rho}_{|G_{\mathbb{Q}_p}}$ is non-scalar.*

We assume that $\bar{\rho}$ is irreducible, odd, p-ordinary and p-distinguished and we fix a p-stabilization. Finally we suppose that $\bar{\rho}$ is modular and that $\mathbb{F}$ is equal to the field generated by the traces of $\bar{\rho}$.

Let $N(\bar{\rho})$ denote the tame conductor of $\bar{\rho}$. If $\ell \neq p$ is a prime, write

$$m_\ell = \dim_k \overline{V}_{I_\ell}$$

and for any finite set of primes Σ that does not contain p, write

$$N(\Sigma) = N(\bar{\rho}) \prod_{\ell \in \Sigma} \ell^{m_\ell}$$

.

Remark 2.0.1. *Let L be a field of characteristic zero, $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(L)$ a Galois representation of tame conductor W and V the L-vector space on which ρ acts; suppose $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ is its residual representation. Then we may write $W = \prod_{\ell|W} \ell^{n_\ell(\rho)}$ and set $\Sigma = \{\ell \text{ prime} | \ \ell \mid W\}$. If we let as above, $N(\Sigma) = \prod_{\ell \in \Sigma_W} \ell^{n_\ell(\bar{\rho}) + \dim \overline{V}_{I_\ell}}$, we have that $W \mid N(\Sigma)$ and they have the same prime factors, but they are not equal in general. In fact, the exponent of the conductor at a prime $\ell \mid W$ is*

$$n_\ell(\rho) = n_\ell(\bar{\rho}) + \left(\dim \overline{V}_{I_\ell} - \dim V_{I_\ell}\right)$$

and the quantity between the parentesis is positive, but it might happen that $\dim V_{I_\ell} > 0$.

Definition 2.0.2. *For any level W, we let $\mathfrak{h}_W^{'}$ denote the Λ-subalgebra of $\mathfrak{h}_W^{\mathsf{ord}}$ generated by the Hecke operators $T_{\ell,\mathsf{Alb}}$ for $\ell \nmid Wp$, together with the operator $U_{p,\mathsf{Alb}}$.*

If $U \mid W$ then restricting the action of the prime to W Hecke operators to p-ordinary forms of level dividing U yields a surjective map of Λ-algebras $\mathfrak{h}_W^{'} \longrightarrow \mathfrak{h}_U^{'}$. Composing it with $\mathfrak{h}_U^{'} \hookrightarrow \mathfrak{h}_U^{\mathsf{ord}} \longrightarrow \mathfrak{h}_U^{\mathsf{new}}$ gives a map $\mathfrak{h}_W^{'} \longrightarrow \mathfrak{h}_U^{\mathsf{new}}$ and taking the product over all divisors U of W we get

$$\mathfrak{h}_W^{'} \longrightarrow \prod_{U|W} \mathfrak{h}_U^{\mathsf{new}}.$$

This map is an isomorphism after tensoring by the fraction field $\mathcal{L}$ of Λ (Proposition 2.3.2 [1]) and this allows us to associate to $\mathfrak{h}'_W \underset{\Lambda}{\otimes} \mathcal{L}$ the Galois representation

$$\mathsf{V} = \left(\prod_{U|W} \mathbf{Ta}_U^{\mathsf{ord}} \underset{\mathfrak{h}_U^{\mathsf{ord}}}{\otimes} \mathfrak{h}_U^{\mathsf{new}} \right) \underset{\Lambda}{\otimes} \mathcal{L}.$$

For every maximal ideal $\mathfrak{m}$ of $\mathfrak{h}'_W$ we can consider the Galois representation of the local factor $\mathsf{V}_{\mathfrak{m}}$ at $\mathfrak{m}$. If the residual representation is absolutely irreducible then there is a uniquely determined representation $\mathsf{T}_{\mathfrak{m}}$ defined over $(\mathfrak{h}'_W)_{\mathfrak{m}}$ such that $\mathsf{T}_{\mathfrak{m}} \underset{\Lambda}{\otimes} \mathcal{L} \cong \mathsf{V}_{\mathfrak{m}}$.

Theorem 2.0.1. *There is a unique maximal ideal $\mathfrak{m}$ of $\mathfrak{h}'_{N(\Sigma)}$ such that $\mathsf{T}_{\mathfrak{m}}/\mathfrak{m}\mathsf{T}_{\mathfrak{m}}$, with its canonical p-stabilization (Definition 2.2.11 [1]), is isomorphic to $\bar{\rho}$, with its p-stabilization.*

Proof. This is Theorem 2.4.1 [1]. ☐

Proposition 2.0.1. *If $\mathfrak{m}$ denotes the maximal ideal of $\mathfrak{h}'_{N(\Sigma)}$ of the preceeding theorem, then there is a unique maximal ideal $\mathfrak{n}$ of $\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}$ satisfying the following conditions:*

- $\mathfrak{n}$ *lifts* $\mathfrak{m}$.
- $U_{\ell,\mathsf{Alb}} \in \mathfrak{n}$ *for each* $\ell \in \Sigma$.
- *The natural map of localizations* $(\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}} \to (\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ *is an isomorphism of* Λ*-algebras.*

In particular, $(\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}}$ is a finite flat Λ-algebra. Also, the image of $U_{\ell,\mathsf{Alb}}$ in the localization $(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ vanishes for each $\ell \in \Sigma$.

Proof. This is Proposition 2.4.2 [1]. ☐

Definition 2.0.3. *We let $\mathfrak{h}_\Sigma(\bar{\rho})$ (or simply $\mathfrak{h}_\Sigma$ when $\bar{\rho}$ is understood) denote the localization of $\mathfrak{h}'_{N(\Sigma)}$ at the maximal ideal whose existence is guaranteed by Theorem* 2.0.1 *above.*

2.0.2 Branches

Let $\mathfrak{a}'$ denote a minimal prime of $\mathfrak{h}^{\mathsf{new}}_W$ that is contained in a maximal ideal whose corresponding residual Galois representation is isomorphic to $\bar{\rho}$ with its given p-stabilization and such that $W \mid N(\Sigma)$.

We call such a prime an admissible minimal prime associated to $\mathfrak{h}_\Sigma$.

Proposition 2.0.2. *Let $\mathfrak{a}' \subset \mathfrak{h}^{\mathsf{new}}_W$ be an admissible minimal prime associated to $\mathfrak{h}_\Sigma$. Then there is a unique minimal prime $\mathfrak{a}$ of $\mathfrak{h}_\Sigma$ such that the following diagram commutes*

$$\begin{array}{ccccc} \mathfrak{h}_\Sigma & \longrightarrow & \mathfrak{h}'_{N(\Sigma)} & \longrightarrow & \prod_{U|N(\Sigma)} \mathfrak{h}^{\mathsf{new}}_U \\ \downarrow & & & & \downarrow \\ \mathfrak{h}_\Sigma/\mathfrak{a} & & \longrightarrow & & \mathfrak{h}^{\mathsf{new}}_W/\mathfrak{a}' \end{array} \tag{2.1}$$

.

Proof. By Proposition 2.5.2 [1] the only thing to show is that $\mathfrak{a}'$ is contained in the local component corresponding to the maximal ideal $\mathfrak{m}$ that defines $\mathfrak{h}_\Sigma$. Let $\mathfrak{w} \subset \mathfrak{h}^{\mathsf{new}}_W$ be the unique maximal ideal that contains $\mathfrak{a}'$ and let $\mathfrak{m}'$ be its inverse image under $\mathfrak{h}'_{N(\Sigma)} \to \mathfrak{h}^{\mathsf{new}}_W$, $\mathfrak{m}'$ is maximal because the morphism is finite. Consider the local component $V_{\mathfrak{m}'}$ of the representation V attached to $\mathfrak{h}'_{N(\Sigma)} \otimes_\Lambda \mathcal{L}$. We have the projection

$$V_{\mathfrak{m}'} \longrightarrow \left(\mathbf{Ta}^{\mathsf{ord}}_W \otimes_{\mathfrak{h}^{\mathsf{ord}}_W} \mathfrak{h}^{\mathsf{new}}_W\right)_{\mathfrak{w}} \otimes_\Lambda \mathcal{L}$$

which induces an isomorphism when we take the residual representations. By hypothesis the residual representation associated to $\mathfrak{w}$ is isomorphic to $\bar{\rho}$ with its given p-stabilization, which implies that is the case for the one associated to $\mathfrak{m}'$. Therefore $\mathfrak{m} = \mathfrak{m}'$ by uniqueness of Theorem 2.0.1. ☐

Remark 2.0.2. *Set* $\mathfrak{h}(\mathfrak{a}) := \mathfrak{h}_\Sigma/\mathfrak{a}$ *and* $\mathfrak{h}(\mathfrak{a})^\circ := \mathfrak{h}_W^{\mathsf{new}}/\mathfrak{a}'$. *The embedding described in* (2.1) *is an embedding of local domains*

$$\mathfrak{h}(\mathfrak{a}) \longrightarrow \mathfrak{h}(\mathfrak{a})^\circ.$$

Note that the target is local since it a complete finite Λ*-algebra and hence a product of local rings. Being a domain, it must be local.*

For future use, we conclude this section introducing the Euler factors.

Definition 2.0.4. *Let* $\mathfrak{a}'$ *be an admissible minimal prime of tame conductor* W *associated to* $\mathfrak{h}_\Sigma$*, and let* $\mathfrak{h}(\mathfrak{a})^\circ$ *be the new quotient corresponding to it. For each prime* $\ell \neq p$*, define the reciprocal Euler factor* $E_\ell(\mathfrak{a}', X) \in \mathfrak{h}(\mathfrak{a})^\circ[X]$ *via:*

$$E_\ell(\mathfrak{a}', X) := \begin{cases} 1 - (U_{\ell,\mathsf{Alb}} \mod \mathfrak{a}')X & \ell \mid W \\ 1 - (T_{\ell,\mathsf{Alb}} \mod \mathfrak{a}')X + \ell\langle\ell\rangle X^2 & \ell \nmid W \end{cases}$$

Chapter 3

Comparison morphism

In this section we compare the Galois representations

$$\mathbf{T}^{\Sigma}_{\mathfrak{a}'} := \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} \mathfrak{h}(\mathfrak{a})^{\circ}, \qquad \mathbf{T}_{\mathfrak{a}'} := \mathbf{Ta}^{\mathsf{ord}}_{W} \underset{\mathfrak{h}^{\mathsf{ord}}_{W}}{\otimes} \mathfrak{h}(\mathfrak{a})^{\circ}$$

depending on an admissible minimal prime $\mathfrak{a}'$ of tame level W associated to some $\mathfrak{h}_{\Sigma}$. Recall that $\mathfrak{h}_{\Sigma}$ is the localization of $\mathfrak{h}'_{N(\Sigma)}$ at a maximal ideal $\mathfrak{m}$ which has a unique lift $\mathfrak{n}$ to $\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}$ and such that the map of localization $\mathfrak{h}_{\Sigma} \to (\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ is an isomorphism. Moreover, the admissible minimal prime $\mathfrak{a}'$ is contained in a unique maximal ideal $\mathfrak{w}$ of $\mathfrak{h}^{\mathsf{new}}_{W}$.

Motivation

We want to be able to consider a commutative diagram of the following shape to compare big Heegner points.

$$\begin{array}{ccccc} & & \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} \mathfrak{h}^{\dagger}_{\Sigma} & & \\ & \swarrow & & \searrow & \\ \mathbf{T}^{\dagger}_{f} & & & & \mathbf{T}^{\dagger}_{g} \\ \downarrow & & & & \downarrow \\ T^{\dagger}_{f,\nu} & & \overset{\Upsilon}{\dashrightarrow} & & T^{\dagger}_{g,\nu'} \end{array} \tag{3.1}$$

Here f and g are the cusp forms considered in Section 1.3 and $T^{\dagger}_{\nu}$, $T^{\dagger}_{\nu'}$ are the specializations of the big Galois representations at the intersection of the Hida families; Υ is a suitable isomorphism.

As the big Heegner points $\mathfrak{X}_{f,c}$, $\mathfrak{X}_{g,c}$ live in $H^1(H_c, \mathbf{T}^{\dagger}_{f})$ and $H^1(H_c, \mathbf{T}^{\dagger}_{g})$ respectively, if we found a cohomology class $\widetilde{\mathfrak{X}}_c \in H^1(H_c, \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}^{\dagger}_{\Sigma})$ whose images under the morphisms induced in cohomology by the oblique maps is related to $\mathfrak{X}_{f,c}$ and $\mathfrak{X}_{g,c}$, then also its images $\mathsf{Im}_{\nu}(\widetilde{\mathfrak{X}}_c) \in H^1(H_c, T^{\dagger}_{f,\nu})$ and $\mathsf{Im}_{\nu'}(\widetilde{\mathfrak{X}}_c) \in H^1(H_c, T^{\dagger}_{g,\nu'})$ would be related to the specializations of $\mathfrak{X}_{f,c}$ and $\mathfrak{X}_{g,c}$. By the commutativity of the diagram, we would know that

$$\Upsilon_c : H^1(H_c, T^{\dagger}_{f,\nu}) \xrightarrow{\sim} H^1(H_c, T^{\dagger}_{g,\nu'})$$

maps

$$\mathsf{Im}_{\nu}(\widetilde{\mathfrak{X}}_c) \mapsto \mathsf{Im}_{\nu'}(\widetilde{\mathfrak{X}}_c).$$

This is the way we compare big Heegner points at the intersection of two Hida families.

3.1 Construction

Definition 3.1.1. *Let $N \geq 1$ be an integer, d a divisor of N and d' a divisor of d, then we let $B_{d,d'} : X_s(N) \to X_s(N/d)$ denote the map induced by the map $\tau \mapsto d'\tau$ on the upper half plane. It is defined over $\mathbb{Q}$ and induces a map*

$$(B_{d,d'})_* : \mathsf{Ta}_p(J_s^N) \to \mathsf{Ta}_p(J_s^{N/d})$$

between the p-adic Tate modules of the Jacobians.

If ℓ is a prime different from p, let e_ℓ be the largest power of ℓ dividing $N(\Sigma)/W$. We have $0 \leq e_\ell \leq 2$. We easily see that if $\ell \notin \Sigma$ then $e_\ell = 0$ and if $e_\ell = 2$ then $\ell \in \Sigma$ and $\ell \nmid W$.

Write

$$\epsilon(\ell) := \begin{cases} 1 & e_\ell = 0 \\ \\ \Big[(B_{\ell,1})_* - \ell^{-1} U_{\ell,\mathsf{Alb}} (B_{\ell,\ell})_*\Big] & e_\ell = 1 \\ \\ \Big[(B_{\ell^2,1})_* - \ell^{-1} T_{\ell,\mathsf{Alb}} (B_{\ell^2,\ell})_* + \ell^{-1}\langle \ell \rangle (B_{\ell^2,\ell^2})_*\Big] & e_\ell = 2 \end{cases}$$

Choose an ordering $\Sigma = \{\ell_1, \ldots, \ell_n\}$ and for any $s \geq 1$ define

$$\epsilon_s : \mathsf{Ta}_p^{\mathsf{ord}}(J_s^{N(\Sigma)}) \longrightarrow \mathsf{Ta}_p^{\mathsf{ord}}(J_s^W) \tag{3.2}$$

by $\epsilon_s = \epsilon(\ell_n) \circ \cdots \circ \epsilon(\ell_1)$. To explain this formula, let's write $N_i = N(\Sigma)/\ell_1^{e_1} \ldots \ell_i^{e_i}$, then in the formula for ϵ_s

$$\epsilon(\ell_i) : \mathsf{Ta}_p^{\mathsf{ord}}(J_s^{N_{i-1}}) \longrightarrow \mathsf{Ta}_p^{\mathsf{ord}}(J_s^{N_i})$$

is the map given by the formula above. The symbols $T_{\ell_i,\mathsf{Alb}}$ or $U_{\ell_i,\mathsf{Alb}}$ in the formula for $\epsilon(\ell_i)$ are understood to stand for the corresponding Hecke operators acting in level $N_i p^s$. The map ϵ_s is independent of the ordering and it can be verified through direct computations as it is suggested at the bottom of page 557 of [1].

If regard the source and the target of ϵ_s as $\mathfrak{h}'_{N(\Sigma)}$-modules via the inclusion $\mathfrak{h}'_{N(\Sigma)} \subset \mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}$ and the natural map $\mathfrak{h}'_{N(\Sigma)} \to \mathfrak{h}'_W \subset \mathfrak{h}^{\mathsf{ord}}_W$ then ϵ_s is seen to be $\mathfrak{h}'_{N(\Sigma)}$-linear.

As s varies, the sources and the targets of the maps ϵ_s each form a projective system and the maps ϵ_s are compatible with the projection maps on the source and target. Thus, taking the inverse limit in s, we get an $\mathfrak{h}'_{N(\Sigma)}$-linear map

$$\epsilon_\infty : \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_W.$$

If we localize ϵ_∞ with respect to $\mathfrak{m}$ the maximal ideal of $\mathfrak{h}'_{N(\Sigma)}$ recalled at the beginning of the chapter, we obtain a map

$$\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}'_{N(\Sigma)}}{\otimes} (\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}} \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_W \underset{\mathfrak{h}'_{N(\Sigma)}}{\otimes} (\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}}. \tag{3.3}$$

Now $\mathfrak{n}$ and $\mathfrak{w}$ each pull back to $\mathfrak{m}$ under the maps $\mathfrak{h}'_{N(\Sigma)} \to \mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}$ and $\mathfrak{h}'_{N(\Sigma)} \to \mathfrak{h}^{\mathsf{new}}_W$ and so the localizations $(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ and $(\mathfrak{h}^{\mathsf{new}}_W)_{\mathfrak{w}}$ are local factors of the complete semilocal rings $(\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}} \otimes_{\mathfrak{h}'_{N(\Sigma)}} (\mathfrak{h}^{\mathsf{ord}}_N)_{\mathfrak{n}}$ and $(\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}} \otimes_{\mathfrak{h}'_{N(\Sigma)}} (\mathfrak{h}^{\mathsf{new}}_W)_{\mathfrak{w}}$. Thus the localizations $\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} (\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ and $\mathbf{Ta}^{\mathsf{ord}}_W \otimes_{\mathfrak{h}^{\mathsf{ord}}_W} (\mathfrak{h}^{\mathsf{new}}_W)_{\mathfrak{w}}$ appear naturally as direct factors of $\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} (\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}}$ and $\mathbf{Ta}^{\mathsf{ord}}_W \otimes_{\mathfrak{h}^{\mathsf{ord}}_W} (\mathfrak{h}'_{N(\Sigma)})_{\mathfrak{m}}$ respectively, so the map (3.3) induces a map

$$\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} (\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}} \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_W \underset{\mathfrak{h}^{\mathsf{ord}}_W}{\otimes} (\mathfrak{h}^{\mathsf{new}}_W)_{\mathfrak{w}}.$$

Tensoring the source of this map with $\mathfrak{h}(\mathfrak{a})^\circ$ over $(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}$ and the target with $\mathfrak{h}(\mathfrak{a})^\circ$ over $(\mathfrak{h}^{\mathsf{new}}_W)_{\mathfrak{w}}$, we obtain a $\mathfrak{h}(\mathfrak{a})^\circ$-linear map

$$\Xi_{\mathfrak{a}'} : \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_W \underset{\mathfrak{h}^{\mathsf{ord}}_W}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ. \tag{3.4}$$

Theorem 3.1.1. *The map (3.4) is an isomorphism of $\mathfrak{h}(\mathfrak{a})^\circ[G_{\mathbb{Q}}]$-modules*

$$\Xi_{\mathfrak{a}'} : \mathbf{T}^{\Sigma}_{\mathfrak{a}'} \xrightarrow{\sim} \mathbf{T}_{\mathfrak{a}'}.$$

Proof. This is the analogous result to Theorem 3.6.2 [1]. Following [1], we define for any positive integer A

$$\mathfrak{M}_A := \lim_{\leftarrow,\ s} H_1\big(X_s(A)_{/\overline{\mathbb{Q}}}, \{\mathsf{cusps}\}; \mathcal{O}\big)^{\mathsf{ord}}.$$

When we localize at a maximal ideal $\mathfrak{a}$ of $\mathfrak{h}^{\mathsf{ord}}_A$ corresponding to a p-ordinary irreducible residual representation we find that

$$\big(\mathfrak{M}_A\big)_{\mathfrak{a}} \cong \big(\lim_{\leftarrow,\ s} H_1\big(X_s(A)_{/\overline{\mathbb{Q}}}; \mathcal{O}\big)^{\mathsf{ord}}\big)_{\mathfrak{a}}$$

as $(\mathfrak{h}^{\mathsf{ord}}_A)_{\mathfrak{a}}$-modules ((3.1) page 544 and page 547 [1]).

Theorem 3.6.2 [1] states that the morphism

$$\big(\mathfrak{M}_{N(\Sigma)}\big)_{\mathfrak{n}} \otimes_{(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}} \mathfrak{h}(\mathfrak{a})^\circ \longrightarrow \big(\mathfrak{M}_W\big)_{\mathfrak{w}} \otimes_{(\mathfrak{h}^{\mathsf{ord}}_W)_{\mathfrak{w}}} \mathfrak{h}(\mathfrak{a})^\circ \tag{3.5}$$

constructed exactly as our map (3.4), is an isomorphism of $\mathfrak{h}(\mathfrak{a})^\circ$-modules. By (1.6.1) [5], for every positive integer A we have

$$H^1_{\mathsf{et}}\big(X_s(A)_{/\overline{\mathbb{Q}}}, \mathcal{O}\big) = \mathsf{Hom}_{\mathcal{O}}\big(\mathsf{Ta}_p(J^A_s) \otimes_{\mathbb{Z}_p} \mathcal{O}(1), \mathcal{O}\big)$$

which, taking $\mathsf{Hom}_{\mathcal{O}}\big(-, \mathcal{O}\big)$, implies

$$H_1\big(X_s(A)_{/\overline{\mathbb{Q}}}, \mathcal{O}\big) = \mathsf{Ta}_p(J^A_s) \otimes_{\mathbb{Z}_p} \mathcal{O}(1).$$

We then deduce an isomorphism of $\mathfrak{h}^{\mathsf{ord}}_A[G_{\mathbb{Q}}]$-modules

$$\mathbf{Ta}^{\mathsf{ord}}_A(1) \cong \lim_{\leftarrow,\ s} H_1\big(X_s(A)_{/\overline{\mathbb{Q}}}; \mathcal{O}\big)^{\mathsf{ord}}$$

such that the diagram

$$\begin{array}{ccc} \big(\mathfrak{M}_{N(\Sigma)}\big)_{\mathfrak{n}} \underset{(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ & \longrightarrow & \big(\mathfrak{M}_W\big)_{\mathfrak{w}} \underset{(\mathfrak{h}^{\mathsf{ord}}_W)_{\mathfrak{w}}}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ \\ \downarrow & & \downarrow \\ \big(\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)}\big)_{\mathfrak{n}}(1) \underset{(\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)})_{\mathfrak{n}}}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ & \xrightarrow{\Xi_{\mathfrak{a}'}} & \big(\mathbf{Ta}^{\mathsf{ord}}_W\big)_{\mathfrak{w}}(1) \underset{(\mathfrak{h}^{\mathsf{ord}}_W)_{\mathfrak{w}}}{\otimes} \mathfrak{h}(\mathfrak{a})^\circ \end{array}$$

commutes, where the horizontal upper map is (3.5) and the horizontal lower map is $\Xi_{\mathfrak{a}'}$ after a Tate twist on the source and on the target. □

3.2 Isomorphism of specializations

We are interested to apply the previous constructions to the situation where f and g are the p-ordinary p-stabilized modular forms of tame conductors N and M respectively, whose residual representations are isomorphic and with a unique p-stabilization, let's call it $\bar{\rho}$. Let Σ be a finite set of primes, $p \notin \Sigma$, such that $N \mid N(\Sigma)$ and $M \mid N(\Sigma)$ and consider $\mathfrak{h}_\Sigma := \mathfrak{h}_\Sigma(\bar{\rho})$. In this setting the minimal primes

$$\mathfrak{a}' \subset \mathfrak{h}^{\mathsf{new}}_N, \qquad \mathfrak{b}' \subset \mathfrak{h}^{\mathsf{new}}_M \tag{3.6}$$

corresponding to the families passing through f and g are admissible for $\mathfrak{h}_\Sigma$, thus we get the corresponding minimal primes $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{h}_\Sigma$. Moreover, we have

$$R_f = \mathfrak{h}(\mathfrak{a})^\circ, \qquad R_g = \mathfrak{h}(\mathfrak{b})^\circ. \tag{3.7}$$

Remark that if $\nu : R_f \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p}$ and $\nu' : R_g \longrightarrow \mathcal{O}_{\overline{\mathbb{Q}}_p}$ are the continuous $\mathcal{O}$-algebra homomorphisms corresponding to an intersection point of the Hida families passing through f and g, then $R_f \otimes_\nu \mathcal{O}_{\overline{\mathbb{Q}}_p} = R_g \otimes_{\nu'} \mathcal{O}_{\overline{\mathbb{Q}}_p}$. We will denote this ring by S.

Corollary 3.2.1. *There is an isomorphism of $S[G_{\mathbb{Q}}]$-modules*

$$\Upsilon : T^{\dagger}_{f,\nu} \xrightarrow{\sim} T^{\dagger}_{g,\nu'}$$

such that the diagram

$$\begin{array}{ccccc} & & \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} \mathfrak{h}^{\dagger}_{\Sigma} & & \\ & \swarrow & & \searrow & \\ \mathbf{T}^{\dagger}_{f} & & & & \mathbf{T}^{\dagger}_{g} \\ \downarrow & & & & \downarrow \\ T^{\dagger}_{f,\nu} & & \xrightarrow{\quad\Upsilon\quad} & & T^{\dagger}_{g,\nu'} \end{array}$$

commutes, where the oblique morphisms are defined as the composites

$$\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}^{\dagger}_{\Sigma} \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}(\mathfrak{a})^{\circ,\dagger} \xrightarrow{\Xi_{\mathfrak{a}'}} \mathbf{T}^{\dagger}_{f}$$

and

$$\mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}^{\dagger}_{\Sigma} \longrightarrow \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}(\mathfrak{b})^{\circ,\dagger} \xrightarrow{\Xi_{\mathfrak{b}'}} \mathbf{T}^{\dagger}_{g}.$$

Proof. Υ is the unique map that makes the diagram

$$\begin{array}{ccc} \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} S^{\dagger} & \xrightarrow{\quad\mathsf{id}\quad} & \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \underset{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}}{\otimes} S^{\dagger} \\ \sim\downarrow & & \downarrow\sim \\ T^{\dagger}_{f,\nu} & & T^{\dagger}_{g,\nu'} \end{array}$$

commutes, where the left vertical isomorphism is obtain from

$$\Xi_{\mathfrak{a}'} : \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}(\mathfrak{a})^{\circ,\dagger} \xrightarrow{\sim} \mathbf{T}^{\dagger}_{f}$$

by tensoring over R_f with $S = R_f \otimes_{\nu} \mathcal{O}_{\overline{\mathbb{Q}}_p}$ and the right vertical isomorphism from

$$\Xi_{\mathfrak{b}'} : \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}(\mathfrak{b})^{\circ,\dagger} \xrightarrow{\sim} \mathbf{T}^{\dagger}_{g}$$

by tensoring over R_g with $S = R_g \otimes_{\nu'} \mathcal{O}_{\overline{\mathbb{Q}}_p}$ □

Chapter 4

Effect on cohomology classes

In this chapter we calculate the effect on big Heegner points of the comparison maps. We consider an admissible minimal prime $\mathfrak{a}'$ of tame level W associated to some $\mathfrak{h}_\Sigma$. Fix an imaginary quadratic field K where all the prime divisors of $N(\Sigma)$ are split. Recall the construction of the comparison morphism in Section 3.1. As the map ϵ_s (3.2) does not depend on the ordering of Σ, we can assume that $\Sigma = \{\ell\}$. Then, $N(\Sigma) = W\ell^e$ where $e \in \{0, 1, 2\}$. If $e = 0$, the map is the identity so everything is clear. Let's consider the cases where $e \in \{1, 2\}$.

For each prime $q \mid N(\Sigma)$, $q \neq \ell$ let b_q be the greatest power of q that divides $N(\Sigma)$ and choose one prime $\mathfrak{Q}$ in K above it. Similarly let b_ℓ the greatest power of ℓ that divides $N(\Sigma)$ and choose a prime $\mathfrak{L}$ above it in K.

Let $\mathfrak{W}$ be the ideal of $\mathcal{O}_K = \mathbb{Z} + \varpi\mathbb{Z}$ defined as

$$\Bigg(\prod_{q \mid N(\Sigma), q \neq \ell} \mathfrak{Q}^{b_q}\Bigg)\mathfrak{L}^{b_\ell - e},$$

and $\mathfrak{N}(\Sigma)$ the ideal $\mathfrak{W}\mathfrak{L}^e$ then $\mathcal{O}_K/\mathfrak{W} \cong \mathbb{Z}/W\mathbb{Z}$ and $\mathcal{O}_K/\mathfrak{N}(\Sigma) \cong \mathbb{Z}/N(\Sigma)\mathbb{Z}$.

For every positive integer c prime to $N(\Sigma)$ and $s \geq 0$ consider the point

$$\widetilde{Q}_{c,s} = [(\mathbb{C}/\mathcal{O}_{cp^s}, \mathfrak{N}(\Sigma)_{c,s}^{-1}/\mathcal{O}_{cp^s}, [c\varpi])] \in X_s(N(\Sigma))(H_{cp^s})$$

where $\mathfrak{N}(\Sigma)_{c,s} = \mathfrak{N}(\Sigma) \cap \mathcal{O}_{cp^s}$ and

$$Q_{c,s} = [(\mathbb{C}/\mathcal{O}_{cp^s}, \mathfrak{W}_{c,s}^{-1}/\mathcal{O}_{cp^s}, [c\varpi])] \in X_s(W)(H_{cp^s})$$

where $\mathfrak{W}_{c,s} = \mathfrak{W} \cap \mathcal{O}_{cp^s}$.

Then the projection map

$$X_s(N(\Sigma)) \longrightarrow X_s(W)$$

that simply forgets the $N(\Sigma)/W$-torsion structure maps

$$\widetilde{Q}_{c,s} \mapsto Q_{c,s}.$$

From now on, we will adorne with a tilde all the objects that appear in the construction of big Heegner points at level $N(\Sigma)$.

We want to understand the effect of the maps $B_{i,j}$ (Definition 3.1.1). Consider the map

$$\lambda_{\ell^e} : X_s(N(\Sigma)) \longrightarrow X_s(W)$$

defined on moduli by $(E, C, P) \mapsto (E/C', f(C), f(P))$ where C is a cyclic subgroup of order $N(\Sigma)$ of E, P is a point of exact order p^s of E, $C' \subset C$ is the unique subgroup of C of order ℓ^e and $f : E \to E/C'$ is the quotient map.

Lemma 4.0.1. *We have*

$$\langle\ell\rangle^e B_{\ell^e,\ell^e} = \lambda_{\ell^e}, \qquad \langle\ell\rangle B_{\ell^2,\ell} = \lambda_\ell.$$

Proof. B_{l^e,l^e} acts on the points of the modular curve by

$$\left(\mathbb{C}/\Lambda_\tau, \left\langle \frac{1}{N(\Sigma)} \right\rangle, \frac{1}{p^s}\right) \mapsto \left(\mathbb{C}/\Lambda_{\ell^e\tau}, \left\langle \frac{\ell^e}{N(\Sigma)} \right\rangle, \frac{1}{p^s}\right).$$

The ℓ^e-isogeny $\mathbb{C}/\Lambda_\tau \to \mathbb{C}/\Lambda_{\ell^e\tau}$ induced by $z \mapsto \ell^e z$ has as kernel the subgroup $\langle 1/\ell^e\rangle$ so λ_{ℓ^e} acts as

$$\left(\mathbb{C}/\Lambda_\tau, \left\langle \frac{1}{N(\Sigma)} \right\rangle, \frac{1}{p^s}\right) \mapsto \left(\mathbb{C}/\Lambda_{\ell^e\tau}, \left\langle \frac{\ell^e}{N(\Sigma)} \right\rangle, \frac{\ell^e}{p^s}\right).$$

Therefore we get the equality $\langle\ell\rangle^e B_{\ell^e,\ell^e} = \lambda_{\ell^e}$. A similar argument works for the equality $\langle\ell\rangle B_{\ell^2,\ell} = \lambda_\ell$. □

For a positive integer c prime to $N(\Sigma)$ denote $L_c^\infty := \bigcup_s H_{cp^s}(\mu_{p^s})$ and let $\mathsf{Frob}_{\mathfrak{L}}$ be the Frobenius at $\mathfrak{L}$ in the abelian group $\mathrm{Gal}(L_c^\infty/K)$.

Lemma 4.0.2. *Case $e = 1$.*

- $B_{l,1}(\widetilde{Q}_{c,s}) = Q_{c,s}$
- $\langle l\rangle B_{l,l}(\widetilde{Q}_{c,s}) = (Q_{c,s})^{\mathsf{Frob}_{\mathfrak{L}}}$

Proof. Let $\mathfrak{N}_{c,s} = \mathfrak{N}(\Sigma) \cap \mathcal{O}_{cp^s}$ where $\mathfrak{N}(\Sigma) = \mathfrak{L}\mathfrak{W}$ and let $x \in \hat{K}^\times$ be a finite idele which is a uniformizer at $\mathfrak{L}$ and has trivial components at all other primes, then the fractional ideal generated by x is $\mathfrak{L}$. The effect of $\langle l\rangle B_{l,l} = \lambda_l$ on $\widetilde{Q}_{c,s}$ is

$$\begin{aligned}\widetilde{Q}_{c,s} &= [(\mathbb{C}/\mathcal{O}_{cp^s}, \mathfrak{N}_{c,s}^{-1}/\mathcal{O}_{cp^s}, c\varpi)]\\ &\mapsto [(\mathbb{C}/x^{-1}\mathcal{O}_{cp^s}, \mathfrak{N}_{c,s}^{-1}/x^{-1}\mathcal{O}_{cp^s}, c\varpi)].\end{aligned}$$

On the other hand, the main theorem of complex multiplication gives us an isomoprhism

$$\begin{aligned}(Q_{c,s})^{\mathsf{Frob}_{\mathfrak{L}}} &= [(\mathbb{C}/\mathcal{O}_{cp^s}, \mathfrak{W}_{c,s}^{-1}/\mathcal{O}_{cp^s}, c\varpi)^{\mathsf{Frob}_{\mathfrak{L}}}]\\ &\cong [(\mathbb{C}/x^{-1}\mathcal{O}_{cp^s}, (x\mathfrak{W}_{c,s})^{-1}/x^{-1}\mathcal{O}_{cp^s}, x^{-1}c\varpi)].\end{aligned}$$

But $x\mathfrak{W}_{c,s} = (\mathfrak{L}\mathfrak{W})_{c,s} = \mathfrak{N}_{c,s}$ and as x as trivial component at p, $c\varpi$ and $x^{-1}c\varpi$ determine the same element of $K/x^{-1}\mathcal{O}_{cp^s}$. □

Lemma 4.0.3. *Case $e = 2$.*

- $B_{l^2,1}(\widetilde{Q}_{c,s}) = Q_{c,s}$
- $\langle l\rangle B_{l^2,l}(\widetilde{Q}_{c,s}) = (Q_{c,s})^{\mathsf{Frob}_{\mathfrak{L}}}$
- $\langle l\rangle^2 B_{l^2,l^2}(\widetilde{Q}_{c,s}) = (Q_{c,s})^{\mathsf{Frob}_{\mathfrak{L}}^2}$

Proof. The proof is analogous to that of Lemma 4.0.2. □

For the next proposition it will be useful to compute $e_{k+j-2}\langle\ell\rangle$. Write $\langle\ell\rangle = \langle\delta_\ell\rangle\langle\gamma_\ell\rangle$ according to the decomposition $\mathbb{Z}_p^\times \cong \mu_{p-1} \times \Gamma$ and let $\xi_\ell = \omega(\delta_\ell)$. Then

$$\begin{aligned}e_{k+j-2}\langle\ell\rangle &= \omega^{k+j-2}(\delta_\ell)\langle\gamma_\ell\rangle e_{k+j-2}\\ &= \xi_\ell^{k+j-2}\langle\gamma_\ell\rangle e_{k+j-2}\\ &= \Theta(\mathsf{Frob}_\ell)^2 e_{k+j-2}.\end{aligned}$$

Proposition 4.0.1. *Case $e = 1$.*

- $(B_{l,1})_* \widetilde{\mathfrak{X}}_c = \mathfrak{X}_c$
- $(B_{l,l})_* \widetilde{\mathfrak{X}}_c = \Theta(\ell)^{-1} \big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell}$

Proof. By Lemma 4.0.2 we have

$$B_{l,l}\big(\widetilde{Q}_{c,s}\big) = \langle l \rangle^{-1} \big(Q_{c,s}\big)^{\mathsf{Frob}_{\mathfrak{L}}}.$$

As $B_{\ell,\ell}$ is Hecke equivariant on the Jacobian and the ordinary projector and the idempotent e_{k+j-2} are defined over $\mathbb{Q}$ we get that

$$B_{\ell,\ell}\big(\widetilde{y}_{c,s}\big) = \Theta(\mathsf{Frob}_\ell)^{-2} \big(y_{c,s}\big)^{\mathsf{Frob}_{\mathfrak{L}}}.$$

Twisting by ζ_s we get

$$\begin{aligned} B_{\ell,\ell}\big(\widetilde{y}_{c,s} \otimes \zeta_s\big) &= \Theta(\mathsf{Frob}_\ell)^{-2} \big(y_{c,s}\big)^{\mathsf{Frob}_{\mathfrak{L}}} \otimes \zeta_s \\ &= \Theta(\mathsf{Frob}_\ell)^{-2} \Theta(\mathsf{Frob}_{\mathfrak{L}}) \big(y_{c,s} \otimes \zeta_s\big)^{\mathsf{Frob}_{\mathfrak{L}}}. \end{aligned}$$

Then we take corestriction from H_{cp^s} to H_c

$$\begin{aligned} B_{\ell,\ell}(\widetilde{x}_{c,s}) &= B_{\ell,\ell}\Bigg(\sum_{\eta \in G(H_{cp^s}/H_c)} (\widetilde{y}_{c,s} \otimes \zeta_s)^\eta \Bigg) \\ &= \Bigg(\sum_{\eta \in G(H_{cp^s}/H_c)} B_{\ell,\ell}^\eta (\widetilde{y}_{c,s} \otimes \zeta_s)^\eta \Bigg) \\ &= \Theta(\mathsf{Frob}_\ell)^{-2} \Theta(\mathsf{Frob}_{\mathfrak{L}}) \Bigg(\sum_{\eta \in G(H_{cp^s}/H_c)} \big[\big(y_{c,s} \otimes \zeta_s\big)^{\mathsf{Frob}_{\mathfrak{L}}} \big]^\eta \Bigg) \\ &= \Theta(\mathsf{Frob}_\ell)^{-2} \Theta(\mathsf{Frob}_{\mathfrak{L}}) \big(x_{c,s}\big)^{\mathsf{Frob}_{\mathfrak{L}}} \end{aligned}$$

because $\mathrm{Gal}(L_{c,s}/K)$ is abelian.

As ℓ is split in K, $\mathsf{Frob}_{\mathfrak{L}} = \mathsf{Frob}_\ell$ in $\mathsf{Gal}(L_c^\infty/\mathbb{Q})$ which implies $\epsilon_{cyc}(\mathsf{Frob}_{\mathfrak{L}}) = \ell$. Identifying Θ with a character on $\mathbb{Z}_p^\times$ by factoring through the cyclotomic character, we get $\Theta(\mathsf{Frob}_{\mathfrak{L}}) = \Theta(\ell)$. By the $G_{\mathbb{Q}}$-equivariance of the twisted Kummer map we also have that

$$(B_{\ell,\ell})_* \widetilde{\mathfrak{X}}_{c,s} = \Theta(\ell)^{-1} \big(\mathfrak{X}_{c,s}\big)^{\mathsf{Frob}_\ell}$$

and taking the inverse limit over s we conclude. □

Proposition 4.0.2. *Case $e = 2$.*

- $(B_{\ell^2,1})_* \widetilde{\mathfrak{X}}_c = \mathfrak{X}_c$
- $(B_{\ell^2,\ell})_* \widetilde{\mathfrak{X}}_c = \Theta(\ell)^{-1} \big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell}$
- $(B_{\ell^2,\ell^2})_* \widetilde{\mathfrak{X}}_c = \Theta(\ell)^{-2} \big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell^2}$

Proof. The proof is analogous to that of Proposition 4.0.1. □

4.1 Precise results

Finally, we have proved the following theorem.

Theorem 4.1.1. *Let h be a p-ordinary newform in $S_t(\Gamma_0(Wp), \omega^u)$ whose residual representation is absolutely irreducible and p-distinguished. We assume that the prime factors of W split in the imaginary quadratic field K. Let Σ be a finite set of rational primes that split in K which contains $\{\ell$ prime$|$ ℓ divides $W\}$ and let $\mathfrak{a}'$ be the minimal prime of $\mathfrak{h}_W^{\mathsf{new}}$ corresponding to the Hida family passing through h. Then for every $c \in \mathbb{N}$ prime to $N(\Sigma)$ there exists an isomorphism*

$$\Xi_{h,c}^{\Sigma} : H^1\big(H_c, \mathbf{T}_{\mathfrak{a}'}^{\Sigma,\dagger}\big) \longrightarrow H^1\big(H_c, \mathbf{T}_h^{\dagger}\big)$$

such that

$$\Xi_{h,c}^{\Sigma}\big(\widetilde{\mathfrak{X}}_c\big) = \prod_{\ell|(N(\Sigma)/W)} E_\ell\big(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell\big)\mathfrak{X}_c$$

where the Euler factors $E_\ell(\mathfrak{a}', X) \in \mathfrak{h}(\mathfrak{a})^\circ[X]$ were defined in Definition 2.0.4.

Proof. The isomorphism $\Xi_{h,c}^{\Sigma}$ is induced by the map (3.4), thus we just have to put together the calculations of the last chapter to check that the effect on big Heegner points is the one claimed. Let $\ell \mid (N(\Sigma)/W)$.

If $\ell \mid W$, then $E_\ell(\mathfrak{a}', X) = 1 - (U_{\ell,\mathsf{Alb}} \mod \mathfrak{a}')X$ and $e_\ell = 1$. Therefore

$$\begin{aligned}\Big[(B_{\ell,1})_* - \ell^{-1}U_{\ell,\mathsf{Alb}}(B_{\ell,\ell})_*\Big]\widetilde{\mathfrak{X}}_c = \\ &= \mathfrak{X}_c - \ell^{-1}U_{\ell,\mathsf{Alb}}\Theta(\ell)^{-1}\big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell} \\ &= E_\ell(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell)\mathfrak{X}_c\end{aligned}$$

If $\ell \nmid W$, then $E_\ell(\mathfrak{a}', X) = 1 - (T_{\ell,\mathsf{Alb}} \mod \mathfrak{a}')X + \ell\langle\ell\rangle X^2$ and $e_\ell = 2$. Therefore

$$\begin{aligned}\Big[(B_{\ell^2,1})_* - \ell^{-1}T_{\ell,\mathsf{Alb}}(B_{\ell^2,\ell})_* + \ell^{-1}\langle\ell\rangle(B_{\ell^2,\ell^2})_*\Big]\widetilde{\mathfrak{X}}_c = \\ &= \mathfrak{X}_c - \ell^{-1}T_{\ell,\mathsf{Alb}}\Theta(\ell)^{-1}\big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell} + \ell^{-1}\langle\ell\rangle\Theta(\ell)^{-2}\big(\mathfrak{X}_c\big)^{\mathsf{Frob}_\ell^2} \\ &= E_\ell(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell)\mathfrak{X}_c\end{aligned}$$

□

Recall our cusp forms f and g of tame conductors N and M. Putting together Theorem 4.1.1 and the isomorphism Υ of Corollary 3.2.1 we get the following corollary.

Corollary 4.1.1. *Let $f \in S_k(\Gamma_0(Np), \omega^j)$ and $g \in S_r(\Gamma_0(Mp), \omega^s)$ be p-ordinary p-stabilized newforms with isomorphic residual representation $\bar{\rho}$ absolutely irreducible and p-distinguished and let $\mathfrak{a}' \subset \mathfrak{h}_N^{\mathsf{new}}$, $\mathfrak{b}' \subset \mathfrak{h}_M^{\mathsf{new}}$ be the minimal primes corresponding to the intersecting Hida families passing through f and g. Let $\Sigma = \{\ell$ prime$|$ $\ell \mid NM\}$, and consider $\mathfrak{h}_\Sigma = \mathfrak{h}_\Sigma(\bar{\rho})$. For every positive interger c prime to NM denote by $\mathfrak{X}_{f,c}$ and $\mathfrak{X}_{g,c}$ the big Heegner points of conductor c. Then there is an isomorphism*

$$\Upsilon_c : H^1\big(H_c, T_{f,\nu}^{\dagger}\big) \longrightarrow H^1\big(H_c, T_{g,\nu'}^{\dagger}\big)$$

such that

$$\nu\left(\prod_{\ell|(N(\Sigma)/N)} (E_\ell\big(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell\big)\mathfrak{X}_{f,c}\right) \mapsto \nu'\left(\prod_{\ell'|(N(\Sigma)/M)} E_{\ell'}\big(\mathfrak{b}', (\ell')^{-1}\Theta(\ell')^{-1}\mathsf{Frob}_{\ell'}\big)\mathfrak{X}_{g,c}\right)$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} & & H^1\big(H_c, \mathbf{Ta}_{N(\Sigma)}^{\mathsf{ord}} \otimes_{\mathfrak{h}_{N(\Sigma)}^{\mathsf{ord}}} \mathfrak{h}_\Sigma^{\dagger}\big) & & \\ & \swarrow & & \searrow & \\ H^1\big(H_c, \mathbf{T}_f^{\dagger}\big) & & & & H^1\big(H_c, \mathbf{T}_g^{\dagger}\big) \\ \downarrow & & & & \downarrow \\ H^1\big(H_c, T_{f,\nu}^{\dagger}\big) & & \xrightarrow{\quad \Upsilon_c \quad} & & H^1\big(H_c, T_{g,\nu'}^{\dagger}\big) \end{array}$$

and $\widetilde{\mathfrak{X}}_c \in H^1\big(H_c, \mathbf{Ta}^{\mathsf{ord}}_{N(\Sigma)} \otimes_{\mathfrak{h}^{\mathsf{ord}}_{N(\Sigma)}} \mathfrak{h}^{\dagger}_{\Sigma}\big)$. Its image in $H^1\big(H_c, \mathbf{T}^{\dagger}_f\big)$ is

$$\prod_{\ell|(N(\Sigma)/N)} E_\ell\big(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell\big)\mathfrak{X}_{f,c}$$

and in $H^1\big(H_c, \mathbf{T}^{\dagger}_g\big)$ is

$$\prod_{\ell'|(N(\Sigma)/M)} E_{\ell'}\big(\mathfrak{b}', (\ell')^{-1}\Theta(\ell')^{-1}\mathsf{Frob}_{\ell'}\big)\mathfrak{X}_{g,c}$$

by further specializing to $H^1\big(H_c, T^{\dagger}_{f,\nu}\big)$ and $H^1\big(H_c, T^{\dagger}_{g,\nu'}\big)$ we get that Υ_c maps

$$\nu\left(\prod_{\ell|(N(\Sigma)/N)} (E_\ell\big(\mathfrak{a}', \ell^{-1}\Theta(\ell)^{-1}\mathsf{Frob}_\ell\big)\mathfrak{X}_{f,c}\right) \mapsto \nu'\left(\prod_{\ell'|(N(\Sigma)/M)} E_{\ell'}\big(\mathfrak{b}', (\ell')^{-1}\Theta(\ell')^{-1}\mathsf{Frob}_{\ell'}\big)\mathfrak{X}_{g,c}\right)$$

by the commutativity of the diagram. □

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