DIAMETER AND BROADCAST TIME OF THE KNÖDEL GRAPH

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Abstract

Diameter and Broadcast Time of the Knödel graph

Efficient dissemination of information remains a central challenge for all types of networks. There are two ways to handle this issue. One way is to compress the amount of data being transferred and the second way is to minimize the delay of information distribution. Well-received approaches used in the second way either design efficient algorithms or implement reliable network architectures with optimal dissemination time. Among the well-known network architectures, the Knödel graph can be considered a suitable candidate for the problem of information dissemination. The Knödel graph $W_{d,n}$ is a regular graph, of an even order n and degree d, $1 \leq d \leq \lfloor \log_2 n \rfloor$. The Knödel graph was introduced by W. Knödel almost four decades ago as network architecture with good properties in terms of broadcasting and gossiping in interconnected networks. Although the Knödel graph has a highly symmetric structure, its diameter is only known for $W_{d,2^d}$. Recently, the general upper and lower bounds on diameter and broadcast time of the Knödel graph have been presented.

In this thesis, our motivation is to find the diameter, the number of vertices at a particular distance and the broadcast time of the Knödel graph. Theoretically, we succeed to prove the diameter and broadcast time of the Knödel graph $W_{3,n}$. We also claim that the Knödel graph $W_{3,n}$ for $n = 4 \; mod \; 6$ and $n > 16$ is a diametral broadcast graph. We present that $W_{3,22}$ is a broadcast graph. Experimentally, however, we obtain the following results; (a) the diameter of some specific Knödel graphs, and (b) the propositions on the number of vertices at a particular distance. We also construct a new graph, denoted as $HW_{d,2^d}$, by connecting Knödel graph $W_{d-1,2^{d-1}}$ to hypercube H_{d-1} and experimentally show that $HW_{d,2^d}$ has even a smaller diameter than Knödel graph $W_{d,2^d}$.

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Chapter 1

Introduction

Since the birth of the Internet, our world has become a global village where almost all commercial, social, private, public, and research and development networks are covered under the umbrella of the Internet. Fast and reliable dissemination of information remains the central issue in all types of real networks such as ad-hoc, wireless, satellite communications, supercomputers, Internet, cloud-based infrastructure. Much effort, money and time has been spent improving dissemination of information. There are two ways to approach this issue. One way would be to compress the amount of data that is being transferred and the second way would be to minimize the delay of information distribution. The well-received approaches used in the second way either design efficient algorithms or implement reliable network architectures with optimal dissemination time. Network architecture can be defined as the logical and structural layout of the network. Regular network architectures provide the platform to implement the powerful algorithms related to routing, broadcasting and parallel and distributed computing [42].

1.1 Network architecture design

There are some important aspects of network architecture design i.e., (i) network implementation cost (ii) support to create or extend a network to any size, (iii) performance of the network architecture in terms of information dissemination. Along with other aspects of network architecture design, the above three play an important role in the design of network architecture. There are many network architectures available for dissemination of information, each with its own advantages and limitations. For instance, some networks are less expensive in terms of implementation cost but they lack the ability to provide better performance. In another case, the network architecture may provide a better performance, but it may not support the creation or extension of a network to any size. A network architecture is needed that not only provides better performance, low implementation costs, and the support to create a network of any size, but also appropriately addresses and handles other important issues related to the dissemination of information.

Among well-known network architectures, the Knödel graph can be considered a suitable candidate for the problem of information dissemination. The Knödel graph not only supports important aspects of network architecture design, but it also contains a wide class of graphs. Using the Knödel graph one can design any type of network, where either distribution of information is on high priority or with less implementation cost.

1.2 Communication model for dissemination of information

In this thesis, we focus on the problem of broadcasting. Broadcasting is a process of information distribution in an interconnected network by which messages are transmitted from the originator to the remaining nodes of the network. To broadcast in a network, we consider the classical communication model. This model is simple and can be utilized when small messages are exchanged. Additionally, this model is also suitable for the type of networks, where the nodes of the network have very limited processing power and resources.

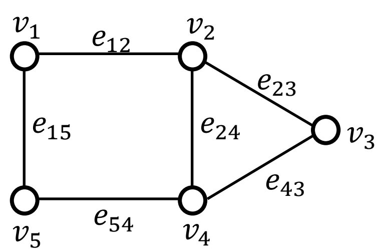
We studied the problem of information dissemination under these constraints:

- Each call requires one unit of time.
- A vertex can participate in only one call per unit of time.
- Each call involves only one informed vertex and one of its uninformed neighbors.

The following section provides definitions and notations helpful to understand the research work provided in this thesis.

1.3 Definitions and notations

In general, any interconnected network can be modeled as a graph *G = (V, E)*, where *V* is the set of vertices (nodes) and *E* is the set of edges (communication links) as depicted by Figure 1.1.



$$G = (V, E)$$

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E = \{e_{12}, e_{23}, e_{43}, e_{24}, e_{54}, e_{15},\}$$

Figure 1.1 Graph *G* with 5 vertices and degree 3

Two vertices $u, v \in V$, are adjacent, if there is an edge $e \in E$, such that $e = (u, v)$. In this case, we can say that u and v are neighbors. The degree of vertex v, $deg(v)$, is the number of neighbors of this vertex. The degree of graph G, $\Delta(G)$, is the maximum degree among all vertices, formally written as:

$$\Delta(G) = max\{\ deg(v) \mid v \in V\}$$

Figure 1.2 demonstrates that $\Delta(G) = 3$. A graph *G,* where each vertex has the same degree, is called a regular graph. A path *P* in a graph *G,* is a sequence of edges which

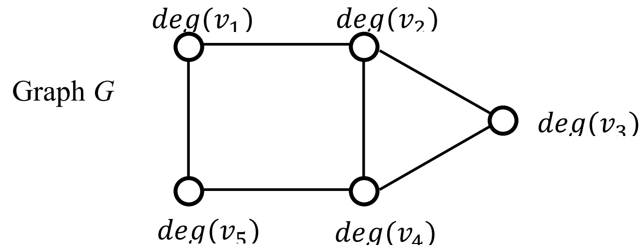


Figure 1.2 The degree of graph *G* is 3

connect a sequence of vertices. Generally it is of the form $P = (v_1, v_1, \dots, v_n)$, $n > 1$, where the length of the path is the number of edges of P. The length of the shortest path between two vertices v and u is the distance between them, $dist(v, u)$. The diameter of the graph is the maximum distance between any pair of vertices of the graph:

$$D(G) = max\{\, dist(v, u) \mid v, u \in V\}$$

A graph is connected, if there is a path between every two nodes in G.

Broadcasting and gossiping are two problems of information dissemination described for a group of individuals connected by a communication network. In broadcasting, an individual has a piece of information which needs to be communicated to everyone else. In gossiping, each person in the network has a unique piece of information and needs to communicate it to everyone else [33].

The message broadcasting from originator v in a graph $G = (V, E)$ is a sequence of vertex sets $\{v\} = S_0 \subset S_1 \subset \cdots \subset S_k = V$, where each S_i represents the set of informed vertices after the *i-th* time unit. All vertices from $S_i \backslash S_{i-1}$ are connected by disjoint edges with S_{i-1}. Given an originator v, the broadcast time, $b(v)$, is defined as the minimum number of time units required to complete broadcasting from vertex v. It is easy to conclude that for any vertex v in a connected graph G with n vertices, $\lceil \log_2 n \rceil \leq b(v) \leq n - 1$, since during each time unit the number of informed vertices can at most double. The broadcast time of graph G, $b(G)$, is defined as the maximum broadcast time among all the vertices, formally written as:

$$b(G) = max\{\, b(v) \mid v \in V\}$$

The process of broadcasting and the broadcast time of graph G are demonstrated in Figure 1.3, where v_1 is the originator of broadcasting.

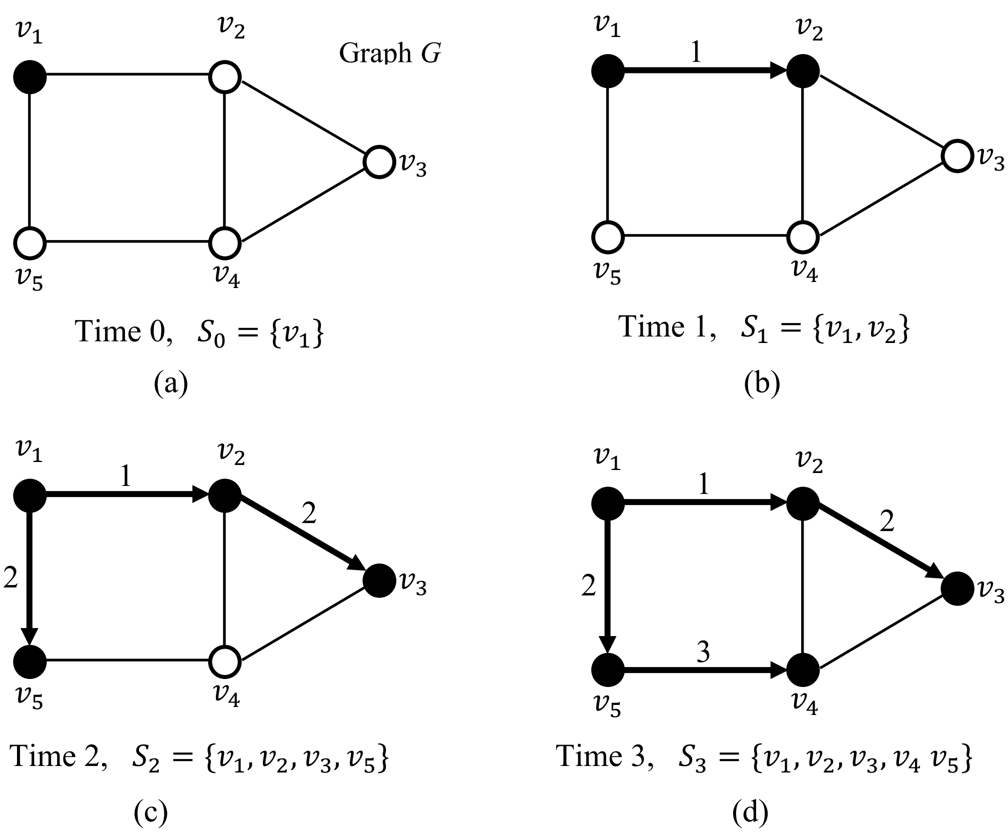


Figure 1.3 The process of broadcasting in graph *G*, where *b(G)* = 3

A graph on *n* vertices with $b(G) = \lceil \log_2 n \rceil$ is called a broadcast graph. A broadcast graph with the minimum possible number of edges is called minimum broadcast graph (mbg). The broadcast function, denoted $B(n)$, is defined as the number of edges in an *n* vertex mbg. A minimum gossip graph is a gossip graph with a minimum number of edges. The graph, where the broadcast time equals to its diameter, is called the diametral broadcast graph.

1.4 Motivation

Many interconnection networks for efficient communication are considered in the literature i.e., Path P_n, Cycle C_n, Complete tree T_k^m, Complete graph K_n, Hypercube H_m, Cube-connected cycles CCC_m, Butterfly BF_m, Shuffle-exchange SE_m, DeBruijn DB_m, Grid $G[a_1 \mathrm{X}\, a_2 \mathrm{X} \dots \mathrm{X}\, a_d]$ and Recursive circulant $G(2^m, 4)$ [36][37]. All these network architectures either have constant degree and relatively small diameter or they have logarithmic degree and logarithmic diameter. Also all of the above mentioned interconnection networks can be designed only for specific number of nodes. In particular the network architectures i.e., T_k^m, H_m, CCC_m, BF_m, SE_m, DB_m, $G[a_1 \mathrm{X}\, a_2 \mathrm{X} \dots \mathrm{X}\, a_d]$ and $G(2^m, 4)$ have $(K^{m+1} - 1)/(k - 1)$, 2^m, $m2^m$, $m2^m$, 2^m, 2^m, $a_1\, a_2 \dots a_d$ and 2^m number of nodes, respectively.

Compared to all of the networks, Knödel graph is the only network that can be designed for any even number of nodes. Moreover, the degree of every node in Knödel graph on n nodes can be any value between 2 and $\lfloor \log_2 n \rfloor$. When the degree of Knödel graph is 2, then it becomes the well-known cycle. When the degree is equal to $\lfloor \log_2 n \rfloor$, then Knödel graph is a broadcast and gossip graph, in which the main communication tasks can be performed, theoretically in minimum possible time.

The above properties make the Knödel graph the largest possible unique interconnection network, which could be sparse (when degree is constant) or dense (when degree is logarithmic of n). This way Knödel graph can be suitable for all possible applications based on communication time, network design or implementation cost.

All this gives us the motivation to study the Knödel graph. Therefore, in this thesis we studied the diameter, broadcast time and also the number of nodes at particular distance in Knödel graph for all possible even number of nodes and degree of any node.

1.5 Contribution of this thesis

In this thesis, our motivation is to find the diameter, the number of vertices at a particular distance and the broadcast time of the Knödel graph. Theoretically, we succeed in proving the diameter and the broadcast time of the Knödel graph $W_{3,n}$. We claim that the Knödel graph $W_{3,n}$ for $n = 4\ mod\ 6$ and $n > 16$, is the first infinite family of diametral broadcast graphs in the Knödel graph $W_{d,n}$. Experimentally, however, we obtain the following results: (a) the diameter of some specific Knödel graphs, and (b) the propositions on the number of vertices at a particular distance. The obtained results increase the list of explored communication properties of the Knödel graph.

1.6 Thesis outline

The rest of the thesis is structured as follows: Chapter 2 is divided in two sections; the first section covers the brief review of commonly used interconnection topologies. The second section surveys the Knödel graph in the light of known important results from the previous research work.

Chapter 3 is divided in three sections. In the first section, the diameter of the Knödel graph $W_{3,n}$, for $n > 8$, is given through a constructive proof. In the second section, we give the diameters of some specific Knödel graphs through extensive simulation. In the last section, we present three propositions for the number of vertices at a particular distance in some specific Knödel graphs.

In Chapter 4, we present the broadcast time of Knödel graph $W_{3,n}$. We also present that Knödel graph $W_{3,n}$, for $n = 4\ mod\ 6$ and $n > 16$, is the first infinite family of the diametral broadcast graphs in the Knödel graph $W_{d,n}$.

In chapter 5, we construct a new graph, denoted as $HW_{d,2^d}$, by connecting the vertices of the Knödel graph $W_{d-1,2^{d-1}}$ to hypercube H_{d-1}. We investigate the communication properties of $HW_{d,2^d}$ in terms of number of vertices, degree, edges, diameter, and broadcast time. With the use of extensive simulation, we provide diameter and broadcast time of $HW_{d,2^d}$ for all $d \leq 24$.

Chapter 6 concludes the thesis and lists the future work.

Chapter 2

Literature Review

This chapter is divided in two sections. The first section of this chapter briefly reviews the commonly used interconnection topologies. A topology is a schematic or geometric description of the arrangement of a network (graph), including its nodes (vertices) and connecting lines (edges). The second section of this chapter surveys the Knödel graph in the light of known important results from the previous research work.

2.1 Commonly used topologies

This section reviews the commonly used topologies on basis of three important communication parameters: (i) the degree, (ii) the diameter, and (iii) the broadcast time.

The Path P_n

The path P_n is a tree with two end nodes of vertex degree 1, and the remaining $n-2$ nodes of vertex degree 2, thus the maximum degree of P_n is 2. The $D(P_n) = b(P_n) = n - 1$. A path is therefore a graph that can be drawn so that all of its vertices and edges lie on a single straight line [24]. Figure 2.1 shows a path with seven vertices, where $D(P_7) = b(P_7) = 6$.

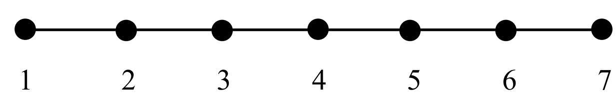


Figure 2.1 Path P_7

The Cycle C_n

Cycle C_n, $n \geq 3$, is a simple graph with vertices $v_1, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$. In other words cycle C_n is a path such that the start vertex and end vertex are also connected by an edge. C_n has *n* vertices and the maximum degree is 2. The $D(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and the $b(C_n) = \left\lceil \frac{n}{2} \right\rceil$. Figure 2.2 demonstrates C_6, where the diameter and the broadcast time of C_6 is 3.

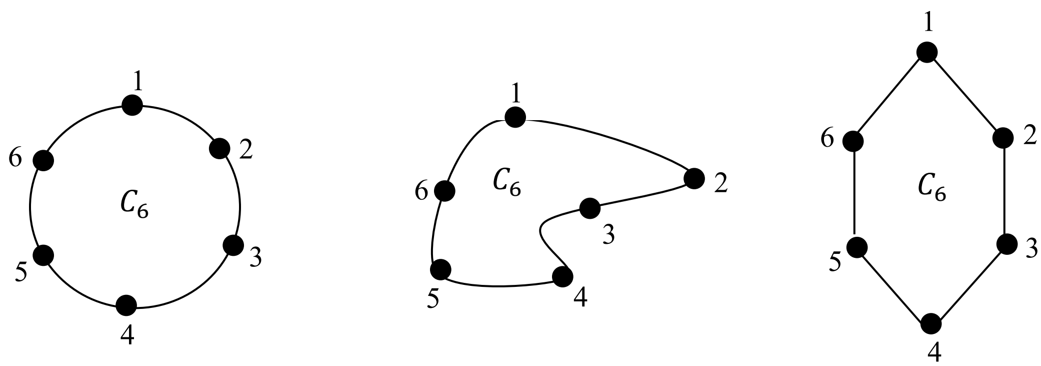


Figure 2.2 Cycle C_6 in three different shapes

The Complete graph K_n

A complete graph K_n is a simple graph with exactly one edge between any pair of distinct vertices. K_n has *n* vertices and degree *n* – 1. The diameter of K_n is 1. K_n is a broadcast graph because during each time unit the number of informed vertices is doubled, thus $b(K_n) = \lceil \log_2 n \rceil$. Figure 2.3 shows a complete graph K_6, where $b(K_6) = 3$.

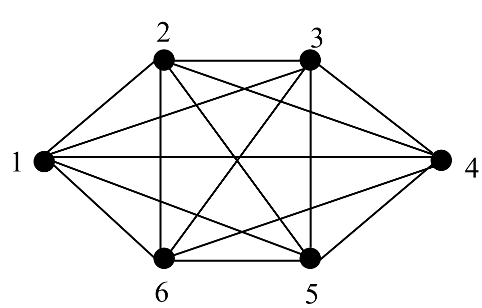


Figure 2.3 The Complete graph K_6

The Hypercube H_n

The hypercube of dimension n, denoted by H_n, is a simple graph with vertices representing 2^n bit strings of length n, $n \geq 1$ such that adjacent vertices have bit strings differing in exactly one bit position. H_n has 2^n vertices and $n \cdot 2^{n-1}$ edges. The diameter of H_n is n and each vertex has exactly degree n. A (n+1)-dimensional hypercube can be constructed from two n-dimensional hypercubes by connecting each pair of the corresponding vertices. H_n is the minimum broadcast graph. The $b(H_n) = \lceil \log_2 2^n \rceil = n$. Figure 2.4 illustrates three hypercubes of dimensions 1, 2 and 3.

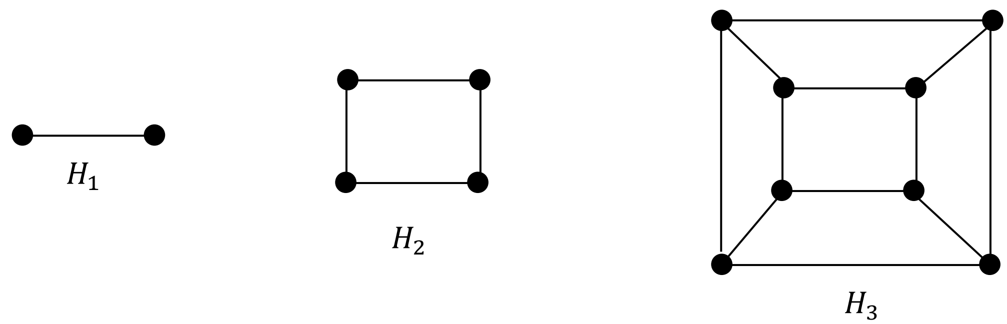


Figure 2.4 Hypercubes of dimensions 1, 2 and 3

The Cube-Connected Cycles CCC_n

CCC_n is a modification of the hypercube H_n by replacing each vertex of the hypercube with a cycle of n vertices. The i-th dimensional edge incident to a node of the hyper-node is then connected to the i-th node of corresponding cycle of the CCC_n. Thus, CCC_n has $n \cdot 2^n$ nodes and the maximum degree is 3. The $D(CCC_n) = 2n + \left\lfloor \frac{n}{2} \right\rfloor - 2$. The $b(CCC_n) = \left\lceil \frac{5n}{2} \right\rceil - 1$ [9], first every informed vertex sends the message to the hypercube neighbor, then to the right neighbor on the ring, and finally to the left one. Figure 2.5 shows a 3-dimensional cube connected cycle.

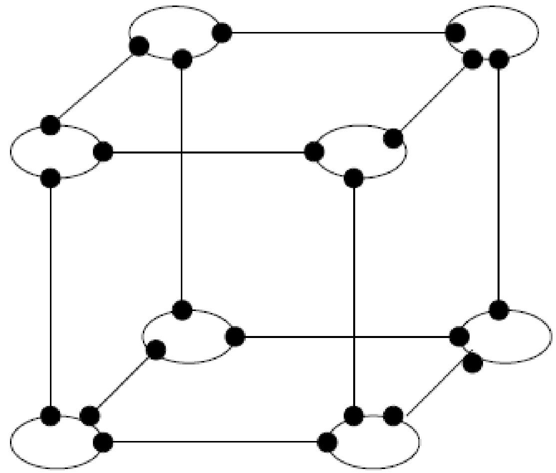


Figure 2.5 Cube Connected Cycle CCC_3

The Shuffle-Exchange SE_n

SE_n is the graph whose vertices can be represented by binary strings of length n. Each edge of SE_n connects vertex βa, where β is a binary string of length $n-1$ and a is in {0, 1}, with vertex βc and vertex βa, where c is the binary complement of a. SE_n has 2^n vertices and the maximum degree is 3. The $D(SE_n) = 2n-1$ and in [38], it is provided that $b(SE_n) \leq 2n-1$. Figure 2.6 presents a Shuffle-Exchange graph SE_3.

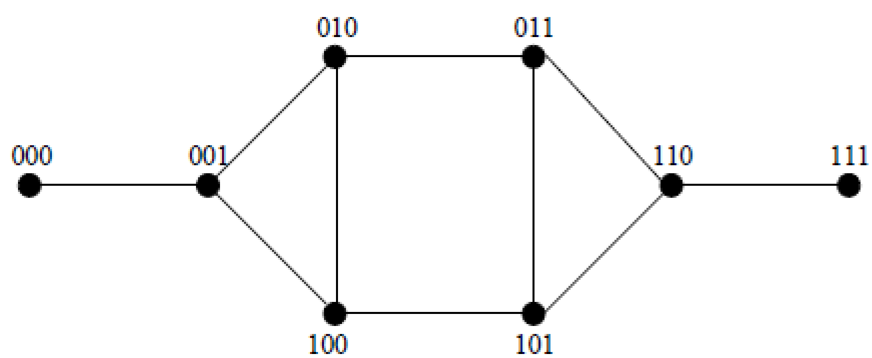


Figure 2.6 Shuffle-Exchange graph SE_3

The DeBruijn DB_n

DB_n is the graph, whose nodes can be represented by binary strings of length n and whose edges connect each string βa, where β is a binary string of length $n-1$ and a is in {0, 1}, with the strings βb, where b is a symbol in $\{0, 1\}$. DB_n has 2^n vertices with the maximum degree 4 and the diameter is n. [43] provides the lower bound $b(DB_n) \geq 1.3171n$, and [4] proves the upper bound, $b(DB_n) \leq 1.5n + 1.5$. Figure 2.7 illustrates a DeBruijn graph of dimension 3.

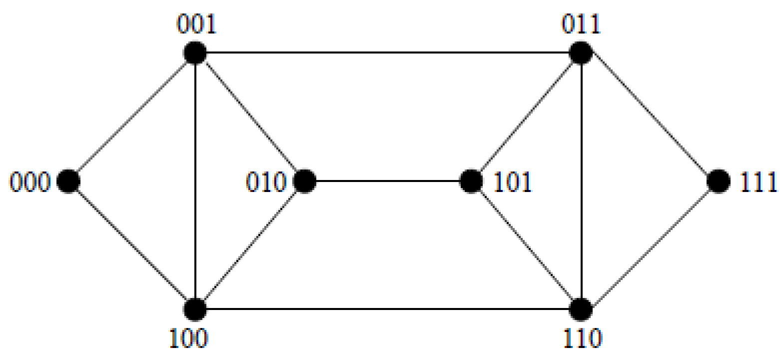


Figure 2.7 DeBruijn graph DB_3

The d-Grid $G[a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d]$

The d-dimensional grid (or mesh) is the graph whose nodes are all d-tuples of positive integers $(z_1, z_2, \ldots, z_d)$, where $0 \leq z_i < a_i$ for all i $(1 \leq i \leq d)$, and whose edges connect d-tuples, which differ in exactly by coordinate one. For example, in $G[3,3]$, vertex (1, 1) is connected to vertices (0, 1), (2, 1), (1, 0) and (1, 2). $G[a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d]$ has $a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d$ vertices with the maximum degree $2d$, if each a_i is at least 3. The diameter of d-Grid $G[a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d]$ is $(a_1 - 1) + (a_2 - 1) + \cdots + (a_d - 1)$ and [33] provides the $b(G[a_1 \text{ X } a_2]) = a_1 + a_2 - 2$. Figure 2.8 shows a 2-Grid graph $G[4 \text{ x } 5]$.

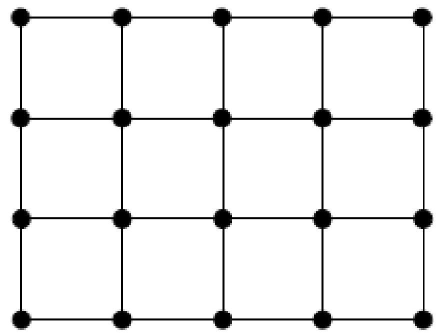


Figure 2.8 2-Grid graph $G[4 \text{ x } 5]$

The *d*-Torus *T*

A d-Torus graph is a d-grid graph with both ends of rows and columns connected. $T[a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d]$ denotes the d-Torus graph. The diameter of k x k X-Torus is given in [24], that is $\lfloor k/2 \rfloor + 1$ if k is odd, and $\lfloor k/2 \rfloor$ if k is even. It is proven in [11] that the optimal broadcast time of 2-Torus graph is $\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil$, when a_1 or a_2 is even; and it is $\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil - 1$, when both a_1 and a_2 are odd. The bounds on the broadcast time of Torus are $D \leq b(T[a_1 \text{ X } a_2 \text{ X} \ldots \text{ X } a_d]) \leq D + \max(0, m-1)$, where $D = \sum_{i=1}^{d} a_i - d$, and m is the number of odd a_i. Figure 2.9 shows a 2-Torus graph $T[4 \text{ x } 3]$.

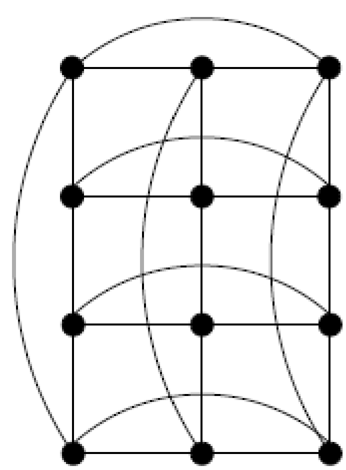
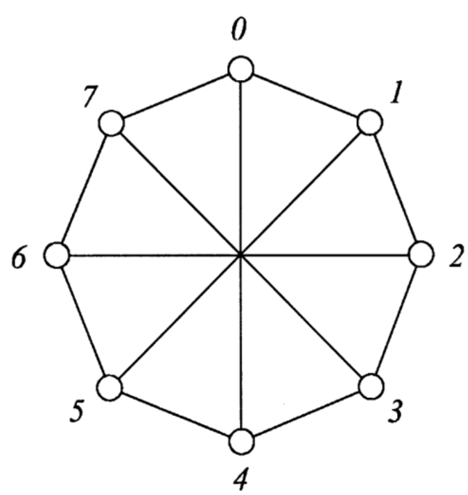


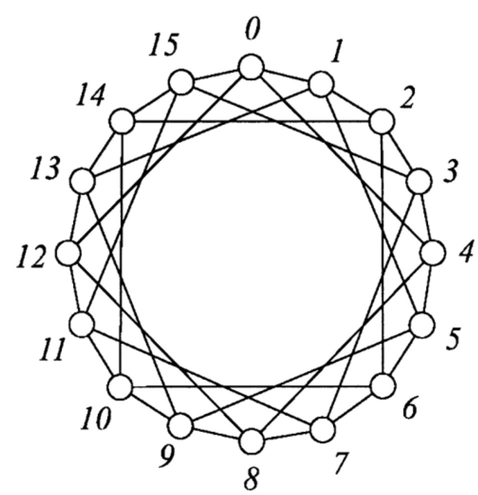
Figure 2.9 2-Torus graph $T[4 \text{ x } 3]$

Recursive Circulant graph $G(n, d)$

The recursive circulant graph $G(n, d)$ is introduced by Park and Chwa [37]. We define the recursive circulant graph $G(n, d) = (V, E)$ with $d \geq 2$, to be a graph where, $V = \{0,1, \dots, n-1\}$, and the edge set $E = \{uv | \exists i, 0 \leq i \leq \lceil \log_d(n) \rceil - 1,$ such that $u + d^i \equiv v (\bmod n)\}$. $G(N, d)$ has recursive structure when $N = cd^m$, $1 \leq c < d$. The [37] provides the diameter as follows: if *d* is odd, $D\big(G(cd^m, d)\big) = \lfloor d/2 \rfloor m + \lfloor c/2 \rfloor$. When *d* is even and *c* is odd, the diameter is $\left\lceil \frac{d-1}{2} m \right\rceil + \lfloor c/2 \rfloor$. Finally, when both *d* and *c* are even, the diameter is $\left\lfloor \frac{d-1}{2} m \right\rfloor + \lfloor c/2 \rfloor$. $G(2^m, 4)$, whose degree is *m*, compares favorably to the hypercube H_m. $G(2^m, 4)$ has the maximum possible connectivity, and its diameter is $\lceil 3m - 1/4 \rceil$. The broadcast time of $G(2^m, 4)$ is *m*. Figure 2.10 shows the two recursive circulant graphs, *G*(8,4) and *G*(16,4).



(a) $G(8, 4)$



(b) $G(16, 4)$

Figure 2.10 Recursive circulant graphs $G(8, 4)$ and $G(16, 4)$

The summary of the communication properties (i.e., the degree, the diameter and the broadcast time, of reviewed commonly used topologies), is provided in Table 2.1.

Table 2.1 Summary of commonly used topologies

Graph	Degree	Diameter	Broadcast time
Path graph P_n	2	$n-1$	$n-1$
Cycle C_n	2	$\left\lfloor \frac{n}{2} \right\rfloor$	$\left\lceil \frac{n}{2} \right\rceil$
Complete graph K_n	$n-1$	1	$\lceil \log_2 n \rceil$
Hypercube H_n (n = dimension)	n	n	$n = \lceil \log_2 2^n \rceil$
Cube-Connected Cycles CCC_n	3	$2n + \left\lfloor \frac{n}{2} \right\rfloor - 2$	$\left\lceil \frac{5n}{2} \right\rceil - 1$
Shuffle-Exchange SE_n	3	$2n - 1$	$2n - 1$
DeBruijn DB_n	4	n	$1.3171n \leq b(DB_n) \leq 1.5n + 1.5$
d-Grid $G[a_1 \text{ x } a_2 \text{ x} \dots \text{x } a_d]$	$2d$	$(a_1 - 1) + (a_2 - 1) + \cdots + (a_d - 1)$	Broadcast time of a 2-grid $b(G[a_1 \text{ X } a_2]) = a_1 + a_2 - 2$
d-Torus graph $[a_1 \text{ x } a_2 \text{ x} \dots \text{ x } a_d]$	$2d$	Diameter of kxk d-Torus, is $\lfloor k/2 \rfloor + 1$ if k is odd and $\lfloor k/2 \rfloor$ if k is even	The bounds on the broadcast time of d-Torus are $D \leq b(T[a_1 \text{ x } a_2 \text{ x} \dots \text{ x } a_d]) \leq D + \max(0, m-1)$, where $D = \sum_{i=1}^{d} a_i - d$, and m is the number of odd a_i.
Recursive Circulant $G(2^m, 4)$	m	$\lceil 3m - 1/4 \rceil$	m

2.2 Survey of the Knödel graph

The Knödel graph $W_{d,n}$ is a regular graph of even order *n* and degree d, $1 \leq d \leq \lfloor \log_2 n \rfloor$. It was introduced by W. Knödel for $d = \lfloor \log_2 n \rfloor$ in 1975 and was used in an optimal gossiping algorithm [40]. For smaller *d,* the family of Knödel graphs has been defined formally by Fraigniaud and Peters [19]. Since 1994 a lot of research has been done on Knödel graph, especially because some subfamilies of the Knödel graph tend to have good properties in terms of broadcasting and gossiping [16]. Many graphs introduced as minimum broadcast (resp. gossip) graphs, such as in [7] [39] [41], were in fact isomorphic to the Knödel graphs [17].

In particular, for any $n = 2^d$, the Knödel graph of order *n* and degree *d*, $W_{d,2^d}$, turns out to be minimum broadcast (resp. gossip, linear gossip) graph [16]. In that way $W_{d,2^d}$ competes to the hypercube of dimension *d*, H_d, and the recursive circulant graph $G(2^d, 4)$ [37]. These three topologies are comparable because they all have good properties in terms of interconnection networks. Moreover, they are of the same order 2^d, and regular of the same degree *d.* The Knödel graph become famous due to its smallest known diameter among all regular and minimum broadcast graphs on 2^d vertices with degree *d* [26].

Definitions of the Knödel graph

The Knödel graph has been formally defined in [19] as follows:

Definition 1: (Knödel graph – one layer representation)

The Knödel graph $W_{d,n}$ of an even order n and degree d is the graph G = (V, E) with number of vertices, V={0, 1,…, n-1} and $E = \{(i,j) |\ i + j = 2^r - 1 \ mod\ n,$ $0 \leq i,j \leq n-1,\ 1 \leq r \leq d\}$.

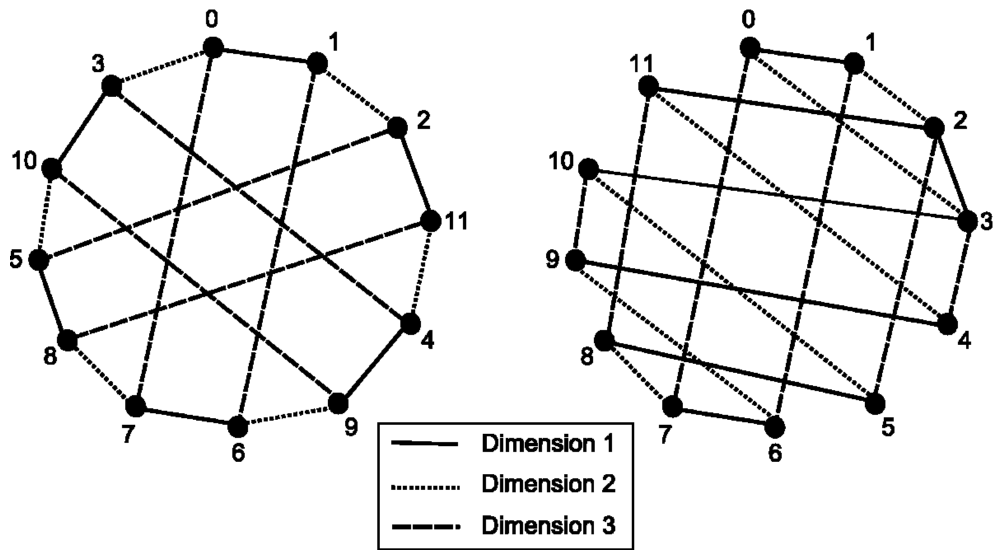


Figure 2.11 One layer representations of the Knödel graph $W_{3,12}$

It is clear from the above definition that the Knödel graph is a regular graph of degree *d*. Through-out this thesis we will refer to this definition as one-layer representation. Figure 2.11 demonstrates the one-layer representations of the Knödel graph $W_{3,12}$.

Definition 2: (Knödel graph – Bipartite or two-layer representation)

The Knödel graph on n ≥ 2 vertices (n even) and degree d, 1 ≤ d ≤ $\lfloor log_2 n \rfloor$, *is denoted by* $W_{d,n}$. *The vertices of* $W_{d,n}$ *are the pairs* (i, j) *with* $i = 0, 1$ *and* $0 \leq j \leq \frac{n}{2} - 1$, *and the set of edges:* $E = \{(0, i), (1, j) \mid j = i + 2^r - 1 \; mod \; \frac{n}{2}, \; 0 \leq i, j \leq \frac{n}{2} - 1, \; 0 \leq r \leq d - 1\}$.

Dimensions of the Knödel graph

An edge of $W_{d,n}$ that connects a vertex $(0, j)$ to vertex $\left(1, i + 2^r - 1 \; mod \; \frac{n}{2}\right)$ is said to be in dimension r, where $0 \leq r \leq d - 1$. Figure 2.12 illustrates an example of the Knödel graph $W_{3,12}$ in a bipartite representation.

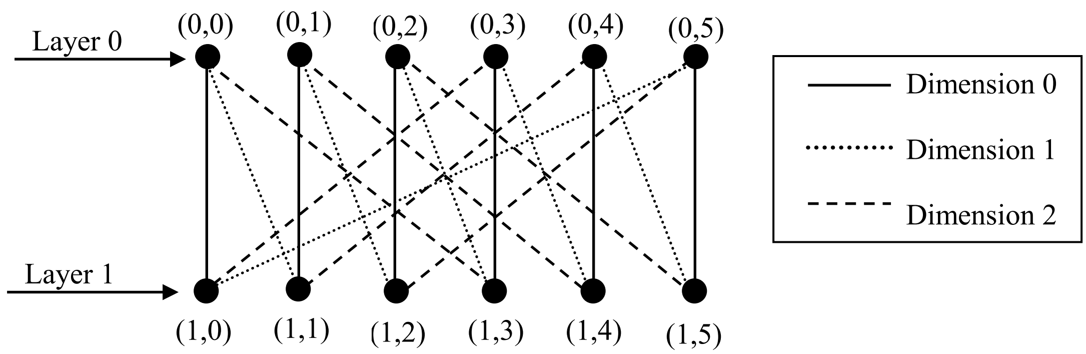


Figure 2.12 Bipartite or two-layer representation of the Knödel graph $W_{3,12}$

It is clear from the above definition, that the Knödel graph $W_{d,n}$ is a bipartite graph. Knödel graph $W_{d,n}$ is connected, iff $d \geq 2$, since in that case it suffices to alternate edges in dimensions 0 and 1 to get Hamiltonian cycle [16].

The Knödel graph $W_{d,n}$ can also be defined as a Cayley graph [34] [35], as stated in Proposition 1 below.

Proposition 1 [35]

For any even n and $1 \leq d \leq \lfloor \log n \rfloor$, $W_{d,n}$ is a Cayley graph on the semi-direct product $G = \mathbb{Z}_2 \ltimes \mathbb{Z}_{n/2}$ for the multiplicative law: $(x, y)(x', y') = (x+x', y+(-1)^x y')$, with $x, x' \in \mathbb{Z}_2$ and $y, y' \in \mathbb{Z}_{n/2}$ with the set of generators $S = \{(1, 2^i\text{-}1), 0 < i < \Delta - 1\}$.

Corollary 1 [16]

For any even n and $1 \leq d \leq \lfloor \log n \rfloor$, $W_{d,n}$ is *vertex-transitive*.

Proof: This follows directly from Proposition 1 above because it is well known that any Cayley graph is vertex-transitive (see [44]).

It has been proven in [16], that for any n and $1 \leq d \leq \lfloor \log n \rfloor$, it is possible to construct $W_{d+1,2n}$ by taking two copies of $W_{d,n}$ and linking the vertices of each copy by a certain perfect matching.

Knödel graphs, along with their routing, broadcasting and gossiping performances, have been studied in [2], where at each step only edges in a certain dimension k are being used. Such graphs are called *Modified Knödel Graphs*, which turn out to be isomorphic to Knödel graphs $W_{\lfloor \log_2(n) \rfloor, n}$ according to Definition 2, for any n not a power of 2 [16]. Their main goal was to study the performances of these graphs, when dimensions are used alternatively. They proved that the dimensions of the Knödel graphs had a similar role than the ones of hypercubes, with respect to routing, broadcasting and gossiping.

Shortest path problem in the Knödel graph

In [31] 2-approximation algorithm with the logarithmic time complexity is proposed for the shortest path problem in the Knödel graph $W_{d,2^d}$.

Diameter of the Knödel graph

Despite being a highly symmetric and widely studied graph, the diameter of the Knödel graph $W_{d,n}$ is only known for $n = 2^d$ and degree d. In [13], it was proven that $D(W_{d,2^d}) = \left\lceil \frac{d+2}{2} \right\rceil$. The nontrivial proof of this result is algebraic and the actual diametral path is not presented. The diameter of $W_{d,2^d}$ is the smallest among all known regular and broadcast networks on 2^d vertices with degree d [26].

The diameter is one of the parameters for which we can say that $W_{d,2^d}$ can compete with Hypercube H_d and Recursive circulant $G(2^d, 4)$ graphs. Table 2.2 [16] provides the comparison between $W_{d,2^d}$, H_d, and $G(2^d, 4)$.

Table 2.2 Comparison between $W_{d,2^d}$, H_d, and $G(2^d, 4)$

Properties	H_d	$G(2^d, 4)$	$W_{d,2^d}$
Number of vertices	2^d	2^d	2^d
Degree	d	d	d
Diameter	d	$\lceil (3d-1)/4 \rceil$	$\lceil (d+2)/2 \rceil$
Vertex-transitivity	Yes	Yes	Yes
Edge-transitivity	Yes	No	No
Hamiltonian cycle	Yes	Yes	Yes
Binomial Tree	Yes	Yes	Yes

The upper and lower bounds on the diameter of the Knödel graph $W_{d,n}$ have recently been proved in [22]. These bounds are given by the following Theorem 1 and 2.

Theorem 1 [22]*:* **(Upper bound on Diameter)** *Let* $a = \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor$ *and* $b = \Delta - 2 (\Delta \geq 3)$.

$$\text{If } a \geq b \text{ then } D(W_{\Delta,n}) \leq 2a + 3 = 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 3, \text{otherwise}$$

$$D(W_{\Delta,n}) \leq 2a + 3\lceil (\Delta - 2 - a)/4 \rceil + 7 \ \leq \frac{3}{4}\Delta + \frac{5}{4}a + \frac{17}{2}.$$

Theorem 2 [22]*:* **(Lower bound on Diameter)** $D(W_{\Delta,n}) \geq 2 \left\lfloor \frac{1}{2} \left\lceil \frac{n-2}{2^{\Delta}-2} \right\rceil \right\rfloor + 1.$

Broadcasting in the Knödel graph

The Knödel graph has been studied since long time in terms of broadcasting and gossiping. The Knödel graph $W_{\lfloor \log_2(n) \rfloor, n}$ is a broadcast and gossip graph [2] [13] [18]. The $W_{\lfloor \log_2(n) \rfloor, n}$ is used to construct sparse broadcast graphs of a bigger size by interconnecting several smaller copies or by adding and deleting vertices [3] [8] [23] [27] [28] [29] [30] [31]. The broadcast time of the Knödel graph is known only for $W_{d,2^d}$ and for $W_{d-1,2^d-2}$. It is shown that $b(W_{d,2^d}) = d$ $(d \geq 1)$ [12] [37] [40] and that

$b(W_{d-1,2^d-2}) = d\ (d \geq 2)$ [7] [39]. Table 2.3 [42] provides a summary of known broadcast and gossiping properties of the Knödel graph.

Table 2.3 Broadcast and gossip properties of the Knödel graphs

Type of graph	Properties
$W_{d,2^d}$	Minimum broadcast graph [12] Minimum gossip graph [40] Minimum linear gossip graph [19]
$W_{d-1,2^d-2}$	Minimum broadcast graph [7] [39] Minimum gossip graph [41] Minimum linear gossip graph [19]
$W_{d-1,2^d-4}$	Minimum gossip graph [41] Minimum linear gossip graph [19]
$W_{d-1,2^d-6}$	Minimum linear gossip graph [19]
$W_{d-2,n}$ $2^{d-1}+2 \leq n \leq 3 \cdot 2^{d-2}-4$	Broadcast graph [14] Gossip graph [14] Linear gossip graph [15]
$W_{d-1,n}$ $3 \cdot 2^{d-2}-4 \leq n \leq 2^d-2$	Broadcast graph [14] Gossip graph [14] Linear gossip graph [15]

It is shown in [2] that the edges of the Knödel graph can be grouped into dimensions that are similar to hypercube dimensions. This allows these dimensions to be used in a similar manner to hypercube for broadcasting [27].

The broadcast graphs on odd number of vertices have been constructed in [1], by applying a vertex deletion method to the Knödel graph. This construction provides an improved general upper bound on $B(n)$ for all odd *n* except when $n = 2^d - 1$.

The general upper and lower bounds on the broadcast time of the Knödel graph $W_{d,n}$, have recently been proven in [23], which are as follows:

$$2\left\lfloor\frac{1}{2}\left\lceil\frac{n-2}{2^d-2}\right\rceil\right\rfloor + 1 \leq b(W_{d,n}) \leq \left\lceil\frac{n-2}{2^d-2}\right\rceil + d - 1$$

Chapter 3

The Diameter of the Knödel graph

This chapter is divided in three sections. In the first section, the diameter of the Knödel graph $W_{3,n}$, for $n > 8$, is given through a constructive proof. In the second section, we find the diameters of some specific Knödel graphs through extensive simulation. In the last section, we present three propositions for the number of vertices at a particular distance in some specific Knödel graphs.

3.1 The Diameter of Knödel graph $W_{3,n}$

In this section, we present the diameter of the Knödel graph $W_{3,n}$. Our proof is constructive and we provide an actual diametral path in $W_{3,n}$. The distance between vertices u and v is denoted by $dist(u, v)$. Using these notations and the vertex transitivity of the Knödel graph, we state that

$$D(W_{3,n}) = max\ \{dist(0, y) \mid 0 \leq y \leq n - 1\}.$$

3.1.1 Paths in the Knödel graph $W_{3,n}$

Recall that according to the Definition 1 of the Knödel graph $W_{d,n}$, in $W_{3,n}$ three different paths can be formed using the dimensions: (i) 1 and 2, (ii) 2 and 3, and (iii) 1 and 3. In the first path the 1 and 2-dimensional edges "move" forward by only two vertices. In the second path 2 and 3-dimensional edges, "move" forward by only four vertices. In fact, a shorter path can be formed by the 1 and 3-dimensional edges, where every "move" is of six vertices.

We construct three different paths between two vertices in the Knödel graph $W_{3,n}$. These paths have certain properties, which will be used to determine the diameter of $W_{3,n}$. We first discuss the set of vertices in $W_{3,n}$ that can be reached from vertex 0, using 1 and 3-dimensional edges. It is clear that in $W_{3,n}$ we can "move" either clockwise or anti-clockwise from vertex 0, as shown in Figure 3.1. We can choose the path

$$0 \xrightarrow{dim:1} 1 \xrightarrow{dim:3} 6 \xrightarrow{dim:1} n-5 \xrightarrow{dim:3} 6(2) \xrightarrow{dim:1} \dots \xrightarrow{dim:3} 6x \qquad \text{(clockwise)}$$

or the path

$$0 \xrightarrow{dim:3} 7 \xrightarrow{dim:1} n-6 \xrightarrow{dim:3} 13 \xrightarrow{dim:1} n-6(2) \xrightarrow{dim:3} \dots \xrightarrow{dim:1} n-6x \ \ (\text{anti}-\text{clockwise})$$

by alternating the 1 and 3-dimensional edges. The 3-dimensional edges move "forward" by 5 vertices, whereas the 1-dimensional edges by 1 vertex only. So, in the each iteration, the 1 and 3-dimensional edges move forward by 6 vertices. These two paths will eventually intersect or overlap, somewhere near the vertex $n/2$. There are six possible cases of $W_{3,n}$, depending on the number of vertices. The 1 and 3-dimensional edges will split $W_{3,n}$ into $2x$ segments, where

$$2x = \frac{number\ of\ vertices}{6},$$

each having length 6, except the one containing vertex $n/2$. We can perform only

$$x = \frac{1}{2}\left(\frac{number\ of\ vertices}{6}\right) = \frac{number\ of\ vertices}{12}$$

1 and 3-dimensional passes in each of these two paths, (i.e., clockwise and anti-clockwise), before they intersect. Therefore, we will never use more than x 1 and 3-dimensional passes to reach a vertex in $W_{3,n}$.

Using 1 and 3-dimensional edges, we can reach the vertices $6, 6(2), 6(3), \ldots, 6(x-1), 6x$ from vertex 0, in the clockwise direction. Similarly, anti-clockwise, we reach the vertices $n-6,\ n-6(2),\ n-6(3), \ldots,\ n-6(x-1),\ n-6x$.

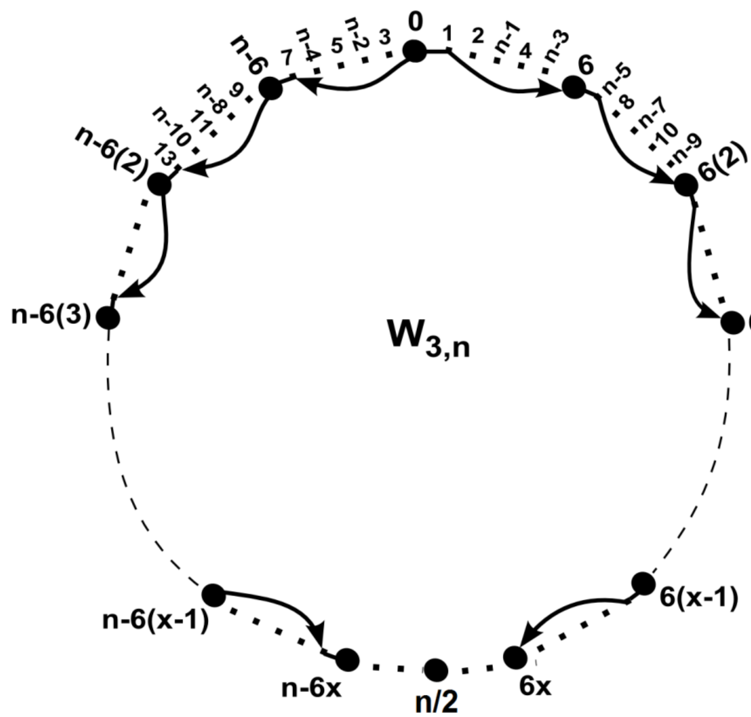


Figure 3.1 Schematic illustration of paths (clockwise and anti-clockwise) in $W_{3,n}$.

Once, we arrive at the vertices $v_1 = 6x$ and $v_2 = n - 6x$, then our goal is to find and reach the diametral vertices. Since, these diametral vertices of $W_{3,n}$ cannot be reached using 1 and 3-dimensional edges. Therefore, these vertices are reached by the small moves of 1, 2 or 3 dimensional edges, either in "forward" or "backward" directions, from vertices v_1 and v_2.

3.1.2 Six cases of the Knödel graph $W_{3,n}$

In this section, we consider six different cases depending on number of vertices of the Knödel graph $W_{3,n}$, in order to determine the diameter of $W_{3,n}$.

Case 1: $n = 0 \; mod \; 6$ and $\frac{n}{2}$ is even

$n = 0 \; mod \; 6$ and $\frac{n}{2}$ is even, can be written as $n = 0 \; mod \; 12$. Let us consider $n = 12x$, for some $x \in \mathbb{Z}^+$ ($\mathbb{Z}^+$: set of positive integers). We can perform only $x = \frac{n}{12}$, 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. Figure 3.2 illustrates the discussed path. The vertices v_1 and v_2 can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n}{12}\right) = \frac{n}{2} \qquad \text{(clockwise)}$$

and $$v_2 = n - 6x = n - 6\left(\frac{n}{12}\right) = \frac{n}{2} \qquad \text{(anti-clockwise)}$$

In this case, in either direction, we reach the vertex $v = \frac{n}{2}$, using 1 and 3-dimensional edges. The distance between the vertices 0 and v,

$$dist\left(0, \frac{n}{2}\right) = 2x = 2\left(\frac{n}{12}\right) = \frac{n}{6}$$

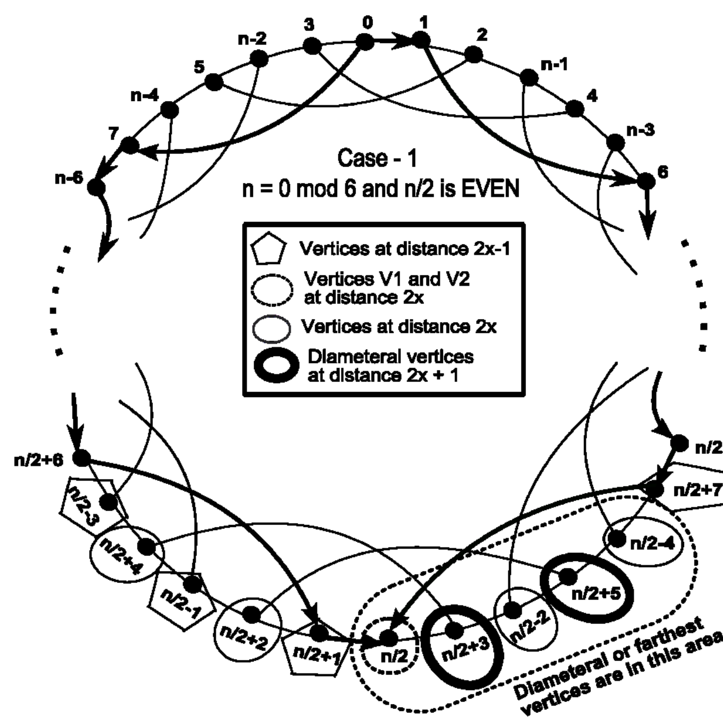


Figure 3.2 The diameter of the Knödel graph $W_{3,n}$, where n = 0 mod 6 and n/2 is even

Two vertices, $\frac{n}{2}+3$ and $\frac{n}{2}+5$, are the neighbors of the vertices those are at distance $2x$, from vertex 0. Therefore, their distance from vertex 0 is $2x+1$. Since, there is no any other vertex in the graph, whose distance is greater than the vertices $\frac{n}{2}+3$ and $\frac{n}{2}+5$, from vertex 0. Therefore, these are the diametral vertices of $W_{3,n}$, when $n = 0\ mod\ 6$ and $\frac{\mathrm{n}}{2}$ is even. Each diametral vertex can be reached from three vertices of distance $2x$ as follows:

$$\frac{n}{2}-2 \xrightarrow{dim:1} \frac{n}{2}+3, \qquad \frac{n}{2} \xrightarrow{dim:2} \frac{n}{2}+3 \quad \text{and} \quad \frac{n}{2}+4 \xrightarrow{dim:3} \frac{n}{2}+3$$

$$\frac{n}{2}-4 \xrightarrow{dim:1} \frac{n}{2}+5, \qquad \frac{n}{2}-2 \xrightarrow{dim:2} \frac{n}{2}+5 \quad \text{and} \quad \frac{n}{2}+2 \xrightarrow{dim:3} \frac{n}{2}+5$$

Since the diametral vertices are at distance $2x+1 = \frac{n}{6}+1$ from vertex 0, thus,

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6}+1} \qquad \text{for } n = 0\ mod\ 6 \text{ and } \frac{n}{2} \text{ is even}$$

Case 2: $\boldsymbol{n = 0\ mod\ 6}$ **and** $\boldsymbol{\frac{\mathrm{n}}{2}}$ **is odd**

$n = 0\ mod\ 6$ and $\frac{\mathrm{n}}{2}$ is odd, can be written as $n = 6\ mod\ 12$. Let us consider $n = 12x+6$, for some $x \in \mathbb{Z}^{+}$. We can perform only $x = \frac{n-6}{12}$ 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. The discussed path is demonstrated in Figure 3.3. The vertices v_1 and v_2 can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n-6}{12}\right) = \frac{n-6}{2} = \frac{n}{2}-3 \qquad \text{(clockwise)}$$

and $$v_2 = n-6x = n-6\left(\frac{n-6}{12}\right) = n-\left(\frac{n-6}{2}\right) = \frac{n}{2}+3 \qquad \text{(anti-clockwise)}$$

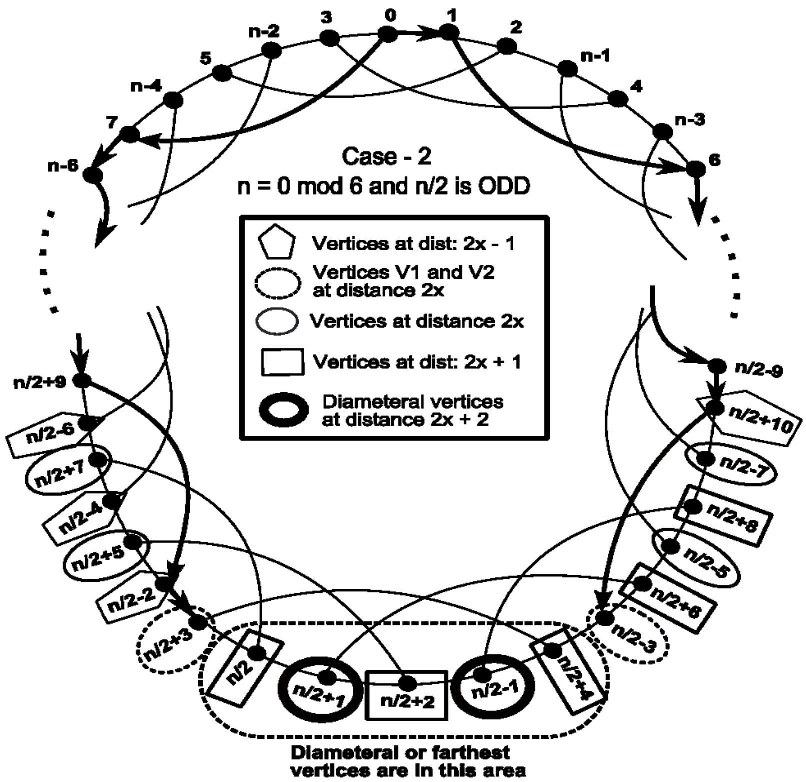


Figure 3.3 The diameter of the Knödel graph $W_{3,n}$, where *n* = 0 *mod* 6 and *n*/2 is odd

The vertices v_1 and v_2 are at distance $2x$ from vertex 0.

$$dist\left(0,\ \frac{n}{2} - 3\right) = 2x \ = 2\left(\frac{n-6}{12}\right) \ = \frac{n-6}{6} \ = \frac{n}{6} - 1$$

$$dist\left(0,\ \frac{n}{2} + 3\right) = 2x \ = 2\left(\frac{n-6}{12}\right) \ = \frac{n-6}{6} \ = \frac{n}{6} - 1$$

Once we reach the vertices $v_1 = \frac{n}{2} - 3$ and $v_2 = \frac{n}{2} + 3$ then our goal is to find and reach the diametral vertices, using the paths discussed in Section 3.1.1. The vertices $\frac{n}{2} - 7$, $\frac{n}{2} - 5$, $\frac{n}{2} + 5$ and $\frac{n}{2} + 7$ are also at distance $2x$ because they are the neighbors of the vertices of distance $2x - 1$ from vertex 0. The vertices $\frac{n}{2}$, $\frac{n}{2} + 2$, $\frac{n}{2} + 4$, $\frac{n}{2} + 6$, and $\frac{n}{2} + 8$ are the neighbors of the vertices of distance $2x$ from vertex 0. Therefore, their distance from vertex 0 is $2x + 1$. Two vertices labeled with $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$, are only connected to the vertices of distance $2x + 1$. Therefore, these two vertices are at distance

$2x+2$, from vertex 0. Since there is no any other vertex in the graph whose distance is greater than the distance of vertices $\frac{n}{2}+3$ and $\frac{n}{2}+5$, from vertex 0. Therefore, these are the diametral vertices of $W_{3,n}$, when $n = 0 \ mod\ 6$ and $\frac{n}{2}$ is odd. Each diametral vertex can be reached from three vertices of distance $2x+1$ as follows:

$$\frac{n}{2}+2 \xrightarrow{dim:\,1} \frac{n}{2}-1, \qquad \frac{n}{2}+4 \xrightarrow{dim:\,2} \frac{n}{2}-1 \quad \text{and} \quad \frac{n}{2}+8 \xrightarrow{dim:\,3} \frac{n}{2}-1$$

$$\frac{n}{2} \xrightarrow{dim:\,1} \frac{n}{2}+1, \qquad \frac{n}{2}+2 \xrightarrow{dim:\,2} \frac{n}{2}+1 \quad \text{and} \quad \frac{n}{2}+6 \xrightarrow{dim:\,3} \frac{n}{2}+1$$

Since the diametral vertices are at distance $2x+2$ from vertex 0, thus

$$D(W_{3,n}) = 2x+2 = \left(\frac{n}{6}-1\right)+2 = \frac{n}{6}+1 \qquad \text{where } 2x = \frac{n}{6}-1$$

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6}+1} \qquad \text{for } n = 0 \ mod\ 6 \text{ and } \frac{n}{2} \text{ is odd}$$

Case 3: $\quad$ **$n = 2 \ mod\ 6$ and $\frac{n}{2}$ is even**

$n = 2 \ mod\ 6$ and $\frac{n}{2}$ is even, can be written as $n = 8 \ mod\ 12$. Let us consider $n = 12x+8$, for some $x \in \mathbb{Z}^+$. We can perform only $x = \frac{n-8}{12}$ 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. The discussed path is demonstrated in Figure 3.4. The vertices v_1 and v_2, can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n-8}{12}\right) = \frac{n-8}{2} = \frac{n}{2}-4 \qquad \text{(clockwise)}$$

and $\quad v_2 = n-6x = n-6\left(\frac{n-8}{12}\right) = n-\left(\frac{n-8}{2}\right) = \frac{n}{2}+4 \qquad$ (anti-clockwise)

From vertex 0, the vertices v_1 and v_2 are at the distance:

$$dist\left(0,\ \frac{n}{2}-4\right) = 2x = 2\left(\frac{n-8}{12}\right) = \frac{n-8}{6} = \frac{n}{6}-\frac{4}{3}$$

$$dist\left(0,\ \frac{n}{2}+4\right) = 2x = 2\left(\frac{n-8}{12}\right) = \frac{n-8}{6} = \frac{n}{6}-\frac{4}{3}$$

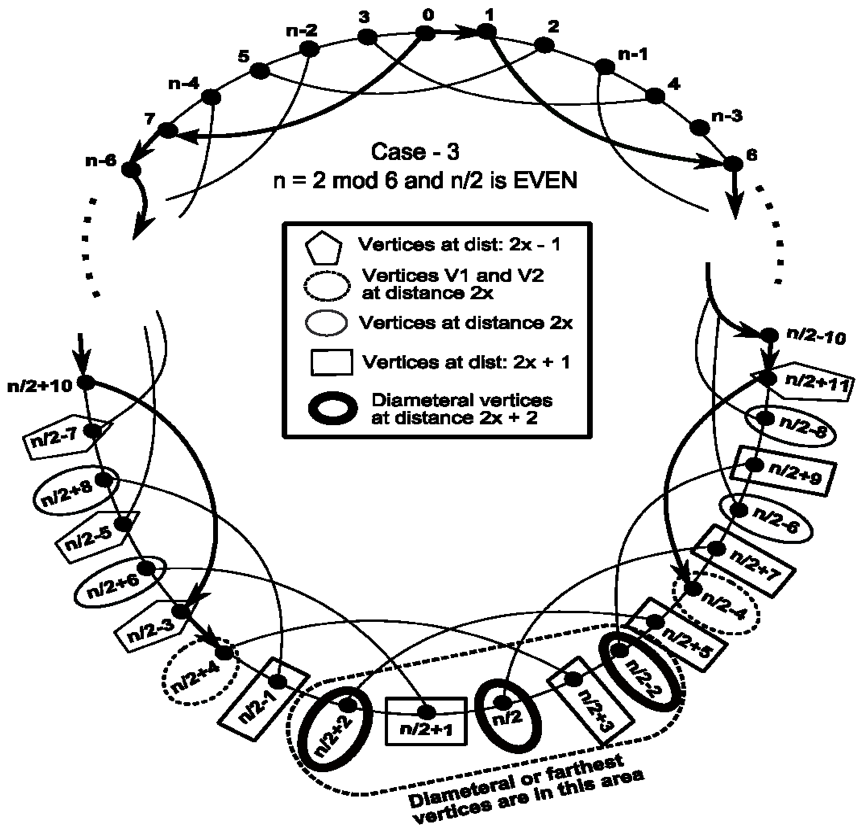


Figure 3.4 The diameter of the Knödel graph $W_{3,n}$ where n = 2 mod 6 and n/2 is even

Once we reach the vertices $v_1 = \frac{n}{2} - 4$ and $v_2 = \frac{n}{2} + 4$ then our goal is to find and reach the diametral vertices, using the paths discussed in Section 3.1.1. The vertices $\frac{n}{2} - 8$, $\frac{n}{2} - 6$, $\frac{n}{2} + 6$ and $\frac{n}{2} + 8$ are also at distance $2x$ because they are the neighbors of the vertices of distance $2x - 1$ from vertex 0. The vertices $\frac{n}{2} - 1$, $\frac{n}{2} + 1$, $\frac{n}{2} + 3$, $\frac{n}{2} + 5$, $\frac{n}{2} + 7$ and $\frac{n}{2} + 9$ are the neighbors of the vertices of distance $2x$, from vertex 0. Therefore, their distance from vertex 0 is $2x + 1$.

Three vertices labeled with $\frac{n}{2}$, $\frac{n}{2} - 2$ and $\frac{n}{2} + 2$, are only connected to the vertices of distance $2x + 1$, from vertex 0. Therefore, their distance from vertex 0, is $2x + 2$. Since there is no any other vertex in the graph, whose distance is greater than the distance of vertices $\frac{n}{2}$, $\frac{n}{2}-2$ and $\frac{n}{2}+2$, from vertex 0. Therefore, these are the diametral vertices of $W_{3,n}$, when $n = 2\ mod\ 6$ and $\frac{n}{2}$ is even. Each diametral vertex can be reached from three vertices of distance $2x+1$, as follows:

$$\frac{n}{2}+1 \xrightarrow{dim:1} \frac{n}{2}, \qquad \frac{n}{2}+3 \xrightarrow{dim:2} \frac{n}{2} \qquad \text{and} \qquad \frac{n}{2}+7 \xrightarrow{dim:3} \frac{n}{2}$$

$$\frac{n}{2}+3 \xrightarrow{dim:1} \frac{n}{2}-2, \qquad \frac{n}{2}+5 \xrightarrow{dim:2} \frac{n}{2}-2 \qquad \text{and} \qquad \frac{n}{2}+9 \xrightarrow{dim:3} \frac{n}{2}-2$$

$$\frac{n}{2}-1 \xrightarrow{dim:1} \frac{n}{2}+2, \qquad \frac{n}{2}+1 \xrightarrow{dim:2} \frac{n}{2}+2 \qquad \text{and} \qquad \frac{n}{2}+5 \xrightarrow{dim:3} \frac{n}{2}+2$$

Since the diametral vertices are at distance $2x+2$ from vertex 0, thus,

$$D(W_{3,n}) = 2x+2 = \left(\frac{n}{6}-\frac{4}{3}\right)+2 = \frac{n}{6}+\frac{2}{3} \qquad \text{where } 2x = \frac{n}{6}-\frac{4}{3}$$

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6}+\frac{2}{3}} \qquad \text{for } n > 8,\ n = 2\ mod\ 6 \text{ and } \frac{n}{2} \text{ is even}$$

Case 4: $\boldsymbol{n = 2\ mod\ 6}$ **and $\frac{n}{2}$ is odd**

$n = 2\ mod\ 6$ and $\frac{n}{2}$ is odd, can be written as $n = 2\ mod\ 12$. Let us consider $n = 12x+2$, for some $x \in \mathbb{Z}^+$. We can perform only $x = \frac{n-2}{12}$ 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. Figure 3.5 illustrates the discussed path. The vertices v_1 and v_2, can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n-2}{12}\right) = \frac{n-2}{2} = \frac{n}{2}-1 \qquad \text{(clockwise)}$$

and $$v_2 = n-6x = n-6\left(\frac{n-2}{12}\right) = n-\left(\frac{n-2}{2}\right) = \frac{n}{2}+1 \qquad \text{(anti-clockwise)}$$

From vertex 0, the vertices v_1 and v_2 are at the distance:

$$dist\left(0, \frac{n}{2}-1\right) = 2x = 2\left(\frac{n-2}{12}\right) = \frac{n-2}{6}$$

$$dist\left(0, \frac{n}{2}+1\right) = 2x = 2\left(\frac{n-2}{12}\right) = \frac{n-2}{6}$$

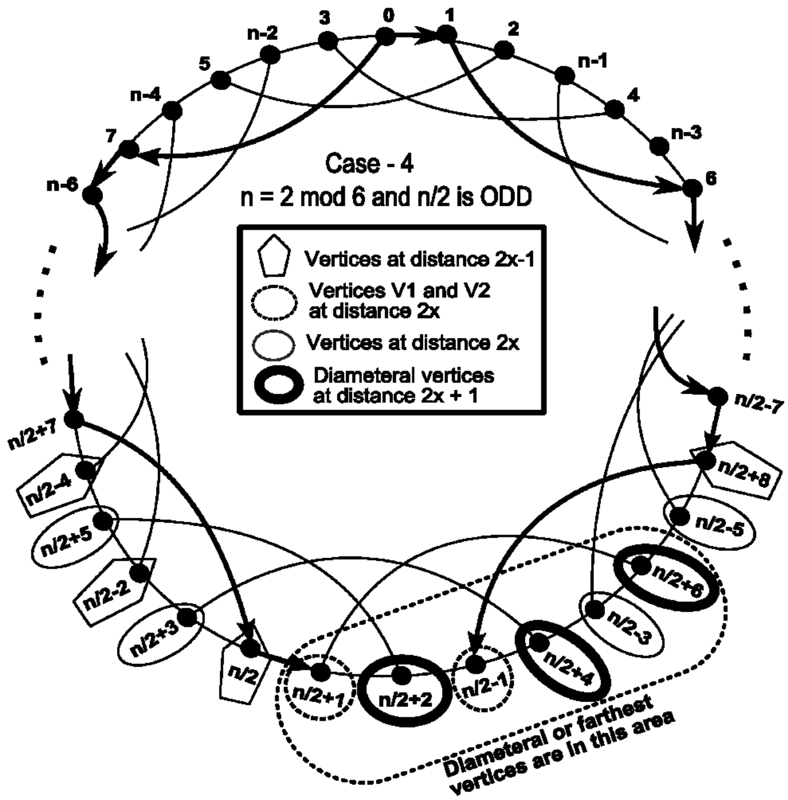


Figure 3.5 The diameter of the Knödel graph $W_{3,n}$, where $n = 2$ *mod* 6 and $n/2$ is odd

Once we reach the vertices $v_1 = \frac{n}{2} - 1$ and $v_2 = \frac{n}{2} + 1$ then our goal is to find and reach the diametral vertices, using the paths discussed in Section 3.1.1. The vertices $\frac{n}{2} - 5$, $\frac{n}{2} - 3$, $\frac{n}{2} + 3$ and $\frac{n}{2} + 5$ are also at distance $2x$ because they are the neighbors of the vertices of distance $2x - 1$ from vertex 0. There are three vertices, $\frac{n}{2} + 2$, $\frac{n}{2} + 4$ and $\frac{n}{2} + 6$, those are only connected to the vertices of distance $2x$. Therefore, their distance from vertex 0, is $2x + 1$. Since there is no any other vertex in the graph whose distance is greater than the distance of vertices $\frac{n}{2} + 2$, $\frac{n}{2} + 4$ and $\frac{n}{2} + 6$, from vertex 0. Therefore, these are the diametral vertices of $W_{3,n}$, when $n = 2\ mod\ 6$ and $\frac{\mathrm{n}}{2}$ is odd. Each diametral vertex can be reached from three vertices of distance $2x$, as follows:

$$\frac{n}{2}+1 \xrightarrow{dim:1} \frac{n}{2}+2, \qquad \frac{n}{2}+3 \xrightarrow{dim:2} \frac{n}{2}+2 \quad \text{and} \quad \frac{n}{2}+7 \xrightarrow{dim:3} \frac{n}{2}+2$$

$$\frac{n}{2}+3 \xrightarrow{dim:1} \frac{n}{2}+4, \qquad \frac{n}{2}+5 \xrightarrow{dim:2} \frac{n}{2}+4 \quad \text{and} \quad \frac{n}{2}+9 \xrightarrow{dim:3} \frac{n}{2}+4$$

$$\frac{n}{2}-1 \xrightarrow{dim:1} \frac{n}{2}+6, \qquad \frac{n}{2}+1 \xrightarrow{dim:2} \frac{n}{2}+6 \quad \text{and} \quad \frac{n}{2}+5 \xrightarrow{dim:3} \frac{n}{2}+6$$

Since the diametral vertices are at distance $2x+1$ from vertex 0, thus,

$$D(W_{3,n}) = 2x+1 = \frac{n-2}{6} + 1 = \frac{n-2+6}{6} = \frac{n+4}{6} \qquad \text{where } 2x = \frac{n-2}{6}$$

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6} + \frac{2}{3}} \qquad \text{for } n = 2 \; mod \; 6 \text{ and } \frac{n}{2} \text{ is odd}$$

Case 5: $\boldsymbol{n = 4 \; mod \; 6}$ **and** $\boldsymbol{\frac{n}{2}}$ **is even**

$n = 4 \; mod \; 6$ and $\frac{n}{2}$ is even, can be written as $n = 4 \; mod \; 12$. Let us consider $n = 12x + 4$, for some $x \in \mathbb{Z}^+$. We can perform only $x = \frac{n-4}{12}$ 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. The discussed path is demonstrated in Figure 3.6. The vertices v_1 and v_2, can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n-4}{12}\right) = \frac{n-4}{2} = \frac{n}{2} - 2 \qquad \text{(clockwise)}$$

and $$v_2 = n - 6x = n - 6\left(\frac{n-4}{12}\right) = n - \left(\frac{n-4}{2}\right) = \frac{n}{2} + 2 \qquad \text{(anti-clockwise)}$$

From vertex 0, the vertices v_1 and v_2 are at the distance:

$$dist\left(0, \frac{n}{2} - 2\right) = 2x = 2\left(\frac{n-4}{12}\right) = \frac{n-4}{6}$$

$$dist\left(0, \frac{n}{2} + 2\right) = 2x = 2\left(\frac{n-4}{12}\right) = \frac{n-4}{6}$$

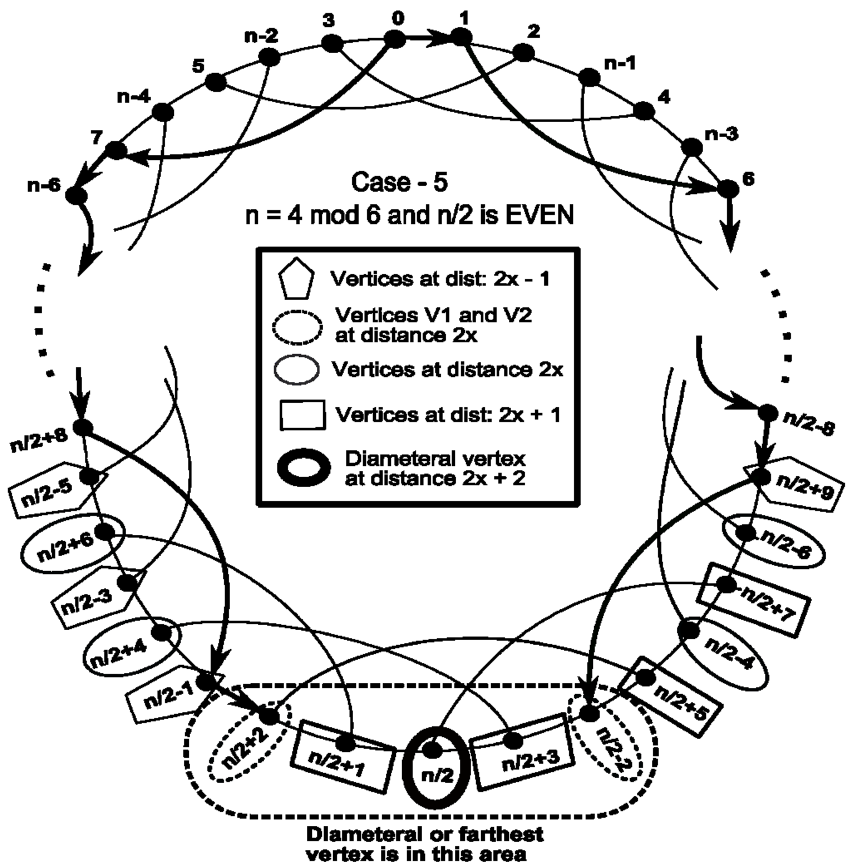


Figure 3.6 The diameter of Knödel graph $W_{3,n}$, where *n* = 4 *mod* 6 and *n*/2 is even

Once we reach the vertices $v_1 = \frac{n}{2} - 2$ and $v_2 = \frac{n}{2} + 2$ then our goal is to find and reach the diametral vertices, using the paths discussed in Section 3.1.1. The vertices $\frac{n}{2} - 6$, $\frac{n}{2} - 4$, $\frac{n}{2} + 4$ and $\frac{n}{2} + 6$ are also at distance $2x$ because they are the neighbors of the vertices of distance $2x - 1$ from vertex 0. The vertices labeled with $\frac{n}{2} + 1$, $\frac{n}{2} + 3$, $\frac{n}{2} + 5$ and $\frac{n}{2} + 7$ are the neighbors of the vertices of distance $2x$ from the vertex 0. Therefore, they are at distance $2x + 1$, from vertex 0.

There is a vertex, $\frac{n}{2}$ that is only connected to the three vertices of distance $2x + 1$, from vertex 0. Therefore, the distance of the vertex $\frac{n}{2}$ is $2x + 2$, from vertex 0. Since there is no any other vertex in the graph whose distance is greater than the distance of vertex $\frac{n}{2}$, from vertex 0. Therefore, this is the diametral vertex of $W_{3,n}$, when $n = 4\ mod\ 6$ and $\frac{n}{2}$ is even. This diametral vertex can be reached, by any of the three vertices of distance $2x+1$, as follows:

$$\frac{n}{2}+1 \xrightarrow{dim:\,1} \frac{n}{2}, \qquad \frac{n}{2}+3 \xrightarrow{dim:\,2} \frac{n}{2} \qquad \text{and} \qquad \frac{n}{2}+7 \xrightarrow{dim:\,3} \frac{n}{2}$$

Since the diametral vertex is at distance $2x+2$ from vertex 0, thus,

$$D(W_{3,n}) = 2x+2 = \left(\frac{n-4}{6}\right)+2 \qquad \text{where } 2x = \frac{n-4}{6}$$

$$= \frac{n-4+12}{6} = \frac{n+8}{6}$$

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6}+\frac{4}{3}} \qquad \text{for } n = 4\ mod\ 6 \text{ and } \frac{n}{2} \text{ is even}$$

Case 6: $\boldsymbol{n = 4\ mod\ 6}$ **and** $\boldsymbol{\frac{n}{2}}$ **is odd**

$n = 4\ mod\ 6$ and $\frac{n}{2}$ is odd, can be written as $n = 10\ mod\ 12$. Let us consider $n = 12x+10$, for some $x \in \mathbb{N}$ ($\mathbb{N}$: set of natural numbers). We can perform only $x = \frac{n-10}{12}$ 1 and 3-dimensional passes in each of these two paths (i.e., clockwise and anti-clockwise), before they intersect. The discussed path is demonstrated in Figure 3.7. The vertices v_1 and v_2, can be determined as follows:

$$v_1 = 6x = 6\left(\frac{n-10}{12}\right) = \frac{n-10}{2} = \frac{n}{2}-5 \qquad \text{(clockwise)}$$

and $$v_2 = n-6x = n-6\left(\frac{n-10}{12}\right) = n-\left(\frac{n-10}{2}\right) = \frac{n}{2}+5 \qquad \text{(anti-clockwise)}$$

From vertex 0, the vertices v_1 and v_2 are at the distance:

$$dist\left(0, \frac{n}{2}-5\right) = 2x = 2\left(\frac{n-10}{12}\right) = \frac{n-10}{6}$$

$$dist\left(0, \frac{n}{2}+5\right) = 2x = 2\left(\frac{n-10}{12}\right) = \frac{n-10}{6}$$

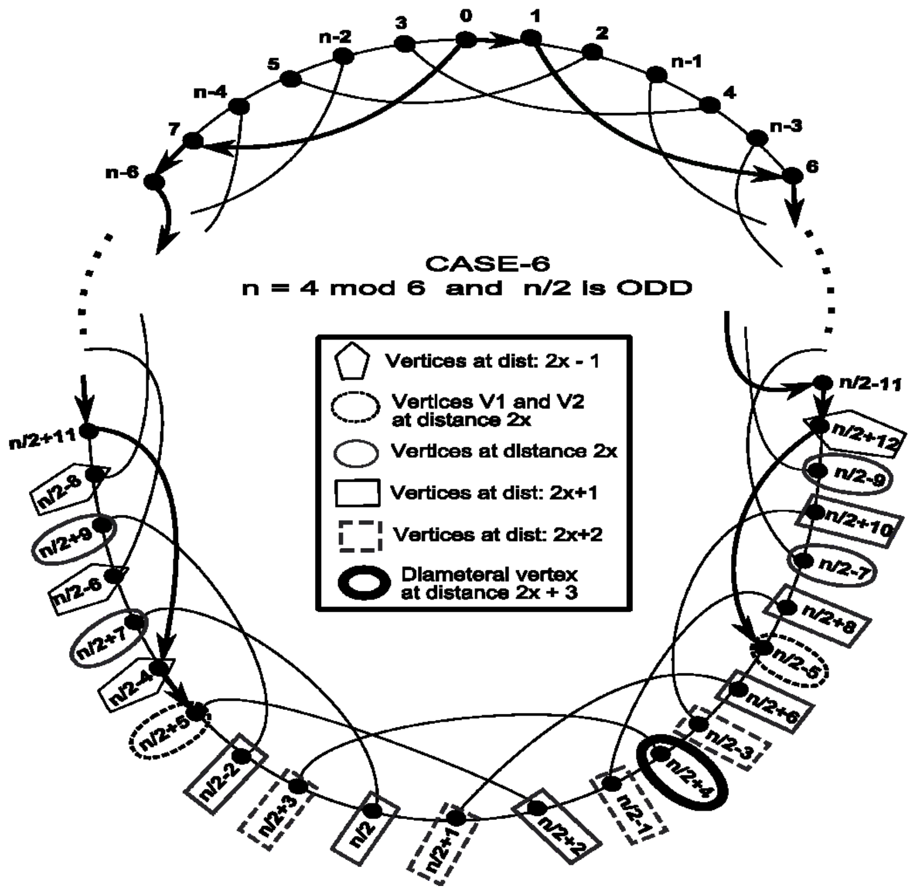


Figure 3.7 The diameter of the Knödel graph $W_{3,n}$, where n = 4 *mod* 6 and n/2 is odd

Once we reach the vertices $v_1 = \frac{n}{2} - 5$ and $v_2 = \frac{n}{2} + 5$ then our goal is to find and reach the diametral vertices, using the paths discussed in Section 3.1.1. The vertices $\frac{n}{2} - 7$, $\frac{n}{2} - 9$, $\frac{n}{2} + 7$ and $\frac{n}{2} + 9$ are also at distance $2x$ because they are the neighbors of the vertices of distance $2x - 1$ from vertex 0. The vertices $\frac{n}{2} - 2$, $\frac{n}{2}$, $\frac{n}{2} + 2$, $\frac{n}{2} + 6$, $\frac{n}{2} + 8$ and $\frac{n}{2} + 10$, are one edge far from the vertices of distance $2x$, from vertex 0. So, they are at distance $2x + 1$, from vertex 0. The vertices $\frac{n}{2} - 1$, $\frac{n}{2} - 3$, $\frac{n}{2} + 1$ and $\frac{n}{2} + 3$, can only be reached by the vertices of distance $2x + 1$, from vertex 0. Therefore, these vertices are at distance $2x + 2$, from vertex 0.

There is a vertex, $\frac{n}{2}+4$, that is only connected to the three of vertices of distance $2x+2$. Since there is no any other vertex in the graph whose distance is greater than the distance of vertex $\frac{n}{2}+4$, from vertex 0. Therefore, this is the diametral vertex of $W_{3,n}$, when $n = 4\ mod\ 6$ and $\frac{n}{2}$ is odd. This diametral vertex can be reached, from any of the three vertices of distance $2x+2$, as follows:

$$\frac{n}{2}-3 \xrightarrow{dim:1} \frac{n}{2}+4, \qquad \frac{n}{2}-1 \xrightarrow{dim:2} \frac{n}{2}+4 \qquad \text{and} \qquad \frac{n}{2}+3 \xrightarrow{dim:3} \frac{n}{2}+4$$

Since the diametral vertex is at distance $2x+3$ from vertex 0, thus,

$$D(W_{3,n}) = 2x+3 = \left(\frac{n-10}{6}\right)+3 \qquad \text{where } 2x = \frac{n-10}{6}$$

$$= \frac{n-10+18}{6} = \frac{n+8}{6}$$

$$\boldsymbol{D(W_{3,n}) = \frac{n}{6}+\frac{4}{3}} \qquad \text{for } n = 4\ mod\ 6 \text{ and } \frac{n}{2} \text{ is odd}$$

3.1.3 Generalized Expression for Diameter of the Knödel graph $W_{3,n}$

We get the following expressions for the diameter of the Knödel graph $W_{3,n}$.

(i) $D(W_{3,n}) = \frac{n}{6}+1$ for $n = 0\ mod\ 6$

(ii) $D(W_{3,n}) = \frac{n}{6}+\frac{2}{3}$ for $n = 2\ mod\ 6$ and $n > 8$

(iii) $D(W_{3,n}) = \frac{n}{6}+\frac{4}{3}$ for $n = 4\ mod\ 6$

From the above three expressions, the generalized expression for the diameter of the Knödel graph $W_{3,n}$ can be obtained as,

$$\boxed{\boldsymbol{D(W_{3,n}) = \left\lceil\frac{n-2}{6}\right\rceil + 1}} \qquad \text{for } n > 8$$

3.2 The diameter of some specific Knödel graphs

In this section, we provide the exact diameter of some specific Knödel graphs. The diameter of these graphs is obtained, by the extensive use of simulation. The simulation uses the *Breadth-First-Search* technique. Due to limited computer memory, we went up to the certain number of vertices and degrees.

Diameter of the Knödel graph $W_{d-1,\, 2^d-2}$

$$D\left(W_{d-1,\, 2^d-2}\right) = \left\lceil \frac{d+2}{2} \right\rceil \qquad \text{for} \quad 3 \le d \le 24.$$

Example 1.1 $\qquad D\left(W_{4-1,\, 2^4-2}\right) = \left\lceil \frac{4+2}{2} \right\rceil = 3$

Diameter of the Knödel graph $W_{d-1,\, 2^d}$

$$D\left(W_{d-1,\, 2^d}\right) = \left\lceil \frac{d+2}{2} \right\rceil \qquad \text{for} \quad 5 \le d \le 24$$

Example 1.2: $\qquad D\left(W_{8-1,\, 2^8}\right) = \left\lceil \frac{8+2}{2} \right\rceil = 5$

Diameter of the Knödel graph $W_{d,\, 2^d+2}$

$$D\left(W_{d,\, 2^d+2}\right) = \left\lfloor \frac{d+2}{2} \right\rfloor \qquad \text{for} \quad 4 \le d \le 24$$

Example 1.3: $\qquad D\left(W_{7,\, 2^7+2}\right) = \left\lfloor \frac{7+2}{2} \right\rfloor = 4$

Diameter of the Knödel graph $W_{d,\, 2^d+4}$

$$D\left(W_{d,\, 2^d+4}\right) = \left\lceil \frac{d+2}{2} \right\rceil \qquad \text{for} \quad 5 \le d \le 24$$

Example 1.4: $\qquad D\left(W_{10,\, 2^{10}+4}\right) = \left\lceil \frac{10+2}{2} \right\rceil = 6$

Diameter of the Knödel graph $W_{d,\, 2^d+2^{d-1}-2}$

$$D\left(W_{d,\, 2^d+2^{d-1}-2}\right) = \left\lceil \frac{d+2}{2} \right\rceil \qquad \text{for} \quad 3 \le d \le 24$$

Example 1.5: $\qquad D\left(W_{9,\, 2^9+2^8-2}\right) = \left\lceil \frac{9+2}{2} \right\rceil = 6$

3.3 The number of vertices at a particular distance in Knödel graph

In this section, we present three propositions, regarding the number of vertices at a particular distance, denoted as N_i, where $0 \leq i \leq D(W_{d,n})$, from vertex 0, for six specific Knödel graphs. The N_i can be obtained, using the *Breadth-First-Search (BFS)* operation on the Knödel graph.

The massive experimental work, enables us to obtain the N_i for the six Knödel graphs i.e.,$W_{d,\,2^d}$, $W_{d,\,2^d+2}$, $W_{d,\,2^d+4}$, $W_{d,\,2^d+6}$, $W_{d,\,2^d+8}$ and $W_{d-1,\,2^d-2}$. The N_i for the specified Knödel graphs is presented in Tables 3.1 to 3.6. Due to limited computer memory, we went up to the $2^{24} + 8 = 16777224$ vertices and degree 24.

Proposition 3.1

Let N_i denotes the number of vertices of the Knödel graph $W_{d,\,2^d}$ at distance i, where $0 \leq i \leq D(W_{d,\,2^d})$. Then $N_0 = 1$, $N_1 = d$, $N_2 = (d-1) + (d-2)^2$ and $N_3 = \frac{(d-2)^2(d-3)}{2} + 2$, for $4 \leq d \leq 24$.

The careful study of the N_i values for $W_{d,\,2^d}$ presented in Table 3.1, enables us to give the N_i for N_0 to N_3. Using vertex transitivity of the Knödel graph, we consider the vertex labeled 0, as the root vertex. $N_0 = 1$ because at distance 0, there is only one root vertex. $N_1 = d$, because, the Knödel graph $W_{d,\,2^d}$ is a regular graph of degree d, therefore, vertex 0 is connected to the d vertices. $N_2 = (d-1) + (d-2)^2$ and $N_3 = \frac{(d-2)^2(d-3)}{2}$, are determined by observing the data presented in Table 3.1.

Table 3.1 Number of vertices at particular distance from vertex 0 in $W_{d,2^d}$

# of Vertices	Number of vertices at distance													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^3	1	3	3	1										
2^4	1	4	7	4										
2^5	1	5	13	11	2									
2^6	1	6	21	26	10									
2^7	1	7	31	52	32	5								
2^8	1	8	43	92	84	28								
2^9	1	9	57	149	192	98	6							
2^{10}	1	10	73	226	386	276	52							
2^{11}	1	11	91	326	702	673	230	14						
2^{12}	1	12	111	452	1182	1459	754	125						
2^{13}	1	13	133	607	1874	2869	2070	607	18					
2^{14}	1	14	157	794	2832	5214	4980	2170	222					
2^{15}	1	15	183	1016	4116	8891	10790	6426	1294	36				
2^{16}	1	16	211	1276	5792	14393	21470	16593	5294	490				
2^{17}	1	17	241	1577	7932	22319	39832	38476	17484	3147	46			
2^{18}	1	18	273	1922	10614	33384	69730	81758	49628	13990	826			
2^{19}	1	19	307	2314	13922	48429	116280	161624	125346	49670	6288	88		
2^{20}	1	20	343	2756	17946	68431	186100	300752	288184	150599	31714	1730		
2^{21}	1	21	381	3251	22782	94513	287570	531707	613116	404710	124614	14374	112	
2^{22}	1	22	421	3802	28532	127954	431112	899776	1222248	987186	411968	78412	2870	
2^{23}	1	23	463	4412	35304	170199	629490	1466278	2305560	2222841	1195796	330347	27690	204
2^{24}	1	24	507	5084	43212	222869	898130	2312384	4147588	4679309	3130098	1163154	169072	5784

Table 3.2 Number of vertices at a particular distance from vertex 0 in $W_{d,\,2^d+2}$

# of Vertices	Number of vertices at distance													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^3+2	1	3	4	2										
2^4+2	1	4	8	5										
2^5+2	1	5	16	12										
2^6+2	1	6	26	27	6									
2^7+2	1	7	38	58	26									
2^8+2	1	8	52	107	76	14								
2^9+2	1	9	68	176	188	72								
$2^{10}+2$	1	10	86	268	406	235	20							
$2^{11}+2$	1	11	106	386	770	628	148							
$2^{12}+2$	1	12	128	533	1328	1459	592	45						
$2^{13}+2$	1	13	152	712	2134	3006	1810	366						
$2^{14}+2$	1	14	178	926	3248	5634	4702	1619	64					
$2^{15}+2$	1	15	206	1178	4736	9804	10750	5388	692					
$2^{16}+2$	1	16	236	1471	6670	16083	22210	15067	3652	132				
$2^{17}+2$	1	17	268	1808	9128	25154	42312	36983	13828	1575				
$2^{18}+2$	1	18	302	2192	12194	37826	75478	81896	42920	9141	178			
$2^{19}+2$	1	19	338	2626	15958	55044	127562	166861	115482	37595	2804			
$2^{20}+2$	1	20	376	3113	20516	77899	206110	317494	278114	125414	19172	349		
$2^{21}+2$	1	21	416	3656	25970	107638	320640	570679	612558	360586	88992	5997		
$2^{22}+2$	1	22	458	4258	32428	145674	482942	977748	1253554	924570	327304	44881	466	
$2^{23}+2$	1	23	502	4922	40004	193596	707398	1608170	2412368	2161999	1023654	225595	10378	
$2^{24}+2$	1	24	548	5651	48818	253179	1011322	2553786	4406962	4685323	2831108	889781	89850	865

Table 3.3 Number of vertices at a particular distance from vertex 0 in $W_{d,\,2^d+4}$

# of	Number of vertices at distance													
Vertices	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^3+4	1	3	5	3										
2^4+4	1	4	8	6	1									
2^5+4	1	5	14	13	3									
2^6+4	1	6	24	28	9									
2^7+4	1	7	36	55	29	4								
2^8+4	1	8	50	101	79	21								
2^9+4	1	9	66	168	189	81	2							
$2^{10}+4$	1	10	84	258	393	246	36							
$2^{11}+4$	1	11	104	374	741	635	180	6						
$2^{12}+4$	1	12	126	519	1279	1436	644	83						
$2^{13}+4$	1	13	150	696	2061	2926	1880	463	6					
$2^{14}+4$	1	14	176	908	3147	5463	4730	1809	140					
$2^{15}+4$	1	15	204	1158	4603	9502	10626	5696	952	15				
$2^{16}+4$	1	16	234	1449	6501	15604	21750	15398	4284	303				
$2^{17}+4$	1	17	266	1784	8919	24446	41246	37032	15088	2259	18			
$2^{18}+4$	1	18	300	2166	11941	36831	73440	81009	44898	11050	494			
$2^{19}+4$	1	19	336	2598	15657	53698	124078	163881	117696	41915	4378	35		
$2^{20}+4$	1	20	374	3083	20163	76132	200586	310588	278810	133450	24356	1017		
$2^{21}+4$	1	21	414	3624	25561	105374	312350	557144	607698	372582	102508	9833	46	
$2^{22}+4$	1	22	456	4224	31959	142831	471016	953799	1235436	937195	356652	59083	1634	
$2^{23}+4$	1	23	500	4886	39471	190086	690810	1568709	2367834	2163911	1077286	266607	18404	84
$2^{24}+4$	1	24	546	5613	48217	248908	988878	2492150	4315164	4650128	2911614	988533	124190	3254

Table 3.4 Number of vertices at a particular distance from vertex 0 in $W_{d,\,2^d+6}$

# of	Number of vertices at distance													
Vertices	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^3+6	1	3	6	4										
2^4+6	1	4	10	7										
2^5+6	1	5	16	14	2									
2^6+6	1	6	26	29	8									
2^7+6	1	7	38	59	28	1								
2^8+6	1	8	52	111	78	12								
2^9+6	1	9	68	183	190	67								
$2^{10}+6$	1	10	86	278	418	227	10							
$2^{11}+6$	1	11	106	399	808	617	112							
$2^{12}+6$	1	12	128	549	1402	1472	520	18						
$2^{13}+6$	1	13	152	731	2254	3103	1692	252						
$2^{14}+6$	1	14	178	948	3424	5900	4574	1333	18					
$2^{15}+6$	1	15	206	1203	4978	10345	10800	4824	402					
$2^{16}+6$	1	16	236	1499	6988	17027	22822	14189	2724	40				
$2^{17}+6$	1	17	268	1839	9532	26651	44136	36174	11602	858				
$2^{18}+6$	1	18	302	2226	12694	40048	79488	82356	38546	6427	44			
$2^{19}+6$	1	19	338	2663	16564	58185	135110	171162	108800	30118	1334			
$2^{20}+6$	1	20	376	3153	21238	82175	218982	330259	271436	108598	12258	86		
$2^{21}+6$	1	21	416	3699	26818	113287	341112	599375	613706	329482	66526	2715		
$2^{22}+6$	1	22	458	4304	33412	152956	513836	1033593	1280454	878982	268888	27298	106	
$2^{23}+6$	1	23	502	4971	41134	202793	752138	1707154	2499484	2117922	896876	161444	4172	
$2^{24}+6$	1	24	548	5703	50104	264595	1073990	2717807	4613846	4695835	2600068	704452	50054	195

Table 3.5 Number of vertices at a particular distance from vertex 0 in $W_{d,\,2^d+8}$

# of	Number of vertices at distance													
Vertices	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^4+8	1	4	11	8										
2^5+8	1	5	16	15	3									
2^6+8	1	6	24	30	11									
2^7+8	1	7	36	59	31	2								
2^8+8	1	8	50	103	81	21								
2^9+8	1	9	66	169	193	82								
2^{10}+8	1	10	84	259	405	247	26							
2^{11}+8	1	11	104	375	753	641	170	1						
2^{12}+8	1	12	126	520	1293	1463	632	57						
2^{13}+8	1	13	150	697	2077	2973	1872	417						
2^{14}+8	1	14	176	909	3165	5534	4774	1739	80					
2^{15}+8	1	15	204	1159	4623	9600	10764	5613	796	1				
2^{16}+8	1	16	234	1450	6523	15732	22040	15404	3974	170				
2^{17}+8	1	17	266	1785	8943	24607	41750	37298	14580	1833				
2^{18}+8	1	18	300	2167	11967	37028	74224	81810	44350	10053	234			
2^{19}+8	1	19	336	2599	15685	53934	125214	165600	117612	39995	3300	1		
2^{20}+8	1	20	374	3084	20193	76410	202152	313724	280268	130591	21304	463		
2^{21}+8	1	21	414	3625	25593	105697	314430	562321	612586	369768	95556	7148		
2^{22}+8	1	22	456	4225	31993	143202	473700	961774	1246716	937484	343672	50449	618	
2^{23}+8	1	23	500	4887	39507	190508	694194	1580382	2389820	2173945	1057906	244562	12380	1
2^{24}+8	1	24	546	5614	48255	249384	993064	2508575	4353808	4682129	2891370	941718	101568	1168

Table 3.6 Number of vertices at a particular distance from vertex 0 in $W_{d-1,\,2^d-2}$

# of	Number of vertices at distance													
Vertices	0	1	2	3	4	5	6	7	8	9	10	11	12	13
2^4-2	1	3	6	4										
2^5-2	1	4	12	11	2									
2^6-2	1	5	20	26	10									
2^7-2	1	6	30	52	32	5								
2^8-2	1	7	42	92	84	28								
2^9-2	1	8	56	149	196	98	2							
2^{10}-2	1	9	72	226	396	276	42							
2^{11}-2	1	10	90	326	720	680	212	7						
2^{12}-2	1	11	110	452	1210	1496	726	88						
2^{13}-2	1	12	132	607	1914	2962	2046	514	2					
2^{14}-2	1	13	156	794	2886	5395	5018	1989	130					
2^{15}-2	1	14	182	1016	4186	9198	11032	6146	982	9				
2^{16}-2	1	15	210	1276	5880	14870	22160	16353	4516	253				
2^{17}-2	1	16	240	1577	8040	23016	41328	38712	15924	2214	2			
2^{18}-2	1	17	272	1922	10744	34357	72522	83436	47192	11339	340			
2^{19}-2	1	18	306	2314	14076	49740	121008	166515	122844	43545	3908	11		
2^{20}-2	1	19	342	2756	18126	70148	193572	311733	288496	138985	23750	646		
2^{21}-2	1	20	380	3251	22990	96710	298780	553095	623028	387223	103394	8276	2	
2^{22}-2	1	21	420	3802	28770	130711	447258	937706	1255058	969451	364790	55460	854	
2^{23}-2	1	22	462	4412	35574	173602	651992	1529121	2384558	2224673	1107612	262460	14104	13
2^{24}-2	1	23	506	5084	43516	227010	928648	2411205	4310476	4748350	2995290	995394	110170	1541

Proposition 3.2

In $W_{d,\,2^d}$, $W_{d,\,2^d+2}$, $W_{d,\,2^d+4}$, $W_{d,\,2^d+6}$, $W_{d,\,2^d+8}$ and $W_{d-1,\,2^d-2}$, the maximum number of vertices, are at distance $1+\left\lfloor \frac{\boldsymbol{d}}{\boldsymbol{3}} \right\rfloor$, for $3 < d \le 24$.

Tables 3.1 to 3.6 present the number of vertices at a particular distance from vertex 0, for Knödel graphs, $W_{d,\,2^d}$, $W_{d,\,2^d+2}$, $W_{d,\,2^d+4}$, $W_{d,\,2^d+6}$, $W_{d,\,2^d+8}$ and $W_{d-1,\,2^d-2}$, respectively. It can be observed from the data that the maximum numbers of vertices are at a distance $1+\left\lfloor \frac{\boldsymbol{d}}{\boldsymbol{3}} \right\rfloor$ from vertex 0.

Proposition 3.3

Let N_x denotes the number of vertices at diametral distance of the Knödel graphs $W_{d,\,2^d}$, $W_{d,\,2^d+2}$, $W_{d,\,2^d+4}$, $W_{d,\,2^d+6}$, $W_{d,\,2^d+8}$ and $W_{d-1,\,2^d-2}$. Then, $N_0 < N_1 < \cdots < N_{\left\lfloor \frac{d}{3} \right\rfloor} < \boldsymbol{N}_{\boldsymbol{1}+\left\lfloor \frac{\boldsymbol{d}}{\boldsymbol{3}} \right\rfloor} > N_{2+\left\lfloor \frac{d}{3} \right\rfloor} > \cdots > N_x$, where $3 < d \le 24$.

This proposition is based on the data presented in Tables 3.1 to 3.6, where the number of vertices increases from distance 0 until the distance is $1+\left\lfloor \frac{d}{3} \right\rfloor$. Once the vertex count reaches its maximum at distance $1+\left\lfloor \frac{d}{3} \right\rfloor$, then the number of vertices starts to decrease from the distance $2+\left\lfloor \frac{d}{3} \right\rfloor$ till N_x. Here N_x denotes the number of vertices at diametral distance of the Knödel graphs $W_{d,\,2^d}$, $W_{d,\,2^d+2}$, $W_{d,\,2^d+4}$, $W_{d,\,2^d+6}$, $W_{d,\,2^d+8}$ and $W_{d-1,\,2^d-2}$.

3.4 Summary

Table 3.7 provides the summary of contributions, regarding the diameter of some specific Knödel graphs.

Table 3.7 Diameter of some specific Knödel graphs

Knödel graph	Diameter	Tested ranges
$W_{3,n}$	$\left\lceil \frac{n-2}{6} \right\rceil + 1$	$n > 8$
$W_{d-1,\, 2^d-2}$	$\left\lceil \frac{d+2}{2} \right\rceil$	$3 \le d \le 24$
$W_{d-1,\, 2^d}$	$\left\lceil \frac{d+2}{2} \right\rceil$	$5 \le d \le 24$
$W_{d,\, 2^d+2}$	$\left\lfloor \frac{d+2}{2} \right\rfloor$	$5 \le d \le 24$ and d is odd
$W_{d,\, 2^d+2}$	$\left\lceil \frac{d+2}{2} \right\rceil$	$4 \le d \le 24$ and d is even
$W_{d,\, 2^d+4}$	$\left\lceil \frac{d+2}{2} \right\rceil$	$5 \le d \le 24$
$W_{d,\, 2^d+\, 2^{d-1}-2}$	$\left\lceil \frac{d+2}{2} \right\rceil$	$3 \le d \le 24$

Three propositions, regarding the number of vertices at a particular distance, in the Knödel graphs $W_{d,\, 2^d}$, $W_{d,\, 2^d+2}$, $W_{d,\, 2^d+4}$, $W_{d,\, 2^d+6}$, $W_{d,\, 2^d+8}$ and $W_{d-1,\, 2^d-2}$:

- **Proposition 3.1:** *Let N_i denotes the number of vertices of Knödel graph $W_{d,\, 2^d}$ at distance i, where $0 \le i \le D(W_{d,\, 2^d})$. Then $N_0 = 1$, $N_1 = d$, $N_2 = (d-1) + (d-2)^2$ and $N_3 = \frac{(d-2)^2(d-3)}{2} + 2$, for $4 \le d \le 24$.*

- **Proposition 3.2:** *In $W_{d,\, 2^d}$, $W_{d,\, 2^d+2}$, $W_{d,\, 2^d+4}$, $W_{d,\, 2^d+6}$, $W_{d,\, 2^d+8}$ and $W_{d-1,\, 2^d-2}$, the maximum number of vertices, are at distance $1 + \left\lfloor \frac{\boldsymbol{d}}{\mathbf{3}} \right\rfloor$, for $3 < d \le 24$.*

- **Proposition 3.3:** *Let N_x denotes the number of vertices at diametral distance of the Knödel graphs $W_{d,\, 2^d}$, $W_{d,\, 2^d+2}$, $W_{d,\, 2^d+4}$, $W_{d,\, 2^d+6}$, $W_{d,\, 2^d+8}$ and $W_{d-1,\, 2^d-2}$. Then, $N_0 < N_1 < \cdots < N_{\left\lfloor \frac{d}{3} \right\rfloor} < \boldsymbol{N}_{\mathbf{1}+\left\lfloor \frac{\boldsymbol{d}}{\mathbf{3}} \right\rfloor} > N_{2+\left\lfloor \frac{d}{3} \right\rfloor} > \cdots > N_x$, where $3 < d \le 24$.*

Chapter 4

The Broadcasting in the Knödel graph $W_{3,n}$

In this chapter, we present the broadcast time of the Knödel graph $W_{3,n}$ for all even n and degree 3. There are six possible cases of $W_{3,n}$, depending on the number of vertices. We present a broadcast scheme for six cases of the Knödel graph $W_{3,n}$. We show that the Knödel graph $W_{3,n}$ for $n = 4 \; mod \; 6$ and $n > 16$, is the first infinite family of diametral broadcast graphs in the Knödel graph $W_{d,n}$.

4.1 Broadcasting in the Knödel graph $W_{3,n}$

In this section, we study the broadcast problem in the Knödel graph $W_{3,n}$. In general the broadcast problem can be defined as follows:

> Let $G = (V, E)$ be a graph and let v be a vertex in G. Now consider that v knows a piece of information, $I(v)$, that is unknown to all other vertices in $V = \{v\}$. The broadcast problem is to find a communication strategy, called broadcast scheme, such that all nodes from G learn the piece of information $I(v)$ in the minimum possible time [42].

Now we present the broadcast scheme for the Knödel graph $W_{3,n}$. For that we consider only the 1 and 3-dimensional edges of the Knödel graph $W_{3,n}$. Recall that 1 and 3-dimensional edges split $W_{3,n}$ into $2x$ segments, where

$$2x = \frac{number \; of \; vertices}{6}.$$

Each segment is of length 6, except the one containing the vertex $n/2$. We can perform exactly

$$x = \frac{1}{2}\left(\frac{number\ of\ vertices}{6}\right) = \frac{number\ of\ vertices}{12}$$

1 and 3-dimensional passes clockwise and anti-clockwise, before they intersect. Using 1 and 3-dimensional edges, we can reach the vertices $6,\ 6(2),\ 6(3), \dots, 6(x-1),\ 6x$ from vertex 0, in the clockwise direction. Similarly, going anti-clockwise, we reach the vertices $n-6,\ n-6(2),\ n-6(3), \dots,\ n-6(x-1),\ n-6x$.

Using the vertex transitivity of the Knödel graph, we consider the vertex labeled 0 as the originator (initially informed vertex) in broadcasting. The broadcasting in the Knödel graph $W_{3,n}$ will be performed in two steps.

Step – 1:

In step – 1 our goal is to pass the message from the vertex 0 to the vertices $6x$ and $n-6x$ as early as possible. To achieve this goal, we use the long "moves" of 1 and 3-dimensional edges (see Figure 4.1). We start the broadcasting from vertex 0. At time 1 the vertex 0 sends the message to the vertex 1 in the clockwise direction. At time 2, the vertex 0 informs the vertex 7 in the anticlockwise direction. Since the vertices $6x$ and $n-6x$ are at distance $2x$ from the vertex 0, and we start broadcasting in a clockwise direction first, the vertex $6x$ will be informed at the time $T = 2x$. Subsequently, the vertex $n-6x$ will receive the message one time unit later than the vertex $6x$, so the vertex $n-6x$ will be informed at time $T+1$.

Recall that to pass the message to the vertex $6x$, in the clockwise direction, we use the path and time slots as follows:

$$0 \xrightarrow{Time\ 1} 1 \xRightarrow{Time\ 2} 6 \xrightarrow{Time\ 3} n-5 \xRightarrow{Time\ 4} 6(2) \xrightarrow{Time\ 5} \ldots \xRightarrow{Time\ T} 6x.$$

Similarly, to pass the message to the vertex $n-6x$, in the anti-clockwise direction, we use the path and time slots as follows:

$$0 \xRightarrow{Time\ 2} 7 \xrightarrow{Time\ 3} n-6 \xRightarrow{Time\ 4} 13 \xrightarrow{Time\ 5} n-6(2) \xRightarrow{Time\ 6} \ldots \xrightarrow{Time\ T+1} n-6x.$$

Note that in the above specified two paths, the single line arrow "⟶" is representing the 1-dimensional edge and the double line arrow "⟹" is representing the 3-dimensional edge.

When we broadcast in $W_{3,n}$ from the vertex 0 using the above specified paths, formed by 1 and 3-dimensional edges, the 3-dimensional edge forms a "cycle" of length 6. As the "cycle" of length 6 is formed, it starts to broadcast within itself, and in parallel broadcasting continues on the specified path that forms other "cycles" of length 6. Now consider the "cycle" in the clockwise direction where the 3-dimensional edge is labeled with time unit 2. Recall that at time 2, the vertex 1 informs the vertex 6 using 3-dimensional edge, that forms the cycle of length 6 (see Figure 4.1). At time 3, the vertex 1 informs the vertex 2 and the vertex 6 sends the message to the vertex $n-5$ that is out of this "cycle" but it is on the specified path. At time 4, the vertices labeled with 2 and 6 inform the vertices $n-1$ and $n-3$, respectively. In parallel at time 4, the vertex $n-5$ sends the message to the vertex 6(2) using 3-dimensioal edge, this forms another "cycle" of length 6 on above specified path. At time 5, the vertex $n-1$ informs the vertex 4, this way broadcasting in this "cycle" finishes in 4 time units (i.e., from time 2 to 5). And also at time 5 the vertex $n-3$ sends the message using 3-dimensional edge to the vertex 10 that is in the next "cycle" of length 6 where 3-dimensional edge is labeled with time 4.

Similarly, in anti-clockwise direction the "cycle", where the 3-dimensional edge is labeled with time 2, completes the broadcasting in 4 time units (i.e., from time 2 to 5). Here we can see that the second "cycle" on each path is formed at time 4, whereas the broadcasting in first "cycle" where 3-dimensional edge is labeled with time 2 completes broadcasting at time 5.

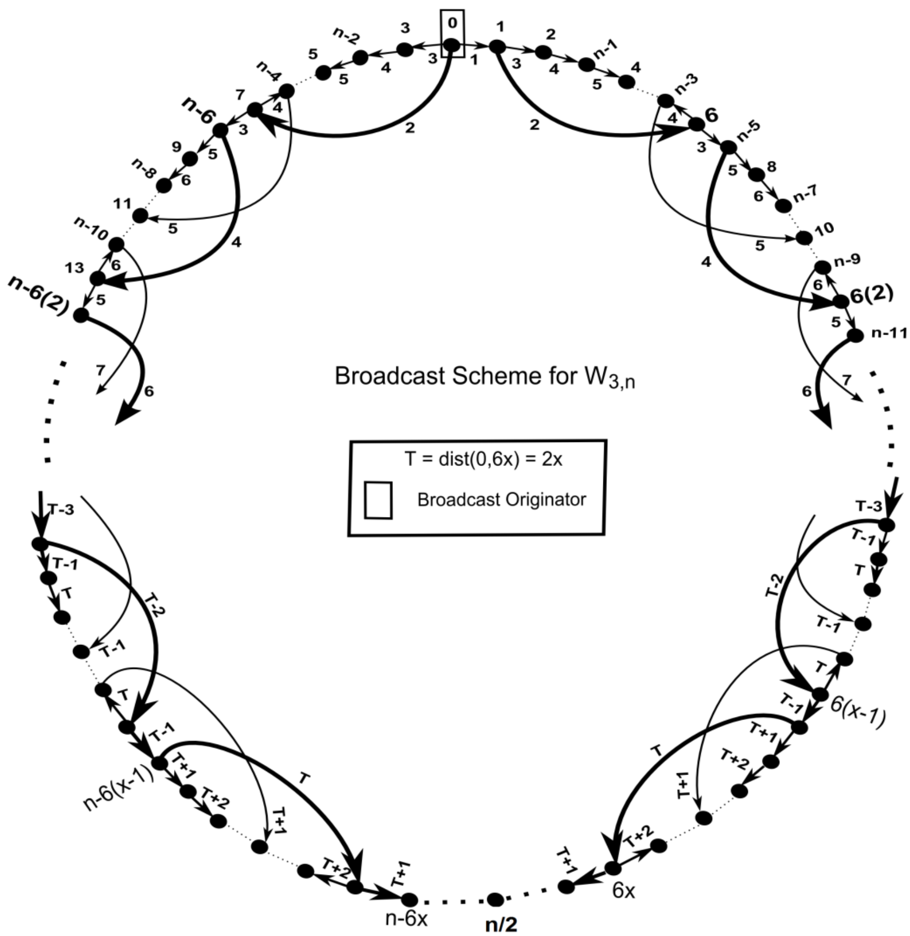


Figure 4.1 The general broadcast scheme for the Knödel graph $W_{3,n}$

The broadcasting in the remaining $2x - 2$ "cycles" of length 6 is performed in a similar way as described above, except that the one of the vertices in each of these "cycles" is informed by a vertex from the previous "cycle" of length 6, using 3-dimensional edge. Now consider the "cycle" in clockwise direction where 3-dimensional edge is labeled with time 4. Recall that at time 4, the vertex $n-5$ informs the vertex $6(2)$ using 3-dimensional edge, that forms the "cycle" of length 6 (see Figure 4.1). At time 5, the vertex $n-5$ informs the vertex 8, the vertex $n-3$, the informed vertex from the previous "cycle", informs the vertex 10, and the vertex $6(2)$ sends the message to the vertex $n-11$ that is out of this "cycle" but it is on the specified path. At time 6, the vertices labeled with $6(2)$ and 8 inform the vertices $n-9$ and $n-7$, this way the broadcasting in this "cycle" finishes in 3 time units (i.e., from time 4 to 6). Also in parallel at time 6 the vertex $n-11$, that is out of this "cycle", informs the vertex $6(3)$ using 3-dimensional edge, that forms another "cycle" of length 6 on specified path. At time 7 the vertex $n-9$ will inform one of the vertices of a newly formed "cycle" of length 6 using 3-dimensional edge. Similarly, in anti-clockwise direction the "cycle" where the 3-dimensional edge is labeled with time 4, completes the broadcasting in 3 time units (i.e., from time 4 to 6). And in parallel at time 6, the vertex $n-6(2)$ informs a certain vertex using 3-dimensional edge that forms another "cycle" of length 6 on specified path. The process of broadcasting as described above continues till the x-*th* (the last) "cycle" of length 6 on each path (clockwise and anti-clockwise). It follows that, except 2 "cycles" where 3-dimensioal edges are labeled with time 2, the broadcasting in each of the remaining $2x-2$ "cycles" of length 6 finishes at the same time when the next "cycle" of length 6 is formed on the paths specified above (see Figure 4.1).

Moreover, except 2 "cycles", where 3-dimensioal edges are labeled with time 2, the broadcasting in each of the remaining $2x-2$ "cycles" of length 6 is performed in 3 time units. Because one of the vertices in each of these $2x-2$ "cycles" is informed by a vertex from the previous "cycle" of length 6, using 3-dimensional edge, therefore it is taking 3 time units. In general this can be expressed as follows: "If any "cycle" from $2x-2$ "cycles" of length 6, where the 3-dimensional edge is labeled with time y, it takes 3 time units (i.e., $y,\ y+1$ and $y+2$) to finish the broadcasting within it". Now consider the last two "cycles" of length 6, where the 3-dimensional edges are labeled with time T, the broadcasting in these two "cycles" finishes in 3 time units (i.e. $T,\ T+1$ and $T+2$).

Recall that we can perform only x 3-dimensional passes on each path (clockwise and anti-clockwise), that is formed by 1 and 3-dimensional edges. The x-th 3-dimensional pass on each path forms the last "cycle" of length 6. Since x-th "cycles" of length 6 on each path are formed at the same time when $(x-1)$-*th* "cycles", where 3-dimensional edges are labeled with time $T-2$, finish the broadcasting within themselves. It follows that when the x-*th* "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T.

Step – 2:

The second step of broadcasting in $W_{3,n}$ starts when the vertices $6x$ and $n-6x$ have been informed at time units T and $T+1$, respectively. Recall that, there are six cases of $W_{3,n}$, depending on the number of vertices. In the following, we present the six broadcast schemes, one for each case, to inform the remaining vertices starting from time $T+1$ onwards. We also obtain the upper bound on the broadcast time for each of these six cases of $W_{3,n}$ using the following broadcast schemes.

Case 1: $n = 0 \; mod \; 6$ and $\frac{n}{2}$ is even

Recall from Case 1 under Section 3.1, that after $x = \frac{n}{12}$ 1 and 3-dimensional passes in each direction, we the reach the vertex labeled $\frac{n}{2}$. The vertex $\frac{n}{2}$ is at distance $\frac{n}{6}$ from vertex 0. If we start broadcasting in a clockwise direction, the vertex $\frac{n}{2}$ will receive the message at time $T = \frac{n}{6}$. Figure 4.2 demonstrates the broadcast scheme for Case 1.

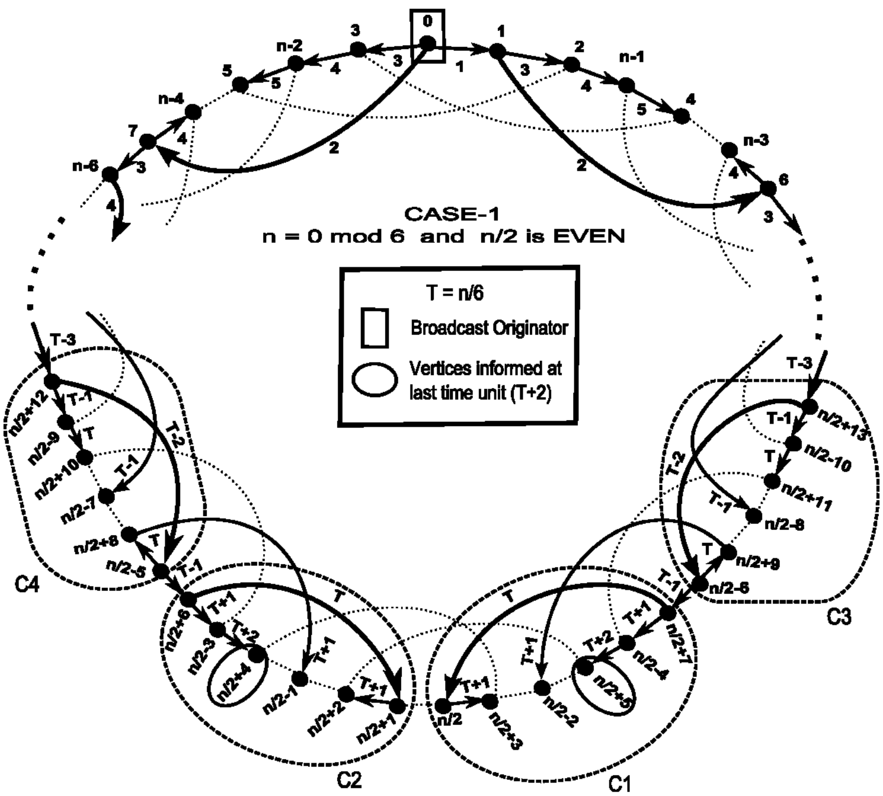


Figure 4.2 The broadcast scheme for $W_{3,n}$, where n = 0 mod 6 and n/2 is even

Also recall that when the x-th "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T. Therefore all the vertices in any "cycles" of length 6 except C_1 and C_2 will receive the message by time T. The broadcasting in C_1 will be completed at time $T + 2$ in the following way.

Since the vertex $\frac{n}{2}+7$ is on the specified path, therefore at time $T-1$ it is informed by the vertex $\frac{n}{2}-6$ using 1-dimensional edge. At time T, the vertex $\frac{n}{2}+7$ sends the message to the vertex $\frac{n}{2}$, that forms the x-*th* "cycle" of length 6 in the clockwise direction. Now at time $T+1$ the vertices $\frac{n}{2}+7$ and $\frac{n}{2}$ send the message to the vertices $\frac{n}{2}-4$ and $\frac{n}{2}+3$, respectively. Also at time $T+1$ the vertex $\frac{n}{2}+9$ (the informed vertex from $(x-1)$-th "cycle") informs the vertex $\frac{n}{2}-2$ using 3-dimensional edge. At time $T+2$ the vertex $\frac{n}{2}-4$ informs the vertex $\frac{n}{2}+5$ using 3-dimensional edge, this completes the broadcasting in x-*th* "cycle" on specified path (clockwise direction).

Similarly, the broadcasting in C_2, the x-*th* "cycle" on specified path in anti-clockwise direction, will be completed at time $T+2$ in a similar way as described above. Since there is no other vertex left to be informed, this proves that, $b(0, W_{3,n}) \leq$ T + 2, where $T=\frac{n}{6}$, $n=0 \ mod\ 6$ and $\frac{n}{2}$ is even. Thus,

$$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n})+1=\frac{n}{6}+2=\left\lceil\frac{n-2}{6}\right\rceil+2}, \quad \text{for } n=0 \ mod\ 6 \text{ and } \frac{n}{2} \text{ is even}$$

Case 2: $\boldsymbol{n=0 \ mod\ 6}$ and $\boldsymbol{\frac{n}{2}}$ is odd

Recall from Case 2 under Section 3.1, that after $x=\frac{n-6}{12}$ 1 and 3-dimensional passes in clockwise and anti-clockwise directions, we reach the vertices $\frac{n}{2}-3$ and $\frac{n}{2}+3$, respectively. Both of these vertices are at distance $\frac{n}{6}-1$ from vertex 0. If broadcasting is started in a clockwise direction the vertex $\frac{n}{2}-3$ receives the message at time $T=\frac{n}{6}-1$. The vertex $\frac{n}{2}+3$ receives the message one time unit later than vertex $\frac{n}{2}-3$, at time $T+1$. Figure 4.3 demonstrates the broadcast scheme for Case 2.

Recall that when the x-th "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T. Also recall that the broadcasting in the x-th "cycles" of length 6 on each path will finish by time $T + 2$. Therefore all the vertices in any "cycles" of length 6 except two vertices of C_1 will receive the message by time $T + 2$. The broadcasting in C_1 will be completed at time $T + 3$ in the following way.

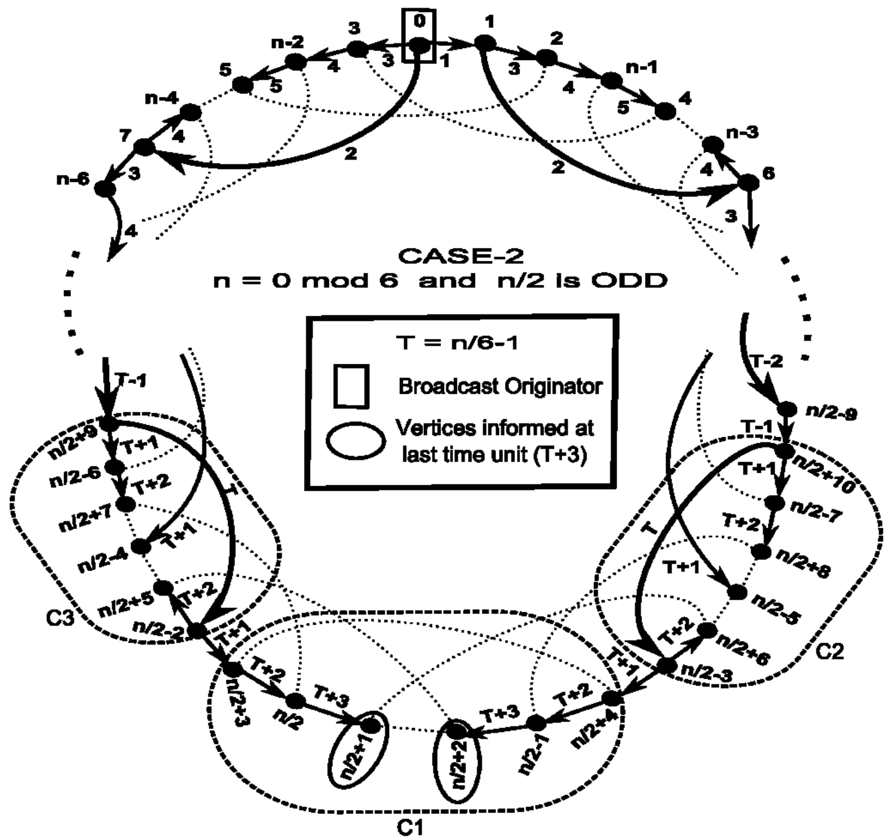


Figure 4.3 The broadcast scheme for $W_{3,n}$, where *n* = 0 *mod* 6 and *n*/2 is odd

Since the vertices $\frac{n}{2} - 3$ and $\frac{n}{2} + 3$ are informed at time T and $T + 1$, respectively, therefore at time $T + 1$ the vertex $\frac{n}{2} - 3$ sends the message to the vertex $\frac{n}{2} + 4$. Now at time $T + 2$ the vertices $\frac{n}{2} + 3$ and $\frac{n}{2} + 4$ send the message to the vertices $\frac{n}{2}$ and $\frac{n}{2} - 1$, respectively. There are only two vertices left to be informed in C_1. Finally at time $T+3$ vertices $\frac{n}{2}-1$ and $\frac{n}{2}$ send the message to the vertices $\frac{n}{2}+2$ and $\frac{n}{2}+1$, respectively. Since there is no other vertex left to be informed, this proves that $b(0, W_{3,n}) \leq \mathrm{T}+3$, where $T=\frac{n}{6}-1$, $n=0 \bmod 6$ and $\frac{n}{2}$ is odd. Thus,

$$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n})+1=\frac{n}{6}+2=\left\lceil\frac{n-2}{6}\right\rceil+2}, \quad \text{for } n=0 \bmod 6 \text{ and } \frac{n}{2} \text{ is odd}$$

Case 3: $\boldsymbol{n=2 \bmod 6}$ **and** $\boldsymbol{\frac{n}{2}}$ **is even**

Recall from Case 3 under Section 3.1, that after $x=\frac{n-8}{12}$ 1 and 3-dimensional passes in clockwise and ant-clockwise directions, we the reach the vertices $\frac{n}{2}-4$ and $\frac{n}{2}+4$ respectively. Both of these vertices are at distance $\frac{n}{6}-\frac{4}{3}$ from vertex 0. If broadcasting is started in a clockwise direction the vertex $\frac{n}{2}-4$ will receive the message at time $T=\frac{n}{6}-\frac{4}{3}$. The vertex $\frac{n}{2}+4$ will receives the message one time unit later then the vertex $\frac{n}{2}-4$, at time $T+1$. Figure 4.4 depicts the broadcast scheme for Case 3.

Recall that when the x-*th* “cycles” of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous “cycles” will receive the message by time T. Also recall that the broadcasting in the x-*th* “cycles” of length 6 on each path will finish by time $T+2$. Therefore all the vertices in any “cycles” of length 6 except four vertices of C_1 will receive the message by time $T+2$. The broadcasting in C_1 will be completed at time $T+3$ in the following way.

Since the vertices $\frac{n}{2}-4$ and $\frac{n}{2}+4$ are informed at time T and $T+1$, respectively, therefore at time $T+1$ the vertex $\frac{n}{2}-4$ sends the message to the vertex $\frac{n}{2}+5$. At time $T+2$ the vertices $\frac{n}{2}-2$ and $\frac{n}{2}-1$ receive the message from the vertices $\frac{n}{2}+5$ and $\frac{n}{2}+4$, respectively. Now there are only four vertices left to be informed in C_1. At time $T+3$ the vertices $\frac{n}{2}-2$ and $\frac{n}{2}-1$ inform the vertices $\frac{n}{2}+3$ and $\frac{n}{2}+2$, respectively. As the vertices $\frac{n}{2}+7$ and $\frac{n}{2}+6$ are from the *x-th* "cycles" and they get informed by time $T+2$, so they can participate in the broadcasting process. Therefore at time $T+3$ the vertices $\frac{n}{2}+7$ and $\frac{n}{2}+6$ send the message to the vertices $\frac{n}{2}$ and $\frac{n}{2}+1$ of C_1, respectively. Since there is no other vertex left to be informed, this proves that, $b(0, W_{3,n}) \leq T+3$, where $T=\frac{n}{6}-\frac{4}{3}$, $n=2\ mod\ 6$ and $\frac{n}{2}$ is even. Thus,

$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n})+1=\frac{n}{6}+\frac{5}{3}=\left\lceil\frac{n-2}{6}\right\rceil+2}$, for $n>8$, $n=2\ mod\ 6$ and $\frac{n}{2}$ is even

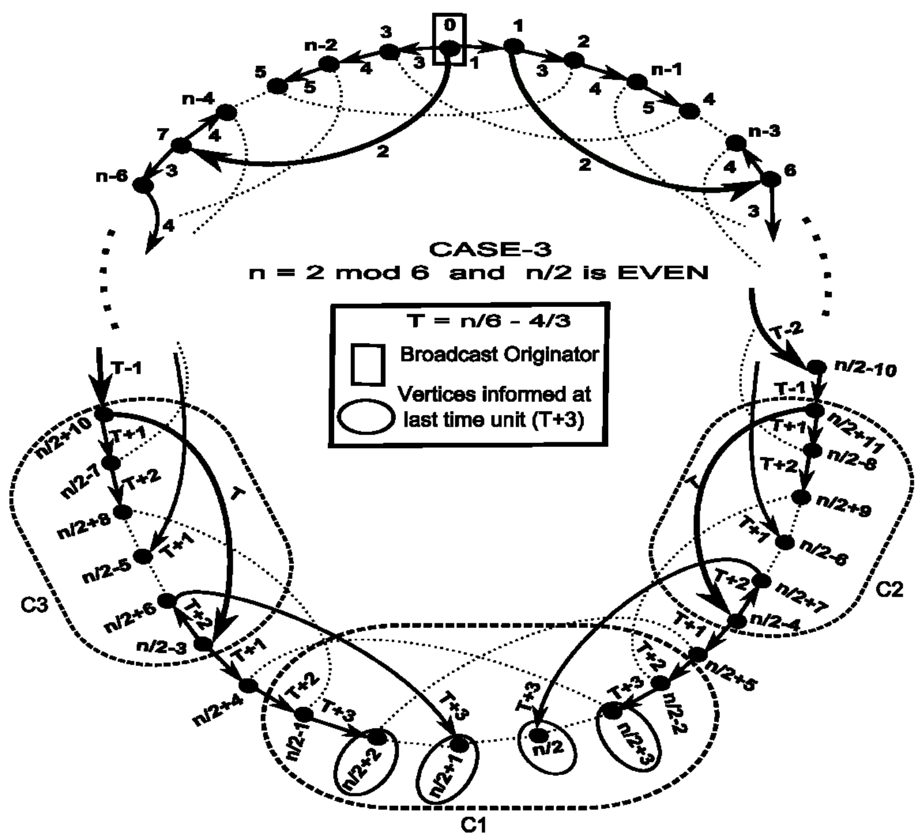


Figure 4.4 The broadcast scheme for $W_{3,n}$, where *n* = 2 *mod* 6 and *n*/2 is even

Case 4: $\quad n = 2\ mod\ 6$ and $\frac{n}{2}$ is odd

Recall from Case 4 under Section 3.1, that after $x = \frac{n-2}{12}$ 1 and 3-dimensional passes in clockwise and anti-clockwise directions, we reach the vertices $\frac{n}{2} - 1$ and $\frac{n}{2} + 1$, respectively. Both of these vertices are at distance $\frac{n}{6} - \frac{1}{3}$ from vertex 0. If broadcasting is started in a clockwise direction the vertex $\frac{n}{2} - 1$ will receive the message at time $T = \frac{n}{6} - \frac{1}{3}$. The vertex $\frac{n}{2} + 1$ will receive the message one time unit later then the vertex $\frac{n}{2} - 1$, at time $T + 1$. Figure 4.5 illustrates the broadcast scheme for Case 4.

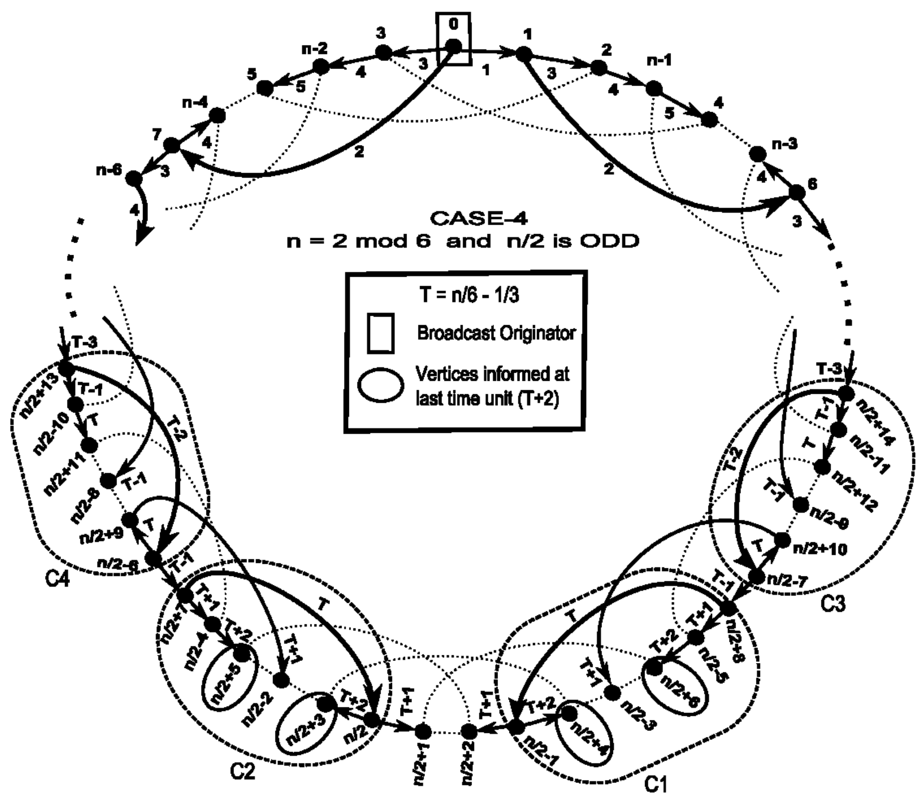


Figure 4.5 The broadcast scheme for $W_{3,n}$, where $n = 2\ mod\ 6$ and $n/2$ is odd

Recall that when the x-th "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T. Therefore all the vertices in any "cycles" of length 6 except C_1 and C_2 (the x-*th* "cycles") will receive the message by time T. The broadcasting in C_1 will be completed at time $T+2$ in the following way.

Since the vertex $\frac{n}{2}+8$ is on the specified path, therefore at time $T-1$ it is informed by the vertex $\frac{n}{2}-7$ using 1-dimensional edge. At time T the vertex $\frac{n}{2}+8$ sends the message to the vertex $\frac{n}{2}-1$ that forms the x-*th* "cycle" of length 6 in the clockwise direction. At time $T+1$ the vertices $\frac{n}{2}+8$ and $\frac{n}{2}-1$ send the message to the vertices $\frac{n}{2}-5$ and $\frac{n}{2}+2$, respectively. As the vertex $\frac{n}{2}+10$ is from the $(x-1)$-*th* "cycle" and it get informed by time T, so this can participate in the broadcasting process. Therefore at time $T+1$ the vertex $\frac{n}{2}+10$ sends the message to the vertices $\frac{n}{2}-3$ using 3-dimensional edge. Now there are two vertices left to be informed in C_1. At time $T+2$ the vertices $\frac{n}{2}-5$ and $\frac{n}{2}-1$ inform the vertices $\frac{n}{2}+6$ and $\frac{n}{2}+4$, respectively, this completes the broadcasting in C_1.

Similarly, the broadcasting in C_2, the x-*th* "cycle" on specified path in anti-clockwise direction, will be completed at time $T+2$ in a similar way as described above. Since there is no other vertex left to be informed, this proves that, $b(0, W_{3,n}) \leq$ $T+2$, where $T=\frac{n}{6}-\frac{1}{3}$, $n=2 \; mod \; 6$ and $\frac{n}{2}$ is odd. Thus,

$$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n})+1=\frac{n}{6}+\frac{5}{3}=\left\lceil\frac{n-2}{6}\right\rceil+2}, \text{ for } n=2 \; mod \; 6 \text{ and } \frac{n}{2} \text{ is odd}$$

Case 5: $\boldsymbol{n=4 \; mod \; 6}$ **and** $\boldsymbol{\frac{n}{2}}$ **is even**

Recall that in Case 5 under Section 3.1 after $x=\frac{n-4}{12}$ 1 and 3-dimensional passes in clockwise and anti-clockwise directions, we reach the vertices $\frac{n}{2}-2$ and $\frac{n}{2}+2$, respectively. Both of these vertices are at distance $\frac{n}{6}-\frac{2}{3}$ from vertex 0. If the broadcasting is started in a clockwise direction the vertex $\frac{n}{2}-2$ will receive the message at time $T=\frac{n}{6}-\frac{2}{3}$. The vertex $\frac{n}{2}+2$ will be informed one time unit later then the vertex $\frac{n}{2}-2$, at time $T+1$. Figure 4.6 shows the broadcast scheme for Case 5.

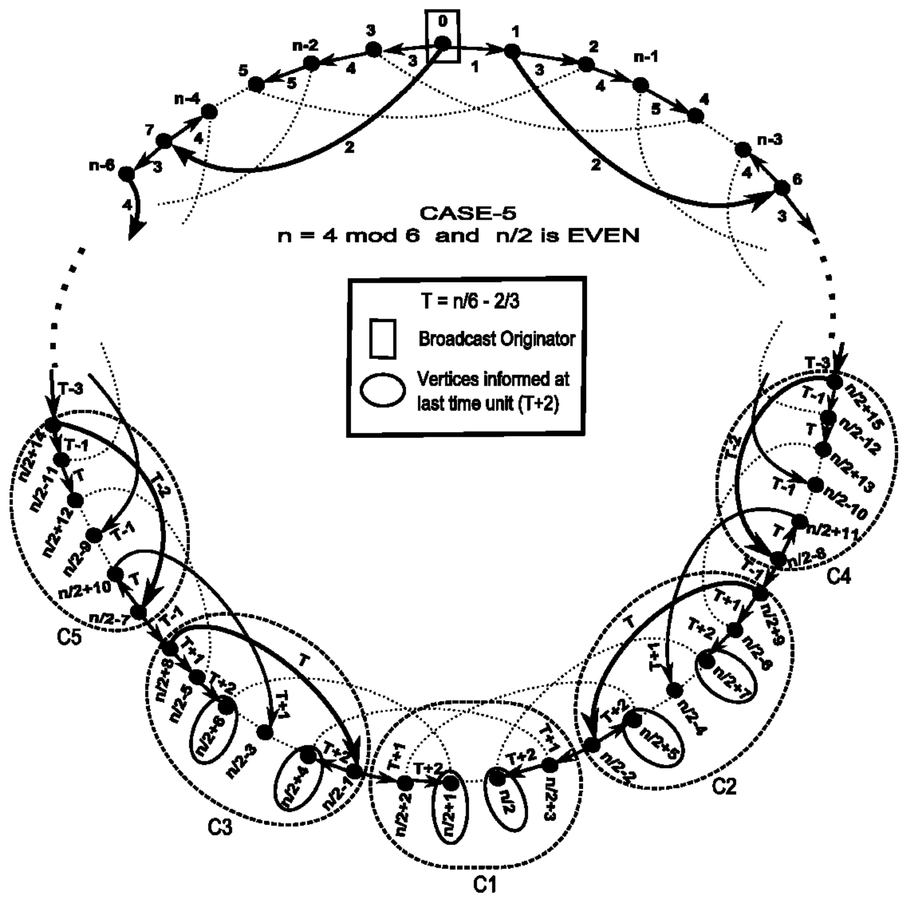


Figure 4.6 The broadcast scheme for $W_{3,n}$, where *n* = 4 *mod* 6 and *n*/2 is even

Recall that when the x-th "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T. In other words, all the vertices from "cycle" 1 to $(x-1)$ of length 6 on each path will receive the message by time T. The broadcasting in x-th "cycles" of length 6 on each path and the vertices between these two "cycles" will be completed at time $T+2$.

Now consider the C_2 (x-*th* "cycle" of length 6 in clockwise direction). Since the vertex $\frac{n}{2}+9$ is on the specified path, therefore at time $T-1$ it is informed by the vertex $\frac{n}{2}-8$ using 1-dimensional edge. At time T the vertex $\frac{n}{2}+9$ sends the message to the vertex $\frac{n}{2}-2$, that forms the x-*th* "cycle" of length 6 in the clockwise direction. At time $T+1$ the vertices $\frac{n}{2}+9$ and $\frac{n}{2}-2$ send the message to the vertices $\frac{n}{2}-6$ and $\frac{n}{2}+3$, respectively. As the vertex $\frac{n}{2}+11$ is from the $(x-1)$-*th* "cycle" and it get informed by time T, so this can participate in the broadcasting process. Therefore at time $T+1$ the vertex $\frac{n}{2}+11$ sends the message to the vertices $\frac{n}{2}-4$ using 3-dimensional edge. Now there are two vertices left to be informed in C_2. At time $T+2$ the vertices $\frac{n}{2}-6$ and $\frac{n}{2}-2$ inform the vertices $\frac{n}{2}+7$ and $\frac{n}{2}+5$, respectively, this completes the broadcasting in C_2. Similarly the broadcasting in C_3, the x-*th* "cycle" on specified path in anti-clockwise direction, will be completed at time $T+2$ in a similar way as described above.

C_1 consists of the vertices between x-*th* "cycles" of length 6 on each path. The broadcasting in C_1 will also be completed at time $T+2$ in the following way. Recall that at time $T+1$ the vertices $\frac{n}{2}-2$ and $\frac{n}{2}-1$ send the message to the vertices $\frac{n}{2}+3$ and $\frac{n}{2}+2$, respectively. Now at time $T+2$ the vertices $\frac{n}{2}+3$ and $\frac{n}{2}+2$ inform the vertices $\frac{n}{2}$ and $\frac{n}{2}+1$, respectively, this way all the vertices of the graph are informed. Since there is no other vertex left to be informed, this proves that $b(0, W_{3,n}) \leq \mathrm{T}+2$, where $T = \frac{n}{6}-\frac{1}{3}$, $n = 4 \, mod \, 6$ and $\frac{n}{2}$ is even. Thus,

$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n}) = \frac{n}{6}+\frac{5}{3} = \left\lceil\frac{n-2}{6}\right\rceil + 1}$, for $n > 16$, $n = 4 \, mod \, 6$ and $\frac{n}{2}$ is even

Case 6: $n = 4\ mod\ 6$ and $\frac{n}{2}$ is odd

Recall that in Case 6 under Section 3.1 after $x = \frac{n-10}{12}$ 1 and 3-dimensional passes in clockwise and anti-clockwise directions, we reach the vertices $\frac{n}{2} - 5$ and $\frac{n}{2} + 5$, respectively. Both of these vertices are at distance $\frac{n}{6} - \frac{5}{3}$ from vertex 0. If the broadcasting is started in a clockwise direction the vertex $\frac{n}{2} - 5$ will receive the message at time $T = \frac{n}{6} - \frac{5}{3}$. The vertex $\frac{n}{2} + 5$ will receives the message one time unit later than the vertex $\frac{n}{2} - 5$, at time $T + 1$. Figure 4.7 depicts the broadcast scheme for Case 6.

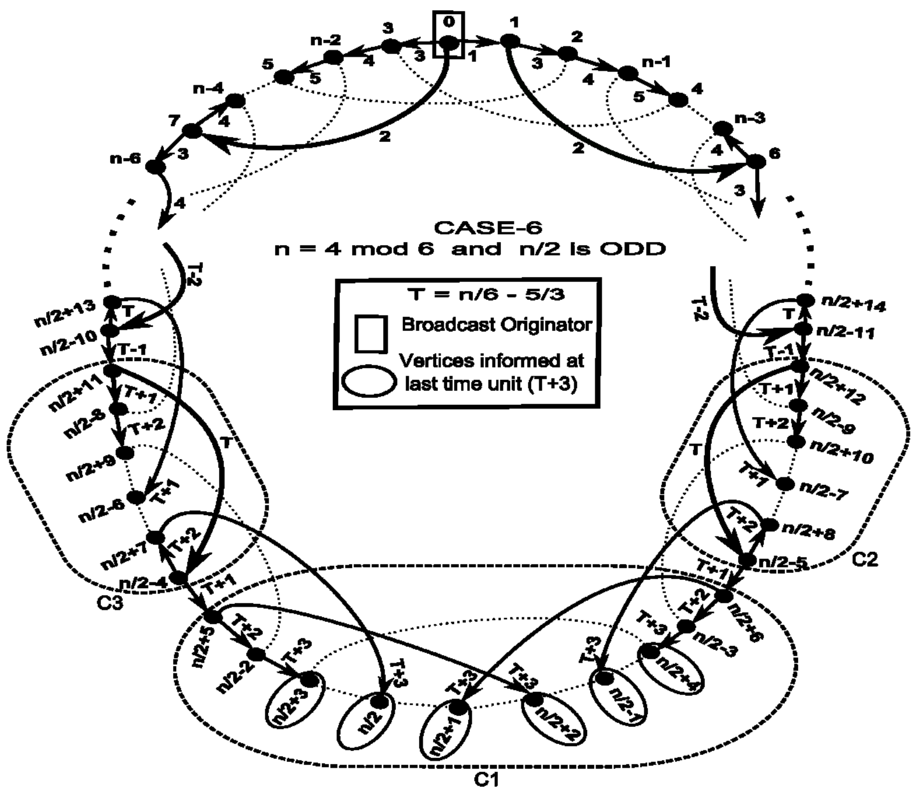


Figure 4.7 The broadcast scheme for $W_{3,n}$, where n = 4 mod 6 and n/2 is odd

Recall that when the x-th "cycles" of length 6 on each path are formed, where 3-dimensional edges are labeled with time T, the vertices in all previous "cycles" will receive the message by time T. Also recall that the broadcasting in the x-th "cycles" of length 6 on each path finish by time $T + 2$. Therefore all the vertices from the "cycle" 1 to x-th of length 6 on each path will receive the message by time $T + 2$. The vertices between x-th "cycles" of length 6 on each path (C_1) will be informed by time $T + 3$ in the following way.

Recall that, at time $T + 1$, the vertices $\frac{n}{2} - 5$ and $\frac{n}{2} - 4$ send the message to the vertices $\frac{n}{2} + 6$ and $\frac{n}{2} + 5$, respectively. Now, at time $T + 2$, the vertices $\frac{n}{2} + 6$ and $\frac{n}{2} + 5$ inform the vertices $\frac{n}{2} - 3$ and $\frac{n}{2} - 2$, respectively. After time $T + 2$, there are 6 vertices yet to be informed in C_1. At time $T + 3$, the vertices $\frac{n}{2} - 3$ and $\frac{n}{2} - 2$ using 1-dimensional edge inform the vertices $\frac{n}{2} + 4$ and $\frac{n}{2} + 3$, respectively. Also at time $T + 3$ the vertices $\frac{n}{2} + 6$, $\frac{n}{2} + 8$, $\frac{n}{2} + 5$ and $\frac{n}{2} + 7$ using 3-dimensional edge inform the vertices $\frac{n}{2} + 1$, $\frac{n}{2} - 1$, $\frac{n}{2} + 2$ and $\frac{n}{2}$, respectively. Since there is no other vertex left to be informed, this proves that $b(0, W_{3,n}) \leq T + 3$, where $T = \frac{n}{6} - \frac{5}{3}$, $n = 4 \ mod \ 6$ and $\frac{n}{2}$ is odd. Thus,

$\boldsymbol{b(W_{3,n}) \leq D(W_{3,n}) = \frac{n}{6} + \frac{5}{3} = \left\lceil \frac{n-2}{6} \right\rceil + 1}$, for $n > 10$, $n = 4 \ mod \ 6$ and $\frac{n}{2}$ is odd

4.2 Broadcast time of $W_{3,n}$

In this section, we present the $b(W_{3,n})$ for all even n and degree 3. We know that $b(G) \geq D(G)$, for any connected graph G. The following lemma provides the lower bound on $b(G)$, when at least two vertices are at diametral distance D, from vertex u.

Lemma 4.1 [18]

If there exists at least two vertices at diameteral distance D from vertex u in graph G, then $b(G) \geq D + 1$.

Recall that the Knödel graph $W_{3,n}$, when $n = 0\ mod\ 6$, has two vertices at diametral distance D from vertex 0. Also recall that $W_{3,n}$, when $n > 8$ and $n = 2\ mod\ 6$, has three vertices at diametral distance D from vertex 0. Based on the Lemma 4.1, it follows that $b(W_{3,n}) \geq D(W_{3,n}) + 1$, where $n > 8$ and $n = 0, 2\ mod\ 6$. In Section 4.1 we derived that $b(W_{3,n}) \leq D(W_{3,n}) + 1$, where $n > 8$ and $n = 0, 2\ mod\ 6$. Combining these two inequalities we get,

$$\boxed{\boldsymbol{b(W_{3,n}) = D(W_{3,n}) + 1 = \left\lceil \frac{n-2}{6} \right\rceil + 2}} \quad \text{for} \quad n = 0, 2\ mod\ 6$$

The lower bound on $b(W_{3,n})$, when $n = 4\ mod\ 6$, follows from its diameter. We need at least $D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1$, time units to inform a vertex at distance $\left\lceil \frac{n-2}{6} \right\rceil + 1$, from the broadcast originator. Thus,

$$\boldsymbol{b(W_{3,n}) \geq D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1} \quad \text{for} \quad n = 4\ mod\ 6$$

In Section 4.1, we derived that $b(W_{3,n}) \leq D(W_{3,n})$, where $n = 4\ mod\ 6$. Combining these two inequalities we get,

$$\boxed{\boldsymbol{b(W_{3,n}) = D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1}} \qquad \text{for} \ n > 16,\ n = 4\ mod\ 6$$

$W_{3,10}$ and $W_{3,16}$ are the only two graphs in $W_{3,n}$, for $n = 4\ mod\ 6$, where the broadcasting cannot be done in diametral time. The $D(W_{3,10}) = 3$, and $b(W_{3,10}) = 4$. In $W_{3,10}$, 8 vertices will be informed in 3 time units, because a Knödel graph of order 8 and degree 3 is a broadcast graph, where $b(W_{3,8}) = \lceil \log_2 8 \rceil = 3$. The remaining 2 vertices of $W_{3,10}$ will be informed at time 4. Similarly, the $D(W_{3,16}) = 4$, and $b(W_{3,16}) = 5$. In $W_{3,16}$, 8 vertices will be informed in 3 time units. After time 3, 2 out of 8 informed vertices will not participate in the rest of the broadcasting process, since all of their neighbors are already informed. So, at time 4, at most 6 vertices can be informed and the remaining 2 vertices will be informed at time 5, thus $b(W_{3,16}) = 5$. Figures 4.8 and 4.9 illustrate the diametral broadcasting in $W_{3,22}$ and $W_{3,28}$, respectively.

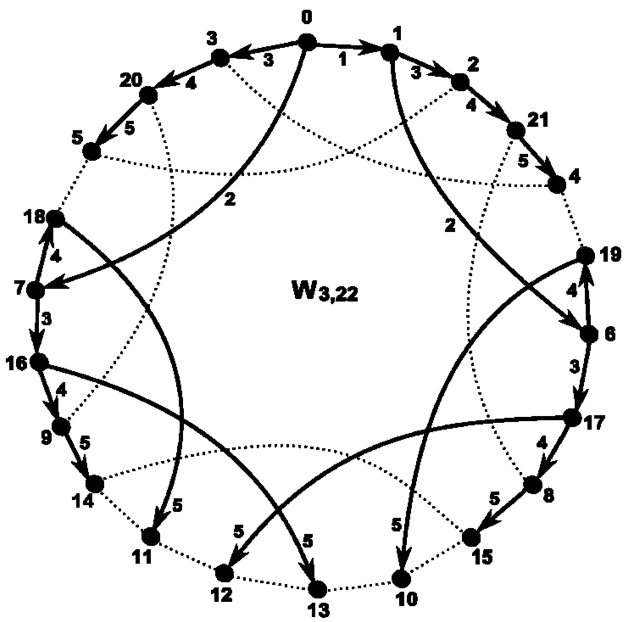


Figure 4.8 The diametral broadcast graph, the Knödel graph $W_{3,22}$

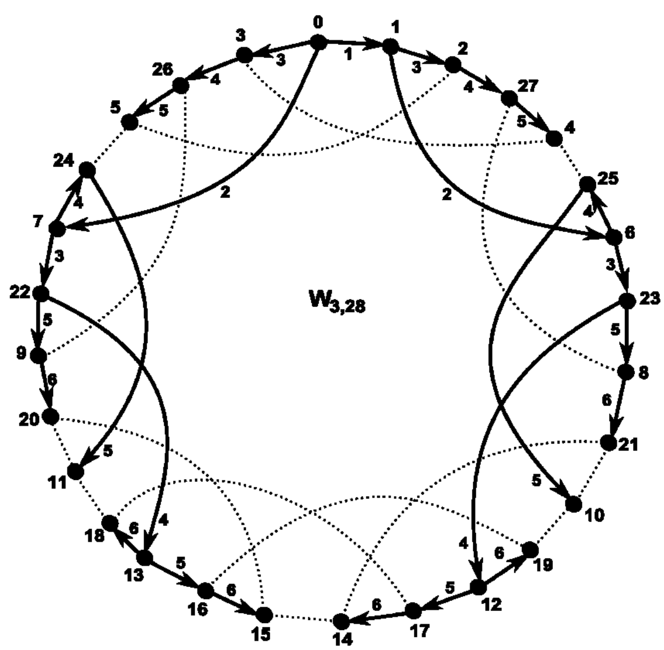


Figure 4.9 The diametral broadcast graph, the Knödel graph $W_{3,28}$

4.3 The first diametral broadcast graph family in $W_{d,n}$

Recall that for the lower bound on the broadcast time of any graph G, we have $b(G) \geq \lceil \log_2 n \rceil$. Another obvious lower bound on the broadcast time is $b(G) \geq D(G)$. Also recall that the graph, where the broadcast time equals to its diameter, is known as the diametral broadcast graph.

The graphs with $b(G) = D(G)$ have been studied in [23], where the problem of existence of graphs with broadcast time equal to their diameter was introduced. The *diametral broadcast graph (dbg) problem* is to answer the question whether for a given n and d, a graph on n vertices can be constructed whose diameter and broadcast time are equal to d [23]. They also defined the diametral broadcast function $DB(n, d)$ as the minimum possible number of edges in a *dbg* on n vertices and diameter d. In [23], the following three different constructions were presented to solve the diametral broadcast graph problem for all possible values of n and d.

In [23], the first construction was based on trees and provided the exact value of $DB(n, d) = n - 1$, for a few values of n and d. The second construction was based on a hypercube and binomial subtrees attached to it, where they obtained that the $DB(n, d) = \frac{1}{2}n(\lceil \log_2 n \rceil - 1)$. In the last construction a *dbg* was obtained by removing certain vertices with adjacent edges from the hypercube and they obtained $DB(n, d) = \frac{1}{2}n\lceil \log_2 n \rceil$.

However, in section 4.2, we have presented and proved that the broadcasting in the Knödel graph $W_{3,n}$, where $n > 16$ and $n = 4 \; mod \; 6$, can be performed in diametral time. Since the Knödel graph $W_{d,n}$ has degree d and n vertices, and it is bipartite, therefore the number of edges are equal to $|E| = \frac{dn}{2}$. Subsequently, the number of edges in $W_{3,n}$, where $n > 16$ and $n = 4 \; mod \; 6$, are $|E| = \frac{3n}{2}$. Moreover, $W_{3,n}$, where $n >$

16 and $n = 4 \bmod 6$, is the first infinite diametral broadcast graph family in Knödel graph $W_{d,n}$.

4.4 The Broadcast graph $W_{3,22}$

The construction of the graphs with $b(G) = \lceil \log_2 n \rceil$ (i.e., broadcast graphs) is a well-studied problem in literature. See e.g. [3] [5] [6] [7] [10] [12] [20] [21] [27] [29] [30]. In terms of Knödel graph, it is presented in [14] that $W_{d-2,n}$, where $2^{d-1} + 2 \leq n \leq 3 \cdot 2^{d-2} - 4$, is a broadcast graph. Recall that a graph on n vertices with $b(G) = \lceil \log_2 n \rceil$ is known as a broadcast graph. For the Knödel graph $W_{3,n}$ the number of vertices can be calculated from the range of n given above in [14]. Since the degree in $W_{d-2,n}$ is $d - 2$, therefore $d = 5$ in $W_{3,n}$. Thus,

$$2^{d-1} + 2 \leq n \leq 3 \cdot 2^{d-2} - 4$$

$$2^{5-1} + 2 \leq n \leq 3 \cdot 2^{5-2} - 4$$

$$18 \leq n \leq 20$$

For $W_{3,n}$ the interval of n presented in [14] is calculated as $18 \leq n \leq 20$. Because of the the interval of n from [14], the Knödel graph $W_{3,22}$, cannot be considered as a broadcast graph. But we show that $W_{3,22}$ is the broadcast graph. In Figure 4.8, we present the broadcast scheme for $W_{3,22}$, which demonstrates that the broadcasting in $W_{3,22}$ can be done in $\lceil \log 22 \rceil = 5$ time units. The broadcasting of $W_{3,22}$ is also presented in Table 4.1, where the information, like 0→7(3), can be interpreted as follows: the vertex 0 informs the vertex 7 using 3-dimensional edge. Moreover $b(W_{3,22}) = D(W_{3,22})$. Therefore this is a diametral broadcast graph too.

Table 4.1 Broadcasting in the Knödel graph $W_{3,22}$

Time	Informed vertices
0	0
1	0→1(1)
2	0→7(3), 1→6(3)
3	0→3(2), 1→2(2), 6→17(1), 7→16(1)
4	2→21(1), 3→20(1), 6→19(2), 7→18(2), 16→9(2), 17→8(2)
5	8→15(1), 9→14(1), 16→13(3), 17→12(3), 18→11(3), 19→10(3), 20→5(2), 21→4(2)

4.5 Summary

In this chapter, we studied the problem of broadcasting in the Knödel graph $W_{3,n}$, for all even n and degree 3. Our contributions to this chapter are the following:

- $\boldsymbol{b(W_{3,n}) = D(W_{3,n}) + 1 = \left\lceil \frac{n-2}{6} \right\rceil + 2}$ for $n = 0, 2 \; mod \; 6$
- $\boldsymbol{b(W_{3,n}) = D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1}$ for $n > 16,\ n = 4 \; mod \; 6$
- We showed that $W_{3,n}$, for $n > 16$ and $n = 4 \; mod \; 6$, is the first diametral broadcast graph family in the Knödel graph $W_{d,n}$.
- Using the broadcast scheme, we proved that, $W_{3,22}$, is a broadcast graph. Moreover, $b(W_{3,22}) = D(W_{3,22})$, so this is also a diametral broadcast graph.
- Since $\boldsymbol{b(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1}$ for $n > 16$ and $n = 4 \; mod \; 6$. Therefore, it turns out that the conjecture $\boldsymbol{b(W_{d,n}) = \left\lceil \frac{n-2}{2^d-2} \right\rceil + d - 1}$ for all even n and degree d given in [23] is not valid anymore.

Chapter 5

New graph construction and its communication properties

In this chapter we construct a new graph, denoted as $HW_{d,2^d}$, by connecting the vertices of the Knödel graph $W_{d-1,2^{d-1}}$ to hypercube H_{d-1}. We investigate the communication properties of $HW_{d,2^d}$ in terms of number of vertices, degree, edges, diameter, and broadcast time. With the use of extensive simulation, we provide diameter and broadcast time of $HW_{d,2^d}$ for all $d \leq 24$.

5.1 The construction of $HW_{d,2^d}$

We construct the $HW_{d,2^d}$ graph of order 2^d and degree d by connecting the vertices of the Knödel graph $W_{d-1,2^{d-1}}$ and hypercube H_{d-1}. The construction of $HW_{d,2^d}$ is as follows:

Consider the Knödel graph $W_{d-1,2^{d-1}}$ and hypercube H_{d-1}, we use the definition 1 for Knödel graph. Now connect the vertices of Knödel graph $W_{d-1,2^{d-1}}$ and hypercube H_{d-1} using d-dimensional edges in a way that the vertex i, for all $0 \leq i \leq 2^{d-1} - 1$, of the Knödel graph is connected to the vertex of H_{d-1} whose binary label is equal to i. Connecting both of these graphs using d-dimensional edges, we have a $HW_{d,2^d}$ graph of order 2^d and degree d.

To demonstrate the construction of $HW_{d,2^d}$ graph, let's consider $HW_{4,2^4}$, where the vertex labeled 0 of Knödel graph $W_{3,2^3}$ connects to the vertex labeled 000 of hypercube H_3 using 4-dimensioanl edge. Now the rest of the vertices labeled 1, 2, 3, 4, 5, 6, 7 of Knödel graph $W_{3,2^3}$ connect to the vertices $001, 010, 011, 100, 101, 110, 111$ of hypercube H_3, respectively, using 4-dimensional edges. Figure 5.1 illustrates the construction of $HW_{4,2^4}$ graph by connecting the vertices of $W_{3,2^3}$ and H_3 using 4-dimensional edges.

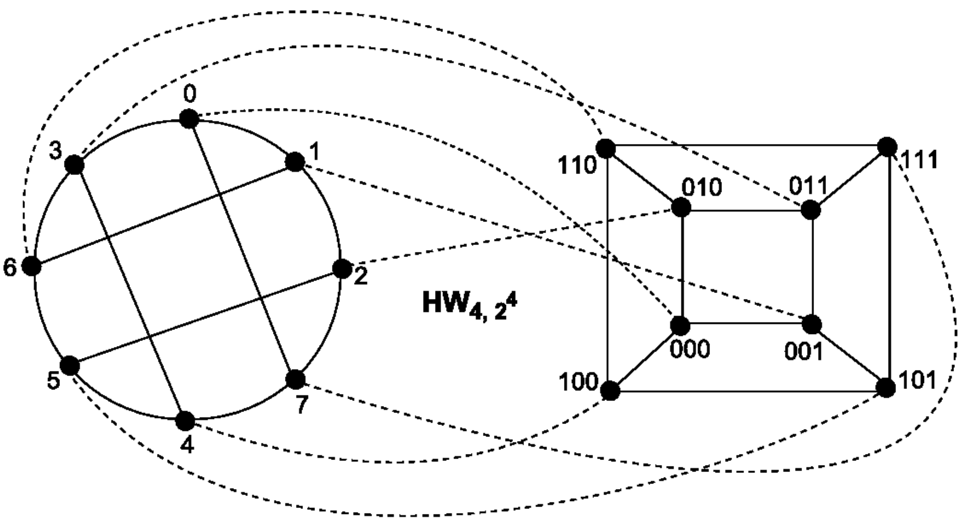


Figure 5.1 The construction of $HW_{4,2^4}$ graph by connecting the vertices of $W_{3,2^3}$ and H_3 using 4-dimensional edges.

5.2 The communication properties of $HW_{d,2^d}$

In this section we investigate the communication properties of $HW_{d,2^d}$ graph in terms of number of vertices, degree, edges. The $HW_{d,2^d}$ graph has 2^d vertices and the degree *d*. The number of edges of $HW_{d,2^d}$ graph is obtained from the number of edges of $W_{d-1,2^{d-1}}$, H_{d-1} and the edges those connects these two graphs. Thus, $HW_{d,2^d}$ graph has $|E| = d \cdot 2^{d-1}$.

5.3 Diameter of $HW_{d,2^d}$ graph

In order to obtain $D(HW_{d,2^d})$, we have performed an experiment, where the *Breadth-First-Search* operation is applied on the $HW_{d,2^d}$ graph. The simulation results suggest that $\boldsymbol{D(HW_{d,2^d}) = \left\lfloor \frac{d+2}{2} \right\rfloor}$, for $1 \le d \le 24$.

5.4 Broadcast time of $\boldsymbol{HW_{d,2^d}}$ graph

This section provides the broadcast time of $HW_{d,2^d}$ graph that is based on the simulation results. The broadcasting in $HW_{d,2^d}$ graph is performed using classical broadcast model. The careful study of generated data regarding $b(HW_{d,2^d})$ demonstrates that the broadcasting in $HW_{d,2^d}$ graph is performed in $\lceil \log_2 n \rceil$ time. Recall that a graph on *n* vertices with $b(G) = \lceil \log_2 n \rceil$ is called a broadcast graph. Thus, $HW_{d,2^d}$ is a broadcast graph of order 2^d and degree *d*.

$$\boldsymbol{b(HW_{d,2^d}) = \lceil \log_2 2^d \rceil = d} \qquad \text{for } 1 \leq d \leq 24$$

5.5 Comparison of $\boldsymbol{H_d}$, $\boldsymbol{G(2^d, 4)}$, $\boldsymbol{W_{d,2^d}}$ and $\boldsymbol{HW_{d,2^d}}$

For any $n = 2^d$ and degree *d*, hypercube H_d, recursive circulant $G(2^d, 4)$ and the Knödel graph $W_{d,2^d}$ are the three non-isomorphic infinite graph families known to be minimum broadcast and gossip graphs [16][22]. Recall that a broadcast graph with the minimum possible number of edges is called minimum broadcast graph. Since $HW_{d,2^d}$ graph has 2^d vertices and degree *d,* and the broadcasting is performed in $\lceil \log_2 n \rceil$ time (as in H_d, $G(2^d, 4)$ and $W_{d,2^d}$), therefore, $HW_{d,2^d}$, when $1 \leq d \leq 24$, is also a minimum broadcast graph.

These four topologies are comparable because they all have good communication properties in terms of interconnection networks. Moreover, they are of the same order 2^d and regular graphs with the same degree *d.* Table 5.1 provides the comparison between H_d, $G(2^d, 4)$, $W_{d,2^d}$ and $HW_{d,2^d}$ graphs.

Table 5.1 Comparison between $\boldsymbol{H_d}$, $\boldsymbol{G(2^d, 4)}$, $\boldsymbol{W_{d,2^d}}$ and $\boldsymbol{HW_{d,2^d}}$

Properties	H_d	$G(2^d, 4)$	$W_{d,2^d}$	$HW_{d,2^d}$
Number of vertices	2^d	2^d	2^d	2^d
Degree	d	d	d	d
Edges	$d \cdot 2^{d-1}$	$d \cdot 2^{d-1}$	$d \cdot 2^{d-1}$	$d \cdot 2^{d-1}$
Diameter	d	$\left\lceil \frac{3d-1}{4} \right\rceil$	$\left\lceil \frac{d+2}{2} \right\rceil$	$\left\lfloor \frac{d+2}{2} \right\rfloor$ for $1 \leq d \leq 24$
Broadcast time	d	d	d	d
Minimum broadcast graph	Yes	Yes	Yes	Yes

It is observed from Table 5.1, that diameter of $HW_{d,2^d}$ and $W_{d,2^d}$ is equal to $\frac{d+2}{2}$ for even degree d and $2 \leq d \leq 24$. Whereas, when degree d is odd and $3 \leq d \leq 24$ then $D(HW_{d,2^d}) = \left\lfloor \frac{d+2}{2} \right\rfloor$, that is smaller than the $D(W_{d,2^d}) = \left\lceil \frac{d+2}{2} \right\rceil$, for any odd degree d.

5.6 Summary

In this chapter, we provided the construction of a new graph, denoted as $HW_{d,2^d}$, by connecting the vertices of the Knödel graph $W_{d-1,2^{d-1}}$ to hypercube H_{d-1}. Our contributions to this chapter are the following:

- The construction of a new graph, denoted as $HW_{d,2^d}$
- $D(HW_{d,2^d}) = \left\lfloor \frac{d+2}{2} \right\rfloor$, for $1 \leq d \leq 24$
- $b(HW_{d,2^d}) = \lceil \log_2 2^d \rceil = d$, for $1 \leq d \leq 24$
- $HW_{d,2^d}$ graph is a minimum broadcast graph

Chapter 6

Conclusion and Future Work

In this thesis we studied three inter-related communication properties, i.e., diameter, number of vertices at a particular distance and the broadcast time of the Knödel graph. We can divide our work in three parts.

In the first part, the Knödel graph is studied in terms of diameter. Theoretically, we provide $\boldsymbol{D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1}$ for $n > 8$. Moreover, the massive experimental work on the Knödel graph, and careful studies of the observed properties lead us to give the exact diameter of some other specific Knödel graphs, i.e., $W_{d-1,\, 2^d-2}$, $W_{d-1,\, 2^d}$, $W_{d,\, 2^d+2}$, $W_{d,\, 2^d+4}$, and $W_{d,\, 2^d+\, 2^{d-1}-2}$.

In the second part, we studied the Knödel graph in terms of the number of vertices at a particular distance. In this regard, the experiment was conducted, where *Breadth-First-Search* operation was performed on Knödel graphs $W_{d,\, 2^d}$, $W_{d,\, 2^d+2}$, $W_{d,\, 2^d+4}$, $W_{d,\, 2^d+6}$, $W_{d,\, 2^d+8}$ and $W_{d-1,\, 2^d-2}$. The comprehensive study of the data and observed properties enables us to give three propositions for the number of vertices at a particular distance in specific Knödel graphs.

The problem of determining the broadcast time of the Knödel graph $W_{3,n}$ for all even n and degree 3 has been undertaken in the last part of our work. Regarding the broadcast time of Knödel graph $W_{3,n}$, we obtained the following results:

- $\boldsymbol{b(W_{3,n}) = D(W_{3,n}) + 1 = \left\lceil \frac{n-2}{6} \right\rceil + 2}$ for $n = 0, 2 \ mod\ 6$
- $\boldsymbol{b(W_{3,n}) = D(W_{3,n}) = \left\lceil \frac{n-2}{6} \right\rceil + 1}$ for $n > 16$ and $n = 4\ mod\ 6$
- We showed that $W_{3,n}$, for $n > 16$ and $n = 4\ mod\ 6$, is the first diametral broadcast graph family in the Knödel graph $W_{d,n}$.

- Using the broadcast scheme, we proved that $W_{3,22}$ is a broadcast graph. Moreover, $b(W_{3,22}) = D(W_{3,22})$, so this is also a diametral broadcast graph.
- Since $\boldsymbol{b(W_{3,n})} = \left\lceil \frac{\boldsymbol{n-2}}{\boldsymbol{6}} \right\rceil + \boldsymbol{1}$ for $n > 16$ and $n = 4 \ mod \ 6$. Therefore, it turns out that the conjecture $\boldsymbol{b(W_{d,n})} = \left\lceil \frac{\boldsymbol{n-2}}{\boldsymbol{2^d-2}} \right\rceil + \boldsymbol{d-1}$ for all even n and degree d given in [23] is not valid anymore.

In Chapter 5, we provided the construction of a new graph, denoted as $HW_{d,2^d}$, and also investigated it's communication properties. It turns out that this is a minimum broadcast graph on order 2^d and degree *d*, $1 \leq d \leq 24$. Also when degree *d* is odd and $3 \leq d \leq 24$ then $D(HW_{d,2^d}) = \left\lfloor \frac{d+2}{2} \right\rfloor$, that is smaller than the $D(W_{d,2^d}) = \left\lceil \frac{d+2}{2} \right\rceil$, for any odd degree *d*.

Following this we would like to mention few open questions for future research:

- The exact value of diameter is given in this thesis for some specific Knödel graphs. To find the diameter for other families of Knödel graph $W_{d,n}$ is still an open and a challenging question.
- We presented that $W_{3,n}$, where $n > 16, \ n = 4 \ mod \ 6$, is the first diametral broadcast graph family in the Knödel graph $W_{d,n}$. But we are expecting that there might be other families of diametral graphs in Knödel graph $W_{d,n}$, where $4 \leq d \leq \lfloor \log_2 n \rfloor$.

The remarkable number of vertices to diameter ratio characteristic enables Knödel graph to compete with hypercube and circulant graphs of same order and degree. Knödel graph with its known characteristics in terms of dissemination of information becomes a suitable candidate for communication networks, where parallel algorithms are heavily employed.

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