Set-Valued Maps And Their Applications

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ABSTRACT

Set-Valued Maps and Their Applications

Joe Pharaon

The serious investigation of set-valued maps began only in the mid 1900s when mathematicians realized that their uses go far beyond a mere generalization of single-valued maps. We explore their fundamental properties and emphasize their continuity. We present extensions of fixed point theorems to the set-valued case and we conclude with an application to Game Theory.

ACKNOWLEDGEMENT AND DEDICATION

I would like to thank Dr. Pawel Gora and Dr. Abraham Boyarsky for their academic guidance. I would like to thank friends for their continuous encouragement and enthusiasm, especially Judy. I would like to thank all my family for being always available and providing the support I needed. Special thanks to my father and mother Elie and Mirna.

I dedicate this thesis to everyone who wants to learn and make our world a better place.

Contents

Chapter 1

Introduction

In this thesis, we attempt to understand set-valued maps, a class of maps that was considered irrelevant until Kuratowski [8] gave them their proper status. They are an extension and a generalization of single-valued maps. Their use in applications such as in control theory, economics and management, biology and systems sciences, artificial intelligence, etc... [2] helped them gain unprecedented attention and encouraged a lot of mathematicians, like J.P. Aubin, to develop theories about their calculus. J.P. Aubin, in his book "Set-Valued Analysis" [2], expands on the properties of sequences of sets and their limits [11], known as Kuratowski limits. We will present these results in the second chapter of this thesis.

In the third chapter, we move from the discrete case of a sequence of sets to the continuous case to get a set-valued map. We discuss fundamental properties and we emphasize, in our discussion, the definition of continuity. The sequential definition of continuity for single-valued maps fails to hold in the set-valued case. It wasn't until 1932 that the concepts of semi-continuous maps had been introduced by G. Bouligand [3] and K. Kuratowski [8].

In the fourth chapter, the class of upper semi-continuous set-valued maps is used to extend the Brouwer Fixed Point Theorem to its analogue in the set-valued case. The theorem due to Kakutani [6] provides sufficient conditions for a set-valued map to have a fixed point. We also state the Ky-Fan inequality [4], an important theorem discovered in 1972, used to prove existence of equilibria.

In the final chapter, we discuss an application of set-valued maps and the existence of equilibria in the context of game theory. Game theory became a field of its own after the publication of a paper by John Von Neumann in 1928 [13]. It is the study of strategic decision making through building mathematical models that describe the process of decision making. We start the chapter with fundamental concepts about a two-person non-cooperative normal game. In this game, we have two players who do not cooperate and their decision rules and evaluation functions are represented by matrices. In 1951, John Nash proved a theorem of existence of an equilibrium for non-cooperative games with finitely many players [10]. We conclude the thesis with a statement of the theorem and an elegant proof [1].

Chapter 2

Sequences Of Sets And Their Limits

2.1 Sequences of Sets

Think about a sequence of sets $S_n := S(n)$ as a rule that assigns to every natural number n, a subset of a well-defined space. In most of our discussion, the space under consideration is $\mathbb{R}^d$, the set of ordered d-tuples whose elements are real numbers. There are many reasons behind this choice. We observe its frequent use in the modelling of real world problems and its elegant topological properties.
We refer to the collection of all subsets of $\mathbb{R}^d$ as the power set of $\mathbb{R}^d$ and we denote it by $\mathcal{P}(\mathbb{R}^d)$. Therefore, formally speaking, we will define a sequence of sets S_n as follows:

$$S : \mathbb{N} \longrightarrow \mathcal{P}(\mathbb{R}^d)$$

Example: Suppose a virus is released in the world causing each country to have a certain number of infected individuals and non-infected individuals. We might be interested in the evolution of this virus in time. If we let n be a unit of time, we can create a sequence S_n that sends each unit of time to a subset of $\mathbb{R}^2$ that represents the number of infected individuals I and non-infected individuals N in 193 countries. Therefore, if we let $n = 0$ be the moment of the release of the virus, we can let $S_0 = \{(0, p_i) | i = 1 \ldots 193\}$ where p_i represents the population of the ith country. In this example each element of the sequence S_n has precisely 193 elements of $\mathbb{R}^2$.

We will now look at an abstract example in $\mathbb{R}^2$ since it is very easy to graph.

$$S_n = \left\{\frac{1}{n}\right\} \times [0,1]$$

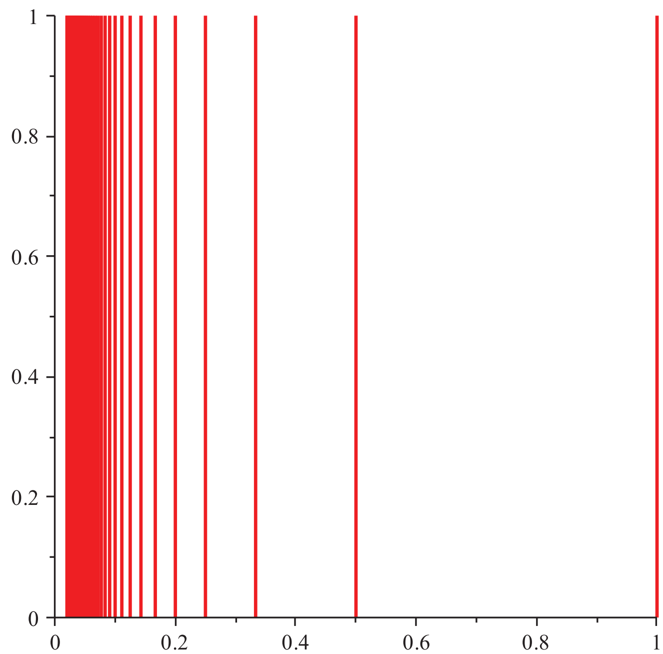

Figure 2.1: Sequence $S_n = \{\frac{1}{n}\} \times [0,1]$

In this example, $S_1 = \{(1,y)|0 \leq y \leq 1\}$.

Just as we are interested in finding the distance that separates two points in a Euclidean space, we are also interested in defining a distance function to measure how far two sets are apart from each other. We will start by defining the distance from a point in $\mathbb{R}^d$ to a subset of $\mathbb{R}^d$.
Let $x \in \mathbb{R}^d$ and let $S \subset \mathbb{R}^d$ such that $S \neq \emptyset$. Then the distance between x and S is given as follows:

$$\operatorname{dist}(x,S) = \inf\{d(x,y)|y \in S\} = \inf\{\|x-y\||y \in S\}$$

For metric spaces that are not normed, we use their well-defined metric. Also note that if S is closed and bounded in $\mathbb{R}^d$, then we can write minimum instead of infimum. As a convention we define the distance from x to the empty set as:

$$\mathrm{dist}(x, \emptyset) = +\infty$$

Example: Consider the line $y = x + 1$ that is represented by the set $L = \{(x, y)|y = x + 1, x \in \mathbb{R}\}$ and the point $p(2, 0)$.

$$\mathrm{dist}(p, L) = \min\{\sqrt{(x-2)^2 + y^2}|(x, y) \in L\} = \min\{\sqrt{(x-2)^2 + (x+1)^2}|x \in \mathbb{R}\} = \frac{3}{\sqrt{2}}$$

Later, we will discuss distance between two sets which describes the work of the German mathematician Felix Hausdorff.

In the next section we develop an understanding of the work of the Polish mathematician Kazimierz Kuratowski who characterized the convergence of sets.

2.2 Kuratowski Convergence

We start by considering a sequence of real numbers s_n and reviewing two fundamental concepts: limit point and accumulation point.

We call s a limit point of the sequence s_n if the tail of the sequence can be grouped inside a ball of small radius around s. More formally, s is a limit point of the sequence s_n if for every small positive real number ϵ we can find a natural number N such that all the elements following s_N are at most ϵ far away from s.

We call s an accumulation point of the sequence s_n if we can find infinitely many elements of s_n close enough to s. More formally, s is an accumulation point of s_n if, for every small positive real number ϵ and for every natural number N, we can find $n \geq N$ such that s_n is at most ϵ away from s.

For sequences of real numbers, any limit point is an accumulation point but the converse is not true. The proof merely uses the definitions.

Example 1: $s_n = \frac{1}{n}$. The limit point is 0 since the tail of the sequence can be grouped around 0 in a ball with a very small radius. The accumulation point is also 0 since we can find infinitely many elements of this sequence close enough to 0.

Example 2: $s_n = \frac{(-1)^n}{n}$. The limit point is 0, and so is the accumulation point.

Example 3: $s_n = (-1)^n$. In this case the elements of this sequence are -1 and 1. Those two numbers are accumulation points since we can find infinitely many elements of s_n around -1 and 1. In fact there are infinitely many 1's and -1's in this sequence. On the other hand, this sequence has no limit point since there does not exist a real number to which the tail of the sequence is infinitesimally close.

It is worth noting that for sequences of real numbers, if the limit point exists then it is unique. On the other hand, we can have an arbitrary number of accumulation points.

Very often, sequences of real numbers do not have limit points. Therefore, it is very useful to define a "limit superior" and a "limit inferior" of a sequence of real numbers. If we are working in the extended reals, those two values always exist and they are frequently used in analysis. We will use soon limit superior and limit inferior in the definition of the Kuratowski limits. But first, we have to understand the concepts.

Consider a sequence of real numbers s_n. s is the limit superior of s_n if it is the supremum of the limits of all (convergent) subsequences of s_n. On the other hand, s would be the limit inferior of s_n if it is the infimum of all (convergent) subsequences of s_n. More formally, let L be the set of all limits of all convergent subsequences of s_n. Then limit superior and limit inferior are defined as follows:

$$\limsup_{n\to\infty} s_n = \sup\{L\}$$

$$\liminf_{n\to\infty} s_n = \inf\{L\}$$

We now provide two examples.

Example 1: $s_n = (1, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \dots)$. If we pick all the convergent subsequences, we get two limits: 0 and 1. An example of a subsequence that converges to 1 is $(1, 1, 1, 1, 1, 1, 1, 1, \dots)$ and a subsequence that converges to 0 is $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots)$. Therefore $L = \{0, 1\}$ and $\limsup_{n\to\infty} s_n = \sup\{0, 1\} = 1$ and $\liminf_{n\to\infty} s_n = \inf\{0, 1\} = 0$.

Example 2: $s_n = \sin(\frac{n\pi}{3})$. Writing this sequence explicitly, we can deduce that $L = \{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\}$. Hence, $\limsup_{n\to\infty} s_n = \sup\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\} = \frac{\sqrt{3}}{2}$ and $\liminf_{n\to\infty} s_n = \inf\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\} = -\frac{\sqrt{3}}{2}$.

We are now ready to define the "Upper Kuratowski Limit" and the "Lower Kuratowski Limit" of a sequence of sets. Consider a sequence of sets S_n. Then, the upper Kuratowski limit (sometimes called the upper limit) is defined as follows:

$$\mathrm{UL}(S_n) = \limsup_{n\to\infty} S_n = \{y \in \mathbb{R}^d | \liminf_{n\to\infty} \mathrm{dist}(y, S_n) = 0\}$$

The lower Kuratowski limit (sometimes called the lower limit) is defined as follows:

$$\mathrm{LL}(S_n) = \liminf_{n\to\infty} S_n = \{y \in \mathbb{R}^d | \lim_{n\to\infty} \mathrm{dist}(y, S_n) = 0\}$$

An equivalent way of thinking about the UL and LL of S_n is the following: the upper limit of S_n is the set of all accumulation points of all sequences s_n such that $s_n \in S_n$. The lower limit of S_n is the set of all limit points of all sequences s_n such that $s_n \in S_n$.

An interesting property is that the Lower Kuratowski limit (as a set) is a subset of the Upper Kuratowski limit. This is true since any limit point is an accumulation point.

Before we provide a few examples, we must note that if the upper limit and lower limit coincide then the sequence of sets S_n has a Kuratowski limit S. This is precisely Kuratowski convergence of S_n.

Example 1: Consider the following sequence of sets:

$$S_n = \begin{cases} \frac{1}{n} \times [0,1] & \text{if} \quad n \quad \text{is odd} \\ \\ \frac{1}{n} \times [-1,0] & \text{if} \quad n \quad \text{is even} \end{cases}$$

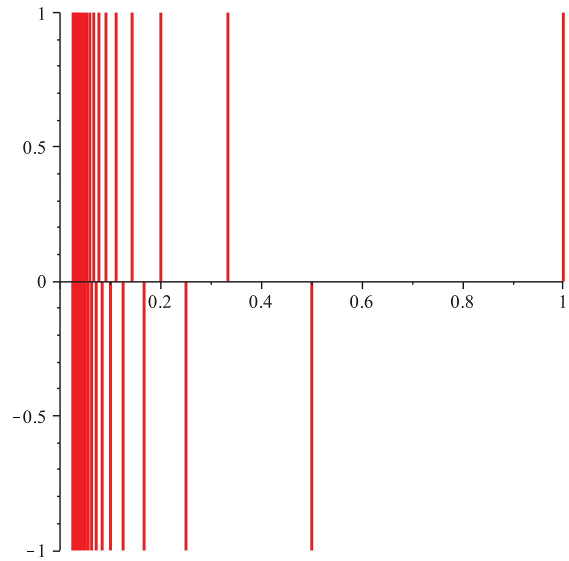

Figure 2.2: Sequence S_n from Example 1

We claim that the Lower Kuratowski limit of this sequence is the singleton $\{(0,0)\}$. First we consider the sequence $s_n^{(0)} \in S_n$ defined as $s_n^{(0)} = (\frac{1}{n}, 0)$. Elements of this sequence clearly belong to the sequence of sets and $s_n^{(0)}$ converges to the point $\{(0,0)\}$. Another sequence that converges to $\{(0,0)\}$ is for example $s_n^{(1)} = (\frac{1}{n}, \frac{(-1)^{n+1}}{n})$.

Next we show that if there exists a convergent sequence $s_n \in S_n$ then it can only converge to $\{(0,0)\}$. This last statement holds because whatever sequence we choose, we are always alternating between the positive side of the x-axis and the negative side. This fact does not allow convergence to any other point of the form $(0,y)$ where y is strictly positive or strictly negative, otherwise, we could draw a small ball around $(0,y)$ and infinitely many elements of s_n will not be contained in it.

We now find the Upper Kuratowski limit of this sequence. We know that $\{(0,0)\}$ belongs to the upper limit since it is a superset of the lower limit. We claim that the line segment joining the points $(0,1)$ and $(0,-1)$ (i.e. the set $\{(0,y) | -1 \leq y \leq 1\}$) is the Upper Kuratowski limit. Indeed, any point of this set is an accumulation point of all possible sequences of $s_n \in S_n$. For instance, if we pick the sequence $s_n^{(2)} = (\frac{1}{n}, \frac{(-1)^{n+1}}{2})$, there exists a subsequence $s_{n_k}^{(2)} = (\frac{1}{2n-1}, \frac{1}{2}) \in S_{n_k}$ that converges to $(0, \frac{1}{2})$. In other words, $(0, \frac{1}{2})$ is an accumulation point of $s_n^{(2)}$.

Finally notice how the sequence S_n "accumulates" around the vertical line segment described above. The Upper Kuratowski limit is validated.

Example 2: We can represent the graph of a sequence of functions as a sequence of sets since for any function $f : X \to Y$ we can associate the set Graph$(f) = \{(x, y)|f(x) = y\} \subset X \times Y$.

Consider the graphs of the sequence of functions $f_n : [0, 1] \to [-1, 1]$ defined by:

$$f_n(x) = (-1)^n(-2x^3 + 3x^2 + x - 1)^{2n-1}$$

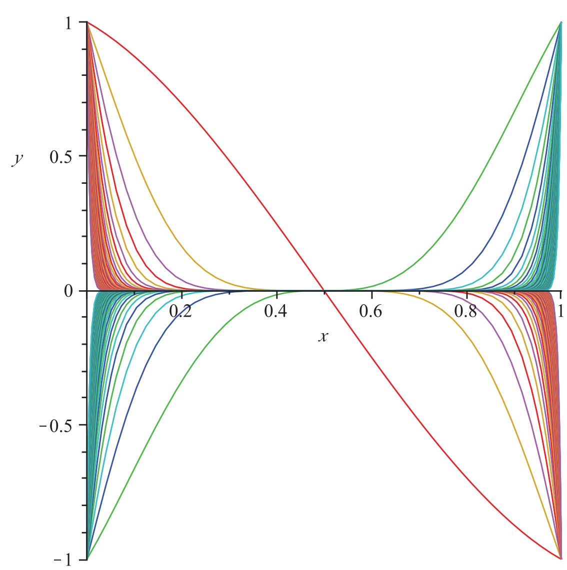

Figure 2.3: Sequence S_n from Example 2

Notice that $f_{2n-1}(x)$ starts at the point $(0, 1)$ and ends at the point $(1, -1)$ whereas $f_{2n}(x)$ starts at the point $(0, -1)$ and ends at the point $(1, 1)$. Note also that each $f_n(x)$ goes through the point $(\frac{1}{2}, 0)$. It is worthy mentioning that often we guess the upper and lower limits and then try to prove we have the correct answer. In this example, we claim that the lower limit is the unit interval on the x-axis. More formally, the set $LL = \{(x, 0)|0 \leq x \leq 1\}$. Let $S_n = \text{Graph}(f_n)$. Fix $x_0 \in (0, 1)$ and the sequence $s_n^0 = (x_0, f_n(x_0))$. Since $x_0 \in (0, 1)$ and $| -2x_0^3 + 3x_0^2 + x_0 - 1 |< 1$ it is easy to show that s_n^0 converges to $(x_0, 0)$.

In the limit, and in a small neighborhood of $(0, 0)$ we can approximate the sequence of sets by that from example 1 so $(0, 0)$ belongs to the lower limit and by symmetry $(1, 0)$ also belongs to the lower limit. We claim that the upper limit is the union of three segments. The first segment is derived from the lower limit. The other two segments are $V1 = \{(0, y)| -1 \leq y \leq 1\}$ and $V2 = \{(1, y)| -1 \leq y \leq 1\}$ so that $UL = LL \cup V1 \cup V2$. Indeed, If we fix $\bar{y} \in (0, 1)$ then the subsequence $(f_{2n-1}^{-1}(\bar{y}), \bar{y})$ converges to $(0, \bar{y})$ (all the f_n's are one-to-one) and the subsequence $(f_{2n}^{-1}(\bar{y}), \bar{y})$ converges to $(1, \bar{y})$. On the other hand, fixing $\bar{\bar{y}} \in (-1, 0)$ we use a similar argument to get similar results and the subsequences converge to $(0, \bar{\bar{y}})$ and $(1, \bar{\bar{y}})$. Consider the graph and notice how the sequence of sets accumulates at the union of the three segments.

We end this section with a statement of a theorem due to Zarankiewicz [14].The proof follows [2].

Theorem 1. *Every sequence of subsets S_n of a separable metric space X contains a subsequence which has a (possibly empty) limit.*

Proof. Since X is a separable metric space, there exists a countable collection of open subsets U_m in X that satisfies the following:

$$\forall \quad \text{open subset} \quad U, \forall x \in U, \exists U_m \quad \text{such that} \quad x \in U_m \subset U.$$

Consider any sequence of subsets S_n. We construct a sequence of subsequences of S_n by induction and denote it by $(S_n^{(m)})_{n>0}$. For $m = 0$ we define $(S_n^{(0)})_{n>0} = S_n$ i.e. the first element of this sequence is the sequence S_n itself. The construction is achieved through the process of induction. Assume that the $m - 1$ first subsequences have been constructed: $(S_n^{(p)})_{n>0}$ for $0 \leq p \leq m - 1$. We proceed to construct the mth term of the sequence.

Consider the mth open subset U_m. Then either for every subsequence n_j of natural numbers, we have $U_m \cap \limsup_{j\to\infty} S_{n_j}^{(m-1)} \neq \emptyset$ in which case $S_n^{(m)} = S_n^{(m-1)}$

OR there exists a subsequence n_j such that $U_m \cap \limsup_{j\to\infty} S_{n_j}^{(m-1)} = \emptyset$ in which case $S_n^{(m)} = S_{n_j}^{(m-1)}$.

Having constructed the sequence of subsequences, we now extract the diagonal subsequence and define it $D_n = S_n^{(n)}$. We claim that D_n has a set limit.

We prove our claim by contradiction. Assume it does not have a set limit. This means that its upper limit and lower limit do not coincide. Therefore there exists $x_0 \in X$ such that $x_0 \in \limsup_{n\to\infty} D_n$ and $x_0 \notin \liminf_{n\to\infty} D_n$. If $x_0 \notin \liminf_{n\to\infty} D_n$, there exists a neighborhood U of x_0 and a subsequence D_{n_j} such that $U \cap D_{n_j} = \emptyset$ for any j.

Let us fix U_m such that $x_0 \in U_m \subset U$, we thus deduce $U_m \cap \limsup_{j\to\infty} D_{n_j} = \emptyset$. Since for $n_j \geq m$ we have $D_{n_j} = S_{n_j}^{(n_j)} = S_{p_j}^{(m-1)}$ for some p_j, we observe that D_{n_j} is a subsequence of $(S_{n_j}^{(m-1)})_{n>0}$ the upper limit of which is disjoint from U_m.

By the construction of the sequence of subsequences, we have $S_j^{(m)} = S_{p_j}^{(m-1)}$ and therefore $U_m \cap \limsup_{j\to\infty} S_j^{(m)} = U_m \cap \limsup_{j\to\infty} S_{p_j}^{(m-1)} = \emptyset$.

Since $D_n = S_n^{(n)} = S_{p_n}^{(m)}$ for some p_n, we deduce that $(D_n)_{n\geq m}$ is a subsequence of $(S_j^{(m)})_{j>0}$ and $x_0 \in \limsup_{n\to\infty} D_n \subset \limsup_{j\to\infty} S_j^{(m)} \subset X \backslash U_m$ which contradicts $x_0 \in U_m$ and therefore such x_0 does not exist.

$\square$

As a corollary to this theorem, every sequence of sets in $\mathbb{R}^d$ has a convergent subsequence.

2.3 Hausdorff Convergence

We start this section by defining the Hausdorff space of $\mathbb{R}^d$. We follow this definition with a metric construction. The main purpose of this section is to be able to measure how far two subsets of $\mathbb{R}^d$ are from each other. We conclude the section with an interesting theorem.

We will call the collection of all non-empty compact subsets of $\mathbb{R}^d$ the Hausdorff space of $\mathbb{R}^d$ denoted by $\mathcal{H}(\mathbb{R}^d)$. Since $\mathbb{R}^d$ is a complete space then so is $\mathcal{H}(\mathbb{R}^d)$. A proof can be found in [7]. The elements of $\mathcal{H}(\mathbb{R}^d)$ are subsets of $\mathbb{R}^d$. For example, any closed disk is an element of $\mathcal{H}(\mathbb{R}^2)$. We now construct a metric that calculates the distance between two elements of $\mathcal{H}(\mathbb{R}^d)$.

Let d denote the Euclidean metric. Consider two elements $A, B \in \mathcal{H}(\mathbb{R}^d)$. We fix $x \in A$ and recall from the previous section the distance from x to B:

$$\text{dist}(x, B) = \inf\{d(x, y)|y \in B\}$$

Next we define the Hausdorff semi-distance as:

$$D(A, B) = \sup\{\text{dist}(x, B)|x \in A\} = \sup\{\inf\{d(x, y)|y \in B\}|x \in A\}$$

The reason why D fails to be a proper metric on $\mathcal{H}(\mathbb{R}^d)$ is because it fails to satisfy the symmetric property of metrics. In particular, it is not always true that $D(A, B) = D(B, A)$. Consider for example the two sets A and B in $\mathbb{R}^2$ defined as follows:

$$A = \{(x, y)|(x + 4)^2 + y^2 \leq 1\}, \qquad B = \{(x, y)|(x - 4)^2 + y^2 \leq 4\}$$

They are both closed disks in $\mathbb{R}^2$. A simple picture and calculation concludes that $7 = D(A, B) \neq D(B, A) = 9$.

Finally, we define the Hausdorff distance between A and B as follows:

$$H(A, B) = \max\{D(A, B), D(B, A)\}$$

$(\mathcal{H}(\mathbb{R}^d), H)$ inherits the properties of $(\mathbb{R}^d, d)$; It is complete and separable. The non-negativity and symmetry conditions are easily deduced. It remains to prove the triangle inequality. We first prove the following lemma:

Lemma. Let m, n, p and q be non-negative real numbers. Then the following holds:

$$\max\{m + n, p + q\} \leq \max\{m, p\} + \max\{n, q\}$$

Proof. Since $m \leq \max\{m, p\}$ and $n \leq \max\{n, q\}$ then: $m + n \leq \max\{m, p\} + \max\{n, q\}$. Similarly we can say that $p + q \leq \max\{m, p\} + \max\{n, q\}$. The two inequalities give the desired result. $\square$

Proposition. $D(A, B) \leq D(A, C) + D(C, B)$.

Proof. Let $a \in A$ be arbitrary. There exists $c \in C$ such that $d(a,c) \leq D(A,C)$. For this c, we can find $b \in B$ such that $d(c,b) \leq D(C,B)$. Therefore $d(a,b) \leq d(a,c) + d(c,b) \leq D(A,C) + D(C,B)$. But $\text{dist}(a,B) \leq d(a,b)$ so $\text{dist}(a,B) \leq D(A,C) + D(C,B)$.

Since the choice of $a \in A$ was arbitrary, we can take the supremum over all $a \in A$ and conclude $D(A,B) \leq D(A,C) + D(C,B)$. □

We use a similar argument to conclude that $D(B,A) \leq D(B,C) + D(C,A)$.

Finally, using the lemma above, we get the following result:

$H(A,B) = \max\{D(A,B), D(B,A)\} \leq \max\{D(A,C)+D(C,B), D(B,C)+D(C,A)\} \leq \max\{D(A,C), D(C,A)\}+ \max\{D(C,B), D(B,C)\} = H(A,C) + H(C,B)$.

Definition. A sequence of sets $A_n \in \mathcal{H}(\mathbb{R}^d)$ converges to a set $A \in \mathcal{H}(\mathbb{R}^d)$ if and only if:

$$\lim_{n\to\infty} H(A_n, A) = 0$$

We say that A_n converges to A in the Hausdorff metric. This is precisely Hausdorff convergence.

We conclude this section with an interesting result (without a proof) that marks the connection between the Kuratowski convergence and Hausdorff convergence [7]:

Theorem 2. *Let X be a metric space. Let $A_n \in \mathcal{H}(X)$ be a sequence of sets. Assume that there exists $K \subset X$ such that K is compact and $A_n \subset K$, $\forall n \in \mathbb{N}$. Then the Hausdorff convergence of A_n is equivalent to the Kuratowski convergence of A_n.*

2.4 Calculus of Limits

In the last section of this chapter, we state some properties of sequences of sets in the form of a lemma.

Lemma: Assume $S_n^{(1)}$ and $S_n^{(2)}$ are two sequences of subsets of $\mathbb{R}^d$. Then the following holds:

$$\limsup_{n\to\infty}(S_n^{(1)} \cap S_n^{(2)}) \subset \limsup_{n\to\infty} S_n^{(1)} \cap \limsup_{n\to\infty} S_n^{(2)}$$

$$\liminf_{n\to\infty}(S_n^{(1)} \cap S_n^{(2)}) \subset \liminf_{n\to\infty} S_n^{(1)} \cap \liminf_{n\to\infty} S_n^{(2)}$$

$$\limsup_{n\to\infty}(S_n^{(1)} \cup S_n^{(2)}) = \limsup_{n\to\infty} S_n^{(1)} \cup \limsup_{n\to\infty} S_n^{(2)}$$

$$\liminf_{n\to\infty}(S_n^{(1)} \cup S_n^{(2)}) \subset \liminf_{n\to\infty} S_n^{(1)} \cup \liminf_{n\to\infty} S_n^{(2)}$$

Proof.

- Suppose $x \in \limsup_{n\to\infty}(S_n^{(1)} \cap S_n^{(2)})$ then there exists a sequence $x_n \in (S_n^{(1)} \cap S_n^{(2)})$ with at least one subsequence $x_{n_k} \in (S_{n_k}^{(1)} \cap S_{n_k}^{(2)})$ such that $x_{n_k} \to x$. This means that $x_{n_k} \in S_{n_k}^{(1)}$ and $x \in \limsup_{n\to\infty} S_n^{(1)}$. Similarly, $x_{n_k} \in S_{n_k}^{(2)}$ and $x \in \limsup_{n\to\infty} S_n^{(2)}$. We conclude that $x \in \limsup_{n\to\infty} S_n^{(1)} \cap \limsup_{n\to\infty} S_n^{(2)}$.

- Suppose $x \in \liminf_{n\to\infty}(S_n^{(1)} \cap S_n^{(2)})$ then there exists a sequence $x_n \in (S_n^{(1)} \cap S_n^{(2)})$ such that $x_n \to x$. But this means that $x_n \in S_n^{(1)}$ and $x_n \in S_n^{(2)}$ and therefore $x \in \liminf_{n\to\infty} S_n^{(1)}$ as well as $x \in \liminf_{n\to\infty} S_n^{(2)}$. We conclude that $x \in \liminf_{n\to\infty} S_n^{(1)} \cap \liminf_{n\to\infty} S_n^{(2)}$.

- Suppose $x \in \limsup_{n\to\infty} S_n^{(1)} \cup \limsup_{n\to\infty} S_n^{(2)}$. Then x is either in $\limsup_{n\to\infty} S_n^{(1)}$ or in $\limsup_{n\to\infty} S_n^{(2)}$. Without loss of generality, consider $x \in \limsup_{n\to\infty} S_n^{(1)}$. x is therefore an accumulation point of a sequence $x_n \in S_n^{(1)}$. So x is a cluster point of $x_n \in S_n^{(1)} \cup S_n^{(2)}$ and finally $x \in \limsup_{n\to\infty}(S_n^{(1)} \cup S_n^{(2)})$. On the other hand, suppose that that $x \in \limsup_{n\to\infty}(S_n^{(1)} \cup S_n^{(2)})$. Then, x is a cluster point of some sequence $x_n \in S_n^{(1)} \cup S_n^{(2)}$. There exists a subsequence x_{n_k} converging to x that belongs to the union of the two sequences of sets. But x_{n_k} contains infinitely many elements that belong to $S_n^{(1)}$ (wlog). We index those elements and create a further subsequence $x_{n_{k_j}}$ which converges to x since x_{n_k} converges to x and therefore x is a cluster point of some sequence in $S_n^{(1)}$ and so $x \in \limsup_{n\to\infty} S_n^{(1)}$. Finally $x \in \limsup_{n\to\infty} S_n^{(1)} \cup \limsup_{n\to\infty} S_n^{(2)}$.

- Suppose $x \in \liminf_{n\to\infty}(S_n^{(1)} \cup S_n^{(2)})$. Then x is a limit point of some sequence $x_n \in S_n^{(1)} \cup S_n^{(2)}$. But x_n contains infinitely many elements in $S_n^{(1)}$ (wlog). We index those elements by k. We create a subsequence $x_{n_k} \in S_n^{(1)}$ that converges to x since x_n converges to x. Therefore $x \in \liminf_{n\to\infty} S_n^{(1)}$. Finally, we conclude that $x \in \liminf_{n\to\infty} S_n^{(1)} \cup \liminf_{n\to\infty} S_n^{(2)}$.

□

Consider now a single-valued continuous function $f(x)$ defined on $\mathbb{R}^d$. Recall, if $x_n \in \mathbb{R}^d$ such that x_n converges to $\bar{x}$ then by the continuity of f we have:

$$f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

If also f is one-to-one, then if $y_n = f(x_n)$ is given such that y_n converges to $\bar{y}$ then the following holds:

$$f^{-1}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} f^{-1}(y_n)$$

We can generalize the above to sequences of sets. Under the same assumptions on f (continuity), the following holds:

$$f(\limsup_{n\to\infty} S_n) \subset \limsup_{n\to\infty} f(S_n)$$

$$f(\liminf_{n\to\infty} S_n) \subset \liminf_{n\to\infty} f(S_n)$$

$$f^{-1}(\limsup_{n\to\infty} S_n) \supset \limsup_{n\to\infty} f^{-1}(S_n)$$

$$f^{-1}(\liminf_{n\to\infty} S_n) \supset \liminf_{n\to\infty} f^{-1}(S_n)$$

Proof.

- Let $y \in f(\limsup_{n\to\infty} S_n)$ then there exists $x \in \limsup_{n\to\infty} S_n$ such that $f(x) = y$. Since $x \in \limsup_{n\to\infty} S_n$ then x is the accumulation point of some sequence $s_n \in S_n$ and therefore there exists a subsequence s_{n_k} that converges to x. By the continuity of f, we deduce that $f(s_{n_k})$ converges to $f(x) = y$ and therefore $y \in \limsup_{n\to\infty} f(S_n)$.

- Let $y \in f(\liminf_{n\to\infty} S_n)$. There exists $x \in \liminf_{n\to\infty} S_n$ such that $f(x) = y$. There exists a sequence $x_n \in S_n$ such that $x_n \to x$. By the continuity of f, we observe that $f(S_n) \ni f(x_n) \to f(x) = y$. Finally, we conclude that $y \in \liminf_{n\to\infty} f(S_n)$.

- Let $x \in \limsup_{n\to\infty} f^{-1}(S_n)$. There exists $x_n \in f^{-1}(S_n)$ such that x is the cluster point of this sequence. But $x_n = f^{-1}(y_n)$ and so $f(x_n) = y_n \in S_n$. By the continuity of f, $f(x)$ is a cluster point of $f(x_n) = y_n \in S_n$. Hence, $f(x) \in \limsup_{n\to\infty} S_n$ and finally, $x \in f^{-1}(\liminf_{n\to\infty} S_n)$.

- Let $x \in \liminf_{n\to\infty} f^{-1}(S_n)$. There exists $x \in f^{-1}(S_n)$ such that $x_n \to x$. There exists $y_n \in S_n$ such that $x_n = f^{-1}(y_n)$. By the continuity of f, we have $f(x_n) = y_n \to f(x)$. Therefore, $f(x) \in \liminf_{n\to\infty} S_n$ and finally $x \in f^{-1}(\liminf_{n\to\infty} S_n)$.

$\square$

Chapter 3

Set-Valued Maps And Continuity

3.1 Set-valued Maps

We have previously seen that sequences of sets map a natural number to an element of $\mathcal{P}(\mathbb{R}^d)$. We can view them as discrete functions defined on the discrete metric space of natural numbers $\mathbb{N}$. From there we can extend this concept to the continuous case and define a set-valued map F as a function that maps a metric space ($\mathbb{R}^d$ in our discussion) to an element of $\mathcal{P}(\mathbb{R}^d)$:

$$F : \mathbb{R}^d \longrightarrow \mathcal{P}(\mathbb{R}^d)$$

We can regard a set-valued map as a correspondence or a multifunction that maps a point in $\mathbb{R}^d$ to a subset of $\mathbb{R}^d$. It is represented as:

$$F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$$

Set-valued maps are a generalization of single-valued maps. Their relevance was neglected until the late parts of the 20th century. Many applied mathematicians realized that they could be used to answer questions in different sciences such as economics, biology, physics, etc... In the last chapter of this thesis, we will see how set-valued maps are used for modelling in game theory. Mathematicians have tried to extend the properties of single-valued maps (functions) to the set-valued case. New concepts had to be formulated. Among those who contributed to this theory is the French mathematician Jean-Pierre Aubin in his book on set-valued analysis published in 1990.

In this chapter, we will present some of the significant properties starting with the basic concept of domain and ending with a detailed study of continuity which is crucial to further analysis.

We start by formally defining the domain of a set-valued map F:

$$\mathrm{Dom}(F) = \{x \in \mathbb{R}^d | F(x) \neq \emptyset\}$$

The domain of a set-valued map is any point in the d-dimensional Euclidean space, for which F admits a non-empty image. In order not to reduce it to the single valued case, we require that at least one $x \in \mathbb{R}^d$ admits an image with a cardinality strictly greater than 1. The image of F is defined as follows:

$$\mathrm{Im}(F) = \bigcup_{x \in \mathrm{Dom}(F)} F(x)$$

Suppose we want to follow an air molecule's path in space. It is natural to model the dynamics as a set-valued map since at each instant, the molecule occupies a particular volume in space (a subset of $\mathbb{R}^3$). It would be plausible to express the dynamics using the following motion map:

$$M : \mathbb{T} \rightsquigarrow \mathbb{R}^3,$$

where $\mathbb{T} = [0, \infty)$ represents time. Questions like continuity of motion may arise.
Let us consider a few abstract examples with their graphs.
Consider the set-valued map $F : [0, 1] \rightsquigarrow [0, 1]$ defined as follows:

$$F(x) = \{y \in [0, 1] | x^2 \leq y \leq \sqrt{x}\}$$

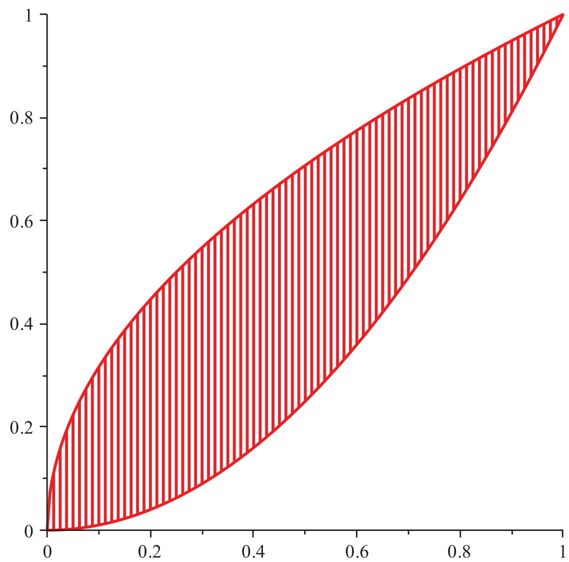

Figure 3.1: Set-Valued Map example 1

In this example, $F(\frac{1}{2}) = [\frac{1}{4}, \sqrt{\frac{1}{2}}]$.

Another example is $G : [0,1] \rightsquigarrow [0,1]$ defined as follows:

$$G(x) = \begin{cases} [\frac{1}{4}, \frac{3}{4}] & \text{if } x \in [0, \frac{1}{4}) \\ [0,1] & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ [\frac{1}{4}, \frac{3}{4}] & \text{if } x \in (\frac{3}{4}, 1] \end{cases}$$

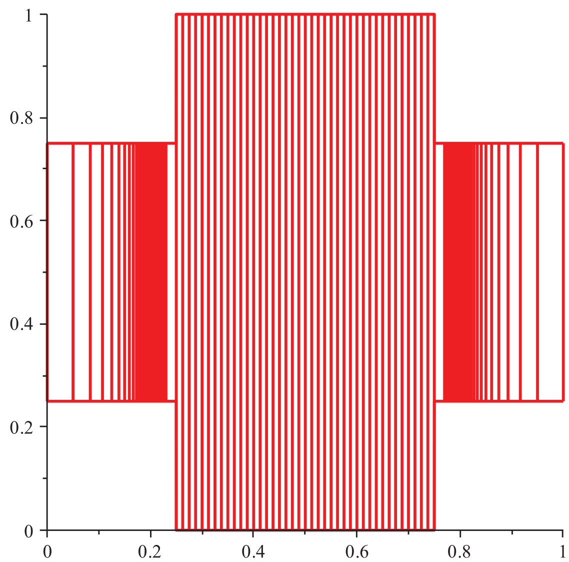

Figure 3.2: Set-Valued Map example 2

In this last example, $G(\frac{1}{8}) = [\frac{1}{4}, \frac{3}{4}]$ and $G(\frac{1}{2}) = [0, 1]$.

Finally note that in the second example there is a "jump" at $x = \frac{1}{4}$. Later in this chapter, we learn that $G(x)$ graphed above is not a continuous set-valued map.

3.2 Properties of Set-Valued Maps

In this section, we consider some properties of set-valued maps. Consider $S, S_1, S_2 \subset \mathbb{R}^d$ and a correspondence F between Euclidean spaces then the following holds:

$$F(S_1 \cup S_2) = F(S_1) \cup F(S_2)$$

$$F(S_1 \cap S_2) \subset F(S_1) \cap F(S_2)$$

$$\mathrm{Im}(F) \backslash F(S) \subset F(\mathbb{R}^d \backslash S)$$

$$S_1 \subset S_2 \Longrightarrow F(S_1) \subset F(S_2)$$

We prove the first statement in the above list:

Let $y \in F(S_1 \cup S_2)$. Then there exists $x \in S_1 \cup S_2$ such that $y \in F(x)$. Therefore, $x \in S_1$ or $x \in S_2$. Without loss of generality, assume $x \in S_1$. Therefore $y \in F(x) \subset F(S_1)$ and we have: $y \in F(S_1) \cup F(S_2)$. Note that the proof works in both directions and therefore both inclusions $F(S_1 \cup S_2) \subset F(S_1) \cup F(S_2)$ and $F(S_1 \cup S_2) \supset F(S_1) \cup F(S_2)$ are true. Hence $F(S_1 \cup S_2) = F(S_1) \cup F(S_2)$.

We say a set-valued map is described by its graph. First recall that the graph of a set-valued map F is defined as follows:

$$\mathrm{Graph}(F) = \{(x, y) | y \in F(x)\} \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$$

The graph of a set-valued map F is a subset of the product space and whatever (topological) property the graph displays, we name F after this property. For example, if the graph of F is closed then we say that F is a closed set-valued map. If the graph of F is compact we say that F is a compact set-valued map.

A very important and interesting property is convexity. A set-valued map F is convex if the following holds:

$$\begin{cases} \forall x_1, x_2 \in \mathrm{Dom}(F), \quad \forall \lambda \in [0,1] \\ \lambda F(x_1) + (1-\lambda)F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) \end{cases}$$

Referring to the two graphs above, the first shows a convex set-valued map but the second one is not.

We end this section with the definition of the inverse of a set-valued map. Unlike single-valued maps, set-valued maps admit two distinct types of inverses. The first one is the "strong inverse" and the second is the "weak inverse". We will follow the terminology used in Jean-Pierre Aubin's book (Inverse and Core).

Let $S \subset \mathbb{R}^d$ and let $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ be a set-valued map. Then the inverse of S by F is defined as follows:

$$F^{-}(S) = \{x \in \mathbb{R}^d | F(x) \cap S \neq \emptyset\}$$

.

The core of S by F is defined as follows:

$$F^{+}(S) = \{x \in \mathbb{R}^d | F(x) \subset S\}$$

Notice that the core is a subset of the inverse because whenever $F(x) \subset S$ then $F(x)$ intersects S. Consider the first example above:

$$F(x) = \{y \in [0,1] | x^2 \leq y \leq \sqrt{x}\}$$

Let $S = [0, \frac{1}{2}]$. Then the inverse of S by F is:

$$F^{-}(S) = [0, \frac{1}{\sqrt{2}}]$$

The core of S by F is:

$$F^{+}(S) = [0, \frac{1}{4}]$$

3.3 Continuity of Set-valued Maps

We start by recalling the concept of continuity for a single-valued map. We call f continuous at x if the following holds:

$$\forall \epsilon > 0, \exists \delta > 0 \quad \text{such that} \quad \forall y \quad \text{satisfying} \quad \|x - y\| < \delta, \text{we have} \quad \|f(x) - f(y)\| < \epsilon$$

In simple terms, a single-valued map f is continuous at x if a small deviation from x does not induce a large deviation from $f(x)$. There is another equivalent way of defining continuity of single-valued maps. f is continuous at x if for any sequence x_n converging to x, we have $f(x_n)$ converging to $f(x)$.

Problems arise when we want to extend the definition of continuity to set-valued maps. The sequential definition no longer holds true. Therefore in 1932, the concept of semicontinuous maps was introduced by two famous mathematicians G. Bouligand and K. Kuratowski.

Definition. A set-valued map F is called upper semicontinuous at $x \in \mathrm{Dom}(F)$ if and only if for any open set $V \supset F(x)$, there exists a neighbourhood $U \ni x$ such that $U \subset F^{+}(V)$. F is upper semicontinuous if it is upper semicontinuous on its domain.

Definition. A set-valued map F is called lower semicontinuous at $x \in \mathrm{Dom}(F)$ if and only if for any open set V such that $V \cap F(x) \neq \emptyset$, there exists a neighbourhood $U \ni x$ such that $U \subset F^{-}(V)$. F is lower semicontinuous if it is lower semicontinuous on its domain.

There are other equivalent definitions that can be found in [2].
A set-valued map is said to be continuous if it is both lower and upper semicontinuous.

Maps that are either lower or upper semicontinuous exist and we present an example of each case.

Consider the set-valued map from the example above $G : [0,1] \rightsquigarrow [0,1]$ defined as follows:

$$G(x) = \begin{cases} [\frac{1}{4}, \frac{3}{4}] & \text{if } x \in [0, \frac{1}{4}) \\ [0,1] & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ [\frac{1}{4}, \frac{3}{4}] & \text{if } x \in (\frac{3}{4}, 1] \end{cases}$$

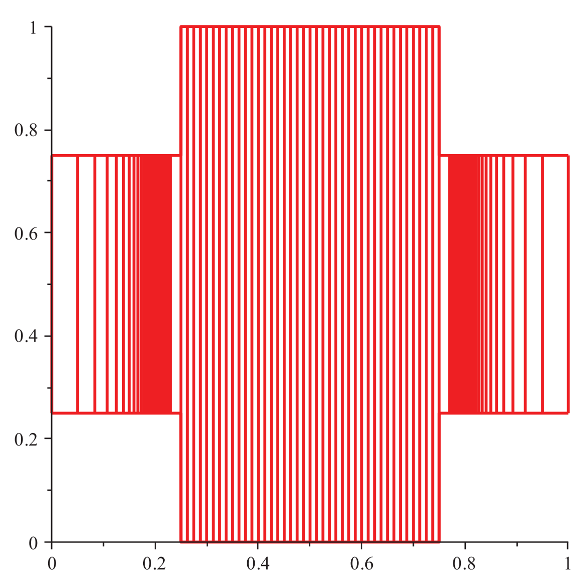

Figure 3.3: Upper but not Lower semicontinuous

G is an upper semicontinuous map but not lower semicontinuous. Indeed G is upper semicontinuous at $x = \frac{1}{4}$. This is true because $F(\frac{1}{4}) = [0, 1]$. Let $V = (-\epsilon, 1+\epsilon) \supset F(\frac{1}{4})$ be an open set. Now $F^+(V) = [0, 1]$ and from here we see that any neighbourhood U of $x = \frac{1}{4}$ satisfies $U \subset F^+(V)$. Also G is not lower semicontinuous at $x = \frac{1}{4}$. Consider $V = (1-\epsilon, 1+\epsilon)$ and note that $F(\frac{1}{4}) \cap V \neq \emptyset$. Now $F^-(V) = [\frac{1}{4}, \frac{3}{4}]$. There does not exist a neighbourhood $U \ni x$ such that $U \subset F^-(V)$.

Consider on the other hand the set-valued map $\overline{G} : [0, 1] \rightsquigarrow [0, 1]$ defined as follows:

$$\overline{G}(x) = \begin{cases} [0,1] & , x \in [0, \frac{1}{4}) \\ [\frac{1}{4}, \frac{3}{4}] & , x \in [\frac{1}{4}, \frac{3}{4}] \\ [0,1] & , x \in (\frac{3}{4}, 1] \end{cases}$$

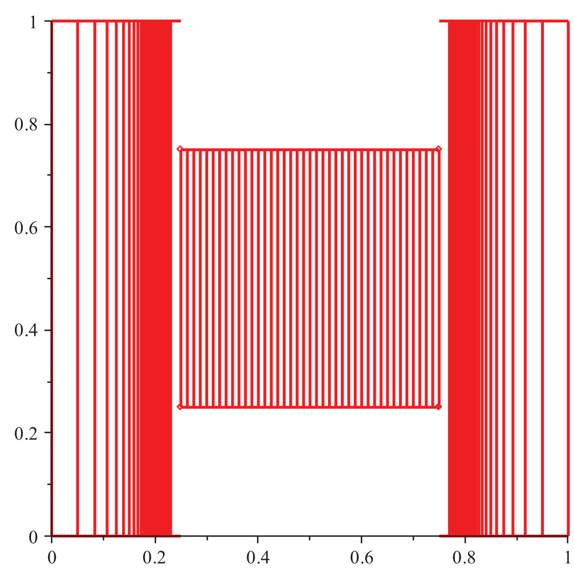

Figure 3.4: Lower but not Upper semicontinuous

$\overline{G}$ is a lower semicontinuous map but not upper semicontinuous. Indeed, $\overline{G}$ is lower semicontinuous at $x = \frac{1}{4}$. This is true because $F(\frac{1}{4}) = [\frac{1}{4}, \frac{3}{4}]$. Choosing any open set V that intersects $F(\frac{1}{4})$, we will always get $F^-(V) = [0, 1]$ which definitely intersects any image of $\bar{x} \in U$ where U is a neighbourhood of $x = \frac{1}{4}$. For any $x \in [0, 1]$, we have $F(x) \cap V \neq \emptyset$. Also $\overline{G}$ is not upper semicontinuous at $x = \frac{1}{4}$. Letting $V = (\frac{1}{4} - \epsilon, \frac{3}{4} + \epsilon)$, we notice that $F(\frac{1}{4}) \subset V$ and that $F^+(V) = [\frac{1}{4}, \frac{3}{4}]$. There exists no neighbourhood of $U \ni x = \frac{1}{4}$ such that

$U \subset F^+(V)$.

We can similarly create set-valued maps that are neither upper nor lower semicontinuous but they are not of analytical interest. When working in Euclidean spaces, upper semicontinuous maps form a very important class of maps that is helpful in fixed point and equilibrium theory. We shall discuss fixed points and equilibrium points in the next chapter. In the last section of this chapter, we will cover topics that will help us in our study of the following chapter.

3.4 Cones and Upper Hemicontinuity

In this section we define the tangent cone of a set at a point and the polar cone of a set. These two sets are-essential to understand the next chapter. Before we do so, we have to introduce the concept of a "support function". It is much easier to visualize a "support function" in the Euclidean space $\mathbb{R}^d$. Recall if we regard the space $\mathbb{R}^d$ as the set of all column vectors of d-components, then the dual of $\mathbb{R}^d$ is the set of all row vectors of d-components.

Definition. Let S be a non-empty subset of $\mathbb{R}^d$. We associate with every row vector v in the dual space of $\mathbb{R}^d$ the following:

$$\sigma_S(v) = \sup_{x \in S} < v, x > \in \mathbb{R} \cup \{+\infty\}$$

The function $\sigma_S : (\mathbb{R}^d)^* \to \mathbb{R} \cup \{+\infty\}$ is called the support function of S.

Here $< v, x >$ denotes the regular dot product of two vectors. If we fix v in the dual space of $\mathbb{R}^d$, then the set S is contained in the closed half space $\{y \in \mathbb{R}^d | < y, v > \leq \sigma_S(v)\}$ and there is at least one point of S on the boundary of the closed half space.

We illustrate with an example. Consider the closed disk S centred at $(3, 0)$ and with radius 1 in $\mathbb{R}^2$. Let $v = [1, 1]$ then:

$$\sigma_S([1, 1]) = \sup_{x \in S} \{x_1 + x_2\}$$

Here x_1 and x_2 are the components of the vectors in the closed disk. Since S is compact we can replace supremum with maximum and we proceed to solve the following problem:

$$\begin{cases} \text{maximize:} \quad x_1 + x_2 \\ \text{subject to:} \quad (x_1-3)^2 + x_2^2 \leq 1 \end{cases}$$

Solving for x_2 in the inequality and choosing positive values of x_2 we get: $x_2 \leq \sqrt{1-(x_1-3)^2}$ which implies that $x_1 + x_2 \leq x_1 + \sqrt{1-(x_1-3)^2}$. We define $m(x_1) = x_1 + \sqrt{1-(x_1-3)^2}$. This function admits a maximum for $x_1 = 3 + \frac{1}{\sqrt{2}}$. Therefore, $m(3+\frac{1}{\sqrt{2}}) = 3 + \sqrt{2}$, and finally $\sigma_S([1,1]) = \max_{x \in S}\{x_1 + x_2\} \leq \max\{m(x_1)\} = 3 + \sqrt{2}$.

This means that we can contain S in the closed half-plane determined by $\{y \in \mathbb{R}^2 | y_1 + y_2 \leq 3 + \sqrt{2}\}$ and S touches the boundary of the closed half plane at the point $(3 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, as shown in Figure 5:

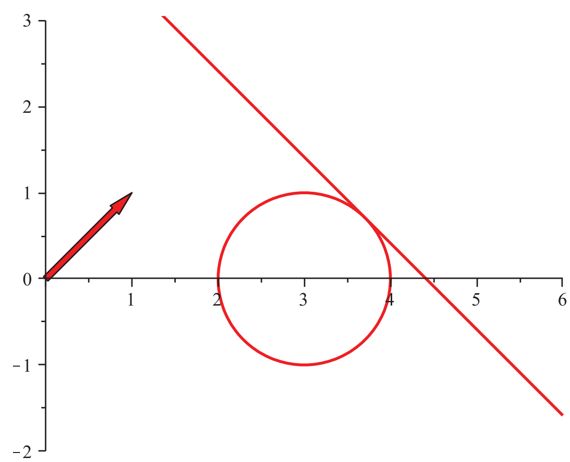

Figure 3.5: Support function example

We use support functions in $\mathbb{R}^d$ to define the polar cone of a set. Let S be a subset of $\mathbb{R}^d$ (later in our discussion, we will require S to be convex and compact). The polar cone of S is defined as:

$$S^- := \{y \in \mathbb{R}^d | \sigma_S(y) \leq 0\}$$

We regard elements of the set S as vectors. Then the polar cone of S is the set of vectors whose dot product with any vector from S is non-positive. Specifically, in $\mathbb{R}^2$, S^- contains all the vectors that form an angle greater than 90° with any vector in S. We present an example.

Consider the closed disk $S = \{(x_1, x_2)|(x_1 - 3)^2 + (x_2 - 3)^2 \leq 4\}$. The polar cone of S is then given by: $S^- = \{(y_1, y_2)|x_1y_1 + x_2y_2 \leq 0\}$.

After simple algebra, we can deduce that: $S^- = \{(y_1, y_2)|y_2 \leq \frac{\sqrt{14}-7}{\sqrt{14}+7}y_1 \quad \text{and} \quad y_2 \leq \frac{\sqrt{14}+7}{\sqrt{14}-7}y_1\}$. See Figure 6, where the polar cone of S is the shaded region. The lines that determine the boundaries of the polar cone are described in S^- above.

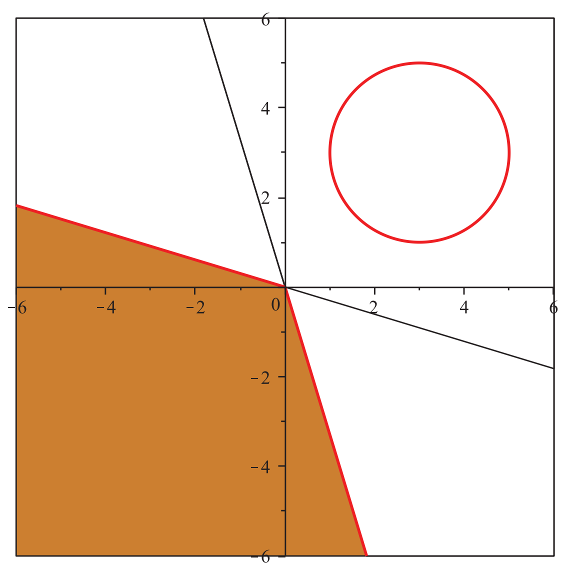

Figure 3.6: Polar cone of the disk

We now introduce the tangent cone to a set at a point. Notice that it is important to specify at what point we are forming our tangent cone. It is similar to the idea of drawing a tangent line to a function at a particular point (tangent lines will look differently at different points of the function). For that, we consider a set $S \subset \mathbb{R}^d$ that we require to be compact. We define the tangent cone $T_S(x)$ of S at x as follows:

$$T_S(x) = \overline{\bigcup_{h>0} \frac{S - x}{h}}$$

Consider a compact set $S \subset \mathbb{R}^2$. Fix a point x on the boundary of S so that $x \in \partial S$. In order to draw the tangent cone of S at x, draw all the vectors that start at point x and whose terminal points are all other points of S and then take all their positive scalar multiples. If we choose x from the interior of S so that $x \in Sn\partial S$ then the tangent cone of S at x is generally the whole space. In fact this is true whenever we can draw a ball around x with a radius $\epsilon > 0$ such that the ball is entirely contained in S.

We finish our discussion of tangent cones with an illustrative example. Consider the boundary and the interior of the lemniscate S ("sleeping eight" figure) defined by: $(x^2 + y^2)^2 \leq 8(x^2 - y^2)$. We are interested in drawing the tangent cone of S at the origin. In Figure 7, the union of the lines passing through the origin and whose slope is bounded by real numbers whose absolute values are less than or equal to 1, forms the tangent cone to S at the origin .

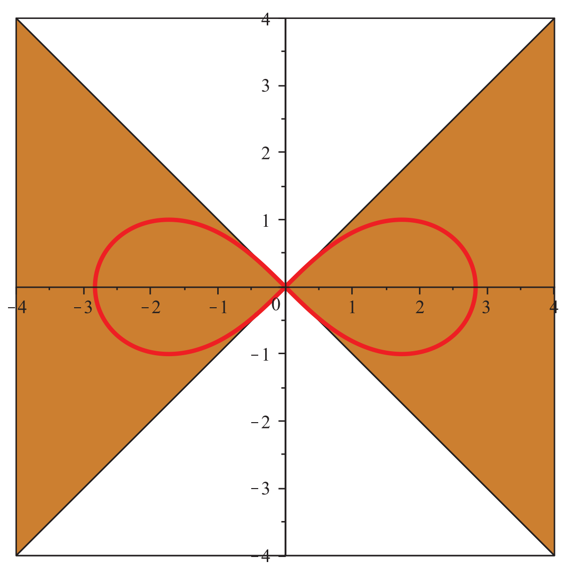

Figure 3.7: Tangent cone to the lemniscate

We end this chapter with an introduction to upper hemicontinuous maps. We will define this class of maps and state a theorem that is relevant to our study.

Definition. A set-valued map $F : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is upper hemicontinuous at $x \in \text{Dom}(F)$ if and only if for any $y \in \mathbb{R}^d$ the function $x \to \sigma_{F(x)}(y)$ is upper semicontinuous at x. F is upper hemicontinuous if and only if it is upper hemicontinuous on its domain.

Theorem 3. *Let X be a topological space and Y a metric space and $F : X \rightsquigarrow Y$.*

If F is upper semicontinuous then F is upper hemicontinuous.

Theorem 4. *Let X be a topological space and Y be a metric space and $F : X \rightsquigarrow Y$.*

If F is upper hemicontinuous and compact-valued then F is upper semicontinuous.

The proofs of the these theorems can be found in [12].

Corollary. Let F be a compact set-valued map between Euclidean spaces. Upper semicontinuity is equivalent to upper hemicontinuity.

Chapter 4

Equilibria And Fixed Points

4.1 Definitions

In this chapter, we present the most important part of this thesis. We state and prove 3 theorems that will guarantee the existence of an equilibrium and/or a fixed point. These points play a major role in game theory as we will see in Chapter 5. Before we state and prove them, we start with some definitions. Throughout this chapter, $X = \mathbb{R}^d$ unless otherwise stated.

Definition. Let $F : X \rightsquigarrow X$ be a set-valued map. We call $x \in \mathrm{Dom}(F)$ an equilibrium point of F if and only if $0 \in F(x)$.

Example 1: Let $F(x) = \{\sqrt{x}, -\sqrt{x}\}$ for any non-negative real number x. It is clear from the definition that $x = 0$ is the unique equilibrium point.

Example 2: Consider the set-valued map F defined by the area bounded between these two single-valued maps:

$$T_1(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}] \\ 2 - 2x & \text{if } x \in (\frac{1}{2}, 1] \end{cases} \qquad T_2(x) = \begin{cases} 2x - \frac{1}{4} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{7}{4} - 2x & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

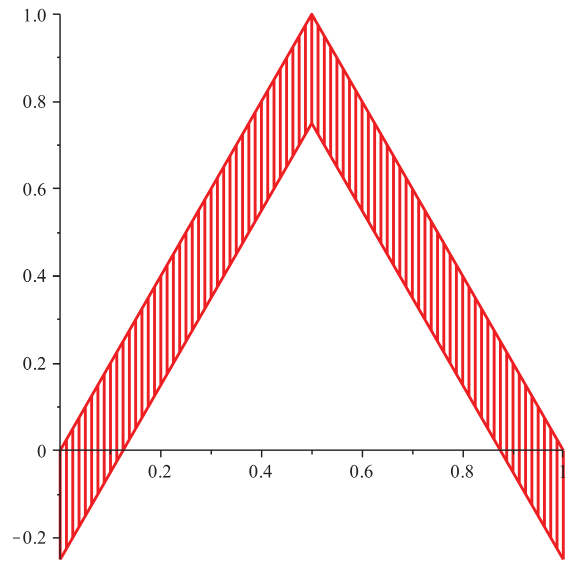

Figure 4.1: Equilibrium points for example 2

A careful study shows that x is an equilibrium point if and only if $x \in [0, \frac{1}{8}] \cup [\frac{7}{8}, 1]$. In fact, the set-valued map intersects the x-axis at $x = 0$ and $x = 1$ but also where $T_2(x)$ touches the x-axis. This happens at $x = \frac{1}{8}$ and $x = \frac{7}{8}$.

We now proceed to define a fixed point for a set-valued map. It is a natural extension of the single-valued case. It is also important to understand the difference between an equilibrium point and a fixed point.

Definition. Let $F : X \rightsquigarrow X$ be a set-valued map. We call $x \in \mathrm{Dom}(F)$ a fixed point of F if and only if $x \in F(x)$.

Example 1: Let $F(x) = \{\sqrt{x}, -\sqrt{x}\}$ for any non-negative real number x. It is clear from the definition that $x = \{0, 1\}$ are the only 2 fixed points.

Example 2: Consider the set-valued map F defined by the area bounded between $T_1(x)$ and $T_2(x)$ from above. In order to find the fixed points of F, it suffices to find the intersection of the line $y = x$ with the lines $\{y = 2x - \frac{1}{4}, y = \frac{7}{4} - 2x, y = 2 - 2x\}$ which gives the x-values $\{\frac{1}{4}, \frac{7}{12}, \frac{2}{3}\}$ respectively. Therefore, x is a fixed point of F if and only if $x \in [0, \frac{1}{4}] \cup [\frac{7}{12}, \frac{2}{3}]$.

In [1], a viability domain is defined for a set-valued map F. The viability domain of F, call it V, is a subset of the domain and satisfies the following condition:

$$\forall x \in V, \quad F(x) \cap T_V(x) \neq \emptyset,$$

where $T_V(x)$ is the tangent cone to V at x. This means that for any $x \in V$, there exists $y \in F(x)$ such that y is tangent to V at x. In [2], viability domains are used to show that for any initial state $x_0 \in V$, there exists a solution $x(t)$ to the differential inclusion $x' \in F(x)$ which is *viable* in V in the sense that $x(t) \in V$ for any $t \geq 0$.

We end this section with Schauder's fixed point theorem that will be essential in proving the theorems of this chapter.

Theorem 5. *A continuous function f on a compact convex set K admits a fixed point, i.e. there exists $x \in K$ such that $x = f(x)$.*

4.2 Ky-Fan's Inequality

The Ky-Fan Inequality is a fundamental inequality in fixed point theory. Discovered in 1972, it was used in the proof of many theorems, in particular the Constrained Equilibrium Theorem. The statement of the theorem and the proof follows [2].

Theorem 6. *Let K be a compact convex subset of a Banach space X and $\phi : X \times X \to \mathbb{R}$ be a function satisfying*

$$\begin{cases} i)\forall y \in K, & x \to \phi(x,y) \textit{is lower semicontinuous}; \\ ii)\forall x \in K, & y \to \phi(x,y) \textit{is concave}; \\ iii)\forall y \in K, & \phi(y,y) \leq 0. \end{cases}$$

Then there exists $\overline{x} \in K$*, a solution to*

$$\forall y \in K, \quad \phi(\overline{x}, y) \leq 0$$

Proof. Our proof is by contradiction. Assume that the conclusion does not hold. Then, for any $x \in K$, there exists $y \in K$ such that $\phi(x, y) > 0$.

We define new subsets of K in the following way:

$$\nu_y = \{x \in K | \phi(x, y) > 0\}$$

These subsets cover K and they are open sets since ϕ is lower semicontinuous in x. So, $\{\nu_y\}_y$ is an open covering of K. But K is compact so it has a finite subcover $\{\nu_{y_i}\}$ where $i = 1, .., n$.

We consider a continuous partition of unity $\{\alpha_i\}$ associated with the open covering $\{\nu_{y_i}\}$. This means that for each $x \in K$, $\sum_{i=1}^{n} \alpha_i(x) = 1$ and for any i, $\alpha_i(x) \geq 0$ and support$(\alpha_i) \subset \nu_{y_i}$.
We define a new function $f : K \to X$ by:

$$\forall x \in K, \quad f(x) = \sum_{i=1}^{n} \alpha_i(x) y_i,$$

f maps K to itself since K is convex and $y_i \in K$. f is also continuous. So we can apply the *Schauder's fixed point theorem* to conclude the existence of a fixed point $\overline{y} = f(\overline{y}) \in K$ of f. Therefore, the second assumption of the theorem (concavity in the second argument) implies:

$$\phi(\overline{y}, \overline{y}) = \phi(\overline{y}, \sum_{i=1}^{n} \alpha_i(\overline{y}) y_i) \geq \sum_{i=1}^{n} \alpha_i(\overline{y}) \phi(\overline{y}, y_i)$$

We introduce the following set:

$$I(\overline{y}) = \{i = 1, ..., n | \alpha_i(\overline{y}) > 0\}$$

It is not empty because $\sum_{i=1}^{n} \alpha_i(\overline{y}) = 1$ so at least one of the values is strictly positive. Furthermore

$$\sum_{i=1}^{n} \alpha_i(\overline{y}) \phi(\overline{y}, y_i) = \sum_{i \in I(\overline{y})} \alpha_i(\overline{y}) \phi(\overline{y}, y_i) > 0$$

The last inequality holds because whenever $i \in I(\overline{y})$ then $\alpha_i(\overline{y}) > 0$ which means that $\overline{y} \in \nu_{y_i}$ and by the very definition of these sets $\phi(\overline{y}, y_i) > 0$. We conclude that:

$$\phi(\overline{y}, \overline{y}) > 0$$

This contradicts the third assumption of the theorem. □

4.3 Constrained Equilibrium Theorem

We are now ready to state and prove the Constrained Equilibrium Theorem [2]. We show that for an upper semicontinuous map F that is closed and convex defined on a Banach space X, with constraints belonging to a closed subset V, there exists $x \in V$ such that $0 \in F(x)$ or in other words x is an equilibrium point.

Theorem 7. *Let X be a Banach space and $F : X \rightsquigarrow X$ is an upper hemicontinuous set-valued map with closed and convex images. If $K \subset X$ is a convex compact viability domain of F, then it contains an equilibrium of F.*

Proof. We prove by contradiction. We assume that there exists no equilibrium point of F. Hence, for any

$x \in K$, $0 \notin F(x)$. Since the images of F are closed and convex, the geometric Hahn-Banach Separation Theorem [9] implies the existence of $p_x \in X^*$ such that $\sigma(F(x), p_x) < 0$.

We define the following subsets:

$$\nu_p = \{x \in K | \sigma(F(x), p) < 0\}$$

The negation of the existence of an equilibrium implies that K can be covered by the subsets ν_p. These subsets are open by the very definition of upper hemicontinuity of F. Since K is compact, there exists a finite subcover $\{\nu_{p_i}\}$ where $i = 1, ...n$. We consider a continuous partition of unity $\{\alpha_i\}$ associated with the subcover and we define a new function $\phi : K \times K \to \mathbb{R}$ by:

$$\phi(x, y) = \sum_{i=1}^{n} \alpha_i(x) < p_i, x - y >$$

It is continuous with respect to x so it satisfies the first assumption of Ky-Fan's inequality. It is affine with respect to y and so it satisfies the second assumption of the inequality. Finally $\phi(y, y) = 0$ and the third assumption is also satisfied. Therefore, there exists $\overline{x} \in K$ such that

$$\forall y \in K, \quad \phi(\overline{x}, y) = \sum_{i=1}^{n} \alpha_i(\overline{x}) < p_i, \overline{x} - y >=< \overline{p}, \overline{x} - y >\leq 0$$

Here, $\overline{p} = \sum\limits_{i=1}^{n} \alpha_i(\overline{x}) p_i$. The above inequality can be rewritten as: $< -\overline{p}, y - \overline{x} >\leq 0$, thus $-\overline{p}$ belongs to the polar cone of the tangent cone to K at $\overline{x}$, i.e., $-\overline{p} \in T_K(\overline{x})^-$.

Since K is a viability domain of F, there exists $v \in F(\overline{x}) \cap T_K(\overline{x})$. Thus,

$$\sigma(F(\overline{x}), \overline{p}) = \sup_{y \in F(x)} < \overline{p}, y >\geq< \overline{p}, v >\geq 0$$

The last inequality holds because $v \in T_K(\overline{x})$ and $-\overline{p} \in T_K(\overline{x})^-$, so we have $< -\overline{p}, v >\leq 0 \Rightarrow< \overline{p}, v >\geq 0$.

We now show that $\sigma(F(\overline{x},\overline{p}) < 0$ which contradicts the last series of inequalities. To do so, we first define:

$$I(\overline{x}) = \{i = 1, ..., n | \alpha_i(\overline{x}) > 0\}$$

which is not empty. Hence,

$$\begin{aligned}
\sigma(F(\overline{x}),\overline{p}) &= \sup_{\overline{y}\in F(\overline{x})} < \overline{y},\overline{p} > \\
&= \sup_{\overline{y}\in F(\overline{x})} < \overline{y}, \sum_{i=1}^{n} \alpha_i(\overline{x}) p_i > \\
&= \sup_{\overline{y}\in F(\overline{x})} \sum_{i=1}^{n} \alpha_i(\overline{x}) < \overline{y}, p_i > \\
&\leq \sum_{i=1}^{n} \alpha_i(\overline{x}) \sup_{\overline{y}\in F(\overline{x})} < \overline{y}, p_i > \\
&= \sum_{i=1}^{n} \alpha_i(\overline{x}) \sigma(F(\overline{x}), p_i) \\
&= \sum_{i\in I(\overline{x})} \alpha_i(\overline{x}) \sigma(F(\overline{x}), p_i) \\
&< 0
\end{aligned}$$

The last inequality holds because $i \in I(\overline{x})$ implies $\alpha_i(\overline{x}) > 0$, which means that $\overline{x} \in \nu_{p_i}$ and therefore $\sigma(F(\overline{x}), p_i) < 0$ by the definition of ν_{p_i}. We have created the desired contradiction. □

4.4 Kakutani Fixed Point Theorem

The Kakutani Fixed Point Theorem is a generalization of the Brouwer Fixed Point Theorem [5] in the single-valued case. It establishes conditions under which a set-valued map F admits a fixed point. The statement and proof of the theorem follow [2].

Theorem 8. *Let K be a convex compact subset of a Banach space X and $G : X \rightsquigarrow K$ be an upper hemicontinuous set-valued map with nonempty closed convex images. Then G has a fixed point $\overline{x} \in G(\overline{x})$.*

Proof. We define the new map $F(x) = G(x) - x$ which is also upper hemicontinuous being the sum of two upper hemicontinuous maps. Also since G is convex then $\forall\lambda \in [0, 1]$ and $\forall x_1, x_2 \in \mathrm{Dom}(G)$ we have

$$G(\lambda x_1 + (1-\lambda)x_2) \subset \lambda G(x_1) + (1-\lambda)G(x_2)$$

Therefore,

$$\begin{aligned}
F(\lambda x_1 + (1-\lambda)x_2) &= G(\lambda x_1 + (1-\lambda)x_2) - (\lambda x_1 + (1-\lambda)x_2) \\
&\subset \lambda G(x_1) + (1-\lambda)G(x_2) - (\lambda x_1 + (1-\lambda)x_2) \\
&= \lambda(G(x_1) - x_1) + (1-\lambda)(G(x_2) - x_2) \\
&= \lambda F(x_1) + (1-\lambda)F(x_2)
\end{aligned}$$

Since K is convex then $K - x \subset T_K(x)$ and since G is convex then $G(K) \subset K$. We deduce that K is a viability domain of F because

$$F(x) = G(x) - x \subset G(K) - x \subset K - x \subset T_K(x)$$

Hence there exists an equilibrium point $\overline{x} \in K$ of F, i.e. $0 \in F(\overline{x}) = G(\overline{x}) - \overline{x}$ which implies that $\overline{x} \in G(\overline{x})$ which is the desired fixed point of G. $\square$

Chapter 5

Game Theory Application

5.1 Fundamental Concepts

We start by considering two players: Elie and Mirna. The game requires Elie to pick a strategy $x \in E$ and Mirna to pick a strategy $y \in M$. The pair $(x, y) \in E \times M$ is called a strategy pair or a bistrategy. A natural mechanism for the selection of strategies by the two players is coming up with *decision rules*.

Definition. A decision rule for Elie is a set-valued map $C_E : M \rightsquigarrow E$ which associates each strategy $y \in M$ played by Mirna with the strategies $x \in C_E(y)$ which may be played by Elie. Similarly, a decision rule for Mirna is a set-valued map $C_M : E \rightsquigarrow M$ which associates each strategy $x \in E$ played by Elie with the strategies $y \in C_M(x)$ which may be played by Mirna.

Once Elie and Mirna come up with their decision rules C_E and C_M, respectively, we become interested in pairs of strategies $(\overline{x}, \overline{y})$ that are in static equilibrium, in the sense that:

$$\overline{x} \in C_E(\overline{y}) \quad \text{and} \quad \overline{y} \in C_M(\overline{x})$$

This leads to the following definition:

Definition. A pair of strategies $(\overline{x}, \overline{y})$ which is in static equilibrium is called a consistent pair of strategies or a consistent bistrategy.

The set of consistent bistrategies may be empty or very large. From the point of view of a game theorist, it would be only interesting if it is non-empty and small (at best consisting of one element). The problem of finding consistent bistrategies is a fixed point problem. We use C to denote the set-valued map from $E \times M$ into itself:

$$\forall (x, y) \in E \times M, \qquad C(x, y) := C_E(y) \times C_M(x).$$

And so, we are looking for pairs $(\overline{x}, \overline{y})$ that satisfy the following condition:

$$(\overline{x}, \overline{y}) \in C(\overline{x}, \overline{y})$$

We recall the powerful Brouwer's Fixed Point Theorem which provides sufficient conditions for a map defined on a certain set to have a fixed point.

Theorem 9. *Let K be a convex compact subset of a finite-dimensional space. Any continuous mapping of K into itself has a fixed point.*

We state the following corollary:

Corollary. Suppose that the behaviours of Elie and Mirna are described by one-to-one continuous decision rules and that the strategy sets E and M are convex compact subsets of finite-dimensional vector spaces. Then there is at least one consistent bistrategy.

We generalize this Corollary later to the multi-valued case.

Most of fixed point theorems require the sets to be convex and compact but in practice, this assumption might fail to be satisfied. Over the years mathematicians had to find a way to work around that problem since finite strategy sets are not convex. We first start by identifying the finite strategy set $E = \{1, \ldots, n\}$ of n elements and we associate E with the $(n-1)-$simplex of $\mathbb{R}^n$, given by:

$$S^n = \{\lambda \in \mathbb{R}^n_+ | \sum_{i=1}^{n} \lambda_i = 1\}.$$

Note that a simplex is a convex compact set. We next embed E in S^n by the mapping δ given by:

$$\delta : i \in E \longrightarrow e^i \in \mathbb{R}^n,$$

where $(e^1, \ldots, e^n)$ is the canonical basis of $\mathbb{R}^n$.

John Von Neumann proposed interpreting the elements of S^n as mixed strategies. A player does not choose one strategy but chooses only the probabilities with which he plays all the strategies. By doing so, the player is disguising his intentions from his opponents.

Any set-valued map C from $E = \{1, \ldots, n\}$ to a vector space X may be extended to a set-valued map $\overline{C}$ from $\mathbb{R}^n$ to X as follows:

$$\forall \lambda \in \mathbb{R}^n, \quad \overline{C}(\lambda) = \sum_{i=1}^{n} \lambda_i C(i).$$

Elie and Mirna will each use an *evaluation function* to classify their strategies. We will call it a loss function f_i since it will be representing Elie's and Mirna's losses if a bistrategy (x, y) is played. The function f_i is associated with a partial order of preference as follows:

$$(x_1, y_1) \in E \times M \quad \text{is preferred to} \quad (x_2, y_2) \in E \times M \quad \text{if and only if} \quad f_i(x_1, y_1) \leq f_i(x_2, y_2).$$

This means that a bistrategy is preferred if the loss is minimal.

$f_E(x, y)$ represents Elie's loss if Elie plays strategy x and Mirna plays strategy y.
$f_M(x, y)$ represents Mirna's loss if Mirna plays strategy y and Elie plays strategy x.

We set:

$$f(x, y) = (f_E(x, y), f_M(x, y)) \in \mathbb{R}^2$$

Definition. A two-person game in normal (strategic) form is defined by a mapping f from $E \times M$ into $\mathbb{R}^2$ called a biloss mapping.

If Elie knows that Mirna is playing strategy $y \in M$, then he may be tempted to choose a strategy $\overline{x} \in E$ that minimizes his loss. From this idea, we create the canonical decision rule $\overline{C}_E$ in the following way:

$$\overline{C}_E(y) = \{\overline{x} \in E | f_E(\overline{x}, y) = \inf_{x \in E} f_E(x, y)\}$$

Similarly, if Mirna knows that Elie is playing strategy $x \in E$, then she may be inclined to choose strategy $\overline{y} \in M$ that minimizes her loss. Her canonical decision rule is:

$$\overline{C}_M(x) = \{\overline{y} \in M | f_M(x, \overline{y}) = \inf_{y \in M} f_M(x, y)\}$$

Definition. A consistent pair of strategies $(\overline{x}, \overline{y})$ based on the canonical decision rules is called a **non-cooperative equilibrium** (or a Nash equilibrium) of the game.

In other words $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if

$$f_E(\overline{x}, \overline{y}) = \inf_{x \in E} f_E(x, \overline{y})$$

$$f_M(\overline{x}, \overline{y}) = \inf_{y \in M} f_M(\overline{x}, y).$$

A convenient way to find non-cooperative equilibria is to introduce the following functions ($\flat$ is read "flat"):

$$f_E^\flat(y) = \inf_{x \in E} f_E(x, y)$$

$$f_M^\flat(x) = \inf_{y \in M} f_M(x, y)$$

And so $(\overline{x}, \overline{y})$ is a non-cooperative equilibrium if and only if

$$f_E^{\flat}(\overline{y}) = f_E(\overline{x}, \overline{y}) \quad \text{and} \quad f_M^{\flat}(\overline{x}) = f_M(\overline{x}, \overline{y}).$$

5.2 Pareto Optima

In this context, we assume that players communicate and cooperate so there may exist bistrategies (x, y) such that

$$f_E(x, y) < f_E(\overline{x}, \overline{y}) \quad \text{or} \quad f_M(x, y) < f_M(\overline{x}, \overline{y})$$

Definition. $(x_\star, y_\star)$ is a bistrategy that is Pareto optimal if there are no other pairs $(x, y) \in E \times M$ such that $f_E(x, y) < f_E(x_\star, y_\star)$ and $f_M(x, y) < f_M(x_\star, y_\star)$.

If there exists a bistrategy that minimizes Elie's loss: α_E, then this bistrategy is Pareto optimal. For this to happen, Mirna's only goal would be to please Elie. On the other hand, if there exists another bistrategy that minimizes Mirna's loss: α_M, then this bistrategy is also Pareto optiomal. For this to happen, Elie's only goal would be to please Mirna. It rarely occurs that the virtual minimum of the game (α_E, α_F) is a non-cooperative equilibrium.

5.3 Conservative Strategies

There is a behaviour where Mirna's only goal is to annoy Elie and Elie is aware of this. Therefore, it would be wise for Elie to evaluate the loss associated with a strategy $x \in E$ using the function $f_E^\sharp$ (f_E sharp) given by:

$$f_E^\sharp(x) = \sup_{y \in M} f_E(x, y).$$

This is called the worst-loss function. In this case, Emil's behaviour consists of finding $x^\sharp \in E$ which minimizes his worst loss, namely:

$$f_E^\sharp(x^\sharp) = \inf_{x\in E} f_E^\sharp(x) = \inf_{x\in E} \sup_{y\in M} f_E(x,y).$$

This strategy is conservative and $v_E^\sharp := \inf_{x\in E} f_E^\sharp(x)$ is Elie's conservative value.

Similarly, $v_M^\sharp := \inf_{y\in M} f_M^\sharp(y)$ is Mirna's conservative value. the vector $v^\sharp = (v_E^\sharp, v_M^\sharp)$ is called the conservative vector of the game. Thus the bistrategies of interest are contained in the rectangle $[\alpha_E, v_E^\sharp]\times[\alpha_M, v_M^\sharp]$.

5.4 Examples of Finite Games

5.4.1 Prisoner's dilemma

Suppose that Elie and Mirna are accomplices to a crime which leads to their imprisonment. Each has to choose between the strategies of confession (”I” for Elie and ”1” for Mirna) or accusation (”II” for Elie and ”2” for Mirna).

The strategy sets are therefore $E = \{I, II\}$ and $M = \{1, 2\}$.

If neither confesses, moderate sentences are given (b years in prison).
If Elie confesses and Mirna accuses him, Mirna is freed and Elie is sentenced to $c > b$ years in prison.
If Mirna confesses and Elie accuses her, Elie is freed and Mirna is sentenced to $c > b$ years in prison.
If both confess, they will each serve a years in prison where $a < b < c$.
We summarize the bilosses of the different bistrategies in the following table:

		Mirna	
		1	2
Elie	I	(a,a)	(c,0)
	II	(0,c)	(b,b)

We compute the following:

$$f_E^\sharp(I) = c, \quad f_E^\sharp(II) = b, \quad f_M^\sharp(1) = c, \quad f_M^\sharp(2) = b.$$

Whence, $v_E^\sharp = v_M^\sharp = b$ and the bistrategy $(II, 2)$ is the conservative strategy.

Also note that:

$$f_E^\flat(1) = 0, \quad f_E^\flat(2) = b, \quad f_M^\flat(I) = 0, \quad f_M^\flat(II) = b.$$

It turns out that the bistrategy $(II, 2)$ is also a non-cooperative equilibrium because:

$$f_E(II, 2) = b < f_E(I, 2) \quad \text{and} \quad f_M(II, 2) = b < f_M(II, 1) = c.$$

5.4.2 Battle of the sexes

The strategies of Elie and Mirna consist of watching a political debate or going to the mall. Mirna prefers going to the mall while Elie prefers watching a political debate but they both prefer to be together. Elie's strategies are I and II for watching a political debate and going to the mall, respectively. As for Mirna, her strategies are 1 and 2, in the same order as well. Here is a table that summarizes the bilosses incurred based on the bistrategies played ($0 < a < b$):

		Mirna	
		1	2
Elie	I	(0,a)	(b,b)
	II	(b,b)	(a,0)

We compute the following:

$$f_E^\sharp(I) = f_E^\sharp(II) = f_M^\sharp(1) = f_M^\sharp(2) = b.$$

Therefore all 4 bistrategies are conservative.

Also note that:

$$f_E^\flat(1) = 0, \quad f_E^\flat(2) = a, \quad f_M^\flat(I) = 0, \quad f_M^\flat(II) = a.$$

We conclude that the bistrategies $(I, 1)$ and $(II, 2)$ are the non-cooperative equilibria, since:

$$f_E(I, 1) = 0 < f_E(II, 1) = b$$

$$f_M(I, 1) = a < f_M(I, 2) = b$$

$$f_E(II, 2) = a < f_E(II, 1) = b$$

$$f_M(II, 2) = 0 < f_M(I, 2) = b.$$

5.5 N-person Games

We extend the concept of a two-person game to an n-person game. The ith player is denoted by $i = 1, \dots, n$. Each player may choose a strategy $x^i \in E^i$. The set of multistrategies $x = (x^1, \dots, x^n)$ is denoted by $E = \prod_{i=1}^{n} E^i$.

In the perspective of the ith player, the set of multistrategies E is considered to be the product of E^i (his own set of strategies) and $E^{\hat{i}} = \prod_{j \neq i} E^j$ of strategies $x^{\hat{i}} = (x^1, \dots, x^n)$. Thus from his point of view $x = (x^i, x^{\hat{i}})$, the set $E = E^i \times E^{\hat{i}}$.

Definition. A decision rule of the ith player is a set-valued map C^i from $E^{\hat{i}}$ to E^i which associates multistrategies $x^{\hat{i}} \in E^{\hat{i}}$ determined by the other players with a strategy set $C^i(x^{\hat{i}}) \subset E^i$.

Definition. Consider an n-person game described by n decision rules C^i from $E^{\hat{i}}$ to E^i. We shall say that a multistrategy $x \in E$ is consistent if $\forall i = 1, \ldots, n$, we have $x^i \in C^i(x^{\hat{i}})$.

The set of consistent multistrategies is the set of fixed points of the set-value map C from E to E defined by: $C(x) = \prod_{i=1}^{n} C^i(x^{\hat{i}})$.

Kakutani's Fixed Point Theorem immediately provides an existence result of consistent multistrategies since the maps are assumed to be continuous and the set E to be convex and compact.

We shall suppose that the decision rules C^i of the players are determined by loss functions f^i.

Definition. A game in normal (strategic) form is a game in which the behaviour of the ith player is defined by a loss function $f^i : E \to \mathbb{R}$ which evaluates the loss $f^i(x)$ inflicted on the ith player by each multistrategy x.

The multiloss mapping is therefore defined by:

$$\forall x \in E, \quad f(x) = (f^1(x), f^2(x), \ldots, f^n(x)) \in \mathbb{R}^n$$

The associated decision rules are:

$$\overline{C}^i(x^{\hat{i}}) = \{x^i \in E^i | f^i(x^i, x^{\hat{i}}) = \inf_{y^i \in E^i} f^i(y^i, x^{\hat{i}})\}$$

Definition. The decision rules $\overline{C}^i$ associated with the loss function f^i are called the canonical decision rules. A multistrategy $\overline{x} \in E$ which is consistent for the canonical decision rules is called a non-cooperative equilibrium (Nash equilibrium).

This leads us to introduce the following map:

$$\phi : (x, y) \in E \times E \longrightarrow \sum_{i=1}^{n} (f^i(x^i, x^{\hat{i}}) - f^i(y^i, x^{\hat{i}})) \in \mathbb{R}$$

Proposition. The following assertions are equivalent:

1. $\overline{x} \in E$ is a non-cooperative equilibrium.

2. $\forall i = 1, \ldots, n$, $\forall y^i \in E^i$, we have $f^i(\overline{x}^i, \overline{x}^{\hat{\imath}}) \leq f^i(y^i, \overline{x}^{\hat{\imath}})$.

3. $\forall y \in E$, we have $\phi(\overline{x}, y) \leq 0$.

Proof. The equivalence of (1) and (2) is a consequence of the definition of non-cooperative equilibria and the infimum.

(2) $\Rightarrow$ (3) is obtained by adding up the n inequalities from (2).

(3) $\Rightarrow$ (2): Suppose that $\phi(\overline{x}, y) \leq 0$, $\forall y \in E$. We fix i and let $y = (y^i, \overline{x}^{\hat{\imath}})$. And so (3) can be rewritten as

$$f^i(\overline{x}^i, \overline{x}^{\hat{\imath}}) - f^i(y^i, \overline{x}^{\hat{\imath}}) + \sum_{j \neq i} (f^j(\overline{x}^j, \overline{x}^{\hat{\jmath}}) - f^j(y^j, \overline{x}^{\hat{\jmath}})) \leq 0$$

But $\overline{x}^j = y^j$ whenever $j \neq i$ because of the definition of y. This concludes the proof. □

Theorem 10. *(Nash). Suppose that $\forall i \in N$, the sets E^i are convex and compact and f^i are continuous and $y^i \to f^i(y^i, x^{\hat{\imath}})$ are convex, then there exists a non-cooperative equilibrium $\overline{x}$.*

Proof. It follows from the Ky-Fan inequality. The product of convex, compact sets is also convex and compact.

$\phi(x, y)$ is continuous in x because f^i's are continuous in x.

$\phi(x, y)$ is concave in y because f^i's are convex in y and $f^i(y^i, .)$'s are preceded by a negative sign.

$\phi(y, y) = 0$ for any $y \in E$.

All three assumptions of the Ky-Fan inequality are satisfied and therefore there exists $\overline{x} \in E$ such that

$$\phi(\overline{x}, y) \leq 0$$

Finally we use the above proposition that states the equivalence of statements (1) and (3) to conclude the existence of the non-cooperative equilibrium. □

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