Riemann-Hilbert approach to Gap Probabilities of Determinantal Point Processes

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ABSTRACT

Riemann-Hilbert Approach to Gap Probabilities of Determinantal Point Processes

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In this work, we study problems related to gap probabilities of certain universal determinantal point processes. The study of gap probabilities can be addressed in two directions: derivation of a Lax formulation of PDEs, as in the first two works presented here, and study of asymptotic behaviour, as in the last work. In order to achieve such results, the powerful theory of Riemann-Hilbert problem will be widely implemented.

We first consider the gap probability for the Bessel process in the single-time and multi-time case. We prove that the scalar and matrix Fredholm determinants of such process can be expressed in terms of determinants of integrable kernels in the sense of Its-Izergin-Korepin-Slavnov and thus related to suitable Riemann-Hilbert problems. In the single-time case, we construct a Lax pair formalism and we derive a Painlevé III equation related to the Fredholm determinant.

Next, we consider the problem of the gap probabilities for the Generalized Bessel process in the single-time and multi-time case, a determinantal process which arises as critical limiting kernel in the study of self-avoiding squared Bessel paths. As in the Bessel case, we connect the gap probability to a Riemann-Hilbert problem (derived from an IIKS kernel) on one side and to the isomonodromic τ-function on the other side. In particular, in the single-time case we construct a Lax pair formalism and in the multi-time case we explicitly define a completely new multi-time kernel and we proceed with the study of gap probabilities as in the single-time case.

Finally, we investigate the gap probabilities of the single-time Tacnode process. Through steepest descent analysis of a suitable Riemann-Hilbert problem, we show that under appropriate scaling regimes the gap probability of the Tacnode process degenerates into a product of two independent gap probabilities of the Airy process.

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Οὐδεμία ἡμέρα ἄνευ γραφῆς.
Not a day without a line.
(Apelles of Kos)

At the beginning it was a chant. Before all of this happened, before I packed and shipped my life 6,500 kilometres away from my home country, I was only a student, singing in a Gregorian Choir, with a white tunic.

Serendipity caught me on a Fall Sunday afternoon. “Manuela, come. I want to introduce you to Giorgio Pederzoli, professor in Mathematics.” Prof. Pederzoli was very happy to meet me and, because of our common interests, singing and Mathematics, we soon felt deep sympathy for each other.

“Why don’t we have a chat some time?”

I went to visit him in his office at Cattolica University several times. We discussed about a wide range of topics from Mathematics, to Opera music. One day he asked me about my post-graduation plans and, hearing that doing Mathematical research might have been one of the options, he shared with me many precious advices and thoughts that he had developed during his long academic career.

A few days before Christmas, I received a handwritten letter from Prof. Pederzoli. He was writing from Montréal and telling me about Concordia University, where he taught himself long time ago, and about its Mathematics department. From that Christmas, my life had an unexpected turn.

I am and will always be extremely grateful to him. I have been honoured to enjoy his friendship and his support.

Four years have passed, yet this time has lasted a blink.

The teachings of Prof. Matteo Zindo and Prof. Francesco Maj, carved like rock in my memory, were my good companions. The knowledge acquired from Prof. Franco Gallone, Prof. Dietmar Klemm and Prof. Elisabetta Rocca was my primary tool.

Four years have passed, yet so many events have happened to fill a man’s life.

I would like to express my gratitude to my supervisor Prof. Marco Bertola from whom I learned the most about integrable systems. I am grateful for his support and guidance through my doctoral studies.

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I wish to thank all my friends that lived and shared these wonderful years with me. With their presence I truly felt like home in this foreign land and they are the best friends one can wish for.

I left as last the most important people in my life, my family. Along the good events and the misadventures that happened to me, my family has always been a safe spot I could rely on, seek advice, help and joy.

I wish to thank my parents, Giuseppina Micheletto and Luigi Girotti for all the sacrifices they made for me, for all the support they gave me from far away and for always believing in me. I wish to thank Peyman Eslami for being so close and supportive with me. I thank him for his patience and for constantly inspiring me to be a better person. His presence illuminated every moment for all these years.

I hope I can copiously requite and be worthy of all this love.

If I turn back to gaze at the footprints that led me where I am now, I see a mosaic of precious friends, fundamental experiences, new knowledge, hard times and little but important successes.

I have been rich and lucky for living through such a blessed time.

To Prof. Giorgio Pederzoli

Multas per gentes et multa per æquora vectus
advenio has miseras, frater, ad inferias,
ut te postremo donarem munere mortis
et mutam nequiquam alloquerer cinerem,
quandoquidem fortuna mihi tete abstulit ipsum,
heu miser indigne frater adempte mihi.
Nunc tamen interea hæc prisco quæ more parentum
tradita sunt tristi munere ad inferias,
accipe fraterno multum manantia fletu,
atque in perpetuum, frater, ave atque vale.

(G.V. Catullus, *Carmina*)

یک چند به کودکی به استاد شدیم
یک چند ز استادی خود شاد شدیم
پایان سخن شنو که ما را چه رسید
از خاک درآمدیم و بر باد شدیم

With them the Seed of Wisdom did I sow,
And with my own hand labour'd it to grow:
And this was all the Harvest that I reap'd –
"I came like Water, and like Wind I go."

(Omar Khayyām, *Rub'áyyāt*, XXVIII,
translated by Edward FitzGerald)

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Chapter 1

Introduction

The work presented in this thesis springs from a specific class of stochastic processes, called Determinantal Point Processes, which arises in many mathematical and physical contexts. Nevertheless, the probabilistic setting is just the starting point for a study that involves tools from many other fields of Mathematics, like Analysis and Complex Geometry.

Many models in Mathematical Physics rely on the notion of Determinantal random Point Processes. A few examples are offered by the statistical distribution of the eigenvalues of random matrix models pioneered by Dyson ([31]), certain models of random growth of crystals ([7], [34], [59]), and mutually avoiding random walkers, usually referred as Dyson's processes.

To give an intuitive idea of what a Determinantal Point Process is, we can consider the following toy-model. Consider a given number n of points (or particles) on the real line $\mathbb{R}$, moving in a "chaotic" and random way, say as a Brownian motion, maintaining nevertheless their mutual order (see Figure 1.1).

We introduce now a probability measure on the space of configurations, in other words a function (called probability density)

$$\rho_k(x_1, \dots, x_k) \quad \forall\ k = 1, \dots, n \tag{1.0.1}$$

that evaluates which scenario is the most probable and which one is the least probable for a subset of k points ($k = 1, \dots, n$). Moreover, assume that the initial configuration of the points as well as the final configuration, after a certain time T, are known (see Figure 1.2).

A fundamental theorem due to S. Karlin and J. McGregor states that such probability density has a very specific shape as the determinant of a suitable function K, called correlation kernel.

Theorem 1.1 (Karlin-McGregor, 1959; [68]). *The probability density of the physical system*

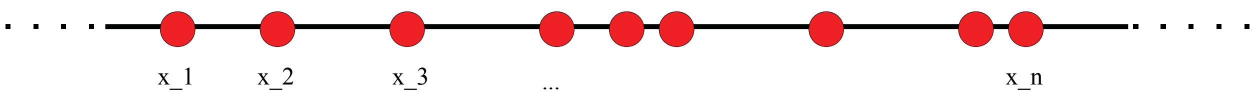

Figure 1.1: Visual realization of the toy-model for a determinantal point process.

under consideration is equal to

$$\rho_k(x_1, \ldots, x_k) = \det \left[K(x_i, x_j)\right]_{i,j=1}^k \quad \forall k = 1, \ldots, n \tag{1.0.2}$$

where $K(x, y)$ is a function of two variables which can be built out of the law regulating the particles' motion.

From this result it is clear why such system of points was called "determinantal" in the literature. This toy-model is just an example of a very general notion that will be detailed in the following Chapter 2 of this thesis.

From Theorem 1.1 it follows that every information about the system of points is contained in the function K and all the quantities of interest, in particular how much a given configuration is likely to happen or not, can be expressed in terms of K. Our focus, in particular, will be on the so called "gap probability", i.e. the probability that there are no points or particles in a prescribed region of the space, e.g. an interval on the real line $\mathbb{R}$ in our model above. The reason for considering this type of probability is because it is a natural first step to study a particle system. Moreover, as it will be clear in the next Chapter (Chapter 2), all the other quantities can be derived from the so called "generating function", of which the gap probability is a particular value.

We point out that the same considerations hold true whenever we consider an infinite number of particles, which means that we can also consider a limit physical system where the number of particles n tends to infinity. In this case, the discrete system becomes a continuum. The precise description of this limiting procedure is explained in Section 2.1.2 of the coming chapter.

The original motivation for studying this particular class of point processes dates back to the Fifties and it is due to the physicist E.P. Wigner.

In the field of Nuclear Physics, Wigner wished to describe the general properties of the

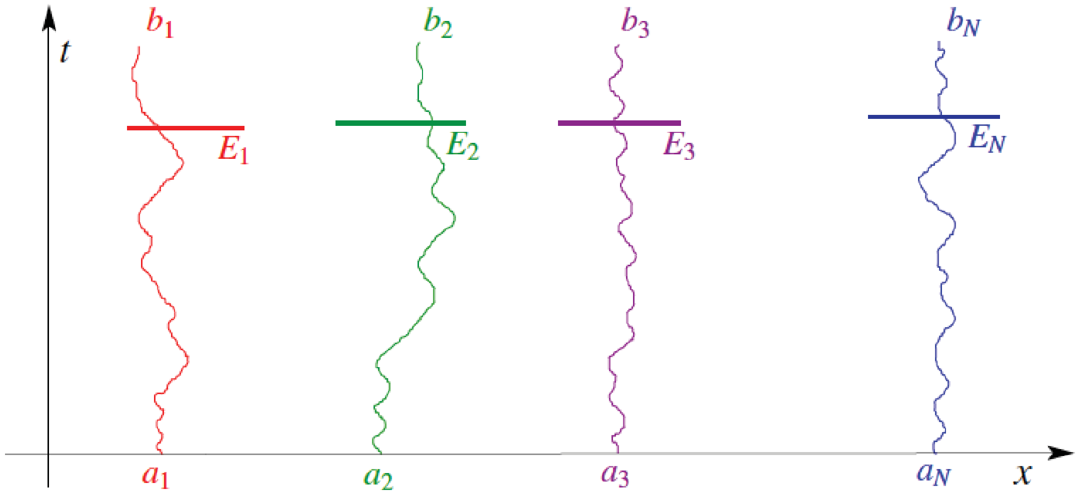

Figure 1.2: Visual realization of the toy-model for a determinantal point process. The points $\{a_k\}_{k=1,\ldots,N}$ are the starting points and $\{b_k\}_{k=1,\ldots,N}$ are the ending points after a given time T. A quantity of interest may be the probability that each particle x_i belongs to a given interval E_i, for all i's.

energy levels of highly excited states of heavy nuclei, as measured in nuclear reactions ([109]). In particular, he wanted to study the spacings between those energy levels. Such a complex nuclear system is usually represented by a Hermitian operator $\mathcal{H}$, called the Hamiltonian, defined on an infinite-dimensional Hilbert space and governed by physical laws. However, except for very specific and simple cases, $\mathcal{H}$ is unknown or very hard to compute. On the other hand, the real quantities of interest are the eigenvalues of $\mathcal{H}$, which represent the energy levels, defined by the so called Schrödinger equation

$$\mathcal{H}v = \lambda v \tag{1.0.3}$$

where v is the eigenfunction associated to the eigenvalue λ.

Wigner argued that one should regard a specific Hamiltonian $\mathcal{H}$ as behaving like a large-dimension random matrix (i.e. a matrix with random entries). Such matrix is thought as a member of a large class of Hamiltonians, all of which would have similar general properties as the specific Hamiltonian $\mathcal{H}$ in question ([108]). As a consequence, the eigenvalues of $\mathcal{H}$ could then be approximated by the eigenvalues of a large random matrix and the spacings between energy levels of heavy nuclei could be modelled by the spacings between successive eigenvalues of a random $n \times n$-matrix as $n \to +\infty$.

The ensemble of the random eigenvalues is precisely a Determinantal Point Process. Therefore, studying the spacings or gaps between eigenvalues means studying the gap probabilities of the determinantal system. Furthermore, the distribution of the largest eigenvalue obeys a different law on its own and is governed by the so called "Tracy-Widom" distribution ([100]), which can still be considered as a gap probability on an interval of the type $[a, +\infty]$, $a \in \mathbb{R}$ (the eigenvalues, or in general the points of a Determinantal Process, are always confined in finite positions on the real line).

This was the starting point of a powerful theory in Mathematical Physics called Random Matrix Theory, which was developed since the 1960s by Wigner and his colleagues, including F. Dyson and M. L. Mehta, and many other mathematicians (see [85]).

The theory of Determinantal Point Processes has not only applications in Physics, but also in many other areas. As an example, we can cite a diffusion process called Squared Bessel Process (BESQ), which will be analyzed in Chapter 6. A set of non-intersecting particles undergoing diffusion according to BESQ describe a determinantal point process. The BESQ is the underlying structure for the Cox-Ingersoll-Ross (CIR) model in Finance, which describes the short term evolution of interest rates, and for many models of the Growth Optimal Portfolio (GOP; [44], [91]). Moreover, a collection of non-intersecting BESQ play a very important role in the so called "principal components analysis" (PCA) of multivariate data, a technique that is used in detecting hidden patterns in data and image processing ([110], [42], [92]).

The purpose of the present thesis is to establish a connection between certain gap probabilities and a particular class of boundary value problems in the complex plane, generally referred to as "Riemann-Hilbert problems" (see e.g. [18]), or Wiener-Hopf method in older literature.

This is the first basic step that we will perform in all our works. Indeed, reformulating the study of gap probabilities as a suitable boundary value problem allows an effective analysis of such quantities. In particular, we can perform either a quantitative or a qualitative study.

Starting from the Riemann-Hilbert problem, it will be possible to derive a system of differential equations whose solution describes the behaviour of the gap probabilities as the gaps themselves vary. More specifically, it will be possible to express the gap probabilities in terms of the theory of equation of Painlevé type; this relationship is quite well-known originally in two dimensional statistical physics ([83]) and it was extensively studied in the Eighties and Nineties ([53], [54], [57], [94], [100], [101]).

In order to frame our results in a narrower context, we recall the Tracy-Widom distribution ([100]), which, as we wrote earlier, expresses the fluctuations of the largest eigenvalue of a random matrix with Gaussian entries; such distribution is defined in terms of the solution of a specific nonlinear ODE, the Painlevé II equation. Similarly in [101] the authors connected the fluctuation of the smallest eigenvalue of another set of random matrices called

"Laguerre ensemble" to the third member of the Painlevé hierarchy. Our results are closely related to these and they will extend this connection to two cases: the "Bessel process" (Chapter 5) and the "Generalized Bessel process"(Chapter 6).

We will first consider the gap probability for the Bessel process in the single-time and multi-time case. The multi-time Bessel process is simply a multi-dimensional version of the "single-time" process where we introduced a new parameter representing the time (see Section 2.2 of the following chapter). We will prove that the scalar and matrix Fredholm determinants of such process, which coincide with the respective gap probabilities, can be expressed in terms of Fredholm determinants of integrable operators in the sense of Its-Izergin-Korepin-Slavnov (IIKS). Such types of operator are related to a Riemann-Hilbert problem in a natural way. In the single-time case, we will construct a Lax pair formalism from the given Riemann-Hilbert problem and we will derive a Painlevé III equation related to the Fredholm determinant. Similar calculations are performed for the Generalized Bessel process.

On the other hand, the presence of a Riemann-Hilbert problem may allow also a qualitative study of gap probabilities in certain critic regimes using the method of (non-linear) Steepest Descent (see Chapter 4). The focus in this case is not to give an exact form to the gap probabilities, but rather to study their asymptotic behaviour in the limit as a given parameter converges to a critical value. A straightforward example is the asymptotic behaviour of the Pearcey process ([2], [11], [12]): in the setting of a large finite gap, the Pearcey gap probability factorizes into a product of two gap probabilities of the Airy process for semi-infinite gaps. Along the same lines as [11], our work will investigate the limiting behaviour of the gap probabilities of the tacnode process (Chapter 7). We will firstly show that the Fredholm determinant of this process can be described by the Fredholm determinant of an IIKS integrable operator, as in the Bessel and Generalized Bessel case, and through the steepest descent analysis of the associated Riemann-Hilbert problem, we wil show that under appropriate scalings the gap probability of the tacnode process degenerates into a product of two independent gap probabilities of the Airy process.

The present thesis is organized as follows. For the sake of completeness and self-containedness, in the first coming Chapters 2, 3 and 4 we will review all the crucial results that will be used in order to achieve our study of gap probabilities. In particular, we will formally define a determinantal point process and describe its properties in Chapter 2, while in Chapter 3 we will explain the connection between a specific class of integral operators (to which the Bessel, Generalized Bessel and tacnode process belong) and the well-known Jimbo-Miwa-Ueno τ-function via a suitably constructed Riemann-Hilbert problem. In Chapter 4 we will recall the powerful technique of Steepest Descent, first introduced by Deift and Zhou ([24]), for the study of asymptotic behaviour of a given Riemann-Hilbert problem, that will in turn allows to draw meaningful conclusions on the asymptotic behaviour of the gap probabilities we started with.

The new and original results which represent the core of this thesis are exposed in Chapter 5 for the Bessel process, Chapter 6 for the Generalized Bessel process and Chapter 7 for the tacnode process. Conclusions and important remarks are discussed at the end of every Chapter and collected in the conclusive Chapter 8. In the appendix A, we briefly describe some numerical methods that have been used to obtain some of the figures appearing along the thesis.

Chapter 2

Determinantal Point Processes

In the present chapter we will review the main concepts about Determinantal Point Processes in timeless and dynamic regimes.

Determinantal point processes are of considerable current interest in Probability theory and Mathematical Physics. They were first introduced by Macchi ([82]) and they arise naturally in Random Matrix theory, non-intersecting paths, certain combinatorial and stochastic growth models and representation theory of large groups, see e.g. Deift [18], Johansson [58], Katori and Tanemura [71], Borodin and Olshanski [16], and many other papers cited therein. For surveys on determinantal processes, we refer to the papers by Hough *et al.* [48], Johansson [61], König [73] and Soshnikov [96].

2.1 Point Processes

Consider a random collection of points on the real line. A configuration $\mathcal{X}$ is a subset of $\mathbb{R}$ that locally contains a finite number of points, i.e. $\#(\mathcal{X} \cap [a,b]) < +\infty$ for every bounded interval $[a,b] \subset \mathbb{R}$.

Definition 2.1. A (locally finite) **point process** on $\mathbb{R}$ is a probability measure on the space of all configurations of points $\{\mathcal{X}\}$.

Loosely speaking, given a point process on $\mathbb{R}$, it is possible to evaluate the probability of any given configuration. Moreover, the mapping $A \mapsto \mathbb{E}[\#(\mathcal{X} \cap A)]$, which assigns to a Borel set A the expected value of the number of points in A under the configuration $\mathcal{X}$, is a measure on $\mathbb{R}$.

Let us assume there exists a density ρ_1 with respect to the Lebesgue measure and we call it 1-point correlation function for the point process. Then, we have

$$\mathbb{E}[\#(\mathcal{X} \cap A)] = \int_A \rho_1(x)\mathrm{d}x. \tag{2.1.1}$$

and $\rho_1(x)dx$ represents the probability to have a point in the infinitesimal interval $[x, x+\mathrm{d}x]$. In general, a k-point correlation function ρ_k (if it exists) is a function of k variables such that for distinct points

$$\rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{2.1.2}$$

is the probability to have a point in each infinitesimal interval $[x_j, x_j + \mathrm{d}x_j]$, $j = 1, \ldots, k$. Thus, given disjoints sets $A_1, \ldots, A_k$, we have

$$\mathbb{E}\left[\prod_{j=1}^{k} \#(\mathcal{X} \cap A_j)\right] = \int_{A_1} \cdots \int_{A_k} \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{2.1.3}$$

i.e. the expected number of k-tuples $(x_1, \ldots, x_k) \in \mathcal{X}^k$ such that $x_j \in A_j$ for every j. In case the A_j's are not disjoint it is still possible to define the quantity above, with little modifications. For example, if $A_j = A$ for every j, then

$$\frac{1}{k!}\int_A \cdots \int_A \rho_k(x_1, \ldots, x_k)\mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{2.1.4}$$

is the expected number of ordered k-tuples $(x_1, \ldots, x_k)$ such that $x_1 < x_2 < \ldots < x_k$ and $x_j \in A$ for every $j = 1, \ldots, k$.

If $P(x_1, \ldots, x_n)$ is a probability density function on $\mathbb{R}^n$, invariant under permutations of coordinates, then we can build an n-point process on $\mathbb{R}$ with correlation functions

$$\rho_k(x_1, \ldots, x_k) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} P(x_1, \ldots, x_n)\mathrm{d}x_{k+1} \ldots \mathrm{d}x_n. \tag{2.1.5}$$

The problem of existence and uniqueness of a random point field defined by its correlation functions was studied by Lenard in [80] and [81].

Definition 2.2. A point process with correlation functions ρ_k is **determinantal** if there exists a kernel $K(x, y)$ such that for every k and every $x_1, \ldots, x_k$ we have

$$\rho_k(x_1, \ldots, x_k) = \det[K(x_i, x_j)]_{i,j=1}^k. \tag{2.1.6}$$

The kernel K is called correlation kernel of the determinantal point process.

Remark 2.3. *The correlation kernel is not unique. If K is a correlation kernel, then* *the conjugation of K with any positive function $h(\cdot)$ gives an equivalent correlation kernel $\widetilde{K}(x,y) := h(x)K(x,y)h(y)^{-1}$ describing the same point process.*

Determinantal processes became quite common as a model describing (random) points that tend to exclude one another. Indeed, it is possible to show that in a determinantal process there is a repulsion between nearby points and this is the reason why in physics literature a determinantal point process is sometimes called a Fermionic point process (see e.g [82], [32], [97]).

Examples of determinantal processes can be constructed thanks to the following result. We refer to [96] for a thorough exposition.

Theorem 2.4. *Consider a kernel K with the following properties:*

- *trace-class:* $\operatorname{Tr} K = \int_{\mathbb{R}} K(x,x)\mathrm{d}x = n < +\infty$*;*
- *positivity:* $\det[K(x_i,x_j)]_{i,j=1}^n$ *is non-negative for every* $x_1,\dots,x_n \in \mathbb{R}$*;*
- *reproducing property:* $\forall\ x,y \in \mathbb{R}$

$$K(x,y) = \int_{\mathbb{R}} K(x,s)K(s,y)\mathrm{d}s. \tag{2.1.7}$$

Then,

$$P(x_1,\dots,x_n) := \frac{1}{n!}\det[K(x_i,x_j)]_{i,j=1}^n \tag{2.1.8}$$

is a probability measure on $\mathbb{R}^n$, invariant under coordinates permutations. The associated point process is a determinantal point process with K as correlation kernel.

In a determinantal process all information is contained in the correlation kernel and all quantities of interest can be expressed in terms of K. In particular, given a Borel set A, we are interested in the so called gap probability, i.e. the probability to find no points in A.

Consider a point process on $\mathbb{R}$ with correlation function ρ_k and let A be a Borel set such that, with probability 1, there are only finitely may points in A (for example, A is bounded). Denote by $p_A(n)$ the probability that there are exactly n points in A. If there are n points in A, then the number of ordered k-tuples in A is $\binom{n}{k}$. Therefore, the following equality holds

$$\frac{1}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k = \sum_{n=k}^{\infty}\binom{n}{k}p_A(n). \tag{2.1.9}$$

Assume the following alternating series is absolutely convergent, then

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k\binom{n}{k}p_A(n)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\right)p_A(n);\end{aligned} \tag{2.1.10}$$

on the other hand, $\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}$ vanishes unless n is zero. In conclusion,

$$p_A(0)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\rho_k(x_1,\dots,x_k)\mathrm{d}x_1\dots\mathrm{d}x_k, \tag{2.1.11}$$

where we call $p_A(0)$ **gap probability**, i.e. the probability to find no points in A. In particular, when a point process is determinantal, we have

$$p_A(0)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{A^k}\det\left[K(x_i,x_j)\right]_{i,j=1}^{k}\mathrm{d}x_1\dots\mathrm{d}x_k, \tag{2.1.12}$$

which is clearly the Fredholm determinant

$$\det\left(\mathrm{Id}-\mathbf{K}\Big|_A\right) \tag{2.1.13}$$

of the (trace class) integral operator $\mathbf{K}$ defined by

$$\mathbf{K}(\phi)(x)=\int_{\mathbb{R}}K(x,y)\phi(y)\mathrm{d}y \tag{2.1.14}$$

and restricted to the Borel set A.

It is actually possible to prove a more general result, which reduces to the one above when considering zero particles.

Theorem 2.5 (Theorem 2, [96])**.** *Consider a determinantal point process with kernel K. For any finite Borel sets $B_j\subseteq\mathbb{R}$, $j=1,\dots,n$, the generating function of the probability distribution of the occupation number $\#_{B_j}:=\#\{x_i\in B_j\}$ is given by*

$$\mathbb{E}\left(\prod_{i=1}^{n}z_j^{\#_{B_j}}\right)=\det\left(\mathrm{Id}-\sum_{j=1}^{n}(1-z_j)K\Big|_{B_j}\right). \tag{2.1.15}$$

In particular, the probability of finding any number of points k_j in the correspondent set B_j $\forall j$ is given by a suitable derivative of the generating function at the origin. We refer again to [96] for a detailed proof of the Theorem.

2.1.1 Examples of Determinantal Point Processes

In this section, we will briefly review some of the main examples of Determinantal point processes, which also provide a physical motivation for the study of such type of processes. For more details, we cite standard references as [32] and [97] for the Fermi gas model and [18] and [85] for the Random Matrix models.

Fermi gas. Consider the Schrödinger operator $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ with V a real-valued function, acting on the space $L^2(E)$, E is a separable Hausdorff space (for the sake of simplicity, E will be $\mathbb{R}$ or S^1), and let $\{\varphi_k\}_{k=0}^{\infty}$ be a set of orthonormal eigenfunctions for the operator H. The n^{th} exterior power of H is an operator $\bigwedge^n H := \sum_{i=1}^{n} \left[-\frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + V(x_i) \right]$ acting on $\bigwedge^n L^2(E)$ (the space of antisymmetric L^2-functions of n variables) and it describes the Fermi gas with n particle, i.e. an ensemble of n fermions.

The ground state of the Fermi gas is given by the so called Slater determinant

$$\begin{aligned} \psi(x_1, \dots, x_n) &= \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} \varphi_{i-1}(x_{\sigma(i)}) \\ &= \frac{1}{\sqrt{n!}} \det \left[\varphi_{i-1}(x_j) \right]_{i,j=1}^{n} . \end{aligned} \tag{2.1.16}$$

It is known that the absolute value squared of the ground state defines the probability distribution of the particles. Therfore, we have

$$p_n(x_1, \dots, x_n) = |\psi(x_1, \dots, x_n)|^2 = \frac{1}{n!} \det \left[K_n(x_i, x_j) \right]_{i,j=1}^{n} \tag{2.1.17}$$

$$K_n(x, y) := \sum_{i=0}^{n-1} \varphi_{i-1}(x) \overline{\varphi_{i-1}(y)} \tag{2.1.18}$$

and $K_n(x, y)$ is the kernel of the orthogonal projection onto the subspace spanned by the first n eigenfunctions $\{\varphi_j\}$ of H.

The formula above defines a determinantal process. Indeed, it can be shown (see [85]) that the correlation function are

$$\rho_k^{(n)}(x_1, \dots, x_k) = \frac{n!}{(n-k)!} \int p_n(x_1, \dots, x_n) \mathrm{d}x_{k+1} \dots \mathrm{d}x_n = \det \left[K_n(x_i, x_j) \right]_{i,j=1}^{k} . \tag{2.1.19}$$

To give some practical examples, let's focus on two special cases of H. The first case is the harmonic oscillator on the real line $\mathbb{R}$

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2; \tag{2.1.20}$$

its eigenfunctions are the Weber-Hermite functions

$$\varphi_k(x) = \frac{(-1)^k}{\pi^{\frac{1}{4}}}(2^k k)^{\frac{1}{2}} e^{\frac{x^2}{2}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2} \tag{2.1.21}$$

and the correlation kernel is (using the Christoffel-Darboux formula)

$$K_n(x,y) = \left(\frac{n}{2}\right)^{\frac{1}{2}} \frac{\varphi_n(x)\varphi_{n-1}(y) - \varphi_n(y)\varphi_{n-1}(x)}{x-y}. \tag{2.1.22}$$

The second case is the free particle on a circle S^1

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \tag{2.1.23}$$

and its correlation kernel is

$$K_n(\theta,\eta) = \frac{\sin\left(\frac{n}{2}(\theta-\eta)\right)}{2\pi \sin\left(\frac{\theta-\eta}{2}\right)}. \tag{2.1.24}$$

The two examples above can also be interpreted as the equilibrium distribution of n unit charges confined to the line $\mathbb{R}$ or to the unit circle S^1 respectively, repelling each other according to the Coulomb law of two-dimensional electrostatic.

Random Matrix Ensembles. The probability distribution in the previous example allows another interpretation.

Consider the space of $n \times n$ complex Hermitian matrices

$$\mathfrak{H}_n = \{\, M \in \mathrm{Mat}_n(\mathbb{C}) \mid M = M^\dagger \,\}\,. \tag{2.1.25}$$

This is a n^2-dimensional vector space with the real diagonal entries $\{M_{ii}\}_{i=1}^n$ and the real and imaginary part of the upper diagonal elements $\{\Re M_{ij},\ \Im M_{ij}\}_{i<j}$ as independent coordinates. The flat Lebesgue measure on $\mathfrak{H}_n$ is

$$\mathrm{d}M = \prod_{i=1}^{n} \mathrm{d}M_{ii} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathrm{d}\Re M_{ij} \mathrm{d}\Im M_{ij}. \tag{2.1.26}$$

Definition 2.6. The **Gaussian Unitary Ensemble** (GUE) is the probability measure

$$p_n(M) = \frac{1}{Z_n} e^{-\operatorname{Tr} M^2} \mathrm{d}M \tag{2.1.27}$$

on the space $\mathfrak{H}_n$.

The above definition is equivalent to the requirement that all the entries $\{\Re M_{ij},\ \Im M_{ij}\}_{i<j}$ and $\{M_{ii}\}$ are mutually independent random variables; more precisely, $\Re M_{ij}$, $\Im M_{ij}$ and M_{ii} are normal random variable with zero mean and variance equal to $\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2}$ respectively: $\Re M_{ij},\ \Im M_{ij} \sim N\left(0, \frac{1}{4}\right)$ and $M_{ii} \sim N\left(0, \frac{1}{2}\right)$.

Definition 2.7. A **Unitary Ensemble** is the probability measure

$$p_n(M) = \frac{1}{Z_n} e^{-\operatorname{Tr} V(M)} \mathrm{d}M \tag{2.1.28}$$

on the space $\mathfrak{H}_n$, where $V : \mathbb{R} \to \mathbb{R}$ is a given function, called the potential, with suitable growth condition at $\pm\infty$ to guarantee that the probability measure above is well defined.

Remark 2.8. *A sufficient condition for the probability (2.1.28) to be well-defined is that $V(x)$ grows faster than $\ln(1+x^2)$, for $|x| \gg 1$, which is certainly satisfied if V is a polynomial of even degree, with positive leading coefficient.*

In general, the entries of a Unitary Ensembles are not independent, but strongly correlated. The name "Unitary Ensemble" comes from the fact that the probability distribution (2.1.28) is invariant under conjugation with a unitary matrix $M \mapsto UMU^{-1}$, $U \in U(n)$.

In random matrix theory, one is interested in the distribution of the eigenvalues of the (random) matrix M. For the case of GUE, the eigenvalues are real random variables.

According to the spectral theorem, any Hermitian matrix M can be written as $M = U\Lambda U^{-1}$, where $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ is the matrix of eigenvalues and $U \in U(n)$. Therefore, we can perform the following change of variables

$$\begin{gathered} M \mapsto (\Lambda, U) \\ \{M_{ii},\ i=1,\ldots,n;\ \Re M_{ij}, \Im M_{ij},\ i<j\} \mapsto \{\lambda_1, \ldots, \lambda_n;\ u_{ij}\}, \end{gathered} \tag{2.1.29}$$

where u_{ij} are the parameters that parametrize the unitary group. Under such transformation, the Lebesgue measure reads (thanks to the Weyl integration formula)

$$\mathrm{d}M = c_n \prod_{i<j} (\lambda_i - \lambda_j)^2 \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \mathrm{d}U \tag{2.1.30}$$

where c_n is a suitable normalization constant and $\mathrm{d}U$ is the Haar measure on $U(n)$.

Since $\operatorname{Tr} V(M) = \sum_j V(\lambda_j)$, we can conclude that the probability measure on the space of matrices (2.1.28) induces a joint probability density on the eigenvalues given by

$$\frac{1}{\widetilde{Z}_n} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{j=1}^{n} e^{-V(\lambda_j)} = \frac{1}{n!} \det \left[K_n(x_i, x_j)\right]_{i,j=1}^{n} \tag{2.1.31}$$

with $\widetilde{Z}_n$ a normalization constant and

$$K_n(x, y) = e^{-\frac{V(x)+V(y)}{2}} \sum_{j=0}^{n-1} \phi_j(x)\phi_j(y) \tag{2.1.32}$$

$\{\phi_j\}$ being the set of orthonormal polynomials with respect to $\exp\left\{-\frac{V(\cdot)}{2}\right\}$.

In particular, in the GUE case ($V(x) = x^2$), the polynomials ϕ_j are the Hermite polynomials and the kernel K_n is the Hermite kernel already described in the previous example (2.1.22).

Non-intersecting path ensemble. Let $p_t(x; y)$ be the transition probability density from point x to point y at time t of a one-dimensional strong Markov process with continuous sample paths. A classical theorem by S. Karlin and J. McGregor [68] gives a determinantal formula for the probability that a number of paths with given starting and ending positions fall in certain sets at some later time without intersecting in the intermediate time interval (see Figure 1.2).

Theorem 2.9 ([68])**.** *Consider n independent copies $X_1(t), \ldots, X_n(t)$ of a one-dimensional strong Markov process with continuous sample paths, conditioned so that*

$$X_j(0) = a_j \tag{2.1.33}$$

for given values $a_1 < a_2 < \ldots < a_n \in \mathbb{R}$. Let $p_t(x, y)$ be the transition probability function of the Markov process and let $E_1, \ldots, E_n \subseteq \mathbb{R}$ be disjoint Borel sets (more precisely, we assume $\sup E_j < \inf E_{j+1}$). Then,

$$\frac{1}{Z_n} \int_{E_1} \cdots \int_{E_n} \det \left[p_t(a_i, x_j)\right]_{i,j=1}^{n} \mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{2.1.34}$$

is equal to the probability that each path X_j belongs to the set E_j at time t, without any intersection between paths in the time interval $[0, t]$, for some normalizing constant Z_n.

Sketch of the proof (a heuristic argument). Let $n = 2$, then

$$\begin{aligned}\frac{1}{Z_2}\int_{E_1}\int_{E_2} p_t(a_1,x_1)p_t(a_2,x_2) - p_t(a_1,x_2)p_t(a_2,x_1)\,\mathrm{d}x_1\mathrm{d}x_2 \\ = P\left(X_1(t)\in E_1\right)P\left(X_2(t)\in E_2\right) - P\left(X_1(t)\in E_2\right)P\left(X_2(t)\in E_1\right) \\ =: P\left(\mathcal{A}\right) - P\left(\mathcal{B}\right).\end{aligned} \tag{2.1.35}$$

On the other hand,

$$P\left(\mathcal{A}\right) - P\left(\mathcal{B}\right) = P\left(\mathcal{A}_1\right) + P\left(\mathcal{A}_2\right) - P\left(\mathcal{B}_1\right) - P\left(\mathcal{B}_2\right) \tag{2.1.36}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$ represent the following events:

$$\begin{aligned}\mathcal{A}_1 &= \{\, X_i(t)\in E_i \text{ respectively and the paths did not intersect}\,\}\\ \mathcal{A}_2 &= \{\, X_i(t)\in E_i \text{ respectively and the paths did intersect at least once}\,\}\\ \mathcal{B}_1 &= \{\, X_1(t)\in E_2,\ X_2(t)\in E_1 \text{ and the paths did not intersect}\,\}\\ \mathcal{B}_2 &= \{\, X_1(t)\in E_2,\ X_2(t)\in E_1 \text{ and the paths did intersect at least once}\,\}\end{aligned}$$

Clearly $P\left(\mathcal{B}_1\right) = 0$. Moreover, consider the event $\mathcal{A}_2$: at the first time when the two path collide, we can interchange the labels. This is a bijection $\Psi : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{B}_2$. Since the process is Markovian and the two particles act independently, we have

$$P\left(\mathcal{A}_2\right) = P\left(\mathcal{B}_2\right). \tag{2.1.37}$$

In conclusion,

$$\frac{1}{Z_2}\int_{E_1}\int_{E_2} \det\left[p_t(a_i,x_j)\right]_{i,j=1,2}\,\mathrm{d}x_1\mathrm{d}x_2 = P\left(\mathcal{A}_1\right). \tag{2.1.38}$$

□

However, this is not a determinantal process, since the correlation functions are not expressible in terms of the determinant of a kernel. If we additionally condition the paths to end at time $T > 0$ at some given points $b_1 < \ldots < b_n$, without any intersection between the paths along the whole time interval $[0,T]$, then it can be shown (using an argument again based on the Markov property) that the random positions of the n paths at a given time $t \in [0,T]$ have the joint probabilty density function

$$\frac{1}{Z_n}\det\left[p_t(a_i,x_j)\right]_{i,j=1}^n \det\left[p_{T-t}(x_i,b_j)\right]_{i,j=1}^n = \frac{1}{Z_n}\det\left[K_n(x_i,x_j)\right]_{i,j=1}^n \tag{2.1.39}$$

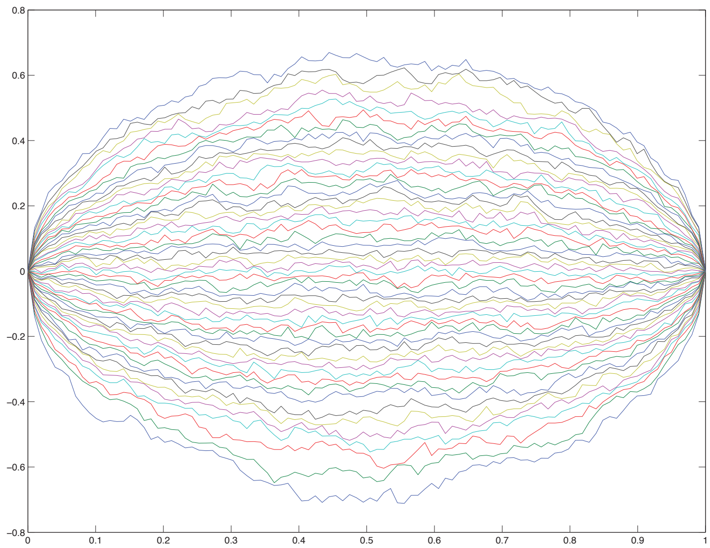

Figure 2.1: Numerical simulation of 50 non-intersecting Brownian paths in the confluent case with one starting and one ending point.

with a suitable normalizing constant Z_n and kernel

$$K_n(x,y) := \sum_{j=1}^{n} \phi_j(x)\psi_j(y) \tag{2.1.40}$$

$$\phi_j \in \operatorname{span}\{p_t(a_1,x),\ldots,p_t(a_n,x)\} \tag{2.1.41}$$

$$\psi_k \in \operatorname{span}\{p_{T-t}(x,b_1),\ldots,p_{T-t}(x,b_n)\} \tag{2.1.42}$$

$$\int_{\mathbb{R}} \phi_j(x)\psi_k(x)\mathrm{d}x = \delta_{jk}. \tag{2.1.43}$$

Remark 2.10. *The model we just constructed is known in the literature as biorthogonal ensemble. We refer to [15] for a thorough exposition on the subject.*

Of interest is also the confluent case when two or more starting (or ending) points collapse together. For example, in the confluent limit as $a_j \to a$ and $b_j \to b$, for all j's (see Figure 2.1), applying l'Hôpital rule to (2.1.39) gives

$$\frac{1}{\widetilde{Z}_n} \det\left[\frac{\mathrm{d}^{i-1}}{\mathrm{d}a^{i-1}} p_t(a,x_j)\right]_{i,j=1}^{n} \det\left[\frac{\mathrm{d}^{j-1}}{\mathrm{d}x^{j-1}} p_{T-t}(x_i,b)\right]_{i,j=1}^{n} \tag{2.1.44}$$

which is still a determinantal point process with kernel derived along the same method as in (2.1.40)-(2.1.43).

2.1.2 Limit of Determinantal processes and universality.

Suppose that for each n we can construct a (finite) determinantal point process $\mathbb{P}_n$ with correlation kernel K_n. If the sequence of kernels $\{K_n\}$ converge in some sense to a limit kernel K as $n \to \infty$, one can expect that also the point processes $\mathbb{P}_n$ will converge to a new determinantal point process $\mathbb{P}$ with correlation kernel K.

This is indeed the case provided some mild assumptions.

Proposition 2.11. *Let $\mathbb{P}$ and $\mathbb{P}_n$ be determinantal point processes with kernels K and K_n respectively. Let K_n converge pointwise to K*

$$\lim_{n\to\infty} K_n(x,y) = K(x,y) \tag{2.1.45}$$

uniformly in $x,\ y$ over compact subsets of $\mathbb{R}$. Then, the point processes $\mathbb{P}_n$ converge to $\mathbb{P}$ weakly.

Remark 2.12. *The condition of uniform convergence on compact sets may be relaxed.*

Suppose we have a sequence of kernels K_n and a fixed reference point x_*. Before taking the limit, we first perform a centering and rescaling of the form

$$x \mapsto Cn^\alpha (x - x_*) \tag{2.1.46}$$

with suitable values of $C,\ \alpha > 0$. Then in many cases of interest the rescaled kernels have a limit

$$\lim_{n\to\infty} \frac{1}{Cn^\alpha} K_n\left(x_* + \frac{x}{Cn^\alpha}, x_* + \frac{y}{Cn^\alpha}\right) = K(x,y) \tag{2.1.47}$$

Therefore, the scaling limit K is a kernel that corresponds to a determinantal point process with an infinite number of points.

The physical meaning of this scaling and limiting procedure is the following: as the number of points tends to infinity, one is interested in the local (microscopic) behaviour of the system in specific points of the domain where the particles may lie, upon suitable rescaling: for example, in an infinitesimal neighbourhood entirely contained in the domain (the so-called "bulk") or in an infinitesimal neighbourhood only including the left-most or right-most particles on the line (the so-called "edge").

In many different situations the same scaling limit K may appear. The phenomenon is known as **universality** in Random Matrix Theory. Instances of limiting kernels are the sine kernel ([99])

$$K_{\text{sine}} = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{2.1.48}$$

and the Airy kernel ([100])

$$K_{\mathrm{Ai}} = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \tag{2.1.49}$$

where Ai is the Airy function. Other "universal" kernels are the Bessel kernel, the Generalized Bessel kernel and the tacnode kernel, which will be exhaustively described in the present thesis.

The sine kernel appears, for example, as a scaling limit in the bulk of the spectrum in GUE (see [85, Ch. 5]) and the Airy kernel appears as a scaling limit at the edge of the spectrum in GUE and at the soft (right) edge of the spectrum in the Laguerre ensemble (introduced by Bronk [17]), while the Bessel kernel appears at the hard (left) edge in the Laguerre ensemble (see [100], [101]).

2.2 Multi-time processes. Introduction of time

In this last section on point processes, we want to propose a generalization of the model of non-intersecting random paths, introducing a time variable in the physical process. In this way, the model will not only describe a static model in timeless thermodynamic equilibrium (as before), but also a dynamical system which may be in an arbitrary non-equilibrium state changing with time.

The first implementation of this dynamic regime was proposed by Dyson in [31] for the study of the random eigenvalues of a Gaussian Unitary Ensemble.

Let consider again N non-intersecting moving particles $X_1(t), \ldots, X_N(t)$ on $\mathbb{R}$ with a certain transition probability, conditioned to start at time $t = 0$ from points $\{a_i\}$ and to end at final time $t = 1$ at points $\{b_j\}$.

Now, consider a set of times $0 < \tau_1 < \ldots < \tau_n < 1$ and corresponding Borel sets $E_1, \ldots, E_n \subset \mathbb{R}$. The quantity of interest is the joint probability that for all $k = 1, \ldots, n$ no curve passes through E_k at time τ_k. We call again this quantity "gap probability" and it is still a Fredholm determinant for a (matrix) operator, generalizing the one used in the scalar case (formulæ (2.1.13)-(2.1.14)).

More precisely, the joint probability distribution of the N paths is described in terms of product of $n+1$ determinants, as proved by S. Karlin and J. McGregor.

Theorem 2.13 ([68])**.** *Consider N independent copies $X_1(t), \ldots, X_N(t)$ of a one-dimensional strong Markov process with continuous paths and transition probability $p_t(x, y)$, conditioned so that*

$$X_j(0) = a_j \quad \text{and} \quad X_j(1) = b_j \tag{2.2.1}$$

where $\{a_j\}_1^N$ and $\{b_j\}_1^N$ are given values.

Then, the joint probability density of the process can be written as a product of $n+1$ determinants:

$$p\left(\vec{X}(\tau_1)=\vec{x}^1,\dots,\vec{X}(\tau_n)=\vec{x}^n\right)=$$
$$\frac{1}{Z_{N,n}}\det\left[p_{\tau_1}(a_i,x_j^1)\right]_{i,j=1}^N\cdot\prod_{k=1}^{n-1}\det\left[p_{\tau_{k+1}-\tau_k}(x_i^k,x_j^{k+1})\right]_{i,j=1}^N\cdot\det\left[p_{1-\tau_n}(x_i^n,b_j)\right]_{i,j=1}^N \tag{2.2.2}$$

with $\vec{x}^{\,i}=(x_1^i,\dots,x_N^i)\in\mathbb{R}^N$ the vector of positions of the particles.

This structure does not change in the limit case $a_i\to 0$, $b_j\to 0$; we simply have to modify with some *caveat* the expression of the first and last determinants.

Applying a classical result due to B. Eynard and M.L. Mehta ([33]), it is possible to prove that such point process is actually a determinantal process and its gap probability is a suitable Fredholm determinant.

Theorem 2.14 ([33], [46]). *A measure on $\left(\mathbb{R}^N\right)^n$ of the form (2.2.2) induces a (determinantal) point process on the space $\{1,\dots,n\}\times\mathbb{R}$ with correlation kernel entries*

$$K_{ij}(x,y)=\tilde{K}_{ij}(x,y)-\varphi_{ij}(x,y)\delta_{i<j} \tag{2.2.3}$$

with

$$\tilde{K}_{pq}(x,y)=\sum_{i,j=1}^N\phi_{p,n+1}(x,x_i^{n+1})\left(A^{-1}\right)_{ij}\phi_{0,q}(x_j^0,y) \tag{2.2.4}$$

$$A_{ij}=\phi_{0,n+1}(x_i^0,x_j^{n+1}),\quad \phi_{ij}(x,y)=p(x,y;\tau_j-\tau_i) \tag{2.2.5}$$

$$\varphi_{ij}(x,y)=\left(\phi_{i,i+1}*\dots*\phi_{j-1,j}\right)(x,y). \tag{2.2.6}$$

Remark 2.15. *We denote by $*$ the usual convolution operation between two or more kernels $w_1,\dots,w_k$ (see [68, Formula 2.7]):*

$$\left(w_1*\dots*w_k\right)(\xi,\eta):=\int_{\mathbb{R}^{k-1}}w_1(\xi,\xi_1)w_2(\xi_1,\xi_2)\dots w_k(\xi_{k-1},\eta)\mathrm{d}\xi_1\dots\mathrm{d}\xi_{k-1} \tag{2.2.7}$$

Once verified that this point process is indeed determinantal, our quantity of study (i.e. the gap probability) is simply a Fredholm determinant, as in the scalar case.

Theorem 2.16. *In the previous hypotheses stated in Theorems 2.13 and 2.14, given a col-*

lection of sets $E_1, \dots, E_n \subseteq \mathbb{R}$*, the gap probability of the process is*

$$P\left(X_i(\tau_k) \notin E_k, \forall\, i\ \forall\, k\right) = \det\left(\mathrm{Id} - [K_{ij}]\Big|_{E_1,\dots,E_n}\right) \tag{2.2.8}$$

with $[K_{ij}]$ *the matrix operator with kernel entries defined in (2.2.3)-(2.2.6).*

The study of gap probabilities of some relevant determinantal point process is the topic of the present thesis. Such investigation can be addressed into two directions: finding a system of differential equations that explicitly describe the gap probabilities themselves or studying their asymptotic behaviour in some critical scaling regime. As claimed in the introduction (Chapter 1), the common starting point for both of the goals is the formulation of a suitable Riemann-Hilbert problem as it will be explained in the following Chapters 3 and 4.

Chapter 3

Isomonodromic Theory and Integrable operators

In the present chapter we will review the main results on the Theory of Systems of Ordinary Differential Equations with rational coefficients and Isomonodromy Theory, that will be used in the following Chapters 5, 6 and 7. The main references are the book by Fokas, Its *et al.* [36] and the papers by the Japanese School [53], [54], [57].

We will show how it is possible to effectively study and explicitly calculate the gap probability of a determinantal point process via results borrowed from Isomonodromy Theory. The connection bridge is the Riemann-Hilbert formalism.

On one hand, the Riemann-Hilbert formalism is an integral part of the Monodromy Theory. It is fundamental for the study of the direct and inverse monodromy map, i.e. the map associating the set of so-called "monodromy" data to the set of singular data of a system of ordinary differential equations. The Hilbert's twenty-first problem, which is about the existence of a linear differential equation having a prescribed monodromic group, is commonly called Riemann-Hilbert problem, precisely for the massive use of the Riemann-Hilbert formalism in the developments of such problem.

On the other hand, we will see that for a particular class of integral operators (called "integrable" operators) there exist a natural Riemann-Hilbert formulation which will allow to study the variation of their Fredholm determinant. Many universal determinantal point processes are defined through correlation kernels that belong to such types of integral operator, therefore their gap probabilities (i.e. the Fredholm determinants of the respective integral operator) can be linked to a suitable Riemann-Hilbert problem and can be explicitly described in terms of a solution to a systems of ODE.

Generally speaking, we define a Riemann-Hilbert problem as a jump problem for piecewise analytic functions. Consider an oriented smooth contour $\mathcal{C}$ in the complex plane. The contour might have points of self-intersection and it might have more than one connected component. The orientation induces a $+$-side and a $-$-side on $\mathcal{C}$, where the $+$-side lies to the left and the $-$-side to the right if one traverses the contour according to the orientation. Suppose, in addition, that we are given a map $J : \mathcal{C} \to \mathrm{GL}_N(\mathbb{C})$, where $\mathrm{GL}_N(\mathbb{C})$ is the set of $N \times N$ invertible matrices. A Riemann-Hilbert problem consists in finding an $N \times N$ matrix-valued function $\Gamma = \Gamma(\lambda)$ with the following properties

- **analyticity**: Γ is analytic on the whole complex plane off $\mathcal{C}$.
- **jump**: the limit Γ_- of Γ from the minus side of $\mathcal{C}$ and the limit Γ_+ from the plus side of $\mathcal{C}$ are related by
$$\Gamma_+(\lambda) = \Gamma_-(\lambda)J(\lambda) \qquad \lambda \in \mathcal{C}. \tag{3.0.1}$$
- **normalization**: Γ tends to the identity matrix as $\lambda \to \infty$ (in general, it is possible to fix the value of Γ at a given point $z_0 \in \mathbb{C}\backslash\mathcal{C}$: $\Gamma(z_0) = \Gamma_0$, $\Gamma_0 \in \mathrm{GL}_N(\mathbb{C})$).

More details on specific Riemann-Hilbert problems will be given in this chapter, in particular in Section 3.3 and 3.4, and in the next Chapter 4.

3.1 Systems of ODEs with Rational Coefficients

We will state here the first results on existence of a solution to an ordinary differential equation with rational coefficient in the complex plane. The main reference will be [36, Chapter 1] The main purpose of this section is the introduction and definition of the "monodromy data" which will play an important role in the subsequent section on Isomonodromy Theory.

Let consider a first-order linear ODE in the complex plane

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = A(\lambda)\Psi(\lambda) \tag{3.1.1}$$

where $\Psi(\lambda)$ and $A(\lambda)$ are both $N \times N$ matrix-valued functions and, in particular, A is a meromorphic function with rational entries. The (local) behaviour of the solution Ψ near a given point $\lambda_0 \in \mathbb{C}P^1$ depends on the type of point we are considering. We have three types of scenario.

The first case occurs when the coefficient matrix $A(\lambda)$ is holomorphic at the point λ_0 and, if $\lambda_0 = \infty$, $A(\lambda)$ has a zero of second order or higher. In this case, existence of solutions can be easily proved

Theorem 3.1 ([41])**.** *Let $A(\lambda)$ be a $N \times N$ matrix-valued function holomorphic in a suitable neighbourhood B_{λ_0} of λ_0. Given a constant matrix Ψ_0, there exists a unique solution $\Psi(\lambda)$ of the equation (3.1.1), holomorphic in B_{λ_0} and such that $\Psi(\lambda_0) = \Psi_0$.*

The second case occurs when the coefficient matrix $A(\lambda)$ has a simple pole at λ_0 or, if $\lambda_0 = \infty$, $A(\lambda)$ has a simple zero. Let assume, for the sake of simplicity, that all the eigenvalues of the residue matrix $A_0 := \lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)A(\lambda)$ are distinct modulo $\mathbb{Z}\backslash\{0\}$, i.e.

$$P^{-1}A_0P = \Lambda_0, \qquad (\Lambda_0)_{ij} = \alpha_i\delta_{ij}, \qquad \alpha_i - \alpha_j \notin \mathbb{Z}\backslash\{0\}. \tag{3.1.2}$$

Then, the following result on existence of a solution holds.

Theorem 3.2 ([107])**.** *Given $A(\lambda)$ a $N\times N$ matrix-valued function holomorphic in a suitable punctured disk $B_{\lambda_0}\backslash\{\lambda_0\}$ and let $\lambda_0 \in \mathbb{C}P^1$ be a simple pole for $A(\lambda)$, in the generic non-resonant hypotesis (3.1.2), then there exists a fundamental solution $\Psi(\lambda)$ to the equation (3.1.1) of the form*

$$\Psi(\lambda) = \hat{\Psi}(\lambda)\xi^{\Lambda_0} \tag{3.1.3}$$

where $\hat{\Psi}(\lambda)$ is holomorphic and invertible in a B_{λ_0} and ξ is a local coordinate: $\xi := \lambda - \lambda_0$ if $\lambda_0 \in \mathbb{C}$ or $\xi := 1/\lambda$ if $\lambda_0 = \infty$.

Remark 3.3. *In the general case, where the eigenvalues of $A(\lambda)$ may coincide modulo a non-zero integer, the fundamental solution shows an extra term involving a constant nilpotent matrix determined by the eigenvalues themselves.*

The last case occurs when the coefficient matrix $A(\lambda)$ has a multiple pole at λ_0 or, if $\lambda_0 = \infty$, $A(\lambda_0)$ does not vanish; in particular, the order of the pole minus 1 is called Poincaré rank of the singularity λ_0. Let assume, again for the sake of simplicity, that the coefficient A_{-r} (where r is the Poincaré rank) of the leading order of singularity in the Laurent series of $A(\lambda)$ in the neighbourhood B_{λ_0} of λ_0 has distinct eigenvalues, i.e.

$$P^{-1}A_{-r}P = \Lambda_{-r}, \qquad \det P \neq 0, \qquad (\Lambda_{-r})_{ij} = \alpha_i\delta_{ij}, \quad \alpha_i \neq \alpha_j \text{ for } i \neq j. \tag{3.1.4}$$

Then, we have the following result.

Theorem 3.4 ([57], [107])**.** *Under the generic condition (3.1.4), there exists a unique formal* *fundamental solution $\Psi(\lambda)$ to the equation (3.1.1) of the form*

$$\Psi_f(\lambda) = P\left(\sum_{k=0}^{\infty} \Psi_k \xi^k\right) e^{\Lambda(\xi)}, \qquad \Psi_0 = I \tag{3.1.5}$$

$$\Lambda(\xi) = \sum_{k=-r}^{-1} \frac{\Lambda_k}{k} \xi^k + \Lambda_0 \ln \xi \tag{3.1.6}$$

where all the matrices Λ_k, $k = -r, \ldots, 0$, are diagonal and Λ_{-r} is the Jordan form of the coefficient matrix A_{-r}; moreover, all the coefficients Ψ_k and the diagonal exponent $\Lambda(\xi)$ can be determined recursively from the Laurent expansion of $A(\lambda)$ in the neighbourhood B_{λ_0}.

In general, the series in (3.1.5) does not converge (thus the denomination "formal"). Nevertheless, it can be interpreted as the asymptotics of a genuine fundamental solution of (3.1.1) as $\lambda \to \lambda_0$ along any path belonging to specific sectors Ω of B_{λ_0}, called Stokes sectors, which will ensure uniqueness of of the solution:

$$\Psi(\lambda) \sim \Psi_f(\lambda) \qquad \lambda \to \lambda_0, \quad \lambda \in \Omega. \tag{3.1.7}$$

Remark 3.5. *By asymptotic behaviour, we mean the following (see [107]). Let $\Psi_f(\lambda) := \sum_{k=0}^{\infty} \Psi_k \lambda^k$ be a formal power series. We say that Ψ_f is the asymptotic series (or expansion) of the function $\Psi(\lambda)$ at a point λ_0, i.e.*

$$\Psi(\lambda) \sim \Psi_f(\lambda), \qquad \lambda \to \lambda_0, \quad \lambda \in \Omega \tag{3.1.8}$$

being Ω a given subset of $\mathbb{C}P^1$, if for every positive number $m \in \mathbb{N}$ there exists a positive constant $C_{m,\Omega}$ such that

$$\left\| \Psi(\lambda) - \sum_{k=0}^{m-1} \Psi_k \lambda^k \right\| \leq C_{m,\Omega} |\lambda - \lambda_0|^m \tag{3.1.9}$$

for every λ within a compact subset $\Omega' \subset \Omega$.

A neighbourhood of a singular point λ_0 with Poincaré rank r can always be covered by $2r$ different Stokes sectors, in a canonical way. For sufficiently small $\delta > 0$ any sector of the form

$$\Omega = \{\, \xi \in \mathbb{C} \mid 0 < |\xi| < \rho,\ \theta_1 < \arg \xi < \theta_2 \,\} \qquad \theta_2 - \theta_1 = \frac{\pi}{r} + \delta \tag{3.1.10}$$

is a Stokes sector; then, each of the sectors

$$\Omega_n = \{ \xi \in \mathbb{C} \mid 0 < |\xi| < \rho, \ \theta_1 + \frac{\pi}{r}(n-1) < \arg \xi < \theta_2 + \frac{\pi}{r}(n-1) \}$$

$$= e^{i\frac{\pi}{r}(n-1)}\Omega \quad n = 1, \dots, 2r \tag{3.1.11}$$

$$\Omega_1 = \Omega_{2r+1} = \Omega \tag{3.1.12}$$

is a Stokes sector as well. Therefore, we can associate to a given formal solution (3.1.5) $2r$ genuine solutions Ψ_j, $j = 1, \dots, 2r$ (one for each Stoke sector) such that

$$\Psi_j(\lambda) \sim \Psi_f(\lambda) \qquad \lambda \to \lambda_0, \quad \lambda \in \Omega_j, \quad j = 1, \dots, 2r. \tag{3.1.13}$$

Moreover, these solutions differ by a non-trivial matrix S_j (Stokes matrix) whenever two consecutive Stokes sectors overlap

$$S_j := \Psi_j^{-1}(\lambda)\Psi_{j+1}(\lambda) \qquad j = 1, \dots, 2r. \tag{3.1.14}$$

For more details on the Stokes phenomenon we refer to [36] and [107].

Theorems 3.1, 3.2 and 3.4 guarantee the existence of a local solution to the system (3.1.1) in a neighbourhood of either a regular point or a singular point. We will now state the fundamental Monodromy Theorem, which will allow us to build a global solution to a given system of ordinary differential equations with rational coefficients, and we will then focus on the concept of monodromy data.

Given a linear ODE of the form (3.1.1), let denote $\alpha_\nu \in \mathbb{C}P^1$, $\nu = 1, \dots, m$, the poles of the coefficient matrix $A(\lambda)$. Given a curve $\gamma : [0, 1] \to \mathbb{C}P^1 \backslash \{a_\nu\}_{\nu=1,\dots,m}$, $t \mapsto \gamma(t)$, the following result holds.

Theorem 3.6 (Monodromy theorem, [41])**.** *Let $\Psi(\lambda) = \sum_{k=0}^{\infty} \Psi_k \xi^k$ be the germ of a solution of equation (3.1.1) at the point $a = \gamma(0)$. Then, $\Psi(\lambda)$ can be analytically continued along γ to the point $b = \gamma(1)$, the continuation depending only on the homotopy class of γ.*

Even more,

Corollary 3.7. Let $\Psi(\lambda)$ be a germ of solution of equation (3.1.1), then $\Psi(\lambda)$ can be analytically continued on the universal covering of the punctured Riemann sphere $\mathbb{C}P^1 \backslash \{a_\nu\}_{\nu=1,\dots,m}$.

Recapping all the results described so far, given a linear ODE as (3.1.1), to each of the singular points of $A(\lambda)$ we can associate a certain set of data. In particular, we have

- the formal monodromy exponent $\Lambda_0^{(\nu)}$ (formula (3.1.3)), which may be paired with a nilpotent matrix in the general case, as in Remark 3.3, if a_ν is a simple pole;

- the Stokes phenomenon (formulæ (3.1.6)-(3.1.14))

$$\mathcal{S}^{(\nu)} := \left\{\Lambda_{-r}^{(\nu)}, \ldots, \Lambda_{-1}^{(\nu)}, \Lambda_0^{(\nu)}; S_1, \ldots, S_{2r}\right\} \tag{3.1.15}$$

where S_j are the Stokes matrices, if a_ν is a singular point with Poincaré rank r.

All these quantities are called "local monodromy data" for each singular point a_ν.

If we fix a point $a_0 \in \mathbb{C}P^1 \backslash \{a_\nu\}_{\nu=1,\ldots,m}$ and matrix $\Psi_0 \in \mathrm{GL}_N(\mathbb{C})$ as initial condition to the equation (3.1.1) $\Psi(a_0) = \Psi_0$, then Theorems 3.2 and 3.4 it is possible to build a local solution to (3.1.1) in a neighbourhood of each singular point a_ν and, thanks to the Monodromy Theorem 3.6, each of these solutions admits analytic continuation on the universal covering of $\mathbb{C}P^1 \backslash \{a_\nu\}$. Therefore, all these solutions must differ from each other by a constant (right) matrix multiplier called "connection matrix", i.e. denoting $\Psi(\lambda)$ the fundamental solution determined by the initial condition $\Psi(a_0) = \Psi_0$, then

$$\Psi(\lambda) = \Psi^{(\nu)}(\lambda) C_\nu = \left(\hat{\Psi}(\lambda)\xi_\nu^{\Lambda_0}\right) C_\nu \quad \text{if } a_\nu = \text{ simple pole} \tag{3.1.16}$$

$$\Psi(\lambda) = \Psi_1^{(\nu)}(\lambda) C_\nu \qquad \text{if } a_\nu = \text{ sing. point with rank } r \tag{3.1.17}$$

where $\Psi_1^{(\nu)}(\lambda) \sim \Psi_f^\nu(\lambda)$ (as $\lambda \to a_\nu$) is the canonical solution in the first Stokes sector Ω_1 and C_ν is the connection matrix.

Definition 3.8. Given a linear ODE (3.1.1) with $a_1, \ldots, a_p$ simple poles and $a_{p+1}, \ldots, a_m$ singular point with Poincaré rank r_k ($k = p+1, \ldots, m$), the **global monodromy data** is the set of the following data

$$\mathcal{M} := \left\{a_1, \ldots, a_m; \Lambda_0^{(1)}, \ldots, \Lambda_0^{(p)}; \mathcal{S}^{(p+1)}, \ldots, \mathcal{S}^{(m)}; C_1, \ldots, C_m\right\}. \tag{3.1.18}$$

The global monodromy data $\mathcal{M}$ completely characterizes the global behaviour of the solutions of a linear system of the type (3.1.1) and determines uniquely the system itself.

3.2 Isomonodromic Deformations

In this section we will briefly illustrate the theory of Isomonodromic deformation. We will report only the main facts that are stategic for our purposes (see Chapters 5, 6 and 7), since the subject is very wide. We refer to the triad of papers [53], [54], [57] as well as to the book [36, Chapter 4] for an exhaustive exposition.

Consider again a linear ODE with rational coefficients of the form

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi(\lambda) \tag{3.2.1}$$

Let assume that the coefficient matrix $A(\lambda)$ depends holomorphically on certain additional parameters $s_1, \ldots, s_q$, called "times", belonging to some parameter space $S \subseteq \mathbb{C}$:

$$A(\lambda) = A(\lambda, \vec{s}) \qquad \vec{s} = (s_1, \ldots, s_q), \quad q \geq 1, \quad \vec{s} \in S \tag{3.2.2}$$

Definition 3.9. The holomorphic family (3.2.2) is an **admissible deformation** of the linear equation (3.2.1) if

1. the number of singular points does not depend on the times: $a_\mu(\vec{s}) \neq a_\nu(\vec{s})$, $\forall\ \mu, \nu\ \forall\ \vec{s}$. Moreover, we require the points to be always separable:

$$\exists\ \text{ disks}\,\{B_\nu\} \quad \text{such that}\quad a_\nu(\vec{s}) \in B_\nu\ (\forall \vec{s}), \quad \overline{B_\mu} \cap \overline{B_\nu} = \emptyset \quad \mu \neq \nu; \tag{3.2.3}$$

2. the type of Jordan form of the leading coefficient of the Laurent series of the matrices $A(\lambda, \vec{s})$ at each singular point $a_\nu(\vec{s})$ does not depend on $\vec{s} \in S$;

3. at each singular point $a_\nu(\vec{s})$ with rank r_ν, the set of Stokes sectors $\{\Omega_n^{(\nu)}\}_{n=1,\ldots,2r_\nu}$ is holomorphically equivalent under the map $\lambda \mapsto \lambda - a_\nu(\vec{s})$;

4. canonical solutions are holomorphic with respect to the times and the asymptotic condition (3.1.5) at a singular point with positive rank holds uniformly with respect to $\vec{s}$.

Moreover, we highlight the following important class of deformations.

Definition 3.10. An admissible deformation is called **isomonodromic deformation** if the set of canonical solutions can be chosen in such a way that

- the formal monodromy exponents Λ_0^ν,
- the Stokes matrices $\mathcal{S}_n^{(\nu)}$,
- the connection matrices C_ν

are independent on the times $\vec{s} \in S$.

The fundamental fact about isomonodromic deformations is that they can be described by suitable systems of nonlinear differential equations.

Consider the logarithmic derivative with respect to times

$$U(\lambda,\vec{s}) := \mathrm{d}\Psi\Psi^{-1}, \qquad \mathrm{d}\Psi = \sum_j \frac{\partial\Psi}{\partial s_j}\mathrm{d}s_j. \tag{3.2.4}$$

Without lost of generality let assume $a_m = a_\infty = \infty$. Then, the following result holds.

Theorem 3.11. *The differential form $U(\lambda,\vec{s})$ (3.2.4) is a rational matrix-valued function with respect to the variable λ. Its poles coincide with the singular points a_ν ($\nu = 1,\dots,m-1,\infty$) and $U(\lambda,\vec{s})$ is determined uniquely and explicitly as a differential matrix-valued form on the manifold $\mathcal{A}$ of the linear systems having the given number m of singularities (with given Poincaré rank r_ν at each point):*

$$U(\lambda) = U\left(\lambda;\{A_j^{(\nu)}\},\{a_\nu\}\right) \tag{3.2.5}$$

where the matrices $A_j^{(\nu)}$ are the coefficients of the decomposition of the rational function $A(\lambda)$ over its principal part, i.e.

$$A(\lambda) = A^{(\infty)}(\lambda) + \sum_{\nu=1}^{m-1} A^{(\nu)}(\lambda) \tag{3.2.6}$$

$$A^{(\nu)}(\lambda) = \sum_{k=1}^{r_\nu+1}(\lambda - a_\nu)^{-k}A_{-k+1}^{(\nu)} \qquad \nu = 1,\dots,m-1 \tag{3.2.7}$$

$$A^{(\infty)}(\lambda) = -\sum_{k=0}^{r_\infty-1}\lambda^k A_{-k-1}^{(\infty)} \qquad \textit{if}\ \ r_\infty > 0 \tag{3.2.8}$$

$$A^{(\infty)}(\lambda) \equiv 0 \qquad \textit{if}\ \ r_\infty = 0. \tag{3.2.9}$$

Therefore, the function $\Psi(\lambda,\vec{s})$, in addition to the basic λ-equation (3.2.1), also satisfies an auxiliary linear system with respect to the parameters $\vec{s}$:

$$\mathrm{d}\Psi(\lambda) = U(\lambda,\vec{s})\Psi \tag{3.2.10}$$

or, equivalently,

$$\frac{\partial\Psi}{\partial s_j} = U_j(\lambda)\Psi \tag{3.2.11}$$

given $U(\lambda) = \sum_j U_j(\lambda)\mathrm{d}s_j$.

Cross-differentiating the overdetermined system (3.2.1)-(3.2.10), we find the following compatibility conditions, also called “zero curvature equations”:

$$\mathrm{d}A - \frac{\partial U}{\partial \lambda} + [A, U] = 0 \tag{3.2.12}$$

identically in λ. Since the above equation is a rational function in λ, it is possible to find a finite system of non-linear differential equations for the matrix coefficients $A_k^{(\nu)} = A_k^{(\nu)}(\vec{s})$ by simply equating to zero the corresponding principal parts. The system obtained is also called **isomonodromic deformation equations**.

The equation (3.2.12) is an overdetermined system of nonlinear differential equations. It is possible to show (see [57]) that such system is integrable in the sense of Frobenius and that the largest independent set of deformation parameters $\vec{s}$ can be chosen as the set of singular points $a_1, \ldots, a_{m-1}$ plus the matrix entries of the diagonal matrices $\Lambda_k^{(\nu)}$ ($k = 1, \ldots, r_\nu > 0$, $\nu = 1, \ldots, m-1, \infty$).

Moreover, in the overdetermined system

$$\frac{\partial \Psi}{\partial \lambda} = A(\lambda)\Psi \qquad\qquad \mathrm{d}\Psi = U(\lambda)\Psi \tag{3.2.13}$$

we can recognize a Lax representation for the nonlinear system (3.2.12), thus linking the theory of isomonodromy deformations to the Soliton Theory (see e.g. [30]).

The use of the Lax pair above in the analysis of the solutions of the isomonodromic deformation equations (3.2.12) follows from the fact that the monodromy data of the λ-equation, i.e.

$$\{\, \Lambda_0^{(\nu)}; \mathcal{S}^{(\nu)}, E_\nu \,\}\,, \tag{3.2.14}$$

forms a complete set of first integrals of the system (3.2.12). Therefore, the problem of the integration of the nonlinear equations (3.2.12) is reduced to the analysis of the direct and inverse monodromy maps of the system (3.2.1), i.e. the map that associates to the ODE (3.2.1) the global monodromy data $\mathcal{M}$ (3.1.18) and viceversa.

The τ-function

To each solution of the deformation equations (3.2.12) it is possible to canonically associate the following 1-form, introduced for the first time by M. Jimbo, T. Miwa and K. Ueno in

[57]:

$$\begin{aligned}\omega &:= \sum_{\nu=1,\ldots,m-1,\infty} \omega_\nu \\ \omega_\nu &:= -\operatorname*{res}_{\lambda=a_\nu} \operatorname{Tr}\left(\Psi^{(\nu)}(\lambda)^{-1}\frac{\partial\Psi^{(\nu)}}{\partial\lambda}(\lambda)\mathrm{d}\Lambda^{(\nu)}(\lambda)\right)\end{aligned} \tag{3.2.15}$$

where $\Lambda^{(\nu)}$ is defined in (3.1.6) and d is the exterior differentiation with respect to the times $\vec{s}$.

The fundamental property of the 1-form ω is the following theorem.

Theorem 3.12 ([57]). *For any solution of the isomonoromic deformation equations (3.2.12), the 1-form (3.2.15) is closed* $\mathrm{d}\omega = 0$.

Therefore, there exists a scalar function of the deformation parameteres τ satisfying

$$\omega = \mathrm{d}\ln\tau. \tag{3.2.16}$$

Moreover, the 1-form ω enjoys the Painlevé property (the only movable singularities are poles), which in turns translates to the fact that the τ function is holomorphic everywhere on the universal covering manifold of $\mathbb{C}^q\backslash\mathcal{V}$, where q is the number of deformation parameters and $\mathcal{V}$ is the set of following critical varieties

$$a_\nu(\vec{s}) = a_\mu(\vec{s}) \qquad \text{for some } \mu \neq \nu,\ \text{for some } \vec{s} \tag{3.2.17}$$

$$\alpha_i^{(\nu)} - \alpha_j^{(\nu)} \in \mathbb{Z}\backslash\{0\} \qquad \text{if } \ a_\nu = \text{ simple pole} \tag{3.2.18}$$

$$\alpha_i^{(\nu)} = \alpha_j^{(\nu)} \qquad \text{if } \ a_\nu = \text{ sing. point with rank } r_\nu \tag{3.2.19}$$

which we excluded at the beginning of our exposition: see (3.1.2) and (3.1.4) together with point 1. in Definition 3.9. We refer to the article by T. Miwa [86] for a detailed proof of the above facts.

3.3 Integrable kernels

In this section, we temporarily detach from the subject of Monodromy Theory and we analyze a special class of integral operators, with a collection of curves in the complex plane as domain. Such operators are called "integrable operators" and they were first introduced by A. R. Its, A. G. Izergin, V. E. Korepin and N.A. Slavnov in the paper [50].

Its, Izergin, Korepin and Slavnov developed their theory to establish a connection between certain Fredholm determinants representing quantum correlation functions for Bose gas and the Painlevé V equation.

The peculiarity of these Its-Izergin-Korepin-Slavnov (IIKS) operators resides in the fact that their solvability, i.e. the existence of the operator $(\mathrm{Id} - \mathbf{K})^{-1}$ ($\mathbf{K}$ being an IIKS operator), is equivalent to solving a suitable boundary-value Riemann-Hilbert problem in the complex plane. Thanks to the Jacobi formula (see (3.4.1)), it will be possible to study the Fredholm determinant of the operator $\mathbf{K}$ through the Riemann-Hilbert problem constructed above, leading the way to a powerful connection between the gap probability (i.e. the Fredholm determinant) and the theory of isomonodromy deformations, as it will later be explained.

We refer to the original paper [50] as well as to the paper [45] to review the concepts of IIKS operators and their application to the present case.

Consider a $p \times p$ matrix Fredholm integral operator acting on $\mathbb{C}^p$-valued functions $\phi(\lambda)$,

$$\mathbf{K}(\phi)(\lambda) = \int_{\Sigma} K(\lambda, \mu)\phi(\mu)\,\mathrm{d}\mu \tag{3.3.1}$$

defined along a piecewise smooth, oriented curve Σ in the complex plane (possibly extending to ∞), with integral kernel of the special form

$$K(\lambda, \mu) = \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda - \mu} \tag{3.3.2}$$

where $\mathbf{f}, \mathbf{g}$ are rectangular $r \times p$ matrix valued functions, $p < r$. The most common case is $p = 1$, $r = 2$ defining a scalar integral operator $\mathbf{K}$. Let assume that $\mathbf{f}$ and $\mathbf{g}$ are smooth functions along the connected components of Σ, such that

$$\mathbf{f}^T(\lambda)\mathbf{g}(\lambda) = 0, \tag{3.3.3}$$

in order to ensure that $\mathbf{K}$ is nonsingular and the diagonal values are given by $K(\lambda, \lambda) = \mathbf{f}'^T(\lambda)\mathbf{g}(\lambda) = -\mathbf{f}^T(\lambda)\mathbf{g}'(\lambda)$.

For the sake of simplicity, let also assume that the functions $\mathbf{f}$ and $\mathbf{g}$ can be analytically continued to a neighborhood of each of the connected components of Σ.

Fredholm determinants of some operators of this type appear as eigenvalue distributions for random matrix ensembles ([85], [98], [99]), as in the case at hand, and as generating functions for correlators in many integrable quantum field theory models ([50], [56]).

An crucial observation is that the resolvent operator

$$\mathbf{R} := (\mathrm{Id} - \mathbf{K})^{-1}\mathbf{K} \tag{3.3.4}$$

is also in the same class., i.e. $\mathbf{R}$ may also be expressed as an integral operator of the form (3.3.1)-(3.3.2)

$$\mathbf{R}(\mathbf{v})(\lambda) = \int_\Gamma R(\lambda,\mu)\mathbf{v}(\mu)\,\mathrm{d}\mu \tag{3.3.5}$$

$$R(\lambda,\mu) := \frac{\mathbf{F}^T(\lambda)\mathbf{G}(\mu)}{\lambda-\mu} \tag{3.3.6}$$

where the matrix-valued functions $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F}^T = (\mathrm{Id}-\mathbf{K})^{-1}\mathbf{f}^T = (\mathrm{Id}+\mathbf{R})\mathbf{f}^T \tag{3.3.7}$$

$$\mathbf{G} = \mathbf{g}(\mathrm{Id}-\mathbf{K})^{-1} = \mathbf{g}(\mathrm{Id}+\mathbf{R}), \tag{3.3.8}$$

the operator $(\mathrm{Id}-\mathbf{K})^{-1}$ acting to the right in the first line and to the left in the latter. Similarly, these quantities satisfy the non-singularity condition $\mathbf{F}^T(\lambda)\mathbf{G}(\lambda) = 0$, so that the resolvent may be defined for diagonal values as well: $R(\lambda,\lambda) = \mathbf{F}'^T(\lambda)\mathbf{G}(\lambda) = -\mathbf{F}^T(\lambda)\mathbf{G}'(\lambda)$.

Given such an integrable operator $\mathbf{K}$ (3.3.1)-(3.3.2), it turns out that that determining its resolvent $\mathbf{R}$ is equivalent to solving a Riemann-Hilbert problem. Let start by defining the following $r \times r$ matrix valued function $\chi(\lambda)$

$$\chi(\lambda) := \mathbf{I}_r + \int_\Sigma \frac{\mathbf{F}(\mu)\mathbf{g}^T(\mu)}{\lambda-\mu}\,\mathrm{d}\mu \tag{3.3.9}$$

with $\mathbf{F}$ given in (3.3.7). By construction, $\chi(\lambda)$ is analytic on the complement of Σ and extends to infinity off Σ, with asymptotic expansion

$$\chi(\lambda) = \mathbf{I}_r + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty \tag{3.3.10}$$

$$\chi(\lambda) = \mathcal{O}\left(\ln(\lambda-\alpha)\right) \quad \text{as } \lambda \to \alpha \tag{3.3.11}$$

with α any endpoint of a connected component of Σ. Moreover, it is easy to see that the function $\chi(\lambda)$ has jump discontinuities across Σ given by

$$\chi_+(\lambda) = \chi_-(\lambda)J(\lambda) \quad \lambda \in \Sigma \tag{3.3.12}$$

where χ_+ and χ_- are the limiting values of χ as Σ is approached from the left and the right, respectively, according to the orientation of Σ. The $r \times r$ invertible jump matrix $J(\lambda)$ is defined as the following rank-p perturbation of the identity matrix $\mathbf{I}_r$:

$$J(\lambda) := \mathbf{I}_r - 2\pi i\mathbf{f}(\lambda)\mathbf{g}^T(\lambda). \tag{3.3.13}$$

Collecting the considerations above, we can state that if the operator Id $-\mathbf{K}$ is invertible, then the function (3.3.9) defines a solution to a Riemann-Hilbert problem withs jump condition (3.3.12)-(3.3.13) and asymptotic behaviour (3.3.10)-(3.3.11). Furthermore, it can be proved that the converse is also true: if there exists a matrix-valued function χ solution to the Riemann-Hilbert problem (3.3.10)-(3.3.13), then χ can be written in the form (3.3.9) and Id $-\mathbf{K}$ is invertible.

In conclusion, there exists equivalence between the inversion of the operator Id $-\mathbf{K}$, with $\mathbf{K}$ an integrable kernel in the sense of Its-Izergin-Korepin-Slavnov (3.3.1)-(3.3.2), and the solution to the Riemann-Hilbert problem (3.3.10)-(3.3.13). The general theorem states as follow.

Theorem 3.13 (Section 1.2, [45])**.** *Consider an IIKS integrable operator* $\mathbf{K}$ *defined on a collection of orientend contours* Σ*. The operator* Id $-\boldsymbol{K}$ *is invertible if and only if there exist a solution to the following Riemann-Hilbert problem: find a matrix valued function* Γ *such that*

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)J(\lambda) \qquad \lambda \in \Sigma \tag{3.3.14}$$

$$\Gamma(\lambda) = \mathbf{I}_r + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty \tag{3.3.15}$$

$$\Gamma(\lambda) = \mathcal{O}\left(\ln(\lambda - \alpha)\right) \qquad \lambda \to \alpha \tag{3.3.16}$$

with α *any endpoint of a connected component of* Σ *and the jump matrix*

$$J(\lambda) := \mathbf{I}_r - 2\pi i\, \boldsymbol{f}(\lambda)\boldsymbol{g}(\lambda)^T. \tag{3.3.17}$$

Moreover, the resolvent of the operator $\mathbf{K}$ *will be an integral operator of IIKS form as well, with kernel*

$$R(\lambda,\mu) = \frac{\boldsymbol{f}(\lambda)^T\Gamma(\lambda)^T\left(\Gamma^{-1}(\mu)\right)^T\boldsymbol{g}(\mu)}{\lambda-\mu}. \tag{3.3.18}$$

3.4 Fredholm determinants as Isomonodromic τ functions

The purpose of the coming section is to get to the core of the connection between gap probabilities of determinantal point processes and integrable systems. The linking ring is precisely the theory of Riemann-Hilbert problem and IIKS operators that we introduced above.

The IIKS theory has been lately used extensively in the theory of random matrices and random processes. Indeed these are some of its general features:

- The Riemann-Hilbert problem typically has jumps which are conjugated to constant jumps, therefore the solution of the Riemann-Hilbert problem solves an ODE with meromorphic coefficients (connecting to the theory of isomonodromic deformations);

- in some interesting cases, the Fredholm determinant coincides with the isomonodromic τ function of Jimbo, Miwa and Ueno.

First of all we recall a basic deformation formula that relates the Fredholm determinant to the resolvent operator called Jacobi formula for the variation of determinants

$$\mathrm{d} \ln \det(\mathrm{Id} - \mathbf{K}) = - \operatorname{Tr}\left((\mathrm{Id} - \mathbf{K})^{-1} \mathrm{d}\mathbf{K} \right) = \mathrm{d}\left((\mathrm{Id} + \mathbf{R})\, \mathrm{d}\mathbf{K} \right) \tag{3.4.1}$$

where d is the differential with respect to any auxiliary parameters on which $\mathbf{K}$ may depend.

Such relation is the key formula that will allow to describe gap probabilities of determinantal processes in terms of explicit quantities that will have a precise geometric meaning.

We will show that the Fredholm determinant of a given IIKS integrable kernel, thought of as a function of a set of parameters, is a τ function (in the sense of [53], [54], [57]) of the corresponding isomonodromy problem. In other words, it can be expressed through a solution of a system of differential equations, which is completely integrable.

We refer to the papers [9] and [11] for a thorough exposition on the connection between Fredholm determinants and isomonodromic τ function. We will report here only the principal facts that will be functional to the present thesis.

We start by considering a general notion of τ function associated to any Riemann-Hilbert problem (RHP) depending on parameters and which will reduce to that of Jimbo-Miwa-Ueno ([53], [54], [57]) in case such a Riemann-Hilbert problem coincides with the one associated to a rational ODE.

Consider a Riemann-Hilbert problem defined on a collection of oriented contours Σ and depending on additional deformation parameters $\vec{s} \in S$. For the sake of simplicity, we assume that the contours are either loops or they extends to infinity, so that there are no endpoints.

$$\Gamma_+(\lambda; \vec{s}) = \Gamma_-(\lambda; \vec{s}) J(\lambda; \vec{s}) \quad \lambda \in \Sigma \tag{3.4.2}$$

$$\Gamma(\lambda; \vec{s}) = \mathbf{I}_r + \mathcal{O}\left(\frac{1}{\lambda} \right) \quad \lambda \to \infty \tag{3.4.3}$$

where the jump matrix $J(\lambda; \vec{s}) : \Sigma \times S \to SL_r(\Sigma)$ is a suitably smooth functions of λ and $\vec{s}$.

On the space of deformation parameters S, we introduce the following one-form

$$\omega(\partial) := \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_\partial(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \tag{3.4.4}$$

$$\Xi_\partial(\lambda) := \partial J(\lambda) J^{-1}(\lambda) \tag{3.4.5}$$

where $'$ is the derivative with respect to λ (the dependence on $\vec{s}$ is implicit).

We point out that the definition of ω is valid for arbitrary jump matrices. In the case of the Riemann-Hilbert problem built up from an IIKS integrable kernel, where the jump matrix reads

$$J(\lambda, \vec{s}) = \mathbf{I}_r - 2\pi i \mathbf{f}(\lambda, \vec{s})\mathbf{g}^T(\lambda, \vec{s}), \tag{3.4.6}$$

we can advance our study of such one-form and we will be able to relate it to a Fredholm determinant up to a certain explicit correction term.

Theorem 3.14 (Theorem 2.1, [11])**.** *Let $\boldsymbol{f}(\lambda;\vec{s})$, $\boldsymbol{g}(\lambda;\vec{s}) : \Sigma \times S \to \mathrm{Mat}_{r\times p}(\mathbb{C})$ be sufficiently smooth functions and consider the Riemann-Hilbert problem (3.4.2)-(3.4.3) with jump matrix (3.4.6). Given any vector field ∂ in the space of the parameters S, the following equality holds*

$$\omega(\partial) = \partial \ln \det(\mathrm{Id} - \boldsymbol{K}) - H(J) \tag{3.4.7}$$

where $\mathbf{K}$ is the IIKS integrable operator with kernel

$$K(\lambda, \mu) = \frac{\boldsymbol{f}^T(\lambda)\boldsymbol{g}(\mu)}{\lambda - \mu} \qquad \lambda, \mu \in \Sigma \tag{3.4.8}$$

and the correction term is

$$H(J) := \int_\Sigma \left(\partial \boldsymbol{f}'^{\,T} \boldsymbol{g} + \boldsymbol{f}'^{\,T} \partial \boldsymbol{g}\right) \mathrm{d}\lambda - 2\pi i \int_\Sigma \boldsymbol{g}^T \boldsymbol{f}' \partial \boldsymbol{g}^T \boldsymbol{f} \mathrm{d}\lambda. \tag{3.4.9}$$

Proof. The result follows from the use of the Jacobi formula applied to the specific case of an IIKS integrable kernel, where the definitions of both the kernel and the resolvent are explicit in terms of the Riemann-Hilbert problem (3.3.10)-(3.3.13). □

On the other hand, it is possible to show that ω is also the logarithmic total differential of the isomonodromic τ-function introduced by Jimbo, Miwa and Ueno, in the case when the Riemann-Hilbert problem corresponds to a rational ODE.

Theorem 3.15 (Theorem 5.1 and Proposition 5.1, [9])**.** *The one-form ω restricted to the (sub)-manifold of isomonodromic deformations is closed and coincides with the Jimbo-Miwa-Ueno differential ω_{JMU} ([53, 54, 57]).*

It is thus possible to define, up to a nonzero multiplicative constant, the isomonodromic τ-function

$$\tau_{JMU} = \exp\left\{\int \omega\right\}. \tag{3.4.10}$$

In the special case where the extra term $H(J) \equiv 0$, the connection between Fredholm determinant and τ-function becomes linear and explicit.

Corollary 3.16. In the same hypotheses of Theorems 3.14 and 3.15, if $H(J) \equiv 0$, then the isomonodromic τ-function coincides with the Fredholm determinant of the IIKS integrable operator $\mathbf{K}$

$$\tau_{JMU} = \det(\mathrm{Id} - \mathbf{K}). \tag{3.4.11}$$

The above powerful results have been applied to several settings, where well-known IIKS integrable operators arose in the description of certain universal behaviours in Random Matrix Theory, self-avoiding random walks or growing models. The first applications were originally carried out on the Airy process and the Pearcey process by M. Bertola and M. Cafasso ([11], [10]). The present thesis deals with other well-known (universal) processes, which will be later described in details, namely the Bessel (Chapter 5), the Generalized Bessel (Chapter 6) and tacnode processes (Chapter 7).

Chapter 4

Asymptotic Analysis. The Steepest Descent Method

Over the last three decades, the nonlinear Steepest Descent Method for the asymptotic analysis of Riemann-Hilbert problems has been successfully applied to prove rigorous results on long time, long range and semiclassical asymptotics for solutions of completely integrable equations and correlation functions of exactly solvable models ([22], [23], [63], [64], [66]), asymptotics for orthogonal polynomials of large degree ([19], [20]), the eigenvalue distribution of random matrices of large dimension and related universality results ([21]), important results in combinatorial probability ([6]).

A preliminary application of the stationary phase idea was first performed on a Riemann-Hilbert problem related to a nonlinear integrable equation by Its in [51], but the method became systematic and rigorous in the work of Deift and Zhou [24] and [25].

In analogy to the linear stationary-phase and steepest-descent methods (see for example [1, Section 6]), where one asymptotically reduces an exponential integral to another which can be exactly evaluated up to a small error term, in the nonlinear case one asymptotically reduces the given Riemann-Hilbert problem to an exactly solvable one up to a small error term as well. On the other hand, the nonlinear asymptotic theory shows an extra feature which is peculiar of this method, i.e. the Lax-Levermore variational problem ([79]), closely related to the so-called "g-function", which is crucial in many situations in order to transform Riemann-Hilbert problems into others which can be solved in exact form.

The steepest descent method that will be used in the present work (Chapter 7) is the original version of the non-linear Deift-Zhou method, where the use of the g-function is not needed. However, all the main ingredients are present: identifying stationary points, deforming contours to contours of steepest descent and approximating the original problem with a solvable one.

We refer to [65] for an introductory description of the theory and to [1, Section 6] for the first results on the linear steepest descent method. We will develop here only the main ideas of the nonlinear method, which will be used later, while we refer to the original paper by Deift and Zhou [24] as main reference. Nevertheless, we will first recall some guidelines on the linear method, in order to give some motivations and a general idea of asymptotics evaluation and approximations of quantities in certain critical regimes.

4.1 The linear method

Consider the following integral

$$I(k) = \int_C f(z)e^{kV(z)}\mathrm{d}z \tag{4.1.1}$$

where C is a contour in the complex z-plane and $f(z)$, $V(z)$ are sufficiently smooth functions (for the sake of simplicity we can assume them to be analytic), $V(z)$ decaying at infinity sufficiently fast so as to guarantee the convergence of the integral. We are interested in evaluating its asymptotic behaviour as $k \to +\infty$, in particular the order of magnitude at which $I(k)$ vanishes for large k.

A motivation for such study comes from the analysis of solutions to differential equations which are given in closed form as an exponential integral. A well-known example is the solution to the Schrödinger equation of a free particle

$$i\psi_t + \psi_{xx} = 0 \tag{4.1.2}$$

$$\psi(x,t) = \int_{\mathbb{R}} \hat{\psi}_0(\xi)e^{i\xi x - i\xi^2 t}\frac{\mathrm{d}\xi}{2\pi} \tag{4.1.3}$$

where $\hat{\psi}_0(\xi)$ is the Fourier transform of the initial data $\psi(x,0) = \psi_0(x)$. Although such integral provides the exact solution, its true content is not very explicit. In order to better understand the properties of the solution, it may be useful to study its behaviour for large time t or for large space variable x; frequently, the interesting limit is $t \to \infty$ with ratio x/t fixed.

The basic idea to evaluate (4.1.1) is to deform the given contour C, using the fact that the integrand functions are analytic, into a new contour $\widetilde{C}$ such that the path $\widetilde{C}$ passes through a point z_0 for which $V'(z_0) = 0$ (saddle point) and the phase has constant imaginary part $\Im(V) =$ constant on $\widetilde{C}$. Thanks to this deformation, we are now dealing with an integral which can be analyzed directly, using the Laplace method (see [1, Chapter 6.2.3]) and we can recover asymptotics valid to all orders. Indeed, performing a suitable change of variables at the stationary point z_0, one can prove that the major contribution to the integral is given by the points that are near z_0.

Theorem 4.1 (see Section 6.4.1, [1]). *Consider the integral (4.1.1) and assume that the contour C can be deformed into a contour $\widetilde{C}$ passing through the saddle point z_0 of order $n-1$, i.e.*

$$\left.\frac{\mathrm{d}V}{\mathrm{d}z^j}\right|_{z=z_0} = 0, \qquad \forall j = 1, \ldots, n-1 \tag{4.1.4}$$

$$\left.\frac{\mathrm{d}V}{\mathrm{d}z^n}\right|_{z=z_0} = \left|V^{(n)}(z_0)\right| e^{i\alpha}, \qquad \alpha > 0. \tag{4.1.5}$$

Assuming that $f(z) \sim \beta(z-z_0)^{\gamma-1}$ in a neighbourhood of z_0 ($\Re(\gamma) > 0$), then

$$\int_C f(z)e^{kV(z)}\mathrm{d}z \sim \frac{\beta(n!)^{\frac{\gamma}{n}} e^{i\gamma\theta}}{n} \frac{e^{kV^{(n)}(z_0)}\,\Gamma\left(\frac{\gamma}{n}\right)}{\left(k\left|V^{(n)}(z_0)\right|\right)^{\frac{\gamma}{n}}}. \tag{4.1.6}$$

Remark 4.2. *It is worth pointing out that, even if the evaluation of the integral (4.1.1) reduces to the evaluation of a local quantity in a neighbourhood of the saddle points, the choice of the new contour of integration $\widetilde{C}$ requires a study of the global behaviour of the phase V.*

Remark 4.3. *The name "steepest descent method" comes from the fact that, thanks to the Cauchy-Riemann equations, the paths defined by the relation $\Im(V) = \text{constant}$ coincide with those along which either the decrease of the corresponding real part is minimal (paths of steepest descent) or the increase of the real part is maximal (paths of steepest ascent). In evaluating the integral (4.1.1) one will consider the former type of paths.*

The nonlinear steepest descent method generalizes the ideas above, but also employs new ones.

4.2 The non-linear method

Suppose we are given a Riemann-Hilbert problem on a collection of contours Σ, depending on a parameter k:

$$\Gamma_+(\lambda, k) = \Gamma_-(\lambda, k)J(\lambda, k), \quad \lambda \in \Sigma, \tag{4.2.1}$$

$$\Gamma(\lambda, k) = I + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \tag{4.2.2}$$

We are again interested in studying the asymptotic behaviour of the solution as $k \to +\infty$.

Writing the entries of the jump matrix as exponentials, the first step to perform is to identify the stationary points of the phases appearing in J.

It was first realized by Its ([51], [52]) and then fully implemented in the work of Deift and Zhou ([24]) that an accurate estimate of the asymptotic behaviour of the solution Γ in the regime $k \to +\infty$ can be achieved by replacing the problem (4.2.1)-(4.2.2) by "local" model Riemann-Hilbert problems located in a small neighbourhood of the stationary phase points.

Therefore, the non-linear method borrows from the linear one the same idea of focusing on the neighbourhoods of specific critical points of the problem, which govern the leading behaviour of the quantity under consideration (Γ in this case) in the regime $k \to +\infty$. On the other hand, the non-linear steepest descent method shows also a completely new feature, the so-called finite-gap g-function mechanism.

The g-function was introduced in [26] and in [22] but the powerfulness of such idea and the connection to the Lax-Levermore variational problem ([79]) was first explored in the analysis of the KdV equation in [23].

The introduction of a g-function in our asymptotic analysis becomes necessary when the Riemann-Hilbert problem shows some particular singularities that depends on the parameter k and cannot be factored away via a suitable rescaling or conjugation of the problem (4.2.1)-(4.2.2).

As an example, we describe the well-known classic problem of the asymptotics of orthogonal polynomials ([37], [18, Chapter 7]). First of all, we state the Riemann-Hilbert problem for orthogonal polynomials with respect to a given measure $e^{-\Lambda V(x)}\mathrm{d}x$, where Λ is a suitable parameter that will later be sent to infinity and $V(x)$ is a polynomial of some even degree with positive leading coefficient. $V(x)$ is generically called potential or external field in the literature, for reasons that will be clear in a moment.

We want to find a 2×2 matrix-valued function $Y(z) = Y_n(z)$, analytic on $\mathbb{C}\backslash\mathbb{R}$, such that

$$Y_+(z) = Y_-(z) \begin{bmatrix} 1 & e^{-\Lambda V(x)} \\ 0 & 1 \end{bmatrix} \qquad z \in \mathbb{R} \tag{4.2.3}$$

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{n\sigma_3} \qquad z \to \infty,\ \arg(z) \in (0,\pi) \cup (\pi, 2\pi). \tag{4.2.4}$$

Theorem 4.4. *The above Riemann-Hilbert problem admits a unique solution of the form*

$$Y_n(z) = \begin{bmatrix} p_n(z) & \int_{\mathbb{R}} \frac{p_n(x)e^{-\Lambda V(x)}}{x-z}\frac{\mathrm{d}x}{2\pi i} \\ \frac{-2\pi i}{h_{n-1}} p_{n-1}(z) & \frac{-1}{h_{n-1}} \int_{\mathbb{R}} \frac{p_{n-1}(x)e^{-\Lambda V(x)}\mathrm{d}x}{x-z} \end{bmatrix} \tag{4.2.5}$$

where p_n, p_{n-1} are the monic orthogonal polynomials for the measure $e^{-\Lambda V(x)}\mathrm{d}x$ and $h_{n-1} = \|p_{n-1}\|^2_{L^2}$.

Sketch of the proof. The uniqueness follows from standard considerations on the determinant. As for the form of the solution (4.2.5), it follows from considerations on the jump matrix and the Sokhotski-Plemelj's formula. In order to identify the polynomials as monic orthogonal polynomials with respect to the given measure, one can easily reach the conclusion by studying the asymptotic behaviour of the matrix Y_n. □

The interest is on the behaviour of the set of polynomials as their degree goes to infinity $n \to +\infty$ and at the same time also the parameter Λ diverges, say $\Lambda = Tn \to \infty,\ T > 0$. As first remark, we can notice that in this regime the singularity at infinity increases, due to the factor $z^{n\sigma_3}$. A *naive* attempt would be to remove the singularity by defining $W(z) := Y(z)e^{-n\sigma_3 \ln z}$; in this way the asymptotic behaviour at infinity looks more regular $W(z) = I + \mathcal{O}\left(z^{-1}\right)$, but on the other hand it does not solve the problem, since the same singularity issue appears now in the origin.

The problem originates from the logarithm $\ln z$ which is unbounded at the origin, even if it helped remove the singularity at infinity. The ideal approach would therefore be the following: transforming the original Riemann-Hilbert problem for Y into a new Riemann-Hilbert problem for W, with

$$W(z) := Y(z)e^{-ng(z)\sigma_3} \tag{4.2.6}$$

where $g(z)$ is a function (still to be determined) such that

- g is analytic everywhere away from the jump contour $\mathbb{R}$;
- g is bounded on any compact set of $\mathbb{C}$;
- g has a logarithmic behaviour at infinity $g(z) = \ln z + \mathcal{O}\left(z^{-1}\right)$.

The g-function satisfying conditions above can be written as

$$g(z) = \int_{\mathbb{R}} \ln(z - \eta)\mathrm{d}\mu(\eta) \tag{4.2.7}$$

where μ is a suitable continuous measure supported on some subsets of $\mathbb{R}$.

The new Riemann-Hilbert problem is then the following

$$W_+(z) = W_-(z) \begin{bmatrix} e^{n(g_+-g_-)} & e^{-n(TV-g_--g_+-\ell)} \\ 0 & e^{-n(g_+-g_-)} \end{bmatrix} \qquad z \in \mathbb{R} \tag{4.2.8}$$

$$W(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty. \tag{4.2.9}$$

where we require also the following constraints:

- there exists a constant ℓ (Robin's constant) such that

$$\varphi(z) := TV(z) - \Re\left(g_-(z) + g_+(z) - \ell\right) \geq 0, \qquad z \in \mathbb{R} \tag{4.2.10}$$

  so that the off-diagonal entry of the jump is also bounded;

- the jumps are purely imaginary, $g_+(z) - g_-(z) \in i\mathbb{R}$ when $z \in \mathbb{R}$, so that the diagonal entries of the jump are oscillatory but not growing;

- $\frac{1}{i}\left(g_+(z) - g_-(z)\right)$ is decreasing on $\mathbb{R}$.

In a sense, the reduction of the given Riemann-Hilbert problem Y to an explicitly solvable one W depends on the existence of a particular measure $\mathrm{d}\mu$ that defines the g-function. The conditions above on g turn out to be equivalent to a maximization problem for logarithmic potentials under external field depending on the potential $V(x)$ over positive measures with an upper constraint. This is related to the so-called Lax-Levermore variational problem [79]. We refer to [18], [67] and [93] for a detailed discussion about this topic. As conclusion remark, we point out that a g-function, provided it exists, may be either explicitly defined (as in [22]) or only implicitly defined via the conditions above (as in [23]).

In the case of orthogonal polynomials for example, it can be proved that such measure $\mathrm{d}\mu$ exists (therefore, also the g-function) and in general is supported on a collection of finite intervals ("cuts"). In this setting, we can notice that along the real line, but outside the intervals, the jump (4.2.8) tends to the identity matrix as $n \to \infty$, since the off-diagonal term tends to zero.

The Riemann-Hilbert problem can be now solved explicitly and the solving method involve three steps.

- The "lens"-argument: auxiliary contours are introduced near the pieces of the real line (one below and one above each cut) and appropriate factorizations of the jumps and analytic extensions are used. This will simplify the expression of the jump along the support of $\mathrm{d}\mu$, while the new jumps along the "lenses" will be close to the identity in the limit $n \to \infty$. In general, we want to identify those contours (provided they exist and provided that the original contour can be deformed into them) along which the jump matrix is asymptotically close to the identity.

- Dealing with singularities creating local parametrices near them. In our case the growth of the entries of the jump matrix is not bounded in near the endpoints of the intervals: thus, one introduces a new contour, homeomorphic to a small circle, centered at each of the endpoints and builds an exact solution to the local Riemann-Hilbert problem inside the circles.

- Small Norm Theorem (see for example [49, Section 5.1.3]): one considers a "model" Riemann-Hilbert problem, where only the jumps that do not tend to the identity (in the limit $n \to \infty$) are considered. Such a problem can be solved explicitly and it approximates the original Riemann-Hilbert problem in the n-limit.

The "small norm theory" will be widely used in the present thesis, in particular in Chapter 7. Therefore, we will now give a detailed description of the results that will be applied later. We refer to [49, Section 5.1.3] as a standard reference.

Given a collection of oriented contours Σ in the complex plane, denote by $|\mathrm{d}z|$ the arclength, assuming for the sake of simplicity each arc to be sufficiently smooth.

Let $f \in L^p(\Sigma, |\mathrm{d}z|)$ $(1 \leq p < \infty)$ be a (possibly matrix-valued) function and define the following Cauchy boundary operators

$$\mathcal{C}_\pm : L^p(\Sigma, |\mathrm{d}z|) \to L^p(\Sigma, |\mathrm{d}z|) \tag{4.2.11}$$

$$f \mapsto \mathcal{C}_\pm \left[f\right](s) := \lim_{z \to s^\pm} \frac{1}{2i\pi} \int_\Sigma \frac{f(\lambda)|\mathrm{d}\lambda|}{\lambda - z} \tag{4.2.12}$$

where the notation $s^\pm$ indicates that the limit is taken as z approaches $s \in \Sigma$ from the left or the right side of the oriented curve, within a nontangential cone.

The Cauchy boundary operators enjoy the following properties.

Theorem 4.5. *Let $f \in L^p(\Sigma, |\mathrm{d}z|)$, $1 \leq p < \infty$, then*

- *$\mathcal{C}_\pm \left[f\right]$ exists almost everywhere for $s \in \Sigma$;*

- *the Cauchy operator is bounded $\forall\ p > 1$*

$$\left\|\mathcal{C}_\pm \left[f\right]\right\|_{L^p} \leq C_p \left\|f\right\| \tag{4.2.13}$$

for some positive constant $C_p = C_p(\Sigma, f)$;

- *the following formula holds*

$$\mathcal{C}_{\pm} = \pm\frac{1}{2}\,\mathrm{Id} - \frac{1}{2}PV \tag{4.2.14}$$

where PV *is the Cauchy Principal Value operator*

$$PV\left[f\right](s) := \frac{1}{i\pi}P.V.\int_{\Sigma}\frac{f(w)\mathrm{d}w}{z-w} := \frac{1}{i\pi}\lim_{\epsilon\to 0}\int_{\Sigma_\epsilon}\frac{f(w)\mathrm{d}w}{z-w} \tag{4.2.15}$$

$$\Sigma_\epsilon := \Sigma\backslash\left\{|z-s|<\epsilon\right\}; \tag{4.2.16}$$

in particular,

$$\mathcal{C}_+ - \mathcal{C}_- = \mathrm{Id}\,. \tag{4.2.17}$$

Remark 4.6. *The last point in the above Theorem is just a restatement of the well-known Sokhotski-Plemelj formulæ(see for example [87]).*

Consider the following Riemann-Hilbert problem: given a matrix function $J(\lambda)$ defined over the collection of curves Σ, find a matrix $\mathcal{E}(\lambda)$ such that

1. $\mathcal{E}(\lambda)$ is analytic on $\mathbb{C}\backslash\Sigma$;

2. $\mathcal{E}(\lambda)$ has nontangential boundary values on Σ and they satisfy

$$\mathcal{E}_+(\lambda) = \mathcal{E}_-(\lambda)J(\lambda) \qquad \lambda\in\Sigma \tag{4.2.18}$$

3. $\mathcal{E}(\lambda)$ is asymptotically equal to the identity matrix in any norm:

$$\|\mathcal{E}(\lambda) - I\| = \mathcal{O}\left(\frac{1}{\lambda}\right) \tag{4.2.19}$$

alternatively

$$\mathcal{E}(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda\to\infty. \tag{4.2.20}$$

Suppose that the jump matrix $J(\lambda)$ is a small perturbation of the identity, then the Small Norm Theorem will allow to give some pointwise estimates on the solution $\mathcal{E}(\lambda)$.

Theorem 4.7 (Small Norm Theorem)**.** *Assume* $J(\lambda) = I + \delta J(\lambda)$ *is close to the identity jump, i.e. the norm*

$$\|\delta J(\lambda)\|_{L^2(\Sigma)\cap L^\infty(\Sigma)} := \max\left(\|\delta J(\lambda)\|_{L^2(\Sigma)}, \|\delta J(\lambda)\|_{L^\infty(\Sigma)}\right) \ll 1 \tag{4.2.21}$$

is small enough, with $\delta J \in L^p(\Sigma)$, $p = 1, 2, \infty$. Then, the solution $\mathcal{E}$ to the above Riemann-Hilbert problem exists and it satisfies the following pointwise estimate

$$\|\mathcal{E}(\lambda) - I\| \leq \frac{C}{\operatorname{dist}(\lambda, \Sigma)} \tag{4.2.22}$$

for some constant $C = C(\|\delta J(\lambda)\|) \to 0$, as $\|\delta J(\lambda)\|_{L^1(\Sigma)\cap L^2(\Sigma)} \to 0$.

Remark 4.8. *The condition $\delta J \in L^1(\Sigma)$ could be weakened. On the other hand, if $J(\lambda)$ is analytic, then it is possible to prove a stronger estimate of the form*

$$\|\mathcal{E}(\lambda) - I\| \leq \frac{C}{1 + \operatorname{dist}(\lambda, \Sigma)}. \tag{4.2.23}$$

We will give here a sketch of the proof.

Proof. The solution to the Riemann-Hilbert problem (4.2.18)-(4.2.19) can be written as

$$\mathcal{E}(\lambda) = I + \frac{1}{2i\pi} \int_\Sigma \frac{\mathcal{E}_+(s) - \mathcal{E}_-(s)}{s - \lambda} \mathrm{d}s. \tag{4.2.24}$$

Indeed, both sides have the same jump and the same asymptotic behaviour at ∞, thanks to the Sokhotski-Plemelj formula. On the other hand, since $\mathcal{E}_+(s) - \mathcal{E}_-(s) = \mathcal{E}_-(s)(J(\lambda) - I) = \mathcal{E}_-(s)\delta J(\lambda)$, we have

$$\mathcal{E}(\lambda) = I + \frac{1}{2i\pi} \int_\Sigma \frac{\mathcal{E}_-(s)\delta J(\lambda)}{s - \lambda} \mathrm{d}s. \tag{4.2.25}$$

Therefore, it is clear that $\mathcal{E}$ is uniquely determined by its boundary value $\mathcal{E}_-$. Taking the limit as λ approaches the curves Σ on the left, we have

$$\mathcal{E}_-(\lambda) = I + \frac{1}{2i\pi} \int_\Sigma \frac{\mathcal{E}_-(s)\delta J(\lambda)}{s - \lambda} \mathrm{d}s = I + \mathcal{C}_- \left[\mathcal{E}\delta J\right](\lambda). \tag{4.2.26}$$

Thus, solving the Riemann-Hilbert problem (4.2.18)-(4.2.19) is equivalent to solving a linear inhomogeneous equation for the matrix-valued function $f := \mathcal{E}_- - I \in L^2(\Sigma)$

$$f(\lambda) = \mathcal{C}_- \left[(I + f)\delta J\right](\lambda) = \mathcal{C}_- \left[\delta J\right](\lambda) + \mathcal{C}_- \left[f\delta J\right](\lambda) \tag{4.2.27}$$

or equivalently

$$(\mathrm{Id} - \mathcal{L})f = v_0 \tag{4.2.28}$$

$$\mathcal{L} := \mathcal{C}_- \left[\,\cdot\,\delta J\right], \qquad v_0 := \mathcal{C}_- \left[\delta J\right] \tag{4.2.29}$$

The next step is to prove that the operator norm of $\mathrm{Id} - \mathcal{L}$ is smaller than 1. Indeed, if this is the case, then the invertibility of the operator $\mathrm{Id} - \mathcal{L}$ is guaranteed and therefore also the existence of the solution to the Riemann-Hilbert problem (4.2.18)-(4.2.19) is proved.

Given the operator norm as $|||\mathcal{L}||| := \sup_{\|f\|_2=1} \|\mathcal{L}f\|_{L^2}$, then performing standard estimates we get

$$|||\mathcal{L}||| < |||\mathcal{C}_-||| \cdot \|\delta J\|_{L^\infty(\Sigma)} \tag{4.2.30}$$

which implies that the operator norm is smaller than one if the essential sup of δJ is smaller than the inverse of the operator norm of $\mathcal{C}_-$, which is indeed the case thanks to (4.2.21). The same conclusion can be said about the norm of the matrix v_0

$$\|v_0\| \leq |||\mathcal{C}_-||| \cdot \|\delta J\|_{L^2(\Sigma)} \ll 1. \tag{4.2.31}$$

We are finally able to derive an estimate for the solution $\mathcal{E}$ and conclude the proof of the theorem.

$$2\pi \left|\mathcal{E}(\lambda) - I\right| \leq \left| \int_\Sigma \frac{\delta J(\lambda)}{s-\lambda} \mathrm{d}s \right| + \left| \int_\Sigma \frac{f(s)\delta J(\lambda)}{s-\lambda} \mathrm{d}s \right| \leq \frac{1}{\mathrm{dist}(\lambda, \Sigma)} \left(\|\delta J\|_{L^1} + \|\delta J\|_{L^2} \|f\|_{L^2} \right) \tag{4.2.32}$$

Using the fact that $f = (\mathrm{Id} - \mathcal{L})^{-1} v_0 = (\mathrm{Id} - \mathcal{L})^{-1} \mathcal{C}_- [\delta J]$, we can estimate its norm by

$$\|f\|_{L^2} \leq \frac{1}{1 - |||\mathcal{L}|||} |||\mathcal{C}_-||| \cdot \|\delta J\|_{L^2(\Sigma)}; \tag{4.2.33}$$

in conclusion,

$$2\pi \left|\mathcal{E}(\lambda) - I\right| \leq \frac{1}{\mathrm{dist}(\lambda, \Sigma)} \left(\|\delta J\|_{L^1} + \frac{|||\mathcal{C}_-||| \cdot \|\delta J\|^2_{L^2(\Sigma)}}{1 - |||\mathcal{C}_-||| \cdot \|\delta J\|_{L^\infty(\Sigma)}} \right). \tag{4.2.34}$$

□

In the applications, one usually deals with a Riemann-Hilbert problem where a parameter k is very large

$$\Gamma_+(\lambda, k) = \Gamma_-(\lambda, k) J(\lambda, k), \quad \lambda \in \Sigma, \tag{4.2.35}$$

$$\Gamma(\lambda, k) = I + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \tag{4.2.36}$$

The idea is to perform a sequences of transformations, which may involve the introduction of a g-function, from the original Riemann-Hilbert problem into a final problem

$$\widetilde{\Gamma}_+(\lambda,k) = \widetilde{\Gamma}_-(\lambda,k)\widetilde{J}(\lambda,k), \quad \lambda \in \widetilde{\Sigma}, \tag{4.2.37}$$

$$\widetilde{\Gamma}(\lambda,k) = I + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty. \tag{4.2.38}$$

such that $\widetilde{\Gamma}$ is an explicit and approximate solution to the original problem. Indeed, if the jumps J and $\widetilde{J}$ are such that $J\widetilde{J}^{-1} = I + \delta\mathcal{J}$, where $\delta\mathcal{J}$ is sufficiently small in the L^p-norms, $p = 1, 2, \infty$, then one can build the "error" matrix $\mathcal{E}(\lambda) := \Gamma(\lambda)\widetilde{\Gamma}(\lambda)^{-1}$ which satisfies a Riemann-Hilbert problem with jump matrix $\widetilde{\Gamma}_-(I + \delta\mathcal{J}(\lambda))\widetilde{\Gamma}_-^{-1}$, plus the usual normalization at infinity. The Small Norm Theorem can therefore be applied and the estimate (4.2.23) gives the order of approximation of $\widetilde{\Gamma}$ with respect to Γ, in the setting $k \gg 1$.

The Small Norm Theorem will be the main tool used in Chapter 7 in order to prove the degeneracy of the tacnode Riemann-Hilbert problem into two Airy Riemann-Hilbert problems in the scaling limit as the "pressure" parameter σ tends to infinity or as the "time" parameter τ tends to either plus or minus infinity.

Chapter 5

Gap probabilities for the Bessel Process

5.1 Introduction

The Bessel process is a determinantal point process as detailed above in Chapter 2 defined in terms of a trace-class integral operator acting on $L^2(\mathbb{R}_+)$, with kernel

$$K_B(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_{\nu+1}(\sqrt{y}) - J_{\nu+1}(\sqrt{x})\sqrt{x}J_\nu(\sqrt{y})}{2(x-y)} \tag{5.1.1}$$

where J_ν are Bessel functions with parameter $\nu > -1$.

The Bessel kernel K_B arose originally as the correlation function in the scaling limit of the Laguerre and Jacobi Unitary Ensembles near the hard edge of their spectrum at zero ([38], [88], [89]) as well as of generalized LUEs and JUEs ([78], [104]).

Both these ensembles consist of complex self-adjoint matrices equipped with a certain probability measure, invariant under unitary transform. In particular, the LUE consists of positive self-adjoint complex $N \times N$ random matrices such that the joint probability density function of the (positive) eigenvalues is given by

$$\rho^{\mathrm{Lag}}_{\nu,N}(\lambda_1,\ldots,\lambda_N) = c_{\nu,N}\prod_{k=1}^{N}\lambda_k^\nu e^{-\lambda_k}\prod_{1\le j<k\le N}|\lambda_j-\lambda_k|^2 = \det\left[K_N(\lambda_i,\lambda_j)\right]_{i,j=1}^N, \tag{5.1.2}$$

where

$$K_N(x,y) := \sum_{k=0}^{N-1}\phi_k(x)\phi_k(y), \tag{5.1.3}$$

and $\{\phi_k(x)\}_{k=0}^\infty$ is the sequence obtained by orthonormalizing the functions $\left\{x^k x^{\frac{\nu}{2}} e^{-\frac{x}{2}}\right\}$ on

$(0, \infty)$, with $\nu > -1$.

The JUE consists of all contractive (i.e. its eigenvalues are smaller than 1 in absolute value) self-adjoint complex $N \times N$ random matrices with joint probability density function of the eigenvalues given by

$$\begin{aligned} \rho^{\mathrm{Jac}}_{\nu,\mu,N}(\lambda_1, \dots, \lambda_N) &= c_{\nu,\mu,N} \prod_{k=1}^{N} (1-\lambda_k)^{\nu} (1+\lambda_k)^{\mu} \prod_{1 \le j < k \le N} |\lambda_j - \lambda_k|^2 \\ &= \det \left[K_N(\lambda_i, \lambda_j) \right]_{i,j=1}^{N}, \end{aligned} \tag{5.1.4}$$

with $-1 < \lambda_1, \dots, \lambda_N < 1$, where K_N is given as in (5.1.3) with functions $\{\phi_k(x)\}_{k=0}^{\infty}$ obtained by orthonormalizing $\left\{x^k (1-x)^{\frac{\nu}{2}} (1+x)^{\frac{\mu}{2}}\right\}$ on $(-1, 1)$, with $\nu, \mu > -1$.

In both cases and for finite N, the probability that no eigenvalue lies in a subinterval I of $\mathbb{R}_+$ or $[-1, 1]$, respectively, can be written as a Fredholm determinant

$$\det \left(\mathrm{Id} - K_N \Big|_{\mathcal{I}} \right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{I^k} \det \left[K_N(x_i, x_j) \right]_{i,j=1}^{k} \mathrm{d}x_1 \dots \mathrm{d}x_k \tag{5.1.5}$$

where K_N stands for the orthogonal projection onto the subspace of $L^2(\mathbb{R}_+)$ or $L^2([-1, 1])$, respectively, spanned by the first N Laguerre or Jacobi functions, respectively, and $K_N(x, y)$ is the corresponding integral kernel of the form (5.1.3).

Let $P_N^{\mathrm{Lag},\nu}(s)$ and $P_N^{\mathrm{Jac},\nu,\mu}(s)$ denote the probabilities that no eigenvalues lie in the interval $[0, s] \subset \mathbb{R}_+$ (Laguerre case) or $[1-s, 1] \subset [-1, 1]$ (Jacobi case), respectively. We can notice that these probabilities describe also the behaviour of the eigenvalue that is closest to the hard edges of the ensembles. With the appropriate scaling these probabilities converge (as $N \nearrow \infty$) to the Fredholm determinant of the Bessel kernel:

$$\lim_{N \to \infty} P_N^{\mathrm{Lag},\nu} \left(\frac{s^2}{4N} \right) = \det \left(\mathrm{Id} - K_B \Big|_{[0,s]} \right) \tag{5.1.6}$$

$$\lim_{N \to \infty} P_N^{\mathrm{Jac},\nu,\mu} \left(\frac{s^2}{2N^2} \right) = \det \left(\mathrm{Id} - K_B \Big|_{[1-s,1]} \right). \tag{5.1.7}$$

In fact, the Laguerre and Jacobi kernels converge themselves, after the hard edge rescaling, to the Bessel kernel $K_B(x, y)$. This is also true for certain modified Laguerre and Jacobi random matrix ensembles.

In this chapter we focus on the study of the gap probabilities of the limit process, i.e. the Bessel process. In particular, we will be concerned with the Fredholm determinant of such an operator on a collection of (finite) intervals $I := \bigcup_{i=1}^{N} [a_{2i-1}, a_{2i}]$, i.e. the quantity

$\det\left(\mathrm{Id} - K_B\Big|_I\right)$, and the emphasis is on the determinant thought of as function of the endpoint a_i, $i = 1, \ldots, 2N$.

The gap probabilities for the Bessel process were originally studied by Tracy and Widom in their article [101]; we refer to this paper for a comparison with the differential equations showed in the present work (in particular, Theorem 5.16 and formula (5.2.47)). We point out that such equations are not the same as those shown in [101] and they are derived through a completely different method.

The second part of this chapter will examine the Bessel process in a time-dependent regime. Consider n times $\tau_1, \ldots, \tau_n$ in a given time interval $(0, T)$; the so called multi-time or extended Bessel process (see [72] and [102]) is a determinantal point process with matrix kernel $[K_B]_{ij}$ with entries

$$[K_B]_{ij}(x,y) = \begin{cases} \int_0^1 e^{u\Delta} J_\nu(\sqrt{xu}) J_\nu(\sqrt{yu})\, \mathrm{d}u & i \geq j \\ -\int_1^\infty e^{u\Delta} J_\nu(\sqrt{xu}) J_\nu(\sqrt{yu})\, \mathrm{d}u & i < j \end{cases} \tag{5.1.8}$$

$i, j = 1, \ldots, n$; with $\Delta := \Delta_{ij} = \tau_i - \tau_j$ the time gap between two times and $\nu > -1$.

Remark 5.1. *In the case $T = \tau_1 = \ldots = \tau_n = 0$, we can recover the time-less Bessel kernel (5.1.1).*

As shown by Forrester, Nagao and Honner in [39], the multi-time Bessel process (with its correspondent kernel) appears as scaling limit of the Extended (multi-time)Laguerre process at the hard edge of the spectrum.

Although the multi-time Bessel process has been known since a long time, the study of its gap probabilities has never been performed before and it is addressed in this chapter. Again, we will focus on the Fredholm determinant of such process on a collection of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$, I_j refers to time τ_j for all j. The result is a set of relations that describes the Fredholm determinant as a function of the endpoints of the intervals and of the n times.

The Fredholm determinant of the time-less Bessel kernel and, as it will be clear in the chapter, the Fredholm determinant of its multi-time counterpart will be related to Fredholm determinants of integrable operators in the sense of Its-Izergin-Korepin-Slavnov ([50], see Section 3.3). We point out that , while the definition of the Bessel kernel already shows an IIKS structure, the multi-time Bessel kernel (5.1.8) is not of integrable form. Nevertheless, it will be possible to reduce its Fredholm determinant to a determinant of an integrable operator of such form.

The main steps in our study of the gap probabilities for the Bessel process are the following: we will first find an IIKS integrable operator, acting on $L^2(\Sigma)$, with Σ a suitable collection of contours. Through an appropriate Fourier transform, we will prove that such an operator has the same Fredholm determinant as the Bessel process. We will then set up a Riemann-Hilbert problem for this integrable operator and connect it to the Jimbo-Miwa-Ueno τ function.

This strategy will be applied separately to both the single-time and the multi-time Bessel process. Our approach derives from the one used in [10] and [11] for the Airy and Pearcey processes in the dynamic and time-less regime respectively.

Whereas the part dedicated to the single-time process is mostly a review of known results (see [36], [53] and [101]), re-derived using an alternative approach, the results on the multi-time Bessel are genuinely new and never appeared in the literature before.

The present chapter is organized as follows: in section 5.2 we will deal with the single-time Bessel process in the general case of several intervals; in the subsection 5.2.3 we will focus on the process restricted to a single interval $[0, a]$: we will find a Lax pair and we will be able to make a connection between the Fredholm determinant and the third Painlevé transcendent. This provides a different and direct proof of this known connection ([53], [101]); in particular our approach directly specifies the monodromy data of the associated isomonodromic system and allows to use the steepest descent method to investigate asymptotic properties, if so desired. In section 5.3 we will study the gap probabilities for the multi-time Bessel process. Although the results of section 5.3 strictly include those of section 5.2, we have decided to separate the two cases for the benefit of a clearer exposition.

5.2 The single-time Bessel process and the Painlevé Transcendent

5.2.1 Preliminary results

We recall the definition of the Bessel kernel

$$K_B(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_{\nu+1}(\sqrt{y}) - J_{\nu+1}(\sqrt{x})\sqrt{x}J_\nu(\sqrt{y})}{2(x-y)}; \tag{5.2.1}$$

writing the Bessel functions as explicit contour integrals, it is possible to show, through some suitable manipulations and integrations by parts, that the Bessel kernel can be written also in the following form

$$K_B(x,y) = \left(\frac{y}{x}\right)^{\nu/2} \iint_{\gamma\times\hat{\gamma}} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s} \left(\frac{s}{t}\right)^\nu \frac{\mathrm{d}t}{2\pi i}\frac{\mathrm{d}s}{2\pi i} \tag{5.2.2}$$

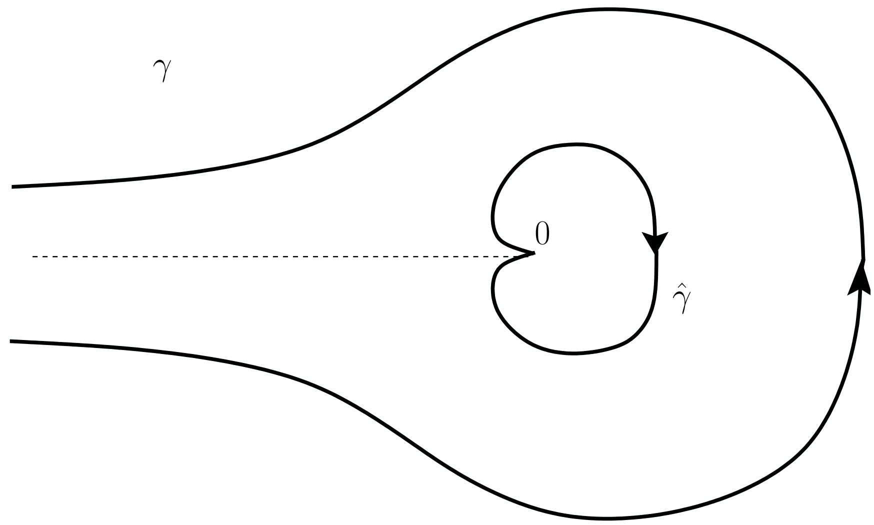

Figure 5.1: The contours appearing in the definition of the Bessel kernel (5.2.2).

with $\nu > -1$, $x, y > 0$ and γ a curve that extends to $-\infty$ and winds around the zero counterclockwise, while the curve $\hat{\gamma}$ is simply the transformed curve under the map $t \to 1/t$; the logarithmic cut is on $\mathbb{R}_-$. The contours are as in Figure 5.1.

We want to study the Fredholm determinant of the Bessel operator; in particular, we will focus on the following quantity

$$\det\left(\mathrm{Id} - K_B\Big|_I\right) \tag{5.2.3}$$

where $I := [a_1, a_2] \cup [a_3, a_4] \cup \ldots \cup [a_{2N-1}, a_{2N}]$ is a collection of finite intervals ($0 \leq a_1 < \ldots < a_{2N}$).

Remark 5.2. *The Bessel operator is not trace-class on an infinite interval. Thus, it is meaningless to consider the operator restricted to such interval.*

Remark 5.3. *Defining* $K_a := K_B(x, y)\Big|_{[0,a]}$*, then we have*

$$K_B(x, y)\Big|_I := \sum_{j=1}^{2N} (-1)^j K_{a_j}(x, y). \tag{5.2.4}$$

Our goal is to set up a Riemann-Hilbert problem associated to the Fredholm determinant of $K_B\Big|_I$.

Theorem 5.4. *The following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - K_B\Big|_I\right) = \det\left(\mathrm{Id} - \mathbb{B}\right) \tag{5.2.5}$$

where $\mathbb{B}$ is a trace-class integrable operator acting on $L^2(\gamma \cup \hat{\gamma})$ with kernel

$$\mathbb{B}(s,t) := \frac{\vec{f}(s)^T \cdot \vec{g}(t)}{s-t} \tag{5.2.6a}$$

$$\vec{f}(s) = \frac{1}{2\pi i}\begin{bmatrix} e^{\frac{a_1 s}{2} - \frac{1}{4s}} s^{-\nu} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \chi_\gamma(s) + \frac{1}{2\pi i}\begin{bmatrix} 0 \\ e^{-a_1 s + \frac{1}{4s}} s^{\nu} \\ -e^{-a_2 s + \frac{1}{4s}} s^{\nu} \\ \vdots \\ (-1)^{2N} e^{-a_{2N-1} s + \frac{1}{4s}} s^{\nu} \\ (-1)^{2N+1} e^{-a_{2N} s + \frac{1}{4s}} s^{\nu} \end{bmatrix} \chi_{\hat{\gamma}}(s) \tag{5.2.6b}$$

$$\vec{g}(t) = \begin{bmatrix} 0 \\ e^{\frac{a_1 t}{2}} \\ e^{t\left(a_2 - \frac{a_1}{2}\right)} \\ \vdots \\ e^{t\left(a_{2N} - \frac{a_1}{2}\right)} \end{bmatrix} \chi_\gamma(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \chi_{\hat{\gamma}}(t). \tag{5.2.6c}$$

Remark 5.5. *The space $L^2(\gamma \cup \hat{\gamma})$ is the space of square integrable functions in the arclength measure, defined on the curves $\gamma \cup \hat{\gamma}$.*

Proof. We work on a single kernel K_{a_j} and we will later sum them up (as in Remark 5.3), thanks to the linearity of the operations that we are going to perform.

First of all, we can notice that, if $x < 0$ or $y < 0$, $K_B(x,y) \equiv 0$; in particular, if $x < 0$, then a simple residue calculation shows that the kernel vanishes. Similar arguments lead to the same conclusion for $y < 0$. Then, using Cauchy's theorem, we can write

$$\begin{aligned} K_{a_j}(x,y) &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{\xi(a_j - y)} \iint_{\hat{\gamma}\times\gamma} \frac{e^{xt - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)\,(t-s)} \left(\frac{s}{t}\right)^{\nu} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} = \\ &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i}\, e^{-\xi y} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i}\, e^{xt} \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i}\, \frac{e^{\xi a_j - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^{\nu} \end{aligned} \tag{5.2.7}$$

where $i\mathbb{R} + \epsilon$ ($\epsilon > 0$) is a translated imaginary axis; thanks to the analyticity of the kernel, we continuously deformed the curve γ into such translated imaginary axis, in order to make the Fourier operator defined below more explicit. We also discarded the conjugation term $\left(\frac{y}{x}\right)^{\nu/2}$, due to the invariance of the Fredholm determinant under conjugation by a positive function.

Defining the following Fourier transform operators:

$$\begin{array}{c|c} \mathcal{F} : L^2(\mathbb{R}) \to L^2(i\mathbb{R}+\epsilon) & \mathcal{F}^{-1} : L^2(i\mathbb{R}+\epsilon) \to L^2(\mathbb{R}) \\ f(x) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} f(x) e^{\xi x} \mathrm{d}x & h(\xi) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{i\mathbb{R}+\epsilon} h(\xi) e^{-\xi x} \mathrm{d}\xi \end{array} \tag{5.2.8}$$

it is straightforward to deduce that

$$K_B\Big|_I = \mathcal{F}^{-1} \circ \overline{\mathcal{K}_B} \circ \mathcal{F} \tag{5.2.9}$$

with $\overline{\mathcal{K}_B} = \sum_j (-1)^j \overline{\mathcal{K}_{a_j}}$ and $\forall\, j = 1, \dots, N$ $\overline{\mathcal{K}_{a_j}}$ is an operator on $L^2(i\mathbb{R}+\epsilon)$ with kernel

$$\overline{\mathcal{K}_{a_j}}(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi a_j - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^{\nu}.$$

In order to ensure convergence of the Fourier-transformed Bessel kernel, we conjugate $\overline{\mathcal{K}_B}$ by a suitable function

$$\begin{aligned} \mathcal{K}_B(\xi, t) &:= e^{\frac{a_1 t}{2} - \frac{a_1 \xi}{2}} \overline{\mathcal{K}_B}(\xi, t) \\ = \sum_{j=1}^{2N} (-1)^j \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} & \frac{e^{\xi\left(a_j - \frac{a_1}{2}\right) + \frac{a_1 t}{2} - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^{\nu} =: \sum_{j=1}^{2N} (-1)^j \mathcal{K}_{a_j}(\xi, t) \end{aligned} \tag{5.2.10}$$

and we continuously deform the translated imaginary axis $i\mathbb{R}+\epsilon$ into its original shape γ; note that $a_j - \frac{a_1}{2} > 0$, $\forall\, j = 2, \dots, 2N$ and $\frac{a_1}{2} \geq 0$.

Lemma 5.6. *For each $j = 1, \dots, 2N$, the operator $\mathcal{K}_{a_j}$ with kernel*

$$\mathcal{K}_{a_j}(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi\left(a_j - \frac{a_1}{2}\right) + \frac{a_1 t}{2} - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^{\nu} \tag{5.2.11}$$

is trace-class. Moreover, the following decomposition holds $\mathcal{K}_{a_j} = \mathcal{A} \circ \mathcal{B}_{a_j}$, with

$$\begin{array}{c|c} \mathcal{A} : L^2(\hat{\gamma}) \to L^2(\gamma) & \mathcal{B}_{a_j} : L^2(\gamma) \to L^2(\hat{\gamma}) \\ h(s) \mapsto t^{-\nu} e^{\frac{a_1 t}{2} - \frac{1}{4t}} \int_{\hat{\gamma}} \frac{h(s)}{t-s} \frac{\mathrm{d}s}{2\pi i} & f(t) \mapsto s^{\nu} e^{-a_j s + \frac{1}{4s}} \int_{\gamma} \frac{e^{t\left(a_j - \frac{a_1}{2}\right)}}{t-s} f(t) \frac{\mathrm{d}t}{2\pi i}. \end{array} \tag{5.2.12}$$

$\mathcal{A}$ and $\mathcal{B}_{a_j}$ are trace-class operators as well.

Proof. It is easy to verify that $\mathcal{A}$ and $\mathcal{B}_{a_j}$ are Hilbert-Schmidt and that their composition gives $\mathcal{K}_{a_j}$.

Moreover, we have the following decomposition of kernels. Introducing an additional contour $i\mathbb{R}+\delta$ not intersecting either of $\gamma, \hat{\gamma}$, we have $\mathcal{A} = \mathcal{P}_2 \circ \mathcal{P}_1$ with

$$\mathcal{P}_1 : L^2(\hat{\gamma}) \to L^2(i\mathbb{R}+\delta) \qquad \mathcal{P}_2 : L^2(i\mathbb{R}+\delta) \to L^2(\gamma)$$

$$\mathcal{P}_1[f](u) = \int_{\hat{\gamma}} \frac{f(s)}{u-s}\frac{\mathrm{d}s}{2\pi i} \qquad \mathcal{P}_2[h](t) = \frac{e^{\frac{a_1 t}{2}-\frac{1}{4t}}}{t^{\nu}} \int_{i\mathbb{R}+\delta} \frac{h(u)}{t-u}\frac{\mathrm{d}u}{2\pi i}.$$

Analogously, $\mathcal{B}_{a_j} = \mathcal{O}_{2,j} \circ \mathcal{O}_{1,j}$ with

$$\mathcal{O}_{1,j} : L^2(\gamma) \to L^2(i\mathbb{R}+\delta) \qquad \mathcal{O}_{2,j} : L^2(i\mathbb{R}+\delta) \to L^2(\hat{\gamma})$$

$$\mathcal{O}_{1,j}[f](w) = \int_{\gamma} \frac{e^{t\left(a_j-\frac{a_1}{2}\right)} f(t)}{t-w}\frac{\mathrm{d}t}{2\pi i} \qquad \mathcal{O}_{2,j}[h](s) = s^{\nu} e^{-a_j s+\frac{1}{4s}} \int_{i\mathbb{R}+\delta} \frac{h(w)}{w-s}\frac{\mathrm{d}s}{2\pi i}.$$

It is straightforward to check that $\mathcal{P}_i$ and $\mathcal{O}_{i,j}$ are Hilbert-Schmidt operators, $i = 1, 2$ and $j = 1, \ldots, N$. Therefore, $\mathcal{A}$ and $\mathcal{B}_{a_j}$ are trace-class. $\square$

Remark 5.7. *The kernel $\mathcal{A}$ does not depend on the set of parameters $\{a_j\}_2^{2N}$, but only on the first endpoint a_1.*

Before proceeding further, we notice that any operator acting on the Hilbert space $H := L^2(\gamma \cup \hat{\gamma}) \simeq L^2(\gamma) \oplus L^2(\hat{\gamma}) = H_1 \oplus H_2$ can be written as a 2×2 matrix of operators with (i, j)-entry given by an operator $H_j \to H_i$.

According to such split and using matrix notation, we can thus write $\det(\mathrm{Id} - \mathcal{K}_B)$ as

$$\det\left(\mathrm{Id}_{L^2(\gamma)} - \sum_{j=1}^{2N} (-1)^j \mathcal{A} \circ \mathcal{B}_{a_j}\right)$$

$$= \det\left(\mathrm{Id}_{L^2(\gamma)} \otimes \mathrm{Id}_{L^2(\hat{\gamma})} - \left[\begin{array}{c|c} 0 & \mathcal{A} \\ \hline \sum_{j=1}^{2N}(-1)^j \mathcal{B}_{a_j} & 0 \end{array}\right]\right) = \det(\mathrm{Id}_{L^2(\gamma\cup\hat{\gamma})} - \mathbb{B}). \qquad (5.2.13)$$

The first identity comes from multiplying the right hand side on the left by the following matrix (with determinant equal 1)

$$\mathrm{Id}_{L^2(\gamma)\oplus L^2(\hat{\gamma})} + \left[\begin{array}{c|c} 0 & -\mathcal{A} \\ \hline 0 & 0 \end{array}\right].$$

□

5.2.2 The Riemann-Hilbert problem for the Bessel process.

Thanks to Theorem 5.4 we can relate the computation of the Fredholm determinant of the Bessel operator to the theory of isomonodromic equations. We start by setting up a suitable Riemann-Hilbert problem which is naturally related to the Fredholm determinant of the operator $\mathbb{B}$.

Proposition 5.8. *Given the integrable kernel (5.2.6a)-(5.2.6c), the associated Riemann-Hilbert problem is the following:*

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda)\,(I - J(\lambda)) & \lambda \in \Sigma := \gamma \cup \hat{\gamma} \\ \Gamma(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) & \lambda \to \infty \end{cases} \tag{5.2.14}$$

where Γ is a $(2N+1)\times(2N+1)$ matrix such that it is analytic on $\mathbb{C}\backslash\Sigma$, bounded near $\lambda = 0$ and satisfies the jump conditions above with

$$J(\lambda) := \begin{bmatrix} 0 & e^{\theta_1} & e^{\theta_2} & \dots & e^{\theta_{2N}} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \chi_\gamma(\lambda) + \begin{bmatrix} 0 & 0 & \dots & 0 \\ e^{-\theta_1} & 0 & \dots & 0 \\ -e^{-\theta_2} & 0 & \dots & 0 \\ \vdots & & & \vdots \\ (-1)^{2N+1}e^{-\theta_{2N}} & 0 & \dots & 0 \end{bmatrix} \chi_{\hat{\gamma}}(\lambda) \tag{5.2.15}$$

$\theta_j := a_j\lambda - \frac{1}{4\lambda} - \nu\ln\lambda$, $\forall\, j = 1, \dots, 2N$, *where χ_γ, $\chi_{\hat{\gamma}}$ are the characteristic functions on the contour γ and $\hat{\gamma}$ respectively.*

Proof. It is straightforward to verify that $I - J(\lambda) = I - \vec{f}(\lambda)\cdot\vec{g}(\lambda)^T$. □

Theorem 5.9. *The Tracy-Widom distribution of the Bessel process, i.e. the Fredholm determinant* $\det\left(\mathrm{Id} - K_B\Big|_I\right)$*, is equal to the isomonodromic τ-function related to the Riemann-Hilbert problem defined in Proposition 5.8. In particular, $\forall\, j = 1, \dots, 2N$*

$$\partial_{a_j}\ln\det\left(\mathrm{Id} - K_B\Big|_I\right) = \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_{\partial_{a_j}}(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{5.2.16a}$$

$$\Xi_\partial(\lambda) := -\partial J(\lambda)\cdot(I - J(\lambda))^{-1} \tag{5.2.16b}$$

where $I = [a_1, a_2]\cup\dots[a_{2N-1}, a_{2N}]$ is a collection of finite intervals and $\Sigma = \gamma\cup\hat{\gamma}$; we denote by $'$ the derivative with respect to the spectral parameter λ.

Proof. Referring to at the Theorem 3.14 from Section 3.4, we just need to verify that the extra term $H(I - J(\lambda)) \equiv 0$.

Moreover, we notice that the jump matrix $J(\lambda)$ can be written as

$$J(\lambda, \vec{a}) = e^{T(\lambda,\vec{a})} J_0 e^{-T(\lambda,\vec{a})} \tag{5.2.17}$$

where J_0 is a constant matrix, consisting only on 0 and ± 1, and

$$\begin{gathered} T(\lambda, \vec{a}) = \operatorname{diag}\left(T_0, T_1, \ldots, T_N\right) \\ T_0 = \frac{1}{N+1} \sum_{j=1}^{N} \theta_j \qquad T_j = T_0 - \theta_j . \end{gathered} \tag{5.2.18}$$

Therefore, the matrix $\Psi(\lambda, \vec{a}) := \Gamma(\lambda, \vec{a}) e^{T(\lambda,\vec{a})}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE. This guarantees the identification of the one-form above with the one defined by Jimbo, Miwa and Ueno ([53], [54], [57]), as explained in Chapter 3.4. □

Starting from Theorem 5.9, it it possible to derive more explicit differential identities by the use of the Jimbo-Miwa-Ueno residue formula adapted to the case at hand.

$$\int_\Sigma \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_{\partial_{a_j}}(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} = -\operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}(\lambda)\Gamma'(\lambda)\partial_{a_j} T(\lambda)\right). \tag{5.2.19}$$

In conclusion,

Proposition 5.10. *For all $j = 1, \ldots, N$, the Fredholm determinant satisfies*

$$\partial_{a_j} \ln\det\left(\mathrm{Id} - K_B\Big|_I\right) = -\Gamma_{1;j+1,j+1} \tag{5.2.20}$$

with $\Gamma_{1;j+1,j+1}$ the $(j+1, j+1)$ component of the residue matrix $\Gamma_1 = \lim_{\lambda\to\infty} \lambda\,(I - \Gamma(\lambda))$ and Γ is the solution to the Riemann-Hilbert problem (5.2.14).

Proof. The proof will follow the same guidelines described in [11, Proposition 3.2]. Given the definition of $T(\lambda)$,

$$\partial_{a_j} T(\lambda, \vec{a}) = \lambda \left(\frac{1}{N+1} I - E_{j+1,j+1}\right) \tag{5.2.21}$$

and plugging into (6.3.17), we have

$$\partial_{a_j} \ln\det\left(\mathrm{Id} - K_B\Big|_I\right) = \frac{\operatorname{Tr}\Gamma_1}{N+1} - \Gamma_{1;j+1,j+1} = -\Gamma_{1;j+1,j+1} \tag{5.2.22}$$

since $\det \Gamma(\lambda) \equiv 1$, thus $\operatorname{Tr} \Gamma_1 = 0$. □

5.2.3 The single-interval case for the Bessel process and the Painlevé III equation

We consider now the case in which the Bessel kernel is restricted to a single finite interval $[0, a]$.

We will see that from the 2×2 Bessel Riemann-Hilbert problem we can derive a suitable Lax pair which matches with the Lax pair of the Painlevé III transcendent, as shown in [36]. The Lax pair described in [36] is slightly different from the one found in our present thesis, but it can be shown that the two formulations are equivalent.

In order to make the connection with the Painlevé transcendent more explicit, we will work on a rescaled version of the Bessel kernel, which can be easily derived from our original definition (5.2.2) through suitable scalings.

By specializing the results of the previous section, we get a (Fourier transformed) Bessel operator on $L^2(\gamma)$ with the following kernel

$$\mathcal{K}_B(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\frac{\xi x}{4} + \frac{x}{2}\left(\frac{t}{2} - \frac{1}{t}\right) - \frac{x}{2}\left(s - \frac{1}{s}\right)}}{(\xi - s)(t - s)} \left(\frac{s}{t}\right)^{\nu} \tag{5.2.23}$$

where $x := \sqrt{a}$.

It can be easily shown that $\mathcal{K}_B$ is a trace-class operator, since product of two Hilbert-Schmidt operators $\mathcal{K}_B = \mathcal{A}_2 \circ \mathcal{A}_1$ with kernels

$$\mathcal{A}_1(t, s) = \frac{1}{2\pi i} \frac{\exp\left\{\frac{tx}{4} - \frac{x}{2}\left(s - \frac{1}{s}\right)\right\}}{t - s} s^{\nu} \cdot \chi_\gamma(t)\chi_{\hat{\gamma}}(s) \tag{5.2.24}$$

$$\mathcal{A}_2(s, t) = -\frac{1}{2\pi i} \frac{\exp\left\{\frac{x}{2}\left(\frac{s}{2} - \frac{1}{s}\right)\right\}}{t - s} s^{-\nu} \cdot \chi_\gamma(s)\chi_{\hat{\gamma}}(t). \tag{5.2.25}$$

Proposition 5.11. *The operators $\mathcal{A}_j$, $j = 1, 2$, are trace-class.*

Proof. The proof follows the same arguments as the proof of Lemma 5.6. □

Theorem 5.12. *Consider the interval $[0, x]$, then the following identity holds*

$$\det \left(\mathrm{Id} - K_B \Big|_{[0,x]} \right) = \det \left(\mathrm{Id} - \mathbb{B}\right) \tag{5.2.26}$$

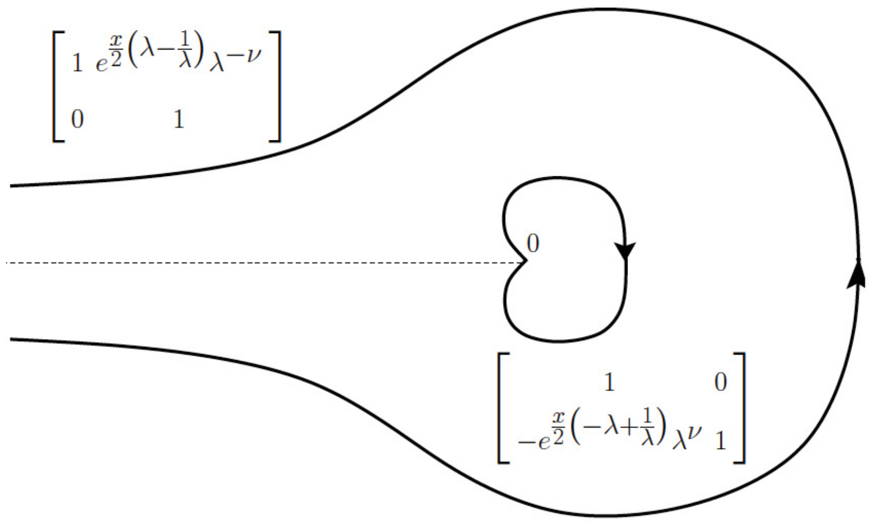

Figure 5.2: The jump matrices for the Bessel Riemann-Hilbert Problem in the single-time case.

with $\mathbb{B}$ a trace-class integrable operator with kernel defined as follows

$$\begin{aligned}\mathbb{B}(t,s) &= \frac{1}{2\pi i}\frac{e^{\frac{tx}{4}-\frac{x}{2}\left(s-\frac{1}{s}\right)}s^{\nu}\cdot\chi_{\gamma}(t)\chi_{\hat{\gamma}}(s) - e^{\frac{x}{2}\left(\frac{s}{2}-\frac{1}{s}\right)}s^{-\nu}\cdot\chi_{\gamma}(s)\chi_{\hat{\gamma}}(t)}{t-s} \\ &= \frac{\vec{f}(t)^T\cdot\vec{g}(s)}{t-s} \end{aligned} \tag{5.2.27a}$$

with

$$\vec{f}(t) = \frac{1}{2\pi i}\begin{bmatrix} e^{\frac{tx}{4}} \\ 0 \end{bmatrix}\chi_{\gamma}(t) + \frac{1}{2\pi i}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\chi_{\hat{\gamma}}(t) \tag{5.2.27b}$$

$$\vec{g}(s) = \begin{bmatrix} e^{\frac{x}{2}\left(-s+\frac{1}{s}\right)}s^{\nu} \\ 0 \end{bmatrix}\chi_{\hat{\gamma}}(s) + \begin{bmatrix} 0 \\ -e^{\frac{x}{2}\left(\frac{s}{2}-\frac{1}{s}\right)}s^{-\nu} \end{bmatrix}\chi_{\gamma}(s). \tag{5.2.27c}$$

The associated 2×2 Riemann-Hilbert problem has jump matrix $M(\lambda) := I - J(\lambda)$ on $\Sigma := \gamma\cup\hat{\gamma}$ with

$$J(\lambda) = \begin{bmatrix} 0 & -e^{\frac{x}{2}\left(\lambda-\frac{1}{\lambda}\right)}\lambda^{-\nu} \\ 0 & 0 \end{bmatrix}\chi_{\gamma}(\lambda) + \begin{bmatrix} 0 & 0 \\ e^{\frac{x}{2}\left(-\lambda+\frac{1}{\lambda}\right)}\lambda^{\nu} & 0 \end{bmatrix}\chi_{\hat{\gamma}}(\lambda). \tag{5.2.28}$$

and the solution Γ to the RHP is bounded near the origin when $x = 0$. See Figure 5.2 for a sketch of the jumps.

It is easy to see that

$$M(\lambda) = e^{T(\lambda)} M_0 e^{-T(\lambda)}$$
$$\text{with } \ T(\lambda) := \frac{\theta_x}{2}\sigma_3, \quad \theta_x := \frac{x}{2}\left(\lambda - \frac{1}{\lambda}\right) - \nu \ln \lambda \tag{5.2.29}$$

where M_0 is a constant matrix. Thus, the matrix $\Psi(\lambda) := \Gamma(\lambda) e^{T_x(\lambda)}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Applying again Theorem 5.9 and Jimbo-Miwa-Ueno residue formula, we get

$$\begin{aligned}\partial_x \ln\det(\mathrm{Id} - \mathbb{B}) &= \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma'_-(\lambda)\Xi_x(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \\ &= -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_x T\right) + \operatorname*{res}_{\lambda=0} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_x T\right).\end{aligned} \tag{5.2.30}$$

Proposition 5.13. *The Fredholm determinant of the single-interval Bessel operator satisfies the following identity*

$$\partial_x \ln\det\left(\mathrm{Id} - \mathbb{B}\right) = -\frac{1}{2}\Gamma_{1;22} + \frac{1}{2}\tilde{\Gamma}_{1;2,2} \tag{5.2.31}$$

where $\Gamma_{1;22}$ is the $(2,2)$-entry of the residue matrix at infinity, while $\tilde{\Gamma}_{1;2,2}$ is the $(2,2)$-entry of residue matrix at zero.

Proof. As in the proof of Proposition 5.10, we can easily get the result by calculating the derivative of the conjugation matrix

$$\partial_x T(\lambda) = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)\left(\frac{1}{2}I - E_{2,2}\right) \tag{5.2.32}$$

and by keeping into account that, since $\det\Gamma(\lambda) \equiv 1$, $\mathrm{Tr}\,\Gamma_1 = \mathrm{Tr}\,\tilde{\Gamma}_1 = 0$. □

The Lax pair and the Third Painlevé Transcendent

From the asymptotic behaviour at infinity of the matrix Ψ, we can calculate the Lax pair associated to our Riemann-Hilbert problem.

$$A := \partial_\lambda \Psi \cdot \Psi^{-1}(\lambda) = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \tag{5.2.33}$$
$$B := \partial_x \Psi \cdot \Psi^{-1}(\lambda) = B_0 + \lambda B_1 + \frac{B_{-1}}{\lambda} \tag{5.2.34}$$

with coefficients

$$
\begin{aligned}
A_0 &= \frac{x}{4}\sigma_3 \\
A_{-1} &= \frac{x}{4}[\Gamma_1, \sigma_3] - \frac{\nu}{2}\sigma_3 \\
A_{-2} &= \frac{x}{4}[\Gamma_2, \sigma_3] + \frac{x}{4}[\sigma_3\Gamma_1, \Gamma_1] - \frac{\nu}{2}[\Gamma_1, \sigma_3] + \frac{x}{4}\sigma_3 - \Gamma_1 \\
B_1 &= \frac{1}{4}\sigma_3 \\
B_0 &= \frac{1}{4}[\Gamma_1, \sigma_3] \\
B_{-1} &= \frac{1}{4}\left([\Gamma_2, \sigma_3] + [\sigma_3\Gamma_1, \Gamma_1] - \sigma_3 + 4\frac{\mathrm{d}\Gamma_1}{\mathrm{d}x}\right).
\end{aligned}
\tag{5.2.35}
$$

The form of the coefficients matches with the results in [36, Chapter 5, Section 3, Formulæ (5.3.32) and (5.3.34)]. In particular, using the same notation as in [36, Chapter 5, Section 3, Formulæ (5.3.7) and (5.3.8)], we have

$$
\begin{aligned}
A_0 = \begin{bmatrix} \frac{x}{4} & 0 \\ 0 & -\frac{x}{4} \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} -\frac{\nu}{2} & Y(x) \\ V(x) & \frac{\nu}{2} \end{bmatrix}, \\
B_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & \frac{Y(x)}{x} \\ \frac{V(x)}{x} & 0 \end{bmatrix},
\end{aligned}
\tag{5.2.36}
$$

$$
A_{-2} = \begin{bmatrix} \frac{x}{4} - U(x) & -W(x)U(x) \\ \frac{U(x)-\frac{x}{2}}{W(x)} & -\frac{x}{4} + U(x) \end{bmatrix}, \quad B_{-1} = -\frac{1}{x}A_{-2}. \tag{5.2.37}
$$

Calculating the compatibility equation, we get the following system of ODEs

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -2\,\frac{WUV}{x} - 2\,\frac{UY}{xW} + \frac{U}{x} + \frac{Y}{W} \tag{5.2.38a}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{\nu\,V}{x} - \frac{U}{W} + \frac{x}{2W} \tag{5.2.38b}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} = 2\,\frac{VW^2}{x} - \frac{\nu\,W}{x} - 2\,\frac{Y}{x} \tag{5.2.38c}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -WU + \frac{\nu\,Y}{x} \tag{5.2.38d}$$

and the constant

$$\frac{\Theta_0}{4} := -\frac{\nu}{4} + \frac{UY}{xW} - \frac{WUV}{x} - \frac{Y}{2W} + \frac{\nu U}{x} \tag{5.2.38e}$$

which can be proven to be the monodromy exponent at 0 and equal to $-\nu$.

Setting now

$$F(x) := \frac{Y(x)}{W(x)U(x)} \tag{5.2.39}$$

and substituting in the equations above, we get that F and L satisfy

$$x\frac{\mathrm{d}F}{\mathrm{d}x} = (4U - x)F^2 + (2\nu - 1)F - x \tag{5.2.40}$$

$$x\frac{\mathrm{d}U}{\mathrm{d}x} = -4FU^2 + 2xUF - (2\nu - 1)U. \tag{5.2.41}$$

Remark 5.14. *The latter equation for the function U is a Bernoulli 1st-order ODE with $n = 2$.*

Remark 5.15 (Behaviour as $x \to 0_+$). *To inspect the behaviour of the functions U, F as $x \to 0_+$, consider the matrix*

$$Y(\lambda, x) := x^{-\frac{\nu}{2}\sigma_3}\Gamma(\lambda, x)x^{\frac{\nu}{2}\sigma_3} \tag{5.2.42}$$

which solves a similar RHP with jumps on contours like in Fig. 5.2 but where the off diagonal terms are multiplied by a factor $x^{\pm\nu}$.

For the sake of simplicity we consider only the case $\nu > 0$. Given $\mathbb{D}$ a disk containing $\hat{\gamma}$ and entirely contained in γ, define the following matrix

$$\Phi(\lambda, x) := \begin{cases} \left(I + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right)(I + A(\lambda, x)\sigma_+) =: \Phi_\infty & \lambda \in \mathbb{C}\backslash\mathbb{D} \\ \Phi_\infty\left[I + \left(B(\lambda, x) - \frac{x^\nu J_{\nu+1}(x)}{\lambda} - \frac{x^\nu J_{\nu+2}(x)}{\lambda^2}\right)\sigma_-\right] & \lambda \in \mathbb{D} \end{cases} \tag{5.2.43}$$

where

$$A(\lambda, x) = x^{-\nu}\int_\gamma \frac{s^{-\nu}e^{\frac{x}{2}\left(s-\frac{1}{s}\right)}\mathrm{d}s}{(s-\lambda)2\pi i}, \qquad B(\lambda, x) = x^{\nu}\int_{\hat{\gamma}} \frac{s^{\nu}e^{-\frac{x}{2}\left(s-\frac{1}{s}\right)}\mathrm{d}s}{(s-\lambda)2\pi i}, \tag{5.2.44}$$

$\sigma_\pm$ is a 2×2 matrix where the only non-zero entry is the upper (respectively, lower) diagonal entry equal to 1 and the matrices C_1, C_2 can be explicitly computed by inspecting the behaviour of Φ inside the disk. Such matrix has the same jumps on γ and $\hat{\gamma}$ as for Y and it displays an extra jump on the circle $\partial\mathbb{D}$ (counterclockwise oriented). The "error" matrix $\mathcal{E} := Y\Phi^{-1}$

has only a jump on $\partial\mathbb{D}$ by construction:

$$\mathcal{E}_+ = \mathcal{E}_- \left(\Phi_- \Phi_+^{-1}\right) \quad \lambda \in \partial\mathbb{D}, \qquad \mathcal{E} = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \lambda \to \infty. \tag{5.2.45}$$

It is possible to show that the surviving jump on $\partial\mathbb{D}$ is a perturbation of the identity of the order $|x|^{2\nu+3}$. Therefore, thanks to a small norm argument ([49], Chapter 4), the matrix Φ can be considered as an explicit approximant of the matrix Y (and of the original matrix Γ). Inspecting its behaviour at ∞, it is possible to recover the functions Y, V, U, W and $F := \frac{Y}{WU}$ appearing in the Lax pair; in particular,

$$U(x) = C_\nu x^{2\nu+1} + \mathcal{O}\left(x^{2\nu+2}\right), \qquad F(x) = \frac{-2\nu}{x} + \mathcal{O}\left(1\right), \tag{5.2.46}$$

where C_ν is a constant depending on the parameter ν.

For $-1 < \nu \leq 0$ the argument is similar, but one needs to be more careful with the asymptotic expansion and the rate of convergence.

Differentiating (5.2.40) and using (5.2.41), we get the following Painlevé III equation:

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} = \frac{1}{F}\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}F}{\mathrm{d}x} + \frac{2}{x}\left(\Theta_0 F^2 + \nu - 1\right) + F^3 - \frac{1}{F}. \tag{5.2.47}$$

Given the expression of the matrix A, we can find an expression for the residue matrix Γ_1. Focusing on the residue at 0, we can perform similar calculation with the already known Lax pair (5.2.33)-(5.2.34) and obtain

$$\begin{aligned} &\operatorname*{res}_{\lambda=0} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_x T\right) = -\operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_x T\right) \\ &= \frac{1}{2x}\left[-2U^2F^2 + \left(xF^2 - 2\nu F + x\right)U - \frac{x^2}{4}\right]. \end{aligned} \tag{5.2.48}$$

In conclusion,

Theorem 5.16. *The gap probability of the Bessel process restricted to a single interval satisfies the following identity*

$$\det(\mathrm{Id} - \mathbb{B}) = \exp\left\{\int_0^x H_{\mathrm{III}}(s)\,\mathrm{d}s\right\} \tag{5.2.49}$$

where H_{III} is the Hamiltonian associated to the Painlevé III equation (see [53])

$$H_{\mathrm{III}}(F, U; x) = \frac{1}{x}\left[-2U^2F^2 + \left(xF^2 - 2\nu F + x\right)U - \frac{x^2}{4}\right]. \tag{5.2.50}$$

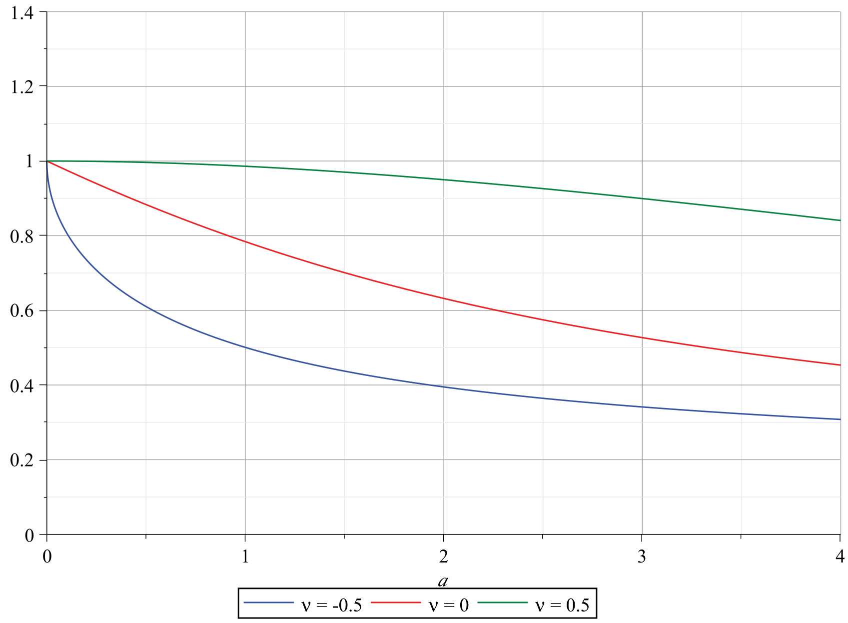

Figure 5.3: Numerical computation of the Fredholm determinant $\det(\mathrm{Id} - K_B\chi_{[0,a]})$ as function of a with different values of the parameter ν. The outcome has been obtained by directly calculating the Fredholm determinant of the Bessel operator (following the ideas of [14]).

Remark 5.17. *The Hamiltonian H_{III} is singular at 0, but it is integrable in a (right) neighborhood of the origin: $H_{\mathrm{III}} \in L^1(0,\epsilon)$, $\epsilon > 0$. Given that $\partial \ln \tau = H_{\mathrm{III}}$ and τ is continuous, this yields $\tau(0) = \det(\mathrm{Id}) = 1$, as expected.*

5.3 The multi-time Bessel process

5.3.1 Preliminary results

The multi-time Bessel process on $L^2(\mathbb{R}_+)$ with times $\tau_1 < \ldots < \tau_n$ is governed by the matrix operator $[K_B] := [\tilde{K}_B] + [H_B]$ with kernels $[K_B]$, $[\tilde{K}_B]$ and $[H_B]s$ given as follows

$$[K_B]_{ij}(x,y) := [\tilde{K}_B]_{ij}(x,y) + [H_B]_{ij}(x,y) \tag{5.3.1a}$$

$$[\tilde{K}_B]_{ij}(x,y) := \frac{1}{(2\pi i)^2}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \iint_{\gamma\times\hat{\gamma}_j} \frac{\mathrm{d}t\,\mathrm{d}s}{ts} \frac{e^{\Delta_{ij}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{ij}}\left(\frac{s}{t}\right)^{\nu} \tag{5.3.1b}$$

$$[H_B]_{ij}(x,y) := \chi_{\tau_i<\tau_j}\frac{1}{\Delta_{ji}}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \int_\gamma e^{\frac{x}{4\Delta_{ji}}(t-1)+\frac{y}{4\Delta_{ji}}\left(\frac{1}{t}-1\right)} t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i} \tag{5.3.1c}$$

with the same curve γ as in the single-time Bessel kernel (a contour that winds around zero counterclockwise an extends to $-\infty$) and $\hat{\gamma}_j := \frac{1}{\gamma+4\tau_j}$, $\forall\, i,j = 1,\ldots,n$, $\Delta_{ij} := \tau_i - \tau_j$.

Remark 5.18. *The matrix $H_{B;ij}$ is strictly upper triangular.*

Remark 5.19. *The integral expression (5.3.1b)-(5.3.1c) for the multi-time Bessel kernel is equivalent to the one given in the introduction (5.1.8) (see [72] and [102]). To prove the equivalence, one simply needs to write the Bessel functions as contour integrals and perform some suitable integrations by parts.*

As in the single-time case, we are interested in the following quantity

$$\det\left(\mathrm{Id} - K_B\Big|_{\mathcal{I}}\right) \tag{5.3.2}$$

which is equal to the gap probability of the multi-time Bessel kernel restricted to a collection of multi-intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$,

$$I_j := [a_1^{(j)}, a_2^{(j)}] \cup \ldots \cup [a_{2k_j-1}^{(j)}, a_{2k_j}^{(j)}], \qquad 0 \le a_1^{(j)} < \ldots < a_{2k_j}^{(j)}. \tag{5.3.3}$$

Remark 5.20. *The multi-time Bessel operator fails to be trace-class on infinite intervals.*

For the sake of clarity, we will focus on the simple case $I_j = [0, a^{(j)}]$, $j = 1, \ldots, n$. The general case follows the same guidelines described below.

Theorem 5.21. *The following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - K_B\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - \mathbb{K}_B\right) \tag{5.3.4}$$

where $\mathcal{I}$ is defined as in (5.3.3). The operator $\mathbb{K}_B$ is an integrable operator with a $2n \times 2n$ matrix kernel of the form

$$\mathbb{K}_B(t, \xi) = \frac{\boldsymbol{f}(t)^T \cdot \boldsymbol{g}(\xi)}{t - \xi} \tag{5.3.5}$$

acting on the Hilbert space

$$H := L^2\left(\gamma \cup \bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \sim L^2\left(\bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \oplus L^2(\gamma, \mathbb{C}^n), \tag{5.3.6}$$

with $\gamma_{-k} := \frac{1}{\gamma} - 4\tau_k$.

The functions $\boldsymbol{f}$, $\boldsymbol{g}$ are the following $2n \times 2n$ matrices

$$\boldsymbol{f}(t) = \left[\begin{array}{c|c} \operatorname{diag} \mathcal{N}(t) & 0 \\ \hline 0 & A(\mathcal{M}(t)) \\ \hline 0 & B(\mathcal{H}(t)) \end{array}\right] \tag{5.3.7}$$

$$\boldsymbol{g}(\xi) = \left[\begin{array}{c|c} 0 & \operatorname{diag} \mathcal{N}(\xi) \\ \hline C(\mathcal{M}(\xi)) & 0 \\ \hline 0 & D(\mathcal{H}(\xi)) \end{array}\right] \tag{5.3.8}$$

where $\operatorname{diag} \mathcal{N}$ *is a $n \times n$ matrix, A and C are two rows with n entries and B and D are $(n-1) \times n$ matrices,*

$$\operatorname{diag} \mathcal{N}(t) := \operatorname{diag}\left(-4e^{-\frac{a^{(1)}}{t_1}}\chi_\gamma(t), \dots, -4e^{-\frac{a^{(n)}}{t_n}}\chi_\gamma(t)\right) \tag{5.3.9}$$

$$A(\mathcal{M}(t)) := \left[e^{-\frac{t}{4}}t_1^\nu \chi_{\gamma_{-1}}(t), \dots, e^{-\frac{t}{4}}t_n^\nu \chi_{\gamma_{-n}}(t)\right] \tag{5.3.10}$$

$$B(\mathcal{H}(t)) := \begin{bmatrix} -4e^{-\frac{a^{(2)}}{t_2}}\frac{t_1^\nu}{t_2^\nu}\chi_{\gamma_{-1}} & 0 & & & \\ -4e^{-\frac{a^{(3)}}{t_3}}\frac{t_1^\nu}{t_3^\nu}\chi_{\gamma_{-1}} & -4e^{-\frac{a^{(3)}}{t_3}}\frac{t_2^\nu}{t_3^\nu}\chi_{\gamma_{-2}} & & & \\ \vdots & \vdots & \ddots & & \\ -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_1^\nu}{t_n^\nu}\chi_{\gamma_{-1}} & -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_2^\nu}{t_n^\nu}\chi_{\gamma_{-2}} & & -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_{n-1}^\nu}{t_n^\nu}\chi_{\gamma_{-(n-1)}} & 0 \end{bmatrix} \tag{5.3.11}$$

$$\operatorname{diag} \mathcal{N}(\xi) := \operatorname{diag}\left(e^{\frac{a^{(1)}}{\xi_1}}\chi_{\gamma_{-1}}(\xi), \dots, e^{\frac{a^{(n)}}{\xi_n}}\chi_{\gamma_{-n}}(\xi)\right) \tag{5.3.12}$$

$$C(\mathcal{M}(\xi) := \left[e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_\gamma(\xi), \dots, e^{\frac{\xi}{4}}\xi_n^{-\nu}\chi_\gamma(\xi)\right] \tag{5.3.13}$$

$$D(\mathcal{H}(\xi)) = \begin{bmatrix} 0 & e^{\frac{a^{(2)}}{\xi_2}}\chi_{\gamma_{-2}}(\xi) & & & & \\ & 0 & e^{\frac{a^{(3)}}{\xi_3}}\chi_{\gamma_{-3}}(\xi) & & & \\ & & 0 & e^{\frac{a^{(4)}}{\xi_4}}\chi_{\gamma_{-4}}(\xi) & & \\ & & & & \ddots & \\ & & & & 0 & e^{\frac{a^{(n)}}{\xi_n}}\chi_{\gamma_{-n}}(\xi) \end{bmatrix} \tag{5.3.14}$$

with $\xi_k := \xi + 4\tau_k$, $t_k := t + 4\tau_k$, for $k = 1, \dots, n$.

Remark 5.22. *The naming of Fredholm determinant in the theorem above needs some clar-*

ification: by "det" *we denote the determinant defined through the Fredholm expansion*

$$\det(\mathrm{Id} - K) := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{X^k} \det[K(x_i, x_j)]_{i,j=1}^{k} \mathrm{d}\mu(x_1) \dots \mathrm{d}\mu(x_k) \tag{5.3.15}$$

with K an integral operator acting on the Hilbert space $L^2(X, \mathrm{d}\mu(x))$, with kernel $K(x, y)$.

In our case, the operator $[K_B]\Big|_{\mathcal{I}} = ([\tilde{K}_B] + [H_B])\Big|_{\mathcal{I}}$ is actually the sum of a trace-class operator $[\tilde{K}_B]\Big|_{\mathcal{I}}$ and a Hilbert-Schmidt operator $[H_B]\Big|_{\mathcal{I}}$ whose kernel is diagonal-free, as it will be clear along the proof.

Thus, to be precise, we have the following chain of identities

$$\begin{aligned} \text{“}\det\text{”} \left(\mathrm{Id} - K_B\Big|_{\mathcal{I}} \right) &= \text{“}\det\text{”} \left(\mathrm{Id} - \tilde{K}_B\Big|_{\mathcal{I}} - H_B\Big|_{\mathcal{I}} \right) \\ &= e^{\mathrm{Tr}\, \tilde{K}_B} \det{}_2 \left(\mathrm{Id} - \tilde{K}_B\Big|_{\mathcal{I}} - H_B\Big|_{\mathcal{I}} \right) \end{aligned} \tag{5.3.16}$$

where $\det_2$ denotes the regularized Carleman determinant (see [95] for a detailed description of the theory).

Proof. Thanks to the invariance of the Fredholm determinant under kernel conjugation, we can discard the term $\left(\frac{y}{x}\right)^{\nu/2}$ in our further calculations.

We will work on the entry (i, j) of the kernel. We can notice that for $x < 0$ or $y < 0$ the kernel is identically zero, $K_B(x, y) \equiv 0$, as in the single-time case. Then, applying Cauchy's theorem and after some suitable calculations, we have

$$\begin{aligned} \tilde{K}_{B;ij}(x, y)\Big|_{[0, a^{(j)}]} &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{\xi(a^{(j)} - y)}}{\xi - s} \iint_{\gamma \times \hat{\gamma}_j} \frac{\mathrm{d}t\, \mathrm{d}s}{(2\pi i)^2 ts} \frac{e^{\Delta + xt - \frac{1}{4t} - a^{(j)}s + \frac{1}{4s}}}{\frac{1}{4t} - \frac{1}{4s} - \Delta} \left(\frac{s}{t}\right)^{\nu} \\ &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-\xi y} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{xt} \int_{\hat{\gamma}_j} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\Delta + \xi a^{(j)} - \frac{1}{4t} - a^{(j)}s + \frac{1}{4s}}}{(\xi - s)\left(\frac{1}{4t} - \frac{1}{4s} - \Delta\right)} \left(\frac{s}{t}\right)^{\nu} \frac{1}{ts} \\ &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-\xi y} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{xt} \int_{\gamma} \frac{\mathrm{d}s}{2\pi i} \frac{4 e^{\tau_i + \xi a^{(j)} - \frac{1}{4t} - \frac{a^{(j)}}{s + 4\tau_j} + \frac{s}{4}}}{\left(\frac{1}{\xi} - 4\tau_j - s\right)\left(\frac{1}{t} - 4\tau_i - s\right)} \left(\frac{1}{(s + 4\tau_j)t}\right)^{\nu} \frac{1}{t\xi} \end{aligned} \tag{5.3.17}$$

where we deformed γ into a translated imaginary axis $i\mathbb{R} + \epsilon$ ($\epsilon > 0$) in order to make Fourier transform operator more explicit; the last equality follows from the change of variable on s: $s \to 1/(s + 4\tau_j)$ (thus the contour $\hat{\gamma}_j$ becomes similar to γ and can be continuously deformed into that).

On the other hand

$$
\begin{aligned}
H_{B;ij}(x,y)\Big|_{[0,a^{(j)}]} &= \frac{-1}{\Delta_{ji}}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}\frac{e^{\xi(a^{(j)}-y)}}{\xi-\frac{1}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}\int_{\gamma}e^{\frac{x}{4\Delta_{ji}}(t-1)-\frac{a^{(j)}}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i}\\
&= \frac{-1}{\Delta_{ji}}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}\frac{e^{\xi a^{(j)}+\frac{x}{4\Delta_{ji}}(t-1)-\frac{a^{(j)}}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}}{\xi-\frac{1}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i}\\
&= -4\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}t}{2\pi i}\,e^{xt}\frac{e^{a^{(j)}\left(\xi-\frac{t}{4\Delta_{ji}t+1}\right)}}{t\xi\left(4\Delta_{ji}+\frac{1}{t}-\frac{1}{\xi}\right)}(4\Delta_{ji}t+1)^{-\nu}.
\end{aligned}
\tag{5.3.18}
$$

It is easily recognizable the conjugation with a Fourier-like operator as in (6.3.7), so that

$$
\left(K_B\Big|_{\mathcal{I}}\right)_{ij} = \mathcal{F}^{-1}\circ\left(\mathcal{B}_{ij}+\chi_{i<j}\mathcal{H}_{ij}\right)\circ\mathcal{F} \tag{5.3.19}
$$

with

$$
\mathcal{B}_{ij}(t,\xi) = \int_{\gamma}\frac{\mathrm{d}s}{2\pi i}\frac{4e^{\tau_i+\xi a^{(j)}-\frac{1}{4t}-\frac{a^{(j)}}{s+4\tau_j}+\frac{s}{4}}}{\left(\frac{1}{\xi}-4\tau_j-s\right)\left(\frac{1}{t}-4\tau_i-s\right)}\left(\frac{1}{(s+4\tau_j)t}\right)^{\nu}\frac{1}{t\xi} \tag{5.3.20}
$$

$$
\mathcal{H}_{ij}(t,\xi) := -4\frac{e^{a^{(j)}\left(\xi-\frac{t}{4\Delta_{ji}t+1}\right)}}{4\tau_j-4\tau_i+\frac{1}{t}-\frac{1}{\xi}}(4\Delta_{ji}t+1)^{-\nu}\frac{1}{\xi t}. \tag{5.3.21}
$$

Now we will perform a change of variables on the Fourier-transformed kernel $\mathcal{B}_{ij}+\chi_{i<j}\mathcal{H}_{ij}$: $\xi_j := \frac{1}{\xi}-4\tau_j$ and $\eta_i := \frac{1}{t}-4\tau_i$. This will lead to the following expression for the (Fourier-transformed) multi-time Bessel kernel

$$
\mathcal{K}_{B;ij} = \mathcal{B}_{ij}(\eta,\xi)+\chi_{\tau_i<\tau_j}\mathcal{H}_{ij}(\eta,\xi) =
$$

$$
4\int_{\gamma}\frac{\mathrm{d}t}{2\pi i}\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{\eta}{4}-\frac{a^{(j)}}{t+4\tau_j}+\frac{t}{4}}}{(\xi-t)(\eta-t)}\left(\frac{\eta+4\tau_i}{t+4\tau_j}\right)^{\nu}+\chi_{\tau_i<\tau_j}\cdot 4\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{a^{(j)}}{\eta+4\tau_j}}}{\xi-\eta}\left(\frac{\eta+4\tau_j}{\eta+4\tau_i}\right)^{-\nu} \tag{5.3.22}
$$

with $\xi\in\frac{1}{\gamma}-4\tau_j =: \gamma_{-j}$ and $\eta\in\frac{1}{\gamma}-4\tau_i =: \gamma_{-i}$, $\forall\, i,j=1,\ldots,n$. Such operator is acting on the Hilbert space $L^2\left(\bigcup_{k=1}^{n}\gamma_{-k},\mathbb{C}^n\right)\sim\bigoplus_{k=1}^{n}L^2\left(\gamma_{-k},\mathbb{C}^n\right)$.

Lemma 5.23. *The operator $\mathcal{B}$ is trace-class and the operator $\mathcal{H}$ is Hilbert-Schmidt. More-*

over, the following decomposition holds $\mathcal{K}_B = \mathcal{M}\circ\mathcal{N} + \mathcal{H}$, *where*

$$\begin{gathered}\mathcal{M}:\ L^2(\gamma,\mathbb{C}^n)\to L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right), \qquad \mathcal{N}:\ L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right)\to L^2(\gamma,\mathbb{C}^n)\\ \mathcal{H}:\ L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right)\to L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right)\end{gathered} \tag{5.3.23}$$

with entries

$$\mathcal{M}_{ij}(t,\eta) := \frac{1}{2\pi i}\frac{e^{-\frac{\eta}{4}+\frac{t}{4}}}{\eta - t}\left(\frac{\eta_i}{t_j}\right)^{\nu}\cdot\chi_\gamma(t)\cdot\chi_{\gamma_{-i}}(\eta) \tag{5.3.24a}$$

$$\mathcal{N}_{ij}(\xi,t;a^{(j)}) = 4\delta_{ij}\cdot\frac{e^{a^{(j)}\left(\frac{1}{\xi_j}-\frac{1}{t_j}\right)}}{\xi - t}\cdot\chi_{\gamma_{-j}}(\xi)\cdot\chi_\gamma(t) \tag{5.3.24b}$$

$$\mathcal{H}_{ij}(\xi,\eta) = \chi_{\tau_i<\tau_j}\cdot 4\frac{e^{a^{(j)}\left(\frac{1}{\xi_j}-\frac{1}{\eta_j}\right)}}{\xi-\eta}\left(\frac{\eta_j}{\eta_i}\right)^{-\nu}\cdot\chi_{\gamma_{-i}}(\eta)\cdot\chi_{\gamma_{-j}}(\xi) \tag{5.3.24c}$$

$\zeta_k := \zeta + 4\tau_k$ *(*$\zeta = \xi, t, \eta$*) and* $\gamma_{-k} := \frac{1}{\gamma} - 4\tau_k$, $\forall\, k = 1,\ldots,n$.

Proof. All the kernels are of the general form $H(z,w)$ with z and w on disjoint supports, that we indicate now temporarily by S_1, S_2. It is then simple to see that in each instance $\int_{S_1}\int_{S_2}|H(z,w)|^2|\mathrm{d}z||\mathrm{d}w| < +\infty$ and hence each operator is Hilbert-Schmidt. Then $\mathcal{B}$ is trace class because it is the composition of two HS operators. $\square$

Now consider the Hilbert space

$$H := L^2\left(\gamma\cup\bigcup_{k=1}^n\frac{1}{\gamma}-4\tau_k,\mathbb{C}^n\right)\sim L^2\left(\bigcup_{k=1}^n\frac{1}{\gamma}-4\tau_k,\mathbb{C}^n\right)\otimes L^2(\gamma,\mathbb{C}^n), \tag{5.3.25}$$

and the matrix operator $\mathbb{K}_B : H\to H$ defined as

$$\mathbb{K}_B = \left[\begin{array}{c|c} 0 & \mathcal{N}\\ \hline \mathcal{M} & \mathcal{H}\end{array}\right] \tag{5.3.26}$$

due to the splitting of the space H into its two main addenda.

For now, we denote by “det” the determinant defined by the Fredholm expansion (6.4.6); then, “det”$(\mathrm{Id}-\mathbb{K}_B) = \det_2(\mathrm{Id}-\mathbb{K}_B)$, since its kernel is diagonal-free. Moreover, we introduce another Hilbert-Schmidt operator

$$\mathbb{K}'_B = \left[\begin{array}{c|c} 0 & -\mathcal{N} \\ \hline 0 & 0 \end{array}\right]$$

which is only Hilbert-Schmidt, but nevertheless its Carleman determinant ($\det_2$) is well defined and $\det_2 (I - \mathbb{K}'_B) \equiv 1$.

Collecting all the results we have seen so far, we perform the following chain of equalities

$$\begin{aligned} \text{“}\det\text{”}\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_B\Big|_{\mathcal{I}}\right) &= \det{}_2\left(\mathrm{Id} - K_B\Big|_{\mathcal{I}}\right) e^{-\operatorname{Tr}(\tilde{K})} \\ = \det{}_2\left(\mathrm{Id}_{L^2\left(\bigcup_{k=1}^n \gamma_{-k}\right)} - \mathcal{K}_B\right) e^{-\operatorname{Tr}(\mathcal{B})} &= \det{}_2\left(\mathrm{Id}_H - \mathbb{K}_B\right)\det{}_2\left(\mathrm{Id}_H - \mathbb{K}'_B\right) \\ &= \det{}_2\left(\mathrm{Id}_H - \mathbb{K}_B\right) = \text{“}\det\text{”}(\mathrm{Id}_H - \mathbb{K}_B). \end{aligned} \tag{5.3.27}$$

The first equality follows from the fact that $K_B - \tilde{K}_B$ is diagonal-free; the second equality follows from invariance of the determinant under Fourier transform; the first identity on the last line is just an application of the following result: given $\mathbb{K}_B$, $\mathbb{K}'_B$ Hilbert-Schmidt operators, then

$$\det{}_2\left(\mathrm{Id} - \mathbb{K}_B\right)\det{}_2\left(\mathrm{Id} - \mathbb{K}'_B\right) = \det{}_2\left(\mathrm{Id} - \mathbb{K}_B - \mathbb{K}'_B + \mathbb{K}_B\mathbb{K}'_B\right)e^{\operatorname{Tr}(\mathbb{K}_B\mathbb{K}'_B)}.$$

It is finally just a matter of computation to show that $\mathbb{K}_B$ is an integrable operator of the form (5.3.5)-(5.3.14). ☐

Example 2×2. For the sake of clarity, let us consider a simple example of the multi-time Bessel process with two times τ_1, τ_2, restricted to the finite intervals $I_1 := [0, a]$ and $I_2 := [0, b]$:

$$\begin{aligned} &K_B(x,y)\Big|_{I_1,I_2} = \\ &\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \left\{ \begin{bmatrix} 4\chi_{[0,a]}(y)\int_{\gamma\times\hat{\gamma}_j} \frac{dt\,ds}{(2\pi i)^2} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s}\left(\frac{s}{t}\right)^\nu & \frac{\chi_{[0,b]}(y)}{(2\pi i)^2}\int_{\gamma\times\hat{\gamma}_j} \frac{dt\,ds}{ts} \frac{e^{\Delta_{12}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{12}}\left(\frac{s}{t}\right)^\nu \\ \frac{\chi_{[0,a]}(y)}{(2\pi i)^2}\int_{\gamma\times\hat{\gamma}_j} \frac{dt\,ds}{ts} \frac{e^{\Delta_{21}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{21}}\left(\frac{s}{t}\right)^\nu & 4\chi_{[0,b]}(y)\int_{\gamma\times\hat{\gamma}_j} \frac{dt\,ds}{(2\pi i)^2} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s}\left(\frac{s}{t}\right)^\nu \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} 0 & -\chi_{[0,b]}(y)\frac{1}{\Delta_{12}}\int_\gamma e^{\frac{x}{4\Delta_{12}}(1-t)+\frac{y}{4\Delta_{12}}\left(1-\frac{1}{t}\right)} t^{-\nu-1}\frac{dt}{2\pi i} \\ 0 & 0 \end{bmatrix} \right\}. \end{aligned} \tag{5.3.28}$$

Then, the integral operator $\mathbb{K}_B : H \to H$ on the space $H := L^2\left(\gamma \cup \gamma_{-1} \cup \gamma_{-2}, \mathbb{C}^2\right)$ has the following expression

$$
\mathbb{K}_B = \left[\begin{array}{c|c} 0 & \mathcal{N} \\ \hline \mathcal{M} & \mathcal{H} \end{array}\right] =
$$

$$
\left[\begin{array}{cc|cc}
0 & 0 & -4e^{\frac{a}{\xi_1}}\chi_{\gamma-1}e^{-\frac{a}{t_1}}\chi_\gamma & 0 \\
0 & 0 & 0 & -4e^{\frac{b}{\xi_2}}\chi_{\gamma-2}e^{-\frac{b}{t_2}}\chi_\gamma \\
\hline
e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_\gamma e^{-\frac{t}{4}}t_1^\nu\chi_{\gamma-1} & e^{\frac{\xi}{4}}\xi_2^{-\nu}\chi_\gamma e^{-\frac{t}{4}}t_1^\nu\chi_{\gamma-1} & 0 & -4e^{\frac{b}{\xi_2}}\chi_{\gamma-2}e^{-\frac{b}{t_2}}\frac{t_1^\nu}{t_2^\nu}\chi_{\gamma-1} \\
e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_\gamma e^{-\frac{t}{4}}t_2^\nu\chi_{\gamma-2} & e^{\frac{\xi}{4}}\xi_2^{-\nu}\chi_\gamma e^{-\frac{t}{4}}t_2^\nu\chi_{\gamma-2} & 0 & 0
\end{array}\right] \tag{5.3.29}
$$

and the equality between Fredholm determinants holds

$$
\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+,\mathbb{C}^2)} - K_B\Big|_{I_1,I_2}\right) = \det\left(\mathrm{Id}_H - \mathbb{K}_B\right). \tag{5.3.30}
$$

5.3.2 The Riemann-Hilbert problem for the multi-time Bessel process.

As explained in the introduction, we can relate the computation of the Fredholm determinant of the matrix Bessel operator to the theory of isomonodromic equations, through a suitable Riemann-Hilbert problem.

Proposition 5.24. *Given the integrable kernel (5.3.5)-(5.3.14), the associated Riemann-Hilbert problem is the following:*

$$
\Gamma_+(\lambda) = \Gamma_-(\lambda)\left(I - 2\pi i J_B(\lambda)\right) \qquad \lambda \in \Sigma \tag{5.3.31a}
$$

$$
\Gamma(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty \tag{5.3.31b}
$$

where Γ *is a* $2n \times 2n$ *matrix* Γ *such that it is analytic on the complex plane except at* $\Sigma := \gamma \cup \bigcup_{k=1}^n \frac{1}{\gamma} - 4\tau_k$*; the jump matrix* $J_B(\lambda) := \boldsymbol{f}(\lambda)\cdot\boldsymbol{g}(\lambda)^T$ *has the expression*

$$
J_B(\lambda) := \left[\begin{array}{c|c|c} 0 & \star_1 & 0 \\ \hline \star_2 & 0 & \star_3 \\ \hline \star_4 & 0 & \star_5 \end{array}\right]
$$

$$\star_1 := \left[-4e^{\theta_1}\chi_\gamma, \ldots, -4e^{\theta_n}\chi_\gamma\right]^T$$
$$\star_2 := \left[e^{-\theta_1}\chi_{\gamma_{-1}}, \ldots, e^{-\theta_n}\chi_{\gamma_{-n}}\right]$$
$$\star_3 := \left[e^{-\theta_2}\chi_{\gamma_{-2}}, \ldots, e^{-\theta_n}\chi_{\gamma_{-n}}\right]$$

$$\star_4 := \begin{bmatrix} -4e^{-\theta_1+\theta_2}\chi_{\gamma_{-1}} & 0 & & & \\ -4e^{-\theta_1+\theta_3}\chi_{\gamma_{-1}} & -4e^{-\theta_2+\theta_3}\chi_{\gamma_{-2}} & 0 & & \\ -4e^{-\theta_1+\theta_4}\chi_{\gamma_{-1}} & -4e^{-\theta_2+\theta_4}\chi_{\gamma_{-2}} & -4e^{-\theta_3+\theta_4}\chi_{\gamma_{-3}} & 0 & \\ \vdots & & & \ddots & \\ -4e^{-\theta_1+\theta_n}\chi_{\gamma_{-1}} & \ldots & & -4e^{-\theta_{n-1}+\theta_n}\chi_{\gamma_{-(n-1)}} & 0 \end{bmatrix}$$

$$\star_5 := \begin{bmatrix} 0 & & & \\ -4e^{-\theta_2+\theta_3}\chi_{\gamma_{-2}} & & & \\ -4e^{-\theta_2+\theta_4}\chi_{\gamma_{-2}} & -4e^{-\theta_3+\theta_4}\chi_{\gamma_{-3}} & & \\ \vdots & \vdots & & \\ & & & \\ -4e^{-\theta_2+\theta_n}\chi_{\gamma_{-2}} & & -4e^{-\theta_{n-1}+\theta_n}\chi_{\gamma_{n-1}} & 0 \end{bmatrix}$$

$$\theta_i := \frac{\lambda}{4} - \frac{a_i}{\lambda_i} - \nu \ln \lambda_i, \qquad \lambda_i = \lambda + 4\tau_i. \tag{5.3.32}$$

We recall that we are considering the simple case $\mathcal{I} = \bigsqcup_j I_j$ with $I_j := [0, a^{(j)}]$, $\forall\, j = 1, \ldots, n$.

Applying again the results stated in [9] and [11], we can claim the following.

Theorem 5.25. *Given n times $\tau_1 < \tau_2 < \ldots < \tau_n$ and given the multi-interval $\mathcal{I} = \{I_1, \ldots, I_n\}$, the Tracy-Widom distribution of the multi-time Bessel operator, i.e. the Fredholm determinant* $\det\left(\mathrm{Id} - [K_B]\Big|_{\mathcal{I}}\right)$*, is equal to the isomonodromic τ-function related to the above Riemann-Hilbert problem.*

In particular, we have

$$\partial \ln \det\left(\mathrm{Id} - [K_B]\Big|_{\mathcal{I}}\right) = \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_\partial(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{5.3.33a}$$
$$\Xi_\partial(\lambda) = -2\pi i\, \partial J_B\left(I + 2\pi i J_B\right) \tag{5.3.33b}$$

the $'$ notation means differentiation with respect to λ, while with ∂ we denote any of the *derivatives with respect to times ∂_{τ_k} or endpoints $\partial_{a^{(k)}}$ $(k = 1, \ldots, n)$.*

Proof. Keeping into account Theorem 3.14 from Section 3.4, it is enough to verify that $H(I - J_B(\lambda)) = 0$. ☐

Example 2×2. In the simple 2-times case, the jump matrix is

$$J_B(\lambda) = \mathbf{f}(\lambda) \cdot \mathbf{g}(\lambda)^T =$$
$$\left[\begin{array}{cc|c|c} 0 & 0 & -4e^{\frac{\lambda}{4} - \frac{a}{\lambda_1}} \lambda_1^{-\nu} \chi_\gamma & 0 \\ 0 & 0 & -4e^{\frac{\lambda}{4} - \frac{b}{\lambda_2}} \lambda_2^{-\nu} \chi_\gamma & 0 \\ \hline e^{-\frac{\lambda}{4} + \frac{a}{\lambda_1}} \lambda_1^{\nu} \chi_{\gamma-1} & e^{-\frac{\lambda}{4} + \frac{b}{\lambda_2}} \lambda_2^{\nu} \chi_{\gamma-2} & 0 & e^{-\frac{\lambda}{4} + \frac{b}{\lambda_2}} \lambda_2^{\nu} \chi_{\gamma-2} \\ \hline -4e^{\frac{a}{\lambda_1} - \frac{b}{\lambda_2}} \left(\frac{\lambda_1}{\lambda_2}\right)^{\nu} \chi_{\gamma-1} & 0 & 0 & 0 \end{array}\right]. \tag{5.3.34}$$

Thanks to Theorem 6.25, it is possible to derive some more explicit differential identities by using the Jimbo-Miwa-Ueno residue formula (see [9]).

First we notice that the jump matrix is equivalent up to conjugation with a constant matrix J_0:

$$J_B(\lambda) = e^{T_B(\lambda)} J_B^0 e^{-T_B(\lambda)} \tag{5.3.35}$$

with

$$T_B(\lambda) := \operatorname{diag}\left[\theta_1 - \frac{\kappa}{2n}, \ldots, \theta_n - \frac{\kappa}{2n}, 1 - \frac{\kappa}{2n}, \theta_2 - \frac{\kappa}{2n}, \ldots, \theta_n - \frac{\kappa}{2n}\right]$$
$$\kappa := \theta_1 + 2\sum_{k=2}^{n} \theta_k. \tag{5.3.36}$$

Therefore, the matrix $\Psi_B(\lambda) = \Gamma(\lambda) e^{T_B(\lambda)}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Theorem 5.26. *The quantity (5.3.33a) can be computed explicitly*

$$\int_\Sigma \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma'_-(\lambda)\Xi_\partial(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} = -\operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}\Gamma' \partial T_B\right) +$$
$$+ \sum_{i=1}^{n} \operatorname*{res}_{\lambda=-4\tau_i} \operatorname{Tr}\left(\Gamma^{-1}\Gamma' \partial T_B\right). \tag{5.3.37}$$

More specifically, regarding the derivative with respect to the endpoints $a^{(i)}$ $(i = 1, \ldots, n)$, *we have*

$$\operatorname*{res}_{\lambda=-4\tau_1} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(1)}}T_B\right) = \left(\frac{1}{2n}-1\right)\left(\Gamma_0^{-1}\Gamma_1\right)_{(1,1)} \tag{5.3.38a}$$

$$\operatorname*{res}_{\lambda=-4\tau_i} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(i)}}T_B\right) = \left(\frac{1}{n}-1\right)\left[\left(\Gamma_0^{-1}\Gamma_1\right)_{(i,i)} + \left(\Gamma_0^{-1}\Gamma_1\right)_{(i+n,i+n)}\right] \tag{5.3.38b}$$

and, regarding the derivative with respect to the times τ_i ($i = 1, \ldots, n$), we have

$$\begin{aligned}\operatorname*{res}_{\lambda=-4\tau_1} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_1}T_B\right) &= 4\nu\left(\frac{1}{2n}-1\right)\left(\Gamma_0^{-1}\Gamma_1\right)_{(1.1)} + \\ &+4a^{(1)}\left(1-\frac{1}{2n}\right)\left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(1,1)}\end{aligned} \tag{5.3.39a}$$

$$\begin{aligned}\operatorname*{res}_{\lambda=-4\tau_i} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_i}T_B\right) &= 4\nu\left(\frac{1}{n}-1\right)\left[\left(\Gamma_0^{-1}\Gamma_1\right)_{(i.i)} + \left(\Gamma_0^{-1}\Gamma_1\right)_{(i+n,i+n)}\right] + \\ &+4a^{(i)}\left(1-\frac{1}{n}\right)\left[\left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(i,i)} + \right. \\ &\left. + \left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(i+n,i+n)}\right]\end{aligned} \tag{5.3.39b}$$

where the Γ_i's are coefficients of the asymptotic expansion of the matrix Γ near ∞ and $-4\tau_j$. We recall that each asymptotic expansion (the Γ_i's) is different in a neighborhood of each point $-4\tau_j$ and it's different from the one near ∞.

The residue at infinity does not give any contribution in either case.

Proof. We calculate the derivatives of the conjugation factor

$$\begin{aligned}\partial_{a^{(1)}}T_B(\lambda) &= \operatorname{diag}\left[\partial_{a^{(1)}}\left(\theta_1 - \frac{\kappa}{2n}\right), 0, \ldots, 0\right] = \operatorname{diag}\left[\frac{1}{\lambda_1}\left(\frac{1}{2n}-1\right), 0, \ldots, 0\right] \\ &= \frac{1}{\lambda_1}\left(\frac{1}{2n}-1\right)\cdot E_{(1,1)}\end{aligned} \tag{5.3.40}$$

$$\begin{aligned}\partial_{a^{(i)}}T_B(\lambda) &= \operatorname{diag}\left[0, \ldots, \partial_{a^{(i)}}\left(\theta_i - \frac{\kappa}{2n}\right), \ldots, \partial_{a^{(i)}}\left(\theta_i - \frac{\kappa}{2n}\right), \ldots, 0\right] \\ &= \frac{1}{\lambda_i}\left(\frac{1}{n}-1\right)\cdot E_{(i,i),\,(i+n,i+n)}\end{aligned} \tag{5.3.41}$$

$$\partial_{\tau_1} T_B(\lambda) = \mathrm{diag}\left[\partial_{\tau_1}\left(\theta_1 - \frac{\kappa}{2n}\right), 0, \ldots, 0\right] = \mathrm{diag}\left[\left(\frac{4a^{(1)}}{\lambda_1^2} - \frac{4\nu}{\lambda_1}\right)\left(1 - \frac{1}{2n}\right), 0, \ldots, 0\right]$$
$$= \left(\frac{4a^{(1)}}{\lambda_1^2} - \frac{4\nu}{\lambda_1}\right)\left(1 - \frac{1}{2n}\right) \cdot E_{(1,1)} \tag{5.3.42}$$

$$\partial_{\tau_i} T_B(\lambda) = \mathrm{diag}\left[0, \ldots, \partial_{\tau_i}\left(\theta_i - \frac{\kappa}{2n}\right), \ldots, \partial_{\tau_i}\left(\theta_i - \frac{\kappa}{2n}\right), \ldots, 0\right]$$
$$= \left(\frac{4a^{(i)}}{\lambda_i^2} - \frac{4\nu}{\lambda_i}\right)\left(1 - \frac{1}{n}\right) \cdot E_{(i,i),\,(i+n,i+n)} \tag{5.3.43}$$

where $E_{(i,i)\,(i+n,i+n)}$ is the zero matrix with only two non-zero entries (which are 1's) in the (i,i) and $(i+n, i+n)$ positions.

Then, recalling the (formal) asymptotic expansion of the matrix Γ near ∞ and $-4\tau_i$ for all i (see [107] for a detailed discussion on the topic), the results follow from straightforward calculations. □

5.4 Conclusions and further developments

In this chapter we discussed the gap probabilities for the Bessel process restricted to a collection of intervals in both the timeless and dynamic regime.

As far as the timeless Bessel process is concerned, we were able to express its Fredholm determinant as a Jimbo-Miwa-Ueno τ-function and give a quite explicit formulation in terms of the solution of a suitable Riemann-Hilbert problem which defines the τ function.

It is known that the gap probability restricted to a finite interval $[0, x]$, $x > 0$, can be interpreted as the distribution of the smallest eigenvalue of the Laguerre ensemble near the hard edge (when $x = 0$). In this work we showed that such quantity is linked in a non linear way to the Painlevé III equation as already shown in [101]. On the other hand, the method employed in this work allows to not only identify the Painlevé equation, but also to identify the monodromy data of the associated isomonodromic system.

The study of the gap probabilities for the multi-time process has never been performed before and the connection with the τ-function allows the formulation of differential identities which might lead to differential equations in the spirit of [5] and [106], if one desires to do so. In particular, a first step in this direction could be the recovery of the system of PDEs showed in [102] for the multi-time Bessel process, using the Lax pair formalism.

Chapter 6

Gap probabilities for the Generalized Bessel Process

6.1 Introduction

In this chapter we deal with a relatively new determinantal point process which arises in the setting of mutually avoiding random paths, called Generalized Bessel process. As in the previous Chapter, we will be interested in studying certain "gap" probabilities of the possible configurations of the system and we will connect them with suitable Riemann-Hilbert problems (RHP, see Chapter 3.3).

As discussed in Chapter 2, gap probabilities of determinantal processes are equal to Fredholm determinants of suitable integral operators. Therefore, the main goal of the present Chapter will be the analysis of such Fredholm determinants and, possibly, their calculation in a quite explicit or more manageable form. These gap probabilities may be also seen as instances of "Tracy-Widom" distribution ([100], [101]), in the sense of quantities describing a "last particle" behaviour, as in the Bessel case (Chapter 5), thus establishing a connection with the theory of Random Matrices and equations of Painlevé type. Our results on the Generalized Bessel process fit in the same setting; in particular, we will be able to set a connection between gap probabilities and a member of some Painlevé hierarchy, using the same method performed in Chapter 5 (see [10], [11]), through the identification of the Lax pair. However, the explicit ODE is still object of investigation.

The Generalized Bessel process is a determinantal point process defined in terms of a trace-class integral operator acting on $L^2(\mathbb{R}_+)$, with kernel

$$K^{\mathrm{GEN}}(x,y) = \int_\Gamma \frac{\mathrm{d}s}{2\pi i} \int_\Sigma \frac{\mathrm{d}t}{2\pi i} \frac{e^{xs+\frac{\tau}{s}+\frac{1}{2s^2}-yt-\frac{\tau}{t}-\frac{1}{2t^2}}}{t-s} \left(\frac{s}{t}\right)^\nu \tag{6.1.1}$$

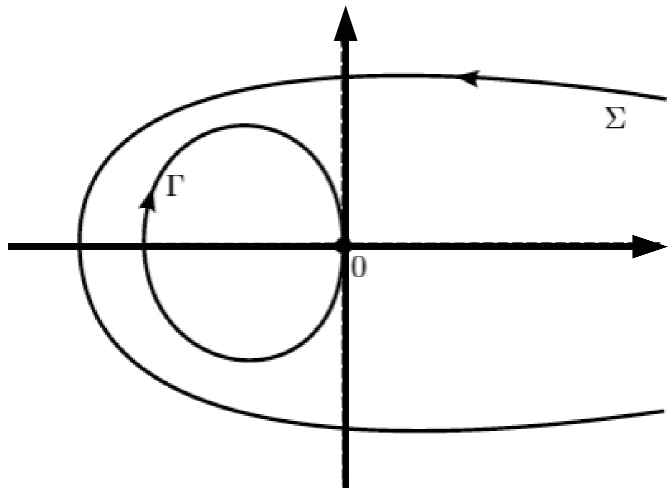

Figure 6.1: The original contours for the Generalized Bessel kernel defined in [76].

with $\nu > -1$, $\tau \in \mathbb{R}$; the logarithmic cut is on $\mathbb{R}_-$. The curve Γ and Σ are described in Figure 6.1.

The Generalized Bessel kernel was first introduced as a critical kernel by Kuijlaars *et al.* in [76] and [77]. Let consider a model of n non-intersecting squared Bessel processes and let study the scaling limit as the number of paths goes to infinity. We recall that if $\left\{\vec{X}(t)\right\}_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, then the diffusion process

$$R(t) = \|\mathbf{X}(t)\|_2 := \sqrt{X_1(t)^2 + \ldots + X_n(t)^2} \quad t \geq 0 \tag{6.1.2}$$

is called *Bessel process* with parameter $\nu = \frac{d}{2} - 1$, while $R^2(t)$ is the *squared Bessel process* usually denoted by BESQ^d (see e.g. [69, Ch. 7], [74]). As stated in the Chapter 1, these are an important family of diffusion processes which have applications in finance and other areas. The Bessel process $R(t)$ for $d = 1$ reduces to the Brownian motion reflected at the origin, while for $d = 3$ it is connected with the Brownian motion absorbed at the origin ([70], [71]).

In particular, we want to consider a system of n particles performing BESQ^d conditioned never to collide with each other and conditioned to start at time $t = 0$ at the same positive value $x = \kappa > 0$ and end at 0. Of particular interest here is the interaction of the non-intersecting paths with the hard edge at 0. Due to the nature of the squared Bessel process, the paths starting at a positive value remain positive, but they are conditioned to end at time $t = 1$ at $x = 0$.

The positions of the paths at any given time $t \in (0, 1)$ are a determinantal point process

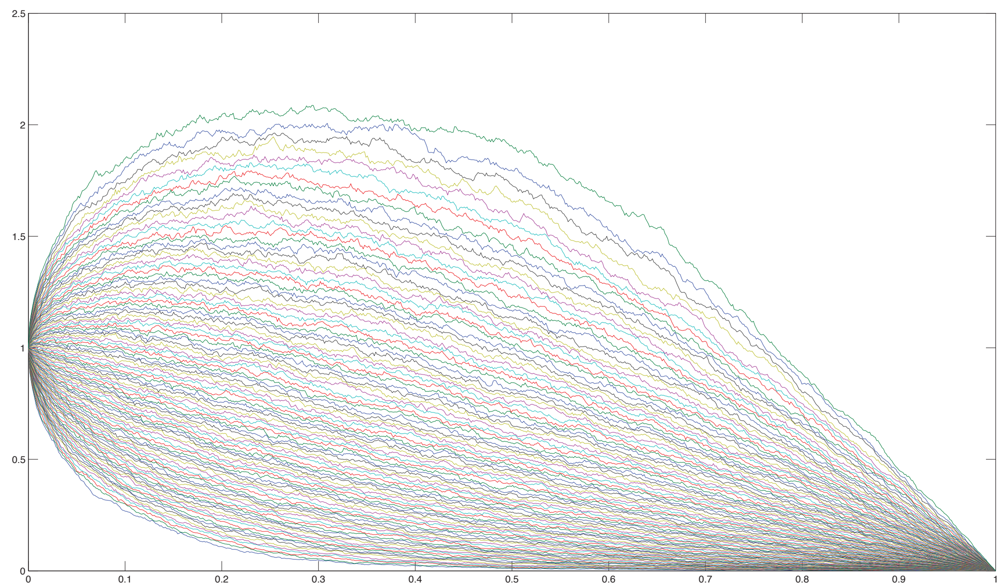

Figure 6.2: Numerical simulation of 100 non-intersecting Squared Bessel Paths with starting point $\kappa = 1$.

with correlation kernel built out of the transition probability density of the squared Bessel process. In [76], it was proven that, after appropriate rescaling in the limit as $n \to +\infty$, the paths fill out a region in the tx-plane as in Figure 6.2: the paths stay initially away from the axis $x = 0$, but at a certain critical time t^*, which depends only on the position of the starting point κ, the smallest paths hit the hard edge and remain close to it. In particular, the domain of the non-intersecting paths is a simply connected region in the tx-plane, bounded by two curves which are the *loci* of the zeros of a certain algebraic equation.

As the number of paths tends to infinity, the local scaling limits of the correlation kernel are the usual universal kernels appearing in Random Matrix Theory: the sine kernel appears in the bulk, the Airy kernel at the soft edges, i.e. the upper boundary for all $t \in (0, 1)$ and the lower boundary of the limiting domain for $t < t^*$, while for $t > t^*$, the Bessel kernel appears at the hard edge $x = 0$, see [76, Theorems 2.7-2.9]. It is interesting to notice that neither the boundary of the domain filled by the scaled paths, nor the behaviour in the bulk or at the soft edge depends on the parameter ν related to the dimension d of the BESQ^d. This dependency appears only in the interaction with the hard edge at $x = 0$. A possible interpretation may be that ν is a measure for the interaction with the hard edge. It does not influence the global behavior as $n \to +\infty$, but only the local behaviour near 0 (for more details we refer to [76]).

At the critical time $t = t^*$, there is a transition between the soft and the hard edges and the dynamics at that point is described by the new kernel (6.1.1), which we call Generalized Bessel kernel.

In this chapter we will focus on the gap probability of the Generalized Bessel process on a collection of intervals $I := [a_1, a_2] \cup \ldots \cup [a_{2N-1}, a_{2N}]$ and the emphasis is again on the probabilities thought of as functions of the endpoints $\{a_j\}$:

$$P\left(\text{no points in } I\right) = \det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right) \tag{6.1.3}$$

where, with abuse of notation, we called K^{GEN} the integral operator with kernel (6.1.1).

As seen in the introductory chapters, Section 2.2, it is possible to introduce a more general concept of gap probability, introducing a time parameter in the point process. Thus, the point process becomes a dynamical system and one can study the behaviour of the points evolving with time.

Given a collection of n consecutive times $\{\tau_1, \ldots, \tau_n\}$, within the time interval $[0, 1]$, and subsets $I_k \subset \mathbb{R}$, $k = 1, \ldots, n$, we are interested again in the probability that at time τ_k no points lie in I_k (for all $k = 1, \ldots, n$), i.e. the gap probability in a multi-time setting:

$$P\left(\text{no points in } I_k \text{ at time } \tau_k, \forall\, k\right) = \det\left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right) \tag{6.1.4}$$

where the operator $[K]^{\mathrm{GEN}}$ is the multi-time "counter-part" of the Generalized Bessel operator K^{GEN} (6.1.1), with matrix kernel $[K]^{\mathrm{GEN}}_{i,j=1,\ldots,n}$ of dimension $n \times n$, restricted to the sets $\mathcal{I} = I_1 \sqcup \ldots \sqcup I_n$. The kernel of the Generalized Bessel operator is defined as $[K]^{\mathrm{GEN}}_{ij} = H_{ij} + \chi_{i<j} P_{ij}$ $(i, j = 1, \ldots, n)$ with

$$H_{ij}(x, y) := 4 \iint_{\hat{\gamma}\times\gamma} \frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2} \frac{e^{-xs+yt+\frac{1}{2}\left(\tau-\frac{1}{s}+4\Delta_{ji}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{(t-s-4\Delta_{ji}ts)} \left(\frac{s}{t}\right)^{\nu} \tag{6.1.5a}$$

$$P_{ij}(x, y) = -\frac{1}{\Delta_{ji}} \int_{\gamma} e^{\frac{x}{4\Delta_{ji}}\left(\frac{1}{w}-1\right)+\frac{y}{4\Delta_{ji}}(w-1)} w^{-\nu} \frac{\mathrm{d}w}{(2\pi i)w} \tag{6.1.5b}$$

where the curve γ is the same one as in the definition of the Bessel process (Chapter 5) and it appears also in an equivalent definition of the single-time Generalized Bessel kernel (see formula (6.3.1a)); $\hat{\gamma} := \frac{1}{\gamma}$, $\Delta_{ji} := \tau_j - \tau_i$ $(i, j = 1, \ldots, n)$.

The formulation of the multi-time Generalized Bessel kernel is a completely new result and its derivation will be addressed in the next section. An equivalent formulation has been autonomously derived by S. Delvaux and B. Vető ([105]) and it is shown here.

In order to accomplish our study of gap probabilities, we will show again that the gap probabilities of the Generalized Bessel operator (single-time and multi-time) can be expressed in terms of Fredholm determinants of a suitable integral operator K (matrix-valued in the multi-time case) in the sense of Its-Izergin-Korepin-Slavnov ([50], see Section 3.3). Moreover, through the study of the corresponding Riemann-Hilbert problem, it will be possible to link the gap probabilities to the τ-function, as we did with the Bessel process.

The main steps will be the following: we will find an IIKS integrable operator, which will have the same Fredholm determinant as the Generalized Bessel process, up to conjugation with a Fourier-like operator. We will then set up a Riemann-Hilbert problem for such integrable operator and connect it to the Jimbo-Miwa-Ueno τ function. For the sake of clearness, we will apply this strategy to the single-time and the multi-time Generalized Bessel process separately.

We point out that although the single time operator can be formulated in an IIKS form (see the alternative definition in [77, Formula 1.33]), the corresponding multi-time process is not of this type and its restriction to a collection of intervals is crucial to find a new IIKS operator with equivalent Fredholm determinant.

As an example of possible applications we will describe how to obtain a system of isomonodromic Lax equations for the single-time process. Moreover, having a Riemann-Hilbert formulation for such Fredholm determinants would allow the study of asymptotics of Generalized Bessel gap probabilities and their connection with Airy and Bessel gap probabilities, using steepest descent methods, along the lines of [11].

Remark 6.1. *We preferred to refer to the process under consideration as "Generalized Bessel process" because of several analogies with the Bessel process (see Chapter 5) appearing in our study. As will be clear, the contours setting is similar to the one for the Bessel kernel; many of the calculations performed in Chapter 5 for the Bessel kernel are here reproduced with very few adjustments. Moreover, as it will be clear in Section 6.3, gap probabilities of the Generalized Bessel operator are related to a Lax pair that shows similar properties to the one associated with the Painlevé III transcendent, which is known to be related to the gap probabilities for the Bessel process (see Chapter 5 and [101]). On the other hand, such a Lax pair has a higher order pole at zero and this fact suggests that its compatibility equations might lead to an ODE belonging to some Painlevé hierarchy.*

The chapter is organized as follows: in Section 6.3 we will deal with the single-time Generalized Bessel operator restricted to a generic collection of intervals; in the subsection 6.3.2 we will focus on the single-time Generalized Bessel process restricted to a single interval and we will find a corresponding Lax pair In the following Section 6.4 we will study the gap probabilities for the multi-time Bessel process.

In the coming Section 6.2, we show how we found the multi-time Generalized Bessel kernel and we compare it with the one found by Delvaux and Vető ([105]). We prove that these two kernels are equivalent up to a transposition of the operator and a translation of the parameter τ.

6.2 Building the multi-time Generalized Bessel kernel

The starting point of this investigation is the known single-time kernel derived by Kuijlaars *et al.* ([77])

$$K_\nu^{\mathrm{KMW}}(x,y;\tau) = \iint_{\gamma\times\hat\gamma} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2}\,\frac{e^{-xs-\frac{\tau}{s}+\frac{1}{2s^2}+yt+\frac{\tau}{t}-\frac{1}{2t^2}}}{s-t}\left(\frac{s}{t}\right)^\nu \tag{6.2.1}$$

where the curve γ is an unbounded curve that extends from $-\infty$ to zero and then back to $-\infty$, encircling the origin in a counterclockwise way, and $\hat\gamma := \frac{1}{\gamma}$; the logarithmic cut is on $\mathbb{R}_-$, as shown in Figure 6.3.

The diffusion kernel related to the Squared Bessel Paths is

$$p(x,y,\Delta) := \left(\frac{y}{x}\right)^{\frac{\nu}{2}}\frac{1}{\Delta}e^{-\frac{x+y}{4\Delta}}I_\nu\left(\frac{\sqrt{xy}}{2\Delta}\right) \tag{6.2.2}$$

where $\Delta>0$ represents the gap between two given times τ_i and τ_j and I_ν is the modified Bessel function of first kind (the same diffusion kernel appears in the definition of the multi-time Bessel kernel; see Chapter 5).

The extended multi-time kernel is given by

$$[K] = H - P \tag{6.2.3}$$

where in particular P is a strictly upper-triangular matrix with (i,j)-entry $P_{ij} := \chi_{i<j}p(x,y,\Delta_{ij})$ ($\Delta_{ij} := |\tau_i-\tau_j|>0$). This is essentially the derivation in [33] applied to case at hand.

Theorem 6.2. *The multi-time Generalized Bessel operator on $L^2(\mathbb{R}_+)$ with times $\tau_1 < \ldots < \tau_n$ is defined through a matrix kernel with the following entries $[K]_{ij}^{\mathrm{GEN}} := H_{ij}+\chi_{i<j}P_{ij}$ ($i,j=1,\ldots,n$)*

$$H_{ij}(x,y) := 4\iint_{\hat\gamma\times\gamma}\frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2}\,\frac{e^{-xs+yt+\frac{1}{2}\left(\tau-\frac{1}{s}+4\Delta_{ji}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{(t-s-4\Delta_{ji}ts)}\left(\frac{s}{t}\right)^\nu \tag{6.2.4a}$$

$$P_{ij}(x,y) = -\frac{1}{\Delta_{ji}}\int_\gamma e^{\frac{x}{4\Delta_{ji}}\left(\frac{1}{w}-1\right)+\frac{y}{4\Delta_{ji}}(w-1)}w^{-\nu}\frac{\mathrm{d}w}{(2\pi i)w} \tag{6.2.4b}$$

the curve γ is the same one as in the single-time Generalized Bessel kernel (a contour that winds around zero counterclockwise an extends to $-\infty$; Figure 6.3) and $\hat{\gamma} := \frac{1}{\gamma}$; $\Delta_{ji} := \tau_j - \tau_i$.

The proof is based on the verification that the definition of the kernel above satisfies the theorem due to Eynard and Mehta on multi-time kernels ([33]).

First of all, we define a convolution operation (see [33, formula (3.2)]).

Definition 6.3. Given two functions f, g with suitable regularity, we define the convolution $f * g$ as

$$(f * g)(\xi, \eta) = \int f(\xi, \zeta) g(\zeta, \eta) \,\mathrm{d}\zeta. \tag{6.2.5}$$

Recalling formulæ (3.12)-(3.13) from [33], we will verify the following relations between the diffusion kernel P and the kernel H:

$$H_{ij} * P_{jk} = \begin{cases} H_{ik} & j < k \\ 0 & j \geq k \end{cases} \tag{6.2.6a}$$

$$P_{ij} * H_{jk} = \begin{cases} H_{ik} & i < j \\ 0 & i \geq j. \end{cases} \tag{6.2.6b}$$

Proof. We set $\Delta := |\tau_i - \tau_j| > 0$. Regarding the upper diagonal terms ($i < j$)

$$H_{ij}(x, y) = \int_0^\infty P_{ij}(x, z) H_{jj}(z, y) \,\mathrm{d}z =$$

$$\int_0^\infty \frac{\mathrm{d}z}{\Delta} \int_\gamma \frac{\mathrm{d}w}{2\pi i w} e^{\frac{x}{4\Delta}\left(\frac{1}{w}-1\right)+\frac{z}{4\Delta}(w-1)} w^{-\nu} \iint_{\gamma\times\hat{\gamma}} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{-zs+yt+\frac{1}{2}\left(\tau-\frac{1}{s}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{t-s} \left(\frac{s}{t}\right)^\nu.$$

Integrating in z and taking calculating a residue, we have

$$\frac{1}{\Delta} \iint_{\gamma\times\hat{\gamma}_u} \frac{\mathrm{d}t\,\mathrm{d}u}{(2\pi i)^2} \frac{e^{-xu+yt+\frac{1}{2}\left(\tau+4\Delta-\frac{1}{u}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{(u-t+4\Delta ut)} \left(\frac{u}{t}\right)^\nu.$$

As for the lower diagonal term, we need to verify that

$$\int_0^\infty P_{ij}(x, z) H_{ji}(z, y) \,\mathrm{d}z = H_{ii}(x, y) \quad j > i \tag{6.2.7}$$

with

$$H_{ji}(x, y) = \frac{4}{(2\pi i)^2} \iint_{\hat{\gamma}\times\gamma} \frac{\mathrm{d}u\,\mathrm{d}t}{ut} \frac{e^{-xu+yt+\frac{1}{2}\left(\tau-\frac{1}{u}-4\Delta_{ij}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{\left(\frac{1}{u}-\frac{1}{t}+4\Delta_{ij}\right)} \left(\frac{u}{t}\right)^\nu. \tag{6.2.8}$$

Again ,we set $\Delta := |\tau_i - \tau_j| > 0$.

$$\int_0^\infty \frac{4\,\mathrm{d}z}{\Delta} \int_\gamma \frac{\mathrm{d}w}{(2\pi i)w} e^{\frac{x}{4\Delta}\left(\frac{1}{w}-1\right)+\frac{z}{4\Delta}(w-1)} w^{-\nu} \iint_{\hat{\gamma}\times\gamma} \frac{\mathrm{d}u\,\mathrm{d}t}{(2\pi i)^2} \frac{e^{-zu+yt+\frac{1}{2}\left(\tau-\frac{1}{u}-4\Delta\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{(u-t-4\Delta tu)} \left(\frac{u}{t}\right)^\nu$$

we integrate in z and calculate a residue to get

$$\frac{4}{\Delta} \iint_{\hat{\gamma}\times\gamma} \frac{\mathrm{d}v\,\mathrm{d}t}{(2\pi i)^2} \frac{e^{-xv+yt+\frac{1}{2}\left(\tau-\frac{1}{v}\right)^2-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2}}{(t-v)} \left(\frac{v}{t}\right)^\nu.$$

□

Independently from the present work and almost simultaneously, Vető and Delvaux introduced another version of the multi-time Generalized Bessel operator, called Hard-edge Pearcey process ([105]).

The kernel of the Hard-edge Pearcey reads $L^\nu := W - P$ with entries

$$W_{ij}(x,y,\sigma) :=$$
$$\left(\frac{y}{x}\right)^\nu \int_{\Gamma_{-\tau_i}} \frac{\mathrm{d}\eta}{2\pi i} \int_{i\mathbb{R}+\delta} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{-\frac{1}{2}(\eta-\sigma)^2+\frac{x}{\eta+\tau_i}-\frac{1}{2}(\xi-\sigma)^2-\frac{y}{\xi+\tau_j}}}{(\eta-\xi)(\eta+\tau_i)(\xi+\tau_j)} \left(\frac{\eta+\tau_i}{\xi+\tau_j}\right)^\nu \tag{6.2.9}$$

and P the usual transition density. $\Gamma_{-\tau_i}$ is a clockwise oriented closed loop which intersects the real line at a point to the right of $-\tau_i$, and also at $-\tau_i$ itself, where it has a cusp at angle π; $\delta > 0$ is chosen such that the contour $i\mathbb{R}+\delta$ passes to the right of the singularity at $-t$ and to the right of the contour $\Gamma_{-\tau_i}$. The logarithmic branch is cut along the negative half-line.

Proposition 6.4. *The Hard-edge Pearcey operator is the transpose of the Generalized Bessel operator (6.2.4a)-(6.2.4b) defined above. More precisely,*

$$L^\nu(x,y;\sigma) = \left(\frac{y}{x}\right)^\nu [K]^{\mathrm{GEN}}(y,x;\tau_i+\sigma). \tag{6.2.10}$$

Proof. The results come from straightforward changes of variables. □

Corollary 6.5. In the single-time case ($\tau_i = \tau_j$), both the Generalized Bessel kernel and the Hard-edge Pearcey kernel coincide with the single-time kernel defined in [77], up to a transposition:

$$L^\nu(x,y;\sigma)\big|_{\Delta_{ij}=0} = \left(\frac{y}{x}\right)^\nu K_\nu^{\mathrm{KMW}}(y,x;\tau_i+\sigma) = \left(\frac{y}{x}\right)^\nu [K]^{\mathrm{GEN}}(y,x;\tau_i+\sigma)\big|_{\Delta_{ij}=0}. \tag{6.2.11}$$

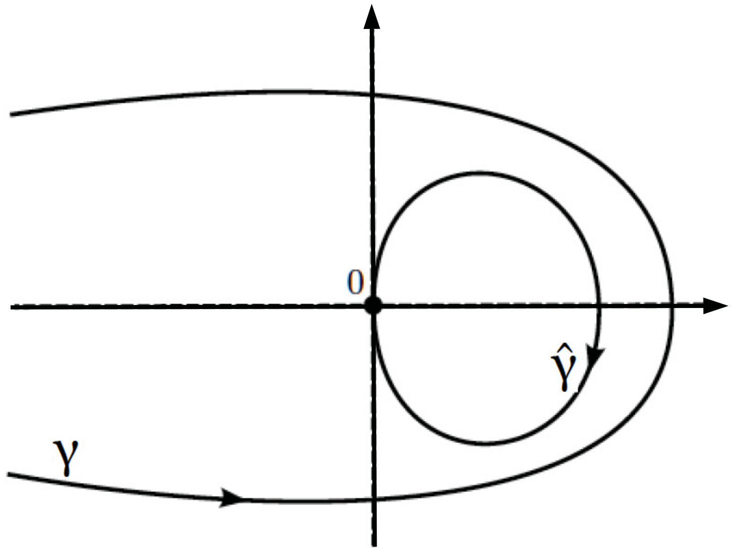

Figure 6.3: The curves appearing in the definition of the Generalized Bessel kernel.

Remark 6.6. *Since the Fredholm determinant is invariant under transposition, we prefer to work on the version given by Vető and Delvaux for the multi-time Generalized Bessel operator, because of more straightforward calculations which reminds more closely the ones performed for the Bessel process (Chapter 5).*

6.3 The Single-time Generalized Bessel

The Generalized Bessel kernel is

$$K^{\mathrm{GEN}}(x,y) = \int_{\gamma\times\hat{\gamma}} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{\phi_\tau(y,t)-\phi_\tau(x,s)}}{s-t} \left(\frac{s}{t}\right)^\nu \tag{6.3.1a}$$

$$\phi_\tau(z,t) := zt + \frac{\tau}{t} - \frac{1}{2t^2} \tag{6.3.1b}$$

where $\tau \in \mathbb{R}$ is a fixed parameter, the contour $\hat{\gamma}$ is a closed loop in the right half-plane tangent to the origin and oriented clockwise, while the contour γ is an unbounded loop oriented counterclockwise and encircling $\hat{\gamma}$; the logarithmic cut lies on $\mathbb{R}_-$ (see Figure 6.3).

Remark 6.7. *The curve setting is equivalent to the curve setting appearing in the definition of the Bessel kernel (Chapter 5). Moreover, the phase appearing in the exponential (6.3.1b) resembles the Bessel kernel one $\psi(z,t) := zt - \frac{1}{4t}$ with an extra term which introduces a higher singularity at 0.*

Our interest is focused on the gap probability of such operator restricted to a collection of intervals I, i.e. the quantity

$$\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right). \tag{6.3.2}$$

Remark 6.8. *Given a multi-interval $I := \bigcup_{k=1}^{N}[a_{2k-1}, a_{2k}]$, we define $K_a^{\mathrm{GEN}} := K^{\mathrm{GEN}}(x,y)\Big|_{[0,a]}$; then we have*

$$K^{\mathrm{GEN}}(x,y)\Big|_I = \sum_{j=1}^{2N}(-1)^j K_{a_j}^{\mathrm{GEN}}(x,y). \tag{6.3.3}$$

Remark 6.9. *The Generalized Bessel operator is not trace class on an infinite interval.*

As mentioned in the above introduction, the first step in our study is to establish a relation between the Generalized Bessel operator and a suitable IIKS integrable operator (Section 3.3, [50]).

Theorem 6.10. *Given a collection of (disjoint) intervals $I := \bigcup_{k=1}^{N}[a_{2k-1}, a_{2k}]$, the following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right) = \det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{GEN}}\right) \tag{6.3.4}$$

where $\mathbb{K}^{\mathrm{GEN}}$ is an IIKS integrable operator acting on $L^2(\gamma \cup \hat{\gamma})$ with kernel

$$\mathbb{K}^{\mathrm{GEN}}(t,s) = \frac{\vec{f}^T(t)\cdot\vec{g}(s)}{t-s} \tag{6.3.5a}$$

$$\vec{f}(t) = \frac{1}{2\pi i}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}\chi_{\hat{\gamma}}(t) + \frac{1}{2\pi i}\begin{bmatrix} 0 \\ e^{\frac{ta_1}{2}} \\ e^{t\left(a_2 - \frac{a_1}{2}\right)} \\ \vdots \\ e^{t\left(a_{2N} - \frac{a_1}{2}\right)} \end{bmatrix}\chi_\gamma(t) \tag{6.3.5b}$$

$$\vec{g}(s) = \begin{bmatrix} 0 \\ -e^{-a_1 s - \frac{\tau}{s} + \frac{1}{2s^2}} s^\nu \\ e^{-a_2 s - \frac{\tau}{s} + \frac{1}{2s^2}} s^\nu \\ \vdots \\ (-1)^{2N} e^{-a_{2N} s - \frac{\tau}{s} + \frac{1}{2s^2}} s^\nu \end{bmatrix}\chi_{\hat{\gamma}}(s) + \begin{bmatrix} e^{\frac{sa_1}{2} + \frac{\tau}{s} - \frac{1}{2s^2}} s^{-\nu} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}\chi_\gamma(s). \tag{6.3.5c}$$

Proof. Since the preliminary calculations are linear, we will start working on the single term

$K^{\mathrm{GEN}}_{a_j}$ and we will later sum them up over the $j = 1, \dots, 2N$, as in formula (6.3.3).

$$
\begin{aligned}
K^{\mathrm{GEN}}_{a_j} &:= \chi_{[0,a_j]}(x) K^{\mathrm{GEN}}(x, y; \tau) \\
&= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{\xi(a_j - x)} \int_{\gamma\times\hat{\gamma}} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{-a_j s - \frac{\tau}{s} + \frac{1}{2s^2} + yt + \frac{\tau}{t} - \frac{1}{2t^2}}}{(\xi - s)\,(s - t)} \left(\frac{s}{t}\right)^{\nu} \\
&= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-x\xi} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{yt} \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi a_j + \frac{\tau}{t} - \frac{1}{2t^2} - a_j s - \frac{\tau}{s} + \frac{1}{2s^2}}}{(\xi - s)(s - t)} \left(\frac{s}{t}\right)^{\nu}
\end{aligned}
\tag{6.3.6}
$$

where we continuously deformed the contour γ into a suitably translated imaginary axis $i\mathbb{R} + \epsilon$, $\epsilon > 0$ big enough such that the vertical line lays on the right of the curve $\hat{\gamma}$.

Introducing the following Fourier transform operators

$$
\begin{array}{l|l}
\mathcal{F} : L^2(\mathbb{R}) \to L^2(i\mathbb{R} + \epsilon) & \mathcal{F}^{-1} : L^2(i\mathbb{R} + \epsilon) \to L^2(\mathbb{R}) \\
 & \\
f(x) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} f(x) e^{\xi x} \mathrm{d}x & h(\xi) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{i\mathbb{R}+\epsilon} h(\xi) e^{-\xi x} \mathrm{d}\xi
\end{array}
\tag{6.3.7}
$$

we can claim that

$$
K^{\mathrm{GEN}} = \mathcal{F}^{-1} \circ \overline{\mathcal{K}^{\mathrm{GEN}}} \circ \mathcal{F}, \tag{6.3.8}
$$

$\overline{\mathcal{K}^{\mathrm{GEN}}} := \sum_j (-1)^j \overline{\mathcal{K}^{\mathrm{GEN}}_{a_j}}$ being an operator acting on $L^2(i\mathbb{R} + \epsilon)$ with kernels

$$
\overline{\mathcal{K}^{\mathrm{GEN}}}_{a_j}(\xi, t; \tau) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi a_j + \frac{\tau}{t} - \frac{1}{2t^2} - a_j s - \frac{\tau}{s} + \frac{1}{2s^2}}}{(\xi - s)(s - t)} \left(\frac{s}{t}\right)^{\nu} \tag{6.3.9}
$$

$\forall\, j = 1, \dots, 2N$, $\xi, t \in i\mathbb{R} + \epsilon$.

In order to ensure the convergence of the kernel, we conjugate it with the function $f(z) := e^{\frac{a_1 z}{2}}$

$$
\begin{aligned}
\mathcal{K}^{\mathrm{GEN}}(\xi, t; \tau) &:= e^{\frac{a_1 t}{2} - \frac{a_1 \xi}{2}} \overline{\mathcal{K}^{\mathrm{GEN}}}(\xi, t; \tau) \\
&= \sum_{j=1}^{2N} (-1)^j \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi\left(a_j - \frac{a_1}{2}\right) + \frac{t a_1}{2} + \frac{\tau}{t} - \frac{1}{2t^2} - a_j s - \frac{\tau}{s} + \frac{1}{2s^2}}}{(\xi - s)(s - t)} \left(\frac{s}{t}\right)^{\nu} \\
&=: \sum_{j=1}^{2N} (-1)^j \mathcal{K}^{\mathrm{GEN}}_{a_j}(\xi, t; \tau).
\end{aligned}
\tag{6.3.10}
$$

Remark 6.11. *We recall that Fredholm determinants are invariant under conjugation by bounded invertible operators.*

We continuously deform the translated imaginary axis $i\mathbb{R}+\epsilon$ into its original shape γ; note that $a_j - \frac{a_1}{2} > 0$, $\forall j = 1, \ldots, 2N$. It can be easily shown that the operator $\mathcal{K}^{\mathrm{GEN}}_{a_j}$ is the composition of two operators for every $j = 1, \ldots, 2N$; moreover, it is trace-class.

Lemma 6.12. *The operators $\mathcal{K}^{\mathrm{GEN}}_{a_j}$ are trace-class operators, $\forall\, j = 1, \ldots, 2N$, and the following decomposition holds $\mathcal{K}^{\mathrm{GEN}}_{a_j} = \mathcal{B}_1 \circ \mathcal{A}_{j,1}$, with*

$$\left.\begin{array}{c} \mathcal{A}_{j,1} : L^2(\gamma) \to L^2(\hat{\gamma}) \\ h(t) \mapsto s^{\nu} e^{-a_j s - \frac{\tau}{s} + \frac{1}{2s^2}} \int_{\gamma} \frac{e^{t\left(a_j - \frac{a_1}{2}\right)}}{t-s} h(t) \, \frac{\mathrm{d}t}{2\pi i} \end{array}\right| \begin{array}{c} \mathcal{B}_1 : L^2(\hat{\gamma}) \to L^2(\gamma) \\ f(s) \mapsto t^{-\nu} e^{\frac{t a_1}{2} + \frac{\tau}{t} - \frac{1}{2\tau^2}} \int_{\hat{\gamma}} \frac{f(s)}{s-t} \frac{\mathrm{d}s}{2\pi i}. \end{array} \tag{6.3.11}$$

$\mathcal{A}_{j,1}$ and $\mathcal{B}_1$ are trace-class operators themselves.

Proof. We introduce an additional translated imaginary axis $i\mathbb{R}+\delta$ ($\delta > 0$), not intersecting with γ and $\hat{\gamma}$, and we decompose $\mathcal{A}_{j,1}$ and $\mathcal{B}_1$ in the following way: $\mathcal{A}_{j,1} = \mathcal{O}_{j,2} \circ \mathcal{O}_{j,1}$ and $\mathcal{B}_1 = \mathcal{P}_2 \circ \mathcal{P}_1$ with

$$\left.\begin{array}{c} \mathcal{O}_{j,1} : L^2(\gamma) \to L^2(i\mathbb{R}+\delta) \\ f(\xi) \mapsto \displaystyle\int_{\gamma} \frac{\mathrm{d}\xi}{2\pi i} e^{\xi\left(a_j - \frac{a_1}{2}\right)} \frac{f(\xi)}{\xi - w} \end{array}\right| \begin{array}{c} \mathcal{O}_{j,2} : L^2(i\mathbb{R}+\delta) \to L^2(\hat{\gamma}) \\ g(w) \mapsto s^{\nu} e^{-a_j s - \frac{\tau}{s} + \frac{1}{2s^2}} \displaystyle\int_{i\mathbb{R}+\delta} \frac{\mathrm{d}w}{2\pi i} \frac{g(w)}{w-s} \end{array}$$

and

$$\left.\begin{array}{c} \mathcal{P}_1 : L^2(\hat{\gamma}) \to L^2(i\mathbb{R}+\delta) \\ f(s) \mapsto \displaystyle\int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{f(s)}{s-u} \end{array}\right| \begin{array}{c} \mathcal{P}_2 : L^2(i\mathbb{R}+\delta) \to L^2(\gamma) \\ g(u) \mapsto t^{-\nu} e^{\frac{t a_1}{2} + \frac{\tau}{t} - \frac{1}{2t^2}} \displaystyle\int_{i\mathbb{R}+\delta} \frac{\mathrm{d}u}{2\pi i} \frac{g(u)}{u-t}. \end{array}$$

All the kernels involved are of the form $K(z,w)$ with z and w on two disjoint curves, say C_1 and C_2. It is sufficient to check that $\iint_{C_1 \times C_2} |K(z,w)|^2 |\mathrm{d}z||\mathrm{d}w| < \infty$ to ensure that the operator belongs to the class of Hilbert-Schmidt operators. This implies that $\{\mathcal{A}_{j,1}\}_j$, $\mathcal{B}_1$ and $K^{\mathrm{GEN}}_{a_j}$ are trace-class (for all $j = 1, \ldots, 2N$), since composition of two HS operators. $\square$

Now we recall that any operator acting on a Hilbert space of the type $H = H_1 \oplus H_2$ can be decomposed as a 2×2 matrix of operators with (i,j)-entry given by an operator

$H_j \to H_i$. Thus, we can perform a chain of equalities

$$\det\left(\mathrm{Id}_{L^2(\gamma)} - \mathcal{K}^{\mathrm{GEN}}\right) = \det\left(\mathrm{Id}_{L^2(\gamma)} - \sum_{j=1}^{2N}(-1)^j \mathcal{B}_1 \circ \mathcal{A}_{j,1}\right)$$

$$= \det\left(\mathrm{Id}_{L^2(\gamma)} \otimes \mathrm{Id}_{L^2(\hat{\gamma})} - \left[\begin{array}{c|c} 0 & \mathcal{B}_1 \\ \hline \sum_{j=1}^{2N}(-1)^j \mathcal{A}_{j,1} & 0 \end{array}\right]\right) = \det\left(\mathrm{Id}_{L^2(\gamma\cup\hat{\gamma})} - \mathbb{K}^{\mathrm{GEN}}\right); s \quad (6.3.12)$$

the second equality follows from the multiplication on the left by the matrix (with determinant equal to 1)

$$\mathrm{Id}_{L^2(\gamma)\otimes L^2(\hat{\gamma})} + \left[\begin{array}{c|c} 0 & -\mathcal{B}_1 \\ \hline 0 & 0 \end{array}\right]$$

and the operator $\mathbb{K}^{\mathrm{GEN}}$ is an integrable operator with kernel as in the statement of Theorem 6.10. □

6.3.1 Riemann-Hilbert problem and τ-function

We can proceed now with building a Riemann-Hilbert problem associated to the integrable kernel we just found in Theorem 6.10. This will allow us to find some explicit identities for its Fredholm determinant.

Definition 6.13. Given the integrable kernel (6.3.5a)-(6.3.5c), the correspondent Riemann-Hilbert problem is the following: finding an $(2N+1)\times(2N+1)$ matrix Γ such that it is analytic on $\mathbb{C}\backslash\Xi$ ($\Xi := \gamma \cup \hat{\gamma}$) and

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda) & \lambda \in \Xi \\ \Gamma(\lambda) = I + \mathcal{O}(1/\lambda) & \lambda \to \infty \end{cases} \quad (6.3.13)$$

with jump matrix $M(\lambda) := I - J(\lambda)$,

$$J(\lambda) := 2\pi i \vec{f}(\lambda) \cdot \vec{g}^T(\lambda)$$

$$= \begin{bmatrix} 0 & -e^{\theta_{a_1}}\chi_{\hat{\gamma}} & e^{\theta_{a_2}}\chi_{\hat{\gamma}} & \dots & (-1)^{2N}e^{\theta_{a_{2N}}}\chi_{\hat{\gamma}} \\ e^{-\theta_{a_1}}\chi_\gamma & 0 & 0 & \dots & 0 \\ e^{-\theta_{a_2}}\chi_\gamma & 0 & \dots & & \vdots \\ \vdots & & & & \\ e^{-\theta_{a_{2N}}}\chi_\gamma & 0 & 0 & \dots & 0 \end{bmatrix} \quad (6.3.14a)$$

$$\theta_{a_j} := -a_j\lambda - \frac{\tau}{\lambda} + \frac{1}{2\lambda^2} + \nu \ln\lambda \quad \forall j = 1, \dots, 2N. \quad (6.3.14b)$$

It is easy to see that the jump matrix is conjugate to a matrix with (piece-wise) constant entries

$$M(\lambda) = e^{T(\lambda)} M_0 e^{-T(\lambda)}, \qquad T(\lambda, \vec{a}) = \operatorname{diag}\left(T_0, T_1, \dots, T_N\right), \tag{6.3.15}$$

$$T_0 = \frac{1}{N+1} \sum_{j=1}^{2N} \theta_{a_j}, \qquad T_j = T_0 - \theta_{a_j} \tag{6.3.16}$$

with $\vec{a}$ the collection of all endpoints $\{a_j\}$.

Thus, considering the matrix $\Psi(\lambda, \vec{a}) := \Gamma(\lambda, \vec{a}) e^{T(\lambda, \vec{a})}$, Ψ satisfies a RH-problem with constant jumps, thus it's (sectionally) a solution to a polynomial ODE.

Referring to the results stated in Section 3.4 (see also [9] and [11]) and adapted to the case at hand, we can claim that

Theorem 6.14. *Given a collection of intervals $I = \bigcup_k [a_{2k-1}, a_{2k}]$, the Fredholm determinant of the Generalized Bessel process $\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right)$ is equal to the isomonodromic τ-function related to the RHP in Definition 6.13.*

Moreover, for every parameter ρ, on which the Generalized Bessel operator may depend,

$$\partial_\rho \ln \det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right) = \int_\Xi \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Theta_{\partial_\rho}(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \tag{6.3.17}$$

$$\Theta_{\partial_\rho}(\lambda) := \partial_\rho M(\lambda) M^{-1}(\lambda) \tag{6.3.18}$$

with $\Xi = \gamma \cup \hat{\gamma}$. Thanks to the Jimbo-Miwa-Ueno residue formula (see [11]), $\forall\, j = 1, \dots, 2N$ the Fredholm determinant satisfies

$$\partial_{a_j} \ln \det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right) = -\operatorname{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a_j} T\right) = \Gamma_{1;j+1,j+1} \tag{6.3.19}$$

i.e. the $(j+1, j+1)$ component of the residue matrix $\Gamma_1 = \lim_{\lambda\to\infty} \lambda\left(I - \Gamma(\lambda)\right)$ at infinity.

As far as the parameter τ is concerned, the following result holds

$$\partial_\tau \ln \det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_I\right) = \operatorname{res}_{\lambda=0} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_\tau T\right) = -\left(\tilde{\Gamma}_0^{-1}\tilde{\Gamma}_1\right)_{1,1} \tag{6.3.20}$$

where $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_1$ are coefficients appearing in the asymptotic expansion of the matrix Γ in a neighbourhood of zero.

Proof. First of all, recalling Theorem 3.14 (Chapter 3), it is easy to verify that $H(M) \equiv 0$. Subsequently, we can calculate (6.3.19) and (6.3.20). The phases θ_{a_j} are linear in a_j, exactly as in the Bessel kernel case (Chapter 5).

$$\partial_{a_j} T(\lambda, \vec{a}) = \lambda \left(\frac{1}{2N+1} I - E_{j+1,j+1} \right) \tag{6.3.21}$$

Then, we plug this expression into (6.3.19)

$$\operatorname{res}_{\lambda=\infty} \operatorname{Tr} \left(\Gamma^{-1} \Gamma' \partial_{a_j} T \right) = \frac{\operatorname{Tr} \Gamma_1}{2N+1} - \Gamma_{1;j+1,j+1}. \tag{6.3.22}$$

Regarding the residue at zero, we recall the asymptotic expansion of $\Gamma \sim \tilde{\Gamma}_0 + \lambda \tilde{\Gamma}_1 + \dots$ near zero (see [107]) and we calculate

$$\partial_\tau T = -\frac{1}{\lambda} \left[E_{1,1} - \frac{1}{2N+1} I \right] \tag{6.3.23}$$

thus

$$\operatorname{res}_{\lambda=0} \operatorname{Tr} \left(\Gamma^{-1} \Gamma' \partial_\tau T \right) = \frac{\operatorname{Tr} \left(\tilde{\Gamma}_0^{-1} \tilde{\Gamma}_1 \right)}{2N+1} - \left(\tilde{\Gamma}_0^{-1} \tilde{\Gamma}_1 \right)_{1,1}. \tag{6.3.24}$$

The result follows from $\operatorname{Tr} \Gamma_1 = \operatorname{Tr} \left(\tilde{\Gamma}_0^{-1} \tilde{\Gamma}_1 \right) = 0$, since $\det \Gamma(\lambda) \equiv 1$. □

6.3.2 The single-interval case

In case we consider a single interval $I = [0, a]$, we are able to perform a deeper analysis on the gap probability of the Generalized Bessel operator and link it to an explicit Lax pair.

We will see that the Lax pair $\{A, U\}$ will recall the Painlevé III Lax pair very closely (see Section 5.2.3 and [36, Chapter 5, Section 3]), except for the presence of an extra term for the spectral matrix A. Such term will introduce a higher order Poincaré rank at $\lambda = 0$ as it will be clear in the following calculations. Moreover, thanks to the presence of the parameter τ other than the endpoint a, we can actually calculate an extra matrix, complementary to the Lax pair.

First of all, we reformulate Theorems 6.10 and 6.14, focusing on our present case.

Theorem 6.15. *Given $I = [0, a]$, the following equality between Fredholm determinants holds*

$$\det \left(I_{L^2(\gamma)} - K^{\mathrm{GEN}} \Big|_{[0,a]} \right) = \det \left(I_{L^2(\gamma \cup \hat{\gamma})} - \mathbb{K}^{\mathrm{GEN}} \right) \tag{6.3.25}$$

with $\mathbb{K}^{\mathrm{GEN}}$ an IIKS integrable operator with kernel

$$\mathbb{K}^{\mathrm{GEN}}_{\nu,\tau}(t,s) = \frac{\vec{f}^T(t)\cdot\vec{g}(s)}{t-s} \tag{6.3.26a}$$

$$\vec{f}(t) = \frac{1}{2\pi i}\begin{bmatrix} e^{\frac{ta}{2}} \\ 0 \end{bmatrix}\chi_\gamma(t) + \frac{1}{2\pi i}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\chi_{\hat{\gamma}}(t) \tag{6.3.26b}$$

$$\vec{g}(s) = \begin{bmatrix} 0 \\ s^{-\nu}e^{\frac{sa}{2}+\frac{\tau}{s}-\frac{1}{2s^2}} \end{bmatrix}\chi_\gamma(s) + \begin{bmatrix} s^{\nu}e^{-sa-\frac{\tau}{s}+\frac{1}{2s^2}} \\ 0 \end{bmatrix}\chi_{\hat{\gamma}}(s). \tag{6.3.26c}$$

The associated Riemann-Hilbert problem reads as follows:

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda) & \lambda \in \Xi := \hat{\gamma}\cup\gamma \\ \Gamma(\lambda) = I + \mathcal{O}(1/\lambda) & \lambda\to\infty \end{cases}$$

with Γ a 2×2 matrix, analytic on analytic on the complex plane except on the collection of curves Ξ, along which the above jump condition is satisfied with jump matrix $M(\lambda) := I - J(\lambda)$

$$\begin{aligned} M(\lambda) &= \begin{bmatrix} 1 & -e^{\lambda a+\frac{\tau}{\lambda}-\frac{1}{2\lambda^2}-\nu\ln\lambda}\chi_\gamma(\lambda) \\ -e^{-\lambda a-\frac{\tau}{\lambda}+\frac{1}{2\lambda^2}+\nu\ln\lambda}\chi_{\hat{\gamma}}(\lambda) & 1 \end{bmatrix} \\ &= e^{T_a(\lambda)}M_0e^{-T_a(\lambda)}. \end{aligned} \tag{6.3.27}$$

Thus the jump matrix M is equivalent to a matrix with constant entries, via the conjugation $e^{T_a(\lambda)}$, $T_a(\lambda) = \frac{1}{2}\theta_a\sigma_3$, where $\theta_a := -\lambda a - \frac{\tau}{\lambda} + \frac{1}{2\lambda^2} + \nu\ln\lambda$ and σ_3 is the third Pauli matrix. This allows us to define the matrix $\Psi(\lambda) := \Gamma(\lambda)e^{T_a(\lambda)}$ which solves a Riemann-Hilbert problem with constant jumps and is (sectionally) a solution to a polynomial ODE.

Applying Theorem 6.14, we get

Theorem 6.16.

$$\partial_\rho \ln\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_{[0,a]}\right) = \int_\Xi \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Theta_{\partial_\rho}(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{6.3.28a}$$

$$\Theta_\partial(\lambda) := \partial M(\lambda)M^{-1}(\lambda), \quad \Xi := \gamma\cup\hat{\gamma} \tag{6.3.28b}$$

for every parameter ρ on which the operator K^{GEN} depends.

In particular, thanks to the Jimbo-Miwa-Ueno residue formula, we have

$$\partial_a \ln\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_{[0,a]}\right) = -\operatorname{res}_{\lambda=\infty}\ \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_a T_a\right) = \Gamma_{1;2,2} \tag{6.3.29a}$$

$$\partial_\tau \ln\det\left(\mathrm{Id} - K^{\mathrm{GEN}}\Big|_{[0,a]}\right) = \operatorname{res}_{\lambda=0}\ \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_\tau T_a\right) = \left(\tilde{\Gamma}_0^{-1}\tilde{\Gamma}_1\right)_{2,2} \tag{6.3.29b}$$

with $\Gamma_{1;2,2}$ *the* $(2,2)$*-entry of the residue matrix* Γ_1 *at* ∞*, while the* $\tilde{\Gamma}_j$*'s appear in the asymptotic expansion of* Γ *near zero.*

We can now calculate the Lax "triplet" associated to the Riemann-Hilbert problem above:

$$A := \partial_\lambda \Psi \cdot \Psi^{-1} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3} \tag{6.3.30a}$$

$$U := \partial_a \Psi \cdot \Psi^{-1} = U_0 + \lambda U_1 \tag{6.3.30b}$$

$$V := \partial_\tau \Psi \cdot \Psi^{-1} = V = V_0 + \frac{V_{-1}}{\lambda} \tag{6.3.30c}$$

with coefficients

$$\begin{aligned}
A_0 &= \frac{a}{2}\sigma_3, \quad A_{-1} = -\frac{\nu}{2}\sigma_3 + \frac{a}{2}[\Gamma_1,\sigma_3] \\
A_{-2} &= -\frac{a}{2}[\Gamma_1,\sigma_3\Gamma_1] + \frac{a}{2}[\Gamma_2,\sigma_3] - \frac{\nu}{2}[\Gamma_1,\sigma_3] - \frac{\tau}{2}\sigma_3 - \Gamma_1 \\
A_{-3} &= \Gamma_1^2 - 2\Gamma_2 + \frac{a}{2}[\sigma_3\Gamma_2,\Gamma_1] + \frac{a}{2}[\Gamma_1,\sigma_3\Gamma_1^2] + \frac{a}{2}[\sigma_3\Gamma_1,\Gamma_2] + \frac{a}{2}[\Gamma_3,\sigma_3] \\
&\quad + \frac{\nu}{2}\sigma_3\Gamma_2 + \frac{\nu}{2}[\Gamma_1,\sigma_3\Gamma_1] + \frac{\tau}{2}\sigma_3\Gamma_1 + \frac{1}{2}\sigma_3 \\
U_0 &= \frac{1}{2}[\Gamma_1,\sigma_3], \quad U_1 = \frac{1}{2}\sigma_3 \\
V_0 &= 0, \quad V_{-1} = \frac{1}{2}\sigma_3
\end{aligned}$$

where $\sigma_3 = \operatorname{diag}\{1,-1\}$ is the third Pauli matrix.

We point out that $\lambda = 0$ is an irregular point of Poincaré rank 2. The behaviour at zero shows a higher order rank with respect to the Lax pair for the Painlevé III transcendent (associated to the Bessel operator; Section 5.2.3 and [36, Chapter 5, Section 3]) where the point $\lambda = 0$ was of rank 1. Moreover, the matrix U is the same as the one appearing in the Painlevé III Lax pair (in the non-rescaled case, see Chapter 5.2).

Remark 6.17. *The expression of the Lax pair* A *and* U *suggests that their compatibility equation, together with some constraint induced by the additional matrix* V*, will lead to a* *higher order ODE belonging to some Painlevé hierarchy. Nevertheless, the reduction of the system of* 1^{st} *order ODEs (originated from the compatibility equation of the Lax pair) to a unique higher order ODE, which can describe the gap probability of the Generalized Bessel process, is not straightforward and it is still under investigation.*

6.4 The Multi-time Generalized Bessel

The multi-time Generalized Bessel operator on $L^2(\mathbb{R}_+)$ with times $\tau_1 < \ldots < \tau_n$ is defined through a $n \times n$ matrix kernel with entries $[K]^{\text{GEN}}_{ij} := H_{ij} + \chi_{i<j} P_{ij}$

$$H_{ij}(x,y) = -4\left(\frac{y}{x}\right)^{\nu} \int_{\gamma\times\hat{\gamma}} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2+xt+\frac{1}{2}\left(\tau-\frac{1}{s}+\Delta_{ji}\right)^2-ys}}{(s-t+\Delta_{ji}ts)} \left(\frac{s}{t}\right)^{\nu} \tag{6.4.1a}$$

$$\begin{aligned} P_{ij}(x,y) &= \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \frac{1}{\Delta_{ji}} e^{-\frac{x+y}{4\Delta_{ji}}} I_{\nu}\left(\frac{\sqrt{xy}}{2\Delta_{ji}}\right) \\ &= -\left(\frac{y}{x}\right)^{\nu} \frac{1}{\Delta_{ji}} \int_{\gamma} e^{\frac{x}{4\Delta_{ji}}(t-1)+\frac{y}{4\Delta_{ji}}\left(\frac{1}{t}-1\right)} t^{-\nu-1} \frac{\mathrm{d}t}{2\pi i} \end{aligned} \tag{6.4.1b}$$

the curve γ is the same one as in the single-time Generalized Bessel kernel (6.3.1a) (a contour that winds around zero counterclockwise an extends to $-\infty$) and $\hat{\gamma} := \frac{1}{\gamma}$; $\Delta_{ji} := \tau_j - \tau_i$ and I_ν is the modified Bessel function of first kind.

Remark 6.18. *The matrix with entries $\chi_{i<j}P_{ij}$ $(i,j = 1,\ldots,n)$ is strictly upper triangular, by construction.*

Remark 6.19. *The above definition of the multi-time kernel is the one given by Delvaux and Vető ([105]). We preferred to use this one because the study of the gap probability with the above expression involves less complicated calculations than with the equivalent version given in Section 6.2.*

As in the single-time case, we are again interested in the gap probability of the operator restricted to a collection intervals I_j at each time τ_j $(\forall\, j)$, i.e.

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - [K]^{\text{GEN}}\Big|_{\mathcal{I}}\right) \tag{6.4.2}$$

where $\mathcal{I} = I_1 \sqcup \ldots \sqcup I_n$ is a collection of Borel sets of the form

$$I_j := [a_1^{(j)}, a_2^{(j)}] \cup \ldots \cup [a_{2k_j-1}^{(j)}, a_{2k_j}^{(j)}] \quad \forall\, j = 1,\ldots,n.$$

Remark 6.20. *The multi-time Bessel operator fails to be trace-class on infinite intervals.*

For the sake of clarity, we will focus on the simple case $I_j = [0, a^{(j)}]$, $\forall\, j$. The general case follows the same guidelines described below; the only difficulties are mostly technical, due to heavy notation, and not theoretical.

As in the single-time case, we start by establishing a link between the multi-time Generalized Bessel operator and a suitable IIKS operator, which we will examine deeper in the next subsection.

Theorem 6.21. *The following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - [\mathbb{K}]^{\mathrm{GEN}}\right) \tag{6.4.3}$$

with where $\mathcal{I} = I_1 \sqcup \ldots \sqcup I_n$ is a collection of disjoint intervals $I_j := [0, a^{(j)}]$, $\forall\, j = 1, \ldots, n$. The operator $[\mathbb{K}]^{\mathrm{GEN}}$ is an integrable operator acting on the Hilbert space

$$H := L^2\left(\gamma \cup \bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \sim L^2\left(\bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \oplus L^2(\gamma, \mathbb{C}^n), \tag{6.4.4}$$

with $\gamma_{-k} := \frac{1}{\gamma} - 4\tau_k$, mutually disjoint.

Its kernel is a $2n \times 2n$ matrix of the form

$$[\mathbb{K}]^{\mathrm{GEN}}(v, \xi) = \frac{\boldsymbol{f}(v)^T \cdot \boldsymbol{g}(\xi)}{v - \xi} \tag{6.4.5a}$$

$$\boldsymbol{f}(v)^T = \frac{1}{2\pi i}\left[\begin{array}{c|c|c} \mathrm{diag}\,\mathcal{N}(v) & 0 & 0 \\ \hline 0 & \mathrm{diag}\,\mathcal{M}(v) & \mathcal{A}(v) \end{array}\right] \tag{6.4.5b}$$

$$\boldsymbol{g}(\xi) = \left[\begin{array}{c|c} 0 & \mathrm{diag}\,\mathcal{N}(\xi) \\ \hline \mathcal{M}(\xi) & 0 \\ \hline 0 & \mathcal{B}(\xi) \end{array}\right] \tag{6.4.5c}$$

where $\boldsymbol{f}, \boldsymbol{g}$ *are* $N \times 2n$ *matrices, with* $N = 2n + (n-1) = 3n-1$.

$$\operatorname{diag} \mathcal{N}(v) := \operatorname{diag}\left[-4e^{-\frac{a^{(1)}}{v_1}}\chi_\gamma, \ldots, -4e^{-\frac{a^{(n)}}{v_n}}\chi_\gamma\right]$$
$$\operatorname{diag} \mathcal{N}(\xi) := \operatorname{diag}\left[e^{\frac{a^{(1)}}{\xi_1}}\chi_{\gamma-1}, \ldots, e^{-\frac{a^{(n)}}{\xi_n}}\chi_{\gamma-n}\right]$$
$$\operatorname{diag} \mathcal{M}(v) := \operatorname{diag}\left[e^{-\frac{(v_{1,\tau})^2}{2}}v_1^\nu\chi_{\gamma-1}, \ldots, e^{-\frac{(v_{n,\tau})^2}{2}}v_n^\nu\chi_{\gamma-n}\right]$$
$$\mathcal{M}(\xi) := \begin{bmatrix} e^{\frac{(\xi_{1,\tau})^2}{2}}\xi_1^{-\nu}\chi_\gamma & \ldots & e^{\frac{(\xi_{1,\tau})^2}{2}}\xi_n^{-\nu}\chi_\gamma \\ \vdots & & \vdots \\ e^{\frac{(\xi_{n,\tau})^2}{2}}\xi_1^{-\nu}\chi_\gamma & \ldots & e^{\frac{(\xi_{n,\tau})^2}{2}}\xi_n^{-\nu}\chi_\gamma \end{bmatrix}$$

$$\mathcal{A}(v) =$$
$$\begin{bmatrix} -4e^{-\frac{a^{(2)}}{v_2}}\frac{v_1^\nu}{v_2^\nu}\chi_{\gamma-1} & -4e^{-\frac{a^{(3)}}{v_3}}\frac{v_1^\nu}{v_3^\nu}\chi_{\gamma-1} & -4e^{-\frac{a^{(4)}}{v_4}}\frac{v_1^\nu}{v_4^\nu}\chi_{\gamma-1} & \ldots & -4e^{-\frac{a^{(n)}}{v_n}}\frac{v_1^\nu}{v_n^\nu}\chi_{\gamma-1} \\ 0 & -4e^{-\frac{a^{(3)}}{v_3}}\frac{v_2^\nu}{v_3^\nu}\chi_{\gamma-2} & -4e^{-\frac{a^{(4)}}{v_4}}\frac{v_2^\nu}{v_4^\nu}\chi_{\gamma-2} & \ldots & -4e^{-\frac{a^{(n)}}{v_n}}\frac{v_2^\nu}{v_n^\nu}\chi_{\gamma-2} \\ & 0 & -4e^{-\frac{a^{(4)}}{v_4}}\frac{v_3^\nu}{v_4^\nu}\chi_{\gamma-3} & \ldots & \\ & & \vdots & & \\ & & & 0 & -4e^{-\frac{a^{(n)}}{v_n}}\frac{v_{n-1}^\nu}{v_n^\nu}\chi_{\gamma-(n-1)} \\ & & & & 0 \end{bmatrix}$$

$$\mathcal{B}(\xi) = \begin{bmatrix} 0 & e^{\frac{a^{(2)}}{\xi_2}}\chi_{\gamma-2} & & & & \\ & 0 & e^{\frac{a^{(3)}}{\xi_3}}\chi_{\gamma-3} & & & \\ & & 0 & e^{\frac{a^{(4)}}{\xi_4}}\chi_{\gamma-4} & & \\ & & & \ddots & & \\ & & & & 0 & e^{\frac{a^{(n)}}{\xi_n}}\chi_{\gamma-n} \end{bmatrix}$$

$\zeta_k := \zeta + 4\tau_k$, $\zeta_{k,\tau} := \zeta + 4\tau_k - \tau$ $(\zeta = v, \xi,\ k = 1, \ldots, n)$.

Remark 6.22. *By Fredholm determinant* "det" *we denote the determinant defined through the usual series expansion*

$$\det(\mathrm{Id} - K) := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{X^k} \det[K(x_i, x_j)]_{i,j=1}^k \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_k) \tag{6.4.6}$$

with K *an integral operator acting on the Hilbert space* $L^2(X, d\mu(x))$ *and kernel* $K(x,y)$.

In the case at hand, we will see that the operator $[K]^{\mathrm{GEN}}\Big|_{\mathcal{I}} = (H + P_\Delta)\Big|_{\mathcal{I}}$ *is the sum of a trace-class operator* ($H\Big|_{\mathcal{I}}$) *plus a Hilbert-Schmidt operator* ($P_\Delta\Big|_{\mathcal{I}}$) *with diagonal-free kernel.*

Therefore the naming of Fredholm determinant refers to the following expression:

$$\text{“}\det\text{”}\left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right) = e^{\operatorname{Tr} H} \det{}_2 \left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right) \tag{6.4.7}$$

where $\det_2$ *denotes the regularized Carleman determinant (see [95]).*

Proof. Thanks to the invariance of the Fredholm determinant under kernel conjugation, we can discard the term $\left(\frac{y}{x}\right)^\nu$ in formulæ (6.4.1a)-(6.4.1b) for our further calculations.

We will work on the entry (i,j) of the kernel. We can notice that for $x < 0$ or $y < 0$ the kernel is identically zero, $[K]^{\mathrm{GEN}}(x,y) \equiv 0$. Then, applying Cauchy's theorem, we have

$$\begin{aligned} H_{ij}(x,y)\Big|_{[0,a^{(j)}]} &= 4\int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} \frac{e^{\xi(a^{(j)}-y)}}{\xi - s} \int_{\hat{\gamma}\times\gamma} \frac{\mathrm{d}s\,\mathrm{d}t}{(2\pi i)^2} \frac{e^{-a^{(j)}s + xt + \frac{1}{2}\left(\tau - \frac{1}{s} + 4\Delta_{ji}\right)^2 - \frac{1}{2}\left(\tau - \frac{1}{t}\right)^2}}{\left(\frac{1}{s} - \frac{1}{t} - 4\Delta_{ji}\right)} \left(\frac{s}{t}\right)^\nu \frac{1}{st} \\ &= -4\int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-y\xi} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{xt} \int_\gamma \frac{\mathrm{d}v}{2\pi i} \frac{e^{a^{(j)}\xi - \frac{1}{2}\left(\tau - \frac{1}{t}\right)^2 - \frac{a^{(j)}}{v + 4\tau_j} + \frac{1}{2}(\tau - 4\tau_i - v)^2}}{\left(\frac{1}{\xi} - 4\tau_j - v\right)\left(\frac{1}{t} - 4\tau_i - v\right)} \left(\frac{1}{(v + 4\tau_j)t}\right)^\nu \frac{1}{\xi t} \end{aligned} \tag{6.4.8}$$

where we deformed γ into a translated imaginary axis $i\mathbb{R}+\epsilon$ ($\epsilon > 0$) in order to make Fourier operator defined below more explicit; the last equality follows from the change of variable on $s = 1/(v + 4\tau_j)$, thus the contour $\hat{\gamma}$ becomes similar to γ and can be continuously deformed into it.

On the other hand, as $i < j$

$$\begin{aligned} P_{ij}(x,y)\Big|_{[0,a^{(j)}]} &= \frac{-1}{\Delta_{ji}} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-\xi y} \int_\gamma \frac{e^{\xi a^{(j)} + \frac{x}{4\Delta_{ji}}(t-1) - \frac{a^{(j)}}{4\Delta_{ji}}\left(1 - \frac{1}{t}\right)}}{\xi - \frac{1}{4\Delta_{ji}}\left(1 - \frac{1}{t}\right)} t^{-\nu-1} \frac{\mathrm{d}t}{2\pi i} \\ &= -4\int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-\xi y} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{xt} \frac{e^{a^{(j)}\left(\xi - \frac{t}{4\Delta_{ji}t + 1}\right)}}{t\xi\left(4\Delta_{ji} + \frac{1}{t} - \frac{1}{\xi}\right)} (4\Delta_{ji}t + 1)^{-\nu}. \end{aligned} \tag{6.4.9}$$

It is easily recognizable the conjugation with a Fourier-like operator as in (6.3.7), so that

$$\left([K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right)_{ij} = \mathcal{F}^{-1} \circ (\mathcal{H}_{ij} + \chi_{i<j}\mathcal{P}_{ij}) \circ \mathcal{F} \tag{6.4.10}$$

with

$$\mathcal{H}_{ij}(\xi,t) := -4\int_\gamma \frac{\mathrm{d}v}{2\pi i}\frac{e^{a^{(j)}\xi-\frac{1}{2}\left(\tau-\frac{1}{t}\right)^2-\frac{a^{(j)}}{v+4\tau_j}+\frac{1}{2}(\tau-4\tau_i-v)^2}}{\left(\frac{1}{\xi}-4\tau_j-v\right)\left(\frac{1}{t}-4\tau_i-v\right)}\left(\frac{1}{(v+4\tau_j)t}\right)^{\nu}\frac{1}{\xi t} \tag{6.4.11a}$$

$$\mathcal{P}_{ij}(\xi,t) := -4\frac{e^{a^{(j)}\left(\xi-\frac{t}{4\Delta_{ji}t+1}\right)}}{4\tau_j-4\tau_i+\frac{1}{t}-\frac{1}{\xi}}(4\Delta_{ji}t+1)^{-\nu}\frac{1}{\xi t}. \tag{6.4.11b}$$

Now we can perform the following change of variables on the Fourier-transformed kernel

$$\xi_j := \frac{1}{\xi}-4\tau_j, \quad \eta_i := \frac{1}{t}-4\tau_i \tag{6.4.12}$$

so that the kernel will have the final expression

$$\begin{aligned}\mathcal{K}^{\mathrm{GEN}}_{ij}(\xi,\eta) = \mathcal{H}_{ij}+\chi_{\tau_i<\tau_j}\mathcal{P}_{ij} = \\ -4\int_\gamma \frac{\mathrm{d}v}{2\pi i}\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{1}{2}(\tau-4\tau_i-\eta)^2-\frac{a^{(j)}}{v+4\tau_j}+\frac{1}{2}(\tau-4\tau_i-v)^2}}{(\xi-v)(\eta-v)}\left(\frac{\eta+4\tau_i}{v+4\tau_j}\right)^{\nu} \\ +4\chi_{\tau_i<\tau_j}\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{a^{(j)}}{\eta+4\tau_j}}}{\xi-\eta}\left(\frac{4\Delta_{ji}}{\eta+4\tau_i}+1\right)^{-\nu}\end{aligned} \tag{6.4.13}$$

with $\xi \in \frac{1}{\gamma}-4\tau_j =: \gamma_{-j}$ and $\eta \in \frac{1}{\gamma}-4\tau_i =: \gamma_{-i}$. The obtained (Fourier-transformed) Generalized Bessel operator is an operator acting on $L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right) \sim \bigoplus_{k=1}^n L^2\left(\gamma_{-k},\mathbb{C}^n\right)$.

Lemma 6.23. *The following decomposition holds $\mathcal{K}^{\mathrm{GEN}} = \mathcal{M}\circ\mathcal{N}+\mathcal{P}$, with $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$ Hilbert-Schmidt operators*

$$\mathcal{M} : L^2\left(\gamma,\mathbb{C}^n\right) \to L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right) \tag{6.4.14a}$$

$$\mathcal{N} : L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right) \to L^2\left(\gamma,\mathbb{C}^n\right) \tag{6.4.14b}$$

$$\mathcal{P} : L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right) \to L^2\left(\bigcup_{k=1}^n \gamma_{-k},\mathbb{C}^n\right) \tag{6.4.14c}$$

with kernel entries

$$\mathcal{M}_{ij}(v,\eta) = \frac{e^{-\frac{1}{2}(\tau-4\tau_i-\eta)^2+\frac{1}{2}(\tau-4\tau_i-v)^2}}{(\eta-v)}\left(\frac{\eta+4\tau_i}{v+4\tau_j}\right)^{\nu}\chi_\gamma(v)\chi_{\gamma_{-i}}(\eta) \tag{6.4.15a}$$

$$\mathcal{N}_{ij}(\xi,v;a^{(j)}) = 4\delta_{ij}\frac{e^{a^{(j)}\left(\frac{1}{\xi+4\tau_j}-\frac{1}{v+4\tau_j}\right)}}{\xi-v}\chi_{\gamma_{-j}}(\xi)\chi_\gamma(v) \tag{6.4.15b}$$

$$\mathcal{P}_{ij}(\xi,\eta;a^{(j)}) = 4\chi_{\tau_i<\tau_j}\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{a^{(j)}}{\eta+4\tau_j}}}{\xi-\eta}\left(\frac{\eta+4\tau_j}{\eta+4\tau_i}\right)^{-\nu}\chi_{\gamma_{-i}}(\eta)\chi_{\gamma_{-j}}(\xi). \tag{6.4.15c}$$

Proof. As in Lemma 6.12, all the kernels involved are of the form $K(z,w)$ with z and w on two disjoint curves, say C_1 and C_2. The Hilbert-Schmidt property it thus ensured by simply checking that $\iint_{C_1\times C_2}|K(z,w)|^2|\mathrm{d}z||\mathrm{d}w| < \infty$. $\square$

We define the Hilbert space

$$H := L^2\left(\gamma\cup\bigcup_{k=1}^n \frac{1}{\gamma}-4\tau_k,\mathbb{C}^n\right) \sim L^2\left(\bigcup_{k=1}^n \frac{1}{\gamma}-4\tau_k,\mathbb{C}^n\right)\oplus L^2(\gamma,\mathbb{C}^n), \tag{6.4.16}$$

and the matrix operator $[\mathbb{K}]^{\mathrm{GEN}}: H\to H$

$$[\mathbb{K}]^{\mathrm{GEN}} = \left[\begin{array}{c|c} 0 & \mathcal{N} \\ \hline \mathcal{M} & \mathcal{P}\end{array}\right]. \tag{6.4.17}$$

For now, we denote by "det" the determinant defined by the Fredholm expansion (6.4.6); then, "det"$(\mathrm{Id}-[\mathbb{K}]^{\mathrm{GEN}}) = \det_2(\mathrm{Id}-[\mathbb{K}]^{\mathrm{GEN}})$, since its kernel is diagonal-free. We also introduce another Hilbert-Schmidt operator

$$[\mathbb{K}]^{\mathrm{GEN},2} = \left[\begin{array}{c|c} 0 & -\mathcal{N} \\ \hline 0 & 0\end{array}\right]$$

whose Carleman determinant ($\det_2$) is still well defined and $\det_2(I-[\mathbb{K}]^{\mathrm{GEN},2})$ is identically 1.

We finally perform the following chain of equalities

$$
\begin{aligned}
``\det" \left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - [K]^{\mathrm{GEN}} \Big|_{\mathcal{I}} \right) &= \det{}_2 \left(\mathrm{Id} - [K]^{\mathrm{GEN}} \Big|_{\mathcal{I}} \right) e^{-\operatorname{Tr}(H)} \\
&= \det{}_2 \left(\mathrm{Id}_{L^2\left(\bigcup_{k=1}^n \gamma_{-k}\right)} - \mathcal{K}^{\mathrm{GEN}} \right) e^{-\operatorname{Tr}(\mathcal{H})} \\
= \det{}_2 \left(\mathrm{Id}_H - [\mathbb{K}]^{\mathrm{GEN}}\right) \det{}_2 \left(\mathrm{Id}_H - [\mathbb{K}]^{\mathrm{GEN},2}\right) &= \det{}_2 \left(\mathrm{Id}_H - [\mathbb{K}]^{\mathrm{GEN}}\right) \\
&= ``\det"\left(\mathrm{Id}_H - [\mathbb{K}]^{\mathrm{GEN}}\right).
\end{aligned}
\tag{6.4.18}
$$

The first equality follows from the fact that $[K]^{\mathrm{GEN}} - H$ is diagonal-free; the second equality follows from invariance of the determinant under Fourier transform; the third identity is an application of the following result: given A, B Hilbert-Schmidt operators, then

$$
\det{}_2 (\mathrm{Id} - A) \det{}_2 (\mathrm{Id} - B) = \det{}_2 (\mathrm{Id} - A - B + AB) e^{\operatorname{Tr}(AB)}.
$$

It is finally just a matter of computation to show that $[\mathbb{K}]^{\mathrm{GEN}}$ is an integrable operator of the form (6.4.5a)-(6.21). $\square$

Example: 2×2 case. As an explanatory example, let's consider a Generalized Bessel process with two times $\tau_1 < \tau_2$ and two intervals $I_1 := [0, a]$ and $I_2 := [0, b]$.

$$
\begin{aligned}
[K]^{\mathrm{GEN}}(x,y)\Big|_{[0,a],[0,b]} &= \left(\frac{y}{x}\right)^{\nu} \left\{ \begin{bmatrix} -4 \int_\Sigma \frac{dt\, ds}{(2\pi i)^2} \frac{e^{-\frac{1}{2}\left(\tau - \frac{1}{t}\right)^2 + xt + \frac{1}{2}\left(\tau - \frac{1}{s}\right)^2 - ys}}{(s-t) t^{\nu} s^{-\nu}} \Big|_{[0,a]} & 0 \\ -4 \int_\Sigma \frac{dt\, ds}{(2\pi i)^2} \frac{e^{-\frac{1}{2}\left(\tau - \frac{1}{t}\right)^2 + xt + \frac{1}{2}\left(\tau - \frac{1}{s} + \Delta_{12}\right)^2 - ys}}{(s-t+\Delta_{12} ts) t^{\nu} s^{-\nu}} \Big|_{[0,a]} & 0 \end{bmatrix} \right. \\
&+ \left. \begin{bmatrix} 0 & \left[-4 \int_\Sigma \frac{dt\, ds}{(2\pi i)^2} \frac{e^{-\frac{1}{2}\left(\tau - \frac{1}{t}\right)^2 + xt + \frac{1}{2}\left(\tau - \frac{1}{s} + \Delta_{21}\right)^2 - ys}}{(s-t+\Delta_{21} ts) t^{\nu} s^{-\nu}} - \frac{1}{\Delta_{21}} \int_\gamma e^{\frac{x}{4\Delta_{21}}(t-1) + \frac{y}{4\Delta_{21}}\left(\frac{1}{t} - 1\right)} t^{-\nu-1} \frac{dt}{2\pi i} \right] \Big|_{[0,b]} \\ 0 & -4 \int_\Sigma \frac{dt\, ds}{(2\pi i)^2} \frac{e^{-\frac{1}{2}\left(\tau - \frac{1}{t}\right)^2 + xt + \frac{1}{2}\left(\tau - \frac{1}{s}\right)^2 - ys}}{(s-t) t^{\nu} s^{-\nu}} \Big|_{[0,b]} \end{bmatrix} \right\}
\end{aligned}
\tag{6.4.19}
$$

with $\Sigma := \gamma \times \hat{\gamma}$.

Then, the integral operator $[\mathbb{K}]^{\mathrm{GEN}} : H \to H$ on the space $H := L^2\left(\gamma \cup \gamma_{-1} \cup \gamma_{-2}, \mathbb{C}^2\right)$ has the following expression

$$
[\mathbb{K}]^{\mathrm{GEN}} = \left[\begin{array}{c|c} 0 & \mathcal{N} \\ \hline \mathcal{M} & \mathcal{P} \end{array} \right]
\tag{6.4.20a}
$$

$$\mathcal{N} = \frac{1}{\xi - v}\begin{bmatrix} -4e^{\frac{a}{\xi_1}-\frac{a}{v_1}}\chi_{\gamma_{-1}}(\xi)\chi_\gamma(v) & 0 \\ 0 & -4e^{\frac{b}{\xi_2}-\frac{b}{v_2}}\chi_{\gamma_{-2}}(\xi)\chi_\gamma(v) \end{bmatrix} \tag{6.4.20b}$$

$$\mathcal{M} = \begin{bmatrix} \frac{e^{\frac{(\xi_{1,\tau})^2}{2}-\frac{(v_{1,\tau})^2}{2}}}{\xi-v}\frac{v_1^\nu}{\xi_1^\nu}\chi_\gamma(\xi)\chi_{\gamma_{-1}}(v) & \frac{e^{\frac{(\xi_{1,\tau})^2}{2}-\frac{(v_{1,\tau})^2}{2}}}{\xi-v}\frac{v_1^\nu}{\xi_2^\nu}\chi_\gamma(\xi)\chi_{\gamma_{-1}}(v) \\ \frac{e^{\frac{(\xi_{2,\tau})^2}{2}-\frac{(v_{2,\tau})^2}{2}}}{\xi-v}\frac{v_2^\nu}{\xi_1^\nu}\chi_\gamma(\xi)\chi_{\gamma_{-2}}(v) & \frac{e^{\frac{(\xi_{2,\tau})^2}{2}-\frac{(v_{2,\tau})^2}{2}}}{\xi-v}\frac{v_2^\nu}{\xi_2^\nu}\chi_\gamma(\xi)\chi_{\gamma_{-2}}(v) \end{bmatrix} \tag{6.4.20c}$$

$$\mathcal{P} = \frac{1}{\xi - v}\begin{bmatrix} 0 & -4e^{\frac{b}{\xi_2}-\frac{b}{v_2}}\frac{v_1^\nu}{v_2^\nu}\chi_{\gamma_{-2}}(\xi)\chi_{\gamma_{-1}}(v) \\ 0 & 0 \end{bmatrix} \tag{6.4.20d}$$

and the equality between Fredholm determinants holds

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+,\mathbb{C}^2)} - [K]^{\mathrm{GEN}}\Big|_{I_1,I_2}\right) = \det\left(\mathrm{Id}_H - [\mathbb{K}]^{\mathrm{GEN}}\right).$$

6.4.1 Riemann-Hilbert problem and τ-function

We can now relate the Fredholm determinant of the multi-time Generalized Bessel operator to the isomonodromy theory by defining a suitable Riemann-Hilbert problem.

Definition 6.24. The Riemann-Hilbert problem associated to the integrable kernel (6.4.5a)-(6.21) is the following:

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda) \quad \lambda \in \Xi := \gamma \cup \left(\bigcup_{j=1}^{n}\gamma_{-j}\right) \tag{6.4.21a}$$

$$\Gamma(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \lambda \to \infty \tag{6.4.21b}$$

$$M(\lambda) := I - 2\pi i J^{\mathrm{GEN}}(\lambda) \tag{6.4.21c}$$

with Γ a $(3n-1)\times(3n-1)$ matrix which is analytic on $\mathbb{C}\backslash\Xi$ and along the collection of curves Σ satisfies the above jump condition with

$$J^{\mathrm{GEN}}(\lambda) = \mathbf{f}(\lambda)\mathbf{g}(\lambda)^T =$$

$$\left[\begin{array}{c|c|c} 0 & \operatorname{diag}\mathcal{N}_f(\lambda)\mathcal{M}_g(\lambda)^T & 0 \\ \hline \operatorname{diag}\mathcal{M}_f(\lambda)\operatorname{diag}\mathcal{N}_g(\lambda) & 0 & \operatorname{diag}\mathcal{M}_f(\lambda)\mathcal{B}(\lambda)^T \\ \hline \mathcal{A}(\lambda)^T\operatorname{diag}\mathcal{N}_g(\lambda) & 0 & \mathcal{A}(\lambda)^T\mathcal{B}(\lambda)^T \end{array}\right] \tag{6.4.22}$$

$$\operatorname{diag}\mathcal{N}_f \cdot \mathcal{M}_g^T = \begin{bmatrix} -4e^{-\theta_1+\theta_{1,\tau}}\chi_\gamma & \dots & -4e^{-\theta_1+\theta_{n,\tau}}\chi_\gamma \\ \vdots & & \vdots \\ -4e^{-\theta_n+\theta_{1,\tau}}\chi_\gamma & \dots & -4e^{-\theta_n+\theta_{n,\tau}}\chi_\gamma \end{bmatrix} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$

$$\operatorname{diag}\mathcal{M}_f \cdot \operatorname{diag}\mathcal{N}_g = \operatorname{diag}\left[e^{\theta_1-\theta_{1,\tau}}\chi_{\gamma_{-1}}, \dots, e^{\theta_n-\theta_{n,\tau}}\chi_{\gamma_{-n}}\right] \in \mathrm{Mat}_{n\times n}(\mathbb{C})$$

$$\begin{aligned} &\operatorname{diag}\mathcal{M}_f \cdot \mathcal{B}^T \in \mathrm{Mat}_{n\times(n-1)}(\mathbb{C}) \\ &= \begin{bmatrix} 0 & & & \\ e^{\theta_2-\theta_{2,\tau}}\chi_{\gamma_{-2}} & 0 & & \\ & e^{\theta_3-\theta_{3,\tau}}\chi_{\gamma_{-3}} & \ddots & \\ & & & e^{\theta_n-\theta_{n,\tau}}\chi_{\gamma_{-n}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\mathcal{A}^T \cdot \operatorname{diag}\mathcal{N}_g \in \mathrm{Mat}_{(n-1)\times n}(\mathbb{C}) \\ &= \begin{bmatrix} -4e^{\theta_1-\theta_2}\chi_{\gamma_{-1}} & 0 & & & \\ -4e^{\theta_1-\theta_3}\chi_{\gamma_{-1}} & -4e^{\theta_2-\theta_3}\chi_{\gamma_{-2}} & 0 & & \\ -4e^{\theta_1-\theta_4}\chi_{\gamma_{-1}} & -4e^{\theta_2-\theta_4}\chi_{\gamma_{-2}} & -4e^{\theta_3-\theta_4}\chi_{\gamma_{-3}} & 0 & \\ \vdots & & & \ddots & \\ & & & & \\ -4e^{\theta_1-\theta_n}\chi_{\gamma_{-1}} & \dots & & -4e^{\theta_{n-1}-\theta_n}\chi_{\gamma_{-(n-1)}} & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &\mathcal{A}^T \cdot \mathcal{B}^T \in \mathrm{Mat}_{(n-1)\times(n-1)}(\mathbb{C}) \\ &= \begin{bmatrix} 0 & & & \\ -4e^{\theta_2-\theta_3}\chi_{\gamma_{-2}} & & & \\ -4e^{\theta_2-\theta_4}\chi_{\gamma_{-2}} & -4e^{\theta_3-\theta_4}\chi_{\gamma_{-3}} & & \\ \vdots & \vdots & & \\ & & & \\ -4e^{\theta_2-\theta_n}\chi_{\gamma_{-2}} & & -4e^{\theta_{n-1}-\theta_n}\chi_{\gamma_{n-1}} & 0 \end{bmatrix} \end{aligned}$$

with $\theta_k = \frac{a^k}{\lambda_k} + \nu \ln \lambda_k$ and $\theta_{h,\tau} = \frac{(\lambda_{h,\tau})^2}{2}$, $k, h = 1, \dots, n$.

Example: 2×2 case. In the simple 2-times case, the jump matrix reads

$$J^{\mathrm{GEN}}(\lambda) =$$
$$\begin{bmatrix} 0 & 0 & -4e^{-\theta_1+\theta_{1,\tau}}\chi_\gamma & -4e^{-\theta_1+\theta_{2,\tau}}\chi_\gamma & 0 \\ 0 & 0 & -4e^{-\theta_2+\theta_{1,\tau}}\chi_\gamma & -4e^{-\theta_2+\theta_{2,\tau}}\chi_\gamma & 0 \\ e^{\theta_1-\theta_{1,\tau}}\chi_{\gamma_{-1}} & 0 & 0 & 0 & 0 \\ 0 & e^{\theta_2-\theta_{2,\tau}}\chi_{\gamma_{-2}} & 0 & 0 & e^{\theta_2-\theta_{2,\tau}}\chi_{\gamma_{-2}} \\ -4e^{\theta_1-\theta_2}\chi_{\gamma_{-1}} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.4.23}$$

The jump matrix, though it might look complicated, is equivalent to a matrix with constant entries

$$e^{T^{\mathrm{GEN}}}J_0e^{-T^{\mathrm{GEN}}} = J^{\mathrm{GEN}}$$
$$T^{\mathrm{GEN}} = \operatorname{diag}\left[-\theta_1,\dots,-\theta_n,-\theta_{1,\tau},\dots,-\theta_{n,\tau},-\theta_2,\dots,-\theta_n\right] \tag{6.4.24}$$

so that the matrix $\Psi^{\mathrm{GEN}}(\lambda) = \Gamma(\lambda)e^{T^{\mathrm{GEN}}(\lambda)}$ solves a Riemann-Hilbert problem with constant jumps and it is a solution to a polynomial ODE.

Referring to the theorems described in Section 3.4 (see also [9], [10] and [11]), we can claim

Theorem 6.25. *Given n times $\tau_1 < \tau_2 < \ldots < \tau_n$ and given the collection of intervals $\mathcal{I} = \{I_1,\dots,I_n\}$ with*

$$I_j := \left[a_1^{(j}, a_2^{(j)}\right] \cup \left[a_3^{(j)}, a_4^{(j)}\right] \cup \ldots \cup \left[a_{2k_j-1}^{(j)}, a_{2k_j}^{(j)}\right], \tag{6.4.25}$$

the Fredholm determinant $\det\left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right)$ *is equal to the isomonodromic τ-function related to the Riemann-Hilbert problem in Definition 6.24.*

In particular, $\forall\, j = 1,\dots,n$ and $\forall\, \ell = 1,\dots,2k_j$ we have

$$\partial \ln\det\left(\mathrm{Id} - [K]^{\mathrm{GEN}}\Big|_{\mathcal{I}}\right) = \int_\Xi \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Theta_\partial(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{6.4.26a}$$
$$\Theta_\partial(\lambda) := \partial M(\lambda)M(\lambda)^{-1} = -2\pi i\,\partial J^{\mathrm{GEN}}\left(I + 2\pi i J^{\mathrm{GEN}}\right) \tag{6.4.26b}$$

$\Xi := \gamma \cup \gamma_{-1} \cup \ldots \cup \gamma_{-n}$; the $'$ notation means differentiation with respect to λ, while with ∂ we denote any of the partial derivatives ∂_{τ_j}, $\partial_{a_\ell^{(j)}}$, ∂_τ.

Proof. The following formula holds in general (see Theorem 3.14 in Chapter 3)

$$\omega(\partial) = \partial \ln \det \left(\mathrm{Id} - [K]^{\mathrm{GEN}} \Big|_{\mathcal{I}} \right) - H(M) \tag{6.4.27}$$

where

$$\begin{aligned} \omega(\partial) &:= \int_{\Xi} \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma'_-(\lambda)\Theta_\partial(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \\ H(M) &:= \int_{\Xi} \left(\partial \mathbf{f}'^T \mathbf{g} + \mathbf{f}'^T \partial \mathbf{g}\right) \mathrm{d}\lambda - 2\pi i \int_{\Sigma} \mathbf{g}^T \mathbf{f}' \partial \mathbf{g}^T \mathbf{f} \,\mathrm{d}\lambda. \end{aligned}$$

Therefore, it is enough to verify that $H(M) \equiv 0$ with $M(\lambda) = I - J^{\mathrm{GEN}}(\lambda)$. □

Moreover, recalling of the Jimbo-Miwa-ueno residue formula, it can be shown that

Theorem 6.26. *The following equality holds*

$$\begin{aligned} &\int_{\Xi} \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma'_-(\lambda)\Theta_\partial(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \\ &= -\operatorname{res}_{\lambda=\infty} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial T^{\mathrm{GEN}}\right) + \sum_{i=1}^{n} \operatorname{res}_{\lambda=-4\tau_i} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial T^{\mathrm{GEN}}\right). \end{aligned} \tag{6.4.28}$$

In particular, regarding the derivative with respect to the endpoints $a^{(j)}$ $(j = 1, \dots, n)$

$$\operatorname{res}_{\lambda=-4\tau_k} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(k)}} T^{\mathrm{GEN}}\right) = -\left(\Gamma_0^{-1}\Gamma_1\right)_{(k,k)} - \chi_{k>1}\left(\Gamma_0^{-1}\Gamma_1\right)_{(2n-1+k,2n-1+k)}. \tag{6.4.29}$$

Regarding the derivative with respect to τ

$$\operatorname{res}_{\lambda=\infty} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_\tau T^{\mathrm{GEN}}\right) = -\sum_{k=1}^{n} \Gamma_{1;n+k,n+k}. \tag{6.4.30}$$

Finally, regarding the derivative with respect to the times τ_j $(j = 1, \dots, n)$

$$\operatorname{res}_{\lambda=\infty} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_k} T^{\mathrm{GEN}}\right) = 4\Gamma_{1;n+k,n+k} \tag{6.4.31}$$

$$\begin{aligned} \operatorname{res}_{\lambda=-4\tau_k} \mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_k} T_B\right) &= -4\nu\left(\Phi_{0;k,k} + \chi_{k>1}\Phi_{0;2n-1+k,2n-1+k}\right) \\ &\quad + 4a^{(k)}\left(\Phi_{1;k,k} + \chi_{k>1}\Phi_{1;2n-1+k,2n-1+k}\right) \end{aligned} \tag{6.4.32}$$

where, given the asymptotic expansion of the matrix $\Gamma \sim \Gamma_0 + \lambda_k\Gamma_1 + \lambda_k^2\Gamma_2 + \cdots$ in a neighbourhood of $-4\tau_k$, we defined $\Phi_0 := \Gamma_0^{-1}\Gamma_1$ and $\Phi_1 := 2\Gamma_0^{-1}\Gamma_2 - \left(\Gamma_0^{-1}\Gamma_1\right)^2$.

Remark 6.27. *We stated the second part of the theorem above in the simple case $\mathcal{I} =$*

$\{[0, a^{(1)}], \ldots, [0, a^n]\}$ *in order to avoid heavy notation. The general case follows the same guidelines shown in the proof.*

Proof. We will calculate the residues separately and we will focus on the different parameters ($a^{(j)}$, τ_j and τ).

Residue at ∞. There's no contribution from the residue at infinity when we consider the derivative with respect to the endpoints $a^{(j)}$. On the other hand, taking the derivative with respect to the times τ_k gives:

$$\partial_{\tau_k} T^{\text{GEN}} = \left(\frac{4a^{(k)}}{\lambda_k^2} - \frac{4\nu}{\lambda_k} \right) (E_{k,k} + \chi_{k>1} E_{2n-1+k, 2n-1+k}) - 4\lambda_{k,\tau} E_{n+k,n+k} \tag{6.4.33}$$

thus the residue is

$$\text{res}_{\lambda=\infty} \operatorname{Tr} \left(\Gamma^{-1} \Gamma' \partial_{\tau_k} T^{\text{GEN}} \right) = 4\Gamma_{1;n+k,n+k} \quad \forall\, k = 1, \ldots, n. \tag{6.4.34}$$

We follow a similar argument for the parameter τ:

$$\partial_\tau T^{\text{GEN}} = \sum_{k=1}^{n} \lambda_{k,\tau} E_{n+k,n+k} \tag{6.4.35}$$

Thus,

$$\text{res}_{\lambda=\infty} \operatorname{Tr} \left(\Gamma^{-1} \Gamma' \partial_\tau T^{\text{GEN}} \right) = -\sum_{k=1}^{n} \Gamma_{1;n+k,n+k}. \tag{6.4.36}$$

Residue at $4\tau_k$. We recall the asymptotic expansion of the matrix Γ in a neighbourhood of $-4\tau_k$:

$$\Gamma \sim \Gamma_0 + \lambda_k \Gamma_1 + \lambda_k^2 \Gamma_2 + \cdots \qquad \lambda \to -4\tau_k,\ \forall\, k = 1, \ldots, n. \tag{6.4.37}$$

Remark 6.28. *Note that the asymptotic expansion near $-4\tau_k$ is, in general, different for each k, but we wrote them in this way in order to avoid heavy notation.*

Regarding the derivative with respect to the endpoints $a^{(k)}$, we have

$$\partial_{a^{(k)}} T^{\text{GEN}} = -\frac{1}{\lambda_k} \left[E_{k,k} + \chi_{k>1} E_{2n-1+k, 2n-1+k} \right] \tag{6.4.38}$$

which implies

$$\operatorname{res}_{\lambda=-4\tau_k}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(k)}}T^{\mathrm{GEN}}\right) = -\left(\Gamma_0^{-1}\Gamma_1\right)_{(k,k)} - \chi_{k>1}\left(\Gamma_0^{-1}\Gamma_1\right)_{(2n-1+k,2n-1+k)} \tag{6.4.39}$$

and regarding the derivative with respect to the times τ_k, we have

$$\partial_{\tau_k}T^{\mathrm{GEN}} = \left(\frac{4a^{(k)}}{\lambda_k^2} - \frac{4\nu}{\lambda_k}\right)\left[E_{k,k} + \chi_{k>1}E_{2n-1+k,2n-1+k}\right] - 4\lambda_{k,\tau}E_{n+k,n+k} \tag{6.4.40}$$

thus,

$$\begin{aligned}\operatorname{res}_{\lambda=-4\tau_k}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_k}T_B\right) = &-4\nu\left(\Phi_{0;k,k} + \chi_{k>1}\Phi_{0;2n-1+k,2n-1+k}\right)\\ &+4a^{(k)}(\Phi_{1;k,k} + \chi_{k>1}\Phi_{1;2n-1+k,2n-1+k}).\end{aligned} \tag{6.4.41}$$

There is no contribution from the residue at $-4\tau_k$ $(k = 1, \dots, n)$ when taking the derivative with respect to τ. □

6.5 Conclusions and future developments

In the present Chapter we have analyzed gap probabilities for the so-called Generalized Bessel process ([76], [77]) restricted to a collection of disjoint intervals.

We stress out that two completely new contributions were introduced along the present work: a Lax pair for the single-time Generalized Bessel operator and the explicit definition of the multi-time Generalized Bessel kernel.

Both for the single-time and multi-time process, the main result was the connection with a Riemann-Hilbert problem associated to an IIKS integrable operator, whose Fredholm determinant coincide with the aforementioned gap probabilities. The presence of such Riemann-Hilbert problem allows a deeper analysis of these quantities, if desired. It can be the starting point for many possible future developments and we will briefly cite a few here.

The first study that can be done on gap probabilities is the asymptotic behaviour as the size of the intervals go to ∞ or 0; it is, of course, expected that as the Borel set, on which we calculate the gap probability, shrinks to zero, the gap probability tends to $1 = \det(\mathrm{Id})$. The second and more interesting analysis is the degenerative behaviour as $\tau \to \pm\infty$. Indeed, the origin of the Generalized Bessel operator itself suggests that, being a critical kernel depending on a parameter $\tau \in \mathbb{R}$, the gap probabilities may degenerate into gap probabilities of an Airy process or a Bessel process. Physically, this means to start at the critical point at time t^* and move away from it along the soft edge of the boundary of the domain (as $\tau \to -\infty$), where the local behaviour is described by the Airy process, or along the hard edge (as $\tau \to +\infty$) where the local behaviour is given by the Bessel process (see Figure 6.2). In order to achieve the conjectured results, one may consider to perform a steepest descent analysis (see Chapter 4) on the associated Riemann-Hilbert problem, as it has been done in the coming Chapter 7.

In the same spirit as it was done for the Airy (([5], [10], [100], [106]), Pearcey ([103]) and Bessel processes ([101], [102] and Chapter 5), one may wonder whether there exist partial or ordinary differential equations that describe the τ-function (i.e. the gap probabilities) of the Generalized Bessel process. From the given Riemann-Hilbert problem, it is possible to give a formulation of a Lax pair, as we did in Section 6.3.2, and calculate the compatibility equations which will give a system of coupled first order ODEs; then, the system may be reduced to a higher order ODE in one of the dependent variables appearing originally (as we did for the Bessel process in Chapter 5). This approach can be applied in the multi-time setting as well. Another approach can be the following: if it is possible to prove that the τ-function under consideration is a multi-component Kadomtsev-Petviashvili (KP) τ-function, by verifying the Hirota bilinear equations, then it will be possible to manually construct ODEs which are satisfied by the τ-function itself. We refer to the papers [29] and [55] and to the monograph [47] for all the details.

As final remark, we would like to thank Dr. Bálint Vető for the useful exchange of emails on the multi-time Generalized Bessel kernel and for the productive discussions at the ICTP (Trieste, Italy) during the Summer School "Random Matrices and Growth Models" in September 2013.

Chapter 7

Asymptotics of gap probabilities: from the tacnode to the Airy process

7.1 Introduction

In this last chapter we will focus on the gap probabilities of the so-called tacnode process. In particular, we will show that its gap probability restricted to a collection of intervals is again equal to the isomonodromic τ-function; however, we will not derive a system of differential equations for such gap probability, but on the other hand we will focus on its asymptotic behaviour. Indeed, the nature of the tacnode process as a critical transition process suggests that its gap probability can degenerate in the limit as some physical parameters diverge to either plus or minus infinity, as it will be clear below.

Let us start from the model of n non-intersecting Brownian path and let assume that all the paths start at two given fixed points and end at two other points (which may be equal to the starting points). For every time $t \in [0, 1]$ (1 being the end time where the particles collapse in the two final points), the positions of the Brownian paths form a determinantal process. Moreover, as the number of particles tends to infinity, the paths fill a specific limit region which depends on the relative position of the starting and ending points.

There are three possible scenarios: two independent connected components similar to ellipses or one connected component similar to two "merged" ellipses (see Figure 7.1 and 7.2). It is well-known that the microscopic behaviour of such infinite particle system is regulated by the Sine process in the bulk of the particle bundles ([85]), by the Airy process along the soft edges ([28], [61], [60], [75], [100]) and by the Pearcey process in the cusp singularities ([13], [103]), when they occur.

There exist a third critical configuration, which can be seen as a limit of the large sepa-

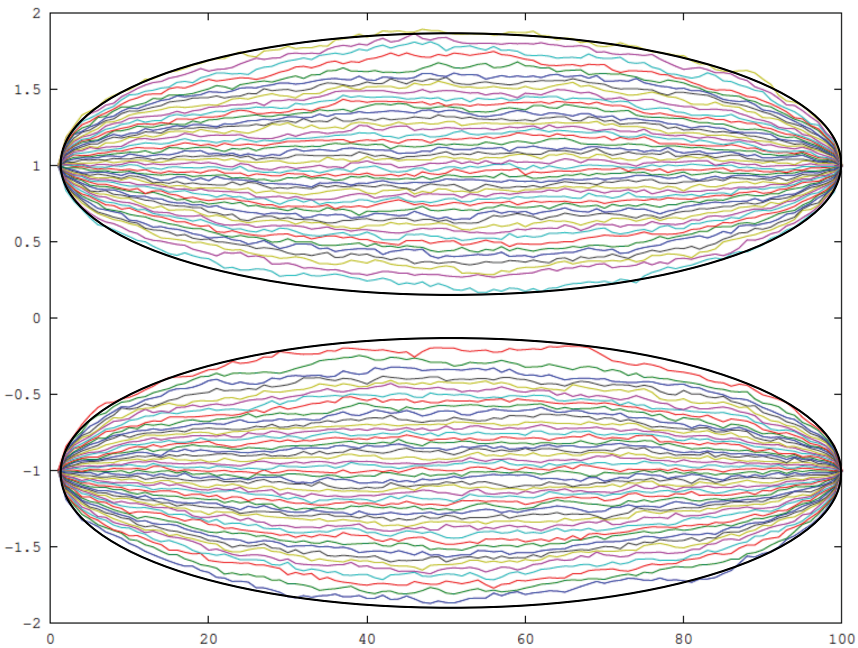

Figure 7.1: Numerical simulation of 90 non-intersecting Brownian motions with two starting points $\pm\alpha = 1$ and two ending positions $\pm\beta = 1$ in case of large separation between the endpoints.

ration case, when the two bundles are tangential to each other in one point, called tacnode point (see Figure 7.3), as well as a limit of the small separation case, when the two cusp singularities coincide at one point. In a microscopic neighbourhood of this point the fluctuations of the particles are described by a new critical process called tacnode process. In this limit setting, a parameter σ appears which controls the strength of interaction between the left-most particles and the right-most ones (σ can be thought as a pressure or temperature parameter).

The kernel of such process in the single-time case has been first introduced by Adler, Ferrari and Van Moerbeke in [3] as a scaling limit of a model of random walks, and shortly after by Delvaux, Kuijlaars and Zhang in [28], where the kernel was expressed in terms of a 4×4 matrix valued Riemann-Hilbert problem. In [62] Johansson formulated the multi-time (or extended) version of the process, remarking nevertheless the fact that this extended version does not automatically reduce to the single-time version given in [28]. In this paper, for the first time, the kernel was expressed in terms of the resolvent and Fredholm determinant of the Airy kernel.

In [4] the authors analyzed the same process as arising from random tilings instead of self-avoiding Brownian paths and they proved the equivalency of all the above formulations. A similar result has been obtained by Delvaux in [27], where a Riemann-Hilbert expression

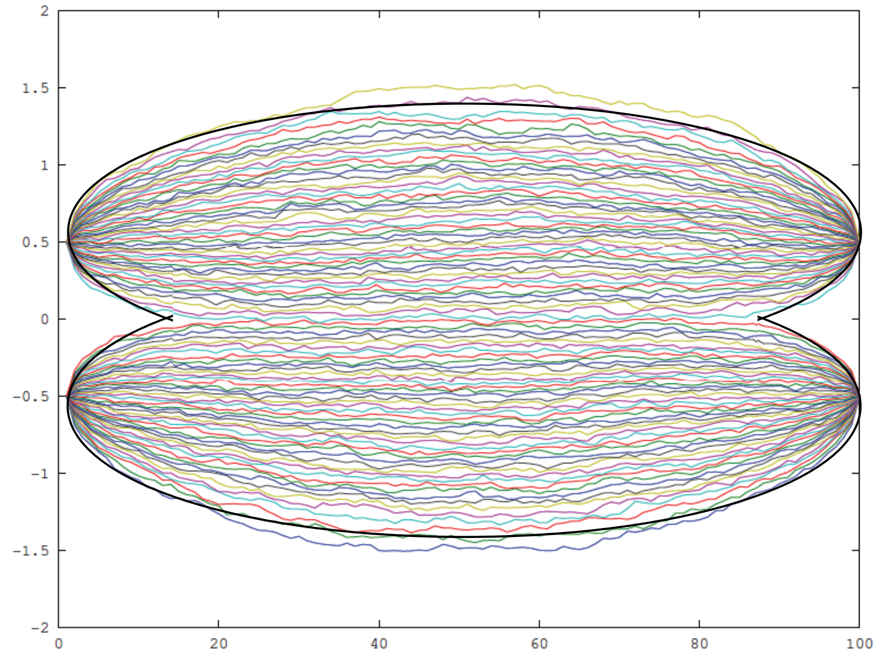

Figure 7.2: Numerical simulation of 90 non-intersecting Brownian motions with two starting points $\pm\alpha = 0.5$ and two ending positions $\pm\beta = 0.5$ in case of small separation between the endpoints.

for the multi-time tacnode kernel is given. A more general formulation of this process has been studied in [35], where the limit shapes of the two groups of particles are allowed to be non-symmetric.

Physically, if we start from the tacnode configuration and we push together the two ellipses, they will merge giving rise to the single connected component in Figure 7.2, while if we pull the ellipses apart, we simply end up with two disjoint ellipses as in Figure 7.1. It is thus natural to expect that the local dynamic around the tacnode point will in either cases degenerates into a Pearcey process or an Airy process, respectively.

The degeneration tacnode-Pearcey has been proven in [43] where the authors showed a uniform convergence of the tacnode kernel to the Pearcey kernel over compact sets in the limit as the two bundles are pushed to merge together. On the other hand, the method used in [43] cannot be extensively applied to the tacnode-Airy degeneration. The Airy process is structurally different from the Pearcey, since it shows the feature of a "last particle" (or largest eigenvalue in the Random Matrix setting), that is described by the well-known Tracy-Widom distribution ([100]). The method above does not allow to recover the emerging of the "last particle" feature from the tacnode-to-Airy degeneration, which, on the other hand, is showed in the present work.

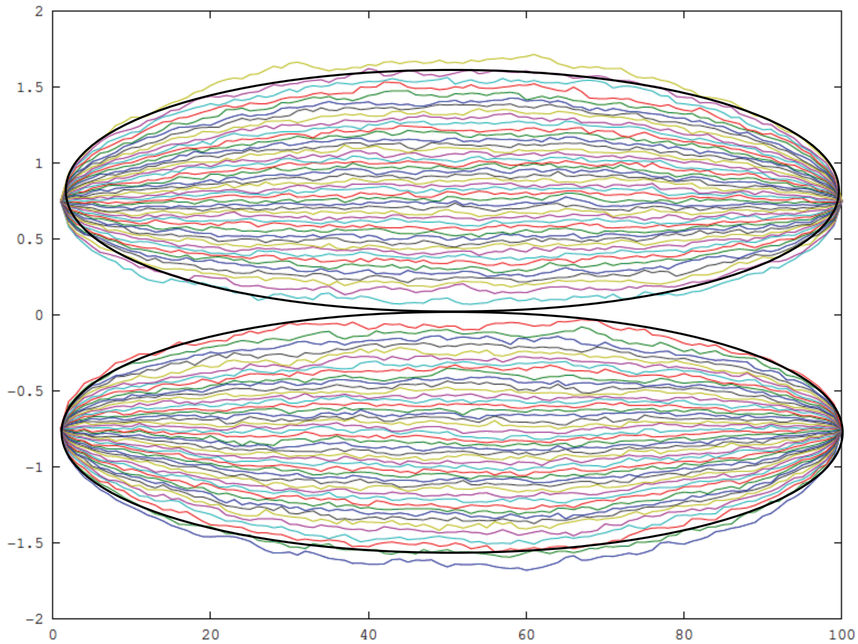

Figure 7.3: Numerical simulation of 90 non-intersecting Brownian motions with two starting points $\pm\alpha = .75$ and two ending positions $\pm\beta = .75$ in case of critical separation between the endpoints.

The purpose of this chapter is to study the asymptotic behaviour of the gap probability of the (single-time) tacnode process and its degeneration into the gap probability of the Airy process. There are two types of regimes in which this degeneration occurs: the limit as $\sigma \to +\infty$ (large separation), which physically corresponds to pulling apart the two sets of Brownian particles touching on the tacnode point (see Figure 7.7), and the limit as $\tau \to \pm\infty$ (large time), which corresponds to moving away from the singular point along the boundary of the space-time region swept out by the non-intersecting paths (see Figure 7.10). Numerical evidences of such degenerations were showed in [12].

An expression for the single-time tacnode kernel is the following (see [4, formula (19)])

$$\mathbb{K}^{\mathrm{tac}}(\tau; x, y) =$$
$$K_{\mathrm{Ai}}^{(\tau,-\tau)}(\sigma - x, \sigma - y) + \sqrt[3]{2} \int_{\widetilde{\sigma}}^{\infty} \mathrm{d}z \int_{\widetilde{\sigma}}^{\infty} \mathrm{d}w \mathcal{A}_{x-\sigma}^{\tau}(w) \left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},+\infty)} \right)^{-1} (z, w) \mathcal{A}_{y-\sigma}^{-\tau}(z) \tag{7.1.1}$$

with $\widetilde{\sigma} := 2^{\frac{2}{3}}\sigma$ and

$$\begin{aligned}
\mathrm{Ai}^{(\tau)}(x) &:= \mathrm{e}^{\tau x+\frac{2}{3}\tau^3}\mathrm{Ai}(x) = \int_{\gamma_R} \frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\frac{\lambda^3}{3}+\lambda^2\tau - x\lambda} \\
\mathrm{Ai}(x) &:= \int_{\gamma_R} \frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\frac{\lambda^3}{3}-x\lambda} = -\int_{\gamma_L} \frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{-\frac{\lambda^3}{3}+x\lambda} \\
\mathcal{A}_x^{\tau}(z) &:= \mathrm{Ai}^{(\tau)}(x+\sqrt[3]{2}z) - \int_0^\infty \mathrm{d}w\, \mathrm{Ai}^{(\tau)}(-x+\sqrt[3]{2}w)\mathrm{Ai}(w+z) \\
K_{\mathrm{Ai}}^{(\tau,-\tau)}(-x,-y) &:= \int_0^\infty \mathrm{d}u \mathrm{Ai}^{(\tau)}(-x+u)\mathrm{Ai}^{(-\tau)}(-y+u) \\
K_{\mathrm{Ai}}(z,w) &:= \int_0^\infty \mathrm{d}u\, \mathrm{Ai}(z+u)\mathrm{Ai}(w+u)
\end{aligned}$$

where the contour γ_R is the contour extending to infinity in the λ-plane along the rays $e^{\pm i\frac{\pi}{3}}$, oriented upwards and entirely contained in the right half plane ($\Re(\lambda) > 0$), and $\gamma_L := -\gamma_R$.

The quantity of interest, i.e. the gap probability of the process, is expressed in terms of the Fredhom determinant of an integral operator with kernel (7.1.1). Given a Borel set $\mathcal{I}$, then

$$P\left(\text{no particles in } \mathcal{I}\right) = \det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{\mathcal{I}}\right). \tag{7.1.2}$$

The first difficulty in studying the tacnode process is the expression of its kernel, since it is highly transcendental and it involves the resolvent of the Airy operator. It it thus necessary to reduce it to a more approachable form.

The first important step was [12, Theorem 3.1] where it was proved that gap probabilities of the tacnode process can be defined as ratio of two Fredholm determinants of explicit integral operators with kernels that only involves contour integrals, exponentials and Airy functions. This result, which will be recalled in Section 7.3, will be our starting point in the investigation of the gap probabilities and their asymptotics. The second step will be to find an appropriate integral operator in the sense of Its-Izergin-Korepin-Slavnov ([50]) whose Fredholm determinant coincides with the quantity (7.1.2). In this way, it will be possible to give a formulation of the gap probabilities of the tacnode in terms of a Riemann-Hilbert (RH) problem, naturally associated to an IIKS integral operator (see Chapter 3.3 and [45]). Finally, applying well-known steepest descent methods (Chapter 4) to the above RH problem along the lines of [11], we will be able to prove the conjectured degeneration into Airy processes.

The outline of the chapter is the following: in Section 7.2 we state the main results of the paper, which will be proved in Sections 7.3, 7.4 and 7.5. In particular, Section 7.3 deals with some preliminary calculations which are necessary to set a Riemann-Hilbert problem on which we shall later perform some steepest descent analysis in the limit as $\sigma \to +\infty$ (Section 7.4) or $\tau \to \pm\infty$ (Section 7.5).

7.2 Results

The first results on asymptotic regime of the tacnode process were stated in [12]. We are recalling them here for the sake of completeness.

Theorem 7.1. *Let $\mathcal{I} := \bigcup_{j=1}^{K}[a_{2j-1}, a_{2j}]$ be collection of intervals, with $a_j = a(s_j) = -\sigma - \tau^2 + s_j$. Keeping the overlap σ fixed, we have*

$$\lim_{\tau\to\pm\infty} \det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J}\right) \tag{7.2.1}$$

with $J = \bigcup_{\ell=1}^{K}[s_{2\ell-1}, s_{2\ell}]$. Analogously, keeping τ fixed, we obtain

$$\lim_{\sigma\to+\infty} \det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J}\right). \tag{7.2.2}$$

Proof. The convergence follows easily by directly studying the kernel of the extended tacnode process (see [4, formula (19)]), since the term involving the resolvent of the Airy kernel tends to zero, uniformly over compact sets of the spatial variables $x - \sigma - \tau^2$. □

The physical interpretation of such results is that if we follow, starting from the tacnode point, only one of the two soft edges (either in the case of large separation or in the case of large times) we can easily see that the tacnode kernel converges to the Airy kernel, therefore the convergence of the process respectively. Nevertheless, a more interesting situation is the one in which, as we are taking the limit, we follow both soft edges and the tacnode process degenerates into a couple of Tracy-Widom distributions, in analogy with the Pearcey-to-Airy transition (see [11]). In this case, half of the space variables (endpoints of the gaps) moves far away from the tacnode following the left branch of the boundary of the space-time region swept by the particles, and the other half goes in the opposite direction. Therefore, it is expected that the gap probability of the tacnode process for a "large gap" factorize into two Fredholm determinants for semi-infinite gaps of the Airy process.

Numerically, these regimes are illustrated in Figure 7.4 and 7.5. The results were already conjectured in [12] and they are here rigorously proved.

In the simple case with only one interval, we have the following theorems.

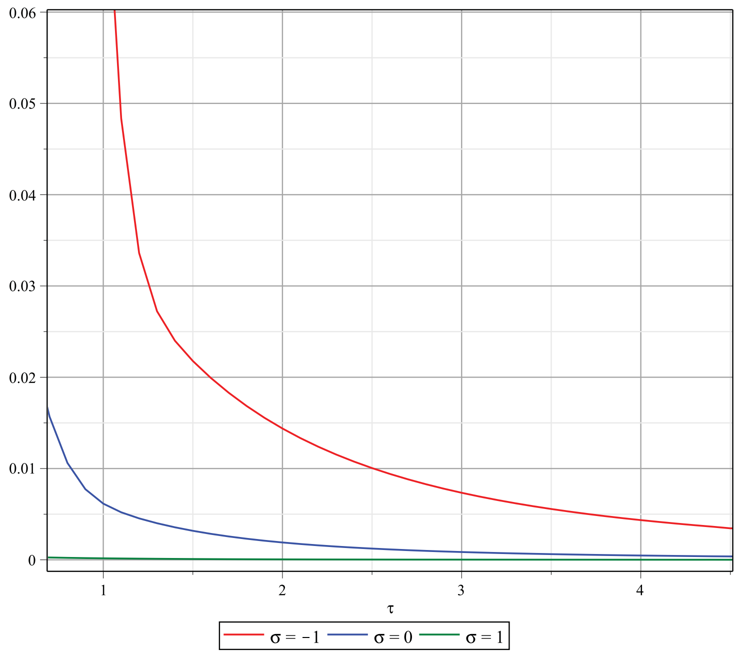

Figure 7.4: The relative values $\frac{\det(\mathrm{Id} - \Pi \mathbb{K}^{\mathrm{tac}} \Pi)}{F_2(a)F_2(b)} - 1$ with Π the projection on the interval $[a_{\mathrm{tac}}, b_{\mathrm{tac}}]$, $a_{\mathrm{tac}} = a - \sigma - \tau^2$ and $b_{\mathrm{tac}} = -b + \sigma + \tau^2$, plotted against τ, showing the convergence of the tacnode gap probability to the product of two Tracy-Widom distributions as $\sigma \to +\infty$. Here $a = -0.2$, $b = 0.4$.

Theorem 7.2 (**Asymptotics as** $\sigma \to +\infty$)**.** *Let $\mathbb{K}^{\mathrm{tac}}$ and K_{Ai} be the kernels associated to the tacnode and Airy process respectively. Let*

$$a = a(t) = -\sigma - \tau^2 + t \quad b = b(s) = \sigma + \tau^2 - s \tag{7.2.3}$$

then as $\sigma \to +\infty$

$$\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{[-\sigma-\tau^2+t, \sigma+\tau^2-s]}\right) = \\ \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[s,+\infty)}\right) \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[t,+\infty)}\right) \left(1 + \mathcal{O}(\sigma^{-1})\right) \tag{7.2.4}$$

and the convergence is uniform over compact sets of the variables s, t provided

$$-\infty < s, t < K_1(\sigma + \tau^2), \quad 0 < K_1 < 1.$$

Theorem 7.3 (**Asymptotics as** $\tau \to \pm\infty$)**.** *Let $\mathbb{K}^{\mathrm{tac}}$ and K_{Ai} be the kernels associated to*

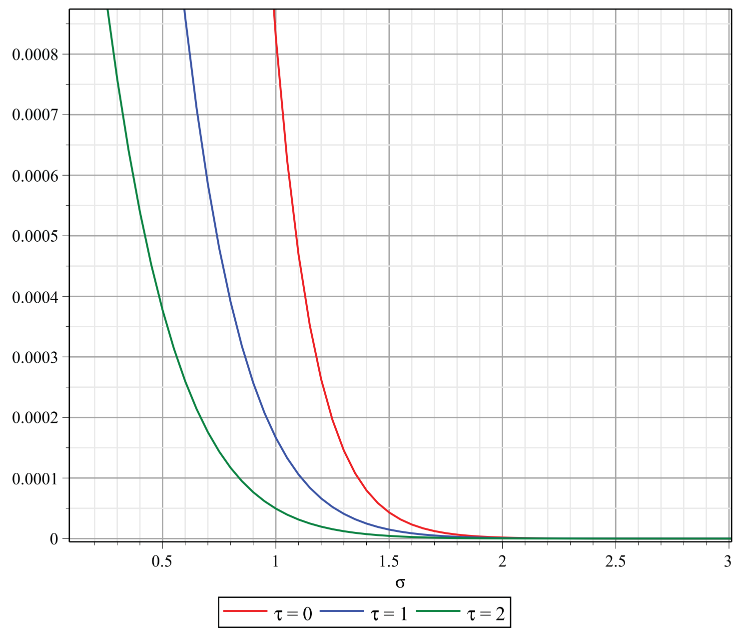

Figure 7.5: The relative values $\frac{\det(\mathrm{Id} - \Pi \mathbb{K}^{\mathrm{tac}} \Pi)}{F_2(a) F_2(b)} - 1$ with Π the projection on the interval $[a_{\mathrm{tac}}, b_{\mathrm{tac}}]$, $a_{\mathrm{tac}} = a - \sigma - \tau^2$ and $b_{\mathrm{tac}} = -b + \sigma + \tau^2$, plotted against σ, showing the convergence of the tacnode gap probability to the product of two Tracy-Widom distributions as $\tau \to +\infty$. Here $a = -0.2$, $b = 0.4$.

the tacnode and Airy process respectively. Let

$$a = a(t) = -\sigma - \tau^2 + t \quad b = b(s) = \sigma + \tau^2 - s \tag{7.2.5}$$

then as $\tau \to \pm\infty$

$$\begin{aligned} &\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}} \Big|_{[-\sigma-\tau^2+t, \sigma+\tau^2-s]} \right) = \\ &\frac{\det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[s,+\infty)} \right) \det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[t,+\infty)} \right) \det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[\tilde{\sigma},\infty)} \right) (1 + \mathcal{O}(\tau^{-1}))}{\det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[\widetilde{\sigma},\infty)} \right)} \\ &= \det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[s,+\infty)} \right) \det\left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[t,+\infty)} \right) \left(1 + \mathcal{O}(\tau^{-1})\right) \end{aligned} \tag{7.2.6}$$

and the convergence is uniform over compact sets of the variables s, t provided

$$-\infty < s, t < K_1(\sigma + \tau^2)$$
$$t = 4\tau^2 - \delta, \quad 0 < \delta < \frac{7}{3}K_2\tau^2; \qquad s = \tau^2 + 2\sigma - \delta, \quad 0 < \delta < K_3\left(2\sigma + \frac{2}{3}\tau^2\right)$$

for some $0 < K_1,\ K_2,\ K_3 < 1$.

More generally, we consider the tacnode process restricted to a collection of intervals.

Theorem 7.4. *Given*

$$\mathcal{I} = \bigcup_{j=1}^{J}[a_{2j-1}, a_{2j}] \cup [a_{2J+1}, b_0] \cup \bigcup_{k=1}^{K}[b_{2k-1}, b_{2k}] \tag{7.2.7}$$

where

$$a_\ell = a(s_\ell) = -\sigma - \tau^2 + t_\ell \quad b_\ell = b(t_{2K+1-\ell}) = \sigma + \tau^2 - s_{2K+1-\ell}, \tag{7.2.8}$$

then as $\sigma \to +\infty$

$$\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J_1}\right)\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J_2}\right)\left(1 + \mathcal{O}(\sigma^{-1})\right) \tag{7.2.9}$$

or as $\tau \to \pm\infty$

$$\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J_1}\right)\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{J_2}\right)\left(1 + \mathcal{O}(\tau^{-1})\right) \tag{7.2.10}$$

where

$$J_1 = \bigcup_{\ell=1}^{J}[t_{2\ell-1}, t_{2\ell}] \cup [t_{2J+1}, +\infty) \quad J_2 = \bigcup_{\ell=1}^{K}[s_{2\ell-1}, s_{2\ell}] \cup [s_{2K+1}, +\infty) \tag{7.2.11}$$

and the convergence is uniform over compact sets of the variables s, t provided

$$-\infty < s_\ell, t_\ell < K_1(\sigma + \tau^2)$$
$$t_\ell = 4\tau^2 - \delta, \quad 0 < \delta < \frac{7}{3}K_2\tau^2; \qquad s_\ell = \tau^2 + 2\sigma - \delta, \quad 0 < \delta < K_3\left(2\sigma + \frac{2}{3}\tau^2\right)$$

for some $0 < K_1,\ K_2,\ K_3 < 1$.

The parametrization of the endpoints a and b in Theorems 7.2 and 7.3 (and of a_ℓ and b_ℓ in Theorem 7.4) has the following meaning. At the critical time $0 < t_{\mathrm{tac}} < 1$, the two bulks tangentially touch at the tacnode point P_{tac}. From the common tacnode point $a(t_{\mathrm{tac}}) = b(t_{\mathrm{tac}})$, two new endpoints $[a(t), b(t)]$ emerge and move away along the branches of the boundary.

The tacnode point process describes the statistics of the random walkers in a scaling neighborhood of $t = t_{\mathrm{tac}}$ and $a = b = P_{\mathrm{tac}}$. The asymptotics as $\tau \to \pm\infty$ given in Theorem 7.3 is the regime where we look "away" from the critical point (either in the future for $\tau > 0$ or in the past for $\tau < 0$) and it is expected to reduce to two Airy point processes, which describe the edge-behavior of the random walkers. Similarly, when we take the limit as $\sigma \to +\infty$ (Theorem 7.2) we are physically pushing away the two bulks from each other and the expected regime around the not-any-more critical time will be again a product of two Airy point processes.

The proof of these theorems relies essentially upon the construction of a Riemann-Hilbert problem deduced from a suitable IIKS integrable kernel and the steepest descent method. In the next section we will show how to deduce such integrable kernel from the tacnode kernel. We will start with considerations that apply to the more general case, but then we will specialize to the single interval case (Theorems 7.2 and 7.3) in order to avoid unnecessary complications, which are purely notational and not conceptual.

7.3 The Riemann-Hilbert setting for the gap probabilities of the tacnode process

We recall the definition of the tacnode kernel, referring to the formula given by Adler, Johansson and Van Moerbeke in [4].

The single-time tacnode kernel reads (see [4, formula (19)])

$$\mathbb{K}^{\mathrm{tac}}(\tau; x, y) =$$
$$K_{\mathrm{Ai}}^{(\tau,-\tau)}(\sigma - x, \sigma - y) + \sqrt[3]{2} \int_{\widetilde{\sigma}}^{\infty} \mathrm{d}z \int_{\widetilde{\sigma}}^{\infty} \mathrm{d}w \mathcal{A}_{x-\sigma}^{\tau}(w) \left(\mathrm{Id} - K_{\mathrm{Ai}} \Big|_{[\widetilde{\sigma}, +\infty)} \right)^{-1} (z, w) \mathcal{A}_{y-\sigma}^{-\tau}(z) \tag{7.3.1}$$

where $\widetilde{\sigma} := 2^{\frac{2}{3}}\sigma$ and the functions appearing in the above definition are specified below:

$$\begin{aligned}
\mathrm{Ai}^{(\tau)}(x) &:= \mathrm{e}^{\tau x+\frac{2}{3}\tau^3}\mathrm{Ai}(x) = \int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\frac{\lambda^3}{3}+\lambda^2\tau-x\lambda}\\
\mathrm{Ai}(x) &:= \int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\frac{\lambda^3}{3}-x\lambda} = -\int_{\gamma_L}\frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{-\frac{\lambda^3}{3}+x\lambda}\\
\mathcal{A}_x^\tau(z) &:= \mathrm{Ai}^{(\tau)}(x+\sqrt[3]{2}z) - \int_0^\infty \mathrm{d}w\,\mathrm{Ai}^{(\tau)}(-x+\sqrt[3]{2}w)\mathrm{Ai}(w+z)\\
K_{\mathrm{Ai}}^{(\tau,-\tau)}(-x,-y) &:= \int_0^\infty \mathrm{d}u\mathrm{Ai}^{(\tau)}(-x+u)\mathrm{Ai}^{(-\tau)}(-y+u)\\
K_{\mathrm{Ai}}(z,w) &:= \int_0^\infty \mathrm{d}u\,\mathrm{Ai}(z+u)\mathrm{Ai}(w+u)
\end{aligned}$$

The contour γ_R is a contour extending to infinity in the λ-plane along the rays $e^{\pm i\frac{\pi}{3}}$, oriented upwards and entirely contained in the right half plane ($\Re(\lambda) > 0$), and $\gamma_L := -\gamma_R$.

First of all, since only the combination $x-\sigma, y-\sigma$ appears, we shift the variables and we perform a spatial rescaling of the form $u = \sqrt[3]{2}u'$. The resulting kernel is

$$\begin{aligned}
\widetilde{\mathbb{K}}(x,y) := \sqrt[3]{2}\mathbb{K}^{\mathrm{tac}}(\sqrt[3]{2}x,\sqrt[3]{2}y) &= \sqrt[3]{2}\int_0^\infty \mathrm{d}u\mathrm{Ai}^{(\tau)}(\sqrt[3]{2}(u-x))\mathrm{Ai}^{(-\tau)}(\sqrt[3]{2}(u-y))+\\
&+\sqrt[3]{2}\int_{\widetilde{\sigma}}^\infty \mathrm{d}z\int_{\widetilde{\sigma}}^\infty \mathrm{d}w\mathcal{A}^\tau_{\sqrt[3]{2}x}(w)\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{\widetilde{\sigma}}\right)^{-1}(z,w)\mathcal{A}^{-\tau}_{\sqrt[3]{2}y}(w). \qquad (7.3.2\mathrm{a})
\end{aligned}$$

For the sake of brevity, we shall introduce the operators K_{Ai}, $K_{\mathrm{Ai}}^{(\tau,-\tau)}$, $\mathfrak{A}_\tau$ (with abuse of notation) as the operators with the kernels,

$$\begin{aligned}
K_{\mathrm{Ai}}^{(\tau,-\tau)} \quad &:= K_{\mathrm{Ai}}^{(\tau,-\tau)}(\sqrt[3]{2}x,\sqrt[3]{2}y)\\
&= \sqrt[3]{2}\int_0^\infty \mathrm{d}u\mathrm{Ai}^{(\tau)}(\sqrt[3]{2}(u-x))\mathrm{Ai}^{(-\tau)}(\sqrt[3]{2}(u-y)) && (7.3.2\mathrm{b})\\
K_{\mathrm{Ai}} \quad &:= K_{\mathrm{Ai}}(x,y)\Big|_{[\widetilde{\sigma},\infty)} && (7.3.2\mathrm{c})\\
\mathcal{B}_\tau(x,z) \quad &:= 2^{\frac{1}{6}}\mathrm{Ai}^{(\tau)}\left(\sqrt[3]{2}(x+z)\right), \quad \mathcal{A}(z,w) := \mathrm{Ai}(z+w) && (7.3.2\mathrm{d})\\
\mathfrak{A}_\tau(x,z) \quad &:= \mathcal{A}^\tau_{\sqrt[3]{2}x}(z) = \mathcal{B}_\tau(x,z) - \int_0^\infty \mathrm{d}w\,\mathcal{B}_\tau(-x,w)\mathcal{A}(w,z) && (7.3.2\mathrm{e})
\end{aligned}$$

moreover, we set π as the projector on the interval $[\widetilde{\sigma},\infty)$.

Given the above definitions, we can rewrite the tacnode kernel in the following way

Proposition 7.5. *The kernel $\widetilde{\mathbb{K}}$ can be represented as*

$$\widetilde{\mathbb{K}}(x,y) = K_{\mathrm{Ai}}^{(\tau,-\tau)}(x,y) + \int_{[\widetilde{\sigma},\infty)} \mathrm{d}z \int_{[\widetilde{\sigma},\infty)} \mathrm{d}w\, \mathfrak{A}_\tau(x,z)\mathcal{R}(z,w)\mathfrak{A}_{-\tau}(z,y) \tag{7.3.3}$$

$$\mathcal{R}(z,w) := \left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right)^{-1}(z,w).$$

Alternatively,

$$\widetilde{\mathbb{K}} = K_{\mathrm{Ai}}^{(\tau,-\tau)} + \mathfrak{A}_\tau \pi (\mathrm{Id} - K_{\mathrm{Ai}})^{-1} \pi \mathfrak{A}_{-\tau}^T \tag{7.3.4}$$

where we recall that $\widetilde{\mathbb{K}}$ is the transformed of the kernel $\mathbb{K}^{\mathrm{tac}}$ under the change of variables $u' = 2^{-\frac{1}{3}}(u-\sigma)$.

Let $\mathcal{I} = [a_1,a_2] \sqcup [a_3,a_4] \cdots \sqcup [a_{2K-1},a_{2K}]$ and denote by Π the projector on $\mathcal{I}$. We will denote with $\tilde{\Pi}$ the projection on the rescaled and translated collection of intervals $[\tilde{a}_1,\tilde{a}_2] \sqcup \ldots \sqcup [\tilde{a}_{2K-1},\tilde{a}_{2K}]$, where $\tilde{a}_j := 2^{-\frac{1}{3}}(a_j - \sigma)$. We are interested in studying the gap probability of the tacnode process restricted to this collection of intervals, namely

$$\det(\mathrm{Id} - \Pi\mathbb{K}^{\mathrm{tac}}\Pi) = \det\left(\mathrm{Id} - 2^{\frac{1}{3}}\tilde{\Pi}\left(K_{\mathrm{Ai}}^{(\tau,-\tau)} + \mathfrak{A}_\tau\pi\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right)^{-1}\pi\mathfrak{A}_{-\tau}^T\right)\tilde{\Pi}\right). \tag{7.3.5}$$

The following proposition is a restatement of Theorem 3.1 from [12], adapted to the single-time case which we are examining.

Proposition 7.6. *The gap probability of the tacnode process admits the following equivalent representation*

$$\det(\mathrm{Id} - \Pi\mathbb{K}^{\mathrm{tac}}\Pi) = F_2(\widetilde{\sigma})^{-1}\det\left(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}\right) :=$$

$$= F_2(\widetilde{\sigma})^{-1}\det\left(\mathrm{Id} - \left[\begin{array}{c|c} \pi K_{\mathrm{Ai}}\pi & -\sqrt[6]{2}\pi\mathfrak{A}_{-\tau}^T\tilde{\Pi} \\ \hline -\sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau\pi & \sqrt[3]{2}\tilde{\Pi}K_{\mathrm{Ai}}^{(\tau,-\tau)}\tilde{\Pi}\end{array}\right]\right) \tag{7.3.6}$$

where $\hat{\Pi}\mathbb{H}\hat{\Pi}$ is an operator acting on the Hilbert space $L^2([\tilde{\sigma},\infty)) \oplus L^2(\mathbb{R})$, $\hat{\Pi} := \pi \oplus \tilde{\Pi}$ and $F_2(\tilde{\sigma})$ is the Tracy–Widom distribution

$$F_2(\widetilde{\sigma}) := \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right). \tag{7.3.7}$$

Remark 7.7. *The projection π in (7.3.6) is redundant since by definition the operator acts on the Hilbert space $L^2([\widetilde{\sigma},\infty))$, but we will keep it for convenience.*

The gap probabilities of the tacnode process are expressible as ratio of two Fredholm determinants. Therefore, we can interpret the tacnode process as a (formal) conditioned process: its gap probabilities are the gap probabilities of the process $\mathbb{H}$ conditioned such that there are no points in the interval $[\widetilde{\sigma}, \infty)$. We refer to [12, Remark 3.1 and Appendix] for a discussion about possible probabilistic interpretations of such result.

Proof. The identity is based on the following operator identity (all being trace-class perturbations of the identity)

$$\det\left(\mathrm{Id} - \left[\begin{array}{c|c} \pi K_{\mathrm{Ai}}\pi & -\sqrt[6]{2}\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi} \\ \hline -\sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau\pi & \sqrt[3]{2}\tilde{\Pi}K^{(\tau,-\tau)}_{\mathrm{Ai}}\tilde{\Pi}\end{array}\right]\right) = \det\left[\begin{array}{c|c} \mathrm{Id} - \pi K_{\mathrm{Ai}}\pi & 0 \\ \hline 0 & \mathrm{Id}\end{array}\right] \det\left[\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline \sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau\pi & \mathrm{Id}\end{array}\right] \times$$

$$\times \det\left[\begin{array}{c|c} \mathrm{Id} & \sqrt[6]{2}(\mathrm{Id} - K_{\mathrm{Ai}})^{-1}_{\widetilde{\sigma}}\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi} \\ \hline 0 & \mathrm{Id} - \sqrt[3]{2}\left\{\tilde{\Pi}K^{(\tau,-\tau)}_{\mathrm{Ai}}\tilde{\Pi} - \tilde{\Pi}\mathfrak{A}_\tau\pi(\mathrm{Id} - K_{\mathrm{Ai}})^{-1}_{\widetilde{\sigma}}\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi}\right\}\end{array}\right]$$

$$= \det\left(\mathrm{Id} - \pi K_{\mathrm{Ai}}\pi\right)\det(\mathrm{Id} - \tilde{\Pi}\widetilde{\mathbb{K}}\tilde{\Pi}).$$

□

Our next goal is to find suitable Fourier representations of the various operators appearing in (7.3.6). In order to do that, we will rewrite the kernels involved, with their projections respectively, in terms of contour integrals. The results are shown in the following two lemmas. Their proof is just a matter of straightforward calculations using Cauchy's residue theorem.

Lemma 7.8. *The kernels involved in the definitions (7.3.2b)-(7.3.2e) can be represented as the following contour integrals*

$$\mathcal{B}^\tau(x,z) = 2^{-\frac{1}{6}}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\widetilde{\theta}_\tau(\lambda;x+z)}, \quad \mathcal{A}(z,w) = \int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\mathrm{e}^{\theta(\lambda;z+w)} \tag{7.3.8a}$$

$$\mathfrak{A}_\tau(x,z) = 2^{-\frac{1}{6}}\left[-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu;x+z)} - \int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu;-x)+\theta(\lambda;z)}}{\mu-\lambda}\right] \tag{7.3.8b}$$

$$K^{(\tau,-\tau)}_{\mathrm{Ai}}(x,y) = \int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu,-x)+\widetilde{\theta}_{-\tau}(\lambda,-y)}}{\sqrt[3]{2}(\mu-\lambda)} \tag{7.3.8c}$$

$$K_{\mathrm{Ai}}(z,w) := \int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{\theta(\lambda,z)-\theta(\mu,w)}}{\mu-\lambda} \tag{7.3.8d}$$

with $\widetilde{\theta}_\tau(\lambda;x) := \frac{\lambda^3}{6} + \frac{\tau}{2^{2/3}}\lambda^2 - x\lambda$ *and* $\theta(\lambda;x) := \frac{\lambda^3}{3} - x\lambda$.

Moreover, if $\tilde{\Pi}$ is the projector on the collection of intervals $\bigcup_j[\tilde{a}_{2j-1}, \tilde{a}_{2j}]$ and π is the projector on the interval $[\widetilde{\sigma}, +\infty)$, a simple application of Cauchy's theorem yields the following identities

$$\pi K_{\mathrm{Ai}}\pi(z,w)=\int_{i\mathbb{R}}\frac{\mathrm{d}\xi}{2i\pi}\mathrm{e}^{\xi(z-\widetilde{\sigma})}\int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\mathrm{e}^{\zeta(\widetilde{\sigma}-w)}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{\theta(\lambda,\widetilde{\sigma})-\theta(\mu,\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)} \tag{7.3.9}$$

Indeed, if $w>\widetilde{\sigma}$ we can close the ζ-integration with a big semicircle on the right-half plane, picking up the residue at $\lambda\in\gamma_R$; viceversa, if $w<\widetilde{\sigma}$ we close the ζ-integration with a big semicircle in the left-half plane, which yields zero since there are no singularities within this contour of integration; the same argument applies for the variable z.

Similarly,

$$\begin{aligned}\tilde{\Pi}\mathfrak{A}_\tau\pi(x,w)=\sum_j^{2K}(-1)^j\int_{i\mathbb{R}}\frac{\mathrm{d}\xi}{2i\pi}\mathrm{e}^{\xi(\tilde{a}_j-x)}\int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\mathrm{e}^{\zeta(\widetilde{\sigma}-w)}\Bigg[\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{\widetilde{\theta}_\tau(\lambda;\tilde{a}_j+\widetilde{\sigma})}}{(\xi-\lambda)(\lambda-\zeta)}-\\-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu;-\tilde{a}_j)+\theta(\lambda;\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}\Bigg]\end{aligned} \tag{7.3.10}$$

$$\begin{aligned}\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi}(z,y)=\sum_j(-1)^j\int_{i\mathbb{R}}\frac{\mathrm{d}\xi}{2i\pi}\mathrm{e}^{\xi(z-\widetilde{\sigma})}\int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\mathrm{e}^{\zeta(y-\tilde{a}_j)}\Bigg[-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\frac{\mathrm{e}^{-\widetilde{\theta}_\tau(\mu;\tilde{a}_j+\widetilde{\sigma})}}{(\xi-\mu)(\mu-\zeta)}-\\-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{\widetilde{\theta}_{-\tau}(\lambda;-\tilde{a}_j)-\theta(\mu;\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}\Bigg]\end{aligned} \tag{7.3.11}$$

$$\begin{aligned}&\tilde{\Pi}K^{(\tau,-\tau)}_{\mathrm{Ai}}(x,y)\tilde{\Pi}= \qquad (7.3.12)\\&=\sum_{j,k}(-1)^{j+k}\int_{i\mathbb{R}}\frac{\mathrm{d}\xi}{2i\pi}\mathrm{e}^{\xi(\tilde{a}_j-x)}\int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\mathrm{e}^{\zeta(y-\tilde{a}_k)}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu,-\tilde{a}_j)+\widetilde{\theta}_{-\tau}(\lambda,-\tilde{a}_k)}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}.\end{aligned}$$

Lemma 7.9. *The Fourier representation of the previous operators is the following*

$$\begin{aligned}\mathcal{F}(\tilde{\Pi}\mathfrak{A}_\tau\pi)(\xi,\zeta)=\sum_j\frac{(-1)^j}{2i\pi}\mathrm{e}^{\xi\tilde{a}_j+\zeta\widetilde{\sigma}}\Bigg[\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{\widetilde{\theta}_\tau(\lambda;\tilde{a}_j+\widetilde{\sigma})}}{(\xi-\lambda)(\lambda-\zeta)}-\\-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu;-\tilde{a}_j)+\theta(\lambda;\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}\Bigg]\end{aligned} \tag{7.3.13}$$

$$\mathcal{F}(\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi})(\xi,\zeta) = \sum_k \frac{(-1)^k}{2i\pi}\mathrm{e}^{-\widetilde{\sigma}\xi-\tilde{a}_k\zeta}\left[-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\frac{\mathrm{e}^{-\widetilde{\theta}_\tau(\mu;\tilde{a}_k+\widetilde{\sigma})}}{(\xi-\mu)(\mu-\zeta)}-\right.$$
$$\left.-\int_{\gamma_L}\frac{\mathrm{d}\mu}{2\pi i}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2\pi i}\frac{\mathrm{e}^{\widetilde{\theta}_{-\tau}(\lambda;-\tilde{a}_k)-\theta(\mu;\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}\right] \tag{7.3.14}$$

$$\mathcal{F}(\tilde{\Pi}K^{(\tau,-\tau)}_{\mathrm{Ai}}\tilde{\Pi})(\xi,\zeta) = \sum_{j,k}\frac{(-1)^{j+k}}{2i\pi}\mathrm{e}^{\tilde{a}_j\xi-\tilde{a}_k\zeta}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{-\widetilde{\theta}_{-\tau}(\mu,-\tilde{a}_j)+\widetilde{\theta}_{-\tau}(\lambda,-\tilde{a}_k)}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)} \tag{7.3.15}$$
$$\mathcal{F}(\pi K_{\mathrm{Ai}}\pi)(\xi,\zeta) = \frac{1}{2i\pi}\mathrm{e}^{\widetilde{\sigma}(\zeta-\xi)}\int_{\gamma_R}\frac{\mathrm{d}\lambda}{2i\pi}\int_{\gamma_L}\frac{\mathrm{d}\mu}{2i\pi}\frac{\mathrm{e}^{\theta(\lambda,\widetilde{\sigma})-\theta(\mu,\widetilde{\sigma})}}{(\mu-\lambda)(\xi-\mu)(\lambda-\zeta)}. \tag{7.3.16}$$

All these kernels act on $L^2(i\mathbb{R})$.

With the convention that $\rho,\zeta,\xi\in i\mathbb{R}$ and $\lambda\in\gamma_R,\ \mu\in\gamma_L$, we have the following result.

Lemma 7.10. *The operators in Lemma 7.9 can be represented as the composition of several operators:*

$$\mathcal{F}(\pi K_{\mathrm{Ai}}\pi)(\xi,\zeta) = A(\xi,\mu)C(\mu,\lambda)B(\lambda,\zeta) \tag{7.3.17}$$
$$A(\xi,\mu) := \frac{\mathrm{e}^{(\mu-\xi)\widetilde{\sigma}-\frac{\mu^3}{4}}}{2i\pi(\xi-\mu)}\qquad C(\mu,\lambda) := \frac{\mathrm{e}^{\frac{\lambda^3-\mu^3}{12}}}{2i\pi(\mu-\lambda)}\qquad B(\lambda,\zeta) := \frac{\mathrm{e}^{\frac{\lambda^3}{4}+(\zeta-\lambda)\widetilde{\sigma}}}{2i\pi(\lambda-\zeta)}$$

$$\mathcal{F}(\tilde{\Pi}K^{(\tau,-\tau)}_{\mathrm{Ai}}\tilde{\Pi})(\xi,\zeta) := A_j(\xi,\mu)C(\mu,\lambda)B_k(\lambda,\zeta) \tag{7.3.18}$$
$$A_j(\xi,\mu) := \sum_j\frac{(-1)^j\mathrm{e}^{(\xi-\mu)\tilde{a}_j-\frac{\mu^3}{12}+\frac{\tau}{2^{2/3}}\mu^2}}{2i\pi(\xi-\mu)}\qquad B_k(\lambda,\zeta) := \sum_k\frac{(-1)^k\mathrm{e}^{\frac{\lambda^3}{12}-\frac{\tau}{2^{2/3}}\lambda^2+(\lambda-\zeta)\tilde{a}_k}}{2i\pi(\lambda-\zeta)}$$

$$\mathcal{F}(\tilde{\Pi}\mathfrak{A}_\tau\pi)(\xi,\zeta) := H_j(\xi,\lambda)Q_R(\lambda,\zeta) - A_j(\xi,\mu)C(\mu,\lambda)B(\lambda,\zeta) \tag{7.3.19}$$
$$H_j(\xi,\lambda) := \sum_j(-1)^j\frac{\mathrm{e}^{(\xi-\lambda)\tilde{a}_j-\widetilde{\sigma}\lambda+\frac{\lambda^3}{12}+\frac{\tau}{2^{2/3}}\lambda^2}}{2i\pi(\xi-\lambda)}\ ,\qquad Q_R(\lambda,\zeta) := \frac{\mathrm{e}^{\frac{\lambda^3}{12}+\widetilde{\sigma}\zeta}}{2i\pi(\lambda-\zeta)}$$

$$
\mathcal{F}(\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi})(\xi,\zeta) := Q_L(\xi,\mu)\widetilde{H}_k(\mu,\zeta) - A(\xi,\mu)C(\mu,\lambda)B_k(\lambda,\zeta) \tag{7.3.20}
$$

$$
\widetilde{H}_k(\mu,\zeta) := \sum_k (-1)^{k+1}\frac{\mathrm{e}^{(\mu-\zeta)\tilde{a}_k+\mu\widetilde{\sigma}-\frac{\mu^3}{12}-\frac{\tau}{2^{2/3}}\mu^2}}{2i\pi(\mu-\zeta)}\ , \qquad Q_L(\xi,\mu) := \frac{\mathrm{e}^{-\frac{\mu^3}{12}-\widetilde{\sigma}\xi}}{2i\pi(\xi-\mu)}
$$

with

$$
\begin{aligned}
B, B_k &: L^2(i\mathbb{R}) \to L^2(\gamma_R)\\
A, A_j &: L^2(\gamma_L) \to L^2(i\mathbb{R})\\
C &: L^2(\gamma_R) \to L^2(\gamma_L)\\
H_j : L^2(\gamma_R) \to L^2(i\mathbb{R}) \quad Q_R &: L^2(i\mathbb{R}) \to L^2(\gamma_R)\\
Q_L : L^2(\gamma_L) \to L^2(i\mathbb{R}) \quad \widetilde{H}_k &: L^2(i\mathbb{R}) \to L^2(\gamma_L)
\end{aligned}
$$

Finally,

Proposition 7.11. *The following identity of determinants holds*

$$
\det\left(\mathrm{Id} - \left[\begin{array}{c|c} ACB & -Q_L\widetilde{H}_k + ACB_k \\ \hline -H_jQ_R + A_jCB & A_jCB_k \end{array}\right]\right) = \det\left[\begin{array}{c|c|c} \mathrm{Id} & B & B_k \\ \hline AC & \mathrm{Id} & Q_L\widetilde{H}_k \\ \hline A_jC & H_jQ_R & \mathrm{Id}\end{array}\right] =
$$

$$
\det\left[\begin{array}{c|c|c|c|c|c}
\mathrm{Id}_{L_1} & 0 & 0 & 0 & 0 & \widetilde{H}_k \\ \hline
0 & \mathrm{Id}_{R_1} & 0 & 0 & Q_R & 0 \\ \hline
0 & 0 & \mathrm{Id}_{L_2} & C & 0 & 0 \\ \hline
0 & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_1} & B & B_k \\ \hline
-Q_L & 0 & -A & 0 & \mathrm{Id}_{i\mathbb{R}_2} & 0 \\ \hline
0 & -H_j & -A_j & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_3}
\end{array}\right] = \det\left[\begin{array}{c|c|c|c}
\mathrm{Id}_{L_1} & \widetilde{H}_kH_j & \widetilde{H}_kA_j & 0 \\ \hline
Q_RQ_L & \mathrm{Id}_{R_1} & Q_RA & 0 \\ \hline
0 & 0 & \mathrm{Id}_{L_2} & C \\ \hline
BQ_L & B_kH_j & BA + B_kA_j & \mathrm{Id}_{R_2}
\end{array}\right]. \tag{7.3.21}
$$

where by the Id_{X_j} *we denote the identity operator on* $L^2(X,\mathbb{C})$ *and the further subscript distinguishes orthogonal copies of the same space.*

Proof. We start by noticing that all operators introduced in Lemma 7.10 are Hilbert–Schmidt. Since a product of two such operators is a trace class operator, the first two determinants and the last one are ordinary Fredholm determinants; the third determinant should be understood as Carleman regularized $\det_2$ determinant. However, since the operator whose determinant is computed is diagonal-free, the formal definition coincides with the usual Fredholm determinant. The first identity is seen by multiplying on the left by a proper lower triangular matrix, while the second one is given by multiplying the matrix

$$\mathcal{M} = \left[\begin{array}{c|c|c|c|c|c|c} \mathrm{Id}_{L_1} & 0 & 0 & 0 & 0 & 0 & \widetilde{H}_k \\ \hline 0 & \mathrm{Id}_{R_1} & 0 & 0 & 0 & Q_R & 0 \\ \hline 0 & 0 & \mathrm{Id}_{L_2} & 0 & C & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{Id}_{R_2} & 0 & B & B_k \\ \hline 0 & 0 & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_1} & 0 & 0 \\ \hline -Q_L & 0 & -A & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_2} & 0 \\ \hline 0 & -H_j & -A_j & 0 & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_3} \end{array}\right]$$

on the left by

$$\mathcal{N} = \left[\begin{array}{c|c|c|c|c|c|c} \mathrm{Id}_{L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \mathrm{Id}_{R_1} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{L_2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathrm{Id}_{R_2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_1} & 0 & 0 \\ \hline Q_L & 0 & A & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_2} & 0 \\ \hline 0 & H_j & A_j & 0 & 0 & 0 & \mathrm{Id}_{i\mathbb{R}_3} \end{array}\right]$$

where 0_j is a copy of the imaginary axis $i\mathbb{R}$. We now multiply the two matrices in reverse order, as we know that $\det(\mathcal{M}\mathcal{N}) = \det(\mathcal{N}\mathcal{M})$. In conclusion, we obtain the operator

$$\det \left[\begin{array}{c|c|c|c} \mathrm{Id}_{L_1} & \widetilde{H}_k H_j & \widetilde{H}_k A_j & 0 \\ \hline Q_R Q_L & \mathrm{Id}_{R_1} & Q_R A & 0 \\ \hline 0 & 0 & \mathrm{Id}_{L_2} & C \\ \hline B Q_L & B_k H_j & BA + B_k A_j & \mathrm{Id}_{R_2} \end{array}\right]$$

where we have removed the trivial part involving the three copies of $i\mathbb{R}$. □

Collecting all the results found so far, we have

Theorem 7.12. *The gap probability of the tacnode process at single time is*

$$\det(\mathrm{Id} - \Pi\widetilde{\mathbb{K}}\Pi) = F_2(\widetilde{\sigma})^{-1} \det\left(\mathrm{Id} - \mathbb{M}\right) \tag{7.3.22}$$

where

$$\mathbb{M} := \left[\begin{array}{c|c|c|c} 0_{L_1} & -\widetilde{H}_k H_j & -\widetilde{H}_k A_j & 0 \\ \hline -Q_R Q_L & 0_{R_1} & -Q_R A & 0 \\ \hline 0 & 0 & 0_{L_2} & -C \\ \hline -BQ_L & -B_k H_j & -(BA + B_k A_j) & 0_{R_2} \end{array}\right] \tag{7.3.23a}$$

with

$$Q_R Q_L(\lambda, \mu) = \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{12}}}{2i\pi(\lambda - \mu)}, \qquad BQ_L(\lambda, \mu) = \frac{\mathrm{e}^{\frac{\lambda^3}{4} - \lambda\widetilde{\sigma} - \frac{\mu^3}{12}}}{2i\pi(\lambda - \mu)} \tag{7.3.23b}$$

$$Q_R A(\lambda, \mu) = \frac{\mathrm{e}^{\mu\widetilde{\sigma} - \frac{\mu^3}{4} + \frac{\lambda^3}{12}}}{2i\pi(\lambda - \mu)}, \qquad C(\mu, \lambda) = \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{12}}}{2i\pi(\mu - \lambda)} \tag{7.3.23c}$$

$$\widetilde{H}_k H_j(\mu, \lambda) = \frac{\sum_{j=1}^{2K} (-1)^{j+1} h_j^{-1}(\mu) h_j(\lambda)}{2i\pi(\mu - \lambda)} \tag{7.3.23d}$$

$$\widetilde{H}_k A_j(\mu_1, \mu_2) = \sum_{j=1}^{2K} (-1)^{j+1} \frac{h_j^{-1}(\mu_1) g_j(\mu_2)}{2i\pi(\mu_1 - \mu_2)} \tag{7.3.23e}$$

$$B_k H_j(\lambda_2, \lambda_1) = \sum_{j=1}^{2K} (-1)^j \frac{g_j^{-1}(\lambda_2) h_j(\lambda_1)}{2i\pi(\lambda_2 - \lambda_1)} \tag{7.3.23f}$$

$$\left(BA + B_k A_j\right)(\lambda, \mu) = \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{4} + (\mu - \lambda)\widetilde{\sigma}}}{2i\pi(\lambda - \mu)} + \sum_j (-1)^j \frac{g_j^{-1}(\lambda) g_j(\mu)}{2i\pi(\lambda - \mu)} \tag{7.3.23g}$$

and

$$h_j(\zeta) := \mathrm{e}^{\zeta^3/12 + \frac{\tau}{2^{2/3}}\zeta^2 - (\tilde{a}_j + \widetilde{\sigma})\zeta}, \qquad g_j(\zeta) := \mathrm{e}^{-\zeta^3/12 + \frac{\tau}{2^{2/3}}\zeta^2 - \zeta\tilde{a}_j}. \tag{7.3.23h}$$

Proof. The first three kernels and the kernel BA follow from easy computations.

$$\begin{aligned} Q_R Q_L(\lambda, \mu) &= \int_{i\mathbb{R}} \frac{\mathrm{d}\zeta}{2i\pi} \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{12}}}{2i\pi(\lambda - \zeta)(\zeta - \mu)} = \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{12}}}{2i\pi(\lambda - \mu)} \\ BQ_L(\lambda, \mu) &= \int_{i\mathbb{R}} \frac{\mathrm{d}\zeta}{2i\pi} \frac{\mathrm{e}^{\frac{\lambda^3}{4} - \lambda\widetilde{\sigma} - \frac{\mu^3}{12}}}{2i\pi(\lambda - \zeta)(\zeta - \mu)} = \frac{\mathrm{e}^{\frac{\lambda^3}{4} - \lambda\widetilde{\sigma} - \frac{\mu^3}{12}}}{2i\pi(\lambda - \mu)} \\ Q_R A(\lambda, \mu) &= \int_{i\mathbb{R}} \frac{\mathrm{d}\zeta}{2i\pi} \frac{\mathrm{e}^{\mu\widetilde{\sigma} - \frac{\mu^3}{4} + \frac{\lambda^3}{12}}}{2i\pi(\lambda - \zeta)(\zeta - \mu)} = \frac{\mathrm{e}^{\mu\widetilde{\sigma} - \frac{\mu^3}{4} + \frac{\lambda^3}{12}}}{2i\pi(\lambda - \mu)} \\ BA(\lambda, \mu) &= \int_{i\mathbb{R}} \frac{\mathrm{d}\zeta}{2i\pi} \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{4} + (\mu - \lambda)\widetilde{\sigma}}}{2i\pi(\lambda - \zeta)(\zeta - \mu)} = \frac{\mathrm{e}^{\frac{\lambda^3 - \mu^3}{4} + (\mu - \lambda)\widetilde{\sigma}}}{2i\pi(\lambda - \mu)} \end{aligned}$$

Next, we recall that the endpoints are ordered $\tilde{a}_j < \tilde{a}_{j+1}$, so that we can pick up residues accordingly to the sign of $\tilde{a}_j - \tilde{a}_k$ $(j,k=1,\dots,2K)$.

$$
\widetilde{H}_k H_j(\mu,\lambda) = \sum_{j,k}(-1)^{j+k+1}\int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\mathrm{e}^{\zeta(\tilde{a}_j-\tilde{a}_k)}\frac{\mathrm{e}^{\mu\tilde{a}_k+\mu\widetilde{\sigma}-\frac{\mu^3}{12}-\frac{\tau}{2^{2/3}}\mu^2}}{2i\pi(\mu-\zeta)}\frac{\mathrm{e}^{-\lambda\tilde{a}_j-\widetilde{\sigma}\lambda+\frac{\lambda^3}{12}+\frac{\tau}{2^{2/3}}\lambda^2}}{(\zeta-\lambda)} =
$$

$$
\sum_{j<k}(-1)^{j+k}\frac{\mathrm{e}^{(\mu-\lambda)\tilde{a}_k+(\mu-\lambda)\widetilde{\sigma}+\frac{\lambda^3-\mu^3}{12}+\frac{\tau}{2^{2/3}}(\lambda^2-\mu^2)}}{2i\pi(\mu-\lambda)}+\sum_{k<j}(-1)^{j+k}\frac{\mathrm{e}^{(\mu-\lambda)\tilde{a}_j+(\mu-\lambda)\widetilde{\sigma}+\frac{\lambda^3-\mu^3}{12}+\frac{\tau}{2^{2/3}}(\lambda^2-\mu^2)}}{2i\pi(\mu-\lambda)}+
$$

$$
+\sum_{j=1}^{2K}\frac{\mathrm{e}^{(\mu-\lambda)\tilde{a}_j+(\mu-\lambda)\widetilde{\sigma}+\frac{\lambda^3-\mu^3}{12}+\frac{\tau}{2^{2/3}}(\lambda^2-\mu^2)}}{2i\pi(\mu-\lambda)}.
$$

Thanks to some cancellations, we are left with

$$
\widetilde{H}_k H_j(\mu,\lambda) = \sum_{j=1}^{2K}(-1)^{j+1}\frac{\mathrm{e}^{(\mu-\lambda)\tilde{a}_j+(\mu-\lambda)\widetilde{\sigma}+\frac{\lambda^3-\mu^3}{12}+\frac{\tau}{2^{2/3}}(\lambda^2-\mu^2)}}{2i\pi(\mu-\lambda)}.
$$

Similarly,

$$
B_k A_j(\lambda,\mu) := \int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\sum_{k,j}\frac{(-1)^{k+j}\mathrm{e}^{\frac{\lambda^3}{12}-\frac{\tau}{2^{2/3}}\lambda^2+(\lambda-\zeta)\tilde{a}_k}}{2i\pi(\lambda-\zeta)}\frac{\mathrm{e}^{(\zeta-\mu)\tilde{a}_j-\frac{\mu^3}{12}+\frac{\tau}{2^{2/3}}\mu^2}}{(\zeta-\mu)} =
$$

$$
= \sum_{j=1}^{2K}\frac{(-1)^j\mathrm{e}^{\frac{\lambda^3-\mu^3}{12}-\frac{\tau}{2^{2/3}}(\lambda^2-\mu^2)+(\lambda-\mu)\tilde{a}_j}}{2i\pi(\lambda-\mu)}.
$$

In the next computation, we set $\lambda_1,\lambda_2\in\gamma_R$:

$$
B_k H_j(\lambda_2,\lambda_1) = \int_{i\mathbb{R}}\frac{\mathrm{d}\zeta}{2i\pi}\sum_{k,j}\frac{(-1)^{k+j}\mathrm{e}^{\frac{\lambda_2^3}{12}-\frac{\tau}{2^{2/3}}\lambda_2^2+(\lambda_2-\zeta)\tilde{a}_k}}{2i\pi(\lambda_2-\zeta)}\frac{\mathrm{e}^{(\zeta-\lambda_1)\tilde{a}_j-\widetilde{\sigma}\lambda_1+\frac{\lambda_1^3}{12}+\frac{\tau}{2^{2/3}}\lambda_1^2}}{(\zeta-\lambda_1)} =
$$

$$
= \sum_{j\le k}\frac{(-1)^{k+j}\mathrm{e}^{\frac{\lambda_2^3}{12}-\frac{\tau}{2^{2/3}}\lambda_2^2-\widetilde{\sigma}\lambda_1+\frac{\lambda_1^3}{12}+\frac{\tau}{2^{2/3}}\lambda_1^2}}{2i\pi(\lambda_2-\lambda_1)}\left(\mathrm{e}^{(\lambda_2-\lambda_1)\tilde{a}_j}-\mathrm{e}^{(\lambda_2-\lambda_1)\tilde{a}_k}\right);
$$

the first term contributes only with the terms with even j (with positive sign) , the second only those with odd k with a negative sign so that

$$
B_k H_j(\lambda_2,\lambda_1) = \sum_{j=1}^{2K}\frac{(-1)^j\mathrm{e}^{\frac{\lambda_2^3}{12}-\frac{\tau}{2^{2/3}}\lambda_2^2-\widetilde{\sigma}\lambda_1+\frac{\lambda_1^3}{12}+\frac{\tau}{2^{2/3}}\lambda_1^2}}{2i\pi(\lambda_2-\lambda_1)}\mathrm{e}^{(\lambda_2-\lambda_1)\tilde{a}_j}.
$$

Note that the kernel is regular at $\lambda_1=\lambda_2$ because the sum vanishes.

In a similar way

$$\widetilde{H}_k A_j(\mu_1,\mu_2) = \int_{i\mathbb{R}} \frac{\mathrm{d}\zeta}{2i\pi} \sum_{k,j} (-1)^{k+j+1} \frac{\mathrm{e}^{(\mu_1-\zeta)\tilde{a}_k+\mu_1\widetilde{\sigma}-\frac{\mu_1^3}{12}-\frac{\tau}{2^{2/3}}\mu_1^2}}{2i\pi(\mu_1-\zeta)} \frac{\mathrm{e}^{(\zeta-\mu_2)\tilde{a}_j-\frac{\mu_2^3}{12}+\frac{\tau}{2^{2/3}}\mu_2^2}}{(\zeta-\mu_2)} =$$

$$= \sum_{j\geq k} (-1)^{k+j+1} \frac{\mathrm{e}^{\mu_1\widetilde{\sigma}-\frac{\mu_1^3}{12}-\frac{\tau}{2^{2/3}}\mu_1^2}\mathrm{e}^{-\frac{\mu_2^3}{12}+\frac{\tau}{2^{2/3}}\mu_2^2}}{2i\pi(\mu_1-\mu_2)} \left(\mathrm{e}^{(\mu_1-\mu_2)\tilde{a}_k} - \mathrm{e}^{(\mu_1-\mu_2)\tilde{a}_j}\right) =$$

$$= \sum_{j=1}^{2K} (-1)^{j+1} \frac{\mathrm{e}^{\mu_1\widetilde{\sigma}-\frac{\mu_1^3}{12}-\frac{\tau}{2^{2/3}}\mu_1^2-\frac{\mu_2^3}{12}+\frac{\tau}{2^{2/3}}\mu_2^2}}{2i\pi(\mu_1-\mu_2)} \mathrm{e}^{(\mu_2-\mu_1)\tilde{a}_j}.$$

□

Now we recall that any operator acting on a Hilbert space of the type $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$ can be decomposed as a 4×4 matrix of operators with (i,j)-entry given by an operator $H_j \to H_i$. In conclusion, the kernel can be written as an integrable kernel in the sense of Its-Izergin-Korepin-Slavnov ([50]):

$$\mathbb{M}(\xi,\zeta) = \frac{\mathbf{f}(\xi)^T \cdot \mathbf{g}(\zeta)}{2\pi i(\xi-\zeta)} \tag{7.3.24}$$

with

$$\mathbf{f}(\xi) = \begin{bmatrix} -\mathrm{e}^{-\xi^3/12}\chi_{L_2} \\ -\mathrm{e}^{\xi^3/4-\xi\widetilde{\sigma}}\chi_{R_2} - \mathrm{e}^{\xi^3/12}\chi_{R_1} \\ g_1^{-1}(\xi)\chi_{R_2} - h_1^{-1}(\xi)\chi_{L_1} \\ \vdots \\ -(-1)^{2K} g_{2K}^{-1}(\xi)\chi_{R_2} + (-1)^{2K} h_{2K}^{-1}(\xi)\chi_{L_1} \end{bmatrix} \tag{7.3.25}$$

$$\mathbf{g}(\zeta) = \begin{bmatrix} \mathrm{e}^{\zeta^3/12}\chi_{R_2} \\ \mathrm{e}^{-\zeta^3/12}\chi_{L_1} + \mathrm{e}^{-\zeta^3/4+\zeta\widetilde{\sigma}}\chi_{L_2} \\ g_1(\zeta)\chi_{L_2} + h_1(\zeta)\chi_{R_1} \\ \vdots \\ g_{2K}(\zeta)\chi_{L_2} + h_{2K}(\zeta)\chi_{R_1} \end{bmatrix}. \tag{7.3.26}$$

It is thus natural to associate to it the following RH problem. We refer to Section 3 for a detailed explanation.

Proposition 7.13. *The Fredholm determinant* $\det(\mathrm{Id} - \mathbb{M})$ *is linked through IIKS corre-*

spondence to the following $(2K+2)\times(2K+2)$ Riemann-Hilbert problem

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)J(\lambda), \quad \lambda \in \Sigma := \gamma_L \cup \gamma_R$$

$$\Gamma(\lambda) = I + \mathcal{O}(\lambda^{-1}), \quad \lambda \to \infty \tag{7.3.27}$$

$$J(\lambda) := I - 2i\pi \boldsymbol{f}(\lambda)\boldsymbol{g}(\lambda)^T = \tag{7.3.28}$$

$$\left[\begin{array}{c|c|c|c|c|c}
1 & \mathrm{e}^{-\Theta_{\tilde{\sigma}}}\chi_L & \mathrm{e}^{-\Theta_{\tau,-a_1}}\chi_L & \dots & \dots & \mathrm{e}^{-\Theta_{\tau,-a_{2K}}}\chi_L \\
\hline
\mathrm{e}^{\Theta_{\tilde{\sigma}}}\chi_R & 1 & \mathrm{e}^{\Theta_{-\tau,a_1}}\chi_R & \dots & \dots & \mathrm{e}^{\Theta_{-\tau,a_{2K}}}\chi_R \\
\hline
-\mathrm{e}^{\Theta_{\tau,-a_1}}\chi_R & \mathrm{e}^{-\Theta_{\tau,a_1}}\chi_L & 1 & \dots & \dots & 0 \\
\hline
\vdots & \vdots & 0 & \dots & \dots & \vdots \\
\hline
\vdots & \vdots & 0 & \dots & \dots & \vdots \\
\hline
(-1)^{2K}\mathrm{e}^{\Theta_{-\tau,-a_{2K}}}\chi_R & (-1)^{2K+1}\mathrm{e}^{-\Theta_{\tau,a_{2K}}}\chi_L & 0 & \dots & \dots & 1
\end{array}\right]$$

with

$$\Theta_{\widetilde{\sigma}}(\lambda) = \frac{\lambda^3}{3} - \widetilde{\sigma}\lambda, \qquad \Theta_{\tau,a_i}(\lambda) = \frac{\lambda^3}{6} - 2^{-\frac{2}{3}}\tau\lambda^2 - 2^{-\frac{1}{3}}(a_i+\sigma)\lambda. \tag{7.3.29}$$

Proof. It is simply a matter of straightforward calculations: starting from the formula $J(\lambda) := I - 2i\pi\mathbf{f}(\lambda)\mathbf{g}(\lambda)^T$ and writing explicitly the endpoints $\tilde{a}_i$ as functions of the original endpoints a_i (the change of variables is defined in Proposition 7.5), we can get the jump matrix as in (7.3.28), but with two distinct copies of γ_R and γ_L, as specified in (7.3.25)-(7.3.26). On the other hand, it is easy to show that the jumps on - say - γ_{R_1} and γ_{R_2} commute, hence we can identify the two contours. □

In particular, let's consider the simplest case where $\mathcal{I} = [a,b]$ $(K = 1)$, then the RH problem is 4×4 with jump matrix

$$J(\lambda) = \begin{bmatrix}
1 & e^{-\Theta_{\tilde{\sigma}}}\chi_{L_1} & e^{-\Theta_{\tau,-a}}\chi_{L_2} & e^{-\Theta_{\tau,-b}}\chi_{L_2} \\
e^{\Theta_{\tilde{\sigma}}}\chi_{R_1} & 1 & e^{\Theta_{-\tau,a}}\chi_{R_3} & e^{\Theta_{-\tau,b}}\chi_{R_3} \\
-e^{\Theta_{\tau,-a}}\chi_{R_2} & e^{-\Theta_{-\tau,a}}\chi_{L_3} & 1 & 0 \\
e^{\Theta_{\tau,-b}}\chi_{R_2} & -e^{-\Theta_{-\tau,b}}\chi_{L_3} & 0 & 1
\end{bmatrix} \tag{7.3.30}$$

where

$$\Theta_{\widetilde{\sigma}}(\lambda) = \frac{\lambda^3}{3} - \widetilde{\sigma}\lambda, \qquad \Theta_{\tau,a_i}(\lambda) = \frac{\lambda^3}{6} - 2^{-\frac{2}{3}}\tau\lambda^2 - 2^{-\frac{1}{3}}(a_i+\sigma)\lambda. \tag{7.3.31}$$

The contour configuration can be seen in Figure 7.6, where we have renamed the contours R_1, R_2, R_3 and L_1, L_2, L_3).

We will now focus exclusively on the single-interval case and we will apply a steepest

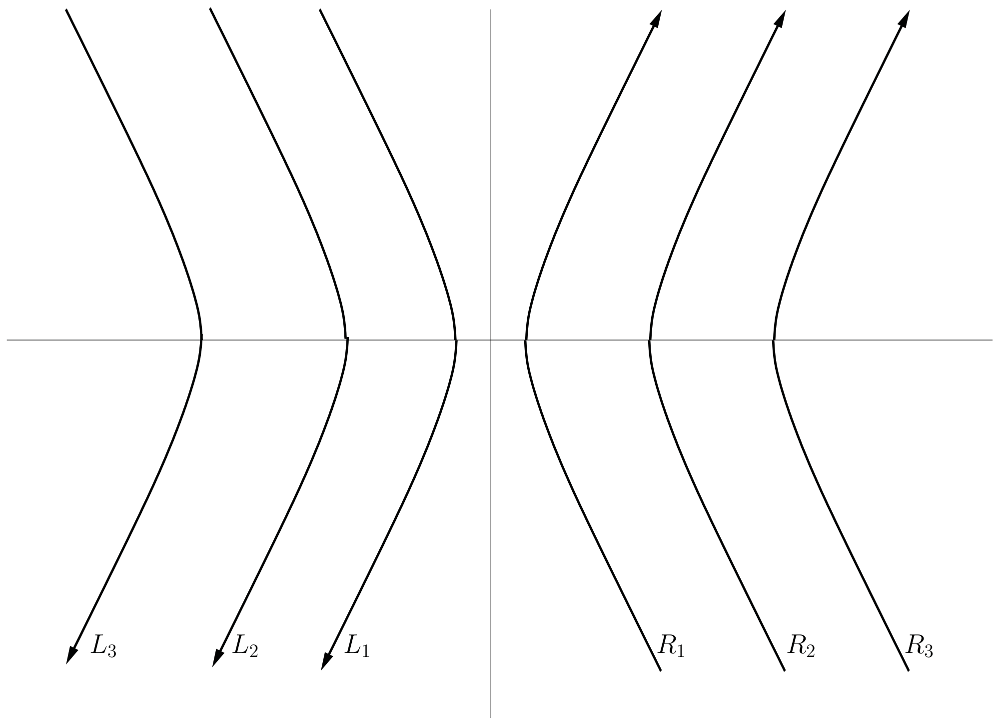

Figure 7.6: The contour configuration of the tacnode Riemann-Hilbert problem in the case $\mathcal{I} = [a, b]$.

descent method in order to prove the factorization of the gap probability of tacnode process into two gap probabilities of the Airy process. The starting point is the 4×4 Riemann-Hilbert problem (7.3.30) with contour configuration as in Figure 7.8 or Figure 7.11, depending on the scaling regime we are considering.

7.4 Proof of Theorem 7.2

From now on, we are assuming $\tau > 0$. For $\tau \leq 0$ the calculations follow the same guidelines as below.

The phase functions $\Theta_\tau(\lambda, -b)$ and $\Theta_{-\tau}(\lambda, a)$ (appearing in the entries of the 2×2 off-diagonal blocks of the jump matrix (7.3.30)) have inflection points with zero derivative when the discriminant of the derivative vanishes, which occurs when

$$a_{\text{crit}} + \sigma + \tau^2 = 0, \qquad b_{\text{crit}} - \sigma - \tau^2 = 0 \tag{7.4.1}$$

with critical values $\Theta_\tau(\lambda, -b_{\text{crit}}) = 2^{1/3}\tau$ and $\Theta_{-\tau}(\lambda, a_{\text{crit}}) = -2^{1/3}\tau$. The neighbourhood of

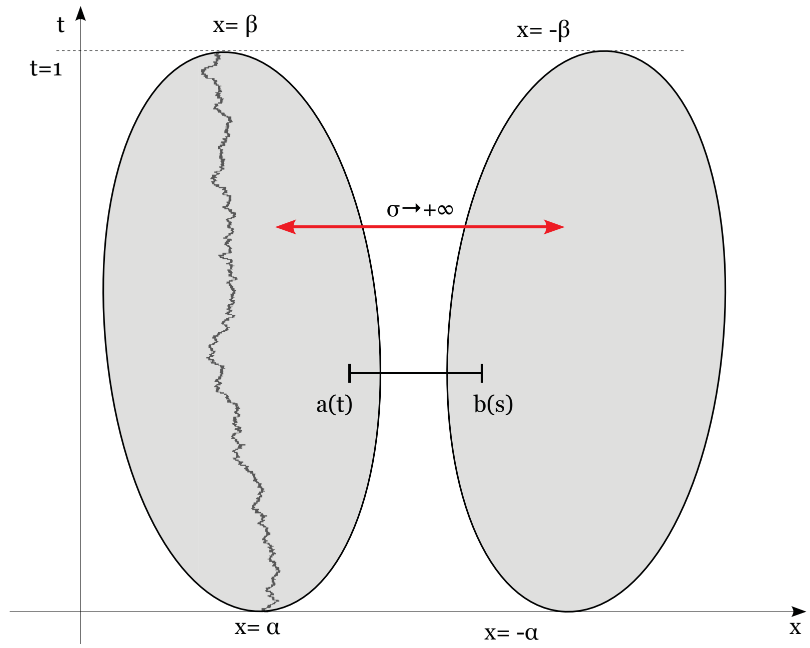

Figure 7.7: The large separation case. As $\sigma \to +\infty$, the two bundles are pulled apart causing the tacnode process to degenerate into two Airy processes.

the discriminant is parametrizable as follows

$$a = a(t) = -\sigma - \tau^2 + t \tag{7.4.2}$$

$$b = b(s) = \sigma + \tau^2 - s. \tag{7.4.3}$$

Thus, from (7.3.31) and substituting (7.4.2)-(7.4.3), we have the following expressions

$$\Theta_\tau(\lambda, -b) = \frac{\xi_-^3}{3} - s\xi_- + \frac{\tau^3}{3} - s\tau, \quad \xi_- := \frac{\lambda - 2^{\frac{1}{3}}\tau}{2^{\frac{1}{3}}} \tag{7.4.4}$$

$$\Theta_{-\tau}(\lambda, a) = \frac{\xi_+^3}{3} - t\xi_+ - \frac{\tau^3}{3} + t\tau, \quad \xi_+ := \frac{\lambda + 2^{\frac{1}{3}}\tau}{2^{\frac{1}{3}}}. \tag{7.4.5}$$

On the other hand, the phase $\Theta_{\tilde{\sigma}}$ in the entries $(1,2)$ and $(2,1)$ of (7.3.30) has critical point at $\pm\sqrt{\tilde{\sigma}} = \pm\sqrt{2^{\frac{2}{3}}\sigma}$.

Preliminary step. We conjugate the matrix Γ by the constant (with respect to λ) diagonal matrix

$$D := \operatorname{diag}(1, 1, -K(t), -K(s)) \tag{7.4.6}$$

where $K(u) := \frac{\tau^3}{3} - u\tau$. As a result, also the jump matrices (7.3.30) are similarly conjugated and this has the effect of replacing the phases $\Theta_{\pm\tau,\mp a}$ and $\Theta_{\pm\tau,\mp b}$ by $\Theta_{\pm\tau,\mp a} \mp K(t)$ and $\Theta_{\pm\tau,\mp b} \mp K(s)$ respectively, so that their critical value is zero. We denote by a hat the new matrix and respective jump:

$$\hat{\Gamma} := e^{-D}\Gamma e^{D} \quad \hat{J} := e^{-D} J e^{D}. \tag{7.4.7}$$

Thus, the resulting jump $\hat{J}$ has the following form:

$$\begin{bmatrix} 1 & e^{-\Theta_{\tilde{\sigma}}} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on } L_1, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ e^{\Theta_{\tilde{\sigma}}} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on } R_1, \tag{7.4.8}$$

$$\begin{bmatrix} 1 & 0 & e^{-\Theta_{\tau,-a}+K(t)} & e^{-\Theta(\xi_-,s)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on } L_2, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -e^{\Theta_{\tau,-a}-K(t)} & 0 & 1 & 0 \\ e^{\Theta(\xi_-,s)} & 0 & 0 & 1 \end{bmatrix} \text{ on } R_2, \tag{7.4.9}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & e^{-\Theta(\xi_+,t)} & 1 & 0 \\ 0 & -e^{-\Theta_{-\tau,b}-K(s)} & 0 & 1 \end{bmatrix} \text{ on } L_3, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\Theta(\xi_+,t)} & e^{\Theta_{-\tau,b}+K(s)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on } R_3, \tag{7.4.10}$$

where

$$\Theta(\xi_\pm, u) := \frac{\xi_\pm^3}{3} - \xi_\pm u \qquad \xi_\pm := \frac{\lambda \pm \sqrt[3]{2}\tau}{\sqrt[3]{2}} \tag{7.4.11}$$

$$\Theta_{-\tau,b}(\lambda, s) := \frac{\lambda^3}{6} + \frac{\tau\lambda^2}{2^{2/3}} - 2^{2/3}\sigma\lambda - \frac{\tau^2\lambda}{\sqrt[3]{2}} + \frac{s\lambda}{\sqrt[3]{2}} \tag{7.4.12}$$

$$\Theta_{\tau,-a}(\lambda, t) := \frac{\lambda^3}{6} - \frac{\tau\lambda^2}{2^{2/3}} - 2^{2/3}\sigma\lambda - \frac{\tau^2\lambda}{\sqrt[3]{2}} + \frac{t\lambda}{\sqrt[3]{2}}. \tag{7.4.13}$$

We choose the contours according to the following configuration (see Figure 7.8):

- L_2 and R_2 are centred around the critical point $P_R := 2^{\frac{1}{3}}\tau$
- L_3 and R_3 are centred around the critical point $P_L := -2^{\frac{1}{3}}\tau$

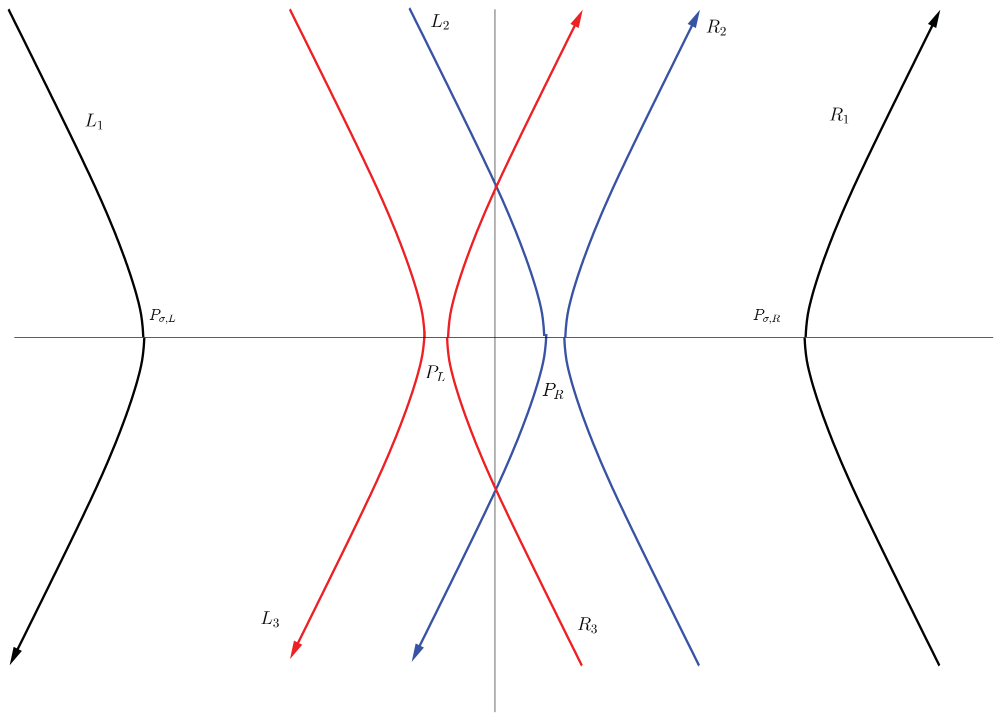

Figure 7.8: The contour setting in the asymptotic limit as $\sigma \to +\infty$.

- L_1 passes through the critical point $P_{\sigma,L} := -\sqrt{\widetilde{\sigma}}$ and R_1 passes through the critical point $P_{\sigma,R} := \sqrt{\widetilde{\sigma}}$; these points are thought as very far from the origin, in the limit as $\sigma \gg 1$.

Remark 7.14. *All the left jumps commute with themselves and similarly all the right jumps. Moreover, the jump matrices L_2 and R_3 commute.*

The proof now proceeds along the following scheme (as $\sigma \to +\infty$):

1. the matrices L_1 and R_1 are exponentially close to the identity in every L^p norm (Lemma 7.15);

2. regarding the matrices L_2 and R_2, the entries of the form $\pm(\Theta_{\tau,-a} - K(t))$ are exponentially small in every L^p norm; the same behaviour will appear for the entries of the type $\pm(\Theta_{-\tau,b} + K(s))$ in the matrices L_3 and R_3 (Lemma 7.16);

3. for the remaining entries in the jumps $L_{2,3}$ and $R_{2,3}$ we will explicitly and exactly solve a (model) Riemann-Hilbert problem which will approximate the problem at hand.

7.4.1 Estimates on the phases

The proof of the first two points relies on the following lemmas.

Lemma 7.15. *The jumps on the curves L_1 and R_1 are exponentially suppressed in any L^p norm, $1 \leq p \leq \infty$, as $\sigma \to +\infty$.*

Proof. A parametrization for the curves L_1 and R_1 is the following $\lambda = \pm 2^{1/3}\sqrt{\sigma} + u\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i\right)$. Therefore, we have (for both signs)

$$\Re\left[\Theta_{\tilde{\sigma};R_1}\right] = \Re\left[-\Theta_{\tilde{\sigma};L_1}\right] = -\frac{4}{3}\sigma^{3/2} - \frac{\sqrt{\sigma}}{2^{2/3}}u^2 - \frac{u^3}{3}$$

which implies

$$\left\|e^{\Theta_{\tilde{\sigma}}}\right\|^p_{L^p(R_1)} = 2\int_0^\infty e^{p\Re[\Theta_{\tilde{\sigma}}]} du \leq Ce^{-\frac{4}{3}p\sigma^{3/2}}, \quad \left\|e^{\Theta_{\tilde{\sigma}}}\right\|_{L^\infty(R_1)} = e^{-\frac{4}{3}\sigma^{3/2}}. \tag{7.4.14}$$

The same results holds for the contour L_1. □

Lemma 7.16. *Given $0 < K_1 < 1$ fixed and $s < K_1(\sigma + \tau^2)$, then the function $e^{\Theta(-\tau,b)+K(s)}$ tends to zero exponentially fast in any $L^p(R_3)$ norm ($1 \leq p \leq \infty$) as $\sigma \to +\infty$:*

$$\left\|e^{\Theta(-\tau,b)+K(s)}\right\|_{L^p(R_3)} \leq Ce^{-2\tau(1-K_1)\sigma}. \tag{7.4.15}$$

Similarly, the function $e^{-\Theta(-\tau,b)-K(s)}$ is exponentially small in any $L^p(L_3)$ norm ($1 \leq p \leq \infty$).

Moreover, the function $e^{-\Theta_{\tau,-a}+K(t)}$ and $e^{\Theta_{\tau,-a}-K(t)}$ are exponentially small in any $L^p(L_2)$ and $L^p(R_2)$ norms, respectively ($1 \leq p \leq \infty$).

Proof. A parametrization of R_3 is $\lambda = \sqrt[3]{2}\tau + u\left[\frac{1}{2} \pm \frac{2}{\sqrt{3}} i\right]$, $u \geq 0$. This yields

$$\Re\left[\Theta(-\tau,b) + K(s)\right] = -\frac{u^3}{6} - \frac{\delta u}{2^{\frac{4}{3}}} - 2\tau\sigma - 2\tau^3 + 2\tau\delta$$

where we set $s = 2\sigma + 2\tau^2 - \delta$, $0 < \delta < \sigma + \tau^2$, and this is valid for both branches of the curve.

Regarding the $L^p(R_3)$ norms, we have that $\left|e^{\Theta_{-\tau,b}+K(s)}\right| = e^{\Re[\Theta_{-\tau,b}+K(s)]}$; therefore,

$$\begin{aligned}
\left\|e^{\Theta(-\tau,b)+K(s)}\right\|^p_{L^p(R_3)} &\leq 2Ce^{-2p\tau(\sigma+\tau^2-\delta)}\left[\int_0^1 e^{-2^{-\frac{4}{3}}p\delta u}du + \int_1^\infty e^{-p\frac{u^3}{6}}du\right] \\
&\leq Ce^{-2p\tau(1-K_1)\sigma}
\end{aligned} \tag{7.4.16a}$$

$$\left\|e^{\Theta(-\tau,b)+K(s)}\right\|_{L^\infty(R_3)} = e^{-2\tau(\sigma+\tau^2-\delta)} \leq Ce^{-2\tau(1-K_1)\sigma} \tag{7.4.16b}$$

given that $s < K_1(\sigma + \tau^2)$ with $0 < K_1 < 1$.

All the other cases are completely analogous. □

7.4.2 Global parametrix. The model problem

In this subsection we will use the Hasting-McLeod matrix (see [36], but in the normalization of [11]) as parametrix for the RH problem related to $\hat{\Gamma}$.

Let us consider the following model problem:

$$\begin{cases} \Omega_+(\lambda) = \Omega_-(\lambda) J_R(\lambda) & \text{on } L_2 \cup R_2 \\ \Omega_+(\lambda) = \Omega_-(\lambda) J_L(\lambda) & \text{on } L_3 \cup R_3 \\ \Omega(\lambda) = I + \mathcal{O}\left(\lambda^{-1}\right) & \text{as } \lambda \to \infty \end{cases} \tag{7.4.17}$$

with jumps (see Figure 7.9)

$$J_R := \begin{bmatrix} 1 & 0 & 0 & e^{-\Theta(\xi_-,s)}\chi_{L_2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{\Theta(\xi_-,s)}\chi_{R_2} & 0 & 0 & 1 \end{bmatrix} \tag{7.4.18}$$

$$J_L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\Theta(\xi_+,t)}\chi_{R_3} & 0 \\ 0 & e^{-\Theta(\xi_+,t)}\chi_{L_3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.4.19}$$

and we recall $\xi_\pm := \frac{\lambda \pm \sqrt[3]{2}\tau}{\sqrt[3]{2}}$ as defined in (7.4.11).

This model problem can be solved in exact form by considering two solutions of the Hasting-McLeod Painlevé II RH problem, namely

$$\Phi_{HM}(s) \quad \text{and} \quad \tilde{\Phi}_{HM}(t) := \sigma_3\sigma_2\Phi_{HM}(t)\sigma_2\sigma_3, \tag{7.4.20}$$

where σ_2, σ_3 are Pauli matrices and $\Phi_{HM}(u)$ is the solution to a 2×2 RH problem with jump matrix

$$\begin{bmatrix} 1 & e^{\Theta(\lambda,u)}\chi_{\gamma_R} \\ e^{-\Theta(\lambda,u)}\chi_{\gamma_L} & 1 \end{bmatrix}, \quad \Theta(\lambda,u) = \frac{\lambda^3}{3} - u\lambda \tag{7.4.21}$$

and behaviour at infinity normalized to the identity 2×2 matrix; as usual, γ_R is a contour which extends to infinity along the rays $\arg(\lambda) = \pm\frac{i\pi}{3}$ and $\gamma_L = -\gamma_R$ (for more details see

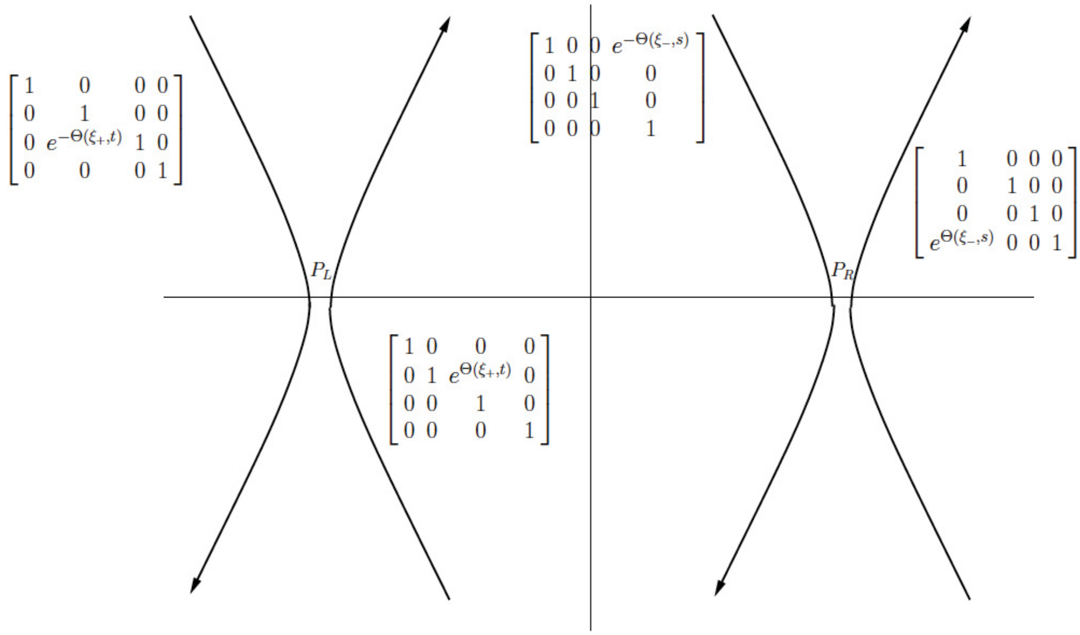

Figure 7.9: The contour setting with the jump matrices in the model problem.

[11]). The asymptotic behaviour of the functions (7.4.20) as $\xi \to \infty$ is

$$\Phi(\xi_+, t) = I + \frac{1}{\xi_+} \begin{bmatrix} p(t) & q(t) \\ -q(t) & -p(t) \end{bmatrix} + \mathcal{O}\left(\frac{1}{\xi_+^2}\right) \tag{7.4.22}$$

$$\begin{aligned} \tilde{\Phi}(\xi_-, s) &= \sigma_3\sigma_2 \left[I + \frac{1}{\xi_-} \begin{bmatrix} p(s) & q(s) \\ -q(s) & -p(s) \end{bmatrix} + \mathcal{O}\left(\frac{1}{\xi_-^2}\right) \right] \sigma_2\sigma_3 \\ &= I + \frac{1}{\xi_-} \begin{bmatrix} -p(s) & -q(s) \\ q(s) & p(s) \end{bmatrix} + \mathcal{O}\left(\frac{1}{\xi_-^2}\right). \end{aligned} \tag{7.4.23}$$

The global parametrix, i.e. the exact solution of the model problem, is then easily verified to be given by

$$\Omega := \begin{bmatrix} \tilde{\Phi}_{11}(\xi_-, s) & 0 & 0 & \tilde{\Phi}_{12}(\xi_-, s) \\ 0 & \Phi_{11}(\xi_+, t) & \Phi_{12}(\xi_+, t) & 0 \\ 0 & \Phi_{21}(\xi_+, t) & \Phi_{22}(\xi_+, t) & 0 \\ \tilde{\Phi}_{21}(\xi_-, s) & & 0 & \tilde{\Phi}_{22}(\xi_-, s) \end{bmatrix}. \tag{7.4.24}$$

7.4.3 Approximation and error term for the matrix $\hat{\Gamma}$

The following relation holds

$$\hat{\Gamma} = \mathcal{E} \cdot \Omega \tag{7.4.25}$$

where $\mathcal{E}$ is the "error" matrix. The goal is to show that the RHP satisfied by the error matrix has jump equal to a small perturbation of the identity matrix $I + \mathcal{O}(\sigma^{-\infty})$, so that a standard small norm argument can be applied (see Chapter 4).

Lemma 7.17. *Given $s, t < K_1(\sigma + \tau^2)$ with $0 < K_1 < 1$, the error matrix $\mathcal{E} = \hat{\Gamma}(\lambda)\Omega^{-1}(\lambda)$ solves a RH problem with jumps on the contours as indicated in Figure 7.8 and of the following orders*

$$\begin{cases} \mathcal{E}_+(\lambda) = \mathcal{E}_-(\lambda) J_{\mathcal{E}}(\lambda) & \text{on } \Sigma \\ \mathcal{E}(\lambda) = I + \mathcal{O}\left(\lambda^{-1}\right) & \text{as } \lambda \to \infty \end{cases} \tag{7.4.26}$$

$$J_{\mathcal{E}} = \begin{bmatrix} 1 & \mathcal{O}(\sigma^{-\infty})\chi_{L_1} & \mathcal{O}(\sigma^{-\infty})\chi_{L_2} & 0 \\ \mathcal{O}(\sigma^{-\infty})\chi_{R_1} & 1 & 0 & \mathcal{O}(\sigma^{-\infty})\chi_{R_3} \\ -\mathcal{O}(\sigma^{-\infty})\chi_{R_2} & 0 & 1 & 0 \\ 0 & -\mathcal{O}(\sigma^{-\infty})\chi_{L_3} & 0 & 1 \end{bmatrix} \tag{7.4.27}$$

and the $\mathcal{O}$-symbols are valid in any L^p norms ($1 \leq p \leq \infty$).

Proof. First of all, we notice that, thanks to Lemma 7.15 and 7.16, all the extra phases that were not included in the model problem Ω behave like $\mathcal{O}(\sigma^{-\infty})$ as $\sigma \to \infty$ in any L^p norm. The jump of the error problem are the remaining jumps appearing in the original $\hat{\Gamma}$-problem conjugated with the Hasting-McLeod solution Ω, which is independent on σ:

$$J_{\mathcal{E}} = \Omega^{-1}\hat{J}\Omega =$$

$$\Omega^{-1} \begin{bmatrix} 1 & e^{-\Theta_{\tilde{\sigma}}}\chi_{L_1} & e^{-\Theta_{\tau,-a}+K(t)}\chi_{L_2} & 0 \\ e^{\Theta_{\tilde{\sigma}}}\chi_{R_1} & 1 & 0 & e^{\Theta_{-\tau,b}+K(s)}\chi_{R_3} \\ -e^{\Theta_{\tau,-a}-K(t)}\chi_{R_2} & 0 & 1 & 0 \\ 0 & -e^{-\Theta_{-\tau,b}-K(s)}\chi_{L_3} & 0 & 1 \end{bmatrix} \Omega$$

$$= \Omega^{-1}\left(I + \mathcal{O}(\sigma^{-\infty})\right)\Omega = I + \mathcal{O}(\sigma^{-\infty})$$

since Ω and Ω^{-1} are uniformly bounded in σ. $\square$

We recall that the Small Norm Theorem says that

$$\|\mathcal{E}(\lambda) - I\| \leq \frac{C}{\operatorname{dist}(\lambda, \Sigma)} \left(\|J_{\mathcal{E}} - I\|_1 + \frac{\|J_{\mathcal{E}} - I\|_2^2}{1 - \|J_{\mathcal{E}} - I\|_\infty} \right) \tag{7.4.28}$$

uniformly on closed sets not containing the contours of the jumps, where Σ is the collection of all contours. Thanks to Lemma 7.17, we conclude

$$\|\mathcal{E}(\lambda) - I\| \leq \frac{C}{\operatorname{dist}(\lambda, \Sigma)} e^{-K\sigma} \tag{7.4.29}$$

for some positive constants C and K. The error matrix $\mathcal{E}$ is then found as the solution to the integral equation

$$\mathcal{E}(\lambda) = I + \int_\Sigma \frac{\mathcal{E}_-(w)\left(J_{\mathcal{E}}(\lambda) - I\right) dw}{2\pi i(w-\lambda)} \tag{7.4.30}$$

and can be obtained by iterations

$$\mathcal{E}^{(0)}(\lambda) = I, \quad \mathcal{E}^{(k+1)}(\lambda) = I + \int_\Sigma \frac{\mathcal{E}_-^{(k)}(w)\left(J_{\mathcal{E}}(\lambda) - I\right) dw}{2\pi i(w-\lambda)}$$

and, thanks to Lemma 7.17 we have

$$\mathcal{E}(\lambda) = I + \frac{1}{\operatorname{dist}(\lambda, \Sigma)} \mathcal{O}\left(\sigma^{-\infty}\right). \tag{7.4.31}$$

7.4.4 Conclusion of the proof of Theorem 7.2

Using known results about Fredholm determinants of IIKS integrable kernels (see [9, Section 5] and [11, Section 2], in particular Theorem 2.1) and adapting them to the case at hand we can state the following theorem.

Theorem 7.18. *The Fredholm determinant* $\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi})$ *of (7.3.6) satisfies the following differential equations*

$$\partial_\rho \ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}) = \omega_{JMU}(\partial_\rho) = \int_\Sigma \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma'_-(\lambda)\partial_\rho\Xi(\lambda)\right)\frac{d\lambda}{2\pi i}. \tag{7.4.32}$$

More specifically,

$$\partial_s \ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_s T\right) = \frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(4,4)} \tag{7.4.33a}$$

$$\partial_t \ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_t T\right) = -\frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(3,3)} \tag{7.4.33b}$$

where $\Gamma_1 := \lim_{\lambda\to\infty}\lambda(\Gamma(\lambda) - I)$.

Proof. We notice that the original RHP for Γ (see (7.3.30)) is equivalent to a RH problem with constant jumps up to a conjugation with the matrix

$$T = \operatorname{diag}\left[\frac{\kappa}{4}, -\Theta_{\tilde{\sigma}} + \frac{\kappa}{4}, -\Theta_{\tau,-a} + \frac{\kappa}{4}, -\Theta_{\tau,-b} + \frac{\kappa}{4}\right] \tag{7.4.34}$$
$$\kappa = \Theta_{\tilde{\sigma}} + \Theta_{\tau,-a} + \Theta_{\tau,-b}.$$

Thus, the matrix $\Psi := \Gamma e^{T}$ solves a RHP with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Applying the Theorem [11, Theorem 2.1] to the case at hand, we have the equality (7.5.23). Moreover, using the Jimbo-Miwa-Ueno residue formula, we can explicitly calculate

$$\partial_s \ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_s T\right) \tag{7.4.35a}$$
$$\partial_t \ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_t T\right). \tag{7.4.35b}$$

Taking into account the asymptotic behaviour at ∞ of the matrix Γ we have

$$\operatorname{Tr}\left[\Gamma^{-1}\Gamma'\partial_s T\right] = \operatorname{Tr}\left[\left(-\frac{\Gamma_1}{\lambda^2} + \mathcal{O}\left(\lambda^{-3}\right)\right)\left(\frac{\partial_s\kappa}{4}I - \partial_s\Theta_{\tau,-b}E_{4,4}\right)\right] = -\frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(4,4)}$$
$$\operatorname{Tr}\left[\Gamma^{-1}\Gamma'\partial_t T\right] = \operatorname{Tr}\left[\left(-\frac{\Gamma_1}{\lambda^2} + \mathcal{O}\left(\lambda^{-3}\right)\right)\left(\frac{\partial_t\kappa}{4}I - \partial_t\Theta_{\tau,-a}E_{3,3}\right)\right] = +\frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(3,3)}$$

since $\det\Gamma \equiv 1$ which implies $\operatorname{Tr}\Gamma_1 = 0$. □

We now use the exact formula in Theorem 7.18 to conclude the proof of Theorem 7.2; recall that

$$\Gamma(\lambda) = e^{D}\mathcal{E}(\lambda)\Omega(\lambda)e^{-D} \tag{7.4.36}$$

and thanks to Lemma 7.17 we have

$$\Gamma_1 = e^{D}\hat{\Gamma}_1 e^{-D} = \Omega_1\left(I + \mathcal{O}(\sigma^{-\infty})\right)$$
$$= \sqrt[3]{2}\begin{bmatrix} -p(s) & 0 & 0 & -q(s) \\ 0 & p(t) & q(t) & 0 \\ 0 & -q(t) & -p(t) & 0 \\ q(s) & & 0 & p(s) \end{bmatrix}\left(I + \mathcal{O}(\sigma^{-\infty})\right) \tag{7.4.37}$$

which yields

$$\Gamma_{1;(4,4)} = \Omega_{1;(4,4)} = \sqrt[3]{2}p(s) + \mathcal{O}(\sigma^{-\infty}) \tag{7.4.38a}$$
$$\Gamma_{1;(3,3)} = \Omega_{1;(3,3)} = -\sqrt[3]{2}p(t) + \mathcal{O}(\sigma^{-\infty}). \tag{7.4.38b}$$

Recall that $p(u)$ is the logarithmic derivative of the gap probability for the Airy process (i.e the Tracy-Widom distribution); collecting all the previous results, we have

$$\begin{aligned} &\mathrm{d}_{s,t} \ln\det\left(\mathrm{Id} - \mathbb{H}\Big|_{[-\sigma-\tau^2+t,\sigma+\tau^2-s]}\right) \\ &= p(s)\mathrm{d}s + p(t)\mathrm{d}t + \mathcal{O}\left(\sigma^{-\infty}\right)\mathrm{d}s + \mathcal{O}\left(\sigma^{-\infty}\right)\mathrm{d}t + \mathcal{O}\left(\sigma^{-\infty}\right)\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{7.4.39}$$

uniformly in s, t within the domain that guarantees the uniform validity of the estimates above as per Lemma 7.17, namely, $s, t < K_1(\sigma+\tau^2)$, $0 < K_1 < 1$.

We now integrate from (s_0, t_0) to (s, t) with $s_0 := a+\sigma+\tau^2$, $t_0 = -b+\sigma+\tau^2$ and we get

$$\begin{aligned} &\ln\det\left(\mathrm{Id} - \mathbb{H}\Big|_{[-\sigma-\tau^2+t,\sigma+\tau^2-s]}\right) \\ &= \ln\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[s,+\infty)}\right) + \ln\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[t,+\infty}\right) + \mathcal{O}(\sigma^{-1}) + C \end{aligned} \tag{7.4.40}$$

with $C = \ln\det\left(\mathrm{Id} - \mathbb{H}\Big|_{[a,b]}\right)$.

In conclusion,

$$\begin{aligned} &\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{[-\sigma-\tau^2+t,\sigma+\tau^2-s]}\right) \\ &= \frac{e^C \det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[s,+\infty)}\right)\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[t,+\infty}\right)(1+\mathcal{O}(\sigma^{-1}))}{\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right)} \end{aligned} \tag{7.4.41}$$

On the other hand, the Fredholm determinant of the Airy kernel appearing in the denominator tends to unity as $\sigma \to \infty$, thus we only need to prove that the constant C is zero. Indeed this is the case:

Lemma 7.19. *The constant of integration C in (7.4.40) is zero.*

Proof. We recall the definition of the integral operator $\hat{\Pi}\mathbb{H}\hat{\Pi}$ acting on $\mathcal{H}_1 \oplus \mathcal{H}_2 = L^2([\widetilde{\sigma}, \infty)) \oplus L^2([\tilde{a}, \tilde{b}])$, with kernel

$$\hat{\Pi}\mathbb{H}\hat{\Pi} = \left[\begin{array}{c|c} \pi K_{\mathrm{Ai}}\pi & -\sqrt[6]{2}\pi\mathfrak{A}^T_{-\tau}\tilde{\Pi} \\ \hline -\sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau\pi & \sqrt[3]{2}\tilde{\Pi}K_{\mathrm{Ai}}^{(\tau,-\tau)}\tilde{\Pi} \end{array}\right] \tag{7.4.42}$$

where $\hat{\Pi} := \pi \oplus \tilde{\Pi}$, π is the projector on $[\widetilde{\sigma}, +\infty)$, Π is the projector on $[\tilde{a}, \tilde{b}]$ and

$$
\begin{aligned}
K_{\mathrm{Ai}}(x,y) &:= \int_0^{\infty} \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)\,du \\
K_{\mathrm{Ai}}^{(\tau,-\tau)}(\sigma-x,\sigma-y) &:= e^{\tau(y-x)}\times \\
\times \int_0^{\infty} du\, \mathrm{Ai}(\sigma - x + \tau^2 &+ \sqrt[3]{2}u)\mathrm{Ai}(\sigma - y + \tau^2 + \sqrt[3]{2}u) \\
\mathfrak{A}_\tau(x,y) :=\ & \mathrm{Ai}^{(\tau)}(x-\sigma+\sqrt[3]{2}y) - \int_0^{\infty} \mathrm{Ai}^{(\tau)}(\sigma - x + \sqrt[3]{2}v)\mathrm{Ai}(v+y)\,dv \\
=\ & 2^{1/6} e^{\tau\left(x-\sigma+\sqrt[3]{2}y\right)+\frac{2}{3}\tau^3}\mathrm{Ai}(x-\sigma+\sqrt[3]{2}y+\tau^2)+ \\
-2^{1/6}\int_0^{\infty} dv\, & e^{\tau\left(\sigma-x+\sqrt[3]{2}v\right)+\frac{2}{3}\tau^3}\mathrm{Ai}(\sigma-x+\sqrt[3]{2}v+\tau^2)\mathrm{Ai}(v+y) \\
\mathfrak{A}_{-\tau}^T(x,y) &:= 2^{1/6} e^{-\tau\left(y-\sigma+\sqrt[3]{2}x\right)-\frac{2}{3}\tau^3}\mathrm{Ai}(y-\sigma+\sqrt[3]{2}x+\tau^2)+ \\
-2^{1/6}\int_0^{\infty} dv\, & e^{-\tau\left(\sigma-y+\sqrt[3]{2}v\right)-\frac{2}{3}\tau^3}\mathrm{Ai}(\sigma-y+\sqrt[3]{2}v+\tau^2)\mathrm{Ai}(v+x).
\end{aligned}
$$

We would like to perform some uniform pointwise estimates on the entries of the kernel in order to prove that as $\sigma \to +\infty$ the trace of the operator $\hat{\Pi}\mathbb{H}\hat{\Pi}$ tends to zero.

Indeed,

$$|\pi K_{\mathrm{Ai}}(u,v)\pi| \leq \frac{C_1}{\sqrt{\sigma}} e^{-\frac{2}{3}u^{3/2}-\frac{2}{3}v^{3/2}} \tag{7.4.43a}$$

$$|\sqrt[3]{2}\Pi K_{\mathrm{Ai}}^{(\tau,-\tau)}(x,y)\tilde{\Pi}| \leq C_2 e^{-\sigma^{3/2}} \tag{7.4.43b}$$

$$|\sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau(x,v)\pi| \leq C_3 e^{-\tau^2\sqrt{\sigma}} e^{\tau\left(\sqrt[3]{2}v-\sigma\right)-\frac{2}{3}\left(\sqrt[3]{2}v-\sigma+\tilde{a}\right)^{3/2}} \tag{7.4.43c}$$

$$|\sqrt[6]{2}\pi\mathfrak{A}_{-\tau}^T(u,y)\tilde{\Pi}| \leq C_4 e^{-\tau^2\sqrt{\sigma}} e^{-\tau\left(\sqrt[3]{2}u-\sigma\right)-\frac{2}{3}\left(\sqrt[3]{2}u-\sigma+\tilde{a}\right)^{3/2}} \tag{7.4.43d}$$

for some positive constants C_j $(j = 1, \ldots, 4)$, where we used the convention that x, y are the variables running in $[\tilde{a}, \tilde{b}]$ and u, v are the variables running in $[\tilde{\sigma}, \infty)$. Such estimates follow from simple arguments on the asymptotic behaviour of the Airy function when its argument is very large.

Collecting all the estimates, we get

$$\left[\begin{array}{c|c} \pi K_{\mathrm{Ai}}\pi & -\sqrt[6]{2}\pi\mathfrak{A}_{-\tau}^T\tilde{\Pi} \\ \hline -\sqrt[6]{2}\tilde{\Pi}\mathfrak{A}_\tau\pi & \tilde{\Pi}K_{\mathrm{Ai}}^{(\tau,-\tau)}\tilde{\Pi} \end{array}\right] \leq C_\sigma \left[\begin{array}{c|c} f(u)f(v) & f(u) \\ \hline f(v) & 1 \end{array}\right] \tag{7.4.44}$$

with $C_\sigma = \frac{\max\{C_j,\ j=1,\ldots,4\}}{\sqrt{\sigma}}$ and $f(z) = e^{\tau\left(\sqrt[3]{2}u-\sigma\right)-\frac{2}{3}\left(\sqrt[3]{2}u-\sigma-2^{-1/3}\sigma+2^{-1/3}a\right)^{3/2}}$. On the right hand side we have a new operator $\mathcal{L}$ acting on the same Hilbert space $L^2([\widetilde{\sigma}, \infty)) \oplus L^2([\tilde{a}, \tilde{b}])$ with

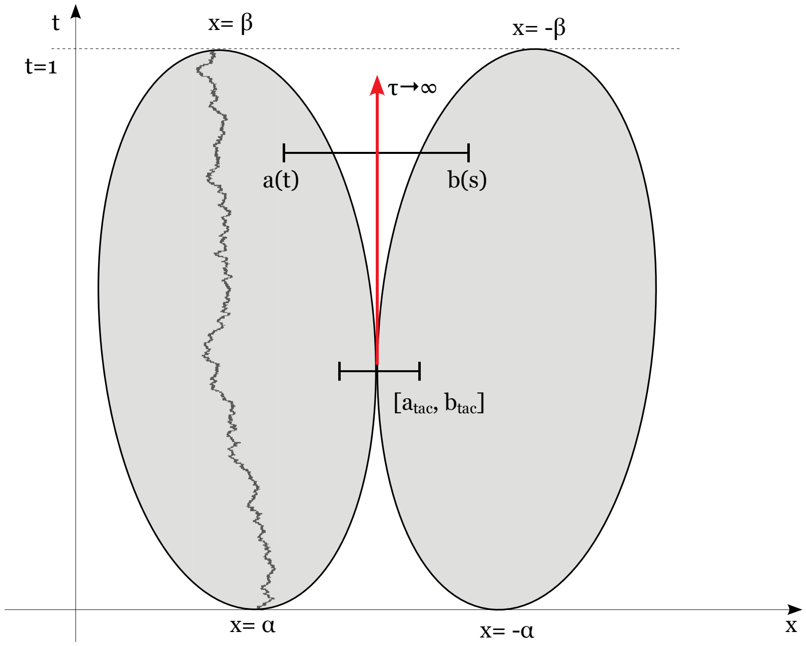

Figure 7.10: The large time case. As $\tau \to +\infty$, we are physically moving away form the tacnode point and along the soft edges of the boundary, where the Airy process occurs. The same result holds when $\tau \to -\infty$.

trace

$$\operatorname{Tr}\mathcal{L} = \|f\|^2_{L^2(\tilde{\sigma},\infty)} + (\tilde{b} - \tilde{a}) \leq C(b-a) \tag{7.4.45}$$

for some positive constant C, since $\|f\|^2_{L^2(\tilde{\sigma},\infty)} \to 0$ as $\sigma \to +\infty$.

Concluding, keeping $[a, b]$ fixed,

$$\begin{aligned} |\ln\det(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi})| &= \sum_{n=1}^{\infty} \frac{\operatorname{Tr}(\hat{\Pi}\mathbb{H}\hat{\Pi}^n)}{n} \\ &\leq \sum_{n=1}^{\infty} \frac{C_\sigma^n (b-a)^n}{n} \leq \frac{C_\sigma(b-a)}{1 - C_\sigma(b-a)} \to 0 \end{aligned} \tag{7.4.46}$$

as $\sigma \to +\infty$. This implies that the constant of integration C must be zero. □

7.5 Proof of Theorem 7.3

We deal now with the case $\tau \to \pm\infty$, i.e. we are moving away from the tacnode point along the boundary curves of the domain so that there is one of the gaps that divaricates as we proceed. From now on, we will only focus on the case $\tau \to +\infty$. The case $\tau \to -\infty$ is analogous.

The RH problem we are considering is the same as for the proof of Theorem 7.2 (7.3.27)-(7.3.30). We conjugates the jumps with the constant diagonal matrix D (see definition (7.4.6)) and we have the same jump matrices as in (7.4.8)-(7.4.13).

The position of the curves is depicted in Figure 7.11:

- L_2 and R_2 are centred around the critical point $P_R := 2^{\frac{1}{3}}\tau$
- L_3 and R_3 are centred around the critical point $P_L := -2^{\frac{1}{3}}\tau$
- L_1 passes through the critical point $P_{\sigma,L} := -\sqrt{\widetilde{\sigma}}$ and R_1 passes through the critcal point $P_{\sigma,R} := \sqrt{\widetilde{\sigma}}$.

The points $P_{R/L} = \pm 2^{\frac{1}{3}}\tau$ are thought as very far from the origin, in the limit as $\tau \gg 1$.

We need to perform certain "contour deformations" and "jump splitting" in the RHP (7.3.27)-(7.3.30). To explain these manipulation consider a general RHP with a jump on a certain contour γ_0 and with jump matrix $J(\lambda)$

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)J(\lambda)\ , \quad \lambda \in \gamma_0.$$

The "contour deformation" procedure stands for the following; suppose γ_1 is another contour such that

- $\gamma_0 \cup \gamma_1^{-1}$ is the positively oriented boundary of a domain D_{γ_0,γ_1}, where γ_1^{-1} stands for the contour traversed in the opposite orientation,
- $J(\lambda)$ and $J^{-1}(\lambda)$ are both analytic in D_{γ_0,γ_1} and (in case the domain extends to infinity) $J(\lambda) \to I + \mathcal{O}(\lambda^{-1})$ as $|\lambda| \to \infty, \lambda \in D_{\gamma_0,\gamma_1}$.

We define $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)$ for $\lambda \in \mathbb{C} \setminus D_{\gamma_0,\gamma_1}$ and $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)J(\lambda)^{-1}$ for $\lambda \in D_{\gamma_0,\gamma_1}$. This new matrix then has jump on γ_1 with jump matrix $J(\lambda)$ ($\lambda \in \gamma_1$) and no jump (i.e. the identity jump matrix) on γ_0. While technically this is a new Riemann Hilbert problem, we shall refer to it with simply as the "deformation" of the original one, without introducing a new symbol.

The "jump splitting" procedure stands for a similar manipulation: suppose that the jump matrix relative to the contour γ_0 is factorizable into two (or more) matrices $J(\lambda) = J_0(\lambda)J_1(\lambda)$. Let γ_1, D_{γ_0,γ_1} be exactly as in the description above. Then define $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)$ for $\lambda \in \mathbb{C} \setminus D_{\gamma_0,\gamma_1}$ and $\widetilde{\Gamma}(\lambda) = \Gamma(\lambda)J(\lambda)^{-1}$ for $\lambda \in D_{\gamma_0,\gamma_1 tacnode}$. Then $\widetilde{\Gamma}$ has jumps

$$\widetilde{\Gamma}_+(\lambda) = \widetilde{\Gamma}_-(\lambda)J_0(\lambda), \quad \lambda \in \gamma_0, \quad \widetilde{\Gamma}_+(\lambda) = \widetilde{\Gamma}_-(\lambda)J_1(\lambda), \quad \lambda \in \gamma_1.$$

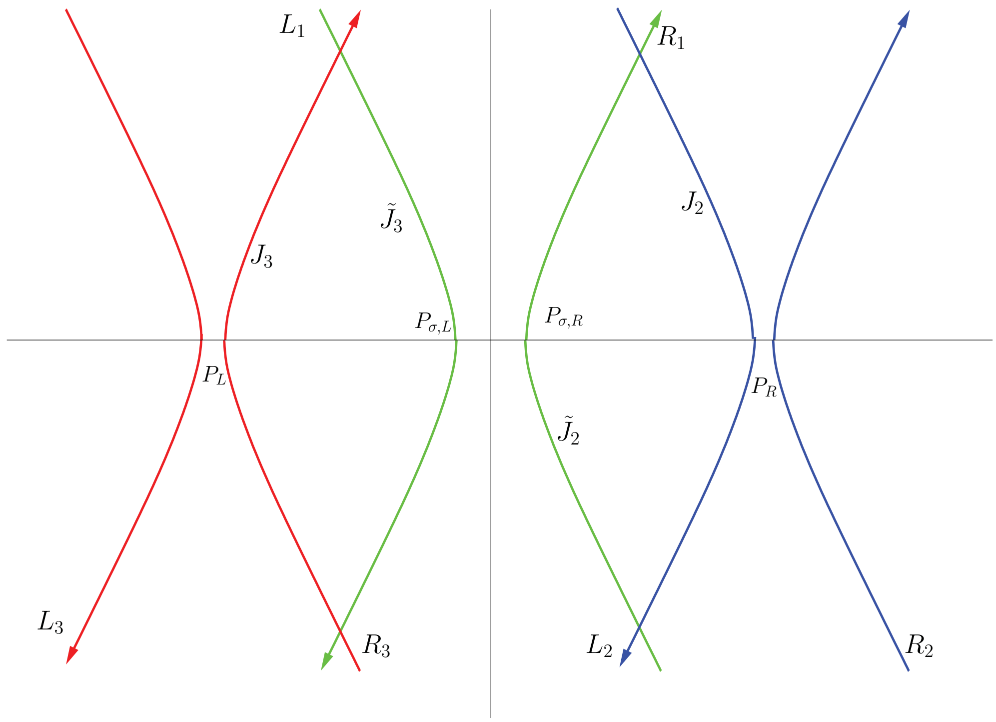

Figure 7.11: The contour setting in the asymptotic limit as $\tau \to +\infty$.

Also in this case, while this is technically a different RHP, we shall refer to it with the same symbol Γ. We will also refer to the inverse operation as "jump merging".

With this terminology in mind, we deform R_3 on the left next to its critical point $-\sqrt[3]{2}\tau$ leads to a new jump matrix on R_3, due to conjugation with the curve L_1 (similarly for L_2)

$$J_3 := L_1 R_3 L_1^{-1} = R_1 L_2 R_1^{-1} =: J_2$$
$$= \begin{bmatrix} 1 & 0 & e^{-\Theta(\tau,-a)+K(t)} & e^{-\Theta(\xi_-,s)} \\ 0 & 1 & e^{\Theta(\xi_+,t)} & e^{\Theta(-\tau,b)+K(s)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.5.1}$$

Again as before, the proof is based on estimating the phases in the jump matrices which are not critical and solving the RH problem by approximation with an exact solution to a model problem.

7.5.1 Estimates of the phases

First of all we notice that a similar version of Lemma 7.15 does not apply here, since the phases on the contours L_1 and R_1 do not depend on τ. On the other hand, we can partially restate Lemma 7.16 applied to the case at hand when $\tau \to \infty$.

Lemma 7.20. *Given $0 < K_1 < 1$ fixed and $s < K_1(\sigma + \tau^2)$, then the function $e^{\Theta(-\tau,b)+K(s)}$ tends to zero exponentially fast in any $L^p(R_3)$ norm ($1 \le p \le \infty$) as $\tau \to +\infty$:*

$$\left\| e^{\Theta(-\tau,b)+K(s)} \right\|_{L^p(R_3)} \le Ce^{-2(1-K_1)\tau^3} \tag{7.5.2}$$

Similarly, the functions $e^{-\Theta(-\tau,b)-K(s)}$, $e^{\Theta_{\tau,-a}-K(t)}$ and $e^{-\Theta_{\tau,-a}+K(t)}$ are exponentially small in any $L^p(L_3)$, $L^p(R_2)$ and $L^p(L_2)$ norms, respectively ($1 \le p \le \infty$).

Proof. Using the same parametrization as in Lemma 7.16, we have

$$\begin{aligned} \left\| e^{\Theta(-\tau,b)+K(t)} \right\|^p_{L^p(R_3)} &\le 2Ce^{-2p\tau(\sigma+\tau^2-\delta)} \left[\int_0^1 e^{-2^{-\frac{4}{3}}\delta p u} du + \int_1^\infty e^{-p\frac{u^3}{6}} du \right] \\ &\le Ce^{-2p(1-K_1)\tau^3} \end{aligned} \tag{7.5.3a}$$

$$\left\| e^{\Theta(\tau,b)-K(t)} \right\|_{L^\infty(R_3)} = e^{-2\tau(\sigma+\tau^2-\delta)} \le Ce^{-2(1-K_1)\tau^3} \tag{7.5.3b}$$

where we set $s = 2\sigma + 2\tau^2 - \delta$, $0 < \delta < \sigma + \tau^2$. The proof for the other phases on the contours L_3, R_2 and L_2 is analogous. ☐

Before estimating the entries of the jump matrices on J_2 and J_3, we factor the jumps in the following way. We split the jump J_2 into two jumps (and two curves): with abuse of notation we call the first one J_2 and we merge the second jump with the jump on R_1. Thus, the new jumps are the following (see Figure 7.11)

$$J_2 = \begin{bmatrix} 1 & 0 & e^{-\Theta(\tau,-a)+K(t)} & e^{-\Theta(\xi_-,s)} \\ 0 & 1 & e^{\Theta(\xi_+,t)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.5.4}$$

$$\tilde{J}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ e^{\Theta_{\tilde{\sigma}}} & 1 & 0 & -e^{\Theta(-\tau,b)+K(s)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.5.5}$$

Analogously, we split the jump J_3 into two jumps: we call the first one again J_3 and we merge the second one with the jump on L_1. The new configuration of jump matrices is illustrated in Figure 7.11.

$$J_3 = \begin{bmatrix} 1 & 0 & 0 & e^{-\Theta(\xi_-,s)} \\ 0 & 1 & e^{\Theta(\xi_+,t)} & e^{\Theta(-\tau,b)+K(s)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.5.6}$$

$$\tilde{J}_3 = \begin{bmatrix} 1 & e^{-\Theta_{\tilde{\sigma}}} & e^{-\Theta(\tau,-a)+K(t)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.5.7}$$

Lemma 7.21. *Let $\kappa := \frac{8}{3} - \frac{p}{6}$. Given $0 < K_2 < 1$ fixed and $t = 4\tau^2 - \delta$, $0 < \delta \leq K_2\kappa\tau^2$, then the $(1,3)$ and $(2,3)$ entries of the jump matrix J_2 are exponentially suppressed as $\tau \to +\infty$ in L^p norms with $p = 1, 2, +\infty$.*

Given $0 < K_3 < 1$ fixed and $s = \tau^2 + 2\sigma - \delta$, $0 < \delta \leq K_3\left(2\sigma + \frac{2}{3}\tau^2\right)$, the $(2,4)$ entry of $\tilde{J}_2$ is exponentially suppressed in any L^p norm $(1 \leq p \leq \infty)$.

Similarly, the same results hold true for the $(1,4)$ and $(2,4)$ entries of J_3 and for the $(1,3)$ entry of $\tilde{J}_3$.

Proof. The first row on J_2 is the same as the one on L_2 and the entry $e^{-\Theta(\tau,-a)+K(t)}$ is exponentially suppressed in any L^p norm, thanks to Lemma 7.20.

Regarding the remaining term on the second row, the real part of the argument in the exponent is

$$\Re\left[\Theta(\xi_+, t)\right] = \frac{u^3}{6} - \frac{\tau}{2^{2/3}}u^2 - \frac{\delta}{2\sqrt[3]{2}}u - \frac{16}{3}\tau^3 + 2\tau\delta$$

where we set $t = 4\tau^2 - \delta$, $\delta > 0$.

Remark 7.22. *A parametrization for the curve J_2 is $\lambda = \sqrt[3]{2}\tau + u\left[\frac{1}{2} \pm \frac{2}{\sqrt{3}}i\right]$, $u \in [0, \sqrt[3]{2}\tau]$. When $u = \sqrt[3]{2}\tau$, the curve J_2 hits the curve R_1 and for $u > \sqrt[3]{2}\tau$ the contour L_2 appears.*

Provided $\delta < \kappa\tau^2$ ($\kappa := \frac{8}{3} - \frac{p}{6}$), it is straightforward to compute the L^p norms $(1 \leq p < 16)$

$$\left\|e^{\Theta(\xi_+,t)}\right\|^p_{L^p(J_2)} = 2e^{-2p\tau\left(\frac{8}{3}\tau^2-\delta\right)}\int_0^{\sqrt[3]{2}\tau} e^{p\left(\frac{u^3}{6}-\frac{\tau}{2^{2/3}}u^2-\frac{\delta}{2\sqrt[3]{2}}u\right)}du$$

$$\leq Ce^{-2p\tau\left[\left(\frac{8}{3}-\frac{p}{6}\right)\tau^2-\delta\right]}\left[\int_1^\infty e^{-\frac{p\tau}{2^{2/3}}u^2}du + \int_0^1 e^{-\frac{p\delta}{2\sqrt[3]{2}}u}du\right] \leq Ce^{-2p\kappa(1-K_2)\tau^3} \tag{7.5.8a}$$

$$\left\|e^{\Theta(\xi_+,t)}\right\|_{L^\infty(J_2)} = e^{-\frac{16}{3}\tau^3+2\tau\delta} \leq Ce^{-2[\kappa(1-K_2)+16]\tau^3} \tag{7.5.8b}$$

for some suitable $0 < K_2 < 1$.

The phase on $\tilde{J}_2$ behaves like

$$\Re\left[\Theta(-\tau,b)+K(s)\right] = -\frac{u^3}{6} - \frac{\tau}{2\,2^{2/3}}u^2 - \frac{\delta}{2\,2^{1/3}}u - \frac{2}{3}\tau^3 - 2\tau\sigma + \tau\delta$$

where we set $s = \tau^2 + 2\sigma - \delta$, $\delta > 0$. Thus, provided $\delta < 2\sigma + \frac{2}{3}\tau^2$, the L^p norms are

$$\left\|e^{\Theta(-\tau,b)+K(s)}\right\|^p_{L^p(\tilde{J}_2)} = 2e^{-p\tau\left(\frac{2}{3}\tau^2+2\sigma-\delta\right)}\int_0^{\sqrt[3]{2}\tau} e^{-p\left(\frac{u^3}{6}+\frac{\tau}{2\,2^{2/3}}u^2+\frac{\delta}{2\,2^{1/3}}u\right)}du$$

$$\leq Ce^{-p\tau\left(\frac{2}{3}\tau^2+2\sigma-\delta\right)}\left[\int_0^1 e^{-\frac{p\delta}{2\,2^{1/3}}u}du + \int_1^\infty e^{-p\frac{u^3}{6}}du\right]$$

$$\leq Ce^{-p(1-K_3)\tau^3} \tag{7.5.9a}$$

$$\left\|e^{\Theta(-\tau,b)+K(s)}\right\|_{L^\infty(\tilde{J}_2)} = e^{-\frac{2}{3}\tau^3-2\tau\sigma+\tau\delta} \leq e^{-C(1-K_3)\tau^3} \tag{7.5.9b}$$

for some suitable $0 < K_3 < 1$.

The arguments for J_3 and $\tilde{J}_3$ are analogous. □

7.5.2 Global parametrix. The model problem

We will now define a new "model" RH problem which will eventually approximate the solution to our original problem $\hat{\Gamma}$.

We define the following RH problem:

$$\begin{cases} \Omega_+(\lambda) = \Omega_-(\lambda)J_{\mathrm{Ai}}(\lambda) & \text{on } L_1 \cup R_1 \\ \Omega_+(\lambda) = \Omega_-(\lambda)J_R(\lambda) & \text{on } L_2 \cup R_2 \\ \Omega_+(\lambda) = \Omega_-(\lambda)J_L(\lambda) & \text{on } L_3 \cup R_3 \\ \Omega(\lambda) = I + \mathcal{O}\left(\lambda^{-1}\right) & \text{as } \lambda \to \infty \end{cases} \tag{7.5.10}$$

with jumps

$$J_{\mathrm{Ai}} := \begin{bmatrix} 1 & e^{-\Theta_{\widetilde{\sigma}}}\chi_{L_1} & 0 & 0 \\ e^{\Theta_{\widetilde{\sigma}}}\chi_{R_1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.5.11a}$$

$$J_R := \begin{bmatrix} 1 & 0 & 0 & e^{-\Theta(\xi_-,s)}\chi_{L_2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e^{\Theta(\xi_-,s)}\chi_{R_2} & 0 & 0 & 1 \end{bmatrix} \tag{7.5.11b}$$

$$J_L := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\Theta(\xi_+,t)}\chi_{R_3} & 0 \\ 0 & e^{-\Theta(\xi_+,t)}\chi_{L_3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.5.11c}$$

Let's denote by $\Psi_{a,b}$ the 4×4 solution to the Airy RHP related to the submatrix formed by the a-th row and column and by the b-th row and column. In particular, we call $\Psi_{1,2}$ the matrix solution to the Hasting-McLeod Airy RHP for the minor $(1,2)$, related to the jump J_{Ai}, with asymptotic solution

$$\Psi_{1,2}(\widetilde{\sigma}) = I_{4\times 4} + \frac{1}{\lambda}\left[\begin{array}{cc|cc} -p(\widetilde{\sigma}) & -q(\widetilde{\sigma}) & 0 & 0 \\ q(\widetilde{\sigma}) & p(\widetilde{\sigma}) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] + \mathcal{O}\left(\frac{1}{\lambda^2}\right). \tag{7.5.12}$$

We consider now the matrix $\Xi := \Omega \cdot \Psi_{1,2}^{-1}(\widetilde{\sigma})$. This matrix doesn't have jumps on L_1 and R_1 by construction, but still has jumps on L_2, R_2 and L_3, R_3:

$$\tilde{J}_L := \Psi_{1,2} J_L \Psi_{1,2}^{-1} \quad \text{and} \quad \tilde{J}_R := \Psi_{1,2} J_R \Psi_{1,2}^{-1}. \tag{7.5.13}$$

On the other hand, as $\tau \to +\infty$ the critical points $\pm\sqrt[3]{2}\tau$ as well as the curves L_2, R_2, L_3, R_3 go to infinity, while the matrix $\Psi_{1,2}$ is asymptotically equal to the identity matrix.

We are left with

$$\Xi = \mathcal{E}_1 \cdot \Psi_{2,3}(t) \cdot \Psi_{1,4}(s) \tag{7.5.14}$$

where $\Psi_{2,3}$ and $\Psi_{1,4}$ where defined in (7.4.20) and $\mathcal{E}_1$ is the error matrix.

Following the previous remark, it is easy to show that the error matrix $\mathcal{E}_1$ is a sufficiently small perturbation of the identity and therefore we can apply the Small Norm Theorem and approximate the global parametrix Ω by simply the product of the matrices $\Psi_{a,b}$ $((a,b) = (1,2),(2,3),(1,4))$

$$\Omega = \Xi \cdot \Psi_{1,2}(\widetilde{\sigma}) \sim \Psi_{2,3}(t) \cdot \Psi_{1,4}(s) \cdot \Psi_{1,2}(\widetilde{\sigma}). \tag{7.5.15}$$

7.5.3 Approximation and error term for the matrix $\hat{\Gamma}$

The relation between our original RH problem $\hat{\Gamma}$ and the global parametrix Ω is the following

$$\hat{\Gamma} = \mathcal{E}_2 \cdot \Omega := \mathcal{E}_2 \cdot \Psi_{2,3}(t) \cdot \Psi_{1,4}(s) \cdot \Psi_{1,2}(\widetilde{\sigma}) \tag{7.5.16}$$

where $\mathcal{E}_2$ is again an error matrix, to which we will apply the small norm argument once again (Chapter 4).

Lemma 7.23. *In the estimates on s,t stated in Lemmas 7.20 and 7.21, the error matrix $\mathcal{E} = \hat{\Gamma}(\lambda)\Omega^{-1}(\lambda)$ solves a RH problem with jumps on the contours as indicated in Figure 7.11 and of the following orders*

$$\begin{cases} \mathcal{E}_+(\lambda) = \mathcal{E}_-(\lambda) J_{\mathcal{E}}(\lambda) & \text{on } \Sigma \\ \mathcal{E}(\lambda) = I + \mathcal{O}\left(\lambda^{-1}\right) & \text{as } \lambda \to \infty \end{cases} \tag{7.5.17}$$

$$J_{\mathcal{E}} = \tag{7.5.18}$$

$$\begin{bmatrix} 1 & 0 & \mathcal{O}(\tau^{-\infty})\chi_{L_2} + \mathcal{O}(\tau^{-\infty})\chi_{\tilde{J}_3} & \mathcal{O}(\tau^{-\infty})\chi_{J_3} \\ 0 & 1 & \mathcal{O}(\tau^{-\infty})\chi_{J_2} & \mathcal{O}(\tau^{-\infty})\chi_{R_3} + \mathcal{O}(\tau^{-\infty})\chi_{\tilde{J}_2} \\ \mathcal{O}(\tau^{-\infty})\chi_{R_2} & 0 & 1 & 0 \\ 0 & \mathcal{O}(\tau^{-\infty})\chi_{L_3} & 0 & 1 \end{bmatrix}$$

where Σ is the collection of all contours and the $\mathcal{O}$-symbols are valid for L^1, L^2 and L^∞ norms.

Proof. Due to Lemmas 7.20 and 7.21, we know from the estimates above that all the extra phases that appear in the original RH problem for $\hat{\Gamma}$ are bounded by an expontential function of the form $C_1 e^{-C_2\tau^3}$. The jumps of the error problem are the remaining jumps appearing in the $\hat{\Gamma}$-problem conjugated with the global parametrix Ω:

$$J_{\mathcal{E}} = \Omega^{-1}\left(I + \mathcal{O}(\tau^{-\infty})\right)\Omega = I + \mathcal{O}(\tau^{-\infty}). \tag{7.5.19}$$

The last equality follows from the fact that the solution Ω depends on τ with a growth that is smaller than the bound $C_1 e^{-C_2\tau^3}$ that we have for the phases. $\square$

Thus, the Small Norm Theorem can be applied

$$\|\mathcal{E}(\lambda)-I\| \leq \frac{C}{\operatorname{dist}(\lambda,\Sigma)}\left(\|J_{\mathcal{E}}-I\|_1+\frac{\|J_{\mathcal{E}}-I\|_2^2}{1-\|J_{\mathcal{E}}-I\|_\infty}\right) \leq \frac{C}{\operatorname{dist}(\lambda,\Sigma)}e^{-K\tau} \tag{7.5.20}$$

where Σ is the collection of all contours, for some positive constants C and K. The error matrix $\mathcal{E}$ is then found as the solution to an integral equation and, thanks to Lemma 7.23 we have

$$\mathcal{E}(\lambda) = I + \frac{1}{\operatorname{dist}(\lambda,\Sigma)}\mathcal{O}\left(\tau^{-\infty}\right). \tag{7.5.21}$$

We need the first coefficient $\hat{\Gamma}_1 = \hat{\Gamma}_1(s,t,\widetilde{\sigma})$ of $\hat{\Gamma}(\lambda)$ at $\lambda=\infty$ and how it compares to the corresponding coefficient Ω_1 of $\Omega(\lambda)$; the error analysis above shows that

$$\hat{\Gamma}_1 = \Omega_1 + \mathcal{O}\left(\tau^{-\infty}\right). \tag{7.5.22}$$

7.5.4 Conclusion of the proof of Theorem 7.3

Theorem 7.24. *The Fredholm determinant* $\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi})$ *is equal to the Jimbo-Miwa-Ueno isomonodromic τ-function of the RH problem (7.3.30). For any parameter ρ on which the integral operator $\hat{\Pi}\mathbb{H}\hat{\Pi}$ may depend, we have*

$$\partial_\rho \ln\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi}) = \omega_{JMU}(\partial_\rho) = \int_\Sigma \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\partial_\rho\Xi(\lambda)\right)\frac{d\lambda}{2\pi i}. \tag{7.5.23}$$

More specifically,

$$\partial_{\widetilde{\sigma}} \ln\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\widetilde{\sigma}}T\right) = \frac{1}{\lambda}\Gamma_{1;(2,2)} \tag{7.5.24a}$$

$$\partial_t \ln\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_t T\right) = -\frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(3,3)} \tag{7.5.24b}$$

$$\partial_s \ln\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_s T\right) = \frac{1}{\sqrt[3]{2}\lambda}\Gamma_{1;(4,4)} \tag{7.5.24c}$$

where $\Gamma_1 := \lim_{\lambda\to\infty}\lambda(\Gamma(\lambda)-I)$.

Proof. The first part of the Theorem is the same as Theorem 7.18. Then, using the Jimbo-Miwa-Ueno residue formula, we have

$$\partial_\rho \ln\det(\mathrm{Id}-\hat{\Pi}\mathbb{H}\hat{\Pi}) = -\operatorname{res}_{\lambda=\infty}\operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_\rho T\right) \tag{7.5.25}$$

with $\rho = \widetilde{\sigma}, s, t$.

Taking into account the definition of the conjugation matrix T (see (7.4.34)) and the asymptotic behaviour of the matrix Γ at infinity, we get again

$$\operatorname{Tr}\left[\Gamma^{-1}\Gamma'\partial_{\widetilde{\sigma}}T\right] = -\frac{\Gamma_{1;(2,2)}}{\lambda}, \quad \operatorname{Tr}\left[\Gamma^{-1}\Gamma'\partial_{s}T\right] = -\frac{\Gamma_{1;(4,4)}}{\sqrt[3]{2}\lambda}, \quad \operatorname{Tr}\left[\Gamma^{-1}\Gamma'\partial_{t}T\right] = \frac{\Gamma_{1;(3,3)}}{\sqrt[3]{2}\lambda}.$$

□

On the other hand, thanks to Lemma 7.23 and the Small Norm Theorem, we can approximate the solution Γ with the global parametrix Ω using (7.5.22) and we get

$$\begin{gathered} \mathrm{d}\ln\det\left(\mathrm{Id}-\mathbb{H}\Big|_{[-\sigma-\tau^2+s,\sigma+\tau^2-t]}\right) = \\ p(s)\mathrm{d}s + p(t)\mathrm{d}t + p(\widetilde{\sigma})\mathrm{d}\widetilde{\sigma} + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}s + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}t + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}\widetilde{\sigma} \\ +\mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}s\,\mathrm{d}t + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}s\,\mathrm{d}\widetilde{\sigma} + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}t\,\mathrm{d}\widetilde{\sigma} + \mathcal{O}\left(\tau^{-\infty}\right)\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\widetilde{\sigma}. \end{gathered} \tag{7.5.26}$$

Integrating from a fixed point $(s_0, t_0, \widetilde{\sigma}_0)$ up to $(s, t, \widetilde{\sigma})$,

$$\begin{gathered} \det\left(\mathrm{Id}-\mathbb{K}^{\mathrm{tac}}\Big|_{[-\sigma-\tau^2+t,\sigma+\tau^2-s]}\right) = \\ \frac{e^C\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[s,+\infty)}\right)\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[t,+\infty)}\right)\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right)(1+\mathcal{O}(\tau^{-1}))}{\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right)} \\ = e^C\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[s,+\infty)}\right)\det\left(\mathrm{Id}-K_{\mathrm{Ai}}\Big|_{[t,+\infty)}\right)\left(1+\mathcal{O}(\tau^{-1})\right) \end{gathered}$$

with s, t within the domain that guarantees the uniform validity of the estimates above (see Lemmas 7.20 and 7.21) and $C = \ln\det(\mathrm{Id}-\mathbb{H}\chi_{[-\sigma_0-\tau^2+t_0,\sigma_0+\tau^2-s_0]})$.

Finally, we need again to show that the constant of integration C is equal zero.

Lemma 7.25. *The constant of integration C in the formula (7.5.4) is zero.*

Proof. First of all we notice that (see Lemma 7.6)

$$
\begin{aligned}
\det(\mathrm{Id} - \Pi \mathbb{K}^{\mathrm{tac}} \Pi) &= \frac{\det\left(\mathrm{Id} - \hat{\Pi} \mathbb{H} \hat{\Pi}\right)}{\det\left(\mathrm{Id} - \pi K_{\mathrm{Ai}} \pi\right)} \\
= \det\left(\left[\begin{array}{c|c} (\mathrm{Id} - \pi K_{\mathrm{Ai}} \pi)^{-1} & 0 \\ \hline 0 & \mathrm{Id} \end{array} \right] \cdot \left[\begin{array}{c|c} \mathrm{Id} - \pi K_{\mathrm{Ai}} \pi & \sqrt[6]{2} \pi \mathfrak{A}^T_{-\tau} \tilde{\Pi} \\ \hline \sqrt[6]{2} \tilde{\Pi} \mathfrak{A}_\tau \pi & \mathrm{Id} - \sqrt[3]{2} \tilde{\Pi} K_{\mathrm{Ai}}^{(\tau,-\tau)} \tilde{\Pi} \end{array} \right] \right) \\
&= \det\left(\mathrm{Id} - \left[\begin{array}{c|c} 0 & -\sqrt[6]{2} (\mathrm{Id} - \pi K_{\mathrm{Ai}} \pi)^{-1} \pi \mathfrak{A}^T_{-\tau} \tilde{\Pi} \\ \hline -\sqrt[6]{2} \tilde{\Pi} \mathfrak{A}_\tau \pi & \sqrt[3]{2} \tilde{\Pi} K_{\mathrm{Ai}}^{(\tau,-\tau)} \tilde{\Pi} \end{array} \right] \right)
\end{aligned} \tag{7.5.27}
$$

where $\hat{\Pi} := \pi \oplus \tilde{\Pi}$, π is the projector on $[\widetilde{\sigma}, \infty)$ and Π is the projector on $[\tilde{a}, \tilde{b}]$.

Along the same guidelines as the proof of Lemma 7.19, we will perform some uniform estimates on the entries of the kernel that will lead to the desired result.

We have

$$
\left| \sqrt[3]{2} \tilde{\Pi} K_{\mathrm{Ai}}^{(\tau,-\tau)}(u,v) \tilde{\Pi} \right| \le \frac{C}{\tau} e^{-\frac{4}{3}\tau^3 - 2\tau\sigma + 2\tau v} \le \frac{C_1}{\sqrt{\tau}} \tag{7.5.28a}
$$

$$
\left| \sqrt[6]{2} \tilde{\Pi} \mathfrak{A}_\tau(x,v) \pi \right| \le \frac{C_2}{\sqrt{\tau}} \tag{7.5.28b}
$$

$$
\begin{aligned}
\left| \sqrt[6]{2} (\mathrm{Id} - \pi K_{\mathrm{Ai}} \pi)^{-1} \pi \mathfrak{A}^T_{-\tau}(u,y) \tilde{\Pi} \right| &\le C_{\mathrm{Ai}} e^{-\frac{4}{3}\tau^3} \left[e^{-2\tau(y - \sigma + \sqrt[3]{2} u)} + \frac{e^{2\tau(y-\sigma)}}{2\sqrt[3]{2}\tau} \right] \\
&= C_{\mathrm{Ai}} e^{-\frac{\tau^3}{3}} \left[e^{-\tau^3 - 2\tau(y - \sigma + \sqrt[3]{2} u)} + \frac{e^{-\tau^3 + 2\tau(y-\sigma)}}{2\sqrt[3]{2}\tau} \right] \\
&\le \frac{C_3}{\sqrt{\tau}} e^{-\tau(u-\sigma)}
\end{aligned} \tag{7.5.28c}
$$

for some positive constants C_j ($j = 1,2,3$), where the variables x, y run in $[\tilde{a}, \tilde{b}]$ and u, v run in $[\widetilde{\sigma}, \infty)$. Such estimates follow again from simple arguments on the asymptotic behaviour of the Airy function when its argument is very large. Moreover, the resolvent of the Tracy-Widom distribution is uniformly bounded and independent on τ; here is the reason for the constant C_{Ai}.

Collecting the above estimates, we have

$$
\left[\begin{array}{c|c} 0 & -\sqrt[6]{2} (\mathrm{Id} - \pi K_{\mathrm{Ai}} \pi)^{-1} \pi \mathfrak{A}^T_{-\tau} \tilde{\Pi} \\ \hline -\sqrt[6]{2} \tilde{\Pi} \mathfrak{A}_\tau \pi & \sqrt[3]{2} \tilde{\Pi} K_{\mathrm{Ai}}^{(\tau,-\tau)} \tilde{\Pi} \end{array} \right] \le C_\tau \left[\begin{array}{c|c} 0 & f(u) \\ \hline 1 & 1 \end{array} \right] \tag{7.5.29}
$$

with $C_\tau := \frac{\max\{C_j,\ j=1,2,3\}}{\sqrt{\tau}}$ and $f(u) = e^{-\tau(u-\sigma)}$. On the right hand side, we have a new operator $\mathcal{M}$ acting on $L^2([\widetilde{\sigma},\infty))\oplus L^2([\tilde{a},\tilde{b}]))$ with bounded trace

$$\operatorname{Tr}\mathcal{M}\leq \hat{C}\left(\|f\|^2_{L^2(\widetilde{\sigma},\infty)}+(\tilde{b}-\tilde{a})\right)\leq \hat{C}(b-a) \tag{7.5.30}$$

for some positive constant $\hat{C}$, since $\|f\|^2_{L^2(\widetilde{\sigma},\infty)}\to 0$ as $\tau\to+\infty$.

Concluding, having $[a,b]$ fixed,

$$\begin{aligned}|\ln\det(\mathrm{Id}-\Pi\mathbb{K}^{\mathrm{tac}}\Pi)| &= \sum_{n=1}^{\infty}\frac{\operatorname{Tr}(\Pi\mathbb{K}^{\mathrm{tac}}\Pi)^n}{n}\leq\sum_{n=1}^{\infty}\frac{C_\tau^n\hat{C}^n(b-a)^n}{n}\\ &\leq\sum_{n=1}^{\infty}C_\tau^n\hat{C}^n(b-a)^n=\frac{C_\tau\hat{C}(b-a)}{1-C_\tau\hat{C}(b-a)}\to 0\end{aligned} \tag{7.5.31}$$

as $\tau\to+\infty$.

Therefore, the constant of integration is equal zero. □

7.6 Conclusions and future developments

In this last chapter we showed how gap probabilities of the critical tacnode process can degenerate, under appropriate scaling regimes, into a product of two independent gap probabilities of the Airy process.

The first connections between the critical configuration of non-intersecting Brownian paths (see Figure 7.3) and the Hastings-McLeod solution to the Painlevé II equation, which describes the Airy gap probability, were established in the papers [28] and [75]. In particular, in the latter paper, the local distribution of the particles along the soft edges, afar from the tacnode critical point, was considered, in the limit as the two disjoint bundles touch each other tangentially. More interesting is the first paper, where a similar setting was considered: given two independent bundles of non-intersecting Brownian paths that under certain limit conditions touches tangentially at a critical point (the tacnode), the authors define a 4×4 Riemann-Hilbert problem which describes the tacnode kernel. Moreover, the residue matrix in the asymptotic series at infinity of such Riemann-Hilbert problem shows the presence of the Hastings-McLeod solution of Painlevé II. However, this connection, though remarkable, is mostly a hint that the tacnode process is somehow related to the Airy one through an appropriate limiting configuration.

In our work, on the other hand, we systematically prove the degeneracy in the setting as the two tangential bundles are pushed afar ($\sigma\to+\infty$; the opposite of the limiting procedure in [28] and [75]) and, moreover, in the new setting as we move away from the tacnode singularity along the soft edges of the bundles ($\tau\to\pm\infty$).

The Riemann-Hilbert formulation given in Proposition 7.13 (Section 7.3) may allow the study of another type of asymptotics of gap probabilities: the degeneration of the tacnode gap probability into the Pearcey gap probability. Physically, we can picture this transition by pushing the two touching ellipses further close so that they would merge; the soft edges would collapse and give rise to two cusp singularities, where the Pearcey process will appear. The scaling limit one need to perform in this situation is allowing the pressure parameter σ to diverge at $-\infty$ and the local time τ to be a function of σ itself (this is a natural assumption, since, as the two bundles get closer and closer, the cusps move vertically away from the original tacnode point). The conjectured asymptotic regime, supported by numerical evidences, has been stated in [12, Section 3.1]. Since the Riemann-Hilbert problem for the Pearcey gap probabilities ([11]) shows a quartic phase in the jump matrix, while the phase in the tacnode case is a cubic, the asymptotic study may require the introduction of a g-function (see Chapter 4) in order to apply the Deift-Zhou steepest descent method. However, we recall that such degeneration has already been proved by Geudens and Zhang in [43].

Another direction that would lead to completely new results is the derivation of differential equations describing the τ-function associated with the tacnode gap probability. This problem has never been addressed before in the literature and, starting form the Riemann-Hilbert formulation given in our work, it could be a natural future development.

Throughout this chapter, we have only focused on the single-time tacnode process. However, it can be of great interest also the study of its multi-time version (see for example [4] for its definition). It has already been proved in [12] that the gap probability of the Extended (multi-time) tacnode process can be expressed as ratio of two Fredholm determinants of explicit, not transcendental integral operators. The main challenge still remains, i.e. the formulation of a Riemann-Hilbert problem derived from a suitable IIKS integrable operator which will allow either the study of asymptotic behaviour or the study of differential equations associated with the gap probabilities.

As final note, we would like to gratefully acknowledge Dr. Bertola and Dr. Cafasso for their fundamental calculations in Section 7.3, without which the present work would not have been possible.

Chapter 8

Conclusions

In this thesis we tackled the problem of studying gap probabilities of specific determinantal point processes of recent interest.

The first important contribution, that affects all the three works presented in the thesis, is the further development of the method introduced by Bertola and Cafasso in [10] and [11], to use the isomonodromic τ-function and Riemann-Hilbert techniques to study such gap probabilities. This approach, compared with earlier ones by, for instance, Tracy and Widom ([100], [101]), Adler and Van Moerbeke *et al.* ([5]), Forrester and Witte ([40]), Basor and Chen *et al.* ([8]), has the advantage of being more systematic.

The "Fourier" method, that has been extensively used in the previous chapters, has been originally applied on the universal kernels of Airy and Pearcey ([10] and [11]) and it has now been successfully applied to other instances of universal kernels. The key point is not so much that the given kernel itself is "integrable" in the IIKS sense, but that the Fourier transform of its restriction to an interval is; the main signal (but not the exclusive one, see e.g. the tacnode kernel) is the double-integral representation with a denominator, as it is the case for the Airy kernel (see [11])

$$K_{\mathrm{Ai}}(x,y) = \int_{\gamma_R} \frac{\mathrm{d}\mu}{2\pi i} \int_{\gamma_L} \frac{\mathrm{d}\lambda}{2\pi i} \frac{e^{\frac{\mu^3}{3} - x\mu - \frac{\lambda^3}{3} + y\lambda}}{\lambda - \mu}. \tag{8.0.1}$$

In our work we first considered the case of the limiting gap probability in the so called "hard edge" of the random matrix theory, characterized by the Bessel kernel. We showed that this gap probability can be expressed in term of the isomonodromic τ function associated to a suitable Riemann-Hilbert problem. Generally the result is not in a simple form; however, in a special case, further simplification is possible and we were able to find a relation to a Painlevé III transcendent. This relation is not new, and was originally derived by Tracy and

Widom with a different method [101].

We were also able to express the gap probability for multi-time Bessel process through a Riemann-Hilbert problem and to analyze it in the same way as the one-time case. This is the first time that the multi-time gap probabilities in the Bessel process are expressed in an integrable way and this is the main contribution of the first work presented here.

The same method has then been applied to the Generalized Bessel process. The study of gap probabilities for such process has never been addressed before and our results are encouraging for future further investigations. In particular, a new Lax pair associated with the gap probabilities has been proposed and its shape suggests a connection with some higher order representative of a Painlevé hierarchy. Moreover, the definition of the multi-time Generalized Bessel process is genuinely new and its gap probabilities, expressed as τ-function, lead the way to additional possible analysis.

Next we turned to the problem of the asymptotic behaviour of the tacnode process. The main difficulty in approaching the problem was the fact that the expression of the tacnode kernel is highly transcendental, involving the resolvent of the Airy operator, and it was not in a double integral form as the Bessel or Generalized Bessel operators. Once the connection with an equivalent IIKS operator had been established, application of standard techniques of steepest descent lead to the expected degeneration into two independent Airy processes in given critical regimes. The tacnode process has been extensively studied in the past couple of years and it is still subject of investigation. The results shown in this thesis are an important contribution in the comprehension of the process and its properties.

In conclusion, the present thesis has attempted to shed light on some features of the gap probability of the above Determinantal Point Processes, by either deriving differential relations regulating this quantity or by studying its behaviour in specific critical settings.

Appendix A

Numerical simulation

In this appendix we will describe the numerical methods that were implemented in order to obtain some of the figures shown along this thesis. To be more precise, the chapter will focus on two subjects: we will first discuss about numerical evaluation of Fredholm determinants and how it is possible to get quite accurate quantities up to a small error term. Next, we will show how to get a realization of generic Dyson processes (i.e. non-intersecting Brownian paths).

A.1 Evaluation of Fredholm determinants

We start by recalling a general definition of Fredholm determinant and by discussing the most common theoretical methods that are used to evaluate such quantity.

Let $(X, \mathrm{d}\mu(x))$ be a (σ-finite) measure space and consider an integral operator K acting on the Hilbert space $L^2(X, \mathrm{d}\mu(x))$ and being trace-class (in general, K may belong to some trace ideal [95]). We define its Fredholm determinant through the Fredholm expansion

$$\det\left(\mathrm{Id} + zK\right) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!} \int_{X^k} \det[K(x_i, x_j)]_{i,j=1}^{k} \mathrm{d}\mu(x_1) \ldots \mathrm{d}\mu(x_k). \tag{A.1.1}$$

where with abuse of notation we called K the kernel of the given operator.

The computation of the Fredholm determinant (or the regularized Fredholm-Carleman determinant [95, Chapter 5] for generic integral operators) is an essentially transcendental problem. Even assuming reasonable regularity properties, in most of the cases an evaluation is possible if either the eigenvalues of the integral operator are explicitly known in a suitable way or if an alternative analytic expression has been found that is numerically more accessible, mostly a differential equation whose solution is related the behaviour of the determinant.

Therefore, theoretically speaking, every case of integral operator seems to require an *ad hoc* study, lacking a general procedure for evaluating its Fredholm determinant.

On the other hand, if one is aiming at a simple numerical evaluation, finding eigenvalues or integrating differential equations requires quite a computational cost for the machine. Nevertheless, in the recent paper by Bornemann [14] such issue has been addressed effectively.

A first approach would be the so called projection method, where, by the use of well-known Galerking techniques, the Hilbert space $\mathcal{H} := L^2(X, \mathrm{d}\mu(x))$ is decomposed into a sequence of finite-dimensional, increasing subspaces V_m (with $m = \dim V_m$ and $V_m \subset V_{m+1}$, $\forall m$) whose union is dense in $\mathcal{H}$, $\overline{\bigcup_{m=1}^{\infty} V_m} = \mathcal{H}$. Projecting the operator on such subspaces reduces the evaluation of its Fredholm determinant to the computation of a finite determinant, which approximates the original quantity up to an error depending on the regularity of the kernel.

A more efficient approach is the Nyström-type quadrature method, especially for analytic kernels, like the ones appearing in Random Matrix Theory. The idea is very simple and it takes inspiration from Nyström's ([90]) classical quadrature method for the numerical solution of the Fredholm equation

$$u(x) + z\int_a^b K(x,y)u(y)\mathrm{d}y = f(x) \qquad x \in [a,b], \tag{A.1.2}$$

where the integral operator $\mathbf{K}$ is defined as

$$\mathbf{K}(\phi)[x] := \int_a^b K(x,y)\phi(y)\mathrm{d}y \qquad \text{in } L^2(a,b) \tag{A.1.3}$$

$$\text{with kernel } K \in C^0([a,b]^2) \quad a, b \in \mathbb{R}. \tag{A.1.4}$$

Given a quadrature rule

$$\int_a^b f(x)\mathrm{d}x \sim \sum_{j=1}^{m} w_j f(x_j) =: Q_m(f) \tag{A.1.5}$$

where w_j are some suitable weights, Nyström discretized the Fredholm equation (A.1.2) as the linear system

$$u_i + z\sum_{j=1}^{m} w_j K(x_i, x_j)u_j = f(x_i) \qquad i = 1, \dots, m \tag{A.1.6}$$

which has to be solved for u_i (i.e. the value $u(x_i)$), $\forall\ i = 1, \dots, m$, and $\{x_i\}_{i=1}^m$ come from the m-point Gauss-Legendre rule.

The above method applied to the case of evaluating a Fredholm determinant

$$d(z) := \det\left(\mathrm{Id} + z\mathbf{K}\right) \tag{A.1.7}$$

is implemented by calculating the determinant of an $m \times m$-matrix

$$d_Q(z) = \det\left[\delta_{ij} + z w_i K(x_i, x_j)\right]_{i,j=1}^m ; \tag{A.1.8}$$

alternatively, if the weights w_j of the quadrature rule are positive, the approximant becomes

$$d_{Q_m}(z) = \det\left[\delta_{ij} + z w_i^{1/2} K(x_i, x_j) w_j^{1/2}\right]_{i,j=1}^m . \tag{A.1.9}$$

The convergence results follow.

Theorem A.1 (Theorem 6.1, [14])**.** *Consider a trace-class integral operator $\boldsymbol{K}$ of the type (A.1.3)-(A.1.4). If a family $\{Q_m\}$ of quadrature rules converges for continuous functions, then the corresponding Nyström-type approximation of the Fredholm determinant converges,*

$$d_{Q_m}(z) \to d(z) \qquad m \to \infty, \tag{A.1.10}$$

uniformly for bounded z.

Theorem A.2 (Theorem 6.2, [14])**.** *If the kernel $K \in C^{k-1,1}([a,b]^2)$[1], then for each quadrature rule Q of order $\nu \geq k$ with positive weights there holds the error estimate*

$$d_Q(z) - d(z) = \mathcal{O}\left(\nu^{-k}\right). \tag{A.1.11}$$

Therefore, whenever an evaluation of gap probabilities for a specific determinantal process (with correlation kernel $\mathfrak{K}$) is needed, one can apply the results above and calculate the approximated Fredholm determinant of the operator

$$K := \mathfrak{K}\Big|_{\mathcal{I}} \tag{A.1.12}$$

with $\mathcal{I}$ the bounded Borel set where the gap probabilities are studied.

Remark A.3. *The method can also be generalized and applied to matrix kernels representing the multi-time counterpart of the timeless process (see [14, Section 8.1 and 8.2]).*

[1]We recall that, given an interval $I \subseteq \mathbb{R}$, $C^{\alpha,1}(I)$ is the space of functions with are differentiable α times with Lipschitz derivatives.

Using Gauss-Legendre or Curtis-Clenshaw quadrature rules, the computational cost of the method is of order $\mathcal{O}(m^3)$. The implementation in MatLab® or Maple® is straightforward and it takes just a few lines of code. For our purposes, we programmed using Maple 17.

The following code is the evaluation of the Bessel process in single time restricted to the interval $[0, s]$ (see Chapter 5, Figure 5.3).

```
> gen:=proc(n)
> local N,W,P;
> N:= `evalf/int/AGQ/AGQ_wr`(n,'W','P');
> if type(n,odd) then
> [seq(P[i],i=1..N), seq(1-P[N-i],i=1..N-1)],
> [seq(W[i],i=1..N),seq(W[N-i], i=1..N-1)];
> else
> [seq(P[i],i=1..N), seq(1-P[N-i+1],i=1..N)],
> [seq(W[i],i=1..N), seq(W[N-i+1],i=1..N)];
> end if;
> end proc:
> BesselKernel:=unapply(
> ( BesselJ(nu, sqrt(x))*sqrt(y)*BesselJ(nu+1,  sqrt(y)) -
> BesselJ(nu, sqrt(y))*sqrt(x)*BesselJ(nu+1,  sqrt(x)))
> /(2*(x-y)),x,y,nu);
> BesselDens:=unapply(simplify(subs(y=x,
> diff(( BesselJ(nu, sqrt(x))*sqrt(y)*BesselJ(nu+1,  sqrt(y))
> - BesselJ(nu, sqrt(y))*sqrt(x)*BesselJ(nu+1,  sqrt(x)))
> /(2),y))),x,nu);
> KB:=(x,y,nu)-> `if`(x=y, BesselDens(x,nu), BesselKernel(x,y,nu));
```

```
> FredBessel:=proc(s,nu,M)
> local P, W, Kern;
> if s>0 then
> (P,W):= gen(M):
> Kern:=(s)->
> IdentityMatrix(M) -
> Matrix(M,M,(i,j)-> evalf(W[i]*s*KB(s*P[i], s*P[j],nu))):
> return(evalf(Determinant(Kern(s))));
> else
> return (1);
> fi;
> end:
```

The same numerical strategy has been used for the evaluation of the tacnode process in the limit regime as $\sigma \to +\infty$ and $\tau \to \pm\infty$, when the process degenerates into two Airy processes (see Figures 7.4 and 7.5, Chapter 7). The original Maple code was written by M. Bertola and M. Cafasso and illustrated in [12]. Such code has been reproduced here and adjusted to the present purpose. In particular, in order to evaluate the Fredholm determinant of the tacnode kernel we use the following formula ([12, Theorem 3.2])

$$\det\left(\mathrm{Id} - \mathbb{K}^{\mathrm{tac}}\Big|_{[a,b]}\right) = \frac{\det\left(\mathrm{Id} - \hat{\Pi}\mathbb{H}\hat{\Pi}\right)}{\det\left(\mathrm{Id} - \pi K_{\mathrm{Ai}}\pi\right)} \tag{A.1.13}$$

where $\hat{\Pi} := \mathrm{Id} \oplus \pi \oplus \Pi$, Π is the projector on $[a,b]$ and π is the projector on $[\widetilde{\sigma}, +\infty)$ ($\widetilde{\sigma} := 2^{\frac{2}{3}}\sigma$) and $\mathbb{H}$ is equal to

$$\left[\begin{array}{c|c|c} \mathbb{H}_{-1,-1}\equiv 0 & \mathbb{H}_{-1,0}(x,y)=-\mathrm{Ai}(x+y) & \mathbb{H}_{-1,1}(x,y)=\mathrm{Ai}^{(-\tau)}\left(\sqrt[3]{2}x+\sigma-y\right) \\ \hline \mathbb{H}_{0,-1}(x,y)=-\mathrm{Ai}(x+y) & \mathbb{H}_{0,0}\equiv 0 & \mathbb{H}_{0,1}(x,y)=\mathrm{Ai}^{(-\tau)}\left(\sqrt[3]{2}x+y-\sigma\right) \\ \hline \mathbb{H}_{1,-1}(x,y)=\mathrm{Ai}^{(\tau)}\left(\sigma-x+\sqrt[3]{2}y\right) & \mathbb{H}_{1,0}(x,y)=\mathrm{Ai}^{(\tau)}\left(x-\sigma+\sqrt[3]{2}y\right) & \mathbb{H}_{1,1}\equiv 0 \end{array}\right]. \tag{A.1.14}$$

On the other hand, the quadrature method described above cannot be applied directly to the Airy process, since the Airy operator is restricted to an infinite interval of the type $[\widetilde{\sigma}, +\infty)$: indeed, the convergence of the Nyström-type approximation is guaranteed only for operators defined on bounded sets (Theorem A.1).

The strategy is therefore to transform the infinite interval into a finite one, following the idea of [14]: by using a monotone, smooth transformation ([14, Formula 7.5])

$$\phi : [0,1] \to [\widetilde{\sigma}, \infty), \qquad \phi_{\widetilde{\sigma}}(\zeta) = \widetilde{\sigma} + 10 \tan\left(\frac{\pi\zeta}{2}\right) \tag{A.1.15}$$

we define a new Airy integral operator pushed back on $[0,1]$ with kernel

$$\widetilde{K_{\mathrm{Ai}}}(\xi,\eta) := \sqrt{\phi'(\xi)\phi'(\eta)} K_{\mathrm{Ai}}\left(\phi(\xi), \phi(\eta)\right) \tag{A.1.16}$$

such that

$$\det\left(\mathrm{Id} - K_{\mathrm{Ai}}\Big|_{[\widetilde{\sigma},\infty)}\right) = \det\left(\mathrm{Id} - \widetilde{K_{\mathrm{Ai}}}\Big|_{[0,1]}\right). \tag{A.1.17}$$

```
> K1:=(x,y)-> evalf((AiryAi(x)*AiryAi(1,y)
> - AiryAi(y)*AiryAi(1,x))/(x-y)):
> K2:=unapply(evalf(-simplify(diff((AiryAi(x)*AiryAi(1,y)
> - AiryAi(y)*AiryAi(1,x)), y),{y=x})) ,x):
> K_Ai:=(x,y)-> `if`(x=y,K2(x), K1(x,y)):
> tpp:=x-> (10*tan(Pi*x/2)):
> dtpp:=unapply(simplify(diff(tpp(x),x)),x):
> phi:=unapply((tpp(x)),x);
> dphi:=unapply((dtpp(x)),x);
> KAi[pushback]:=(x,y,s)
> ->evalf(sqrt(dphi(x)*dphi(y)) *K_Ai(s+phi(x),s+phi(y)));
> FredAiry:=proc(s,M)
> local P, W,Z, Kern;
> (P,W):= gen(M):
> Kern:=(s)->
> IdentityMatrix(M) -
> Matrix(M,M,(i,j)-> evalf(W[i]*KAi[pushback](P[i], P[j],s))):
> return(evalf(Determinant(Kern(s))));
> end:
```

```
> Ai[tau]:= unapply( 2^(1/6)*exp(tau*x+2/3*tau^3)*AiryAi(x+tau^2),(x,tau));
> HH[-1,0]:=(x,y)-> -AiryAi(x+y);
> HH[0,-1]:=(x,y)-> -AiryAi(x+y);
> HH[-1,1]:= unapply( Ai[tau]( x*2^(1/3)+s-y, -t_j),(x,y,t_j,s));
> HH[1,-1]:=unapply( Ai[tau]( s-x+y*2^(1/3), t_j ),(x,y, t_j,s));
> HH[0,1]:= unapply( Ai[tau]( x*2^(1/3)+y-s, -t_j),(x,y, t_j,s));
> HH[1,0]:= unapply( Ai[tau]( x-s+y*2^(1/3), t_j ),(x,y, t_j,s));
> FredTac:=proc (a,b,t,s,N)
> local Kern,P,W,II,W2, ss, NUMtac:
> (P,W):= gen(N):
> W2:= map(evalf,map(dphi,P)):
> ss:=s*2^(2/3):
> Kern[-1,0]:=  Matrix(N,N,(i,j)->evalf(W2[i]*W[i]*
> *HH[-1,0]  (phi(P[i]),ss+phi(P[j])))):
> Kern[-1,1]:=  Matrix(N,N,(i,j)->evalf(W2[i]*W[i]*
> *HH[-1,1] (phi(P[i]),a+P[j]*(b-a),t,s ))):
> Kern[0,-1]:=  Matrix(N,N,(i,j)->evalf(W2[i]*W[i]*
> *HH[0,-1]  (ss+phi(P[i]),phi(P[j])))):
> Kern[0,1]:=   Matrix(N,N,(i,j)->evalf(W2[i]*W[i]*
> *HH[0,1]  (ss+phi(P[i]),a+P[j]*(b-a),t,s ))):
> Kern[1,-1]:=  Matrix(N,N,(i,j)->evalf((b-a)*W[i]*
> *HH[1,-1] (a+P[i]*(b-a) ,phi(P[j]) ,t,s ))):
> Kern[1,0]:=   Matrix(N,N,(i,j)->evalf((b-a)*W[i]*
> *HH[1,0]  (a+P[i]*(b-a) ,ss+phi(P[j]) ,t,s ))):
> II:= IdentityMatrix(N):
> NUMtac:= <
> <II         | -Kern[-1,0]  | -Kern[-1,1]>,
> <-Kern[0,-1] | II          | -Kern[0,1] >,
> <-Kern[1,-1] | -Kern[1,0]  | II         >>;
> return(evalf(Determinant(NUMtac))):
> end:
```

Remark A.4. *The process "FredTac" only calculates the numerator of the Fredholm deter-*

minant of the tacnode process. Therefore, it still needs to be divided by the Tracy-Widom distribution on $[\widetilde{\sigma}, \infty)$.

A.2 Non-intersecting random paths

As mentioned in the introduction of this chapter, we will now focus on the so called Dyson processes. We call a Dyson process any process on ensembles of matrices in which the entries undergo diffusion; in the original paper by Dyson [31], it was the ensemble of $n \times n$ Hermitian matrices M, where the coefficients of each matrix independently executes Brownian motion subject to a simple harmonic force. For the present section, we refer to the book by Mehta [84] for all the details.

Suppose that the coefficients of the matrix have values $\{M_1, \ldots, M_N\}$ $(N = n^2)$ at time t, and values $\{M_1 + \delta M_1, \ldots, M_N + \delta M_N\}$ at time $t + \delta t$. A Brownian motion of M is defined by requiring that each δM_μ is a random variable with first and second moments

$$\mathbb{E}\left(\delta M_\mu\right) = -\frac{M_\mu}{fa^2}\delta t, \qquad \mathbb{E}\left((\delta M_\mu)^2\right) = \frac{g_\mu}{2}\delta t \tag{A.2.1}$$

where $g_\mu = 1 + \delta_{ij}$, $a \in \mathbb{R}$ and the constant f is the friction coefficient which fixes the rate of diffusion. The Fokker-Planck equation corresponding to equations (A.2.1) is

$$f\frac{\partial P}{\partial t} = \sum_\mu \left[\frac{1}{4}g_\mu \frac{\partial^2 P}{\partial M_\mu^2} + \frac{1}{a^2}\frac{\partial}{\partial M_\mu}\left(M_\mu P\right)\right], \tag{A.2.2}$$

where $P(M_1, \ldots, M_N; t)$ is the time-dependent probability density of the entries M_μ. Given an initial condition $M = M'$ at $t = 0$, the solution of equation (A.2.2) can be computed explicitly:

$$P(M;t) = \frac{c}{(1-q^2)^{\frac{N}{2}}} \exp\left\{-\frac{\mathrm{Tr}\ (M - qM')^2}{a^2(1-q^2)}\right\} \tag{A.2.3}$$

$$q = \exp\left\{-\frac{t}{fa^2}\right\} \tag{A.2.4}$$

and c is a suitable normalization constant.

Remark A.5. *It is easy to see that the equilibrium measure as* $t \to +\infty$ *is the stationary Gaussian Unitary Ensemble (GUE) measure from Random Matrix Theory*

$$\frac{1}{Z_{n,\mathrm{GUE}}} e^{-\operatorname{Tr} M^2}. \tag{A.2.5}$$

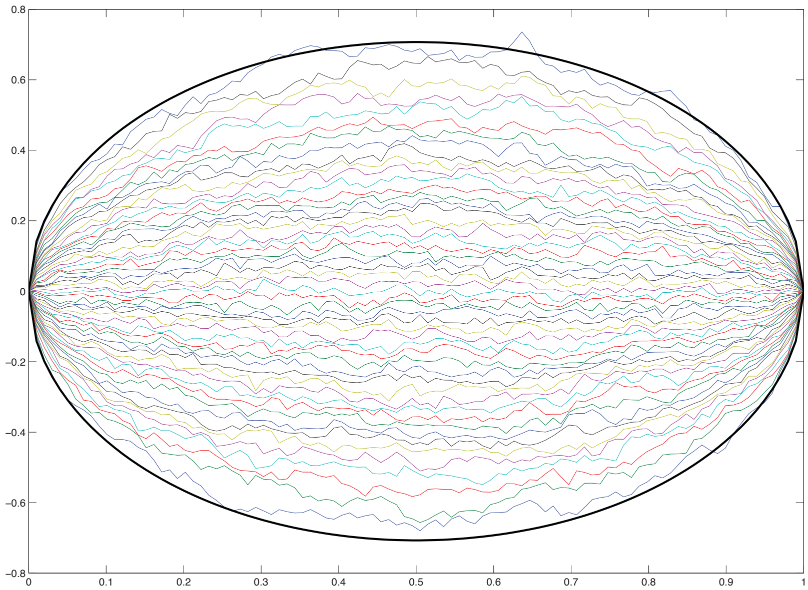

Figure A.1: Numerical simulation of 50 non-intersecting Brownian paths with limiting shape.

with $Z_{n,\mathrm{GUE}}$ the normalization constant, also called partition function, of the GUE.

As described in Chapter 2, given a matrix ensemble, the corresponding eigenvalue distribution is a determinantal point process which can be visualized as a collection of n non-intersecting Brownian paths in the tx-plane.

Theorem A.6 (Theorem II, [31])**.** *When the matrix M executes a Brownian motion according to equations (A.2.1), starting from any initial condition, its eigenvalues $\{x_1, \ldots, x_n\}$ execute a Brownian motion obeying the equation of motion of the time-dependent Coulomb gas: if $F(x_1, \ldots, x_n; t)$ is the time-dependent probability density for finding the particles at the positions x_i at time t, then F satisfies the Fokker-Plank equation*

$$f\frac{\partial F}{\partial t} = \sum_i \left[\frac{1}{2}\frac{\partial^2 F}{\partial x_i^2} - \frac{\partial}{\partial x_i}\left(E(x_i)F\right)\right] \tag{A.2.6}$$

where E is the external electric force

$$E(x_i) = \sum_{i\neq j} \frac{1}{x_i - x_j} - \frac{x_i}{a^2}. \tag{A.2.7}$$

It is straightforward to implement this result in a numerical code using MatLab®. We

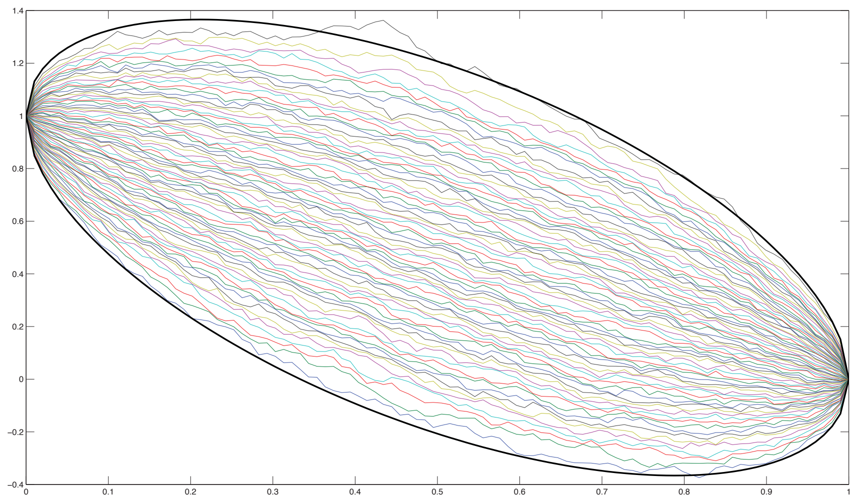

Figure A.2: Numerical simulation of 70 non-intersecting Brownian paths starting at 1 and ending at 0, with limiting shape.

first define the distribution of the matrix, given an initial condition, and then we calculate its corresponding eigenvalues (see Figure A.1). For our purposes, we programmed using MatLab R2014a.

```
H=zeros(N,N,T);

x=linspace(0,1,T);

Init = diag( a1*ones(N,1));
H(:,:,1)= Init;
Final = diag( b1*ones(N,1));

Evals = zeros(T,N);
Evals(1,:)=eig(H(:,:,1));

for t=2:T
    for k=1:N
        H(k,k,t) = (randn(1)*sqrt(1/(T*N)) -1/(T*sqrt(N))*H(k,k,t-1)) + H( k,k,t-1) ;
        for j=k+1:N
            H(j,k,t) = (randn(1) *sqrt(1/(T*(N))/2)
            - 1/(T*sqrt(N))*real(H(j,k,t-1)) ) +real(H(j,k,t-1)
            + 1i*(randn(1) *sqrt(1/(T*(N))/2)-1/T/sqrt(N)*imag(H(j,k,t-1))
            + imag(H(j,k,t-1) ) ));
            H(k,j,t) = H( j,k,t)';
        end;
```

```
    end;
end;

for t=2:T
 H(:,:,t) = H(:,:,t) - t/T*( H(:,:,T) -Final);
 Evals(t,:) = eig(H(:,:,t));
end;


EP=zeros(T,2);
EP(1,:)= [a1 a1];

for k=2:T
     EP(k,1)= (1-x(k))*a1 +x(k)*b1 - 2*sqrt(1/2*x(k)*(1-x(k)));
     EP(k,2)= (1-x(k))*a1 +x(k)*b1 + 2*sqrt(1/2*x(k)*(1-x(k)));
end;


plot(x,Evals);
hold on
plot(x,EP,'k', 'LineWidth',2);
hold off
```

The limiting hull in the tx-plane consists of an ellipse-like shape which can be explicitly described. Indeed, for any $t \in [0,1]$, the limiting distribution of the positions of the paths at time t is supported on an interval $[\alpha_t, \beta_t]$, where the endpoints satisfy

$$\alpha_t = (1-t)a + tb - 2\sqrt{Kt(1-t)}, \tag{A.2.8}$$

$$\beta_t = (1-t)a + tb + 2\sqrt{Kt(1-t)}, \tag{A.2.9}$$

where K is a parameter depending on the constant a and the friction f. We recall that the limiting density of the particles is given by the Wigner's semicircle law on that interval (see [84]).

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