STOCHASTIC FLOW AND FBSDE APPROACHES TO QUADRATIC TERM STRUCTURE MODELS

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ABSTRACT

Stochastic Flow and FBSDE Approaches to Quadratic Term Structure

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We study the stochastic flow method and Forward-backward Stochastic Differential Equation (FBSDE) approach to the Quadratic Term Structure Models (QTSMs). Applying the stochastic flow approach, we get a closed form solution for the zero-coupon bond price under a one-dimensional QTSM. However, in the higher dimensional cases, the stochastic flow approach is difficult to implement. Therefore, we solve the n-dimensional QTSMs by implementing the FBSDE approach, which shows that the zero-coupon bond price under QTSM provided some Riccati type equations have global solutions.

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Chapter 1

Introduction

Modeling the term structure of interest rates is an important topic in mathematical finance. Because of the tractability in pricing, affine term structure models (ATSMs) have been widely used in financial modeling. In ATSMs, the price of a zero-coupon bond, which pays \$ 1 at time T, is an exponential-affine function of the factor process X_t

$$P(X_t, t, T) = \exp\Big[B(T, t)'X_t + C(T, t)\Big],$$

at time $0 \leq t \leq T$ where $B(T, t)$ is an $n \times 1$ vector and $C(T, t)$ is a scalar. The study of the ATSMs includes Vasicek and Smith (1977), Cox et al. (1985) and Duffie and Kan (1996). However, ATSMs have some drawbacks. Dai and Singleton (2000) show that ATSMs fail to capture some aspects of swap yield distribution, which suggest that there may be some omitted nonlinearlity in ATSMs. Ahn and Gao (1999) empirically show that non-affine term structure models outperform one-factor affine models.

Recently quadratic term structure models (QTSMs) have been studied by several authors. In QTSMs, the zero-coupon bond prices are exponential-quadratic functions of the factor process X_t

$$P(X_t, t, T) = \exp\Big[X_t' A(T,t) X_t + B(T,t)' X_t + C(T,t)\Big],$$

at time $0 \leq t \leq T$ where $A(T,t)$ is a non-singular $n \times n$ matrix, $B(T,t)$ ia an $n \times 1$ vector, and $C(T,t)$ is a scalar. Ahn et al. (2002) introduce the comprehensive QTSMs and study the characteristic of the models. The pricing problems of QTSMs have been studied by Chen et al. (2004) and Leippold and Wu. Other research on the topic of QTSMs includes Levendorskií (2005) and Boyarchenko and Levendorskií (2007). Compared to ATSMs, QTSMs can capture nonlinearities between economic factors and provide more flexibility when constructing models. Moreover, as shown by Chen et al. (2004), Leippold and Wu, and Leippold and Wu (2002) QTSMs are analytically tractable in that the zero-coupon bond price has an exponential-quadratic form in the state variables and the the prices of European style options can be calculated by Fourier transform methods.

In this thesis, we study QTSMs using two approaches, a stochastic flow approach and a forward-backward stochastic differential equation (FBSDE). The stochastic flows approach to ATSMs has been studied by Elliott and van der Hoek (2001), Grasselli and Tebaldi (2007), Hyndman (2007a) and Hyndman (2009). This method gives a closed-form solution to the pricing problems for certain ATSMs. The FBSDE approach was first introduced by Hyndman (2007b). In Hyndman (2007b), the author adapted a technique from Ma and Yong (1999) to prove existence and uniqueness of a nonlinear FBSDE which is generated from the pricing problem for the Cox, Ingersoll and Ross (CIR) model of Cox et al. (1985). The results in Hyndman (2007b) have been extended to the n-dimensional case by Hyndman (2009). Geman and Yor (1993) have shown that the CIR process is a Bessel process under certain restrictions, which means that the CIR process and the comprehensive QTSMs are equivalent in certain cases. Motivated by this fact, we extend the techniques of the stochastic flows approach and the FBSDE approach to the comprehensive QTSMs.

The thesis is organized as follows. Chapter 2 introduces the models we study. Model A in Chapter 2 is consistent with the QTSMs introduced by Ahn et al. (2002). In Chapter 2, we also study the stochastic flows approach to QTSMs. We give a close-form solution for zero-coupon bond price to our model in one-dimensional case. We also discuss the parametric restrictions required when implementing the flows method to two-dimensional QTSMs. In Chapter 3, we demonstrate the FBSDE approach for the zero-coupon bond price for the class of the comprehensive QTSMs. Chapter 4 concludes and discusses future research directions.

Chapter 2

Stochastic Flows Method

In this thesis, we study two different models for the short interest rate process. Both are formulated on the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$ for $0 \leq t \leq T^*$ where T^* is the investment horizon, $\{\mathcal{F}_t\}$ is a right-continuous and complete filtration, and Q is the risk-neutral measure. As in Shreve (2004) (p 411), the price of the zero-coupon bond at time t for maturity $T \leq T^*$ is given by

$$P(t,T) = E_Q\left[\exp\left(-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right]. \tag{2.1}$$

On the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$, suppose the factor process $X \in \mathbf{R}^n$ is given by

$$dX_t = \{AX_t + B\}dt + \sigma dW_t, \tag{2.2}$$

where

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{bmatrix}$$

and

$$W_t = \begin{bmatrix} W_t^{(1)} \\ \vdots \\ W_t^{(n)} \end{bmatrix}$$

is any n-dimensional Brownian motion with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$. From Friedman (1975), the unique solution of SDE (2.2) satisfies the strong Markov property.

We assume that the riskless interest rate r_t is given by a function $r(X_t)$. In this thesis, we would like to consider two different models of $r(X_t)$. Model A is defined as follows,

$$r(X_t) = X_t'\Gamma X_t + RX_t + k, \tag{2.3}$$

where

$$\Gamma = \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,n} \\ \vdots & \ddots & \vdots \\ \gamma_{n,1} & \cdots & \gamma_{n,n} \end{bmatrix}, \quad R = \begin{bmatrix} r_1, \cdots, r_n \end{bmatrix}$$

and k is a scalar. Γ is required to be positive semidefinite.

Remark 2.1. *Since Γ is positive semidefinite. The lower bound of $r(X_t)$ from (2.3) is $k - \frac{1}{4}R\Gamma^{-1}R'$ when $X_t = -\frac{1}{2}\Gamma^{-1}R'$. So this model can guarantee the positive sign of the short rate process by setting $k - \frac{1}{4}R\Gamma^{-1}R' \geq 0$. To find the minimum of (2.3) we just need to take the partial derivative of $r(x)$, for $x = (x_1, \cdots, x_n)$ with respect to $x_1, \cdots x_n$. Then we know that if X_t satisfies equation*

$$(\Gamma + \Gamma')X_t + R' = 0_{n\times 1}, \tag{2.4}$$

$r(X_t)$ can reach its minimum. Since Γ is positive semidefinite, it is a symmetric matrix. Simplifying equation (2.4) to get $X_t = -\frac{1}{2}\Gamma^{-1}R^T$.

Model B is defined as follows,

$$r(X_t) = CX_tX_t'C' + RX_t + k, \tag{2.5}$$

where

$$c = \begin{bmatrix} c_1, \cdots, c_n \end{bmatrix}, \quad R = \begin{bmatrix} r_1, \cdots, r_n \end{bmatrix}$$

and k is a scalar.

Remark 2.2. *Model B can be seen as a special case of Model A by setting $\Gamma = C'C$. In one-dimensional case, these two models coincide.*

In this chapter we study the stochastic flows approach for quadratic term structure models. In the following discussion we find that the stochastic flows method can solve the one-dimensional model. However, this approach becomes complicated in the higher dimensional cases and it is not clear if the method works unless we add some restrictions on the parameters of our models.

2.1 Introduction to Stochastic Flows

We introduce the definition and some important properties of stochastic flows in this section.

In the general case, consider the stochastic differential equations (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.6}$$

taking values in $\mathbf{R}^n$. Suppose

$$b : [0, \infty) \times \mathbf{R}^n \to \mathbf{R}^n$$

$$\sigma : [0, \infty) \times \mathbf{R}^n \to \mathbf{R}^{n \times m}$$

are Borel-measurable functions.

We are interested in solutions of (2.6) started from an initial condition x at time s.

Definition 2.1. *Fix some $x \in [0, \infty)$. A solution of the SDE (2.6) for the pair (b, σ), starting at s, is a triple*

$$\{(\Omega, \mathcal{F}, \mathcal{F}_t, Q), (X_t, t \geq s), (W_t, t \geq 0)\}$$

with the following properties:

(i)(a) and (b) hold:

(a) $(\Omega, \mathcal{F}, Q)$ is a complete probability space;

(b) $\mathcal{F}_t$ is a filtration with $\{N \in \mathcal{F} | Q(N) = 0\} \in \mathcal{F}_0$.

(ii)$\{W_t, \mathcal{F}_t\}$ is a $\mathbf{R}^m$-valued Wiener process on $(\Omega, \mathcal{F}, Q)$;

(iii) $\{X_t\}$ is a continuous $\mathbf{R}^n$-valued process on $(\Omega, \mathcal{F}, Q)$ such that

(a) $\omega \to X_t(\omega)$ is $\mathcal{F}_t$-measurable for each $t \geq s$;

(b)

$$\int_s^t |b^i(u, X_t)| du + \int_s^t |\sigma^{ij}(u, X_t)|^2 du < \infty$$

Q-a.s. for each $t \in [s, \infty)$;

(c)

$$X_t = X_s + \int_s^t b(u, X_u) du + \int_s^t \sigma(u, X_u) dW_u$$

Q-a.s. for each $t \in [s, \infty)$.

We shall require solutions of (2.6) to have the property of path-wise uniqueness as in the following definition.

Definition 2.2. *Fix an arbitrary $x \in \mathbf{R}^n$. The SDE (2.6) for the pair (b, σ), starting at s, has the property of path-wise uniqueness from $x \in \mathbf{R}^n$ if for any two solutions*

$$\{(\Omega, \mathcal{F}, \mathcal{F}_t^i, Q), (X_t^i, t \geq s), (W_t, t \geq 0)\}, \quad i = 1, 2$$

such that

$$X_s^1 = X_s^2 = x, \quad Q\text{-}a.s.$$

it follows that

$$Q(\{X_t^1 = X_t^2, t \in [s, \infty)\}) = 1.$$

In the most general case our analysis requires the following assumption.

Assumption 2.1. *Assume that the Bórel-measurable functions $b : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ satisfy the following conditions (a) and (b) for $s \in [0, \infty)$:*

(a) for $x \in \mathbf{R}^n$, there exists some solution

$$\{(\Omega, \mathcal{F}, \mathcal{F}_t, Q), (X_t, t \geq s), (W_t, t \geq 0)\}$$

of the stochastic differential equation (SDE) for the pair (b, σ) starting at s, such that $X_s = x$, Q a.s.;

(b)for $x \in \mathbf{R}^n$, the SDE for the pair (b, σ) starting at s, has the property of path-wise uniqueness from x.

Remark 2.3. *The motivation for Assumption 2.1 is so that we can apply the results of Kallenberg (1996). This result is rather technical and is stated in Theorem A.2.7 of Appendix of Hyndman (2005)*

On a probability space $(\Omega, \mathcal{F}, Q)$ consider the stochastic differential equation (2.6) taking values in $\mathbf{R}^n$ with b and σ satisfying conditions (a) and (b) of Assumption 2.1. For each $x \in \mathbf{R}^n$ and for $t \leq u$ write $X_u^{t,x}$ for the solution to

$$X_u^{t,x} = x + \int_t^u b(s, X_s^{t,x})ds + \int_t^u \sigma(s, X_s^{t,x})dW_s. \tag{2.7}$$

The following result allows us to formally differentiate (2.7) with respect to x and shall be used throughout the thesis.

Theorem 2.1 (Diffeomorphism Theorem). *Suppose $b : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^n$ and $\sigma : [0,\infty) \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ are such that equation (2.7) has a solution that is path-wise unique and there is an open set $D \subset \mathbf{R}^n$ such that $x \to b(t,x)$ and $x \to \sigma(t,x)$ are smooth functions of x (up to order two) for all $x \in D$. Then $x \to X_u^{t,x}$ is differentiable and the result of differentiating equation (2.7) formally with respect to x is valid. That is, $\frac{\partial X_u^{t,x}}{\partial x}$ satisfies*

$$\frac{\partial X_u^{t,x}}{\partial x} = I + \int_t^u \frac{\partial b(s,X_s^{t,x})}{\partial t}\frac{\partial X_s^{t,x}}{\partial x} ds + \int_t^u \frac{\partial \sigma(s,X_s^{t,x})}{\partial t}\frac{\partial X_s^{t,x}}{\partial x} dW_s$$

for all $x \in D$.

Proof. This result has its roots in the work of Blagoveščenskiĭ and Freĭdlin (1961). The proof, which is a consequence of the Kolmogorov-Čentsov continuity theorem (Karatzas and Shreve (1991), Theorem 2.2.8), can be found in Kunita (1981). □

A key property of stochastic flows which we shall employ is the semi-group or flow property.

Lemma 2.1 (Flow Property). *If $X_u^{t,x}$ is the solution of (2.7) and $X_u^{s,x}$ is the solution of (2.7) starting at time s with $s \le t \le u$ then*

$$X_u^{s,x} = X_u^{t,X_t^{s,x}}$$

in the sense that one is a modification of the other.

In our models, $b(t,x) = (Ax + B)$ and $\sigma(t,x) = \sigma$ satisfy Assumption 2.1. Therefore, the stochastic flow associated with the factor process X_t given in (2.2) has the diffeomorphism and flow properties.

2.2 One Dimensional Case

We use the following notation in this section. The factor process is given by

$$dX_t = \beta(\alpha - X_t)dt + \sigma dW_t. \tag{2.8}$$

On the risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$, the riskless interest rate r_t is given by the function

$$r(X_t) = cX_t^2 + bX_t + a. \tag{2.9}$$

Define $X_s^{t,x}$ as the solution of (2.8) started from $x \in \mathbf{R}$ at time $t \geq 0$. So that $X_s^{t,x}$ satisfies

$$X_s^{t,x} = x + \int_t^s \beta(\alpha - X_u^{t,x})du + \sigma \int_t^s dW_u, \quad s \in [t, T]. \tag{2.10}$$

The zero coupon bond price with maturity T at time t is

$$P(t,T) = E\left[\exp\left\{-\int_t^T r(X_u)du\right\}\Big|\mathcal{F}_t\right]. \tag{2.11}$$

By the Markov property of X_t,

$$P(t,T) = P(t,T,X_t), \tag{2.12}$$

where

$$P(t,T,x) = E\left[\exp\left\{-\int_t^T r(X_u^{t,x})du\right\}\right]. \tag{2.13}$$

Taking the derivative of (2.13) with respect to x,

$$\begin{aligned}\frac{\partial P(t,T,x)}{\partial x} &= E\left[\left(-\int_t^T r'(X_u^{t,x})\frac{\partial X_u^{t,x}}{\partial x}du\right)\exp\left(-\int_t^T r(X_u^{t,x})du\right)\right]\\ &= E\left[L(t,T.x)\exp\left(-\int_t^T r(X_u^{t,x})du\right)\right],\end{aligned} \tag{2.14}$$

where

$$L(t,T,x) = -\int_t^T r'(X_u^{t,x}) \frac{\partial X_u^{t,x}}{\partial x} du. \tag{2.15}$$

We may exchange the order of expectation and differentiation since $b(x,t) \triangleq \beta(\alpha - x)$ and $\sigma(x,t) \triangleq \sigma$ satisfy the linear growth condition in x and global Lipschitz condition, the partial derivatives of $b(x,t)$ and $\sigma(x,t)$ are continuous and satisfy a polynomial growth condition and the function $\exp\{-\int_t^T r(x)du\}$ has two continuous derivatives satisfying a polynomial growth condition (see Friedman (1975) p 117-123).

In order to factor out the bond price $P(t,T)$ from the expectation in (2.14), we want to work on the forward measure Q^T. Recall that the forward measure is defined as follows:

Definition 2.3. *Let T be the maturity date. The T-forward measure Q^T is defined by*

$$Q^T(A) := E_Q[\Lambda_T 1_A] \qquad \forall A \in \mathfrak{F}_T$$

where

$$\Lambda_T = \left.\frac{dQ^T}{dQ}\right|_{\mathfrak{F}_T} := P(0,T)^{-1} exp\left(-\int_0^T r(X_u)du \right). \tag{2.16}$$

Let

$$\Lambda_t = E[\Lambda_T|\mathfrak{F}_t] = \frac{\exp\left\{ -\int_0^t r(X_u)du \right\} P(t,T)}{P(0,T)}. \tag{2.17}$$

Then for any $\mathfrak{F}_T$-measurable random variable φ with $E_T|\varphi| < \infty$, the Bayes' Theorem, Karatzas and Shreve (1991) (p193, Lemma 5.3), we have

$$E_T[\varphi|\mathfrak{F}_t] = \Lambda_t^{-1} E[\varphi\Lambda_T|\mathfrak{F}_t]. \tag{2.18}$$

So

$$E_T[\varphi|\mathfrak{F}_t]P(t,T) = \Lambda_t^{-1} E[\varphi\Lambda_T|\mathfrak{F}_t]P(t,T). \tag{2.19}$$

Substitute (2.17) and Definition 2.3 into (2.19),

$$
\begin{aligned}
E_T[\varphi|\mathfrak{F}_t]P(t,T) &= \frac{P(0,T)}{\exp\left\{-\int\limits_0^t r(X_u)du\right\}P(t,T)} \cdot E\left[\varphi \cdot \frac{\exp\left(-\int\limits_0^T r(X_u)du\right)}{P(0,T)}\middle|\mathfrak{F}_t\right]P(t,T) \\
&= E\left[\varphi \cdot \frac{\exp\left(-\int\limits_0^T r(X_u)du\right)}{\exp\left(-\int\limits_0^t r(X_u)du\right)}\middle|\mathfrak{F}_t\right] \\
&= E\left[\varphi \cdot \exp\left(-\int_t^T r(X_u)du\right)\middle|\mathfrak{F}_t\right] \\
&= E\left[\varphi \cdot \exp\left\{-\int_t^T r(X_u^{t,X_t})du\right\}\middle|\mathfrak{F}_t\right].
\end{aligned} \tag{2.20}
$$

So

$$
\begin{aligned}
\left.\frac{\partial P(t,T,x)}{\partial x}\right|_{x=X_t} &= P(t,T)E_T\left[L(t,T,X_t)\middle|\mathfrak{F}_t\right] \\
&= P(t,T)E_T\left[-\int_t^T (2cX_u^{t,x}+b)\left(\frac{\partial X_u^{t,x}}{\partial x}\right)\Big|_{x=X_t} du\middle|\mathfrak{F}_t\right].
\end{aligned} \tag{2.21}
$$

Now we want to find a Brownian Motion under the forward measure Q^T. Let $f(X_t) = \Lambda_t$. By Itô's formula, we have

$$
\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t f'(X_u)dX_u + \frac{1}{2}\int_0^t f''(X_u)d\langle X\rangle_u \\
&= f(X_0) + \int_0^t f'(X_u)\{\beta(\alpha - X_u)du + \sigma dW_u\} + \frac{1}{2}\int_0^t f''(X_u)\sigma^2 du \\
&= f(X_0) + \int_0^t \{f'(X_u)[\beta(\alpha - X_u)] + \frac{1}{2}f''(X_u)\sigma^2\}du + \int_0^t f'(X_u)\sigma dW_u.
\end{aligned}
$$

By Definition 2.3, Λ_t is a martingale in the probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t, t \geq 0\}, Q)$.

Then we have

$$
\int_0^t \{f'(X_u)[\beta(\alpha - X_u)] + \frac{1}{2}f''(X_u)\sigma^2\}du = 0. \tag{2.22}
$$

So

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)\sigma dW_u, \tag{2.23}$$

where

$$\begin{aligned} f'(X_t) &= \frac{\exp\left\{-\int_0^t r(X_u)du\right\}}{P(0,T)} \cdot \frac{\partial P(t,T,x)}{\partial x}\Big|_{x=X_t} \\ &= \frac{\exp\left\{-\int_0^t r(X_u)du\right\}}{P(0,T)} \cdot P(t,T)E_T\left[L(t,T,X_t)\Big|\mathfrak{F}_t\right] \\ &= f(X_t)E_T\left[L(t,T,X_t)\Big|\mathfrak{F}_t\right]. \end{aligned} \tag{2.24}$$

Define

$$\Theta_u = -\sigma E_T\left[L(u,T,X_u)\Big|\mathfrak{F}_u\right]. \tag{2.25}$$

Then

$$\Lambda_t = \Lambda_0 - \int_0^t \Theta_u \Lambda_u dW_u. \tag{2.26}$$

By Girsanov's Theorem,

$$W_t^T = W_t + \int_0^t \Theta_u du \tag{2.27}$$

is a Brownian Motion with respect to the forward measure Q^T. The dynamics of $X_s^{t,x}$ under the forward measure becomes

$$X_s^{t,x} = x + \int_t^s \{\beta(\alpha - X_v^{t,x}) - \sigma\Theta_v\}dv + \sigma \int_t^s dW_v^T. \tag{2.28}$$

Consider this expectation $E_T\left[\int_t^T X_u^{t,x} \frac{\partial X_u^{t,X_t}}{\partial x}\Big|_{x=X_t} du\Big|\mathfrak{F}_t\right]$. We want to construct an ODE of it in the following discussion. Define

$$D_{ts}(x) = \frac{\partial X_s^{t,x}}{\partial x}. \tag{2.29}$$

Since our factor process X_t is Gaussian, differentiating equation (2.28) with respect to x gives that $D_{ts}(x)$ satisfies, by Protter (2005) (Theorem 39, p 250),

$$D_{ts}(x) = 1 - \beta \int_t^s D_{tv}(x)dv. \tag{2.30}$$

which can be solved independent of x. The solution to (2.30) is

$$D_{ts}(x) = e^{-\beta(s-t)}.$$

Applying Itô's product rule using the dynamics of $D_{ts}(x)$ given by (2.30) and $X_s^{t,x}$ given by (2.28), since $D_{ts}(x)$ is of finite variation, we have

$$\begin{aligned} X_s^{t,x} D_{ts}(x) &= x + \int_t^s D_{tv}(x) dX_v^{t,x} + \int_t^s X_v^{t,x} dD_{tv}(x) \\ &= x + \alpha\beta \int_t^s D_{tv}(x)dv - 2\beta \int_t^s X_v^{t,x} D_{tv}(x)dv + \sigma \int_t^s D_{tv}(x)dW_v^T \\ &\quad - \sigma^2 \int_t^s D_{tv}(x) E_T\left[\int_v^T (2cX_{v_1}^{v,x} + b) D_{vv_1}(x)\Big|_{x=X_v} dv_1 \Big| \mathfrak{F}_v \right] dv. \end{aligned} \tag{2.31}$$

Take the expectation of equation (2.31) under the forward measure Q^T,

$$\begin{aligned} E_T[X_s^{t,X_t} D_{ts}(X_t)|\mathfrak{F}_t] &= X_t + \alpha\beta \int_t^s E_T[D_{tv}(X_t)|\mathfrak{F}_t]dv - 2\beta \int_t^s E_T[X_v^{t,X_t} D_{tv}(X_t)|\mathfrak{F}_t]dv \\ &\quad - \sigma^2 \int_t^s \int_v^T E_T[D_{tv}(X_t) E_T[(2cX_{v_1}^{v,X_v} + b) D_{vv_1}(X_v) dv_1 |\mathfrak{F}_v]|\mathfrak{F}_t]dv \end{aligned}$$

$$\tag{2.32}$$

By the tower property of conditional expectation, since $t \leq v \leq s \leq T$, in equation (2.32), the conditional expectation becomes

$$\begin{aligned} & E_T[D_{tv}(X_t) E_T[(2cX_{v_1}^{v,X_v} + b) D_{vv_1}(X_v)|\mathfrak{F}_v]|\mathfrak{F}_t] \\ =& E_T[D_{tv}(X_t)(2cX_{v_1}^{v,X_v} + b) D_{vv_1}(X_v)|\mathfrak{F}_t] \\ =& 2cE_T[D_{tv}(X_t) X_{v_1}^{v,X_v} D_{vv_1}(X_v)|\mathfrak{F}_t] + bE_T[D_{tv}(X_t) D_{vv_1}(X_v)|\mathfrak{F}_t] \end{aligned} \tag{2.33}$$

By the flow property we have, for $t \le v \le v_1 \le T$,

$$X_{v_1}^{v,X_v} = X_{v_1}^{v,X_v^{t,X_t}} = X_{v_1}^{t,X_t}. \tag{2.34}$$

Then applying the chain rule we find

$$D_{tv}(X_t)D_{vv_1}(X_v) = D_{tv_1}(X_t). \tag{2.35}$$

Then, substitute (2.33) and (2.35) into (2.32) we find

$$\begin{aligned} E_T[X_s^{t,X_t}D_{ts}(X_t)|\mathfrak{F}_t] = X_t + \alpha\beta \int_t^s E_T[D_{tv}(X_t)|\mathfrak{F}_t]dv - 2\beta \int_t^s E_T[X_v^{t,X_t}D_{tv}(X_t)|\mathfrak{F}_t]dv \\ - \sigma^2 b \int_t^s \int_v^T E_T[D_{tv_1}(X_t)|\mathfrak{F}_t]dv_1 dv \\ - 2\sigma^2 c \int_t^s \int_v^T E_T[X_{v_1}^{t,X_t}D_{tv_1}(X_t)|\mathfrak{F}_t]dv_1 dv. \end{aligned} \tag{2.36}$$

Define

$$g(s) = E_T[X_s^{t,X_t}D_{ts}(X_t)|\mathfrak{F}_t]. \tag{2.37}$$

Substituting (2.30) in (2.36), we have an ODE

$$\begin{aligned} g(s) = X_t + \alpha\beta \int_t^s e^{-\beta(v-t)}dv - 2\beta \int_t^s g(v)dv \\ - \sigma^2 b \int_t^s \int_v^T e^{-\beta(v_1-t)}dv_1 dv - 2\sigma^2 c \int_t^s \int_v^T g(v_1)dv_1 dv. \end{aligned} \tag{2.38}$$

Differentiating equation (2.38) with respect to s, we obtain the following ODE with boundary conditions

$$\begin{cases} g''(s) = -\alpha\beta^2 e^{-\beta(s-t)} - 2\beta g'(s) + b\sigma^2 e^{-\beta(s-t)} + 2c\sigma^2 g(s) \\ g(t) = X_t \\ g'(t) = \alpha\beta - 2\beta X_t - \sigma^2 b \int_t^T e^{-\beta(v_1-t)}dv_1 - 2c\sigma^2 \int_t^T g(v_1)dv_1. \end{cases} \tag{2.39}$$

Equation (2.39) has general solution

$$g(s) = c_1 e^{(-\beta+\sqrt{\beta^2+2c\sigma^2})s} + c_2 e^{(-\beta-\sqrt{\beta^2+2c\sigma^2})s} + \frac{b\sigma^2 - \alpha\beta^2}{-\beta^2 - 2c\sigma^2} \cdot e^{-\beta(s-t)}. \tag{2.40}$$

Applying the boundary conditions allows us to show that

$$\begin{aligned}
c_1 = & \Big\{ \alpha\beta + \frac{\sigma^2 b\eta^2 - 2c\sigma^2(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1) + X_t(-\beta+\eta)e^{(-\beta-\eta)(T-t)} \\
& + \frac{(b\sigma^2 - \alpha\beta^2)\beta}{\eta^2} + \frac{(b\sigma^2 - \alpha\beta^2)(-\beta+\eta)}{\eta^2} \cdot e^{(-\beta-\eta)(T-t)} \Big\} \Big/ \Big\{ e^{(-\beta+\eta)T} \\
& \times \Big[\beta + \eta + (-\beta+\eta)e^{2\eta(t-T)} \Big] \Big\}, \\
c_2 = & \Big\{ \alpha\beta + \frac{\sigma^2 b\eta^2 - 2c\sigma^2(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1) + X_t(-\beta-\eta)e^{(-\beta+\eta)(T-t)} \\
& + \frac{(b\sigma^2 - \alpha\beta^2)\beta}{\eta^2} + \frac{(b\sigma^2 - \alpha\beta^2)(-\beta-\eta)}{\eta^2} \cdot e^{(-\beta+\eta)(T-t)} \Big\} \Big/ \Big\{ e^{(-\beta-\eta)T} \\
& \times \Big[\beta - \eta + (-\beta-\eta)e^{2\eta(T-t)} \Big] \Big\},
\end{aligned}$$

where

$$\eta = \sqrt{\beta^2 + 2c\sigma^2}.$$

Therefore $E_T[X_s^{t,X_t} D_{ts}(X_t)|\mathfrak{F}_t]$ is a deterministic function of X_t. Hence, by the Fubini's Theorem, we have

$$\begin{aligned}
& E_T\Big[L(t,T,X_t) \Big| \mathfrak{F}_t \Big] \\
= & -\int_t^T E_T\Big[(2cX_u^{t,X_t} + b) D_{tu}(X_t) du \Big| \mathfrak{F}_t \Big] \\
= & -b\int_t^T E_T[D_{tu}(X_t)|\mathfrak{F}_t] du - 2c\int_t^T g(u) du \\
= & \frac{c_1(-\beta-\eta)}{\sigma^2} \cdot (e^{(-\beta+\eta)T} - e^{(-\beta+\eta)t}) + \frac{c_2(-\beta+\eta)}{\sigma^2} \cdot (e^{(-\beta-\eta)T} - e^{(-\beta-\eta)t}) \\
& + \frac{b\eta^2 - 2c(b\sigma^2 - \alpha\beta^2)}{\beta\eta^2} \cdot (e^{-\beta(T-t)} - 1).
\end{aligned} \tag{2.41}$$

Define

$$A(\tau) = \frac{2c(e^{2\eta\tau}-1)}{\beta-\eta+(-\beta-\eta)e^{2\eta\tau}}, \tag{2.42}$$

$$\begin{aligned}B(\tau) = &\left\{\left[\alpha\beta + \frac{(b\sigma^2-\alpha\beta^2)\beta}{\eta^2}\right]\left[(-\beta-\eta)(e^{2\eta\tau}-1)-2\eta\right]\right.\\ &\left.+\frac{(b\sigma^2-\alpha\beta^2)(\beta^2-\eta^2)}{\eta^2}\cdot(e^{2\eta\tau}-1)+2\eta e^{\eta\tau}\cdot\frac{b\sigma^2\eta^2-2c\sigma^2(b\sigma^2-\alpha\beta^2)}{\beta\eta^2}\right\}\\ &\div\left\{\sigma^2\left[(\beta+\eta)(e^{2\eta\tau}-1)+2\eta\right]\right\},\end{aligned} \tag{2.43}$$

where

$$\tau = T-t.$$

Substitute (2.41) in (2.21).

$$\left.\frac{\partial P(t,T,x)}{\partial x}\right|_{x=X_t} = P(t,T)\Big\{A(\tau)X_t + B(\tau)\Big\}. \tag{2.44}$$

Let

$$P(t,T,X_t) = \exp\Big\{\frac{1}{2}\cdot A(\tau)X_t^2 + B(\tau)X_t + C(T,t)\Big\}, \tag{2.45}$$

where $C(T,t)$ is a differentiable function from $\mathbf{R}^2$ to $\mathbf{R}$. By Feynman-Kac Theorem Karatzas and Shreve (1991), $P(t,T,x)$ defined in (2.13) satisfies the Cauchy problem

$$\begin{aligned}\frac{\partial P(t,T,x)}{\partial t} &+ \beta(\alpha-x)\cdot\frac{\partial P(t,T,x)}{\partial x}\\ &+\frac{1}{2}\sigma^2\cdot\frac{\partial^2 P(t,T,x)}{(\partial x)^2} - (cx^2+bx+a)P(t,T,x) = 0,\end{aligned} \tag{2.46}$$

and the boundary condition

$$P(T,T,x) = 1.$$

Substituting (2.45) in ODE (2.46) and dividing the whole equation by $P(t,T,X_t)$, we have

$$\begin{aligned}&\frac{1}{2}X_t^2\frac{\partial A(\tau)}{\partial t} + X_t\cdot\frac{\partial B(\tau)}{\partial t} + \frac{\partial C(T,t)}{\partial t} + (\beta\alpha-\beta X_t)[A(\tau)X_t+B(\tau)]\\ &+\frac{1}{2}\sigma^2\Big[(A(\tau)X_t+B(\tau))^2 + A(\tau)\Big] = cX_t^2 + bX_t + a\end{aligned} \tag{2.47}$$

Comparing the coefficients on both sides of equation (2.47), we get an ODE of $C(T,t)$

$$\begin{cases} \dfrac{\partial C(T,t)}{\partial t} + \beta\alpha B(\tau) + \dfrac{1}{2}\sigma^2 B(\tau)^2 + \dfrac{1}{2}\sigma^2 A(\tau) - a = 0 \\ C(T,T) = 0. \end{cases} \tag{2.48}$$

Solving this ODE, we find that $C(T,t)$ is a function of τ. We denote $C(\tau) = C(T,t)$.

$$\begin{aligned} C(\tau) = & \left[\frac{b^2\sigma^2 - 2\alpha\beta^2 b - 2\alpha^2\beta^2 c}{2\eta^2}\right]\tau + \frac{1}{2}\log\left\{\frac{2\eta e^{(\eta+\beta)\tau}}{(\beta+\eta)e^{2\eta\tau} + \eta - \beta}\right\} \\ & + \frac{2c(b\sigma^2 - \alpha\beta^2)^2 + 2\beta(b+2c\alpha)(b\sigma^2 - \alpha\beta^2)(\beta+\eta)e^{\eta\tau} - \beta^2\sigma^2(b+2c\alpha)^2}{\eta^3(\beta+\eta)\left[(\beta+\eta)e^{2\eta\tau} + \eta - \beta\right]} \\ & + \frac{\beta^2\sigma^2(b+2c\alpha)^2 - 2c(b\sigma^2 - \alpha\beta^2)^2 - 2\beta(b+2c\alpha)(b\sigma^2 - \alpha\beta^2)(\beta+\eta)}{2\eta^4(\beta+\eta)} - a\tau. \end{aligned} \tag{2.49}$$

Summarizing the material of this section we have the following Theorem.

Theorem 2.2. *For $t \in [0,T]$ and for all $x \in \mathbf{R}$,*

$$P(t,T,x) = exp\left\{\frac{1}{2}\cdot A(\tau)x^2 + B(\tau)x + C(T,t)\right\},$$

where $A(\tau)$, $B(\tau)$ and $C(\tau)$ is given by (2.42), (2.43) and (2.49), respectively.

Corollary 2.1. *If the factor process is given by (2.8) and the short rate is represented by the function (2.9), the zero-coupon bond price is*

$$P(t,T) = exp\left\{\frac{1}{2}\cdot A(\tau)X_t^2 + B(\tau)X_t + C(T,t)\right\},$$

where $A(\tau)$, $B(\tau)$ and $C(\tau)$ is given by (2.42), (2.43) and (2.49), respectively.

This agrees with the results in Nawalkha et al. (2007)(p 487-488).

2.3 Two Dimensional Case

Next, we discuss the application of the stochastic flows approach to the two-dimensional quadratic term structure models. Consider the Model B of two dimension. The factor process is given by

$$dX_t = \begin{bmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{bmatrix} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_t^{(1)} \\ X_t^{(2)} \end{bmatrix} + \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix} \right\} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix}. \quad (2.50)$$

The short rate process is

$$r(X_t) = CX_tX_t'C' + RX_t + k,$$

where

$$C = \begin{bmatrix} c_1, & c_2 \end{bmatrix}, R = \begin{bmatrix} r_1, & r_2 \end{bmatrix}$$

and $k \in \mathbf{R}$. Recall that $P(t,T,x)$ is defined in (2.13). In the two-dimensional case

$$x' = \begin{bmatrix} x_1, & x_2 \end{bmatrix}$$

$$\left.\frac{\partial P(t,T,x)}{\partial x}\right|_{x=X_t} = \begin{bmatrix} \left.\frac{\partial P(t,T,x)}{\partial x_1}\right|_{x=X_t} \\ \left.\frac{\partial P(t,T,x)}{\partial x_2}\right|_{x=X_t} \end{bmatrix}$$

$$= \begin{bmatrix} P(t,T)E_T\left[-\int_t^T \left\{ 2 \begin{bmatrix} X_{(1),u}^{t,X_t} \\ X_{(2),u}^{t,X_t} \end{bmatrix}' C'C + R \right\} \begin{bmatrix} \frac{\partial X_{(1),u}^{t,X_t}}{\partial x_1} \\ \frac{\partial X_{(2),u}^{t,X_t}}{\partial x_1} \end{bmatrix} du \right] \\ P(t,T)E_T\left[-\int_t^T \left\{ 2 \begin{bmatrix} X_{(1),u}^{t,X_t} \\ X_{(2),u}^{t,X_t} \end{bmatrix}' C'C + R \right\} \begin{bmatrix} \frac{\partial X_{(1),u}^{t,X_t}}{\partial x_2} \\ \frac{\partial X_{(2),u}^{t,X_t}}{\partial x_2} \end{bmatrix} du \right] \end{bmatrix} \quad (2.51)$$

Following similar steps to the one-dimensional case, we prefer to work under the forward measure Q^T. The Brownian motion under the forward measure is given by

$$W_t^T = \begin{bmatrix} W_t^{(1)T} \\ W_t^{(2)T} \end{bmatrix} = \int_0^t \sigma' \begin{bmatrix} \Theta_u^{(1)} \\ \Theta_u^{(2)} \end{bmatrix} du + W_t, \quad (2.52)$$

where

$$\Theta_u^{(1)} = E_T\left[\int_u^T \left\{2\begin{bmatrix} X_{(1),v}^{u,X_u} \\ X_{(2),v}^{u,X_u}\end{bmatrix}' C'C + R\right\} \begin{bmatrix} \frac{\partial X_{(1),v}^{u,X_u}}{\partial x_1} \\ \frac{\partial X_{(2),v}^{u,X_u}}{\partial x_1}\end{bmatrix} dv \middle| \mathfrak{F}_u\right], \tag{2.53}$$

$$\Theta_u^{(2)} = E_T\left[\int_u^T \left\{2\begin{bmatrix} X_{(1),v}^{u,X_u} \\ X_{(2),v}^{u,X_u}\end{bmatrix}' C'C + R\right\} \begin{bmatrix} \frac{\partial X_{(1),v}^{u,X_u}}{\partial x_2} \\ \frac{\partial X_{(2),v}^{u,X_u}}{\partial x_2}\end{bmatrix} dv \middle| \mathfrak{F}_u\right]. \tag{2.54}$$

Then under the forward measure, the dynamics of $X_s^{t,x}$ are

$$X_s^{t,x} = x + \int_t^s \left\{AX_v^{t,x} + B - \sigma\sigma' \begin{bmatrix} \Theta_v^{(1)} \\ \Theta_v^{(2)} \end{bmatrix}\right\} dv + \int_t^s \sigma dW_v^T. \tag{2.55}$$

We also know that $\frac{\partial X_s^{t,x}}{\partial x}$ is deterministic

$$\frac{\partial X_s^{t,x}}{\partial x} = D_{tu}(x) = e^{A(u-t)} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

And the dynamics of $\frac{\partial X_s^{t,x}}{\partial x}$ are

$$\begin{aligned} D_{ts}(x) &= \begin{bmatrix} \frac{\partial X_{(1)s}^{t,x}}{\partial x_1} & \frac{\partial X_{(1)s}^{t,x}}{\partial x_2} \\ \frac{\partial X_{(2)s}^{t,x}}{\partial x_1} & \frac{\partial X_{(2)s}^{t,x}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} D_{ts}^{11}(x) & D_{ts}^{12}(x) \\ D_{ts}^{21}(x) & D_{ts}^{22}(x) \end{bmatrix} \\ &= I + \int_t^s AD_{tu}(x)du \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \int_t^s \begin{bmatrix} a_{11}D_{tu}^{11}(x) + a_{12}D_{tu}^{21} & a_{11}D_{tu}^{12}(x) + a_{12}D_{tu}^{22} \\ a_{21}D_{tu}^{11}(x) + a_{22}D_{tu}^{21} & a_{21}D_{tu}^{12}(x) + a_{22}D_{tu}^{22} \end{bmatrix} du \end{aligned} \tag{2.56}$$

Now consider the expectations

$$E_T\left[\int_t^T \begin{bmatrix} X_{(1)u}^{t,x} \\ X_{(2)u}^{t,x} \end{bmatrix}' C'C \begin{bmatrix} D_{tu}^{11}(x) \\ D_{tu}^{21}(x) \end{bmatrix} du \middle| \mathfrak{F}_t\right], \tag{2.57}$$

$$E_T\left[\int_t^T \begin{bmatrix} X_{(1)u}^{t,x} \\ X_{(2)u}^{t,x} \end{bmatrix}' C'C \begin{bmatrix} D_{tu}^{12}(x) \\ D_{tu}^{22}(x) \end{bmatrix} du \middle| \mathfrak{F}_t\right]. \tag{2.58}$$

We want to show that (2.57) and (2.58), are deterministic functions of X_t. From (2.55), we have, evaluated at $x = X_t$,

$$c_1 X^{t,x}_{(1)u} + c_2 X^{t,x}_{(2)u} = c_1 x_1 + c_2 x_2 + \int_t^u \Big\{ (c_1 a_{11} + c_2 a_{21}) X^{t,x}_{(1)v} + (c_1 a_{12} + c_2 a_{22}) X^{t,x}_{(2)v} + c_1 \tilde{b}_1$$
$$+ c_2 \tilde{b}_2 \Big\} dv + \int_t^u (c_1 \sigma_{11} + c_2 \sigma_{21}) dW^{(1)T}_v + \int_t^u (c_1 \sigma_{12} + c_2 \sigma_{22}) dW^{(2)T}_v$$
$$- \int_t^u \Big\{ [c_1(\sigma_{11}^2 + \sigma_{12}^2) + c_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})] \Theta^{(1)}_v$$
$$+ [c_1(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) + c_2(\sigma_{21}^2 + \sigma_{22}^2)] \Theta^{(2)}_v \Big\} dv, \tag{2.59}$$

Similarly, we could get the dynamics of $c_1 D^{11}_{tu}(x) + c_2 D^{21}_{tu}$ and $c_1 D^{12}_{tu}(x) + c_2 D^{22}_{tu}$,

$$c_1 D^{11}_{tu}(x) + c_2 D^{21}_{tu} = c_1 + \int_t^u \{ (c_1 a_{11} + c_2 a_{21}) D^{11}_{tv}(x) + (c_1 a_{12} + c_2 a_{22}) D^{21}_{tv}(x) \} dv \tag{2.60}$$

$$c_1 D^{12}_{tu}(x) + c_2 D^{22}_{tu} = c_1 + \int_t^u \{ (c_1 a_{11} + c_2 a_{21}) D^{12}_{tv}(x) + (c_1 a_{12} + c_2 a_{22}) D^{22}_{tv}(x) \} dv \tag{2.61}$$

So by Itô's product rule, we have

$$\begin{bmatrix} X^{t,x}_{(1)u} \\ X^{t,x}_{(2)u} \end{bmatrix}' C'C \begin{bmatrix} D^{11}_{tu}(x) \\ D^{21}_{tu}(x) \end{bmatrix}$$

$$=(c_1 X^{t,x}_{(1)u} + c_2 X^{t,x}_{(2)u})(c_1 D^{11}_{tu}(x) + c_2 D^{21}_{tu})$$

$$=c_1^2 x_1 + c_1c_2x_2 + \int_t^u (c_1 D^{11}_{tv}(x) + c_2 D^{21}_{tv}(x))\big[(c_1a_{11} + c_2a_{21})X^{t,x}_{(1)v} + (c_1a_{12} + c_2a_{22})X^{t,x}_{(2)v}$$

$$+ c_1\tilde{b}_1 + c_2\tilde{b}_2\big]dv + \int_t^u (c_1 D^{11}_{tv}(x) + c_2 D^{21}_{tv}(x))(c_1\sigma_{11} + c_2\sigma_{21})dW^{(1)T}_v$$

$$+ \int_t^u (c_1 D^{11}_{tv}(x) + c_2 D^{21}_{tv}(x))(c_1\sigma_{12} + c_2\sigma_{22})dW^{(2)T}_v$$

$$- \int_t^u (c_1 D^{11}_{tv}(x) + c_2 D^{21}_{tv}(x))(k_1\Theta^{(1)}_v + k_2\Theta^{(2)}_v)dv$$

$$+ \int_t^u (c_1 X^{t,x}_{(1)v} + c_2 X^{t,x}_{(2)v})\big[(c_1a_{11} + c_2a_{21})D^{11}_{tv}(x) + (c_1a_{12} + c_2a_{22})D^{21}_{tv}(x)\big]dv, \tag{2.62}$$

where

$$k_1 = c_1(\sigma_{11}^2 + \sigma_{12}^2) + c_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})$$

$$k_2 = c_1(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) + c_2(\sigma_{21}^2 + \sigma_{22}^2).$$

Consider $(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t))(k_1\Theta^{(1)}_v + k_2\Theta^{(2)}_v)$. From (2.53) and (2.54), we have

$$(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t))(k_1\Theta^{(1)}_v + k_2\Theta^{(2)}_v)$$

$$=E_T\Bigg[\int_v^T \Big\{2 \begin{bmatrix} X^{v,X_v}_{(1)v_1} \\ X^{v,X_v}_{(2)v_1} \end{bmatrix}' C'C + R\Big\}$$

$$\times \begin{bmatrix} (k_1 D^{11}_{vv_1}(X_v) + k_2 D^{12}_{vv_1}(X_v))(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t)) \\ (k_1 D^{21}_{vv_1}(X_v) + k_2 D^{22}_{vv_1}(X_v))(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t)) \end{bmatrix} dv_1 \Bigg| \mathcal{F}_v\Bigg] \tag{2.63}$$

By the flow property, we have

$$X^{t,X_t}_{(1)v_1} = X^{v,X_v}_{(1)v_1} \quad X^{t,X_t}_{(2)v_1} = X^{v,X_v}_{(2)v_1},$$

and

$$D^{11}_{tv_1}(X_t) = D^{11}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t) + D^{12}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t), \tag{2.64}$$

$$D^{21}_{tv_1}(X_t) = D^{21}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t) + D^{22}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t), \tag{2.65}$$

$$D^{12}_{tv_1}(X_t) = D^{11}_{vv_1}(X_v)\cdot D^{12}_{tv}(X_t) + D^{12}_{vv_1}(X_v)\cdot D^{22}_{tv}(X_t), \tag{2.66}$$

$$D^{22}_{tv_1}(X_t) = D^{21}_{vv_1}(X_v)\cdot D^{12}_{tv}(X_t) + D^{22}_{vv_1}(X_v)\cdot D^{22}_{tv}(X_t). \tag{2.67}$$

But in (2.63), we have

$$\begin{aligned}
&(k_1 D^{11}_{vv_1}(X_v) + k_2 D^{12}_{vv_1}(X_v))(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t))\\
=&k_1c_1 D^{11}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t) + k_1c_2 D^{11}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t) + k_2c_1 D^{12}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t)\\
&+ k_2c_2 D^{12}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t),
\end{aligned} \tag{2.68}$$

and

$$\begin{aligned}
&(k_1 D^{21}_{vv_1}(X_v) + k_2 D^{22}_{vv_1}(X_v))(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t))\\
=&k_1c_1 D^{21}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t) + k_1c_2 D^{21}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t) + k_2c_1 D^{22}_{vv_1}(X_v)\cdot D^{11}_{tv}(X_t)\\
&+ k_2c_2 D^{22}_{vv_1}(X_v)\cdot D^{21}_{tv}(X_t).
\end{aligned} \tag{2.69}$$

In order to apply the flow property to (2.68) and (2.69), we need to add the following restrictions

$$k_1c_1 = k_2c_2 = k, \tag{2.70}$$

$$k_1c_2 = k_2c_1 = 0, \tag{2.71}$$

where $k \in \mathbf{R}$ is any constant. So

$$\begin{aligned}
&(c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t))(k_1\Theta^{(1)}_v + k_2\Theta^{(2)}_v)\\
=&E_T\left[\int_v^T k\left\{2\begin{bmatrix} X^{t,X_t}_{(1)v_1}\\ X^{t,X_t}_{(2)v_1}\end{bmatrix}' C'C + R\right\}\begin{bmatrix} D^{11}_{tv_1}(X_t)\\ D^{21}_{tv_1}(X_t)\end{bmatrix} dv_1 \middle| \mathcal{F}_v\right].
\end{aligned} \tag{2.72}$$

Moreover, we have to put some restrictions on the coefficients in the equation (2.62), which is

$$m = \frac{c_1 a_{11} + c_2 a_{21}}{c_1} = \frac{c_1 a_{12} + c_2 a_{22}}{c_2}. \tag{2.73}$$

With the restrictions (2.70), (2.71) and (2.73), we can get an ODE for

$$E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)v_1} \\ X^{t,X_t}_{(2)v_1} \end{bmatrix}' C'C \begin{bmatrix} D^{11}_{tv_1}(X_t) \\ D^{21}_{tv_1}(X_t) \end{bmatrix} \middle| \mathfrak{F}_v\right] \tag{2.74}$$

from equation (2.62). That is

$$\begin{aligned}
&E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)u} \\ X^{t,X_t}_{(2)u} \end{bmatrix}' C'C \begin{bmatrix} D^{11}_{tu}(X_t) \\ D^{21}_{tu}(X_t) \end{bmatrix} \middle| \mathfrak{F}_t\right] = c_1^2 x_1 + c_1 c_2 x_2 \\
&+ 2m \int_t^u E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)v} \\ X^{t,X_t}_{(2)v} \end{bmatrix}' C'C \begin{bmatrix} D^{(11)}_{tv}(X_t) \\ D^{21}_{tv}(X_t) \end{bmatrix} \middle| \mathfrak{F}_t\right] dv \\
&+ \int_t^u \left[c_1 D^{11}_{tv}(X_t) + c_2 D^{21}_{tv}(X_t)\right](c_1 \tilde{b}_1 + c_2 \tilde{b}_2) dv \\
&- 2k \int_t^u \int_v^T E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)v_1} \\ X^{t,X_t}_{(2)v_1} \end{bmatrix}' C'C \begin{bmatrix} D^{(11)}_{tv_1}(X_t) \\ D^{21}_{tv_1}(X_t) \end{bmatrix} \middle| \mathfrak{F}_t\right] dv_1 dv - k \int_t^u \int_v^T R \begin{bmatrix} D^{11}_{tv_1}(X_t) \\ D^{21}_{tv_1}(X_t) \end{bmatrix} dv_1 dv.
\end{aligned}$$

(2.75)

Similarly, we have

$$
\begin{bmatrix} X^{t,x}_{(1)u} \\ X^{t,x}_{(2)u} \end{bmatrix}' C'C \begin{bmatrix} D^{12}_{tu}(x) \\ D^{22}_{tu}(x) \end{bmatrix}
$$

$$
\begin{aligned}
=&(c_1 X^{t,x}_{(1)u} + c_2 X^{t,x}_{(2)u})(c_1 D^{12}_{tu}(x) + c_2 D^{22}_{tu}) \\
=&c_1 c_2 x_1 + c_2^2 x_2 + \int_t^u (c_1 D^{12}_{tv}(x) + c_2 D^{22}_{tv}(x)) \big[(c_1 a_{11} + c_2 a_{21}) X^{t,x}_{(1)v} + (c_1 a_{12} + c_2 a_{22}) X^{t,x}_{(2)v} \\
&+ c_1 \tilde{b}_1 + c_2 \tilde{b}_2\big] dv + \int_t^u (c_1 D^{12}_{tv}(x) + c_2 D^{22}_{tv}(x))(c_1 \sigma_{11} + c_2 \sigma_{21}) dW^{(1)T}_v \\
&+ \int_t^u (c_1 D^{12}_{tv}(x) + c_2 D^{22}_{tv}(x))(c_1 \sigma_{12} + c_2 \sigma_{22}) dW^{(2)T}_v \\
&- \int_t^u (c_1 D^{12}_{tv}(x) + c_2 D^{22}_{tv}(x))(k_1 \Theta^{(1)}_v + k_2 \Theta^{(2)}_v) dv \\
&+ \int_t^u (c_1 X^{t,x}_{(1)v} + c_2 X^{t,x}_{(2)v}) \big[(c_1 a_{11} + c_2 a_{21}) D^{12}_{tv}(x) + (c_1 a_{12} + c_2 a_{22}) D^{22}_{tv}(x)\big] dv,
\end{aligned} \tag{2.76}
$$

and

$$
\begin{aligned}
&(c_1 D^{12}_{tv}(X_t) + c_2 D^{22}_{tv}(X_t))(k_1 \Theta^{(1)}_v + k_2 \Theta^{(2)}_v) \\
=&E_T\Bigg[\int_v^T \Big\{2 \begin{bmatrix} X^{v,X_v}_{(1)v_1} \\ X^{v,X_v}_{(2)v_1} \end{bmatrix}' C'C + R\Big\} \\
&\times \begin{bmatrix} (k_1 D^{11}_{vv_1}(X_v) + k_2 D^{12}_{vv_1}(X_v))(c_1 D^{12}_{tv}(X_t) + c_2 D^{22}_{tv}(X_t)) \\ (k_1 D^{21}_{vv_1}(X_v) + k_2 D^{22}_{vv_1}(X_v))(c_1 D^{12}_{tv}(X_t) + c_2 D^{22}_{tv}(X_t)) \end{bmatrix} dv_1 \Bigg| \mathcal{F}_v \Bigg].
\end{aligned} \tag{2.77}
$$

We need the following restriction to implement the flows method.

$$
k_1 c_1 = k_2 c_2 = k \tag{2.78}
$$

$$
k_1 c_2 = k_2 c_1 = 0, \tag{2.79}
$$

where $k \in \mathbf{R}$ is any constant. The ODEs is

$$\begin{aligned}
&E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)u} \\ X^{t,X_t}_{(2)u} \end{bmatrix}' C'C \begin{bmatrix} D^{12}_{tu}(X_t) \\ D^{22}_{tu}(X_t) \end{bmatrix} \Bigg| \mathfrak{F}_t\right] = c_1c_2x_1 + c_2^2x_2 \\
&+ 2m\int_t^u E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)v} \\ X^{t,X_t}_{(2)v} \end{bmatrix}' C'C \begin{bmatrix} D^{(12)}_{tv}(X_t) \\ D^{22}_{tv}(X_t) \end{bmatrix} \Bigg| \mathfrak{F}_t\right] dv \\
&+ \int_t^u \left[c_1D^{12}_{tv}(X_t) + c_2D^{22}_{tv}(X_t)\right](c_1\tilde{b}_1 + c_2\tilde{b}_2)dv \\
&- 2k\int_t^u\int_v^T E_T\left[\begin{bmatrix} X^{t,X_t}_{(1)v_1} \\ X^{t,X_t}_{(2)v_1} \end{bmatrix}' C'C \begin{bmatrix} D^{(12)}_{tv_1}(X_t) \\ D^{22}_{tv_1}(X_t) \end{bmatrix} \Bigg| \mathfrak{F}_t\right] dv_1dv - k\int_t^u\int_v^T R\begin{bmatrix} D^{12}_{tv_1}(X_t) \\ D^{22}_{tv_1}(X_t) \end{bmatrix} dv_1dv.
\end{aligned} \tag{2.80}$$

Hence, we can get two ODEs only if we put the following restrictions on our model.

$$\begin{cases} k_1c_1 = k_2c_2 \\ k_1c_2 = k_2c_1 = 0 \\ m = \dfrac{c_1a_{11} + c_2a_{21}}{c_1} = \dfrac{c_1a_{12} + c_2a_{22}}{c_2}. \end{cases} \tag{2.81}$$

Simplifying (2.81), we have

$$\begin{cases} c_1^2(\sigma_{11}^2 + \sigma_{12}^2) = c_2^2(\sigma_{21}^2 + \sigma_{22}^2) \\ c_1(\sigma_{11}^2 + \sigma_{12}^2) + c_2(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) = 0 \\ c_1(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) + c_2(\sigma_{21}^2 + \sigma_{22}^2) = 0 \\ m = \dfrac{c_1a_{11} + c_2a_{21}}{c_1} = \dfrac{c_1a_{12} + c_2a_{22}}{c_2}. \end{cases} \tag{2.82}$$

That is, with the restriction (2.82) we obtain the ODEs given by (2.75) and (2.80) which allow us to show the zero-coupon bond price is of the form

$$\frac{\partial P(t,T,x)}{\partial x}\bigg|_{x=X_t} = \begin{bmatrix} P(t,T)l_1(t) \\ P(t,T)l_2(t) \end{bmatrix},$$

where $l_1(t)$ and $l_2(t)$ satisfy ODEs (2.75) and (2.80) respectively. The Feynman-Kac formula can then be applied. However, the restrictions (2.82) are quite strong and as such we turn our attention to the FBSDE method where such restrictions are not needed.

Chapter 3

FBSDEs Method

From the previous chapter we see that the stochastic flows approach is relatively straightforward in the one dimensional case. However, in higher dimensional cases, the stochastic flows method requires the addition of some parametric restrictions to our models. In this chapter, we implement the forward-backward SDEs approach to Model A. The result shows that the FBSDE approach, first introduced by Hyndman (2007b), can solve our quadratic term structure model A without restrictions. We first derive the FBSDE satisfied by the factors process and zero coupon bond price.

3.1 Connections between QTSMs and FBSDEs

In this section we discuss the connection between our problem and solving a FBSDE. As we introduced in Chapter 2, the factor process is given by equation (2.2)

$$dX_t = \{AX_t + B\}dt + \sigma dW_t, \tag{3.1}$$

and the short rate process is given by equation (2.3)

$$r(X_t) = X_t'\Gamma X_t + RX_t + k. \tag{3.2}$$

Define

$$H_s = \exp\Big(-\int_0^s r(X_u)du\Big) \text{ and} \tag{3.3}$$

$$V_s = E_Q\Big[\exp\Big(-\int_0^T r(X_u)du\Big)|\mathfrak{F}_s\Big], \tag{3.4}$$

where $s \in [0,T]$ and $\mathfrak{F}_s$ is the right-continuous and complete filtration discussed in Chapter 2. Using Itô's formula we can show that H_s satisfies the dynamics

$$dH_s = -r(X_s)H_s ds. \tag{3.5}$$

and is of finite variation. By the definition of V_s in equation (3.4), we know that V_s is a martingale with respect to the risk neutral probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t, t \geq 0\}, Q)$. By the Martingale Representation Theorem (see Shreve (2004) Theorem 5.4.2), there exists a $\mathfrak{F}_s$-adapted process $J_s = [J_s^{(1)} \cdots J_s^{(n)}]$ such that

$$V_s = V_0 + \int_0^s [J_s^{(1)} \cdots J_s^{(n)}]dW_u. \tag{3.6}$$

With $Y_s = V_s/H_s$ we have from equations (3.3) and (3.4)

$$\begin{aligned} Y_s = V_s/H_s &= \frac{E_Q\Big[\exp\Big(-\int_0^T r(X_u)du\Big)|\mathfrak{F}_s\Big]}{\exp\Big(-\int_0^s r(X_u)du\Big)} \\ &= E_Q\Big[\exp\Big(-\int_s^T r_u du\Big)|\mathfrak{F}_s\Big] = P(s,T). \end{aligned} \tag{3.7}$$

So Y_s is the price of zero-coupon bond with maturity T at time s. To get the dynamics of Y_s, consider the function $f(x,y) = \frac{x}{y}$. Apply Itô's formula to $f(x,y)$ by using the dynamics of H_s in (3.5) and V_s in equation (3.6),

$$\begin{aligned}
Y_s =& Y_0 + \int_0^s H_u^{-1} dV_u + \int_0^s (-V_u H_u^{-2}) dH_s + \frac{1}{2}\int_0^s 2V_u H_u^{-3} d\langle H\rangle_u \\
&+ \int_0^s (-H_u^{-2}) d\langle V, H\rangle_u \\
=& Y_0 + \int_0^s V_u H_u^{-2} r(X_u) H_u du + \int_0^s H_u^{-1} J_u dW_u \\
=& Y_0 + \int_0^s Y_u r(X_u) du + \int_0^s Z_u dW_u, \qquad (3.8)
\end{aligned}$$

where

$$Z_u = \frac{J_u}{H_u} = [\frac{J_u^{(1)}}{H_u} \cdots \frac{J_u^{(n)}}{H_u}]. \qquad (3.9)$$

Subtracting the dynamics of Y_T from (3.8) we have

$$Y_s = Y_T - \int_s^T Y_u r(X_u) du - \int_s^T Z_u dW_u. \qquad (3.10)$$

Since Y_T is the zero-coupon bond price with maturity T at time T, we have from (2.13) $Y_T = 1$. Therefore, in the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$, X_s and Y_s satisfy the system

$$\begin{cases} X_s = X_0 + \int_0^s (AX_u + B) du + \int_0^s \sigma dW_u \\ Y_s = Y_T - \int_s^T Y_u r(X_u) du - \int_s^T Z_u dW_u \end{cases} . \qquad (3.11)$$

for $s \in [0, T]$. Equations (3.11) constitute a forward-backward stochastic differential equation (FBSDE) with adapted solution (X_s, Y_s, Z_s), $s \in [0, T]$ as defined in Ma and Yong (1999), E.Pardoux and S.Peng (1992) and Karoui et al. (1997).

In order to simplify things, as in Hyndman (2009), we choose to solve FBSDE under the forward measure. Recall that the forward measure is defined as in Definition 2.3.

Let

$$\Lambda_t = \frac{dQ^T}{dQ}\bigg|_{\mathfrak{F}_t} = E_Q[\Lambda_T|\mathfrak{F}_t].$$

We have

$$\begin{aligned}
\Lambda_t =& E_Q[\Lambda_T|\mathfrak{F}_t] \\
=& E_Q\left[\exp\left(-\int_0^T r(X_u)du\right)P(0,T)^{-1}\middle|\mathfrak{F}_t\right] \\
=& E_Q\left[\exp\left(-\int_0^T r(X_u)du\right)\frac{H_0}{V_0}\middle|\mathfrak{F}_t\right] \\
=& V_0^{-1}E_Q\left[\exp\left(-\int_0^T r(X_u)du\right)\middle|\mathfrak{F}_t\right] = V_0^{-1}V_t.
\end{aligned} \tag{3.12}$$

Substitute the result of (3.12) into (3.6) to find

$$V_0\Lambda_s = V_0\Lambda_0 + \int_0^s J_u dW_u. \tag{3.13}$$

Dividing both sides of (3.13) by V_0 we have

$$\begin{aligned}
\Lambda_s =& \Lambda_0 + \int_0^s \frac{J_u}{V_0}dW_u \\
=& 1 + \int_0^s \frac{J_u}{V_0}\frac{H_u}{H_u}\frac{V_u}{V_u}dW_u \\
=& 1 + \int_0^s Y_u^{-1}Z_u\Lambda_u dW_u.
\end{aligned} \tag{3.14}$$

Then, by Girsanov's Theorem, we have that

$$W_t^T = W_t - \int_0^t \frac{Z'_u}{Y_u}du \tag{3.15}$$

is a standard Brownian Motion under the forward measure Q^T. Therefore, under the forward measure Q^T, the FBSDE in (3.11) becomes

$$\begin{cases} X_s = X_0 + \int_0^s (AX_u + B + \sigma\frac{Z'_u}{Y_u})du + \int_0^s \sigma dW_u^T \\ Y_s = 1 - \int_s^T \{Y_u r(X_u) + \frac{Z_u Z'_u}{Y_u}\}du - \int_s^T Z_u dW_u^T \end{cases} \tag{3.16}$$

for $s \in [0, T]$. In the QTSM, with $r(X_t)$ given by (2.3), we have

$$\begin{cases} X_s = X_0 + \int_0^s (AX_u + B + \sigma\frac{Z'_u}{Y_u})du + \int_0^s \sigma dW_u^T \\ Y_s = 1 - \int_s^T \{Y_u[X'_u \Gamma X_u + RX_u + k] + \frac{Z_u Z'_u}{Y_u}\}du - \int_s^T Z_u dW_u^T \end{cases} \tag{3.17}$$

for $s \in [0, T]$. In the next section we give a heuristic derivation of an explicit solution for the FBSDE (3.17).

3.2 Heuristic Derivation of Explicit Solution

Following Hyndman (2007b) and Hyndman (2009), we adapt the technique for linear FBSDEs from Ma and Yong (1999) to solve our FBSDE (3.17). Consider the dynamics of $\log Y_s$. Apply Itô's formula to the function $f(x) = \log x$ using the dynamics of Y_s from (3.17) to find

$$\log Y_s = \log Y_0 + \int_0^s \{\frac{1}{2}\frac{Z_u Z'_u}{Y_u^2} + X'_u \Gamma X_u + RX_u + k\}du + \int_0^s \frac{Z_u}{Y_u} dW_u^T. \tag{3.18}$$

Similarly, we have

$$\log Y_T = \log Y_0 + \int_0^T \{\frac{1}{2}\frac{Z_u Z'_u}{Y_u^2} + X'_u \Gamma X_u + RX_u + k\}du + \int_0^T \frac{Z_u}{Y_u} dW_u^T. \tag{3.19}$$

But $Y_T = P(T, T) = 1$. So $\log Y_T = 0$. Subtracting (3.19) from (3.18) we have

$$\begin{cases} X_s = X_0 + \int_0^s (AX_u + B + \sigma\frac{Z'_u}{Y_u})du + \int_0^s \sigma dW_u^T \\ \log Y_s = -\int_s^T \{\frac{1}{2}\frac{Z_u Z'_u}{Y_u^2} + X'_u \Gamma X_u + RX_u + k\}du - \int_s^T \frac{Z_u}{Y_u} dW_u^T \end{cases} \tag{3.20}$$

for $s \in [0, T]$. It's well known (see Ahn et al. (2002)) that for the quadratic term structure models, the price of zero-coupon bond has exponential quadratic form of the factor process X_t. So we assume that

$$\log Y_s = X_s' R_2(s) X_s + R_1(s) X_s + R_0(s) \tag{3.21}$$

where

$$R_2(s) = \begin{bmatrix} R_2^{(11)}(s) & \cdots & R_2^{(1n)}(s) \\ \vdots & \ddots & \vdots \\ R_2^{(n1)}(s) & \cdots & R_2^{(nn)}(s) \end{bmatrix}, \quad R_1'(s) = \begin{bmatrix} R_1^{(1)}(s) \\ \vdots \\ R_1^{(n)}(s) \end{bmatrix} \tag{3.22}$$

and $R_0(s)$ is one dimensional and satisfies the ODE

$$dR_0(s) = \alpha(s) ds. \tag{3.23}$$

Moreover, we require that $R_2(s)$, $R_1(s)$ and $R_0(s)$ satisfy the boundary condition

$$R_2(T) = 0_{n \times n}, \quad R_1(T) = 0_{1 \times n} \quad \text{and} \quad R_0(T) = 0, \tag{3.24}$$

since $\log Y_T = 0$.

Using this assumption, we obtain the dynamics of $d \log Y_s$. Apply Itô's formula to function $f(t, x) = x' R_2(t) x + R_1(t) x + R_0(t)$ using the dynamics of X_t from (3.17), where $x' = [x_1 \cdots x_n] \in R^n$. We have

$$\begin{aligned} d \log Y_t &= df(t, X_t) \\ &= f_t(t, X_t) dt + \sum_{i=1}^{n} f_{x_i}(t, X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{x_i x_j}(t, X_t) d\langle X_i, X_j \rangle, \end{aligned} \tag{3.25}$$

where

$$f_t(t,x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \dot{R}_2^{(ji)}(t)x_j x_i + \sum_{i=1}^{n} \dot{R}_1^{(i)}(t)x_i + \dot{R}_0(t),$$

$$f_{x_i}(t,x) = \sum_{k=1}^{n}[R_2^{(ki)}(t) + R_2^{(ik)}(t)]x_k + R_1^{(i)}(t),$$

$$f_{x_i x_j}(t,x) = R_2^{(ji)}(t) + R_2^{(ij)}(t)$$

$$\dot{R}_i(t) = \partial R_i/\partial t.$$

The simplified result is

$$\begin{aligned} d\log Y_t =& \{X_t'\dot{R}_2(t)X_t + \dot{R}_1(t)X_t + \dot{R}_0(t) \\ &+ X_t'(R_2'(t) + R_2(t))AX_t + R_1(t)AX_t + B'(R_2(t) + R_2'(t))X_t + R_1(t)B \\ &+ [X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma\frac{Z_t'}{Y_t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime}\} \\ &+ [X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma dW_t^T, \end{aligned} \tag{3.26}$$

where

$$\sigma = \begin{bmatrix} \sigma^{r_1} \\ \vdots \\ \sigma^{r_n} \end{bmatrix} = \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,n} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,n} \end{bmatrix}.$$

Comparing (3.26) with the dynamics of $d\log Y_t$ in (3.20), we have

$$[X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma = \frac{Z_t}{Y_t} \tag{3.27}$$

and

$$
\begin{aligned}
& X_t'\dot{R}_2(t)X_t + \dot{R}_1(t)X_t + \dot{R}_0(t) \\
& + X_t'(R_2'(t) + R_2(t))AX_t + R_1(t)AX_t + B'(R_2(t) + R_2'(t))X_t + R_1(t)B \\
& + [X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma\frac{Z_t'}{Y_t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} \\
& = \frac{1}{2}\frac{Z_tZ_t'}{Y_t^2} + X_t'\Gamma X_t + RX_t + k. \qquad (3.28)
\end{aligned}
$$

Substitute $\dot{R}_0(t) = \alpha(t)$ and (3.27) into (3.28).

$$
\begin{aligned}
& X_t'[\dot{R}_2(t) + (R_2'(t) + R_2(t))A - \Gamma + \frac{1}{2}(R_2(t) + R_2'(t))\sigma\sigma'(R_2(t) + R_2'(t))]X_t \\
& + [\dot{R}_1(t) + R_1(t)A + B'(R_2(t) + R_2'(t)) - R + R_1(t)\sigma\sigma'(R_2(t) + R_2'(t))]X_t \\
& + R_1(t)B + \alpha(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} - k \\
& + \frac{1}{2}R_1(t)\sigma\sigma'R_1'(t) = 0. \qquad (3.29)
\end{aligned}
$$

Consider the Riccati type equations

$$
\begin{cases}
\dot{R}_2(t) + (R_2'(t) + R_2(t))A - \Gamma + \frac{1}{2}(R_2(t) + R_2'(t))\sigma\sigma'(R_2(t) + R_2'(t)) = 0_{n\times n} \\
\dot{R}_1(t) + R_1(t)A + B'(R_2(t) + R_2'(t)) - R + R_1(t)\sigma\sigma'(R_2(t) + R_2'(t)) = 0_{1\times n} \\
R_1(T) = 0_{1\times n} \quad R_2(T) = 0_{n\times n}.
\end{cases} \qquad (3.30)
$$

If the Riccati type equations (3.30) admits a solution $R_1(\cdot)$, $R_2(\cdot)$ over $[0, T]$, then (3.29) gives

$$
\alpha(t) = k - R_1(t)B - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} - \frac{1}{2}R_1(t)\sigma\sigma'R_1'(t). \qquad (3.31)
$$

Hence the ODE of $R_0(\cdot)$ is

$$
\begin{cases}
dR_0(t) = \{k - R_1(t)B - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} - \frac{1}{2}R_1(t)\sigma\sigma'R_1'(t)\}dt \\
R_0(T) = 0.
\end{cases}
$$

(3.32)

The ODE (3.32) admits a solution if $R_2(\cdot)$ and $R_1(\cdot)$ are given. Substitute (3.27) into (3.20) to find,

$$X_t = X_0 + \int_0^t \{[A + \sigma\sigma^T(R_2(u) + R_2^T(u))]X_u + B + \sigma\sigma^T R_1^T(u)\}du + \int_0^t \sigma dW_u^T. \quad (3.33)$$

The SDE (3.33) has a unique solution if $R_2(\cdot)$ and $R_1(\cdot)$ are measurable functions satisfying the globally Lipschitz condition (see Friedman (1975) Volume I, Theorem 2.2 p 104).

We have heuristically shown in this section that we could find the dynamics of $\log Y_t$ and X_t if the Riccati type equations (3.30) admit solution over $[0, T]$. In the next section, we will give the rigorous proof of this result.

3.3 Theorem in One-Dimensional Case

In order to provide an easier way to understand the FBSDEs approach, we first give the proof of the main existence and uniqueness theorem in the one-dimensional case.

Consider the quadratic term structure model in one dimension. The shot rate process is given by

$$r(X_t) = r_2 X_t^2 + r_1 X_t + r_0, \quad (3.34)$$

where

$$X_t = X_0 + \int_0^t (AX_u + B)du + \int_0^t \sigma dW_u \quad (3.35)$$

Theorem 3.1. *If the Riccati equation*

$$\begin{cases} \dot{R}_2(t) + 2AR_2(t) + 2\sigma^2 R_2(t)^2 - r_2 = 0 \\ \dot{R}_1(t) + AR_1(t) + 2BR_2(t) + 2\sigma^2 R_1(t)R_2(t) - r_1 = 0 \\ R_1(T) = 0 \quad R_2(T) = 0. \end{cases} \quad (3.36)$$

admits a unique solution $R_2(\cdot)$, $R_1(\cdot)$ over the interval $[0,T]$, the FBSDE

$$\begin{cases} Y_t =1 - \int_t^T \{Y_u(r_2X_u^2 + r_1X_u + r_0) + \frac{Z_u^2}{Y_u}\}du - \int_t^T Z_u dW_u^T \\ X_t =X_0 + \int_0^t (AX_u + B + \sigma\frac{Z_u}{Y_u})du + \int_0^t \sigma dW_u^T \end{cases} \tag{3.37}$$

for $t \in [0,T]$, admits a unique solution (X, Y, Z) given by

$$X_t = X_0 + \int_0^t \{[A + 2\sigma^2 R_2(u)]X_u + B + \sigma^2 R_1(u)\}du + \int_0^t \sigma dW_u^T, \tag{3.38}$$

$$Y_t = exp\Big(R_2(t)X_t^2 + R_1(t)X_t + R_0(t)\Big), \tag{3.39}$$

$$Z_t = [2X_tR_2(t) + R_1(t)]\sigma exp\Big(R_2(t)X_t^2 + R_1(t)X_t + R_0(t)\Big) \tag{3.40}$$

where $R_0(\cdot)$ satisfies

$$\begin{cases} dR_0(t) = \{r_0 - R_1(t)B - \frac{1}{2}\sigma^2 R_1(t)^2 - \sigma^2 R_2(t)\}dt \\ R_0(T) = 0. \end{cases} \tag{3.41}$$

Proof. First, we must show that $(X,\ Y,\ Z)$ given by (3.38), (3.39) and (3.40) satisfy the FBSDE (3.37). Diving (3.40) by (3.39), we have

$$\frac{Z_t}{Y_t} = [2X_tR_2(t) + R_1(t)]\sigma \tag{3.42}$$

Substituting (3.42) into (3.38), we obtain the dynamics of X_t in (3.37). So X_t given by (3.38) satisfies the FBSDEs (3.37).

Consider the function $f(t,x) = \exp\Big(R_2(t)x^2 + R_1(t)x + R_0(t)\Big)$. Apply Itô's formula to $f(t,x)$ using the dynamics of X_t in (3.37). We have

$$\begin{aligned} dY_t &= df(t, X_t) \\ &= f_t(t, Xt)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)d\langle X.\rangle_t, \end{aligned} \tag{3.43}$$

where

$$f_t(t,x) = [\dot{R}_2(t)x^2 + \dot{R}_1(t)x + \dot{R}_0(t)]f(t,x),$$

$$f_x(t,x) = \{2R_2(t)x + R_1(t)\}f(t,x),$$

$$f_{xx}(t,x) = \{2R_2(t) + [2R_2(t)x + R_1(t)]^2\}f(t,x).$$

$$\begin{aligned}
dY_t &= [\dot{R}_2(t)X_t^2 + \dot{R}_1(t)X_t + \dot{R}_0(t)]Y_t dt \\
&+ \{[2R_2(t)X_t + R_1(t)][(AX_t + B + \sigma\frac{Z_t}{Y_t})dt + \sigma dW_t^T]\}Y_t \\
&+ \frac{1}{2}\{2R_2(t) + [2R_2(t)X_t + R_1(t)]^2\}Y_t\sigma^2 dt \\
&= \{\dot{R}_2(t)X_t^2 + \dot{R}_1(t)X_t + \dot{R}_0(t) + 2R_2(t)AX_t^2 + 2R_2(t)BX_t \\
&+ (2R_2(t)X_t + R_1(t))\sigma\frac{Z_t}{Y_t} + R_1(t)AX_t + R_1(t)B + R_2(t)\sigma^2 \\
&+ \frac{1}{2}[2R_2(t)X_t + R_1(t)]^2\sigma^2\}Y_t dt + [2R_2(t)X_t + R_1(t)]\sigma Y_t dW_t^T. \qquad (3.44)
\end{aligned}$$

Substituting (3.42) into (3.44) we have

$$\begin{aligned}
dY_t &= \{[\dot{R}_2(t) + 2R_2(t)A + 2R_2(t)^2\sigma^2]X_t^2 \\
&+ [\dot{R}_1(t) + 2BR_2(t) + R_1(t)A + 2R_1(t)R_2(t)\sigma^2]X_t \\
&+ \frac{Z_t^2}{Y_t^2} + \dot{R}_0(t) + R_1(t)B + R_2(t)\sigma^2 + \frac{1}{2}R_1(t)^2\sigma^2)\}Y_t dt + Z_t dW_t^T. \qquad (3.45)
\end{aligned}$$

Substitute (3.36) and (3.41) into (3.45) to find

$$dY_t = \{Y_t(r_2X_t^2 + r_1X_t + r_0) + \frac{Z_t^2}{Y_t}\}dt + Z_t dW_t^T$$

That is, Y_t defined by (3.38)-(3.40) satisfies

$$Y_t = Y_T - \int_t^T \{Y_u(r_2X_u^2 + r_1X_u + r_0) + \frac{Z_u^2}{Y_u}\}du - \int_t^T Z_u dW_u^T.$$

By the boundary condition in (3.36) we have

$$Y_T = \exp\{R_2(T)X_T^2 + R_1(T)X_T + R_0(T)\} = \exp\{0X_T^2 + 0X_T + 0\} = 1.$$

Therefore,

$$Y_t = 1 - \int_t^T \{Y_u(r_2X_u^2 + r_1X_u + r_0) + \frac{Z_u^2}{Y_u}\}du - \int_t^T Z_u dW_u^T.$$

Hence (X, Y, Z) given by (3.38), (3.39) and (3.40) satisfy the FBSDE (3.37).

Second, we prove the uniqueness of the solution. Let (X, Y, Z) be any adapted solution of the FBSDEs (3.37). Define

$$\log \bar{Y}_t = R_2(t)X_t^2 + R_1(t)X_t + R_0(t), \tag{3.46}$$

$$\bar{Z}_t = [2R_2(t)X_t + R_1(t)]\sigma\bar{Y}_t, \tag{3.47}$$

then

$$\frac{\bar{Z}_t}{\bar{Y}_t} = [2R_2(t)X_t + R_1(t)]\sigma. \tag{3.48}$$

Consider the function

$$f(t, x) = R_2(t)x^2 + R_1(t)x + R_0(t).$$

Apply Itô's formula to $f(t, x)$ where X_t is given by (3.37) we have

$$\begin{aligned} df(t, X_t) = d\log\bar{Y}_t = \{&\dot{R}_2(t)X_t^2 + \dot{R}_1(t)X_t + \dot{R}_0(t) + 2R_2(t)AX_t^2 \\ &+ R_1(t)AX_t + 2BR_2(t)X_t + R_1(t)B + [2R_2(t)X_t + R_1(t)]\sigma\frac{Z_t}{Y_t} \\ &+ R_2(t)\sigma^2\}dt + [2R_2(t)X_t + R_1(t)]\sigma dW_t^T. \end{aligned} \tag{3.49}$$

Substituting (3.36), (3.41) and (3.48) into (3.49) we have

$$d\log\bar{Y}_t = \{r_2X_t^2 + r_1X_t + r_0 - \frac{1}{2}\frac{\bar{Z}_t^2}{\bar{Y}_t^2} + \frac{\bar{Z}_t}{\bar{Y}_t}\frac{Z_t}{Y_t}\}dt + \frac{\bar{Z}_t}{\bar{Y}_t}dW_t^T \tag{3.50}$$

and

$$\log\bar{Y}_T = 0. \tag{3.51}$$

Then $\bar{Y}_t$ satisfies the BSDE

$$\log \bar{Y}_t = -\int_t^T \{r_2 X_u^2 + r_1 X_u + r_0 - \frac{1}{2}\frac{\bar{Z}_u^2}{\bar{Y}_u^2} + \frac{\bar{Z}_u}{\bar{Y}_u}\frac{Z_u}{Y_u}\}du - \int_t^T \frac{\bar{Z}_u}{\bar{Y}_u} dW_u^T. \tag{3.52}$$

Subtract (3.52) from (3.18) to find

$$\begin{aligned}\log Y_t - \log \bar{Y}_t &= -\int_t^T \{\frac{1}{2}\frac{Z_u^2}{Y_u^2} - \frac{\bar{Z}_u Z_u}{\bar{Y}_u Y_u} + \frac{1}{2}\frac{\bar{Z}_u^2}{\bar{Y}_u^2}\}du - \int_t^T \{\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u}\}dW_u^T \\ &= -\frac{1}{2}\int_t^T \{(\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u})^2\}du - \int_t^T (\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u})dW_u^T. \end{aligned} \tag{3.53}$$

Define

$$\hat{Y}_t = \log Y_t - \log \bar{Y}_t \tag{3.54}$$

$$\hat{Z}_t = \frac{Z_t}{Y_t} - \frac{\bar{Z}_t}{\bar{Y}_t}. \tag{3.55}$$

Then equation (3.53) becomes

$$\hat{Y}_t = -\frac{1}{2}\int_t^T \hat{Z}_u^2 du - \int_t^T \hat{Z}_u dW_u^T. \tag{3.56}$$

By the result of Kobylanski (2000), the BSDE (3.56) admits a unique adapted solution $(\hat{Y}_t, \hat{Z}_t) = (0, 0_{1\times n})$. So we have $Y_t = \bar{Y}_t$ and $Z_t = \bar{Z}_t$. This means that any solution (X, Y, Z) of the FBSDE (3.37) must satisfy (3.38), (3.39) and (3.40). □

Corollary 3.1. *If the factor process is given by (3.35) and the short rate process is represented by (3.34), the zero coupon bond price has exponential quadratic form,*

$$P(t,T) = exp\{R_2(t)X_t^2 + R_1(t)X_t + R_0(t)\}$$

where $R_2(t)$, $R_1(t)$ and $R_0(t)$ are given by equations (3.36) and (3.41), respectively.

In the next section we prove the main existence and uniqueness result in the n-dimensional case. The proof is similar to the one-dimensional case just presented.

3.4 FBSDE Approach

We prove the existence and uniqueness of the FBSDE (3.17) following a technique for linear FBSDEs that was extended by Hyndman (2007b) and Hyndman (2009).

Theorem 3.2. *If the Riccati equations*

$$\begin{cases} \dot{R}_2(t) + (R_2'(t) + R_2(t))A - \Gamma + \frac{1}{2}(R_2(t) + R_2'(t))\sigma\sigma'(R_2(t) + R_2'(t)) = 0_{n\times n} \\ \dot{R}_1(t) + R_1(t)A + B'(R_2(t) + R_2'(t)) - R + R_1(t)\sigma\sigma'(R_2(t) + R_2'(t)) = 0_{1\times n} \\ R_1(T) = 0_{1\times n} \quad R_2(T) = 0_{n\times n}. \end{cases} \tag{3.57}$$

admit unique solutions $R_2(\cdot)$, $R_1(\cdot)$ over the interval $[0,T]$, the FBSDE

$$\begin{cases} Y_t = 1 - \int_t^T \{Y_u[X_u'\Gamma X_u + RX_u + k] + \frac{Z_u Z_u'}{Y_u}\}du - \int_t^T Z_u dW_u^T \\ X_t = X_0 + \int_0^t (AX_u + B + \sigma\frac{Z_u'}{Y_u})du + \int_0^t \sigma dW_u^T \end{cases} \tag{3.58}$$

for $t \in [0,T]$, has a unique adapted solution (X, Y, Z) given by

$$X_t = X_0 + \int_0^t \{[A + \sigma\sigma'(R_2(u) + R_2'(u))]X_u + B + \sigma\sigma' R_1'(u)\}du + \int_0^t \sigma dW_u^T, \tag{3.59}$$

$$Y_t = exp\Big(X_t' R_2(t) X_t + R_1(t)X_t + R_0(t)\Big), \tag{3.60}$$

$$Z_t = [X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma exp\Big(X_t' R_2(t) X_t + R_1(t) X_t + R_0(t)\Big) \tag{3.61}$$

where $R_0(\cdot)$ satisfies

$$\begin{cases} dR_0(t) = \Big\{k - R_1(t)B - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} - \frac{1}{2}R_1(t)\sigma\sigma' R_1'(t)\Big\} dt \\ R_0(T) = 0. \end{cases} \tag{3.62}$$

Proof. First, we must show that (X, Y, Z) given by (3.59), (3.60) and (3.61) satisfy the FBSDE (3.58). Dividing (3.61) by (3.60), we have

$$\frac{Z_t}{Y_t} = [X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma \tag{3.63}$$

Substituting (3.63) into (3.59), we obtain the dynamics of X_t given by equation (3.58). So X_t given by (3.59) satisfies the FBSDE (3.58).

Consider the function $f(t, x) = \exp\Big(x'R_2(t)x + R_1(t)x + R_0(t)\Big)$. Apply Itô's formula to $f(t, x)$ using the dynamics of X_t in (3.58). We have

$$\begin{aligned} dY_t &= df(t, X_t) \\ &= f_t(t, Xt)dt + \sum_{i=1}^{n} f_{x_i}(t, X_t)dX_t^{(i)} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} f_{x_i x_j}(t, X_t)d\langle X^{(i)}, X^{(j)}\rangle_t, \end{aligned} \tag{3.64}$$

where

$$\begin{aligned} f_t(t, x) &= [\sum_{j=1}^{n}\sum_{i=1}^{n} \dot{R}_2^{(ji)}(t)x_j x_i + \sum_{i=1}^{n} \dot{R}_1^{(i)}(t)x_i + \dot{R}_0(t)]f(t, x), \\ f_{x_i}(t, x) &= \{\sum_{k=1}^{n}(R_2^{(ki)}(t) + R_2^{(ik)}(t))x_k + R_1^{(i)}(t)\}f(t, x), \\ f_{x_i x_j}(t, x) &= \{R_2^{(ji)}(t) + R_2^{(ij)}(t) + [\sum_{k=1}^{n}(R_2^{(ki)}(t) + R_2^{(ik)}(t))x_k + R_1^{(i)}(t)] \\ &\quad \times [\sum_{k=1}^{n}(R_2^{(kj)}(t) + R_2^{(jk)}(t))x_k + R_1^{(j)}(t)]\}f(t, x). \end{aligned}$$

That is,

$$\begin{aligned} dY_t &= [X_t'\dot{R}_2(t)X_t + \dot{R}_1(t)X_t + \dot{R}_0(t)]Y_t dt \\ &+ \sum_{i=1}^{n} Y_t\{[X_t'(R_2^{c_i}(t) + R_2^{r_i\prime}(t)) + R_1^{(i)}(t)][(A^{r_i}X_t + \tilde{b}_i + \sigma^{r_i}\frac{Z_t'}{Y_t})dt + \sigma^{r_i}dW_t^T]\}dt \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} Y_t(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime}dt \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} Y_t[X_t'(R_2^{c_i}(t) + R_2^{r_i\prime}(t)) + R_1^{(i)}(t)][(R_2^{c_j\prime}(t) + R_2^{r_j}(t))X_t + R_1^{(j)}(t)]\sigma^{r_i}\sigma^{r_j\prime}dt \end{aligned} \tag{3.65}$$

where $R_2^{c_i}$ is the ith column of square matrix R_2, $R_2^{r_i}$ is the ith row of R_2, σ^{c_i} is the ith column of σ and σ^{r_i} is the ith row of σ. Write (3.65) in matrices notation to obtain

$$\begin{aligned}
dY_t = &\{X_t'\dot{R}_2(t)X_t + \dot{R}_1(t)X_t + \dot{R}_0(t) + X_t'(R_2(t) + R_2'(t))AX_t \\
&+ B'(R_2(t) + R_2'(t))X_t + [X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma\frac{Z_t'}{Y_t} + R_1(t)AX_t \\
&+ R_1(t)B + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} \\
&+ \frac{1}{2}[X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma\sigma'[(R_2'(t) + R_2(t))X_t + R_1'(t)]\}Y_t dt \\
&+ Y_t[X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma dW_t^T.
\end{aligned} \tag{3.66}$$

Substitute (3.63) in (3.66) to find

$$\begin{aligned}
dY_t = &\{X_t'[\dot{R}_2(t) + (R_2(t) + R_2'(t))A + \frac{1}{2}(R_2(t) + R_2'(t))\sigma\sigma'(R_2'(t) + R_2(t))]X_t \\
&+ [\dot{R}_1(t) + B'(R_2(t) + R_2'(t)) + R_1(t)A + R_1(t)\sigma\sigma'(R_2'(t) + R_2(t))]X_t \\
&+ \frac{Z_tZ_t'}{Y_t^2} + \dot{R}_0(t) + R_1(t)B + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime} \\
&+ \frac{1}{2}R_1(t)\sigma\sigma'R_1'(t)\}Y_t dt + Z_t dW_t^T.
\end{aligned} \tag{3.67}$$

Substituting (3.57) and (3.62) into (3.67) gives,

$$dY_t = \{Y_t[X_t^T\Gamma X_t + RX_t + k] + \frac{Z_tZ_t^T}{Y_t}\}dt + Z_t dW_t^T.$$

That is, Y_t defined by (3.59)-(3.61) satisfies

$$Y_t = Y_T - \int_t^T \{Y_u[X_u'\Gamma X_u + RX_u + k] + \frac{Z_uZ_u'}{Y_u}\}du - \int_t^T Z_u dW_u^T. \tag{3.68}$$

By the boundary condition in (3.57) we have

$$Y_T = \exp\{X_T'R_2(T)X_T + R_1(T)X_T + R_0(T)\} = \exp\{X_T'0_{n\times n}X_T + 0_{1\times n}X_T + 0\} = 1.$$

Therefore,

$$Y_t = 1 - \int_t^T \{Y_u[X_u'\Gamma X_u + RX_u + k] + \frac{Z_uZ_u'}{Y_u}\}du - \int_t^T Z_u dW_u^T.$$

Hence (X, Y, Z) given by (3.59), (3.60) and (3.61) satisfy the FBSDE (3.58).

Second, we prove the uniqueness of the solution. Let (X, Y, Z) be any adapted solution of the FBSDE (3.58). Define

$$\log \bar{Y}_t = X_t' R_2(t) X_t + R_1(t) X_t + R_0(t), \tag{3.69}$$

$$\bar{Z}_t = [X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma \bar{Y}_t, \tag{3.70}$$

then

$$\frac{\bar{Z}_t}{\bar{Y}_t} = [X_t'(R_2(t) + R_2'(t)) + R_1(t)]\sigma. \tag{3.71}$$

Consider the function

$$f(t, x) = x' R_2(t) x + R_1(t) x + R_0(t).$$

Apply Itô's formula to $f(t, x)$ where X_t is given by (3.58) we have

$$\begin{aligned} df(t, X_t) = d\log \bar{Y}_t = \{& X_t' \dot{R}_2(t) X_t + \dot{R}_1(t) X_t + \dot{R}_0(t) \\ & + X_t'(R_2'(t) + R_2(t)) A X_t + R_1(t) A X_t + B'(R_2(t) + R_2'(t)) X_t + R_1(t) B \\ & + [X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma \frac{Z_t'}{Y_t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(R_2^{(ji)}(t) + R_2^{(ij)}(t))\sigma^{r_i}\sigma^{r_j\prime}\} dt \\ & + [X_t'(R_2'(t) + R_2(t)) + R_1(t)]\sigma dW_t^T. \end{aligned} \tag{3.72}$$

Substitute (3.57), (3.62) and (3.71) into (3.72) to find

$$d\log \bar{Y}_t = \{X_t' \Gamma X_t + R X_t + k - \frac{1}{2}\frac{\bar{Z}_t \bar{Z}_t'}{\bar{Y}_t^2} + \frac{\bar{Z}_t}{\bar{Y}_t}\frac{Z_t'}{Y_t}\} dt + \frac{\bar{Z}_t}{\bar{Y}_t} dW_t^T \tag{3.73}$$

and

$$\log \bar{Y}_T = 0. \tag{3.74}$$

So

$$\log \bar{Y}_t = -\int_t^T \{X_u' \Gamma X_u + R X_u + k - \frac{1}{2}\frac{\bar{Z}_u \bar{Z}_u'}{\bar{Y}_u^2} + \frac{\bar{Z}_u}{\bar{Y}_u}\frac{Z_u'}{Y_u}\} du - \int_t^T \frac{\bar{Z}_u}{\bar{Y}_u} dW_u^T. \tag{3.75}$$

Subtract (3.75) from (3.18) to find

$$\log Y_t - \log \bar{Y}_t = -\int_t^T \{\frac{1}{2}\frac{Z_u Z'_u}{Y_u^2} - \frac{\bar{Z}_u Z'_u}{\bar{Y}_u Y_u} + \frac{1}{2}\frac{\bar{Z}_u \bar{Z}'_u}{\bar{Y}_u^2}\}du - \int_t^T \{\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u}\}dW_u^T$$
$$= -\frac{1}{2}\int_t^T \{(\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u})(\frac{Z'_u}{Y_u} - \frac{\bar{Z}_u^T}{\bar{Y}_u})\}du - \int_t^T (\frac{Z_u}{Y_u} - \frac{\bar{Z}_u}{\bar{Y}_u})dW_u^T. \tag{3.76}$$

Define

$$\hat{Y}_t = \log Y_t - \log \bar{Y}_t \tag{3.77}$$
$$\hat{Z}_t = \frac{Z_t}{Y_t} - \frac{\bar{Z}_t}{\bar{Y}_t}. \tag{3.78}$$

Then equation (3.76) becomes

$$\hat{Y}_t = -\frac{1}{2}\int_t^T \hat{Z}_u \hat{Z}'_u du - \int_t^T \hat{Z}_u dW_u^T. \tag{3.79}$$

By the result of Kobylanski (2000), the BSDE (3.79) admits a unique adapted solution $(\hat{Y}_t, \hat{Z}_t) = (0, 0_{1\times n})$. So we have $Y_t = \bar{Y}_t$ and $Z_t = \bar{Z}_t$. This means that any solution (X, Y, Z) of FBSDEs (3.58) must satisfy (3.59), (3.60) and (3.61). □

As we have discussed in the previous material, Y_t is the zero coupon bond price $P(t,T)$, which is given by the Theorem 3.2. So we have the following Corollary.

Corollary 3.2. *If the factor process is given by (2.2) and the short rate process is represented by Model A, the zero coupon bond price has exponential quadratic form,*

$$P(t,T) = exp\{X'_t R_2(t) X_t + R_1(t) X_t + R_0(t)\},$$

where $R_2(t)$, $R_1(t)$ and $R_0(t)$ solve equations (3.57) and (3.62), respectively.

Chapter 4

Conclusion

In this thesis we have implemented the stochastic flows approach and the FBSDE approach to quadratic term structure models. By using the stochastic flows approach, we obtain a closed-form solution for the zero-coupon bond price in the one dimensional QTSM. This result is consistent with that in Nawalkha et al. (2007). However, as shown in Chapter 2, the stochastic flow method is difficult to generalize to the higher dimensional QTSMs without restrictions. We discuss the necessary restrictions to implement the flows method in the two-dimensional Model B in Chapter 2. In Chapter 3 we prove that the zero-coupon bond price is an exponential quadratic function of the factor variables by using the FBSDEs approach. The main result of the thesis is an existence and uniqueness theorem for a FBSDE satisfied by the zero-coupon bond price. The existence and uniqueness theorem also provides an explicit solution which gives the bond price as a corollary. This is consistent with Chen et al. (2004) and Leippold and Wu. The FBSDEs approach can be implemented to the n-dimensional QTSMs without restrictions. In this regard, the FBSDEs approach is more powerful than the stochastic flows approach, hence it has the potential to solve other pricing problem. In the future, following Hyndman (2009), we would like to implement the FBSDE approach to price futures, forward contracts, options and other derivatives under the QTSMs. Also, we might be able to relax the restrictions we have discussed in Chapter 2 in the future study.

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