Breaking Customs in an Algebra Classroom for Mature Students and Providing Them With Opportunities to Engage in Theoretical Thinking

Steven Pileggi

A Thesis
In
The Department
Of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master in the Teaching of
Mathematics at
Concordia University
Montreal, Quebec, Canada

January 2015

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Steven Pileggi

Entitled: Breaking Customs in an Algebra Classroom for Mature Students and Providing them with Opportunities to Engage in Theoretical Thinking

and submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final Examining Committee:

Fred Szabo Chair

Anna Sierpinksa Examiner

Fred Szabo Examiner

Nadia Hardy Supervisor

Approved by

Chair of Department or Graduate Program Director

Dean of Faculty

Date January 9th, 2015

ABSTRACT

Breaking Customs in an Algebra Classroom for Mature Students and Providing them with Opportunities to Engage in Theoretical Thinking

Steven Pileggi

When an adult student returns to mathematics classes, it can sometimes be after a years-long absence, and often they may hold a negative attitude towards the subject as a whole. They arrive with a clear belief about what a mathematics classroom should look like, specifically with regards to the role of the teacher and the role of the student – a belief that has been formed over many years of exposure to 'traditional' mathematics classrooms. There are also certain customs that the students have come to believe should be features of all mathematics courses – that the teacher will lecture on how to solve the different kinds of problems they will encounter, that the teacher will always tell them whether their work is correct or incorrect, and that there is only one 'acceptable' method of solving a given mathematics problem. In this thesis we discuss a teaching approach that was taken in an algebra course designed for adults – a teaching approach that tried to break away from these customs. The Teachers of the course felt that by breaking custom they would be better able to succeed in achieving the three goals that they had set up for their course: First, they wanted to engage their students in theoretical thinking, following Sierpinska et. al's (2002) model. Second, they wanted to respect and acknowledge the students' different (mathematics) backgrounds and life goals. Third, they wanted their students to succeed in the institutional sense. As Researchers, we will be investigating the design and implementation of this teaching approach to discuss whether or not the Teachers were successful in achieving their goals.

Acknowledgements

There are many people that I would like to thank, who helped me to reach this ‘finish line’ of my Master’s degree, but I cannot thank any of them before I offer a special thanks to my fantastic thesis supervisor - Dr. Nadia Hardy

I confess that Dr. Hardy used to frighten me. Almost three years ago now, in January 2012, she was the professor for my first ever MTM course. I was shy (which I now find hard to believe), nervous, and did not know what to expect. She waltzed in to our conference room with a commanding air about her and announced to the class “If this is the first ever MTM course you’re taking, good luck…you’ll need it!” As you’ll imagine, for a nervous first-time graduate student these words did not constitute the most comforting welcome. Plus, I learned that she is some sort of karate-wizard…this did not lessen my fear of her. As the semester went on, however, I began to see the multitude of ways in which she supports her students as well as her sense of humor, which is remarkably similar to my own, and I started to think “OK, maybe she’s not that scary after all.” I’m sure that you’re all worried, but rest assured that I did pass her course (and with an A+ no less). After this course was finished she became the first professor to suggest to me that writing a thesis might be a good idea, and a few short months later I had cemented her as my supervisor.

Since becoming my supervisor Dr. Hardy (or Nadia, as she commanded I address her) has provided me with so many amazing opportunities. She encouraged me to attend and present at conferences – experiences that have been both educational and fun, and where I was able to make many new friends. She opened the door for me to get involved in many different courses at Concordia – not just the section of MATH 200 that we will be discussing at length here. Quite often we would be working together, and we would always have a good time (even if she forgot that it was her day to pick up coffee). Plus, she is able to put up with my jokes…so that is a plus.

I could go on and on, but I'll restrain myself and say that her guidance over the last two years has been invaluable – I wouldn't be writing this right now if it weren't for her – and I will always be thankful for all the help and support that she has shown me. MTM students of the future, if you are reading this thesis looking for inspiration then let me tell you something – do a thesis, it's worth it. Also, get over your fear of Nadia and do everything you can to get her to be your supervisor – you will absolutely not regret it.

I would also like to thank Dr. Anna Sierpinska and Dr. Fred Szabo for the time that they spent reading my thesis, as well as their feedback. It has been very much appreciated.

To my friends in Concordia's Mathematics and Statistics department, thank you for keeping me sane throughout this process. The countless games of cribbage (and euchre, I suppose…) definitely did not do much for my productivity levels, but they did serve as a wonderful distraction when I needed to look at something other than my computer screen.

To my non-Concordia friends and my family, who have always supported me in anything that I have done and were my reminder that there *is* a life outside the Mathematics and

Statistics department, I’m not sure what I can really say. I love you all, and thank you so much for everything.

Contents

Chapter 1. Introduction

1.1 What prompted the study?

In January of 2012 a professor and her teaching assistants at a large, urban university designed and implemented an introductory course on proofs, offered to first year students registered in the major in Mathematics and Statistics program, using what they called a *discussion-oriented approach*. For the professor and her assistants, this meant abandoning the lecture-based approach to teaching mathematics and the focus on ‘reproducing’ or ‘mimicking’ techniques, traditionally used in university classrooms, to develop an approach where students engage in *doing mathematics* and in *behaving mathematically*. They called it *discussion-oriented* because students spent most of the class time engaged in group discussion about the activities proposed by the professor. The design of the teaching approach included the design of activities for the students to work on in class and in assignments and the design of interaction protocols (how the professor and TAs will interact with the students, face-to-face during class and online in discussion forums). These activities and interaction protocols were aimed at engaging students in mathematical behaviours. Therefore, the design of the approach included a characterization of mathematical behaviour; in particular, the professor and TAs considered *theoretical thinking* (Sierpinska et al., 2002), *self-regulation* (Schoenfeld, 1987), and *self-efficacy* (Selden and Selden, 2013) as mathematical behaviours that they wanted to foster and provoke in their students. An analysis of students’ work seems to indicate that the approach was somehow ‘successful’ at engaging students in these mathematical behaviours (Hardy et al., 2013).

This thesis was born from the question of what such a discussion-oriented approach, focused on engaging students in mathematical behaviours, would look like in the context of other courses. The challenge, the professor thought, was that while the introduction to proofs course did not have any fixed mathematical content that needed to be covered, all other courses (seem to) do. Thus, the professor believed that the challenge would be trying to ensure that specific mathematics content (content that students are expected to be familiar with in subsequent mathematics courses) be covered (and learned) through a discussion-oriented approach. From here the idea was born to attempt to teach another course using the same teaching style, a course that serves as an introduction to algebra (MATH 200).

MATH 200, subtitled ‘Fundamental Concepts of Algebra’, is a course designed to reacquaint mature students with mathematics. It covers many topics that are typically seen in secondary school including (but not limited to) operations on real numbers, algebraic expressions, and linear and quadratic equations. The students who register in this course are often engaging in mathematics for the first time in a while, many years in some cases, and it is not always easy for them to engage in the ‘school-traditional’ activities of ‘solving’, and perhaps less, engage in ‘non-traditional activities’ as the professor expected to propose.

Based on their previous experiences teaching both this and other courses, the Teachers believed that the 'traditional' or 'institutional' approach so typically seen in mathematics courses does not allow for a classroom environment that is conducive to engaging in 'mathematical behaviour'. In some sense, this thesis tells the story of this first attempt to teach MATH 200 in a way that departs from this traditional approach in favor of a teaching style that could foster students' engagement in *doing* mathematics, in behaving mathematically. In "telling this story," we first describe the teaching approach and its goals, and present and discuss the challenges that the professor and TA faced and the assumptions and choices they made. Then, we analyze the course design and implementation in terms of the goals the professor and TA had set for their approach.

In what remains of this introduction, we give brief overviews of the teaching approach and of the analyses. Finally, we present the thesis structure.

1.2 Teachers and Researchers

For the purposes of this thesis we refer to 'Teachers' and 'Researchers' as separate entities, although we filled the role of both (there were always two teachers in the classroom and although one was the professor in charge and the other was the teaching assistant, their roles were quite the same all along the semester). When we talk about the Teachers, we refer to our role as teachers of the course: designing the activities, teaching the class, facilitating the discussions between students, marking, etc. When referring to the Researchers we are speaking of our role as researchers investigating the potential of the designed activities and the teaching approach at helping the students to succeed at the goals set by the Teachers.

The distinction, Teachers vs. Researchers, is sometimes useful but sometimes, it feels counter-productive or, at least, confusing. As it may be expected, the moment at which one person changes from one positioning (in the sense of Ostrom, 2005) to another, cannot be always, easily pinpointed, and one person's behavior at a certain point in time may originate from him or her shifting (or alternating) from one position to another. Thus, although we may write Teachers or Researchers, sometimes it was very clear that we were occupying one and only one of the positions, while sometimes it was not. For example, because the Teachers were researchers as well, many of their decisions and assumptions were triggered by their experiences in research.

Despite the fact that, perhaps even too often, the 'who' was not absolutely clear, we found useful to keep, whenever possible, this distinction.

1.3 Assumptions, goals and challenges regarding the teaching approach

At the heart of the teaching design is the Teachers' conviction that teaching mathematics is not about "showing" students how to solve a list of too-alike exercises and that learning mathematics is not about "mimicking" techniques. In particular, in their weakly-defined ideal of what teaching and learning mathematics at the university level should look like, the Teachers considered that theoretical thinking is a pillar. However, as suggested above, designing a discussion-based approach to MATH 200, one that would focus on *doing mathematics* and *behaving mathematically*, required very different considerations than those necessary for the introductory course on proving methods. On top of what was mentioned above regarding the difference in content (see p. 11), there was also a big difference in terms of who the students are. While the introduction to proofs course is a university entry-level course designed for students majoring in Mathematics, MATH 200 is the first of the university's prerequisite level mathematics courses, intended for students who are not necessarily mathematically literate and who need to either refresh their knowledge of, or complete for the first time, high school level mathematics before entering in to the program of their choice. The type of activities and the interaction protocols, the Teachers thought, had to be conceived taking into account who are the students and not just the goal of fostering mathematical behaviours.

Thus, based on their experiences teaching MATH 200, the 'academic' goals of the course, their goal of fostering mathematical behaviours, and research literature on mature students' learning of mathematics, the Teachers made the following assumptions about the students enrolled in the course. These assumptions governed their goals for, and the design of, the teaching approach.

The Teachers assumed that more often than not, students would have a somewhat negative attitude towards mathematics as a whole (Gustaffson & Mouwitz, 2004; Evans & Wedege, 2006). These attitudes may have been formed over the course of many years and could be the result of many different things: experiences with poor mathematics teachers, not being able to succeed in previous mathematics courses, etc. In addition, many of the students would be returning to studying mathematics after not having done so for many years, in which time the negative feelings may have 'festered'. *The Teachers wanted to design a course that could somehow make students if not 'like' mathematics, at least, not fear it.*

The Teachers assumed that students would each have very different backgrounds and different previous experiences with mathematics. Similarly, the students would have very different life goals: a student hoping to major in Psychology sitting next to someone who wants to be an engineer sitting next to someone who is simply taking the course as an elective. Despite these differences, the Teachers assumed that their students would have had, at some point, some previous exposure to algebra. *The Teachers wanted to design a course that could somehow acknowledge these different backgrounds and life goals.*

The teachers assumed that after many years of schooling (and in some cases, experiences with the schooling of their children) students enrolled in MATH 200 would have developed expectations of what teaching and learning mathematics looks like. In particular, the Teachers assumed that the students would expect (1) a lecture-based approach where the instructor shows them what to do and how, for every problem they may encounter in exams; (2) that it is the duty of the instructor to tell the students if they are right or wrong; and (3) that for each problem, there is one and only one (accepted) way of solving it. The Teachers assumed that as a consequence of these expectations, students would believe that learning mathematics (at least in the context of school) means memorizing algorithms and formulas and applying them. The Teachers assumed that these expectations and the consequent behavior had been engrained in the students' minds over many years of exposure to certain 'traditions' that are oftentimes featured in a typical mathematics classroom. *The Teachers wanted to design a course that could somehow break from these expectations and to foster (and provoke) a different behaviour, consistent with the Teachers' ideas about mathematical behaviours.*

Based on these assumptions, the Teachers reflected on the discussion-based approach to the introduction to proofs course and on the objectives they had for that approach, contrasting these with the 'academic' goals of MATH 200. Considering their experiences teaching or tutoring for the course and available literature on mature students' learning of mathematics, the Teachers outlined, at the beginning of the semester, three main goals for their discussion-based approach to teaching MATH 200.

The first goal was to engage students in one particular aspect of mathematical behaviour, namely, theoretical thinking. There were multiple reasons for this decision. The Teachers wanted the students to be able to take ownership of their work, and be able to act as their own authority as to whether their work was correct or not. They also wanted to emphasize that there is often more than one correct method for solving a problem, and wanted the students to take on the role of active thinkers, as opposed to simply passive 'receivers' of knowledge. The Teachers chose to focus on this aspect following the framework provided by Sierpinska et al. (2002). (The Researchers would also choose this framework for their analysis of the activities designed by the Teachers.)

The second goal was to respect and acknowledge students' previous mathematical knowledge.

The third goal was that the Teachers wanted their students to succeed in the 'institutional' sense, which is to say that they wanted their students to be able to pass the course. This was to be sure that it is possible to 'get away with' breaking classroom tradition without compromising the students' academic careers.

The Teachers surmised that negative attitudes and certain expectations could present obstacles for their goals. In particular, they were convinced that students' expectations about what teaching and learning mathematics looks like would (and had to be) shattered in order to provoke situations where theoretical thinking was possible. The challenges were, they presumed, to design tasks that will acknowledge students' diverse backgrounds in mathematics, all the while shattering certain expectations (about the role of teachers and students in the mathematics classroom and about mathematics in general) without shattering, or actually while boosting, their confidence and without compromising academic success.

1.4 Overview of the teaching approach

Based on their assumptions and goals, the Teachers designed a collection of activities for the students to engage in during class time and for homework. These were to coexist with the 'institutional' activities (graded assignments and recommended problems from the textbook).

In terms of the functioning of the class, instead of lectures, the majority of class time was spent working on the activities the Teachers had designed. At the beginning of each class a list of problems would be projected on to a screen at the front of the class, and the students would use this time to try and work them out (either in small groups or on their own). During this time, the Teachers would be circulating throughout the class to offer clarifications for any student(s) who might need them. At the discretion of the Teachers, there were instances of whole class discussion and reverting temporarily back to the traditional chalkboard-lecture paradigm.

1.5 Goals and overview of the research

The goal of this research is to analyze and reflect on the teaching approach the Teachers designed and implemented as a tool to help the Teachers achieve their goals as described above.

We propose an analysis of the classroom interactions and, to some extent, the activities, using notions developed within the Theory of Didactic Situations developed by Brousseau (1997), namely the notion of *didactic contract* and of *fundamental situation*. We also consider the idea of *didactic customs* brought forward by Balacheff (1999), and how this idea 'complements' the notion of the didactic contract.

We consider Sierpinska, Nnadozie, and Oktaç (2002)'s framework for Theoretical Thinking (TT) in the analysis of the classroom activities to help us identify moments where the students of MATH 200 might have engaged in TT.

1.6 Overview of the conclusions

We believe that, in the end, the Teachers were successful in achieving (at least) two of their goals. They were able to provide their students with opportunities to engage in theoretical thinking, and their students were able to succeed in the course in the institutional sense – there were no significant disparities in the failure rates between the Teachers' section of MATH 200 and the three other sections that were being taught at the same time using a 'traditional' approach.

While the Teachers believe that they did the best that they could to acknowledge and respect the students' different previous experiences and life goals, there is no way to know if the students themselves felt that this was the case.

1.7 Structure of the Thesis

Although there is plenty of research out there concerning the teaching and learning of algebra, and some research on teaching mathematics to adult learners, there is very little research available on the teaching of algebra to adult learners. In Chapter 2 of this thesis, we focus mainly on the previous research concerning adult learners of mathematics, with an emphasis on their attitudes towards the subject matter.

In Chapter 3 we provide a brief overview of the theories and perspectives that are considered in this thesis (both by the Teachers and by the Researchers).

In Chapter 4 we describe MATH 200 as it has been previously taught (the 'traditional' or 'institutional' approach), and then outline the Teachers approach, and how this might serve the goals that they had set up for the students at the beginning of the semester.

Chapter 5, the analysis, will be presented in two parts. The first part will be an analysis of how breaking specific classroom customs influenced the way that mathematics knowledge was presented to the students, and how breaking these customs in particular might serve the Teachers' goals. The second part is an analysis of the activities proposed to the students of MATH 200. As will be elaborated on in Chapter 5, these activities were classified into 7 categories. We examine examples from each category to look for opportunities to engage in theoretical thinking. We also look more in-depth at two activities, using Brousseau's notion of fundamental situation and examining students' responses to see whether or not they did engage in theoretical thinking.

In Chapter 6 we present conclusions and discuss possibilities for future work.

Chapter 2. Literature Review

Although much of the analysis in this thesis was done considering Brousseau's theory of didactic situations (TDS), there is not yet (to our knowledge) any literature on this theory as it pertains to adult learners studying elementary algebra. For this reason, we will not be discussing TDS-oriented research in this literature review. Instead, the goal of this literature review is to provide a general overview of the current landscape of research in adult mathematics education as we felt it relates to our study. In particular, we organized this literature review around two of the assumptions made by the Teachers (see chapter 1, p. 14), students' attitudes and their mathematics background and life goals; sections 2.2 and 2.3, respectively. In section 2.1 we briefly refer to the adults' learning mathematics sub-field of mathematics education.

Conducting this literature review played an essential role for designing the teaching approach; in particular, for (a) designing tasks to engage students in theoretical thinking without shattering their confidence, or, actually, while boosting their confidence; and (b) for designing tasks that would acknowledge their previous knowledge.

2.1 Adults' Learning of Mathematics

Adult students are defined by Gustaffson and Mouwitz as a people who attend classes at the compulsory or upper secondary school level in adult education programs that have been arranged by the government, or study at an adult educational facility.

We shall also define *school mathematics*, as the mathematics taught in the formal education system from pre-school up until the end of secondary school. This subject (mathematics) is often given a high status: it is difficult to learn and has a high value in society (Gustaffson & Mouwitz, 2004).

In the early 1990's, adults' learning of mathematics was opened up as a new field of research. The ground for the emergence of this field was located in two different processes: institutionalizing processes, where schools for adults are being subject to the same regulations as schools for children, and de-institutionalizing processes, where the focus is on the mathematics that adults learn in areas outside of school, such as the workplace (Wedege, 2010). Interestingly, the 'traditional' or 'institutional' approach to teaching MATH 200 (which we describe in detail in chapter 4) is (or looks like) the result of an institutionalizing process: the regulations and norms that govern the approach seem to be based on the beliefs that the targeted student lacks motivation, is inclined to procrastination, and doesn't know any algebra - all beliefs that may result from imagining the MATH 200 student as an adolescent taking a first algebra course in high school. Furthermore, the approach seems to be blind to (or to deny) students' previous knowledge (which quite often seems to have been learned, or largely practiced, in de-institutionalized contexts).

Some of the first research issues to be raised in this 'new' research domain were mathematics and gender, mathematics and anxiety, and mathematics implicit in the traditional crafts of different cultures (Wedege, 2010).

In 1994, with the help of Diana Coben, a research forum was formed to discuss the issues of adult learning of mathematics. The forum itself was named ALM, **A**dults' **L**earning of **M**athematics, and the International Handbook of Mathematics Education made reference to adult students for the first time in 1996 (Wedege, 2010). ALM as a research field, however, is still quite young and a review of the research conducted in 2003 by Coben et al. concluded that although the field is fast developing, it is also under-researched, under-theorized, and under-developed.

In what follows, we describe the two aspects that have been discussed in the ALM literature that had a significant impact in our reflections as Teachers and Researchers in this study, namely adult students' attitudes towards mathematics and their mathematics backgrounds and life-goals.

2.2 Adults' attitudes towards mathematics and learning mathematics

'Attitude' is a word that is used frequently when speaking of adult mathematics learning. Before jumping in to a discussion of studies that have dealt with attitudes and affect in adult mathematics education, it will be beneficial to first speak of them in a broader context.

There has been an agreement that attitude should be considered as an important factor in mathematics learning (Leder, 1985). However, the subject has been problematic as researchers have not been able to settle on one concrete definition for the word 'attitude'. Thomas and Znaniecki defined attitude in 1918 as "a process of individual consciousness which determines real or possible activities of the individual in the social world", while Thurstone defined the concept of attitude in 1928 as "the sum total of man's inclinations and feelings, prejudice or bias, preconceived notions, ideas, fears, threats, and convictions about any specific topic." Another definition was put forth by Allport (1935), who said that an attitude is "a mental and neural state of readiness, organized through experience, exerting a directive and dynamic influence upon the individual's response to all objects and situations with which it is related" (Leder, 1985).

A more recent, 'three dimensional' definition was constructed by Ruffel, Mason and Allen (1998). The three dimensions of attitude as they describe are:

Cognitive – Expressions of beliefs about an attitude object

Affective – Expression of feelings towards an attitude object

Conative – Expressions of behavioral intention.

They also talk about how attitudes, in particular towards mathematics, are usually spoken of in terms of 'positive' or 'negative'. This supports yet *another* definition put forth by Ajzen in 1988, who defined attitude as "a disposition to respond favorably or unfavorably to an object, person, institution, or event."

Evans & Wedege (2006) said that adults experience education as a field of tension between what one wants (or has) to learn, and various constraints. They also identified three different dimensions of adult learning: the cognitive dimension of knowledge and skills, the affective dimension of feelings and motivation, and the social dimension of communication and cooperation. In order to accommodate these different dimensions, many theoretical frameworks have to be 're-constructed'. Lave's theory of situated learning and Engeström's theory of expansive learning are examples of existing theoretical frameworks that have been adapted to better suit the situation of adults learning mathematics (Evans & Wedege, 2006).

The ALM literature discusses several attitudinal aspects that relate to adults learning mathematics. Gustaffson & Mouwitz (2004), discuss affective factors, social factors, and previous educational experiences. Evans & Wedege (2006) also discuss the role of previous experiences and add to the picture the role of motivation.

Affective factors in adults' mathematics education, as these have been discussed in the literature, refer to the need of perceiving learning as meaningful. It must appear relatable and relevant and of practical use in situations the adult might face throughout life. The feeling that the learning is irrelevant (to one's own experiences) may likely lead students to conclude that the subject (to be learned) is meaningless (Gustaffson & Mouwitz, 2004).

ALM researchers have argued that the social aspects of education, both teacher-student and student-student interactions, have a strong impact on possibilities for learning, as does the pace at which a course is taught (Gustaffson & Mouwitz, 2004). Many adult students fail to complete their given courses because the material is condensed and time is tight. Mathematics courses are often cited as an example of these issues. Taking a more flexible approach to focus on the informal knowledge that the adults already possess can help build self-confidence and motivation, and may help alleviate any anxieties or learning blocks the student may have towards mathematics.

Negative experiences from past mathematics course are often cited as having inhibiting effects on adult students studying mathematics. They can have feelings of anxiety and disassociation which can lead to the student closing doors on life goals that they no longer view as possible to achieve. Some examples of past negative experiences given by adults students are the inability of their teacher to provide adequate explanations, a lack of 'caring' on the part of the teacher, a hurried pace in covering course material, and the apparent incomprehensibility of the material. Many students were able to give concrete examples of times when they felt particularly humiliated or embarrassed in a past math class, and these experiences led to a lifelong bitterness. However, in light of this bitterness, adults nevertheless still return to study again, and are prepared to make considerable personal sacrifices to engage in learning mathematics. They may feel like they need to overcome their feelings of inadequacy, as many see success in mathematics as a sign of intelligence and an ability to learn (Gustaffson & Mouwitz, 2004).

Many adults often experience 'resistance' when coming back to school to learn mathematics, and this resistance is often explained on the basis of a lack of motivation on the part of the learner (Evans & Wedege, 2006). Resistance in math education will usually result from the situation where the learner has found themselves to be competent in everyday life without the use of mathematics, or that they have yet to experience mathematics as something that is relevant to their everyday life (Evans & Wedge, 2006). The goal of this resistance, then, is to protect oneself and one's self-perception.

Other forms of resistance can be seen in the notions of non-consideration and rejection (Jarvis, 2001). Non-consideration refers to situations where learners realize that there is a discrepancy between their ability and what is expected of them, but they do not adapt and learn something new. Rejection refers to situations where learners have an experience but deliberately reject the possibility of learning (Evans & Wedege, 2006).

The resistance of adults in mathematics education also can result from a range of social processes, and can be understood as the result of the 'positioning' that the learner has taken in the past. These can be 'positions' that a learner has been put in (e.g., identified by the teacher as the 'lazy' student) or 'positions' that the learner has put himself or herself in (Evans & Wedege, 2006).

Teaching for Different Learning Styles

A study conducted in 2000 by Duffy & Simpson sought to understand the tensions between the cognitive and the affective in adult students returning to school in order to learn mathematics. When pupils were prompted to offer judgments about their own mathematical abilities, responses were along the lines of 'I was thick', or 'I'm mathematically dyslexic'. Amongst researchers, these attitudes are thought to be common in adult learners. Similar attitudes were even reported by people who have degrees in mathematics – they consider themselves to be unintelligent in spite of what many would consider to de an excellent background in mathematics. The authors hope was to find a way to combat these attitudes, and they arrived at a dual approach which would tackle both the cognitive and affective aspects of learning (Duffin & Simpson, 2000).

The theoretical framework built by the authors classifies learning experiences in to three categories: The 'natural' experience which fits in with what the learner already thinks and knows and can be easily assimilated into pre-existing knowledge, the 'alien' experience which does *not* fit, and does not seem to be related to any previous knowledge, and the 'conflicting' experience, which seems to contradict one's earlier experiences and forces one to think further. Learners respond differently to each of these learning experiences. Natural experiences are 'comfortable' and reinforce what the student already knows, whereas conflicting experiences might limit the student's current way of thinking, but lead to a more 'connected' way of thinking later on. Different responses to alien experiences have been reported: learners may choose to ignore it completely, avoid it by going back to a more familiar topic, or they may internalize it as a completely separate experience – in no way linked to any prior knowledge (Duffin & Simpson, 2000).

In addition to categorizing different learning experiences, the authors also categorized two different ways of learning: a learner may prefer to seek connections between new experiences and old ones in order to facilitate the learning of the new experience, or they might prefer to learn each piece of mathematics as a distinct entity and only make connections between them later on. In both scenarios, however, some form of conflict (and subsequent resolution of the conflict) is required to advance the knowledge of the learner. Based on this, the authors define two types of learners: the natural/conflicting learner and the alien/conflicting learner. Based on these classifications some difficulties can arise since natural/conflicting learners may become frustrated by an alien/conflicting teaching approach, and vice versa. Perry (1970) referred to this problem as ‘different worlds in the same classroom’.

In response to this difficulty, there are teachers who have tried to take a balanced approach in their classrooms. The authors were able to place these teachers in to one of three groups depending on their teaching method. The groups are:

Discovery – Teachers who tend to treat all methods of calculation as equally acceptable. What is important is that the answer to a problem is obtained using some method that is understood by the learner. These teachers generally put emphasis on student ‘readiness’ and interpret misconceptions as evidence that students are not prepared to move on to a new idea.

Transmission – Teachers who view mathematics as an acquisition of procedures and routines. The emphasis here is places on the use of the ‘correct method’, as well as on efficiency.

Connectionist – Attempt to place emphasis on the links between different topics. The teaching attempts to build on students’ existing strategies, but the teacher also has a responsibility to improve the efficiency of some of the more naïve strategies that the student might be using.

A transmission teacher may more often employ the use of alien experiences, while a discovery teacher may be more suited to a natural environment. Reports have suggested connectionist teachers, the more flexible of the three, are generally more successful. It is suggested that this is because they are able to model and respond to the different ways in which their students learn (Duffin & Simpson, 2000).

Positions and Beliefs of Adults Learning Mathematics

While preparing to teach MATH 200, the Teachers wanted to try and foster an environment where all sorts of adult learners would be able to come together to learn about and discuss mathematics; a 'safe' environment where the students could feel comfortable asking questions without feeling inadequate. Although they expected that perhaps some of their students would feel comfortable with the material, they believed that there would be many more who were not comfortable with it.

According to the literature, there are three main positions, or mindsets, taken by adults who are in the process of learning mathematics. The first is the position that they are 'not here to learn mathematics'. While mathematics is often part of a 'package deal' in adult education, many adults are surprised to learn that they need to study it to move forward ("I'm here to study to be a nurse, not a mathematician"). This attitude is usually attributed to poor experiences with math in the past, but this is not always the case. An example given by Wedege was that of engineers who were training to become secondary school teachers. They were surprised to learn that they had to study math because they believed that their mathematical background was sufficient to cover the material taught at secondary school (Evans & Wedege, 2006).

The second position is 'Mathematics – That is what I cannot do'. This position deals with the phenomenon that as soon as someone succeeds in applying a piece of mathematics to their everyday life, it ceases to become mathematics and becomes 'common sense', therefore they never see themselves as successful at mathematics. The mathematics that these people do, that is not recognized as mathematics, is called unrecognized or invisible mathematics (Evans & Wedege, 2006). Mathematics becomes something unattainable for these people, and this negative self-image affects their self-confidence and perpetuates the idea in society that mathematics, as a field, is open only to a select group of people.

The third position often taken by adult mathematics learners is 'No, I don't use mathematics at work'. Often times, if asked if they use mathematics at work an adult will answer with a flat 'No'. However, in many of the cases they simply do not realize that the math that they *do* use is simply hidden (in computer processes, in technology, etc.) (Evans & Wedege, 2006).

Duffin and Simpson (2000) identify a 'quartet' of emotions that are typically associated with adults who are in the process of learning mathematics. These are confidence/frustration and security/anxiety. These emotions are regulated by the learner's perception of their ability to move towards their goal, but initially many adult mathematics learners have come so see mathematics as an 'anti goal': something which is to be avoided, and through this avoidance they gain a sense of relief. In order to design a 'support system' for teaching mathematics to adults in a university setting, it was recognized as important that learners should be able to draw on their pre-existing abilities and methods to help form arguments within the subject, with the hope that students would gain an increased ability to reflect and control their own learning process. Teachers can use the prior knowledge of the students together with constructive conflict to build stronger understanding, with the hope that mathematics would after be viewed as a goal that could be moved towards, instead of an anti-goal (Duffin & Simpson, 2000).

When designing MATH 200, the Teachers wanted to reduce the negative attitudes towards mathematics typically held by mature students. They believed that their discussion based approach, and having the students take a more active role in their own learning, was a good way to try and achieve this.

2.3 Role of adult students' previous mathematical knowledge and life goals

Lifelong learning and life-wide learning stems from formal, informal, and non-formal educational experiences (Wedege, 2010). What kinds of learning do adults get from these different environments, and how can we place value on these different kinds of learning? The small number of studies that have been carried out in this area show that today's 'school mathematics' do not necessarily provide a good foundation for adult students who return to school. Quite the opposite, school mathematics may even interfere with the informal knowledge that adults hold and can result in lower test scores when the student reattempts the course (Gustaffson & Mouwitz, 2004).

The informal knowledge that an adult acquires in various circumstances can be used to add new dimensions and context to teaching materials used in school mathematics. Recognition of prior learning is a key issue here, as is guidance. The knowledge that an adult has acquired throughout their life is an integral part of their identity and self-regard.

Adults have complex internal structures, and this may lead to higher chances for a mismatch between learning preference and teaching style. Adults respond to these mismatches differently than adolescents in the secondary school system. An adult learner *chooses* to attend class, and an unresolved mismatch can cause them to choose to not attend just as easily. To avoid any problems caused by these mismatches, it is important that educators have an idea of how adults learn mathematics and more importantly, and understanding of how adults may *understand* mathematics. In order to have a definition of understanding that fit with their theoretical perspective, Duffin & Simpson (2000) came up with a three-pronged definition of understanding:

i) *Building Understanding* is the formation of connections between internal mental structures.

ii) *Having Understanding* is the state a learner is in by virtue of the connections that they have formed at any particular time.

iii) *Enacting Understanding* is the use of these connections to solve a problem, or construct a response to a question.

While adult students may enter a classroom with some connections already formed, many do not have a sufficient number to feel they have the internal aspects of understanding (such as comfort/confidence when it comes to the course material) or exhibit the external behaviors from which we may infer understanding (such as being able to explain answers, and derive consequences from these answers). One suggestion offered is that teachers should provide situations to their students where enacting understanding is the only real way to approach the problem. Asking for an explanation for why a method works, a question set in a strange context, etc… make it less likely that the student will be able to rely on memory alone for the answer, and will thus have to use their pre-existing connections to build a solution (Duffin & Simpson, 2000). This notion seems to be linked to reflective thinking, one of the aspects of theoretical thinking in which the Teachers' hoped to engage their students.

The Teachers expected that the students they would encounter in MATH 200 would all have very different backgrounds: they would be different ages, have attended different schools, and all had different previous experiences with mathematics. By breaking from the customs ‘The teacher will tell me how to solve each type of problem’, ‘The teacher will tell me if I am right or wrong’, and ‘There is only one correct way of solving a mathematics problem’ the Teachers hoped that the students would instead build on their preexisting knowledge to develop strategies to solve the problem(s) at hand. By allowing the students to construct their own methods using the knowledge that they already had, the Teachers hoped that the students would feel like their preexisting knowledge was valued in the MATH 200 classroom.

Chapter 3. Theoretical perspective and framework for analysis

As mentioned in the introduction, for the purposes of this thesis we refer to 'Teachers' and 'Researchers' as separate entities, although we filled the role of both. When we talk about the Teachers, we refer to our role as teachers of the course: designing the activities, teaching the class, facilitating the discussions between students, marking, etc. When referring to the Researchers we are speaking of our role as researchers analyzing and reflecting on the teaching approach and its implementation with regards to the goals set up by the Teachers (see Chapter 4, section 4.2).

In this chapter, we introduce the frameworks used to analyze the activities designed by the Teachers and the discussion-based teaching approach. To analyze the activities, we consider

a. a model of Theoretical Thinking (TT), developed by Sierpinska, Nnadozie, and Oktaç (2002); and
b. the notion of fundamental situation and didactic variable (Brousseau, 1997).

It is important to note that the model of TT was known by the Teachers and in some informal way was used by them as a guide to develop the class activities (see section 4.2).

A discussion of the discussion-based teaching approach is presented in terms of the notion of *didactic contract* (Brousseau 1997) and *didactic customs* (Balacheff, 1988).

The chapter will end with a few brief notes on the different methodologies used by the Researchers as they performed their analysis.

3.1 Didactic contract and didactic customs

If the didactic situation is the game that takes place between the teachers, the students, and the didactic milieu, then the *didactic contract* can be viewed as the norms[1] and strategies that are appropriate for this game (Brousseau, 1997)

The norms of a didactic contract are generally not explicitly given, and are essentially local to a classroom or even to a didactic situation in a given classroom; they correspond to a (perceived) implicit agreement between the teacher and the students to play a given game in a given didactic milieu. Although the norms of the didactic contract are not made explicit (the teacher and students don't sign any form of binding document) they are most definitely there and both students and teachers know when they are being broken.

[1] In his work, Brousseau used the word "regles". We have chosen to translate this as "norms" to emphasize the non-legislative character. We follow Ostrom (2005) and consider norms as, typically implicit, mechanisms that regulate participants behavior – breaking norms does not entail "legal" consequences, but social ones (e.g., being orally reprimanded, appearing as someone who does not understand what has to be done or what is being said, etc.). In the context of adult education at the university level, we feel that "norms" better describe the role of the didactic contract.

When someone, either student or teacher, breaks the didactic contract there are not generally any 'official' penalties. For example, if a student was enrolled in a class where the didactic contract had been negotiated in such a way that questions were only to be asked at the end of class, this student would be breaking the contract if they were to interrupt the teacher while he or she was explaining a proof at the board to ask for further explanation. Conversely, if a teacher enters the classroom, calls a student to the blackboard, and asks them to teach the day's material, then the teacher is the one who is breaking the contract. A student may be prepared to go to the blackboard to solve a problem, but they would most probably not be prepared to teach an entire topic to their peers.

Balacheff (1999) suggested that the notion of didactic contract, as it had been described in the literature up to then was insufficient to account for the complete set of social phenomena that regulate the functioning of knowledge in a class. In analyzing the mechanisms that regulate this functioning, Balacheff refers to 'laws'[2] in a classroom, which (by their legislative nature) do not belong to the didactic contract, and customs. An example of a student breaking a 'law' would be if they hand in an assignment past the deadline. The institution often controls assessment dates and deadlines, and the student may be subject to some sort of penalty (for example, a grade deduction) for not respecting them. A teacher, on the other hand, would be breaking a law if she arrives to class 20 minutes late and completely unprepared to give a lecture. Her students, if they so desired, could file an official

[2] What Sierpinska et al. (2008) call "rules" (see also, Hardy, 2009).

grievance with the dean, department chair, etc… and the teacher could be subject to disciplinary measures.

As the norms in a didactic contract, customs too are a set of normative practices established as such by their use, and which, in the majority of cases, are established implicitly (Balacheff, 1999, p. 25). These customary practices, however, have a long-term character, established by repeated use, in opposition to the temporary nature of the norms that pertain to a didactic contract. After years of participating in an educational system (or culture), teachers and students perceive (or co-construct?) a number of norms they abide by regardless of the particular classroom or didactic situation they find themselves in.

Laws are the result of formalizing and making explicit the norms of a customary society; this process of formalization and of explicitly stating certain norms transforms a customary society in a legal society. The difference between the two is that customs are almost unconsciously followed, whereas laws require a figure of authority that makes them explicit and enforces them. Laws make judges necessary, and Balacheff notes that 'to satisfy the law might mean, in the first place, to satisfy the judge' (Balacheff, 1988, p. 27). Today's classrooms at the college and university level are legal societies: there are laws, there are customs and there are didactic contracts. The tensions that may arise among these three different mechanisms that together regulate the functioning of the classroom are important to our analysis. As it will be discussed later on, the situations that the Teachers designed often required re-negotiations of the didactic contract, but, in some perhaps more challenging sense for both the students and the Teachers, they required an abandonment of customary practices.

Customs are more global in character than contracts. In (mathematics) education, customs will probably not vary too much from classroom to classroom, but instead may vary as a function of the level of education. An example is how the definition of 'acceptable proof' varies from elementary levels of education to more advanced ones (ibid, p. 26). In an Analysis course, the customs dictate that a certain level of rigour will be required when proving a theorem. In an eighth grade classroom, customs may require a far lesser degree of rigour.

The didactic contract, in contrast, is more local. It is specific to each individual classroom and is often 'negotiated' when a particular task or lesson requires that the rules for social interactions be defined in a new way. Customs are important when the contract is being negotiated, as tension may arise in negotiating norms that are against customs; moreover, in some situations, customs may even dictate what is negotiable and what is not.

To illustrate the importance of distinguishing customs from the local norms in the didactic contract, in particular in relation to our analysis of the situations designed by the Teachers for MATH 200, we discuss the 'paradox' presented by Brousseau as inherent to the didactic contract: Students today tend to believe that it is a norm of the didactic contract that their teacher will show them, often quite explicitly, how to solve different types of problems. Mathematics has, then, become more of a routine – an activity where one needs to only look at the problem, determine which formula to use, and then apply that formula. However, if a teacher complies with this expectation (or 'norm'), then the students have not really learned anything in the context of the theory of didactic situations since the knowledge that they obtain was not something that they constructed themselves. This paradox is summed up with the following quote from the text:

> 'So the didactic contract puts the teacher in front of a paradox: everything that the teacher undertakes to make the student produce expected behaviour tends to deprive the student of the necessary conditions for understanding and learning of the notion she aims at; if the teacher tells the student what she wants, she can no longer obtain it' (p. 41).

However if we keep in mind the distinction between contract and custom, this is no longer a paradox at all. The expectation that a teacher needs to explicitly tell the students what to do, how to present their answers, etc., is a norm that has been established in the minds of students over the course of their studies (which in the case of university students, accounts for *many* years of exposure to customary practices). Thus, although the student is still not constructing the knowledge on their own, the 'paradox' is no longer because what is being broken is not a feature of the didactic contract but a classroom custom.

In the case of this research, the Teachers, using their previous experiences teaching introductory algebra courses to adults, assumed that students would be arriving to their class with certain didactic customs that they, the Teachers, wanted to break away from. These norms, the Teachers assumed, have become ingrained in the minds of the students over years of exposure to mathematics classes that take a 'transmission of knowledge' approach to teaching and learning and the idea that to succeed in school mathematics they need simply to replicate the problem solving methods shown by the teacher. The students will have come to think of mathematics tasks as routine – all they need to do is recognize which formula they need to apply, and then use it to solve the problem at hand. Related to this is the notion that it is the duty of the teacher alone to determine the correctness/incorrectness of a student's work, and as a result the student has neither authority nor responsibility or sense of authorship over their own work.

There were three didactic, customary practices that the Teachers assumed the students would bring with them into the MATH 200 classroom (see p. 14): that the Teachers would lecture to the students explicitly on how to solve different kinds of problems, that the Teachers would tell the students whether their work was correct or incorrect, and there is only one acceptable method for solving a mathematics problem. A detailed discussion of these is presented in section 4.2 from the perspective of the Teachers. In section 5.1, we (the Researchers) discuss the interactions in the MATH 200 classroom studied here in light of the notions of laws, didactic contract and didactic customs presented above.

3.2 Fundamental situations and didactic variables

One of the questions that we (the Researchers) had when considering the activities of MATH 200 was 'What features of a classroom activity make it a "good" activity?', in the sense of its potential to promote the goals of the Teachers. In order to address this question, we needed a framework that would allow us to look in further detail at the composition of the activities to be analyzed. We felt that the notion of fundamental situations as well as the concept of didactic variables, both developed by Brousseau, could provide a context to this task.

The notion of fundamental situation was developed within the Theory of Didactic Situations (TDS) (Brousseau, 1997). In what follows, we present some elements of this theory as there are relevant to our study. Our interpretation of these elements is based on the notes prepared by Sierpinska (1999).

In TDS, didactic situations are described using the metaphor of a game that takes place between the teacher, the students, and the didactic milieu (which can be understood as the learning environment). The goal of the teacher is to engage the student in the game, with a particular piece of mathematics knowledge as the focus. For a student in a didactic situation, knowledge is defined as having an understanding of the ground norms and winning strategies particular to the game and being able to put those strategies in place. It is important that the learners[3] themselves construct the knowledge. In doing this, the hope is that the learners gain a deeper understanding of the material; if a teacher simply tells a student what to do, they are not really learning a piece of knowledge. Rather, they are learning how to apply a piece of knowledge that has been given to them.

Within a didactic situation, learning is not as simple as copying down the notes that a teacher may write on the blackboard. Instead, students learn by making sense of the different situations they find themselves in within the didactic milieu, and by developing ways of coping with these situations. In order to teach a particular piece of knowledge, it is the job of the teacher to design the situation so that this knowledge becomes *essential* for the student to 'survive' the game.

Students and teachers view didactic situations differently. A student may view a didactic situation as a means to achieve a life goal (for example, "I want to be an engineer"), whereas teachers might look at a didactic situation from the perspective of a designer/researcher, with the goal of attaining a certain objective (a curricular objective, research objective, assessment objective, etc…). In the case in question, the Teachers had three particular goals: to engage students in theoretical thinking, to respect and acknowledge students' different mathematical backgrounds and life goals, and to prepare them to pass the course.

[3] TDS is a theory conceived and developed considering elementary, pre-university education; below we discuss how the relevant concepts can be thought in the context of MATH 200, a course with the (official) goal of "re-teaching" mathematics topics to which students have already been exposed to.

Four types of didactic situations have been described in the context of TDS, depending on the type of game that the teacher wants to play with their students. The different types account for the level of involvement of the teacher.

1) The situation of institutionalization. In these situations, the teacher acts as a representative of some 'official curriculum', or 'official mathematics' as the ministry of education or the textbook that is in use presents it. The teacher imparts the 'correct' definitions, theorems, and terminology. Knowledge is pre-established, as opposed to an answer to some scientific inquiry and is validated by someone in a position of authority rather than by having the students check their work for logical inconsistencies.

2) The situation of validation. Here, the students take on the role of 'theoreticians' whose task is to explain some phenomenon or to verify some conjecture. The teacher evaluates their work as an equal, who intervenes only to try and steer the students in the right direction (to help them 'survive' in the game). In this situation, knowledge is dynamic (more like a theory in the making, as opposed to an institutionalized theory).

3) The situation of formulation. The students exchange and share observations between themselves, which allows these situations to be based on a shared experience. They might not yet have the correct language to express what they mean, so part of the situation involves coming up with a language that

they can agree on. The teacher oversees the exchanges, and makes sure that everyone is on the same page. Knowledge here is personal, and to be able to communicate it in a way that can be understood by others it needs to be de-personalized.

4) The situation of action. Here, the teacher creates a milieu in which students can engage with but then completely withdraws from the situation. The milieu has to be created so that students both want to engage with it and are interested to satisfy their own curiosities, and so that they already have the requisite knowledge to construct the solution by themselves. Here, knowledge appears as a means to solve a problem.

An interesting observation that can be made is that when new mathematics knowledge is being constructed the process of formalizing this new knowledge often works from action to institutionalization (Sierpinska, 1999, Lecture 1 Page 4). In today's classes, at least at the college and university level, however, lectures are typically dominated by situations of institutionalization, which present the knowledge to the students while almost never discussing how that knowledge came to be and without providing students with opportunities to engage in validation, formulation or action, in the sense described above.

One of the main aims of TDS is the design, and study of the design, of 'good' activities through which students might construct for themselves the knowledge required to 'survive' in the 'game' set up by the teacher; activities in which the targeted knowledge is essential to 'survive' the 'game'. Within the TDS, the framework used to design these activities is called 'didactic engineering'.

By analyzing the knowledge that one wishes to teach, and by listing all the variables of a problem-situation (not simply the problem itself, but how the problem has appeared, what is at stake in solving the problem, the aims of solving the problem, etc…) pertinent to the situation, one gets a *fundamental situation* associated with that piece of knowledge. The fundamental situation for a piece of knowledge to be taught can act as a 'blueprint' when designing classroom activities.

For example, when teaching the concept of number you may have to consider the following variables: the size of the sets you'll be dealing with, whether the sets are continuous or discrete, how the concept is to be used, and what representation of number will be used. When you assign 'values' to these variables, you get a *specific situation*. For example, the specific situation for teaching children the names of natural numbers for everyday counting purposes would be dealing with small, discrete sets. The context of use would be to compare the sizes of different sets, and the representation of number that is used is that of 'oral numerals' (one, two, three…).

A teacher engaging in didactic engineering needs to carefully construct their didactic situations: The problem-situation that the students will be faced with needs to be subject to certain constraints –these constraints need to satisfy the variables of the specific situation that they are dealing with, and may change depending on what subject matter the teacher wishes to teach.

One has to be careful when designing the game. If you are teaching an algebra class and want your students to come up with the distributive property on their own, it might not be a good idea to only give them activities of the type 'Perform the multiplication: $5(3+2)$'. In this case, it would not be unexpected for the students to first add the 3 and the 2, and simply multiply 5 by 5. They will likely obtain the correct answer, but without constructing the desired knowledge.

The goal of the specific didactic situation, from the perspective of TDS, is to propose students a game they can *only* win (or survive) if they construct winning strategies (thus *learning* the knowledge at stake).

Research done from this perspective has dealt mostly (if not always) with the construction of *new* knowledge. However, when considering a classroom of adult learners who are not innumerate (they may have different levels of numeracy, but they are not mathematically-illiterate), this goal may have to be rephrased. One of the assumptions the Teachers made about the MATH 200 students (see p. 14 and section 4.2) was that they have already been exposed to arithmetic and algebra, at some point in the past, in schooling and also, perhaps, in working environments, and therefore already have developed strategies for dealing with some of the topics covered in the course. Therefore, we have modified the notion of fundamental situation to take into consideration the mathematical background of the MATH 200 students, and the role of this background in the (re)learning of algebra concepts.

In the following section, we present the notion of fundamental situation as it has been modified for this study and reflect on the implications of considering the notion of didactic situation in the context of the teaching of algebra to (not innumerate) adults.

3.3 Fundamental situations in the context of this study

A fundamental situation is, essentially, a blueprint for a class activity. The main idea is that activities can be designed so that the constraints of the situation are such that the student who engages in the activity will construct the new knowledge that the teacher wishes them to build. As mentioned above, however, this was not our case. It was assumed that the students of MATH 200, while not strong mathematically, would have had some previous exposure to the concepts being taught to them. The goal of fundamental situations, as described by Brousseau, would not apply for the students of MATH 200 as they were not necessarily, or all the time, constructing *new* knowledge. They were, instead, *rebuilding* or *refining* their knowledge of concepts that they had previously seen. For the analysis, the Researchers needed to adapt the notion of the fundamental situation in order to account for this difference. When designing the activities, the Teachers, based on their experiences as researchers, assumed that many of their students would hold some common misconceptions when it comes to the subject matter (for example, they may believe that to add two fractions you need only add the numerators and denominators together). The Teachers believed that to adapt the notion of the fundamental situation for adult learners only a small modification would need to be made – their classroom activities would be subject to a new variable. In addition to the normal difficulties that students encounter when learning new material, these students have these common misconceptions as an extra obstacle to overcome. The Teachers wanted their activities to bring these misconceptions to the surface so that they could be exposed an eliminated, and so the extra variable that they considered for their activities became 'the common misconception to be addressed'.

We denote this particular version of fundamental situation, where the 'common-misconception' variable is always present, by FS*.

In this context, the goal of a didactic situation becomes slightly less clear. Is the goal to construct new knowledge, or to expand the knowledge that the student already has? Or is the goal to identify and then eliminate common misconceptions? We cannot really say what the goal is with any certainty, as the goal (or goals, in some cases) may vary from student to student. For a student who is returning to mathematics classes after a short break, the goal may be to simultaneously expand on their knowledge while eliminating misconceptions. However, a student enrolled in MATH 200 who has never taken an algebra course before wouldn't hold any misconceptions to begin with, so instead the goal becomes constructing new knowledge.

These misconceptions were characterized based on previous research and on the Teachers' experiences as teachers and researchers. Using their experience as researchers, the Teachers hoped that they would be able to design their activities in such a way that they would be able to 'draw out' and eliminate common misconceptions, all the while students are building or adapting their 'winning strategy' for the task at hand. Thus, the FS* will not only make necessary the piece of knowledge at stake but will also draw out the associated misconceptions, if any – making them visible to both the teacher and the student.

In this sense, FS* can be seen as having two different goals. For students who already have pre-existing strategies for dealing with the topic in question, the goal will be to refine their strategies and address misconceptions. For students who do not have any pre-existing strategies, FS* will function much in its traditional sense, as a fundamental situation, while possibly 'preventing' some common, well-known misconceptions.

Some of the variables that we will be considering when analyzing the class activities are the purpose of the problem, the topic addressed, and the type of numbers involved (abstract or concrete), as well as the 'new' variable of the misconception to be addressed.

3.4 Theoretical Thinking

Before the semester started the Teachers, using their previous experiences as teachers and researchers, designed the course activities with the goal of providing students opportunities to engage in theoretical thinking (TT) as formulated by Sierpinska, Nnadozie, and Oktaç (2002). The Researchers considered this framework to analyze these activities' potential at achieving this goal. In what follows, we briefly introduce Sierpinska et al.'s model and how it was used in the analysis of the activities.

There are three main features of TT: 'reflective', 'systemic', and 'analytic' thinking. Reflective thinking is characterized by students taking an investigative attitude towards the problems that they are faced with. This can manifest itself as a student either reflecting on their solution, looking for a more efficient way to solve a problem, or noticing links between previously solved problems. It can be thought of as the opposite of when a student merely applies a memorized procedure to solve a problem, and then forgets about the problem once it is finished. For an algebra student, reflective thinking may manifest itself in a number of ways. For example, the student might be investigative in considering different approaches that could be taken to solve a word problem, and then deciding which of these approaches they feel to be the most efficient.

The second feature of TT is systemic thinking. Systemic thinking is characterized by thinking about systems of concepts. It is described as definitional, proof-based, and hypothetical. Being 'definitional' means that the concepts are 'defined by reference to other concepts within the system'. Being 'proof-based' means that decisions about the truth of a statement are made by means of proofs, which rely on previously accepted definitions, and conceptual and logical relations within a system. The hypothetical character of systemic thinking manifests itself when the 'theoretical thinker' becomes aware of the conditional character of the statements they are looking at, and when they try to identify the implicit assumptions that are being made. Once these assumptions are identified, the 'thinker' can then study all logically possible cases. A student studying algebra has many opportunities to engage in systemic thinking. As an example, consider the manipulations that need to be made in order to solve a linear equation. Each step in the process needs to be justified by reference to some property or definition (for example the commutative, associative, or distributive properties of real numbers, or the definition of the '=' symbol).

The third feature of TT, analytic thinking, can be broken down in to two different types of sensitivities: linguistic and meta-linguistic. Linguistic sensitivity is characterized by sensitivity to formal symbolic notation, as well as to specialized terminology, while meta-linguistic sensitivity is defined as being sensitive to the structure and logic of mathematical language. An example of how an algebra student might need to engage in the analytic thinking is in being able to distinguish between a variable and an unknown. This would be an example of linguistic sensitivity, whereas a student would need meta-linguistic sensitivity to recognize when a symbolic expression is nonsensical (for example, $2 < x < 1$).

Using the theoretical behaviors described by Challita (2013) as a guide, we analyze certain MATH 200 activities for their potential to foster TT. To do this we consider an 'ideal solution' to the problems in question, a solution generated by us but one that could reasonably be expected (where reasonably is defined within the framework of the Teachers' experiences teaching the course) from a student at the MATH 200 level. In some cases we will also be looking directly at student responses, to see whether or not they did engage in TT.

The following table, from Challita's 2013 Master's thesis, is a summary of the main components of TT as described by Sierpinska et al. (2002).

Category of TT • **Feature of TT** ○ **Sub-feature of TT**	**General description**
TT1 Reflective	Theoretical thinking is aimed at reflecting on, investigating, and extending ideas. Its aim is not merely to accomplish tasks, rather to reflect on curiosities and mental challenges
TT2 Systemic	Theoretical thinking is thinking about systems of concepts, where the meaning of a concept is established based on its relations with other concepts and not with things or events
• TT21 Definitional	• The meanings of concepts are stabilized by means of definitions
• TT22 Proving	• Theoretical thinking is concerned with the internal coherence of conceptual systems
• TT23 Hypothetical	• Theoretical thinking is aware of the conditional character of its statements; it seeks to uncover implicit assumptions and study all logically conceivable cases
TT3 Analytic	Theoretical thinking has an analytical approach to signs
• TT31 Linguistic sensitivity	
○ TT311 Sensitivity to formal symbolic notations	
○ TT312 Sensitivity to specialized terminology	
• TT32 Meta-linguistic sensitivity	
○ TT321 Awareness of the symbolic distance between sign and object	
○ TT322 Sensitivity to the structure and logic of mathematical language	

3.5 Methods for analysis

Methodology for analyzing the negotiation of a new didactic contract

To analyze how a new didactic contract was negotiated in the MATH 200 classroom, we will be operating in both of our roles – Teachers and Researchers.

As the Teachers we will be recalling ‘moments’ that seem to describe a typical scenario that was found in the classroom after the implementation of the new didactic contract – specifically moments that highlight the customs that the Teachers wanted to break from in their classroom. Then we will look at these moments through the eyes of the Researchers to assess whether or not we believe that breaking the classroom customs achieved what the Teachers wanted to achieve in terms of the goals they had set up for their classroom.

At the end of the semester, the Teachers ran a survey with their students about their overall experience in MATH 200. While much of the data from this survey did not end up being used in this thesis, we refer once or twice to comments that students made on the survey as ‘anecdotal data’.

Methodology for analyzing the class activities

To analyze the class activities, and to investigate their potential to foster theoretical thinking, we begin by describing the activity in question using our adapted notion of fundamental situations. Fundamental situations are usually associated with specific pieces of knowledge, and not to specific activities. For this reason it may be better to say that our analysis and model for the classroom activities was *inspired* by Brousseau’s notion of fundamental situations, rather than a direct application of his ideas.

Recall that a fundamental situation is something of a blueprint for a class activity. By listing all of the variables associated with that activity, and then assigning 'values' to these variables, you obtain the specific situation associated with the activity you are dealing with.

Before analyzing the activities, we wanted to come up with a list of variables that could characterize a very general fundamental situation for the activities designed by the Teachers, variables that could then be given values specific to each problem. We came up with six of these variables:

V1 – Category: Describes how the activity fits in to the activity categories, which will be described in section 5.2. These categories are not disjoint so it may be possible for a single activity to be assigned more than one value for this particular variable.

V2 – Topic: Describes where the activity falls in the list of topics to be covered in MATH 200. This variable may also take on more than one value – the first being the unit of the course (linear/non-linear), the second being the topic, and the third being the subtopic within the topic. For example, V2 for a particular problem may look something like 'non-linear; radical expressions; rationalizing denominators'.

V3 – Misconception: Describes the common misconception to be addressed by the problem. This variable may take a value of ‘null’ if no misconception is being targeted. May also take more than one value, if a problem is trying to ‘draw out’ two or more misconceptions simultaneously.

V4 – Feature of TT: Describes which of the three features of theoretical thinking the problem is trying to target. This variable may take a value of ‘null’ if no specific aspect of TT is being targeted. This is not to say that students might not engage in TT while solving the problem, simply that the Teachers did not have a specific aspect of TT in mind when designing the activity. The variable may also take more than one value depending on the problem at hand.

V5 – Solution expected: Problems in MATH 200 were not always asked in the same way. Some problems required the students to perform calculations to arrive at an answer, while other problems provided the students with worked out solutions that they needed to discuss. This variable can take one of these two ‘values’.

V6 – Group Activity/Individual Activity: Any given problem can be assigned one of these two values.

When values are assigned to each variable of a given fundamental situation, FS*, we obtain a specific situation (SS*) for the problem we are considering. In our analysis, more often than we had expected, V3 and V4 took values of 'Null'. This might be considered distressing to some readers, as these variables were the ones most closely tied to the Teachers' goals. As Researchers, we did not consider this to be a large cause for concern. Just because a certain activity is not designed to target a particular feature of TT, this does not mean that the activity is void of opportunities for students to engage in TT. Similarly, student misconceptions may reveal themselves even if a student is engaging with an activity not designed to draw out any one misconception in particular, and students might also have misconceptions that were not expected by the Teachers.

Our analysis of the opportunities to engage in TT that a given activity affords is based on an 'expected solution'; one that we (the Researchers) expected from students at the MATH 200 level.

The last two activities that we analyze in this thesis (see section 5.2) are activities that took place after the didactic contract for the class had been briefly renegotiated to allow for individual work that would be collected by the Teachers. For these activities, we analyze actual student responses to see if they did, in fact, engage in TT. The responses of four students are analyzed. These students were selected at random from the group of 24 students who had submitted both activities.

Chapter 4. An introductory algebra course at the university level

In this chapter we, in our capacity as the Teachers[4] of MATH 200, explain the design of the course, our goals and assumptions, contrasting them with the institutional approach - by which we mean the teaching approach that can be gleaned from he outline, textbook, and assessments that were in place when the Teachers were assigned to teach one section of the course. The Teachers had previous experiences, both in this and other courses, following the institutional approach. We start by describing the course itself and some of its institutional features, its role in the university and the "typical" student enrolled in it.

4.1 MATH 200

MATH 200, Fundamental concepts of algebra, is a pre-university course (high school level) that is designed to provide students with the background knowledge required for future courses in algebra, functions, and calculus. It is a course that is open to all students enrolled in the university, however, as it is the lowest level mathematics course that is offered, a student is not able to take it for credit if they have completed or received credits for any higher-level mathematics course. Also, students who are enrolled in programs that lead to a BSc or BA in Mathematics and

[4] Recall that, in this thesis, Teachers refers to the two instructors (professor and teaching assistant) in charge of the MATH 200 section in question – other instructors were teaching other sections of the same course, following the institutional approach, during the same academic semester.

Statistics may not take MATH 200 for credit to be applied to their program of concentration, according to the university's academic calendar.

MATH 200 runs over 13 weeks (fall and winter sessions). Every session, the Department of Mathematics and Statistics offers between 3 and 5 sections of the course. Each section consists of two lectures and one tutorial per week. The lectures for a given section run for one hour and fifteen minutes, and are typically overseen by the 'instructor' assigned to that section. 'Instructors' can be full time professors, part-time professors or graduate students. Tutorials take place once a week for one hour. They are not mandatory, and are designed as a place where students can come to ask extra questions if they are having trouble with the material. Instead of being run by the 'instructor', the department typically assigns a graduate student to oversee the tutorial.

MATH 200 is typically taught using what will be referred to in this thesis as the 'institutional' approach. This approach is lecture based, where the teacher showcases different techniques that can be used to solve problems and, as a consequence, the activity of problem solving becomes something of a 'routine'. That is, students can evaluate a problem and quickly know which procedure to apply in order to solve it.

Following departmental policies, this multi-section course has a unique outline that describes the topics to be covered, in which order and how much time is allotted to them. The outline also enforces an official textbook and common assessment (eleven assignments, one midterm and one comprehensive final exam). The course has a course examiner (or coordinator) who is in charge of choosing the exercises in the assignments and preparing the midterm and final exam. While the course examiner has a certain degree of leeway for modifying the course outline (topics have to abide by topics listed in the MELS (Ministère de l'éducation, du Loisir et du Sport) outline for the equivalent high school and CEGEP course), the instructor of the course may feel like they are limited in terms of being able to make changes or adjustments (either to their teaching approach or to the material itself) because so much of their classroom is strictly controlled by the institution.

In section 4.3, it is discussed what changes (and why) were made to the course corresponding to the assumptions that the Teachers held about their students before the semester began, as well as the changes that were made in order to help the Teachers achieve the goals that they had set up for their section of MATH 200.

The outline that was in place when the Teachers were assigned to the section of the course studied in this thesis listed the topics in the following order, which reproduces the order in which topics are presented in typical college algebra textbooks (Hardy and Sierpinska, 2011):

- Real Numbers
- Fundamentals of Algebra

- Linear Equations
- Equations and Inequalities
- Exponents and Polynomials
- Factoring
- Rational Expressions and Equations
- Systems of Linear Equations
- Roots and Radicals

The assignments, also common to all sections, are given weekly via an online system called WeBWorK.

The midterm exam is a one and a half hour, pencil-and-paper examination common to all sections. It covers material from the 'linear section' of the course – operations on integers and rational numbers, linear expressions and problem solving, linear equations and graphing, and systems of linear equations. The exam is typically held after the sixth week of classes.

The final examination is a three-hour pencil-and-paper exam, common to all sections and coordinated by the university's Examinations Office. The final exam is cumulative, and covers all of the material presented in the course.

In her 2012 thesis, Challita wrote "in a review of the assessment materials (such as assignments and exams), one can notice certain 'constants' that mark these materials from year to year (such as the type of function whose limit students are asked to calculate)." These 'constant' aspects of the course have, over time, become *norms* of these courses and are expected by the students. By extension, they become constraints by which the teacher is bound.

Most typically, students enrolled in this course are independent students (students who are missing the necessary requirements to be granted admission into the program of their choice) or mature students (as defined by the University, among other criteria, these students are 21 years or older and have been out of full-time studies for at least 2 consecutive years since age 18). The University has a special entry program for mature students in which, depending on the aimed career, MATH 200 may be a required course. In any case, the people who register in MATH 200 come from a variety of different backgrounds. There are some students who never completed their mathematics courses in high school, there are some who are coming back to school for the first time in many years and need a refresher before moving on to other courses, and there are a few, already registered in BA or BSc non-mathematics programs, who take the course for extra credits.

4.2 The Teachers' assumptions, goals and design

The assumptions

Based on their previous teaching experiences, on anecdotal data about who the students are and what their needs are, and on research on the teaching and learning of mathematics concerning mature students (see chapter 2, Literature review), the Teachers made the following three assumptions.

First, they believed that that the students they would be teaching would have *varied attitudes towards mathematics* as a whole. If a student had poor experiences with mathematics in their previous education, then the Teachers expected that this student might be less than enthusiastic about returning to a mathematics classroom. On the other hand, there may also be students who are quite motivated – students who know that they need to take MATH 200 to enter in to the program of their choosing and as such want to do the best that they can.

The second of the Teachers' assumptions, somewhat linked to the first, is the notion that the students who enroll in MATH 200 will all have *different backgrounds and life goals*. Their past experiences with school mathematics will have been different. Some students will have had good experiences, and some will have had bad ones. Some students' exposure to school mathematics may come solely from helping their child with their homework, while some students may not have encountered school mathematics at all in their lives. Additionally, regardless of where these students are coming from, they may all be working towards very different goals – for example, it is quite possible to have one student who is hoping to be a psychologist sitting together with one who wants to be an engineer, who is in turn sitting next to a student who is simply taking the course as an elective.

Finally, the Teachers assumed that the students would be arriving to class with several different *customs as to what a mathematics classroom should look like*. More specifically, there were three customs that the Teachers were expecting that they wanted to directly address and break from through the teaching approach. The first of these is the notion that it is the Teacher's job and responsibility to show the students, quite explicitly, how to solve different types of problems. The second custom they wanted to break from was the belief that puts the Teacher in a position of absolute authority in the classroom when it comes to the validation of student work – they wanted their students to be able to assess whether or not their work was correct on their own, without any external validation. The last custom that the Teachers wanted to break from was the idea that there is only one 'correct' method for solving different types of problems, and that the teacher will expect this 'correct' solution.

At the beginning of the semester, the Teachers circulated an informal questionnaire to their students in an effort to check that their assumptions held for the particular course that was to be taught. The results from this questionnaire were mainly in line with the Teachers' assumptions, and a copy of this questionnaire can be found in the appendices.

The goals

With these assumptions in mind, the Teachers set up the following goals for their course:

- Engaging students in Theoretical Thinking
- Respect and acknowledge students' different backgrounds and life-goals
- That the students will succeed at the course in the institutional sense

It was the Teachers' belief that any mathematics course should provide students opportunities to engage in theoretical thinking. Even if the students of MATH 200 will not continue to pursue studies in mathematics, the behaviours described in the TT framework are somewhat universal and learning to think theoretically would be beneficial for the students regardless of what career path they take. It was a concern that the large volume of material that needs to be covered in only thirteen weeks in pre-university courses such as MATH 200 would not leave much room for these opportunities. The Teachers felt that, due to the negative attitudes towards mathematics held by some of the students, it would be easy for some people to fall in to the trap of just 'going through the motions' of the course material – i.e. learning how to properly perform a procedure and learning where to perform it without truly understanding what it is that they are doing.

In order to mitigate this, the Teachers wanted to provide students chances to think more deeply about what they were doing, and to make connections between the different topics that were to be covered in the course. They also hoped that engaging students in Theoretical Thinking (TT) would be beneficial for the students in that it might help them to notice inconsistencies in their own work, promoting student autonomy in the process. The Teachers had had previous research experiences with a model for TT (Sierpinska et al, 2002) that they used informally in order to help them structure the activities. We say 'informally' because the Teachers did not design their activities using this model as a guide to help them foster the three categories of TT (reflective, systemic, and analytic – see Chapter 3). Rather, they had vague ideas in their minds of what each category of TT entailed and tried to design a wide range of activities that would help the students to engage in TT.

Their second goal was to respect and acknowledge the different backgrounds of their students. As was stated in Chapter 2, the knowledge that an adult has acquired throughout their life is an integral part of their identity and self-regard (Gustaffson & Mouwitz, 2004). The Teachers did not want their students to arrive to class and feel like the knowledge that they had gained in their previous education needed to be cast aside in favor of the instructor's 'correct' version of mathematics. Instead they wanted to create an environment where all methods and strategies would be considered valid, as long as they were logically consistent and led to the correct answer.

Finally, it was important for the Teachers that their students succeed in the course at a similar rate (or hopefully higher) than the one typically obtained in the context of the institutional approach, despite the significant changes they were making in their section.

Of course, a preferred course of action would have been to make changes to the way that the assessments were done, but the multi-section nature of MATH 200 compelled the Teachers to use the same scheme as the other sections.

The Teachers believed that, of the three assumptions that they made about their students, the negative attitudes and the customs that they bring to class with them would be the biggest obstacles for them in working towards their goals. The student attitudes needed to be 'shifted', while the customs needed to be broken if they were to succeed in achieving their goals.

The design

To achieve these goals the Teachers considered many different means and methods, and one of the first decisions that were made regarding the new approach to MATH

200 was to change the order in which the course material was presented to the students.

The Teachers wanted to design the course in such a way that the students would be able to see the links between the different topics that are covered and then begin to see mathematics as a 'whole', as a system, instead of a list of disjoint, unrelated, compartmentalized topics. This change would particularly help the students to engage in TT, specifically the systemic aspect of TT. The pre-established, long used order of topics seemed to play against this goal. For example, students would be exposed to natural exponents of real numbers in Week 1 when covering Chapter 1 of the textbook ("The Real Number System"). Then, they would encounter integer exponents of generalized numbers or unknowns in Week 6, when covering Chapter 5 of the textbook ("Exponents and Polynomials"). Next, they would encounter rational exponents in Week 13 when covering Chapter 9 ("Roots and Radicals"). The Teachers felt that the span of time between the exposures to different types of exponents was not helpful in proposing a systemic view of the topic.

Another example in the same vein would be the gap between linear equations and systems of linear equations. Students would learn how to solve linear equations in Week 5, while systems of linear equations are only covered in Week 11.

When the Teachers reorganized the material, they did it in such a way that the course would have two distinct units. After being introduced to the basics of the real number system and how to perform operations (minus exponentiation) on both integers and rational numbers, the material was covered in the following order:

- Linear expressions and problem solving
- Linear equations and graphing
- Systems of linear equations
- Linear inequalities

These topics made up the first unit of the course (the 'linear' unit), and the midterm covered everything up until this point. After the midterm, they entered the 'non-linear' unit. The non-linear unit is comprised of:

- Exponents, polynomial expression, and polynomial equations (includes factoring)
- Rational expressions and equations
- Radicals and rational exponents

The Teachers felt that this order would allow them to present a systemic view of the topics in question. Polynomials should follow from linear equations, but only once all topics that deal with linear equations have been dealt with. Rational expressions follow from polynomials, since a rational expression is a division of two polynomials. Radicals and rational exponents were left for the end mostly because it was a compromise with the other sections of the course.

Copies of the “traditional” outline, the one used by the Teachers, and an updated one (after revising the one proposed by the Teachers), all can be found in the appendices.

Keeping in mind that they expected their students to have varying attitudes towards mathematics, the Teachers also wanted to create an environment where any student would feel comfortable getting re-acquainted with the fundamentals of algebra while at the same time be able to having opportunities to succeed at the three goals that the Teachers had set up for them. It was decided, based on the past experiences of one of the Teachers who had previously taught an introductory course on proofs using a discussion-based approach, that teaching MATH 200 in a similar fashion would be the best way to meet all of these criteria.

Instead of the traditional chalkboard-lecture paradigm, where the teacher writes notes and examples on the board for students to copy, the majority of the learning in MATH 200 took place through group work and discussion. It was the Teachers’ hope that running the class in this way would help to account for the different backgrounds and needs of the students, while providing them with opportunities to engage in TT. From the beginning of the first class the students came to understand that each day they would be presented with a list of problems, and that it was up to them (with the help of their peers if they wanted, although some chose to work alone) to come up with the answers. The main goal of allowing the students to take ownership of the course material in this way was to acknowledge and give a role to their knowledge of the material. Aside from writing a new formula or property on the board whenever the material required it (such as when introducing the associative, commutative, and distributive properties of algebra, or when fist encountering the quadratic formula), the Teachers spent very little time at the chalkboard. Instead, they would be walking through the class offering assistance to any student (or group of students) that needed it. The tutorials acted as an extension of the lectures, and had the same format. The only difference was that no new material was ever covered in the tutorials, since they were not mandatory and some students could never attend due to scheduling conflicts. One of the initial goals of the tutorials was to provide extra 'drill-and-practice' problems that the students need to master in order to succeed in the course in the institutional sense.

There were, however, certain topics that required the Teachers to revert temporarily to the 'institutional' chalkboard-lecture method. An example of one such topic was the subject of graphing on the Cartesian plane, where the students need to be shown on the board the different graphing conventions before they can go off and start graphing equations on their own.

After reviewing the literature, and keeping in mind their assumption that the students had previous exposure to algebra, the Teachers decided that the activities should not be designed so that there would be only one 'acceptable' way of arriving at the final answer. Recall that one of the Teachers' goals was to respect and acknowledge the different backgrounds of their students - it is important for adult learners to feel that their previous knowledge has value, and as such the Teachers wanted the activities to help them build on their pre-existing strategies for solving problems instead of imposing one 'correct' method on the students.

For example, in the section of the textbook that teaches how to solve systems of linear equations the problems are all set up in a very specific way and there are separate sections for solving by graphing, solving by elimination, and solving by substitution. In the textbook, for each problem, the student is told exactly which method to use when solving. This deprives the student of the chance to look at the problems critically and then decide for themselves, based on their previous knowledge, which method would be most appropriate for solving the problem at hand.

The following is an example of an activity from the lecture problems when the teachers introduced systems of linear equations:

A system of linear equations is a set of 2 or more linear equations. The goal is to find the points (x, y) that satisfy all the equations in the system. In this course, we will work with systems of two linear equations in two variables.

6. *Solve the system:* $x + y = 4$, $3x + y = 6$.

7. *Graph the two equations in problem 6 in a Cartesian plane; what point represents the solution to the system?*
8. *Thinking about problem 7, what are the possible solutions to a system of two linear equations in two variables?*
9. *Give examples of SLE for each of the possibilities found in problem 8.*

These problems were given to the students to work on before the methods of substitution, elimination, and solving graphically were discussed. Instead, the students were left to their own devices to try and come to a solution. They also tried to include some more open-ended questions (like 8 and 9 above) to try and get the students to think in a more abstract manner.

In the context of acknowledging and giving a place to students' previous knowledge of the subject matter, the Teachers wanted to address the common misconceptions that students may have. These misconceptions are not very different from the ones we may find in any mathematics classrooms, but the difference is that in many cases these students have been living many years with these misconceptions, and it might therefore take some work to reveal them and help them overcome them.

For example, many students are under the impression that if they are given a generalized number preceded by a minus sign, such as $-a$, that this number must be negative. They tend to not consider the case where a itself is negative, making $-a$ a positive number. To try and address this, the Teachers posed the following question on the first day of classes:

Let $'a'$ be a number; explain the meaning of $|a|$. Is $-a$ positive or negative? Is $|a|$ positive or negative?

The examples of classroom activities given so far are typical examples of the problems that students encounter in the 'lectures' as well as in the tutorials, where they are able to team up with peers to try and tackle the problems. While the Teachers would always be present in the class to address any of the questions that the students had about the material, they still wanted to try and find a way to give the students of MATH 200 individualized feedback.

To do this, a few times throughout the semester the didactic contract would be renegotiated to require individual work. In these instances, student responses would be collected by the Teachers and 'corrected' as a means to provide individual feedback. Note that we use the word 'corrected' here, but this is simply for the lack of a better term; These activities were not for marks, nor were the 'corrections' meant to simply show the student how to go about arriving at the correct answer. Instead, the feedback that was given took the form of prodding questions that would try and get the students to realize their own mistakes rather than having them explicitly pointed out to them. The Teachers provided the students with five of these activities over the course of the first half of the semester. The following is an example of one of these activities.

Question:

Is it true that if $|\boldsymbol{b}| > |\boldsymbol{a}|$, ***then*** $\boldsymbol{a} - \boldsymbol{b} < 0$***?***

Answer:

Yes, this is true. Since $|\boldsymbol{b}| > |a|$***, b is farther away from 0 than a on the number line, therefore when you subtract b from a the result is a negative number.***

- *Is the answer above correct or incorrect? Explain your reasoning.*

- *If the answer above is* incorrect*, can you change the wording of the question (or add something to the question) so that the given answer is correct?*

An analysis and discussion of the classroom activities can be found in section 5.2.

Finally, to achieve their goals the Teachers felt it was important to break from certain didactic customs. Breaking from these customs would help them to set up the didactic contract that they wanted to put in place – one in which students have a sense of authorship over their work, promoting autonomy and a shift in their perception of mathematical activity, towards one in which there are different ways of dealing with problems and where once can discuss their preference for, or the efficiency of, one method over another. At the heart of the Teachers' intentions were the convictions that 'mathematical autonomy' is essential for progressing in the learning and understanding of mathematics and that any teaching approach should be planned and implemented so as to give an active role to students' previous knowledge of the subject,

The first of the customs the Teachers wanted to break from was the notion that the teacher would explicitly show the students how to solve different classes of problems. The main mechanism that the Teachers used to break this custom was to almost entirely remove lectures from their classroom – this will be further discussed in section 5.1.

The second custom to be broken was the idea that the Teachers will always let the students know if their work is correct, or incorrect. During the class time when students are actively solving problems, the teachers could usually be found walking around the classroom and addressing any concerns that the students might have. More often than not, a student would call one of the teachers over to ask 'Is this right?' Instead of giving them a straight up answer, the teachers would try and provide methods for the students to decide for themselves whether or not they had arrived at the right answer.

For example, if the teacher could see that the answer was correct, but did not want to say this outright, they might suggest methods that the student could use to verify their answer. These methods might be substituting their answers back in to the original equations when the task is to solve a linear or quadratic equation, or if a student is asked to factor a polynomial the teachers might suggest that they multiply out their answer in order to see if it is equal to the original expression.

If the teachers could see that the answers were *incorrect*, they would try and guide the student back through their work to have them check for any logical inconsistencies or calculation errors. When dealing with word problems, the teachers also tried to make sure that the students had a clear understanding of what their answers meant in the context of the problem. With this in mind, it should be easy for a student to spot an incorrect answer (for example, if the question is something along the lines of 'How many hours does it take for Susan to make $500?' and the student arrives at an answer of -5, they should be able to become aware that something is wrong).

Finally, the Teachers wanted to dispel the notion that there is one, and only one, 'correct' method for solving a given mathematics problem. To do this, the Teachers would be careful to not provide a single solution method for any problem. If solutions were given, the Teachers tried to provide a variety of different ones (often taken directly from student work) and would then try to foster a discussion about the different methods – Which method did the students feel most comfortable with? Which one did they feel is the more efficient solution and why? They might also discuss why a solution would no longer be valid if certain changes were made to a problem. For example, the technique of 'cross multiplying' would be a valid first step in solving the equation

$$\frac{2x+1}{4} = \frac{3}{5},$$

but would be an inappropriate first step if this problem were changed to

$$\frac{2x+1}{4} - \frac{3}{5} = 4.$$

There were some situations (such as when working on practice final exams) where worked out solutions were not given at all. Instead, the students would often be given a list of answers to the problems so that as long as they arrived at the correct answer they would know that their method was valid. The Teachers did not want to provide a single worked out solution for these problems so that the students would not feel obligated to use that same method when solving similar problems.

There were a few things that the Teachers would have liked to be able to change about MATH 200, but were unable to due to institutional constraints. The most obvious of these seems to be the assessments used in MATH 200.

As they are, the assessments that are in place really only address one of the goals that the Teachers had set up for their course – succeeding in the institutional sense. The Teachers might have liked to adjust the assessments so that they better address their three goals, but were unable to due to the fact that assessments were required to be common amongst all sections of MATH 200.

This 'common' requirement is true for the midterm, final exam, and weekly assignments. The assignments were run via an online system called WebWorK, and although the teachers had some freedom before the semester began to choose some questions for the assignments that *might* be better at evoking TT, they were still limited to choosing problems from the online question bank. They could not create their own questions without proper knowledge of computer programming, something that neither Teacher possessed.

Chapter 5. Analysis and Discussion

This chapter is presented in two parts. In the first part, we analyze and discuss the customs that were fostered in the classroom, the ways in which students were initiated in to these customs, the didactic contract(s) that emerged, etc., and we discuss how these related to the assumptions and goals of the Teachers. In the second part we analyze the activities designed by the Teachers which the students engaged in. In this second part, our analysis focuses first on modelling these activities considering a model inspired by the notion of fundamental situation and didactic variable, and we then investigate the potential of these activities to engage students in theoretical thinking (TT).

5.1 Customs and didactic contract(s)

As we have already mentioned, the class that is the focus of this thesis was not a traditional chalkboard-and-lecture classroom. The Teachers of MATH 200 sought instead to run their course using a 'discussion-based' approach, where most of the learning would take place through students actively solving problems.

As it was discussed in section 3.1, interactions in the classroom are governed by certain 'laws' (legislated rules), didactic customs and didactic norms.

In the particular case of this research, the Teachers, based on their teaching and research experiences, assumed that students would arrive to the MATH 200 classroom with certain didactic customs (see section 4.3) resulting from years of exposure to a mathematics teaching and learning style based on 'transmission of knowledge' and a conception of (school) mathematics as mimicking the institutional (the teacher's) way of writing a solution to a standardized (routinized) exercise. In this model, assessing the correctness/incorrectness ('right or wrong') of a solution is the prerogative of the teacher – the student has no authority (nor authorship) over his or her work, and there is only one acceptable way of solving each particular problem.

The Teachers assumed that these customs would become obstacles for negotiating the didactic contract for their classroom. They wanted to break from these customs, but in doing so they wanted to make sure that they did not shatter their students' confidence in their previous knowledge or their abilities. Instead, they wanted to break from these customs while increasing the students' confidence in their abilities. They wanted their students to feel a sense of authorship over their work, and arrive at a place where they would realize that there are many different ways to tackle a problem and would be able to discuss the efficiency of, and their preference for, one method or another.

As it was discussed in section 4.3, the customs the teachers assumed students will bring with them to the MATH 200 classroom, and which they were particularly interested in breaking, were: (1) that the students would be expecting a lecture based approach, where the instructor explicitly shows them how to solve all different kinds of problems; (2) that the instructor is the authority in the classroom for determining whether or not student work is correct or incorrect; and (3) that for any given problem there is one, and only one, acceptable way of solving it.

In what follows, we (the Researchers) reflect on the Teachers' and students' actions related to the negotiation of a didactic contract. What follows is our perception of how these customs may have been shattered and how this may have affected the classroom environment. Our perceptions and reflections discussed below are the result of our observations in the classroom (where we were most of the time Teachers, but sometimes Researchers).

The teacher will lecture on how to solve each type of problem

The Teachers of MATH 200 assumed that their students, based on their previous experiences in mathematics classrooms, would expect their teachers to "explicitly show" them how to solve each "type" of problem (exercise) students would have to do in MATH 200 (and, by extension, all algebra problems they will need to solve in future math courses). The Teachers wanted to break from this custom because, based on their experiences as teachers and researchers, they believed that this approach reinforces the routinization of mathematics problems, and does not help the students' ability to develop generalizations nor to engage in TT.

The main mechanism the Teachers put in place in the MATH 200 classroom to break away from this expectation was to remove the lecture format to negotiate a didactic contract in which classroom activities and interactions followed the discussion-based approach (see section 4.3).

For a number of the students it was not easy to be thrust in to this different kind of learning environment. This was especially true for those students who were taking MATH 200 as their first mathematics class in many years – they were not yet confident enough in their abilities to tackle the problems on their own. In addition, at the beginning of the semester the students in the class did not yet know each other well and some students were reluctant to ask others for help. This could have been for a number of reasons – perhaps they were embarrassed about needing to ask questions in the first place, or perhaps some students were self-conscious about the age gap between themselves and the other students (the age range of students was from 17 to 50+). There may be a dozen more, very interesting reasons why students might feel uncomfortable asking questions in a mathematics class, but these reasons are not necessarily the focus of this thesis. For these reasons, in the first few classes the students would mostly sit and wait for guidance from the Teachers.

This (somewhat to be expected) rough start did not last very long; by the second week the students had already formed groups based on where they usually sat in class and were more comfortable working together. They had also been reintroduced to enough of the mathematical content that they had a base from which they were able to work. Once this happened the Teachers were able to take on more of the role that they had envisioned, 'floating' throughout the class asking prodding questions about the students' methods and helping to nudge those who might need in the 'right direction'.

Anecdotally, many students expressed that they enjoyed the teaching approach overall. They felt that they were better able to absorb the material by actually solving the problems, instead of sitting and copying notes the entire time. There were students who felt that the approach did not match their learning style, however. For example, one student felt that spending all class time on problem solving was a waste since she could solve problems on her own at home – she felt like the class time would have been better spent with the Teachers explaining formulas and demonstrating how to solve problems (essentially, she prefers the 'institutional' approach).

The Teachers believed that breaking this custom would fall in line with the goals they had outlined at the beginning of the semester. The adjustment in the role of the teacher helps to promote student autonomy, while allowing the students to decide for themselves how to solve the problems they are faced with may help to shift their perception of mathematical activity. Beforehand, the students' notion of 'mathematical activity' may simply be deciding which algorithm to apply, as a result of the routinization of mathematics problems found in mathematics courses that follow the 'institutional approach'. With this approach that focuses on problem solving they might begin viewing the mathematics activities as more investigative in nature – instead of the different 'types' of problems being laid out for them on a silver platter, they need to learn them through experience. Instead of being shown which method to use to solve a problem, they are able to choose (or even construct) the method that they would be most comfortable using.

The Teachers also believed that having the students spend most of their time working on problems might allow for more opportunities to engage in TT. This will be further discussed in section 5.2.

The teacher will tell me if I am right or wrong

The second custom that the teachers wanted to break was the idea that it is the job of the teacher to be the authority on mathematics knowledge in the classroom who must decide for the students whether or not their work is right or wrong.

Breaking this custom can be seen as having the express purpose of fostering autonomy in the students of MATH 200; by removing the authority of the teacher in this regard, the students need to make their own decisions concerning the validity of their work.

For many students, this custom was a frustrating one to break from. It is a 'natural' thing for students to want their teachers to provide them with some reassurance that their work is correct. So, when a student calls the teacher over to ask if their answers are correct or not, and are greeted instead with questions such as 'Do you think it is correct?' or 'How can you check if it is correct?' they may get frustrated. However, the students eventually came to understand what was expected of them and that the Teachers would not be telling them whether or not their work was right or wrong. Rather, it was *their* job to convince the Teachers, and themselves, that the work that they had done to arrive at their answer was valid.

Some students did not fully accept this break in custom. While they understood that the Teachers would not tell them if they were right or wrong, they might instead have used one of their peers to fill that role – i.e. "I know that my work is correct because I got the same answer as my peer". This is not entirely out of line with the Teachers' goals, however. Recall that one of their goals was to respect and acknowledge their students' various backgrounds and previous knowledge. By allowing certain students, who perhaps had a stronger background than other MATH 200 students, to take on a role of 'tutor' in the classroom the Teachers hoped that these students would feel that their prior knowledge was being acknowledged and respected. They would also have the added bonus of engaging in reflective thinking, as they need to reflect on their own methods before they can explain them to another person.

There is only one correct way of solving a mathematics problem

The final custom that the teachers of MATH 200 tried to break away from was the idea that there is one (and only one) correct method to use in order to solve a given mathematics problem. This is a notion that the Teachers assumed may have been ingrained in the students minds from past experiences where the teacher acted as the absolute authority in the classroom, and whose solutions to classroom problems acted as a guide for students when they were to solve similar problems in the future.

More than once throughout the semester the students would engage in small-group discussions amongst themselves about the different methods that could be used to solve a task at hand, and which of these methods might be most efficient. Also, since answers were not provided for the problems that the students were to work on in class, more than once the teacher would ask students to write their worked out solutions on the blackboard. A whole-class discussion might follow on the different ways that people had arrived at the answer or on why the answer was or not correct. In any case, the discussion would be rich; students would debate which methods they preferred and why, which ones they thought were more efficient. If there were an answer on the board that they disagreed with they would argue against it (sometimes quite vehemently) until they either convinced the class that the answer was wrong, or accepted that there had been a flaw in their own methods and it was their approach that was incorrect.

Many interesting discussions also took place between small groups of students. By necessity, students who attended the classes had to abandon the role of the 'passive observer' that is common to so many algebra classrooms and become actively engaged in both the material and the discussions. The Teachers did not always prompt these discussions, either. As an example, we will describe an interaction that happened between two of the students while working on the following problem:

Peter works two jobs, one as a waiter that pays $9.50 per hour and one as a music teacher that pays $14 per hour. Last week he worked 35 hours and his salary after 12% deductions, was $352. How many hours did he work each job?

Student A decided to approach the problem by first denoting the number of hours Peter worked as a waiter as x, and then denoting by $35 - x$ the hours Peter worked as a music teacher. By multiplying these each by his hourly wage at each respective job, Student A arrived at the equation $9.5(x) + 14(35 - x) = 352$. Student A solved the equation to find the number of hours Peter worked as a waiter and then used the expression $35 - x$ to determine the number of hours that Peter worked as a music teacher.

Student B approached the problem differently, by setting up a system of two linear equations. He let x be the number of hours Peter worked as a waiter, and y, the number of hours Peter worked as a music teacher. He then constructed and solved the following system:

$$x + y = 35$$

$$9.5x + 14y = 352$$

Both Student A and Student B arrived at the same answer, but both seemed confused. Their discussion seemed to be along the lines of how was it possible that one of them (A) could solve the problem using only one variable, and the other (B) needed two variables, and have both of them get the same answer. It took a few minutes of looking over each other's work (to make sure that the other person had not made mistakes) before they realized that once Student B began to solve their system by substitution her method became identical to the one used by Student A. This led them in to a discussion about the benefits of each method, Student A believing hers was better by virtue of the fact that there were fewer steps, while Student B preferred hers as she believed that the problem was easier to keep track of when you assign a symbol (such as x or y) to each unknown quantity. The Teachers did not need to intervene in this discussion at all, and many students who were sitting near to the discussion were drawn in also. All of this had been spontaneous, and was the kind of classroom interaction that the Teachers had been hoping to foster. To be able to debate their points of view regarding different solution paths means that they have taken the time to reflect on the path that they themselves took, engaging in reflective thinking in the process.

In breaking this custom, the teachers also had hoped to show that they were open to both acknowledging and valuing the students' previous experiences in mathematics. Many of the students in MATH 200, either in high school or elsewhere, already had experiences with the material covered throughout the course of the semester. They did not all learn at the same pace, however, and may have developed different strategies for dealing with different kinds of problems. By not being able to reference 'official solutions' to the classroom problems, the students could feel free to use whichever methods and strategies they have previously developed, or those with which they felt comfortable. This allows them the chance to compare and contrast their solutions with those of other students, as well as boosts their confidence in their previous abilities.

The Teachers had hoped that breaking these three customs would help them to implement the didactic contract they had wanted to set up for their classroom. Although some students managed to find a way around the fact that the Teachers would not provide validation for their answers (instead using a peer as a surrogate) most, if not all, students embraced (if only grudgingly) the removal of lectures from the class in favor of problem solving time. As the semester went on and students came to know what was expected of them, they also seemed to understand that more often than not in mathematics there are many ways of approaching the same problem – each as valid (if, perhaps, not as efficient) as another. It seems, then, that the Teachers were more successful than not in putting in place a different didactic contract, and based on what we (the Researchers) have observed it seems like breaking the already mentioned didactic customs helped the Teachers to move towards achieving their goals.

5.2 Mathematical activities

During the semester, the Teachers prepared and ran more than 140 classroom activities with their students. For the most part, these activities were designed to fit within the new didactic contract that the Teachers had set up in their class – allowing for group work and discussion. There were, however, a few (4) instances throughout the semester where the contract was briefly renegotiated in a way that required students to work individually, after which their work would be collected and "corrected" by the Teachers. Note that we place the word corrected in quotation marks because we are not using it in the traditional sense – it is merely for lack of a better term. The Teachers would not say if the work was right or wrong, instead they would ask probing questions – i.e., "Have you considered all possible cases here?" – if the student was off track.

Before the semester started, when designing the activities, the Teachers had a vague image in their minds that there would be two main groups of activities. 'Direct Computation' problems would essentially serve as 'drill-and-practice' for the institutional assessments (weekly assignments, midterm, and final exam), while the 'Goal Oriented' problems would either address the assumptions made by the Teachers, or help to further their goals for the classroom (in particular, engaging in TT).

When we (as Researchers) began our analysis of these activities' potential to foster TT we found several (often not disjoint) sub-categories under the umbrellas of 'Direct Computation' and 'Goal Oriented'. This section will begin with a description of each of these categories, followed by analysis of a few example problems using our adapted notion of the fundamental situation.

Categories for classifying the activities

In this section we describe the seven different categories of problems that we came up with when analyzing the activities developed by the Teachers. Again, it is important to note that these categories are not necessarily disjoint – there were often problems that had characteristics of two or more of the following categories. However, we felt that classifying the problems was important in that we could explain with greater ease how certain categories of problems might have helped to serve the Teachers' needs based on their assumptions and goals.

We will begin first with the two subcategories from the 'Direct Computation' group.

I. Institutional Problems

Recall that one of the three goals outlined by the Teachers at the beginning of the semester was that they wanted their students to be able to succeed in the institutional sense – i.e., they wanted their students to be able to pass the course.

The institutional problems are those 'drill-and-practice' problems that will help the students to become comfortable with the techniques and calculations they will need to perform on assignments and exams in order to succeed in achieving this goal. Most of the problems that the students encountered in MATH 200 - during class time, as well as in the weekly assignments and examinations - were of this type, but the teachers hoped that by exposing their students to problems from the other categories they would be afforded more opportunities (or, more opportunities than they would have in a traditional classroom) to engage in TT.

The main feature of the problems in this category is that they explicitly state what students have to do (i.e. 'Factor the following algebraic expressions' or 'Solve the following system of linear equations'). However, the students were never told *which method* they were to use to solve the problem; they could use whichever method they felt most comfortable with.

Typically, the problems assigned in MATH 200 are very explicit as to which method is to be used when solving problems. For example, instead of 'Factor the following algebraic expression', typical textbooks or common assessment will state 'Factor the following difference of squares', or 'Solve the following system of equations by elimination' instead of simply 'Solve the following system of equations'. The Teachers felt that this contributed heavily to the routinization of problems, and wanted to remove this characteristic from their classroom activities to promote the notion that there is more than one way to solve a given mathematics problem. This was not always entirely possible for the WebWorK problems because, as was previously mentioned, the Teachers were restricted to choosing problems from the online question bank when creating the assignment.

II. Word Problems

Word problems are a very important component of MATH 200. Instead of being given an equation to solve, or a polynomial to factor, students are presented with some sort of contextualized problem. They need to be able to read the problem and decide for themselves what information is important, what they need to obtain in order to solve the problem, and what method they will be using to arrive at a solution.

Again, these problems mainly help the Teachers' goal of having their students succeed in the institutional sense. Word problems are a standard feature of assignments and exams in any algebra classroom, and in order to succeed the students need to be adept at understanding what a word problem is asking of them.

The categories that follow are those that fall under the umbrella of 'Goal Oriented' activities.

III. Multiple Methods

To be grouped in to the ‘Multiple Methods’ subcategory, the problem in question needs to explicitly ask the students to arrive at a solution using two or more different strategies.

Problems from this category quite directly address the assumption made by the Teachers that students will arrive to class believing that a given problem can be solved in one – and only one – way. This was a custom that the Teachers wanted to break from, and problems of this type were especially helpful (particularly at the beginning of the semester) in doing so. Initially, if students *did* believe that problems could only be solved in one way seeing a problem that states ‘Solve this problem using two different strategies’ might get them to think “OK, if the teacher is asking me to solve this using more than one method, then it must be possible to do so.” As the semester went on and the students came to understand what was expected of them these problems became less and less necessary.

IV. Conceptual

There are two different kinds of conceptual activities that the students of MATH 200 encountered – we will call them ‘Abstract Conceptual’ and ‘Concrete Conceptual’.

A problem was classified as 'Abstract Conceptual' if solving it requires an understanding of the conceptual underpinnings of (some of) the course topics. In addition, problems in this category do not deal with particular numbers, but with generalized ones.

'Concrete Conceptual' problems are quite similar; however the problems would concern particular numbers, instead of generalized ones.

It seemed natural to the Teachers that problems of a conceptual nature might be more effective than others in fostering TT, and as such they wanted to include as many problems of this type as possible. Students may engage in reflective thinking by thinking critically about how a property applies to a more general case, or they may engage in systemic thinking by linking different concepts together. Abstract conceptual problems in particular may also be of more use in fostering analytic thinking as they tend to require students to interpret symbolic expressions with a certain degree of 'rigour'.

V. Symbol Manipulation

For a problem to be classified in this category, it needs to ask the students to perform some of the operations that they have become familiar with (such as addition, subtraction, multiplication, division, etc.) on a mathematical object that consists solely of generalized numbers.

These problems differ from those that are classified as Abstract Conceptual mainly in that in these problems the student would be performing operations directly on the object. In Abstract Conceptual problems, calculations are not always required – in many cases the student is simply given a bit of information and then asked what can be inferred from what is given – for example, 'Explain why the product of an even integer and any other integer is even. What conclusions can you make about the product of two odd integers?'.

These problems address the Teachers' goal of providing students with opportunities to engage in TT. In particular, problems of this type might encourage analytic thinking.

VI. Different Representations

In this final category, the students are asked to work with (or provide) two or more different representations of the same mathematical object.

These problems are similar to the 'Multiple Methods' problems in that they were only mostly necessary for the beginning of the course, when the students may not have yet realized that there are often different ways to represent the same mathematical object. As the semester moved forward, instead of asking the students to provide different representations of the same object, the Teachers would provide problems with varied representations of similar objects and would expect the students to know how to alter the representation as they saw fit to most efficiently solve the problem in question.

The following table is a breakdown of how many activities, both from class time and from the tutorials, were placed in each category. Each activity, even if it had characteristics of more than one category, seemed to have one ‘main’ characteristic that was more prominent than others. It was this ‘main’ feature that was used to classify the activities for the purpose of this thesis:

			Class Problems	Tutorial Problems	Total
Direct Computation	Institutional Problems		18	22	40
	Word Problems		7	14	21
Goal Oriented	Conceptual	Abstract	25	7	32
		Concrete	18	5	23
	Multiple Methods		10	3	13
	Symbol Maniuplation		5	2	7
	Different Representations		7	0	7
Total			90	53	143

Analysis of class activities

Now we proceed to analyze a few examples of class activities, looking at their structure in terms of fundamental situations, as well as looking for opportunities that students may have had to engage in TT. We used Challita’s (2013) list of theoretical behaviours as a guide when looking for these opportunities. Recall that on occasion, the didactic contract had been renegotiated for individual work. When the Teachers ran these individual activities, they collected responses from the students, and so in the last two examples we look at student work to see if we can find any evidence that these students did, in fact, engage in TT.

Recall that for our FS* we consider six variables:

V1 – Category

V2 – Topic

V3 – Misconception to be addressed

V4 – Feature of TT

V5 – Solution expected

V6 – Group/Individual Activity.

In what follows we will be using these variables to describe the specific situations, SS*, associated with five of the in-class activities, as well as analyzing these activities' potential to foster TT.

Example 1

The following table represents the quantity of shoes that consumers would demand at each given price. Do the values correspond to a linear expression? If yes, write the linear expression in the form y = mx + b. What is the slope? (It is negative, why??)

P	***Q (in thousands)***
$140	***0***
$120	***5***
$100	***10***
$80	***15***
$60	***20***
$40	***25***
$20	***30***
$0	**35**

To obtain SS_1*, we assign our variables the following values:

V1 – Different representations

V2 – Linear Equations and Graphing; Equations of Lines

V3 – Null

V4 – Null

V5 – Provide solution

V6 – Group activity

We expect that students would first identify P as the independent variable and Q as the dependent variable, and then note that the rate of change remains constant –

specifically that for each \$20 increase in P, Q decreases by 5 meaning that the rate of change is $-\frac{1}{4}$. Knowing the slope, they could then choose any point (P, Q) to put in to the point-slope form of the equation of a line $m(P - P_1) = (Q - Q_1)$. After performing some manipulations they'd arrive at the desired answer of $Q = -\frac{1}{4}P + 35$.

Before the students can attempt this problem, they need to be able to recognize which variable is the independent variable and which is the dependent variable. Once this is done, they may engage in systemic thinking by recalling the definition of a linear equation and remembering that for an equation to be linear the slope needs to remain constant.

Once they find the slope, they need to identify the y-intercept. To do this, the students again need to be able to represent the table in a 'different mathematical register'. They are given a linear equation as a table, but need to translate this table to an equation of the form $y = mx + b$. They need to recognize that the entry in the \$0 row is equivalent to they y-intercept of the linear equation, and use this information to construct their answer.

The students may also engage in reflective thinking near the end of the problem, by saying that this y-intercept does not actually make sense in the context of the problem since no company would sell shoes for \$0. This could also lead in to some discussion on domains of functions.

In addition to having to be sensitive to specialized terminology, the students also need to be aware of 'conventional terminology', or the meaning of certain words in a given context. In this problem, for example, the words 'negative' and 'slope' are being used. These are words that most people have in their vocabulary, and they both have well-defined meanings outside of the realm of mathematics. For a student to succeed at this problem, they need to be able to contextualize the information they are being given.

Example 2

Simplify the following expression as much as possible: $\dfrac{\frac{a*c*b}{c*a}}{\frac{d*b*a}{c*d}}$

For SS_2*, we assign our variables the following values

V1 - Symbol Manipulation

V2 - Operations on integers and rational numbers

V3 - Null

V4 - Analytic thinking

V5 - Provide solution

V6 - Group activity

The provided expression is ambiguous, and depending on the students' interpretation many different answers and solution paths are possible. One of the expected ways that a student might tackle this problem is to look at the dividend and the divisor first as separate expressions. They would recognize that the dividend $\frac{acb}{ca}$ could be simply reduced to b, while the divisor becomes $\frac{ba}{c}$. The expression now becomes $\frac{b}{\frac{ba}{c}}$. From here, the student remembers that a division of two fractions is equal to the dividend multiplied by the reciprocal of the divisor. They would rewrite the expression as $b * \frac{c}{ba}$, and realize that the b's can be 'cancelled out'. This leaves the student with the final answer of $\frac{c}{a}$.

First and foremost for this problem, students need to exercise analytic thinking by rigorously interpreting this symbolic expression. Requiring particular attention here is the bar symbol. Students will see this symbol in many places used as a symbol for division, but it is also used as a symbol to represent fractions and students need to be sensitive to which use the symbol has in a given context. In this problem, the bar symbol takes on both meanings as the student will interpret the problem as a division of two fractions. The students also need to be sensitive to the conventional meaning of the word 'simplify' in an algebraic context.

Students may also use systemic thinking in using the known property $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} * \frac{d}{c}$ to help them fully simplify the expression. There is also an opportunity for students to exercise the hypothetical character of systemic thinking – they need to consider the different interpretations of the expression, and may also consider the conditions under which the expression makes sense (for example, what happens if *b* is equal to 0?).

Example 3

Does $\frac{2}{3} + \frac{3}{2} = \frac{2+3}{3+2} = 1$? Explain your answer.

To obtain SS_3* we assign the following values to our variables

V1 – Concrete Conceptual

V2 – Operations on integers and rational numbers; addition of rational numbers

V3 – To add two rational numbers you add the numerators and the denominators

V4 – Null

V5 – Discuss Solution

V6 – Group activity

Here we would expect students to realize two things; First, they need to recognize that $\frac{3}{2}$ represents a number that is greater than 1. Second, they must see that $\frac{2}{3}$ is a positive number. The solution follows from here – if you add a positive number to a number than is already greater than 1, the answer cannot possibly be 1.

This problem directly addresses the misconception that to add two rational numbers you need simply to add the numerators and denominators together. Hopefully this activity elicits 'proving' type behaviour from the students – a feature of systemic thinking, where they can see that this cannot possibly be the case. Depending on their past experiences, some students may also be able to link this problem to their previous mathematics knowledge and immediately recall that in order to combine two rational numbers (resulting in a single rational number), they need to have the same denominator – another systemic behaviour.

Example 4

Distribute and combine like terms: $\boldsymbol{x^2(x-1)+2x(3-2x)}$

To obtain SS_4* we assign the following values to our variables

V1 – Institutional Problem

V2 – Polynomial expressions; multiplication and combining like terms

V3 – Null

V4 – Null

V5 – Provide solution

V6 – Group activity

To solve this problem it would be expected that students would remember the distributive property and rewrite the expression as $x^2(x) - x^2(1) + 2x(3) - 2x(2x)$. They would then apply their knowledge of multiplication and exponents to expand this to $x^3 - x^2 + 6x - 4x^2$. Finally, they would combine like terms to arrive at $x^3 - 3x^2 + 6x$.

It was expected by the Researchers that activities from this category would offer fewer chances to engage in TT. That being said, this example that we have chosen is not completely void of opportunities to do so. Students need to be sensitive to the meanings of both 'distribute' and 'combine like terms' when reading this problem. Similar to the words 'negative' and 'slope' mentioned above, 'distribute' has a well defined meaning outside of the realm of mathematics. The students need to be able to understand the meaning of the word in context. They may also have an opportunity to link their previous knowledge about the distributive property with the task at hand, engaging in systemic thinking.

The last two examples are those activities that took place when the didactic contract was briefly renegotiated to allow for individual work. These activities were designed by the Teachers to help students to 'control' their progress and understanding – the Teachers would give them feedback on their responses.

There were five of these activities given over the course of the first five weeks of class, with approximately one activity given per week. For our final two examples we will be considering the first of these activities as well as the fifth to see if there was any perceptible change in the way that the students responded to the activities. We start by looking at the design of the activity using our slightly modified version of the fundamental situation (see section 3.5). We then identify opportunities to engage in TT. Finally, we consider the responses of four different students to see whether or not they did, in fact, engage in any TT. The four students were selected at random from the pool of 24 students who had submitted both activities.

Example 5

Question:

Is it true that if $|\boldsymbol{b}| > |\boldsymbol{a}|$, then $\boldsymbol{a} - \boldsymbol{b} < 0$?

Answer:

Yes, this is true. Since $|\boldsymbol{b}| > |a|$, $\boldsymbol{b}$ is farther away from 0 than $\boldsymbol{a}$ on the number line, therefore when you subtract $\boldsymbol{b}$ from $\boldsymbol{a}$ the result is a negative number.

- Is the answer above correct or incorrect? Explain your reasoning.
- If the answer above is *incorrect*, can you change the wording of the question (or add something to the question) so that the given answer is correct?

To obtain SS_5*, our variables are assigned the following values

V1 – Abstract conceptual

V2 – Operations on integers and rational numbers; absolute values

V3 – Abstract numbers without the '–' symbol in front of them must be positive; proof by example is always sufficient.

V4 – Null (no specific aspect is targeted)

V5 – Discuss the validity of a given solution

V6 – Individual activity

Note that the misconceptions themselves are not explicitly addressed by the problem. Rather, the problem provides an opportunity for a discussion about these misconceptions to come up in the class (and in the feedback individually provided to students by the Teachers).

As a side note here, this blueprint could be used to create a number of different problems that would be appropriate for the MATH 200 classroom. By changing the value for just one of the variables, for example changing V5 from 'discussing the validity of solution' to 'provide a solution', the problem becomes an entirely different (yet still interesting) one.

This problem gives students opportunities to engage in all three types of TT. To come up with a proper solution to this problem, the students need to be investigative and consider all the possible cases where $|b| > |a|$. There are four possible cases: a positive/b positive, a positive/b negative, a negative/b positive, and a negative/b negative. They may also choose to generalize their solution, i.e. "This statement is true if both a and b are positive numbers". These are both behaviors associated with reflective thinking.

The students may also engage in the proving feature of systemic thinking. This activity is, in essence, a proving activity, and the students needed to give arguments to support their claims whether or not they believed that the statement was incorrect. Those who believe the statement to be false may also use a counterexample to support their claims. This action could also be seen as hypothetical thinking (a feature of systemic thinking), as they would be considering a particular case to negate the statement.

Finally, the students need to be able to interpret the symbolic expressions involved in order to come to some meaningful answer. They need to understand the meaning of absolute value, as well as the function of the '<' and '>' symbols.

S1

- Is the answer above correct or incorrect? Explain your reasoning.
- If the answer above is *incorrect*, can you change the wording of the question (or add something to the question) so that the given answer is correct?

$|b| > |a|$; $a - b < 0$

★ $4 - 5 < 0$ If a and b are positive numbers; then $a - b < 0$

$-1 < 0$

★ $(-5) - (-4) < 0$

$-5 + 4 < 0$

$-1 < 0$ If a and b are negative numbers; then $a - b < 0$

★ $(-5) - (8) < 0$

$-5 - 8 < 0$

$-13 < 0$

For this problem, S1 did not completely fall prey to the misconception that was being targeted. She did consider cases other than the case where both numbers were positive, however in her work for the case where both numbers are negative she subtracted *a* from *b* instead of the other way around as is asked in the question. There are many reasons why she may have done this – perhaps by 'switching the signs' she felt as though she also needed to switch the order in which the numbers were subtracted? Without more information we cannot be sure.

In the both cases that she considered, she did offer a generalized solution, though, when she wrote that 'If a and b are positive numbers, then $a - b < 0$'. While she did give a generalized solution, her explanation itself is not generalized – she is instead relying on her example. This idea that proofs by example are adequate is a misconception that was evoked by the problem, even though the problem itself was not targeting this particular misconception. It does, however, give the Teachers an opportunity to talk to the students about it and provide feedback on this issue.

S2

Yes it is correct, as an example if $b = 5$ since $|b| > |a|$
$a = 3$
it will always be a negative number if you subtract $|b|$ from $|a|$

It seems fair to assume that S2 has fallen in to the trap of assuming that the variables represent (only) positive numbers. As a consequence, S2 misses out on most of the opportunities for TT in this problem by stating right away that the statement is correct. This could be viewed as a failure in the design of the problem; once a student falls in to the 'trap' of the misconception that is being targeted, they have no way of realizing their mistake.

Although the conclusion is wrong, S2 does offer an argument to support their claim. However, because S2 did not consider any scenarios where a or b were less than zero we cannot be certain whether or not they are truly sensitive to the meaning of the absolute value symbol.

<u>S3</u>

- The answer above is incorrect because although the absolute value of a number is always positive, we don't know if the number itself is either positive or negative.

Example: $|b| > |a|$ with $|b| = 5$ & $|a| = 4$ if $b = 5$ and $a = 4$, than the answer is correct.
But if $b = -5$ and $a = 4$, than $-5 < 4$ so
$4 - (-5) = 4 + 5$
$\boxed{= 9}$. $9 > 0$, so the answer is incorrec

- I would change the question by transforming the expression $a - b < 0$ to $|a| - |b| < 0$

↳ Is it true that if $|b| > |a|$, then $|a| - |b| < 0$?

Of the four students whose work was selected, S3 provided the closest to what could be considered an ideal response to this question. She did not fall in to the trap of assuming a and b must be positive, and although she did not consider all four possible cases, only one of the cases (a positive and b negative) was necessary for exposing the contradiction in the question, and conclude that the statement was false.

They were also able to provide a satisfactory answer to the second part of the question by saying that the statement would be true if you were asked instead to perform $|a| - |b|$.

S4

$|B| > |a|$

$|-4| - |2|$

$4 - 2 = 2.$

- No, it is not correct, $|b|$ is an absolute value therefore even if it is negative it is looked at as a positive number.

- If b was not absolute.

While S4 does correctly state that the given statement is not true, their reasoning is off. They select -4 as their value for b and 2 as their value for a, but then instead of performing the subtraction $a - b$ as asked they base their conclusion off of the subtraction $b - a$.

Based on what they have written, however, we might be able to conclude that they are at least sensitive to the meaning of the absolute value signs (even if their given 'definition' is not by any means formal).

S4 was able to come up with a way of changing the question so that the statement would be true, but as their written work did not include any mention of their thought process it is unclear what kind of TT they might have engaged in while coming to this answer (if they engaged at all).

Example 6

Two students were asked to solve the following linear equation:

$$\mathbf{4x + 8 = 2(5 - x)}$$

One of these students did some calculations and arrived at $-1 = -3x$**, to then find that *x* = 1/3.**

The other student approached the problem in a different way and arrived at $6x = 2$**, to then find that *x* = 2/6.**

1) **Can we say that** $-1 = -3x$ **and** $6x = 2$ **are different representations of the same equation? Explain your answer.**
2) **Try and replicate these students' work. What are the possible steps that they could have taken to arrive at their respective answers?**

3) **Discuss which method, in your opinion, is more efficient. The first method, the second method, or some other method entirely? Explain your answer.**

To obtain SS_6* our variables take the following values:

V1 – Different representations

V2 – Linear equations and expressions; solving linear equations

V3 – A linear equation can be solved in only one way

V4 – Null

V5 – Discuss solutions

V6 – Individual activity

Much like the first week activity, the misconception is not directly addressed by the problem. Rather, it opens avenues for students to engage in a discussion about it.

The first part of this activity offers students some opportunity to engage in systemic thinking by participating in a definitional and proving-type activity. For students to be able to adequately state that the two equations are different representations of the same thing, they need to be able to recall the definition of equivalent equations (namely, that you can perform valid manipulations on one of the equations to arrive at the other). They can then use this definition to prove that the two equations are, in fact, equivalent.

The second part of this activity is more reflective in nature as it requires the student to investigate different possible solution paths for a single problem. The third part of the activity is an extension of the second, where the students are almost explicitly asked to reflect on what they've done. They may feel that one method is more efficient over the other based on their comfort level with previous topics. It also offers students a chance to discuss the misconception that is being addressed.

Student Responses

Part 1

S1

> Yes, they are, because at the end we get the same result.

Although she doesn't explicitly state the definition of equivalent equations here, it does seem as though it is clear that S1 understands the definition and what it means. Even though she does not engage in any proving activity (that we can see on paper), she does note that the equations give 'the same result'. This may imply that she has mentally carried out a proving activity, since it is relatively simple to recognize that; all that needs to be done is to multiply the first equation by -2. We cannot be certain, however.

S2

Yes, if multiplied by -2 for $-1=-3x$
it's the same equation

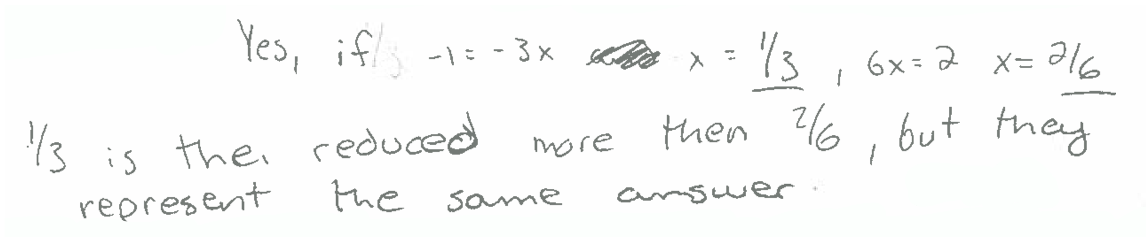
S2 has given a close to ideal response here. While she does not fully flesh out her work by multiplying the equation $-1 = -3x$ by -2, she clearly seems to understand the definition of equivalent equations, and recognizes the operation that is required to show that the two equations are, in fact, equivalent.

S3

Yes. ① $3x = 1$ then If you multipy a side of an
equation by a number, you have to multiply the othe side, too. In
in this case, both sides were multiplied by 2. So, $3x \times 2 = 6x$ & $1 \times 2 = 2$
We can conclude that $-1=-3x$ & $6x=2$ are equal

S3's response is very similar to S2's, although she fleshes out her explanation a bit more. In any case, it is clear that S3 understands the definition of equivalent equations, and knows how to back up her claim that the two equations represent the same thing.

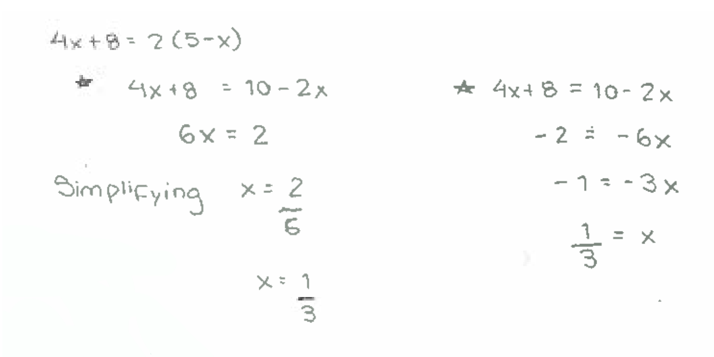
S4



S4's answer is interesting in that her reasoning is different from the others, although still correct. Instead of multiplying or dividing one of the equations to make it the same as the other, she opts instead to solve both equations for x and correctly states that both equations have the same answer (namely, that $x = 1/3$).

Part 2

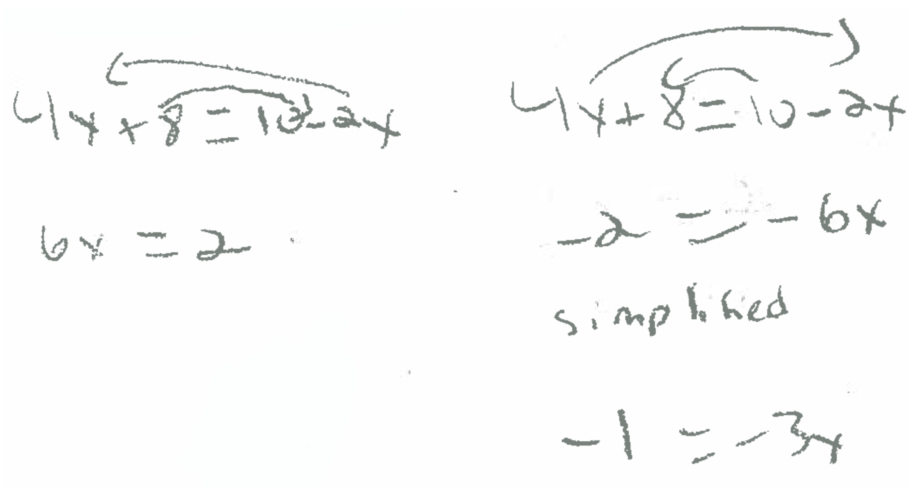
S1



While S1 did solve the equation in two different ways, each of which arrived at the equations indicated in the problems, we are not sure that she *truly* did engage in investigating different solution paths. Both of her methods are very similar, with the only distinction being that after the first step of distributing the 2 on the right hand side of the equation she proceeds to collect the x's on opposite sides of the equals sign.

That being said, she did perform what the question asked of her. All that was required was to show *possible* steps in solving the equation, and her methods are definitely valid.

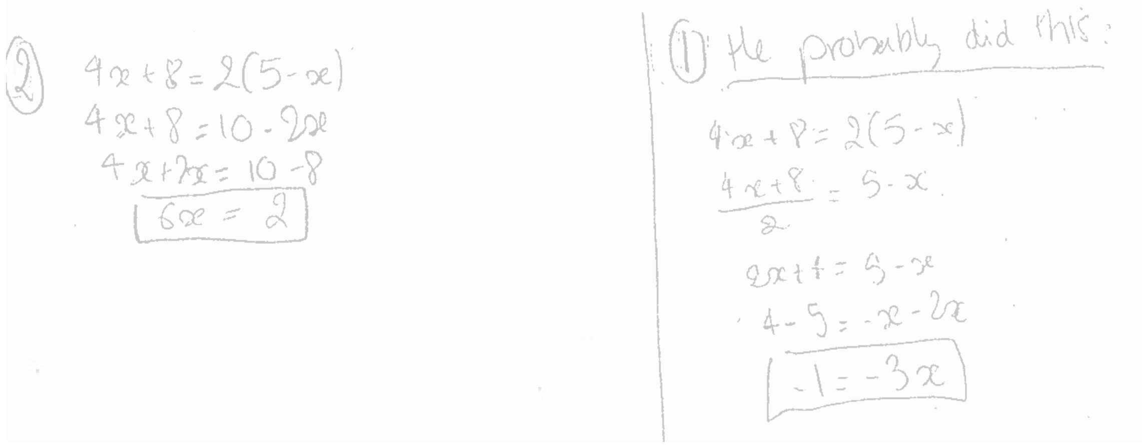
S2



S2's response is almost identical to S1's. She immediately distributed the 2 on the right hand side, and then isolated the unknown on opposite sides of the equals sign.

Like S1, while the methods are very similar there is the *one* difference that shows S2 investigating different solution paths.

<u>S3</u>



S3, in a more clear fashion than S1 and S2, can be seen investigating different solution paths. It is clearer in her case because each method starts off in a distinctly different manner. In her first solution she recognizes that both sides can immediately be divided by 2, and then proceeds to isolate the unknowns, while her second solution is similar to those offered by S1 and S2.

S4

$$4x+8=2(5-x)$$

$$\cancel{-4x+4x+8=2(5-2x)-4x}$$

$$4x+8-8=2((5-x)(-5+x))$$

$$4x-5-x=2-8$$

$$-x=$$

While S4 was able to recognize in the first part of the activity that we were dealing with two equivalent equations, she proved unable to arrive at these equations on her own when starting from scratch.

There are some deep-seeded issues here with the basic ability to perform operations on abstract numbers, which are stopping S4 from being able to engage in the TT that this part of the activity offers. From what little of her work we can see, it seems that she may be confused by the difference between multiplying an abstract number by 2 and squaring it as evidenced by her moving from $2(5-x)$ to $2((5-x)(-5+x))$, but even then there are mistakes with the signs, and if this was the case then the 2 should have disappeared.

All in all, we cannot conclude anything meaningful from S4's work here that would help us to determine whether or not she engaged in TT.

Part 3

S4 opted to not complete the third part of this activity, so we will only be looking at the responses of S1, S2, and S3 here.

<u>S1</u>

The second method, because you are using positive numbers and it would be easier to solve the equation and then just simplify.

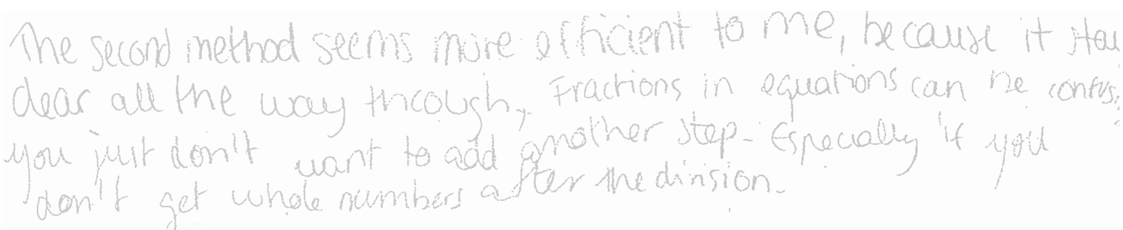
S1 seems to be aware of what she is comfortable with, and what she is less comfortable with, and as a consequence she decides that, for her, the second method is more efficient. She claims that this is because you are only using positive numbers, and presumably she believes that as soon as you begin to deal with negative numbers (especially at the MATH 200 level) you begin to introduce more opportunities to make mistakes.

<u>S2</u>

Second method perhaps is easier because the numbers are simplified

S2’s response is slightly confusing. She states that she believes that the second method is easier because ‘the numbers are simplified’, but if we go back and look at her two different solutions the only difference between the two of them is that the second method has an extra ‘simplification’ step. This extra step is one that could easily be applied to the first method as well, which suggests that perhaps S2 is not thinking too deeply about what the question is asking (which process is the most efficient). Instead, she seems to be basing her judgment on the appearances of the two equations to be dealt with in part 1.

<u>S3</u>


The second method seems more efficient to me, because it stau
clear all the way through. Fractions in equations can be confus
you just don't want to add another step. Especially if you
don't get whole numbers after the division.

Parts of S3’s work for this question were cut off in the process of scanning. Her text reads:

‘The second method seems more efficient to me, because it stays clear all the way through. Fractions in equations can be confusing. You just don’t want to add another step. Especially if you don’t get whole numbers after the division.’

S3 once again provides us with one of the more interesting responses to the question that she is being asked. Unlike S1 and S2, S3 really seems to be reflecting on the *method* used to solve the equation, and is clearly stating why she prefers the second method to the first method. Somewhat understandably she wants to avoid situations where she has to work with fractions, and prefers the method with the fewest steps. She also states that the second method is more ‘clear’, although she doesn’t say what it is about the first method that she finds unclear.

It seems like we can say that these four students did engage in some (if not all) of the opportunities to engage in TT that were provided by these problems. However, it is difficult for us to truly discuss a students’ method just by looking at their written work. We may be able to deduce certain things, but without more thorough data (for example, recording the students as they solve problems and asking them to detail their thought processes) there is a limit to what we can say.

Chapter 6. Conclusion and future work

Over the course of this thesis, we have continually been switching between our roles as Teachers and Researchers as they relate to our MATH 200 classroom.

As Teachers we had made three main assumptions about our students before the semester had even started – that they would have, to some degree, negative attitudes towards mathematics in general; that they would all have very different backgrounds and previous experiences with mathematics, as well as different life goals; and that they would be arriving to class with certain customs in mind – features that they feel should be a part of any mathematics classroom.

The Teachers were particularly interested in three customs that they assumed students would be arriving to class with. The first of these customs is the notion that the teacher will lecture to the students and show them, often quite explicitly, how to solve the different types of problems that they will be encountering throughout the course. The second of these customs was the idea that it is the responsibility of the teacher to tell the students whether or not their work is correct. The third custom the teachers were interested in was the belief that, for any given mathematics problem, there is only one 'correct' way to solve it.

Taking these assumptions into account, the Teachers set up three goals for their students. First, they wanted their students to engage in theoretical thinking; second, they wanted to be sure to respect and acknowledge the students' different backgrounds and life goals; third, they wanted their students to be able to succeed at the course in the institutional sense – this is, they wanted them to pass the course.

The Teachers felt that the best way to accomplish these three goals was through the implementation of a teaching approach that deviated from the 'institutional' approach so often seen in mathematics classroom. The teaching approach that was taken instead focused on learning through problem solving and group discussion. Implementing a new teaching approach in this way required a renegotiation of the didactic contract, and to help the Teachers put their new contract in place they pointedly decided to break from the three customs that they assumed that their students would be bringing with them to the classroom.

To break from the first custom, the idea that the teachers would lecture on how to solve different kinds of problems, the Teachers decided to remove lectures (almost) entirely from their classroom. In doing so they hoped to allow students room to build on their previous knowledge, and to develop their own strategies for solving problems instead of imposing any single 'institutional' solution method on them. This was also mostly their mechanism for breaking the third custom, the idea that any given problem has only a single correct solution. To break from the second custom, the notion that the teacher will tell the students whether their work is right or wrong, the Teachers would mostly reply to student inquiries with probing questions such as 'Is there anything that you can do to verify whether your solution is correct?' or 'Does this answer make sense in the context of the problem?'

As Researchers, our job was to evaluate the 'efficacy' of the teaching approach as a means to achieve the goals set up by the Teachers.

The first of the Teachers' goals was to provide students with opportunities to engage in theoretical thinking. Based on what we have seen, the discussion-based approach offered students more opportunities to engage than might be seen in a traditional mathematics course, particularly in the way of reflective thinking. We also analyzed some of the in-class activities to look for opportunities that the students might have had to engage in TT. We began by describing the design of the activities in terms of fundamental situations as described by Brousseau, and then used an 'expected solution' to look for instances where students might have engaged in TT. In a few cases, we were able to examine actual student responses to determine whether or not the students had, in fact, engaged. Based on what we have seen, we can say that the Teachers were successful in providing the students with opportunities to engage in TT.

The Teachers' second goal was to try and respect and acknowledge the different backgrounds of their students. While we cannot say whether or not the students themselves felt that their backgrounds and previous knowledge were valued, the Teachers felt that they did the best they could in this regard. They (the Teachers) did not discredit any (valid) solution methods that students used in class, and in doing so they hoped to show that they valued their previous knowledge. They also allowed stronger students to take on a sort of 'mentor' role in the classroom, and hope that this made these students feel like their previous knowledge and experiences were valued. It was, perhaps, a shortcoming of this study that there was no way to assess whether or not the students felt as though their previous experiences were valued.

Finally, the Teachers wanted their students to succeed in the 'institutional sense'. They wanted them to be able to pass the course, performing (at least) as well as the students in the other 'traditional' sections of the course. In this, we can say that the Teachers succeeded. The following table compares the distribution of the final grades from both the discussion-based sections and the 'traditional' sections.

Grade Range	% of Students	
	Discussion Based (n=49)	Traditional (n=159)
0-49	22.45%	25.79%
50-59	18.37%	13.20%
60-69	8.16%	12.57%
70-79	22.45%	20.75%
80-89	20.40%	18.86%
90-100	8.16%	8.80%

For each grade range, there were no significant differences in proportion found between the two groups ($\alpha = 0.05$).

Following the Teachers' work with MATH 200 the order of the topics was slightly rearranged for the Fall 2014 semester, after a set of notes had been created to replace the textbook.

Future Work

This work opens up many avenues for some interesting future research. Perhaps, as it was the most inconclusive of the Teachers' three goals, there could be a study relating specifically to a discussion-based approach and how it might be used to expose, value, and build on an adult algebra student's previous knowledge of mathematics.

Also, perhaps a more thorough work could be done on providing adult algebra students with opportunities to engage in TT. Interviewing students while they are solving problems designed to illicit the different aspects of TT maybe interesting as they might offer a deeper insight in to the students' though process. As researchers, we can only glean so much information from analyzing student responses before we have to resort to statements such as 'perhaps they were thinking this…' or 'they might have done this because…'

Finally, we wonder how this teaching approach could be further adapted to apply to more and more mathematics courses. Making the changes necessary to adapt this approach from one that was acceptable for a proof-based course (with no fixed mathematical content to be taught) to one acceptable for a course in algebra (with a very rigid list of topics that need to be covered) was an interesting process. Now that these adaptations have been made, perhaps it would be easier to apply this approach in other courses that have a fixed set of content that needs to be taught. Of course, it is important to remember that MATH 200 was the lowest-level mathematics course offered at the university. Adapting the approach for yet another course would mean having to teach more advanced topics in a similar fashion. This most likely would mean that to effectively communicate these topics to students while keeping lecturing to a minimum many other considerations would need to be taken.

References

Allport, G.W. (1935). Attitudes. In C. Murchison (Ed) Handbook of Social Psychology, Worcester, Mass: Clark University Press.

Balacheff, N. (1988). Le contrat et la coutume: Deux registres des interactions didactiques. In C. Laborde (Ed.), *Actes du premier colloque Franco-Allemand de didactique des mathématiques et de l'informatique* (pp. 15-26). Grenoble, France: La Pensée Sauvage. (1999, English version: Contract and custom: Two registers of didactical interactions. *The Mathematics Educator*, 9(2), pp. 23-29).

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics 1970-1990*. Translation from French: M. Cooper, N. Balacheff, R. Sutherland & V. Warfield. Dodrecht, The Netherlands: Kluwer Academic Publishers (1998, French version: *Théorie des situations didactiques*. Grenoble, France: La Pensée Sauvage).

Challita, D (2013). *Providing college level calculus students with opportunities to engage in theoretical thinking*. M.T.M. Thesis. Concordia University, Canada.

Duffin, Janet & Simpson, Adrian. (2000). Understanding their Thinking: the tension between the Cognitive and the Affective. In D. Coben (Ed.), *Perspectives on Adults Learning Mathematics* (pp. 83-100). Norwell, MA: Kluwer Academic Publishers.

Evans, Jeff & Wedege, Tine. (2006). *Adults' Resistance to Learning in School versus Adults' Competences in Work: The Case of Mathematics. Adults Learning Mathematics, 1 (2),* 28-44

Hardy, N. (2009). Students' perceptions of institutional practices: The case of limits of functions in college level Calculus courses. *Educational Studies in Mathematics, 72*(3): 341-358.

Hardy, N. and Sierpinska, A. (2011).Mathematical organization of some French and English algebra textbooks used in remedial courses in colleges and universities in North America. *Proceedings of the Colloquium "Formation à la recherche en didactique des maths"*, UQAM, March 24-26. (pp. 239-250)

Hardy, N., Beddard, C. and Boileau, N. (2013). Teaching first-year undergraduates to think mathematically. *Proceedings of the Psychology of Mathematics Education North American Chapter* (PME-NA 35). Chicago, Illinois. November 14-17, 2013.

Jarvis, P. (2001). *Learning in Later Life – an introductions for educators and carers*. Kogan Page, London.

Leder, Gilah C. (1985). Measurement of attitude to mathematics. *For the Learning of Mathematics*. 5(3). 18-21.

Gustaffson, Lars & Mouwitz, Lars. (2004). *Adults and Mathematics – a vital subject.* National Center for Mathematics Education Göteborg, Sweden.

Ostrom, E. (2005). *Understanding institutional diversity*. Princeton, NJ: Princeton University Press.

Perry, W.G. (1970). *Forms of intellectual and ethical development in the college years*. New York: Holt Rhinehart and Winston

Ruffell, Moyra, Mason, John, & Allen, Barbara. (1998). Studying Attitude to Mathematics. *Educational Studies in Mathematics*. 35(1). 1-18.

Schoenfeld, A. H. (1987). What's All the Fuss About Metacognitlon?. *Cognitive science and mathematics education*, 189.

Selden, A., Selden, J., Hauk, S., & Mason, A. (1999). *Do Calculus students eventually learn to solve non-routine problems*. Technical Report: Department of Mathematics (5), Tennessee Technological University.

Sierpinska, A. (1999). *MATH 645: Theory of Situations/Lecture 1 'Introduction to the concept of didactic situation and the distinction between situations of action, formulation, and validation'*. Personal Collection of Anna Sierpinska, Concordia University, Montreal, Quebec.

Sierpinska, A. (1999). *MATH 645: Theory of Situations/Lecture 3 'The notion of "didactic contract"'*. Personal Collection of Anna Sierpinska, Concordia University, Montreal, Quebec.

Sierpinska, A. (1999). *MATH 645: Theory of Situations/Lecture 6 'Didactic Engineering 1. The notion of fundamental situation'*. Personal Collection of Anna Sierpinska, Concordia University, Montreal, Quebec.

Sierpinska, A., Nnadozie, A., & Oktac, A. (2002). A study of relationships between theoretical thinking and high achievement in Linear Algebra. Unpublished manuscript. Retrieved on May 30th, 2014 from http://www.annasierpinska.rowbeca.net/index.php?page=publications.

Sierpinska, A., Bobos, G., & Knipping, C. (2008). Sources of students' frustration in pre- university level, prerequisite mathematics courses. *Instructional Science*, *36*(4), 289- 320.

Thomas, W. I. and Znaniecki, Florian (1918/1920). The Polish Peasant in Europe and America Volumes I and II. New York: Dover Publications, 1958.

Thurstone, L.L. (1928). Attitudes Can Be Measured. *American Journal of Sociology*. 33. 529-554.

Wedege, Tine. (2010). *Proceedings of the 16th International Conference of Adults Learning Mathematics – A Research Forum (ALM)* (pp. 13-24). London: Adults Learning Mathematics (ALM) and LLU+, London South Bank University.

Appendix A: Course Outline(s)

Original Course Outline

Department of Mathematics & Statistics
Concordia University

MATH 200
Fundamental Concepts of Algebra

Winter 2013

Instructor*: ______________________________

Office/Tel No.: ______________________________

Office Hours: ______________________________

*Students should get the above information from their instructor during class time. The instructor is the person to contact should there be any questions about the course.

Course Examiner: Dr. N. Hardy

Textbook: *Elementary Algebra,* 5th Edition, Larson/Hostetler (Brooks Cole).

Credit: This is an introductory course in Algebra. Students with credits for any Concordia Math course will not receive credit for this course.

Moodle Site: Information pertaining to your section will be provided by your instructor. General information pertaining to the course (such as deadlines for assignments, dates for midterm, alternate, final exam, quizzes, etc.) will be posted in the Moodle Meta Site of the course.

Office Hours: Your professor will announce her/his office hours during which she/he also will be available to give a reasonable amount of help. However, if you missed a class, it is not reasonable to expect your professor to cover the missed material for you.

Tutorials: The material in this course requires a lot of practice. There is not enough class time to do all the examples and problems needed to learn the material thoroughly. The Department has therefore organized special **Tutorials** conducted once per week for one hour for every section of this course to provide additional support to students outside the lecture room environment. Tutorials are conducted by graduate students who will help with solving problems on the topics learned in class that week, with particular emphasis on the material that students may have difficulties with in this course. Students are strongly encouraged to participate and be active at these problem-solving sessions. They are an important new resource to help you succeed in this course.

Math Help Centre: In addition to Tutorial sessions, a Math Help Centre staffed by graduate students is available. The schedule of its hours of operation and its location will be posted in the Department.

Assignments: Students are expected to submit assignments online using **WeBWorK** (see below). Late assignments **will not** be accepted. Working regularly on the assignments is essential for success in this course. Students are also strongly encouraged to do as many problems on their own as their time permits from the list of recommended problems included in this outline as well as the practice problems in WeBWorK. A solutions manual for all odd-numbered questions is packaged with the textbook.

WeBWorK: Every student will be given access to an online system called **WeBWorK**. The system provides you with many exercises and practice problems. Students will use this system to do *online* **assignments**. In addition, before the midterm test and a few weeks before the end of the course, a number of practice problems will be posted in WeBWorK to help you review the material.

Calculators: Only calculators approved by the Department are permitted in the class tests and final examination. The calculators are the **Sharp EL 531** and the **Casio FX 300MS**, available at the Concordia Bookstore. See www.mathstat.concordia.ca for more information.

Midterm Test: There will be one common midterm test based on the material covered on chapters 1 to 4, inclusive. It will be held on **Sunday February 24, 2013 at 2:00 P.M.** Students who will not be able to write the test that day for a valid reason, e.g. religious or illness (medical note is required), may write an alternate midterm test on **Saturday March 2, 2013 at 10:00 A.M.**
NOTE: It is the Department's policy that tests missed for any reason, including illness, cannot be made up. If you miss both the midterm and alternate test because of illness (a medical note is required) the final exam can count for 90% of your final grade; the remaining 10% will be determined by the WebWork assignments.

Final Exam: The final examination will be three hours long and will cover all the material in the course.
NOTE: Students are responsible for finding out the date and time of the final exam once the schedule is posted by the Examinations Office. Any conflict or problems with the scheduling of the final exam must be reported to the Examinations Office, **not** to your instructor. It is the Department's policy and the Examinations Office's policy that **students**

are to be available until the end of the final examinations period. Conflicts due to travel plans will **not** be accommodated.

Grading Scheme: The final grade will be based, in all cases, on the **higher** of the two options:

a) 10% for the assignments
25% for the midterm test
65% for the final exam.

b) 10% for the assignments,
10% for the midterm test,
80% for the final exam.

IMPORTANT: THERE IS NO "100% FINAL EXAM" OPTION IN THIS COURSE.

Approximate # of Lectures	Chapters/Topics	Sections	Recommended problems
2	Chapter 1 Real Numbers	1.1	9, 15, 19, 25, 29, 41, 45, 49, 55, 61, 65
		1.2	15, 17, 45, 55, 65, 77, 79, 83, 85, 91, 105
		1.3	3, 13, 25, 31, 39, 41, 45, 51, 61, 69, 91, 105, 115, 117
		1.4	3, 7, 15, 19, 23, 27, 37, 45, 51, 65, 75, 79, 87, 95, 105, 113, 135, 141, 159
		1.5	5, 13, 21, 23, 49, 71, 73, 109, 115, 121, 127, 137, 149
2	Chapter 2 Fundamentals of Algebra	2.1	Concept check: 1, 3 -- 3, 45, 47, 59, 69, 77, 91, 93, 97
		2.2	Concept check: 1, 2 -- 1, 3, 9, 11, 13, 43, 59, 79, 85, 89, 97, 123, 129, 151, 161
		2.3	7, 11, 19, 23, 25, 55, 59, 71, 83
		2.4	11, 15, 21, 67, 75, 85, 88
3	Chapter 3 Linear Equations	3.1	Concept check: 1, 2, 3, 4 -- 1, 7, 27, 35, 65, 69, 75, 83, 91
		3.2	27, 29, 53, 63, 77, 83, 85, 92
		3.3	Concept check: 1, 3 -- 3, 11, 23, 29, 43, 47, 51, 55, 83, 87, 99, 100
		3.4	15, 21, 27, 39, 45, 55, 67, 83
		3.5	3, 11, 31, 35, 45, 72
		3.6	9, 11, 15, 21, 49, 63, 69, 85
3	Chapter 4 Equations & Inequalities	4.1	Concept check: 1, 3, -- 9, 13, 15, 25, 37, 41, 53, 55, 59, 81, 83
		4.2	1, 7, 13, 15, 25, 35, 45, 51, 65, 73
		4.3	Concept check: 1, 2, 3, 4 -- 5, 7, 17, 19, 21, 33, 37, 45, 47, 57, 75, 103, 106
		4.4	Concept check: 1, 3, 4--5, 9, 31, 33, 37, 55, 57, 67, 73, 83, 111, 114, 115
1	Chapter 5 Exponents & Polynomials	5.1	Concept check: 3 -- 7, 21, 31, 37, 45, 51, 57, 65, 69, 75, 79, 95, 99, 103, 105, 131, 147, 191, 197
		5.2	25, 29, 33, 41, 45, 53, 65, 69, 77, 89, 93, 101
4	Chapters 5 & 6 Factoring	5.3	9, 13, 23, 31, 45, 65, 71, 87, 101, 109, 135, 136, 137
		6.1	Concept check: 1, 3 -- 11, 13, 29, 51, 65, 67, 83
		6.2	Concept check: 1, 2, 4 -- 5, 15, 21, 25, 41, 47, 59
		6.3	7, 23, 31, 43, 57, 65, 95, 107, 111, 117, 118
		6.4	1, 11, 23, 31, 43, 53, 71, 75, 85, 101, 117, 135, 136
		6.5	3, 7, 13, 25, 31, 41, 55, 61, 71, 73, 75

2	Chapter 7 Rational Expressions & Equations	7.1 7.2 7.3 7.4 7.5	Concept check: 1, 2 -- 1, 3, 9, 13, 21, 33, 35, 43, 51, 75, 110, 114 5, 11, 51, 55, 59, 65, 73, 83, 95 3, 7, 15, 45, 51, 55, 61, 73, 88 1, 5, 9, 13, 23, 27, 35, 44 1, 5, 9, 13, 33, 39, 53, 59, 63, 73, 97, 101, 103
4	Chapter 8 Systems of Linear Equations	8.1 8.2 8.3 8.4	1, 9, 15, 17, 23, 31, 47, 49, 51, 59, 61, 63 17, 37, 41, 43, 73 7, 17, 25, 27, 43, 47, 1, 3, 29, 49, 57, 59, 71, 72
2	Chapter 9 Roots & Radicals (Quadratic formula for solving quadratic equations)	9.1 9.2 9.3	Concept check: 1, 2, 4 -- 1, 5, 13, 17, 21, 59, 61, 73, 77, 95, 105, 109 Concept check: 1, 2, 4 -- 3, 9, 13, 17, 21, 29, 35, 41, 49, 59, 63, 73, 87, 95, 125, 127 Concept check: 1, 2, 3, 4 -- 9, 13, 25, 51, 63, 71, 81, 95, 101, 115, 123, 137

- For sections taught once a week, 2 lectures correspond to 1 day of class.
- There would be at least 1 lecture dedicated to reviewing for the midterm and at least 1 lecture dedicated to reviewing for the final exam.

Revised Course Outline

Department of Mathematics & Statistics
Concordia University

MATH 200 Fundamental Concepts of Algebra

Fall 2013

Instructor*: ____

Office/Tel No.: ____

Office Hours: ____

*Students should get the above information from their instructor during class time. The instructor is the person to contact should there be any questions about the course.

Course Examiner: Dr. N. Hardy

Textbook: *Elementary Algebra,* 5th Edition, Larson/Hostetler (Brooks Cole).

Credit: This is an introductory course in Algebra. Students with credits for any Concordia Math course will not receive credit for this course.

Moodle Site: Information pertaining to your section will be provided by your instructor. General information pertaining to the course (such as deadlines for assignments, dates for midterm, alternate, final exam, etc.) will be posted in the Moodle Meta Site of the course.

Office Hours: Your professor will announce her/his office hours during which she/he also will be available to give a reasonable amount of help. However, if you missed a class, it is not reasonable to expect your professor to cover the missed material for you.

Tutorials: The material in this course requires a lot of practice. There is not enough class time to do all the examples and problems needed to learn the material thoroughly. The Department has therefore organized special **Tutorials** conducted once per week for one hour for every section of this course to provide additional support to students outside the lecture room environment. Tutorials are conducted by graduate students who will help with solving problems on the topics learned in class that week, with particular emphasis on the material that students may have difficulties with in this course. Students are strongly encouraged to participate and be active at these problem-solving sessions. They are an important new resource to help you succeed in this course.

Math Help Centre: In addition to Tutorial sessions, a Math Help Centre staffed by graduate students is available. The schedule of its hours of operation and its location will be posted in the Department.

Assignments: Students are expected to submit assignments online using **WeBWorK** (see below). Late assignments **will not** be accepted. Working regularly on the assignments is essential for success in this course. Students are also strongly encouraged to do as many problems on their own as their time permits from the list of recommended problems included in this outline as well as the practice problems in WeBWorK. A solutions manual for all odd-numbered questions is packaged with the textbook.

WeBWorK: Every student will be given access to an online system called **WeBWorK**. The system provides you with many exercises and practice problems. Students will use this system to do **online assignments**. In addition, before the midterm test and a few weeks before the end of the course, a number of practice problems will be posted in WeBWorK to help you review the material. It is still essential that you work on the recommended problems from the textbook – see below.

Calculators: Only calculators approved by the Department are permitted in the midterm test and final examination. The calculators are the **Sharp EL 531** and the **Casio FX 300MS**, available at the Concordia Bookstore. See www.mathstat.concordia.ca for more information.

Midterm Test: There will be one midterm test based on the material covered on chapters 1 to 4, inclusive. It will be held on **Sunday February 24, 2013 at 2:00 P.M.** Students who will not be able to write the test that day for a valid reason, e.g. religious or illness (medical note is required), may write an alternate midterm test on **Saturday March 2, 2013 at 10:00 A.M.**
NOTE: It is the Department's policy that tests missed for any reason, including illness, cannot be made up. If you miss both the midterm and alternate test because of illness (a medical note is required) the final exam can count for 90% of your final grade; the remaining 10% will be determined by the WebWork assignments.

Final Exam: The final examination will be three hours long and will cover all the material in the course.
NOTE: Students are responsible for finding out the date and time of the final exam once the schedule is posted by the Examinations Office. Any conflict or problems with the scheduling of the final exam must be reported to the Examinations Office, **not** to your instructor. It is the Department's policy and the Examinations Office's policy that **students are to be available until the end of the final examinations period**. Conflicts due to travel plans will **not** be accommodated.

Grading Scheme: The final grade will be based, in all cases, on the **higher** of the two options:

a) 10% for the assignments
25% for the midterm test

65% for the final exam.

b) 10% for the assignments,

10% for the midterm test,

80% for the final exam.

IMPORTANT: THERE IS NO "100% FINAL EXAM" OPTION IN THIS COURSE.

Approximate # of Lectures	Chapters/Topics	Sections	Recommended problems
2	Operations on Integers and Rational Numbers	1.1	9, 15, 19, 25, 29, 41, 45, 49, 55, 61, 65
		1.2	17, 55, 65, 77, 79, 83, 85, 91, 105, 112, 114
		1.3	3, 13, 25, 45, 51, 61, 69, 91, 105, 115, 117
		1.4	3, 7, 15, 19, 23, 27, 37, 45, 51, 65, 75, 79, 87, 95, 105, 113, 159
		1.5	33, 37, 39, 45, 51, 59, 69, 75, 93, 111, 115, 121, 127, 149
2	Linear Expressions and Problem Solving	2.1	Concept check: 1, 3 -- 3, 45, 47, 59, 69, 77, 91, 93, 97, 99
		2.2	Concept check: 1, 2 -- 1, 3, 9, 11, 13, 43, 59, 79, 85, 89, 97, 123, 129, 151, 161
		2.3	7, 11, 19, 23, 25, 55, 59, 71, 83, 85
		2.4	11, 15, 21, 67, 75, 85, 88
4	Linear Equations and Graphing	3.1	Concept check: 1, 2, 3, 4 -- 1, 7, 27, 35, 65, 69, 75, 83, 91
		3.2	27, 29, 53, 63, 77, 83, 85, 92
		3.3	Concept check: 1, 3 -- 3, 11, 23, 29, 43, 47, 51, 55, 83, 87, 99, 100
		3.4	15, 21, 27, 39, 45, 55, 67, 83
		3.5	3, 11, 31, 35, 45, 72
		4.1	Concept check: 1, 3, -- 9, 13, 15, 25, 37, 41, 53, 55, 59, 81, 83
		4.3	Concept check: 1, 2, 3, 4 -- 5, 7, 17, 19, 21, 33, 37, 45, 47, 57, 75, 103, 106
		4.4	Concept check: 1, 3, 4--5, 9, 31, 33, 37, 55, 57, 67, 73, 83, 111, 114, 115
3	Systems of Linear Equations	4.2	1, 5, 11, 15, 25, 35, 45, 51, 65, 71, 73
		8.1	1, 9, 15, 17, 23, 31, 47, 49, 51, 59, 61, 63
		8.2	17, 37, 41, 43, 73
		8.3	7, 17, 25, 27, 43, 47, 67
		8.4	1, 3, 29, 49, 57, 59, 71, 72
2	Linear Inequalities	3.6	Concept check: 3 -- 9, 11, 15, 21, 49, 63, 69, 85
5	Exponents, Polynomial Expressions and Equations	1.5	5, 13, 21, 23, 49, 71, 73, 137
		5.2	25, 29, 33, 41, 45, 53, 65, 69, 77, 89, 93, 101
		5.3	9, 13, 23, 31, 45, 65, 71, 87, 101, 109, 135, 136, 137
		6.1	Concept check: 1, 3 -- 11, 13, 29, 51, 65, 67, 83
		6.2	Concept check: 1, 2, 4 -- 5, 15, 21, 25, 41, 47, 59, 79
		6.3	Concept check: 2, 3 -- 7, 23, 31, 43, 57, 65, 95, 107, 111, 117, 118
		6.4	1, 11, 23, 31, 43, 53, 71, 75, 85, 101, 117, 135, 136
		6.5	Concept check: 3, 4 -- 3, 7, 13, 25, 31, 41, 55, 61, 71, 73, 75
3	Rational Expressions & Equations	5.1	Concept check: 3 -- 7, 21, 31, 37, 45, 51, 57, 65, 69, 75, 79, 95, 99, 103, 105, 131, 147, 191, 197
		7.1	Concept check: 1, 2 -- 1, 3, 9, 13, 21, 33, 35, 43, 51, 75, 110, 114
		7.2	5, 11, 51, 55, 59, 65, 73, 83, 95
		7.3	3, 7, 15, 45, 51, 55, 61, 73, 88
		7.4	1, 5, 9, 13, 23, 27, 35, 44
		7.5	1, 5, 9, 13, 33, 39, 53, 59, 63, 73, 97, 101, 103
2	Radicals and Rational Exponents	9.1	Concept check: 1, 2, 4 -- 1, 5, 13, 17, 21, 59, 61, 73, 77, 95, 105, 109
		9.2	Concept check: 1, 2, 4 -- 3, 9, 13, 17, 21, 29, 35, 41, 49, 59, 63, 73, 87, 95, 125, 127
		9.3	Concept check: 1, 2, 3, 4 -- 9, 13, 25, 51, 63, 71, 81, 95, 101, 115, 123, 137,

			139, 140

- For sections taught once a week, 2 lectures correspond to 1 day of class.
- There would be at least 1 lecture dedicated to reviewing for the midterm and at least 1 lecture dedicated to reviewing for the final exam.

Appendix B: MATH 200 Activities

Institutional Problems

Direct Computation

In-Class Problems

Evaluate the following algebraic expressions:

a. $a^2 - 3$ for $a = 1$, for $a = 4$, and for $a = -2$

b. $(a - 3)^2$ for $a = 1$, for $a = 4$, and for $a = -2$

c. $\frac{x^2+6}{5x-2}$ for $x = 2$, for $x = -1$, and for $x = ½$

Distribute and combine like terms:

$5(x - 3) + 4(1 + x)$
$5(x - y) + 4(y + x)$
$x(8 - 2) + 2(x - 3)$
$2x(x - 1) + x(2 - x)$
$x(y - x) + y(x + x) - 2x + 2y$

Evaluate $|3x^2 - y|$ for $x = 3$ and $y = -2$.

Distribute and combine like terms:

a. $(5x - 2)(x + 4) - 3x$

b. $x^2(x - 1) + 2x(3 - 2x)$

Solve for x the following linear equations:

a. $3x - 2 = 5$

b. $x + 1 = 1$

What is the 50% of 40?

What is the 50% of 50?

What is the 20% of 50?

What is the 25% of 66?

Write a linear expression that represents that when x = 4 then y = 2/3, and the rate of change is 5. (We can also say that the point (4, 2/3) belongs to the line with slope 5.)

Solve the system: $x + y = 4$, $3x + y = 6$.

Graph the two equations in problem 6 (above) in a Cartesian plane; what point represents the solution to the system?

Problem 1. Factor completely.

$x^2 - 16$	$x^2 - 3x + 9$	$x^3 + 4x^2 + 4x$	$x^3 + 2x^2 - (x + 2)$
$x^4 - 1$	$x^5 - 3x^4 - x + 3$		

Problem 2. For each of the expressions in problem 1, solve the correspondent homogeneous equation (the expression equal zero).

Problem 3. Solve the following equations

a. $x^2 + 3x + 2 = 0$

b. $\dfrac{x^2 + x}{x + 1} = -2$

Problem 4. Factor completely:

$x^3 - 1$	$x^3 - 8$	$x^3 + 8$	$3x^3 + 3x^2 + 3x$	$x^4 - 9$	$x^4 + 9$
$x^2 + 3x - 18$	$5x^2 - 4x - 2$				

Problem 5. For each of the expressions above, solve the corresponding homogeneous equation

Problem 6. Solve

$$\frac{4}{x+3} = x$$

$$\frac{4}{x} = x + 3$$

$$x^2 + 3x - 4 = 0$$

Problem 7. Solve

$$\frac{x(x+1)}{x-1} = 4$$

$$x = 4\frac{x-1}{x+1}$$

1. Write as a simple fraction in lowest terms.

a. $\frac{\frac{2}{3}}{\frac{1}{7}}$ b. $\frac{1}{\frac{-2}{5}}$ c. $\frac{\frac{2}{3}}{4}$ d. $\frac{\frac{2}{x}}{\frac{x^2}{3}}$ e. $\frac{\frac{2}{3}+\frac{1}{2}}{\frac{1}{4}}$

f. $\frac{1+\frac{7}{2}}{\frac{2}{3}+3}$

g. $\frac{x+1}{1+\frac{1}{x}}$ h. $\frac{\frac{1}{x}+\frac{1}{y}}{3}$ i. $\frac{1}{x}+\frac{x^2+1}{\frac{1}{x}}$ j. $\frac{15x+21}{9x+15}$ k. $\frac{x^2+2x}{x^2-x}$

l. $\frac{x^2-11x+28}{x^2+2x-24}$ m. $\frac{x^7}{x^2+5x-50}$ n. $\frac{\frac{1}{2a}+\frac{3}{a^2}}{\frac{1}{3a}-\frac{4}{a^2}}$

2. Solve the homogeneous equations corresponding to items d, g, j, k, l and m in problem 1.

3. Solve the following equations:

a. $\frac{3}{2b-1}=\frac{4}{3b}$ b. $\frac{4}{3b}+\frac{1}{6b}=\frac{7}{2b}+\frac{1}{3}$ c. $\frac{4}{3}+\frac{7}{x-4}=\frac{x-1}{3x-12}$ d. $\frac{x^2+8x+6}{x^2+3x-4}=\frac{3}{x-1}-\frac{2}{x+4}$

e. $\frac{6}{z^2-1}=\frac{5}{z-1}-\frac{3}{z+1}$ f. $\frac{2}{x-1}-\frac{4}{x^2-1}=-2$ g. $\frac{x^3+8}{x^2+2}=x$ h. $x^3+20-\frac{1}{x}=\frac{-7x-1}{x}$

<u>Tutorial Problems</u>

Evaluate $\frac{a^2-4}{a^2-8a+16}$ for a = 2, for a = -3 and for a = 4.

Use the distributive property and combine like terms:

a. $-(6u + 12y - 13u)$
b. $18a - (6 + 9a)$
c. $(5y - 6) - (11y + 8)$
d. $3a + 4b - 7a + 3(a - 2b)$; then evaluate for a = 2 and b = 5
e. $(5x - 2x^2)(x - 1)$

Solve for z

a. 3z + 1 = 3z – 1

b. $3(z + 1) - 2z + 3 = z + 6$
c. $3z + 1 = 2z$

Graph the linear expression $y = -4x + 2$; in your graph, the points should represent the relation between the independent variable x and the dependent variable y.

In the expression $y = -4x + 2$, what is the value of y associated with the x-value -2 ?

In the expression $y = -4x + 2$, what is the value of x associated with the y-value 0 ? And with the y-value 3 ?

Solve the following systems of linear equations:

a. $3x - 3y = -2$
   $6x - 6y = 4$

b. $x + y = 1$
   $-x - y = -1$

c. $2y = 6x + 4$
   $3x - y + 2 = 0$

Solve the following inequalities (present your answer graphically and in interval notation):

a. $-x + 2(x - 1) < 4$
b. $-4x \geq -1$
c. $|x + 1| > 0$
d. $|2x - 3| \leq 5$

Graph the following inequalities:

a. $x + y < 0$
b. $2x - y + 4 > 1$

Graph the following system of inequalities:

$x + y > 0$
$y - x < 1$
$x < 2$

Consider the inequality $5x + 2 > 7x - 1$.

a. Write the solution in interval notation.
b. Graph the inequality.

The inequality | 2x + 1 | > 1. Has solution (∞, *a*) U (*b*, ∞). What are the values of *a* and *b*?

Use the rules of exponents to "simplify" the notation; for the constants, "simplify" so that they don't have common factors.

a. $\frac{(2xy)^3x^2}{2y}$

b. $(3xy^2)^4x^2$

c. $\frac{6a^2(ab)b^4}{2(3ab)^3ab^2}$

Factor completely.

$x^2 - 16$	$x^2 - 3x + 9$	$x^3 + 4x^2 + 4x$
$x^3 + 2x^2 - (x + 2)$	$x^4 - 1$	$x^5 - 3x^4 - x + 3$

For each of the expressions in problem 1 (above), solve the correspondent homogeneous equation (the expression equal zero).

Solve the following equations

a. $x^2 + 3x + 2 =0$

b. $\dfrac{x^2 + x}{x + 1} = -2$

1. Write as a rational expression in lowest terms

a. $\frac{1}{x+2} + \frac{1}{x-2} + 1$

b. $\frac{1}{2x-2} + \frac{1}{2x+2}$

c. $\frac{x^3-4x}{x^3+x^2-x}$

d. $\frac{\frac{x^2-2x}{x+2}}{\frac{x^2-x}{x+2}}$

e. $\frac{c}{\frac{a}{b} + \frac{m}{n}}$

f. $\frac{\frac{a}{c}}{\frac{a}{b} + \frac{m}{n}}$

g. $\frac{\frac{x}{y} + \frac{p}{q}}{\frac{a}{b} + \frac{m}{n}}$

2. Solve the homogeneous equations corresponding to items a. to d. above

1. Write as a single fraction, rationalize, and write in lowest terms – when applicable.

   a. $\frac{2}{3-\sqrt{5}}$ b. $3-\frac{2}{\sqrt{5}}$ c. $\frac{5\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

2. $\sqrt{2}+\sqrt{18}$ can be written as $a\sqrt{2}$. What is the value of a?

3. $\sqrt{18x}+3\sqrt{8x}$ can be written as $a\sqrt{bx}$. What are the values of a and b?

4. Solve the following equations

   a. $x^2 = 25$ b. $\sqrt{x} = 5$ c. $\sqrt{x-5} = 2$ d. $\sqrt{x+5} = 2$

   e. $\sqrt{x+1} = 2$ f. $\sqrt{x+2} = \sqrt{2x}$ g. $\sqrt{x} - \sqrt{2x+1} = 0$

   h. $\sqrt{x+9} = x+4$

Word Problems

In-Class Problems

Maria had d dollars to invest. She split the money into two accounts. In one of the accounts, she invested $3000. Write the algebraic expression that represents how much she invested in the other account.

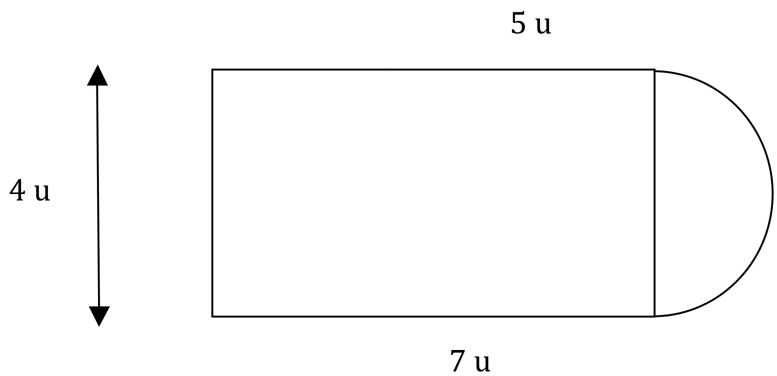
A wire with total length (5x + 4.6) metres is cut into two pieces such that one piece measures (3x – 2.5) metres. Write an expression that represents the length of the other piece. What is the result of this other piece if x = 4.5?

A single serving of cereal contains 13 grams of fat. Of this, 2 grams are saturated fat. What percentage of fat grams is saturated fat in a single serving of cereal? Do you know how the label in the package would read?

In a recent poll, it was found that out of 1200 adult men, 725 don't like basketball. What is the percentage of adult men that don't like basketball?

How long will it take Steven to drive 365 km if his average speed is 117 km/h?

Consider the figure below. Find its area and its perimeter.



Suppose the line y = 0.75x represents cost per unit. What is the cost of 10 units? How many units cost $10?

Tutorial Problems

The record high temperature in Fort Yukon was 100^0F recorded in 1915. The record low was -80^0F in 1971. What is the difference between these two records?

In an algebra test, 5 points were awarded for each correct multiple-choice question and 8 points for each correct free-response question. A student got 11 multiple-choice questions correct and 4 free-response questions correct. What was the student's test score?

James owns Maria $100 and Maria owns James $50. Also, Paul owns James $70 and James owns Carl $30. When James pays what he borrowed and collects what he lent, how much money would he have?

In 2010, first-class postage for a standard postcard was $0.28. At this price, how much would it cost to mail 9 postcards?

A serving of soup of brand K contains 1.5 grams of fat. If a can contains 2.5 servings, how many total grams of fat are in the can?

A rectangle has width one less than twice its length. Write an algebraic expression that represent the perimeter of the rectangle and an algebraic expression that represents the area of the rectangle, both in terms of the length.

At a certain museum, tickets for adults cost $25, tickets for seniors cost $15, and tickets for children under 17 cost $13. Write an expression that gives the revenue for each day. If Mondays there's a 50% discount, what the revenue expression for those days would be?

The sum of three consecutive integers is 66. Find the largest of the three.

The sum of three consecutive odd integers is 39. Find the three integers.

An 80cm wire is cut into two pieces so that one piece is four times as long as the other. Find the length of each piece.

Peter works two jobs, one as a waiter that pays \$9.50 per hour and one as a music teacher that pays \$14 per hour. Last week he worked 35 hours and his salary after 12% deductions, was \$352. How many hours did he work each job?

How much principal did I invest in my account that pays simple interest 2% if at the end of the year I had \$2,550 in the account?

A taxicab charges a flat rate of \$4.15 plus \$0.12 per km.

a. Write a linear expression in which the independent variable is the number of km driven and the dependent variable is the cost of the ride.
b. How much will it cost you to ride 10km?
c. If at the end of a ride you have to pay \$10, who many km was the ride?
d. Graph the expression in a Cartesian plane.
e. If you are 35km from the airport and taking the bus will cost you \$8, what is less expensive, the bus or the taxicab? How can you "see" the answer in the graph?

You are going on a trip to the US and want a plan to use your cell phone there. There's a package in which you pay \$10 so that calls will cost you \$0.35 per minute (<u>no</u>, the \$10 don't include any minutes – you pay for the right of paying \$0.35 per minute; <u>yes</u>, this is how these packages typically work). If you don't take any package, your calls will cost \$0.55 per minute.

a. Write a linear expression that gives you cost in terms of minutes for the package and another linear expression that gives you cost in terms of minutes for the case of not buying any package.
b. What is more convenient? Clearly explain (at least to yourself!) how you assess which one is more convenient.
c. Graph both expressions in the same Cartesian plane. Can you "see" which one is more convenient?

Goal-Oriented Problems

Conceptual – Abstract

In-Class Problems

Let a be a number; explain the meaning of $|a|$. Is $-a$ positive or negative? Is $|a|$ positive or negative?

When is the sum of a positive integer and a negative integer a positive integer? Explain.

Is it possible that the difference of two negative integers is a positive integer? Explain.

Explain why the product of an even integer and any other integer is even. What conclusions can you make about the product of two odd integers?

An integer n is divided by 2 and the quotient is an even integer. What does this tell you about n?

Consider the following statement (it is a theorem): Let a be a number, if no prime number less than $\sqrt{a}$ divides a, then a is prime. Convince yourself that it is true... convince someone else that it is true?

Are the following statements true or false? Explain your choices.

- "The reciprocal of every nonzero rational number is a rational number"

- "The reciprocal of every nonzero integer is an integer"

- "The product of two positive rational numbers is greater than either factor."

What is the additive inverse of a? What is the multiplicative inverse of a? (The multiplicative inverse is also called "reciprocal").

Express symbolically:

a. 5 times a number

b. The square of a number

c. The square of a number, increased by 7

d. Three less than five times a number

e. Twice the sum of two numbers

What is the price of purchasing *x* many apps if each app costs \$0.99?

What number do you have to add to x to get 0?
What number do you have to divide x by to get 1?

What number do you have to subtract from 3x to get 0?

What number do you have to multiply -5x by to get 1?

These are not linear equations: $3x^2 + 4x = 1$; $3x + 1$; $\frac{4}{x+1} = 7$; **Why not?**

These are linear equations: $3x - 2 = -5x$; $\frac{x+1}{4} = 7$; $\frac{1}{2}x + 7 = 4x - \frac{x-1}{3}$; **Why?**

What is the 50% of A?

What is the 25% of A?

What is the 23% of A?

What does it mean that the slope of a given line is 0?

What is the relation between the slopes of two lines that are parallel?

It is important that you know that if two lines are perpendicular, the slope of one is the negative multiplicative inverse of the other. So if a line has slope m, what is the slope of a line perpendicular to it?

What are the y-intercept and x-intercept of the line $y = mx + b$?

Lines with a vertical representation in a Cartesian plane cannot be written in the form $y = mx + b$. Why not? What would be a linear expression representing a vertical line?

What does it mean, in terms of rate of change, that a line has slope 0?

Thinking about problem 7(*), what are the possible solutions to a system of two linear equations in two variables?

(*) 6. Solve the system: $x + y = 4$, $3x + y = 6$.
7. Graph the two equations in problem 6 in a Cartesian plane; what point represents the solution to the system?

Tutorial Problems

Let $a < 0$ and $b > 0$, is $a - b$ always negative? Is $a + b$ always positive?

Order of operations: explain in a brief paragraph the order of operations.

Is $2x^2 + 4x^2 = 6x^2$?

Is $(2x^2)(4x^2) = 8x^4$?

In the expression y = -4x + 2, what is the y-value associated with the x-value a ? and what is the x-value associated with the y-value b ?

In the expression y = -4x + 2 it is easy to find the value of y for any given value of x; solve for x to find an expression in which it is easy to find the value of x for any given value of y.

Are the following identities true?

a. $(x - 1)(x + 1) = x^2 - 1$

b. $\frac{x+1}{(x+1)(x+2)} = \frac{1}{(x+2)}$

c. $\frac{(x^2+1)^2}{x^2+1} = x^2 + 1$

Conceptual – Concrete

In-Class Problems

How many numbers are 3 units away from 0 on the number line? Explain.

Which real number lies farther from -4 on the real number line, 3 or -10? Explain.

Propose an argument to justify that 6 subtracted from 10 is 4.

Propose an argument to justify that 10 subtracted from 6 is -4.

Do you remember the column algorithm for multiplication? Try it with a few numbers, explain how it works, and propose an argument to explain why it works.

Propose an argument to justify that 2 divided by 4 is 0.5 and 2 divided by 3 is 0.66666...

Propose an argument to justify that 3 divided by 2 is 1.5.

Does $\frac{2}{3}+\frac{3}{2}=\frac{2+3}{3+2}=1$? Explain your answer.

Is the following statement correct? If not, how would you fix it?
5 – (4 – 2)
= 5 – 4 – 2
= 5 – 6
= - 1

(Follow up)

We agreed in class that the answer should be 3 (and not -1). We discussed two ways of looking at this:

5 – (4 – 2) = 5 – 2 = 3, Or
5 – (4 – 2) = 5 – 4 + 2 = 1 + 2 = 3

Why do we have to change the – sign in front of 2 by a + sign when removing the parentheses?

What number do you have to multiply x by to get 0?

What number do you have to multiply x by to get 1?

We stated that in an expression such as 3x + 1, x is called a variable; it can take the value of any integer and for each value it takes, 3x + 1 represents a different value. In an expression such as 3x + 1 = 2, x is called an unknown; we can solve for x (which means to find the actual value of that unknown) – in this case, x = 1/3. We will focus now on expressions such as 3x + 1 (a linear algebraic expression). We will see that in many situations, we are interested in representing all the possible values that an expression such as 3x + 1 can take. What are the possible values that 3x + 1 can take?

In referring to the points in the line, we mean the pair of points (x, y) that satisfy that y depends on x. For example, if we consider the line y = 2x + 3, the points (0, 3) and (1, 5) are in the line but the points (0, 1) and (5, 1) are not in the line. Give two other examples of points in this line and two other examples of points that are not in this line.

Given the equation $y = -2x + 1/3$, give examples of solutions and of non-solutions.

Write a linear equation in two variables such that (1, -4) is a solution but (3, 2) is not.

Tutorial Problems

Do the following operations:

a. $3 – (-2 + 5)$
b. $-3 – 2 + 5$
c. $3 + (-2 – 5)$
d. $-3 + (-2 – 5)$
c. $-3 – (-2 – 5)$

How can you check that your answers are correct?

Explain how the column algorithm for addition works. Explain it in the case of adding 234 and 907. Propose other strategies for doing the addition.

1. Write in symbols:
   a. 5 more than the difference of 2 and 7
   b. Two-thirds subtracted from 5/9
   c. 8 subtracted from 5, decreased by 3
   d. Three times the sum of 10 and 4
   e. The difference of 2 and the product of 8 and 15
   f. 3 increased by 15, times 4
   g. The difference of 5 and 9, divided by the difference of 10 and 4

Perform each calculation.

Write an inequality that has for solution the interval [*a*, *b*], where *a* and *b* are the values you found above. (*)

* The inequality $| 2x + 1 | > 1$. Has solution $(\infty, a) \cup (b, \infty)$. What are the values of *a* and *b*?

"Guess" the solutions of the following equations (don't do any calculations, except checking that your guesses are correct). If there are no solutions, justify why not.

a. $x^2 = 0$ (there's only one solution)
b. $x^2 – 1 = 0$ (there are two solutions)

c. $x^2 + 1 = 0$ (there are no solutions)
d. $x^3 - x = 0$ (there are three solutions)
e. $x^3 + 1 = 0$ (there's only one solution)
f. $x(x - 1)^2 = 0$ (there are two solutions; what's the coefficient of x^3? What's the coefficient of x^2, of x, of x^0?

Multiple Methods

In-Class Problems

Propose three different strategies for mentally adding 16 and 17. Clearly explain what the strategies are.

Propose two different strategies for adding 151 and 23 without using the calculator. Do the same for subtracting 23 from 151.

Calculate (can you try by hand first?)
a. 32 divided by 8
b. 32 divided by 4 (can you use what you find in a.?)
c. 320 divided by 8 (can you use what you find in a.?)
d. 320000 divided by 8
e. 325 divided by 3
f. 401 divided by 34
In each case, propose at least two strategies to verify your answer (one could be using the calculator).

Each integer has a unique representation as an integer, for example, 2, as an integer, can be represented by the symbol 2 and nothing else. However, rational numbers have many representations and this, among other properties, makes them complicated objects. For example, 2, as a rational number, can be represented by the symbols 2, 4/2, 8/4, etc. These symbols are called 'fractions'.

a. Two different fractions that represent the same rational number are said to be equivalent. For example, 2 and 4/2 are equivalent fractions because both represent the same rational number. How can you check whether two fractions are equivalent? For example, are 2/5 and 12/30 equivalent? Propose two different strategies for verifying this (at this point, consider exercise 16!) – explain why the strategies work (that is, why the strategies accomplished what you are trying to do).

Perform the following operations: (try by hand first!)
a. 0.24 + 1.1 – 3.2

b. 0.24 x 2.7

In each case, propose at least two strategies to verify your answer (one could be using the calculator).

Calculate (try these by hand – without the calculator):

a. 4 – 3 + 7 – 19 + 8

b. -1 + 1 -1 + 1 – 1

c. 3(4 + 7) – 2(10x21) + (-4)

In each case, propose at least two strategies to verify your answer (one could be using the calculator).

Calculate (try these by hand – without the calculator):

d. 4 – 3 + 7 – 19 + 8 It was said in class that the answer is: -2. Is it correct?

e. -1 + 1 -1 + 1 – 1 It was said in class that the answer is: -1. Is it correct?

f. 3(4 + 7) – 2(10x21) + (-4) It was said in class that the answer is: -391. Is it correct?

In each case, propose at least two strategies to verify your answer (one could be using the calculator).

Calculate 5(2 – 3) + 4(1 + 7).
Do it in two different forms: 1. first calculate the operations between parentheses and then continue; and 2. first calculate the products and then continue.

Give examples of linear equations in one unknown of the three possible types (1 sol, none, inf. many)

Tutorial Problems

Propose two different strategies for adding 151 and 23 without using the calculator. Do the same for subtracting 23 from 151.

Explain how the column algorithm for addition works. Explain it in the case of adding 234 and 907. Propose other strategies for doing the addition.

Perform the following calculations (try without the calculator, by hand, in two different 'efficient' ways, then use the calculator to verify your answers)

a. $\frac{-3}{2} + 5(2 - \frac{2}{3})$

b. $\frac{-3}{2} - 5(-2 + \frac{2}{3})$

c. $\frac{3}{2} \div [5\left(2 - \frac{2}{3}\right)]$

d. $\frac{3}{2} \div 5\left(2 - \frac{2}{3}\right)$

e. $\frac{\frac{2}{5}-3}{3-\frac{2}{5}}$

Symbol Manipulation

In-Class Problems

Simplify the following expression as much as possible:

$$\frac{\frac{a * c * b}{c * a}}{\frac{d * b * a}{c * d}}$$

Perform the following operation:

$$\frac{c}{b} - \frac{a * c}{d * b}$$

Solve for x in the following linear equations

a. $x + 2a = 3x$

b. $-5x + 10p = 5q$ (Solve also for p and then solve for q.)

Solve for x, for p and for W:

$$2x - \frac{2 - p}{3} = 5W + 1$$

Solve for W:

a. R = W + 3W

b. R = W + PW

Tutorial Problems

Solve for p, then for q and finally for r:

p = 2q + 2r

Solve for x, then for y and finally for z:

$$\frac{z-2}{-3x+4} = 1 + \frac{5}{3-2y}$$

Different Representations

In-Class Problems

A prime number is a positive integer that is (wholly) divisible only by itself and by 1. Each integer can be represented as the product of its prime factors. Represent 8, 10, 21, and 210 as a product of prime factors.

Each integer has a unique representation as an integer, for example, 2, as an integer, can be represented by the symbol 2 and nothing else. However, rational numbers have many representations and this, among other properties, makes them complicated objects. For example, 2, as a rational number, can be represented by the symbols 2, 4/2, 8/4, etc. These symbols are called 'fractions'.

b. Give three different fractional representations of the rational number 3.

c. Give three different fractional representations of the rational number ½.

d. Give three different fractional representations of the rational number -4/3.

Decimal expansions are another form of representing rational numbers.
Represent ½ and -4/3 with decimal expansion. Is 1.23234234523456... a rational number?

The following table represents the quantity of shoes that consumers would demand at each given price. Do the values correspond to a linear expression? If yes, write the linear expression in the form y = mx + b. What is the slope? (It is negative, why??)

P	Q (in thousands
$140	0
$120	5
$100	10
$80	15

\$60	20
\$40	25
\$20	30
\$0	35

Write the equation y = 3x + 1 in the form ay + bx = c.

Write the equation 2y – 5x = 1 in the form y = mx + b.

Individualized Activities

Activity 1

Question:

Is it true that if $|b| > |a|$, then $a - b < 0$?

Answer:

Yes, this is true. Since $|b| > |a|$, b is farther away from 0 than a on the number line, therefore when you subtract b from a the result is a negative number.

- Is the answer above correct or incorrect? Explain your reasoning.
- If the answer above is *incorrect*, can you change the wording of the question (or add something to the question) so that the given answer is correct?

Activity 2

Consider the following rational algebraic expression:

$$\frac{t-2}{3t+3}$$

Now consider the following statement:

We can evaluate this expression by replacing t with any real number.

- Is the statement above correct? Why, or why not? Explain your answer.

- Can you add something to the expression so that t has one (and only one) possible value?

Activity 3

Steven is very smart. He attempted to solve a linear equation and wrote all of his work below for you to look at. Examine every step. If the work is correct, explain what is being done. If the work is incorrect, fix it!

$$\frac{1}{x+2} - \frac{3}{4} = 7$$

$$4\left(\frac{1}{x+2}\right) - 4\left(\frac{3}{4}\right) = 7$$

$$\frac{4}{x+2} - 3 = 7$$

$$(x+2)\left(\frac{4}{x+2}\right) - 3(x+2) = 7$$

$$4 - 3x + 2 = 7$$

$$6 - 3x = 7$$

$$6 - 6 - 3x = 7 - 6$$

$$-3x = 1$$

$$x = -\frac{1}{3}$$

Activity 4

Two students were asked to solve the following linear equation:

$$4x + 8 = 2(5 - x)$$

One of these students did some calculations and arrived to $\boldsymbol{-1 = -3x}$, to then find that $x = 1/3$.

The other student approached the problem in a different way and arrived to $\boldsymbol{6x = 2}$, to then find that $x = 2/6$.

1) Can we say that $-\mathbf{1} = -\mathbf{3}\boldsymbol{x}$ and $\mathbf{6}\boldsymbol{x} = \mathbf{2}$ are different representations of the same equation? Explain your answer.

2) Try and replicate these students' work. What are the possible steps that they could have taken to arrive at their respective answers?

3) Discuss which method, in your opinion, is more efficient. The first method, the second method, or some other method entirely? Explain your answer.

Appendix C: Beginning-of-Semester Questionnaire

Name, Last Name:
Circle your age range: 18-20 21-25 26-30 More than 31

<u>Part I</u>

Why are you taking this course?

Have you ever taken an algebra course before? If so, when and at what level (high school, university, etc.)?

What program are you in and/or what program are you working towards?

Are you taking this class as an elective or as a requirement for your program?

In a few words, explain what your expectations about this course are:

Do you like mathematics? Circle one: YES NO

Please, in a few words, explain why you like (don't like) mathematics and what is it about mathematics that you like (don't like).

When you solve a math problem,

a. Do you always check whether your answer is correct?

b. How do you know if you answer is correct?

Part II

1) (Do not use your calculator.) Evaluate the following expressions:

a. $13 - (5 - 12) + 4$

How do you know/What can you do to know that your answer is correct?

b. $\left(\frac{3+2}{7} + 2\left(\frac{1}{7}\right)\right) \div \frac{1}{2}$

How do you know/What can you do to know that your answer is correct?

c. $5.2 + 0.3 - 3.4 \text{ x } 2$

How do you know/What can you do to know that your answer is correct?

2) Consider the statement "if $x < 2$, then $-x < -2$". Is it a true or a false statement? How do you know your answer is correct?

3) If you're working a job that pays you \$12 and hour, and your boss decides to give you a 14% raise, what is your new hourly wage? Show and justify all your work.

How do you know/What can you do to know that your answer is correct?

4) A cab driver charges a flat rate of \$4.00 plus an extra \$1.75 for every kilometer driven. If your cab ride cost you \$37.25, how many kilometers did you travel? Show and justify all your work.

How do you know/What can you do to know that your answer is correct?