# The search for the compactified Kerr solution

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# Abstract


## The search for the compactified Kerr solution

### Borislav Mavrin

Due to the complexity of the Einstein equations the general solution to these equations remains unknown. Currently there exist quite a few special solutions, which were obtained by assuming some symmetries of the solution, which allows one to reduce the complexity of these equations. That is one of the reasons why any exact solution is important. It may shed some light on the general problem.

There is also demand from string theories for a special type of solutions - compactified solutions. String theories use more than 4 dimensions and in order for these theories to make physical sense the extra dimensions must be compactified. Therefore the search for the compactified analogs of the known solutions became an important task.

The well known and widely used in physics non compactified solutions are the Schwarzschild [15] and Kerr [11] solutions, which are discussed in detail in Chapter 2 of this thesis. Chapter 2 also provides a description of the compactified analog of the Schwarzschild solution obtained independently by Korotkin and Nicolai [12] and by Myers [14]. However the compactified analog of the Kerr solution remains unknown.

In Chapter 3 the asymptotic behaviour of the compactified analog of the Kerr solution is investigated. Two possible ways of solving this problem are discussed.

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# Contents

# Chapter 1

# Introduction

Special Theory of Relativity was originated by Albert Einstein in [5] in 1905. Ten years later he generalized his result and established General Theory of Relativity in [6]. Einstein's equations are a system of coupled nonlinear equations, which in general cannot be solved explicitly. After one hundred years the search for physically meaningful solutions is still under way. Surprisingly, the Einstein's equations in vacuum have a very simple form

$$R_{\mu\nu} = 0 \tag{1.1}$$

which is to say that the Ricci tensor vanishes.

## 1.1 Problem statement and research objectives

Some solutions are more important from a mathematical point of view and some are more physically significant. Furthermore, some solutions (like the Gowdy model, discussed in Subsection 2.2.3) are important in verifying the methodology of numerical relativity.

Probably the most famous and physically relevant solutions are due to Schwarzschild and the Kerr. Compactified (in one or more dimensions) analogs of the classical solutions are interesting from the point of view of higher-dimensional theories, in particular string theory.

The periodic analog of the Schwarzschild solution was constructed in [14], [12]. The problem of constructing the periodic analog of Kerr solution was suggested in [12]. However the construction of the general solution to this problem is rather complicated due to nonlinearities of the corresponding equations as stated in [12]. As a first step toward understanding the periodic Kerr solution it is reasonable to study its asymptotic behaviour at infinity.

The Kerr metric is a stationary axisymmetric solution; the general form of a metric possessing this symmetry looks as follows (in the Weyl-Papapetrou coordinates form):

$$ds^2 = f^{-1}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt + Ad\phi)^2 \tag{1.2}$$

where $(x,\rho)$ are Weyl-Papapetrou canonical coordinates: $x$ measures the distance along the symmetry axis; $\rho$ is the distance from the symmetry axis; $t$ is the timelike variable as long as $f > 0$, and $\phi$ is the azimuthal angle.

The Einstein's equations can be restated in the following equivalent form using Ernst potential [7] $\mathscr{E}(x,\rho)$:

$$(\mathscr{E} + \bar{\mathscr{E}})(\mathscr{E}_{xx} + \frac{1}{\rho}\mathscr{E}_\rho + \mathscr{E}_{\rho\rho}) = 2(\mathscr{E}_x^2 + \mathscr{E}_\rho^2) \tag{1.3}$$

where

$$f = Re\mathscr{E} \qquad A_\xi = 2\rho\frac{(\mathscr{E} - \overline{\mathscr{E}})_\xi}{(\mathscr{E} + \overline{\mathscr{E}})^2} \qquad k_\xi = 2i\rho\frac{\mathscr{E}_\xi\overline{\mathscr{E}}_\xi}{(\mathscr{E} + \overline{\mathscr{E}})^2} \tag{1.4}$$

with $\xi = x + i\rho$.

Let us denote the $\{t,\phi\}$ block of the metric by $g$. Then from (1.2) it can be seen that

$$g = \begin{pmatrix} -f & -fA \\ -fA & -fA^2 + \rho^2/f \end{pmatrix} \tag{1.5}$$

The complete metric (1.2) can be constructed either from the Ernst potential $\mathscr{E}$ or from $g$ through (1.4). In term of the matrix $g$ the Einstein's equations reduce to the following matrix equation:

$$(\rho g_x g^{-1})_x + (\rho g_\rho g^{-1})_\rho = 0 \tag{1.6}$$

Therefore two equivalent formulations of the Einstein's equations are obtained, i.e. (1.3) and (1.6).

Asymptotically, as $\rho \to \infty$, the periodic Kerr solution should coincide with a solution which is translationally invariant in the $x$-direction.

Since there are two equivalent forms of the same equations: (1.3) and (1.6), the problem of finding such an $x$-independent solution can be approached in two different ways:

1. Assuming that the Ernst potential is independent of $x$ reduces (1.6) to:

$$(\mathscr{E}+\bar{\mathscr{E}})(\frac{1}{\rho}\mathscr{E}_\rho+\mathscr{E}_{\rho\rho})=2\mathscr{E}_\rho^2 \tag{1.7}$$

2. Assuming that the metric is independent of $x$ reduces (1.6) to:

$$(\rho g_\rho g^{-1})_\rho=0 \tag{1.8}$$

The objective of the current thesis was to analyze the possible asymptotic behaviour of the hypothetic periodic analog of the Kerr solution by solving (1.7) and (1.6). Since the asymptotic behaviour of the periodic analog of the Schwarzschild solution is described by the Kasner metric [12], it is natural to expect that asymptotically the periodic analog of the Kerr solution should be a rotating analog of the Kasner solution [10].

## 1.2 Organization of thesis

The thesis is structured as follows: in Section 2 the solutions of the Einstein's equations relevant to the present research are discussed; in Section 3 the solutions which can be regarded as possible asymptotic behaviour of the periodic analog of the Kerr solution are presented, and finally in Section 4 the results of the thesis are summarized and some related issues are discussed.

# Chapter 2

# Background

## 2.1 Classical solutions

Mathematically, the Einstein equations represent a system of coupled nonlinear partial differential equations. That makes the construction of exact solutions very hard. Several important classical solutions are discussed below.

### 2.1.1 Schwarzschild metric

The Schwarzschild metric [15] is a spherically symmetric static vacuum solution. It describes the gravitational field created by a spherically symmetric non-rotating black hole. In spherical coordinates the metric has the following form:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.1}$$

where $d\Omega^2$ is the metric on a unit two-dimensional sphere,

$$d\Omega^2 = d\theta^2 + \sin\theta^2 d\phi^2 \tag{2.2}$$

and the constant $M$ has a physical meaning of the mass of the black hole. [4]

A rather simple derivation of the Schwarzschild metric can be found in [4], which is outlined below.

The metric outside the black hole satisfies the Einstein's equations in vacuum:

$$R_{\mu\nu} = 0. \tag{2.3}$$

Let us assume the source of gravity to be static (i.e. non-rotating) and spherically symmetric. Static implies that: (i) the components of the metric are time independent (ii) all time-space cross terms are zero. Equivalently, the property of the metric being static can be restated as invariance of the metric under time reversal, $t \to -t$. Under time inversion the term $dt^2$ won't change, but time-space terms will, hence the corresponding metric components must vanish.

Let us first rewrite the ordinary Minkowski metric in spherical coordinates $(t, r, \theta, \phi)$ as follows:

$$ds_M^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{2.4}$$

with $d\Omega$ as in (2.2) .

The general form of the metric with spherical symmetry has the following form:

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2. \tag{2.5}$$

The exponential function is used here to keep the signature of the metric.

It is possible to find the unknown functions $\alpha(r)$ and $\beta(r)$ by using the Einstein's equations. First step is to compute the Christoffel symbols. The non-vanishing Christoffel symbols are:

$$\begin{aligned}
&\Gamma^t_{tr} = \partial_r \alpha, \quad &&\Gamma^r_{tt} = e^{2(\alpha-\beta)} \partial_r \alpha, \quad &&\Gamma^r_{rr} = \partial_r \beta, \\
&\Gamma^\theta_{r\theta} = \frac{1}{r}, &&\Gamma^r_{\theta\theta} = -re^{-2\beta}, &&\Gamma^\phi_{r\phi} = \frac{1}{r}, \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta} \sin^2\theta, &&\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, &&\Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{2.6}$$

Hence the non vanishing components of the Riemann tensor are:

$$
\begin{aligned}
R^t_{rtr} &= \partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 \\
R^t_{\theta t\theta} &= -r^{-2\beta}\partial_r\alpha \\
R^t_{\phi t\phi} &= -r^{-2\beta}\sin^2\theta\partial_r\alpha \\
R^r_{\theta r\theta} &= re^{-2\beta}\partial_r\beta \\
R_{\phi r\phi} &= re^{-2\beta}\sin^2\theta\partial_r\beta \\
R^\theta_{\phi\theta\phi} &= (1-e^{-2\beta})\sin^2\theta
\end{aligned}
\tag{2.7}
$$

The Ricci tensor is obtained by contracting the Riemann tensor:

$$
\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta \\
R_{\theta\theta} &= e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \\
R_{\phi\phi} &= \sin^2(\theta)R_{\theta\theta}
\end{aligned}
\tag{2.8}
$$

and the curvature scalar is

$$
R = -2e^{-2\beta}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}(\partial_r\alpha - \partial_r\beta) + \frac{1}{r^2}(1-e^{2\beta})\right] \tag{2.9}
$$

By the Einstein's equations the Ricci tensor must vanish. Therefore:

$$
0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}(\partial_r\alpha + \partial_r\beta) \tag{2.10}
$$

Hence $\alpha + \beta = c$, where $c$ is a constant of integration. By rescaling the time coordinate by transformation $t \to e^{-c}t$ we can assume $c = 0$, i.e.

$$
\alpha = -\beta \tag{2.11}
$$

The condition $R_{\theta\theta} = 0$ now implies

$$
e^{2\alpha}(2r\partial_r\alpha + 1) = 1, \tag{2.12}
$$

or, equivalently,

$$\partial_r(re^{2\alpha}) = 1 \tag{2.13}$$

Integration of (2.13) yields:

$$e^{2\alpha} = 1 - \frac{R_S}{r} \tag{2.14}$$

where $R_S$ is a constant of integration.

As a result of this derivation we arrive at the following metric:

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.15}$$

describing the gravitational field outside a spherically symmetric black hole.

The only two remaining Einstein's equations are $R_{tt} = 0$ and $R_{rr} = 0$; they are automatically satisfied by the metric (2.15) for any value of $R_S$.

To understand the physical meaning of the constant $R_S$ let us look at the Newtonian limit, which is based on three assumptions: the particles' speed is small compared to the speed of light; the gravitational field is weak; the field is static. In that case the $g_{tt}$ component of the metric around the point mass particle is:

$$g_{tt} = -\left(1 - \frac{2GM}{r}\right). \tag{2.16}$$

As $r \to \infty$ the spherically symmetric black hole can be assumed to behave as a point mass particle and the gravitational tug of the black hole becomes small. Therefore in order to make physical sense the metric (2.15) must reduce to the Newtonian limit and the $g_{tt}$ component of the metric (2.15) should coincide with (2.16). Hence we can make the identification $R_S = 2GM$, where $M$ is the mass of the black hole.

It is easily seen that as $M \to 0$ the metric reduces to the Minkowski flat space metric [13] as expected.

According to Birkhoff's theorem the Schwarzschild metric is the unique spherically symmetric asymptotically flat static solution of the Einstein's equations in vacuum [4].

The components of the metric (2.1) become infinite at two points: $r = 0$ and $r = R_S = 2GM$. To find out whether these are intrinsic singularities of the manifold consider the Kretschmann scalar invariant:

$$W = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \tag{2.17}$$

If W diverges at a point, this point is an intrinsic singularity of a manifold (the converse is not true in general since one can construct several invariants of the Riemann tensor).

For the Schwarzschild metric

$$W = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}; \tag{2.18}$$

thus $r = 0$ is indeed an intrinsic singularity. As for the surface $r = 2GM$, the scalar $W$ is well behaved there. Thus in order to show that $r = 2GM$ is not a singularity it is necessary to find some coordinate system where this singularity disappears.

Such a coordinate system was found by Kruskal and Szekeres; it consists of coordinates $\{T, R, \theta, \phi\}$ defined as follows:

$$T = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \tag{2.19}$$

$$R = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \tag{2.20}$$

for $r > 2GM$, and:

$$T = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \tag{2.21}$$

$$R = -\left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \tag{2.22}$$

for $0 < r < 2GM$. In these coordinates the metric (2.1) looks as follows:

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dT^2 + dR^2) + r^2 d\Omega^2 \tag{2.23}$$

where $r$ is defined implicitly as

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} \tag{2.24}$$

In $\{T, R, \theta, \phi\}$ coordinates point $r = 2GM$ corresponds to $T = 0$, $R = 0$, which is a regular point. Hence $r = 2GM$ is just a coordinate singularity. However the value $r = 2GM$ is special in some other way: it corresponds to the so-called event horizon.

The (compact) event horizon is a null hypersurface, which separates the points of the Pseudo-

Riemannian manifold that are connected to spatial infinity by a timelike path from those that are not [4]. Since it is a global characteristic of the pseudo manifold, it can be a rather complicated task to derive the event horizon from a given metric. But in the case of stationary asymptotically flat metrics which contain event horizons with spherical topology (i.e. Schwarzschild and Kerr solutions), the situation is much simpler. It can be shown (see [4]) that in this case there exists a coordinate system with the property that the event horizon is described by the following condition:

$$g^{rr}(r_H) = 0 \tag{2.25}$$

where $r$ is the radial spherical coordinate and $r_H$ is a certain value. In this thesis these specially chosen coordinates for the Schwarzschild and Kerr solutions are used. Thus in the case of the Schwarzschild metric the event horizon is described by $r = 2GM$. The value $r_H$ corresponding to the Kerr metric is discussed in the next subsection.

### 2.1.2 Kerr metric

The Kerr metric [11] is an axially symmetric static vacuum solution which describes the gravitational field created by an axially symmetric rotating black hole.

In Boyer-Lindquist $(t, r, \theta, \phi)$ coordinates the metric has the following form:

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 - \frac{2GMar\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \\ &\frac{\sin^2\theta}{\rho^2}[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]d\phi^2 \end{aligned} \tag{2.26}$$

where

$$\Delta = r^2 - 2GMr + a^2 \tag{2.27}$$

and

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2.28}$$

Here $M$ and $a$ are constants. Physically $M$ is the mass of the rotating black hole and $a = J/M$ is the angular momentum per unit mass.

The Boyer-Lindquist spatial coordinates $\{r, \theta, \phi\}$ (ellipsoidal coordinates) are related to Cartesian coordinates $\{x, y, z\}$ by:

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2+a^2}\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2.29}$$

Checking that the Kerr metric satisfies the Einstein equations is much more complicated than in the case of the Schwarzschild solution; the complexity stems from the off-diagonal terms of the metric.

If the black hole is not rotating, i.e. $a \to 0$ the Kerr metric reduces to the Schwarzschild metric. On the other hand if $a$ is kept constant and $M \to 0$ the result is:

$$\begin{aligned} ds^2 = &-dt^2 + \frac{(r^2+a^2\cos^2\theta)}{r^2+a^2}dr^2 \\ &(r^2+a^2\cos^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \end{aligned} \tag{2.30}$$

In order for the metric to make physical sense this limit must represent flat spacetime. Indeed, using (2.29) the metric (2.30) represents Minkowski flat spacetime in Cartesian coordinates $\{t, x, y, z\}$.

If a metric is independent of some coordinate $x$, then the vector $\partial_x$ is a Killing vector. Since the metric components in (2.26) are independent of $t$ and $\phi$ the vectors

$$K = \partial_t \qquad R = \partial_\phi \tag{2.31}$$

are Killing vectors. In other words those two Killing vectors tell us that the metric is axially symmetric ($R$) and stationary ($K$). It is worth noting that the metric is stationary since the black hole is spinning at a constant rate.

As discussed in Subsection 2.1.1 the coordinates in (2.26) are chosen so that the event horizon is described by the equation $g^{rr} = 0$. Therefore, the condition for the event horizon is

$$\frac{\Delta}{\rho^2} = 0 \qquad \Delta(r) = r^2 - 2GM + a^2 \tag{2.32}$$

One can consider the following three cases: $GM > a$, $GM = a$ and $GM < a$. The only case which is physically important is the case $GM > a$, which corresponds to two solutions of the equation $g^{rr} = 0$: [4]. Hence there are two values for $r$:

$$r_{\pm} = GM \pm \sqrt{G^2M^2 - a^2} \tag{2.33}$$

at which $\Delta = 0$. These two values correspond to two event horizons; and the singularities at these values of $r$ are coordinate singularities which can be removed by a suitable change of coordinates. Currently there exist several different coordinate systems used to study various properties of the Kerr solution. [18].

Another related notion is the Killing horizon. The Killing horizon is the null hypersurface along which the Killing vector field is null. In the case of the Schwarzschild solution the event horizon coincides with the Killing horizon, since it is stationary asymptotically flat spacetime. But for the Kerr metric the Killing horizon for $K = \partial_t$ does not coincide with the event horizon, since the norm of $K^\mu$ is

$$K^\mu K_\mu = -\frac{1}{\rho^2}(\Delta - a^2 \sin^2\theta) \tag{2.34}$$

which is non-zero at $\Delta = 0$.

At $r = r_+$

$$K^\mu K_\mu = \frac{a^2}{\rho^2}\sin^2\theta \tag{2.35}$$

which is to say that the Killing vector $K$ is spacelike except at the points $\theta = 0, \pi$.

The Killing horizon is determined by the condition $K^\mu K_\mu = 0$, which is described by the following stationary limit surface (on which $g_{tt} = 0$):

$$(r - GM)^2 = G^2M^2 - a^2\cos^2\theta \tag{2.36}$$

While the outer event horizon $(r = r_+)$ is the surface

$$(r_+ - GM)^2 = G^2M^2 - a^2. \tag{2.37}$$

The region between those two surfaces is called the ergosphere; this is the region not present in Schwarzschild spacetime. The characteristic feature of this region is the so-called inertial frame dragging effect: the particle in this region must move in the same direction as the rotation of the black hole.

As a simple example let us consider a photon. The photon moves along light-like trajectories, i.e. along its trajectory the line element has zero length: $ds^2 = 0$. Let us assume that the photon starts its trajectory in the plane $\theta = \frac{\pi}{2}$ with initial $dr = 0$ and $d\theta = 0$. There the equation describing the trajectory is

$$ds^2 = 0 = g_{tt}dt^2 + g_{t\phi}(dtd\phi + d\phi dt) + g_{\phi\phi}d\phi^2 \tag{2.38}$$

Solving for the angular velocity yields:

$$\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \tag{2.39}$$

On the stationary limit surface ($g_{tt} = 0$) the equation takes one of the following two forms, depending on the choice of the sign:

$$\frac{d\phi}{dt} = 0 \quad or \quad \frac{d\phi}{dt} = \frac{a}{2G^2M^2 + a^2} \tag{2.40}$$

Therefore either the photon is not moving in the radial direction at all or spins about the axis of symmetry in the same direction as the black hole does (the velocity has the same sign as $a$).

As in the case of the Schwarzschild solution, locating curvature singularities is non trivial. Namely, consider the following scalar [18]:

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48M^2(r^2 - a^2\cos^2\theta)[(r^2 + a^2\cos^2\theta)^2 - 16r^2a^2\cos^2\theta]}{(r^2 + a^2\cos^2\theta)} \tag{2.41}$$

Since both terms in the denominator are non negative the curvature singularity occurs when

$$r = 0, \qquad \theta = \pi/2 \tag{2.42}$$

which represents a ring. Hence the singularity in the case of the Kerr solution has the ring-like geometry.

It is interesting to describe what happens if one analytically continues the metric inside of the ring. Such an analytic continuation of the metric [4] shows that the spacetime inside the ring is also asymptotically flat and is described by the Kerr metric with $r < 0$.

There is another counterintuitive result associated with the Kerr metric, that is the existence of closed timelike curves. Assume for simplicity that the trajectory starts in the spatial plane with fixed $r$, $\theta$, and $t$. Then for small negative values of $r$ (inside the singularity ring)

$$ds^2 \approx a^2\left(1+\frac{2GM}{r}\right)d\phi^2 \tag{2.43}$$

the proper time is negative. By assumption this path is closed. Hence theoretically the particle travels to the past.

### 2.1.3 Kasner solution

The 4-dimensional Kasner metric has the following form [10]:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{2.44}$$

with two conditions on constants $p_i$:

$$p_1 + p_2 + p_3 = 1 \qquad (p_1)^2 + (p_2)^2 + (p_3)^2 = 1 \tag{2.45}$$

A few physical observations can be drawn by directly studying the metric (2.44). Each time slice $(t = const)$ is flat 3-dimensional space. Another important conclusion is that the volume element is time dependent:

$$\sqrt{-g} = t \tag{2.46}$$

Therefore the universe described by (2.44) is expanding. And the rate of expansion is different in different direction, i.e it is anisotropically expanding universe. Since there are two conditions, only one of the $p_i$ is free. The case $p_1 = p_2 = p_3$ is impossible. And the equality of the two $p_i$ is possible only in the triples: $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and $(0,0,1)$. In the latter case (2.44) is reduced to Minkowski flat-space by the coordinate transformation [2]:

$$t\sinh z = \xi \qquad t\cosh z = \tau \tag{2.47}$$

In [2] the authors also show that $p_i$ can be parametrized in the following way:

$$p_1 = \frac{-u}{1+u+u^2} \qquad p_2 = \frac{1+u}{1+u+u^2} \qquad p_3 = \frac{u(1+u)}{1+u+u^2} \tag{2.48}$$

where $u \geq 1$. Hence

$$-\frac{1}{3} \leq p_1 \leq 0 \qquad 0 \leq p_2 \leq \frac{2}{3} \qquad \frac{2}{3} \leq p_3 \leq 1 \tag{2.49}$$

Physically it means that if the universe is expanding along two dimensions, then it must contract in the remaining dimension.

The metric (2.44) obtained by Kasner in 1921 [10] was important as an exact solution of Einstein's equations. But its physical significance was not apparent at that time. Later, as for example in [2] and [3], it was rediscovered as an asymptotic behaviour of a homogenous spacetime metric as $t \to \infty$.

In [2] the authors show that the asymptotic behaviour of (2.44) near the time singularity, i.e. at $t \to \infty$ is:

$$ds^2 = -dt^2 + t^{2p'_1}dx^2 + t^{2p'_2}dy^2 + t^{2p'_3}dz^2 \tag{2.50}$$

where

$$p'_1 = \frac{|p_1|}{1-2|p_1|} \qquad p'_2 = -\frac{2|p_1| - p_2}{1-2|p_1|} \qquad p'_3 = \frac{p_3 - 2|p_1|}{1-2|p_1|} \tag{2.51}$$

That is if $p_1 < 0$, then $p'_2 < 0$. Physically this implies that the direction in which the universe is contracting changes its direction. The authors call this process *Kasner epoch switching*. Parametrically this result can be represented as

$$u' = u - 1 \qquad p'_1 = p_2(u') \qquad p'_2 = p_1(u') \qquad p'_3 = p_3(u') \tag{2.52}$$

Consider (2.44) in a more general form with a slightly different notation more suitable for further discussion:

$$ds^2 = -t^{2p_0}dt^2 + t^{2p_1}d\rho^2 + t^{2p_2}dx^2 + t^{2p_3}d\phi^2 \tag{2.53}$$

where

$$p_0 + 1 = p_1 + p_2 + p_3 \qquad (p_0+1)^2 = (p_1)^2 + (p_2)^2 + (p_3)^2 \tag{2.54}$$

Consider the following coordinate transformation called Wick rotation:

$$\rho \to -it \tag{2.55}$$

Under the Wick rotation the metric (2.54) becomes (with the adjusted signature):

$$ds^2 = -\rho^{2p_1}dt^2 + \rho^{2p_0}d\rho^2 + \rho^{2p_2}dx^2 + \rho^{2p_3}d\phi^2 \tag{2.56}$$

In this form the Kasner metric plays an important role in describing the asymptotic behaviour of the periodic analog of the Schwarzschild solution as will be shown in 2.2.1.

### 2.1.4 van Stockum solution

Another axisymmetric stationary exact solution of Einstein's equations is the van Stockum solution [17]. The general metric is given by [16]:

$$ds^2 = \rho^{-1/2}(d\rho^2 + dx^2) - 2\rho d\phi dt + \rho\Omega dt^2 \tag{2.57}$$

where $\Omega$ is an arbitrary solution of the Euler-Darboux equation

$$\Omega_{xx} + \frac{1}{\rho}\Omega_\rho + \Omega_{\rho\rho} = 0 \tag{2.58}$$

The source of gravity in this model is the infinite cylinder of pressureless fluid (*dust*) rotating about an axis of symmetry. As it can be observed the metric has an off diagonal time-space cross term which implies the presence of rotation.

The original van Stockum metric in [17] admits closed timelike curves, which limits its applicability as a physically realistic model.

In this work we rediscover a special case of the van Stockum metric during the investigation of the asymptotic behaviour of the periodic analog of the Kerr solution.

## 2.2 Periodic analogs of classical solutions

The periodic solutions of Einstein's equations attracted attention due to the following reasons:

1. Any exact solution of the Einstein equations is important in its own right.

2. String theories need more than 4 dimensions. In order for those theories to make physical sense the number of dimensions must be reduced by compactifying some of them.

3. In numerical relativity one of the active fields is black hole binaries. The nonlinearity of Einsteins' equations make it hard to design stable numerical algorithms for the approximation of its solutions. One of the popular test bed models to test the algorithms is the Gowdy [9] model, whose space slices are three-dimensional tori. [1].

### 2.2.1 Periodic analog of Schwarzschild solution

The periodic analog of the Schwarzschild solution was found in works of R. Myers [14], and D. Korotkin and H. Nicolai [12].

The present subsection is based on [12].

The authors start with a general form of a stationary axisymmetric spacetime with the following metric:

$$ds^2 = f^{-1}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt + Ad\phi)^2 \tag{2.59}$$

where $(x, \rho)$ are Weyl canonical coordinates (cylindrical coordinates): $x$ is measured along the axis of symmetry and $\rho$ is the distance from the axis of symmetry. The Einstein's equations can be restated in the following equivalent form using Ernst potential [7] $\mathscr{E}(x, \rho)$:

$$(\mathscr{E} + \bar{\mathscr{E}})(\mathscr{E}_{xx} + \frac{1}{\rho}\mathscr{E}_\rho + \mathscr{E}_{\rho\rho}) = 2(\mathscr{E}_x^2 + \mathscr{E}_\rho^2) \tag{2.60}$$

such that the metric coefficients can be reproduced from the following equations:

$$f = Re\mathscr{E} \qquad A_\xi = 2\rho\frac{(\mathscr{E} - \overline{\mathscr{E}})_\xi}{(\mathscr{E} + \overline{\mathscr{E}})^2} \qquad k_\xi = 2i\rho\frac{\mathscr{E}_\xi\overline{\mathscr{E}}_\xi}{(\mathscr{E} + \overline{\mathscr{E}})^2} \tag{2.61}$$

with $\xi = x + i\rho$.

If $\mathscr{E}$ is real-valued then coefficient $A$ in(2.61) is zero and the metric (2.59) is static, that is without rotation. By the change of variable:

$$w = \ln \mathscr{E} \tag{2.62}$$

the equation (2.60) reduces to the Euler-Darboux equation:

$$w_{xx} + \frac{1}{\rho} w_\rho + w_{\rho\rho} = 0, \tag{2.63}$$

and the metric (2.59) reduces to

$$ds^2 = e^{-w}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] + e^w dt^2 \tag{2.64}$$

where

$$k_\xi = \frac{i\rho}{2}(w_\xi)^2 \tag{2.65}$$

or in real coordinates $(x, \rho)$:

$$k_\rho = \frac{\rho}{4}(w_\rho^2 - w_x^2) \qquad k_x = \frac{\rho}{2} w_x w_\rho \tag{2.66}$$

In order to construct the periodic analogue of the Schwarzschild metric, the authors made use of the standard averaging procedure used in particular in construction of meromorphic functions on the torus $\mathbb{T} = \mathbb{C}/\{L_1, L_2\}$, where $L_1$ and $L_2$ are the periods. Namely, consider a meromorphic function $f_0(\xi)$, where $\xi \in \mathbb{C}$. If there exist constants $\{a_{mn}\}_{m,n=-\infty}^{\infty}$, such that the series:

$$f(\xi) = \sum_{m,n=-\infty}^{\infty} \{f_0(\xi + mL_1 + nL_2) + a_{mn}\} \tag{2.67}$$

converges, then $f(\xi)$ is a meromorphic function on the torus $\mathbb{T}$. Equivalently $f(\xi)$ is a doubly periodic analog of $f_0(\xi)$.

In a similar way it is possible to construct a doubly periodic analog of a real-valued harmonic function $w(\xi, \overline{\xi}) = Ref(\xi)$ for some locally holomorphic function $f(\xi)$. Therefore $w(\xi, \overline{\xi})$ satisfies the Laplace equation:

$$w_{\xi\overline{\xi}} = 0 \tag{2.68}$$

Note that if there exists some solution $w_0(\xi,\overline{\xi})$ of (2.68), then (i) for any $L \in \mathbb{C}$, $w_0(\xi+L)$ is also a solution and (ii) any linear combination of solutions of (2.68) is also a solution. Therefore if the following series:

$$\sum_{m,n=-\infty}^{\infty} \{w_0(\xi + mL_1 + nL_2) + b_{mn}\} \tag{2.69}$$

converges for some $\{b_{mn}\} \in \mathbb{R}$, then it is also a solution of (2.68).

If we apply this idea to the Euler-Darboux equation (2.63) in complex coordinates $(\xi,\overline{\xi})$ with $\xi = x + i\rho$, we have

$$w_{\xi\overline{\xi}} - \frac{w_\xi - w_{\overline{\xi}}}{2(\xi-\overline{\xi})} = 0 \qquad w(\xi,\overline{\xi}) \in \mathbb{R} \tag{2.70}$$

Even though the Euler-Darboux looks more complicated it still satisfies the same two properties as (2.68) does with one particularity: the invariance holds only under real translations, that is $\xi \to \xi + L$, for any $L \in \mathbb{R}$.

This allows to construct a solution periodic along the $x$-direction. The authors prove the following result:

**Theorem 1.** *Let $w_0(x,\rho)$ be any solution of the Euler-Darboux equation corresponding to an asymptotically flat metric* (2.64)*, i.e.*

$$w_0(x,\rho) = \frac{\beta}{r} + O(r^{-2}) \quad as \quad r \to \infty \tag{2.71}$$

*where $r = \sqrt{x^2+\rho^2}$; $M = -\frac{1}{2}\beta$ is the mass. Let*

$$a_n = -\frac{\beta}{L|n|}, \quad n \neq 0, \quad a_0 = 0 \quad L \in \mathbb{R} \tag{2.72}$$

*Then series*

$$w(x,\rho) = \sum_{n=-\infty}^{\infty} \{w_0(x+nL,\rho) + a_n\} \tag{2.73}$$

*converges for all $(x,\rho)$ except the points $(x_0+nL,\rho_0)$, where the function $w_0(x,\rho)$ is singular $(n \in \mathbb{Z})$, and defines a periodic function with period $L$.*

It is worth noting that this central result of the paper [12] is quite general. Given an arbitrary static asymptotically flat metric it is possible to construct its $x$-periodic analog.

Since the Schwarzschild solution (2.1) satisfies the hypothesis of Theorem 1 the authors immediately obtain its $x$-periodic analog.

The Ernst potential which corresponds to the Schwarzschild solution (2.1) has the following form:

$$w_0 = \ln \mathscr{E}_0 \qquad \mathscr{E}_0(x,\rho) = \frac{\sqrt{(x-M)^2+\rho^2}+\sqrt{(x+M)^2+\rho^2}-2M}{\sqrt{(x-M)^2+\rho^2}+\sqrt{(x+M)^2+\rho^2}+2M} \tag{2.74}$$

where $M \in \mathbb{R}$ is a positive constant, the mass of the black hole.

Therefore by Theorem 1 the periodic analog of the Schwarzschild solution can be represented in the form of the following infinite product:

$$\mathscr{E}(x,\rho) = \mathscr{E}_0(x,\rho)\prod_{n=1}^{\infty}\mathscr{E}_0(x+nL,\rho)\mathscr{E}_0(x-nL,\rho)\exp\left(\frac{4M}{nL}\right) \tag{2.75}$$

Here $\mathscr{E}_0$ is the Ernst potential corresponding to the Schwarzschild metric. And therefore $M$ is the mass or the spherically symmetric non rotating black hole, where $L$ is the period. The infinite product (2.75) converges since the corresponding series (2.73) converges.

The $x$-periodicity of the Ernst potential is not sufficient for the metric to be periodic in the $x$-direction. The periodicity of the metric coefficient is guaranteed by the following result [12]:

**Theorem 2.** *Let $L > 2M$. Then the function $k(x,\rho)$ corresponding to the Ernst potential* (2.75) *is periodic in $x$ with period $L$, i.e.*

$$k(x+L,\rho) = k(x,\rho) \tag{2.76}$$

Hence the metric corresponding to the Ernst potential is indeed periodic in $x$. Since the Ernst potential (2.75) is given by an infinite product obtaining the explicit form of the function $k(x,\rho)$ seems to be a rather complex task. To summarize, the periodic analog of the Schwarzschild solution is given by (2.64), (2.66), (2.62), and (2.75).

Another important result of the paper [12] is the first-order asymptotic behaviour of the periodic analog of the Schwarzschild metric.

**Theorem 3.** *The asymptotic behaviour of the Ernst potential* (2.75) *is given by*

$$\mathscr{E} = C\rho^{4M/L}\big(1+o(1)\big) \quad as \quad \rho \to \infty \tag{2.77}$$

*where $C$ is some constant.*

Hence the metric (2.64) tends to

$$ds^2 = \tilde{C}\rho^{\frac{\alpha^2}{2}-\alpha}(dx^2+d\rho^2)+C^{-1}\rho^{2-\alpha}d\phi^2 - C\rho^{\alpha}dt^2 \quad as \quad \rho \to \infty \tag{2.78}$$

where $\tilde{C}$ and $C$ are constants of integration, and $\alpha = 4ML^{-1}$.

The metric (2.78) is the Kasner metric [10], discussed in more detail in 2.1.3. It is important to note that the periodic analog of the Schwarzschild metric asymptotically, as $\rho \to \infty$, is non flat, which is expected, since the solution is compactified along one of the spatial dimensions.

On the part $\rho = 0$, $M \leq |x| \leq \frac{1}{2}L$ of the symmetry axis, the solution can be represented in terms of the $\Gamma$-function:

$$\mathscr{E}(x,\rho=0) = exp\left(\frac{4\gamma M}{L}\right)\frac{\Gamma\left(\frac{|x|+M}{L}\right)\Gamma\left(1-\frac{|x|-M}{L}\right)}{\Gamma\left(\frac{|x|-M}{L}\right)\Gamma\left(1-\frac{|x|+M}{L}\right)} \tag{2.79}$$

where $\gamma$ is Euler-Mascheroni constant. In the region $\rho = 0$, $|x| \leq M$ the Ernst potential vanishes, $\mathscr{E} \equiv 0$.

As expected, in the limit $L \to \infty$ the metric (2.64) tends to the Schwarzschild metric.

The solution technique described above is applicable in the case of stationary axisymmetric spacetime without rotation, since the Einstein equations in terms of Ernst potential can be linearized. In the case of the metrics with rotation this procedure does not work anymore since in that case the the Einstein equations are non linearizable anymore.

### 2.2.2 Some physical properties of the periodic analog of the Schwarzschild solution

The authors of [12] suggest studying the solution (2.64) on and inside the event horizon. This idea is developed in detail in [8], where the authors study the properties of the deformed (or distorted) event horizon due to mass-period ratio. The content of this section is based on [8] unless stated otherwise.

In [8] the metric for the periodic analog of the Schwarzschild solution from [12] is given in two different forms: integral and series representations.

As well as in the paper [12], the metric in [8] is not given explicitly, but consists of the following three components:

1. function $w(x,\rho)$
2. function $k(x,\rho)$:

$$k_\rho = \frac{\rho}{4}(w_\rho^2 - w_x^2) \qquad k_x = \frac{\rho}{2} w_x w_\rho \tag{2.80}$$

3. the general form of the static stationary axisymmetric line element:

$$ds^2 = e^{-w}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] + e^w dt^2 \tag{2.81}$$

Therefore the function $w(x,\rho)$ completely describes the metric.

In order to obtain the integral representation the authors use the three-dimensional Green's function. The result they obtain is:

$$w(x,\rho) = -\frac{2}{\pi}\int_0^\infty dy\left(\frac{U(\beta,\tilde{x})}{\beta} - \frac{\mu}{\sqrt{\beta^2+b^2}}\right) + \frac{\mu}{\pi}\ln\tilde{\rho}^2 + b^2 \tag{2.82}$$

where

$$\tilde{\rho} = \rho/L \qquad \tilde{x} = x/L \qquad \beta = \sqrt{y^2+\tilde{\rho}^2} \qquad \mu = M/L \tag{2.83}$$

$$U(\beta,\tilde{x}) = V(\beta,\tilde{x}) + V(\beta,-\tilde{x}) \tag{2.84}$$

$$V(\beta,\tilde{x}) = \arctan\left[\frac{\cosh\beta+1}{\sinh\beta}\tan\left(\frac{\mu+\tilde{x}}{2}\right)\right] + \pi\theta(\mu+\tilde{x}-\pi), \tag{2.85}$$

$M$ is the mass of the black hole and $L$ is the period of the metric in the $x$-direction. $\theta(.)$ is the $\theta$-function defined as

$$\theta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2\tau + 2\pi i n z). \tag{2.86}$$

The authors choose $L = 2\pi$.

Similar computations lead to the following integral representation of the $w_S(x,\rho)$ (subscript $S$ here stands for Schwarzschild) function in the original Schwarzschild solution:

$$w_S(x,\rho) = -\frac{2}{\pi}\int_0^\infty dy \frac{U_S(\beta,\tilde{x})}{\beta} \tag{2.87}$$

where

$$U_S(\beta,\tilde{x}) = V_S(\beta,\tilde{x}) + V_S(\beta,-\tilde{x}) \tag{2.88}$$

$$V_S(\beta,\tilde{x}) = \arctan\left(\frac{\mu+\tilde{x}}{\beta}\right). \tag{2.89}$$

Hence in order to study the properties of the event horizon it is convenient to define the following difference (the convenience of this difference will be explained later):

$$\hat{w}(x) = w(x,\rho=0) - w_S(x,\rho=0) = -\frac{2}{\pi}\int_0^\infty dy\Big(\frac{U(y,\tilde{x}) - U_S(y,\tilde{x})}{y} - \frac{\mu}{\sqrt{y^2+1}}\Big). \qquad (2.90)$$

The Fourier series for function $w(x,\rho)$ looks as follows:

$$w(x,\rho) = \frac{2\mu}{\pi}\ln\tilde{\rho} - 4\sum_{k=1}^{\infty}\frac{\sin(k\mu)}{\pi k}\cos(k\tilde{x})K_0(k\tilde{\rho}) \qquad (2.91)$$

where $K_0(.)$ is the MacDonald function (modified Bessel functions of the third kind). For small values of $\tilde{\rho}$:

$$-K_0(k\tilde{\rho}) \sim \ln\Big(\frac{k\tilde{\rho}}{2}\Big) + \gamma_E \qquad (2.92)$$

where $\gamma_E$ is the Euler's constant.

Then as $\rho \to \infty$ in the region $|\tilde{x}| \leq \mu$

$$\begin{aligned} w(x,\rho) &\sim 2\ln\frac{\tilde{\rho}}{\pi} + \frac{2\mu}{\pi}\ln 4\pi \\ &+ \ln\Big|\frac{1}{\pi}\sin\Big(\frac{\mu+\tilde{x}}{2}\Big)\Gamma^2\Big(\frac{\mu+\tilde{x}}{2\pi}\Big)\Big| \\ &+ \ln\Big|\frac{1}{\pi}\sin\Big(\frac{\mu-\tilde{x}}{2}\Big)\Gamma^2\Big(\frac{\mu-\tilde{x}}{2\pi}\Big)\Big| \end{aligned} \qquad (2.93)$$

On the other hand, the asymptotic behaviour of $w_s(x,\rho)$ can be obtained from the representation:

$$w_S(x,\rho) = -\ln\Big[\frac{\sqrt{(\mu-\tilde{x})^2+\tilde{\rho}^2} - \tilde{x} + \mu}{\sqrt{(\mu+\tilde{x})^2+\tilde{\rho}^2} - \tilde{x} - \mu}\Big]. \qquad (2.94)$$

Near the axis of symmetry in the region $|x| \leq \mu$ the first term of the asymptotic expansion is

$$w_S(x,\rho) \sim \ln\frac{\tilde{\rho}^2}{4(\mu^2-\tilde{x}^2)} \qquad |\tilde{x}| \leq \mu. \qquad (2.95)$$

Denoting $\hat{w}(x) = \lim_{\rho\to 0}[w(\rho,x) - w_S(\rho,x)]$ one gets the following asymptotics:

$$\hat{w}(x) \sim \frac{2\mu}{\pi}\ln(4\pi) + \ln\Big[f\Big(\frac{\mu+\tilde{x}}{2}\Big)f\Big(\frac{\mu-\tilde{x}}{2}\Big)\Big] \qquad (2.96)$$

where

$$f(y) = \frac{1}{\pi^2} y \sin(y) \Gamma^2\left(\frac{y}{\pi}\right). \tag{2.97}$$

For $0 \leq y \leq \pi$ it can be approximated by a linear function

$$f(y) \sim 1 - \frac{y}{\pi}. \tag{2.98}$$

In order to obtain the large distance asymptotics ($\rho \to \infty$) the integral representation (2.82) can be used. The first term in this expansion is

$$w(x,\rho) \sim \frac{2\mu}{\pi} \ln \tilde{\rho}. \tag{2.99}$$

In the limit $\rho \to \infty$ the metric has the following asymptotics

$$ds^2 = -\rho^{2\frac{\mu}{\pi}} dt^2 + \rho^{-2\frac{\mu}{\pi}(1-\frac{\mu}{\pi})}(d\rho^2 + dx^2) + \rho^{-2\frac{\mu}{\pi}} \rho^2 d\phi^2 \tag{2.100}$$

which is exactly the Kasner metric (2.78) with $\alpha = 2\frac{\mu}{\pi}$ and $L = 2\pi$.

To see how the properties of the event horizon of (2.64) differ from those of the Schwarzschild solution it is necessary to apply a suitable coordinate transformation.

It can be shown that

$$\hat{k}(x,\rho=0) = 4[\frac{1}{2}\hat{w}(x,\rho=0) - u], \qquad -M \leq x \leq M \tag{2.101}$$

for some constant $u$.

Consider the following coordinate transformation:

$$\begin{aligned} \rho &= e^u \sqrt{r(r-2M_0)} \sin\theta, \\ x &= e^u (r - M_0) \cos\theta, \end{aligned} \tag{2.102}$$

where

$$M_0 = Me^{-u}. \tag{2.103}$$

Applying this transformation to (2.64) one obtains:

$$
\begin{aligned}
ds^2 = & -e^{-\hat{w}}\left(1-\frac{2M_0}{r}\right)dt^2 + e^{\hat{k}-\hat{w}+2u}\left(1-\frac{2M_0}{r}\right)^{-1}dr^2 \\
& + e^{\hat{k}-\hat{w}+2u}r^2(d\theta^2 + e^{-\hat{k}}\sin^2\theta d\phi^2)
\end{aligned}
\tag{2.104}
$$

Hence the event horizon is described by the coordinate singularity $r = 2M_0$ and the 2-dimensional metric on the surface of the event horizon is

$$
d\gamma^2 = 4M_0^2\left[e^{\hat{w}-2u}d\theta^2 + e^{-\hat{w}+2u}\sin^2\theta d\phi^2\right] \tag{2.105}
$$

The area of the event horizon can be computed to give

$$
A = 16\pi M_0^2 \tag{2.106}
$$

It is possible to use the approximations of $\hat{w}$ from (2.96) to compute the constant $u$ from (2.101):

$$
u \sim \frac{\mu}{\pi}\ln 4\pi + \frac{1}{2}\ln f(\mu) \tag{2.107}
$$

The shape function (used to compute the Gaussian curvature) is defined by

$$
\mathscr{F}(x) = \frac{1}{2}\hat{w}(x) - u \tag{2.108}
$$

Multiplication of (2.105) by $(2\mu_0)^{-2}$, where $\mu_0 = \mu e^{-u}$, yields:

$$
d\sigma^2 = e^{2\mathscr{F}}\frac{dx^2}{\mu^2 - x^2} + e^{-\mathscr{F}}(\mu^2 - x^2)\frac{d\phi^2}{\mu^2}. \tag{2.109}
$$

The Gaussian curvature of the event horizon $K = \frac{1}{2}R$, where $R$ is the Ricci scalar curvature is:

$$
K = e^{-2\mathscr{F}(x)}\left[1 + (\mu^2 - x^2)[\mathscr{F}'' - 2(\mathscr{F}')^2] - 4x\mathscr{F}'\right]. \tag{2.110}
$$

The Gauss-Bonnet formula implies:

$$
\int d^2x\sqrt{\sigma}K = 4\pi \tag{2.111}
$$

In the case of the original Schwarzschild solution $K = 1$. In the case of its periodic analog $K > 1$ at $z = \pm M$ and $K < 1$ at $x = 0$. The physical explanation to that result looks as follows: the variation in $K$ is due to the self-attraction of the black hole due to compactification along the $x$ coordinate.

Approximation (2.98) together with (2.96), (2.101) yield the following expressions for the shape function, 2-dimensional metric on the surface of the event horizon and Gaussian curvature:

$$\mathscr{F} = \frac{1}{2}\ln\left[\frac{f\left(\frac{\mu+\tilde{x}}{2}\right)f\left(\frac{\mu-\tilde{x}}{2}\right)}{f(\mu)}\right] \approx \frac{1}{2}\ln\left[1+\frac{\mu^2-\tilde{x}^2}{4\pi(\pi-\mu)}\right] \tag{2.112}$$

$$d\sigma^2 = F(x)dx^2 + \frac{d\phi^2}{\mu^2 F(x)} \tag{2.113}$$

where

$$F(x) \approx \frac{1}{\mu^2 - x^2} + \frac{1}{4\pi^2(1-\mu/\pi)} \tag{2.114}$$

$$K \approx \frac{16\pi^2(\pi-\mu)^2[(2\pi-\mu)^2+3x^2]}{[(2\pi-\mu)^2-x^2]^3}. \tag{2.115}$$

The metric (2.113) can also be obtained as an induced geometry on a surface of revolution $\Sigma$ embedded in $\mathbb{R}^3$. Consider the euclidean line element in $\mathbb{R}^3$ written in cylindrical coordinates:

$$dl^2 = dh^2 + dr^2 + r^2 d\phi^2 \tag{2.116}$$

Let $h = h(r)$ be an equation which determines the surface $\Sigma$. Then the metric on this surface is

$$d\sigma^2 = \left[1+\left(\frac{dh}{dr}\right)^2\right]dr^2 + r^2 d\phi^2 \tag{2.117}$$

which coincides with (2.113) after the identification

$$r = \frac{1}{\sqrt{F(x)}} \tag{2.118}$$

$$\left(\frac{dh}{dx}\right)^2 + \left(\frac{dr}{dx}\right)^2 = F(x) \tag{2.119}$$

Therefore the derivative of the function $h(x)$ should be given by:

$$\frac{dh}{dx} = \sqrt{F - \frac{F'^2}{4F^3}} \tag{2.120}$$

In order to understand the general shape of the curve $h(x)$ the authors use numerical approximations. For small values of $\mu = M/L$ (mass-period ratio) the shape tends to a circle and for large values of $\mu$ the circle deforms into a cigar-like shape.

### 2.2.3 The Gowdy model

Since the research objective of this thesis was to study the periodic analog of the Kerr solution, the discussion of the Gowdy model is relevant here. The Kerr and van Stockum solutions are examples of solutions with rotation, but these solutions are non-periodic. On the other hand the Gowdy solution is periodic in three dimensions. The discussion of this section is based on [1].

The line element corresponding to the Gowdy model has the following form

$$ds^2 = e^{2a}(-dt^2 + dz^2) + R(e^P(dx + Qdy)^2 + e^{-P}dy^2) \tag{2.121}$$

where $a, R, P, Q$ are functions of $t$ and $z$ obeying the Einstein's equations which are periodic with respect to the space variables $x$, $y$ and $z$.

By choosing

$$Q = 0 \qquad R = t \qquad a = \ln t^{-1/4} + \lambda/4 \tag{2.122}$$

the line element (2.121) reduces to

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dz^2) + t(e^P dx^2 + e^{-P}dy^2). \tag{2.123}$$

Then the Einstein's equations reduce to

$$P_{tt} + t^{-1}P_t - P_{zz} = 0 \tag{2.124}$$

$$\lambda_t = t(P_t^2 + P_z^2) \tag{2.125}$$

and

$$\lambda_z = 2tP_zP_t. \tag{2.126}$$

One can choose

$$P = J_0(2\pi)\cos 2\pi z \tag{2.127}$$

as the particular solution to (2.124), where $J_0$ is the Bessel function. That is another simplification, since the equation (2.124) admits series solutions, where each term is a combination of trigonometric functions and Bessel functions:

$$\begin{aligned}\lambda = &-2\pi t J_0(2\pi t)J_1(2\pi t)\cos^2(2\pi z) + 2\pi^2 t^2[J_0^2(2\pi t) + J_1^2(2\pi t)] - \\ &\frac{1}{2}[(2\pi)^2[J_0^2(2\pi) + J_1^2(2\pi)] - 2\pi J_0(2\pi)J_1(2\pi)]\end{aligned} \tag{2.128}$$

This shows that even a simplified ”linearized” version the Gowdy model is still quite non-trivial, and, therefore, can be used as toy test model in numerical relativity.

# Chapter 3

# Results

The goal of this thesis is inspired by the search of the periodic analogue of the Kerr solution with periodicity along the $x$-coordinate (in Weyl canonical coordinates).

As it was discussed in Section 1.1 the study of the asymptotic behaviour of the hypothetical periodic analog of the Kerr solution can be attempted by solving two equivalent formulations of the Einstein's equations for the stationary axisymmetric spacetime metric.

Let us denote the $\{t,\phi\}$ block of the metric by $g$. Then from (2.59) it can be seen that

$$g = \begin{pmatrix} -f & -fA \\ -fA & -fA^2+\rho^2/f \end{pmatrix}. \tag{3.1}$$

In terms of the matrix $g$ the Einstein's equations reduce to the following matrix equation:

$$(\rho g_x g^{-1})_x + (\rho g_\rho g^{-1})_\rho = 0. \tag{3.2}$$

In the limit as $\rho \to \infty$ the period of the metric along the $x$ coordinate becomes negligible and the leading term of the asymptotics of matrix $g$ as $\rho \to \infty$ should be independent of $x$. Then, assuming that the Ernst potential is independent of $x$, the Ernst equation (2.60) reduces to

$$(\mathscr{E}+\bar{\mathscr{E}})(\frac{1}{\rho}\mathscr{E}_\rho + \mathscr{E}_{\rho\rho}) = 2\mathscr{E}_\rho^2; \tag{3.3}$$

the solution of this equation is discussed in 3.1.

On the other hand, if one alternatively assumes that the matrix $g$ given by (3.1) is independent of $x$, the equation (3.2) turns into

$$(\rho g_\rho g^{-1})_\rho = 0; \tag{3.4}$$

solutions of this equation are discussed in 3.2

## 3.1 Equation in terms of Ernst potential

The Lax pair which corresponds to (2.60) is:

$$\Psi_z = \frac{1}{\mathscr{E}+\overline{\mathscr{E}}}\left(\begin{pmatrix} 0 & \mathscr{E}_z \\ \overline{\mathscr{E}}_z & 0 \end{pmatrix}\sqrt{\frac{\lambda-\overline{z}}{\lambda-z}} + \begin{pmatrix} \mathscr{E}_z & 0 \\ 0 & \overline{\mathscr{E}}_z \end{pmatrix}\right)\Psi \tag{3.5}$$

$$\Psi_{\overline{z}} = \frac{1}{\mathscr{E}+\overline{\mathscr{E}}}\left(\begin{pmatrix} 0 & \mathscr{E}_{\overline{z}} \\ \overline{\mathscr{E}}_{\overline{z}} & 0 \end{pmatrix}\sqrt{\frac{\lambda-z}{\lambda-\overline{z}}} + \begin{pmatrix} \mathscr{E}_{\overline{z}} & 0 \\ 0 & \overline{\mathscr{E}}_{\overline{z}} \end{pmatrix}\right)\Psi \tag{3.6}$$

where $z = x + i\rho$. Observe also that

$$\mathscr{E}_z = -\mathscr{E}_{\overline{z}} \qquad \overline{\mathscr{E}}_z = -\overline{\mathscr{E}}_{\overline{z}} \tag{3.7}$$

Hence equations (3.5) and (3.6) can be written as

$$\Psi_z = \left(A\sqrt{\frac{\lambda-\overline{z}}{\lambda-z}} + B\right)\Psi \tag{3.8}$$

$$\Psi_{\overline{z}} = \left(-A\sqrt{\frac{\lambda-z}{\lambda-\overline{z}}} - B\right)\Psi \tag{3.9}$$

where

$$\begin{aligned} A &= \frac{1}{\mathscr{E}+\overline{\mathscr{E}}}\begin{pmatrix} 0 & \mathscr{E}_z \\ \overline{\mathscr{E}}_z & 0 \end{pmatrix} \\ B &= \frac{1}{\mathscr{E}+\overline{\mathscr{E}}}\begin{pmatrix} \mathscr{E}_z & 0 \\ 0 & \overline{\mathscr{E}}_z \end{pmatrix} \end{aligned} \tag{3.10}$$

Using the definition of $z$, the derivatives of $\Psi$ with respect to $x$ and $\rho$ are:

$$\begin{aligned}
\Psi_x &= A\left(\sqrt{\frac{\lambda-x+i\rho}{\lambda-x-i\rho}}-\sqrt{\frac{\lambda-x-i\rho}{\lambda-x+i\rho}}\right)\Psi &(3.11)\\
\Psi_\rho &= i\left(A\left(\sqrt{\frac{\lambda-x+i\rho}{\lambda-x-i\rho}}+\sqrt{\frac{\lambda-x-i\rho}{\lambda-x+i\rho}}\right)+2B\right)\Psi &(3.12)
\end{aligned}$$

Assuming that matrices $A$ and $B$ are independent of $x$, we have that $\Psi_x=-\Psi_\lambda$. Then, setting $x=0$, the Lax pair simplifies to:

$$\begin{aligned}
\Psi_\lambda &= -A\left(\sqrt{\frac{\lambda+i\rho}{\lambda-i\rho}}-\sqrt{\frac{\lambda-i\rho}{\lambda+i\rho}}\right)\Psi &(3.13)\\
\Psi_\rho &= i\left(A\left(\sqrt{\frac{\lambda+i\rho}{\lambda-i\rho}}+\sqrt{\frac{\lambda-i\rho}{\lambda+i\rho}}\right)+2B\right)\Psi. &(3.14)
\end{aligned}$$

The change of variable $(\lambda,\rho)\mapsto(\gamma,\rho)$ defined by

$$\gamma=\frac{1}{i\rho}(\lambda+\sqrt{\lambda^2+\rho^2}) \tag{3.15}$$

simplifies the linear system (3.13), (3.14) to:

$$\begin{aligned}
\Psi_\gamma &= \frac{1}{\gamma}\frac{\rho}{(\mathscr{E}+\overline{\mathscr{E}})}\begin{pmatrix}0 & \mathscr{E}_\rho\\ \overline{\mathscr{E}}_\rho & 0\end{pmatrix}\Psi &(3.16)\\
\Psi_\rho &= \frac{1}{\mathscr{E}+\overline{\mathscr{E}}}\begin{pmatrix}\mathscr{E}_\rho & 0\\ 0 & \overline{\mathscr{E}}_\rho\end{pmatrix}\Psi &(3.17)
\end{aligned}$$

To solve this system we observe that the quantity $|a|^2$, where

$$a=\frac{\rho\mathscr{E}_\rho}{\mathscr{E}+\overline{\mathscr{E}}} \tag{3.18}$$

is a constant of motion, namely differentiating $|a|^2$ with respect to $\rho$ yields:

$$\frac{d}{d\rho}|a|^2 = \frac{2\rho\mathscr{E}_\rho\overline{\mathscr{E}}_\rho + \rho^2(\mathscr{E}_{\rho\rho}\overline{\mathscr{E}}_\rho + \mathscr{E}_\rho\overline{\mathscr{E}}_{\rho\rho})}{(\mathscr{E}+\overline{\mathscr{E}})^2} - \frac{2\rho^2\mathscr{E}_\rho\overline{\mathscr{E}_\rho}(\mathscr{E}_\rho+\overline{\mathscr{E}}_\rho)}{(\mathscr{E}+\overline{\mathscr{E}})^3} \tag{3.19}$$

$$= \frac{\rho^2}{(\mathscr{E}+\overline{\mathscr{E}})^3}\big((\mathscr{E}+\overline{\mathscr{E}})(\frac{1}{\rho}\mathscr{E}_\rho + \mathscr{E}_{\rho\rho})\overline{\mathscr{E}}_\rho + (\mathscr{E}+\overline{\mathscr{E}})(\frac{1}{\rho}\overline{\mathscr{E}}_\rho + \overline{\mathscr{E}}_{\rho\rho})\mathscr{E}_\rho \tag{3.20}$$

$$- \ 2\mathscr{E}_\rho^2\overline{\mathscr{E}}_\rho - 2\overline{\mathscr{E}}_\rho^2\mathscr{E}_\rho\big) \tag{3.21}$$

$$= \frac{\rho^2}{(\mathscr{E}+\overline{\mathscr{E}})^3}(2\mathscr{E}_\rho^2\overline{\mathscr{E}}_\rho + 2\overline{\mathscr{E}}_\rho^2\mathscr{E}_\rho - 2\mathscr{E}_\rho^2\overline{\mathscr{E}}_\rho - 2\overline{\mathscr{E}}_\rho^2\mathscr{E}_\rho) \tag{3.22}$$

$$= 0 \tag{3.23}$$

where the third equality is implied by $(\mathscr{E}+\overline{\mathscr{E}})(\frac{1}{\rho}\mathscr{E}_\rho + \mathscr{E}_{\rho\rho}) = 2\mathscr{E}_\rho^2$, (2.60) with $\mathscr{E}_x = 0$.

If $\mathscr{E}(\rho)$ is real-valued then $\mathscr{E}(\rho) = \rho^\kappa$, that is

$$|a|^2 = \frac{\kappa^2}{4}. \tag{3.24}$$

Now (2.60) can be rewritten as

$$a_\rho = -\frac{1}{\rho}a\overline{a} + \frac{1}{\rho}a^2. \tag{3.25}$$

Representing $a$ in polar form as $a = \frac{\kappa}{2}e^{i\varphi(\rho)}$ leads to a simplified form of (2.60):

$$\varphi_\rho = \frac{\kappa}{\rho}\sin\varphi. \tag{3.26}$$

Solution of this equation yields:

$$\sin\varphi = \frac{2C_0\rho^\kappa}{1+C_0^2\rho^{2\kappa}} \tag{3.27}$$

$$\cos\varphi = \frac{1-C_0^2\rho^{2\kappa}}{1+C_0^2\rho^{2\kappa}} \tag{3.28}$$

where $C_0$ is a constant of integration.

The definition of $a$ (3.18) allows to rewrite one complex equation (3.3) as two real equations:

$$f_\rho = \frac{\kappa}{\rho}f\cos\varphi \tag{3.29}$$

$$g_\rho = \frac{\kappa}{\rho}f\sin\varphi \tag{3.30}$$

where $\mathscr{E} = f + ig$. Combining (3.29), (3.30) with (3.27), (3.28) yields:

$$\begin{aligned} f &= \frac{C_1\rho^\kappa}{1+C_0^2\rho^{2\kappa}} \\ g &= C_2 - \frac{C_1}{C_0(1+C_0^2\rho^{2\kappa})} \end{aligned} \tag{3.31}$$

where $C_1$ and $C_2$ are constants of integration.

Therefore the complex-valued Ernst potential is given by:

$$\mathscr{E}(\rho) = \frac{C_1\rho^\kappa}{1+C_0^2\rho^{2\kappa}} + i\left(C_2 - \frac{C_1}{C_0(1+C_0^2\rho^{2\kappa})}\right) \tag{3.32}$$

By applying the sequence of the Euler's group transformations [16] to (3.32):

$$\mathscr{E} \mapsto \mathscr{E} - iC_2, \quad \mathscr{E} \mapsto \frac{1}{C_1}\mathscr{E}, \quad \mathscr{E} \mapsto \mathscr{E} - \frac{1}{iC_0}, \quad \mathscr{E} \mapsto \frac{\mathscr{E},}{1+iC_0\rho^\kappa} \tag{3.33}$$

solution (3.32) can be transformed to $\mathscr{E} = \rho^\kappa$ is obtained.

Subtracting the constant $iC_2$ from (3.32) and rescaling with $1/C_1$, one gets the simplest rotating generalization of solution $\rho^k$:

$$\mathscr{E} = \frac{\rho^\kappa - \frac{i}{C_0}}{1+C_0^2\rho^{2\kappa}} \tag{3.34}$$

As it was discussed in 2.2.1 the complex-valued Ernst potential is related to the axisymmetric metric by

$$ds^2 = f^{-1}(e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2) - f(dt + Ad\phi)^2 \tag{3.35}$$

where $(x,\rho)$ are Weyl canonical coordinates and

$$f = Re\mathscr{E} \qquad A_z = 2\rho\frac{(\mathscr{E}-\overline{\mathscr{E}})_z}{(\mathscr{E}+\overline{\mathscr{E}})^2} \qquad k_z = 2i\rho\frac{\mathscr{E}_z\overline{\mathscr{E}}_z}{(\mathscr{E}+\overline{\mathscr{E}})^2} \tag{3.36}$$

with $z = x + i\rho$. Substituting (3.34) into the first two expressions of (2.61) gives

$$f = \frac{\rho^\kappa}{1+C_0^2\rho^{2\kappa}} \qquad A = 2C_0\kappa x \tag{3.37}$$

Note that the equation

$$\frac{\rho^2 \mathscr{E}_\rho \overline{\mathscr{E}}_\rho}{(\mathscr{E}+\overline{\mathscr{E}})^2} = |a|^2 = \frac{\kappa^2}{4} \tag{3.38}$$

leads to

$$k_\rho = \frac{\kappa^2}{4\rho} \tag{3.39}$$

Integrating (3.39)

$$k = \ln \rho^{\frac{\kappa^2}{4}} \tag{3.40}$$

Therefore the corresponding metric is given by

$$\begin{aligned} ds^2 = {} & (1+C_0^2\rho^{2\kappa})\rho^{\frac{\kappa^2}{2}-\kappa}(dx^2+d\rho^2) + (1+C_0^2\rho^{2\kappa})\rho^{2-\kappa}d\phi^2 \\ & - \frac{\rho^\kappa}{1+C_0^2\rho^{2\kappa}}(dt+2C_0\kappa x d\phi)^2 \end{aligned} \tag{3.41}$$

This metric has a non zero time-space off-diagonal term, which physically means that this metric describes a spacetime with rotation.

If the constant of integration $C_0$ is chosen to be 0, then

$$f = \rho^\kappa \qquad A = 0 \tag{3.42}$$

and the metric in this case reduces to

$$ds^2 = \rho^{\frac{\kappa^2}{2}-\kappa}(dx^2+d\rho^2) + \rho^{2-\kappa}d\phi^2 - \rho^\kappa dt^2 \tag{3.43}$$

which is the Kasner solution discussed in Subsection 2.1.3. Hence the obtained metric (3.41) can be considered as a possible rotating analog of the Kasner solution.

## 3.2 Equation in terms of metric

On the other hand the Einstein's equations in terms of the metric, which does not depend on $x$ reduce to:

$$(\rho g_\rho g^{-1})_\rho = 0 \tag{3.44}$$

Since $\det(g) = -\rho^2$, relations $\mathrm{Tr}(g_\rho g^{-1}) = \frac{\frac{\partial}{\partial\rho}\det(g)}{\det(g)}$ and $\det(g) = -\rho^2$ imply that $\mathrm{Tr}(g_\rho g^{-1}) = \frac{2}{\rho}$ and $\mathrm{Tr}(\rho g_\rho g^{-1}) = 2$.

Let

$$\rho g_\rho g^{-1} = C \tag{3.45}$$

where $C$ is a constant matrix of integration. Since $\mathrm{Tr}(C) = 2$, it can be parametrized as follows

$$C = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}. \tag{3.46}$$

The eigenvalues of the matrix $C$ are

$$\lambda_{1,2} = 1 \pm \sqrt{1-\det(C)} \tag{3.47}$$

Let us consider three different cases: assuming that matrix $C$ has 2 real eigenvalues, 2 complex eigenvalues, or only one real eigenvalue.

### 3.2.1 The case of two real eigenvalues

The following result describes the solution of (3.44) in the case of 2 real eigenvalues.

**Theorem 4.** *If $C$ has 2 real eigenvalues, then the solution to (3.44) is*

$$g = \begin{pmatrix} \beta\rho^{1+\alpha} & 0 \\ 0 & -\frac{1}{\beta}\rho^{1-\alpha} \end{pmatrix} \tag{3.48}$$

*with arbitrary constants $\beta \neq 0$ and $\alpha = \sqrt{1-\det(C)}$.*

*Proof.* Since $C$ has 2 real eigenvalues, it is diagonalizable as $C = PDP^{-1}$, where

$$P = \begin{pmatrix} v & u \\ -\frac{a-1-\alpha}{b}v & -\frac{a-1+\alpha}{b}u \end{pmatrix}$$

$$D = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix}$$

and $v, u$ are arbitrary non zero constants. Therefore (3.45) can be rewritten as $\rho g_\rho g^{-1} = PDP^{-1}$. Defining $\tilde{g} = P^{-1}gP$, (3.45) is equivalent to $\rho \tilde{g}_\rho \tilde{g}^{-1} = D$. The solution to this equation is

$$\tilde{g} = \begin{pmatrix} c_{11}\rho^{1+\alpha} & c_{12}\rho^{1+\alpha} \\ c_{21}\rho^{1-\alpha} & c_{22}\rho^{1-\alpha} \end{pmatrix}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are constants of integration. By choosing symmetric $\tilde{g}$ with $\det(\tilde{g}) = -\rho^2$, we get (3.48). □

Since matrix $g$ represents the $\{t, \phi\}$ block of the stationary antisymmetric metric, it defines uniquely parameters $f$, $A$ and $k$ in (2.59), which leads to the Kasner solution.

### 3.2.2 The case of one real eigenvalue

In the case of one real eigenvalue the following result holds.

**Theorem 5.** *If $C$ has one real eigenvalue, then the general solution to* (3.44) *is given by*

$$g = \begin{pmatrix} \rho \ln \rho + \beta\rho & \rho \\ \rho & 0 \end{pmatrix} \tag{3.49}$$

*with arbitrary $\beta \neq 0$.*

*Proof.* Since $C$ has one real eigenvalues, it has a Jordan normal form representation. That is

$$C = PJP^{-1} \tag{3.50}$$

where

$$P = \begin{pmatrix} u_1(a-1) + u_2 b & u_1 \\ -\frac{(a-1)}{b}(u_1(a-1) + u_2 b) & u_2 \end{pmatrix} \tag{3.51}$$

and

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{3.52}$$

and $u_1, u_2$ are arbitrary constants, not simultaneously equal to zero.

Therefore (3.45) can be rewritten as

$$\rho g_\rho g^{-1} = PJP^{-1}. \tag{3.53}$$

Introducing

$$\tilde{g} = P^{-1} g P \tag{3.54}$$

allows us to see that (3.45) is equivalent to

$$\rho \tilde{g}_\rho \tilde{g}^{-1} = J. \tag{3.55}$$

The solution to this equation reduces to solving first order linear differential equations with non constant coefficients. The solution is

$$\tilde{g} = \begin{pmatrix} c_{11}\rho + c_{21}\rho \ln \rho & c_{12}\rho + c_{22}\rho \ln \rho \\ c_{21}\rho & c_{22}\rho \end{pmatrix}.$$

By choosing symmetric $\tilde{g}$ with $\det(\tilde{g}) = -\rho^2$, the desired result is obtained. $\square$

In order to obtain all the elements of the metric the relation between the Ernst potential and the metric coefficients can be used. The first equation from (2.61) yields:

$$f = Re\mathscr{E} = -\rho(\ln \rho + \beta). \tag{3.56}$$

The second equation allows to find the imaginary part of the Ernst potential. Let $\mathscr{E} = f + ih$. Then:

$$\begin{aligned} A_z &= \frac{1}{2}A_x + \frac{1}{2i}A_\rho \\ &= 2\rho \frac{(\mathscr{E} - \overline{\mathscr{E}})_z}{(\mathscr{E} + \overline{\mathscr{E}})^2} \\ &= \frac{h_\rho}{2\rho(\ln \rho + \beta)^2} + i\frac{h_x}{2\rho(\ln \rho + \beta)^2}. \end{aligned} \tag{3.57}$$

On the other hand (2.59) and (2.61) imply

$$g_{12} = -Af \qquad A = \frac{1}{\ln\rho + \beta}. \tag{3.58}$$

Therefore

$$0 = A_x = \frac{h_\rho}{2\rho(\ln\rho + \beta)^2} \tag{3.59}$$

$$-\frac{1}{\rho(\ln\rho + \beta)^2} = A_\rho = -\frac{h_x}{\rho(\ln\rho + \beta)^2}. \tag{3.60}$$

Hence $h = x$. The Ernst potential is

$$\mathscr{E}(x,\rho) = -\rho(\ln\rho + \beta) + ix. \tag{3.61}$$

Given the general form of the Ernst potential $\mathscr{E} = f + ih$ the derivative with respect to $z$ can be written as follows:

$$\mathscr{E}_z = \frac{1}{2}f_x + i\left(-\frac{1}{2}f_\rho + \frac{1}{2}h_x + \frac{1}{2i}h_\rho\right). \tag{3.62}$$

In our case this reduces to

$$\mathscr{E}_z = \frac{1}{2i}f_\rho + i\frac{1}{2}h_x \qquad \overline{\mathscr{E}}_z = \frac{1}{2i}f_\rho - i\frac{1}{2}h_x. \tag{3.63}$$

Thus

$$\begin{aligned} k_z &= \frac{1}{2}k_x - i\frac{1}{2}k_\rho \\ &= 2i\rho\frac{\mathscr{E}_z\overline{\mathscr{E}}_z}{(\mathscr{E} + \overline{\mathscr{E}})^2} \\ &= 2i\rho\frac{\frac{1}{4}(-f_\rho^2 + h_x^2)}{4f^2} \\ &= -i\frac{1}{8\rho}\big(1 + \frac{2}{\ln\rho + \beta}\big), \end{aligned} \tag{3.64}$$

and

$$k_x = 0 \qquad k_\rho = \frac{1}{4\rho}\big(1 + \frac{2}{\ln\rho + \beta}\big). \tag{3.65}$$

Integration of (3.65) yields

$$k = \ln\left(\rho^{\frac{1}{4}}(\beta + ln\rho)^{\frac{1}{2}}\right). \tag{3.66}$$

Thus the metric is:

$$ds^2 = -\rho^{-\frac{1}{2}}(dx^2 + d\rho^2) + \rho(\ln\rho + \beta)dt^2 + 2\rho dt d\phi,$$

which is a special case of the van Stockum solution discussed in Subsection 2.1.4 where

$$\Omega = \ln\rho + \beta. \tag{3.67}$$

### 3.2.3 The case of two complex eigenvalues

Finally, in the case of two complex eigenvalues the solution is given by the following result.

**Theorem 6.** *If $C$ has 2 complex eigenvalues, then 2 linearly independent solutions of* (3.44) *are*

$$-\rho \begin{pmatrix} \frac{(a-1)}{\gamma}\sin(\gamma\ln\rho) + \cos(\gamma\ln\rho) & \frac{b}{\gamma}\sin(\gamma\ln\rho) \\ \frac{b}{\gamma}\sin(\gamma\ln\rho) & \frac{(1-a)}{\gamma}\sin(\gamma\ln\rho) + \cos(\gamma\ln\rho) \end{pmatrix} \tag{3.68}$$

$$-\rho \begin{pmatrix} \frac{(a-1)}{\gamma}\cos(\gamma\ln\rho) - \sin(\gamma\ln\rho) & \frac{b}{\gamma}\cos(\gamma\ln\rho) \\ \frac{b}{\gamma}\cos(\gamma\ln\rho) & \frac{(1-a)}{\gamma}\cos(\gamma\ln\rho) - \sin(\gamma\ln\rho) \end{pmatrix}, \tag{3.69}$$

*where*

$$\gamma = \sqrt{-1 + \det(C)} \tag{3.70}$$

*Proof.* Since $C$ has 2 complex eigenvalues, it is diagonalizable in the following way:

$$C = PDP^{-1} \tag{3.71}$$

where

$$P = \begin{pmatrix} v & u \\ -\frac{a-1-i\gamma}{b}v & -\frac{a-1+i\gamma}{b}u \end{pmatrix}, \tag{3.72}$$

$$D = \begin{pmatrix} 1+i\gamma & 0 \\ 0 & 1-i\gamma \end{pmatrix} \tag{3.73}$$

and $v, u$ are arbitrary non zero constants.

Therefore (3.45) can be rewritten as

$$\rho g_\rho g^{-1} = PDP^{-1} \tag{3.74}$$

Defining

$$\tilde{g} = P^{-1} g P \tag{3.75}$$

it can be seen that (3.45) is equivalent to

$$\rho \tilde{g}_\rho \tilde{g}^{-1} = D. \tag{3.76}$$

The solution to this equation is

$$\tilde{g} = \begin{pmatrix} c_{11}\rho^{1+i\gamma} & c_{12}\rho^{1+i\gamma} \\ c_{21}\rho^{1-i\gamma} & c_{22}\rho^{1-i\gamma} \end{pmatrix}$$

where $c_{11}, c_{12}, c_{21}, c_{22}$ are constants of integration.

Since $g$ is symmetric, it is diagonalizable, that is

$$g = B\tilde{D}B^{-1} \tag{3.77}$$

for some diagonal $\tilde{D}$. Hence

$$\tilde{g} = P^{-1}BD(P^{-1}B)^{-1}. \tag{3.78}$$

Thus $\tilde{g}$ is symmetric.

The symmetry of $\tilde{g}$ implies that $c_{12} = 0, c_{21} = 0$. Choosing $\tilde{g}$ with $\det(\tilde{g}) = -\rho^2$ yields $c_{22} = -\frac{1}{c_{11}}$. Direct computation of $g = P\tilde{g}P^{-1}$ leads to the following complex solution of (3.45):

$$g = \frac{1}{2}(c_{11} - \frac{1}{c_{11}})\rho \begin{pmatrix} \frac{(a-1)}{\gamma}\sin(\gamma \ln\rho) + \cos(\gamma \ln\rho) & \frac{b}{\gamma}\sin(\gamma \ln\rho) \\ \frac{b}{\gamma}\sin(\gamma \ln\rho) & \frac{(1-a)}{\gamma}\sin(\gamma \ln\rho) + \cos(\gamma \ln\rho) \end{pmatrix}$$

$$+i\frac{1}{2}(-c_{11}-\frac{1}{c_{11}})\rho\begin{pmatrix}\frac{(a-1)}{\gamma}\cos(\gamma\ln\rho)-\sin(\gamma\ln\rho) & \frac{b}{\gamma}\cos(\gamma\ln\rho)\\ \frac{b}{\gamma}\cos(\gamma\ln\rho) & \frac{(1-a)}{\gamma}\cos(\gamma\ln\rho)-\sin(\gamma\ln\rho)\end{pmatrix}.$$

Even though (3.44) is nonlinear, the real and imaginary parts of this solution satisfy (3.44) as it can be seen from the following computation. The real part of the above complex solution can be rewritten in the following way (omitting the constant scalar):

$$\Re(g)=\rho\Big(\frac{1}{\gamma}\sin(\gamma\ln\rho)(C-I)+\cos(\gamma\ln\rho)I\Big).$$

Then

$$\rho\Re(g)_\rho-C\Re(g)=-\rho\frac{1}{\gamma}\sin(\gamma\ln\rho)\big(C^2-2C+(1+\gamma^2)I\big)+\rho\cos(\gamma\ln\rho)\big(C-C\big)$$

and

$$\begin{aligned}C^2-2C+(1+\gamma^2)I&=P(D^2-2D)P^{-1}+(1+\gamma^2)I\\&=-(1+\gamma^2)PP^{-1}+(1+\gamma^2)I\\&=0.\end{aligned}\tag{3.79}$$

Hence

$$\rho\Re(g)_\rho\Re(g)^{-1}=C\tag{3.80}$$

and therefore

$$\big(\rho\Re(g)_\rho\Re(g)^{-1}\big)_\rho=0.\tag{3.81}$$

The computation for the $\Im(g)$ is similar. □

As it was noted before, the $g$ block of the metric determines uniquely the other components of the metric via (2.61) and (3.1).

# Chapter 4

# Conclusions

The main research objective described in details in Chapter 1 was to solve two equations: (1.7) and (1.8). In this respect the objective was successfully achieved by obtaining the solutions to the above mentioned equations discussed in depth in Chapter 3. Note that the solutions obtained do not describe flat spacetime.

As for the more general question - the asymptotic behaviour of the periodic analog of the Kerr solution - this question was not completely answered. One of the reasons is that our initial assumption in Chapter 1 about neglecting the dependence of the Ernst potential and metric on $x$ might be too strong.

In the end of Section 1.1, it was noted that Dr. Korotkin expected the periodic analog of the Kerr solution to behave asymptotically as some analog of the Kasner solution with rotation. Indeed metric (3.41) has this property. That metric describes spacetime with rotation and if the constant of integration is chosen to be 0, it reduces to the Kasner metric. Therefore, physically that constant of integration might be a measure of angular momentum.

However metric (3.41) depends on $x$. This is an unexpected result, since by our assumption the Ernst potential does not depend on $x$ or the metric does not depend on $x$. It turns out there is some symmetry in this case. Namely, if the metric does not depend on $x$ as in Section 3.2, the corresponding Ernst potential given by (3.61) does depend on $x$.

Therefore, further information on the periodic analog of the Kerr solution is needed to understand the correct asymptotical behaviour of this solution and to decide whether the asymptotical Ernst potential or the asymptotical metric itself must be translationally invariant in the $x$-direction.

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