Formal Analysis of Geometrical Optics using Theorem Proving

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ABSTRACT

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Geometrical optics is a classical theory of Physics which describes the light propagation in the form of rays and beams. One of its main advantages is efficient and scalable formalism for the modeling and analysis of a variety of optical systems which are used in ubiquitous applications including telecommunication, medicine and biomedical devices. Traditionally, the modeling and analysis of optical systems has been carried out by paper-and-pencil based proofs and numerical algorithms. However, these techniques cannot provide perfectly accurate results due to the risk of human error and inherent incompleteness of numerical algorithms. In this thesis, we propose a higher-order logic theorem proving based framework to analyze optical systems. The main advantages of this framework are the expressiveness of higher-order logic and the soundness of theorem proving systems which provide unrivaled analysis accuracy. In particular, this thesis provides the higher-order logic formalization of geometrical optics including the notion of light rays, beams and optical systems. This allows us to develop a comprehensive analysis support for optical resonators, optical imaging and Quasi-optical systems. This thesis also facilitates the verification of some of the most interesting optical system properties like stability, chaotic map generation, beam transformation and mode analysis. We use this infrastructure to build a library of commonly used optical components such as lenses, mirrors and optical cavities. In

order to demonstrate the effectiveness of our proposed approach, we conduct the formal analysis of some real-world optical systems, e.g., an ophthalmic device for eye, a Fabry-Pérot resonator, an optical phase-conjugated ring resonator and a receiver module of the APEX telescope. All the above mentioned work is carried out in the HOL Light theorem prover.

In Loving Memories of my Mother

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APEX	Atacama Pathfinder Experiment
CAS	Computer Algebra System
CNOT	Controlled NOT
CTR	Communications Technology Roadmap
DSP	Digital Signal Processing
FOL	First-Order Logic
FP	Fabry-Pérot
FRL	Fiber Rod Lens
HOL	Higher-Order Logic
HOL4	HOL4 Theorem Prover
HOL Light	Lightweight Implementation of Higher-Order Logic
HH	HOLyHammer
IEC	International Electrotechnical Commission
ISO	International Organization for Standardization
IYL	International Year of Light
LCF	Logic of Computable Function
MEMS	Micro-Electromechanical Systems
OPC	Optical Phase Conjugation
PDA	Personal Digital Assistant
PCM	Phase Conjugate Mirror
PVS	Prototype Verification System
SHeFI	Swedish Heterodyne Facility Instrument

Chapter 1

Introduction

1.1 Motivation

Light brings us the news of the universe.

- Sir William Bragg, *The Universe of Light (1933)*

Optics is one of the main fields of science and engineering incorporating the physical phenomena and technologies that deal with the generation, manipulation, detection and utilization of light. In the last few decades, optics has revolutionized our daily life by providing new functionalities and resolving many bottlenecks in conventional electronic systems. Recent advances in communication technology resulted in the development of sophisticated devices such as multifunction routers and personal digital assistants (PDAs); which brought additional challenges of high-speed, low-power and huge bandwidth requirements. However, traditional electronics technology has already reached a point where addressing these issues is quite difficult if not impossible. On the other hand, optics offers a promising solution to resolve these bottlenecks and provides a better convergence of computation and communication, which is a key for coping with future communication challenges. Even though the complete replacement of existing communication systems is not possible at this point, future communication systems will employ such electronic-optics convergence as mentioned in the MIT's first Communications Technology Roadmap (CTR) [24].

The primary applications of optics have emerged in biomedical imaging [30], communications [19], high-speed computing [76] and aerospace [72] to name just a few (as shown in Figure 1.1). Interestingly, the 68th Session of the UN General Assembly (on December 2013) proclaimed 2015 as the *International Year of Light and Light-based Technologies* (IYL 2015) [6]. The main purpose of celebrating IYL is to consider that the applications of light science and technology are vital for existing and future advances in energy, information and communications, fibre optics, agriculture, mining, astronomy, architecture, archaeology, entertainment, art and culture, as well as many other industries and services [7]. As a result, optics being the mainstream light based technology will gain more awareness about its problem-solving potential among international policymakers and stakeholders.

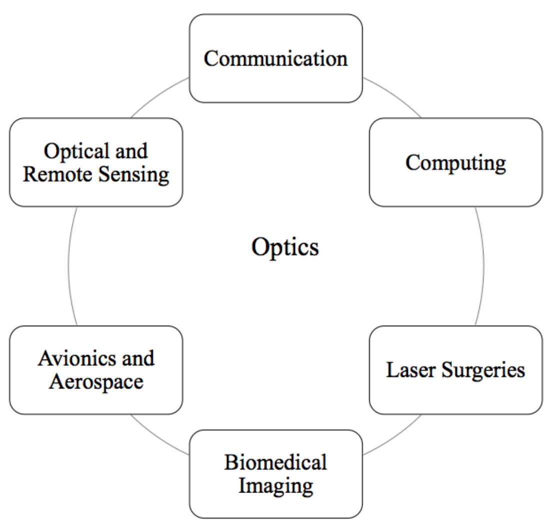

Figure 1.1: Main Applications of Optics

The designing of different optical systems depends heavily on the modeling choices for the light and optical components (e.g., mirrors and lenses). In fact, light can be modeled at different levels of abstraction such as geometrical, wave, electromagnetic and quantum optics. Geometrical optics characterizes light as a set of straight lines or beams that linearly traverse through an optical system [18]. Wave optics [85] and electromagnetic optics [85] describe the scalar and vectorial wave nature of light, respectively. On the other hand, Quantum optics [34] characterizes light as a stream of photons and helps to tackle situations where it is necessary to consider both wave-like and particle-like behaviors of light. In general, each of these theories has been used to model different aspects of the same or different optical components. For example, a phase-conjugate mirror [47] can be modeled using the ray, electromagnetic and quantum optics. The application of each theory is dependent on the type of system properties which needs to be analyzed. The main focus of this thesis is *geometrical optics* which is the foremost modeling approach in the design of a wide class optical systems. Moreover, it provides a convenient way to analyze some important properties of optical systems such as stability of optical and laser resonators used in reconfigurable wavelength division multiplexing [77] and measurement of the refractive index of cancer cells [83]. Similarly, the conditions to produce chaotic maps inside resonators [11], optical imaging in ophthalmic devices [37], coupling efficiency of optical fibers [26] and beam transformation in telescopes [70] can also be analyzed using the concepts of geometrical optics.

Optical systems are considered to be more complicated than many other types of engineering systems. The optical systems development life-cycle involves the physical modeling of optical components, analysis, and production. This process is always subject to time, safety and cost-related constraints. Due to the delays and cost associated with the manufacturing process of optical systems, it is not practical to analyze the impact of design parameters on the system behavior by successive fabrication and characterization of prototypes only. Moreover, overall optical systems characterization is also a time-consuming process and does not unveil all of the internal behaviors of the system design under test, since all properties cannot be directly measured. Therefore, to develop an understanding of the operations of optical systems and the corresponding dependence on the parameters, detailed mathematical models and exhaustive analysis are required. One natural step is to identify some fundamental building-blocks (e.g., mirrors or lenses) used in practical optical systems and then develop universal models characterizing the associated behaviors to process light. Consequently, a significant portion of time is spent in the analysis and verification of these models so that bugs in the design can be detected prior to the manufacturing of the actual system. Even minor bugs in optical systems can lead to disastrous consequences such as the loss of human lives because of their use in biomedical devices (e.g., refractive index measurement of cancer cells [83]), or financial loss because of their use in high budget space missions.

Traditionally, the analysis of geometrical optics has been done using paper-and-pencil based proofs. This technique allows to manipulate physical equations characterizing optical systems (or components) using manual paper-based reasoning. However, the analysis of complex optical systems using this approach is error-prone, particularly for the case of a large number of components and interconnections. Computer simulation is another state-of-the-art technique for the analysis of optical systems. The main principal behind simulation is the utilization of efficient numerical algorithms to assess the behavior of geometrical optics based models under certain initial conditions.

Besides the huge memory and computational time requirements, simulation cannot provide perfectly accurate results due to the discretization of continuous parameters and inability to deal with infinitely many input samples. Another approach to analyze optical systems is the use of computer algebra systems (CAS) (e.g., Mathematica [8]), which provide the symbolic manipulation of complex physical equations. However, the main focus of CASs is performance and user-friendliness which comes at the cost of some drawbacks such as the underlying algorithms and computational procedures, which often times rely on many approximations and heuristics[1].

The above mentioned inaccuracy problems of traditional analysis methods are impeding their usage in designing safety-critical and high-consequence optical systems, where minor bugs can lead to fatal consequences both in terms of monetary loss and human life risk. In particular, it is more important in the applications where failures directly lead to safety issues such as in aerospace as compared to telecommunication where failures can lead to safety problems through some secondary events. An example of such a critical application is Boeing F/A-18E, for which the mission management system is linked using an optical network [89].

Formal methods [45] are computer based reasoning techniques which allow accurate and precise analysis and thus have the potential to overcome the limitations of accuracy, found in traditional approaches. The main idea behind formal methods based analysis of systems is to develop a mathematical model for the given system and analyze this model using computer-based mathematical reasoning, which in turn increases the chances for catching subtle but critical design errors that are often ignored by traditional techniques. The two major formal methods techniques are *model checking* and *theorem proving* (a brief overview of other formal methods techniques can be

[1] A critical overview of traditional analysis approaches (paper-and-pencil based proofs, simulation and computer algebra systems) is provided in Section 1.2.

found in [45]). Model checking [17] is an automated verification technique for systems that can be expressed as finite-state machines. On the other hand, higher-order logic theorem proving [41] is an interactive verification technique, which is mainly based on the notion of formal proofs in a logic (e.g., propositional logic , first-order logic (FOL) or higher-order logic (HOL)). Another important formal verification technique is satisfiability modulo theories (SMT) [27] which deals with the satisfiability of mathematical formulas. There is an increasing interest of integrating SMT solvers with model checkers to verify real-world software and hardware systems. However, model checking cannot be used to analyze hardware aspects of optical systems due to the involvement of multivariate analysis and complex-valued parameters. On the other hand, higher-order logic theorem proving can be applied in optics due to its higher expressibility and the availability of some well-developed theorem proving systems. Moreover, SMT solvers can be used as decision procedures within theorem proving tools [27] to provide an effective automation.

Nowadays, the use of formal methods for high risk and safety-critical systems is recommended in different standards like the general IEC 61508 [58], DO178-C [49] for aviation and ISO 26262 [50] for automotive. The increasing applications of geometrical optics in safety-critical systems suggest applying formal methods in this field as well. To the best of our knowledge, there is no work in the open literature which tackles the analysis of geometrical optics by any formal methods technique. To fill this gap, we propose in this thesis a higher-order logic theorem proving based formal analysis framework for geometrical optics. The main challenging aspect of this thesis is the interdisciplinary nature of the subject as it requires expertise in the underlying physics, mathematical modeling, and formal methods. However, it provides an efficient way for identifying critical design errors that are often ignored by traditional analysis techniques. More details about the proposed framework will be presented in Section 1.3. We next provide a critical overview of the state-of-the-art analysis techniques for geometrical optics.

1.2 Geometrical Optics Analysis Approaches

Due to the vast applications of geometrical optics in different fields of science and engineering, many scientists and engineers use different approaches to analyze corresponding systems based on analytical and numerical models. In this section, we provide an overview of these techniques and highlight their strengths and weaknesses.

1.2.1 Paper-and-Pencil based Proofs

The paper-and-pencil based proofs is a fundamental technique and a starting point to build the model of a physical system and its associated properties using the underlying physical concepts. Then paper-based mathematical reasoning is used to prove whether the system model possesses the desired properties. Due to the analytical nature of this technique, it is widely used by physicists to propose a variety of new optical systems and their corresponding properties and applications. Indeed most of the Optics literature is based on the paper-and-pencil based proofs, e.g., [57, 67, 66, 85]. However, paper-and-pencil based proofs have some serious limitations as described in the following:

- Traditionally, these mathematical proofs are written in a way to make them easily understood by physicist or mathematicians. Usually the fundamental logical steps are omitted and a significant amount of underlying context is assumed on the part of the reader.

- Many examples of erroneous paper-and-pencil proofs in Optics are available in the open literature, a recent one can be found in the paper by Cheng [25] and its identification and correction is reported by Naqvi [68]. The main problem was in the derivation of the polarization vector that led to the erroneous final electric and magnetic field expressions. Interestingly, we have also found a discrepancy in the paper-and-pencil based proofs approach in [65] used to analyze the stability of optical resonators. Particularly, the order of matrix multiplication in Equations (16) and (24) in [65] should be reversed, so as to obtain correct stability constraints.

- Different physicists rely on different fundamental physical assumptions which leads to contrasting mathematical models of the same system. In other words, these proofs cannot be traced down to some unique basic physical or mathematical rules.

In the last decade, many researchers have discussed the limitations and reliability issues of paper-and-pencil based mathematical proofs (e.g., [21, 36, 16]). The proofs involved in the analysis of optical systems are also mathematical in nature and raise the similar questions of trusting them for safety-critical applications.

1.2.2 Computer Simulation

Nowadays computer simulation is a widely used technique to mimic the behavior of complex optical systems due to the availability of a wide range of open-source tools (e.g., reZonator [75] and LASCAD [59]) and commercial tools (e.g., Synopsys CODE V [9], Radiant-Zemax [74] and FRED Optical Engineering Software [5]). The main strength of these tools is the provision of a library of a wide class of optical components and the automatic analysis of different properties such as the propagation of a ray through a series of optical systems, and imaging properties, etc. The theoretical basis of these tools is to encode the mathematical equations corresponding to each optical component and analysis procedures (e.g., ray tracing [86]) in a programming language such as C++ or Fortran. Most of these tools provide the facility to write user-defined components, functions and analysis utilities. The user-friendliness, better visualization and strong automation offered in the above mentioned simulation based optical design tools come at the cost of several problems, including:

- The analysis of optical and laser systems involves complex and vector analysis along with transcendental functions, thus numerical computations cannot provide perfectly accurate results due to the heuristics and approximations (e.g., round-off errors) of the underlying numerical algorithms. Moreover, the complexity of these tools increases exponentially with the size of input data, e.g., the stability of optical resonators requires to consider N round trips (back-and-forth traversal of a ray) and N can be very large. In case of simulation, this type of properties can only be verified for some particular values of N, rather than for all N.

- The core of these tools contains thousands of lines of code characterizing the mathematical models of optical components and numerical algorithms. Although these codes are written by expert programmers, there is still a chance of uncaught errors. One of the best-selling books on testing computer software gives an indication about public bugs (which are encountered in a program after a programmer declares it as error-free) as one error per 100 statements [53]. Another study about the reliability of scientific software based on 100 different codes from 40 different applications identifies that the C codes contained approximately 8 serious static faults (caused by a programmer) for every 1000 lines of executable code, while the Fortran codes contained approximately 12 serious faults per 1000 lines which can lead to serious failures [46].

1.2.3 Computer Algebra Systems

Computer algebra systems (CAS) deal with symbolic computations of mathematical expressions and provide a high degree of automation. Mainly computer algebra systems consist of three main components: 1) user interface; 2) a programming language and 3) simplifying procedures (e.g., `FullSimplify` in Mathematica [8]). The most comprehensive and widely used computer algebra based optical design tool is Optica [71] which provides a rich library of optical components, mirrors and light sources. The main core of Optica is based on the kernel functions of Mathematica and provides both symbolic and numerical computations. Some of the limitations of computer algebra systems are described as follows:

- The internal design of computer algebra systems has a very little concept of logical reasoning and some of the simplification procedures are ill-defined or imprecise which implies that computations performed by computer algebra systems are not reliable [39]. One simple example in Mathematica is the expression $\frac{x}{x}$ for which the simplification function `Simplify[x/x]` provides 1. However, this is only true when $x \neq 0$.

- Another source of inaccuracy in computer algebra systems is the presence of unverified huge symbolic manipulation algorithms in their core, which are quite likely to contain bugs and in case of commercial tools they are not even transparent to users. Recently, a bug in the computation of some determinants with big integers in Mathematica is discussed in [28], where the authors reported this bug to Wolfram Research Inc. (the developers of Mathematica) which was acknowledged by the developers [28]:

> *"It does appear there is a serious mistake on the determinant operation you mentioned. I have forwarded an incident report to our developers with the information you provided."*

It is important to note that the analysis of geometrical optics is heavily dependent on matrix algebra and doing such an analysis using Optica (which relies on Mathematica) can also suffer from similar errors as those mentioned above.

1.2.4 Theorem Proving

Theorem proving is a widely used formal methods technique which is concerned with the construction of mathematical theorems by a computer program (called *theorem prover* or *proof assistant*). Theorem proving systems have been employed in the past to verify generic properties of a wide class of software and hardware systems. For example, a hardware designer can prove different properties of a digital circuit by describing its behavior by some predicates and applying Boolean algebra. Similarly, a mathematician can prove the transitivity of real numbers using the fundamental axioms of real number theory. These properties are described as theorems in a particular logic such as propositional logic, first-order logic (FOL) or higher-order logic (HOL) [41], depending upon the expressibility requirements. For example, the use of higher-order logic is advantageous over first-order logic in terms of the availability of additional quantifiers and its high expressiveness. Moreover, higher-order logic is expressive enough to describe almost all the known concepts from mathematics including topological spaces, real numbers, multivariate calculus and higher transcendental functions. Once such a mathematical theory is expressed inside a theorem prover, we say that it is *formalized*.

HOL theorem provers can be used to formalize mathematical theories with as much accuracy as traditional paper-and-pencil based approach, but with a more precise control of the computer program which ensures that all the steps are consistent and correct. This is achieved by defining a precise syntax of the mathematical sentences by providing some axioms and inference rules which are the only ways to prove the correctness of a sentence. For example, a theorem prover does not allow to conclude that "$\frac{x}{x} = 1$" unless it is first proved that $x \neq 0$, usually computer algebra systems do not consider such subtlety when simplifying mathematical expressions [39]. Indeed, theorem provers allow to check every logical inference all the way back to the fundamental axioms of mathematics. There are two types of provers: automatic and interactive. In an interactive theorem prover, significant user computer interaction is required while automatic theorem provers can perform different proof tasks automatically. The main downside of automatic theorem provers is their limited expressiveness as they rely on decidable subsets of the underlying logic. This in turn limits their usage in the domains where complicated mathematics is involved (e.g., multivariate calculus). Some prominent interactive theorem provers are HOL Light [38], Isabelle/HOL [69], Coq [4], HOL4 [82] and PVS [73].

In the last two decades, theorem proving has been used to verify both hardware and software systems, e.g., verification of digital hardware circuits [10], verification of the floating point algorithms [40], verification of the digital signal processing (DSP) designs [13], full-scale verification of the seL4 operating system [56] and verification of the CompCert compiler [60]. The applications of theorem proving has gone beyond the system verification with the increasing interest of formally verifying mathematics which aims at developing mathematics with greater precision [16]. Large projects in this direction include the completed formal proofs of the Kepler conjecture (Flyspeck) [35], the Odd Order Theorem [33] and the Four Color Theorem [32]. Considering these encouraging applications of theorem proving based formal verification, it is natural to think of applying it to physical systems (e.g., optics or quantum physics). However, this interesting direction of research is largely unexplored and includes only few formalizations which are reported in the open literature.

The first work towards the formal analysis of optical waveguides was reported by Hasan et. al. [44] in 2009. The main target of this work was only one application, i.e., the formalization of electromagnetic equations of a planar optical waveguide and the verification of corresponding eigenvalues using the HOL4 theorem prover. The main problem with this approach was the use of real-analysis to approximate the complex-valued electromagnetic equations which limits its application to verify many optical systems. The principle reason for this approximation was the unavailability of complex-numbers and multivariate analysis in the HOL4 theorem prover. Later on, this seminal work was generalized to build a formal analysis framework for electromagnetic optics [55] using the formalization of complex-valued vectors in the HOL Light theorem prover [54]. The utilization of this work was demonstrated by the verification of the optical intensity for a simple two plane-mirror Fabry-Pérot resonator. However, this work has some limitations as it does not provide the formalization of important optical components such as spherical and phase-conjugated mirrors which are often used in practical optical systems (e.g., laser resonators, optical fiber couplers, etc.). Moreover, this work does not provide a hierarchical formalization of optical systems and it cannot be applied to systems composed of a series of optical components. In [62], an interesting idea about the formalization of quantum theory is reported. The authors provided the formalization of infinite-dimensional linear algebra which is a foremost requirement to reason about quantum systems in HOL. This foundational work has been used to formally reason about quantum optics, including the formalization of coherent light, single-mode, multi-mode along with the formal analysis of different quantum optical components and circuits such as the Flip gate, CNOT gate, and Mach-Zehnder interferometer [61]. This formalization does not provide the support to tackle real-world applications such as astronomical equipments (e.g., telescopes) and optical devices used in ophthalmic medical technology (e.g., eye vision correcting instruments).

While interesting, the above two pioneering projects on electromagnetic and quantum optics handle low abstraction levels of optics and suffer from their application to complex and real-world optical systems. On the other hand, geometrical optics provides a high-level abstraction of light (in terms of rays and beams [86]) and it is widely used to validate the behavioral properties of many critical optical systems, e.g., laser resonators, telescopes and optical imaging devices. The main motivation behind this thesis is to fill the above mentioned gap by providing a comprehensive analysis framework for geometrical optics.

The proposed formalization of geometrical optics is carried out in HOL Light due to two main reasons: First, the formal analysis of geometrical optics is complementary to the related work about electromagnetic and quantum optics. In fact, all three approaches are parts of a larger project[2] which aims at the formal verification of a wide class of practical optical systems [12]. Secondly, an interesting common ground among the three abstract notions of light (i.e., geometrical, electromagnetic and quantum) is the complex-valued linear algebra for which HOL Light provides rich multivariate analysis libraries [43]. Note that the formalization of geometrical optics presented in this thesis is the first of its kind in any of theorem proving or other formal methods

[2] `http://hvg.ece.concordia.ca/projects/optics/`

based systems.

1.3 Formal Framework for Geometrical Optics

The objective of this thesis is mainly targeted towards the development of a theorem proving based analysis framework for geometrical optics that can handle the modeling and analysis of real-world optical systems. In particular, we propose to develop a framework in HOL Light characterizing:

- The ability to formally model the optical systems[3] in a systematic way with no restriction on the number of optical components.
- The ability to formally express the notions of light both as rays and beams which are widely used abstractions at the level of geometrical optics.
- The ability to formally reason about the properties of rays and beams when they traverse through an optical system. Essentially this includes the formalization of commonly used mathematical models such as transforming an optical system model into a matrix-model and composing small optical subsystems to build a complicated system.
- The ability to use the developed infrastructure to analyze different types of optical systems. Mainly, this includes the formalization of properties of interest and reasoning support to verify them with respect to a given optical systems model.

[3] In this thesis, we consider optical systems that can be modeled in the context of geometrical optics, e.g., resonators, telescopes, etc. Moreover, we consider that all optical components in a system are aligned with respect to a fixed optical axis.

The proposed framework, given in Figure 1.2, outlines the above mentioned characteristics and the main idea about the formal analysis of geometrical optics within the sound core of a theorem prover.

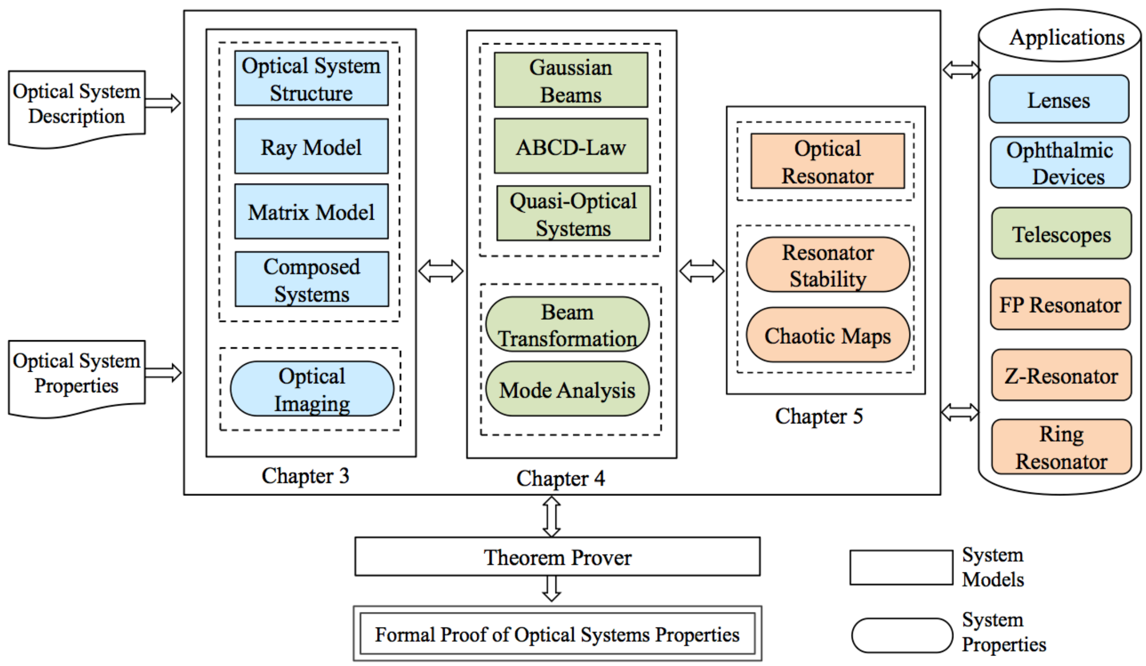

Figure 1.2: Formal Framework for the Analysis of Geometrical Optics

The two inputs to the framework are the description of the optical system and specification, i.e., the spatial organization of various components and their parameters (e.g., radius of curvature of mirrors and distance between the components, etc.). In order to construct a formal model of the given system in higher-order logic, we provide a formalization of optical system structures that consist of definitions of optical interfaces (e.g., plane and spherical) and optical components (e.g., thin lens and thick lens). In this block, we also provide the formalization of the functions that help to evaluate the validity of the parameters of the individual optical components and hence the optical systems. We then provide the formalization of the physical concepts of rays and Gaussian beams including necessary specifications about their physical behavior when they traverse through free space and different optical interfaces (e.g., the behavior of a ray when it incidences on a reflected plane interface).

Building on above fundamentals, we formally derive several models of geometrical optics: 1) A *matrix-model* of the optical systems which is a composition of the matrix models of individual optical components; 2) *ABCD-law* of transformation [86] that describes the mathematical relation between input and output beams or rays; 3) *Composed optical systems* which allow to build complicated optical systems from small or less complicated optical subsystems; 4) *Quasi-optical systems*, which deal with the propagation of a beam of radiations that are used in a variety of critical applications such as radars, commercial telescopes, remote sensing and radiometric optical systems. Indeed, we provide the verification of necessary theorems which state that a composed system inherits all the properties of an individual optical system.

Based on the above formalization infrastructures, we provide a generic approach to formally model *optical resonators*, which are the building blocks of future communication systems, biomedical devices and chaos generating optical systems for energy storage. Moreover, we also develop a reasoning support to reuse the derivation of matrix-models for optical resonators. Finally, we provide the formalization of the most frequently used properties which ensure that the given model of an optical system satisfies some constraints or possess some particular physical behavior. Following are the main properties, which can be analyzed using our proposed framework:

- **Optical imaging** deals with the observation of the image size, location, and orientation of the rays inside an optical system using the notion of *cardinal points* (i.e., pair of points on optical axis).

- **Beam transformation** provides the basis to derive the suitable parameters of Gaussian beams for a given optical system.

- **Mode analysis** deals with the evaluation of field distributions inside an optical system.
- **Stability** ensures the confinement of rays within an optical cavity after N round-trips.
- **Chaotic map generation** ensures that light rays inside an optical resonator possess a chaos, i.e., the ray behavior shows an exponential sensitivity to slight changes to the initial conditions or parameters of the involved optical components.

We develop a library of frequently used optical components such as thin lenses, thick lenses and mirrors. Such a library greatly facilitates the formalization of new optical systems that are composed of these components. The output of the proposed framework is the formal proof certifying that the system implementation meets its specification. The verified systems will then also be available in the library for future use either independently or as part of a larger optical system.

We demonstrate the strength of our proposed framework by conducting the formal analysis of several important and widely used practical systems. In particular, we present the formal analysis of an optical instrument (ophthalmic device) used to compensate the ametropia of an eye. We then utilize the generic formalization of Quasi-optical systems to formally analyze the receiver module of a real-world Atacama Pathfinder Experiment (APEX) telescope[4]. Considering the importance of optical resonators in many domains (e.g., micro-electromechanical system (MEMS), tuned optical filters and optical bio-sensing devices), we formally analyze three application architectures of Fabry Pérot (FP) resonators, i.e., non-symmetric, symmetric and two-dimensional fiber rod lens (FRL) induced cavity. The other applications of optical

[4] `http://www.apextelescope.org/`

resonators that we propose to analyze are the stability of a Z-shaped resonator and the verification of chaotic map generation for a generic ring-resonator that is widely used in optical phase conjugation. Note that all these mentioned applications are chosen due to their wide use in real-world safety and mission critical domains such as biomedical surgeries and space missions.

1.4 Thesis Contributions

The main contribution of this thesis is about the idea of applying formal methods (in particular higher-order-logic theorem proving) for the analysis of geometrical optics. We develop a formal framework on top of the trusted kernel of HOL Light theorem prover which ultimately allows the precise analysis of safety-critical optical systems. Our proposed approach can be considered as a complementary method to other state-of-the-art but less accurate techniques like computer simulation, CAS and paper-and-pencil based analysis. We list below the main contributions of this work with references to related publications provided in the Biography section at the end of this thesis.

- The formalization of the basic notions of optical system structures including different interfaces (e.g., plane and spherical), light rays and corresponding matrix models [Bio-Cf15, Bio-Cf19]. We use this infrastructure to formalize the concepts of cardinal points of optical imaging systems [Bio-Cf10], which lead to the formal analysis of an ophthalmic optical instrument [Bio-Cf6].

- The formalization of Gaussian beams and the paraxial Helmholtz equation and the verification of the ABCD-law for composed optical systems. This development allows us to formalize generic Quasi-optical systems along with the formal analysis of a receiver module of the real-world Atacama Pathfinder Experiment (APEX) telescope [Bio-Cf9, Bio-Jr5].

- The formalization of optical resonators and the verification of generic theorems about the stability conditions in the context of geometrical optics [Bio-Cf14, Bio-Cf16]. We also develop some automation tactics [Bio-Tr1] and conduct the formal stability analysis of Fabry Pérot resonators [Bio-Jr2] along with the verification of two-dimensional chaotic map generation inside a ring resonator [Bio-Jr7].

1.5 Organization of the Thesis

The rest of this thesis is organized as follows: In Chapter 2, we provide some introductory concepts of geometrical optics including the abstractions of light such as rays and beams. We then describe the mathematical treatment of rays and beams for the propagation through an optical system. We also provide an overview of the HOL Light theorem prover along with some of its useful features and notations.

In Chapter 3, we describe the formalization of ray optics and related concepts such as optical interfaces, components, systems, ray model and the verification of ray-transfer matrix transformation of any arbitrary optical system. We also present the development of a component library which includes different types of mirrors and lenses. This chapter also includes the HOL formalization of cardinal points of optical imaging systems. In order to demonstrate the strength of the formalization of ray optics, we present the formal modeling and analysis of an ophthalmic corrective device which is used to treat ametropia of an eye.

In Chapter 4, we present the HOL formalization of light as a beam and related concepts. In particular, we describe the formalization of the q-parameters of Gaussian beams, the formalization of paraxial Helmholtz equation along with the verification that a Gaussian beam satisfies the paraxial Helmholtz equation and the formalization of beam transformation for optical systems and corresponding ABCD-law. We also provide a discussion about the analysis requirements for Quasi-optical systems. We then use this infrastructure to conduct the formal analysis of a real-world APEX telescope receiver.

In Chapter 5, we present a generic formalization of optical resonators and their formal relation with optical systems. This includes the formalization of some useful functions required to analyze the behavior of a ray inside a resonating optical system and the derivation of equivalent matrix relations. This chapter also highlights the development of a reasoning support to formally derive the stability conditions of optical resonators along with the formalization of chaotic maps. Finally, we use our formalization to verify the stability and chaos generation for Fabry-Pérot resonator and ring-resonator, respectively.

Finally, Chapter 6 concludes this thesis by providing some remarks about the developed framework including a description of some challenging aspects of our work and potential future research directions.

Chapter 2

Preliminaries

In this chapter, we provide a brief overview of geometrical optics. We start by describing the different notions of light and optical components in geometrical optics along with the constituent modeling approach. We also provide an overview of the HOL Light theorem prover. The intent is to introduce the basic theories along with some notations that we use in the rest of this thesis.

2.1 Ray Optics

Ray optics describes the propagation of light as rays through different interfaces and mediums. The main principle of ray optics is based on some postulates which can be summed up as follows: Light travels in the form of rays emitted by a source; an optical medium is characterized by its refractive index; light rays follow the Fermat's principle of least time [78]. Generally, the main components of optical systems are lenses, mirrors and a propagation medium which is either a free space or some material such as glass. These components are usually centered about an optical axis, around which rays travel at small inclinations (angle with the optical axis). Such rays are called *paraxial rays* and this assumption provides the basis of *paraxial optics* which is the simplest framework of geometrical optics. When a ray passes through optical components, it undergoes *translation*, *refraction* or *reflection*. In translation, the ray simply travels in a straight line from one component to the next and we only need to know the thickness of the translation. On the other hand, refraction takes place at the boundary of two regions with different refractive indices and the ray obeys the law of refraction, called *Paraxial Snell's law* [78]. Similarly, a ray follows the law of reflection at the boundary of a reflective interface (e.g., mirror). For example, Figure Figure 2.1 (a) shows a ray propagation through a free space of width d with refractive index n, and Figure 2.1 (b) shows a plane interface (with refractive indices n_0 and n_1, before and after the interface, respectively).

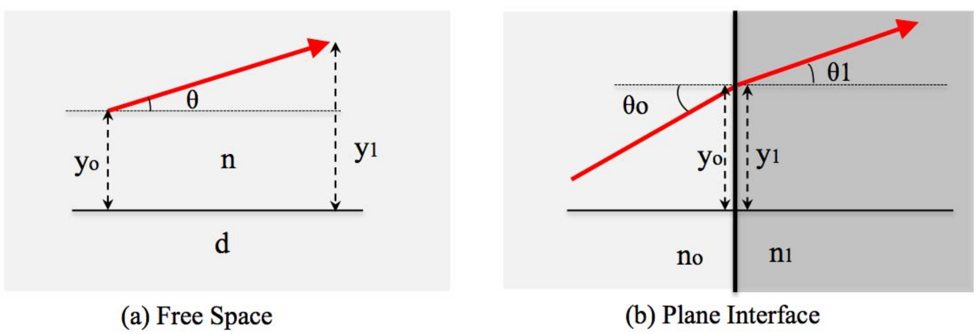

Figure 2.1: Behavior of a Ray at Plane Interface and Free Space

2.1.1 Modeling Approach

The change in position and inclination of a paraxial ray as it travels through an optical system can be described by the use of a matrix algebra. This matrix formalism (called *ray-transfer matrices*) of geometrical optics provides convenient, scalable and systematic analysis of real-world complex optical and laser systems. This is due to the fact that each optical component can be described by a matrix. This helps to use many linear algebraic properties for the analysis of optical systems.

For example, consider the propagation of a ray through a spherical interface with radius of curvature R between two mediums of refractive indices n_0 and n_1, as shown in Figure 2.2. Our goal is to express the relationship between the incident and refracted rays. The trajectory of a ray as it passes through various optical components can be specified by two parameters: its distance from the optical axis and its angle with the optical axis. Here, the distances of the incident and refracted rays are y_1 and y_0, respectively, and $y_1 = y_0$ because the thickness of the surface is assumed to be very small. Here, ϕ_0 and ϕ_1 are the angles of the incident and refracted rays with the normal to the spherical surface, respectively. On the other hand, θ_0 and θ_1 are the angles of the incident and refracted rays with the optical axis.

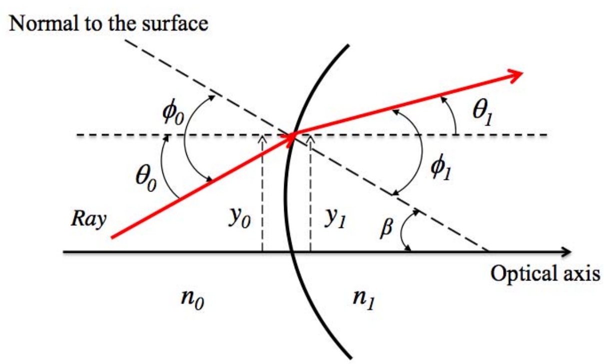

Figure 2.2: Spherical Interface

Applying paraxial Snell's law at the interface, we have $n_0\phi_0 = n_1\phi_1$. We also have $\theta_0 = \phi_0 - \beta$ and $\theta_1 = \phi_1 - \beta$, where β is the angle between the surface normal and the optical axis. Since $\sin(\beta) = \frac{y_0}{R}$, then $\beta = \frac{y_0}{R}$ by paraxial approximation. We can deduce that:

$$\theta_1 = \left(\frac{n_0 - n_1}{n_1 R}\right) y_0 + \left(\frac{n_0}{n_1}\right) \theta_0 \tag{2.1}$$

So, for a spherical surface, we can relate the refracted ray and incident ray by a matrix using Equation (2.1) as follows:

$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_1 R} & \frac{n_0}{n_1} \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Thus the propagation of a ray through a spherical interface can be described by a matrix generally called *ABCD matrix* [86]. Similarly, a general optical system with an input and output ray vector can be described as follows:

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where y_0, θ_0, y_n and θ_n represent the starting and ending points of the ray.

Finally, if we have an optical system consisting of k optical components (C_k), then we can trace the input ray R_i through all optical components using the composition of the matrices of each optical component as follows:

$$R_o = (C_k.C_{k-1} \cdots C_1).R_i \tag{2.2}$$

We can write $R_o = M_s R_i$, where $M_s = \prod_{i=k}^{1} C_i$. Here, R_o is the output ray and R_i is the input ray. Similarly, a composed optical system that consists of N optical systems inherits the same properties as of a single optical system as shown in Figure 2.3. This is a very useful modeling notion for systems that consist of small subsystems, as we can use already available infrastructure with minimal efforts.

2.1.2 Ray Tracing

The propagation of paraxial rays through an optical system is a very useful technique to analyze optical systems. The activity of ray propagation through an optical system

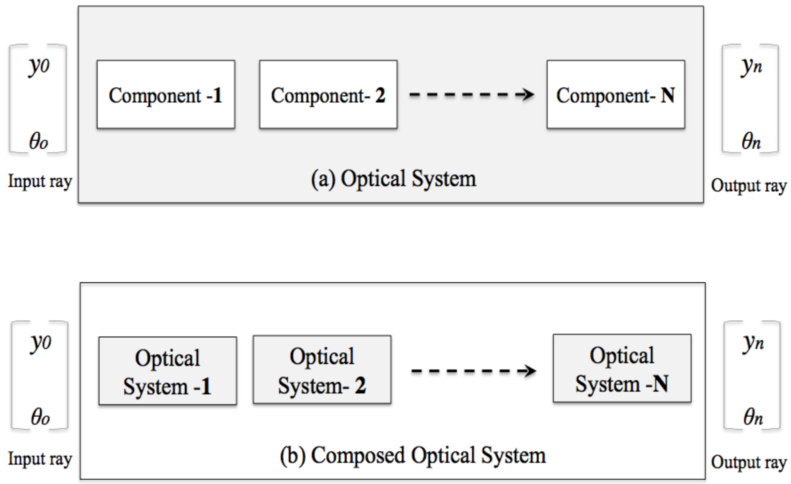

Figure 2.3: Optical System and Composed Optical System

is called *ray tracing* [85], which provides a convenient way for the design optimization along with the assessment of different properties of optical components. Ray tracing can be automated and hence it is a part of almost all optical system design tools such as Radiant-Zemax [74]. There are two types of ray tracing: *sequential* and *non-sequential*. In this thesis, we only consider sequential ray tracing with centered optical components[5], which is based on the following modeling criteria [85] :

1. The type of each interface (e.g., plane or spherical, etc.) is known.

2. The parameters of the corresponding interface (e.g., the radius of curvature in the case of a spherical interface) are known in advance.

3. The spacing between the optical components and misalignment with respect to optical axis are provided by the system specification.

4. Refractive indices of all materials and their dependence on wavelength are available.

[5]In some situations, optical components can be misaligned with respect to a fixed optical axis [81]. We do not consider the effect of misalignment in this thesis.

On the other hand, in case of non-sequential ray tracing the nature of each interface is not predefined, i.e., at each interface, the ray can either be transmitted or reflected. Non-sequential ray tracing is very expensive in terms of its huge computational time and it is only applied when the sequential ray tracing cannot be used. It is sufficient to consider sequential ray tracing to evaluate the performance of most imaging optical systems and hence the main reason of our choice [86].

Some typical applications of ray-tracing are the stability analysis of optical resonators [64], chaotic map generation [11], and the analysis of micro opto-electromechanical systems [90].

2.2 Gaussian Beams

Although ray tracing is a powerful tool for the early analysis of many optical systems, it cannot handle many situations due to the abstract nature of rays. It is important to consider that whether light can travel in free space without the angular spread or not. According to the wave nature of light, it is indeed possible that light can travel in the form of beams which comes close to the spatially localized and non-diverging waves [78]. The behavior of such an abstraction of light (beams) can be explained using the notion of *paraxial waves* whose wave front normals (i.e., the locus of points having the same phase) make very small angles with the axis of propagation. Such wavefront normals are also called paraxial rays as shown in Figure 2.4.

One of the most commonly used method to construct a paraxial wave is to consider a plane wave $Ae^{(-jkz)}$ (where $j = \sqrt{-1}$, $k =$ wave-number and z is the direction of propagation) and modify its complex amplitude A, by making it a slowly varying function of the position, i.e., $A(x, y, z)$. Mathematically, the complex amplitude of a paraxial wave becomes:

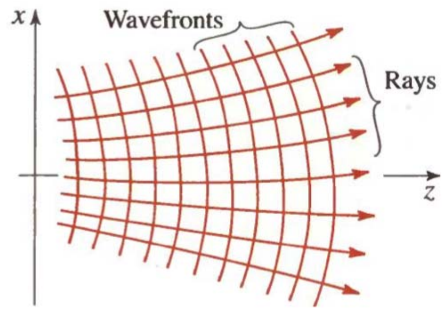

Figure 2.4: The Wavefronts and Wavefront Normal of Paraxial Wave [78]

$$U(x, y, z) = A(x, y, z)e^{(-jkz)} \tag{2.3}$$

For a paraxial wave to be valid in the context of geometrical optics, it should satisfy the paraxial Helmholtz equation [78], which is given as follows:

$$\nabla_T^2 A(x, y, z) - j2k\frac{\partial A(x, y, z)}{\partial z} = 0 \tag{2.4}$$

where $\nabla_T^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator. In general, different solutions can be found which satisfy Equation (2.4). For example, a paraboloidal wave is a solution for which the complex envelope is given as:

$$A(x, y, z) = \frac{A_0}{z}e^{\left[-jk\frac{x^2+y^2}{2z}\right]} \tag{2.5}$$

where $A_0 \in \mathbb{C}$ is a complex-valued constant.

Another solution of the Helmholtz equation provides the Gaussian beam [78] which is obtained from the paraboloidal wave by a simple transformation. Indeed the complex envelope of paraboloidal wave is a solution of the paraxial Helmholtz equation, a shifted version is also a solution, i.e., replacing z by $z - \zeta$ in Equation

(2.5):

$$A(x,y,z) = \frac{A_0}{z-\zeta} e^{\left[-jk\dfrac{x^2+y^2}{2(z-\zeta)}\right]} \tag{2.6}$$

where $\zeta \in \mathbb{C}$ is a constant. Physically, it provides a paraboloidal wave centered about the point $z = \zeta$, rather than $z = 0$. The parameter ζ is very important and produces different properties depending upon the variation of its value, e.g., $\zeta = -jz_R$, which provides the complex envelope of a Gaussian beam that can be compactly described as follows:

$$A(x,y,z) = \frac{A_0}{q(z)} e^{\left[-jk\dfrac{x^2+y^2}{2q(z)}\right]} \tag{2.7}$$

where $q(z) = z + jz_R$ is called the q-parameter of Gaussian beams. The parameter $z_R \in \mathbb{R}$ is known as the Rayleigh range.

In order to study the properties (e.g., phase and amplitude) of Gaussian beams, the above mentioned complex valued q-parameter is expressed as $\dfrac{1}{q(z)} = \dfrac{1}{z + jz_R}$. In the optics literature, this expression is further transformed into a new form by defining two real-valued functions $R(z)$ and $W(z)$, as follows:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi W^2(z)} \tag{2.8}$$

where $W(z)$ and $R(z)$ are measures of the beam width and wavefront radius of curvature, respectively (shown in Figure 2.5). Mathematically, these parameters can be expressed as follows:

$$R(z) = z\left[1 + (\frac{z_R}{z})^2\right] \tag{2.9}$$

$$W(z) = w_0\left[1 + \left(\frac{z_R}{z}\right)^2\right]^{\frac{1}{2}} \tag{2.10}$$

The parameter $w_0 \in \mathbb{R}$ represents the value of the beam width at $z = 0$ which is also called beam waist size or beam waist radius. Finally, substituting Equation (2.8) in

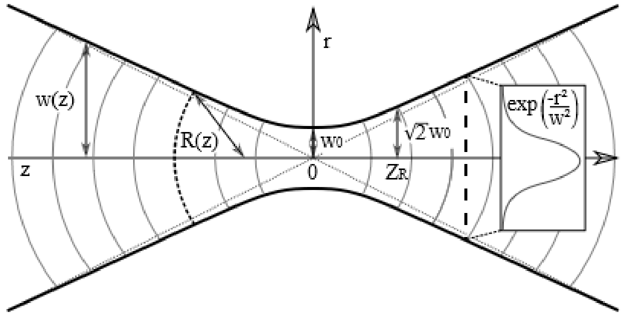

Figure 2.5: Gaussian Beam

Equation (2.7) and using Equation (2.3), we obtain the following complex amplitude $U(r)$ (where $r = (x, y, z)$):

$$U(r) = A_0 \frac{w_0}{W(z)} e^{\left[-\frac{x^2+y^2}{W^2(z)}\right]} e^{\left[-jkz-jk\frac{x^2+y^2}{2R(Z)}+j\xi(z)\right]} \tag{2.11}$$

where $\xi(z) = \tan^{-1}(\frac{z}{z_R})$. The above equation is the main representation of Gaussian beams and describes the important properties of light when it travels from one component to another. For example, the optical intensity, $I(x, y, z) = |U(r)|^2$ can be expressed as follows:

$$I(x, y, z) = |\frac{A_0}{jz_R}|^2 \left(\frac{w_0}{W(z)}\right)^2 e^{\left[-\frac{k\frac{\lambda}{\pi}(x^2+y^2)}{W^2(z)}\right]} \tag{2.12}$$

Note that at each value of z, the intensity is a Gaussian function of the radial distance which leads to the name Gaussian beams.

2.3 ABCD-Law of Beam Transformation

We can completely characterize a Gaussian beam by its q-parameter ($q(z)$), i.e., Equation (2.8) [78]. This provides a convenient way to study the behavior of a Gaussian

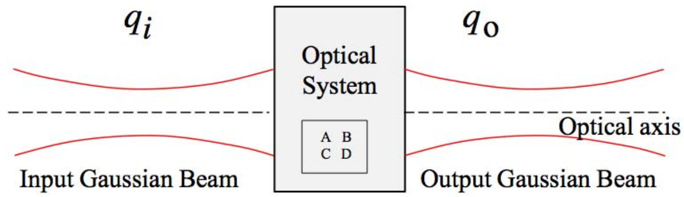

Figure 2.6: Gaussian Beam Transformation

beam when it passes through an optical system. Indeed, it is sufficient to just consider the variations of input q-parameter at each optical component. In paraxial geometrical optics, an optical system is completely characterized by the 2×2 transfer matrix relating the position and inclination of the transmitted ray to the incident ray. Similarly, it is important to find out the effect of an arbitrary optical system (characterized by a matrix M of elements A, B, C, D) on the parameters of an input beam. This can be described by the well-known *ABCD-law* of Gaussian beam transformation [85], given as follows:

$$q_0 = \frac{A.q_i + B}{C.q_i + D} \tag{2.13}$$

where q_i and q_0 represent the input and output beam q-parameters, respectively. The elements A, B, C, and D correspond to the final ray transfer matrix of a geometrical optical system (which indeed represents the composition of the matrices of individual optical components, as shown in Figure 2.6).

The main applications of beam transformation are in the analysis of laser cavities [78], telescopes [70] and the prediction of design parameters for physical experiments [66].

2.4 HOL Light Theorem Prover

HOL Light (an acronym for a lightweight implementation of Higher-Order Logic) [42] is an interactive theorem proving environment for the construction of mathematical proofs. The main implementation of HOL Light is done in Objective CAML (OCaml), which is a functional programming language originally developed to automate mathematical proofs [2]. The main components of the logical kernel of HOL Light (approximately 400 lines of OCaml code) are its types, terms, theorems, rules of inference, and axioms. We present a brief overview of each of them as follows [36]:

- **Types:** The foundation of HOL Light is based on the notion of types and there are only two primitive types, i.e., the Boolean type (`:bool`) and an infinite type (`:ind`). The other types are generated from type variables $:x$; $:y$; $\ldots$ and primitive types (Boolean or infinite) using an arrow $\rightarrow$. For example, `:bool` and `:bool` $\rightarrow$ `x` represent types. Note that a colon (:) is used to specify the corresponding type.

- **Terms:** The terms are the basic objects of HOL Light and their syntax is based on the λ-calculus. We can use λ-terms, also called lambda abstractions, e.g., $\lambda.f(x)$ represents a function which takes x and returns $f(x)$. The collection of terms is constructed from variables x; y; $\ldots$ and constants 0; 1; $\ldots$ using the λ-abstraction ($\lambda x.t$). Each term has a type which can be represented by the notation $x : A$, i.e., the type of term x is A.

- **Inference Rules:** Inference rules are procedures for deriving new theorems. They are represented as OCaml functions. HOL Light has ten inference rules and a mechanism for defining new constants and types. Some of the inference rules are the reflexivity of equality, the transitivity of equality and the fact that equal functions applied to equal arguments are equal [42].

- **Axioms:** The kernel of HOL Light has three mathematical axioms: 1) axiom of extensionality which states that a function is determined by the values that it takes on all inputs; 2) an axiom of infinity which states that the type `:ind` is not finite; and 3) an axiom of choice which states that we can choose a term that satisfies a predicate.

- **Theorems:** A theorem is a formalized statement that is either an axiom or can be deduced from already verified theorems by inference rules. A theorem consists of a finite set Ω of Boolean terms called the assumptions and a Boolean term S called the conclusion. For example, "$\forall x. x \neq 0 \Rightarrow \frac{x}{x} = 1$" represents a theorem in HOL Light.

A HOL Light theory consists of a set of types, constants, definitions, axioms and theorems. HOL theories are organized in a hierarchical fashion and theories can inherit the types, constants, definitions and theorems of other theories as their parents. Proofs in HOL Light are based on the concepts of tactics and tacticals that break goals into simple subgoals. There are many automatic proof procedures and proof assistants available in HOL Light which help the user in directing the proof to the end [42]. We list frequently used HOL Light notations and their corresponding mathematical interpretations in Table 2.1. In the following, we present some examples to show some definitions and theorems in HOL Light.

We consider a function which checks that a given pair (x, y) of real numbers is ordered, i.e., $x > y$. We then use this function to ensure that every element of a list of pairs is an ordered pair. We can define an ordered pair in HOL Light as follows:

Example 2.1.

$\vdash_{def}$ `is_ordered_pair` $=$ `(`λ`((x:real),(y:real)). if x` $>$ `y then T else F)`

where the type of the function `is_ordered_pair` is `(real` $\times$ `real)` $\rightarrow$ `bool`.

We next use this function to define the list of ordered pairs as follows:

Example 2.2.

$\vdash_{def}$ $\forall$ `L. list_of_ordered_pairs L` $\Leftrightarrow$ `ALL is_ordered_pair L`

where the type of the function `list_of_ordered_pairs` is `(real` $\times$ `real)list` $\rightarrow$ `bool`. Note that `ALL` is a HOL Light function from the list theory which ensures that some function holds for every element of that list.

We can use this function to prove that the reverse of a list of ordered pairs remains a list of ordered pairs.

Theorem 2.1.

$\vdash$ $\forall$ `L. list_of_ordered_pairs L` $\Rightarrow$ `list_of_ordered_pairs (REVERSE L)`

where `REVERSE` is a HOL Light function that reverses a given list. The proof of Theorem 2.1 is mainly based on induction on the length of the list `L`.

Theories in HOL Light provide some automated tactics which can prove some intermediate lemmas and reduce the efforts to prove the main theorem. For example, `REAL_FIELD` and `COMPLEX_FIELD` can prove basic field facts over real and complex numbers. In HOL Light, such automation tactics can be composed to perform custom tasks arise during the formalization of a specific theory. We conclude this chapter by providing two examples: one for `REAL_FIELD` and the other for `COMPLEX_FIELD` as follows:

Example 2.3.

```
REAL_FIELD `s pow 2 = b pow 2 - &4 * a * c ⇒
            (a * x pow 2 + b * x + c = &0 ⇔
            if a = &0 then
               if b = &0 then
                  if c = &0 then T else F
               else x = --c / b
            else x = (--b + s) / (&2 * a) ∨
                 x = (--b + --s) / (&2 * a))`;;
```

Example 2.4.

```
COMPLEX_FIELD `∀ (x:complex) u w. ¬(x = u) ∧ ¬(x = w) ⇒
                 Cx(&1) / (x - u) - Cx(&1) / (x - w) =
                 (u - w) / ((x - u) * (x - w))`;;
```

Table 2.1: HOL Light Symbols and Functions

HOL Symbol	**Standard Symbol**	**Meaning**
$\wedge$	and	Logical *and*
$\vee$	or	Logical *or*
$\neg$	not	Logical *negation*
`T`	true	Logical true value
`F`	false	Logical false value
$\Rightarrow$	$\longrightarrow$	Logical *Implication*
$\Leftrightarrow$	$=$	Equality in Boolean domain
$\forall$`x.t`	$\forall x.t$	for all $x : t$
λ`x.t`	$\lambda x.t$	Function that maps x to $t(x)$
`num`	$\{0, 1, 2, \ldots\}$	Positive Integers data type
`real`	All Real numbers	Real data type
`complex`	All complex numbers	Complex data type
`suc n`	$(n + 1)$	Successor of natural number
$--$ `x`	$-x$	Unary negation of x
`exp x`	e^x	Exponential function (real-valued)
`cexp x`	e^x	Exponential function (complex-valued)
`sqrt x`	$\sqrt{x}$	Square root function
`abs x`	$\|x\|$	Absolute function
`a / b`	$\frac{a}{b}$	Division (a and b should have same type)
`a pow b`	a^b	Real or complex power
`Cx a`	$\mathbb{R} \rightarrow \mathbb{C}$	Typecasting from Reals to Complex
&`a`	$\mathbb{N} \rightarrow \mathbb{R}$	Typecasting from Integers to Reals
`A**B`	$[A][B]$	Matrix-Matrix or Matrix-Vector multiplication
`FST`	-	Returns the first element of a pair
`SND`	-	Returns the second element of a pair
[`a`; `b`; $\cdots$]	-	List
`APPEND`	-	Appends two lists
`CONS h t`	-	Appends element h to list t
`LAST`	-	Returns the last element of a list

Chapter 3

Formalization of Ray Optics

In Chapter 2, we provided an introduction to geometrical optics that shows that ray optics is an important formalism to model optical systems and their corresponding properties. This chapter covers in detail the higher-order logic formalization of optical systems and ray optics[6] which is a foundational part in our proposed framework (Figure 1.2). The formalization consists of four parts: 1) fundamental concepts of optical systems structures; 2) formalization of light rays; 3) verification of the ray-transfer matrix model of any arbitrary optical system; and 4) formal development of a component library. We use this infrastructure to formalize the cardinal points of optical imaging systems along with the formal analysis of a visual optical system for human eye.

3.1 Formalization of Optical Systems

Ray optics explains the behavior of light when it passes through a free space and interacts with different optical interfaces. We can model free space by a pair of real

[6]The source codes of the formalizations and proofs presented in this chapter can be found in [87].

numbers (n, d), which are essentially the refractive index and the total width, as shown

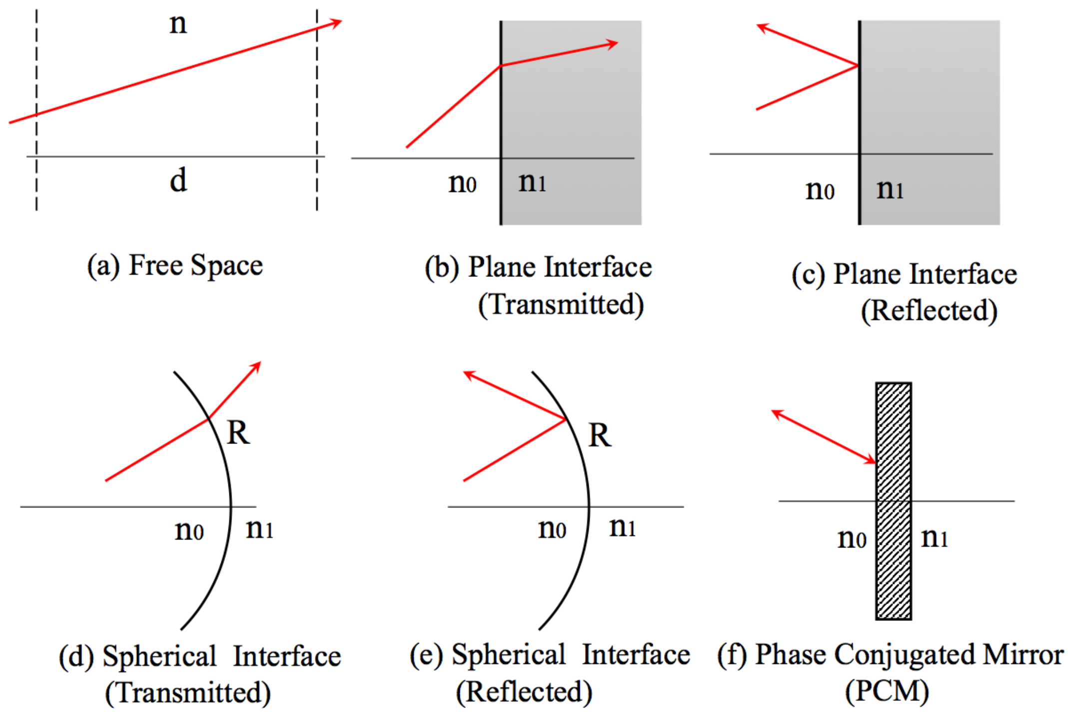

Figure 3.1: Optical Interfaces

in Figure 3.1 (a). We consider five optical interfaces, i.e., plane-transmitted, plane-reflected, spherical-transmitted, spherical-reflected and phase conjugated mirror (PCM) as shown in Figure 3.1 (b)-(f). In geometrical optics, we describe a spherical interface by its radius of curvature (R). In HOL Light, we can use the available types (e.g., Real, Complex, etc.) to abbreviate new types. We use this feature to define a type abbreviation for a free space as follows:

```
new_type_abbrev ("free_space", :real # real);;
```

where `real # real` corresponds to a pair of real numbers $(\mathbb{R} \times \mathbb{R})$.

In many situations, it is convenient to define new types in addition to the ones which are already available in HOL Light theories. One common way is to use enumerated types, where one gives an exhaustive list of members of the new type. In our formalization, we package different optical interfaces in one enumerated type definition to simplify the formal reasoning process. We use HOL Light's `define_type` mechanism to define a new type for optical interface, as follows:

```
interface = plane_transmitted |
            plane_reflected |
            spherical_transmitted real |
            spherical_reflected real |
            pcm |
            unknown complex complex complex complex
```

Note that we also include `unknown` as a part of the type `interface`. The main motivation of considering an unknown element is to tackle the cases where a full description of an optical interface is not known in advance. Moreover, we parameterize the `unknown` element by four complex numbers to consider some prior information such as radius of curvature or aperture, etc. Note that this datatype can easily be extended to many other optical interfaces if needed.

In HOL Light, `define_type` considers the members of the type as constructors and it returns a pair of theorems, one for induction, and one for recursion, as follows:

Theorem 3.1 (Interface Induction)**.**

```
⊢ ∀ P. P plane_transmitted ∧ P plane_reflected ∧
      (∀ a. P (spherical_transmitted a)) ∧
      (∀ a. P (spherical_reflected a)) ∧
       P pcm ∧ (∀ a0 a1 a2 a3. P (unknown a0 a1 a2 a3))
```

where the interface induction theorem states that a property `P` holds for all objects of type `interface` if it holds for all the members of the type `interface`.

Theorem 3.2 (Interface Recursion)**.**

```
⊢ ∀ f0 f1 f2 f3 f4 f5.
   ∃ fn. fn plane_transmitted = f0 ∧ fn plane_reflected = f1 ∧
   (∀ a. fn (spherical_transmitted a) = f2 a) ∧
   (∀ a. fn (spherical_reflected a) = f3 a) ∧
        fn pcm = f4 ∧
   (∀ a0 a1 a2 a3. fn (unknown a0 a1 a2 a3) = f5 a0 a1 a2 a3)
```

where the interface recursion theorem states that given any five values f0, f1, f3, f4 and f5, we can always define a function mapping the five values `plane_transmitted`, `plane_reflected`, `spherical_transmitted`, `spherical_reflected` and `unknown` to those values, respectively.

We model an optical component as a pair of a free space and an optical interface which can be of five different types as shown in Figure 3.1. Consequently, we define an optical system as a list of optical components followed by a free space. In the following, we provide the corresponding formal type abbreviations for optical components and systems:

```
new_type_abbrev("optical_component",`:free_space # interface`);;
new_type_abbrev("optical_system",`
                :optical_component list # free_space`);;
```

Note that the use of a list in the `optical_system` type provides the facility to consider a system with any number of optical components. We formally verify some theorems which state that we can decompose the types `free_space`, `optical_component` and `optical_system` into their constituent components.

Theorem 3.3 (Forall (∀) Theorems for Type Abbreviations)**.**
$\vdash$ `∀P. (∀(fs:free_space). P fs) ⇔ (∀n d. P (n,d))`

$\vdash$ `∀P. (∀(c:optical_component). P c) ⇔ (∀fs i ik. P (fs,i))`

$\vdash$ `∀P. (∀(os:optical_system). P os) ⇔ (∀cs fs. P (cs,fs))`

A value of type `free_space` does represent a real space only if the refractive index is greater than zero. In addition, in order to have a fixed order in the representation of an optical system, we impose that the distance of an optical interface relative to the previous interface is greater or equal to zero. We encode this requirement in the following predicate:

Definition 3.1 (Valid Free Space)**.**

$\vdash_{def}$ `is_valid_free_space (n,d) ⇔ 0 < n ∧ 0 ≤ d`

where the type of `is_valid_free_space` is : `free_space` → `bool`.

We also need to assert the validity of a value of type `interface` by ensuring that the radius of curvature of spherical interfaces is never equal to zero. This yields the following predicate:

Definition 3.2 (Valid Optical Interface)**.**
$\vdash_{def}$ `(is_valid_interface plane_transmitted ⇔ T) ∧`

`(is_valid_interface plane_reflected ⇔ T) ∧`

`(is_valid_interface (spherical_transmitted R) ⇔ ¬(0 = R)) ∧`

`(is_valid_interface (spherical_reflected R) ⇔ ¬(0 = R)) ∧`

`(is_valid_interface pcm ⇔ T) ∧`

`(is_valid_interface (unknown a b c d) ⇔ T)`

where the type of `is_valid_interface` is : `interface` → `bool`.

We now assert the validity of an optical system structure by ensuring the validity of every optical component in a system, as follows:

Definition 3.3 (Valid Optical Component).
$\vdash_{def}$ $\forall$ `fs i.`

`is_valid_optical_component (fs,i)` $\Leftrightarrow$

`is_valid_free_space fs` $\wedge$ `is_valid_interface i`

Definition 3.4 (Valid Optical System).
$\vdash_{def}$ $\forall$ `cs fs.`

`is_valid_optical_system (cs,fs)` $\Leftrightarrow$

`ALL is_valid_optical_component cs` $\wedge$ `is_valid_free_space fs`

where `ALL` is a HOL Light library function which checks that a predicate holds for all the elements of a list.

We conclude our formalization of optical systems by defining a function to retrieve the refractive index of the first free space in an optical system:

Definition 3.5 (Head Index).
$\vdash_{def}$ `head_index ([],n,d) = n` $\wedge$

`head_index (CONS ((n,d),i) cs,nt,dt) = n`

where `[]` represents an empty list of optical components.

3.2 Formalization of Light Rays

One of the important requirements for the formal analysis of optical systems is the formalization of rays which can specify the physical behavior of the light when it passes through an optical system. We only model the points where it hits an optical interface (instead of modeling all the points constituting the ray). So it is sufficient to just provide the distance of each of these hitting points to the optical axis and the angle taken by the ray at these points as shown in Figure 3.2. Consequently, we should have a list of such pairs $(distance, angle)$ for every component of a system.

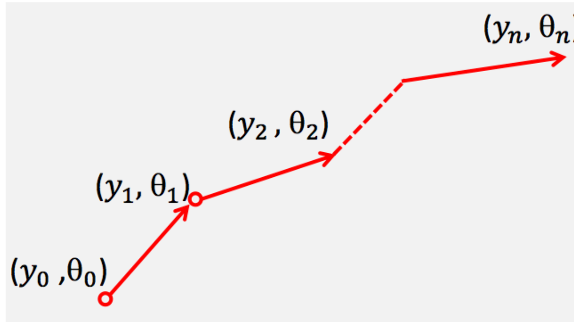

Figure 3.2: Ray Model as Sequence of Points

In addition, the same information should be provided for the source of the ray. For the sake of simplicity, we define a type for a pair $(distance, angle)$ as `ray_at_point` as follows:

```
new_type_abbrev("ray_at_point",`:real # real`);;
new_type_abbrev("ray",
  `:ray_at_point # ray_at_point # (ray_at_point # ray_at_point) list`);;
```

where the first `ray_at_point` is the pair $(distance, angle)$ for the source of the ray, the second one is the one after the first free space, and the list of `ray_at_point` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system.

Once again, we specify what is a valid ray by using some predicates. First of all, we define the behavior of a ray when it is traveling through a free space. This requires the position and orientation of the ray at the previous and current points of observation, and the free space itself. This is shown in Figure 3.1(a).

Definition 3.6 (Behavior of a Ray in Free Space)**.**

$\vdash$ `is_valid_ray_in_free_space` $(\mathtt{y}_0,\theta_0)$ $(\mathtt{y}_1,\theta_1)$ `((n,d):free_space)` $\Leftrightarrow$

$\mathtt{y}_1 = \mathtt{y}_0 + \mathtt{d} * \theta_0 \wedge \theta_0 = \theta_1$

We next define the valid behavior of a ray when hitting a particular interface. This requires the position and orientation of the ray at the previous and current interfaces, and the refractive indices before and after the component. Then the predicate is defined by case analysis on the interface type as follows:

Definition 3.7 (Behavior of a Ray at Given Interface)**.**

$\vdash$ C1:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1

plane_transmitted $\Leftrightarrow$ y_1 = y_0 $\wedge$ n_0 * θ_0 = n_1 * θ_1) $\wedge$

C2:(is_valid_ray_at_interface (y_0,θ_0) (y1,θ_1) n_0 n_1

(spherical_transmitted R) $\Leftrightarrow$ let ϕ_i= θ_0 + $\frac{y_1}{R}$ and ϕ_t = θ_1 + $\frac{y_1}{R}$ in

y_1 = y_0 $\wedge$ n_0 * ϕ_i = n_1 * ϕ_t) $\wedge$

C3:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1

plane_reflected $\Leftrightarrow$ y_1 = y_0 $\wedge$ n_0 * θ_0 = n_0 * θ_1) $\wedge$

C4:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1

(spherical_reflected R) $\Leftrightarrow$ let ϕ_i = $\frac{y1}{R}$ - θ_0 in y_1 = y_0 $\wedge$

θ_1 = -(θ_0 + 2 * ϕ_i))$\wedge$

C5:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 pcm

$\Leftrightarrow$ y_1 = y_0 $\wedge$ θ_1 = -θ_0) $\wedge$

C6:(is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1

(unknown a b c d) $\Leftrightarrow$ y_1 = a*y_0 + b*θ_0 $\wedge$ θ_1 = c*y_0 + d*θ_0)

where each case C1-C6 states some basic geometrical facts about the distance to the axis, and applies paraxial Snell's law and the law of reflection [85] to the orientation of the ray as shown in Figure 3.1.

Finally, we can recursively apply these predicates to define the behavior of a ray going through a series of optical components in an arbitrary optical system, given as follows:

Definition 3.8 (Valid Ray Behavior in an Optical System).

$\vdash \forall$ $\mathtt{sr_1}$ $\mathtt{sr_2}$ h h' fs cs rs i $\mathtt{y_0}$ θ_0 $\mathtt{y_1}$ θ_1 $\mathtt{y_2}$ θ_2 $\mathtt{y_3}$ θ_3 n d n' d'.

C1 : (is_valid_ray_in_system ($\mathtt{sr_1}$,$\mathtt{sr_2}$,[]) (CONS h cs,fs) $\Leftrightarrow$ F) $\wedge$

C2 : (is_valid_ray_in_system ($\mathtt{sr_1}$,$\mathtt{sr_2}$,CONS h' rs) ([],fs) $\Leftrightarrow$F)$\wedge$

C3 : (is_valid_ray_in_system (($\mathtt{y_0}$,θ_0),($\mathtt{y_1}$,θ_1),[]) ([],n,d) $\Leftrightarrow$

is_valid_ray_in_free_space ($\mathtt{y_0}$,θ_0) ($\mathtt{y_1}$,θ_1) (n,d)) $\wedge$

C4 : (is_valid_ray_in_system (($\mathtt{y_0}$,θ_0),($\mathtt{y_1}$,θ_1),

CONS (($\mathtt{y_2}$,θ_2),$\mathtt{y_3}$,θ_3) rs) (CONS ((n',d'),i,ik) cs,n,d) $\Leftrightarrow$

(is_valid_ray_in_free_space ($\mathtt{y_0}$,θ_0) ($\mathtt{y_1}$,θ_1) (n',d') $\wedge$

is_valid_ray_at_interface ($\mathtt{y_1}$,θ_1) ($\mathtt{y_2}$,θ_2) n'

(head_index (cs,n,d)) i)) $\wedge$

(is_valid_ray_in_system (($\mathtt{y_2}$,θ_2),($\mathtt{y_3}$,θ_3),rs) (cs,n,d))

where the first two cases (C1 and C2) describe the two situations where the length of the ray and optical system are not the same. The case C3 describes the situation when the optical system only consists of a free space. The last case recursively ensures the valid behavior of ray at each interface of the optical system. The behavior of a ray going through a series of optical components is thus completely defined.

3.3 Ray-Transfer Matrices of Optical Components

The main strength of ray optics is its matrix formalism [85], which provides an efficient way to model all optical components in the form of a matrix. Indeed, a matrix relates the input and the output ray by a linear relation. For example, in case of free space, the input and output ray parameters are related by two linear equations, i.e., $y_1 = y_0 + d * \theta_0$ and $\theta_1 = \theta_0$, which further can be described as a matrix (also called ray-transfer matrix of free space). We verify this ray-transfer-matrix of free space as follows:

Theorem 3.4 (Ray-Transfer-Matrix for Free Space).

$\vdash \forall$ `fs` $y_0\ \theta_0\ y_1\ \theta_1$. `is_valid_free_space fs` $\wedge$

`is_valid_ray_in_free_space` $(y_0,\theta_0)\ (y_1,\theta_1)$ `fs)` $\Rightarrow$

$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \texttt{free\_space\_matrix fs} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

The first assumption ensures the validity of free space and the second assumption ensures the valid behavior of ray in free space. The proof of Theorem 3.4 is mainly based on the rewriting with the definitions (e.g., `is_valid_free_space`) and properties of matrices. We prove the ray-transfer matrices of all optical interfaces (Definition 3.3), as listed in Table 3.1. The availability of these theorems in our formalization is quite handy as it helps to reduce the interactive verification efforts for the applications. The proof steps for these theorems are quite similar to each other and mainly require rewriting with some properties of vectors and matrices. In order to make the proof of these theorems automatic, we build a tactic `common_prove` which is mainly based on the simplification with the above mentioned definitions and the application of matrix operations.

Out next goal is to formally prove that any optical interface can be described by a general ray-transfer-matrix relation. Mathematically, this relation is described in the following theorem:

Theorem 3.5 (Ray-Transfer-Matrix any Interface).

$\vdash \forall\ n_0\ n_1\ y_0\ \theta_0\ y_1\ \theta_1$ `i`. `is_valid_interface i` $\wedge$

`is_valid_ray_at_interface` $(y_0,\theta_0)\ (y_1,\theta_1)\ n_0\ n_1$ `i` $\wedge$

$$0 < n_0 \wedge 0 < n_1 \Rightarrow \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \texttt{interface\_matrix}\ n_0\ n_1\ \texttt{i} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

Table 3.1: Ray-Transfer Matrices of Optical Components

Component	**HOL Light Formalization**
Plane Interface (Reflection)	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 plane_reflected $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$
Plane Interface (Transmission)	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 plane_transmitted $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n_0}{n_1} \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$
Spherical Interface (Reflection)	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1 R. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ (is_valid_interface (spherical_reflected R) $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 (spherical_reflected R) $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$
Spherical Interface (Transmission)	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1 R. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ (is_valid_interface (spherical_transmitted R) $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 (spherical_transmitted R) $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{R * n_1} & \frac{n_0}{n_1} \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$
Phase Conjugated Mirror	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 pcm $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$
Unknown	$\vdash \forall$ n d y_0 θ_0 y_1 θ_1 a b c d. 0 < n_0 $\wedge$ 0 < n_1 $\wedge$ is_valid_ray_at_interface (y_0,θ_0) (y_1,θ_1) n_0 n_1 (unknown a b c d) $\Rightarrow$ $\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} ** \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$

where interface_matrix accepts the refractive indices ($\mathtt{n_0}$ and $\mathtt{n_1}$) and interface (i), and returns corresponding matrix of the system. In the above theorem, both assumptions ensure the validity of the interface (i) and behavior of ray at each interface, respectively. We prove this theorem using the case splitting on interface i.

3.4 Formalization of Composed Optical Systems

We can trace the input ray R_i through an optical system consisting of n optical components by the composition of ray-transfer matrices of each optical component as described in Equation 2.2. It is important to note that in this equation, individual matrices of optical components are composed in a reverse order. We formalize this fact with the following recursive definition:

Definition 3.9 (Optical System Model).

```
⊢def system_composition ([],fs) = free_space_matrix fs ∧
    system_composition (CONS ((n′,d′),i) cs,n,d) =
    system_composition (cs,n,d) **
    interface_matrix n′ (head_index (cs,n,d)) i **
    free_space_matrix (n′,d′)
```

where the type of system_composition is : optical_system $\rightarrow \mathbb{R}^{2\text{x}2}$, i.e., it takes an optical systems and returns a (2×2) matrix. The function system_composition is defined by two cases, i.e., if an optical system consists of only free space, it returns the corresponding matrix and if an optical system consists of a list of optical components (cs), it returns the product of corresponding matrices in a reversed order. Here, (n′, d′) represents the second free space in the system.

Our next goal is to verify the generalized ray-transfer-matrix relation for an arbitrary optical system which is valid for any optical and ray. We verify this relation in the following theorem:

Theorem 3.6 (Ray-Transfer-Matrix for Optical System).

$\vdash$ $\forall$ `sys ray.  is_valid_optical_system sys` $\wedge$

`is_valid_ray_in_system ray sys` $\Rightarrow$

`let` $(y_0,\theta_0),(y_1,\theta_1)$`,rs = ray in`

`let` y_n,θ_n `= last_ray_at_point ray in`

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition sys **} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where the parameters `sys` and `ray` represent the optical system and the ray, respectively. The function `last_ray_at_point` returns the last `ray_at_point` of the ray in the system. Both assumptions in the above theorem ensure the validity of the optical system and the good behavior of the ray in the system. We prove this theorem using induction on the length of the system and by using previous results and definitions.

The above described model and corresponding ray-transfer matrix relation only hold for a single optical system consisting of different optical components. Our main requirement is to extend this model for a general system which is composed of n optical subsystems as shown in Figure 3.3. We formalize the notion of a composed optical system as follows:

Definition 3.10 (Composed Optical System Model).

$\vdash_{def}$ `composed_system [] = I` $\wedge$

`composed_system (CONS sys cs) =`

`composed_system cs ** system_composition sys`

where `I` represents the identity matrix and the function `composed_system` accepts a list of optical systems `:(optical_system)list` and returns the overall system model by the recursive application of the function `system_composition` (Definition 3.9).

We define the validity of a composed optical system by ensuring the validity of each involved optical system as follows:

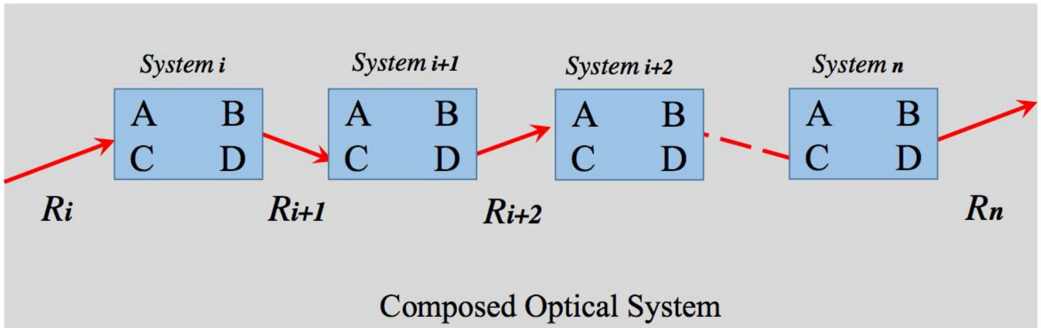

Figure 3.3: Ray Propagation through Composed Optical Systems

Definition 3.11 (Valid Composed Optical System).

$\vdash_{def}$ $\forall$ `sys:(optica_system)list.`

`is_valid_composed_system sys` $\Leftrightarrow$ `ALL is_valid_optical_system sys`

In order to reason about composed optical systems, we need to give some new definitions about the ray behavior inside a composed optical system. One of the easiest ways is to consider n rays corresponding to n optical systems individually and then make sure that each ray is the same as the one applied at the input. This can be done by ensuring that the starting point of each ray is equal to the ending point of the previous ray as shown in Figure 3.3. We encode this physical behavior of ray as follows:

Definition 3.12 (Valid General Ray).

$\vdash_{def}$ `(is_valid_genray []` $\Leftrightarrow$ `F)` $\wedge$

`(is_valid_genray (CONS h t)` $\Leftrightarrow$

`last_single_ray h = fst_single_ray (HD t)` $\wedge$

`is_valid_genray t)`

where fst_single_ray, last_single_ray and HD provide the first and last single ray at a point and the first element of a list, respectively. On the similar lines, we also specify the behavior of a ray when it passes through each optical systems by a function is_valid_gray_in_system. Finally, we verify that the ray-transfer-matrix relation holds for composed optical systems which ensures that all valid properties for a single optical system can be generalized to a composed system as well.

Theorem 3.7 (Ray-Transfer-Matrix for Composed Optical System)**.**

$\vdash$ $\forall$ sys cray.

is_valid_composed_system sys $\wedge$

is_valid_gray_in_system cray sys $\wedge$

is_valid_genray cray $\Rightarrow$

let (y_0, θ_0) = fst_single_ray (HD cray) in

let (y_n, θ_n) = last_single_ray (LAST cray) in

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{composed\_system sys **} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

where sys and cray represent a list of optical systems ((optical_system)list) and a list of rays ((ray)list), respectively.

This concludes our formalization of optical system structures and rays along with the verification of important properties of optical components and optical systems.

3.5 Optical Imaging and Cardinal Points

Optical systems capable of being utilized for imaging (can record or transform objects to an image) are called optical imaging systems. Mainly these systems are divided into two main categories, i.e., mirror-systems (also called *catoptrics*, which deal with reflected light rays) and lens-systems (also called *dioptrics*, which deal with refracted light rays). Examples of such systems are optical fibers and telescopes, for the first and second case, respectively.

An optical imaging system has many cardinal points which are required to analyze imaging properties (e.g., image size, location, and orientation, etc.) of the optical systems. These points are the *principal points*, the *nodal points* and the *focal points*, which are situated on the optical axis. Figure 3.4 describes a general optical imaging system with an object point P_0 with a distance x_0 from the optical axis (called the object height). The image is formed by the optical system at point P_1 with a distance x_1 from the optical axis (called the image height). The refractive indices of object space and image space are n and n', respectively. The points F and F' are the foci in the object space and the image space, respectively. The points N and N' are the nodal points in the object and image space. Finally, the points U and U' are the unit or principal points in the object and image space, respectively [85].

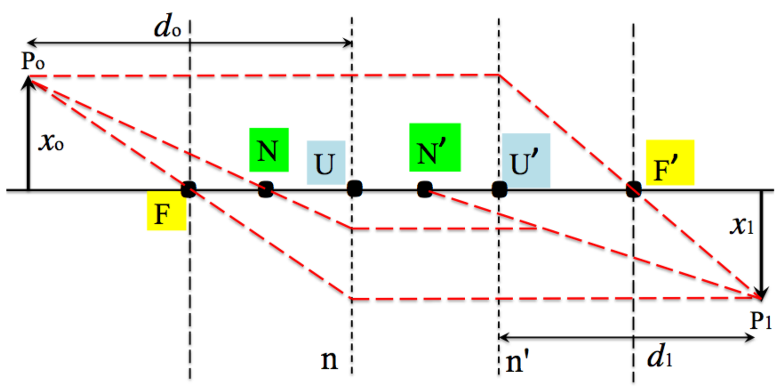

Figure 3.4: Cardinal Points of an Optical System [85]

3.5.1 Formalization of Cardinal Points

We consider a general optical imaging system as shown in Figure 3.5. In this context, the first and the last points of the ray represent the locations of the object and image. As shown in Figure 3.5, the object (P_0) is located at a distance of d_0 from the optical system and image (P_1) is formed at the distance of d_n. The object and image heights are y_0 and y_n, respectively. The ratio of the image height to the object height is called *lateral magnification* which is usually denoted by β. A ray in the object space, which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space in the nodal point N' at the same angle θ'. The ratio of θ and θ' is called *angular magnification*. In our formalization, this corresponds to the angle of the first single and last single ray, respectively. For the sake of generality, we formalize the general notion of optical systems as shown in Figure 3.5, as follows:

Definition 3.13 (General Optical System Model).

$\vdash_{def}$ ∀ni do sys nt dn.

```
    gen_optical_system sys do dn ni nt = [[],ni,do; sys; [],nt,dn]
```

Here, the overall system consists of 3 sub-systems, i.e., free space with (`ni`, `d0`), a general system `sys` and another free space (`nt`, `dn`).

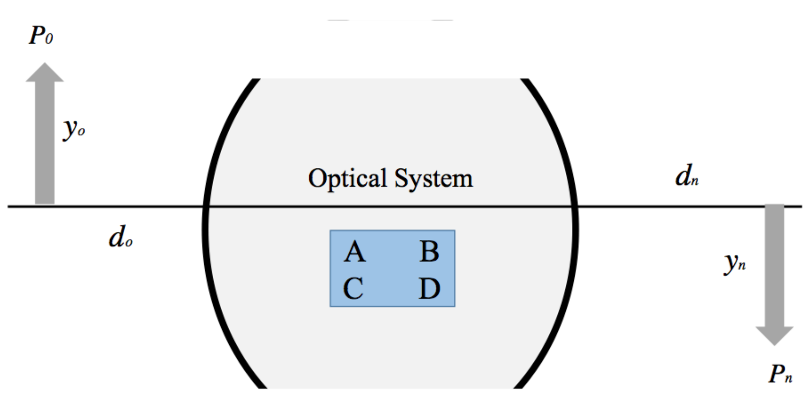

Figure 3.5: General Optical System

Our next step is to verify the ray-transfer matrix relation of general optical systems by using Theorem 3.7, as follows:

Theorem 3.8 (Matrix for General Optical System).

$\vdash \forall$ `sys cray` $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$ `A B C D.`

`is_valid_optical_system sys` $\wedge$ `0 <` $\mathtt{n_i}$ $\wedge$ `0 <` $\mathtt{n_t}$ $\wedge$

`is_valid_genray cray` $\wedge$

`system_composition sys =` $\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix} \wedge$

`is_valid_gray_in_system (gen_optical_system sys` $\mathtt{d_0}$ $\mathtt{d_n}$ $\mathtt{n_i}$ $\mathtt{n_t}$`)` $\Rightarrow$

`let (`$\mathtt{y_0}$`,`θ_0`) = fst_single_ray (HD cray) and`

`(`$\mathtt{y}_n$`,`θ_n`) = last_single_ray (LAST cray) in`

$$\begin{bmatrix} \mathtt{y_n} \\ \theta_{\mathtt{n}} \end{bmatrix} = \begin{bmatrix} \mathtt{A} + \mathtt{Cd_n} & (\mathtt{Ad_0} + \mathtt{B} + \mathtt{Cd_0d_n} + \mathtt{Dd_n}) \\ \mathtt{C} & \mathtt{Cd_0} + \mathtt{D} \end{bmatrix} ** \begin{bmatrix} \mathtt{y_0} \\ \theta_0 \end{bmatrix}$$

Next, we formalize the notion of image and object height, image and object angle, lateral and angular magnification, as follows:

Definition 3.14 (Lateral and Angular Magnification).

$\vdash \forall$ `ray.object_height ray = FST (fst_single_ray (HD ray))`

$\vdash \forall$ `ray.image_height ray = FST (last_single_ray (LAST ray))`

$\vdash \forall$ `ray.object_angle ray = SND (fst_single_ray (HD ray))`

$\vdash \forall$ `ray.image_angle ray = SND (last_single_ray (LAST r))`

$\vdash \forall$ `ray.lateral_magnification ray` $= \dfrac{\texttt{image\_height ray}}{\texttt{object\_height ray}}$

$\vdash \forall$ `ray.angular_magnification ray` $= \dfrac{\texttt{image\_angle ray}}{\texttt{object\_angle ray}}$

where `object_height` and `image_height` accept a `ray` and return the lateral distance of the image and object from the optical axis, respectively. Similarly, `image_angle` and object_angle return the image and object angles, respectively.

The location of all the cardinal points can be found on the optical axis as shown in Figure 3.4. In case of general optical systems (Figure 3.5), these can be defined using the distances $\mathtt{d_o}$ and $\mathtt{d_n}$, by developing some constraints.

Principal Points

In order to find principal points, the image has to be formed at the same height as of the object in the object space, i.e., the lateral magnification should be one. This means that all the rays, starting from a certain height, will have the same height regardless of the incident angle. Mathematically, this leads to the fact that the second element of the 2×2 matrix, representing the optical system has to be 0. We package these constraints into the following predicate:

Definition 3.15 (Principal Points Specification)**.**

```
⊢ ∀ (sys:optical_system list).
  principal_points_spec sys ⇔
  (∀ ray.is_valid_gray_in_system ray sys ∧ is_valid_genray ray ⇒
  (let M = composed_system sys and
       yn = image_height ray and
       y0 = object_height ray in
       y0 ≠ 0 ∧ M(2,1) ≠ 0 ⇒
       M(1,2) = 0 ∧ lateral_magnification ray = 1))
```

The function principal_points_spec accepts an arbitrary composed system sys and ensures that for any ray, the constraints hold as described above. Here, $\mathtt{M}_{(i,j)}$ represents the elements of a square matrix M. Now we can define the principle points as the pair of points (dU,dU') which satisfy the above constraints as follows:

Definition 3.16 (Principle Points of a System)**.**

$\vdash \forall$ (sys:optical_system list) dU dU' $\mathtt{n_i}$ $\mathtt{n_t}$.

principal_points (dU,dU') sys ni nt $\Leftrightarrow$

principal_points_spec (gen_optical_system sys dU dU' $\mathtt{n_i}$ $\mathtt{n_t}$)

We use the reasoning support developed in the last section to prove the analytical expressions for the principal points of the general optical system described in Figure 3.5.

Theorem 3.9 (Principal Points of a General Optical System)**.**

$\vdash \forall \mathtt{n_i}$ $\mathtt{n_t}$ sys.

is_valid_optical_system sys $\wedge$ 0 < $\mathtt{n_i}$ $\wedge$ 0 < $\mathtt{n_t}$ $\Rightarrow$

let M = system_composition sys in

(principle_points

$((\frac{\mathtt{M}_{(2,2)}}{\mathtt{M}_{(2,1)}} * (\mathtt{M}_{(1,1)} - 1) - \mathtt{M}_{(1,2)}), (\frac{1-\mathtt{M}_{(1,1)}}{\mathtt{M}_{(2,1)}}))$ sys $\mathtt{n_i}$ $\mathtt{n_t}$)

Nodal Points

The second cardinal points of an optical system are the nodal points N (in the object space) and N' (in the image space) as shown in Figure 3.4. A ray in the object space which intersects the optical axis in the nodal point N at an angle θ intersects the optical axis in the image space at the nodal point N' at the same angle θ', which implies that the angular magnification should be 1. We encode these constraints as follows:

Definition 3.17 (Nodal Points Specification)**.**

$\vdash \forall$ (sys:optical_system list).

nodal_points_spec sys $\Leftrightarrow$

($\forall$ ray.is_valid_gray_in_system ray sys $\wedge$ is_valid_genray ray $\Rightarrow$

```
(let M = composed_system sys and
     y0 = object_height ray and
     yn = image_height ray and
     θ0 = object_angle ray and
     θn = image_angle ray in
     y0 = 0 ∧ yn = 0 ∧ θ0≠ 0∧ M(2,1) ≠ 0 ⇒
     M(1,2) = 0 ∧ angular_magnification ray = 1))
```

The function `nodal_points_spec` accepts an arbitrary composed system `sys` and ensures that for any ray the constraints hold as described above. Consequently, we can define the nodal points as the pair of points (dN,dN') which satisfies the above constraints as follows:

Definition 3.18 (Nodal Points of a System)**.**

```
⊢ ∀ (sys:optical_system list) dN dN' ni nt.
  nodal_points (dN,dN') sys ni nt⇔
  nodal_points_spec (gen_optical_system sys dN dN' ni nt)
```

The corresponding analytical expressions for the Nodal points of a general optical system are proved in following theorem:

Theorem 3.10 (Nodal Points of General System)**.**

$\vdash \forall\ \mathtt{n_i}\ \mathtt{n_t}\ \mathtt{sys}.$

$\mathtt{is\_valid\_optical\_system\ sys} \wedge 0 < \mathtt{n_i} \wedge 0 < \mathtt{n_t} \Rightarrow$

$\mathtt{let\ \ M = system\_composition\ sys\ in}$

$\mathtt{(nodal\_points}\ ((\frac{1-\mathtt{M}_{(2,2)}}{\mathtt{M}_{(2,1)}}),(\frac{\mathtt{M}_{(1,1)}}{\mathtt{M}_{(2,1)}} * (\mathtt{M}_{(2,2)}-1)-\mathtt{M}_{(1,2)}))\ \mathtt{sys\ n_i\ n_t})$

Focal Points

The focal points F (in the object space) and F' (in the image space) have two properties: A ray starting from the focus F in the object space is transformed into a ray which is parallel to the optical axis in the image space. Similarly, a ray which is parallel to the optical axis in the object space intersects the focus F' in the image space. We define the following predicate using the above description:

Definition 3.19 (Focal Points Specification)**.**

$\vdash$ $\forall$ (sys:optical_system list).

focal_points_spec sys $\Leftrightarrow$

($\forall$ ray.is_valid_gray_in_system ray sys $\wedge$ is_valid_genray ray $\Rightarrow$

(let M = composed_system sys and

y_0 = object_height ray and

y_n = image_height ray and

θ_0 = object_angle ray and

θ_n = image_angle ray in

$M_{(2,1)} \neq 0 \Rightarrow$

(θ_n = 0 $\wedge$ y_0 = 0 $\Rightarrow$ $M_{(1,1)} \neq 0$) $\wedge$

(θ_0 = 0 $\wedge$ y_n = 0 $\Rightarrow$ $M_{(2,2)} \neq 0$)

Finally, we can define the focal points (dF,dF') as follows:

Definition 3.20 (Focal Points of a System)**.**

$\vdash$ $\forall$(sys:optical_system list) dF dF' n_i n_t.

focal_points (dF,dF') sys ni nt$\Leftrightarrow$

focal_points_spec (gen_optical_system sys dF dF' n_i n_t)

We also verify the corresponding analytical expressions for the focal points in the following theorem:

Theorem 3.11 (Focal Points of General System).

$\vdash \forall$ $\texttt{n}_\texttt{i}$ $\texttt{n}_\texttt{t}$ `sys.  is_valid_optical_system sys` $\wedge$ `0 <` $\texttt{n}_\texttt{i}$ $\wedge$ `0 <` $\texttt{n}_\texttt{t}$ $\Rightarrow$

`let  M = system_composition sys in`

`(focal_points` $((\frac{-\texttt{M}_{(2,2)}}{\texttt{M}_{(2,1)}}),(\frac{-\texttt{M}_{(1,1)}}{\texttt{M}_{(2,1)}}))$ `sys` $\texttt{n}_\texttt{i}$ $\texttt{n}_\texttt{t}$`)`

This completes the formalization of cardinal points of optical systems. Theorems 3.9, 3.10 and 3.11 are powerful results as they simplify the calculation of cardinal points to just finding an equivalent matrix of the given optical system.

3.5.2 Ray Optics Component Library

In this section, we present the summary of the formal verification of the cardinal points of widely used optical components. Generally, lenses are characterized by their refractive indices, thickness and radius of curvature in case of a spherical interface. Some of the components are shown in Figure 3.6, i.e., refracting spherical interface, thick lens, ball lens and plano convex lens. Note that all of these components are composed of two kinds of interfaces, i.e., plane or spherical and free spaces of different refractive indices and widths. We use the infrastructure developed in the previous sections to formalize these components and verify the transfer-matrix relation for each model. Consequently, we can easily derive the cardinal points using already verified theorems. Here, we only present the formalization of thick lens and the verification of its principal points. A thick lens is a composition of two spherical interfaces separated by a distance d as shown in Figure 3.6 (b). We formalize thick lenses as follows:

Definition 3.21 (Thick Lens).

$\vdash \forall$ $\texttt{R}_1$ $\texttt{R}_2$ $\texttt{n}_1$ $\texttt{n}_2$ `d.  thick_lens` $\texttt{R}_1$ $\texttt{R}_2$ $\texttt{n}_1$ $\texttt{n}_2$ `d =`

`([(`$\texttt{n}_1$`,0),spherical_transmitted` $\texttt{R}_1$`;`

`(`$\texttt{n}_2$`,d),spherical_transmitted` $\texttt{R}_2$`],(`$\texttt{n}_1$`,0))`

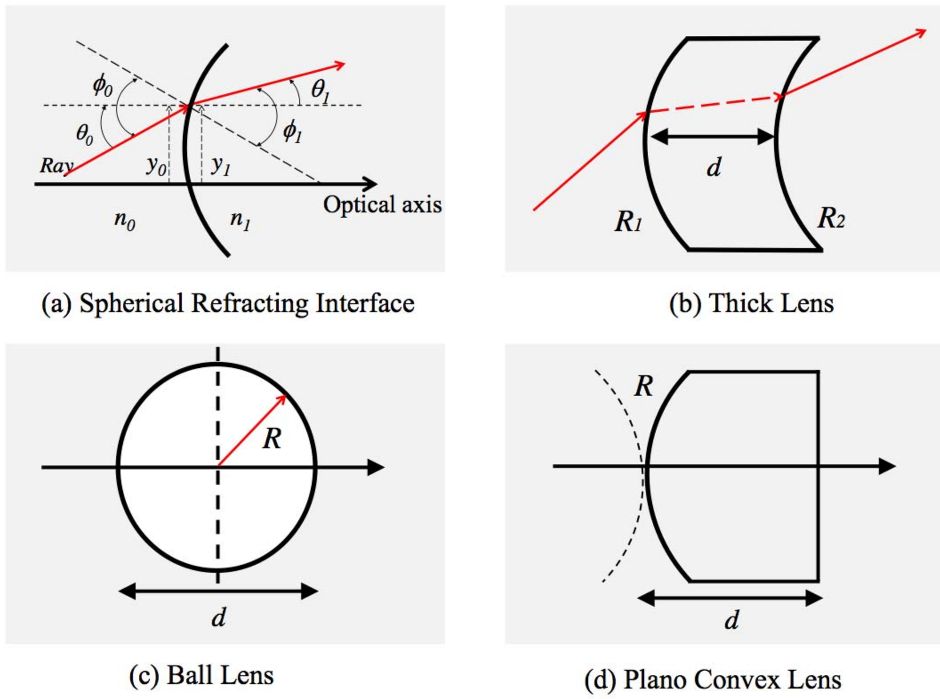

Figure 3.6: Frequently used Optical Components [85]

where $\mathtt{n_1}$ represents the refractive index before and after the first and the second interface, respectively. Whereas $\mathtt{n_2}$ represents the refractive index between the two spherical interfaces which have the radius of curvatures $\mathtt{R_1}$ and $\mathtt{R_2}$, respectively.

We then verify the general expression for the principal points of a thick lens in the following theorem.

Theorem 3.12 (Principal Points of Thick Lens)**.**

$\vdash\forall\ \mathtt{R_1\ R_2\ n_0\ n_1\ d.}\ \ \mathtt{R_1} \neq \mathtt{0} \wedge \mathtt{R_2} \neq \mathtt{0} \wedge \mathtt{0} < \mathtt{n_1} \wedge \mathtt{0} < \mathtt{n_2} \wedge$

$\quad \mathtt{(d * (n_1 - n_2)} \neq \mathtt{-n_2 * (R_1 - R_1))} \Rightarrow$

$\quad \mathtt{(let\ dU = (n * d * R1) / (n_2 * (R_2 - R_1) + (n_2 - n_1) * d)\ and}$

$\qquad \mathtt{dU' = -(n * d * R_2) / (n_2 * (R_2 - R_1) + (n_2 - n_1) * d)\ in}$

$\quad \mathtt{principal\_points\ (dU,dU')\ (thick\_lens\ R1\ R2\ n_1\ n_2\ d)\ n_1\ n_1)}$

Here, the first four assumptions are required to verify the validity of the thick lens structure and the last assumption specifies the condition about thick lens parameters which is required to verify the principal points `dU` and `dU'`. Similarly, we verify the principal points for other optical component as given in Table 3.2. Moreover, we also formalize some other optical components such as thin lens and parallel plate where complete details can be found in the source code [87].

Table 3.2: Principal Points of Some Optical Components

Optical Component	**Principal Points**
Spherical Interface (Transmitted)	$dU = 0 \wedge dU' = 0$
Spherical Interface (Reflected)	$dU = 0 \wedge dU' = 0$
Ball Lens	$dU = -R \wedge dU' = -R$
Meniscus Lens	$dU = \frac{R}{n_L - 1} \wedge dU' = -\frac{R}{n_L - 1}$
Plano Convex Lens	$dU = 0 \wedge dU' = -\frac{d}{n_L}$

3.6 Application: Formal Modeling and Analysis of a Visual Optical System

Human eye is a complex optical system which processes light rays through different biological layers such as cornea, iris and crystalline lens which is located directly behind the pupil. There are different eye diseases; some of them are age related and others are caused by the malfunctioning of some tissues inside the eye. Myopia (or near-sightedness) is a commonly found eye disease which is caused due to the wrong focus of the incoming light inside the eye. In general, myopia is considered as a significant issue due to its high prevalence and the risk for vision-threatening conditions as described in the guidelines by the American Optometric Association [1].

The most commonly used method to avoid this problem is by the use of corrective lenses or eye surgery [1]. Mathematically, different conditions for myopia can be analyzed using geometrical optics and cardinal points [37]. In our work, we consider a general description of a visual optical system of the eye and an optical instrument, as proposed in [37]. In particular, the authors derived the expressions for the axial locations of cardinal points using a paper-and-pencil based approach. However, we intend to model the proposed system in HOL and perform the analysis using our formalized theory of cardinal points. An outline of the complete system in shown in Figure 3.7. The visual optical system of an eye is described by S and an optical device is represented by S_D. The parameter S_G is a homogeneous gap of length z_G between S_D and the eye, S_E is the combination of S_D and S_G. Similarly, S_C is the combination of S_E and S. The points Q_0 and Q_1 are the incident and emergent special points of S and Q_{C0} and Q_{C1} are the corresponding cardinal points (can be either principal, nodal and focal points) of S_C. When we place S_D in front of the eye, it causes Q_0 to be displaced at Q_{C0} and Q_1 at Q_{C1}. The parameters n_0 and n_1 represent the refractive indices. In this design, the entrance plane T_0 is located immediately anterior to the first surface of the tear layer on the cornea and the exit plane T_1 is located immediately anterior to the retina of the eye. Our main goal is to formally derive the cardinal points for this systems description. We proceed by the formal model which consists of three main subsystems:

- The visual optical system of the eye S.
- Homogeneous distance S_G: it can be modeled using a free space of width z_G.
- Any corrective optical device S_D: it can be a contact lens or some surgical equipment.

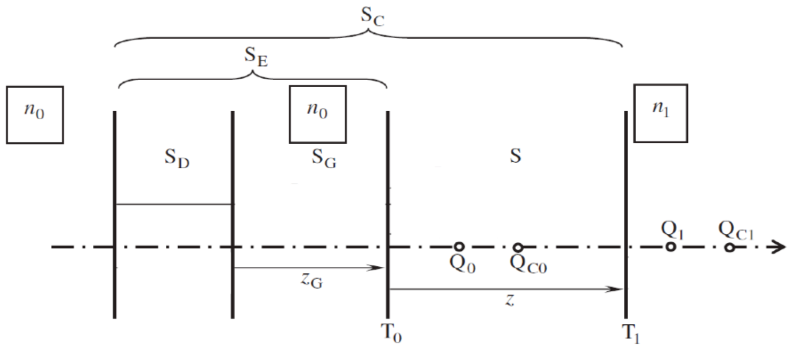

Figure 3.7: Visual Optical System for an Eye [37]

The corresponding HOL Light definition is as follows:

Definition 3.22 (Model of the Optical Corrective Setup for Myopia).

$\vdash \forall$ `system_eye` $\mathtt{z_G}$ `device`.

`eye_corrective_system` $\mathtt{n_0}$ $\mathtt{n_1}$ $\mathtt{z_G}$ =

`let device` $=$ `unknown` $\mathtt{A_D}$ $\mathtt{B_D}$ $\mathtt{C_D}$ $\mathtt{D_D}$ `and`

`eye_model` $=$ `unknown A B C D in`

$[(\mathtt{n_0}, 0); \mathtt{device}; (\mathtt{n_0}, \mathtt{Z_G}); \mathtt{eye\_model}], (\mathtt{n_1}, 0)$

where we model optical device (S_D), homogenous gap (S_G), and model of the eye (S) by `unknown` $\mathtt{A_D}$ $\mathtt{B_D}$ $\mathtt{C_D}$ $\mathtt{D_D}$, $(\mathtt{n_0}, \mathtt{Z_G})$ and `unknown A B C D`, respectively. We now derive the general expressions for the cardinal points as follows:

Theorem 3.13 (Cardinal Points of Visual Optical System).

$\vdash \forall$ $\mathtt{z_G}$ $\mathtt{n_0}$ $\mathtt{n_1}$.

$0 < \mathtt{n_0} \wedge 0 < \mathtt{n_1} \Rightarrow$

`let` $\begin{bmatrix} \mathtt{A_c} & \mathtt{B_c} \\ \mathtt{C_c} & \mathtt{D_c} \end{bmatrix}$ =

`system_composition (eye_corrective_system` $\mathtt{n_0}$ $\mathtt{n_1}$ $\mathtt{z_G}$`) in`

`principle_points` $((\frac{\mathtt{D_c}}{\mathtt{C_c}} * (\mathtt{A_c} - 1) - \mathtt{B_c}), (\frac{1-\mathtt{A_c}}{\mathtt{C_c}}))$

$\mathtt{n_0}$ $\mathtt{n_1}$ `(eye_corrective_system` $\mathtt{n_0}$ $\mathtt{n_1}$ $\mathtt{z_G}$`)` $\wedge$

$\texttt{nodal\_points}\ ((\frac{\texttt{1}-\texttt{A}_\texttt{c}}{\texttt{C}_\texttt{c}}), (\frac{\texttt{A}_\texttt{c}}{\texttt{C}_\texttt{c}} * (\texttt{D}_\texttt{c} - \texttt{1}) - \texttt{B}_\texttt{c}))$

$\texttt{n}_0\ \texttt{n}_1$ (eye_corrective_system $\texttt{n}_0\ \texttt{n}_1\ \texttt{z}_\texttt{G}$) $\wedge$

$\texttt{focal\_points}\ ((-\frac{\texttt{D}_\texttt{c}}{\texttt{C}_\texttt{c}}), (-\frac{\texttt{A}_\texttt{c}}{\texttt{C}_\texttt{c}}))$

$\texttt{n}_0\ \texttt{n}_1$ (eye_corrective_system $\texttt{n}_0\ \texttt{n}_1\ \texttt{z}_\texttt{G}$)

Given the structure of the corrective device, we can easily find the location of Q_{C0} and Q_{C1}, i.e., cardinal points which help to estimate the shifts in the cardinal points of the visual system of eye. Furthermore, different decisions about the diagnoses of a disease can be made based on the equivalent composed system. For example, the element $\texttt{A}_\texttt{c}$ is the direct measure of the myopia of the eye, i.e., the eye is myopic, emmetropic or hyperopic if $\texttt{A}_\texttt{c}$ is negative, zero or positive, respectively [37]. Note that the derived expressions for the cardinal points (Theorem 3.13) are more general than those derived in [37]. Indeed, authors of [37] used the assumption that the determinant of corresponding matrix of optical instrument and visual optical system is always equal to 1. However, this assumption is only valid for a class of optical systems [85]. We can obtain similar expressions by Theorem 3.13 if we consider this assumption. In our analysis, all expressions are derived in a general form which can be directly used for a particular corrective device and parameters of an eye without re-doing manual derivations.

3.7 Summary and Discussions

In this chapter, we proposed a higher-order logic formalization of ray optics in HOL Light. We started with the formalization of basic optical system structure which included formal definitions of new datatypes for optical interfaces, optical components and optical systems. We also formalized the constraints to ensure the valid structure of an optical system. We then specified the physical behavior of a light ray for any arbitrary optical system. We used this infrastructure to verify the classical result of geometrical optics which states that any optical system can be transformed into a matrix model. It is important to note that our formalization of optical systems and light rays is generic and we can model optical systems and rays of any length.

We also formalized the notion of composed optical systems and verified that composed systems inherit the same linear algebraic properties as for the case of a single optical system. Consequently, we formalized the notion of cardinal points of an optical system. Indeed, we verified generic expressions for the cardinal points (principal, nodal and focal), i.e., Theorems 3.9, 3.10 and 3.11. Interestingly, the availability of such a formalized infrastructure significantly reduced the time required to verify the cardinal points of the frequently used optical components (e.g., thin lens, thick lens, parallel plate and ball lens). Finally, we presented the formal analysis of a vision corrective biomedical device to analyze the myopia or nearsightedness.

Apart from the formalization of a number of concepts of ray optics and optical imaging systems, another contribution of our work is to bring out all the hidden assumptions about the physical models of lenses and mirrors which otherwise are not mentioned in the optics literature (e.g., assumptions given in Theorem 3.12 are not stated in [85]). Moreover, we automatized parts of the verification task by introducing new tactics. Some of these tactics are specialized to verify (or simplify) the proofs related to our formalization of ray optics (e.g., `VALID_OPTICAL_SYSTEM_TAC` [87]). However, some tactics are general and can be used in different verification tasks involving matrix/vector operations. An example of such tactic is `common_prove`, which allowed us to verify the ray-transfer matrices in our development. The core formalization described in this chapter took around 2000 lines of HOL Light code including formal definitions, lemmas and theorems. The availability of this infrastructure allowed us to analyze a vision corrective device in less than 50 lines of HOL Light code which demonstrates the effectiveness of our formalization presented in this chapter.

In the optics literature, many systems other than geometrical optics can also be modeled based on the transfer-matrix approach. Some examples of such systems are periodic optical systems [81], frequency division multiplexing/demultiplexing [22] and photonic signal processing applications [91]. The formalization process described in this chapter can be used as a guideline for the formal analysis of above mentioned systems. It may require some modifications about new datatypes for underlying components and corresponding physical behavior. Indeed, we used our experience to formally verify the transmissivity and reflectivity of 2-D lattice photonic filters [80].

In this chapter, we only considered the ray nature of light and developed corresponding reasoning support. Despite the many applications of ray tracing, some of the optical systems can only be analyzed using the notion of light beams. In the next chapter, we will use the infrastructure developed in this chapter to formalize Gaussian beams and associated models of optical systems.

Chapter 4

Formalization of Gaussian Beams

In this chapter, we present the higher-order logic formalization of light as a beam and related concepts described in Chapter 2. In fact, this chapter extends the reasoning support for optical systems by augmenting the formalization of Gaussian beams to ray optics models developed in the previous chapter. We can divide the formalization of Gaussian beams[7] into three parts: 1) formalization of the q-parameters of Gaussian beams and verification of some related properties; 2) formalization of the paraxial Helmholtz equation along with the verification that an envelope of a Gaussian beam satisfies the paraxial Helmholtz equation; and 3) formalization of the Gaussian beams transformation for optical systems and the formal verification of the complex ABCD-law. We then use this infrastructure to build the formal reasoning support for Quasi-optical systems along with the formal analysis of a real-world telescope receiver.

[7]The source codes of the formalizations and proofs presented in this chapter can be found in [87].

4.1 Formalization of q-Parameters

In the optics literature, Gaussian beams are defined in different forms depending upon the application of the beam transformation. They are generally characterized by the corresponding q-parameter, i.e., ($q(z) = z + jz_R$). Furthermore, Rayleigh range ($z_R = \frac{\pi w_0^2}{\lambda}$), can be described by two parameters, i.e., value of the beam width at $z = 0$, (w_0) and wavelength (λ). Thus, the q-parameter can be completely characterized by a triplet (w_0, λ, z) and hence the Gaussian beam. We define the Rayleigh range and q-parameter in HOL Light as follows:

Definition 4.1 (Rayleigh Range and q-parameter)**.**

$\vdash\forall$ $\mathtt{w_0\ lam.\ \ rayleigh\_range\ w_0\ lam} = \dfrac{\pi \mathtt{w_0^2}}{\mathtt{lam}}$

$\vdash\forall$ $\mathtt{z\ w_0\ lam.\ \ qq\ (z,w_0,lam) = z + j\ (rayleigh\_range\ w_0\ lam)}$

where `j` represents the imaginary unit $\sqrt{-1}$.

One of the most important definitions of the q-parameters is given in the form of $R(z)$ and $W(z)$ which are the measures of the beam width and wavefront radius of curvature, respectively. Mathematically, we describe them as follows:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi W^2(z)} \tag{4.1}$$

$$R(z) = z\left[1 + (\frac{z_R}{z})^2\right] \tag{4.2}$$

$$W(z) = w_0\left[1 + \left(\frac{z_R}{z}\right)^2\right]^{\frac{1}{2}} \tag{4.3}$$

We formally define $R(z)$ and $W(z)$ as follows:

Definition 4.2 (Wavefront Radius and Beam Width)**.**

$\vdash\forall$ $\mathtt{z\ w_0\ lam.\ \ RR\ z\ w_0\ lam = z}\left[1 + \left(\dfrac{\mathtt{rayleigh\_range\ w_0\ lam}}{\mathtt{z}}\right)^2\right]$

$\vdash\forall$ $\mathtt{z\ w_0\ lam.\ \ WW\ z\ w_0\ lam = w_0}\left[1 + \left(\dfrac{\mathtt{rayleigh\_range\ w_0\ lam}}{\mathtt{z}}\right)^2\right]^{\frac{1}{2}}$

where the functions RR and WW are both of type $\mathbb{R}\rightarrow\mathbb{R}\rightarrow\mathbb{R}\rightarrow\mathbb{R}$, which take three parameters z, w0 and lam and return a real number corresponding to Equations (4.2) and (4.3), respectively. Next, we use these definitions to verify Equation (4.1) as follows:

Theorem 4.1 (q-Parameter Alternative Form)**.**

$\vdash\forall$ z w_0 lam. $0 < w_0 \ \wedge\ 0 < \texttt{lam} \ \wedge\ z \neq 0$

$$\frac{1}{\texttt{qq}(z, w_0, \texttt{lam})} = \frac{1}{\texttt{RR z } w_0 \texttt{ lam}} - j\ \frac{\texttt{lam}}{\pi(\texttt{WW z } w_0 \texttt{ lam})^2}$$

The proof of this theorem mainly involves complex analysis and some properties of qq, RR, WW and rayleigh_range, which we list here:

Lemma 1 (Properties)**.**

$\vdash\ \forall$ z w_0 lam.

$\quad z \neq 0\ \wedge (\texttt{rayleigh\_range}\ w_0\ \texttt{lam})^2 = z^2 \ \Rightarrow\ (\texttt{RR z}\ w_0\ \texttt{lam}) \neq 0$

$\vdash\ \forall$ z w_0 lam.

$\quad 0 < w_0 \ \wedge\ 0 < \texttt{lam} \ \wedge\ 0 \leq z \ \Rightarrow 0 < \texttt{WW z}\ w_0\ \texttt{lam}$

$\vdash\ \forall$ z w_0 lam.

$\quad 0 < w_0 \ \wedge\ 0 < \texttt{lam} \ \Rightarrow \texttt{qq}\ (z, w_0, \texttt{lam}) \neq 0$

$\vdash\ \forall$ z w_0 lam.

$\quad 0 < w_0 \ \wedge\ 0 < \texttt{lam} \ \Rightarrow 0 < \texttt{rayleigh\_range z}\ w_0\ \texttt{lam}$

The alternative form of the q-parameter proved in Theorem 4.1, is quite helpful to verify the general form of Gaussian beams (Equation (2.11)). Moreover, we can also derive the general expression for the intensity of Gaussian beams (Equation 2.12) using this alternative form. We describe the verification details in the next section.

4.2 Formalization of Paraxial Helmholtz Equation

In this section, our main focus is to formalize the Paraxial Helmholtz Equation and verify that a Gaussian beam satisfies this equation. We then formally derive the general form of Gaussian beams and their intensity from the definition of paraxial wave. Mathematically, the Paraxial Helmholtz Equation is described as a partial differential equation as follows:

$$\nabla_T^2 A(x,y,z) - j2k\frac{\partial A(x,y,z)}{\partial z} = 0 \tag{4.4}$$

Our first step is to formalize the notion of transverse Laplacian operator ($\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$) for arbitrary functions as follows:

Definition 4.3 (Laplacian)**.**

$\vdash$ $\forall$ `f x y.`

`laplacian f` $(\mathtt{x},\mathtt{y}) =$ `higher_complex_derivative 2` $(\lambda$`x. f` $(\mathtt{x},\mathtt{y}))$ `x` $+$

`higher_complex_derivative 2` $(\lambda$`y. f` $(\mathtt{x},\mathtt{y}))$ `y`

where `higher_complex_derivative` represents the n^{th}-order complex derivative of a function:

```
⊢ higher_complex_derivative 0 f = f ∧
   (∀n. higher_complex_derivative (SUC n) f =
        complex_derivative (higher_complex_derivative n f))
```

We use `laplacian` to formalize the Paraxial Helmholtz Equation (i.e., Equation (4.4)) as follows:

Definition 4.4 (Paraxial Helmholtz Equation)**.**

$\vdash$ `Paraxial_Helmholtz_eq A (x,y,z) k` $\Leftrightarrow$

`laplacian`$(\lambda(\mathtt{x},\mathtt{y}).$ `A` $(\mathtt{x},\mathtt{y},\mathtt{z}))$ $(\mathtt{x},\mathtt{y})$ $-$

`2jk complex_derivative` $(\lambda$`z. A` $(\mathtt{x},\mathtt{y},\mathtt{z}))$ `z` $=$ `0` where Paraxial_Helmholtz_eq accepts a function U of type $((\mathbb{C}\times\mathbb{C}\times\mathbb{C}) \rightarrow\mathbb{C})$, a triplet (x,y,z) and a wave number k and returns the Paraxial Helmholtz equation. Next, we formalize the paraxial wave, given in Equation (2.3), as follows:

Definition 4.5 (Paraxial Wave)**.**

$\vdash \forall$ A x y k z.

$$\texttt{paraxial\_wave A x y z k} \;=\; \texttt{A}(\texttt{x},\texttt{y},\texttt{z})\exp(-\texttt{jkz})$$

where $\texttt{A}:((\mathbb{C}\times\mathbb{C}\times\mathbb{C}) \rightarrow\mathbb{C})$ represents the complex amplitude of the paraxial wave. The function cexp represents the complex-valued exponential function in HOL Light.

We need to define the q-parameter based amplitude of the paraxial wave, given in the following equation:

$$A(x,y,z) = \frac{A_0}{q(z)}e^{\left(-jk\dfrac{x^2+y^2}{2q(z)}\right)} \tag{4.5}$$

The corresponding HOL definition is given as follows:

Definition 4.6 (q-parameters Based Solution)**.**

$\vdash \forall$ $\texttt{A}_0$ k x y z $\texttt{w}_0$ lam.

$$\texttt{q\_parameter\_amplitude}\ \texttt{A}_0\ \texttt{z x y k}\ \texttt{w}_0\ \texttt{lam} \;=\; \frac{\texttt{A}_0}{\texttt{qq}\ (\texttt{z},\texttt{w}_0,\texttt{lam})}\exp\left(\frac{-\texttt{jk}(\texttt{x}^2+\texttt{y}^2)}{2\texttt{qq}\ (\texttt{z},\texttt{w}_0,\texttt{lam})}\right)$$

where A0 is a complex-valued constant. The function qq represents the q-parameter as described in Definition 4.1.

Now equipped with above described formal definitions, an important requirement is to verify that the q-parameters based solution (Definition 4.6) satisfies the paraxial Helmholtz equation (Definition 4.4). In other words, this is the main condition for a paraxial wave to be valid in the context of geometrical optics. We establish this result in the following theorem:

Theorem 4.2 (Helmholtz Equation Verified)**.**

$\vdash \forall$ $\mathtt{A_0}$ x y z $\mathtt{w_0}$ lam k.

$0 < \mathtt{w_0} \ \wedge\ 0 < \mathtt{lam} \ \Rightarrow$

Paraxial_Helmholtz_eq $(\lambda(\mathtt{x},\mathtt{y},\mathtt{z}).$

q_parameter_amplitude $\mathtt{A_0}$ z x y k $\mathtt{w_0}$ lam) (x, y, z) k

where both assumptions ensure that the value of qq (q-parameter) is not zero. The proof of this theorem is mainly based on three lemmas about the complex differentiation of q_parameter_amplitude with respect to the parameters x, y and z. The proof of these lemmas is mainly done using the automated tactic called COMPLEX_DIFF_TAC (already available in HOL Light and developed by Harrison), which can automatically compute the complex differentiation of complicated functions. Indeed, this tactic saves a lot of time of user interaction while proving theorems which involve complex differentiation.

Our next step is to derive the expression representing paraxial wave as a Gaussian beam which is described in the following equation:

$$U(r) = \frac{A_0}{jz_r}\frac{w_0}{W(z)}e^{\left(-\dfrac{x^2+y^2}{W^2(z)}\right)}e^{\left(-jkz-jk\dfrac{x^2+y^2}{2R(Z)}+j\xi(z)\right)} \tag{4.6}$$

where $\xi(z) = \tan^{-1}(\frac{z}{z_R})$. The above equation is the main representation of Gaussian beams and describes the important properties of light when it travels from one component to another. Even many laser applications utilize Equation (4.6) as the mathematical model of a laser beam [81]. The formal representation of this equation is given as follows:

Theorem 4.3 (Gaussian Beam)**.**

$\vdash \forall$ x y z $\mathtt{w_0}$ lam $\mathtt{A_0}$ k.

$0 < \mathtt{w_0} \ \wedge\ 0 < \mathtt{lam} \ \wedge\ \mathtt{z} \neq 0 \ \Rightarrow$

$$
\begin{aligned}
&\texttt{paraxial\_wave}\ (\lambda(\texttt{x},\texttt{y},\texttt{z}).\\
&\texttt{q\_parameter\_amplitude A}_0\ \texttt{z x}\ \ \texttt{y k w}_0\ \texttt{lam})\ \texttt{x y z k}\ =\\
&\texttt{let A}_\texttt{c} = \frac{\texttt{A}_0}{\texttt{j(rayleigh\_range w}_0\ \texttt{lam)}}\ \texttt{in}\\
&\texttt{A}_\texttt{c}\frac{\texttt{w}_0}{\texttt{WW z w}_0\ \texttt{lam}}\exp\left[-\frac{\texttt{x}^2+\texttt{y}^2}{(\texttt{WW z w}_0\ \texttt{lam})^2}\right]\\
&\exp\left[-\texttt{jkz}-\texttt{jk}\frac{\texttt{x}^2+\texttt{y}^2}{2(\texttt{RR z w}_0\ \texttt{lam})}+\texttt{j}\arctan\left(\frac{\texttt{z}}{\texttt{rayleigh\_range w}_0\ \texttt{lam}}\right)\right]
\end{aligned}
$$

where atn represents the inverse tangent function in HOL Light. The proof of this theorem mainly requires two lemmas: 1) expressing the $q-$parameter in equivalent form (Equation 4.6); and 2) expressing atn as an argument of cexp. The first lemma can be discharged by Theorem 4.3 and we present the statement of the second lemma here:

Lemma 2 (Arctan as an Argument of exp)**.**

$$
\begin{aligned}
&\vdash \forall\ \texttt{z w}_0\ \texttt{lam}.\\
&\quad 0<\texttt{w}_0\ \wedge\ 0<\texttt{lam}\ \Rightarrow \exp\left[\texttt{j}\arctan\left(\frac{\texttt{z}}{\texttt{rayleigh\_range w}_0\ \texttt{lam}}\right)\right]\ =\\
&\quad \frac{(\texttt{jz}\ +\ \texttt{rayleigh\_range w}_0\ \texttt{lam})}{\sqrt{\texttt{z}^2\ +\ (\texttt{rayleigh\_range w}_0\texttt{lam})^2}}
\end{aligned}
$$

The proof of this lemma mainly involves the properties of transcendental functions in HOL Light. Finally, we define the intensity of a paraxial wave as follows:

Definition 4.7 (Beam Intensity)**.**

$$
\begin{aligned}
&\vdash \forall\ \texttt{A x y z k}.\\
&\quad \texttt{beam\_intensity A x y z k}\ =\ \|\,(\texttt{paraxial\_wave A x y z k})^2\,\|
\end{aligned}
$$

where $\texttt{A}:((\mathbb{C}\times\mathbb{C}\times\mathbb{C})\ \to\mathbb{C})$ represents the complex amplitude of the paraxial wave. The function norm, represents the complex norm of a function in HOL Light. We use the above definition to verify the general expression for the intensity of a Gaussian beam (Equation (2.12)) in the following theorem:

Theorem 4.4 (Intensity of Gaussian Beam Intensity).

$\vdash \forall\ \mathtt{A_0\ x\ y\ z\ k\ w_0\ lam}.$

$\mathtt{0 < w_0\ \wedge\ 0 < lam\ \wedge\ z \neq 0 \Rightarrow}$

$\mathtt{beam\_intensity}\ (\lambda(\mathtt{a,b,z}).$

$\mathtt{q\_parameter\_amplitude\ A_0\ z\ a\ b\ k\ w_0\ lam)\ x\ y\ z\ k}$

$$\left\| \frac{\mathtt{A_0}}{\mathtt{j(rayleigh\_range\ w_0\ lam)}} \right\|^2 \left(\frac{\mathtt{w_0}}{\mathtt{WW\ z\ w_0\ lam}}\right)^2 \exp\left[-\frac{\mathtt{k}\frac{\mathtt{lam}}{\pi}(\mathtt{x}^2+\mathtt{y}^2)}{(\mathtt{WW\ z\ w_0\ lam})^2}\right]$$

The proof of this theorem is mainly based on Theorem 4.3 and properties of complex numbers.

We conclude here the formalization of the Paraxial Helmholtz Equation. The main significance of this section was to verify the validity of Gaussian beams in the paraxial regime, which is a classical result in the literature of geometrical optics [78]. Moreover, we have been able to verify a generic expression for the intensity of a Gaussian beam. We discuss the concepts behind the propagation of Gaussian beams in optical systems along with their HOL formalization in the next section.

4.3 Formalization of Beam Transformation

In our formalization of q-parameter of Gaussian beams, we consider that the size of the beam waist radius w_0 and its location z is already provided by the physicists or optical system design engineers. Indeed these two parameters are sufficient to compute the beam width $W(z)$ and wavefront radius of curvature $R(z)$ because the wavelength λ is fixed throughout the design life-cycle. Mathematically, this notion can be represented as a transformation $w_0, z \to W(z), R(z)$.

Our goal is to formalize the physical behavior of a Gaussian beam when it passes through an optical system. We only model the points where it hits an optical interface (e.g., spherical or plane interface). It is evident from the previous discussion that the q-parameter is sufficient to characterize a Gaussian beam. Furthermore, λ is fixed (i.e., refractive indices are the same) which leads to the requirement of considering only two parameters, i.e., w_0 and z. So we just need to provide the information about (z,w_0) at each interface. Consequently, we should have a list of such pairs for every component of a system. In addition, the same information should be provided for the source of the beam. We define a type for a pair (z, w_0) as `single_q`. This yields the following type definition:

```
new_type_abbrev("single_q",`:real # real`);;
new_type_abbrev("beam",`:single_q # single_q #
                         (single_q # single_q) list`);;
```

where the first `single_q` is the pair (z, w_0) for the source of the beam, the second one is the one after the first free space, and the list of `single_q` pairs represents the same information for the interfaces and free spaces at every hitting point of an optical system.

The transmission of a Gaussian beam in an optical system depends on the nature of components used in that system. It is known that a Gaussian beam remains a Gaussian beam when it is transmitted through a series of optical components aligned with an optical axis [78]. However, only the beam waist and curvature are modified so that the beam is only reshaped as compared to the input beam. If a Gaussian beam is subject to transmission in free space of width d, only one parameter is modified, i.e., z becomes $z + d$. When a beam transmits through a plane interface it only scales with respect to the refractive indices of input and output planes. However, in case of transmission through a spherical interface (Figure 4.1 (a)), the beam width remains the same but the output beam has to satisfy the lens formula, i.e., $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}$, where

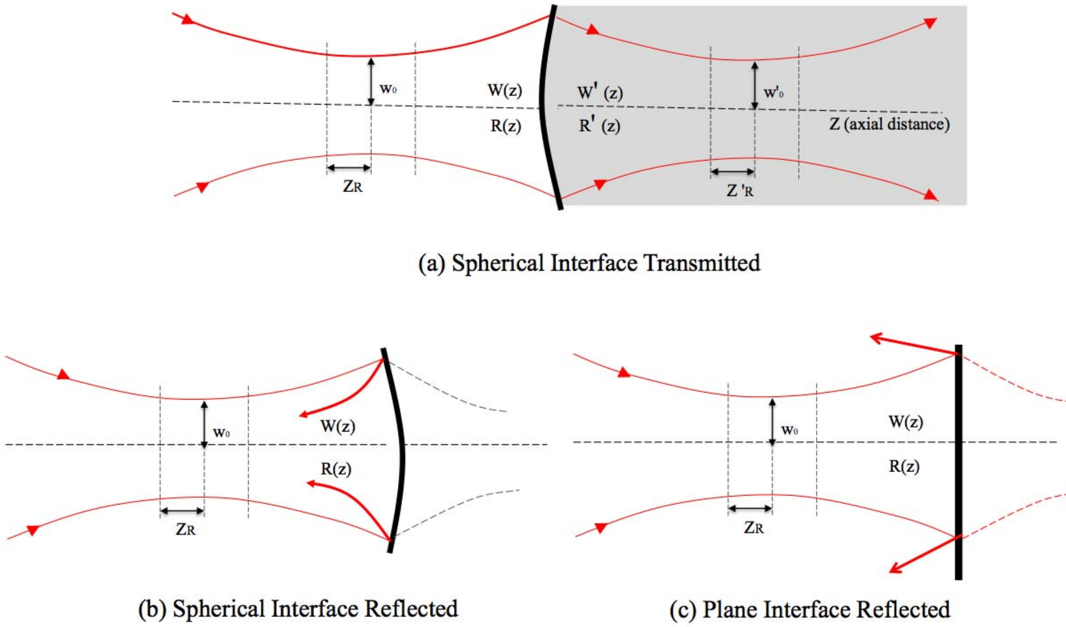

Figure 4.1: Behavior of Gaussian Beam at Different Interfaces

q_1, q_2 and f are the input and output beam q-parameters and the focal length of the spherical interface, respectively. For the case of the reflection from a plane interface, the input Gaussian beam bounces back without any change in its curvature (Figure 4.1 (c)). On the other hand, the reflection from a curved interface results into a modified lens formula, i.e., $\frac{1}{q_2} = \frac{1}{q_1} + \frac{1}{f}$ with no alteration in the beam width as shown in Figure 4.1 (b).

We specify the valid behavior of a beam using some predicates. First of all, we define the behavior of a beam when it is traveling through a free space. This requires the position of the beam at the previous and current point of observation, and the free space itself.

Definition 4.8 (Beam in Free Space)**.**

$\vdash$ `is_valid_beam_in_free_space`

$\quad \texttt{(z,w}_0\texttt{)}\ \texttt{(z}',\texttt{w}_0'\texttt{)}\ \texttt{(n}_0\texttt{,d)}\ \Leftrightarrow\ \texttt{w}_0' = \texttt{w}_0\ \wedge\ \texttt{z}' = \texttt{z} + \texttt{d}$

where $(\mathtt{z}, \mathtt{w_0})$ and $(\mathtt{z'}, \mathtt{w'_0})$ represent the `single_q` at two points. The pair $(\mathtt{n_0}, \mathtt{d})$ represents a free space with refractive index $\mathtt{n_0}$ and width $\mathtt{d}$.

Now we specify the valid behavior of a beam at plane and spherical interfaces as follows:

Definition 4.9 (Beam at Plane Interface)**.**

$\vdash$ (is_valid_beam_at_plane_interface $(\mathtt{z}, \mathtt{w_0})$ $(\mathtt{z'}, \mathtt{w'_0})$ lam n $\mathtt{n'}$

plane_transmitted $\Leftrightarrow$

$$\mathtt{z'} = \mathtt{z}\frac{\mathtt{n'}}{\mathtt{n}} \ \wedge\ \mathtt{w'_0} = \mathtt{w_0}\sqrt{(\frac{\mathtt{n'}}{\mathtt{n}})} \ \wedge\ 0 < \mathtt{n} \ \wedge\ 0 < \mathtt{n'}) \ \wedge$$

(is_valid_beam_at_plane_interface $(\mathtt{z}, \mathtt{w_0})$ $(\mathtt{z'}, \mathtt{w'_0})$ lam n $\mathtt{n'}$

plane_reflected $\Leftrightarrow$

$$\mathtt{z'} = \mathtt{z} \ \wedge\ \mathtt{w'_0} = \mathtt{w_0} \ \wedge\ 0 < \mathtt{n} \ \wedge\ 0 < \mathtt{n'})$$

where `is_valid_beam_at_plane_interface` accepts two `single_q`, wavelength `lam` and two refractive indices `n` and $\mathtt{n'}$ before and after the plane interface (transmitted and reflected), and returns the physical behavior of the beam described above (as shown in Figure 4.1). Similarly, we formally specify the physical behavior of the beam at spherical interface in the following definition:

Definition 4.10 (Beam at Spherical Interface)**.**

$\vdash$ is_valid_beam_at_spherical_interface $(\mathtt{z}, \mathtt{w_0})$ $(\mathtt{z'}, \mathtt{w'_0})$ lam n $\mathtt{n'}$

(spherical_transmitted R) $\Leftrightarrow$

valid_single_q $(\mathtt{z}, \mathtt{w_0})$ $\wedge$ valid_single_q $(\mathtt{z'}, \mathtt{w0'})$ $\wedge$

$$0 < \mathtt{n} \ \wedge\ 0 < \mathtt{n'} \ \wedge\ -\frac{\mathtt{n} - \mathtt{n'}}{\mathtt{nR}} = \frac{1}{\mathtt{qq}\ (\mathtt{z}, \mathtt{w_0}, \mathtt{lam})}$$

$$\frac{1}{\mathtt{RR\ z'\ w'_0\ lam}} = \frac{\mathtt{n'}}{\mathtt{n}}\frac{1}{\mathtt{RR\ z\ w_0\ lam}} + \frac{\mathtt{n} - \mathtt{n'}}{\mathtt{n'R}} \ \wedge$$

$$(\mathtt{WW\ z'\ w'_0\ lam}) = \sqrt{\frac{\mathtt{n'}}{\mathtt{n}}}(\mathtt{WW\ z\ w_0\ lam}) \ \wedge$$

is_valid_beam_at_spherical_interface $(\mathtt{z}, \mathtt{w_0})$ $(\mathtt{z'}, \mathtt{w'_0})$ lam n $\mathtt{n'}$

$$
\begin{aligned}
&\texttt{(spherical\_reflected R)} \ \Leftrightarrow \\
&\texttt{valid\_single\_q} \ (\texttt{z}, \texttt{w}_0) \ \wedge \ \texttt{valid\_single\_q} \ (\texttt{z}', \texttt{w0}') \ \wedge \\
&0 < \texttt{n} \ \wedge \ 0 < \texttt{n}' \ \wedge \ \texttt{qq} \ (\texttt{z}, \texttt{w}_0, \texttt{lam}) \neq \frac{\texttt{R}}{2} \\
&\frac{1}{\texttt{RR z}' \ \texttt{w}_0' \ \texttt{lam}} = \frac{1}{\texttt{RR z w}_0 \ \texttt{lam}} - \frac{2}{\texttt{R}} \ \wedge \ (\texttt{WW z}' \ \texttt{w}_0' \ \texttt{lam}) = (\texttt{WW z w}_0 \ \texttt{lam})
\end{aligned}
$$

where valid_single_q ensures that $\texttt{w}_0$ is positive and z is not equal to zero.

Note that we describe separately the valid behavior of a beam at plane and spherical interfaces for the sake of convenience and finally we combine them into one definition called is_valid_beam_at_interface. On the same lines, we also define the behavior of a beam through an arbitrary optical system (i.e., is_valid_beam_in_system).

In order to ensure the correctness of our definitions and to facilitate the formal analysis of practical systems, we verify three classical results of Gaussian beams theory: (1) Complex ABCD law for each optical interface (i.e., free space, spherical and plane for both reflection and transmission); (2) Complex ABCD law for an arbitrary optical system; and (3) composed optical systems as follows:

Theorem 4.5 (ABCD-Law for Interface)**.**

$$
\begin{aligned}
&\vdash \forall \texttt{i ik z w}_0 \ \texttt{z}' \ \texttt{w}_0' \ \texttt{lam n n}'. \\
&\quad \texttt{is\_valid\_beam\_at\_interface} \ (\texttt{z}, \texttt{w}_0) \ (\texttt{z}', \texttt{w}_0') \ \texttt{lam n n}' \ \texttt{i} \ \wedge \\
&\quad \texttt{is\_valid\_interface i} \ \wedge \ 0 < \texttt{lam} \ \Rightarrow \\
&\quad \texttt{let} \begin{bmatrix} \texttt{A} & \texttt{B} \\ \texttt{C} & \texttt{D} \end{bmatrix} = (\texttt{interface\_matrix n n}' \ \texttt{i ik}) \ \texttt{in} \ \Rightarrow \\
&\quad \texttt{qq} \ (\texttt{z}', \texttt{w}_0', \texttt{lam}) \ = \ \frac{\texttt{Aqq} \ (\texttt{z}, \texttt{w}_0, \texttt{lam}) \ + \ \texttt{B}}{\texttt{Cqq} \ (\texttt{z}, \texttt{w}_0, \texttt{lam}) \ + \ \texttt{D}}
\end{aligned}
$$

where is_valid_beam_at_interface ensures the valid behavior at each interface i. The function is_valid_interface ensures that each interface i is indeed a valid interface. The assumption $0 <$ lam is required to ensure that wavelength is greater than zero. Finally, the function interface_matrix represents the corresponding matrix of each optical component. We next verify the complex ABCD law for an arbitrary optical system as follows:

Theorem 4.6 (ABCD-Law for Optical System)**.**

$\vdash \forall$`sys beam lam A B C D`.

`is_valid_beam_in_system beam lam sys` $\wedge$

`is_valid_optical_system sys` $\wedge$ `0` $<$ `lam` $\wedge$

$\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix} = \mathtt{system\_composition\ sys} \Rightarrow$

$\mathtt{let}\ (\mathtt{z}, \mathtt{w_0}), (\mathtt{z'}, \mathtt{w'_0}), \mathtt{rs} \ = \ \mathtt{beam\ in}$

$\mathtt{let}\ (\mathtt{z_n}, \mathtt{w_{0n}}) \ = \ \mathtt{last\_single\_beam\ beam\ in}$

$$\mathtt{qq}\ (\mathtt{z_n}, \mathtt{w_{0n}}, \mathtt{lam}) \ = \ \frac{\mathtt{Aqq}\ (\mathtt{z}, \mathtt{w_0}, \mathtt{lam}) \ + \ \mathtt{B}}{\mathtt{Cqq}\ (\mathtt{z}, \mathtt{w_0}, \mathtt{lam}) \ + \ \mathtt{D}}$$

where `is_valid_beam_in_system` ensures the valid behavior of the `beam` in optical system `sys`. The function `is_valid_system` ensures the validity of the optical systems structure. Finally, the function `system_composition` represents the corresponding matrix of the optical system. We prove Theorem 4.6 by induction on `sys` and the length of `beam` along with some complex arithmetic reasoning. Similarly, we verify the complex ABCD law for the composed systems where a system is composed of multiple optical systems, given as follows:

Theorem 4.7 (ABCD-Law for Composed System)**.**

$\vdash \forall$`c_sys gbeam lam A B C D`.

`is_valid_gbeam_in_c_system gbeam lam c_sys`

$\wedge$ `is_valid_gen_beam gbeam` $\wedge$

`is_valid_composed_system c_sys` $\wedge$ `0` $<$ `lam` $\wedge$

$$\begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix} = \mathtt{composed\_system\ c\_sys} \ \Rightarrow$$

$$\mathtt{let}\ (\mathtt{z}, \mathtt{w_0}), (\mathtt{z'}, \mathtt{w'_0}) \ = \ \mathtt{beam\_origin\ gbeam\ in}$$

$$\mathtt{let}\ (\mathtt{z_n}, \mathtt{w_{0n}}) \ = \ \mathtt{beam\_end\ gbeam\ in}$$

$$\mathtt{qq}\ (\mathtt{z_n}, \mathtt{w_{0n}}, \mathtt{lam}) \ = \ \frac{\mathtt{Aqq}\ (\mathtt{z}, \mathtt{w_0}, \mathtt{lam}) \ + \ \mathtt{B}}{\mathtt{Cqq}\ (\mathtt{z}, \mathtt{w_0}, \mathtt{lam}) \ + \ \mathtt{D}}$$

This concludes our formalization of the Gaussian beam transformation for arbitrary optical systems. In this section, we mainly specified the physical behavior of beams when passing through free space and interacting with different optical interfaces. We used this infrastructure to verify the ABCD-law which describes the relation between input and output beam parameters. It is important to note that the matrix elements of the ABCD-law and ray-transfer matrices remain the same for optical components which indicates the relationship among ray optics and Gaussian optics. In the next section, we use this development to formalize and build a reasoning support for widely used Quasi-optical systems which involve the prorogation of Gaussian beams.

4.4 Formalization of Quasi-Optical Systems

Quasioptics [31] deals with the propagation of a beam of radiations which is reasonably well collimated (i.e., rays are parallel and their spread is minimal during the propagation, e.g., laser light) and the wavelength is relatively small along the axis of propagation. At a first glance, this looks a restrictive notion of light but it has extraordinarily diverse applications ranging from compact systems in which all components are only a few wavelengths in size to antenna feed systems that illuminate an aperture of thousands or more wavelengths in diameter (e.g., space receiving stations) [31]. It is important to note that ray optics deals with light beams (essentially rays) with wavelength $\lambda \to 0$ and no diffraction effects, whereas quasioptics is concerned with the wavelength $\lambda \simeq system\ \ dimenions$ with diffraction effects. In practice, quasioptics is based on the Gaussian beam theory which provides a convenient formalism to analyze the behavior of a beam and to perform accurate calculations for real optical and laser systems. Some of the successful applications of quasioptics are in critical domains, e.g., millimeter wave lengths to a variety of commercial and military problems such as radars, remote sensing, materials measurement systems [31], radio frequency and radiometric optical systems [23].

4.4.1 Design Requirements for Quasi-Optical Systems

In optical engineering, the high level modeling and analysis of quasi-optical systems is an important design criterion in order to avoid unnecessary design revisions due to high manufacturing costs and critical applications. Given the quasi-optical system design and performance specification, i.e., the information about the size of the overall system, operating frequencies and coupling requirements, we can break the design procedure of such systems into four steps as shown in Figure 4.2.

- **Determination of system architecture and quasi-optical components:** The system architecture means the arrangement of optical components (lenses or mirrors), their nature (i.e., reflective or transmissive) and ability to process frequency bands. In industrial settings, this initial decision is of central importance because of the fact that the choice of the components can only be considered correct after executing all the steps mentioned in Figure 4.2.

- **Beam Waist Radius:** The beam waist radius provides the suitable measure to evaluate how each component modifies the Gaussian beam. In practice, there are many useful quasi-optical components for which the beam waist radius is

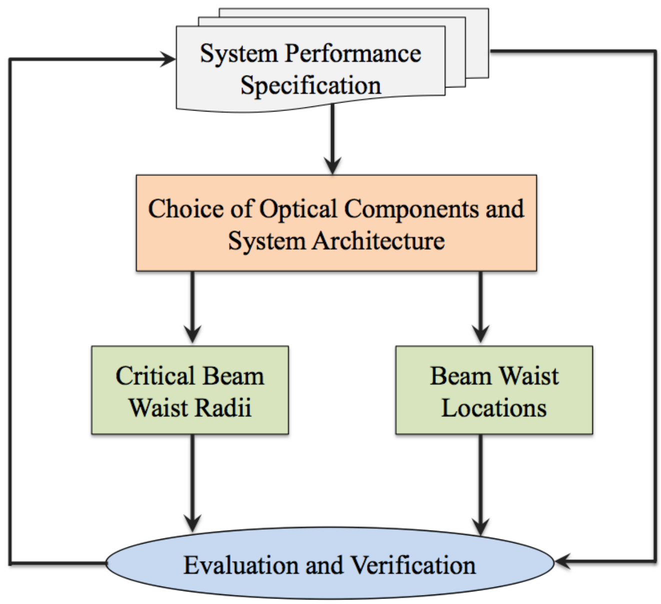

Figure 4.2: Quasi-Optical System Design Flow [31]

not important from the application viewpoint (e.g., polarization rotators, which rotates the polarization axis of the light beams [31]). So one of the important design criterion is the identification of all the components in the system for which the beam waist radius is critical.

- **Beam Waist Location:** The coupling of a Gaussian beam among two optical components is very critical to increase the overall performance of systems such as laser resonators [78] and feed horns [31]. This can be done by the indication of the beam waist location along with the beam waist radius of the Gaussian beam at the input and output of each quasi-optical component.

- **Evaluation and Verification:** Finally, the last step is to evaluate and verify that the selected architecture of quasi-optical system meets the performance specification, i.e., Gaussian beam waist radius and location are suitable for correct operation. Moreover, in some practical situation it is compulsory to evaluate the magnification which is a ratio of minimum beam waist of input and output

Gaussian beam.

We next present the formalization of an arbitrary quasi-optical system along with the derivation of the generalized expressions for the beam waist radius and location of output Gaussian beam. Then we can use our formalization to perform the verification and evaluation as shown in Figure 4.2.

4.4.2 Gaussian Beams in Quasi-Optical Systems

The generalized properties of beam transformation through a quasi-optical system can be analyzed using the complex ABCD law as described in the previous section. We consider a generic case in which a quasi-optical system is modeled as an arbitrary ABCD matrix as shown in Figure 4.3. The input waist radius $w_{0_{in}}$ of the Gaussian beam is at a distance d_{in} from the input reference plane, and the output waist, having the waist radius $w_{O_{out}}$, is located at distance d_{out} from the output reference plane. In this situation, the whole system is composed of three subsystems, i.e., a free space (n_i, d_{in}), a quasi-optical system (which can be modeled as an ABCD matrix), and another free space, i.e., (n_0, d_{out}). Our main goal is to derive the generic expression for the beam waist radius and its location as these are the two critical requirements in the design and analysis of quasi-optical systems as described in Figure 4.2. To this aim, we require three steps: (1) building a formal model of the quasi-optical system described in Figure 4.3 and then verifying the equivalent matrix; (2) deriving the complex ABCD law using the previous step; and (3) computing the general expressions for the output beam width radius and its location, i.e., $w_{0_{out}}$ and d_{out}, respectively. We formally model the quasi-optical system described in Figure 4.3 as follows:

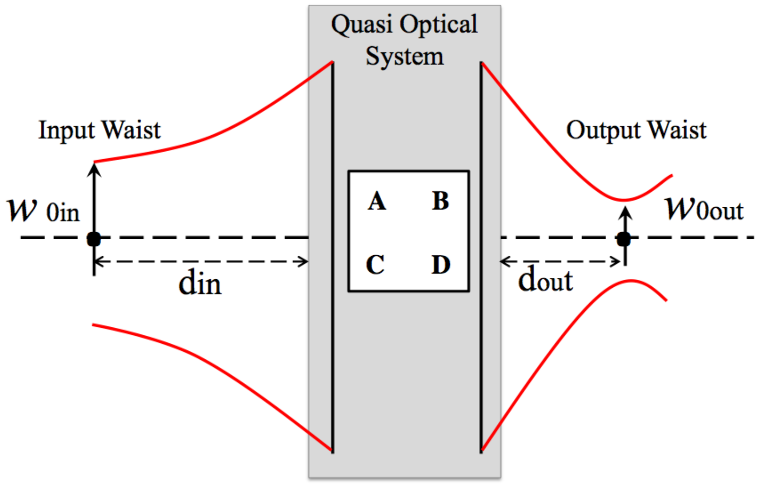

Figure 4.3: Generalized Qausi Optical System

Definition 4.11 (Quasi-Optical System Model).

$\vdash \forall\ \mathtt{n_i}\ \mathtt{d_{in}}\ \mathtt{sys}\ \mathtt{n_0}\ \mathtt{d_{out}}.$

$\quad \mathtt{quasi\_optical\_system\ sys\ d_{in}\ d_{out}\ n_i\ n_0} = [[\ ], \mathtt{n_i}, \mathtt{d_{in}}; \mathtt{sys}; [\ ], \mathtt{n_0}, \mathtt{d_{out}}]$

where `sys` represents the quasi-optical system, the parameters $\mathtt{n_i}$ and $\mathtt{n_0}$ represent the refractive index at the input and output, respectively. We next verify the equivalent matrix relation when the system is represented as an arbitrary ABCD-matrix as follows:

Theorem 4.8 (Matrix of Quasi-Optical System).

$\vdash \forall\ \mathtt{sys}\ \mathtt{d_{in}}\ \mathtt{d_{out}}\ \mathtt{n_i}\ \mathtt{n_0}\ \mathtt{A}\ \mathtt{B}\ \mathtt{C}\ \mathtt{D}.$

$$\mathtt{system\_composition\ sys} = \begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix} \Rightarrow$$

$$\mathtt{composed\_system\ (quasi\_optical\_system\ sys\ d_{in}\ d_{out}\ n_i\ n_0)} =$$

$$\begin{bmatrix} \mathtt{A} + \mathtt{C}\mathtt{d_{out}} & \mathtt{A}\mathtt{d_{in}} + \mathtt{B} + \mathtt{C}\mathtt{d_{in}}\mathtt{d_{out}} + \mathtt{D}\mathtt{d_{dout}} \\ \mathtt{C} & \mathtt{C}\mathtt{d_{in}} + \mathtt{D} \end{bmatrix}$$

The proof of this theorem involves rewriting the definitions of `quasi_optical_system` and `composed_system` along with the corresponding matrices of the input and output free spaces.

Consequently, we verify the ABCD-law of for the quasi-optical system model (Definition 4.11) as follows:

Theorem 4.9 (Quasi-optical System (ABCD))**.**

$\vdash \forall$ $\texttt{sys d}_{\texttt{in}}\ \texttt{d}_{\texttt{out}}\ \texttt{n}_{\texttt{i}}\ \texttt{n}_{0}\ \texttt{gbeam lam A B C D}$.

$\texttt{is\_valid\_gbeam\_in\_c\_system gbeam lam}$

$(\texttt{quasi\_optical\_system sys d}_{\texttt{in}}\ \texttt{d}_{\texttt{out}}\ \texttt{n}_{\texttt{i}}\ \texttt{n}_{0}) \wedge$

$\texttt{is\_valid\_gen\_beam gbeam} \wedge\ 0 < \texttt{lam} \wedge$

$\texttt{is\_valid\_composed\_system (quasi\_optical\_system sys d}_{\texttt{in}}\ \texttt{d}_{\texttt{out}}\ \texttt{n}_{\texttt{i}}\ \texttt{n}_{0}) \wedge$

$$\begin{bmatrix} \texttt{A} & \texttt{B} \\ \texttt{C} & \texttt{D} \end{bmatrix} = \texttt{system\_composition sys} \Rightarrow$$

$\texttt{let}\ (\texttt{z}, \texttt{w}_0), (\texttt{z}', \texttt{w}'_0) = \texttt{beam\_origin gbeam in}$

$\texttt{let}\ (\texttt{z}_{\texttt{n}}, \texttt{w}_{0\texttt{n}}) = \texttt{beam\_end gbeam in}$

$$\texttt{qq}\ (\texttt{z}_{\texttt{n}}, \texttt{w}_{0\texttt{n}}, \texttt{lam}) = \frac{\texttt{Aqq}\ (\texttt{z}, \texttt{w}_0, \texttt{lam}) + \texttt{B}}{\texttt{Cqq}\ (\texttt{z}, \texttt{w}_0, \texttt{lam}) + \texttt{D}}$$

where the first assumption ensures the valid behavior of the beam when it propagates through the quasi-optical system. The proof of this theorem is a direct consequence of Theorem 4.7.

Our next step is to verify the general expressions for the output beam waist radius. Here, one important point is to ensure that we are only interested in the Gaussian beam waist at the input which means that the real part of the input q-parameter should be 0. We include this requirement in the verification of the following main theorem:

Theorem 4.10 (Beam Waist Radius and Location)**.**

$\vdash \forall$ $\texttt{sys gbeam d}_{\texttt{in}}\ \texttt{d}_{\texttt{out}}\ \texttt{lam n}_{\texttt{i}}\ \texttt{n}_{0}\ \texttt{w}_{0\texttt{in}}\ \texttt{w}_{0\texttt{out}}\ \texttt{z z}_{\texttt{n}}\ \texttt{A B C D}$.

$[\texttt{H}_1]$ $\texttt{sys\_constraints (quasi\_optical\_system sys d}_{\texttt{in}}\ \texttt{d}_{\texttt{out}}\ \texttt{n}_{\texttt{i}}\ \texttt{n}_{0}) \wedge$

$$[\mathtt{H_2}]\quad \begin{bmatrix}\mathtt{A} & \mathtt{B}\\ \mathtt{C} & \mathtt{D}\end{bmatrix} = \mathtt{system\_composition\ sys}\ \wedge$$

$$[\mathtt{H_3}]\quad (\mathtt{z}, \mathtt{w_{0_{in}}}) = \mathtt{beam\_origin\ gbeam}\ \wedge\ (\mathtt{z_n}, \mathtt{w_{0_{out}}}) = \mathtt{beam\_end\ gbeam}\ \wedge$$

$$[\mathtt{H_4}]\quad \mathtt{Re(qq\ (z, w_{0_{in}}, lam))} \neq 0\ \wedge \mathtt{Re(qq\ (z_n, w_{0_{out}}, lam))} \neq 0\ \wedge$$

$$(\mathtt{Cd_{in}} + \mathtt{D})^2 + (\mathtt{C\ rayleigh\_range\ w_{0_{in}}\ lam})^2 \neq 0 \Rightarrow$$

$$\mathtt{d_{out}} = -\frac{(\mathtt{Ad_{in}} + \mathtt{B})(\mathtt{Cd_{in}} + \mathtt{D}) + \mathtt{AC}(\mathtt{rayleigh\_range\ w_{0_{in}}\ lam})^2}{(\mathtt{Cd_{in}} + \mathtt{D})^2 + (\mathtt{C}(\mathtt{rayleigh\_range\ w_{0_{in}}\ lam})^2)}\ \wedge$$

$$\mathtt{w_{0_{out}}}^2 = \frac{(\mathtt{AD} - \mathtt{BC})\ \mathtt{w_{0_{in}}}^2}{(\mathtt{Cd_{in}} + \mathtt{D})^2 + (\mathtt{Cw_{0_{in}}}^2\frac{\pi}{\mathtt{lam}})^2}$$

where the first assumption $[\mathtt{H_1}]$ packages three conditions as system constraints, i.e., the validity of the composed optical system architecture, the validity of the general beam and the valid behavior of a general beam in the composed system. The second assumption $[\mathtt{H_2}]$ ensures that the composed system can be described by an arbitrary matrix. Finally, the third and fourth assumptions (i.e., $[\mathtt{H_3}]$ and $[\mathtt{H_4}]$) ensure that the real part of q-parameters are zero and the values $\mathtt{d_{out}}$ and $\mathtt{w_{0_{out}}}$ are finite. The proof of Theorem 4.10 is mainly based on Theorem 4.9 and involves the properties of complex numbers (mainly, equating the real and imaginary parts of the input and output q-parameters).

Note that the expressions obtained in Theorem 4.10 can be applied to any quasi-optical system, and to any Gaussian beam parameters. The given system itself can be arbitrarily complicated, and the analysis reduces the problem of obtaining its overall ABCD matrix from a cascaded representation of its constituent optical components. We apply these results to verify a real-world optical system in the next section.

4.5 Application: APEX Telescope Receiver

The Atacama Pathfinder EXperiment (APEX) [8] is a single dish (12-metre diameter) telescope for millimeter and sub-millimeter astronomy, which operates since its first inauguration in 2005 [70]. The main mission of the APEX is to conduct the astronomical study of cold dust and gas in our own milky way and in distant galaxies. Recent observations based on APEX reveal the cradles of massive star-formation throughout our galaxy [3]. Besides these interesting aspects of the APEX telescope, the other main function is radiometry which helps to provide reliable weather forecasts and environmental dynamics. One of the main modules of the APEX is the Swedish Heterodyne Facility Instrument (SHeFI) receiver which was installed in 2008. In [70], the authors used a Quasi-optics based model for the SHeFI receiver to derive the conditions in terms of beam parameters using a paper-and-pencil based proof approach. Furthermore, these constraints are used to optimize (i.e., minimization of dimensions and distortions) the telescope design for all optical components. In this thesis, we propose to formally analyze the SHeFI receiver within the sound core of HOL Light by using our formalization of Gaussian beams and quasi-optical systems. The main component of the SHeFI receiver is the optical system which is designed to provide the coupling of the SHeFI channels and other instruments within the telescope. The optical layout of the receiving cabin is shown in Figure 4.4. The Points O_1, O_2, and O_3 represent focal points, traced from the original Cassegrain focal point [70]. Here, M_{8s} and M_{10} are ellipsoidal mirrors with focal distances f_2 and f_1 [70], respectively. In this situation, the Gaussian beams transformation is the best possible way to understand the processing of light in the receiver module of the APEX telescope [70].

The main goal is to verify the system magnification which is a ratio of the

[8] `http://www.apex-telescope.org/`

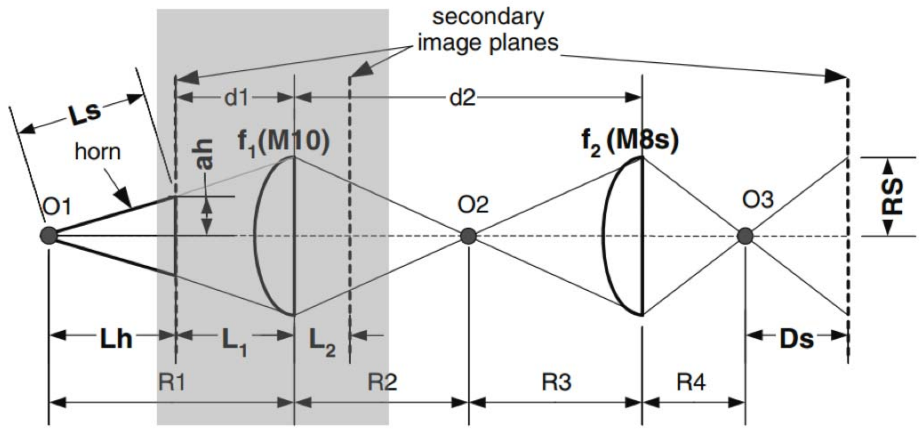

Figure 4.4: Optical Layout of the APEX Telescope Facility Receiver [70]

output and input beam waist radius, i.e., $\frac{w0_{out}}{w0_{in}}$. This can be done by using already verified theorems about the Gaussian beam transformation in an arbitrary Quasi-optical system. We analyze one module of the receiving system, i.e., the gray shaded region in Figure 4.4. Indeed this can be considered as the quasi-optical system with the input and output distances L_1 and L_2 and a thin lens inside as shown in Figure 4.3. Our problem is mainly reduced to the derivation of the equivalent matrix relation for the thin lens and then utilize Theorems 4.9 and 4.10. A thin lens is represented as the composition of two transmitting spherical interfaces such that any variation of the beam parameters is neglected between both interfaces. So, at the end, a thin lens is the composition of two spherical interfaces with a null width free space in between. We formalize a thin lens as follows:

Definition 4.12 (Thin Lens)**.**

$\vdash \forall$ $\mathtt{R_1}$ $\mathtt{R_2}$ $\mathtt{n_0}$ $\mathtt{n_1}$. `thin_lens` $\mathtt{R_1}$ $\mathtt{R_2}$ $\mathtt{n_0}$ $\mathtt{n_1}$ =

([($\mathtt{n_0}$,0),spherical_transmitted $\mathtt{R_1}$;($\mathtt{n_1}$,0),

spherical_transmitted $\mathtt{R_2}$],($\mathtt{n_0}$,0))

where $\mathtt{R_1}$, $\mathtt{R_2}$, $\mathtt{n_1}$, $\mathtt{n_2}$, represent the radius of curvatures of two interfaces and the refractive indices of the input and output planes, respectively. We prove that a thin lens is indeed a valid optical system if the corresponding parameters satisfy some constraints:

Theorem 4.11 (Valid Thin Lens).

$\vdash \forall\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \ \mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \Rightarrow$

$\quad \mathtt{is\_valid\_optical\_system\ (thin\_lens}\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1)$

The proof of this theorem is done automatically by our developed tactic, called `VALID_OPTICAL_SYSTEM_TAC`. Next, we verify the matrix relation of thin lens as follows:

Theorem 4.12 (Thin Lens Matrix).

$\vdash \forall\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1.\ \ \mathtt{R}_1 \neq 0 \wedge \mathtt{R}_2 \neq 0 \wedge 0 < \mathtt{n}_0 \wedge 0 < \mathtt{n}_1 \Rightarrow$

$$\mathtt{system\_composition\ (thin\_lens}\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_0\ \mathtt{n}_1) = \begin{bmatrix} 1 & 0 \\ \frac{\mathtt{n}_1 - \mathtt{n}_0}{\mathtt{n}_0}(\frac{1}{\mathtt{R}_2} - \frac{1}{\mathtt{R}_1}) & 1 \end{bmatrix}$$

At this point, we have all the necessary ingredients to analyze the module of interest of the SHeFI receiver as shown in Figure 4.4. We reuse the definition of generalized quasi-optical system (Definition 4.11) to define the module as follows:

Definition 4.13 (SheFI Receiver Module).

$\vdash \forall\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{L}_1\ \mathtt{L}_2\ \mathtt{n}_1\ \mathtt{n}_2.$

$\quad \mathtt{SHeFI\_receiver\_model}\ \mathtt{L}_1\ \mathtt{L}_2\ \mathtt{n}_1\ \mathtt{n}_2\ \mathtt{R}_1\ \mathtt{R}_2 =$

$\quad \mathtt{quasi\_optical\_system\ (thin\_lens}\ \mathtt{R}_1\ \mathtt{R}_2\ \mathtt{n}_1\ \mathtt{n}_2)\ \mathtt{L}_1\ \mathtt{L}_2\ \mathtt{n}_1\ \mathtt{n}_2$

Finally, we verify the system magnification of the SheFI receiver module as follows:

Theorem 4.13 (APEX Beam Waist).

$\vdash \forall\ \mathtt{gbeam}\ \mathtt{L}_1\ \mathtt{L}_2\ \mathtt{lam}\ \mathtt{n}_1\ \mathtt{n}_2\ \mathtt{R}_1\ \mathtt{R}_2.$

$\quad \mathtt{SHeFI\_constraints\ gbeam}\ \mathtt{L}_1\ \mathtt{L}_2\ \mathtt{lam}\ \mathtt{n}_1\ \mathtt{n}_2\ \mathtt{R}_1\ \mathtt{R}_2 \Rightarrow$

$$\begin{aligned}
&\texttt{let f} = -\frac{1}{\left(\frac{\mathtt{n_2} - \mathtt{n_1}}{\mathtt{n_1}}\frac{1}{\mathtt{R_2}} - \frac{1}{\mathtt{R_1}}\right)} \texttt{ and} \\
&\quad (\mathtt{z}, \mathtt{w_0}) \;=\; \texttt{beam_origin gbeam and} \\
&\quad (\mathtt{z_n}, \mathtt{w_{0n}}) \;=\; \texttt{beam_end gbeam in} \\
&(1 \;-\; \mathtt{L_1}\frac{1}{\mathtt{f}})^2 \;-\; (\frac{1}{\mathtt{f}}(\texttt{rayleigh_range } \mathtt{w_{0in}} \texttt{ lam}))^2 \neq 0 \;\Rightarrow \\
&\frac{\mathtt{w_{0out}}^2}{\mathtt{w_{0in}}^2} \;=\; \frac{1}{\left(1 \;-\; \mathtt{L_1}\frac{1}{\mathtt{f}}\right)^2 \;-\; \left(\frac{1}{\mathtt{f}}(\texttt{rayleigh_range } \mathtt{w_{0in}} \texttt{ lam})\right)^2}
\end{aligned}$$

where `SHeFI_constraints` ensures the validity of `SHeFI_receiver_model` and beam parameters. We verify the above expression using Theorems 4.12 and 4.10. Note that Theorem 4.13 is in a general form and can further be utilized to reason about different cases such as the input and output distances (L_1 and L_2) are equal to f, or $2f$, in order to maximize or minimize the magnification depending upon the practical requirements. We can easily evaluate the real values of the parameters provided by physicists and optical engineers. Another benefit of our approach as compared to paper-and-pencil based derivations (used in [70]) is to identify all assumptions without which the expression for magnification does not hold.

4.6 Summary and Discussions

The mathematical modeling and analysis of many optical systems is based on the notion of Gaussian beams due to their important properties and accurate characterization of light radiations. In this chapter, we proposed the formalization of Gaussian beams in higher-order logic. In particular, we started with the formalization of q-parameters which are sufficient to represent an arbitrary Gaussian beam. We then verified that a Gaussian beam is a solution of the Paraxial Helmholtz Equation. This required us to formalize the notion of transverse Laplacian operator along with the verification of some lemmas about the complex-valued derivatives in HOL Light. Moreover, we formally derived the generic expression for the intensity of Gaussian beams.

We discussed our modeling approach to describe the transformation of Gaussian beams in an optical system. We also formalized some functions specifying the valid behavior of Gaussian beams at each optical interface. We then used these functions to formalize the beam transformation in a series of optical systems. Indeed, we formally proved that the classical ABCD-law of beam transformation is valid in the context of geometrical optics. Building on top of this infrastructure, we formalized widely used quasi-optical systems [31] and formally derived their generic properties related to the beam parameters. Consequently, this allowed us to analyze a cost and safety critical application, i.e., the receiver module of the the APEX telescope. The analysis application carried out in our work is accurate due to the inherent soundness of HOL theorem proving. Note that `SHeFI_constraints` are not mentioned in [70], without which Theorem 4.13 cannot be proved. This improved accuracy comes at the cost of time and efforts spent, while formalizing the underlying theory of Gaussian beams. But the availability of such formalized infrastructure significantly reduced the time required to analyze quasi-optical systems and APEX telescope application. For example, the core formalization of Gaussian beams presented in this chapter took around 2000 lines of HOL Light code. Whereas the analysis of the application, i.e., the modeling and verification of system magnification of the APEX receiver module took less than 100 lines of HOL Light code and a couple of man-hours. This reduction in the number lines of codes demonstrates the utility of our formalization for real-world applications.

In this chapter and Chapter 3, we formalized two notions of light, i.e., rays and Gaussian beams. We also built a reasoning support by verifying some necessary theorems to reason about real-world applications. Moreover, we applied each formalization (i.e., rays or beams) to a particular type of optical systems. Indeed, the analysis of vision correcting device can be performed using ray optics whereas the analysis of telescopic receiver requires the concepts of Gaussian beams. Interestingly, there is another type of optical systems called *optical resonators* that can be analyzed using both ray optics and Gaussian beams, depending upon the properties of interest. In the next chapter, we cover in detail the formalization of optical resonators and corresponding properties of interest.

Chapter 5

Formal Analysis of Optical Resonators

In this chapter, we develop a higher-order logic formalization of optical resonators and related properties[9]. The formalization is done in such a way that theorems (e.g., ray-transfer-matrices and ABCD-Law) proved in Chapters 3 and 4 remain valid for any optical resonator structure. We can divide the contributions of this chapter into four parts: 1) The generic formalization of optical resonators and their formal relation with optical systems; 2) The development of a formal framework to verify the stability conditions of optical and laser resonators; 3) The formalization of chaotic maps and verification of some theorems describing the conditions (relation among the resonator parameters) to generate chaos inside an optical resonator; 4) The applications of our formalization which include the formal stability analysis of a Fabry-Pérot resonator and a Ring resonator.

[9] The source codes of the formalizations and proofs presented in this chapter can be found in [87].

5.1 Optical Resonators

The use of optics yields smaller components, high-speed communication and huge information capacity. This provides the basis of miniaturized complex engineering systems including digital cameras, high-speed internet links, telescopes and satellites. Optoelectronic and laser devices based on optical resonators [78] are fundamental building-blocks for new generation, reliable, high-speed and low-power optical systems. Typically, optical resonators are used in lasers [81], refractometry [83] and reconfigurable wavelength division multiplexing-passive optical network (WDM-PON) systems [77]. An optical resonator usually consists of mirrors or lenses which are configured in such a way that the beam of light is confined in a closed path as shown in Figure 5.1. In general, resonators differ by their geometry and components (interfaces and mirrors) used in their design. Optical resonators are broadly classified as stable or unstable. Stability analysis identifies geometric constraints of the optical components which ensure that light remains inside the resonator. Both stable and unstable resonators have diverse applications, e.g., stable resonators are used in the measurement of the refractive index of cancer cells [83], whereas unstable resonators are used in laser oscillators for high energy applications [81]. In the last few decades, there is an increasing interest in studying the chaotic behavior of optical resonators [11]. In fact, chaotic optical resonators have been used for secure and high-speed transmission of messages in optical-fibre networks [14] and efficient light energy storage [79].

The analysis of optical resonators involves the study of infinite rays, or, equivalently, an infinite set of finite rays. Indeed, a resonator is a closed structure terminated by two reflected interfaces and a ray reflects back and forth between these interfaces. For example, consider a simple plane-mirror resonator as shown in Figure 5.2. Let m_1 be the first mirror, m_2 the second one, and f the free space in between. Then

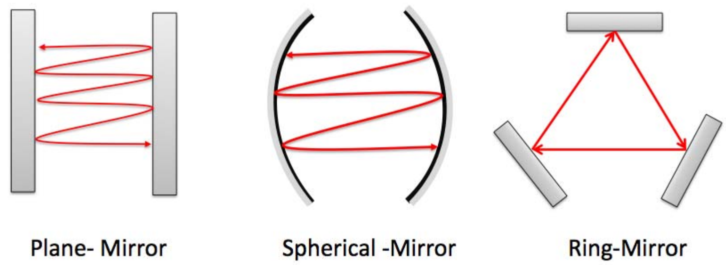

Figure 5.1: Optical Resonators

the analysis involves the study of the ray as it goes through f, then reflects on m_2, then travels back through f, then reflects again on m_1, and starts over. So we have to consider the ray going through the "infinite" path $f, m_2, f, m_1, f, m_2, f, m_1, \ldots$, or, using regular expressions notations, $(f, m_2, f, m_1)^*$. In case of stability analysis, the main purpose is to ensure that this infinite ray remains inside the cavity. On the other hand, in case of chaos generation, the main idea is to reproduce a particular pattern infinitely many times. This is equivalent to consider that, for every n, the ray going through the path $(f, m_2, f, m_1)^n$ remains inside the cavity. This allows to reduce the study of an infinite path to an infinite set of finite paths.

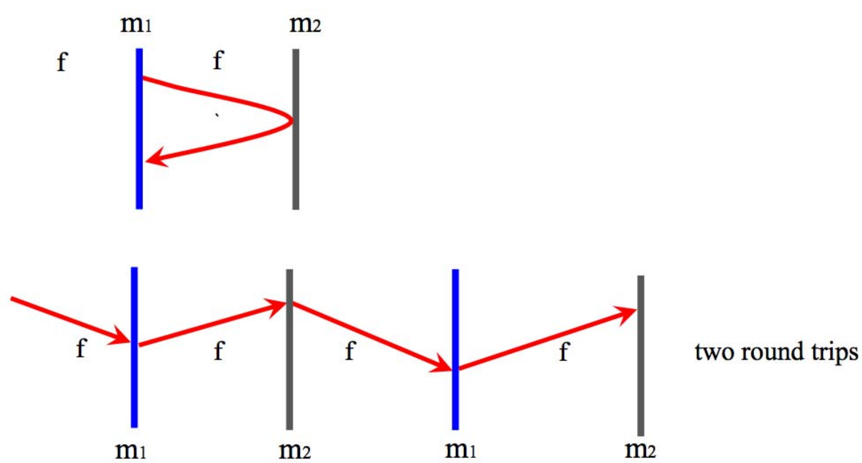

Figure 5.2: Ray Behavior Inside the Resonator

Our formalization (which is inspired by the optics literature), fixes the path of any considered ray. Since we want to consider an infinite set of finite-path rays, we should thus consider an infinite set of optical systems. This has been naturally achieved by optics engineers by "unfolding" the resonator as many times as needed, depending on the considered ray. For instance, consider again the above example of a plane-mirror resonator: if we want to observe a ray going back and forth only once through the cavity, then we should consider the optical system made of f, m_1, f, m_2; however, if we want to study the behavior of rays which make two round-trips through the cavity, then we consider a *new* optical system $f, m_1, f, m_2, f, m_1, f, m_2$ as shown in Figure 5.2; and similarly for more round-trips.

In our formalization, we want the user to provide only the minimum information so that HOL Light generates automatically the unfolded systems. Therefore, we do not define resonators as just optical systems but we define a dedicated type for them. In their most usual form, resonators are made of two reflecting interfaces and a list of components in between. We thus define the following type:

```
new_type_abbrev("resonator",
  `:interface # optical_component list # free_space # interface`);;
```

Note that the additional free space in the type definition is required because the `optical_component` type only contains one free space (the one before the interface, not the one after). For example, we can model the two mirror resonator (i.e., two plane mirrors and free space `fs`) of Figure 5.2 as follows:

Example 5.1 (Two Mirror Plan Resonator).

⊢ ∀ `fs.two_mirror_res fs = (plane,[],fs,plane):resonator`

We formally prove that a variable of type resonator can be decomposed into its constituents, i.e, interfaces, free space and a list of optical components:

Theorem 5.1 (Optical Resonator Decomposition).

```
⊢ ∀P. (∀res. P res) ⇔  (∀i1 cs fs i2. P (i1,cs,fs,i2))
```

Similar to the ray optics formalization (Chapter 3), we introduce a predicate to ensure that a value of type `resonator` indeed models a real resonator:

Definition 5.1 (Valid Optical Resonator).

```
⊢ ∀i₁ cs fs i₂.
 is_valid_resonator ((i₁,cs,fs,i₂):resonator)⇔
  is_valid_interface i₁ ∧ ALL is_valid_optical_component cs ∧
  is_valid_free_space fs ∧ is_valid_interface i₁
```

In our formalization, we develop a tactic `VALID_RESONATOR_TAC` which can automatically verify the validity of an optical resonator [87]. We now present the formalization about the unfolding of a resonator as mentioned above. The first step in this process is to define a function `round_trip` which returns the list of components corresponding to one round-trip in the resonator:

Definition 5.2 (Round Trip).

```
⊢def ∀i2 i1 cs fs.
    round_trip (i1,cs,fs,i2) =
    APPEND cs (CONS (fs,i2)
    let cs′,fs1 = optical_components_shift cs fs in
        MAP (λa. sign_cor_interface a)
            (REVERSE (CONS (fs1,i1) cs′))))
```

where `APPEND` is a HOL Light library function which appends two lists and `REVERSE` reverses the order of elements of a list. The function `optical_component_shift cs fs` shifts the free spaces of `cs` from right to left, introducing `fs` to the right;

the leftmost free space which is "ejected" is also returned by the function. This manipulation is required because unfolding the resonator entails the reversal of the components for the return trip. The function `sign_cor_interface` takes care of the correct sign of radius of curvature of spherical interfaces, i.e., `R` of convex and `-R` for concave interface. Similarly, we can define the notion of half round trip which is important in the study of chaotic optical resonators.

Definition 5.3 (Half Round Trip)**.**

$\vdash_{def}$ ∀fs2 i1 cs fs1 i2.

```
    half_round_trip (i1,cs,fs1,i2) fs2 =
    APPEND (APPEND [fs2,i1] cs) [fs1,i2],&1,&0
```

We can now define the unfolding of a resonator as follows:

Definition 5.4 (Unfold Resonator)**.**

$\vdash$ `unfold_resonator ((`i_1`,cs,fs,`i_2`):resonator) N =`

`list_pow (round_trip (`i_1`,cs,fs,`i_2`)) N,(head_index (cs,fs),0)`

where `list_pow L n` concatenates `n` copies of the list `L`. The argument `N` represents the number of times we want to unfold the resonator. Note that the output type is `optical_system`, therefore all the functions and theorems of Chapter 3 can be used for an unfolded resonator.

We verify a key property which states that `optical_components_shift` always produces a valid structure of a given optical resonator if the list of components and free space are valid. The formal statement of this property is given in the following theorem:

Theorem 5.2 (Valid Optical Component Shift).
$\vdash$ $\forall$cs fs.

```
    ALL is_valid_optical_component cs ∧
    is_valid_free_space fs
    ⇒  (let cs′,fs′ = optical_components_shift cs fs in
         ALL is_valid_optical_component cs′ ∧ is_valid_free_space fs′)
```

In Section 3.1, we described the functions `head_index` and `system_composition` which provide the refractive index of the next optical element and composition of the matrices of optical components, respectively. Here, we provide two properties of these functions which are important to reason about optical resonators:

Theorem 5.3 (Head Index for Round Trip).
$\vdash$ $\forall$i1 cs fs i2.

```
    head_index (round_trip (i1,cs,fs,i2),fs) =
    head_index (cs,fs)
```

Theorem 5.4 (System Composition Append).
$\vdash$ $\forall$cs1 cs2 fs.

```
    system_composition (APPEND cs1 cs2,fs) =
    system_composition (cs2,fs) **
    system_composition (cs1,head_index (cs2,fs),&0)
```

where Theorem 5.3 states that retrieving `head_index` does not depend on the two reflecting interfaces of a resonator whereas Theorem 5.4 describes the application of `system_composition` if the system is made of two appended component lists.

It is important to note that `unfold_resonator` provides the unfolded resonator structure which has the same type as of an optical system. Since an optical systems can be described by a matrix, unfolding a resonator is equivalent to multiplying that matrix n-times. We prove this fact in the following theorem:

Theorem 5.5 (Unfold Resonator Matrix).

```
⊢ ∀n res.
    system_composition (unfold_resonator res n) =
    system_composition (unfold_resonator res 1) pow n
```

We mainly prove this this theorem using the induction on n along with some other already proved theorems (e.g., Theorem 5.1).

This concludes our formalization of optical resonators. In summary, we formalized the basic notions for optical resonators which included the new type definition and corresponding validity constraints and helper functions such as round trip and unfolding of an optical resonator. The notable feature of our formalization is its generic nature, as we can model optical resonators with any number of optical components composed by the basic types of interfaces formalized in Section 3.1. In the next sections, we present the formalization of resonator stability and chaotic maps.

5.2 Formalization of Optical Resonator Stability

Optical resonators are usually designed to provide high quality-factor and little attenuation [78]. One of the most important design requirements is the stability, which states that the beam or ray of light remains within the optical resonator even after N round-trips as shown in Figure 5.3 (a). The stability of a resonator depends on the properties and arrangement of its components, e.g., curvature of mirrors or lenses, and distance between them. In order to determine whether a given optical resonator is stable, we need to analyze the ray behavior after many round trips. To model N round trips of light in the resonator, engineers usually "unfold" N times the resonator description, and compute the corresponding ray-transfer matrix. From the results presented in the previous section, it follows that it is equivalent to take the ray-transfer matrix corresponding to one round-trip and then raise it to the N^{th} power, as shown in Figure 5.3 (b).

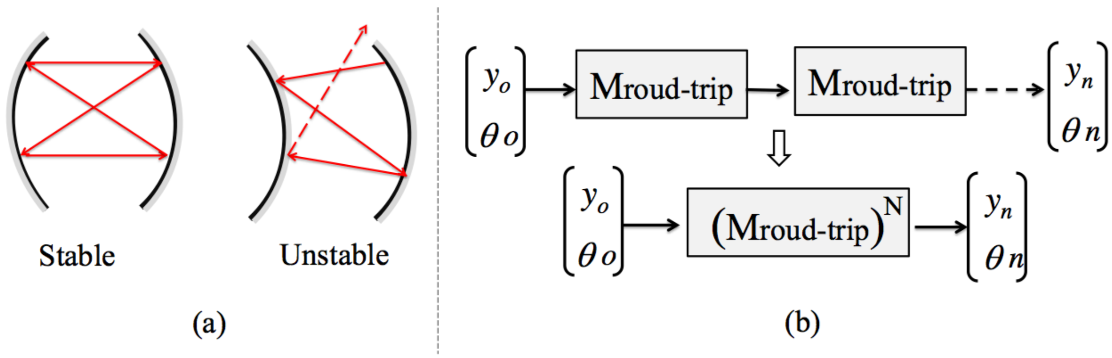

Figure 5.3: (a) Optical Resonator Types (b) Resonator Matrix After N Round-trips

We can now formally define the notion of stability. For an optical resonator to be stable, the distance of the ray from the optical axis and its orientation should remain bounded whatever is the value of N. This is formalized as follows:

Definition 5.5 (Resonator Stability)**.**

$\vdash$ $\forall$ `res.is_stable_resonator res` $\Leftrightarrow$ `(`$\forall$`(r:ray).` $\exists$`y` θ`.` $\forall$`N.`

`is_valid_ray_in_system r (unfold_resonator res N)` $\Rightarrow$

`(let` $\mathtt{y_n},\theta_{\mathtt{n}}$ `= last_single_ray r in abs(`$\mathtt{y_n}$`)` $\leq$ `y` $\wedge$ `abs(`$\theta_{\mathtt{n}}$`) <` θ`))`

where `res` and `abs` represent an optical resonator and absolute value of a real number, respectively. Note that in our definition of stability, a ray is not explicitly provided which implies that a resonator has be to stable for any injected ray.

For an arbitrary optical resonator, proving that a resonator satisfies the abstract condition of Definition 5.5 does not seem trivial at first. However, if the determinant of a resonator matrix M is 1 (which is the case in practice), optics engineers have known for a long time that having $-1 < \frac{M_{11}+M_{22}}{2} < 1$ is sufficient to ensure that the stability condition holds [78]. The obvious advantage of this criterion is that it is immediate to check. This can actually be proved by using Sylvester's Theorem [84], which states that for a matrix M=$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ such that $\mid M \mid = 1$ and $-1 < \frac{A+D}{2} < 1$, the following holds:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \frac{1}{\sin(\theta)} \begin{bmatrix} A\sin[N\theta] - \sin[(N-1)\theta] & B\sin[N\theta] \\ C\sin[N\theta] & D\sin[N\theta] - \sin[(N-1)\theta] \end{bmatrix}$$

where $\theta = cos^{-1}[\frac{A+D}{2}]$. This theorem allows to prove that stability holds under the considered assumptions: indeed, N only occurs under a sine in the resulting matrix; since the sine itself is comprised between -1 and 1, it follows that the components of the matrix are obviously bounded, hence the stability. We formalize Sylvester's theorem as follows:

Theorem 5.6 (Sylvester's Theorem).

$$\vdash \forall \texttt{ N A B C D. } \begin{vmatrix} \texttt{A} & \texttt{B} \\ \texttt{C} & \texttt{D} \end{vmatrix} = 1 \ \wedge \ -1 < \frac{\texttt{A+D}}{2} \ \wedge \ \frac{\texttt{A+D}}{2} < 1 \ \Rightarrow$$

$$\texttt{let } \theta = \texttt{acs}(\frac{\texttt{A+D}}{2}) \texttt{ in}$$

$$\begin{bmatrix} \texttt{A} & \texttt{B} \\ \texttt{C} & \texttt{D} \end{bmatrix}^{\texttt{N}} = \frac{1}{\texttt{sin}(\theta)} \begin{bmatrix} \texttt{A} * \texttt{sin}[\texttt{N}\theta] - \texttt{sin}[(\texttt{N}-1)\theta] & \texttt{B} * \texttt{sin}[\texttt{N}\theta] \\ \texttt{C} * \texttt{sin}[\texttt{N}\theta] & \texttt{D} * \texttt{sin}[\texttt{N}\theta] - \texttt{sin}[(\texttt{N}-1)\theta] \end{bmatrix}$$

We prove Theorem 5.6 by induction on N and using the fundamental properties of trigonometric functions, matrices and determinants. This allows to derive now the generalized stability theorem for any resonator as follows:

Theorem 5.7 (Stability Theorem).

$\vdash \forall$ `res.  is_valid_resonator res` $\wedge$

($\forall$ `N. let M = system_composition (unfold_resonator res 1) in`

`det M = 1` $\wedge$ `-1 <` $\frac{M_{1,1}+M_{2,2}}{2}$ $\wedge$ $\frac{M_{1,1}+M_{2,2}}{2}$ `< 1)` $\Rightarrow$

`is_stable_resonator res`

where $M_{i,j}$ represents the element at column `i` and row `j` of the matrix. The formal verification of Theorem 5.7 requires the definition of stability (Definition 5.5) along with Theorem 5.6 (Sylvester's theorem) and Theorem 5.5. We also require to prove that an unfolded resonator remains structurally valid, as given in the following theorem:

Theorem 5.8 (Valid Unfold Resonator).
$\vdash \forall$ `res. is_valid_resonator res` $\Rightarrow$

($\forall$ `n. is_valid_optical_system (unfold_resonator res n))`

Note that our stability theorem (Theorem 5.7) is quite general and can be used to verify the stability of almost all kinds of optical resonators. In the next section, we present the formalization of chaotic maps and chaos generation inside optical resonators.

5.3 Formalization of Chaos in Optical Resonators

Chaos is a special behavior which is usually observed in dynamical systems where the output response posses a sensitive behavior for minor changes in the initial conditions or system parameters. A chaotic map is a dynamic function that exhibits chaotic behavior. Generally, chaotic maps can be discrete-time or continuous-time. In recent times, chaotic behaviors have been studied in almost all fields of science and engineering, e.g., electrical circuits, chemical, biological and mechanical systems. In optics, the phenomena of chaos is concerned with the dynamic nature of light. For example, chaos can be found in the output of a laser diode and the fluctuations of light inside an optical cavity or resonator [79]. Even though chaotic systems are unpredictable but they can be used for many important performance improvements, e.g., chaos in optical systems has been used for secure and high-speed transmission of messages in optical-fibre networks [14], efficient light energy storage [79] and fast random number generation [88].

In this section, our main focus is to formalize the notion of chaos in optical resonators. The main idea behind this is to find the conditions in terms of the parameters of the resonators so that the trapped light follows some chaotic map. For example, *Duffing Map* and *Tinker Bell Map* are two important two-dimensional discrete-time chaotic maps [11], given as follows:

$$\begin{aligned} y_{n+1} &= \theta_n \\ \theta_{n+1} &= -\beta y_n + \alpha(\theta_n)^3 \end{aligned} \tag{5.1}$$

$$\begin{aligned} y_{n+1} &= (y_0 + \alpha) * y_n + (-\theta_n + \beta) * \theta_n \\ \theta_{n+1} &= (2 * \theta_n + \gamma) * y_n + \delta * \theta_n \end{aligned} \tag{5.2}$$

Equation (5.1) represents the Duffing Map while Equation (5.2) represents the Tinker Bell Map. Note that y_n and θ_n are the scalar state variables and α, β, γ, and δ represent the map parameters.

The half round trip of an optical resonator provides the ray path from one terminating optical interface to the other as formalized in Definition 5.3. Mathematically, an optical resonator is considered to be chaotic if a ray follows a particular chaotic map after every half round trip. We formalize the notion of chaotic resonator as follows:

Definition 5.6 (Chaos in Resonator).

$\vdash \forall$ `res map fs.  is_chaos_in_resonator res map fs` $\Leftrightarrow$

$(\forall$ `ray. is_valid_ray_in_system ray (half_round_trip res fs)`

$\Rightarrow$ (`let` y_0, θ_0 = `fst_single_ray ray`

`and` y_n, θ_n = `last_single_ray ray in`

`map` (y_0, θ_0) $(y_n, \theta_n)))$

where `resonator`, `map` and `fs` represent an optical resonator (`:resonator`), a chaotic map (`:A` $\rightarrow$ `bool`) and a free space, respectively. Our definition of chaotic resonator is general and can be used to model any kind of optical resonator and any type of corresponding two-dimensional chaotic map. We formalize the Duffing Map and Tinker Bell Map in HOL Light as follows:

Definition 5.7 (Duffing Map).

$\vdash \forall\ y_0\ \theta_0\ y_1\ \theta_1\ \alpha\ \beta.$

`duffing_map` $(y_0, \theta_0)\ (y_1, \theta_1)\ \alpha\ \beta \Leftrightarrow$

$y_1 = \theta_0 \wedge \theta_1 = -\beta * y_0 + (\alpha - \theta_0 * \theta_0) * \theta_0$

Definition 5.8 (Tinker Bell Map).

$\vdash \forall\ y_0\ \theta_0\ y_1\ \theta_1\ \alpha\ \beta\ \gamma\ \delta.$

`tinker_bell_map` $(y_0, \theta_0)\ (y_1, \theta_1)\ \alpha\ \beta\ \gamma\ \delta \Leftrightarrow$

$y_1 = (y_0 + \alpha) * y_0 + (-\theta_0 + \beta) * \theta_0$

$\theta_1 = (2 * \theta_0 + \gamma) * y_0 + \delta * \theta_0$

We next formally derive the Duffing Map generation conditions for an arbitrary optical resonator. We start by proving that a Duffing Map can be represented in a matrix-vector form as follows:

Theorem 5.9 (Duffing Map Matrix Form).

$\vdash \forall\ \mathtt{y}_0\ \theta_0\ \mathtt{y}_1\ \theta_1\ \alpha\ \beta.$

$$\mathtt{duffing\_map}\ (\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1)\ \alpha\ \beta \Leftrightarrow$$
$$\begin{bmatrix} \mathtt{y}_1 \\ \theta_1 \end{bmatrix} = \mathtt{duffing\_map\_matrix}\ \alpha\ \beta\ \theta_0\ \mathtt{**} \begin{bmatrix} \mathtt{y}_0 \\ \theta_0 \end{bmatrix}$$

where `duffing_map_matrix` is defined as follows:

Definition 5.9 (Duffing Matrix).

$\vdash \forall\ \alpha\ \beta\ \theta_0.$

$$\mathtt{duffing\_map\_matrix}\ \alpha\ \beta\ \theta_0 = \begin{bmatrix} 0 & 1 \\ -\beta & (\alpha\ -\ \theta_0 * \theta_0) \end{bmatrix}$$

Finally, we can prove the chaos generation inside an optical resonator in the following theorem:

Theorem 5.10 (Duffing Map Conditions).

$\vdash \forall$ `res fs` $\alpha\ \beta$.

$(\forall\ \theta_0.$ `system_composition (res_half_round_trip res fs)` $=$

`duffing_map_matrix` $\alpha\ \beta\ \theta_0) \wedge$

`is_valid_free_space fs` $\wedge$

`is_valid_resonator res` $\Rightarrow$

`is_chaos_in_resonatorres` $(\lambda(\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1).$

`duffing_map` $(\mathtt{y}_0,\theta_0)\ (\mathtt{y}_1,\theta_1)\ \alpha\ \beta)$ `fs`

where the first assumption ensures that the ray-transfer matrix of a half round trip should be equivalent to the matrix of Duffing Map, i.e., ray after each half round trip should follow the duffing map. The second and third assumptions ensure the valid architecture of a given resonator (`res`). We mainly prove this theorem using

Theorem 3.6 which states that any optical system can be described by a matrix and the following theorem, stating the validity of a half round trip of a resonator:

Theorem 5.11 (Valid Half Round Trip)**.**
⊢ ∀ fs res.

```
    is_valid_free_space fs ∧
    is_valid_resonator res ⇒
    is_valid_optical_system (half_round_trip res fs)
```

We conclude here the formalization for chaos generation in optical resonators. Note that the conditions derived in (Theorem 5.10) are general and can be utilized for arbitrarily complex optical resonators. We demonstrate the use of this formalization in the next section.

5.4 Applications

In this section, we consider two real-world optical resonators namely the Fabry Pérot resonator with a fiber rod lens and a ring resonator. We utilize our formalization of optical resonators, stability and chaos in optical resonators to formally verify the sability conditions for the Fabry Pérot resonator and chaotic map generation conditions for the ring resonators.

5.4.1 Formal Stability Analysis of Fabry Pérot Resonator

In order to bring optics technology to the market, a lot of research has been done toward the integration of low cost, low power and portable building blocks in optical systems. One of the most important such building blocks is the Fabry Pérot (FP) resonator [78]. Originally, this resonator was used as a high resolution interferometer in astrophysical applications. Recently, the FP resonator has been realized as a microelectromechanical (MEMS) tuned optical filter for applications in reconfigurable Wavelength Division Multiplexing [77].

Due to diverse applications of FP resonators, different architectures have been proposed in the open literature. The main limitation of traditional designs is the instability of the resonators which prevents their use in many practical applications (e.g., refractometry for cancer cells). Recently, a state-of-the-art FP core architecture has been proposed which overcomes the limitations of existing FP resonators [65, 63]. In the new design, cylindrical mirrors are combined with a fiber rod lens (FRL) inside the cavity, to focus the beam of light in both transverse planes as shown in Figure 5.4 (a). The fiber rod lens is used as light pipe which allows the transmission of light from one end to the other with relatively small leakage. Building a stable FP resonator requires the geometric constraints to be determined in terms of the radius of curvature of mirrors (R) and the free space propagation distance (d_{free_space}) using the stability analysis.

The design shown in Figure 5.4 (a) has a 3-dimensional structure. We can still apply the ray-transfer-matrix approach presented in Section 5.2 to analyze the stability by dividing the given architecture into two planes, i.e., XZ and YZ planes. Now, the stability problem becomes a couple of planar problems which are still valid since the ray focusing behaviors in both directions (XZ and YZ) are decoupled. This is merely a consequence of the decomposition of Euclidean space vectors into a basis. This can be seen in Figures 5.4 (b) and 5.4 (c), where the resonator is divided into two cross-sections.

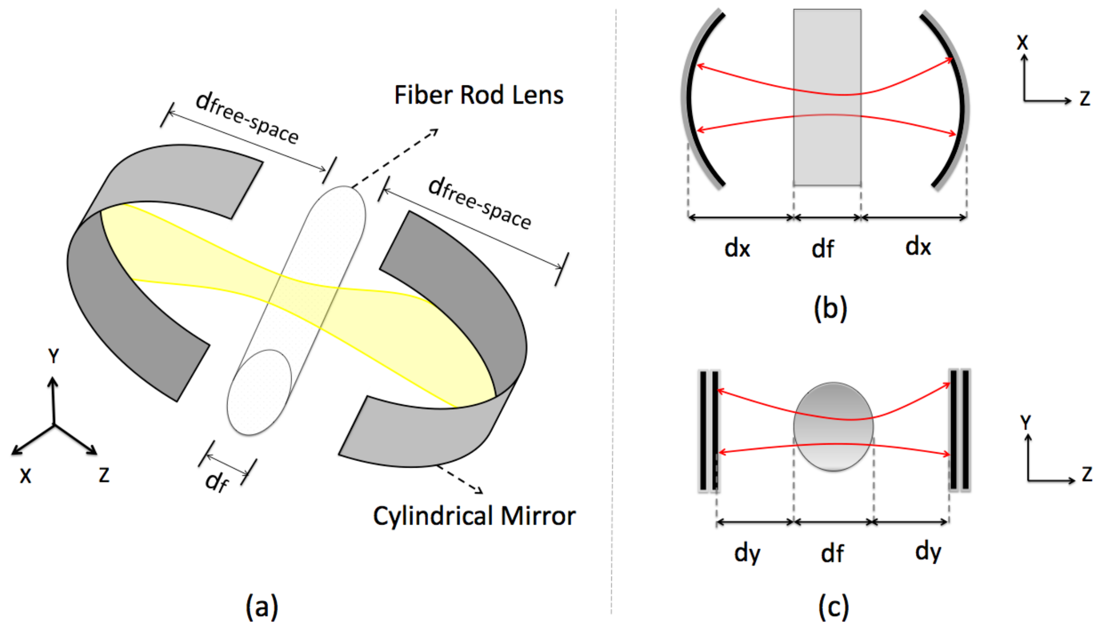

Figure 5.4: FP Resonator with FRL (a) 3-Dimensional Resonator Design (b) Cross-Section view in the XZ Plane (c) Cross-Section view in the YZ Plane [65]

Stability Constraints in XZ-Plane

In the XZ cross-section (Figure 5.4 (b)), the focusing is done by the curved mirrors. The fiber rod lens acts as a refracting slab with width d_f and refractive index n_f. A ray that makes a round-trip in the cavity undergoes (from left to right) the following steps:

- Propagation through free space of length d_x and refractive index 1.
- Refraction from free space to the fiber rod lens.
- Propagation within the fiber rod lens of length d_f and refractive index n_f.
- Refraction from the fiber rod lens to free space.
- Reflection from the spherical interface.

We formally model this system as follows:

Definition 5.10 (Formal Model of FP-FRL in XZ Plane).

```
⊢def ∀nf df dx R.
   fp_frl_resonator_xz R dx df nf =
   spherical_reflected R,
   [(1,dx),plane_transmitted; (nf,df),plane_transmitted],
    (1,dx),spherical_reflected
```

The function `fp_frl_resonator_xz` takes following parameters: radius of curvature of mirror (`R`), free space length (`dx`), length of fiber rod lens (`df`) and refractive index (`nf`) and returns an optical resonator. We check the validity of the model `fp_frl_resonator_xz` under realistic geometric constraints, such as the fact that the refractive index (`nf`) and lengths of free space propagation (`dx` and `df`) should be greater than 0.

Theorem 5.12 (Validity of FP-FRL in XZ-Plane).

```
⊢ ∀R dx df nf.
    ¬(R = 0) ∧ 0 < dx ∧ 0 < df ∧ 0 < nf ⇒
    is_valid_resonator (fp_frl_resonator_xz R dx df nf)
```

We next verify the equivalent matrix expression of the FP resonator in the XZ plane as follows:

Theorem 5.13 (Matrix for FP-FRL in XZ-Plane).

$$\vdash \forall\ \mathtt{R\ dx\ df\ nf.}\ \ 0 < \mathtt{dy} \wedge 0 < \mathtt{df} \wedge 0 < \mathtt{nf} \Rightarrow$$

$$\mathtt{system\_composition\ (fp\_frl\_resonator\_xz\ R\ dx\ df\ nf)} = \begin{bmatrix} 1 - \frac{2\ *\ (\mathtt{df}\ +\ 2*\mathtt{dx}*\mathtt{nf})}{\mathtt{nf}*\mathtt{R}} & 2*\mathtt{dx}\ +\ \frac{\mathtt{df}}{\mathtt{nf}} \\ -\frac{2}{\mathtt{R}} & 1 \end{bmatrix}$$

The verification of this theorem is mainly based on rewriting with the definitions followed by an automated simplification tactic `common_prove` described in Section 3.3. The following theorem is then easy to prove by making use of the results already obtained in Sections 3.4 and 5.2.

Theorem 5.14 (Ray-Transfer-Matrix Model in XZ-Plane).

$\vdash \forall$ R dx df nf. R $\neq$ 0 $\wedge$ 0 < dx $\wedge$ 0 < df $\wedge$ 0 < nf $\Rightarrow$

($\forall$ ray.is_valid_ray_in_system ray (fp_frl_resonator_xz R dx df nf)

$\Rightarrow$ let (y_0,θ_0),(y1,theta1),rs = ray in

(y_n,θ_n) = last_single_ray ray in

$$\begin{bmatrix} y_n \\ \theta_n \end{bmatrix} = \texttt{system\_composition (fp\_frl\_resonator\_xz R dx df nf)**} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix})$$

where `last_single_ray` is a function that takes a `ray` as input and returns the last pair (distance from the optical axis y and the orientation θ) of the `ray`. Finally, we formally verify the stability of the FP resonator in the XZ plane as follows:

Theorem 5.15 (Stability in XZ-Plane).

$\vdash \forall$ R dx df nf. R $\neq$ 0 $\wedge$ 0 < dx $\wedge$ 0 < df $\wedge$ 0 < nf

$$0 < \frac{2 * dx + \frac{df}{nf}}{R} \wedge \frac{2 * dx + \frac{df}{nf}}{R} < 2 \Rightarrow$$

is_stable_resonator (fp_frl_resonator_xz R dx df nf)

where the first four assumptions just ensure the validity of the model description. The following two provide the intended stability criteria. The verification of Theorems 5.15 requires Theorems 5.14 and 5.7 along with some fundamental properties of matrices and arithmetic reasoning.

Stability Constraints in YZ-Plane

In the YZ cross-section (Figure 5.4 (c)), the curved mirrors become straight mirrors and the fiber rod lens acts as a converging lens. In this case, a ray that makes a round-trip in the cavity undergoes (from left to right) the following steps:

- Propagation through free space of length d_y and refractive index 1.
- Refraction through the curved interface with radius of curvature $\frac{d_f}{2}$.
- Propagation through a free space of length d_y.
- Refraction through the curved interface with radius of curvature $-\frac{d_f}{2}$.
- Propagation through free space of length d_y and refractive index 1.
- Reflection from the plane interface.

We formally model this system description as follows:

Definition 5.11 (FP-FRL Resonator in YZ-Plane)**.**

```
⊢def ∀dy nf df.
    fp_frl_resonator_yz dy nf df =
    plane_reflected,
    [(1,dy),spherical_transmitted (df / 2);
     (nf,df),spherical_transmitted (−df / 2)],(1,dy),
    plane_reflected
```

where the function `fp_frl_resonator_yz` takes as parameters the free space of length (`dy`), the length of fiber rod lens (`df`) and the refractive index (`nf`) and returns an optical resonator. We verify that `fp_frl_resonator_yz` indeed represent a valid resonator architecture as follows:

Theorem 5.16 (Validity of FP-FRL in YZ-Plane)**.**

```
⊢ ∀dy nf df.
    0 < dy ∧ 0 < df ∧ 0 < nf ⇒
    is_valid_resonator (fp_frl_resonator_yz dy nf df)
```

We next verify the equivalent matrix expression for the FP resonator in the YZ plane as follows:

Theorem 5.17 (Matrix for FP-FRL in YZ-Plane)**.**

$\vdash \forall$ `dy df nf.` $\;$ `0 < dy` $\wedge$ `0 < df` $\wedge$ `0 < nf` $\Rightarrow$

$$\texttt{system\_composition (fp\_frl\_resonator\_yz dy df nf) =}$$
$$\begin{bmatrix} -\frac{\texttt{df}*(-2 \; + \; \texttt{nf}) \; + \; 4*\texttt{dy}*(-1 \; + \; \texttt{nf}))}{\texttt{df} \; * \; \texttt{nf}} & \frac{(\texttt{df} \; + \; 2*\texttt{dy}) \; * \; (\texttt{df} \; - \; 2*\texttt{dy}*(-1 \; + \; \texttt{nf}))}{\texttt{df} \; * \; \texttt{nf}} \\ \frac{4 \; - \; 4*\texttt{nf}}{\texttt{df} \; * \; \texttt{nf}} & -\frac{\texttt{df}*(-2 \; + \; \texttt{nf}) \; + \; 4*\texttt{dy}*(-1 \; + \; \texttt{nf})}{\texttt{df} \; * \; \texttt{nf}} \end{bmatrix}$$

Finally, we formally verify the stability of the FP resonator in the YZ plane as follows:

Theorem 5.18 (Stability in YZ-Plane)**.**

$\vdash \forall$ `dy df nf.` $\;$ `0 < dy` $\wedge$ `0 < df` $\wedge$ `0 < nf`

$$0 < 1 \; - \; \frac{2}{\texttt{nf}} \; + \; (4 * \frac{\texttt{dy}}{\texttt{df}}) * (1 - \frac{1}{\texttt{nf}}) \; \wedge \; 1 \; - \; \frac{2}{\texttt{nf}} \; + \; (4 * \frac{\texttt{dy}}{\texttt{df}}) * (1 - \frac{1}{\texttt{nf}}) < 1$$

$\Rightarrow$ `is_stable_resonator (fp_frl_resonator_yz dy df nf)`

The first three assumptions just ensure the validity of the model description. The two following ones provide the intended stability criteria. The formal verification of Theorem 5.18 requires Theorems 5.17 and 5.7 along with some fundamental properties of matrices.

It is important to note that for the case of the FP resonator with fiber rod lens, we have obtained two sets of stability constraints, i.e., one in the XZ plane (Theorem 5.15) and another in the YZ plane (Theorem 5.18). In fact, the resonator can be stable in one plane and unstable in the other. Therefore, the stability constraints in both planes have to be satisfied. In real-world scenarios, the most fundamental step is to find the allowable values of the parameters associated with the resonators such as radius of convergence and the width of free space. The verification of above theorems has been done in a generic form, i.e., we derive the stability constraints for arbitrary values of R, d_x, d_f and n_f. This is one of the main advantages of theorem proving based stability analysis of optical resonators.

Automated Tactic for Stability Ranges

We further demonstrate the strength of our approach by the verification of stability constraints used as the guidelines for the fabrication of FP resonators, reported in [64]. In the design, it is considered that $d_x = d_y = d$ and the values of n_f and d_f are fixed and equal to 1.47 and $125\mu m$, respectively. The main goal is to find the ranges of d where the resonator is stable in both planes. We developed a tactic (`STABILITY_PROVE_TAC`) which can automatically verify that the resonator is stable under the given range of parameters. For example, one particular case given as follows:

Input:

```
STABILITY_PROVE_TAC `
(d IN real_interval (#27.5 * #0.000001,#35 * #0.000001)
                 ==> is_stable_resonator (FP_XZ_RES d))`;;
```

Output:

```
CPU time (user): 4.878
val it : thm =
  |- d IN real_interval (#27.5 * #0.000001,#35 * #0.000001)
     ==> is_stable_resonator (FP_XZ_RES d)
```

Table 5.1 provides the typical dimensional ranges, corresponding to different practical situations: in the first case, stability is not reached at all, in the second case, we have stability along the X axis and instability along the Y axis, in the third case, we have instability along the X axis and stability along the Y axis. In the fourth case, we fulfill the stability conditions along both X and Y axis.

Table 5.1: Stability Ranges for FP Resonator

R(μm)	**d(μm)**	**Stability in XZ Plane**	**Stability in YZ Plane**
140	$133 < d$	NO	NO
140	$(27.5, 35)$	YES	NO
140	$(97.5, 132.9)$	NO	YES
140	$(38, 97)$	YES	YES

5.4.2 Chaos Generation Conditions for Ring Resonators

In the last few decades, optical phase conjugation (OPC) has been widely studied in lasers and nonlinear optics [20]. Physically, an OPC describes the relation among light beams propagating in opposite directions with reversed wave front and identical transverse amplitude distributions. The main applications of phase conjugation are the high-brightness laser oscillator/amplifier systems, laser target-aiming systems, long distance optical fiber communications with ultra-high bit-rate. In this section, we consider a ring resonator based OPC [11] which mainly involves the study of two-dimensional chaotic maps. This is usually done by placing an intracavity element which is responsible of generating a particular chaotic map whose state is determined by its previous state. Our main intent is to formally show that the introduction of a specific element within a ring-phase-conjugated resonator can produce a Duffing Map inside the resonator.

The architecture of ring-phase conjugated resonator consisting of two plane mirrors, a phase conjugate mirror along with an unknown optical element is shown in Figure 5.5. As described in Section 3.1, our datatype for optical interfaces is general and we can model an unknown element by a matrix [a,b,c,d]. The half round-trip of a ray is based on the following steps:

- Propagation through free space of length $\frac{L}{2}$ and refractive index n.

- Propagation through an unknown element which has parameters, a, b, c and d.
- Propagation through free space of length $\frac{L}{2}$ and refractive index n.
- Reflection from plane mirror.
- Propagation through free space of length L and refractive index 1.
- Reflection from phase conjugated mirror (PCM).
- Propagation through free space of length L and refractive index 1.

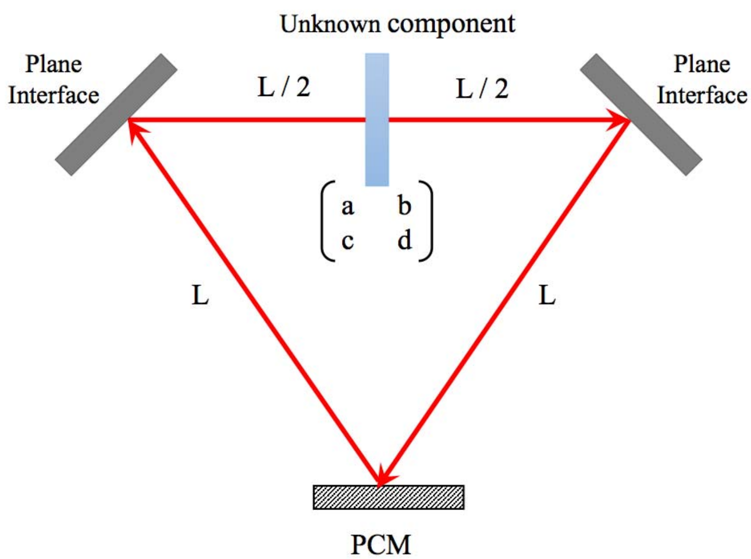

Figure 5.5: Phase Conjugated Ring Resonator

We formally define the structure of a phase-conjugated ring resonator as follows:

Definition 5.12 (Phase Conjugated Ring Resonator)**.**

```
⊢def ∀ a b c d n L.
    ring_resonator a b c d L n =
    plane_reflected,
    [(n,L/2),unknown a b c d; (n,L/2),plane_reflected],(n,L),
    pcm
```

where a, b, c and d represent the parameters of the unknown component whereas n and L represent the refractive index and length of the free space, respectively. We first verify the validity of the ring resonator as follows:

Theorem 5.19 (Valid Ring Resonator).

$\vdash \forall$ n L.

  $0 < n \wedge 0 \leq L \Rightarrow$

  is_valid_resonator (ring_resonator a b c d L n)

The proof of this theorem is automatically done by the tactic VALID_RESONATOR_TAC [87].

At this point, we have already developed the formal model of the ring resonator along with the verification of its structural constraints. Our ultimate goal in the analysis of this resonator is to formally derive the conditions on a, b, c and d, so that the rays inside the resonator follows the duffing map. This leads to the following theorem:

Theorem 5.20 (Chaos Generating Conditions for Ring Resonator).

$\vdash \forall$ n L α β.

$(\ \forall\ \theta_0.\ \mathtt{a} = -\frac{(3*\beta*\mathtt{L})}{2}\ \wedge$

$\mathtt{b} = \frac{1}{4} * (4 + 6 * \alpha * \mathtt{L} + 9 * \beta * \mathtt{L} * \mathtt{L} - 6 * \mathtt{L} * \theta_0 * \theta_0)\ \wedge$

$\mathtt{c} = \beta\ \wedge\ \mathtt{d} = -\alpha - \frac{3}{2} * \mathtt{L} * \beta + \theta_0 * \theta_0)\ \Rightarrow$

is_chaos_in_resonator (ring_resonator a b c d L n)

$(\lambda(y_0,\theta_0)\ (y_1,\theta_1).$duffing_map $(y_0,\theta_0)\ (y_1,\theta_1)\ \alpha\ \beta)\ (n,L)$

The proof of this theorem is mainly based on Theorems 5.10 and 5.19.

In this application, we formally proved that the introduction of a particular map generating device in a ring optical phase-conjugated resonator can generate a ray with the behavior of a specific two-dimensional chaotic map (e.g., Duffing Map).

In particular, we explicitly derived the conditions on the unknown component [a,b,c,d] which are necessary to produce the Duffing Map inside the resonator. The procedure described in this section can be used to derive similar conditions for other chaotic maps such as Tinker Bell Map.

5.5 Summary and Discussions

In this chapter, we described the formalization of optical resonators in HOL Light. We discussed the classical structure of optical resonators along with some analysis techniques. We used the ray optics theory to formalize the notions of optical resonators and some commonly used functions, e.g., round trip and unfolding of the resonator. We then formalized two important concepts, i.e., stability and chaotic maps generation in optical resonators. In particular, we developed a procedure to verify the stability constraints for arbitrary optical resonators. Similarly, we formalized the notion of chaotic resonators and some commonly used two-dimensional chaotic maps, i.e, Duffing Map and Tinker Bell Map. In order to strengthen theorem proving based analysis of optical resonators, we formally verified generic theorems stating important physical and mathematics concepts. For example, we proved that unfolding of a resonator is equivalent to composing the ray-transfer matrix of the round trip of that resonator. We verified the generic stability theorem which is valid for any type of optical resonator in the context of geometrical optics. On the similar lines, we verified the generic conditions to produce Duffing Map inside an optical resonator.

We demonstrated the use of our formalization by analyzing some real-world optical resonators: 1) The stability analysis of a two-dimensional FP resonator with fiber rod lens. This included a detailed verification of the stability constraints in XZ and YZ plane. We also automatized the procedure for the verification of the stability ranges for FP resonators. 2) Formal verification of the chaos generating conditions for optical phase-conjugated ring resonator. Mainly, we verified the analytical expressions for the parameters of the unknown component in ring resonators which are required to produce a Duffing Map. Moreover, we formally analyzed a couple of other important resonator architectures such as FP resonators with curved mirrors and Z-shaped resonators. For the sake of conciseness, we omitted the details of the analysis of these two resonators where more details can be found in the source code of our development [87]. The overall development presented in this chapter took around 1500 lines of HOL code which has been significantly reduced as compared to our initial development. The main reason behind this reduction is the development of automated tactics and our experience of formalizing geometrical optics (Chapter 3). Interestingly, the formal analysis of the applications required very less time and verification efforts as compared to the original formalization of optical resonators, stability and chaotic resonators. For example, the analysis of the phase-conjugated ring resonators took only 20 lines of HOL Light code which demonstrates the strength of our formalization.

It is important to note that all the theorems in our formalization are verified under universal quantification of systems parameter (e.g., radius of curvature and width of free space) unlike the other numerical approaches (e.g., reZonator [75], a numerical analysis software for resonators) where the results hold only for specific values of these parameters. The main benefit of formal proofs is that all the underlying assumptions can be seen explicitly and proof-steps can be verified mechanically using a theorem prover. In spite of the fact that our approach requires significant time to formalize the underlying theories of optics, we believe that our formal development can replace some time consuming simulations and error-prone paper-and-pencil based proofs. For example, verification of the optical resonator stability is time consuming because of the involvement of infinite set of rays. On the other hand, resonator stability can be verified in a very short time using the infrastructure developed in this chapter. Lastly, it is worth mentioning that the formal stability analysis of the FP resonator with fiber rod lens allowed us to find some discrepancy in the paper-and-pencil based proof approach presented in [65]. Particularly, the order of matrix multiplication in Equations (16) and (24) in [65] should be reversed, so as to obtain correct stability constraints. This is one of the main strengths of theorem proving where the soundness is assured for every step during the proof of system properties.

At this point, we have covered all parts of our proposed framework (Figure 1.2) for the analysis of geometrical optics. In the next chapter, we conclude this thesis and highlight some future research directions.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Optical systems are widely used in safety-critical applications such as aerospace, telecommunication and biomedical systems. The verification of such systems is usually performed by informal techniques (e.g., numerical simulation and paper-and-pencil based proofs) which may result in erroneous designs. In the last decade, formal methods have been used to overcome the above mentioned limitations for the verification of a variety of hardware and software systems. However, the use of formal methods, in particular theorem proving, in the analysis of optical systems is very rare and does not support the notions of geometrical optics. In this thesis, we proposed to leverage upon the soundness and accuracy of higher-order-logic theorem proving for the analysis of geometrical optics. The main contribution of the proposed framework for geometrical optics is two-fold: First, the facility to formally model optical systems in a systematic way without any restriction on the number of optical components. Second, the development of an infrastructure to reason about the properties of rays and beams including the formalization of commonly used mathematical models for optical components.

Towards the development of the proposed framework, we formalized the notions of rays and beams along with fundamental optical interfaces and components. We then formalized the mathematical concepts behind the propagation of ray and beams in arbitrarily complex optical systems. We used this infrastructure to build the dedicated HOL theories of optical resonators, optical imaging systems and Quasi-optical systems. During the course of this formalization, we have also made efforts to provide effective automation using derived rules and tactics, so that the application to a particular system does not involve the painful manual proofs often required for interactive (higher- order logic) theorem proving systems.

We demonstrated the strength of our proposed framework by conducting the formal analysis of several important and widely used practical systems. To illustrate the use of our framework in the domain of biomedical systems, we carried out the formal analysis of an optical instrument (ophthalmic device) used to compensate the ametropia of an eye. Optical resonators are widely used in micro-electromechanical system (MEMS), tuned optical filters and optical bio-sensing devices. Considering these critical applications, we formally analyzed three application architectures of Fabry Pérot resonators, i.e., non-symmetric, symmetric and two-dimensional fiber rod lens (FRL) induced cavity. Moreover, we formally verified the chaos generating conditions of a generic optical phase-conjugated ring resonator. Finally, we utilized the generic formalization of quasi-optical systems to analyze the receiver module of the real-world Atacama Pathfinder Experiment (APEX) telescope.

The formal analysis of geometrical optics along with the above mentioned real-world applications provide some thoughtful indications: theorem proving systems have reached to the maturity, where complex physical models can be expressed with less efforts than ever before; and formal methods can assist in the verification of futuristic optical systems which are largely becoming the part of critical applications such as military setups, biomedical surgeries and space missions. Indeed the use of formal methods is more important in the applications, where failures directly lead to safety issues such as in aerospace and biomedical devices. For example, the mission management system of Boeing F/A-18E is linked using a optics technology [89]. However, the question of the utilization of higher-order-logic theorem proving in an industrial settings (particularly, physical systems) still persists due to the huge amount of time required to formalize the underlying theories. We believe that an important factor is the gap between the theorem proving and engineering communities which limits its usage in industrial settings. For example, it is hard to find engineers (or physicists) with theorem proving background and vice-versa. One of the several solutions to tackle this issue is the continuous formal development of optics theories including the libraries of the most frequently used optical components and devices which can ultimately reduce the cost of using formal methods (particularly theorem proving) as an integral part of the physical systems design and verification. The work presented in this thesis can be considered as a one step towards this goal with more efforts to follow in the same or closely related disciplines such as quantum optics, photonic signal processing and optoelectronics.

6.2 Future Work

Geometrical optics is the most fundamental theory of optics which can be used to study some important physical aspects of optical and laser systems. Indeed, almost all optical design analysis tools provide the facility to analyze geometrical optics based models. The formalization and verification results, presented in this thesis, can be used as a complementary approach to less accurate numerical programs and traditional paper-and-pencil based proofs. In the following, we list some future research directions based on our experience and lessons learned during the course of this thesis:

- The work presented in this thesis involves interactive proofs where we needed to supply most of the proof steps to the HOL Light theorem prover. Moreover, sometimes we needed to do the similar proof steps for different theorems which was a cumbersome activity. Recently, a learning-assisted automated reasoning support, HOLyHammer (HH) [51] has been developed for HOL Light, which can also be applied to our formalization to see the future of such automation tools for optics and physics formalization. Indeed, we already performed some experiments to evaluate its efficiency on the formalization of ray optics [52]. The performance of HH was 45% (217 problems solved out of 482 problems) in the fully automated mode when the relevant premises are chosen automatically by machine learning, and seven different combinations of premise selection and automated theorem provers (ATPs) were needed for this [52]. We believe that developing a dedicated automated reasoner for optics formalization is an interesting direction of research. This can further open the door to interdisciplinary research among the formal methodists and physicists.

- The use of higher-order logic theorem proving only ensures the accuracy of the analysis steps as every formal proof can be traced back to the fundamental axioms and inference rules of mathematics. However, the formal analysis of real-world systems involves mathematical models which usually represent an approximated behavior of physical phenomena. In order to formally treat the approximations made by physicists, we need to consider non-standard analysis and asymptotic notations. For example, small angle approximation (or paraxial approximation) entails that $\sin(\theta) \approx \theta$, and it can be treated using asymptotic notations. Interestingly, both non-standard analysis [29] and asymptotic notations [15] are available in Isabelle/HOL [69]. Our work can be extended by using these concepts which will bring more rigor to the formal models of optical systems. Similar concepts can also be used to formally prove that ray optics models are approximations of wave and electromagnetic optics models.

- It is possible to improve the traditional stability analysis method by handling infinite paths of rays by working directly with all possible paths of a ray, and thus avoiding the use of unfolding. In particular, this requires a more general treatment of optical interfaces without explicitly mentioning their behavior, i.e., transmitted or reflected. This is a very interesting direction of research since it would even go beyond what optics engineers currently do.

- In the optics literature, many systems other than geometrical optics can also be modeled based on the transfer-matrix approach. Some examples of such systems are periodic optical systems [81], frequency division multiplexing/demultiplexing [22] and polarization based optical systems [86]. The formalization process described in this thesis can be used as a guide for the formal analysis of above mentioned systems. It may require some modifications about new datatypes for underlying components and corresponding physical behavior. For example, the analysis of polarization requires the formalization of different types of polarizers (e.g., linear polarizer for x and z direction) along with the formalization of Jones and Muller calculus [48].

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• Technical Report

- **Bio-Tr1** U. Siddique and S. Tahar, "Stability Verification of Optical and Laser Resonators in HOL Light", *Technical Report*, Department of Electrical and Computer Engineering, Concordia University, Montreal, QC, Canada, 2014. [12 Pages]
- **Bio-Tr2** S. M. Beillahi, U. Siddique, and S. Tahar, "On the Formalization of Signal-Flow-Graphs in HOL", *Technical Report*, Department of Electrical and Computer Engineering, Concordia University, Montreal, QC, Canada, 2014. [24 Pages]

Honors and Awards

- The Fonds de recherche du Quebec - Nature et technologies (FRQNT), PhD Scholarship for international students (overall rank 5) [2013-2015].
- Concordia Accelerator Award, Concordia University, Montreal, QC, Canada [2015].
- Best doctoral project presentation, Conference of Intelligent Computer Mathematics (CICM-2014), Coimbra, Portugal.
- Federation Logic Conference (FloC) travel grant to attend Vienna Summer of Logic (VSL-2014).
- Travel grant to attend Conference of Intelligent Computer Mathematics (CICM-2014, CICM-2015).
- Travel grant to attend Optical Engineering + Applications, Society for Optics and Photonics (SPIE-2013), San Diego, California, USA.
- Partial tuition fee scholarship from Concordia University, Montreal, QC, Canada [2012-2014].
- PhD scholarship from National University of Sciences and Technology (NUST), Pakistan, *Declined* [2011-2014].
- MS scholarship from National University of Sciences and Technology (NUST), Pakistan [2009-2011].
- Student Certificate of Merit for research paper titled "The Efficient Implementation of S8 AES Algorithm" in International Conference of Computer Science and Engineering, London, UK [2011].