Overconvergent Eichler-Shimura Isomorphisms on Shimura Curves over a Totally Real Field

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A Thesis
In the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy (Mathematics) at
Concordia University
Montreal, Quebec, Canada

September 2016

CONCORDIA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

This is to verify that the thesis prepared

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Entitled: **Overconvergent Eichler-Shimura Isomorphisms on Shimura Curves over a Totally Real Field**

and submitted in partial fulfilment of requirements for the degree of

Docotor of Philosophy (Mathematics)

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ABSTRACT

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Concordia University, 2016

In this work we construct overconvergent Eichler-Shimura isomorphisms on Shimura curves over a totally real field F. More precisely, for a prime $p > 2$ and a wide open disk $\mathfrak{U}$ in the weight space, we construct a Hecke-Galois-equivariant morphism from the space of families of overconvergent modular symbols over $\mathfrak{U}$ to the space of families of overconvergent modular forms over $\mathfrak{U}$. In addition, for all but finitely many weights $\lambda \in \mathfrak{U}$, this morphism provides a description of the finite slope part of the space of overconvergent modular symbols of weight λ in terms of the finite slope part of the space of overconvergent modular forms of weight $\lambda + 2$. Moreover, for classical weights these overconvergent isomorphisms are compatible with the classical Eichler-Shimura isomorphisms.

Acknowledgements

Fist, I would like to thank my supervisors Dr. Adrian Iovita greatly, for the support, guidance, patience and encouragement throughout my study at Concordia University. Dr. Iovita is an enthusiastic and creative researcher who taught and led me to the field of Number Theory, and I have been really enjoying it. I wish him nothing but the best.

I thank the members of the Number Theory Seminars holding at McGill University, Concordia University,University of Montreal and UQAM for helpful discussions. The great and interesting presentations helped my mathematical study a lot.

Thank Dr. D. Barrera sincerely for the help and fruitful discussions. It has been nice to be working with him.

I am grateful to all the members of my thesis examining committee, thank you for your time and work.

I really appreciate the courses and advice from the members in the Department of Mathematics and Statistics. I thank Concordia University for admitting me to the PhD program and for the financial support.

I thank my parents and all other family members, for their care and selfless support. I would like to thank my husband Zhenyang, for the love and support all these years.

Contents

Chapter 1

Introduction

Let us review the classical Eichler-Shimura isomorphism on modular curves. Fix a prime $p \geq 3$, an integer $N \geq 3$ such that $(p, N) = 1$ and let $\Gamma := \Gamma_1(N) \cap \Gamma_0(p) \subseteq \mathrm{SL}_2(\mathbb{Z})$. Let $X := X(N, p)$ be the modular curve for the group Γ over $\mathrm{Spec}(\mathbb{Z}[1/(Np)])$, $E \to X$ the universal semi-abelian scheme and $\omega := \omega_{E/X} = e^*(\Omega^1_{E/X})$ the invertible sheaf on X of invariant 1-differentials, where $e : X \to E$ is the zero section. We have the following theorem.

Theorem 1.0.1. *(Deligne [1971b]) For every nonnegative integer k, we have a natural isomorphism:*

$$\mathrm{H}^1(\Gamma, V_{k,\mathbb{C}}) \cong \mathrm{H}^0(X_{\mathbb{C}}, \omega^{k+2}) \oplus \overline{\mathrm{H}^0(X_{\mathbb{C}}, \omega^k \otimes \Omega^1_{X/\mathbb{C}})},$$

where $V_{k,\mathbb{C}} := \mathrm{Sym}^k(\mathbb{C}^2)$ and the overline on the second term of the right hand side is the complex conjugation. Moreover, this isomorphism is compatible with the action of the Hecke operators.

The elements of $\mathrm{H}^1(\Gamma, V_{k,\mathbb{C}})$ are called classical weight k modular symbols. The elements of $\mathrm{H}^0(X_{\mathbb{C}}, \omega^{k+2})$, respectively $\mathrm{H}^0(X_{\mathbb{C}}, \omega^k \otimes \Omega^1_{X/\mathbb{C}})$ are called classical modular, respectively cusp forms of weight $k+2$. The classical Eichler-Shimura isomorphism describes the space of weight k modular symbols in terms of elliptic modular forms of weight $k+2$. In Faltings [1987] a more arithmetic version of this isomorphism is presented. Now fix a complete discrete valuation field L of characteristic 0, ring of integers $\mathcal{O}_L$ and residue field $\mathbb{L}$, a perfect field of characteristic p. We denote by $\mathbb{C}_p$ the p-adic completion of an algebraic closure $\overline{L}$ of L. Let us consider the modular curve X over the p-adic field L and for a nonnegative integer k, let $V_k := \mathrm{Sym}^k(\mathbb{Q}_p^2)(1)$, where $(\cdot)(k)$ is the Tate twist, with the natural action of Γ and $G_L = \mathrm{Gal}(\overline{L}/L)$. The following theorem is obtained in Faltings [1987].

Theorem 1.0.2. *With the above notations we have a canonical isomorphism compatible with the actions of G_L and all Hecke operators*

$$\mathrm{H}^1\left(\Gamma, V_k\right) \otimes_L \mathbb{C}_p \cong \Big(\mathrm{H}^0(X, \omega^{k+2}) \otimes_L \mathbb{C}_p\Big) \oplus \Big(\mathrm{H}^1(X, \omega^{-k}) \otimes_L \mathbb{C}_p(k+1)\Big).$$

In Coleman [1997] and Coleman and Mazur [1998], the authors show that modular eigenforms of finite slope can be p-adically interpolated. In fact there exists a geometric object parameterizing such modular eigenforms called the *eigencurve*. On the other hand, modular symbols have interesting p-adic properties. The work in Stevens [2015] defines overconvergent modular symbols and shows that classical modular symbols can be interpolated in p-adic families.

A natural question one could raise is if Faltings' Eichler-Shimura isomorphism could be p-adically interpolated in the weight variable. In Andreatta et al. [2015b], the authors answer affirmatively to this question. They show a description of the finite slope part of p-adic families of overconvergent modular symbols, in terms of the finite slope part of p-adic families of overconvergent modular forms, for generic accessible weights. We can think of this result as a comparison between two different approaches to construct eigenvarieties: one using the theory of p-adic and overconvergent modular eigenforms, and the other using cohomology of arithmetic groups (overconvergent modular eigensymbols). More precisely, let $\mathcal{W}$ be the rigid analytic space associated to the complete noetherian semilocal algebra $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, called the weight space. Let $T_0 := \mathbb{Z}_p^\times \times \mathbb{Z}_p$, which can be viewed as a compact subset of $\mathbb{Z}_p^2$, with natural actions of $\mathbb{Z}_p^\times$ and the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z})$. There are two situations:

(a) For any weight $\lambda \in \mathcal{W}(L)$, we denote by D_λ the L-Banach space of analytic distributions on T_0, homogenous of degree λ for the action of $\mathbb{Z}_p^\times$.

(b) Let $U \subset \mathcal{W}^*$ be a wide open disk, where $\mathcal{W}^* \subset \mathcal{W}$ is the rigid subspace of accessible weights, i.e.,weight λ such that $|\lambda(t)^{p-1} - 1| < p^{-1/(p-1)}$. We denote by $\mathcal{O}(U)$ the L-algebra of rigid functions on U, and by $\Lambda_U \subset \mathcal{O}(U)$ the $\mathcal{O}_L$-algebra of bounded by 1 rigid functions, i.e., the set of $f \in \mathcal{O}(U)$ such that $\mid f(\lambda) \mid \leq 1$ for each $\lambda \in U$. We denote by $\lambda_U : \mathbb{Z}_p^\times \to \Lambda_U^\times$ the character defined by $\lambda_U(s)(\lambda) = \lambda(s)$ for each $s \in \mathbb{Z}_p^\times$ and $\lambda \in U(L)$. This is called the universal character associated to U. Similarly denote by D_U the $B_U := \Lambda_U \otimes L$-Banach module of analytic distributions on T_0, with values in B_U, homogenous of degree λ_U for the action of $\mathbb{Z}_p^\times$.

Both D_λ and D_U are Γ-representations. There exists a Γ-equivariant map $D_U \to D_\lambda$ if $\lambda \in U(L)$, called specialization. Similar to the classical case, the elements in $\mathrm{H}^1\left(\Gamma, D_\lambda(1)\right)$ are called overconvergent modular symbols, while the ones in $\mathrm{H}^1\left(\Gamma, D_U(1)\right)$ are called p-adic families of overconvergent modular symbols.

Moreover, for each $w \in \mathbb{Q}$ satisfying $0 < w < p/(p+1)$, we denote by $X(w)$ the strict neighborhood of the component containing the cusp ∞ of the ordinary locus of radius p^w in the rigid analytic curve $(X_{/L})^{\mathrm{an}}$. Then, for any weight $\lambda \in \mathcal{W}(L)$, there exists an invertible modular sheaf $\omega_w^{\dagger,\lambda}$ on $X(w)$ such that if $\lambda = k \in \mathbb{Z}$, then $\omega_w^{\dagger,k} \cong \omega^k|_{X(w)}$. The elements of $H^0(X(w), \omega_w^{\dagger,\lambda})$ are called overconvergent modular forms of weight λ. Similarly, if $U \subset \mathcal{W}$ is a wide open disk with universal weight λ_U, there exists a w and a modular sheaf of B_U-Banach modules $\omega_w^{\dagger,\lambda_U}$, such that the elements of $H^0(X(w), \omega_w^{\dagger,\lambda_U})$ are $p-$adic families of overconvergent modular forms over U.

Fix $U \subset \mathcal{W}^*$, a wide open disk defined over L. In Andreatta et al. [2015b] the authors constructed a geometric $(B_U \hat{\otimes} \mathbb{C}_p)$-linear homomorphism

$$\Psi_U : H^1(\Gamma, D_U) \hat{\otimes}_L \mathbb{C}_p(1) \to H^0\left(X(w), \omega_w^{\dagger,\lambda_U+2}\right) \hat{\otimes}_L \mathbb{C}_p,$$

which is equivariant for the actions of G_L and Hecke operators, also compatible with specializations.

Let $h \geq 0$ be an integer and suppose that U is such that both $H^1(\Gamma, D_U)$ and $H^0\left(X(w), \omega_w^{\dagger,\lambda_U}\right)$ have slope $\leq h$ decompositions and that there exists an integer $k_0 > h-1$ satisfying $k_0 \in U(L)$. Let $\Psi_U^{(h)}$ denote the morphism induced by Ψ_U on slope $\leq h$ parts, we have the following:

Theorem 1.0.3. *(Andreatta et al. [2015b]) Let U, k_0 and h as above. a) There exists a finite set of weight $Z \subset U(\mathbb{C}_p)$ such that for each $\lambda \in U(L) - Z$, we have a natural isomorphism of $\mathbb{C}_p$-vector spaces, which is equivariant for the semilinear G_L-action and the actions of the Hecke operators T_l for $(l, Np) = 1$ and U_l for l dividing Np:*

$$\Psi_\lambda^{\mathrm{ES}} : H^1(\Gamma, D_\lambda)^{(h)} \otimes_L \mathbb{C}_p(1) \cong \left(H^0\left(X(w), \omega_w^{\dagger,\lambda+2}\right)^{(h)} \hat{\otimes}_L \mathbb{C}_p\right) \oplus \left(S_\lambda^{(h)}(\lambda+1)\right).$$

Here $S_\lambda^{(h)}$ is a finite $\mathbb{C}_p$-vector space with trivial semilinear G_L-action and an action of the Hecke operators.

b) We have a family version of a) above: for every wide open disk $V \subset U$ defined over L satisfying $V(\mathbb{C}_p) \cap \mathbf{Z} = \emptyset$, there exists a finite free B_V-module $S_V^{(h)}$ on which the Hecke operators of T_l (for l not dividing pN) and U_l (for l dividing pN) act, and we have a natural G_L and Hecke equivariant isomorphism

$$H^1\left(\Gamma, D_V^{(h)}\right)\hat{\otimes}_L \mathbb{C}_p(1) \cong \left(H^0\left(X(w), \omega_w^{\dagger,\lambda_V+2}\right)^{(h)} \otimes_L \mathbb{C}_p\right) \oplus \left(S_V(\chi_V^{\mathrm{univ}} \cdot \chi)\right),$$

where χ is the cyclotomic character of L and

$$\chi_V^{\mathrm{univ}} : G_L \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{\lambda_V} B_V^\times \longrightarrow (B_V \hat{\otimes} \mathbb{C}_p)^\times$$

is the universal cyclotomic character attached to V, where $B_V := \Lambda_V \otimes L$.

Following the general line of arguments in Andreatta et al. [2015b], we would like to obtain the similar overconvergent Eichler-Shimura isomorphisms on Shimura curves over a totally real field F over $\mathbb{Q}$. There are two cases:

- $F = \mathbb{Q}$: The work was done in Barrera and Gao [2016] following the same argument as in Andreatta et al. [2015b]. In this case, the weight space $\mathcal{W}$ and T_0 are the same as in the modular case. There the authors generalize the result by working with all the weights in the weight space $\mathcal{W}$ not only the accessible ones. Moreover, working on Shimura curves over $\mathbb{Q}$ instead of modular curves, simplifies some problems and complicates others. Namely, the non-existence of cusps simplifies the log structures on Faltings' sites. On the other hand, the universal abelian scheme over the Shimura curve has higher relative dimension and one has to use the quaternionic multiplication in order to obtain objects (Tate modules, sheaves of differentials, canonical subgroups, etc.) of the right size.

- $F \neq \mathbb{Q}$: This is the main goal of this thesis, to develop a similar theory of overconvergent Eichler-Shimura isomorphisms as in Andreatta et al. [2015b] and Barrera and Gao [2016] for modular forms over certain PEL Shimura curves over F. Similarly as in the first case, we have higher relative dimension of the universal abelian scheme hence we need to "cut" certain objects to get the right size. Moreover, in this case, both the weight space $\mathcal{W}$ and T_0 are different and we need to consider more structures to make things work.

Here is a detailed description of the structure of this thesis. We will work mainly on curves with three different level structures (Section 3.1.3.3). For the convenience of the reader, we present the following table, which lists the analogy between the quaternionic curves we are interested in and the classical modular curves. We will consider rigid analytic curves and their corresponding formal models.

Quaternionic curve	Level	Classical curve	Classical level
$M(H)$	$K(H)$	$X_1(N)^{\mathrm{an}}$	$\Gamma_1(N)$
$M(H,\pi^n)$	$K(H,\pi^n)$	$X_1(N;p^n)^{\mathrm{an}}$	$\Gamma_1(N)\cap\Gamma_0(p^n)$
$M(H\pi^n)$	$K(H\pi^n)$	$X_1(Np^n)^{\mathrm{an}}$	$\Gamma_1(Np^n)$

Let F be totally real field of degree $d > 1$ over $\mathbb{Q}$ and denote by $\tau_1, \tau_2, \ldots, \tau_d$ all its real embeddings. Set $\tau = \tau_1$. Let B be a quaternion algebra over F which is split at τ and ramified at all other infinite places $\tau_2, \ldots, \tau_d$. Fix $p \neq 2$ a prime integer. Choose an element

$\lambda \in \mathbb{Q}$, $\lambda < 0$ such that $\mathbb{Q}(\sqrt{\lambda})$ splits at p. Let $E := F(\sqrt{\lambda})$ be an imaginary quadratic extension over F. We denote by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ the primes of F lying above p, denote simply by $\mathcal{P} = \mathcal{P}_1$. Let $F_{\mathcal{P}_i}$ be the completion of F at $\mathcal{P}_i$. Let $\mathcal{O}_{\mathcal{P}}$ be the ring of integers of $F_{\mathcal{P}}$ and denote by e and f its ramification degree and residue degree, respectively. Fix π, a uniformizer of $\mathcal{O}_{\mathcal{P}}$ and let κ be the residue field, with cardinality $q = p^f$ and characteristic p. Let $v(\cdot)$ be the normalized valuation of $F_{\mathcal{P}}$, i.e., $v(\pi) = 1$,

In Chapter 2, we review some basic definitions and properties of log schemes. We defined log smooth and log étale morphisms between fine and saturated log schemes. Then we give a criterion of log smoothness (respectively log étaleness) in terms of charts. Moreover, we introduce Kummer étale morphisms and Kummer étale sites, which play an important role in the construction of Faltings' site later.

Chapter 3 is a brief review of the work of Carayol [1986], Kassaei [2004] and Brasca [2013]. First we define the Shimura curves with different level structures over $\mathbb{C}$ following Carayol [1986], which are Shimura varieties of PEL type (Section 3.1.2). These curves are moduli spaces of abelian schemes with additional structures. We also give explicit description of the moduli problems over both reflex field and local field (Section 3.1.3). Then we recall the definition of the analogue of the Hasse invariant and the theory of canonical subgroup developed in Kassaei [2004]. We also review the definition of the dlog map and the construction of the Hodge-Tate sequence. These are the most important technic to construct the modular sheaves and to define overconvergent (quartenionic) modular forms following Brasca [2013]. Moreover, we introduce a suitable rigid analytic space $\mathcal{W}$ whose L points, for L a finite extension of $F_{\mathcal{P}}$, correspond to continuous characters $\mathcal{O}_{\mathcal{P}}^{\times} \to L^{\times}$. Following Brasca [2013], we recall the construction of modular sheaves ω_w^{λ} on $M(w)$ for any weight $\lambda \in \mathcal{W}$

and for families. At the end of this chapter, we give a construction of the Hecke operators, namely, the U operator and the $\mathrm{T}_{\mathfrak{L}}$ operators, which are analogous to the classical U_p and T_l operators, respectively.

In Chapter 4, first we review the basic construction of Faltings' sites and topoi, introduced by Faltings in Faltings [2002b] and generalized by F. Andreatta and A. Iovita in several papers such as Andreatta and Iovita [2008], Andreatta and Iovita [2013] and Andreatta and Iovita [2012]. Then we define Faltings' sites associated to the Shimura curves which we discuss in Section 4.3.2 and Section 4.3.3. Moreover, we define several continuous functors between these sites which induce morphisms between their corresponding topoi. We also show the localization functors, following Andreatta et al. [2015b], which allows us to calculate or prove things locally.

Chapter 5 is devoted to introducing the right overconvergent cohomology, which can be thought of as (families of) overconvergent modular symbols, to be related with the π-adic families constructed in Brasca [2013]. First we define some modules D_λ called distributions with the right action of the semigroup

$$\Lambda_\pi = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_{\mathcal{P}}) \cap \mathrm{GL}_2(F_{\mathcal{P}}) \mid a \in \mathcal{O}_{\mathcal{P}}^\times,\ c \in \pi\mathcal{O}_{\mathcal{P}},\ d \neq 0 \right\}.$$

The overconvergent modular symbols of weight λ are defined to be $\mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda)$, which can be identified with the group cohomology $\bigoplus_{\mathbf{x}\in \mathrm{CL}_E^+} \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda)$, where $\Gamma_{\mathbf{x}}$ is a certain torsion free arithmetic subgroup of $G(\mathbb{Q})$. Furthermore, this isomorphism is compatible with the action of Hecke operators and G_L. This identification allows us to get slope decompositions on $\mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda)$ by working on $\bigoplus_{\mathbf{x}\in \mathrm{CL}_E^+} \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda)$.

Chapter 6 is the most important part of this thesis. We relate overconvergent modular symbols $\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}},\mathcal{D}_\lambda\right)$ of weight λ with overconvergent modular forms $\mathrm{H}^0\left(M(w),\omega_w^{\lambda+2}\right)$ of weight $\lambda+2$. We use some continuous functors and think of both $\mathcal{D}_\lambda$, and $\omega_w^{\lambda+2}$ as sheaves on $\mathfrak{M}(w)$ which is Faltings' site associated to the pair $(\mathcal{M}(w), M(w))$ (Section 4.3.2 and Section 4.4). By calculating cohomology on $\mathfrak{M}(w)$ (Corollary 6.2.1), we obtain a morphism

$$\Psi : \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}},\mathcal{D}_\lambda\right)\hat{\otimes}_L\mathbb{C}_p(1)\longrightarrow \mathrm{H}^0\left(M(w),\omega_w^{\lambda_{\mathfrak{U}}+2}\right)\hat{\otimes}_L\mathbb{C}_p\,.$$

Eventually, in Chapter 7, we state and prove our main theorem as follows

Theorem 1.0.4. *There exists a finite subset of weights $Z\subset\mathfrak{U}(\mathbb{C}_p)$ such that*

(a) *For each $\lambda\in\mathfrak{U}(L)-Z$, there exists a finite dimensional $\mathbb{C}_p$-vector space $S_\lambda^{\le h}$ endowed with trivial semilinear G_L-action and Hecke operators, such that we have natural G_L and Hecke equivariant isomorphisms*

$$\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et},\mathcal{D}_\lambda\right)^{\le h}\hat{\otimes}_L\mathbb{C}_p(1)\cong\left(\mathrm{H}^0\left(M(w),\omega_w^{\lambda+2}\right)^{\le h}\hat{\otimes}_L\mathbb{C}_p\right)\oplus\left(S_\lambda^{\le h}(\lambda+1)\right),$$

where the first projection is $\Psi_\lambda^{\le h}$.

(b) *For every wide open disk $\mathfrak{V}\subset\mathfrak{U}$ defined over L satisfying $\mathfrak{V}(\mathbb{C}_p)\cap Z=\emptyset$, there exists a finite free $B_{\mathfrak{V}}\hat{\otimes}_L\mathbb{C}_p$ $-$module $S_{\mathfrak{V}}^{\le h}$ endowed with trivial semilinear G_L-action and Hecke operators, for which we have a G_L and Hecke equivariant exact sequence*

$$0\longrightarrow S_{\mathfrak{V}}^{\le h}\left(\chi\cdot\chi_{\mathfrak{V}}^{\mathrm{univ}}\right)\longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et},\mathcal{D}_{\mathfrak{V}}\right)^{\le h}\hat{\otimes}\,\mathbb{C}_p(1)\longrightarrow$$

$$\xrightarrow{\Psi_{\mathfrak{V}}^{\le h}}\mathrm{H}^0\left(M(w),\omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\le h}\hat{\otimes}\,\mathbb{C}_p\longrightarrow 0.$$

Moreover, for any such open disk $\mathfrak{V}$, there exists a finite subset $Z'\subset\mathfrak{V}$ with the property that, for any wide open disk $\mathfrak{V}'\subset\mathfrak{V}$ with $\mathfrak{V}'(\mathbb{C}_p)\cap Z'=\emptyset$, we have a natural

G_L and Hecke equivariant isomorphism

$$\begin{aligned} & \mathrm{H}^1\left(M(H,\pi)^{et}_{\overline{L}}, \mathcal{D}_{\lambda_{\mathfrak{Y}'}}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \\ \cong \quad & \left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{Y}'}+2}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p\right) \oplus \left(S^{\leq h}_{\mathfrak{Y}'}(\chi \cdot \chi^{univ}_{\mathfrak{Y}'})\right), \end{aligned}$$

where the first projection is determined by $\Psi^{\leq h}_{\mathfrak{Y}}$.

Chapter 2

Log Schemes and Log Smoothness

In this chapter, we will briefly recall some basic preliminaries on log schemes and a class of log étale morphisms of log schemes, called Kummer étale morphisms. In particular, let X be an fs log scheme (see Definition 2.2.4), we will describe the associated sites and topos, called the *Kummer étale site* on X, which play an important role in the construction of the Faltings' site. We will only list some basic propositions of log schemes and log étale morphisms and omit most of the proofs. The main references of this chapter are Illusie [2002], Kato [1989], Nakayama [1997] and Ogus [2006].

In the whole chapter, a monoid means a commutative monoid with a unit element 1. (In general, a monoid is written multiplicatively. For some special cases, for example $\mathbb{N}$, the monoid is written additively and the unit element is denoted by 0 in such cases.) A homomorphism of monoids is assumed to preserve the unit. We write **Mon** for the category of monoids and homomorphisms of monoids. Let P be a monoid, there is a universal morphism λ from P to a group P^{gp}, such that any morphism, from P to a gp G, factors uniquely through λ. In other words, there exists a unique group homomorphism $P^{gp} \to G$ such that the following diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\lambda"'] & G \\ P^{gp} \arrow[ur, dashed] & \end{tikzcd}$$

commutes. Moreover, P^{gp} is called the group associated to P and

$$P^{gp} = \{ab^{-1}; a, b \in P\}$$

with the relation that

$$ab^{-1} = cd^{-1} \Leftrightarrow sad = sbc$$

for some $s \in P$. We denote by P^* the subgroup of all invertible elements of P and write $\overline{P} = P/P^*$.

2.1 Monoids

Definition 2.1.1. Let P be a monoid.

- P is called *sharp* if $P^* = \{1\}$.
- P is called *integral* if $ab = ac$ implies $b = c$ in P. This is equivalent to saying that the canonical map $P \to P^{gp}$ is injective.
- P is called *saturated* if P is integral and for any $a \in P^{gp}$, a is in P if and only if there exists an integer $n \geq 1$ such that $a^n \in P$.
- P is said to be *fine* if it is finitely generated and integral. Monoids which are both fine and saturated are often called *fs*-monoids.

Example 2.1.1. (1) A natural example of a monoid is $\mathbb{N}$ with respect to the natural addition. It is a free monoid with generator 1 and integral since $\mathbb{N}^{gp} = \mathbb{Z}$. Moreover, $\mathbb{N}$ is an fs-monoid.

(2) Let A be a commutative ring with identity 1. Then A with respect to its multiplication is a monoid, denoted by $(A, \cdot, 1)$.

Definition 2.1.2. Let $\mathbf{Mon}^{\mathtt{int}}$ denote the full subcategory of $\mathbf{Mon}$ whose objects are the integral monoids. For any monoid M, let $M^{\mathtt{int}}$ denote the image of M in M^{gp} under the universal morphism $\lambda_M : M \to M^{gp}$. Then $M \mapsto M^{\mathtt{int}}$ is left adjoint to the inclusion functor $\mathbf{Mon}^{\mathtt{int}} \to \mathbf{Mon}$.

Definition 2.1.3. Let M be an integral monoid. We define $M^{\mathtt{sat}}$ to be the set

$$M^{\mathtt{sat}} := \left\{ x \in M^{gp} \middle| x^n \in M \text{ for some } n \in \mathbb{Z}, n \geq 1 \right\}.$$

$M^{\mathtt{sat}}$ is a saturated submonoid of M^{gp}, and the functor $M \mapsto M^{\mathtt{sat}}$ is left adjoint to the inclusion functor from the category $\mathbf{Mon}^{\mathtt{sat}}$ of saturated monoids to $\mathbf{Mon}^{\mathtt{int}}$.

Definition 2.1.4. Let P be a monoid.

- A submonoid E of $P \times P$ which is also an equivalence relation on P is called a *congruence* (or *congruence relation*) on P.

- If E is a congruence relation on P, then the set P/E of equivalence classes has a unique monoid structure making the projection $P \to P/E$ a monoid morphism.

- If $\theta : P \to M$ is a homomorphism of monoids, then the set E of pairs $(p_1, p_2) \in P \times P$ such that $\theta(p_1) = \theta(p_2)$ is a congruence relation on P, and if θ is surjective, M can be recovered as the quotient of P by the equivalence relation E.

Definition 2.1.5. The *amalgamated sum* $Q_1 \xrightarrow{v_1} Q \xleftarrow{v_2} Q_2$ of a pair of monoid morphisms $u_i : P \to Q_i$, $i = 1, 2$, often denoted by $Q_1 \oplus_P Q_2$, is the inductive limit of the diagram $Q_1 \xleftarrow{u_1} P \xrightarrow{u_2} Q_2$. That is, the pair (v_1, v_2) universally makes the diagram

$$\begin{tikzcd} P \arrow[r, "u_1"] \arrow[d, "u_2"'] & Q_1 \arrow[d, "v_1"] \\ Q_2 \arrow[r, "v_2"'] & Q \end{tikzcd}$$

commute.

Remark 2.1.1. Indeed, the amalgamated sum Q can be thought of as the coequalizer of the two maps $(u_1, 0)$ and $(0, u_2)$ from P to $Q_1 \oplus Q_2$. As the following proposition shows, the description of Q is considerably simplified if one of the monoids in question is a group.

Proposition 2.1.1. *([Ogus, 2006, Chapter I Proposition 1.1.4]) Let $u_i : P \to Q_i$ be a pair of monoid morphisms, let P be their amalgamated sum and let E be the congruence relation on $Q_1 \oplus Q_2$ given by the natural map $Q_1 \oplus Q_2 \to Q$ (Remark 2.1.1).*

- *Let E' be the set of pairs $((q_1, q_2), (q_1', q_2'))$ of elements of $Q_1 \oplus Q_2$ such that there exist a and b in P with $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$. Then E' is a congruence relation on $Q_1 \oplus Q_2$ containing E, and if any of P, Q_1, or Q_2 is a group, then $E = E'$.*

- *If P is a group, then two elements of $Q_1 \oplus Q_2$ are congruent modulo E if and only if they lie in the same orbit of the action of P on $Q_1 \oplus Q_2$ defined by $p(q_1, q_2) = (q_1 + u_1(p), q_2 + u_2(-p))$.*

- *If P and Q_i are groups, then so is $Q_1 \oplus_P Q_2$, which is in fact just the fibered coproduct (amalgamated sum) in the category of abelian groups.*

Proposition 2.1.2. *([Ogus, 2006, Chapter I Proposition 1.2.2]) Let Q be the amalgamated sum of two homomorphisms $u_i : P \to Q_i$ in **Mon**. Then Q^{int} is the amalgamated sum of $u_i^{int} : P^{int} \to Q_i^{int}$ in the category $\boldsymbol{Mon}^{int}$, and can be naturally identified with the image of Q in $Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$. If P, Q_1, and Q_2 are integral and any of these monoids is a group, then Q is integral.*

Remark 2.1.2. Note that even when P, Q_1 and Q_2 are integral monoids, the amalgamated sum $Q_1 \oplus_P Q_2$ need not be integral. Then same kind of problem arises for saturated monoids.

Example 2.1.2. Fix an $a \in \mathbb{N}$ and consider the following pair of morphisms of monoids:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "h_a"] \arrow[d, "\Delta"'] & \mathbb{N} \\ \mathbb{N}^2 & \end{tikzcd} \tag{2.1}$$

where h_a sends $n \mapsto an$ and $\Delta(n) = (n, n)$.

Let $P := \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ be the amalgamated sum associated to the above diagram. We claim that $P \cong \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$ as a (additive) submonoid of $\mathbb{Q}^2$, where

$$\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 = \left\{ \left(\frac{n}{a} + s, \frac{n}{a} + t\right) \;\middle|\; n, s, t \in \mathbb{N} \right\}.$$

Note that we have natural morphisms of monoids:

$$\begin{aligned} h' : \mathbb{N}^2 &\longrightarrow \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 \\ (s,t) &\longmapsto (\frac{0}{a} + s, \frac{0}{a} + t) = (s,t) \end{aligned}$$

and

$$\begin{aligned} h'' : \mathbb{N} &\longrightarrow \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 \\ n &\longmapsto (\frac{n}{a} + 0, \frac{n}{a} + 0) = (\frac{n}{a}, \frac{n}{a}) \end{aligned}$$

such that the following square is commutative:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "h_a"] \arrow[d, "\Delta"'] & \mathbb{N} \arrow[d, "h''"] \\ \mathbb{N}^2 \arrow[r, "h'"'] & \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 . \end{tikzcd}$$

Then it suffices to show that $\left(\mathbb{N}^2 \xrightarrow{h'} \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 \xleftarrow{h''} \mathbb{N} \right)$ satisfies the universal property of the amalgamated sum of the morphisms in (2.1).

Let Q be a (additive) monoid and suppose we have two morphisms $\alpha : \mathbb{N}^2 \to Q$ and $\beta : \mathbb{N} \to Q$ such that $\alpha \circ \Delta = \beta \circ h_a$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "h_a"] \arrow[d, "\Delta"'] & \mathbb{N} \arrow[d, "h''"] \arrow[ddr, bend left, "\beta"] & \\ \mathbb{N}^2 \arrow[r, "h'"'] \arrow[drr, bend right, "\alpha"'] & \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 \arrow[dr, dashed, "\psi"] & \\ & & Q. \end{tikzcd}$$

Define

$$\begin{aligned} \psi : \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 & \longrightarrow Q \\ (\frac{n}{a} + s, \frac{n}{a} + t) & \longmapsto \beta(n) + \alpha(s,t). \end{aligned}$$

If we have

$$(\frac{n}{a} + s, \frac{n}{a} + t) = (\frac{m}{a} + s', \frac{m}{a} + t')$$

in $\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$, then

$$\frac{n-m}{a} = s' - s = t' - t.$$

Without loss of generality, we may assume $n - m \geq 0$. Then we have

$$\beta(n) + \alpha(s,t) = \beta\left(a(s'-s) + m\right) + \alpha(s,t)$$

$$
\begin{aligned}
&= \beta\left(a(s'-s)\right) + \beta(m) + \alpha(s,t) \\
&= \beta \circ h_a(s'-s) + \beta(m) + \alpha(s,t) \\
&= \alpha \circ \Delta(s'-s) + \beta(m) + \alpha(s,t) \\
&= \beta(m) + \alpha(s'-s, s'-s) + \alpha(s,t) \\
&= \beta(m) + \alpha(s'-s, t'-t) + \alpha(s,t) \\
&= \beta(m) + \alpha(s',t').
\end{aligned}
$$

This shows that ψ is well-defined. It is easy to check that ψ is a monoid homomorphism. Moreover, for any $(s,t) \in \mathbb{N}^2$,

$$(\psi \circ h')(s,t) = \psi(s,t) = \beta(0) + \alpha(s,t) = \alpha(s,t),$$

and for any $n \in \mathbb{N}$,

$$(\psi \circ h'')(n) = \psi(\frac{n}{a}, \frac{n}{a}) = \beta(n) + \alpha(0,0) = \beta(n).$$

The uniqueness of such ψ is obvious. This proves our claim that the amalgamated sum P of the morphisms in (2.1) can be identified with the (additive) monoid $\frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2$. Under this identification, we conclude that P is finitely generated and a set of generators can be given by

$$\left\{ \left(\frac{1}{a}, \frac{1}{a} \right), (1,0), (0,1) \right\}.$$

Moreover, $P^{gp} \cong \frac{1}{a}\Delta(\mathbb{Z}) + \mathbb{Z}^2$ and P is fine and saturated (i.e., an fs-monoid).

Example 2.1.3. (**Monoid Algebras**) Let R be a commutative ring with identity and P a monoid. We denote by 1_R the identity of R and by 1_P the unit element of P. Then we construct an R-algebra $R[P]$ as follows. As an R-module, it is free with basis P, i.e., for any

$f \in R[P]$, f can be written as $f = \sum_{p \in P} r_p \cdot p$, where $r_p \in R$ and $r_p = 0$ for all but finitely many $p \in P$. For $f = \sum_{x \in P} r_x \cdot x$ and $g = \sum_{y \in P} r_y \cdot y$, we define the multiplication of f and g by

$$fg = \sum_{p \in P} \sum_{xy=p} (r_x r_y) \cdot p.$$

The R-algebra structure is given by the natural ring homomorphism $R \to R[P]$ sending $r \mapsto r \cdot 1_P$, for any $r \in R$. This R-algebra $R[P]$ is called the monoid algebra on P over R. If we consider $R[P]$ as a monoid with respect to its multiplication, we have a canonical monoid homomorphism $P \to R[P]$ sending any element $p \in P$ to $1_R \cdot p$ with the following universal property:

For any R-algebra S and a monoid homomorphism $\theta : P \to S$ (here we consider S as a multiplicative monoid), there exists a unique R-algebra homomorphism $R[P] \to S$ making the following diagram commutative:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\theta"'] & R[P] \arrow[dl, dashed, "\exists!"] \\ S. & \end{tikzcd}$$

Monoid algebras have the following properties:

Proposition 2.1.3. *Let P, Q (Q_1, Q_2) be monoids and R a (commutative) ring. Then*

- *P is finitely generated (as a monoid) if and only if $R[P]$ is finitely generated (as an R-algebra).*
- *$R[P]$ is noetherian if and only if R is noetherian and P is finitely generated.*
- *$R[P \oplus Q] \cong R[P] \otimes_R R[Q]$.*
- *$R[Q_1 \oplus_P Q_2] \cong R[Q_1] \otimes_{R[P]} R[Q_2]$.*

- *Moreover, if R is an integral domain and P is integral such that P^{gp} is torsion free, then $R[P]$ is an integral domain.*

Proof. We refer to [Ogus, 2006, Chapter I §3.1, §3.3] for details. $\square$

2.2 Log Schemes

2.2.1 Log structures

Definition 2.2.1. Let $(X, \mathcal{O}_X)$ be a scheme.

- A *prelog structure* on X is a pair (M, α) where M is a sheaf of monoids on the étale site X^{et} and α is a homomorphism from M to the multiplicative monoid of $\mathcal{O}_X$.

- A prelog structure (M, α) is called a *log structure* if the induced map $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism.

- The log structure defined by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$ is called the *trivial log structure* on X.

- A *log scheme* is a triple (X, M, α), usually simply denote by X, consisting of a scheme X and a log structure (M, α) on X. The sheaf of monoids of a log scheme X is generally denoted by M_X, and the sheaf $\mathcal{O}_X^*$ is thought of as a subsheaf of M_X by α.

- A *morphism of log schemes* is a morphism $f : X \to Y$ of the underlying schemes together with a morphism $f^\flat : M_Y \to f_*(M_X)$ such that the following diagram

$$\begin{tikzcd} M_Y \arrow[r, "f^\flat"] \arrow[d, "\alpha_Y"'] & f_*(M_X) \arrow[d, "f_*(\alpha_X)"] \\ \mathcal{O}_Y \arrow[r, "f^\sharp"'] & f_*(\mathcal{O}_X) \end{tikzcd}$$

commutes.

Proposition 2.2.1. *([Ogus, 2006, Chapter II Proposition 1.1.5]) Let X be a scheme. The inclusion functor from the category of log structures to the category of prelog structures on X admits a left adjoint $(M, \alpha) \to (M^a, \alpha^a)$, where $M^a = M \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$ and α^a is the morphism defined by α and the inclusion of $\mathcal{O}_X^*$ in $\mathcal{O}_X$, i.e.,*

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X^* \arrow[d] \arrow[ddr, bend left] & \\ M \arrow[r] \arrow[drr, bend right, "\alpha"'] & M^a \arrow[dr, dashed, "\alpha^a"] & \\ & & \mathcal{O}_X. \end{tikzcd}$$

One calls (M^a, α^a) the log structure associated to the prelog structure (M, α).

Example 2.2.1. Recall R, P in Example 2.1.3 and let $X := \mathrm{Spec}(R[P])$. Endow X with the log structure associated to the prelog structure induced by the canonical monoid morphism $P \to R[P]$. The above log structure is called the canonical log structure on $X = \mathrm{Spec}(R[P])$. Indeed, it is the inverse image of the canonical log structure on $\mathrm{Spec}(\mathbb{Z}[P])$ under the natural morphism $X \to \mathrm{Spec}(\mathbb{Z}[P])$.

Definition 2.2.2. Let $f : X \to Y$ be a morphism of schemes.

- If $\alpha_X : M_X \to \mathcal{O}_X$ is a log structure on X, then the natural pair

$$(f_*(M_X) \times_{f_*(\mathcal{O}_X)} \mathcal{O}_Y, \beta)$$

in the following diagram

$$\begin{tikzcd} f_*(M_X) \times_{f_*(\mathcal{O}_X)} \mathcal{O}_Y \arrow[r, "\beta"] \arrow[d] & \mathcal{O}_Y \arrow[d] \\ f_*(M_X) \arrow[r, "f_*(\alpha_X)"'] & f_*(\mathcal{O}_X) \end{tikzcd}$$

is a log structure on Y, called the *direct image log structure* induced by α_X.

- If $\alpha_Y : M_Y \to \mathcal{O}_Y$ is a log structure on Y, then the composite

$$f^{-1}(M_Y) \xrightarrow{f^{-1}(\alpha_Y)} f^{-1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_X$$

is a prelog structure on X. The associated log structure is called the *inverse image log structure* on X and denoted by $(f^*M_Y, f^*\alpha_Y)$.

Definition 2.2.3. A map of log schemes

$$f : X = (X, M, \alpha) \longrightarrow Y = (Y, N, \beta)$$

is called *strict* if the natural map $f^*N \to M$ is an isomorphism.

2.2.2 Charts

Definition 2.2.4. Let $\alpha : M \to \mathcal{O}_X$ be a log structure on a scheme X.

- A (global) *chart, modeled on* P, of a log scheme X is a strict map of log schemes $X \to \operatorname{Spec} \mathbb{Z}[P]$ for some monoid P, where $\operatorname{Spec} \mathbb{Z}[P]$ is endowed with its canonical log structure. Giving such a chart is the same as giving a monoid P and a homomorphism from the constant sheaf of monoids P_X on X to M inducing an isomorphism on the associated log structures.

- A log structure α is called *quasi-coherent* (resp. *coherent*) if locally on X it admits a chart (resp. a chart modeled on a finitely generated monoid).

- A log scheme X is called *integral* if the stalk of M at each geometric point of X is integral.

- A log scheme X is called *fine* (resp. *fine and saturated*, or fs for short) if it is integral, and locally for the étale topology it admits a chart modeled on a finitely generated and integral (resp. finitely generated and saturated) monoid.

- Let $f : X \to Y$ be a map of log schemes. A *chart* of f is a triple (a, b, u) where $a : X \to \operatorname{Spec} \mathbb{Z}[P]$ and $b : Y \to \operatorname{Spec} \mathbb{Z}[Q]$ are charts of log schemes X and Y and $u : Q \to P$ is a morphism of monoids such that the following square of log schemes

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d] \\ Y \arrow[r, "b"'] & \operatorname{Spec} \mathbb{Z}[Q] \end{tikzcd}$$

  commutes, where the right vertical map is induced by u. Such chart of f is sometimes written as $(P, Q, u : Q \to P)$.

Remark 2.2.1. If $f : X \to Y$ is a map of fine log schemes, a chart of f exists étale locally, and, P and Q can be chosen to be fine and saturated monoids if X and Y are fs log schemes (refer to [Ogus, 2006, Chapter II §2.2] for details).

Example 2.2.2. Let X be a locally noetherian regular scheme and let $D \subset X$ be a divisor with normal crossings. Let $j : U = X - D \hookrightarrow X$ be the corresponding open immersion. Then the inclusion $M_X := \mathcal{O}_X \cap j_* \mathcal{O}_U^* \hookrightarrow \mathcal{O}_X$ is an fs log structure on X, which is called the log structure defined by $X - D$ (or sometimes, by D). Étale locally X has a chart modeled on $\mathbb{N}^r$ (if $\prod_{1 \leq i \leq r} t_i^{a_i}$ is a local equation of D where $(t_i)_{1 \leq i \leq r}$ is a part of a system of local parameters on X, $\mathbb{N}^r \to \mathcal{O}_X$, $(n_i) \mapsto \prod_{1 \leq i \leq r} t_i^{n_i}$ is a local chart).

2.2.3 Fibered products of log schemes

Just as in the case of classical schemes, the existence of products in the category of log schemes (resp. fine log schemes, resp. fs log schemes) has deep consequences. In this section, we list some results for the existence of fibered product in several categories, for details, please refer to [Ogus, 2006, Chapter II §2.4].

Proposition 2.2.2. *([Ogus, 2006, Chapter II Proposition 2.4.2]) The category of log schemes admits fibered products, and the functor $X \to \underline{X}$ taking a log scheme to its underlying scheme commutes with fibered products. The fibered product of coherent log schemes is coherent.*

Remark 2.2.2.

- More explicitly, Let X, Y and Z be log schemes with underlying schemes $\underline{X}$, $\underline{Y}$ and $\underline{Z}$, respectively. And let $f : X \to Z$ and $g : Y \to Z$ be morphisms of log schemes. Their fibered product in the category of log schemes is obtained by endowing the usual fibred product of schemes $\underline{X} \times_{\underline{Z}} \underline{Y}$ with the log structure associated to the prolog structure $p_X^* M_X \oplus_{p_Z^* M_Z} p_Y^* M_Y$, where p_X, p_Y and p_Z are the obvious projections. Consequently, if $(Q_1, P, u_1 : P \to Q_1)$ and $(Q_2, P, u_2 : P \to Q_2)$ are charts for the morphisms $f$and g respectively, then the induced morphism $X \times_Z Y \to \mathrm{Spec}(\mathbb{Z}[Q_1 \oplus_P Q_2])$ is a chart as well.

- If X is a log scheme, let X° denote the log scheme with the same underlying scheme but with trivial log structure. Then there is a natural morphism of log schemes $X \to X^\circ$, and a morphism $f : X \to Y$ of log schemes fitting into a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X^{\circ} \arrow[r, "f^{\circ}"'] & Y^{\circ}. \end{tikzcd}$$

If f is strict, this diagram is Cartesian.

Recall that the amalgamated sum of integral (resp. saturated) monoids need not be integral (resp. saturated) (see Proposition 2.1.2), so the construction of fibered products in the category of fine (or fs) log schemes is more delicate. We have the following properties:

Proposition 2.2.3. *([Ogus, 2006, Chapter II Proposition 2.4.5])*

- *The inclusion functor from the category of fine log schemes to the category of coherent log schemes admits a right adjoint $X \mapsto X^{int}$, and the corresponding morphism of underlying schemes $\underline{(X^{int})} \to \underline{X}$ is a closed immersion.*

- *The inclusion functor from the category of fs log schemes to the category of fine log schemes admits a right adjoint $X \mapsto X^{sat}$, and the corresponding morphism of underlying schemes $\underline{(X^{sat})} \to \underline{X}$ is finite and surjective.*

Remark 2.2.3. One should always keep in mind that the morphisms of topological spaces underlying the maps $X^{\mathtt{int}} \to X$ and $X^{\mathtt{sat}} \to X^{\mathtt{int}}$ are not in general homeomorphisms. In particular, we cannot identify $M_{X^{\mathtt{int}}}$ (resp. $M_{X^{\mathtt{sat}}}$), the log structure on $X^{\mathtt{int}}$ (resp. on $X^{\mathtt{sat}}$), with $(M_X)^{\mathtt{int}}$ (resp. $(M_X)^{\mathtt{sat}}$) in general. Here, $(M_X)^{\mathtt{int}}$ and $(M_X)^{\mathtt{sat}}$ are defined similarly as in the category of monoids (see Definition 2.1.2 and 2.1.3).

Corollary 2.2.1. *([Ogus, 2006, Chapter II Corollary 2.4.6]) The category of fine log schemes (resp. of fs log schemes) admits finite projective limits. Moreover if we have the following* *diagram*

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

in the category of fine (resp. fs) log schemes, then the natural morphism of schemes

$$\underline{(X \times_Z Y)} \longrightarrow \underline{X} \times_{\underline{Z}} \underline{Y}$$

is a closed immersion (resp. a finite morphism).

2.3 Log smooth and log étale morphisms

Definition 2.3.1. A morphism of log schemes $i : (X, M) \to (Y, N)$ is called a *closed immersion* (resp. *exact closed immersion*) if the underlying morphism of schemes $X \to Y$ is a closed immersion and $i^*N \to M$ is surjective (resp. an isomorphism).

Definition 2.3.2. Consider the following commutative diagram of log schemes

$$\begin{tikzcd} (T', L') \arrow[r, "s"] \arrow[d, "i"'] & (X, M) \arrow[d, "f"] \\ (T, L) \arrow[r, "t"'] & (Y, N) \end{tikzcd} \tag{2.2}$$

such that i is an exact closed immersion and T' is defined in T by an ideal I such that $I^2 = 0$. A morphism $f : (X, M) \to (Y, N)$ of fine log schemes is called *log smooth* (resp. *log étale*) if the underlying morphism of schemes $X \to Y$ is locally of finite presentation and if for any commutative diagram as in (2.2), there exists étale locally on T (resp. there exists a unique) $g : (T, L) \to (X, M)$ such that $g \circ i = s$ and $f \circ g = t$.

A standard example of a log smooth (resp. log étale) morphism is given by the following proposition. In Theorem 2.3.1, we shall see that all log smooth (resp. log étale) morphisms are essentially of the type of this standard example.

Proposition 2.3.1. *([Kato, 1989, Proposition (3.4)]) Let P, Q be two fine monoids, $f : Q \to P$ a homomorphism, R a ring, such that the kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups whose orders are invertible in R. Let*

$$X = \mathrm{Spec}(R[P]), \qquad Y = \mathrm{Spec}(R[Q])$$

and endow them with the canonical log structures M_X and M_Y, repectively (see Example 2.2.1). Then the morphism $(X, M_X) \to (Y, M_Y)$ induced by f is log smooth (resp. log étale).

Theorem 2.3.1. *([Kato, 1989, Theorem (3.5)]) Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of fine log schemes. Assume that we are given a chart $Q \to M_Y$ of M_Y. Then the following conditions are equivalent.*

(1) f is log smooth (resp. log étale).

(2) There is a chart $(P \to M_X, Q \to M_Y, Q \to P)$ of f, étale locally on X, extending the given chart $Q \to M_Y$ by satisfying the following conditions (a, b):

(a) The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of orders invertible on X.

(b) The induced morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ is a smooth (resp. étale) map on the underlying schemes.

Example 2.3.1. Let A be a discrete valuation ring and we fix a uniformizer π of A. Let R be an A-algebra satisfying the following conditions:

- $\operatorname{Spec}(R)$ is connected, i.e., R has no nontrivial idempotents;
- There is an $a \in \mathbb{N}$ and an étale morphism $\psi_R : R' \to R$ for which $R' = A[X,Y]/(XY - \pi^a)$.

Denote by $S = (\underline{S}, M)$ the log scheme whose underlying scheme $\underline{S} = \operatorname{Spec}(A)$, the log structure is the one associated to the prelog structure given by the map $\varphi : \mathbb{N} \to A$ sending $1 \mapsto \pi$.

Consider the following commutative diagram of monoids and morphisms of monoids:

$$\begin{tikzcd}
\mathbb{N}^2 \arrow[r, "\varphi_R"] & R' \\
\mathbb{N} \arrow[u, "\Delta"] \arrow[r, "\varphi_a"'] & A, \arrow[u]
\end{tikzcd}$$

where $\varphi_R(m,n) = X^m Y^n$, $\varphi_a(n) = \pi^{an}$ and $\Delta(n) = (n,n)$ for all $m, n \in \mathbb{N}$. Then we can identify R' with $A[\mathbb{N}^2] \otimes_{A[\mathbb{N}]} A$ as A-algebras.

Let $P := \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ be the amalgamated sum as in Example 2.1.2. Consider the log structure on $X = \operatorname{Spec}(R)$ the one associated to the prelog structure induced by the map $P \dashrightarrow R' \xrightarrow{\psi_R} R$, where the maps are shown in the following diagram:

$$\begin{tikzcd}
 & & R' \arrow[r] & R \\
\mathbb{N}^2 \arrow[urr, "\varphi_R"] \arrow[r] & P \arrow[ur, dashed] & & \\
\mathbb{N} \arrow[u, "\Delta"] \arrow[r, "h_a"'] \arrow[rrr, bend right, "\varphi_a"'] & \mathbb{N} \arrow[u, "h"] \arrow[uur] \arrow[rr, "\varphi"] & & A. \arrow[uul] \arrow[uu]
\end{tikzcd}$$

We denote by $X = (\underline{X}, N)$ the corresponding log scheme. Then the triple $(P, \mathbb{N}, h : \mathbb{N} \to P)$ is a chart for the morphism $f : X \to S$. Moreover, we have the following property for the above morphism of log schemes.

Lemma 2.3.1. *The morphism $f : X \to S$ of log schemes in the above example is log smooth.*

Proof. First, note that we have the following identifications from Example 2.1.2:

$$P \cong \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 = \left\{ \left(\frac{n}{a} + s, \frac{n}{a} + t \right) \;\middle|\; n, s, t \in \mathbb{N} \right\},$$

and

$$P^{gp} \cong \frac{1}{a}\Delta(\mathbb{Z}) + \mathbb{Z}^2 .$$

Then $h : \mathbb{N} \to P$ can be replaced by the morphism

$$\begin{aligned} h'' : \mathbb{N} &\longrightarrow \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2, \\ n &\longmapsto \left(\frac{n}{a}, \frac{n}{a} \right), \end{aligned}$$

thus the following commutative square satisfies the universal property of the amalgamated sum

$$\begin{tikzcd} \mathbb{N} \arrow[r, "h_a"] \arrow[d, "\Delta"'] & \mathbb{N} \arrow[d, "h''"] \\ \mathbb{N}^2 \arrow[r, "h'"'] & \frac{1}{a}\Delta(\mathbb{N}) + \mathbb{N}^2 . \end{tikzcd}$$

By Proposition 2.3.1, it is enough to prove that:

(a) The kernel and the torsion part of the cokernel of $(h'')^{gp} : \mathbb{N}^{gp} = \mathbb{Z} \to \frac{1}{a}\Delta(\mathbb{Z}) + \mathbb{Z}^2$ are finite groups of orders invertible in R.

(b) The morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(A) \times_{\mathrm{Spec}(\mathbb{Z}[\mathbb{N}])} \mathrm{Spec}(\mathbb{Z}[P])$ is smooth on the underlying schemes.

Notice that $(h'')^{gp}$ is injective and its image $h''(\mathbb{Z}) = \frac{1}{a}\Delta(\mathbb{Z})$, thus (a) follows immediately. Part (b) can be verified by the fact

$$\begin{aligned} A \otimes_{A[\mathbb{N}]} A[P] &\cong A \otimes_{A[\mathbb{N}]} A\left[\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2\right] \\ &\cong A \otimes_{A[\mathbb{N}]} \left(A[\mathbb{N}] \otimes_{A[\mathbb{N}]} A[\mathbb{N}^2]\right) \qquad \text{(Proposition 2.1.3)} \\ &\cong A \otimes_{A[\mathbb{N}]} A[\mathbb{N}^2] \\ &\cong R' \qquad \text{(Example 2.3.1)}, \end{aligned}$$

and our assumption that $R' \to R$ is étale. This completes the proof. $\square$

Remark 2.3.1. In the above example, when $a = 1$, X is called of semistable reduction over S. In this case $h_a = \mathtt{id}_{\mathbb{N}}$ and $P = \mathbb{N}^2$. Let N' be the log structure on X defined by its special fiber (see Example 2.2.2). Then N' coincides with the log structure N on X defined above. In other words, the triple $\left(\mathbb{N}^2, \mathbb{N}, \Delta : \mathbb{N} \to \mathbb{N}^2\right)$ is a chart for the natural morphism of log schemes $(\underline{X}, N') \to (\underline{S}, M)$.

Log smooth (resp. log étale) morphisms enjoy most of the properties of classical smooth (resp. étale) morphisms in the theory of schemes.

Proposition 2.3.2. *(a) Log smooth (resp. log étale) morphisms are stable under composition and arbitrary base change (either in the category of fine or fs log schemes).*

(b) Suppose we have the following commutative diagram of log schemes and morphisms:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

If f and g are log smooth (resp. log étale), so is h.

(c) If $f : X \to Y$ is a map of schemes, viewed as a map of log schemes with the trivial log structures, then f is log smooth (resp. log étale) if and only if f is classically smooth (resp. étale).

Proof. These are consequences of Theorem 2.3.1. $\square$

2.4 Kummer étale topology

Definition 2.4.1.
- A homomorphism of integral monoids $h : Q \to P$ is said to be *exact* if $Q = (h^{gp})^{-1}(P)$ in Q^{gp}.
- A morphism $f : X \to Y$ of log schemes with integral log structures is said to be *exact* if the homomorphism $(f^* M_Y)_{\overline{x}} \to M_{X,\overline{x}}$ is exact for any $\overline{x} \in X$.
- A morphism $h : Q \to P$ of fs monoids is said to be *Kummer* (or *of Kummer type*) if h is injective and for all $p \in P$ there exists $n \in \mathbb{N}$, $n \geq 1$, such that $np \in h(Q)$ (the monoid laws written additively).
- A morphism $f : X \to Y$ of fs log schemes is said to be *Kummer* (or *of Kummer type*) if for all geometric point $\bar{x}$ of X with image $\bar{y}$ in Y, the natural map

  $$\overline{M}_{Y,\overline{y}} \longrightarrow \overline{M}_{X,\overline{x}}$$

  is Kummer.
- A morphism $f : X \to Y$ of fs log schemes is said to be *Kummer étale* if it is both log étale and Kummer.

Remark 2.4.1. (i) Let $f : X \to Y$ be a morphism of fs log schemes. If f has a chart $(P \to M_X, Q \to M_Y, h : Q \to P)$ such that h is Kummer, then f is Kummer.

(ii) Morphisms of Kummer type are stable under compositions and base changes in the category of fs log schemes.

(iii) A morphism of Kummer type is exact.

(iv) Let $f : X \to Y$ be a log étale morphism of fs log schemes. Then f is Kummer if and only if f is exact.

Another property of Kummer étale morphisms is the following:

Proposition 2.4.1. *(Vidal [2001],1.3) Let $f : Z \to Y$, $g : Y \to X$ be morphisms of fs log schemes and $h = g \circ f$. If g and h are Kummer étale, then f is Kummer étale.*

Definition 2.4.2. Let X be an fs log scheme.

♦ The *Kummer étale site* of X, denoted by $X^{\mathtt{ket}}$, is defined as follows:

- The objects of $X^{\mathtt{ket}}$ are the fs log schemes which are Kummer étale over X.
- If Y, Z are objects of $X^{\mathtt{ket}}$, a morphism from Y to Z is an X-map $Y \to Z$. By Proposition 2.4.1, any such map is again Kummer étale.
- The *Kummer étale topology* is the topology on $X^{\mathtt{ket}}$ generated by the covering families $(f_i : Y_i \to Y)_{i \in I}$ of morphisms of $X^{\mathtt{ket}}$ such that

$$Y = \bigcup_{i \in I} f_i(Y_i)$$

set theoretically.

- The *Kummer étale site* of X is the category $X^{\mathtt{ket}}$ endowed with the Kummer étale topology.

♦ The *Kummer étale topos* of X, denoted by $\mathrm{Top}(X^{\mathtt{ket}})$ (or simply $X^{\mathtt{ket}}$ again if there is no confusion), is the category of sheaves on $X^{\mathtt{ket}}$.

Remark 2.4.2. The datum for each object Y of $X^{\mathtt{ket}}$ and the set of covering families of Y as above define a pretopology on $X^{\mathtt{ket}}$ in the sense of Grothendieck ([Artin et al., 1972, II 1.3]). To verify the axioms of a pretopology the only nontrivial part is checking the stability of covering families under base change. By Proposition 2.3.2 and Remark 2.4.1, Kummer étale morphisms are stable under fs base change, thus it is enough to verify the universal surjectivity of covering families. This follows from the following lemma:

Lemma 2.4.1. *([Nakayama, 1997, Lemma 2.2.2]) Suppose we have the following cartesian square of fs log schemes*

$$\begin{tikzcd} X_4 \arrow[r, "g'"] \arrow[d, "f'"'] & X_3 \arrow[d, "f"] \\ X_2 \arrow[r, "g"'] & X_1. \end{tikzcd}$$

Let $x_3 \in X_3$, $x_2 \in X_2$ be such that $f(x_3) = g(x_2)$. Assume that f or g is exact. Then there exists $x_4 \in X_4$ such that $f'(x_4) = x_2$ and $g'(x_4) = x_3$.

Remark 2.4.3. Let X be a scheme with the trivial log structure. If $f : Y \to X$ is an object of $X^{\mathtt{ket}}$, then f is strict, the log structure on Y is trivial and f is étale in the classical sense. Thus the Kummer étale site $X^{\mathtt{ket}}$ can be identified with the classical étale site $X^{\mathtt{et}}$ of X.

Chapter 3

$\mathcal{P}$-adic Modular Forms over Shimura Curves

In this chapter, we review the theory of $\mathcal{P}$-adic modular forms over Shimura curves over totally real fields of non-integral weights, which was established by R. Brasca in Brasca [2011] and Brasca [2013]. Our main references for this chapter are Carayol [1986], Kassaei [2004], Brasca [2011] and Brasca [2013].

3.1 Shimura Curves

3.1.1 Quaternionic Shimura curves $M_K(G, X)$

First, we introduce some notations. Let F be totally real field of degree $d > 1$ over $\mathbb{Q}$ and denote by $\tau_1, \tau_2, \ldots, \tau_d$ its all real embeddings. Set $\tau = \tau_1$. Let B be a quaternion algebra over F which is split at τ and ramified at all other infinite places $\tau_2, \ldots, \tau_d$ (these assumptions imply that the Shimura datum we are going to construct later gives a Shimura variety of dimension one). In particular, we fix identifications

$$B \otimes_{F,\tau} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R}),$$

and for $i = 2, \ldots, d$

$$B \otimes_{F,\tau_i} \mathbb{R} \cong \mathbb{H},$$

where $\mathbb{H}$ is the Hamilton quaternion algebra over $\mathbb{R}$.

Let G be the reductive group over $\mathbb{Q}$ defined by

$$G := \mathrm{Res}_{F/\mathbb{Q}}(B^\times).$$

Then $G(\mathbb{Q}) = B^\times$ and $G(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H})^{d-1}$. We set $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$, hence $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. The morphism

$$\begin{array}{ccc} \mathbb{C}^\times & \longrightarrow & \mathrm{GL}_2(\mathbb{R}) \times (\mathbb{H})^{d-1} \\ x + \mathbf{i}y & \longmapsto & \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1}, 1, \ldots, 1 \right) \end{array}$$

comes from a morphism

$$h : \mathbb{S} \longrightarrow G_{\mathbb{R}}.$$

Now let X be the $G(\mathbb{R})$-conjugation class of h, then X can be identified with

$$X \cong \mathbb{C} - \mathbb{R} = \mathfrak{H}^+ \bigsqcup \mathfrak{H}^-$$

which is the union of two copies of the Poincaré half plane.

Let K be a compact open subgroup of $G(\mathbb{A}^f)$. We define the following compact Riemann surface to be the double cosets:

$$M_K(G, X)(\mathbb{C}) := G(\mathbb{Q}) \backslash G(\mathbb{A}^f) \times X / K.$$

Here the action of K on X is trivial and via right multiplication on $G(\mathbb{A}^f)$, $G(\mathbb{Q})$ acts by left multiplication on $G(\mathbb{A}^f)$ and via the natural diagonal embedding induced by $\mathbb{Q} \hookrightarrow \mathbb{A}^f$. By the theory of Shimura (Shimura [1970], Deligne [1971a]), there exists a canonical model $M_K(G, X)$ of $M_K(G, X)(\mathbb{C})$, defined over F, where F is thought of as a subfield of $\mathbb{C}$ via the embedding $F \xhookrightarrow{\tau} \mathbb{R} \hookrightarrow \mathbb{C}$. Moreover, $M_K(G, X)$ is smooth and proper. The Shimura curves $M_K(G, X)$ are not of PEL type hence can not be described in terms of abelian varieties. To modify this, we will introduce another reductive group G' with the same derived group as G and the $G'(\mathbb{R})$-conjugation class of a morphism of algebraic groups $h' : \mathbb{S} \to G'_{\mathbb{R}}$ such that the corresponding Shimura curves $M'_{K'}(G', X')$ classify abelian varieties with polarizations, endomorphisms and level structures (see Theorem 3.1.1).

3.1.2 Unitary Shimura curves $M'_{K'}(G', X')$

Let F, B, $\mathbb{S}$, G be as before. Fix $p \neq 2$ a prime integer. Choose an element $\lambda \in \mathbb{Q}$, $\lambda < 0$ such that $\mathbb{Q}(\sqrt{\lambda})$ splits at p. Let $E := F(\sqrt{\lambda})$ be an imaginary quadratic extension over F. By choosing a square root ρ of λ in $\mathbb{C}$, we extend the embeddings $\tau_i : F \hookrightarrow \mathbb{R}$ to embeddings $\tau_i : E \hookrightarrow \mathbb{C}$, via $\tau_i(x + y\sqrt{\lambda}) = \tau_i(x) + \rho\tau_i(y)$, for $1 \leq i \leq d$, where $x, y \in F$. Similarly, we fix the embedding

$$\tau := \tau_1 : E \hookrightarrow \mathbb{C}$$

and consider E as a subfield of $\mathbb{C}$ via τ.

Now let $T := \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$ and Z be the center of G. Then we can identify T by Z. Let $v : G \to T$ be the morphism obtained by the restriction of the reduced norm of B. Then we have an exact sequence of algebraic groups:

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{v} T \longrightarrow 1,$$

where G_1 is the derived group of G. Let $T_E := \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$ and U_E be the subgroup of T_E defined by the equation $z \cdot \bar{z} = 1$, where $\bar{\cdot}$ denotes the natural conjugation of E with respect to F. Let $G'' := G \times_T T_E$ be the amalgamated product of the following pair of morphisms

$$T_E \longleftarrow T \cong Z \longrightarrow G.$$

We have the following commutative diagram of morphisms of algebraic groups:

$$\begin{tikzcd}
T \cong Z \arrow[r] \arrow[d] & G \arrow[d] \arrow[ddr, bend left, "\beta"] & \\
T_E \arrow[r] \arrow[drr, bend right, "\alpha"] & G'' \arrow[dr, dashed, "v'"] & \\
& & T \times U_E,
\end{tikzcd}$$

where

$$\begin{aligned} \alpha : T_E &\longrightarrow T \times U_E, \\ z &\longmapsto \left(z\bar{z}, z/\bar{z}\right), \end{aligned}$$

and

$$\begin{aligned} \beta : G &\longrightarrow T \times U_E, \\ g &\longmapsto (v(g), 1). \end{aligned}$$

This induces a morphism

$$v' : G'' \longrightarrow T \times U_E,$$

$$(g,z) \longmapsto \left(v(g)z\bar{z}, z/\bar{z}\right).$$

Now consider $T' := \mathbb{G}_{m,\mathbb{Q}} \times U_E$, the sub-torus of $T \times U_E$, and let G' be the inverse image of T' via v' in G'', i.e.,

$$\begin{tikzcd} G' = (v')^{-1}(T') \arrow[r, hook] \arrow[d, "v'"'] & G'' = (G \times_T T_E) \arrow[d, "v'"] \\ T' \arrow[r, hook] & T \times U_E. \end{tikzcd} \tag{3.1}$$

The complex embeddings $\tau_1, \ldots, \tau_d$ of E into $\mathbb{C}$ give an isomorphism

$$T_E(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \cong \left(\mathbb{C}^{\times}\right)^d$$

Let $h_E : \mathbb{S} \to (T_E)_{\mathbb{R}}$ be the morphism defined by

$$\begin{aligned} h_E(\mathbb{R}) : \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times} &\longrightarrow \left(\mathbb{C}^{\times}\right)^d = T_E(\mathbb{R}), \\ z &\longmapsto \left(1, z^{-1}, \ldots, z^{-1}\right). \end{aligned}$$

Recall that we have a morphism $h : \mathbb{S} \to G_{\mathbb{R}}$ and the morphism $\mathbb{S} \to (G \times_T T_E)_{\mathbb{R}}$, defined by the composite

$$\mathbb{S} \xrightarrow{h \times h_E} (G \times T_E)_{\mathbb{R}} \xrightarrow{\texttt{proj}} (G \times_T T_E)_{\mathbb{R}},$$

has image in T'. Hence it factors through a morphism $h' : \mathbb{S} \to (G')_{\mathbb{R}}$.

Now let X' be the $G'(\mathbb{R})$-conjugation class of h', which can be identified with $\mathfrak{H}^+$, the Poincaré half plane. For any compact open subgroup K' of $G'(\mathbb{A}^f)$, we associate to (G', X') a Shimura curve over $\mathbb{C}$, defined by

$$M'_{K'}(G', X')(\mathbb{C}) := G'(\mathbb{Q})\backslash G'(\mathbb{A}^f) \times X'/K',$$

which is a compact Riemann surface.

Now we give another description of the reductive group G' as in [Carayol, 1986, §2.2]. We need both descriptions to state the moduli problem of the unitary Shimura curves. Let $D := B \otimes_F E$. Then D is a quaternion algebra over E. We define

$$\begin{aligned} \bar{\cdot} : D &\longrightarrow D, \\ b \otimes z &\longmapsto b' \otimes \bar{z}, \end{aligned}$$

where $\cdot' : B \to B$ is the canonical involution of B, and $z \mapsto \bar{z}$ is the conjugation of E with respect to F. It follows that $\bar{\cdot}$ is an involution on D.

Choose an element $\delta \in D^\times$ such that $\bar{\delta} = \delta$ and define another involution on D by

$$\begin{aligned} \cdot^* : D &\longrightarrow D, \\ d &\longmapsto \delta^{-1}\bar{d}\delta. \end{aligned}$$

Let V be the underlying $\mathbb{Q}$-vector space of D, with the left action of D by left multiplication. We may consider V as a free left D-module of rank 1. Choose a non-zero element $\alpha \in E$ such that $\bar{\alpha} = -\alpha$ and define a $\mathbb{Q}$-bilinear form, for all v and w in V:

$$\begin{aligned} \Theta : V \times V &\longrightarrow \mathbb{Q}, \\ (v, w) &\longmapsto \mathrm{Tr}_{E/\mathbb{Q}}\left(\alpha\, \mathrm{tr}_{D/E}(v\delta w^*)\right). \end{aligned}$$

Then we have

Proposition 3.1.1. Θ *is a symplectic form on* V, *i.e.,*

(i) $\Theta(v, v) = 0$ *for any* $v \in V$ *(alternating).*

(ii) If $\Theta(v, w) = 0$ *for all* $w \in V$, *then* $v = 0$ *(non-degenerate).*

(iii) $\Theta(dv, w) = \Theta(v, d^*w)$*, for any* $v, w \in V$ *and any* $d \in D$*.*

Proof. See [Brasca, 2011, Lemma 1.2.2]. $\square$

Definition 3.1.1. Let W be a free left D-module and let Θ a symplectic (see Prop. 3.1.1) $\mathbb{Q}$-bilinear form on W. We say an element $g \in \mathrm{Aut}_D(W)$ is a D-linear symplectic similitude of (W, Θ) if there exists $\mu_g \in \mathbb{Q}^\times$ such that

$$\Theta\left(g(v), g(w)\right) = \mu_g \Theta(v, w)$$

for all $v, w \in W$.

Remark 3.1.1. In our case, since V is a free left D-module of rank 1, we have $\mathrm{Aut}_D(V) = D^\times$ and a automorphism corresponds to the right multiplication by a unit in D. Then a D-linear symplectic similitude of (V, Θ) is an element $d \in D^\times$ such that for any $v, w \in V$

$$\Theta(vd, wd) = \mu\Theta(v, w) \text{ for some element } \mu \in \mathbb{Q}^\times . \tag{3.2}$$

Then by the $\mathbb{Q}$-linenarity of Θ, the above equality is equivalent to

$$\Theta\left(vd\delta d^*\delta^{-1}, w\right) = \Theta\left(v\mu, w\right),$$

which is equivalent to

$$d\delta d^*\delta^{-1} = \mu$$

since Θ is non-degenerate. Finally, using $d^* = \delta^{-1}\bar{d}\delta$, we conclude that the original equation (3.2) is equivalent to

$$d\bar{d} = \mu \in \mathbb{Q}^\times .$$

Consider the reductive group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra R, its R-points are the D-linear symplectic similitudes of (V_R, Θ_R) (here we extend the above definition to (V_R, Θ_R)). In particular, its $\mathbb{Q}$-points can be identified with

$$\left\{ d \in D^\times \;\middle|\; d\bar{d} \in \mathbb{Q}^\times \right\}.$$

Then we can identify the reductive group G' (diagram (3.1)) with the one defined above. This gives a morphism of algebraic groups $G'_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ and the composite

$$\mathbb{S} \xrightarrow{h'} G'_{\mathbb{R}} \longrightarrow \mathrm{GL}(V_{\mathbb{R}})$$

defines a Hodge structure of type $\{(-1,0),(0,-1)\}$ on $V_{\mathbb{R}}$, where $J = h'(\mathbf{i})$ gives a complex structure on $V_{\mathbb{R}}$ via

$$J \cdot x = x \cdot J^{-1}$$

for any $x \in V_{\mathbb{R}}$. Moreover, we can choose δ so that Θ is a polarization for this Hodge structure, i.e., the form on $V_{\mathbb{R}}$ defined by

$$(x, y) \longmapsto \Theta\left(x, y\,(h'(\mathbf{i}))^{-1}\right)$$

is positive definite (see [Carayol, 1986, §2.2.4] for details).

3.1.3 Moduli problems for unitary Shimura curves

3.1.3.1 The canonical model over the reflex field

For any $d \in D$, define $t : D \to \mathbb{C}$ by

$$t(d) := \mathrm{Tr}_{\mathbb{C}}\left(d; V_{\mathbb{C}} / \mathrm{F}^0(V_{\mathbb{C}})\right),$$

where $\mathrm{F}^i(V_{\mathbb{C}})$ is the Hodge filtration of $V_{\mathbb{C}}$ defined by h'. Let E' be the subfield of $\mathbb{C}$ generated by

$$\Big\{t(d)\big|\ d \in D\Big\}.$$

We have the following results

Theorem 3.1.1. *[Deligne, 1972, §6] The canonical model $M'_{K'}(G', X')$ is defined over E', for any sufficiently small compact open subgroup $K' \subseteq G'(\mathbb{A}^f)$. Moreover, it represents the functor*

$$\mathcal{M}^1_{K'} : \{E'\text{-}algebras\} \longrightarrow \{Sets\}$$

defined as follows:

For any E'-algebra R, $\mathcal{M}^1_{K'}(R)$ is the set of isomorphism classes of quadruples $(A, \iota, \bar{\theta}, \bar{k})$ where

(a) A is an abelian scheme over R, defined up to isogenies, with an action of D via $\iota : D \to \mathrm{End}(A)$ such that for any $d \in D$

$$(*)\ \ \mathrm{Tr}\,(\iota(d); \mathrm{Lie}(A)) = t(d).$$

(b) $\bar{\theta}$ is a homogeneous polarization of A such that the Rosati involution sends $\iota(d)$ to $\iota(d^*)$, for all $d \in D$.

(c) $\bar{k}$ is a class modulo K' of a symplectic D-linear similitudes

$$k : \hat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f,$$

where $\hat{V}(A) = \hat{T}(A) \otimes \mathbb{Q}$, with symplectic structure coming from the Weil pairings and $\hat{T}(A) = \prod T_l(A)$ is the product of the Tate modules of A over all primes.

From now on we always assume that K' is small enough to have a canonical model $M'_{K'}$. By calculating $t(d)$ for element $d \in D$, we have a more explicit description of the reflex field E'. Indeed, we have

Proposition 3.1.2. *The reflex field for the canonical model $M'_{K'}(G', X')$ is E, where E is thought of as a subfield of $\mathbb{C}$ via τ.*

Proof. For $1 \leq i \leq d$, let

$$D_i := D \otimes_{F,\tau_i} \mathbb{R} = D \otimes_{E,\tau_i} \mathbb{C}$$

and let V_i be its underlying $\mathbb{R}$-space. Then we have

$$V_{\mathbb{C}} / \mathrm{F}^0(V_{\mathbb{C}}) = V_{\mathbb{C}} / V_{\mathbb{C}}^{0,-1} \cong V_{\mathbb{C}}^{-1,0}.$$

Using the notations above we obtain the decomposition

$$V_{\mathbb{C}}^{-1,0} = \bigoplus_{i=1}^{d} (V_i \otimes_{\mathbb{R}} \mathbb{C})^{-1,0}.$$

and the trace we want to calculate is just the sum of the various traces in the decomposition above. We calculate the different traces in the following cases:

(a) If $i = 1$, then $V_1 \cong \mathrm{M}_2(\mathbb{C})$ and $V_i \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$ via $v \otimes z \mapsto (vz, \bar{v}z)$. $J = h'(\mathbf{i}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and for any M_1, $M_2 \in \mathrm{M}_2(\mathbb{C})$,

$$J(M_1, M_2) = \left(M_1 J^{-1}, M_2 J^{-1}\right) = \left(M_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}, M_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \right).$$

For any $d_1 \in D_1 = \mathrm{M}_2(\mathbb{C})$,

$$d_1 \cdot (M_1, M_2) = \left(d_1 M_1, \bar{d}_1 M_2\right)$$

where the $\bar{\cdot}$ in the second component is the complex conjugation. By some basic calculation on matrices, we have

$$\begin{aligned}(V_1 \otimes_{\mathbb{R}} \mathbb{C})^{-1,0} &= \left\{(M_1, M_2)\middle| J(M_1, M_2) = \mathbf{i}(M_1, M_2)\right\} \\ &= \left\{\left(\begin{pmatrix} a & -a\mathbf{i} \\ d\mathbf{i} & d \end{pmatrix}, \begin{pmatrix} a & -a\mathbf{i} \\ d\mathbf{i} & d \end{pmatrix}\right)\middle| a, d \in \mathbb{C}\right\}.\end{aligned}$$

Hence

$$\mathrm{Tr}\left(d_1; (V_1 \otimes \mathbb{C})^{-1,0}\right) = \mathrm{tr}(d_1) + \mathrm{tr}(\bar{d_1}) = \mathrm{tr}(d_1) + \bar{\mathrm{tr}}(d_1).$$

(b) For $i \geq 2$, $D_i = \mathrm{M}_2(\mathbb{C})$ and $V_i \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C}) \oplus \mathrm{M}_2(\mathbb{C})$.

$$J = h'(\mathbf{i}) = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}$$

and for any M_1, $M_2 \in \mathrm{M}_2(\mathbb{C})$,

$$J(M_1, M_2) = (\mathbf{i}M_1, -\mathbf{i}M_2).$$

Then

$$\begin{aligned}(V_i \otimes \mathbb{C})^{-1,0} &= \left\{(M_1, M_2)\middle| J(M_1, M_2) = \mathbf{i}(M_1, M_2)\right\} \\ &= \mathrm{M}_2(\mathbb{C}).\end{aligned}$$

Hence $\mathrm{Tr}\left(d_i; (V_i \otimes \mathbb{C})^{-1,0}\right) = 2\,\mathrm{tr}(d_i)$ for any $d_i \in D_i$.

From the above discussion, we deduce that for any $d \in D$

$$\begin{aligned}t(d) &= \mathrm{tr}(d_1) + \bar{\mathrm{tr}}(d_1) + 2\sum_{i=2}^{d} \mathrm{tr}(d_i) \\ &= \tau_1\left(\mathrm{tr}_{D/E}(d)\right) + \bar{\tau}_1\left(\mathrm{tr}_{D/E}(d)\right) + 2\sum_{i=2}^{d} \tau_i\left(\mathrm{tr}_{D/E}(d)\right)\end{aligned}$$

$$= \quad (\tau_1 + \bar{\tau}_1 + 2\tau_2 + \cdots + 2\tau_d)\,(\mathrm{tr}_{D/E}(d)).$$

Define a morphism $\sigma : E \to \mathbb{C}$ by $\sigma := \tau_1 + \bar{\tau}_1 + 2\tau_2 + \cdots + 2\tau_d$. Then for any $d \in D$

$$t(d) = \sigma(\mathrm{tr}_{D/E}(d)).$$

For x, $y \in F$, we have the equalities

$$\begin{aligned} \sigma(x + y\sqrt{\lambda}) &= (\tau_1(x) + \rho\tau_1(y)) + (\tau_1(x) - \rho\tau_1(y)) + 2\sum_{i=2}^{d} (\tau_i(x) + \rho\tau_i(y)) \\ &= 2\,\mathrm{tr}_{F/\mathbb{Q}}(x) + 2\rho\left(\mathrm{tr}_{F/\mathbb{Q}}(y) - y\right), \end{aligned}$$

which imply that the image of σ in $\mathbb{C}$ generates E. Hence $E' = E$. This completes the proof. □

3.1.3.2 The canonical model over a local field

Let F, E, B, p be as before and denote by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ the primes of F lying above p. We denote simply by $\mathcal{P} = \mathcal{P}_1$. Let $F_{\mathcal{P}_i}$ denote the completion of F at $\mathcal{P}_i$. Let $\mathcal{O}_{\mathcal{P}}$ be the ring of integers of $F_{\mathcal{P}}$ and denote by e and f its ramification degree and residue degree, respectively. Fix π, a uniformizer of $\mathcal{O}_{\mathcal{P}}$ and let κ be the residue field, with cardinality $q = p^f$ and characteristic p. Let $v(\cdot)$ be the normalized valuation of $F_{\mathcal{P}}$, i.e., $v(\pi) = 1$, and let $|\cdot|$ be a norm on $F_{\mathcal{P}}$ compatible with $v(\cdot)$.

Choose a square root μ of λ in $\mathbb{Q}_p$, which can be done since we assume that $\mathbb{Q}(\sqrt{\lambda})$ splits at p. The morphism

$$\begin{aligned} E &\longrightarrow F_p \oplus F_p = \left(F \otimes_{\mathbb{Q}} \mathbb{Q}_p\right) \oplus \left(F \otimes_{\mathbb{Q}} \mathbb{Q}_p\right) \\ x + y\sqrt{\lambda} &\longmapsto (x + y\mu, x - y\mu) \end{aligned}$$

extends to an isomorphism

$$E \otimes \mathbb{Q}_p \xrightarrow{\sim} F_p \oplus F_p \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}) \oplus (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}),$$

which allow us to consider $F_{\mathcal{P}}$ as an E-algebra via the following composition

$$E \hookrightarrow E \otimes \mathbb{Q}_p \xrightarrow{\sim} (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}) \oplus (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}) \xrightarrow{\mathrm{pr}_1} (F_{\mathcal{P}_1} \oplus \cdots \oplus F_{\mathcal{P}_m}) \xrightarrow{\mathrm{pr}_1} F_{\mathcal{P}}.$$

From now on we base change the model $M'_{K'}$ to $F_{\mathcal{P}}$ and consider the $F_{\mathcal{P}}$-scheme $M'_{K'}(G', X') \otimes_E F_{\mathcal{P}}$, which is still denoted by $M'_{K'}$.

We assume that the quaternion algebra B is split at $\mathcal{P}$ and fix an isomorphism $B \otimes_F F_{\mathcal{P}} \cong \mathrm{M}_2(F_{\mathcal{P}})$. Let

$$D_p = D \otimes \mathbb{Q}_p = B \bigotimes_F (E \otimes \mathbb{Q}_p).$$

Then the decomposition of $E \otimes \mathbb{Q}_p$ induces a decomposition of D_p as

$$D_p = \left(D_1^1 \oplus \cdots \oplus D_m^1\right) \oplus \left(D_1^2 \oplus \cdots \oplus D_m^2\right),$$

where D_i^k is an $F_{\mathcal{P}_i}$-algebra and $D_i^k \cong B_{\mathcal{P}_i} := B \otimes_F F_{\mathcal{P}_i}$ for $i = 1, \ldots, m$ and $k = 1, 2$. In particular, D_1^1 and D_1^2 are identified with $\mathrm{M}_2(F_{\mathcal{P}})$. The involution $\cdot^* : D \to D$ induces an involution of D_p which switches D_i^1 with D_i^2.

Now suppose Λ is a D_p-module. The decomposition of D_p induces a decomposition of Λ:

$$\Lambda = \left(\Lambda_1^1 \oplus \cdots \oplus \Lambda_m^1\right) \oplus \left(\Lambda_1^2 \oplus \cdots \oplus \Lambda_m^2\right),$$

where D_p acts on Λ_i^k via D_i^k.

Recall that we have fixed isomorphisms $D_1^1 \cong D_1^2 \cong \mathrm{M}_2(F_{\mathcal{P}})$. This allows us to decompose Λ_1^2 into a direct sum of $F_{\mathcal{P}}$-vector spaces:

$$\Lambda_1^2 = \Lambda_1^{2,1} \oplus \Lambda_1^{2,2},$$

where $\Lambda_1^{2,1}$ (resp. $\Lambda_1^{2,2}$) is the kernel of the idempotent $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$) of D_1^2. These two $F_{\mathcal{P}}$-vector spaces are isomorphic and can be switched by conjugation of the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of D_1^2.

Now let R be an $F_{\mathcal{P}}$-algebra and A an abelian scheme over R, defined up to isogenies, with an action of D. Then the relative Lie algebra $\mathrm{Lie}_R(A)$ is an $R \otimes D$-module, in particular is a D_p-module. Then we have a decomposition as before:

$$\mathrm{Lie}(A) = \mathrm{Lie}_1^1(A) \oplus \cdots \oplus \mathrm{Lie}_m^1(A) \oplus \mathrm{Lie}_1^2(A) \oplus \cdots \oplus \mathrm{Lie}_m^2(A),$$

where $\mathrm{Lie}_i^k(A)$ is a projective R-module with an action of D_i^k. Furthermore, the factor $\mathrm{Lie}_1^2(A)$ admits the decomposition

$$\mathrm{Lie}_1^2(A) = \mathrm{Lie}_1^{2,1}(A) \oplus \mathrm{Lie}_1^{2,2}(A)$$

of two projective R-modules with an $F_{\mathcal{P}}$-action. Using such decomposition, we obtain an more explicit description of condition $(*)$ in the moduli problem (see part (a) in Theorem 3.1.1).

Proposition 3.1.3. *Let R, A as above. The condition $(*)$ in Theorem 3.1.1 is equivalent to the followings:*

1. *The relative dimension of A is $4d$.*

2. *The projective R-module $\mathrm{Lie}_1^{2,1}(A)$ has rank 1, with an action of $F_{\mathcal{P}}$ via the natural map $F_{\mathcal{P}} \hookrightarrow R$.*

3. *For $i \geq 2$, $\mathrm{Lie}_i^2(A) = 0$.*

Proof. See [Carayol, 1986, §2.4] □

Now let (V, Θ) be as in section 3.1.2 and consider the D_p-module $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with an decomposition

$$V_p = V_1^1 \oplus \cdots \oplus V_m^1 \oplus V_1^2 \oplus \cdots \oplus V_m^2.$$

We will also give a more explicit description in terms of decompositions of V_p for part (c) of the moduli problem. The space V_p has a symplectic form $\Theta_p = \Theta \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that the components V_i^k and V_j^l are orthogonal unless $i = j$ and $k \neq l$. More generally, we have

Lemma 3.1.1. *Let Λ be a D_p-module with a $\mathbb{Q}_p$-bilinear, alternating, nondegenerate form Φ such that $\Phi(dv, w) = \Phi(v, d^*w)$ for all $d \in D_p$, $v, w \in \Lambda$. Then Λ_i^k and Λ_j^l are orthogonal unless $i = j$ and $k \neq l$.*

The group $G'(\mathbb{Q}_p)$ is identified with the group of D_p-linear symplectic similitudes of (V_p, Θ_p). By the above lemma such a similitude is totally determined by

- its similitude ratio $\mu \in \mathbb{Q}_p^\times$;
- its restriction to V_i^2.

Thus we have

$$\begin{aligned} G'(\mathbb{Q}_p) &\cong \mathbb{Q}_p^\times \times \prod_{i=1}^m \mathrm{Aut}_{D_i^2}(V_i^2) \\ &\cong \mathbb{Q}_p^\times \times \prod_{i=1}^m \left(D_i^2\right)^\times \\ &\cong \mathbb{Q}_p^\times \times \prod_{i=1}^m \left(B \otimes_F F_{\mathfrak{P}_i}\right)^\times \\ &\cong \mathbb{Q}_p^\times \times \mathrm{GL}_2(F_{\mathfrak{P}}) \times \left(B \otimes_F F_{\mathfrak{P}_2}\right)^\times \times \cdots \times \left(B \otimes_F F_{\mathfrak{P}_m}\right)^\times . \end{aligned}$$

Let A be an abelian scheme, defined up to isogenies, with an action of D via $\iota : D \to \mathrm{End}(A)$ and a polarization θ such that the Rosati involution sends $\iota(d)$ to $\iota(d^*)$. Let $\Phi : V_p(A) \times V_p(A) \to \mathbb{Q}_p(1)$ be the pairing associated to θ. Then Φ is a $\mathbb{Q}_p$-bilinear, alternating, nondegenerate from such that for any $d \in D_p$, $v, w \in V_p(A)$,

$$\Phi(dv, w) = \Phi(v, d^* w).$$

Therefore, $V_i^k(A)$ and $V_j^l(A)$ are orthogonal with respect to Φ unless $i = j$ and $k \neq l$. To give a D-linear symplectic similitude

$$k : \hat{V}(A) \xrightarrow{\sim} V \otimes \mathbb{A}^f$$

is equivalent to give the followings:

- a D-linear symplectic similitude

$$k^p : \hat{V}^p(A) = \prod_{l \neq p} V_l(A) \xrightarrow{\sim} V \otimes \mathbb{A}^{f,p};$$

- a similitude ratio $\mu_p \in \mathbb{Q}_p^\times$;

- D_i^2-linear isomorphisms

$$k_i^2 : V_i^2(A) \xrightarrow{\sim} V_i^2$$

  for $1 \leq i \leq m$.

In particular, giving a k_1^2 is equivalent to giving an $F_{\mathfrak{p}}$-linear isomorphism

$$k_1^{2,1} : V_1^{2,1}(A) \xrightarrow{\sim} V_1^{2,1}.$$

It is often necessary to describe the Shimura curve $M'_{K'}(G', X')$ as a moduli problem defined in terms of abelian varieties, rather than isogeny classes of abelian schemes. To do this, we need to choose a maximal order of the quaternion algebra D. Let $\mathcal{O}_D$ be a maximal order of D and denote by $V_{\mathbb{Z}}$ the corresponding lattice in V. The ring $\mathcal{O}_D \otimes \mathbb{Z}_p$ admits a decomposition

$$\begin{array}{ccccccccccccc} \mathcal{O}_D \otimes \mathbb{Z}_p & = & \mathcal{O}_{D_1^1} & \oplus & \cdots & \oplus & \mathcal{O}_{D_m^1} & \oplus & \mathcal{O}_{D_1^2} & \oplus & \cdots & \oplus & \mathcal{O}_{D_m^2} \\ \bigcap & & \bigcap & & & & \bigcap & & \bigcap & & & & \bigcap \\ D \otimes \mathbb{Q}_p & = & D_1^1 & \oplus & \cdots & \oplus & D_m^1 & \oplus & D_1^2 & \oplus & \cdots & \oplus & D_m^2. \end{array}$$

Moreover we can choose $\mathcal{O}_D$, α, δ in such a way that:

(I) $\mathcal{O}_D$ is stable under the involution $d \mapsto d^*$;

(II) each $\mathcal{O}_{D_i^k}$ is a maximal order in D_i^k and $\mathcal{O}_{D_1^2} \subset D_1^2 = \mathrm{M}_2(F_{\mathcal{P}})$ is identified with $\mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$;

(III) the symplectic form $\Theta : V \times V \to \mathbb{Q}$ defined by

$$\Theta(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}\left(\alpha\, \mathrm{tr}_{D/E}(v\delta w^*)\right)$$

takes integer values on $V_{\mathbb{Z}}$;

(IV) Θ induces a perfect pairing Θ_p on $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$.

Then every $\mathcal{O}_{D_p} = \mathcal{O}_D \otimes \mathbb{Z}_p$-module Λ admits a decomposition as

$$\Lambda = \Lambda_1^1 \oplus \cdots \oplus \Lambda_m^1 \oplus \Lambda_1^2 \oplus \cdots \oplus \Lambda_m^2$$

such that each Λ_i^k is an $\mathcal{O}_{D_i^k}$-module. The $\mathcal{O}_{D_1^2} \cong \mathrm{M}_2(\mathcal{O}_{\mathcal{P}})$-module Λ_1^2 decomposes further as the direct sum of two $\mathcal{O}_{\mathcal{P}}$-modules $\Lambda_1^2 = \Lambda_1^{2,1} \oplus \Lambda_1^{2,2}$.

Let K' be an open compact subgroup of $G'(\mathbb{A}^f)$ small enough such that it keeps the adelic lattice $V_{\hat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}} \subset V \otimes \mathbb{A}^f$ invariant. We have the following theorem as in [Carayol, 1986, §2.6.2]:

Theorem 3.1.2. *The functor $\mathcal{M}^1_{K'}$ in Theorem 3.1.1 is isomorphic to the functor*

$$\mathcal{M}^2_{K'} : \{E\text{-}algebras\} \longrightarrow \{Sets\}$$

which is defined as follows:

For any E-algebra R, $\mathcal{M}^2_{K'}(R)$ is the set of isomorphism classes of quadruples $(A, \iota, \theta, \bar{k})$ where

(a) A is an abelian scheme over R of relative dimension $4d$, with an action of D via $\iota : D \to \mathrm{End}(A)$ such that the condition $(*)$ of Theorem 3.1.1 is satisfied.

(b) θ is a polarization of A, of degree prime to p, such that the corresponding Rosati involution sends $\iota(d)$ to $\iota(d^*)$, for all $d \in D$.

(c) $\bar{k}$ is a class modulo K' of $\mathcal{O}_D$-linear symplectic isomorphisms

$$k : \hat{T}(A) \xrightarrow{\sim} V_{\hat{\mathbb{Z}}}.$$

Example 3.1.1. Some level structure we are interested in.

Let

$$\Gamma' := (B \otimes_F F_{\mathcal{P}_2})^\times \times \cdots \times (B \otimes_F F_{\mathcal{P}_m})^\times \times G'(\mathbb{A}^{f,p}).$$

Then the finite adelic points of G' can be described as

$$G'(\mathbb{A}^f) = \mathbb{Q}_p^\times \times \mathrm{GL}_2(F_{\mathcal{P}}) \times \Gamma'.$$

From now on, we will only consider the subgroup $K' \subset G'(\mathbb{A}^f)$ of the form

$$K' = \mathbb{Z}_p^\times \times K_{\mathcal{P}} \times H,$$

where $K_{\mathcal{P}}$ is a subgroup of $\mathrm{GL}_2(F_{\mathcal{P}})$ and H is an open compact subgroup of Γ'.

Let $(A, \iota, \theta, \bar{k})$ be an object of the moduli problem as in Theorem 3.1.2. We will use the following notations to give a more explicit interpretation of a K'-level structure $\bar{k}$. Recall that we have the decomposition of

$$T_p(A) = (T_p(A))_1^1 \oplus \cdots \oplus (T_p(A))_m^1 \oplus (T_p(A))_1^2 \oplus \cdots \oplus (T_p(A))_m^2$$

as an $\mathcal{O}_D \otimes \mathbb{Z}_p$-module. Let

$$\begin{aligned} T_p^{\mathcal{P}}(A) &:= (T_p(A))_2^2 \oplus \cdots \oplus (T_p(A))_m^2, \\ \hat{T}^p(A) &:= \prod_{l \neq p} T_l(A), \\ \hat{T}(A) &:= T_p^{\mathcal{P}}(A) \oplus \hat{T}^p(A), \\ \hat{W}^p &:= V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^p, \\ \hat{W}_p &:= (V_{\mathbb{Z}_p})_2^2 \oplus \cdots \oplus (V_{\mathbb{Z}_p})_m^2. \end{aligned}$$

Then the level structure $\bar{k}$ in the description of the functor $\mathcal{M}_{K'}^2$ can be replaced with the following data:

($\blacklozenge$) $\bar{k}_{\mathcal{P}}$ is a class modulo $K_{\mathcal{P}}$ of isomorphisms of $\mathcal{O}_{\mathcal{P}}$-modules

$$k_{\mathcal{P}} : (T_p(A))_1^{2,1} \xrightarrow{\sim} (V_{\mathbb{Z}_p})_1^{2,1} \cong \mathcal{O}_{\mathcal{P}}^2;$$

($\blacklozenge\blacklozenge$) $\bar{k}^{\mathcal{P}}$ is a class, modulo H, of isomorphisms

$$k^{\mathcal{P}} = k_p^{\mathcal{P}} \oplus k^p : T_p^{\mathcal{P}}(A) \oplus \hat{T}^p(A) \xrightarrow{\sim} W_p^{\mathcal{P}} \oplus \hat{W}^p,$$

with $k_p^{\mathcal{P}} : T_p^{\mathcal{P}}(A) \xrightarrow{\sim} W_p^{\mathcal{P}}$ linear and $k^p : \hat{T}^p(A) \xrightarrow{\sim} \hat{W}^p$ symplectic.

In particular, if $K_{\mathcal{P}} = \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}})$, then condition ($\blacklozenge$) disappears.

3.1.3.3 Special level structures

From now on, we will only consider the following three cases of $K_{\mathcal{P}}$, whose corresponding level structures can be described even more explicitly. We write $A[\pi^n]_1^{2,l}$ for the π^n-torsion in $A[p^n]_1^{2,l}$ and let $A[\pi^n]_1^2 := A[\pi^n]_1^{2,1} \oplus A[\pi^n]_1^{2,2}$. Define

$$K(H) := \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}),$$

$$K(H, \pi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) \middle| c \equiv 0 \bmod \pi^n \right\},$$

$$K(H\pi^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}) \middle| a \equiv 1 \bmod \pi^n \text{ and } c \equiv 0 \bmod \pi^n \right\}.$$

In these cases, the Shimura curves $M'_{K'}$ are denoted, respectively, by $M(H)$, $M(H, \pi^n)$ and $M(H\pi^n)$. They parametrize the following isomorphic classes, respectively:

(1) $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ where

- (A, ι, θ) is as in Theorem 3.1.2;
- $\bar{k}^{\mathcal{P}}$ is as in Example 3.1.1 (♦♦).

(2) $(A, \iota, \theta, C, \bar{k}^{\mathcal{P}})$ where

- (A, ι, θ) is as in Theorem 3.1.2;
- $\bar{k}^{\mathcal{P}}$ is as in Example 3.1.1 (♦♦);
- C is a finite flat subgroup scheme of rank q^n of $A[\pi^n]_1^{2,1}$, stable under $\mathcal{O}_{\mathcal{P}}$.

(3) $(A, \iota, \theta, Q, \bar{k}^{\mathcal{P}})$ where

- (A, ι, θ) is as in Theorem 3.1.2;
- $\bar{k}^{\mathcal{P}}$ is as in Example 3.1.1 (♦♦);
- Q is a point of exact $\mathcal{O}_{\mathcal{P}}$-order π^n in $A[\pi^n]_1^{2,1}$.

3.1.3.4 Integral models

One of the main results of Carayol [1986] is that the Shimura curves $M(H)$, $M(H, \pi^n)$ and $M(H\pi^n)$ over $F_{\mathcal{P}}$ admit canonical proper models over $\mathcal{O}_{\mathcal{P}}$, denoted respectively by $\mathbb{M}(H)$, $\mathbb{M}(H, \pi^n)$ and $\mathbb{M}(H\pi^n)$, which solve the same moduli problems as $M(H)$, $M(H, \pi^n)$ and $M(H\pi^n)$ do, respectively, for $\mathcal{O}_{\mathcal{P}}$-algebras. More explicitly, we have

Theorem 3.1.3. *When H' is small enough, the curve $\mathbb{M}(H)$, $\mathbb{M}(H, \pi^n)$ and $\mathbb{M}(H\pi^n)$ represent the functors $\mathcal{M}_H$, $\mathcal{M}_{H,\pi^n}$ and $\mathcal{M}_{H\pi^n}$, respectivly:*

$$\mathcal{M} : \{\mathcal{O}_{\mathcal{P}}\text{-}algebras\} \longrightarrow \{Sets\}$$

such that for any $\mathcal{O}_{\mathcal{P}}$-algebra R,

(1) $\mathcal{M}_H(R)$ is the set of all isomorphism classes of $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ where

- *A is an abelian scheme over R of relative dimension $4d$ with an action of $\mathcal{O}_D$ via $\iota : \mathcal{O}_D \operatorname{End}_R(A)$ such that*
  - *(a) the projective $R-$module $\operatorname{Lie}_1^{2,1}(A)$ has rank one and $\mathcal{O}_{\mathcal{P}}$ acts on it via $\mathcal{O}_{\mathcal{P}} \hookrightarrow R$,*
  - *(b) for $j \geq 2$, $\operatorname{Lie}_j^2(A) = 0$;*
- *θ is a polarization of A of degree prime to p such that the corresponding Rosati involution sends $\iota(d)$ to $\iota(d^*)$ for any $d \in D$;*

- $\bar{k}^{\mathcal{P}}$ *is a class, modulo* H*, of isomorphisms*

$$k^{\mathcal{P}} = k_p^{\mathcal{P}} \oplus k^p : T_p^{\mathcal{P}}(A) \oplus \hat{T}^p(A) \xrightarrow{\sim} W_p^{\mathcal{P}} \oplus \hat{W}^p,$$

with $k_p^{\mathcal{P}} : T_p^{\mathcal{P}}(A) \xrightarrow{\sim} W_p^{\mathcal{P}}$ *linear and* $k^p : \hat{T}^p(A) \xrightarrow{\sim} \hat{W}^p$ *symplectic (for notations, see Example 3.1.1).*

(2) $\mathcal{M}_{H,\pi^n}(R)$ *is the set of all isomorphism classes of* $(A, \iota, \theta, C, \bar{k}^{\mathcal{P}})$ *where*

- $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ *is as in (1);*
- C *is a finite flat subgroup scheme of rank* q^n *of* $A[\pi^n]_1^{2,1}$*, stable under the action of* $\mathcal{O}_{\mathcal{P}}$*.*

(3) $\mathcal{M}_{H\pi^n}(R)$ *is the set of isomorphism classes of* $(A, \iota, \theta, Q, \bar{k}^{\mathcal{P}})$ *where*

- $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ *is as in (1);*
- Q *is a point of exact* $\mathcal{O}_{\mathcal{P}}$*-order* π^n *in* $A[\pi^n]_1^{2,1}$ *in the sense of Drinfel'd.*

We denote by $A(H)$, $A(H, \pi^n)$ and $A(H\pi^n)$ the universal objects of the moduli problems of the curves $M(H)$, $M(H, \pi^n)$ and $M(H\pi^n)$, respectively. Let $\mathbb{A}(H)$, $\mathbb{A}(H, \pi^n)$ and $\mathbb{A}(H\pi^n)$ be the corresponding canonical integral models, respectively. Now let K be any of the level structures $K(H)$, $K(H, \pi^n)$ and $K(H\pi^n)$ we described in Example 3.1.1, and let $(\mathbb{M}, \mathbb{A})$ be any pair of $(\mathbb{M}(H), \mathbb{A}(H))$, $(\mathbb{M}(H, \pi^n), \mathbb{A}(H, \pi^n))$ and $(\mathbb{M}(H\pi^n), \mathbb{A}(H\pi^n))$ with corresponding level structure K, respectively. The morphism $\mathbb{A} \to \mathbb{M}$ is denoted by ε with zero section $e : \mathbb{M} \to \mathbb{A}$. Consider the sheaf of $\mathcal{O}_{\mathbb{M}}$-modules $\varepsilon_* \Omega^1_{\mathbb{A}/\mathbb{M}}$. It has an action of $\mathcal{O}_D \otimes \mathbb{Z}_p$, which allows us to define

$$\underline{\omega} := \underline{\omega}_K := \left(\varepsilon_* \Omega^1_{\mathbb{A}/\mathbb{M}}\right)_1^{2,1}.$$

The condition on the abelian schemes of the moduli problem (Theorem 3.1.3) implies that $\underline{\omega}$ is a line bundle over $\mathbb{M}$. If R is an $\mathcal{O}_{\mathcal{P}}$-algebra, the pullback of $\underline{\omega}_K$ via the morphism $\mathrm{Spec}(R) \to \mathbb{M}$ will be denoted by $\underline{\omega}_R$. We usually drop the subscript and use $\underline{\omega}$ whenever no confusion arises. We have the following Kodaira-Spencer isomorphism.

Proposition 3.1.4. *Let $\mathbb{M}$, $\mathbb{A}$, $\underline{\omega}$ be as above. Then*

(i) $\underline{\omega}_{\mathbb{A}/\mathbb{M}} \otimes \underline{\omega}_{\mathbb{A}^\vee/\mathbb{M}} \xrightarrow{\sim} \Omega^1_{\mathbb{M}/\mathcal{O}_{\mathcal{P}}}$.

(ii) There is a noncanonical isomorphism $\underline{\omega}^{\otimes 2}_{\mathbb{A}/\mathbb{M}} \xrightarrow{\sim} \Omega^1_{\mathbb{M}/\mathcal{O}_{\mathcal{P}}}$.

Proof. See [Kassaei, 2004, Proposition 4.1]. □

Definition 3.1.2. Let K, $\mathbb{M}$ be as above. Let R be an $\mathcal{O}_{\mathcal{P}}$-algebra and k an integer. The space of *modular forms with respect to D, level K and weight k, with coefficients in R,* is defined as follows:

$$S^D(R, K, k) := \mathrm{H}^0(\mathbb{M}_R, \underline{\omega}_R^{\otimes k}).$$

3.2 Hasse Invariant and Canonical subgroups

3.2.1 Hasse invariant

Definition 3.2.1. Let X be an $\mathcal{O}_{\mathcal{P}}$-scheme (or a formal scheme). A *π-divisible group* $H \to X$ is a Barsotti-Tate group H over X, together with an embedding $\mathcal{O}_{\mathcal{P}} \hookrightarrow \mathrm{End}(H)$ such that the induced action of $\mathcal{O}_{\mathcal{P}}$ on $\mathrm{Lie}(H)$ is the natural action via $H \to X \to \mathrm{Spec}(\mathcal{O}_{\mathcal{P}})$. If X is connected, there is a unique integer $\mathrm{ht}(H)$, called the *height* of H, such that $\mathrm{rk}(H[\pi^n]) = q^{n\mathrm{ht}(H)}$ for all n.

Definition 3.2.2. Let $\mathcal{X}$ be a π-adic formal scheme over $\mathrm{Spf}(\mathcal{O}_{\mathcal{P}})$ and let $\mathcal{G} \to \mathcal{X}$ a smooth formal group. We say that $\mathcal{G}$ is a *formal* $\mathcal{O}_{\mathcal{P}}$*-module* if there is an action of $\mathcal{O}_{\mathcal{P}}$ on $\mathcal{G}$ whose action on $\mathrm{Lie}(\mathcal{G})$ is via the structure map $\mathcal{G} \to \mathcal{X} \to \mathrm{Spf}\,(\mathcal{O}_{\mathcal{P}})$.

Let R be an $\mathcal{O}_{\mathcal{P}}$-algebra and $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}})$ be an object of the moduli problem $\mathbb{M}(H)$ (see Theorem 3.1.3) with $\mathbb{A}$ defined over R. There is a natural action of $\mathcal{O}_D \otimes \mathbb{Z}_p$ on $\mathbb{A}[p^n]$, for all n. Hence $\mathbb{A}[p^n]^l_j$ is defined and we also have $\mathbb{A}[p^n]^{2,i}_1$ for $i = 1, 2$, which has an action of $\mathcal{O}_{\mathcal{P}}$. Let $\mathbb{A}[\pi^n]^{2,i}_1$ be its π^n-torsion and $\mathbb{A}[\pi^n]^2_1 := \mathbb{A}[\pi^n]^{2,1}_1 \oplus \mathbb{A}[\pi^n]^{2,2}_1$. Similarly we can define $\mathbb{A}[\pi^n]^1_1$. Then

$$\mathbb{A}[\pi^\infty]^{2,1}_1 := \varinjlim_n \mathbb{A}[\pi^n]^{2,1}_1$$

is a π-divisible group over $\mathbb{A}$, called the π-divisible group associated to $\mathbb{A}$. Let $\mathcal{A}$ be the π-adic completion of $\mathbb{A}$ and $\hat{\mathcal{A}}$ be the formal completion of $\mathcal{A}$ along its zero section. Then $\hat{\mathcal{A}}^{2,1}_1$ is a formal $\mathcal{O}_{\mathcal{P}}$-module of dimension 1. The formal $\mathcal{O}_{\mathcal{P}}$-module associated to $\mathcal{A}[\pi^\infty]^{2,1}_1$ is $\hat{\mathcal{A}}^{2,1}_1$. We will use the notation $\hat{\mathcal{A}}[\pi^n]^{2,1}_1 := \hat{\mathcal{A}}^{2,1}_1[\pi^n]$. The following proposition is proved in [Kassaei, 2004, §4.3].

Proposition 3.2.1. *Let R, $\hat{\mathcal{A}}^{2,1}_1$ be as above. There exists a coordiate x on $\hat{\mathcal{A}}^{2,1}_1$ such that the action of π takes the following special form*

$$[\pi](x) = \pi x + a x^q + \sum_{j=2}^{\infty} c_j x^{j(q-1)+1},$$

where a, c_j $(j \geq 2) \in R$ and $c_j \in \pi R$ unless $j \equiv 1$ mod q.

Moreover, the height of $\hat{\mathcal{A}}^{2,1}_1$ is either 1 or 2. We say that $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}})$, or simply $\mathbb{A}$, is *ordinary* if $\hat{\mathcal{A}}^{2,1}_1$ has height 1 and say that $\mathbb{A}$ is *supersingular* if $\hat{\mathcal{A}}^{2,1}_1$ has height 2.

Let $W = \mathrm{Spec}(R)$ be an open affine subset of $\mathbb{M}(H) \otimes \kappa$, let ω be the differential dual to the coordinate x and define

$$\mathbf{H}|_W := a\omega^{\otimes(q-1)},$$

where x and a are as in the above proposition. It was showed in [Kassaei, 2004, §6] that the above definition is independent of the choice of the coordinate and the dual differential. Furthermore, these locally defined sections of $\underline{\omega}^{\otimes(q-1)}$ glue together to give a global section $\mathbf{H}$, which is defined to be the Hasse invariant, a modular form of level $K(H)$ and weight $q-1$ over κ.

Proposition 3.2.2. *Let R_0 be a κ-algebra. Then there is an $\mathbf{H} \in S^D(R_0, K(H), q-1)$ which vanishes at a geometric point $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}})$ of $\mathbb{M}(H) \otimes R_0$ exactly when $\mathbb{A}$ is supersingular.*

Proof. This is actually Proposition 6.1 in Kassaei [2004]. □

Moreover, it was proved in [Kassaei, 2004, §7] that the Hasse invariant can be lifted to a modular form of level $K(H)$ and weight $q-1$, defined over $\mathcal{O}_{\mathcal{P}}$.

Proposition 3.2.3. *If H is small enough and $q > 3$, then there exists an element in $S^D(\mathcal{O}_{\mathcal{P}}, K(H), q-1)$, denoted by E_{q-1}, such that*

$$E_{q-1} \equiv \mathbf{H} \quad mod\ \pi.$$

Remark 3.2.1. We want to use this lifting of the Hasse invariant to develop similar theory as in Katz [1973], such as strict neighborhoods, cnonical subgroups. But such element is not unique. Indeed [Kassaei, 2004, Corollary 13.2] shows that all the theory does not depend on the choice of such E_{q-1}.

Now let V be a finite extension of $\mathcal{O}_{\mathcal{P}}$ with fraction field L. Let $0 \leq w < 1$ be a rational number such that V contains an element, denoted by π^w, whose valuation is w. We define

$$\mathbb{M}(H)(w)_V := \mathrm{Spec}_{\mathbb{M}(H)_V}(\mathrm{Sym}(\underline{\omega}^{\otimes(q-1)})/ < E_{q-1} - \pi^w >).$$

Remark 3.2.2. $\mathbb{M}(H)(w)_V$ is a moduli space over V. Indeed, for any V-algebra R, $\mathbb{M}(H)(w)_V(R)$ is naturally in bijection with the set of isomorphism class of $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}}, Y)$, where $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}})$ is as in Theorem 3.1.3 part (1) and Y is a global section of $\underline{\omega}_R^{\otimes(1-q)}$ such that $YE_{q-1} = \pi^w$.

Let $\mathcal{M}(H)(w)$ be the π-adic completion of $\mathbb{M}(H)(w)$. Then the *space of π-adic modular forms with respect to D, level $K(H)$, weight k and growth condition w, with coefficients in* V is defined to be

$$S^D(V, w, K(H), k) := \mathrm{H}^0(\mathcal{M}(H)(w)_V, \underline{\omega}^{\otimes k}).$$

Moreover, the rigidification of the map $\mathcal{M}(H)_V(w) \to \mathcal{M}(H)_V$ is the immersion $\mathcal{M}(H)_V^{\mathrm{rig}}(w) \hookrightarrow \mathcal{M}(H)_V^{\mathrm{rig}}$, where $\mathcal{M}(H)_V^{\mathrm{rig}}(w)$ is the affinoid subdomain of $\mathcal{M}(H)_V^{\mathrm{rig}}$ relative to E_{q-1} and w (see [Kassaei, 2004, Proposition 9.7]). We call $\mathcal{M}(H)_V(0)^{\mathrm{rig}}$ the *ordinary locus*. It is an affinoid subdomain of $\mathcal{M}(H)_V^{\mathrm{rig}}$ and its complement is a finite union of discs, called the *supersingular discs*. The points of the supersingular discs correspond to those objects of the moduli problem that are supersingular.

By rigid GAGA, elements of $S^D(V, K(H), k)_L$ (resp. $S^D(V, w, K(H), k)_L$) correspond to sections of $\underline{\omega}^{\otimes k}$ (after rigidification) over $\mathcal{M}(H)_V^{\mathrm{rig}}$ (resp. $\mathcal{M}(H)_V(w)^{\mathrm{rig}}$). Elements of $S^D(V, w, K(H), k)_L$ are called *overconvergent* (resp. *convergent*) modular forms with coefficients in L if $w > 0$ (resp. $w = 0$).

3.2.2 Quotient of $(\mathbb{A}, \iota, \theta, \bar{k}^{\mathcal{P}})$ by a finite flat subgroup of $\mathbb{A}$

Let $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ be an object of the moduli problem $\mathcal{M}(H)$ over $\mathcal{O}_{\mathcal{P}}$. Let $C \subset A$ be a finite flat subgroup scheme. In addition we assume that C satisfies the following conditions.

- $C \subset A[q]$ is of rank q^{4d} and stable under the action of $\mathcal{O}_D$;
- the isomorphism $\theta : A[q] \xrightarrow{\sim} A[q]^\vee$ takes C onto $(A[q]/C)^\vee \subset A[q]^\vee$;
- $C_p^{\mathcal{P}} := C_2^2 \oplus \cdots \oplus C_m^2 = 0$ or $C_p^{\mathcal{P}} := (A[q])_2^2 \oplus \cdots \oplus (A[q])_m^2$.

Definition 3.2.3. If C satisfies the above conditions, we say that C is of type 1 if $C_p^{\mathcal{P}} = 0$ and of type 2 if $C_p^{\mathcal{P}} = (A[q])_2^2 \oplus \cdots \oplus (A[q])_m^2$. Note that any such C is uniquely determined by $C_1^{2,1}$.

Now let $A' := A/C$. The assumption that C is $\mathcal{O}_D$-invariant implies that A' inherits an action of $\mathcal{O}_D$. We denote this $\mathcal{O}_D$-action on A' by $\iota' : \mathcal{O}_D \to \mathrm{End}(A')$. Moreover, the natural projection $A \xrightarrow{f} A'$ is $\mathcal{O}_D$-equivariant.

Since $\theta : A[q] \xrightarrow{\sim} A[q]^\vee$ takes C to $(A[q]/C)^\vee$, then there is a unique polarization

$$\theta' : A' \longrightarrow (A')^\vee$$

such that the associated Rosati involution sends $\iota'(d)$ to $\iota'(d^*)$ for any $d \in D$ and the following diagram is commutative:

$$\begin{array}{ccc} A' & \xrightarrow{g} & A \\ {\scriptstyle \theta'}\downarrow & & \downarrow{\scriptstyle \theta} \\ (A')^\vee & \xrightarrow[f^\vee]{} & A^\vee, \end{array}$$

where f is the natural projection and g is the unique isogeny such that $g \circ f = [q]$ on A. Furthermore, $\deg(\theta) = \deg(\theta')$.

Remark 3.2.3. Actually, a more general result is proved by Kassaei, see [Kassaei, 2004, Lemma 4.4].

Now since $\mathrm{rk}(C)$ is relatively prime to any prime number $l \neq p$, the map

$$\hat{T}^p(g) : \hat{T}^p(A') \longrightarrow \hat{T}^p(A)$$

induced by g is an isomorphism. Recall that we have a class, modulo H, of isomorphisms $k^p : \hat{T}^p(A) \xrightarrow{\sim} \hat{W}^p$. Define

$$(k^p)' := k^p \circ \hat{T}^p(g),$$

and

$$\left(k_p^{\mathcal{P}}\right)' = \begin{cases} k_p^{\mathcal{P}} \circ \left(T_p^{\mathcal{P}}(f)\right)^{-1} \ , & \text{if } C \text{ is of type 1,} \\ k_p^{\mathcal{P}} \circ T_p^{\mathcal{P}}(g) \ , & \text{if } C \text{ is of type 2.} \end{cases}$$

Finally we define $(\bar{k}^{\mathcal{P}})'$ as the class of $(k_p^{\mathcal{P}})' \oplus (k^p)'$ modulo H. Then $\left(A' = A/C, \iota', \theta', (\bar{k}^{\mathcal{P}})'\right)$ is also a point of $\mathcal{M}(H)$, which is called the quotient of $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ by C.

3.2.3 Canonical subgroups

Now we will briefly recall the theory of canonical subgroup of our abelian schemes, which was developed in Kassaei [2004].

Theorem 3.2.1. *(Canonical subgroups) Let V be an $\mathcal{O}_{\mathcal{P}}$-algebra which is a complete discrete valuation ring of characteristic 0 such that the valuation extends the one on $\mathcal{O}_{\mathcal{P}}$ described at the beginning of section 3.1.3.2. Then*

(1) Let $r \in V$ with $v(r) < q/(q+1)$. There is a canonical way associating to every r-test object $(A, \iota, \theta, \bar{k}^{\mathcal{P}}, Y)$, where

– $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ *is an object of the moduli problem defined over a* V*-algebra* R*,*

– Y *is a section of* $\underline{\omega}_{A/R}^{\otimes(1-q)}$ *satisfying* $Y \cdot E_{q-1} = r$*,*

a finite flat subgroup scheme C *of* A *such that*

- C *has rank* q^{4d} *and is stable under the action of* $\mathcal{O}_D$*,*
- C *depends only on the* R*-isomorphism class of* $(A, \iota, \theta, \bar{k}^{\mathcal{P}}, Y)$*,*
- *the formation of* C *commutes with arbitrary base change of* π*-adically complete* V*-algebras,*
- *if* $\pi/r = 0$ *in* R*, then* C *can be identified with the kernel of Frobenius morphism* $\mathrm{Fr}_q : A \to A^{(q)}$*,*
- $C_p^{\mathcal{P}} := C_2^2 \oplus \cdots \oplus C_m^2 = 0$.

(2) Let $r \in V$ *with* $v(r) < 1/(q+1)$*. There is a canonical way associating to every object* $(A, \iota, \theta, \bar{k}^{\mathcal{P}}, Y)$ *as in part (i), an* r^q*-test object* $(A', \iota', \theta', (\bar{k}^{\mathcal{P}})', Y')$*, where*

– $(A', \iota', \theta', (\bar{k}^{\mathcal{P}})')$ *is the quotient of* $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$ *by* C*,*

– Y' *is a section of* $\underline{\omega}_{A'/R}^{\otimes(1-q)}$ *satisfying* $Y' \cdot E_{q-1} = r^q$*,*

such that

- Y' *depends only on the* R*-isomorphism class of* $(A, \iota, \theta, \bar{k}^{\mathcal{P}})$*,*
- *the formation of* Y' *commutes with arbitrary base change of* π*-adically complete* V*-algebras,*
- *if* $\pi/r^{q+1} = 0$ *in* R*, then* Y' *is equal to* $Y^{(q)}$ *on* $A^{(q)} = A/C$*.*

Proof. This is one of the main results in Kassaei [2004]. We refer [Kassaei, 2004, §10.1] for the proof in details. Here we just give an outline for the construction of C. First, $C_1^{2,1} \subset (A[\pi])_1^{2,1} \subset (A[q])_1^{2,1}$ was constructed to be a subgroup scheme of $(A[\pi])_1^{2,1}$ in a similar way as the construction of the canonical subgroups for elliptic curves in Katz [1973]. Then $C_1^{2,2} \subset (A[\pi])_1^{2,2} \subset (A[q])_1^{2,2}$ was defined to be the image of $C_1^{2,1}$ under the isomorphism $(A[q])_1^{2,1} \xrightarrow{\sim} (A[q])_1^{2,2}$. Then define

$$\begin{aligned} C_1^2 &:= C_1^{2,1} \oplus C_1^{2,2}, \\ C_j^2 &:= 0, \text{ for } 2 \leq j \leq m, \\ C_j^1 &:= \left((A[q])_j^2/C_j^2\right)^\vee \subset \left((A[q])_j^2\right)^\vee \cong (A[q])_j^1. \end{aligned}$$

Finally, C is defined to be

$$C := C_1^1 \oplus \cdots \oplus C_m^1 \oplus C_1^2 \oplus \cdots \oplus C_m^2.$$

□

In the sequel of this section, we fix an integer $r \geq 1$ and suppose that $w < 1/q^{r-2}(q+1)$. Let $\mathcal{A}(H)(w)$ be the base change of $\mathcal{A}(H)$ via the natural map $\mathcal{M}(H)(w) \to \mathcal{M}(H)$. The following proposition is an immediate consequence of the above theorem.

Proposition 3.2.4. *$\mathcal{A}(H)(w)[q^r]$ has a canonical subgroup, $\mathcal{C}_r$, stable under the action of $\mathcal{O}_D$ and $(\mathcal{C}_r)_1^{2,1} \subset (\mathcal{A}(H)(w)[\pi^r])_1^{2,1}$ has order q^r.*

Proof. This is [Brasca, 2013, Proposition 6.30]. □

We write $\mathcal{M}(H)(w)$ simply by $\mathcal{M}(w)$ in the rest of this section. The existence of the canonical subgroup allows us to define a morphism

$$\mathcal{M}(w) \longrightarrow \mathcal{M}(H, \pi^r),$$

whose image is still denoted by $\mathcal{M}(w)$. Its rigidification is a section of the morphism

$$\mathcal{M}(H,\pi^r)^{\mathrm{rig}} \longrightarrow \mathcal{M}(H)^{\mathrm{rig}},$$

defined over $\mathcal{M}(w)^{\mathrm{rig}}$. Now let $\mathcal{M}(H\pi^r)(w)^{\mathrm{rig}}$ be the inverse image of $\mathcal{M}(w)^{\mathrm{rig}}$ under the morphism $\mathcal{M}(H\pi^r)^{\mathrm{rig}} \longrightarrow \mathcal{M}(H,\pi^r)^{\mathrm{rig}}$. It is an affinoid subdomain of $\mathcal{M}(H\pi^r)^{\mathrm{rig}}$ and the map $\mathcal{M}(H\pi^r)(w)^{\mathrm{rig}} \longrightarrow \mathcal{M}(w)^{\mathrm{rig}}$ is finite and étale. Then let $\mathcal{M}(H\pi^r)(w)$ be the normalization of $\mathcal{M}(w)$ in $\mathcal{M}(H\pi^r)(w)^{\mathrm{rig}}$. The rigid analytic fibre of $\mathcal{M}(H\pi^r)(w)$ is $\mathcal{M}(H\pi^r)(w)^{\mathrm{rig}}$ and the rigidification of the morphism $\mathcal{M}(H\pi^r)(w) \longrightarrow \mathcal{M}(w)$ is just the map $\mathcal{M}(H\pi^r)(w)^{\mathrm{rig}} \longrightarrow \mathcal{M}(w)^{\mathrm{rig}}$ described above.

Now we write $\mathcal{M}^r(w)$ simply for $\mathcal{M}(H\pi^r)(w)$. Let $\mathcal{M}$ be any one of $\mathcal{M}(w)$, $\mathcal{M}(H)$, $\mathcal{M}(H,\pi^r)$, $\mathcal{M}(H\pi^r)$ and $\mathcal{M}^r(w)$. We will denote simply by M instead of $\mathcal{M}^{\mathrm{rig}}$ the rigidification of $\mathcal{M}$. We have the following commutative diagram of formal schemes and rigid spaces:

$$\begin{tikzcd}
M(H\pi^r) \arrow[r] & M(H,\pi) \arrow[r] & M(H) \\
M^r(w) \arrow[r] \arrow[u, hook] \arrow[d, "u"'] & M(w) \arrow[r, equal] \arrow[u, hook] \arrow[d, "u"'] & M(w) \arrow[u, hook] \arrow[d, "u"'] \\
\mathcal{M}^r(w) \arrow[r] & \mathcal{M}(w) \arrow[r, equal] \arrow[d, hook] & \mathcal{M}(w) \arrow[d] \\
\mathcal{M}(H\pi^r) \arrow[r] & \mathcal{M}(H,\pi) \arrow[r] & \mathcal{M}(H).
\end{tikzcd}$$

Proposition 3.2.5. *Let S be a normal and π-adically complete V-algebra. For any integer $r \geq 0$, there is a natural bijection between $\mathcal{M}^r(w)(S)$ and the set of isomorphism classes of $(\mathbb{A},\iota,\theta,\bar{k},Y)$, where*

- *$(\mathbb{A},\iota,\theta,\bar{k})$ is an object of moduli problem, with $\mathbb{A}$ defined over S, of $\mathbb{M}(H\pi^r)$. And the canonical S-point of $\mathbb{A}[\pi^r]_1^{2,1}$ generates, as $\mathcal{O}_{\mathcal{P}}$-module, the canonical subgroup of $\mathbb{A}[\pi^r]$;*

- Y *is a section of* $\underline{\omega}_{\mathbb{A}/S}^{\otimes(1-q)}$ *such that* $YE_{q-1} = \pi^w$.

Proof. See Brasca [2011] Propositions 2.3.2 and 2.3.7. □

Definition 3.2.4. We define the space of π-adic modular forms with respect to D, level $K(H\pi^r)$, weight k and growth condition w, with coefficients in V, as

$$S^D\left(V, w, K(H\pi^r), k\right) := \mathrm{H}^0\left(\mathcal{M}^r(w), \underline{\omega}^{\otimes k}\right).$$

Note that we have

$$S^D\left(V, w, K(H\pi^r), k\right)_L := \mathrm{H}^0\left(M^r(w), \underline{\omega}^{\otimes k}\right).$$

3.3 The map dlog and the Hodge-Tate sequence

3.3.1 Group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action

The theory of group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action, which was developed in Faltings [2002a], is needed here, as a generalization of the theory of group schemes, to deal with the $\mathcal{O}_{\mathcal{P}}$-action. In particular, this gives a good duality theory instead of the usual Cartier duality by taking into account of the action of $\mathcal{O}_{\mathcal{P}}$ (since $\mathbb{G}_m$ has no action of $\mathcal{O}_{\mathcal{P}}$). Here we will briefly recall some basic definitions and properties. For more details, please refer to Faltings [2002a], or [Brasca, 2011, §1.7].

Let R be a π-adically complete and π-torsion free $\mathcal{O}_{\mathcal{P}}$-algebra.

Definition 3.3.1. Let G be a finite and flat group scheme over R. We say that G has a *strict* $\mathcal{O}_{\mathcal{P}}$*-action* if there is a ring homomorphism $\mathcal{O}_{\mathcal{P}} \to \mathrm{End}_R(G)$ such that the action on the Lie algebra of G is the natural one. Homomorphisms between group schemes with strict $\mathcal{O}_{\mathcal{P}}$-action are homomorphisms which respect the action of $\mathcal{O}_{\mathcal{P}}$.

Example 3.3.1. Let H be π-divisible group over R. Then the π^n-torsion $H[\pi^n]$ is naturally a group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action for any n.

Example 3.3.2. Consider the ring of power series $R[[x]]$. Then there exists a unique action of $\mathcal{O}_{\mathcal{P}}$ such that the multiplication by π has the form

$$[\pi]x = x^q + \pi x$$

and the action on the Lie algebra is the one induced by the structure map $\mathcal{O}_{\mathcal{P}} \to R$. This is called *Lubin-Tate π-divisible group*, denoted by $\mathcal{LT}$. Then the π^n-torsion of $\mathcal{LT}$ is a group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action for any n.

Now we fix G, a finite flat group scheme with strict $\mathcal{O}_{\mathcal{P}}$-action over R.

Lemma 3.3.1. *G is killed by π^n for some n. In particular, any morphism $G \to \mathcal{LT}$ factors through $\mathcal{LT}[\pi^n]$.*

Proof. This is [Faltings, 2002a, Lemma 7]. □

Theorem 3.3.1. *The functor from the category of π-adically complete and torsion free R-algebras to the category of groups, sending*

$$S \longmapsto \mathrm{Hom}_{\mathcal{O}_{\mathcal{P}}}(G_S, \mathcal{LT}_S),$$

is representable by a finite flat group scheme over R, with strict $\mathcal{O}_{\mathcal{P}}$-action. We will denote this group scheme by $G^\vee$.

Proof. This is [Faltings, 2002a, Theorem 8]. □

Remark 3.3.1. If $\mathcal{O}_{\mathcal{P}} = \mathbb{Z}_p$ and R contains a primitive p-th root of unity, then $\mathcal{L}\mathcal{T} \cong \hat{\mathbb{G}}_{m,R}$. In particular, $G^\vee$ as above coincides with the usual Cartier dual.

3.3.2 The map dlog

Definition 3.3.2. Let R be a V-algebra. We say that R is *small* if:

- R is π-adically complete;
- $\mathrm{Spec}(R)$ is connected, i.e., R has no nontrivial idempotents;
- there is a topologically of finite type and formally étale morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R')$, where $R' := V\{T_1, \ldots, T_s\}/(T_1 \cdots T_j - \pi^a)$ and $a \in \mathbb{N}$.

A *small affine* is a scheme of the form $\mathrm{Spf}(R)$ with R small.

Proposition 3.3.1. *There is an open covering of* $\mathcal{M}(w)$ *by small affines.*

Proof. This is [Brasca, 2011, Proposition 3.1.2]. □

Remark 3.3.2. Moreover, in the proof of [Brasca, 2011, Proposition 3.1.2], the ring R' in the above definition can be taken as $R' = V\{X, Y\}/(XY - \pi^a)$ for some $a \in \mathbb{N}$.

Let $\mathrm{Spf}(R) \subset \mathcal{M}(w)$ be an open small affine and let $\mathrm{Spf}(S_r)$ be the pullback of $\mathrm{Spf}(R)$ to $\mathcal{M}^r(w)$. We assume that $\underline{\omega}_{\mathcal{A}/R} = (\varepsilon_* \Omega^1_{\mathcal{A}/R})^{2,1}_1$ is a free R-module, generated by ω, and we write $E_{q-1}\big|_{\mathrm{Spf}(R)} = E\omega^{\otimes(q-1)}$. Let $\eta = \mathrm{Spec}(\mathbb{K})$ be a generic geometric point of $\mathrm{Spec}(R)$ and denote by $\mathcal{G}$ for $\pi_1(\mathrm{Spec}(R_L), \eta)$. We denote by $\overline{R}$ the direct limit over all normal $R-$algebras $T \subseteq \mathbb{K}$ such that T_L is finite étale over R_L, $\widehat{\overline{R}}$ denotes the π-adic completion of $\overline{R}$. Then $\mathcal{G} = \mathrm{Gal}(\overline{R}_L/RL)$ acts continuously on $\widehat{\overline{R}}$.

Definition 3.3.3. Let G be an abelian group with an $\mathcal{O}_{\mathcal{P}}$-action.

- The *Tate module* of G is defined to be

$$T_\pi(G) := \varprojlim_n G[\pi^n].$$

- If G is a π-divisible group, we define

$$T_\pi(G) := T_\pi\left(G(\overline{R}_L)\right) = \varprojlim_n G[\pi^n](\overline{R}_L).$$

- Let G be a π-divisible group and H a sub $\mathcal{O}_{\mathcal{P}}$-module of $T_\pi\left(G^\vee\right)$. By duality between G and $G^\vee$, we obtain $H^\perp$, the *orthogonal of* H, which is a sub $\mathcal{O}_{\mathcal{P}}$-module of $T_\pi\left(G\right)$.

Now let G be a group scheme with an $\mathcal{O}_{\mathcal{P}}$-action annihilated by π^n and let $\underline{\omega}_{G/R}$ be the module of invariant differential of G. Let W be a normal, Noetherian, π-torsion free $R-$algebra. We define a map

$$\mathrm{dlog}_G := \mathrm{dlog}_{G,W} : G^\vee\left(W_L\right) \longrightarrow \underline{\omega}_{G/R} \otimes_R W/\pi^n W$$

as follows: let x be a W_L-valued point of $G^\vee$, it extends, by normality, to a W-valued point of $G^\vee$, called again x. Such point gives a group scheme homomorphism $f_x : G \to \mathcal{L}\mathcal{T}$, which respects the action of $\mathcal{O}_{\mathcal{P}}$ and we set

$$\mathrm{dlog}_{G,W}(x) := f_x^* d(T).$$

The map dlog satisfies various functoriality properties (see [Brasca, 2011, Lemma 3.1.3]).

Applying the construction above to $G = \mathcal{A}[\pi^n]_1^{2,1}$, for $n \geq 1$, we obtain the map

$$\mathrm{dlog}_{n,W} : (\mathcal{A}[\pi^n]_1^{2,1})^\vee\left(W_L\right) \longrightarrow \underline{\omega}_{\mathcal{A}[\pi^n]_1^{2,1}} \otimes_R W/\pi^n W.$$

Taking the direct limit over all W, we have the map

$$\mathrm{dlog}_{n,\mathcal{A}} : (\mathcal{A}[\pi^n]_1^{2,1})^\vee \left(\overline{R}_L\right) \longrightarrow \underline{\omega}_{\mathcal{A}/R} \otimes_R \overline{R}/\pi^n\overline{R} \cong \underline{\omega}_{\mathcal{A}[\pi^n]_1^{2,1}} \otimes_R \overline{R}/\pi^n\overline{R}.$$

By taking the projective limit, we get the morphism of $\mathcal{G}$-modules

$$\mathrm{dlog}_{\mathcal{A}} : \mathrm{T}_\pi \left((\mathcal{A}[\pi^\infty]_1^{2,1})^\vee\right) \longrightarrow \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}}.$$

Suppose that R is a discrete valuation ring, whose valuation extends that of $\mathcal{O}_{\mathcal{P}}$. From $\mathrm{dlog}_{\mathcal{A}}$, we obtain the maps $\mathrm{dlog}_{n,\widehat{\mathcal{A}}}$ and the map

$$\mathrm{dlog}_{\widehat{\mathcal{A}}} : \mathrm{T}_\pi \left((\widehat{\mathcal{A}}_1^{2,1})^\vee\right) \longrightarrow \underline{\omega}_{\widehat{\mathcal{A}}/R} \otimes_R \widehat{\overline{R}}.$$

3.3.3 The Hodge-Tate sequence

Recall that we have the map

$$\mathrm{dlog}_{\mathcal{A}} : \mathrm{T}_\pi \left((\mathcal{A}[\pi^\infty]_1^{2,1})^\vee\right) \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\overline{R}} \longrightarrow \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}},$$

and its analogue for $\left(\mathcal{A}[\pi^\infty]_1^{2,1}\right)^\vee$,

$$\mathrm{dlog}_{\mathcal{A}^\vee} : \mathrm{T}_\pi \left(\mathcal{A}[\pi^\infty]_1^{2,1}\right) \otimes_{\mathcal{O}_{\mathcal{P}}} \widehat{\overline{R}} \longrightarrow \underline{\omega}_{\mathcal{A}^\vee/R} \otimes_R \widehat{\overline{R}}.$$

Then we have an isomorphism of $\mathcal{G}$-modules

$$\mathrm{T}_\pi \left((\mathcal{A}[\pi^\infty]_1^{2,1})^\vee\right) \cong \mathrm{T}_\pi \left(\mathcal{A}[\pi^\infty]_1^{2,1}\right)^* (1),$$

where $(\cdot)^*$ is the dual module and $(\cdot)(1)$ means the $\mathcal{G}$-action is twisted by the Lubin-Tate character. Let

$$\mathrm{a}_{\mathcal{A}} := (\mathrm{dlog}_{\mathcal{A}})^* (1).$$

Definition 3.3.4. The *Hodge-Tate sequence* of $\mathcal{A}$ is the following sequence of $\widehat{\overline{R}}$-modules with semi-linear action of $\mathcal{G} = \mathrm{Gal}(\overline{R}_L/R_L)$:

$$0 \longrightarrow \underline{\omega}^*_{\mathcal{A}^\vee/R} \otimes_R \widehat{\overline{R}}(1) \xrightarrow{\mathrm{a}_{\mathcal{A}}} \mathrm{T}_\pi\left((\mathcal{A}[\pi^\infty]^{2,1}_1)^\vee\right) \otimes_{\mathcal{O}_\mathcal{P}} \widehat{\overline{R}} \xrightarrow{\mathrm{dlog}_{\mathcal{A}}} \underline{\omega}_{\mathcal{A}/R} \otimes_R \widehat{\overline{R}} \longrightarrow 0.$$

Remark 3.3.3. Actually, we have the fact that $\widehat{\mathcal{A}}^{2,1}_1[\pi]^\vee \cong \widehat{\mathcal{A}}^{1,1}_1[\pi]$ and $\underline{\omega}_{\mathcal{A}^\vee/R} \cong \underline{\omega}_{\mathcal{A}[\pi^\infty]^{1,1}_1/R}$. See [Brasca, 2011, §3.2] for details.

For integer $r \geq 1$, suppose $w < 1/q^{r-2}(q+1)$ and let $v := w/(q-1)$. We denote $\overline{R}_z := \overline{R}/\pi^z\overline{R}$. We have

Theorem 3.3.2. *The homology of the Hodge-Tate sequence is killed by π^v with $v := w/(q-1)$, and we have a commutative diagram of $\mathcal{G}-$modules, with exact rows and vertical isomorphisms:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\,(\mathrm{dlog}_{\mathcal{A}})_{r-v} \arrow[r] \arrow[d, "\wr"] & \mathrm{T}_\pi((\mathcal{A}[\pi^\infty]^{2,1}_1)^\vee) \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{r-v} \arrow[r] \arrow[d, equal] & \mathrm{Im}\,(\mathrm{dlog}_{\mathcal{A}})_{r-v} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & (\mathcal{D}_r)^{2,1}_1 \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{r-v} \arrow[r] & (\mathcal{A}[\pi]^{2,1}_1)^\vee \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{r-v} \arrow[r] & \left((\mathcal{C}_r)^{2,1}_1\right)^\vee \otimes_{\mathcal{O}_\mathcal{P}} \overline{R}_{r-v} \arrow[r] & 0,
\end{tikzcd}$$

where $(\mathcal{D}_r)^{2,1}_1 := ((\mathcal{C}_r)^{2,1}_1)^\perp$. Furthermore, $\mathrm{Im}(\mathrm{dlog}_{\mathcal{A}})$ *and* $\mathrm{Ker}(\mathrm{dlog}_{\mathcal{A}})$ *are free $\widehat{\overline{R}}$-modules of rank 1.*

Proof. See [Brasca, 2013, §5]. □

Recall that there exists a natural morphism $\vartheta_r : \mathcal{M}^r(w) \to \mathcal{M}(w)$, whose rigidification is Galois, with $G_r := (\mathcal{O}_\mathcal{P}/\pi^r\mathcal{O}_\mathcal{P})^\times$ as Galois group. Let $\mathcal{U} = \mathrm{Spf}(R) \subseteq \mathcal{M}(w)$ be an open affine and $\mathcal{V}_r = \mathrm{Spf}(S_r)$ the inverse image of $\mathcal{U}$ under ϑ_r. It follows that $(\mathcal{C}_r)^{2,1}_1$ becomes constant over $S_{r,L}$. Furthermore, there exists a canonical point of $(\mathcal{C}_r)^{2,1}_1$, defined over S_r. We now fix $\{\zeta_n\}_{n\geq 1}$, a sequence of $\mathbb{C}_p-$points of $\mathcal{LT}$ such that the order of ζ_n is exactly π^n. We assume that $\pi\zeta_{n+1} = \zeta_n$ for each n, and that ζ_1 is the fixed $(-\pi)^{1/(q-1)}$. If $\zeta_r \in V$, we obtain γ_r, a canonical S_r-point of $((\mathcal{C}_r)_1^{2,1})^\vee$. By [Brasca, 2013, Proposition 5.6], there exists an element δ_r of $\overline{R}_r$ such that

$$\mathrm{dlog}_{r,\mathcal{A}}(\gamma_r) = \delta_r \omega,$$

where $\omega = \omega \otimes 1$ is a basis of $\underline{\omega}_{\mathcal{A}/R} \otimes \widehat{\overline{R}}$. Let $\tilde{\delta}_r \in \widehat{\overline{R}}$ be a lift of δ_r. Since γ_r is defined over S_r we may assume that $\delta_r \in S_r/\pi^r S_r$ and $\tilde{\delta}_r \in S_r$.

Proposition 3.3.2. *Let $\mathcal{F}(S_r) \subseteq_{\underline{\omega}_{\mathcal{A}/R}} \otimes_R S_r$ be the submodule generated by $\tilde{\delta}_r \omega \otimes 1$. Then we have*

- *$\mathcal{F}(S_r)$ is a free S_r-module of rank 1, with basis $\tilde{\delta}_r \omega \otimes$ and $\mathcal{F}(S_r) \otimes_{S_r} \widehat{\overline{R}} \cong \mathrm{Im}(\mathrm{dlog}_{\mathcal{A}})$.*
- *The S_r-module $\mathrm{Im}(\mathrm{dlog}_{\mathcal{A}})^{\mathcal{H}_r}$ is equal to $\mathcal{F}(S_r)$, where $\mathcal{H}_r := \mathrm{Gal}(\overline{R}_L/S_{r,L})$.*
- *There exists an isomorphism $\mathcal{F}(S_r)_{r-v} \cong \left((\mathcal{C}_r)_1^{2,1}\right)^\vee \otimes_{\mathcal{O}_{\mathcal{P}}} (S_r)_{r-v}$, and its base change to $\widehat{\overline{R}}$ gives the isomorphism of Theorem 3.3.2, via $\mathcal{F}(S_r) \otimes_{S_r} \widehat{\overline{R}} \cong \mathrm{Im}(\mathrm{dlog}_{\mathcal{A}})$.*
- *There is an isomorphism $\mathcal{F}(S_r)^*(1) \otimes_{S_r} \widehat{\overline{R}} \cong \mathrm{Ker}(\mathrm{dlog}_{\mathcal{A}})$.*

Furthermore, all the above isomorphisms are $\mathcal{H}_r$-equivariant.

Proof. See [Brasca, 2013, §6.5] and the Proposition 5.11 there. □

Proposition 3.3.3. *There exists a unique locally free sheaf of $\mathcal{O}_{\mathcal{M}^r(w)}$-modules of rank 1, denoted by $\mathcal{F}_r$, such that*

$$\mathcal{F}_r(\mathrm{Spf}(S_r)) = \mathcal{F}(S_r),$$

for $\mathrm{Spf}(S_r)$ as before. Furthermore, we have isomorphism of sheaves of $\mathcal{O}_{\mathcal{M}^r(w)}$-modules

$$\mathcal{F}_r/\pi^{r-v}\mathcal{F}_r \cong \left((\mathcal{C}_r)_1^{2,1}\right)^\vee \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{M}^r(w)}/\pi^{r-v}\mathcal{O}_{\mathcal{M}^r(w)}.$$

Proof. This follows from the above proposition and [Brasca, 2013, Lemma 5.12]. □

3.4 Modular Sheaves

We assume that $e \leq p-1$. We will remark at the end of this section how to remove this hypothesis.

3.4.1 The weight space

Let L be a finite field extension of $F_{\mathcal{P}}$. Take a L-affinoid algebra A, we consider the $F_{\mathcal{P}}$-locally analytic characters

$$\lambda : \mathcal{O}_{\mathcal{P}}^{\times} = \mu_{q-1} \times (1 + \pi\mathcal{O}_{\mathcal{P}}) \longrightarrow A^{\times}.$$

Let $t \in \mathcal{O}_{\mathcal{P}}^{\times}$. We will use the following notations:

- $[t]$ means $[\cdot]$, the Teichmüler character, applied to the reduction of t modulo π;
- $\langle t \rangle := t/[t]$.

Definition 3.4.1. Let $r \geq 1$ be an integer. A character $\lambda : \mathcal{O}_{\mathcal{P}}^{\times} \to L^{\times}$ is said to be r-*accessible* if it is of the form $t \mapsto [t]^i \langle t \rangle^s := [t]^i \exp(s \log(\langle t \rangle))$ for all t with $v(\langle t \rangle - 1) \geq r$, where

- ◇ $i \in \mathbb{Z}/(q-1)\mathbb{Z}$;
- ◇ $s \in L$ is such that $v(s) > (e/(p-1)) - r$.

The 1-accessible characters are said simply to be accessible. In this case we write $\lambda = (s, i)$. Any integer k can be viewed as the accessible character $t \mapsto t^k$. Note that any locally analytic character is $r-$accessible for some r.

Let $\mathcal{W}$ be the *weight space* for locally analytic characters: it is an $F_{\mathcal{P}}$-rigid analytic space whose A-points, for any $F_{\mathcal{P}}$-affinoid algebra A, are $\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{loc-an}}(\mathcal{O}_{\mathcal{P}}^{\times}, A^{\times})$. There exists a natural bijection between the set of connected components of $\mathcal{W}$ and $\mathbb{Z}/(q-1)\mathbb{Z}$. Let $\mathcal{B}$ be the component corresponding to the identity. We then have $\mathcal{W} = \coprod_{\mathbb{Z}/(q-1)\mathbb{Z}} \mathcal{B}$. By [Schneider and Teitelbaum, 2001, Theorem 3.6], we know that $\mathcal{B}$ is a twisted form, over $\mathbb{C}_p$, of the open disk of radius 1. Note that $\mathcal{B}$ is isomorphic to $\mathcal{B}(1)$ if and only if $F_{\mathcal{P}} = \mathbb{Q}_p$ (see [Schneider and Teitelbaum, 2001, Lemma 3.9]). In general $\mathcal{B}$ is a closed subvariety of $\mathcal{B}^N(1)$, the N-dimensional open polydisk of radius 1, where $N = [F_{\mathcal{P}} : \mathbb{Q}_p]$.

Proposition 3.4.1. *There exists an admissible covering $\{\mathcal{W}_r\}_{r\geq 0}$ of $\mathcal{W}$ by affinoid subdomains such that any $\lambda \in \mathcal{W}_r$ is r-admissible. In particular, any $\lambda \in \mathcal{W}(L)$ lies in some $\mathcal{W}_r(L)$.*

Proof. See [Brasca, 2013, §6.1]. □

3.4.2 A torsor

Let r, w, v, R, S_r, G_r be as in section 3.3.3. Let $\mathcal{F}_r$ be the sheaf as in Proposition 3.3.3. Then we define the sheaf $\mathcal{F}'_{r,v}$ on $\mathcal{M}^r(w)$ to be the inverse image of the constant sheaf of sets which are given by the subset of $\left((\mathcal{C}_r)_1^{2,1}\right)^{\vee}$ of points of order exactly π^r under the natural map

$$\mathcal{F}_r \longrightarrow \mathcal{F}_r/\pi^{r-v}\mathcal{F}_r \cong \left((\mathcal{C}_r)_1^{2,1}\right)^{\vee} \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_{\mathcal{M}^r(w)}/\pi^{r-v}\mathcal{O}_{\mathcal{M}^r(w)}.$$

Now let $\mathcal{S}_{r,v}$ be the sheaf of abelian groups, on $\mathcal{M}^r(w)$, defined by

$$\mathcal{S}_{r,v} := \mathcal{O}_{\mathcal{P}}^{\times}(1 + \pi^{r-v}\mathcal{O}_{\mathcal{M}^r(w)}).$$

Then we have

Proposition 3.4.2. *$\mathcal{F}'_{r,v}$ is a Zariski $\mathcal{S}_{r,v}-$torsor.*

Proof. This is a consequence of Proposition 3.3.2 and 3.3.3. ☐

3.4.3 Modular sheaves

Fix an r-accessible character λ and let $s \in \mathbb{C}_p$ be the element associated to λ. We assume that $\zeta_r \in V$. Since $w < 1/(q^{r-2}(q+1))$, the canonical subgroup of level r exists. Let $x = ub$ be a local section of $\mathcal{S}_{r,v}$ over $\mathrm{Spf}(S_r)$, where u is a section of $\mathcal{O}_{\mathcal{P}}^{\times}$ and b is a section of $1 + \pi^{r-v}\mathcal{O}_{\mathrm{Spf}(S_r)}$. Then $b^s := \exp\left(s \log(b)\right)$ makes sense and we let

$$x^{\lambda} := \lambda(u) b^s,$$

which is also a section of $\mathcal{S}_{r,v}$.

We will write $\mathcal{O}^{\lambda}_{\mathcal{M}^r(w)}$ for the sheaf $\mathcal{O}_{\mathcal{M}^r(w)}$ with the action of $\mathcal{S}_{r,v}$ by multiplication, twisted by λ. Since we have a natural action of $\mathcal{S}_{r,v}$ on $\mathcal{F}'_{r,v}$, we can consider the sheaf

$$\tilde{\Omega}^{\lambda}_w := \mathscr{H}om_{\mathcal{S}_{r,v}}\left(\mathcal{F}'_{r,v}, \mathcal{O}^{\lambda^{-1}}_{\mathcal{M}^r(w)}\right),$$

where $\mathscr{H}om_{\mathcal{S}_{r,v}}(\cdot,\cdot)$ means homomorphisms of sheaves with an action of $\mathcal{S}_{r,v}$. By Proposition 3.4.2, $\tilde{\Omega}^{\lambda}_w$ is an invertible sheaf on $\mathcal{O}_{\mathcal{M}^r(w)}$. Since $\vartheta_r : \mathcal{M}^r(w) \to \mathcal{M}(w)$ is finite, $\vartheta_{r,*}\tilde{\Omega}^{\lambda}_w$ is a coherent sheaf of $\mathcal{O}_{\mathcal{M}(w)}$-modules. The action of G_r on $\mathcal{F}'_{r,v}$ (induced by its action of the subset of $(\mathcal{C}_r)^{2,1}_1$ of points of exactly order π^r) and on $\vartheta_{r,*}\mathcal{O}^{\lambda^{-1}}_{\mathcal{M}^r(w)}$ gives an action of G_r on $\vartheta_{r,*}\tilde{\Omega}^{\lambda}_w$. We define the sheaf Ω^{λ}_w on $\mathcal{M}(w)$ as

$$\Omega^{\lambda}_w := \left(\vartheta_{r,*}\tilde{\Omega}^{\lambda}_w\right)^{G_r}.$$

Definition 3.4.2. The space of π-adic modular forms with respect to D, level $K(H\pi^r)$, weight λ and growth condition w, with cofficients in L, is defined as

$$S^D(L, w, K(H\pi^r), \lambda) := \mathrm{H}^0(\mathcal{M}^r(w), \tilde{\Omega}^\lambda_w)_L.$$

Definition 3.4.3. We define the space of $\pi-$adic modular forms with respect to D, level $K(H)$, weight λ and growth condition w, with coefficients in L, as

$$S^D(L, w, K(H), \lambda) := \mathrm{H}^0(\mathcal{M}(w), \tilde{\Omega}^\lambda_w)_L.$$

Let $w' \geq w$ be a rational number that satisfies the same conditions of w. We have natural morphisms $f_{w,w'} : \mathcal{M}(w) \to \mathcal{M}(w')$ and $g_{w,w'} : \mathcal{M}^r(w) \to \mathcal{M}^r(w')$.

Lemma 3.4.1. *We have a natural isomorphism of $\mathcal{O}_{\mathcal{M}(w)}$-modules $\tilde{\rho}_{w,w'} : g^*_{w,w'}(\tilde{\Omega}^\lambda_{w'}) \cong \tilde{\Omega}^\lambda_w$. Then we have $\tilde{\rho}_{w,w} = \mathrm{id}$ and, if $w'' \geq w'$ satisfies the same conditions of w, we have $\tilde{\rho}_{w,w''} = \tilde{\rho}_{w,w'} g^*_{w,w'} \circ (\tilde{\rho}_{w',w''})$. Furthermore, we obtain a canonical morphism*

$$\rho_{w,w'} : f^*_{w,w'}\left(\Omega^\lambda_{w'}\right) \to \Omega^\lambda_w,$$

which is an isomorphism after rigidification.

Proof. This is [Brasca, 2013, Lemma 6.18]. ☐

Definition 3.4.4. Thanks to the above lemma, we are allowed to define *the space of overconvergent modular forms* with respect to D, level $K(H)$, weight λ and growth condition w, with coefficients in L, as

$$S^D_\dagger(L, K(H), \lambda) := \varinjlim_{w>0} S^D(L, w, K(H), \lambda).$$

Now let h be an integer with $r \geq h$. Suppose that λ is h-accessible. We can repeat the above construction starting with $\mathcal{M}^h(w)$, obtaining another sheaf on $\mathcal{M}(w)$. For $r \geq h$, we consider the natural morphism $\vartheta_{r,h} : \mathcal{M}^r(w) \to \mathcal{M}^h(w)$. The rigidification of $\vartheta_{r,h}$ is Galois. Its Galois group is $G_{r,h} \subseteq G_r$, the image of $1 + \pi^h \mathcal{O}_{\mathcal{P}}$.

Proposition 3.4.3. *We have an isomorphism of $\mathcal{O}_{\mathcal{M}(w)} \otimes L$-modules*

$$\begin{aligned}\sigma_{r,h} : &\left(\vartheta_{h,*}\mathscr{H}om_{\mathcal{S}_{h,v}}\left(\mathcal{F}'_{h,v}\mathcal{O}^{(\lambda^{-1})}_{\mathcal{M}^h(w)}\right) \otimes L\right)^{G_h} \\ &\cong \left(\vartheta_{r,*}\mathscr{H}om_{\mathcal{S}_{r,v}}\left(\mathcal{F}'_{r,v}\mathcal{O}^{(\lambda^{-1})}_{\mathcal{M}^r(w)}\right) \otimes L\right)^{G_r}\end{aligned}$$

Furthermore $\sigma_{r,r} = \mathrm{id}$, and, if $t \leq h$ is an integer, we have $\sigma_{r,t} = \sigma_{h,t} \circ \sigma_{r,h}$.

Proof. This is [Brasca, 2013, Proposition 6.34]. □

Proposition 3.4.4.

- *We have a canonical isomorphism $\vartheta^{\mathrm{rig},*}\Omega^\lambda_w \cong \tilde{\Omega}^\lambda_w$.*
- $\underline{\omega}^{\otimes k,\mathrm{rig}}_{K(H)} = \Omega^{(k,k),\mathrm{rig}}_w$*, for integer k.*

Proof. See [Brasca, 2013, §6.3] and Remark 6.20 there. □

Actually, in [Brasca, 2013, §6.6], it shows that the sheaves Ω^λ_w can be put in families, we have

Proposition 3.4.5. *There exist locally free sheaves of $\mathcal{O}_{\mathcal{W}_r \times M(w)}-$modules of rank 1, denoted by $\Omega_{r,w}$, such that for any $\lambda \in \mathcal{W}_r(L)$, the natural morphism*

$$(\lambda, \mathrm{id})^*(\Omega_{r,w}) \longrightarrow \Omega^\lambda_w$$

is an isomorphism.

Proof. This is [Brasca, 2013, Proposition 6.37]. □

Remark 3.4.1. The assumption $e \leq p-1$ can be removed. For details and differences, see [Brasca, 2013, §6.7]

3.5 Hecke Operators

In this section, we recall the definitions of Hecke operators acting on the space of π-adic modular forms, which was introduced by R. Brasca in [Brasca, 2013, §7]. There, he introduced the U operator and $\mathrm{T}_{\mathcal{L}}$ operators, which are analogous to the classical U_p and T_l operators, respectively. Moreover, he showed that the U operator is a completely continuous operator on the space of overconvergent modular forms. Eventually, he showed that all these operators can be put in families.

3.5.1 The U operator

Let $\lambda : \mathcal{O}_{\mathcal{P}}^{\times} \to L^{\times}$ be a character in $\mathcal{W}_r$ and let $0 < w \leq 1/(q^{r-2}(q+1))$ be positive.

Proposition 3.5.1. *There exists a norm on $S^D(L, w, K(H\pi^r), \lambda)$ making it a potentially orthonormizable L-Banach module.*

Proof. This is [Brasca, 2013, Proposition 7.1]. □

Definition 3.5.1. Let M be a Banach $A-$module, where A is an affinoid $K-$algebra. Following [Buzzard, 2007, §I.2], we say that M satisfies the property (Pr), if there is a Banach $A-$module N such that $M \oplus N$ is potentially orthonormizable.

Corollary 3.5.1. *The subspace $S^D(L, w, K(H), \lambda) \subseteq S^D(L, w, K(H\pi^r), \lambda)$ is a L-Banach module satisfying property (Pr).*

To define the U operator we need to introduce another type of curve. We use the notations of section 3.1.3.2. We define

$$K(H\pi^r, q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K(H\pi^r) \text{ s.t. } b \equiv 0 \bmod \pi \right\}.$$

In the case $K_{\mathcal{P}} = K(H\pi^r, q)$, a choice of a level structure is equivalent to a choice of $(Q, D, \bar{k}^{\mathcal{P}})$, where (here $(A, \iota, \theta, \bar{k})$ is an object of the moduli problem for $F_{\mathcal{P}}-$algebras):

(1) Q is an $R-$point of exact $\mathcal{O}_{\mathcal{P}}$-order π^r in $A[\pi^r]_1^{2,1}$;

(2) D is a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule of $A[\pi^r]_1^{2,1}$ of order q which intersects the $\mathcal{O}_{\mathcal{P}}$-submodule scheme generated by Q trivially;

(3) $\bar{k}^{\mathcal{P}}$ are as in Section 3.1.3.2.

In this case, the curve $M'_{K'}$ will be denoted by $M(H\pi^r, q)$. It is a proper and smooth scheme over L. There exists a natural morphism $\pi_1 : M(H\pi^r, q) \to M(H\pi^r)$, defined by forgeting D and π_1 is finite and flat.

Given D, a finite and flat $\mathcal{O}_{\mathcal{P}}$-submodule of $A[\pi]_1^{2,1}$, we let $t_2(D)$ be the unique subgroup scheme of $A[q]$ satisfying the conditions of Section 3.2.2, of type 2, such that $(t_2(D))_1^{2,1} = D$. We can now define another morphism $\pi_2 : M(H\pi^r, q) \to M(H\pi^r)$ by taking the quotient of $t_2(D)$ on points. Since D intersects the $\mathcal{O}_{\mathcal{P}}$-submodule scheme generated by Q trivially, the image of Q under the natural map $A \to A/t_2(D)$ is a point of exact $\mathcal{O}_{\mathcal{P}}$-order π^r. Passing to the rigidifications and using the same notations, we have morphisms of rigid spaces π_1, $\pi_2 : M(H\pi^r, q) \to M(H\pi^r)$. Furthermore, we write $M_q^r(w)$ for $(\pi_1)^{-1}(M^r(w))$ and define the formal model $\mathcal{M}_q^r(w)$ as the normalization, via π_1, of $\mathcal{M}^r(w)$ in $M_q^r(w)$. This gives a formal model of π_1, denoted by $\mathfrak{p}_1 : \mathcal{M}_q^r(w) \to \mathcal{M}^r(w)$. Moreover, we can define the morphism

$$\mathfrak{p}_2 : \mathcal{M}_q^r(qw) \longrightarrow \mathcal{M}^r(w),$$

by taking the quotient over $\mathcal{D}$, on points (this is well defined by [Brasca, 2013, Lemma 7.5]).

Let $\mathcal{A}_q^r(w)$ be the base change of $\mathcal{A}^r(w)$, via $\mathfrak{p}_1$, to $\mathcal{M}_q^r(w)$. Then $\mathcal{A}_q^r(w)$ has a subgroup of order q of its π^r-torsion, which is denoted by $\mathcal{D}$ and has trivial intersection with its canonical subgroup. The isogeny

$$\pi_{\mathcal{D}} : \mathcal{A}_q^r(qw) \longrightarrow \mathcal{A}_q^r(qw)/\mathcal{D}$$

is defined over $\mathcal{M}_q^r(qw)$. We have the following diagram

$$\begin{tikzcd}
\mathcal{A}_q^r(qw) \arrow[rr, "\pi_{\mathcal{D}}"] \arrow[dr] \arrow[dd] & & \mathcal{A}_q^r(qw)\big/\mathcal{D} \arrow[dd] \\
& \mathcal{A}^r(qw) \arrow[ur] \arrow[dd] & \\
\mathcal{M}_q^r(qw) \arrow[rr, equal] \arrow[dr, "\mathfrak{p}_1"'] & & \mathcal{M}_q^r(qw) \arrow[dl, "g_{w,qw}\circ\mathfrak{p}_2"] \\
& \mathcal{M}^r(qw) &
\end{tikzcd}$$

such that the left and right squares are Carterian and the square on the back is commutative. Since $g_{w,qw}^*\tilde{\Omega}_{qw}^\lambda \cong \tilde{\Omega}_w^\lambda$, we obtain a morphism

$$\tilde{\pi}_{\mathcal{D}}^\lambda : \mathfrak{p}_2^*\tilde{\Omega}_w^\lambda \longrightarrow \mathfrak{p}_1^*\tilde{\Omega}_{qw}^\lambda.$$

Then we can define an operator $\tilde{\mathrm{U}}$ to be the composition

$$\mathrm{H}^0\left(\mathcal{M}^r(qw), \tilde{\Omega}^{\lambda}_{qw} \otimes_V K\right)$$

$$\xrightarrow{\tilde{\rho}^{\mathrm{rig}}_{w,qw}} \mathrm{H}^0\left(\mathcal{M}^r(w), \tilde{\Omega}^{\lambda}_{w} \otimes_V K\right)$$

$$\xrightarrow{\mathfrak{p}_2^*} \mathrm{H}^0\left(\mathcal{M}^r_q(w), \mathfrak{p}_2^*\tilde{\Omega}^{\lambda}_{w} \otimes_V K\right)$$

$$\xrightarrow{\tilde{\pi}^{\lambda}_{\mathcal{D}}} \mathrm{H}^0\left(\mathcal{M}^r_q(w), \mathfrak{p}_1^*\tilde{\Omega}^{\lambda}_{qw} \otimes_V K\right)$$

$$\xrightarrow{\pi_{1,*}} \mathrm{H}^0\left(\mathcal{M}^r(qw), \tilde{\Omega}^{\lambda}_{qw} \otimes_V K\right),$$

where $\pi_{1,*}$ is the map induced by the trace, which is well defined since π_1 is finite and flat. All the maps in the above composition are G_r-equivariant, so is $\tilde{\mathrm{U}}$.

Taking G_r-invariants we obtain a map, denoted still by $\tilde{\mathrm{U}}$,

$$\tilde{\mathrm{U}} : S^D(L, qw, K(H), \lambda) \longrightarrow S^D(L, qw, K(H), \lambda).$$

Then our U operator

$$\mathrm{U} : S^D(L, qw, K(H), \lambda) \longrightarrow S^D(L, qw, K(H), \lambda)$$

is defined by $\mathrm{U} := \frac{1}{q}\tilde{\mathrm{U}}$. Moreover, Brasca shows that

Proposition 3.5.2. *The operator* U *is completely continuous.*

Proof. See [Brasca, 2013, Proposition 7.7]. ☐

Everything we showed above can be repeated for families; in particular, we have the U_r operator and the following proposition.

Proposition 3.5.3. *For any integer $r \geq 1$ and any rational $w \leq 1/(q^{r-2}(q+1))$, $\mathrm{H}^0(\Omega_{r,w}, \mathcal{W}_r \times \mathfrak{M}(H)(w)^{\mathrm{rig}})$ is a Banach $\mathcal{O}_{\mathcal{W}_r}(\mathcal{W}_r)-$module that satisfies the property (Pr). Furthermore the U_r operator is completely continuous.*

Let $\lambda : \mathcal{O}_{\mathcal{P}}^{\times} \to L^{\times}$ be a locally analytic character and let $h \in R$, we have the following proposition.

Proposition 3.5.4. *Let h be in $\mathbb{R}$ and let f be in $S^D(L, w, K(H), \lambda)^{\leq h}$. Then there exists an affinoid $\mathcal{V} \subseteq \mathcal{W}$ such that f can be deformed to a family of modular forms over $\mathcal{V}$. Furthermore, the U-operator acts with slope $\leq h$ on this family.*

3.5.2 Other Hecke operators

We now sketch the definition of other Hecke operators. Let $l \neq p$ be a rational prime such that l splits in $\mathbb{Q}(\sqrt{\lambda})$. Let $\mathcal{L}$ be a prime of F above l such that B is split at $\mathcal{L}$. We have

$$G'(\mathbb{Q}_l) \cong \mathbb{Q}_l^{\times} \times \mathrm{GL}_2(F_{\mathcal{L}}) \times \mathrm{GL}_2(F_{\mathcal{L}_2}) \times \cdots \times \mathrm{GL}_2(F_{\mathcal{L}_k})$$

where $\mathcal{L}_2, \ldots, \mathcal{L}_k$ are the primes of F lying over l different from $\mathcal{L}$, $F_{\mathcal{L}_i}$ is the completion of F at $\mathcal{L}_i$. We assume that the compact open subgroup H is of the form

$$H = \mathbb{Z}_l^* \times \mathrm{GL}_2(\mathcal{O}_{F_{\mathcal{L}}}) \times H'.$$

Let π_l be a uniformizer of $\mathcal{O}_{F_{\mathcal{L}}}$. If A is an abelian scheme as above, we have a decomposition of $A[\varpi_l]$ similar to that of $A[\varpi]$, so $A[\pi_l]_1^{2,1}$ is defined and it has an action of $\kappa_l := \mathcal{O}_{F_{\mathcal{L}}}/\pi_l$.

Let $\lambda : \mathcal{O}_{\mathcal{P}}^{\times} \to L^{\times}$ be an $r-$accessible character. Let $H_{\mathcal{L}}$ be the set of invertible 2×2 matrices with left lower corner congruent to 0 modulo π_l. The Shimura curve corresponding to the case $K_{\mathcal{P}} = K(H\varpi^r)$ and $H = \mathbb{Z}_l^* \times H_{\mathcal{L}} \times H'$ will be denoted by $M_{(l)}(H\pi^r)$. It follows that $M_{(l)}(H\pi^r)$ parametrizes objects of the moduli problem of $M(H\pi^r)$ plus a finite and flat subgroup of $A[\pi_l]_1^{2,1}$ of order $|\kappa_l|$, stable under the action of $\mathcal{O}_{F_\mathfrak{L}}$. If D is such a subgroup, we can define $t_2(D)$ as in the case of subgroups of $A[\pi_l]_1^{2,1}$, and also the quotient of A by $t_2(D)$ can be defined. We can repeat everything we have done for the U operator and define the operator

$$T_\mathfrak{L} : S^D(L, w, K(H\pi^r), \lambda) \to S^D(L, w, K(H\pi^r), \lambda)$$

exactly as in the case of U (using $|\kappa_l| + 1$ as normalization factor). Note that $\tilde{T}_\mathfrak{L}$ is a continuous operator but not completely continuous. The operators $\tilde{T}_\mathfrak{L}$ can also be put in families.

Chapter 4

Faltings' Sites

Let $p > 2$ be a prime integer and L a complete discrete valuation field of characteristic 0 and perfect residue field $\mathbb{L}$ of characteristic p. We denote by $\mathcal{O}_L$ the ring of integers of L and $\overline{L}$ a fixed algebraic closure of L. We set $G_L := \mathrm{Gal}(\overline{L}/L)$.

4.1 Faltings' topos: the smooth case

4.1.1 The algebraic setting

First, we let X be a smooth scheme of finite type over $\mathcal{O}_L$ and let M, $L \subset M \subset \overline{L}$, be an algebraic field extension of L. We denote by $X^{\mathtt{et}}$ the small étale site on X and by $X_M^{\mathtt{fet}}$ the finite étale site on X_M.

Definition 4.1.1. Let E_{X_M} be the category defined as follows.

(1) Objects: the objects of E_{X_M} are the pairs (U, W) where U is an object of $X^{\mathtt{et}}$ and W is an object of $U_M^{\mathtt{fet}}$.

(2) Morphisms: a morphism $(U', W') \to (U, W)$ in E_{X_M} is a pair of morphisms (α, β) where

- $\alpha : U' \to U$ is a morphism in $X^{\mathtt{et}}$;
- $\beta : W' \to W$ is a morphism of schemes such that the following diagram commutes.

$$\begin{tikzcd} W' \arrow[r, "\beta"] \arrow[d] & W \arrow[d] \\ U'_M \arrow[r, "\alpha_M"'] & U_M \end{tikzcd}$$

Remark 4.1.1. The category E_{X_M} has a final object (X, X_M).

Proposition 4.1.1. *The finite projective limits are representable in E_{X_M}. In particular, fibre products exist.*

Proof. It suffices to show that the fibre product of the morphisms

$$(U^2, W^2) \xrightarrow{(\alpha^2, \beta^2)} (U, W) \xleftarrow{(\alpha^1, \beta^1)} (U^1, W^1)$$

exists. We prove this by two steps.

(1) We claim that the pair

$$\big(U^1 \times_U U^2, W^1 \times_W W^2\big)$$

is an object of E_{X_M}.

First of all, by the properties of étale morphisms, it follows that $U^1 \times_U U^2$ is an object of $X^{\mathtt{et}}$. Since the morphisms $W^1 \to U^1_M$, $W^2 \to U^2_M$ are finite étale, the following morphisms are also finite étale,

- $W^1 \times_{U_M} W \longrightarrow U^1_M \times_{U_M} W$,

- $W^2 \times_{U_M} W \longrightarrow U_M^2 \times_{U_M} W$.

Then the morphism

$$\theta' : W^1 \times_{U_M} W \times_{U_M} W^2 \longrightarrow U_M^1 \times_{U_M} U_M^2$$

defined by the composite of finite étale maps

$$\begin{aligned} \big(W^1 \times_{U_M} W\big) \times_{U_M} W^2 &\longrightarrow U_M^1 \times_{U_M} W \times_{U_M} W^2 \\ &\longrightarrow U_M^1 \times_{U_M} W \times_{U_M} U_M^2 \\ &\longrightarrow U_M^1 \times_{U_M} U_M^2 \end{aligned}$$

is again finite étale (the last map of the composite is finite étale since $W \to U_M$ is so).

Let $f : W^1 \to W$ be the unique map such that $p_1 \circ f = \beta^1$, where $p_1 : W \times_{U_M} W \to W$ is the first projection. Then the following commutative diagram

$$\begin{tikzcd} W^1 \arrow[dr, dashed, "\gamma_1"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & W^1 \times_{U_M} W \arrow[r] \arrow[d] & W \times_{U_M} W \arrow[d, "p_1"] \\ & W^1 \arrow[r, "\beta^1"] & W \end{tikzcd}$$

induces a morphism $\gamma_1 : W^1 \to W^1 \times_{U_M} W$ which is finite étale since $W^1 \times_{U_M} W \to W^1$ is. Moreover, there exists a finite étale morphism $\gamma_2 : W^2 \to W^2 \times_{U_M} W$ obtained in a similar way. Now define

$$\theta'' : W^1 \times_W W^2 \longrightarrow W^1 \times_{U_M} W \times_{U_M} W^2$$

as the composite

$$W^1 \times_W W^2 \xrightarrow{\mathrm{id} \times \gamma_2} W^1 \times_W \big(W^2 \times_{U_M} W\big)$$

$$\begin{aligned} &\xrightarrow{\gamma_1 \times \mathrm{id}} \left(W^1 \times_{U_M} W\right) \times_W \left(W^2 \times_{U_M} W\right) \\ &\xrightarrow{\sim} W^1 \times_{U_M} W \times_{U_M} W^2. \end{aligned}$$

Each map in the above composite is finite étale then so is θ''. Hence

$$\theta := \theta' \circ \theta'' : W^1 \times_W W^2 \longrightarrow U_M^1 \times_{U_M} U_M^2$$

is finite étale. Our claim follows immediately by noting that

$$U_M^1 \times_{U_M} U_M^2 \cong \left(U^1 \times_U U^2\right)_M$$

.

(2) We will show that $(U^1 \times_U U^2, W^1 \times_W W^2)$ satisfies the universal property of the fibre product of the given pair of morphisms

$$(U^2, W^2) \xrightarrow{(\alpha^2,\beta^2)} (U, W) \xleftarrow{(\alpha^1,\beta^1)} (U^1, W^1).$$

Suppose we have the following commutative diagram of morphisms in E_{X_M}

$$\begin{tikzcd}
(U^3, W^3) \arrow[drr, bend left, "(f^1,g^1)"] \arrow[ddr, bend right, "(f^2,g^2)"'] & & \\
& (U^1 \times_U U^2, W^1 \times_W W^2) \arrow[r] \arrow[d] & (U^1, W^1) \arrow[d, "(\alpha^1,\beta^1)"] \\
& (U^2, W^2) \arrow[r, "(\alpha^2,\beta^2)"] & (U, W).
\end{tikzcd}$$

Note that we have natural morphisms

$$\begin{aligned} \alpha &: U^3 \longrightarrow U^1 \times_U U^2 \text{ and} \\ \beta &: W^3 \longrightarrow W^1 \times_W W^2 \end{aligned}$$

making the following diagrams commutes

$$\begin{tikzcd}
U^3 \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "f^1"] \arrow[ddr, bend right, "f^2"'] & & \\
& U^1 \times_U U^2 \arrow[r] \arrow[d] & U^1 \arrow[d, "\alpha^1"] \\
& U^2 \arrow[r, "\alpha^2"] & U
\end{tikzcd}$$

$$\begin{tikzcd}
W^3 \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "g^1"] \arrow[ddr, bend right, "g^2"'] & & \\
& W^1 \times_W W^2 \arrow[r] \arrow[d] & W^1 \arrow[d, "\beta^1"] \\
& W^2 \arrow[r, "\beta^2"] & W.
\end{tikzcd}$$

Then it is enough to show that (α, β) is a morphism in E_{X_M}, i.e., the following diagram

$$\begin{tikzcd}
W^3 \arrow[r, "\beta"] \arrow[d] & W^1 \times_W W^2 \arrow[d, "\theta"] \\
U^3_M \arrow[r, "\alpha_M"'] & (U^1 \times_U U^2)_M
\end{tikzcd} \tag{4.1}$$

commutes.

By the universal property of the fibre products, we have the following diagram

$$\begin{tikzcd}
 & & W^3 \arrow[dl, dotted, "\beta"'] \arrow[drr] \arrow[ddd, dotted] & & \\
 & W^1 \times_W W^2 \arrow[rrr] \arrow[dl] \arrow[ddd, dotted, "\theta"] & & & W^2 \arrow[dl] \arrow[ddd] \\
W^1 \arrow[rrr] \arrow[ddd] & & & W \arrow[ddd] & \\
 & & U^3_M \arrow[dl, dotted] \arrow[drr] & & \\
 & (U^1 \times_U U^2)_M \arrow[rrr] \arrow[dl] & & & U^2_M \arrow[dl] \\
U^1_M \arrow[rrr] & & & U_M &
\end{tikzcd}$$

such that each part of the above diagram is commutative except the one we need to check (the square with dot arrows). Hence our diagram (4.1) is also commutative. The propositions follows immediately.

□

Definition 4.1.2. A *site* is given by a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i\in I}$, called coverings (or covering families) of $\mathcal{C}$, satisfying the following axioms.

(1) If $V \to U$ is an isomorphism in $\mathcal{C}$, then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

(2) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in I_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I,\ j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.

(3) If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$, then $U_i \times_U V$ exists for all $i \in I$ and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Definition 4.1.3. Let (U, W) be an object of E_{X_M}

- A family of morphisms $\{(U_i, W_i) \to (U, W)\}_{i\in I}$ in E_{X_M} is called a covering (family) of type (α), respectively type (β) if

  (α) $\{U_i \to U\}_{i\in I}$ is a covering in X^{et} and $W_i \cong W \times_U U_i$ for every $i \in I$. Here the morphism $W \to U$ in the fibre product is the composite $W \to U_M \to U$, or

  (β) $U_i \cong U$ for all $i \in I$ and $\{W_i \to W\}_{i\in I}$ is a covering family in X_M^{et}.

- The topology T_{X_M} generated by the covering families of type (α) and (β) on E_{X_M} is called *Faltings' topology* associated to the data (X, M). The associated site and topos of sheaves of sets are called *Faltings' site* and *Faltings' topos*, and denoted by $\mathfrak{X}_M$, $\mathrm{Sh}(\mathfrak{X}_M)$ respectively.

We now give an alternative definition of the topology T_{X_M}.

Definition 4.1.4. A family $\{(U_{ij}, W_{ij}) \to (U, W)\}_{i\in I,\ j\in J}$ of morphisms in E_{X_M} is called a *strict covering (family)* if the followings hold:

(a) For every $i \in I$, there exists an object U_i in $X^{\mathtt{et}}$ such that $U_{ij} \cong U_i$ for all $j \in J$,

(b) The family of morphisms $\{U_i \to U\}_{i\in I}$ is a covering in $X^{\mathtt{et}}$,

(c) For every $i \in I$, the family $\{W_{ij} \to W \times_{U_M} U_{i,M}\}_{j\in J}$ is a covering in $X_M^{\mathtt{et}}$.

We denote a strict covering family

$$\{(U_{ij}, W_{ij}) \to (U, W)\}_{i\in I,\ j\in J}$$

of (U, W) simply by

$$\{(U_i, W_{ij}) \to (U, W)\}_{i\in I,\ j\in J},$$

where U_i is as in (a).

Remark 4.1.2. (1) Note that the covering families of type (α) and (β) are strict coverings.

(2) Let $\{(U_i, W_{ij}) \to (U, W)\}_{i\in I,\ j\in J}$ be a strict covering of (U, W) in E_{X_M}. It can be obtained by the composite

$$\Big\{\{(U_i, W_{ij}) \to (U_i, W \times_{U_M} U_{i,M})\}_{j\in J}\Big\}_{i\in I} \circ \{(U_i, W \times_{U_M} U_{i,M}) \to (U, W)\}_{i\in I},$$

where the first term is a covering family of $(U_i, W \times_{U_M} U_{i,M})$ of type (β) and the second term is a covering of (U, W) of type (α). Hence a strict covering is a covering family in $\mathfrak{X}_M$.

(3) By the above discussion, the strict coverings also generate the topology T_{X_M}.

Definition 4.1.5. Let $\mathcal{C}$ be a category. A pretopology on $\mathcal{C}$ is: for each object X of $\mathcal{C}$, define a set $\mathrm{Cov}(X)$ of families of morphisms over X satisfying the following axioms:

(PT0) (Existence of fibre product) For all objects X of $\mathcal{C}$, all morphisms $X_0 \to X$ in $\mathrm{Cov}(X)$ and all morphisms $Y \to X$ in $\mathcal{C}$, the fibre product $X_0 \times_X Y$ exists.

(PT1) (Stability under base change) For all objects X of $\mathcal{C}$, all morphisms $Y \to X$ in $\mathcal{C}$, and all $\{X_i \to X\}_{i \in I}$ in $\mathrm{Cov}(X)$, the family $\{X_i \times_X Y \to Y\}_{i \in I}$ is in $\mathrm{Cov}(Y)$.

(PT2) (Local character) If $\{X_i \to X\}_{i \in I}$ in $\mathrm{Cov}(X)$, and for all i, $\{X_{ij} \to X_i\}_{j \in J_i}$ in $\mathrm{Cov}(\mathcal{X})$, then

$$\{X_{ij} \to X_i \to X\}_{i \in I,\ j \in J_i}$$

is also in $\mathrm{Cov}(X)$.

(PT3) (Isomorphisms) $\{X \xrightarrow{\mathrm{id}} X\}$ is in $\mathrm{Cov}(X)$.

Remark 4.1.3. The category E_{X_M} with the strict covering families does not form a pretopology. In fact the strict coverings satisfy (PT0), (PT1) and (PT3) of the above definition but do not satisfy (PT2). However, the covering families of the pretopology PT_{X_M} generated by the strict coverings are composite of a finite number of strict coverings. The following lemma shows that one can use strict covering families to compute the sheaf associated to a presheaf on E_{X_M}.

Lemma 4.1.1. *Let (U, W) be an object of E_{X_M}. The strict coverings of (U, W) are cofinal in the collection of all covering families of (U, W) in* PT_{X_M}.

Proof. This is [Andreatta and Iovita, 2010, Lemma 2.8]. ☐

Note that for any object (U, W) in E_{X_M} we have a natural map $f : W \to U$ given by the composite $W \to U_M \to U$. Then we obtain a morphism of sheaves

$$f^\sharp : \mathcal{O}_U \longrightarrow f_* \mathcal{O}_W.$$

Taking global sections we get a morphism $\Gamma(U, \mathcal{O}_U) \longrightarrow \Gamma(W, \mathcal{O}_W)$.

Definition 4.1.6. Let (U, W) be an object in E_{X_M}. We define the following presheaves on E_{X_M}

- The presheaf of $\mathcal{O}_M$-algebras on E_{X_M}, denoted by $\mathcal{O}_{\mathfrak{X}_M}$, is defined as

$$\mathcal{O}_{\mathfrak{X}_M}(U, W) := \text{ the normalization of } \Gamma(U, \mathcal{O}_U) \text{ in } \Gamma(W, \mathcal{O}_W).$$

- Let $M_0 \subseteq M$ be the maximal absolutely unramified subfield of M and $\mathcal{O}_{M_0}$ be the ring of integers of M_0. We define the sub presheaf of $\mathcal{O}_{M_0}$-algebras $\mathcal{O}^{\mathrm{un}}_{\mathfrak{X}_M}$ of $\mathcal{O}_{\mathfrak{X}_M}$ as follows: $\mathcal{O}^{\mathrm{un}}_{\mathfrak{X}_M}(U, W)$ is the subset of $\mathcal{O}_{\mathfrak{X}_M}(U, W)$ consisting of elements x with the following property: there exist a finite unramified extension $L \subset L' \subset M$, a finite étale morphism $U' \to U_{\mathcal{O}_{L'}}$ and a morphism $W \to U'_M$ over U_M such that x, thought of as an element of $\Gamma(W, \mathcal{O}_W)$, lies in the image of $\Gamma(U', \mathcal{O}_{U'})$.

Proposition 4.1.2. *The presheaves $\mathcal{O}_{\mathfrak{X}_M}$ and $\mathcal{O}^{\mathrm{un}}_{\mathfrak{X}_M}$ are sheaves.*

Proof. See Proposition 2.11 and Remark 2.12 in Andreatta and Iovita [2013]. □

4.1.2 The formal setting

Let L, $\mathcal{O}_L$, π, $\overline{L}$ and M be as before. Now let $\mathcal{X}$ be a formal scheme, flat over $\mathrm{Spf}(\mathcal{O}_L)$ and with ideal of definition generated by π. Let $\mathcal{X}^{\mathrm{et}}$ be the small étale site on $\mathcal{X}$ and let $\mathcal{U} \to \mathcal{X}$

be an étale morphism, topologically of finite type of π-adic formal schemes. We define the following sites.

(1) The site $\mathfrak{U}_{M,\mathtt{fet}}$:

■ The objects of $\mathfrak{U}_{M,\mathtt{fet}}$ are pairs (W, K) where

- K is a finite extension of L contained in M.
- $W \to \left(\mathfrak{U}^{\mathrm{rig}}\right)_K$ is a finite étale cover of K-rigid analytic apaces, where $\mathfrak{U}^{\mathrm{rig}}$ denotes the L-rigid analytic space associated to $\mathfrak{U}$.

■ The morphisms of two objects (W', K'), (W, K) are defined as:

$$\mathrm{Mor}_{\mathfrak{U}_{M,\mathtt{fet}}}((W', K'), (W, K)) := \begin{cases} \emptyset, & \text{if } K \nsubseteq K', \\ \mathrm{Mor}(W', W_{K'}), & \text{if } K \subseteq K'. \end{cases}$$

Here $\mathrm{Mor}(W', W_{K'})$ denotes the morphisms of K'-rigid analytic spaces.

■ The coverings of a pair (W, K) in $\mathfrak{U}_{M,\mathtt{fet}}$ are families of pairs $\{(W_i, K_i)\}_{i\in I}$ over (W, K), where $K \subset K_i$ such that there exists a finite extension K' over K, $K_i \subset K' \subset M$, and the induced map

$$\coprod_{i\in I} W_{i,K'} \longrightarrow W_{K'}$$

is surjective.

Remark 4.1.4. (a) The fibre product of the morphisms

$$\begin{array}{ccc} & & (W_1, K_1) \\ & & \downarrow \\ (W_2, K_2) & \longrightarrow & (W, K) \end{array}$$

exists in $\mathcal{U}_{M,\mathtt{fet}}$ and we have

$$(W_1, K_1) \times_{(W,K)} (W_2, K_2) = (W_1 \times_W W_2, K_3)\,,$$

where K_3 is the composite of K_1 and K_2.

(b) Let $\mathcal{U}^2 \to \mathcal{U}^1$ be a morphism in $\mathcal{X}^{\mathtt{et}}$. Then we have a morphism

$$\begin{aligned} \rho_{\mathcal{U}^1,\mathcal{U}^2} : \mathcal{U}^1_{M,\mathtt{fet}} &\longrightarrow \mathcal{U}^2_{M,\mathtt{fet}} \\ (W, K) &\longmapsto \left(W \times_{\mathcal{U}^{1,\mathrm{rig}}_K} \mathcal{U}^{2,\mathrm{rig}}_K, K\right), \end{aligned}$$

which sends covering families to covering families.

(2) The site $\mathcal{U}^{\mathtt{fet}}_M$:

Proposition 4.1.3. *Let $\mathcal{S}_{\mathcal{U}}$ be the set of morphisms of pairs $(W', K') \to (W, K)$ in $\mathcal{U}_{M,\mathtt{fet}}$ such that $K \subset K'$, and $g : W' \to W_{K'}$ be an isomorphism of K'-rigid analytic spaces. Then we have*

- *$\mathcal{S}_{\mathcal{U}}$ is stable under composition;*
- *$\mathcal{S}_{\mathcal{U}}$ is stable under base change via morphisms in $\mathcal{U}_{M,\mathtt{fet}}$;*
- *given a morphism $\mathcal{U}^2 \to \mathcal{U}^1$ in $\mathcal{X}^{\mathtt{et}}$, we have*

$$\rho_{\mathcal{U}^1,\mathcal{U}^2}(\mathcal{S}_{\mathcal{U}}) \subset \mathcal{S}_{\mathcal{U}},$$

where $\rho_{\mathcal{U}^1,\mathcal{U}^2}$ are defined as in Remark 4.1.4;

- *if we have a commutative diagram of morphisms in $\mathcal{U}_{M,\mathtt{fet}}$*

$$\begin{tikzcd} & (W_1, K_1) \arrow[dl, "h"'] \arrow[d, "f"] \\ (W_2, K_2) \arrow[r, "g"] & (W, K) \end{tikzcd}$$

with f and g in $\mathcal{S}_{\mathcal{U}}$, then h is also in $\mathcal{S}_{\mathcal{U}}$.

Thanks to the above proposition, we may define the underlying category of $\mathcal{U}_M^{\texttt{fet}}$ to be the localization of the underlying category of $\mathcal{U}_{M,\texttt{fet}}$ with respect to $\mathcal{S}_{\mathcal{U}}$. More explicitly we have

- The objects of $\mathcal{U}_M^{\texttt{fet}}$ are pairs (W, K) as in $\mathcal{U}_{M,\texttt{fet}}$.
- The morphisms of two objects (W', K'), (W, K) in $\mathcal{U}_M^{\texttt{fet}}$ are defined as follows

$$\mathrm{Mor}_{\mathcal{U}_M^{\texttt{fet}}}((W', K'), (W, K)) := \varinjlim \mathrm{Mor}_{\mathcal{U}_{M,\texttt{fet}}}((W_1, K_1) \to (W, K)),$$

where the direct limit is taken over all morphisms $(W_1, K_1) \to (W', K')$ in $\mathcal{S}_{\mathcal{U}}$. Equivalently, this is the set of classes of morphisms $(W', K') \leftarrow (W_1, K_1) \to (W, K)$, where $(W_1, K_1) \to (W', K')$ is in $\mathcal{S}_{\mathcal{U}}$. Two such diagrams $(W', K') \leftarrow (W_1, K_1) \to (W, K)$ and $(W', K') \leftarrow (W_2, K_2) \to (W, K)$ are equivalent if and only if there is a third one $(W', K') \leftarrow (W_3, K_3) \to (W, K)$ mapping to the two.

Note that the fibre product of two pairs over a given pair exists in $\mathcal{U}_M^{\texttt{fet}}$ and it coincides with the fibre product in $\mathcal{U}_{M,\texttt{fet}}$. If $(W', K') \leftarrow (W_1, K_1) \to (W, K)$ and $(W'', K'') \leftarrow (W_2, K_2) \to (W', K')$ are two morphisms, their composition $(W'', K'') \leftarrow (W_3, K_3) \to (W, K)$ is defined as:

$$(W_3, K_3) := (W_1, K_1) \times_{(W', K')} (W_2, K_2).$$

The covering families are defined similarly as in $\mathcal{U}_{M,\texttt{fet}}$. From now on we will write an object (W, K) in $\mathcal{U}_M^{\texttt{fet}}$ simply by W but one should keep in mind that W is defined over a finite extension K of L.

If $\mathcal{U}_2 \to \mathcal{U}_1$ ia a morphism in $\mathcal{X}^{\texttt{et}}$, by Proposition 4.1.3, the map $\rho_{\mathcal{U}_1,\mathcal{U}_2}$ extends to the localized categories and defines a map of Grothendieck topologies $\mathcal{U}_{1,M}^{\texttt{fet}} \to \mathcal{U}_{2,M}^{\texttt{fet}}$. We denote this morphism simply by

$$W \longmapsto W \times_{\mathcal{U}_1^{\mathrm{rig}}} \mathcal{U}_2^{\mathrm{rig}}$$

on objects.

Now we give a more explicit description of the site $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$.

- An object in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$ is a pair (W, K) where K is a finite extension of L contained in $\overline{L}$ and W is an object of the finite étale site of $\mathcal{U}_K$.

- Given two objects (W, K) and (W', K') in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$, the morphisms between them are defined to be:

$$\mathrm{Mor}_{\mathcal{U}_{\overline{L}}^{\mathtt{fet}}}\left((W, K), (W', K')\right) := \varinjlim \mathrm{Mor}_{\mathcal{U}_{K''}}\left(W \times_K K'', W' \times_{K'} K''\right),$$

  where the direct limit is taken over all finite extensions K'' of L contained in $\overline{L}$ and containing both K and K'. The morphisms on the right hand side are the ones of rigid analytic spaces over $\mathcal{U}_{K''}$.

(3) The site $\mathfrak{X}_M$:

- The objects are pairs $(\mathcal{U}, W)$ with $\mathcal{U}$ an object of $\mathcal{X}^{\mathsf{et}}$ and W an object of $\mathcal{U}_M^{\mathtt{fet}}$.

- A morphism of $(\mathcal{U}_1, W_1) \longrightarrow (\mathcal{U}, W)$ is a morphism $\mathcal{U}_1 \to \mathcal{U}$ in $\mathcal{X}^{\mathsf{et}}$ and $W_1 \to W \times_{\mathcal{U}^{\mathrm{rig}}} \mathcal{U}_1^{\mathrm{rig}}$ in $\mathcal{U}_{1,M}^{\mathtt{fet}}$. Here the fibre product $W \times_{\mathcal{U}^{\mathrm{rig}}} \mathcal{U}_1^{\mathrm{rig}}$ is taken after applying base change to some finite field extension of L in M.

- The covering families are defined as in the algebraic case, see Definitions 4.1.3 and 4.1.4. Let the Faltings' topology be the topologies generated by the strict covering families.

Remark 4.1.5. We can define the presheaves $\mathcal{O}_{\mathfrak{X}_M}$ and $\mathcal{O}^{\mathrm{un}}_{\mathfrak{X}_M}$ as in Definition 4.1.6. Moreover, these presheaves are sheaves by similar analogue of Proposition 4.1.2.

4.2 Faltings' topos: the semistable case

4.2.1 Assumptions

Recall L, $\mathcal{O}_L$, π, $\mathbb{L}$ defined at the beginning of this chapter. Let $S := \mathrm{Spec}(\mathcal{O}_L)$ and M be the log structure on S associated to the prelog structure given by the map $\mathbb{N} \to \mathcal{O}_L$ sending n to $\pi^n \in \mathcal{O}_L$. We denote by (S, M) the associated log scheme.

Now fix a positive integer a. We assume that we are in one of the following two cases:

(1) (The algebra case)

Let (X, N) be a log scheme. $f : (X, N) \to (S, M)$ is a morphism of log schemes of finite type admitting a covering by étale open subschemes $\mathrm{Spec}(R)$ of the form

- $\mathrm{Spec}(R) \to X$ is étale;
- there is a commutative diagram of $\mathcal{O}_L$-algebras

$$\begin{array}{ccc} \mathcal{O}_L[Q] & \xrightarrow{\psi_R} & R \\ \theta \uparrow & & \uparrow \\ \mathcal{O}_L[\mathbb{N}] & \xrightarrow[\psi_a]{} & \mathcal{O}_L, \end{array}$$

where

(i) $Q = \mathbb{N}^s \times \mathbb{N}^t$;

(ii) θ is the morphism of $\mathcal{O}_L$-algebras induced by the map on monoids $\mathbb{N} \to Q$

sending

$$n \longmapsto ((n, n, \cdots, n), (0, 0, \cdots, 0));$$

(iii) ψ_a is the morphism $\mathcal{O}_L$-algebras sending $n \mapsto \pi^{an}$.

- The induced morphism of $\mathcal{O}_L$-algebras by the above commutative diagram

$$R' := \mathcal{O}_L[Q] \otimes_{\mathcal{O}_L[\mathbb{N}]} \mathcal{O}_L \longrightarrow R$$

  is étale on associated spectra.

- The log structure on $\mathrm{Spec}(R)$ induced by (X, N) is the pullback of the fibre product log structure on $\mathrm{Spec}(R')$.

- For every subset $J_s \subset \{1, \cdots, s\}$ and every subset $J_t \subset \{1, \cdots, t\}$, the ideal of R generated by $\psi_R\left(\mathbb{N}^{J_s} \times \mathbb{N}^{J_t}\right)$ defines an irreducible closed subscheme of $\mathrm{Spec}(R)$.

(2) (The formal case)

We write $(S_n, M_n)_{n \in \mathbb{N}}$ for the compatible system of log schemes given by $S_n := \mathrm{Spec}\left(\mathcal{O}_L \big/ \pi^n \mathcal{O}_L\right)$, and the log structure M_n is the one associated to the prelog structure given by $\mathbb{N} \to \mathcal{O}_L \big/ \pi^n \mathcal{O}_L$ sending $1 \mapsto \pi$.

For every $n \in \mathbb{N}$, suppose we have a log scheme (X_n, N_n) over (S_n, M_n) of finite type, which is denoted by

$$f_n : (X_n, N_n) \longrightarrow (S_n, M_n),$$

such that (X_n, N_n) is isomorphic as log schemes over (S_n, M_n) to the fibre product of the following pairs:

$$\begin{tikzcd} & (X_{n+1}, N_{n+1}) \arrow[d] \\ (S_n, M_n) \arrow[r] & (S_{n+1}, M_{n+1}). \end{tikzcd}$$

Let X_{form} be the formal scheme associated to the X_n's. We require that étale locally on X_n the formal scheme $X_{\text{form}} \to \mathrm{Spf}(\mathcal{O}_L)$ is of the form

$$\begin{tikzcd} \left(\mathcal{O}_L \big/ \pi^n \mathcal{O}_L\right)[Q] \arrow[r, "\psi_{R,n}"] & R \big/ \pi^n R \\ \left(\mathcal{O}_L \big/ \pi^n \mathcal{O}_L\right)[\mathbb{N}] \arrow[u, "\theta"] \arrow[r, "\psi_a"'] & \mathcal{O}_L \big/ \pi^n \mathcal{O}_L, \arrow[u] \end{tikzcd}$$

where Q, θ, ψ_a are as in the algebraic case, $\psi_{R,n}$ induces a morphism

$$R'_n := \mathcal{O}_L[Q] \otimes_{\mathcal{O}_L[\mathbb{N}]} \mathcal{O}_L \big/ \pi^n \mathcal{O}_L \longrightarrow R \big/ \pi^n R$$

which is étale. The log structure on $\mathrm{Spec}(R \big/ \pi^n R)$ induced from (X_n, N_n) is the pullback of the fibre product log structures on $\mathrm{Spec}(R'_n)$.

Moreover, as in the algebraic case, we require that for every subset $J_s \subset \{1, \cdots, s\}$ and $J_t \subset \{1, \cdots, t\}$, the ideal of $R \big/ \pi R$ generated by $\psi_{R,1}\left(\mathbb{N}^{J_s} \times \mathbb{N}^{J_t}\right)$ defines an irreducible closed subscheme of $\mathrm{Spec}(R \big/ \pi R)$.

We have a morphism of sheaves of monoids from

$$N_{\text{form}} := \varprojlim N_n$$

to $\mathcal{O}_{X_{\text{form}}}$, which coincides with the inverse image of N_1 via the canonical map $\mathcal{O}_{X_{\text{form}}} \longrightarrow \mathcal{O}_{X_1}$. N_{form} is called the *formal log structure* on X_{form}. We write (X, N), or sometimes X, for the inductive system of log schemes $\{(X_n, N_n)\}_{n \in \mathbb{N}}$. By the assumption, X_{form} is a noetherian, π-adic formal scheme and has an étale open covering $\mathrm{Spf}(R) \to X_{\text{form}}$ such that

- $\hat{\psi}_R : \mathrm{Spf}(R) \longrightarrow \mathrm{Spf}\left(\mathcal{O}_L[Q] \hat{\otimes}_{\mathcal{O}_L[\mathbb{N}]} \mathcal{O}_L\right)$ is étale, where, ψ_R, Q are as before and $\hat{\otimes}$ is the π-adic completion of the tensor product;

- the formal log structure N_{form} on $\text{Spf}(R)$ is induced by the formal log structure on the fibre product $\text{Spf}\left(\mathcal{O}_L[Q]\hat{\otimes}_{\mathcal{O}_L[\mathbb{N}]}\mathcal{O}_L\right)$.

We end the assumptions by the following remark.

Remark 4.2.1. By Lemma 3.1 of Andreatta and Iovita [2012], the log schemes in both cases ((X, N) in the algebraic case and (X_n, N_n) in the formal case) are fine and saturated log schemes.

4.2.2 Faltings' sites

The notations are as in the previous section.

4.2.2.1 The site $X^{\mathtt{ket}}$

We write $X^{\mathtt{ket}}$ for the Kummer étale site of (X, N) for both cases: the algebraic and the formal. In the former case, $X^{\mathtt{ket}}$ is just the one described in section 2.4. In the latter, we define $X^{\mathtt{ket}}$ as follows.

- The objects are system of Kummer étale morphisms

$$\{g_n : (Y_n, N_{Y_n}) \longrightarrow (X_n, N_n)\}_{n\in\mathbb{N}}$$

  such that g_n is the base change of g_{n+1} via $(X_n, N_n) \to (X_{n+1}, N_{n+1})$ for every $n \in \mathbb{N}$. We simply write $g : (Y, N_Y) \to (X, N)$ for such inductive system.

- The morphisms from one object

$$\{g_n : (Y_n, N_{Y_n}) \longrightarrow (X_n, N_n)\}_{n\in\mathbb{N}}$$

to another

$$\{h_n : (Z_n, N_{Z_n}) \longrightarrow (X_n, N_n)\}_{n\in\mathbb{N}}$$

are the set of systems of morphisms of log schemes

$$\{t_n : (Y_n, N_{Y_n}) \longrightarrow (Z_n, N_{Z_n})\}_{n\in\mathbb{N}}$$

over (X_n, N_n), such that t_n is the base change of t_{n+1} via $(X_n, N_n) \to (X_{n+1}, N_{n+1})$ for every $n \in \mathbb{N}$, which are simply denoted by $t : (Y, N_Y) \to (Z, N_Z)$.

- The coverings are collections of Kummer étale morphisms

$$\left\{\left(Y^i, N_Y^i\right) \longrightarrow (X, N)\right\}_{i\in I}$$

such that X_1 is the set theorectic union of images of Y_1^i's.

Note that we have a natural forgetful functor $X^{\mathtt{ket}} \to X_1^{\mathtt{ket}}$, sending a system

$$\{g_n : (Y_n, N_{Y_n}) \longrightarrow (X_n, N_n)\}_{n\in\mathbb{N}}$$

to $g_1 : (Y_1, N_{Y_1}) \longrightarrow (X_1, N_1)$. Moreover this is an equivalence of categories.

4.2.2.2 Presheaves on $X^{\mathtt{ket}}$

In the algebraic case, we define presheaves $\mathcal{O}_{X^{\mathtt{ket}}}$ and $N_{X^{\mathtt{ket}}}$ on $X^{\mathtt{ket}}$ respectively as: for any object (U, N_U) in $X^{\mathtt{ket}}$,

$$\begin{aligned} \mathcal{O}_{X^{\mathtt{ket}}}(U, N_U) &:= \Gamma\left(U, \mathcal{O}_U\right), \\ N_{X^{\mathtt{ket}}}(U, N_U) &:= \Gamma\left(U, N_U\right). \end{aligned}$$

In the formal case, for every $h \in \mathbb{N}$, we define presheaves $\mathcal{O}_{X^{\mathtt{ket}},h}$ and $N_{X^{\mathtt{ket}},h}$ on $X^{\mathtt{ket}}$ respectively as: for any object $(U_n, N_{U_n})_{n\in\mathbb{N}}$ in $X^{\mathtt{ket}}$,

$$\begin{aligned} \mathcal{O}_{X^{\mathtt{ket}},h}\left((U_n, N_{U_n})_n\right) &:= \Gamma\left(U_h, \mathcal{O}_{U_h}\right), \\ N_{X^{\mathtt{ket}},h}\left((U_n, N_{U_n})_n\right) &:= \Gamma\left(U_h, N_{U_h}\right). \end{aligned}$$

Then Let

$$\mathcal{O}_{X^{\mathtt{ket}}_{\mathrm{form}}} := \varprojlim_h \mathcal{O}_{X^{\mathtt{ket}},h} \quad \text{and} \quad N_{X^{\mathtt{ket}}_{\mathrm{form}}} := \varprojlim_h N_{X^{\mathtt{ket}},h}.$$

Similarly, we can define subpresheaves $\mathcal{O}^{\times}_{X^{\mathtt{ket}}}$ of $\mathcal{O}_{X^{\mathtt{ket}}}$ in the algebraic case, $\mathcal{O}^{\times}_{X^{\mathtt{ket}},h}$ of $\mathcal{O}_{X^{\mathtt{ket}},h}$ and $\mathcal{O}^{\times}_{X^{\mathtt{ket}}_{\mathrm{form}}}$ of $\mathcal{O}_{X^{\mathtt{ket}}_{\mathrm{form}}}$, respectively, in the formal case. We have

Proposition 4.2.1. *(1) In the algebraic case the presheaves $\mathcal{O}_{X^{\mathtt{ket}}}$, $\mathcal{O}^{\times}_{X^{\mathtt{ket}}}$ and $N_{X^{\mathtt{ket}}}$ are sheaves, and*

$$\alpha : N_{X^{\mathtt{ket}}} \longrightarrow \mathcal{O}_{X^{\mathtt{ket}}}$$

is a morphism of sheaves of multiplicative monoids such that

$$\alpha^{-1}\left(\mathcal{O}^{\times}_{X^{\mathtt{ket}}}\right) \xrightarrow{\sim} \mathcal{O}^{\times}_{X^{\mathtt{ket}}}.$$

(2) In the formal case the presheaves

$$\mathcal{O}_{X^{\mathtt{ket}},h}, \quad \mathcal{O}^{\times}_{X^{\mathtt{ket}},h} \text{ and } N_{X^{\mathtt{ket}},h}$$

for every $h \in \mathbb{N}$ and the presheaves

$$\mathcal{O}_{X^{\mathtt{ket}}_{\mathrm{form}}}, \quad \mathcal{O}^{\times}_{X^{\mathtt{ket}}_{\mathrm{form}}} \text{ and } N_{X^{\mathtt{ket}}_{\mathrm{form}}}$$

are sheaves. Moreover,

$$\alpha_h : N_{X^{\mathtt{ket}},h} \longrightarrow \mathcal{O}_{X^{\mathtt{ket}},h} \text{ for every } h \text{ and}$$

$$\alpha : N_{X^{\mathtt{ket}}_{\mathrm{form}}} \longrightarrow \mathcal{O}_{X^{\mathtt{ket}}_{\mathrm{form}}}$$

are morphisms of sheaves of multiplicative monoids such that

$$\alpha_h^{-1}\left(\mathcal{O}^{\times}_{X^{\mathtt{ket}},h}\right) \xrightarrow{\sim} \mathcal{O}^{\times}_{X^{\mathtt{ket}},h} \text{ for every } h, \text{ and}$$
$$\alpha^{-1}\left(\mathcal{O}^{\times}_{X^{\mathtt{ket}}_{\mathrm{form}}}\right) \xrightarrow{\sim} \mathcal{O}^{\times}_{X^{\mathtt{ket}}_{\mathrm{form}}}.$$

4.2.2.3 Faltings' sites

Let U be an object in $X^{\mathtt{ket}}$ and let $L \subset M \subset \overline{L}$. We assume that the log structure on U_L defined by N_U coincides with the trivial log structure. Let $U_M^{\mathtt{fet}}$ be either the site of finite étale covers of U_M in the algebraic case or be the site as defined in Section 4.1.2 in the formal case, respectively. Both are endowed with the trivial log structure. Let E_{X_M} be the category defined as follows:

- the objects are pairs (U, W), where U is an object of $X^{\mathtt{ket}}$ and W is an object of $U_M^{\mathtt{fet}}$;
- a morphism $(U', W') \to (U, W)$ in E_{X_M} is a pair of morphisms (α, β), where $\alpha : U' \to U$ is a morphism in $X^{\mathtt{ket}}$, $\beta : W' \to W \times_{U_L} U'_L$ is a morphism in $(U')_M^{\mathtt{fet}}$.

Remark 4.2.2. Some properties, such as the existence of the fibre products, can be proved exactly the same way as in the smooth case. Moreover, we can also define covering families of type (α), type (β) as well as strict covering in the same way. The associated site and topos of sheaves of sets are denoted by $\mathfrak{X}_M$, $\mathrm{Sh}(\mathfrak{X}_M)$, respectively.

In the algebraic case we define the presheaf of $\mathcal{O}_M$-algebras on $\mathfrak{X}_M$, denoted by $\mathcal{O}_{\mathfrak{X}_M}$, as

$$\mathcal{O}_{\mathfrak{X}_M}(U, W) := \text{ the normalization of } \Gamma\left(U, \mathcal{O}_U\right) \text{ in } \Gamma\left(W, \mathcal{O}_W\right).$$

In the formal case the definition is the same by replacing $\Gamma(U, \mathcal{O}_U)$ with $\Gamma(U_{\text{form}}, \mathcal{O}_{U_{\text{form}}})$. We may also define the subpresheaf of $\mathbb{W}(\mathbb{L})$-algebras $\mathcal{O}^{\text{un}}_{\mathfrak{X}_M}$ of $\mathcal{O}_{\mathfrak{X}_M}$ in the same way as we did in the smooth case. Moreover, these presheaves are sheaves.

4.3 Faltings' site associated to Shimura curves

4.3.1 Log structures

Recall that we have a commutative diagram of formal schemes and rigid analytic spaces (see Section 3.2.3):

$$\begin{tikzcd}
M(H\pi^r) \arrow[r] & M(H,\pi) \arrow[r] & M(H) \\
M^r(w) \arrow[r] \arrow[u, hook] \arrow[d, "u"'] & M(w) \arrow[r, equal] \arrow[u, hook] \arrow[d, "u"'] & M(w) \arrow[u, hook] \arrow[d, "u"'] \\
\mathcal{M}^r(w) \arrow[r] & \mathcal{M}(w) \arrow[r, equal] \arrow[d, hook] & \mathcal{M}(w) \arrow[d] \\
\mathcal{M}(H\pi^r) \arrow[r] & \mathcal{M}(H,\pi) \arrow[r] & \mathcal{M}(H).
\end{tikzcd} \tag{4.2}$$

Fix H, r and w as before. For M a formal scheme or a rigid analytic space , we write $\underline{M}$ for its underlying scheme. We define some log formal schemes and log rigid spaces as follows:

- $S := (\underline{S}, M)$.

  Let $\underline{S} = \mathrm{Spf}(\mathcal{O}_L)$ and let M be the log structure on $\underline{S}$ defined by its closed point, i.e., the log structure associated to the prelog structure given by the morphism $\mathbb{N} \to \mathcal{O}_L$ sending $n \mapsto \pi_n$.

- $\mathcal{M}(w) := (\underline{\mathcal{M}}(w), N)$.

Take a small open affine $\mathcal{U} = \mathrm{Spf}(R) \hookrightarrow \underline{\mathcal{M}}(w)$, i.e., $\mathcal{U}$ is connected and there is a formal étale morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R')$, where $R' := \mathcal{O}_L\{X, Y\}/(XY - \pi^a)$ for some $a \in \mathbb{N}$ (this can be done due to Proposition 3.3.1 and Remark 3.3.2). Then we are in the situation as in Example 2.3.1. Let $P := \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ be the amalgamated sum of the morphisms $\Delta : \mathbb{N} \to \mathbb{N}^2$, $n \to (n, n)$ and $\psi_a : \mathbb{N} \to \mathbb{N}$, $n \to an$. Then we have the following commutative diagram of monoids:

$$\begin{tikzcd}
 & & R' \arrow[r] & R \\
\mathbb{N}^2 \arrow[r] \arrow[urr, "\psi_R"] & P \arrow[ur, dashed] & & \\
\mathbb{N} \arrow[r, "\psi_a"'] \arrow[u, "\Delta"] & \mathbb{N} \arrow[r, "\psi"] \arrow[u, "h"] \arrow[uur] & \mathcal{O}_L, \arrow[uu] \arrow[uur] &
\end{tikzcd}$$

where $\psi_R(m, n) = X^m Y^n$, $\mathcal{O}_L$, R and R' are the multiplicative monoids associated to the respective rings. Let $N_{\mathcal{U}}$ be the log structure on $\mathcal{U}$ associated to the prelog structure given by the composition $P \dashrightarrow R' \to R$. Moreover $\mathcal{M}(w)$ is a fine saturated log scheme, $f : \mathcal{M}(w) \to S$ is log smooth (see Example 2.1.2 and Lemma 2.3.1).

- $\mathcal{M}^r(w) := (\underline{\mathcal{M}}^r(w), N_r)$.

  Here N_r is the inverse image log structure (see Proposition 2.2.2) on $\underline{\mathcal{M}}^r(w)$ via the morphism $\underline{\mathcal{M}}^r(w) \to \underline{\mathcal{M}}(w)$.

- $\mathcal{M}(H, \pi)$ is the log formal scheme whose underlying formal scheme is $\underline{\mathcal{M}}(H, \pi)$, the log structure is defined by its special fibre which is a divisor with normal crossing (since $\underline{\mathcal{M}}(H, \pi)$ has semistable reduction).

- Let $M(w)$ (resp. $M^r(w)$, resp. $M(H, \pi)$) be the log rigid analytic space endowed $\underline{M}(w)$ (resp. $\underline{M}^r(w)$, resp. $\underline{M}(H, \pi)$) with the trivial log structure.

We have the following commutative diagram of log formal schemes and log rigid spaces:

$$\begin{tikzcd} \mathcal{M}(w) \arrow[r, "\nu"] & \mathcal{M}(H,\pi) \\ M(w) \arrow[u, "u"] \arrow[r, hook] & M(H,\pi). \arrow[u, "u"'] \end{tikzcd}$$

4.3.2 The sites $\mathfrak{M}(w)$ and $\mathfrak{M}(H,\pi)$

Both the pairs $(\mathcal{M}(w), M(w))$ and $(\mathcal{M}(H,\pi), M(H,\pi))$ satisfy the conditions of section 4.2.1. Then we denote by $\mathfrak{M}(w)$, respectively $\mathfrak{M}(H,\pi)$ the Faltings' sites associated to these pairs. More explicitly, let $(\mathcal{M}, M)$ be one of the above pairs, the Faltings' site associated to the pair $(\mathcal{M}, M)$ is defined as follows. First, the category $E_{\mathcal{M}_{\overline{L}}}$ is defined by:

(i) the objects consist of pairs $\big(\mathcal{U}, (W,K)\big)$ such that $\mathcal{U} \in \mathcal{M}^{\mathtt{ket}}$ and $(W,K) \in \mathcal{U}_{\overline{L}}^{\mathtt{fet}}$;

(ii) a morphism $\big(\mathcal{U}, (W,K)\big) \to \big(\mathcal{U}', (W',K')\big)$ in $E_{\mathcal{M}_{\overline{L}}}$ is a pair of morphisms (α,β), where $\alpha : \mathcal{U} \to \mathcal{U}'$ is a morphism in $\mathcal{M}^{\mathtt{ket}}$, $\beta : (W,K) \to (W' \times_{\mathcal{U}'_{K'}} \mathcal{U}_{K'}, K')$ is a morphism in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$.

Then we endow the category $E_{\mathcal{M}_{\overline{L}}}$ with the topology generated by the covering families defined as in Definition 4.1.3. Note that this topology on $E_{\mathcal{M}_{\overline{L}}}$ is the same as the one generated by the strict coverings (see Definition 4.1.4). Now let $\mathfrak{M}$ be either the site $\mathfrak{M}(w)$ or $\mathfrak{M}(H,\pi)$. Recall that we have sheaves of $\mathcal{O}_{\overline{L}}$-algebras on $\mathfrak{M}$, denoted by $\mathcal{O}_{\mathfrak{M}}$, defined as:

$$\mathcal{O}_{\mathfrak{M}}(\mathcal{U}, W) := \text{the normalization of } \Gamma(\underline{\mathcal{U}}, \mathcal{O}_{\underline{\mathcal{U}}}) \text{ in } \Gamma(\underline{W}, \mathcal{O}_{\underline{W}}).$$

We also have the subsheaf of $\mathbb{W}(\mathbb{L})$-algebras $\mathcal{O}_{\mathfrak{M}}^{\mathrm{un}}$ of $\mathcal{O}_{\mathfrak{M}}$, whose sections over $(\mathcal{U}, W)$ consist of elements $x \in \mathcal{O}_{\mathfrak{M}}(\mathcal{U}, W)$ for which there exist a finite unramified extension M of L contained in $\overline{L}$, a Kummer log étale morphism $\mathcal{V} \to \mathcal{U} \times_{\mathcal{O}_L} \mathcal{O}_M$ and a morphism $W \to \mathcal{V}_L$ over $\mathcal{U}_L$ such that x, viewed as an element of $\Gamma(\underline{W}, \mathcal{O}_{\underline{W}})$, lies in the image of $\Gamma(\underline{\mathcal{V}}, \mathcal{O}_{\underline{\mathcal{V}}})$.

Then we denote by $\widehat{\mathcal{O}}_{\mathfrak{M}}$ and $\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}}$ the continuous sheaves on $\mathfrak{M}$ defined by the projective systems $\left\{\mathcal{O}_{\mathfrak{M}}/\pi^n\mathcal{O}_{\mathfrak{M}}\right\}_{n\geq 0}$ and $\left\{\mathcal{O}^{\mathrm{un}}_{\mathfrak{M}}/\pi^n\mathcal{O}^{\mathrm{un}}_{\mathfrak{M}}\right\}_{n\geq 0}$, respectively.

4.3.3 Induced sites

Let S be a site whose underlying category is denoted by $\mathcal{C}$, and let E be the category of sheaves of sets on S. Let X be an object in $\mathcal{C}$. We then define the site $S_{/X}$, called the site induced by X, as follows.

- Its underlying category, denoted by $\mathcal{C}_{/X}$, consists of pairs (Y, ϕ), where Y is an object of $\mathcal{C}$ and $\phi : Y \to X$ is a morphism in $\mathcal{C}$. A morphism $(Y', \phi') \to (Y, \phi)$ in $\mathcal{C}_{/X}$ is a morphism $f : Y' \to Y$ in $\mathcal{C}$ such that the following diagram

$$\begin{tikzcd} Y' \arrow[rr, "f"] \arrow[dr, "\phi'"'] & & Y \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

  commutes in $\mathcal{C}$.

- The topology on $\mathcal{C}_{/X}$ is the one induced from S via the forgetful functor $\alpha_X : \mathcal{C}_{/X} \to \mathcal{C}$ sending $(Y, \phi) \mapsto Y$ on objects, i.e., a family of morphisms $\{U_i \to U\}_{i\in I}$ of objects over X is a covering family in $\mathcal{C}_{/X}$ if and only if it is a covering in $\mathcal{C}$.

Moreover, we denote by $E_{/X}$ the sheaves of sets on $S_{/X}$, which are called the topos induced by X. We have natural functors:

$$\alpha_{X,*} : E \longrightarrow E_{/X}, \text{ and}$$

$$\alpha_X^* : E_{/X} \longrightarrow E$$

such that α_X^* is left adjoint to $\alpha_{X,*}$.

Now suppose that the category $\mathcal{C}$ has a final object T and fibre products exist in $\mathcal{C}$. Then we obtain a functor

$$j_X : \mathcal{C} \longrightarrow \mathcal{C}_{/X}$$

defined by $j_X(Z) = (Z \times_T X, pr_2)$ on objects, where pr_2 is the natural morphism $Z \times_T X \to X$.

Definition 4.3.1. Let C and D be sites with underlying categories $\mathcal{C}$ and $\mathcal{D}$, respectively. A functor $u : \mathcal{C} \to \mathcal{D}$ is called *continuous* if for every covering family $\{V_i \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ we have

(1) $\{u(V_i) \to u(V)\}_{i \in I} \in \mathrm{Cov}(\mathcal{D})$;

(2) for any morphism $V' \to V$ in $\mathcal{C}$, the morphism

$$u\left(V' \times_V V_i\right) \longrightarrow u(V') \times_{u(V)} u(V_i)$$

is an isomorphism for each i.

Then j_X is a continuous functor and defines a morphism of sites. Indeed, we have morphism of topos

$$j_X^* : E \longrightarrow E_{/X}, \text{ and}$$
$$j_{X,*} : E_{/X} \longrightarrow E,$$

such that j_X^* is left adjoint to $j_{X,*}$. Moreover, j_X is right adjoint to α_X and we have a canonical isomorphism of functors $j_X^* \cong \alpha_{X,*}$. Then j_X has a canonical left adjoint, namely

α_X^*. We denote this left adjoint of j_X by $j_{X,!}$, which can be described explicitly as follows. For any sheaf $\mathcal{F}$ on $S_{/X}$, $j_{X,!}(\mathcal{F})$ is the sheaf associated to the presheaf on S given by

$$Z \longmapsto \coprod_{g \in \mathrm{Mor}_{\mathcal{C}}(Z,X)} \mathcal{F}(Z,g)\,.$$

Recall that we denote by $(\mathcal{M}, M)$ be one of the two pairs $(\mathcal{M}(w), M(w))$ and $(\mathcal{M}(H,\pi), M(H,\pi))$, and by $\mathfrak{M}$ the Faltings' site associated to the corresponding pair. Let $Z \to M$ be a finite étale morphism of log rigid spaces. Then we get a morphism in $\mathcal{M}_{\overline{L}}^{\mathtt{fet}}$. Thus the pair $(\mathcal{M}, Z)$ is an object of $E_{\mathcal{M}_{\overline{L}}}$. We denote by $\left(E_{\mathcal{M}_{\overline{L}}}\right)_{/(\mathcal{M},Z)}$ the induced category and by $\mathfrak{Z} := \mathfrak{M}_{/(\mathcal{M},Z)}$ the induced Faltings' site. Recall that the site $\mathfrak{M}$ has an final object, namely $(\mathcal{M}, M)$ and fibre products exist in $E_{\mathcal{M}_{\overline{L}}}$. We denote simply by α and j the forgetful functor $\alpha_{/(\mathcal{M},Z)} : \mathfrak{Z} \to \mathfrak{M}$ and the functor $j_{/(\mathcal{M},Z)} : \mathfrak{M} \to \mathfrak{Z}$ defined by $j(\mathcal{U}, W) = (\mathcal{U}, W \times_M Z, pr_2)$, respectively. Then we have the following functors:

$$\begin{aligned}
\alpha_* &: \mathrm{Sh}(\mathfrak{M}) \longrightarrow \mathrm{Sh}(\mathfrak{Z}),\\
\alpha^* &: \mathrm{Sh}(\mathfrak{Z}) \longrightarrow \mathrm{Sh}(\mathfrak{M}),\\
j_* &: \mathrm{Sh}(\mathfrak{Z}) \longrightarrow \mathrm{Sh}(\mathfrak{M}),\\
j^* &: \mathrm{Sh}(\mathfrak{M}) \longrightarrow \mathrm{Sh}(\mathfrak{Z}),\\
j_! &: \mathrm{Sh}(\mathfrak{Z}) \longrightarrow \mathrm{Sh}(\mathfrak{M}),
\end{aligned}$$

such that α^* and j^* are left adjoint to α_* and j_*, respectively, and $j_!$ is left adjoint to j^*.

Now let $(\mathcal{U}, W)$ be an object of $\mathfrak{M}$ and Z as above. We have

Proposition 4.3.1. *There exists an object Z_W in $\mathcal{M}_{\overline{L}}^{\mathtt{fet}}$ such that there exists a canonical* *isomorphism*

$$W \times_M Z \cong \left(\coprod_{g \in \mathrm{Mor}_{\mathcal{M}_{\underline{L}}^{\mathtt{fet}}}(W,Z)} W \right) \coprod Z_W. \tag{4.3}$$

Proof. Let $g : W \to Z$ be a morphism in $\mathcal{M}_{\underline{L}}^{\mathtt{fet}}$. First we claim that there exists an object Z_g in $\mathcal{M}_{\underline{L}}^{\mathtt{fet}}$ such that

$$W \times_M Z \cong W \amalg Z_g,$$

and the following diagram commutes

$$\begin{tikzcd}
W \times_M Z \arrow[r, equal, "\sim"] & W \amalg Z_g \\
W \arrow[u, "\varphi_g"] \arrow[r, equal] & W, \arrow[u, hook]
\end{tikzcd}$$

where the right vertical map is the natural inclusion and φ_g is the unique map (depending on g) induced by the following commutative diagram:

$$\begin{tikzcd}
W \arrow[dr, dashed, "\varphi_g"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "g"'] & & \\
& W \times_M Z \arrow[r] \arrow[d] & W \arrow[d] \\
& Z \arrow[r] & M.
\end{tikzcd}$$

Since $Z \to M$ is finite and étale, so is $W \times_M Z \to W$. Then our claim follows from the following lemma.

Lemma 4.3.1. *Let B be a finite separable A-algebra and $f : B \to A$ an A-algebra homomorphism. Then there exist an A-algebra C and an A-algebra isomorphism $g : B \xrightarrow{\sim} A \times C$ such that $f = p_1 \circ g$, where p_1 is the projection $A \times C \to A$.*

Proof. Clearly, $f \in \mathrm{Hom}_A(B, A)$. Since B is separable,

$$\psi : B \longrightarrow \mathrm{Hom}_A(B, A)$$

$$b \longmapsto \Big(x \mapsto \mathrm{Tr}_{B/A}(bx)\Big)$$

is an isomorphism. Let $e \in B$ be such that $\psi(e) = f$, i.e., $\mathrm{Tr}_{B/A}(ex) = f(x)$ for all $x \in B$. Since f is an A-algebra homomorphism, $\mathrm{Tr}_{B/A}(e) = f(1) = 1$. Furthermore, for all $x, y \in B$,

$$\mathrm{Tr}_{B/A}(exy) = f(xy) = f(x)f(y) = f(x)\,\mathrm{Tr}_{B/A}(ey) = \mathrm{Tr}_{B/A}(f(x)ey),$$

i.e., $\psi(ex) = \psi(f(x)e)$ for all $x \in B$. Since ψ is an isomorphism thus injective, we have $ex = f(x)e$. This implies that $e\,\mathrm{Ker}(f) = 0$. Then the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] \arrow[d, "0"] & B \arrow[r, "f"] \arrow[d, "m_e"] & A \arrow[r] \arrow[d, "m_{f(e)}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(f) \arrow[r] & B \arrow[r, "f"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes with both rows exact, where the first vertical arrow is just $m_e\big|_{\mathrm{Ker}(f)} = 0$ since $e\,\mathrm{Ker}(f) = 0$. Then

$$1 = \mathrm{Tr}_{B/A}(e) = \mathrm{tr}_{\mathrm{Ker}(f)/A}(0) + \mathrm{tr}_{A/A}(f(e)) = 0 + f(e) = f(e).$$

Note that we have $ex = f(x)e$ for all $x \in B$. Taking $x = e$ we get $e^2 = f(e)e = e$, i.e., e is an idempotent of B. $1 - e \in \mathrm{Ker}(f)$ since $f(1-e) = f(1) - f(e) = 0$. Then the map $A \to \mathrm{Ker}(f)$, $a \mapsto a(1-e)$ makes $\mathrm{Ker}(f)$ be an A-algebra. Acturally $1 - e$ is the identity of $\mathrm{Ker}(f)$ since $(1-e)y = y - ey = y - f(y)e = y - 0 = y$ for all $y \in \mathrm{Ker}(f)$. Then the projectivity of A implies $B \cong A \times \mathrm{Ker}(f)$, where the isomorphism $g : B \to A \times \mathrm{Ker}(f)$ is given by $x \mapsto (f(x), x - ef(x))$. Using the identity $ex = f(x)e$ and the fact that f is an A-algebra homomorphism, we have

$$\begin{aligned}
g(xy) &= \Big(f(xy), xy - ef(xy)\Big) \\
&= \Big(f(xy), xy - ef(y)f(x) - ef(x)f(y) + e^2 f(x)f(y)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(f(x)f(y), xy - eyf(x) - exf(y) + e^2f(x)f(y)\Big) \\
&= \Big(f(x)f(y), (x - ef(x))y - (x - ef(x))ef(y)\Big) \\
&= \Big(f(x)f(y), (x - ef(x))(y - ef(y))\Big) \\
&= \big(f(x), x - ef(x)\big)\big(f(y), y - ef(y)\big) \\
&= g(x)g(y),
\end{aligned}
$$

for all $x, y \in B$. Thus g is also an isomorphism of A-algebras. Furthermore, for any $x \in B$, $p_1 \circ g(x) = p_1\big(f(x), x - ef(x)\big) = f(x)$, i.e., $p_1 \circ g = f$. Now $C := \mathrm{Ker}(f)$ as required. Moreover, C is a separable A-algebra. This completes the proof of the lemma. $\square$

Since Z_g is also an object in $\mathcal{M}_{\overline{L}}^{\mathtt{fet}}$, repeating the above argument we get a canonical isomorphism

$$W \times_M Z \cong \left(\coprod_{g \in \mathrm{Mor}_{\mathcal{M}_{\overline{L}}^{\mathtt{fet}}}(W,Z)} W \right) \coprod Z_W$$

for some object Z_W in $\mathcal{M}_{\overline{L}}^{\mathtt{fet}}$. $\square$

Remark 4.3.1. If B is a separable A-algebra, then $B \otimes_A B$ is a separable B-algebra via the second factor. Moreover, the map $f : B \otimes_A B \to B$, $b \otimes b' \mapsto bb'$ is a B-algebra homomorphism. If we apply Lemma 4.3.1 to f, there is a B-algebra C and a B-algebra isomorphism $g : B \otimes_A B \xrightarrow{\sim} B \times C$ making the following diagram

$$
\begin{tikzcd}
B \otimes_A B \arrow[r, "g"] \arrow[dr, "f"'] & B \times C \arrow[d, "p"] \\
& B
\end{tikzcd}
$$

commute, where p is the first projection.

Recall that we have functors $j_!$, $j_* : \mathrm{Sh}(\mathfrak{Z}) \longrightarrow \mathrm{Sh}(\mathfrak{M})$. More precisely, let $\mathcal{F}$ be a sheaf of abelian groups on $\mathfrak{Z}$, for any object $(\mathcal{U}, W)$ in $\mathfrak{M}$,

$$j_*(\mathcal{F})\,(\mathcal{U}, W) = \mathcal{F}\,(j(\mathcal{U}, W)) = \mathcal{F}\,(\mathcal{U}, W \times_M Z, pr_2)\,,$$

and the sheaf $j_!(\mathcal{F})$ on $\mathfrak{M}$ is the sheaf associated to the presheaf

$$(\mathcal{U}, W) \longmapsto \bigoplus_{g \in \mathrm{Mor}_{\mathcal{M}_{\underline{L}}^{\mathtt{fet}}}(W,Z)} \mathcal{F}\,(\mathcal{U}, W, g)\,.$$

By the above proposition, we have a natural morphism

$$\bigoplus_{g \in \mathrm{Mor}_{\mathcal{M}_{\underline{L}}^{\mathtt{fet}}}(W,Z)} \mathcal{F}\,(\mathcal{U}, W, g) \longrightarrow \mathcal{F}\,(\mathcal{U}, W \times_M Z, pr_2)\,,$$

which induces a morphism of sheaves $j_!(\mathcal{F}) \to j_*(\mathcal{F})$. This gives a natural transformation of functors: $j_! \to j_*$. Indeed, we have the following facts.

Proposition 4.3.2. *For any Z as before, the natural transformation $j_! \to j_*$ defined above is an isomorphism of functors.*

Proof. It suffices to prove that for any object $(\mathcal{U}, W)$ in $\mathfrak{M}$, there is a surjective morphism $W' \to W$ in $\mathcal{M}_{\underline{L}}^{\mathtt{fet}}$ such that

$$W' \times_M Z \cong \coprod_{g \in \mathrm{Mor}_{\mathcal{M}_{\underline{L}}^{\mathtt{fet}}}(W',Z)} W',$$

i.e., $Z_{W'} = \emptyset$ in formula (4.3).

Let $V := W \times_M Z$. Then the morphism $V \to W$ is finite étale and we have a map

$$\deg_{V/W} : W \longrightarrow \mathbb{Z}\,.$$

Moreover, we have $\deg_{V/W}(y) = \deg_{Z/M}(y')$ for any $y \in W$, where y' is the image of y under $W \to M$. By restricting to a connected component of M we may assume that $\deg_{Z/M}$ is a constant equal to n. We prove by induction on n. If $n = 0$, then $Z = \emptyset$, our claim follows by taking $W' := W \xrightarrow{\text{id}} W$. Now let $n \geq 1$ and suppose our claim is true when deg $< n$. Consider the following commutative diagram:

$$\begin{tikzcd}
V \arrow[dr, "\Delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& V \times_W V \arrow[r] \arrow[d] & V \arrow[d] \\
& V \arrow[r] & W,
\end{tikzcd}$$

where Δ is the morphism induced by the local natural multiplication $B \otimes B \to B$ sending $b \otimes b' \mapsto bb'$. By Remark 4.3.1, there exists a V' in $\mathcal{M}_{L}^{\mathtt{fet}}$ such that the following diagram commutes

$$\begin{tikzcd}
V \times_W V \arrow[r, equal, "\sim"] & V \amalg V' \\
V \arrow[u, "\Delta"] \arrow[r, equal] & V. \arrow[u, hook]
\end{tikzcd}$$

Moreover $V' \to V$ if finite étale and $\deg_{V'/V} = n - 1$. Then by induction hypothesis, there exists a surjective morphism $h : W' \to V'$ in $\mathcal{M}_{L}^{\mathtt{fet}}$ such that

$$V' \times_V W' \cong \coprod_g W',$$

where g is taken over $\mathrm{Mor}_{\mathcal{M}_{L}^{\mathtt{fet}}}(W', Z)$ such that g is not equivalent to $pr_2 \circ h : W' \to Z$. Then

$$\begin{aligned}
W' \times_M Z &\cong W' \times_W (W \times_M Z) = W' \times_W V \\
&\cong W' \times_V (V \times_W V) \\
&\cong W' \times_V (V \amalg V') \\
&\cong (W' \times_V V) \amalg (W' \times_V V')
\end{aligned}$$

$$\begin{aligned} &\cong W' \amalg \left(\coprod_{g \neq pr_2 \circ h} W' \right) \\ &\cong \coprod_{g \in \mathrm{Mor}_{\mathcal{M}^{\mathtt{fet}}_{\overline{L}}}(W', Z)} W'. \end{aligned}$$

The composition $W' \to V \to W$ is finite étale. Moreover, since $\deg_{V/W} = n \geq 1$, $V \to W$ is also surjective. Hence $W' \to W$ is a surjective morphism in $\mathcal{M}^{\mathtt{fet}}_{\overline{L}}$, which proves the claim and the proposition. □

We have the following consequence immediately.

Corollary 4.3.1. *Let $Z \to M$ be a morphism in $\mathcal{M}^{\mathtt{fet}}_{\overline{L}}$. Then we have*

(a) The functor j_* is exact.

(b) $\mathrm{R}^i j_* = 0$ for all $i \geq 1$.

Proof. Part (a) follows form the fact that $j_* \cong j_! \cong \alpha^*$ and (b) follows immediately from (a). □

By adjunction we obtain a morphism

$$j_* j^*(\mathcal{F}) \cong j_! j^*(\mathcal{F}) \longrightarrow \mathcal{F},$$

which is the functorial on the category of sheaves of abelian groups on the site $E_{\mathcal{M}_{\overline{L}}}$. It is called the *trace map* relative to Z. More explicitly, given a sheaf of abelian groups $\mathcal{F}$ on $E_{\mathcal{M}_{\overline{L}}}$, the above trace map is the morphism of sheaves associated to the morphism of presheaves:

$$\begin{aligned} j_! j^*(\mathcal{F})(\mathcal{U}, W) &= \bigoplus_g j^*(\mathcal{F})(\mathcal{U}, W, g) \\ &= \bigoplus_g \mathcal{F}(\mathcal{U}, W) \longrightarrow \mathcal{F}(\mathcal{U}, W), \end{aligned}$$

where g runs over all the morphisms $\mathrm{Mor}_{\mathcal{M}_{\overline{L}}^{\mathrm{fet}}}(W, Z)$.

Finally, we define the site $\mathfrak{M}^r(w)$ to be the induced site

$$\mathfrak{M}(w)_{/(\mathcal{M}(w), M^r(w))}.$$

4.4 Continuous functors

Now we have defined several sites, namely $\mathcal{M}^{\mathtt{ket}}(w)$, $\mathcal{M}^{\mathtt{ket}}(N, p^r)$, $M_{\overline{L}}^{\mathtt{ket}}$, $\mathfrak{M}(w)$, $\mathfrak{M}(N, p^r)$ and $\mathfrak{M}^r(w)$. We have the following natural functors which send covering families to covering families, commute with fibre products and send final objects to final objects. In particular they induce morphisms of topoi.

(1) $\mu : \mathcal{M}^{\mathtt{ket}}(H, \pi) \longrightarrow \mathcal{M}^{\mathtt{ket}}(w)$ is induced by the natural morphism of formal log schemes $\mathcal{M}(w) \hookrightarrow \mathcal{M}(H, \pi)$. More, explicitly, for any object $\mathcal{U}$ in $\mathcal{M}^{\mathtt{ket}}(H, \pi)$, μ sends

$$\mathcal{U} \longmapsto \mathcal{U} \times_{\mathcal{M}(H,\pi)} \mathcal{M}(w).$$

(2) $\nu : \mathfrak{M}(H, \pi) \longrightarrow \mathfrak{M}(w)$ sending

$$\begin{aligned} ((\mathcal{U}, W)) &\mapsto \left(\mathcal{U} \times_{\mathcal{M}(H,\pi)} \mathcal{M}(w), W \times_{\mathcal{U}_L} (\mathcal{U} \times_{\mathcal{M}(H,\pi)} \mathcal{M}(w))_L\right) \\ &\cong \left(\mathcal{U} \times_{\mathcal{M}(H,\pi)} \mathcal{M}(w), W \times_{M(H,\pi)} M(w)\right). \end{aligned}$$

(3) $v_{\mathfrak{M}} : \mathcal{M}^{\mathtt{ket}} \longrightarrow \mathfrak{M}$ with $v_{\mathfrak{M}}(\mathcal{U}) := (\mathcal{U}, \mathcal{U}_L)$, where $\mathcal{M}$ is either $\mathcal{M}(w)$ or $\mathcal{M}(N, p)$ and $\mathfrak{M}$ is either $\mathfrak{M}(w)$ or $\mathfrak{M}(N, p)$, respectively. Moreover, we have $v_{\mathfrak{M}}^*(\mathcal{O}_{\mathcal{M}^{\mathtt{ket}}}) \cong \mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}$ ([Andreatta and Iovita, 2012, Proposition 2.13]). We also have the following commutative diagram

of sites:

$$\begin{array}{ccc} \mathcal{M}^{\mathtt{ket}}(H,\pi) & \xrightarrow{v_{\mathfrak{M}(H,\pi)}} & \mathfrak{M}(H,\pi) \\ \mu \downarrow & & \downarrow \nu \\ \mathcal{M}^{\mathtt{ket}}(w) & \xrightarrow[v_{\mathfrak{M}(w)}]{} & \mathfrak{M}(w). \end{array}$$

(4) $u : \mathfrak{M} \longrightarrow M_{\overline{L}}^{\mathtt{et}}$ with $(\mathcal{U}, W) \longmapsto W$, where $\mathfrak{M}$ is one of the three sites: $\mathfrak{M}(w)$, $\mathfrak{M}(H,\pi)$ and $\mathfrak{M}^r(w)$.

(5) $j_r : \mathfrak{M}(w) \longrightarrow \mathfrak{M}^r(w)$ sending $(\mathcal{U}, W) \mapsto \big(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\big)$. By the discussion in Section 4.3.3, this morphism of topoi has the following properties

(i) The functor $j_{r,*} : \mathrm{Sh}(\mathfrak{M}^r(w)) \to \mathrm{Sh}(\mathfrak{M}(w))$ is an exact functor.

(ii) $R^i j_{r,*} = 0$ for all $i \geq 1$.

(6) $v_r : \mathcal{M}^{\mathtt{ket}}(w) \longrightarrow \mathfrak{M}^r(w)$, which is defined to be the composite $v_r := j_r \circ v_{\mathfrak{M}(w)}$. Actually $v_r(\mathcal{U}) = \big(\mathcal{U}, \mathcal{U}_K \times_{M(w)} M^r(w), \mathtt{pr}_2\big)$, $v_r(\mathcal{M}(w)) = (\mathcal{M}(w), M^r(w), \mathrm{id})$. Moreover, we have $R^i v_{r,*} = R^i v_* \circ j_{r,*}$.

We denote by $\mathcal{O}_{\mathfrak{M}^r(w)} := j_r^*(\mathcal{O}_{\mathfrak{M}(w)})$ and by $\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)} := j_r^*(\widehat{\mathcal{O}}_{\mathfrak{M}(w)})$. By the construction of $\mathcal{M}^r(w)$, we have natural isomorphisms of sheaves on $\mathcal{M}^{\mathtt{ket}}(w)$:

$$\Big(v_{r,*}(\mathcal{O}_{\mathfrak{M}^r(w)})\Big)^{G_r} \cong \mathcal{O}_{\underline{\mathcal{M}}(w)} \text{ and } \Big(v_{r,*}(\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)})\Big)^{G_r} \cong \widehat{\mathcal{O}}_{\underline{\mathcal{M}}(w)},$$

where $G_r \cong \big(\mathcal{O}_{\mathcal{P}}/\pi^r\mathcal{O}_{\mathcal{P}}\big)^\times$ is the Galois group of $M^r(w)/M(w)$ (see [Andreatta et al., 2015b, Lemma 2.8]).

(7) for any object $\mathcal{U}$ in $\mathcal{M}^{\mathtt{ket}}$, $\beta_{\mathcal{U}} : \mathcal{U}_{\overline{L}}^{\mathtt{fet}} \longrightarrow \mathfrak{M}$ sending $W \longmapsto (\mathcal{U}, W)$, where $\mathcal{M}$ is either $\mathcal{M}(w)$ or $\mathcal{M}(H,\pi)$.

4.5 Localization functors

This section is a brief recall of [Andreatta et al., 2015b, §2.7]. Let $\mathfrak{M}$ be one of the sites $\mathfrak{M}(w)$ or $\mathfrak{M}(N,p)$ and $\mathcal{U} = \big(\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})\big)$ a connected small affine object of $\mathcal{M}^{\mathtt{ket}}$. We denote by $U := \mathcal{U}_L$ the log rigid generic fibre of $\mathcal{U}$. Write $R_{\mathcal{U}} \otimes \overline{L} = \prod_{i=1}^{n} R_{\mathcal{U},i}$ with $\mathrm{Spf}(R_{\mathcal{U},i})$ connected, let $N_{\mathcal{U},i}$ be the monoids giving the respective log structures, and U_i the respective log rigid generic fibres. Let $\mathbb{C}_{\mathcal{U},i} := \overline{\mathrm{Frac}(R_{\mathcal{U},i})}$, and $\eta_{\mathcal{U},i}$ denote the log geometric point of $\mathcal{U}_i := \big(\mathrm{Spf}(R_{\mathcal{U},i}), N_{\mathcal{U},i}\big)$ over $\mathbb{C}_{\mathcal{U},i}$. Let $\mathcal{G}_{\mathcal{U},i}$ be the étale fundamental group of U_i. Then the category $U_i^{\mathtt{fet}}$ is equivalent to the category of finite sets with continuous actions of $\mathcal{G}_{\mathcal{U},i}$. Write $(\overline{R}_{\mathcal{U},i}, \overline{N}_{\mathcal{U},i})$ for the direct limit of all the normal extensions S of $R_{\mathcal{U},i}$ in $\mathbb{C}_{\mathcal{U},i}$ such that $\mathrm{Spm}(S_L) \to U_i$ is finite étale. Also, we let $\overline{R}_{\mathcal{U}} := \prod_{i=1}^{n} \overline{R}_{\mathcal{U},i}$, $\overline{N}_{\mathcal{U}} := \prod_{i=1}^{n} \overline{N}_{\mathcal{U},i}$ and $\mathcal{G}_{U_{\overline{L}}} := \prod_{i=1}^{n} \mathcal{G}_{\mathcal{U},i}$. Then we have an equivalence of categories,

$$\mathrm{Sh}(U_{\overline{L}}^{\mathtt{fet}}) \xrightarrow{\sim} \mathrm{Rep}(\mathcal{G}_{U_{\overline{K}}}), \qquad \mathcal{F} \longmapsto \varinjlim \mathcal{F}(\mathrm{Spm}(S_L)),$$

where $\mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$ is the category of discrete abelian groups with continuous $\mathcal{G}_{U_{\overline{L}}}$-action. Composing with $\beta_{\mathcal{U},*}$, we obtain a localization functor $\mathrm{Sh}(\mathfrak{M}) \longrightarrow \mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$, we donote by $\mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})$ the image of $\mathcal{F}$ in $\mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$. We have

$$\widehat{\mathcal{O}}_{\mathfrak{M}}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}) \cong \widehat{\overline{R}}_{\mathcal{U}}$$

(see [Andreatta and Iovita, 2013, Proposition 2.15]).

Let $\mathcal{F} \in \mathrm{Sh}(\mathfrak{M}^r(w))$. $\mathcal{U} = \big(\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})\big)$ is fixed as a connected small affine object of $\mathcal{M}(w)^{\mathtt{ket}}$ as before. Let

$$\Upsilon_{\mathcal{U}} := \Big\{\text{homomorphisms of } R_{\mathcal{U}} \otimes \overline{L}\text{-algebras } \Gamma_{\mathcal{U}} := \Gamma\big(U^r(w), \mathcal{O}_{U^r(w)}\big) \longrightarrow \overline{R}_{\mathcal{U}}[p^{-1}]\Big\},$$

where $U^r(w) := \mathcal{U}_L \times_{M(w)} M^r(w)$. For any $g \in \Upsilon_{\mathcal{U}}$, write

$$\mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g) := \varinjlim \mathcal{F}(\mathcal{U}, \mathrm{Spm}(S_L)),$$

where the limit is taken over all $\Gamma_{\mathcal{U}}$-subalgebras S of $\overline{R}_{\mathcal{U}}$ (via g) such that $\mathrm{Spm}(S_L) \to U^r(w)$ is finite and étale. Let $\mathcal{G}_{U_{\overline{K}},r,g} \subseteq \mathcal{G}_{U_{\overline{L}}}$ be the subgroup fixing $\Gamma_{\mathcal{U}}$. Similarly as before we obtain a localization functor $\mathrm{Sh}(\mathfrak{M}^r(w)) \longrightarrow \mathrm{Rep}(\mathcal{G}_{U_{\overline{L}},r,g})$ and donote by $\mathcal{F}\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\big)$ the image of $\mathcal{F} \in \mathrm{Sh}(\mathfrak{M}^r(w))$.

Moreover, given a covering of $\mathcal{M}(w)^{\mathtt{ket}}$ by open small affines $\{\mathcal{U}_i\}_{i\in I}$, choosing $g_i \in \Upsilon_{\mathcal{U}_i}$ for every $i \in I$, the map $\mathrm{Sh}(\mathfrak{M}^r(w)) \longrightarrow \prod_{i\in I} \mathrm{Rep}(\mathcal{G}_{U_{i,\overline{K}},r,g_i})$ is faithful. We also have

$$j_{r,*}(\mathcal{F})(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}) \cong \bigoplus_{g\in\Upsilon_{\mathcal{U}}} \mathcal{F}\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\big). \tag{4.4}$$

Chapter 5

Distributions and Overconvergent Cohomology

In this chapter we will introduce the overconvergent cohomology which can be related to the π-adic (families of) overconvergent modular forms constructed in Brasca [2013]. As highlighted by the author in Brasca [2013], to make things work it is necessary "to consider the action of $\mathcal{O}_{\mathcal{P}}$ everywhere". Using this principle we define distributions in the classical way and then we consider its cohomology as usual.

5.1 Distribution

5.1.1 Definitions

We fix an $r \in \mathbb{N}$ and let L be a finite field extension over $F_{\mathcal{P}}$ containing an element $\zeta_r \in \mathbb{C}_p := \widehat{\overline{L}}$, where $\{\zeta_n\}_{n\geq 1}$ is a fixed sequence of $\mathbb{C}_p$ points of $\mathcal{LT}$ satisfying

- the $\mathcal{O}_{\mathcal{P}}$-order of ζ_n is exactly π^n;

- $\pi\zeta_{n+1} = \zeta_n$ for each $n \geq 1$;

- $\zeta_1 = (-\pi)^{\frac{1}{q-1}}$, where $(-\pi)^{\frac{1}{q-1}}$ is a fixed element in $\mathbb{C}_p$.

Let $\mathfrak{U} \subset \mathcal{W}_r$ be a wide open disk defined over L. We denote by $\Lambda_{\mathfrak{U}}$ the $\mathcal{O}_L$-subalgebra of $\mathcal{O}(\mathfrak{U})$ consisting of functions

$$\Lambda_{\mathfrak{U}} = \left\{ f \in \mathcal{O}(\mathfrak{U}) \;\middle|\; |f(x)| \leq 1 \text{ for all } x \in \mathfrak{U} \right\}.$$

We denote by

$$\lambda_{\mathfrak{U}} : \mathcal{O}_{\mathcal{P}}^{\times} \to (\Lambda_{\mathfrak{U}})^{\times}$$

the universal weight attached to $\mathfrak{U}$, i.e., the character defined by

$$\lambda_{\mathfrak{U}}(z)(\lambda) = \lambda(z)$$

for $z \in \mathcal{O}_{\mathcal{P}}^{\times}$ and $\lambda \in \mathfrak{U}$. As $\mathfrak{U}$ is a wide open disk we have a (non canonical) isomorphism

$$\Lambda_{\mathfrak{U}} \cong \mathcal{O}_L[[T]].$$

Thus it follows that $\Lambda_{\mathfrak{U}}$ is a complete, local, noetherian $\mathcal{O}_L$-algebra. Let $\pi_L \in \mathcal{O}_L$ be a uniformizer. We then define a function

$$\mathrm{ord} : \Lambda_{\mathfrak{U}} \to \mathbb{Z} \cup \{\infty\}$$

by

$$\mathrm{ord}(x) = \sup\{n \in \mathbb{N} \,|\, x \in \pi_L^n \Lambda_{\mathfrak{U}}\}.$$

In this section we denote by (B, λ) one of the following pairs:

- $(\mathcal{O}_L, \lambda)$, where $\lambda \in \mathfrak{U}(L)$ is a weight;

- $(\Lambda_{\mathfrak{U}}, \lambda_{\mathfrak{U}})$, where $\Lambda_{\mathfrak{U}}$ and $\lambda_{\mathfrak{U}}$ are as above,

and we denote by $\underline{\mathbf{m}}$ the corresponding maximal ideal of B. In either case, there exists an $s_\lambda \in B \otimes_{\mathcal{O}_L} L$ such that $\lambda(1 + \pi^r z) = \exp(s_\lambda \log(z))$ for all $z \in \mathcal{O}_{\mathcal{P}}$ (refer to [Brasca, 2013, §6]).

Definition 5.1.1. We denote by A_λ^0 the space of functions $f : \mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}} \to B$ satisfying:

- $f(u(w, z)) = \lambda(u) f(w, z)$ for each $u \in \mathcal{O}_{\mathcal{P}}^\times$ and $(w, z) \in \mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}}$;

- the function $\mathcal{O}_{\mathcal{P}} \to B$ defined by $z \mapsto f(1, z)$ is $F_{\mathcal{P}}$-analytic on disks of radius q^{-r}, i.e., for any $z_0 \in \mathcal{O}_{\mathcal{P}}$ there exists a sequence $\{c_m(z_0)\}_{m \in \mathbb{N}}$ in B such that for each $z \in z_0 + \pi^r \mathcal{O}_{\mathcal{P}}$ we have:

$$f(1, z) = \sum_{m \in \mathbb{N}} c_m(z_0) \left(\frac{z - z_0}{\pi^r} \right)^m,$$

  where $\operatorname{ord}(c_m(z_0)) \to \infty$ as $m \to \infty$.

On the B-module A_λ^0, we consider the topology given by the family $\{\underline{\mathbf{m}}^m A_\lambda^0\}_{m \in \mathbb{N}}$. We denote by D_λ^0 the continuous dual of A_λ^0, i.e., the B-module of continuous, B-linear homomorphisms $A_\lambda^0 \to B$. Moreover, we let

$$A_\lambda = A_\lambda^0 \otimes_{\mathcal{O}_L} L$$

and

$$D_\lambda = D_\lambda^0 \otimes_{\mathcal{O}_L} L.$$

The $(B \otimes_{\mathcal{O}_L} L)$-module A_λ is in fact a $(B \otimes_{\mathcal{O}_L} L)$-Banach module with respect to the π-adic topology. We can construct an explicit orthonormal basis of A_λ as follows. Fix S, a set of representatives of $\mathcal{O}_{\mathcal{P}}/\pi^r\mathcal{O}_{\mathcal{P}}$ in $\mathcal{O}_{\mathcal{P}}$. Then for each $z_\eta \in S$ and $m \in \mathbb{N}$, we consider the function $f_{z_\eta,m} \in A_\lambda$ defined by:

$$f_{z_\eta,m}(w,z) = \lambda(w)\left(\frac{\frac{z}{w} - z_\eta}{\pi^r}\right)^m \mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}\left(\frac{z}{w}\right),$$

where $\mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}} : \mathcal{O}_{\mathcal{P}} \to L$ is the characteristic function of the subset $z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}$. By the definition of A_λ^0 it follows that $\{f_{z_\eta,m}\}_{m\in\mathbb{N},z_\eta\in S}$ is a family in A_λ and it is an orthornormal basis for A_λ.

As in Barrera and Gao [2016] and Andreatta et al. [2015b] we have the following fact:

Lemma 5.1.1. *We have an isomorphism of topological B-modules:*

$$\psi : D_\lambda^0 \cong \prod_{z_\eta\in S}\prod_{m\in\mathbb{N}} B,$$

given by $\mu \mapsto (\mu(f_{z_\eta,m}))_{m\in\mathbb{N},z_\eta\in S}$, where D_λ^0 is endowed with its weak ($\underline{\mathbf{m}}$-adic) topology, which corresponds to the product of the $\underline{\mathbf{m}}$-adic topologies on the right side.

These modules of distributions (for either $B = \Lambda_{\mathfrak{U}}$ or $B = \mathcal{O}_L$) are main objects giving rise to the right overconvergent cohomology which will be related to the π-adic (families of) overconvergent modular forms constructed in Brasca [2013]. In order to obtain sheaves on the Shimura curve $M(H,\pi)$ from these modules, we need to endow them with an action of the arithmetic groups described in Section 5.2.1. To do that we consider the following semigroup:

$$\Lambda_\pi = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_{\mathcal{P}}) \cap \mathrm{GL}_2(F_{\mathcal{P}}) \mid a \in \mathcal{O}_{\mathcal{P}}^\times,\ c \in \pi\mathcal{O}_{\mathcal{P}},\ d \neq 0\right\}.$$

We also define the *Iwahori* subgroup $I_\pi \subset \Lambda_\pi$ by

$$I_\pi := \Lambda_\pi \cap \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}}).$$

Now let $T_0 := \mathcal{O}_{\mathcal{P}}^{\times} \times \mathcal{O}_{\mathcal{P}}$, which can be regarded as a compact subset of

$$\mathcal{O}_{\mathcal{P}}^2 = \left\{ (w, z) \;\middle|\; w, z \in \mathcal{O}_{\mathcal{P}} \right\}.$$

There are two natural actions on T_0:

- a left action of $\mathcal{O}_{\mathcal{P}}^{\times}$ by scalar multiplication, i.e., for any $u \in \mathcal{O}_{\mathcal{P}}^{\times}$,

$$u \cdot (w, z) = (uw, uz);$$

- a right action of the semigroup Λ_π by matrix multiplication on the right, i.e., for any $(w, z) \in T_0$, any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda_\pi$,

$$(w, z) \cdot \gamma = (w, z) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (aw + cz, bw + dz).$$

It is obvious that these two actions commute.

Then the semigroup Λ_π acts in a natural way on A_λ^0. More precisely, for any $f \in A_\lambda^0$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda_\pi$, we define

$$\gamma \cdot f : \mathcal{O}_{\mathcal{P}}^{\times} \times \mathcal{O}_{\mathcal{P}} \to B$$

to be the function

$$(\gamma \cdot f)(w, z) = f(aw + cz, bw + dz).$$

Then we have

Lemma 5.1.2. *If $f \in A_\lambda^0$ and $\gamma \in \Lambda_\pi$, then $\gamma \cdot f \in A_\lambda^0$.*

Proof. Let $f \in A^0_\lambda$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda_\pi$. We need to verify that $\gamma \cdot f$ satisfies the two conditions in Definition 5.1.1.

First, let $u \in \mathcal{O}_\mathcal{P}^\times$ and $(w, z) \in \mathcal{O}_\mathcal{P}^\times \times \mathcal{O}_\mathcal{P}$, we have

$$\begin{aligned} (\gamma \cdot f)(u(w,z)) &= f(u(w,z) \cdot \gamma) \\ &= f(u(aw + cz, bw + dz)) \\ &= \lambda(u) f(aw + cz, bw + dz) \\ &= \lambda(u)(\gamma \cdot f)(w, z). \end{aligned}$$

Now we consider the second condition in Definition 5.1.1. For $z \in \mathcal{O}_\mathcal{P}$ we write

$$\gamma(z) = \frac{b + dz}{a + cz}.$$

If $Z \in z_0 + \pi^r \mathcal{O}_\mathcal{P}$ for some $z_0 \in \mathcal{O}_\mathcal{P}$, we denote by

$$z_1 := \gamma(z_0) = \frac{b + dz_0}{a + cz_0},$$

then we have

$$\begin{aligned} & \gamma(z) - z_1 \\ = \; & \frac{b + dz}{a + cz} - \frac{b + dz_0}{a + cz_0} \\ = \; & \frac{\det(\gamma)(z - z_0)}{(a + cz)(a + cz_0)}. \end{aligned}$$

Since $a \in \mathcal{O}_\mathcal{P}^\times$ and $c \in \pi\mathcal{O}_\mathcal{P}$, the denominator in the above equation is a unit of $\mathcal{O}_\mathcal{P}$, hence

$$\gamma(z) - z_1 \in \pi^r \mathcal{O}_\mathcal{P}.$$

Let $c_m(z_1)_{m \in \mathbb{N}}$ be the sequence in B such that for any $z \in z_1 + \pi^r \mathcal{O}_\mathcal{P}$,

$$f(1, z) = \sum_{m \in \mathbb{N}} c_m(z_1) \left(\frac{z - z_1}{\pi^r} \right)^m,$$

with $\operatorname{ord}(c_m(z_1)) \to \infty$ as $m \to \infty$. Thus if we let $u := (a + cz_0)^{-1}$ and $x := \det(\gamma)$, which are elements of $\mathcal{O}_{\mathcal{P}}$, we have

$$\begin{aligned}
& f(1, \gamma(z)) \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) \left(\frac{\gamma(z) - z_1}{\pi^r} \right)^m \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) \left(\frac{u}{a + cz} \right)^m \left(\frac{z - z_0}{\pi^r} \right)^m \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) u^m \left(\frac{1}{a + cz} \right)^m \left(\frac{z - z_0}{\pi^r} \right)^m \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) \cdot (ux)^m \cdot \left(\frac{1}{(a + cz_0) + c(z - z_0)} \right)^m \cdot \left(\frac{z - z_0}{\pi^r} \right)^m \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) \cdot (ux)^m \cdot \left(u \cdot \sum_{i \in \mathbb{N}} (-1)^i (cu)^i (z - z_0)^i \right)^m \cdot \left(\frac{z - z_0}{\pi^r} \right)^m \\
= \quad & \sum_{m \in \mathbb{N}} c_m(z_1) \cdot u^{2m} x^m \cdot \left(\sum_{i \in \mathbb{N}} (-1)^i (cu)^i (z - z_0)^i \right)^m \cdot \left(\frac{z - z_0}{\pi^r} \right)^m ,
\end{aligned}$$

Then

$$\begin{aligned}
& (\gamma \cdot f)(1, z) \\
= \quad & f(a + cz, b + dz) \\
= \quad & \lambda(a + cz) f(1, \gamma(z)) \\
= \quad & \lambda(a + cz) f(1, \gamma(z)) \\
= \quad & \lambda(a + cz) \cdot \sum_{m \in \mathbb{N}} c_m(z_1) \cdot u^{2m} x^m \cdot \left(\sum_{i \in \mathbb{N}} (-1)^i (cu)^i (z - z_0)^i \right)^m \cdot \left(\frac{z - z_0}{\pi^r} \right)^m .
\end{aligned}$$

Since λ is r-accessible, we deduce that the function $z \mapsto (\gamma \cdot f)(1, z)$ is $F_{\mathcal{P}}$-analytic on disks of radius q^{-r}. This completes the proof of the lemma. $\square$

Thus we obtain a well defined left action of Λ_π on A^0_λ, which induces a right action on D^0_λ by duality, i.e., for any $f \in A^0_\lambda$, $\mu \in D^0_\lambda$ and $\gamma \in \Lambda_\pi$ we have

$$\left(\mu\middle|\gamma\right)(f) = \mu(\gamma \cdot f).$$

The following result is important to prove the existence of spectral decompositions for the overconvergent cohomology discussed in Section 5.2.1:

Lemma 5.1.3. *Suppose that $B = \mathcal{O}_L$. Then the L-linear operator on D_λ obtained from the action of* $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ *is compact.*

Proof. We follow the proof of [Urban, 2011, Lemma 3.2.2 and Lemma 3.2.8]. It suffices to verify that the operator on A_λ obtained from the action of $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ is compact. Let $A_r(\mathcal{O}_\mathcal{P}, B)$ be the B-module of the functions $f : \mathcal{O}_\mathcal{P} \to B$ that are $F_\mathcal{P}$-analytic on disks of radius q^{-r}. Then $A_r(\mathcal{O}_\mathcal{P}, B) \otimes L$ is a L-Banach module and we have a natural isomorphism $A_\lambda \to A_r(\mathcal{O}_\mathcal{P}, B) \otimes L$ induced from the map $f(w, z) \mapsto f(1, z)$.

The corresponding operator on $A_r(\mathcal{O}_\mathcal{P}, B) \otimes L$ is given by $f(z) \mapsto f(\pi z)$. This operator factors through the inclusion

$$A_{r-1}(\mathcal{O}_\mathcal{P}, B) \otimes L \subset A_r(\mathcal{O}_\mathcal{P}, B) \otimes L,$$

which is compact by [Urban, 2011, Lemma 3.2.2]. □

Remark 5.1.1. If we consider $\mathfrak{U}$ to be a wide open disk in the weight space and we define the modules of distributions in the same way, we can prove an analogue of the above lemma for families of weights. This remark will be useful to deduce spectral properties for modules over wide open disks (refer to the proof of Proposition 5.2.4).

5.1.2 Filtration

We want to discuss the étale cohomology on the Shimura curve $M(H,\pi)$ whose coefficients are sheaves constructed using the modules of distributions defined in the previous section. To construct these sheaves on the étale site of the Shimura curve, the filtrations defined in this section are crucial. It is useful to remark that in the case $B = \Lambda_{\mathfrak{U}}$, these filtrations have nice properties because of our choice of $\mathfrak{U}$. This is the main reason for the choice of $\mathfrak{U}$ to be a wide open disk not an affinoid as usual in the literature.

Recall that we have fixed a set of representatives S in $\mathcal{O}_{\mathcal{P}}$ for $\mathcal{O}_{\mathcal{P}}/\pi^r\mathcal{O}_{\mathcal{P}}$.

Definition 5.1.2. Let $n \geq 0$ be an integer. We define:

$$\mathrm{Fil}^n_\lambda := \left\{\mu \in D^0_\lambda \middle| \mu(f_{z_\eta,m}) \in \underline{\mathbf{m}}^{n-m} \ \forall\, m = 0,\cdots,n-1 \text{ and } z_\eta \in S\right\}.$$

For each integer $n \in \mathbb{N}$ the set Fil^n_λ is naturally a B-module. Recall that there is a right action of the semigroup Λ_π on D^0_λ. We have

Proposition 5.1.1. *For every $n \in \mathbb{N}$, the B-module Fil^n_λ is stable under the action of Λ_π.*

Proof. First we point out that there is a decomposition of Λ_π as $\Lambda_\pi = \mathrm{N}^- T N^+$, where:

$$N^- = \left\{\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in \pi\mathcal{O}_{\mathcal{P}}\right\},$$

$$N^+ = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathcal{O}_{\mathcal{P}}\right\},$$

$$T = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a \in \mathcal{O}_{\mathcal{P}}^\times,\ d \in \mathcal{O}_{\mathcal{P}} - \{0\}\right\}.$$

Let $\mu \in \mathrm{Fil}_{\lambda}^{n}$ and $\gamma \in \Lambda_{\pi}$. For any $z_{\eta} \in S$ and $m = 0, 1, \cdots, n-1$, we will verify that $(\mu|\gamma)\left(f_{z_{\eta},m}\right) \in \underline{\mathbf{m}}^{n-m}$. We consider the following three cases for γ in each factor of the above decomposition of Λ_{π}.

Case 1: $\gamma = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in N^{-}$.

First, note that for any $(w, z) \in T_0$

$$\begin{aligned} & \frac{z}{w+cz} - z_{\eta} \\ = \quad & \frac{z - wz_{\eta} - czz_{\eta}}{w+cz} \\ = \quad & \frac{z\left(1 - cz_{\eta}\right) - wz_{\eta}}{w\left(1 + c\frac{z}{w}\right)} \\ = \quad & \frac{1 - cz_{\eta}}{1 + c\frac{z}{w}} \cdot \left(\frac{z}{w} - \frac{z_{\eta}}{1 - cz_{\eta}}\right). \end{aligned}$$

Let z'_{η} be the unique element in S satisfying

$$z'_{\eta} \equiv \frac{z_{\eta}}{1 - cz_{\eta}} \bmod \pi^{r}\mathcal{O}_{\mathcal{P}}.$$

Then we have

$$\frac{z}{w+cz} - z_{\eta} \in \pi^{r}\mathcal{O}_{\mathcal{P}} \Longleftrightarrow \frac{z}{w} - z'_{\eta} \in \pi^{r}\mathcal{O}_{\mathcal{P}}.$$

Thus

$$\mathbb{1}_{z_{\eta}+\pi^{r}\mathcal{O}_{\mathcal{P}}}\left(\frac{z}{w+cz}\right) = \mathbb{1}_{z'_{\eta}+\pi^{r}\mathcal{O}_{\mathcal{P}}}\left(\frac{z}{w}\right).$$

Then

$$\begin{aligned} & \left(\gamma \cdot f_{z_{\eta},m}\right)(w, z) \\ = \quad & f_{z_{\eta},m}\left(w + cz, z\right) \\ = \quad & \lambda(w+cz)\left(\frac{\frac{z}{w+cz} - z_{\eta}}{\pi^{r}}\right)^{m} \cdot \mathbb{1}_{z_{\eta}+\pi^{r}\mathcal{O}_{\mathcal{P}}}\left(\frac{z}{w+cz}\right) \end{aligned}$$

$$
\begin{aligned}
&= \lambda(w)\lambda\left(1+c\frac{z}{w}\right)\cdot\left(\frac{\frac{z}{w}-z'_\eta}{\pi^r}\right)^m\cdot\left(\frac{1-cz_\eta}{1+c\frac{z}{w}}\right)^m\cdot\mathbb{1}_{z'_\eta+\pi^r\mathcal{O}_\mathcal{P}}\left(\frac{z}{w}\right)\\
&= \lambda\left(1+c\frac{z}{w}\right)\cdot\left(\frac{1-cz_\eta}{1+c\frac{z}{w}}\right)^m\cdot f_{z'_\eta,m}\left(w,z\right).
\end{aligned}
$$

Let $G_m(x) := \lambda(1+cx)\cdot\left(\frac{1-cz_\eta}{1+c\frac{z}{w}}\right)^m$. Then by the hypothesis on λ, for $x\in z'_\eta+\pi^r\mathcal{O}_\mathcal{P}$ we can write

$$
G_m(x)=\sum_{i\geq 0} d_i\left(\frac{x-z'_\eta}{\pi^r}\right)^i
$$

with $d_i\in\underline{\mathbf{m}}^i$ and $\operatorname{ord}(d_i/\pi^{ri})\to\infty$ as $i\to\infty$.

Moreover, the definition of the filtrations Fil^n_λ is independent of the choice of the set of representatives of $\mathcal{O}_\mathcal{P}/\pi^r\mathcal{O}_\mathcal{P}$. Now, to prove $\left(\mu\middle|\gamma\right)\left(f_{z_\eta,m}\right)\in\underline{\mathbf{m}}^{n-m}$, it suffices to verify the following:

$$
\text{for } l\in\mathbb{N} \text{ and } d\in\underline{\mathbf{m}}^l, \text{ we have } \mu(d\cdot f_{z_\eta,m+l})\in\underline{\mathbf{m}}^{n-m}.
$$

This claim is trivial when $l\geq n-m$. Furthermore, for $l<n-m$ we have $m+l<n$, then

$$
\mu(d\cdot f_{z_\eta,m+l}) = d\cdot\mu(f_{z_\eta,m+l})\in\underline{\mathbf{m}}^{l+(n-m-l)}=\underline{\mathbf{m}}^{n-m}.
$$

Case 2: $\gamma=\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\in T$.

Then we have

$$
\begin{aligned}
&\left(\gamma\cdot f_{z_\eta,m}\right)(w,z)\\
=\ & f_{z_\eta,m}\left(aw,dz\right)\\
=\ & \lambda(aw)\left(\frac{d}{a}\frac{z}{w}-z_\eta\right)^m\mathbb{1}_{z_\eta+\pi^r\mathcal{O}_\mathcal{P}}\left(\frac{d}{a}\frac{z}{w}\right).
\end{aligned}
$$

There exists a subset $S'\subset S$ such that

$$
\left(\gamma\cdot f_{z_\eta,m}\right)(w,z)=\sum_{\zeta\in S'}\sum_{i=0}^{m} b_{\zeta,i}f_{\zeta,m-i}(w,z)
$$

for some $b_{\zeta,i} \in B$. It follows that

$$(\mu|\gamma)\left(f_{z_\eta,m}\right) \in \sum_{i=0}^{m} \underline{\mathbf{m}}^{n-m+i} \subset \underline{\mathbf{m}}^{n-m}.$$

Case 3: $\gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N^{+}$.

In this case we have

$$\begin{aligned}
& \left(\gamma \cdot f_{z_\eta,m}\right)(w,z) \\
= \quad & f_{z_\eta,m}(w, bw+z) \\
= \quad & \lambda(w)\left(\frac{\frac{bw+z}{w} - z_\eta}{\pi^r}\right)^m \mathbb{1}_{z_\eta+\pi^r\mathcal{O}_\mathcal{P}}\left(\frac{bw+z}{w}\right) \\
= \quad & \lambda(w)\left(\frac{\left(b+\frac{z}{w}\right) - z_\eta}{\pi^r}\right)^m \mathbb{1}_{z_\eta+\pi^r\mathcal{O}_\mathcal{P}}\left(b+\frac{z}{w}\right) \\
= \quad & \sum_{i=0}^{m} \binom{m}{i} (b')^i f_{z'_\eta,m-i},
\end{aligned}$$

where $b' \in \pi^r\mathcal{O}_\mathcal{P}$ and z'_η is the unique element in S such that

$$z_\eta - b = z'_\eta - b'.$$

Therefore, we deduce

$$(\mu|\gamma)\left(f_{z_\eta,m}\right) \in \sum_{i=0}^{m} \underline{\mathbf{m}}^{n-m+i} \subset \underline{\mathbf{m}}^{n-m}.$$

□

For any $n \in \mathbb{N}$, we consider the B-module

$$D^0_{\lambda,n} := D^0_\lambda / \mathrm{Fil}^n_\lambda,$$

we have

Proposition 5.1.2. *Let $D^0_{\lambda,n}$ be as above for $n \in \mathbb{N}$. Then*

(a) $D^0_{\lambda,n}$ is a finite $(B/\underline{\mathbf{m}}^n B)$-module.

(b) The natural B-linear morphism

$$D^0_\lambda \longrightarrow \varprojlim_n D^0_{\lambda,n}$$

is an isomorphism.

Proof. For (a), recall that we have a B-linear isomorphism (Lemma 5.1.1)

$$\psi : D^0_\lambda \cong \prod_{z_\eta \in S} \prod_{m \in \mathbb{N}} B,$$

given by $\mu \mapsto \big(\mu(f_{z_\eta,m})\big)_{m\in\mathbb{N}, z_\eta \in S}$.

Then by definition, the image of $\mathrm{Fil}^n_\lambda \subseteq D^0_\lambda$ under the isomorphism ψ is the B-submodule

$$\prod_{z_\eta \in S} \left[\left(\prod_{m=0}^{n-1} \underline{\mathbf{m}}^{n-m} \right) \times \left(\prod_{m \geq n} B \right) \right],$$

i.e.,

$$\mathrm{Fil}^n_\lambda \cong \prod_{z_\eta \in S} \left[\left(\prod_{m=0}^{n-1} \underline{\mathbf{m}}^{n-m} \right) \times \left(\prod_{m \geq n} B \right) \right] \tag{5.1}$$

as B-modules. This implies that the B-linear map ψ induces an isomorphism

$$D^0_{\lambda,n} \cong \prod_{z_\eta \in S} \prod_{m=0}^{n-1} B/\underline{\mathbf{m}}^{n-m} \tag{5.2}$$

of B-modules. Recall that either $B = \mathcal{O}_L$ or $B = \Lambda_{\mathfrak{U}}$, then by the choice of $\mathfrak{U}$, it follows that in both cases $B/\underline{\mathbf{m}}^{n-m}$ are finite sets. This proves the first statement.

For (b), first let us point out that from the definition of Fil^n_λ, we have

$$\bigcap_{n \in \mathbb{N}} \mathrm{Fil}^n_\lambda = \{0\}.$$

Moreover, by the formula (5.2) above, we have

$$\varprojlim_n D^0_{\lambda,n} \cong \varprojlim_n \prod_{z_\eta \in S} \prod_{m=0}^{n-1} B/\underline{\mathbf{m}}^{n-m} \cong \prod_{z_\eta \in S} \prod_{m \in \mathbb{N}} B \cong D^0_\lambda.$$

$\square$

Remark 5.1.2. As an immediate consequence of the above proposition, we see that $D^0_{\lambda,n}$ is an artinian $\mathcal{O}_L$-module for every $n \in \mathbb{N}$.

5.1.3 Specialization

Let r, $\mathfrak{U}$, L be as in Section 5.1.1. We fix π_L a uniformizor of $\mathcal{O}_L$. Recall that we denote by (B, λ) for one of the following two pairs:

- $(\Lambda_{\mathfrak{U}}, \lambda_{\mathfrak{U}})$, where $\mathfrak{U} \subset \mathcal{W}_r$ is a wide open disk with $\Lambda_{\mathfrak{U}}$ being the $\mathcal{O}_L$-algebra of bounded rigid analytic functions on $\mathfrak{U}$, $\lambda_{\mathfrak{U}}$ being the universal weight attached to $\mathfrak{U}$;
- $(\mathcal{O}_L, \lambda)$, where $\lambda \in \mathfrak{U}(L)$.

Moreover, if $B = \Lambda_{\mathfrak{U}}$ (resp. $\mathcal{O}_L$), we denote by $\underline{\mathbf{m}}_{\mathfrak{U}}$ (resp.$\underline{\mathbf{m}}_L$) its maximal ideal. There is a natural structure morphism $\mathcal{O}_L \to \Lambda_{\mathfrak{U}}$. We may also relate $\Lambda_{\mathfrak{U}}$ and $\mathcal{O}_L$ as follows. Let $\lambda \in \mathfrak{U}(L)$ and fix $\pi_\lambda \in \Lambda_{\mathfrak{U}}$ a function which vanishes with order 1 at λ and nowhere else on $\mathfrak{U}$. Such function is called a *uniformizer* at λ. Then

$$(\pi_L, \pi_\lambda) = \underline{\mathbf{m}}_{\mathfrak{U}}.$$

We have an exact sequence

$$0 \longrightarrow \Lambda_{\mathfrak{U}} \xrightarrow{\cdot \pi_\lambda} \Lambda_{\mathfrak{U}} \xrightarrow{\rho_\lambda} \mathcal{O}_L \longrightarrow 0,$$

i.e., $\mathcal{O}_L \cong \Lambda_{\mathfrak{U}}/\pi_\lambda \Lambda_{\mathfrak{U}}$ and $\rho_\lambda(\underline{\mathbf{m}}_{\mathfrak{U}}) = \underline{\mathbf{m}}_L$. Furthermore, for $\lambda \in \mathfrak{U}(L)$, the distributions (for (B, λ) is either $(\Lambda_{\mathfrak{U}}, \lambda_{\mathfrak{U}})$ or $(\mathcal{O}_L, \lambda)$) defined in Section 5.1.1 can also be related by such a uniformizor at λ. We introduce some notations we will use in this section. If $B = \Lambda_{\mathfrak{U}}$ and $\lambda = \lambda_{\mathfrak{U}}$, we set

$$A^0_{\mathfrak{U}} := A^0_{\lambda_{\mathfrak{U}}} \quad , \quad A_{\mathfrak{U}} := A_{\lambda_{\mathfrak{U}}} \quad ,$$

$$D^0_{\mathfrak{U}} := D^0_{\lambda_{\mathfrak{U}}} \quad , \quad D_{\mathfrak{U}} := D_{\lambda_{\mathfrak{U}}},$$

$$\mathrm{Fil}^n_{\mathfrak{U}} := \mathrm{Fil}^n_{\lambda_{\mathfrak{U}}} \quad , \quad D^0_{\mathfrak{U},n} := D^0_{\mathfrak{U}}/\mathrm{Fil}^n_{\mathfrak{U}}.$$

If $B = \mathcal{O}_L$ and $\lambda \in \mathfrak{U}(L)$, we still use the notations as in Section 5.1.1, i.e., A^0_λ, A_λ, D^0_λ, D_λ, Fil^n_λ and $D^0_{\lambda,n}$, respectively.

Now the two $\mathcal{O}_L$-modules of distributions $D^0_{\mathfrak{U}}$ and D^0_λ are related as follows. First we have a natural map

$$\begin{aligned} A^0_\lambda &\longrightarrow A^0_{\mathfrak{U}} \\ f &\longmapsto f_{\mathfrak{U}} \end{aligned}$$

where $f_{\mathfrak{U}}$ is defined by

$$f_{\mathfrak{U}}(w, z) = \lambda_{\mathfrak{U}}(w) f(1, \frac{z}{w}).$$

This induces the so called *specialization* map

$$\begin{aligned} \eta_\lambda : D^0_{\mathfrak{U}} &\longrightarrow D^0_\lambda \\ \mu &\longmapsto \mu_\lambda \end{aligned}$$

defined by

$$\mu_\lambda(f) = \mu(f_{\mathfrak{U}})(\lambda)$$

for any $f \in A_\lambda^0$.

We have the following properties for such specialization maps.

Proposition 5.1.3. *Let $\mathfrak{U} \subset \mathcal{W}_r$ be a wide open disk and $\lambda \in \mathfrak{U}(L)$. Let $\pi_\lambda \in \Lambda_{\mathfrak{U}}$ be a fixed uniformizor at λ.*

(a) We have an exact sequence of Λ_π-modules

$$0 \longrightarrow D_{\mathfrak{U}}^0 \xrightarrow{\cdot \pi_\lambda} D_{\mathfrak{U}}^0 \xrightarrow{\eta_\lambda} D_\lambda^0 \longrightarrow 0.$$

(b) $\eta_\lambda\left(\mathrm{Fil}_{\mathfrak{U}}^n\right) = \mathrm{Fil}_\lambda^n$.

Proof. The proof of part (a) is the same as in [Andreatta et al., 2015b, Proposition 3.11].

For part (b), by Lemma 5.1.1 and equation (5.1) we have the following commutative diagrams.

$$\begin{tikzcd}
D_{\mathfrak{U}}^0 \arrow[r, "\sim"] & \prod_{z_\eta \in S} \prod_{m \geq 0} \Lambda_{\mathfrak{U}} \\
\mathrm{Fil}_{\mathfrak{U}}^n \arrow[u, hook] \arrow[r, "\sim"] & \prod_{z_\eta \in S} \left[\left(\prod_{m=0}^{n-1} \underline{\mathbf{m}}_{\mathfrak{U}}^{n-m}\right) \times \left(\prod_{m \geq n} \Lambda_{\mathfrak{U}}\right)\right] \arrow[u, hook]
\end{tikzcd},$$

and

$$\begin{tikzcd}
D_\lambda^0 \arrow[r, "\sim"] & \prod_{z_\eta \in S} \prod_{m \geq 0} \mathcal{O}_L \\
\mathrm{Fil}_\lambda^n \arrow[u, hook] \arrow[r, "\sim"] & \prod_{z_\eta \in S} \left[\left(\prod_{m=0}^{n-1} \underline{\mathbf{m}}_L^{n-m}\right) \times \left(\prod_{m \geq n} \mathcal{O}_L\right)\right] \arrow[u, hook]
\end{tikzcd}.$$

Moreover the diagram

$$\begin{tikzcd}
D_{\mathfrak{U}}^0 \arrow[r, "\sim"] \arrow[d, "\eta_\lambda"'] & \prod_{z_\eta \in S} \prod_{m \geq 0} \Lambda_{\mathfrak{U}} \arrow[d] \\
D_\lambda^0 \arrow[r, "\sim"] & \prod_{z_\eta \in S} \prod_{m \geq 0} \mathcal{O}_L
\end{tikzcd}$$

is also commutative since $\rho_\lambda\left(\underline{\mathbf{m}}_{\mathfrak{U}}^n\right) = \underline{\mathbf{m}}_L^n$ for any $n \in \mathbb{N}$, where the right vertical map is the product of ρ_λ. Thus we obtain our second statement. $\square$

Now suppose that there exists an integer $k \in \mathbb{N}$ such that $\lambda(t) = t^k$ for each $t \in \mathcal{O}_{\mathcal{P}}$. In this case we say that λ is a *classical weight* and k is said to be *attached* to λ. Let

$$P_\lambda^0 = \left\{ \sum_{m=0}^{k} a_m w^{k-m} z^m \mid a_m \in \mathcal{O}_L \right\} \subset A_\lambda^0$$

be the subset of functions $f : \mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_L$ consisting of homogeneous polynomials of degree k. Moreover, we can verify that this $\mathcal{O}_L$-submodule is invariant under the action of Λ_π. Considering continuous duals we obtain a surjective and Λ_π-equivariant morphism:

$$\varrho_\lambda : D_\lambda^0 \to V_\lambda^0 := \left(P_\lambda^0\right)^\vee = \mathrm{Hom}_{\mathcal{O}_L}\left(P_\lambda^0, \mathcal{O}_L\right).$$

Remark 5.1.3. Let $T := (\mathcal{O}_{\mathcal{P}})^2$. We may identify V_λ^0 with $\mathrm{Sym}^k(T) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}_L$, compatible with the natural right action of Λ_π. The map ϱ_λ induces a filtration on V_λ^0 by

$$\mathrm{Fil}^n\left(V_\lambda^0\right) := \varrho_\lambda\left(\mathrm{Fil}_\lambda^n\right).$$

We denote by $P_\lambda := P_\lambda^0 \bigotimes_{\mathcal{O}_L} L$ and $V_\lambda := V_\lambda^0 \bigotimes_{\mathcal{O}_L} L$. If $\mathfrak{U}$ contains a classical weight λ, we have the following Λ_π-equivariant maps

$$D_{\mathfrak{U}} \xrightarrow{\eta_\lambda} D_\lambda \xrightarrow{\varrho_\lambda} V_\lambda$$

which are compatible with the filtrations.

5.2 Overconvergent cohomology

5.2.1 Definitions

In this section we will work on the Shimura curve $M(H, \pi)$. Our main goal is to relate the group cohomology of its fundamental groups with coefficients in certain modules to its étale cohomology with coefficients in certain sheaves coming from those modules. Recall that the curve $M(H,\pi)$ is not necessarily connected. Moreover, this curve has a canonical model defined on the number field E (Section 3.1.3.1) and we have:

$$M(H,\pi)(\mathbb{C}) = \bigsqcup_{\mathbf{x}\in \mathrm{CL}_E^+} M(H,\pi)_{\mathbf{x}}(\mathbb{C}) \cong \bigsqcup_{\mathbf{x}\in \mathrm{CL}_E^+} \Gamma_{\mathbf{x}} \setminus \mathfrak{H}^+,$$

where the arithmetic group $\Gamma_{\mathbf{x}}$ is defined by $\Gamma_{\mathbf{x}} = g_{\mathbf{x}} K(H,\pi) g_{\mathbf{x}}^{-1} K_\infty \cap G(\mathbb{Q}) \subset G(\mathbb{Q})$, where $\{g_{\mathbf{x}}\}_{\mathbf{x}\in \mathrm{CL}_E^+}$ is a family in $G(\mathbb{A}_f)$ such that each $g_{\mathbf{x}}$ is trivial at p; and $\{\det(g_{\mathbf{x}})\}_{\mathbf{x}\in \mathrm{CL}_E^+}$ is a set of representatives of $\mathbb{A}_E^* / \det(K(H,\pi))\det(K_\infty)$, where K_∞ is certain compact subgroup of $G(\mathbb{R})$.

Let $\mathbf{x} \in \mathrm{CL}_E^+$. We fix a geometric generic point $\eta_{\mathbf{x}} = \mathrm{Spec}(\mathbb{K}_{\mathbf{x}})$ of the corresponding connected component $M(H,\pi)_{\mathbf{x}}/L$ of $M(H,\pi)$. We denote by $\mathcal{G}_{\mathbf{x}}$ the *geometric étale fundamental group* attached to $M(H,\pi)_{\mathbf{x}}$ and $\eta_{\mathbf{x}}$. Let $\mathcal{C} \to M(H,\pi)$ be the level π-subgroup of the universal object $A \to M(H,\pi)$. We denote by $T_{\mathbf{x}} := \varprojlim_n (\mathcal{A}[\pi^n]_1^{2,1})^\vee_{L,\eta_{\mathbf{x}}}$ and let $p_{\mathbf{x}} : T_{\mathbf{x}} \to (\mathcal{C}_1^{2,1})^\vee_{L,\eta_{\mathbf{x}}}$ be the map obtained from the natural morphism $(\mathcal{A}[\pi]_1^{2,1})^\vee \to (\mathcal{C}_1^{2,1})^\vee$. The Tate module $T_{\mathbf{x}}$ is a free $\mathcal{O}_{\mathcal{P}}$-module of rank 2 with continuous action of $\mathcal{G}_{\mathbf{x}}$. Choose a $\mathcal{O}_{\mathcal{P}}$-basis $\{\epsilon_1, \epsilon_2\}$ of $T_{\mathbf{x}}$, such that $p_{\mathbf{x}}(\epsilon_1) = 0$. Let

$$(T_0)_{\mathbf{x}} := \left\{ w\epsilon_1 + z\epsilon_2 \;\middle|\; w \in \mathcal{O}_{\mathcal{P}}^\times, z \in \mathcal{O}_{\mathcal{P}} \right\}.$$

Then $(T_0)_{\mathbf{x}}$ is a compact subset of $T_{\mathbf{x}}$ and can be identified with $\mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}}$. Since the group $\mathcal{G}_{\mathbf{x}}$ acts on $T_{\mathbf{x}}$ and preserves $(T_0)_{\mathbf{x}}$, using this fixed basis, we obtain a homomorphism of groups $\mathcal{G}_{\mathbf{x}} \to \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}})$. Now the special choice of the basis implies that the image of this homomorphism is inside the Iwahori group $I_\pi \subset \mathrm{GL}_2(\mathcal{O}_{\mathcal{P}})$.

Here we will use the same notations as in Section 5.1.1. We denote by (B,λ) be either the pair $(\Lambda_{\mathfrak{U}}, \lambda_{\mathfrak{U}})$ or $(\mathcal{O}_L, \lambda)$ with maximal ideal $\underline{\mathbf{m}}$. Let A^0_λ and D^0_λ be the corresponding B-modules of locally analytic functions and distributions for both cases.

We donote

$$A^0_{\lambda,n} := A^0_\lambda / \underline{\mathbf{m}}^n A^0_\lambda$$

and

$$D^0_{\lambda,n} = D^0_\lambda / \mathrm{Fil}^n_\lambda$$

From Proposition 5.1.3 it follows that these B-modules are in fact finite sets with an action of Λ_π. Then using the above discussion about the action of $\mathcal{G}_{\mathbf{x}}$ on the Tate module, we obtain sheaves on the étale site of $M(H,\pi)_{\mathbf{x}}$, which are denoted by $\mathcal{A}^0_{\lambda,n,\mathbf{x}}, \mathcal{D}^0_{\lambda,n,\mathbf{x}} \in \mathrm{Sh}((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\overline{L}})$, respectively. Putting these sheaves together we obtain sheaves in the entire Shimura curve:

$$\mathcal{A}^0_{\lambda,n} \ , \ \mathcal{D}^0_{\lambda,n} \in \mathrm{Sh}(M(H,\pi)^{\mathtt{et}}_{\overline{L}})$$

$$\mathcal{A}^0_\lambda := (\mathcal{A}^0_{\lambda,n})_{n\in\mathbb{N}} \ , \ \mathcal{D}^0_\lambda := (\mathcal{D}^0_{\lambda,n})_{n\in\mathbb{N}} \in \mathrm{Sh}(M(H,\pi)^{\mathtt{et}}_{\overline{L}})^{\mathbb{N}} \text{ and}$$

$$\mathcal{A}_\lambda \ , \ \mathcal{D}_\lambda \in \mathrm{Ind}-\mathrm{Sh}(M(H,\pi)^{\mathtt{et}}_{\overline{L}})^{\mathbb{N}}.$$

Recall that in this chapter we are going to relate two different kinds of π-adic objects both of which are helpful to construct eigenvarieties. In Section 3.4.3 we introduced the overconvergent modular forms constructed by Kasseai and Brasca. Now we are in position to define the other p-adic object relevant in this chapter:

Definition 5.2.1. The space of *Overconvergent Cohomology* is defined to be the B-module $\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_\lambda)$.

The space of overconvergent cohomology is a B-module endowed with an action of $G_L := \mathrm{Gal}(\overline{L}/L)$, since the curve $M(H,\pi)$ is defined over L.

We can define Hecke operators acting on the overconvergent cohomology. Similarly as in Section 3.5, we define the U-operator:

$$\mathrm{U} : \mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda) \to \mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda),$$

and the operator:

$$T_{\mathfrak{L}} : \mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda) \to \mathrm{H}^1(M(H,\pi)^{\mathsf{et}}_{\overline{L}}, \mathcal{D}_\lambda).$$

5.2.2 Group cohomology

Recall that we have

$$M(H,\pi)(\mathbb{C}) \cong \bigsqcup_{\mathbf{x}\in \mathrm{CL}_E^+} \Gamma_{\mathbf{x}} \setminus \mathfrak{H}^+,$$

where $\Gamma_{\mathbf{x}}$ is a certain torsion free arithmetic subgroup of $G(\mathbb{Q})$. Each variety $\Gamma_{\mathbf{x}} \setminus \mathfrak{H}^+$ is compact. Moreover, note that the image of each group $\Gamma_{\mathbf{x}}$ in $G(F_{\mathfrak{P}})$ is contained in Λ_π, thus the spaces of functions and distributions defined in Section 5.1.1 can be regarded as $\Gamma_{\mathbf{x}}$-modules.

Proposition 5.2.1. *We have the following isomorphism of B-modules:*

$$\mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_\lambda) \cong \varprojlim_n \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \cong \mathrm{H}^1_{\mathrm{cont}}(\Gamma_{\mathbf{x}}, (D^0_{\lambda,n})_{n\in\mathbb{N}}).$$

Proof. We first prove the second isomorphism. Recall that as in [Andreatta et al., 2015b, §3.2] we have the following exact sequence of R-modules:

$$0 \to \lim{}^{(1)}\, \mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \longrightarrow \mathrm{H}^1_{\mathrm{cont}}(\Gamma_{\mathbf{x}}, (D^0_{\lambda,n})_{n\in\mathbb{N}}) \longrightarrow \varprojlim_n \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \to 0,$$

where $\lim^{(1)}$ is the first right derived functor of the inverse limit functor (refer to [Weibel, 1994, §3.5]). From Proposition 5.1.3 we know that each $R[\Gamma_{\mathbf{x}}]$-module $D^0_{\lambda,n}$ is a finite set, thus it follows that the projective system $(\mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}))_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition (Remark 5.2.1) and then $\lim^{(1)} \mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) = 0$. This shows the second isomorphism.

For the first isomorphism we follow the arguments of [Andreatta et al., 2015b, lemma 3.13]. Let $B^1(\Gamma_{\mathbf{x}}, D^0_\lambda)$ be the B-module of 1-coboundaries and $Z^1(\Gamma_{\mathbf{x}}, D^0_\lambda)$ be the B-module of the 1-cocycles with coefficients in D^0_λ; in the same way we define $B^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n})$ and $Z^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n})$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_\lambda) \arrow[r] \arrow[d, "\beta"] & B^1(\Gamma_{\mathbf{x}}, D^0_\lambda) \arrow[r, "d"] \arrow[d, "\delta"] & Z^1(\Gamma_{\mathbf{x}}, D^0_\lambda) \arrow[r] \arrow[d, "\theta"] & \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_\lambda) \arrow[r] \arrow[d, "\kappa"] & 0 \\
\varprojlim_n \mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \arrow[r] & \varprojlim_n B^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \arrow[r, "\varprojlim_n d_n"] & \varprojlim_n Z^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}) \arrow[r, "\alpha"] & \varprojlim_n \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}).
\end{tikzcd}$$

Now using Proposition 5.1.3, it follows that the projective system

$$(B^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n})/\, \mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}))_{n\in\mathbb{N}}$$

satisfies the Mittag-Leffler condition. Then the exact sequence

$$0 \longrightarrow \mathrm{H}^0(\Gamma_{\mathbf{x}}, D^0_{\lambda,M}) \longrightarrow B^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,M}) \xrightarrow{d_M} Z^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,M}) \xrightarrow{\alpha} \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,M}) \longrightarrow 0$$

implies that α is surjective. We then prove that β, δ and θ are isomorphisms. To do this, we basically apply Proposition 5.1.3 and the fact that $\Gamma_{\mathbf{x}}$ is a finitely generated group. The proof of the first isomorphism is then completed applying Five Lemma. $\square$

Remark 5.2.1. A tower $\{A_i\}$ of abelian groups satisfies the Mittag-Leffler condition if for each k there exists a $j \geq k$ such that the image of $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$.

On the group cohomology we can define the action of the Hecke operators in the classical way (refer to [Urban, 2011, §4.2]) by the following recipe. Let $\gamma \in \Lambda_\pi$. We define $[\Gamma_{\mathbf{x}}\gamma\Gamma_{\mathbf{x}}] : \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda) \to \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda)$ by the formula:

$$[\Gamma_{\mathbf{x}}\gamma\Gamma_{\mathbf{x}}] := \mathrm{Cor}_{\Gamma_{\mathbf{x}}}^{\Gamma_{\mathbf{x}}\cap\gamma\Gamma_{\mathbf{x}}\gamma^{-1}} \circ [\gamma] \circ \mathrm{Res}^{\Gamma}_{\Gamma_{\mathbf{x}}\cap\gamma^{-1}\Gamma_{\mathbf{x}}\gamma},$$

where $\mathrm{Cor}_{\Gamma_{\mathbf{x}}}^{\Gamma_{\mathbf{x}}\cap\gamma\Gamma_{\mathbf{x}}\gamma^{-1}}$ and $\mathrm{Res}^{\Gamma}_{\Gamma_{\mathbf{x}}\cap\gamma^{-1}\Gamma_{\mathbf{x}}\gamma}$ are the corestriction map and the restriction map, respectively; and

$$[\gamma] : \mathrm{H}^1(\Gamma_{\mathbf{x}} \cap \gamma^{-1}\Gamma_{\mathbf{x}}\gamma, D_\lambda) \to \mathrm{H}^1(\Gamma_{\mathbf{x}} \cap \gamma\Gamma_{\mathbf{x}}\gamma^{-1}, D_\lambda)$$

is the map given by the action of γ on D_λ.

Now considering $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$, we obtain the operator U. Moreover, using the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{L}} \end{pmatrix}$ we obtain the operator $T_{\mathfrak{L}}$ (where $\pi_{\mathfrak{L}}$ is as in Section 3.5.2).

As a corollary of Proposition 5.2.1, we can interpret the overconvergent cohomology in terms of group cohomology:

Corollary 5.2.1. *We have an isomorphism of B-modules:*

$$\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_\lambda) \cong \bigoplus_{\mathbf{x}\in \mathrm{CL}_E^+} \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda),$$

which are compatible with the action of Hecke operators

Proof. By construction we have

$$\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_\lambda) = \bigoplus_{\mathbf{x}\in \mathrm{CL}_E^+} \mathrm{H}^1((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda,\mathbf{x}}),$$

where $\mathcal{D}_{\lambda,\mathbf{x}} \in \mathrm{Ind}-\mathrm{Sh}((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\overline{L}})^{\mathbb{N}}$ are defined using the sheaves $\mathcal{D}^0_{\lambda,M,\mathbf{x}}$.

For any embedding of $\overline{L}$ in $\mathbb{C}$, the curve $(M(H,\pi)_{\mathbf{x}})_{\mathbb{C}}$ has fundamental group $\Gamma_{\mathbf{x}}$, then by the fact that the curve $(M(H,\pi)_{\mathbf{x}})_{\mathbb{C}}$ is $K(\pi,1)$, we obtain

$$\mathrm{H}^1((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\overline{L}}, \mathcal{D}^0_{\lambda,n,\mathbf{x}}) \cong \mathrm{H}^1((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\mathbb{C}}, \mathcal{D}^0_{\lambda,n,\mathbf{x}}) \cong \mathrm{H}^1(\Gamma_{\mathbf{x}}, D^0_{\lambda,n}).$$

Now it follows from Proposition 5.2.1 that:

$$\mathrm{H}^1((M(H,\pi)_{\mathbf{x}})^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda,\mathbf{x}}) \cong \mathrm{H}^1(\Gamma_{\mathbf{x}}, D_{\lambda}).$$

□

Remark 5.2.2. This corollary is useful in the following sense. On one hand, we will deduce spectral properties for the étale cohomology from those proved for the group cohomology. On the other hand, we can obtain a Galois action on $\mathrm{H}^1(\Gamma_{\mathbf{x}}, D_{\lambda})$ from the above identification.

5.2.3 Slope decomposition

Using a well known construction, we deduce good spectral properties for the group cohomology and then the overconvergent cohomology by using corollary 5.2.1. Before stating the results we recall a classical construction of complexes.

Fix $\mathbf{x} \in \mathrm{CL}^+_E$. Recall that $\Gamma_{\mathbf{x}} \setminus \mathfrak{H}^+$ is a compact variety which is smooth and C^∞. By Munkres [1967] there exists a finite triangulation of it. We fix one of those triangulations, then using the natural projection $\mathfrak{H}^+ \to \Gamma_{\mathbf{x}} \setminus \mathfrak{H}^+$ we obtain a triangulation of $\mathfrak{H}^+$. We denote by $\triangle_t$ the set of simplexes of degree $t \in \mathbb{N}$ of this triangulation, it follows that the action of $\Gamma_{\mathbf{x}}$ on $\triangle_t$ has a finite number of orbits, each of which is bijective with $\Gamma_{\mathbf{x}}$ (since the group $\Gamma_{\mathbf{x}}$ is torsion free). Let $C_t(\Gamma_{\mathbf{x}}) := \mathbb{Z}[\triangle_t]$ be the free $\mathbb{Z}$-module generated by $\triangle_t$. Then $C_t(\Gamma_{\mathbf{x}})$ is a free $\mathbb{Z}[\Gamma_{\mathbf{x}}]$-module of finite rank. Applying the standard boundary operators we obtain the following exact sequence of $\mathbb{Z}[\Gamma_{\mathbf{x}}]$-modules:

$$0 \to C_2(\Gamma_{\mathbf{x}}) \to C_1(\Gamma_{\mathbf{x}}) \to C_0(\Gamma_{\mathbf{x}}) \to \mathbb{Z} \to 0.$$

Now, let M be a module endowed with a left $\Gamma_{\mathbf{x}}$-action. We define the complex $C^\bullet(\Gamma_{\mathbf{x}}, M)$ by:

$$C^t(\Gamma_{\mathbf{x}}, M) := \mathrm{Hom}_\Gamma(C_t(\Gamma), M).$$

This complex satisfies the following properties:

(I) The cohomology of $C^\bullet(\Gamma_{\mathbf{x}}, M)$ computes the cohomology of $\Gamma_{\mathbf{x}}$ i.e. the groups $\mathrm{H}^\bullet(\Gamma_{\mathbf{x}}, M)$;

(II) $C^t(\Gamma_{\mathbf{x}}, M)$ is isomorphic to M^{r_t}, where r_t is the number of orbits of the action of $\Gamma_{\mathbf{x}}$ on $\triangle_t$.

Now, suppose that M admits an action of Λ_π. Following the construction in [Urban, 2011, §4.2.5 and §4.2.6] we obtain Hecke operators on the complex which are liftings of the Hecke operators defined on the group cohomology:

$$\mathrm{U} : C^t(\Gamma_{\mathbf{x}}, M) \to C^t(\Gamma_{\mathbf{x}}, M),$$

$$\mathrm{T}_{\mathfrak{L}} : C^t(\Gamma_{\mathbf{x}}, M) \to C^t(\Gamma_{\mathbf{x}}, M).$$

Proposition 5.2.2. *Let $\lambda \in \mathcal{W}(L)$. The L-vector space $\mathrm{H}^1\left(M(H,\pi)^{et}_{\overline{L}}, \mathcal{D}_\lambda\right)$ admits a $\leq h$-decomposition with respect to the operator* U.

Proof. Using corollary 5.2.1 it suffices to prove that we have $\leq h$-decomposition with respect to the operator U_π on each space $\mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda)$. From general results about slope decompositions in [Urban, 2011, §2], it follows that it suffices to prove the slope decomposition for the action on every term of the complex $C^t(\Gamma_{\mathbf{x}}, D_\lambda)$. This is a consequence of the fact that $C^t(\Gamma_{\mathbf{x}}, D_\lambda)$ is isomorphic to a finite number of copies of D_λ, together with applying Lemma 5.1.3 and [Urban, 2011, Theorem 2.3.8]. $\square$

Now suppose that λ is a classical weight with $k \in \mathbb{N}$ being the integer attached to it. Thus, we have a Λ_π-equivariant surjective map of L-vector spaces $D_\lambda \to V_\lambda$. Applying the construction described in Section 5.2.1 to V_λ we obtain a sheaf:

$$\mathcal{V}_\lambda \in \mathrm{Ind} - \mathrm{Sh}(M(H,\pi)^{\mathtt{et}}_{\overline{L}})^{\mathbb{N}}.$$

From the functoriality of the construction, we obtain a Hecke and Galois equivariant morphism of L-vector spaces:

$$\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_\lambda) \longrightarrow \mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_\lambda).$$

Proposition 5.2.3. *Let $h \in \mathbb{Q}$ be such that $h < k+1$. If we consider the slope decomposition with respect to U_π, the above morphism induces an isomorphism of vector spaces compatible with the action of the Hecke operators and* $\mathrm{Gal}(\overline{L}/L)$*:*

$$\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_\lambda)^{\leq h} \cong \mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_\lambda)^{\leq h}.$$

Proof. We prove the proposition component by component. From Corollary 5.2.1, it suffices to show that, for each $\mathbf{x} \in \mathrm{CL}_E^+$, we have:

$$\mathrm{H}^1(\Gamma_{\mathbf{x}}, D_\lambda)^{\leq h} \cong \mathrm{H}^1(\Gamma_{\mathbf{x}}, V_\lambda)^{\leq h}.$$

To prove this statement we follow the classical arguments in Pollack and Stevens [2013].

For any $i \in \mathbb{Z}$ we denote by $A_i(L)$ the $L[\Lambda_\pi]$-module of $F_{\mathcal{P}}$-analytic functions $f : \mathcal{O}_{\mathcal{P}} \to L$, i.e., there exists a sequence $\{c_m\}_{m\in\mathbb{N}}$ in L satisfying $c_m \to 0$ as $m \to \infty$, such that we have

$f(z)=\sum_{m\in\mathbb{N}} c_m z^m$ for any $z\in\mathcal{O}_{\mathcal{P}}$. The action of Λ_π is given by the following rule:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot f\right)(z)=(a+cz)^i f(\frac{b+zd}{a+zc}).$$

The assignation $f(w,z)\to f(1,z)$ induces an isomorphism of $L[\Lambda_\pi]$-modules $A_\lambda\cong A_k(L)$. Note that here we consider $r=0$ in the definition of A_λ.

The operator $(d/dz)^{k+1}$ induces a morphism of $L[\Lambda_\pi]$-modules

$$A_k(L)\to A_{-2-k}(L)(k+1),$$

where the notation $(k+1)$ refers to the action of Λ_π twisted by the $(k+1)^{\text{st}}$ power of the determinant. It follows from the definition that the kernel of this morphism is V_λ under the identification $A_\lambda\cong A_k(L)$. Dualizing this morphism and considering the identification, we obtain the following exact sequence of $L[\Lambda_\pi]$-modules:

$$0\to D_{-2-k}(L)(k+1)\to D_\lambda\to V_\lambda\to 0.$$

Noticing the long exact sequence attached to this exact sequence and taking slope decomposition with respect to U, we obtain an exact sequence:

$$\mathrm{H}^1(\Gamma_{\mathbf{x}},D_{-2-k}(L))^{\leq h-k-1}\to \mathrm{H}^1(\Gamma_{\mathbf{x}},D_\lambda)^{\leq h}\cong \mathrm{H}^1(\Gamma_{\mathbf{x}},V_\lambda)^{\leq h}\to \mathrm{H}^2(\Gamma_{\mathbf{x}},D_{-2-k}(L))^{\leq h-k-1}.$$

Note that we have the number $h-k-1$ for the cohomology of $D_{-2-k}(L)$ from the twist in the map $(d/dz)^{k+1}$. Finally notice that $D_{-2-k}(L)$ has a natural $\mathcal{O}_{\mathcal{P}}$-lattice stable under the action of Γ_π, thus the condition $h<k-1$ implies:

$$\mathrm{H}^2(\Gamma_{\mathbf{x}},D_{-2-k}(L))^{\leq h-k-1}=\mathrm{H}^1(\Gamma_{\mathbf{x}},D_{-2-k}(L))^{\leq h-k-1}=\{0\}.$$

$\square$

Now, we have a result analogous to Proposition 5.2.3 for any degree of the cohomology following a similar proof.

Corollary 5.2.2. *Let $h \in \mathbb{Q}$ be such that $h < k+1$. For any $i \in \mathbb{N}$ and any $\mathbf{x} \in \mathrm{CL}_E^+$, we have an isomorphism of vector spaces compatible with the action of the Hecke operators and* $\mathrm{Gal}(\overline{L}/L)$*:*

$$\mathrm{H}^i(\Gamma_{\mathbf{x}}, D_\lambda)^{\leq h} \cong \mathrm{H}^i(\Gamma_{\mathbf{x}}, V_\lambda)^{\leq h}.$$

Now we deal with the question about the spectral properties of the modules for families of weights. Following the proofs of Lemma 3.5 and Corollary 3.6 in Andreatta et al. [2015b] we obtain:

Lemma 5.2.1. *Let $\{\mu_j\}_{j\in J}$ be a family of elements in $D_{\mathfrak{U}}^0$ such that its image in $D_{\mathfrak{U}}^0/\underline{\mathbf{m}}D_{\mathfrak{U}}^0$ is a basis of this $\mathbb{L} := \Lambda_{\mathfrak{U}}/\underline{\mathbf{m}}\Lambda_{\mathfrak{U}} \cong \mathcal{O}_L/\pi_L\mathcal{O}_L$-vector space. Then for each $m > 0$ the natural morphism*

$$\oplus_{j\in J}\left(\Lambda_{\mathfrak{U}}/\underline{\mathbf{m}}^m\Lambda_{\mathfrak{U}}\right)\mu_j \longrightarrow D_{\mathfrak{U}}^0/\underline{\mathbf{m}}^m D_{\mathfrak{U}}^0$$

is an isomorphism of $\Lambda_{\mathfrak{U}}$-modules. Moreover, for each $\mu \in D_{\mathfrak{U}}^0$ there exists a unique family $\{a_j\}_{j\in J}$ in $\Lambda_{\mathfrak{U}}$ such that

(i) $a_j \to 0$ in the filter of complements of finite sets in J, in the weak topology, and

(ii) $\mu = \sum_{j\in J} a_j\mu_j$.

Proposition 5.2.4. *For each weight $\lambda \in \mathcal{W}(L)$, there exists a wide open disk $\mathfrak{U} \subset \mathcal{W}$ defined over L and containing λ such that the $\Lambda_{\mathfrak{U}} \otimes_{\mathcal{O}_L} L$-module $\mathrm{H}^1(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}})$ admits a $\leq h$-decomposition with respect to the operator U_π.*

Proof. By the same argument in Proposition 5.2.2, it suffices to prove that we have $\leq h$-decomposition with respect to the operator U_π for the complex $C^\bullet(\Gamma_j, \mathcal{D}_{\mathfrak{U}})$. Since we work with wide open disks, this proposition is not a direct consequence of the theory developed in Ash and Stevens [2008], however, the same proof of [Andreatta et al., 2015b, Theorem. 3.17] can be adapted to our context, combining Lemma 5.2.1 above, our complexes $C^\bullet(\Gamma_j, M)$, remark after Lemma 5.1.3 and the theory described in Ash and Stevens [2008]. □

5.2.4 Sheaves on Faltings' Site

In Section 5.2.2 we described the overconvergent cohomology in terms of group cohomology. Let λ be either $\lambda_{\mathfrak{U}}$ or $\lambda \in \mathfrak{U}(L)$ for some wide open disk in $\mathcal{W}_r$. In this section, we will explain how to regard the étale sheaves $\mathcal{A}^0_\lambda$, $\mathcal{D}^0_\lambda$ (respectively $\mathcal{A}_\lambda$, $\mathcal{D}_\lambda$ and $\mathcal{V}_\lambda$ if λ is an integer) on $M(H,\pi)^{\mathtt{et}}_{\overline{L}}$ as continuous sheaves(respectively, ind-continuous sheaves) on Faltings' site $\mathfrak{M}(H,\pi)$ associated to the pair $(\mathcal{M}(H,\pi), M(H,\pi))$. This identification will be useful to compare the sheaves defining overconvergent modular forms with those used to define the overconvergent cohomology.

Recall that we have a functor (Section 4.4)

$$\begin{aligned} u : \mathfrak{M}(H,\pi) &\longrightarrow M(H,\pi)^{\mathtt{et}}_{\overline{L}} \\ (\mathcal{U}, W) &\longmapsto W. \end{aligned}$$

This functor u sends the final object to the final object, commutes with fiber products and sends covering families to covering families. Hence it defines a morphisms of topoi. In particular, we have

$$u_* : \mathrm{Sh}(M(H,\pi)^{\mathtt{et}}_{\overline{L}}) \longrightarrow \mathrm{Sh}(\mathfrak{M}(H,\pi)),$$

which extends to inductive systems of continuous sheaves. Using this functor we obtain continuous sheaves $u_*\left(\mathcal{A}_\lambda^0\right)$, $u_*\left(\mathcal{D}_\lambda^0\right)$ and ind-continuous sheaves $u_*\left(\mathcal{A}_\lambda\right)$, $u_*\left(\mathcal{D}_\lambda\right)$ and $u_*\left(\mathcal{V}_\lambda\right)$ on $\mathfrak{M}(H,\pi)$. For abuse of notations we omit u_*. Moreover, we can define Hecke operators on $\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right)$ (Section 6.4):

$$\mathrm{U}:\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right)\longrightarrow\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right),$$

$$\mathrm{T}_{\mathcal{L}}:\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right)\longrightarrow\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right).$$

Using the same argument as in [Andreatta et al., 2015b, Proposition 3.19] and [Faltings, 2002b, Theorem 9], we obtain the following proposition:

Proposition 5.2.5. *The natural morphism*

$$\mathrm{H}^1\left(\mathfrak{M}(H,\pi),\mathcal{D}_\lambda\right)\longrightarrow\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et},\mathcal{D}_\lambda\right)$$

is an isomorphism compatible with the action of Hecke operators and the Galois group $G_L:=\mathrm{Gal}(\overline{L}/L)$. *Moreover, it is also compatible with specializations.*

Recall that we have a natural continuous functor

$$\nu:\mathfrak{M}(H,\pi)\longrightarrow\mathfrak{M}(w)$$

induced by the natural morphism of log formal schemes $\mathcal{M}(w)\to\mathcal{M}(H,\pi)$ (Section 4.4). This fact allows us to obtain ind-continuous sheaves $\nu^*\left(\mathcal{A}_\lambda\right)$ and $\nu^*\left(\mathcal{D}_\lambda\right)$ on $\mathfrak{M}(w)$.

Chapter 6

The Morphism

We fix an $r \in \mathbb{N}$ and let L be a finite field extension over $F_{\mathcal{P}}$ containing an element $\zeta_r \in \mathbb{C}_p := \widehat{\overline{L}}$, where $\{\zeta_n\}_{n\geq 1}$ is a fixed sequence of $\mathbb{C}_p$ points of $\mathcal{LT}$ satisfying

- the $\mathcal{O}_{\mathcal{P}}$-order of ζ_n is exactly π^n;
- $\pi\zeta_{n+1} = \zeta_n$ for each $n \geq 1$;
- $\zeta_1 = (-\pi)^{\frac{1}{q-1}}$, where $(-\pi)^{\frac{1}{q-1}}$ is a fixed element in $\mathbb{C}_p$.

Let $w > 0$ be a rational number such that $w < 1/q^{r-2}(q+1)$. Such w is said to be adapted to r. Let $v := w/(q-1)$. Suppose L contains an element of valuation w, denoted by π^w. We will carry out the analogous construction of modular sheaves as in Section 3.4. For convenience, we assume $e \leq p - 1$.

6.1 Modular sheaves on $\mathfrak{M}(w)$

6.1.1 The map dlog

Let $\mathcal{A}(w) \to \mathcal{M}(w)$ be the universal abelian scheme over $\mathcal{M}(w)$. Let

$$\mathcal{T} := \mathrm{T}_\pi\left((\mathcal{A}(w)[\pi^\infty]_1^{2,1})^\vee\right),$$

which can be thought of as a continuous sheaf

$$\mathcal{T} = \left\{\left(\mathcal{A}(w)[\pi^\infty]_1^{2,1}\right)^\vee\right\}_{n\in\mathbb{N}}$$

on $M(w)_L^{\mathtt{et}}$.

Let $\underline{\omega} := \left(\varepsilon_*\Omega^1_{\mathcal{A}(w)/\mathcal{M}(w)}\right)_1^{2,1}$, where $\varepsilon : \mathcal{A}(w) \to \mathcal{M}(w)$ is the natural morphism. Recall that we have continuous functors (refer to Section 4.4 for details)

$$\begin{aligned} u : \mathfrak{M}(w) &\longrightarrow M(w)_L^{\mathtt{et}} \\ (\mathcal{U}, W) &\longmapsto W \end{aligned}$$

and

$$\begin{aligned} v_{\mathfrak{M}(w)} : \mathcal{M}(w)^{\mathtt{ket}} &\longrightarrow \mathfrak{M}(w) \\ \mathcal{U} &\longmapsto (\mathcal{U}, \mathcal{U}_L). \end{aligned}$$

Let $\underline{\omega}_{A/\mathfrak{M}(w)} := v^*_{\mathfrak{M}(w)}(\underline{\omega})$. Since $v^*_{\mathfrak{M}(w)}\left(\mathcal{O}_{\mathcal{M}(w)^{\mathtt{ket}}}\right) \cong \mathcal{O}^{\mathtt{un}}_{\mathfrak{M}(w)}$, $\underline{\omega}_{A/\mathfrak{M}(w)}$ can be thought of as a locally free $\widehat{\mathcal{O}}^{\mathtt{un}}_{\mathfrak{M}(w)}$-module of rank 1, a continuous sheaf on $\mathfrak{M}(w)$. We also see $\mathcal{T}$ as a continuous sheaf on $\mathfrak{M}(w)$, via u_*, which will be omitted for abuse of notations. The usual dlog map (refer to Section 3.3.2)

$$\mathrm{dlog}_{n,\mathcal{A}} : \left(\mathcal{A}(w)[\pi^n]_1^{2,1}\right)^\vee\left(\overline{R}_L\right) \longrightarrow \underline{\omega}_{\mathbb{A}/R} \otimes_R \overline{R}/\pi^n\overline{R}$$

induces a morphism of $\widehat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules,

$$\mathrm{dlog}_{\mathfrak{M}(w)}:\mathcal{T}\otimes\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\longrightarrow\underline{\omega}_{\mathcal{A}/\mathfrak{M}(w)}\otimes_{\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}(w)}}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}.$$

Similarly as in Section 3.3.3, we obtain a Hodge-Tate sequence of continuous sheaves and morphisms of sheaves of $\widehat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules

$$0\longrightarrow\underline{\omega}^{-1}_{\mathcal{A}^\vee/\mathfrak{M}(w)}\otimes_{\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}(w)}}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}(1)\longrightarrow\mathcal{T}\otimes\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\xrightarrow{\mathrm{dlog}_{\mathfrak{M}(w)}}\underline{\omega}_{\mathcal{A}/R}\otimes_{\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}(w)}}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\longrightarrow 0,$$

where $(\cdot)^{-1}$ denotes the dual module. Moreover, we have the following properties.

Lemma 6.1.1. *For every connected, small affine open object* $\mathcal{U}=(\mathrm{Spf}(R_{\mathcal{U}}),N_{\mathcal{U}})$ *of* $\mathcal{M}(w)^{\mathtt{ket}}$*, the localization of the above Hodge-tate sequence of sheaves at* $\mathcal{U}$ *is just the Hodge-Tate sequence of continuous* $\mathcal{G}_{\mathcal{U}}=\mathrm{Gal}(\overline{R}_{\mathcal{U},L}/R_{\mathcal{U},L})$*-representations as in Section 3.3.3:*

$$0\longrightarrow\underline{\omega}^{*}_{\mathcal{A}^\vee/R}\otimes_R\widehat{\overline{R}}(1)\longrightarrow\mathrm{T}_\pi\left((\mathcal{A}[\pi^\infty]^{2,1}_1)^\vee\right)\otimes_{\mathcal{O}_{\mathcal{P}}}\widehat{\overline{R}}\longrightarrow\underline{\omega}_{\mathcal{A}/R}\otimes_R\widehat{\overline{R}}\longrightarrow 0.$$

Proof. This follows immediately by the definition of $\mathcal{T}$ and the fact that

$$\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}}\right)=\widehat{\overline{R}}_{\mathcal{U}}.$$

□

Now let $\mathcal{F}^0:=\mathrm{Im}\left(\mathrm{dlog}_{\mathfrak{M}(w)}\right)$ and $\mathcal{F}^1:=\mathrm{Ker}\left(\mathrm{dlog}_{\mathfrak{M}(w)}\right)$. Recall that we have defined a functor:

$$\begin{aligned} j_r:\mathfrak{M}(w)&\longrightarrow\mathfrak{M}^r(w)\\ (\mathcal{U},W)&\longmapsto\left(\mathcal{U},W\times_{M(w)}M^r(w),\mathtt{pr}_2\right).\end{aligned}$$

Let $\mathcal{F}^{i,r}:=j_r^*\mathcal{F}^i$, $i=0,1$. Our assumption on w (adapted to r) implies the existence of the canonical subgroup of $\mathcal{A}(w)[q^r]$ (see Proposition 3.2.4). We denote this subgroup by $\mathcal{C}_r$

and we have $(\mathcal{C}_r)_1^{2,1} \subset \mathcal{A}(w)[\pi^r]_1^{2,1}$ of order q^r. Consider $(\mathcal{C}_r)_1^{2,1}$ and $\left((\mathcal{C}_r)_1^{2,1}\right)^\vee$ as the group of points of their corresponding group schemes over $M^r(w)$, and we denote by the same symbols the locally constant sheaves on $(M^r(w)_{\overline{L}})^{\mathrm{et}}$. Similarly as before, they can be also viewed as continuous sheaves on $\mathfrak{M}^r(w)$, via u_*. Then we have

Lemma 6.1.2. *Let r, w, v be at the beginning of this chapter, and define $\mathcal{F}^i$, $\mathcal{F}^{i,r}$ as above for $i = 0, 1$.*

(1) $\mathcal{F}^0$ and $\mathcal{F}^1$ are locally free sheaves of $\widehat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules of rank 1.

(2) We have the following isomorphisms of $\widehat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules:

- $\mathcal{F}^0 \Big/ \pi^{r-v}\mathcal{F}^0 \cong \left((\mathcal{C}_r)_1^{2,1}\right)^\vee \otimes \mathcal{O}_{\mathfrak{M}(w)} \Big/ \pi^{r-v}\mathcal{O}_{\mathfrak{M}(w)}$;
- $\mathcal{F}^1 \Big/ \pi^{r-v}\mathcal{F}^1 \cong (\mathcal{D}_r)_1^{2,1} \otimes \mathcal{O}_{\mathfrak{M}(w)} \Big/ \pi^{r-v}\mathcal{O}_{\mathfrak{M}(w)}$

where $(\mathcal{D}_r)_1^{2,1} = \left((\mathcal{C}_r)_1^{2,1}\right)^\perp$ (refer to Definition 3.3.3).

(3) $\mathcal{F}^{0,r}$ and $\mathcal{F}^{1,r}$ are locally free sheaves of $\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)}$-modules of rank 1, where $\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)} := j_r^*\left(\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\right)$.

(4) We have a natural isomorphism of $\mathcal{O}_{\mathcal{M}^r(w)}$-modules with G_r-action:

$$v_{r,*}\left(\mathcal{F}^{0,r}\right) \cong \mathfrak{F}_r \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_p},$$

where $\mathfrak{F}_r$ is the sheaf as in Proposition 3.3.3 and G_r is the Galois group of $M^r(w) \to M(w)$. Here we consider $v_{r,*}\left(\mathcal{F}^{0,r}\right) \in \mathrm{Sh}\left(\mathcal{M}(w)^{\mathtt{ket}}\right)$ as a sheaf on $\mathcal{M}^r(w)^{\mathtt{ket}}$ via the natural morphism $\vartheta_r : \mathcal{M}^r(w) \to \mathcal{M}(w)$. Moreover,

$$\mathcal{F}^{0,r} \Big/ \pi^{r-v}\mathcal{F}^{0,r} \cong \left((\mathcal{C}_r)_1^{2,1}\right)^\vee \otimes \mathcal{O}_{\mathfrak{M}^r(w)} \Big/ \pi^{r-v}\mathcal{O}_{\mathfrak{M}^r(w)}.$$

Proof. Let $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}), N_{\mathcal{U}})$ be a connected, small affine open object of $\mathcal{M}(w)^{\mathtt{ket}}$. The localization of $\mathcal{F}^i$ at $\mathcal{U}$ are:

- $\mathcal{F}^0\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right) = \mathrm{Im}(\mathrm{dlog}_{\mathcal{U}})$;

- $\mathcal{F}^1\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right) = \mathrm{Ker}(\mathrm{dlog}_{\mathcal{U}})$,

where the map $\mathrm{dlog}_{\mathcal{U}}$ is the usual dlog map. Then (1) follows immediately by Theorem 3.3.2.

Taking localizations at $\mathcal{U}$, (2) is a consequence of Theorem 3.3.2.

(3) follows from (1), (4) follows from (2) and the construction of $\mathcal{F}_r$, refer to Section 3.3.3.

□

6.1.2 A torsor

Let

$$S_{\mathfrak{M}^r(w)} := \mathcal{O}_{\mathcal{P}}^{\times}(1 + \pi^{r-v}\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)})$$

be the sheaf of abelian groups, let $\mathcal{F}'_{\mathfrak{M}^r(w)}$ be the inverse image of the constant sheaf of subsets of $\left((\mathcal{C}_r)_1^{2,1}\right)^{\vee}$ of points of order exactly π^r under the natural map

$$\mathcal{F}^{0,r} \longrightarrow \mathcal{F}^{0,r}\Big/\pi^{r-v}\mathcal{F}^{0,r} \xrightarrow{\sim} \left((\mathcal{C}_r)_1^{2,1}\right)^{\vee} \otimes \mathcal{O}_{\mathfrak{M}^r(w)}\Big/\pi^{r-v}\mathcal{O}_{\mathfrak{M}^r(w)}.$$

Then we have

Lemma 6.1.3. *We have that $\mathcal{F}'_{\mathfrak{M}^r(w)}$ is a $S_{\mathfrak{M}^r(w)}$-torsor. Moreover, it is trivial over a covering of the type $\{(\mathcal{U}_i, \mathcal{U}_i \times \mathrm{M}^r(w))\}_{i\in I}$, where $\{\mathcal{U}_i\}_{i\in I}$ is a covering of $\mathcal{M}(w)$ by small affine objects.*

Proof. This is an immediate consequence of Theorem 3.3.2. □

Let $\mathcal{T}_0 \subset \mathcal{T}$ be the inverse image of the subset of $\left((\mathcal{C}_r)_1^{2,1}\right)^\vee$ of points of order exactly π^r via the natural morphism

$$\mathcal{T} \longrightarrow \left(\mathcal{A}(w)[\pi^r]_1^{2,1}\right)^\vee \longrightarrow \left((\mathcal{C}_r)_1^{2,1}\right)^\vee,$$

which can be also thought of as a sheaf on $M(w)_{\overline{L}}^{\mathrm{et}}$ hence a continuous sheaf on $\mathfrak{M}(w)$ via u_*. Moreover, we have a natural morphism induced by the dlog map, which is denoted by the same symbol:

$$\mathrm{dlog} : j_r^*(\mathcal{T}_0) \longrightarrow \mathcal{F}'_{\mathfrak{M}^r(w)},$$

compatible with the action of $\mathcal{O}_{\mathcal{P}}^\times$ on both sides.

Now we fix $(B, \underline{\mathbf{m}})$, λ, r and w such that

- $(B, \underline{\mathbf{m}})$ is a complete, regular, local, noetherian $\mathcal{O}_L$-algebra with $\underline{\mathbf{m}}$ its maximal ideal. B is complete and separated for its $\underline{\mathbf{m}}$-adic topology, hence, also for the π-adic topology.
- $\lambda \in \mathcal{W}(B_L)$.
- $r \in \mathbb{N}$, $r > 0$ is minimal such that $\lambda \in \mathcal{W}_r(B_L)$. Then there is an element $s_\lambda \in B_L$ such that $\lambda(1 + \pi^r y) = \exp\left(s_\lambda \log(y)\right)$ for $y \in \mathcal{O}_{\mathcal{P}}$.
- $0 < w < 1/q^{r-2}(q+1)$ and $w < (q-1)\left(\mathrm{ord}(s_\lambda) + r - \frac{e}{p-1}\right)$.

Consider the following continuous sheaf on $\mathfrak{M}^r(w)$ defined by

$$\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B := \left\{\left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)_n\right\}_{n \in \mathbb{N}},$$

where

$$\left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)_n := \left(\mathcal{O}_{\mathfrak{M}^r(w)} / \pi^n \mathcal{O}_{\mathfrak{M}^r(w)}\right) \otimes \left(B / \underline{\mathbf{m}}^n B\right).$$

We denote by $\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)^{\lambda}$ the continuous sheaf $\left\{\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n^{\lambda}\right\}_{n\in\mathbb{N}}$ endowed with the action of $S_{\mathfrak{M}^r(w)}$, which is twisted by λ and defined as follows. Let $(\mathcal{U}, W, u)$ be an object of $\mathfrak{M}^r(w)$, for

$$ax \in S_{\mathfrak{M}^r(w)}(\mathcal{U}, W, u) = \mathcal{O}_{\mathcal{P}}^{\times}\left(1+\pi^{r-v}\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)}(\mathcal{U}, W, u)\right),$$

and $y \in \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)(\mathcal{U}, W, u)$, we define

$$(ax).y := \lambda(a)x^{s_\lambda}y \in \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)(\mathcal{U}, W, u),$$

where $x^{s_\lambda} = \exp\left(s_\lambda \log(x)\right)$ makes sense by the assumption on w.

Now let

$$\Omega^{\lambda}_{\mathfrak{M}^r(w)} := \mathscr{H}om_{S_{\mathfrak{M}^r(w)}}\left(\mathcal{F}'_{\mathfrak{M}^r(w)}, \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)^{\lambda^{-1}}\right).$$

It is a locally free $\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B$-module of rank 1 by Lemma 6.1.3. Moreover, we have a natural isomorphism of $\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B$-modules

$$\mathscr{H}om_{\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B}\left(\Omega^{\lambda}_{\mathfrak{M}^r(w)}, \mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right) \cong \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)}.$$

Now let $\omega^{\lambda}_{\mathfrak{M}^r(w)}$ be the ind-continuous sheaf

$$\omega^{\lambda}_{\mathfrak{M}^r(w)} := \Omega^{\lambda}_{\mathfrak{M}^r(w)}[1/\pi].$$

If $B = \mathcal{O}_L$ and $\lambda \in \mathcal{W}_r(L)$, we have the following isomorphism of sheaves

$$v_{r,*}\left(\Omega^{\lambda}_{\mathfrak{M}^r(w)}\right) \cong \tilde{\Omega}^{\lambda}_{w} \otimes_{\mathcal{O}_L} \mathcal{O}_{\mathbb{C}_p}.$$

6.1.3 Action of G_r

Let G_r be the Galois group of $M^r(w) \to M(w)$. For any $\sigma \in G_r$, we may consider it as a functor

$$\sigma : \left(E_{\mathcal{M}(w)_{\overline{L}}}\right)_{/(\mathcal{M}(w), M^r(w))} \longrightarrow \left(E_{\mathcal{M}(w)_{\overline{L}}}\right)_{/(\mathcal{M}(w), M^r(w))}$$

defined by sending $(\mathcal{U}, W, u) \mapsto (\mathcal{U}, W, \sigma \circ u)$ on objects and by identity on the morphisms. This functor on the category $\left(E_{\mathcal{M}(w)_{\overline{L}}}\right)_{/(\mathcal{M}(w), M^r(w))}$ induces a continuous functor on the site $\mathfrak{M}^r(w)$. Moreover, if $\mathcal{F}$ is a sheaf (or a continuous sheaf) on $\mathfrak{M}^r(w)$, we denote by $\mathcal{F}^\sigma$ the sheaf

$$(\mathcal{U}, W, u) \longmapsto \mathcal{F}(\sigma(\mathcal{U}, W, u)) = \mathcal{F}(\mathcal{U}, W, \sigma \circ u).$$

Then we have

Lemma 6.1.4. *Let $j_r : \mathfrak{M}(w) \longrightarrow \mathfrak{M}^r(w)$ be the functor as in Section 4.4.*

(1) If $\mathcal{F}$ is a sheaf of abelian groups on $\mathfrak{M}(w)$ and $\mathcal{H} = j_r^* \mathcal{F}$, then $\mathcal{H}^\sigma = \mathcal{H}$ for any $\sigma \in G_r$.

(2) For all $\sigma \in G_r$, then $\mathcal{H}^\sigma = \mathcal{H}$ if $\mathcal{H}$ is one of the following sheaves:

$$\mathcal{F}'_{\mathfrak{M}^r(w)}, \quad \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)^\lambda, \quad \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)^{\lambda^{-1}}, \quad \Omega^\lambda_{\mathfrak{M}^r(w)} \textit{ and}$$
$$\mathcal{A}^{\lambda^{-1}}_{\mathfrak{M}^r(w)} := \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^\times}\left(j_r^*(\mathcal{T}_0), \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)^\lambda\right).$$

(3) Let $\mathcal{H}$ be a sheaf on $\mathfrak{M}^r(w)$ such that $\mathcal{H}^\sigma = \mathcal{H}$ for all $\sigma \in G_r$. Then each σ defines a canonical automorphism

$$j_{r,*}(\mathcal{H}) \longrightarrow j_{r,*}(\mathcal{H}).$$

In other words, we have a canonical action of the group G_r on the sheaf $j_{r,*}(\mathcal{H})$.

Proof.

(1) Since $j_r^* \cong \alpha_{r,*}$, where

$$\alpha_{r,*} : \left(E_{\mathcal{M}(w)_{\overline{L}}}\right)_{/(\mathcal{M}(w), M^r(w))} \longrightarrow E_{\mathcal{M}(w)_{\overline{L}}}$$

is the forgetful functor (see section 4.3.3), we have

$$(j_r^* \mathcal{F})(\mathcal{U}, W, u) = \mathcal{F}(\mathcal{U}, W)$$

for any object $(\mathcal{U}, W, u)$ in $\mathfrak{M}^r(w)$. This proves (1).

(2) This is true by the construction of the sheaves.

(3) Recall that $j_r : \mathfrak{M}(w) \longrightarrow \mathfrak{M}^r(w)$ sending $(\mathcal{U}, W) \mapsto \left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right)$. Then

$$(j_{r,*}\mathcal{H})(\mathcal{U}, W) = \mathcal{H}\left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right).$$

For any $\sigma \in G_r$, $\sigma : M^r(w) \to M^r(w)$ is an automorphism over $M(w)$, after base change to a larger field (finite over L). We have the following commutative diagram

$$\begin{tikzcd}
 & & W \times_{M(w)} M^r(w) \arrow[dll, dashed] \arrow[dd, "\mathtt{pr}_2"] \arrow[drr] & & \\
W \times_{M(w)} M^r(w) \arrow[rrrr, equal] \arrow[ddd, "\mathtt{pr}_2"] & & & & W \arrow[ddd] \\
 & & & & \\
 & & M^r(w) \arrow[ddrr] & & \\
 & & & & \\
M^r(w) \arrow[uurr, "\sigma"] \arrow[rrrr] & & & & M(w).
\end{tikzcd}$$

Hence the morphism induced by σ gives an automorphism

$$\left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right) \longrightarrow \left(\mathcal{U}, W \times_{M(w)} M^r(w), \sigma^{-1} \circ \mathtt{pr}_2\right)$$

in $\mathfrak{M}^r(w)$. This gives an automorphism, denoted still by σ,

$$\begin{tikzcd}
\mathcal{H}\left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right) \arrow[r] \arrow[d, equal] & \mathcal{H}\left(\mathcal{U}, W \times_{M(w)} M^r(w), \sigma^{-1} \circ \mathtt{pr}_2\right) \arrow[d, equal] \\
\left(j_{r,*}\mathcal{H}\right)(\mathcal{U}, W) & \mathcal{H}^{\sigma^{-1}}\left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right) \arrow[d, equal] \\
& \mathcal{H}\left(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\right) \arrow[d, equal] \\
& \left(j_{r,*}\mathcal{H}\right)(\mathcal{U}, W).
\end{tikzcd}$$

Moreover, such σ is compatible with morphisms in $\mathfrak{M}(w)$ hence gives an automorphism of sheaves

$$\sigma : j_{r,*}\mathcal{H} \longrightarrow j_{r,*}\mathcal{H}.$$

This completes the proof of the lemma. $\square$

6.1.4 Modular sheaves

Recall that we have defined a continuous sheaf

$$\Omega^{\lambda}_{\mathfrak{M}^r(w)} := \mathscr{H}om_{S_{\mathfrak{M}^r(w)}}\left(\mathcal{F}'_{\mathfrak{M}^r(w)}, \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)^{\lambda^{-1}}\right),$$

and an ind-continuous sheaf

$$\omega^{\lambda}_{\mathfrak{M}^r(w)} := \Omega^{\lambda}_{\mathfrak{M}^r(w)}[1/\pi].$$

on $\mathfrak{M}^r(w)$. Thanks to Lemma 6.1.4, we can define sheaves on $\mathfrak{M}(w)$ to be

$$\Omega^{\lambda}_{\mathfrak{M}(w)} := \left(j_{r,*}\Omega^{\lambda}_{\mathfrak{M}^r(w)}\right)^{G_r},$$

and

$$\omega^{\lambda}_{\mathfrak{M}(w)} := \left(j_{r,*}\omega^{\lambda}_{\mathfrak{M}^r(w)}\right)^{G_r}.$$

We have the following properties

Proposition 6.1.1. *Let B, λ, r and w be as before. Then*

(i) $\omega^{\lambda}_{\mathfrak{M}(w)}$ *is a locally free* $(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B)[1/\pi]$*-module of rank* 1.

(ii) $v_{\mathfrak{M}(w),*}\left(\omega^{\lambda}_{\mathfrak{M}(w)}\right) \cong \omega^{\lambda}_{w}\hat{\otimes}_L \mathbb{C}_p$, *where* ω^{λ}_{w} *is the rigidification of the sheaf* Ω^{λ}_{w} *defined in Section 3.4.3.*

(iii) $\omega^{\lambda}_{\mathfrak{M}(w)} \cong \omega^{\lambda}_{w} \otimes_{\widehat{\mathcal{O}}_{\mathcal{M}(w)}} \widehat{\mathcal{O}}_{\mathfrak{M}(w)}$.

Proof. Part (i) follows from Lemma 6.1.3. Let $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}), N_{\mathcal{U}})$ be a connected, small object of $\mathcal{M}(w)^{\mathtt{ket}}$. After localization at $\mathcal{U}$, (ii) and (iii) are clear by the construction of the above sheaves. $\square$

Remark 6.1.1. Similarly as in Section 3.4.3, the constructions of such sheaves are compatible for various r's and w's in the sense of Lemma 3.4.1 and Proposition 3.4.3.

6.2 Cohomology of the sheaf $\omega^{\lambda}_{\mathfrak{M}(w)}$

Let $\vartheta : Z \to M(w)$ be a morphism in $M(w)^{\mathtt{fet}}_{L}$ and let $\mathfrak{Z} := \mathfrak{M}(w)_{/(\mathcal{M}(w),Z)}$ be the associated induced site (refer to Section 4.3.3 for details). Recall that we have a continuous functor

$$\begin{aligned} j : \mathfrak{M}(w) &\longrightarrow \mathfrak{Z} \\ (\mathcal{U}, W) &\longmapsto \left(\mathcal{U}, W \times_{M(w)} Z, \mathtt{pr}_2\right), \end{aligned}$$

which induces a morphism of topoi. In this section, we will give a formula for the i-th cohomology of

$$\mathrm{H}^i\left(\mathfrak{Z}, j^*\left(\omega^{\lambda}_{\mathfrak{M}(w)}\right)\right)$$

for all $i \geq 0$. In particular, if $\vartheta : M(w) \to M(w)$ is the identity map, we get an explicit formula for $\mathrm{H}^i\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right)$ for all $i \geq 0$. To calculate the cohomology, we need the following lemma:

Lemma 6.2.1. *Let $\mathcal{F}$ be a locally free $(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B)[1/\pi]$-module of finite rank. Then the sheaf*

$$\mathrm{R}^b\, v_{\mathfrak{M}(w),*}(\mathcal{F})$$

is the one associated to the presheaf

$$\mathcal{U} = (\mathrm{Spf}(R_\mathcal{U}), N_\mathcal{U}) \longmapsto \mathrm{H}^b\left(\mathcal{G}_\mathcal{U}, \mathcal{F}(\overline{R}_\mathcal{U}, \overline{N}_\mathcal{U})\right)$$

on $\mathcal{M}(w)^{\mathtt{ket}}$, where $\mathcal{G}_\mathcal{U}$ is the Kummer étale geometric fundamental group of $\mathcal{U}$, for a choice of a geometric generic point, i.e., $\mathcal{G}_\mathcal{U} = \mathrm{Gal}\left(\overline{R}_\mathcal{U}[1/\pi]/\overline{R}_\mathcal{U}\overline{L}\right)$.

Proof. This is [Andreatta and Iovita, 2012, Proposition 2.10]. □

Theorem 6.2.1. *We have the following isomorphisms of G_L-modules.*

(1) $\mathrm{H}^0\left(\mathfrak{Z}, j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)\right) \cong \mathrm{H}^0\left(Z, \vartheta^*\left(\omega^\lambda_w\right)\right)\hat{\otimes}_L \mathbb{C}_p(1)$;

(2) $\mathrm{H}^1\left(\mathfrak{Z}, j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)\right) \cong \mathrm{H}^0\left(Z, \vartheta^*\left(\omega^{\lambda+2}_w\right)\right)\hat{\otimes}_L \mathbb{C}_p$;

(3) $\mathrm{H}^i\left(\mathfrak{Z}, j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)\right) = 0$ *for $i \geq 2$.*

Proof. First, by exactness of the functor j, we have

$$\mathrm{H}^i\left(\mathfrak{Z}, j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)\right) \cong \mathrm{H}^i\left(\mathfrak{M}(w), j_*j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)\right)$$

for $i \geq 0$. We set $\mathcal{F} := j_*j^*\left(\omega^\lambda_{\mathfrak{M}(w)}(1)\right)$. In order to calculate $\mathrm{H}^i\left(\mathfrak{M}(w), \mathcal{F}\right)$, we will use the following Leray spectral sequence

$$\mathrm{H}^p\left(\mathcal{M}(w)^{\mathtt{ket}}, \mathrm{R}^q\, v_{\mathfrak{M}(w),*}(\mathcal{F})\right) \Longrightarrow \mathrm{H}^{p+q}\left(\mathfrak{M}(w), \mathcal{F}\right).$$

By the above lemma, the sheaf $\mathrm{R}^q\, v_{\mathfrak{M}(w),*}(\mathcal{F})$ on the left hand side is just the sheaf associated to the presheaf

$$\mathcal{U} \longmapsto \mathrm{H}^q\left(\mathcal{G}_{\mathcal{U}}, \mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})\right),$$

for $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}), N_{\mathcal{U}})$ a connected, small affine object of $\mathcal{M}(w)^{\mathtt{ket}}$ and

$$\mathcal{G}_{\mathcal{U}} = \mathrm{Gal}\left(\overline{R}_{\mathcal{U}}[1/\pi]/\overline{R}_{\mathcal{U}}\overline{L}\right)$$

. We first calculate $\mathcal{F}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right)$.

Using the formula (4.4) and note that $j^* = \alpha_*$, where α is the forgetful functor from $\mathfrak{Z}$ to $\mathfrak{M}(w)$, we have

$$\mathcal{F}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right) = \omega_w^{\lambda}(\mathcal{U}) \otimes_{R_{\mathcal{U},L}} \vartheta_* \mathcal{O}_{\mathcal{Z}}(\mathcal{U}_L) \otimes_{R_{\mathcal{U}}} \widehat{\overline{R}}_{\mathcal{U}}(1)$$

Then by [Faltings, 1987, Theorem 3] (refer to Remark 6.2.1) and the Kodaira-Spencer isomorphism (refer to Proposition 3.1.4), we have

$$\mathrm{H}^q\left(\mathcal{G}_{\mathcal{U}}, \widehat{\overline{R}}_{\mathcal{U},L}\right) \cong \begin{cases} R_{\mathcal{U},L}\hat{\otimes}_L \mathbb{C}_p(1) & \text{if } q = 0, \\ \omega_w^2(\mathcal{U}) \otimes R_{\mathcal{U},L}\hat{\otimes}_L \mathbb{C}_p(-1) & \text{if } q = 1, \\ 0 & \text{if } q \geq 2. \end{cases}$$

Thus we can deduce the following formulas for $\mathrm{H}^q\left(\mathcal{G}_{\mathcal{U}}, \mathcal{F}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right)\right)$,

$$\mathrm{H}^q\left(\mathcal{G}_{\mathcal{U}}, \mathcal{F}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}\right)\right) = \begin{cases} \omega_w^{\lambda}(\mathcal{U}) \otimes_{R_{\mathcal{U},L}} \vartheta_* \mathcal{O}_{\mathcal{Z}}(\mathcal{U}_L) \otimes_L \mathbb{C}_p(1) & \text{if } q = 0, \\ \omega_w^{\lambda+2}(\mathcal{U}) \otimes_{R_{\mathcal{U},L}} \vartheta_* \mathcal{O}_{\mathcal{Z}}(\mathcal{U}_L) \otimes_L \mathbb{C}_p & \text{if } q = 1, \\ 0 & \text{if } q \geq 2. \end{cases}$$

Now (1) and (3) follows immediately.

For (2), if $p+q=1$, the Leray spectral sequence degenerates to the following exact sequence:

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathrm{H}^1\left(\mathcal{M}(w)^{\mathtt{ket}}, \mathrm{R}^0\, v_{\mathfrak{M}(w),*}\mathcal{F}\right) \longrightarrow \\ &\longrightarrow \quad \mathrm{H}^1\left(\mathfrak{M}(w), \mathcal{F}\right) \longrightarrow \\ &\longrightarrow \quad \mathrm{H}^0\left(\mathcal{M}(w)^{\mathtt{ket}}, \mathrm{R}^1\, v_{\mathfrak{M}(w),*}\mathcal{F}\right) \longrightarrow \\ &\longrightarrow \quad \mathrm{H}^2\left(\mathcal{M}(w)^{\mathtt{ket}}, \mathrm{R}^0\, v_{\mathfrak{M}(w),*}\mathcal{F}\right). \end{aligned}$$

Since the sheaf $\omega_w^\lambda \otimes_{\mathcal{O}_{M(w)}} \vartheta_*\mathcal{O}_{\mathcal{Z}} \otimes_L \mathbb{C}_p(1)$ on $M(w)$ is locally isomorphic to $(\vartheta_*\mathcal{O}_{\mathcal{Z}} \otimes B_L \otimes_L \mathbb{C}_p)$, it is a sheaf of L-banach modules on $M(w)$, which is an affinoid. Then by Kiehl's vanishing theorem (refer to [Andreatta et al., 2015a, Appendix]), we obtain that

$$\begin{aligned} &\mathrm{H}^i\left(\mathcal{M}(w)^{\mathtt{ket}}, \omega_w^\lambda \otimes_{\mathcal{O}_{M(w)}} \vartheta_*\mathcal{O}_{\mathcal{Z}} \otimes_L \mathbb{C}_p(1)\right) \\ =&\mathrm{H}^i\left(M(w), \omega_w^\lambda \otimes_{O_{M(w)}} \vartheta_*\mathcal{O}_{\mathcal{Z}} \otimes_L \mathbb{C}_p(1)\right) = 0, \end{aligned}$$

for all $i \geq 1$. Therefore we have

$$\mathrm{H}^1\left(\mathfrak{M}(w), \mathcal{F}\right) \cong \mathrm{H}^0\left(\mathcal{M}(w)^{\mathtt{ket}}, \omega_w^{\lambda+2} \otimes_{\mathcal{O}_{M(w)}} \vartheta_*\mathcal{O}_{\mathcal{Z}} \otimes_L \mathbb{C}_p\right)$$

and (2) follows. $\square$

Remark 6.2.1. In fact, if we denote by $\mathbb{C}_p := \widehat{\overline{L}}$, by

$$\chi : G_L := \mathrm{Gal}(\overline{L}/L) \longrightarrow \mathcal{O}_{\mathcal{P}}^\times$$

the Lubin-Tate character and by (n) the twist of Galois modules with the n-th power of χ.

In [Faltings, 2002a, § 9], the author shows the following

$$\mathrm{H}^i\left(G_L, \mathbb{C}_p\right) \cong \begin{cases} L & \text{if } i = 0, \\ L \cdot \chi & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Moreover the nontrivial twists $\mathbb{C}_p(n)$ have trivial Galois cohomology. Then the arguments of [Faltings, 1987, Theorem 3] can be generalized in a similar way to our situation by replacing p-divisible groups with π-divisible groups.

In particular, if $Z = M(w)$, we obtain the following corollary.

Corollary 6.2.1. *We have the following isomorphisms of G_L-modules.*

(1) $\mathrm{H}^0\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)\right) \cong \mathrm{H}^0\left(M(w), \omega^{\lambda}_w\right) \hat{\otimes}_L \mathbb{C}_p(1)$*;*

(2) $\mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)\right) \cong \mathrm{H}^0\left(M(w), \omega^{\lambda+2}_w\right) \hat{\otimes}_L \mathbb{C}_p$*;*

(3) $\mathrm{H}^i\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)\right) = 0$ *for* $i \geq 2$.

6.3 The morphism

6.3.1 Notations

We fix some notations for the rest of this chapter.

Let

$$\mathcal{D}^0_{\lambda} := \left\{\mathcal{D}^0_{\lambda,n}\right\}_{n\in\mathbb{N}},$$

$$\mathcal{A}^0_{\lambda} := \left\{\mathcal{A}^0_{\lambda,n}\right\}_{n\in\mathbb{N}}$$

be the continuous sheaves and

$$\mathcal{D}_\lambda := \mathcal{D}^0_\lambda[1/\pi],$$

$$\mathcal{A}_\lambda := \mathcal{A}^0_\lambda[1/\pi]$$

be the ind-continuous sheaves on $\mathfrak{M}(H,\pi)$ described as in Section 5.2.1.

Recall that we have a continuous functor $\nu : \mathfrak{M}(H,\pi) \to \mathfrak{M}(w)$ induced by the natural map $\mathcal{M}(w) \to \mathcal{M}(H,\pi)$ (refer to Section 4.4). Applying ν^* to the sheaves described above, we obtain continuous and ind-continuous sheaves on $\mathcal{M}(w)$, namely

$$\mathcal{A}^{0,\lambda}_w = \left\{\mathcal{A}^{0,\lambda}_{w,n}\right\}_{n\in\mathbb{N}} := \left\{\nu^*\mathcal{A}^0_{\lambda,n}\right\}_{n\in\mathbb{N}} = \nu^*\mathcal{A}^0_\lambda,$$

$$\mathcal{D}^{0,\lambda}_w = \left\{\mathcal{D}^{0,\lambda}_{w,n}\right\}_{n\in\mathbb{N}} := \left\{\nu^*\mathcal{D}^0_{\lambda,n}\right\}_{n\in\mathbb{N}} = \nu^*\mathcal{D}^0_\lambda,$$

and $\mathcal{A}^\lambda_w := \nu^*\mathcal{A}_\lambda$, $\mathcal{D}^\lambda_w := \nu^*\mathcal{D}_\lambda$.

Now let $\mathfrak{M}$ be any one of the sites $\mathfrak{M}(w)$, $\mathfrak{M}(H,\pi)$ or $\mathfrak{M}^r(w)$. We denote by $\mathcal{O}_\mathfrak{M}\hat{\otimes}B$ the inverse system

$$\left\{\left(\mathcal{O}_\mathfrak{M}\hat{\otimes}B\right)_n\right\}_{n\in\mathbb{N}} := \left\{(\mathcal{O}_\mathfrak{M}/\pi^n\mathcal{O}_\mathfrak{M}) \otimes (B/\underline{\mathbf{m}}^n B)\right\}_{n\in\mathbb{N}},$$

and by $\left(\mathcal{O}_\mathfrak{M}\hat{\otimes}B\right)^\lambda$ the system

$$\left\{\left(\mathcal{O}_\mathfrak{M}\hat{\otimes}B\right)^\lambda_n\right\}_{n\in\mathbb{N}}$$

with an action of $\mathcal{O}^\times_\mathcal{P}$ (or sometimes $S_{\mathfrak{M}^r(w)}$) twisted by λ.

Recall that we have the following sheaves of $\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)$-modules on $\mathfrak{M}^r(w)$:

$$\begin{aligned}\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)} &= \left\{\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n}\right\}_{n\in\mathbb{N}} \\ &:= \left\{\mathscr{H}om_{S_{\mathfrak{M}^r(w)}}\left(\mathcal{F}'_{\mathfrak{M}^r(w)}, \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)^\lambda_n\right)\right\}_{n\in\mathbb{N}},\end{aligned}$$

and

$$\mathcal{A}^{\lambda^{-1}}_{\mathfrak{M}^r(w)} = \left\{\mathcal{A}^{\lambda^{-1}}_{\mathfrak{M}^r(w),n}\right\}_{n\in\mathbb{N}}$$

$$:= \left\{ \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^{\times}} \left(j_r^* \mathcal{T}_0, \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B \right)_n^{\lambda} \right) \right\}_{n \in \mathbb{N}}.$$

6.3.2 Construction of the morphism

The goal of this section is to construct a morphism

$$\mathrm{H}^1 \left(\mathfrak{M}(H, \pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B \right) \right) \longrightarrow \mathrm{H}^1 \left(\mathfrak{M}(w), \omega_{\mathfrak{M}(w)}^{\lambda} \right).$$

To this end we proceed in the following four steps.

Step 1 :

First, let us point out that we have a natural morphism of continuous sheaves $\widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)} \longrightarrow \nu_* \left(\widehat{\mathcal{O}}_{\mathfrak{M}(w)} \right)$ obtained by adjunction a morphism $\nu^* \left(\widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)} \right) \longrightarrow \widehat{\mathcal{O}}_{\mathfrak{M}(w)}$. Since ν commutes with tensor products, we obtain a morphism of ind-continuous sheaves on $\mathfrak{M}(w)$:

$$\nu^* \left(\mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B \right) \right) \longrightarrow \mathcal{D}_w^{\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}(w)} \hat{\otimes} B \right).$$

Passing to cohomology and composing with the morphism

$$\mathrm{H}^1 \left(\mathfrak{M}(H, \pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B \right) \right) \longrightarrow \mathrm{H}^1 \left(\mathfrak{M}(w), \nu^* \left(\mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B \right) \right) \right)$$

induced by ν, we obtain a morphism

$$\mathrm{H}^1 \left(\mathfrak{M}(H, \pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B \right) \right) \longrightarrow \mathrm{H}^1 \left(\mathfrak{M}(w), \mathcal{D}_w^{\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}(w)} \hat{\otimes} B \right) \right). \tag{6.1}$$

Step 2 :

Let $\mathcal{T}_0$ and $\mathcal{F}'_{\mathfrak{M}^r(w)}$ be the continuous sheaves defined as in Section 6.1.2. Then we have a natural map

$$\mathrm{dlog} : j_r^*(\mathcal{T}_0) \longrightarrow \mathcal{F}'_{\mathfrak{M}^r(w)}.$$

The above map induces a morphism

$$\alpha^r : \Omega_{\mathfrak{M}^r(w)}^{\lambda^{-1}} \longrightarrow \mathcal{A}_{\mathfrak{M}^r(w)}^{\lambda^{-1}},$$

which is G_r-invariant.

Moreover, for every $n \in \mathbb{N}$, we have an inclusion of $\mathcal{O}_L$-modules

$$A_\lambda^0 / \underline{\mathbf{m}}^n A_\lambda^0 \subset \mathrm{Hom}_{\mathcal{O}_{\mathcal{P}}^\times}\left(\mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}},\ (B/\underline{\mathbf{m}}^n B)^\lambda\right),$$

where $\mathcal{O}_{\mathcal{P}}^\times$ acts on $\mathcal{O}_{\mathcal{P}}^\times \times \mathcal{O}_{\mathcal{P}}$ by scalar multiplication and acts on $(B/\underline{\mathbf{m}}^n B)^\lambda$ via λ. Then we have an injective morphism of sheaves on $\mathfrak{M}(w)$

$$\mathcal{A}_{w,n}^{0,\lambda} \hookrightarrow \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^\times}\left(\mathcal{T}_0,\ (B/\underline{\mathbf{m}}^n B)^\lambda\right),$$

hence a morphism of sheaves of $\left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B/\right)_n$-modules

$$\beta_n^r : j_r^* \mathcal{A}_{w,n}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)_n \longrightarrow \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^\times}\left(j_r^* \mathcal{T}_0, \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)_n^\lambda\right) = \mathcal{A}_{\mathfrak{M}^r(w)}^{\lambda^{-1}}.$$

These morphisms are compatible for varying n and give a morphism of continuous sheaves of $\left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)$-modules

$$\beta^r : j_r^* \mathcal{A}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right) \longrightarrow \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^\times}\left(j_r^* \mathcal{T}_0, \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right)^\lambda\right) = \mathcal{A}_{\mathfrak{M}^r(w)}^{\lambda^{-1}}.$$

Now consider the following diagram of morphisms of sheaves:

$$\begin{tikzcd} \Omega_{\mathfrak{M}^r(w)}^{\lambda^{-1}} \arrow[r, "\alpha^r"] & \mathcal{A}_{\mathfrak{M}^r(w)}^{\lambda^{-1}} \\ & j_r^* \mathcal{A}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right). \arrow[u, "\beta^r"'] \end{tikzcd}$$

Proposition 6.3.1. *Under the above notations, we have*

(i) β^r is injective and G_r-invariant.

(ii) *The map α^r factors via β^r.*

Proof. To prove the statements, we consider first the following diagram of sheaves for each $n \in \mathbb{N}$,

$$\begin{tikzcd} \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} \arrow[r, "\alpha^r_n"] & \mathcal{A}^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} \\ & j_r^*\mathcal{A}^{0,\lambda}_{w,n} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n . \arrow[u, "\beta^r_n"'] \end{tikzcd}$$

Then it is equivalent to prove that for all $n \in \mathbb{N}$,

(a) β^r_n is injective and G_r-invariant;

(b) the map α^r_n factors via β^r_n.

The G_r-invariance of β^r_n follows immediately by its construction and Lemma 6.1.4. We prove the above assertions by localizing at small affine objects of $\mathcal{M}(w)^{\mathtt{ket}}$ covering $\mathcal{M}(w)$. Let $\mathcal{U} = (\mathrm{Spf}(R_{\mathcal{U}}), N_{\mathcal{U}})$ be a connected small affine object of $\mathcal{M}(w)^{\mathtt{ket}}$. Let $g \in \Upsilon_{\mathcal{U}}$ and $\eta := \mathrm{Spec}(\mathbb{K})$ be a geometric generic point of $\mathrm{Spm}(R_L)$ (refer to Section 4.5 for details). Consider the $\mathcal{O}_{\mathcal{P}}$-module

$$T := \mathrm{T}_{\pi}\left((\mathcal{A}(w)[\pi^{\infty}]_1^{2,1})^{\vee}\right),$$

let $T_0 \subset T$ be the inverse image of the subset of $\left((\mathcal{C}_r)_1^{2,1}\right)^{\vee}$ of points of $\mathcal{O}_{\mathcal{P}}$-order exactly π^r under the natural map

$$\theta^r : T \longrightarrow (\mathcal{A}(w)[\pi^r]_1^{2,1})^{\vee} \longrightarrow \left((\mathcal{C}_r)_1^{2,1}\right)^{\vee}.$$

Moreover, we fix an $\mathcal{O}_{\mathcal{P}}$-basis $\{\varepsilon_0, \varepsilon_1\}$ of T such that $\theta^r(\varepsilon_1) = 0$ and $\theta^r(\varepsilon_0)$ is a point of $\mathcal{O}_{\mathcal{P}}$-order exactly π^r.

Let $x, y : T \to \mathcal{O}_{\mathcal{P}}$ be the $\mathcal{O}_{\mathcal{P}}$-linear map defined by

$$x(a\varepsilon_0 + b\varepsilon_1) = a, \text{ and } y(a\varepsilon_0 + b\varepsilon_1) = b,$$

for any $a, b \in \mathcal{O}_{\mathcal{P}}$. Then we can identify $T_0 = \left\{a\varepsilon_0 + b\varepsilon_1 \mid a \in \mathcal{O}_{\mathcal{P}}^{\times}, b \in \mathcal{O}_{\mathcal{P}}\right\} \subset T$.

Fix S, a set of representatives of $\mathcal{O}_{\mathcal{P}}/\pi^r\mathcal{O}_{\mathcal{P}}$, the discussion in Section 5.1.1 implies that

$$j_r^*\mathcal{A}_{w,n}^{0,\lambda}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right) = \bigoplus_{z_\eta \in S}\bigoplus_{h=0}^{\infty}(B/\underline{\mathbf{m}}^n)x^\lambda\left(\frac{(y/x) - z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}}(y/x).$$

Set $D := \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right)$, we have

$$\left(j_r^*\mathcal{A}_{w,n}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n\right)\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right) = \bigoplus_{z_\eta \in S}\bigoplus_{h=0}^{\infty}Dx^\lambda\left(\frac{(y/x) - z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}}(y/x),$$

where x^λ is the map $T \to \mathcal{O}_{\mathcal{P}}$ such that $x^\lambda(a\varepsilon_0 + b\varepsilon_1) = \lambda(a)$, and

$$\begin{aligned}
&\mathcal{A}_{\mathfrak{M}^r(w),n}^{\lambda^{-1}}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right)\\
&= \mathscr{H}om_{\mathcal{O}_{\mathcal{P}}^{\times}}\left(j_r^*\mathcal{T}_0, \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n^\lambda\right)\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right)\\
&= \left\{f : T_0 \to D \middle| f \text{ is continuous and } f(cz) = \lambda(c)f(z), \text{ for } c \in \mathcal{O}_{\mathcal{P}}, z \in T_0\right\},
\end{aligned}$$

as D-modules.

After localization, the map β_n^r is just the one sending

$$\sum_{z_\eta \in S}\sum_{h=0}^{\infty}\alpha_{z_\eta,h}x^\lambda\left(\frac{(y/x) - z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}}(y/x)$$

to the map

$$\left(a\varepsilon_0 + b\varepsilon_1 \longmapsto \sum_{z_\eta \in S}\sum_{h=0}^{\infty}\alpha_{z_\eta,h}\lambda(a)\left(\frac{(b/a) - z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}}(b/a)\right).$$

It is obvious that if the above map is zero then

$$\sum\sum\alpha_{z_\eta,h}x^\lambda\left(\frac{(y/x) - z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta + \pi^r\mathcal{O}_{\mathcal{P}}}(y/x) = 0.$$

Hence β_n^r is injective.

Now we have

$$\begin{tikzcd}
\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} \arrow[r, "\alpha^r_n"] & \mathcal{A}^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} \\
& j_r^*\mathcal{A}^{0,\lambda}_{w,n} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n . \arrow[u, hook, "\beta^r_n"']
\end{tikzcd}$$

By the injectivity of β^r_n, to prove α^r_n factoring through β^r_n, it is enough to prove that the image of α^r_n is contained in the image of β^r_n.

Recall that we have the following commutative diagram for the map dlog

$$\begin{tikzcd}
0 \arrow[r] & \frac{\mathrm{Ker}(\mathrm{dlog})}{\pi^{r-v}\,\mathrm{Ker}(\mathrm{dlog})} \arrow[r] \arrow[d, "\wr"] & T\otimes \frac{\widehat{\overline{R}}_{\mathcal{U}}}{\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}} \arrow[r] \arrow[d, equal] & \frac{\mathrm{Im}(\mathrm{dlog})}{\pi^{r-v}\,\mathrm{Im}(\mathrm{dlog})} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & (\mathcal{D}_r)_1^{2,1}\otimes \frac{\widehat{\overline{R}}_{\mathcal{U}}}{\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}} \arrow[r] & (\mathcal{A}[\pi^r]_1^{2,1})^\vee \otimes \frac{\widehat{\overline{R}}_{\mathcal{U}}}{\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}} \arrow[r] & \left((\mathcal{C}_r)_1^{2,1}\right)^\vee \otimes \frac{\widehat{\overline{R}}_{\mathcal{U}}}{\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}} \arrow[r] & 0,
\end{tikzcd}$$

with exact row and vertical isomorphisms (refer to Theorem 3.3.2).

Let $\{e_0, e_1\}$ be a $\widehat{\overline{R}}_{\mathcal{U}}$-basis of $T\otimes_{\mathcal{O}_{\mathcal{P}}}\widehat{\overline{R}}_{\mathcal{U}}$ such that

- $\mathrm{dlog}(e_0)$ is a $\widehat{\overline{R}}_{\mathcal{U}}$-basis of $\mathrm{Im}\,(\mathrm{dlog})$ and $e_0\equiv \varepsilon_0 \bmod \pi^{r-v}$;
- e_1 is a $\widehat{\overline{R}}_{\mathcal{U}}$-basis of $\mathrm{Ker}\,(\mathrm{dlog})$ and $e_1\equiv \varepsilon_1 \bmod \pi^{r-v}$.

Let X and Y denote the $\widehat{\overline{R}}_{\mathcal{U}}$-linear maps

$$T\otimes_{\mathcal{O}_{\mathcal{P}}}\widehat{\overline{R}}_{\mathcal{U}} \longrightarrow \widehat{\overline{R}}_{\mathcal{U}}$$

defined by

$$X(e_1)=Y(e_0)=0, \quad \text{and} \quad Y(e_1)=X(e_0)=1,$$

respectively. Then

$$\begin{aligned}
&\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n}\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)\\
=\quad &\mathscr{H}om_{S_{\mathfrak{M}^r(w)}}\left(\mathcal{F}'_{\mathfrak{M}^r(w)},\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)^\lambda_n\right)\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)
\end{aligned}$$

$$= \quad \mathrm{Hom}_{S_{r,v}}\left(S_{r,v}\cdot \underline{v},\ D^{\lambda}\right),$$

where

$$S_{r,v} = \mathcal{O}_{\mathcal{P}}^{\times}\left(1+\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}\right), \underline{v} = \mathrm{dlog}(e_0)$$

and D^{λ} is the module $S_{r,v}$-module D with the action of $S_{r,v}$ twisted by λ. Let X^{λ} denote the map

$$\begin{aligned} X^{\lambda} : S_{r,v}\cdot \underline{v} &\longrightarrow D^{\lambda} \\ ax\cdot \underline{v} &\longmapsto \lambda(a)x^{s_{\lambda}}, \end{aligned}$$

for $a \in \mathcal{O}_{\mathcal{P}}^{\times}$, $x \in \left(1+\pi^{r-v}\widehat{\overline{R}}_{\mathcal{U}}\right)$, where $s_{\lambda} \in B_L$ such that

$$\lambda(1+\pi^r y) = \exp\left(s_{\lambda}\log(y)\right),$$

for $y \in \mathcal{O}_{\mathcal{P}}$. Then we have the identification

$$\mathrm{Hom}_{S_{r,v}}\left(S_{r,v}\cdot \underline{v},\ D^{\lambda}\right) = D\cdot X^{\lambda},$$

i.e.,

$$\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n}\left(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\right) = D\cdot X^{\lambda}.$$

As a summary, after localization, we are now in the following situation:

$$\begin{tikzcd} D\cdot X^{\lambda} \arrow[r, "\alpha_n^r"] & \mathrm{Hom}_{\mathcal{O}_{\mathcal{P}}^{\times}}\left(\mathcal{O}_{\mathcal{P}}^{\times}\cdot\varepsilon_0 + \mathcal{O}_{\mathcal{P}}\cdot\varepsilon_1, D^{\lambda}\right) \\ & \displaystyle\bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{\infty} Dx^{\lambda}\left(\frac{(y/x)-z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}(y/x). \arrow[u, hook, "\beta_n^r"'] \end{tikzcd}$$

Then it suffices to show that X^{λ}, thought of as a map $T_0 \to D^{\lambda}$, can be written as a power series of x^{λ} and $\left(\frac{(y/x)-z_\eta}{\pi^r}\right)^h \mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}(y/x)$.

We write $X = sx + ty$ for some $s, t \in \widehat{\overline{R}}_{\mathfrak{U}}$. By our assumption we have

$$1 = X(e_0) = sx(e_0) + ty(e_0) \equiv sx(\varepsilon_0) + ty(\varepsilon_0) = s \mod \pi^{r-v},$$

$$0 = X(e_1) = sx(e_1) + ty(e_1) \equiv sx(\varepsilon_1) + ty(\varepsilon_1) = t \mod \pi^{r-v},$$

i.e., $s \in 1 + \pi^{r-v}\widehat{\overline{R}}_{\mathfrak{U}} \mod \pi^{r-v}$ and $t \in \pi^{r-v}\widehat{\overline{R}}_{\mathfrak{U}}$. Then we have

$$\alpha_n^r(X^\lambda) = x^\lambda \left(s + ty/x\right)^{s_\lambda} \in \mathrm{Im}(\beta_n^r).$$

This completes the proof. $\square$

Now we obtain a G_r-invariant morphism of sheaves of $\mathcal{O}_{\mathfrak{M}^r(w)} \otimes B$-modules on $\mathfrak{M}^r(w)$:

$$\gamma^r : \Omega_{\mathfrak{M}^r(w)}^{\lambda^{-1}} \longrightarrow j_r^* \mathcal{A}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)} \hat{\otimes} B\right).$$

Step 3 :

Applying $\mathscr{H}om_{\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B}\left(-, \mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)$ to the above map γ^r and using the identification

$$\mathscr{H}om_{\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B}\left(\Omega_{\mathfrak{M}^r(w)}^{\lambda^{-1}}, \mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right) \cong \Omega_{\mathfrak{M}^r(w)}^{\lambda},$$

we have a G_r-invariant morphism

$$\delta^r : \mathscr{H}om_{\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B}\left(j_r^* \mathcal{A}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right), \mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right) \longrightarrow \Omega_{\mathfrak{M}^r(w)}^{\lambda}.$$

In particular, for each $n \in \mathbb{N}$, we have a morphism

$$\delta_n^r : \mathscr{H}om_{\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n}\left(j_r^* \mathcal{A}_{w,n}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n, (\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)_n\right) \longrightarrow \Omega_{\mathfrak{M}^r(w),n}^{\lambda}.$$

We now have the following lemma.

Lemma 6.3.1. *For each $n \in \mathbb{N}$, there exist an integer $k_n \geq n$ and a morphism*

$$j_r^* \mathcal{D}_{w,k_n}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n \longrightarrow \Omega_{\mathfrak{M}^r(w),n}^{\lambda},$$

such that the following diagram is commutative

$$\begin{tikzcd}
\mathscr{H}om_{\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_{k_n}}\left(j_r^*\mathcal{A}^{0,\lambda}_{w,k_n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_{k_n},(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)_{k_n}\right) \arrow[r] & j_r^*\mathcal{D}^{0,\lambda}_{w,k_n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n \arrow[d, dashed] \\
\mathscr{H}om_{\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n}\left(j_r^*\mathcal{A}^{0,\lambda}_{w,n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n,(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)_n\right) \arrow[u] \arrow[r, "\delta_n^r"'] & \Omega^\lambda_{\mathfrak{M}^r(w),n},
\end{tikzcd}$$

where the maps will be described in the following remark.

Remark 6.3.1. Note that we can identify the $\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n$-module

$$\mathscr{H}om_{\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n}\left(j_r^*\mathcal{A}^{0,\lambda}_{w,n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n,(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)_n\right)$$

with

$$\mathrm{Hom}_B\left(j_r^*\mathcal{A}^{0,\lambda}_{w,n}, B/\underline{\mathbf{m}}^nB\right)\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n$$

for each $n\in\mathbb{N}$. Hence the left vertical map is the natural one induced by

$$A^0_\lambda/\underline{\mathbf{m}}^{k_n}A^0_\lambda\longrightarrow A^0_\lambda/\underline{\mathbf{m}}^nA^0_\lambda$$

for $k_n\geq n$. Moreover, by the construction, we also have the identification of $B/\underline{\mathbf{m}}^nB$-modules

$$D^0_\lambda/\underline{\mathbf{m}}^nD^0_\lambda\cong\mathrm{Hom}_B\left(A^0_\lambda/\underline{\mathbf{m}}^nA^0_\lambda, B/\underline{\mathbf{m}}^nB\right).$$

Then the top horizontal map is just the one induced by the quotient

$$\left(D^0_\lambda/\underline{\mathbf{m}}^{k_n}D^0_\lambda\right)\Big/\left(\mathrm{Fil}^{D^0_\lambda}/\underline{\mathbf{m}}^{k_n}D^0\right)\cong D^0_\lambda/\,\mathrm{Fil}^{k_n}D^0_\lambda.$$

Proof. It suffices to prove the above lemma by localizing at small affine objects of $\mathcal{M}(w)^{\mathtt{ket}}$ covering $\mathcal{M}(w)$. With the same notations as before and using the identification (equation 5.2)

$$D^0_{\lambda,n}:=D^0_\lambda/\,\mathrm{Fil}^nD^0_\lambda\cong\bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{n-1}\left(B/\underline{\mathbf{m}}^{n-h}B\right),$$

we have

$$\Omega^{\lambda}_{\mathfrak{M}^r(w),n}\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)=D\cdot\left(X^{\lambda}\right)^{\vee},$$

$$\begin{aligned}&j_r^*\mathcal{D}^{0,\lambda}_{w,n}\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)\\ =\quad&\bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{n-1}\left(B/\underline{\mathbf{m}}^{n-h}B\right)\left(x^{\lambda}\left(\frac{(y/x)-z_\eta}{\pi^r}\right)^h\mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}(y/x)\right)^{\vee},\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}_B\left(j_r^*\mathcal{A}^{0,\lambda}_{w,n},B/\underline{\mathbf{m}}^nB\right)\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)\\ =\quad&\bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{\infty}\left(B/\underline{\mathbf{m}}^{n}B\right)\left(x^{\lambda}\left(\frac{(y/x)-z_\eta}{\pi^r}\right)^h\mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}(y/x)\right)^{\vee}.\end{aligned}$$

Let $D':=\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_{k_n}\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)$. We are now in the following situation:

$$\begin{tikzcd}\bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{\infty}\left(B/\underline{\mathbf{m}}^{k_n}B\right)\left(Z^{\lambda}\right)^{\vee}\otimes D' \arrow[r] & \bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{k_n-1}\left(B/\underline{\mathbf{m}}^{k_n-h}B\right)\left(Z^{\lambda}\right)^{\vee}\otimes D' \\ \bigoplus_{z_\eta\in S}\bigoplus_{h=0}^{\infty}\left(B/\underline{\mathbf{m}}^{n}B\right)\left(Z^{\lambda}\right)^{\vee}\otimes D \arrow[u] \arrow[r, "\delta_n^r"'] & D\cdot\left(X^{\lambda}\right)^{\vee},\end{tikzcd}$$

where $Z^{\lambda}:=\left(x^{\lambda}\left(\frac{(y/x)-z_\eta}{\pi^r}\right)^h\mathbb{1}_{z_\eta+\pi^r\mathcal{O}_{\mathcal{P}}}(y/x)\right)$ and $D\to D'$ is the natural map.

Recall that in the proof of Proposition 6.3.1, we showed that X^{λ} can be written as a power series of Z^{λ}. Let N_n be the maximal number such that the coefficient of X^{λ} with respect to Z^{λ} is nonzero. Next we take an integer k_n such that $k_n\geq n+N_n$, then the map δ_n^r localized at $\mathcal{U}$ factors through $j_r^*\mathcal{D}^{0,\lambda}_{w,k_n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_n\left(\overline{R}_{\mathcal{U}},\overline{N}_{\mathcal{U}},g\right)$ as required. This completes the proof of the lemma. $\square$

Step 4 :

Now for any $n\in\mathbb{N}$, there is an integer $k_n\geq n$ and a morphism

$$j_r^*\mathcal{D}^{0,\lambda}_{w,k_n}\otimes\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)_{k_n}\longrightarrow\Omega^{\lambda}_{\mathfrak{M}^r(w),n}.$$

We thus have a morphism of continuous sheaves of $\left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)$-modules

$$\delta_w^{0,\lambda} : j_r^*\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right) \longrightarrow \Omega^\lambda_{\mathfrak{M}^r(w)},$$

hence a morphism of continuous sheaves of $\left(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B\right)$-modules

$$\left(j_{r,*}\left(j_r^*\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)\right)\right)^{G_r} \longrightarrow \left(j_{r,*}\Omega^\lambda_{\mathfrak{M}^r(w)}\right)^{G_r} = \Omega^\lambda_{\mathfrak{M}(w)}.$$

Since $\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)} = j_r^*\widehat{\mathcal{O}}_{\mathfrak{M}(w)}$, we have a natural morphism

$$\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B\right) \longrightarrow \left(j_{r,*}\left(j_r^*\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B\right)\right)\right)^{G_r}.$$

Composing the above two morphisms of continuous sheaves of $\left(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B\right)$-modules, and passing to ind-continuous sheaves, we obtain

$$\delta_w^\lambda : \mathcal{D}_w^\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B\right) \longrightarrow \omega^\lambda_{\mathfrak{M}(w)}. \tag{6.2}$$

Finally, the above morphism gives a morphism on cohomology

$$\mathrm{H}^1\left(\mathfrak{M}(w), \mathcal{D}_w^\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes}B\right)\right) \longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right).$$

Composing with the morphism in formula (6.1), we obtain the morphism

$$\mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)}\hat{\otimes}B\right)\right) \longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right)$$

as we want from the beginning of this section, which is the main goal of this chapter.

6.4 Hecke operators

First, we define the U operator following the same line as in Section 3.5.1. Let $\mathcal{M}(H\pi^r, q)$, $M(H\pi^r, q)$, $\mathcal{M}_q^r(w)$ and $M_q^r(w)$ be as described in Section 3.5.1. Recall that we have two morphisms

$$\begin{aligned}\pi_1 : M_q^r(w) &\longrightarrow M^r(w),\\ \pi_2 : M_q^r(qw) &\longrightarrow M^r(w),\end{aligned}$$

where π_1 is defined by forgeting the level struction given by the group D, π_2 is defined by taking quotient by D (refer to Section 3.5.1 for details). Similar as in [Andreatta et al., 2014, §3.2], we can prove that π_1 is finite and étale. Then $(\mathcal{M}(w), M_q^r(w))$ is an object of $\mathfrak{M}^r(w)$ hence also an object of $\mathfrak{M}(w)$. Let $\mathfrak{M}_q^r(w)$ denote the induced site

$$\mathfrak{M}_q^r(w) := \mathfrak{M}(w)_{/(\mathcal{M}(w), M_q^r(w))} = \mathfrak{M}^r(w)_{/(\mathcal{M}(w), M_q^r(w))}.$$

Then we have natural continuous functors

$$\begin{aligned}\mathfrak{p}_1 : \mathfrak{M}^r(qw) &\longrightarrow \mathfrak{M}_q^r(qw)\\ (\mathcal{U}, W) &\longmapsto \left(\mathcal{U}, W \times_{M^r(qw)} M_q^r(qw), pr_2\right),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{p}_2 : \mathfrak{M}^r(w) &\longrightarrow \mathfrak{M}_q^r(qw)\\ (\mathcal{U}, W, u) &\longmapsto \left(\mathcal{U}, W \times_{M^r(w)} M_q^r(qw), pr_2\right).\end{aligned}$$

Note that $\mathfrak{p}_1$ is just the functor j defined in Section 4.3.3, hence $\mathfrak{p}_{1,*}$ is exact.

Moreover, for fixed r, let $w \leq w'$ be two rational numbers adapted to r such that they satisfy the same conditions as w. Then we have natural morphisms and the following commutative diagram:

$$\begin{tikzcd} M^r(w) \arrow[r, "g_{w,w'}"] \arrow[d] & M^r(w') \arrow[d] \\ M(w) \arrow[r, "f_{w,w'}"'] & M(w'). \end{tikzcd}$$

Let $(\mathcal{U}', W', u')$ be an object on $\mathfrak{M}^r(w')$, we have a natural commutative diagram:

$$\begin{tikzcd}
& W' \times_{M^r(w')} M^r(w) \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & M^r(w) \arrow[dl, "g_{w,w'}"'] \arrow[dd] \\
W' \arrow[rr] \arrow[dd] & & M^r(w') \arrow[dd] & \\
& \mathcal{U}'_L \times_{M(w')} M(w) \arrow[rr] \arrow[dl] & & M(w) \arrow[dl, "f_{w,w'}"'] \\
\mathcal{U}'_L \arrow[rr] & & M(w'). &
\end{tikzcd}$$

If we denote by

$$\begin{aligned}
\mathcal{U} &:= \mathcal{U}' \times_{\mathcal{M}(w')} \mathcal{M}(w), \\
W &:= W' \times_{M^r(w')} M^r(w),
\end{aligned}$$

then $W \to \mathcal{U}_L \cong \mathcal{U}'_L \times_{M(w')} M(w)$ is finite étale. In other words, $(\mathcal{U}, W, pr_2)$ is also an object of $\mathfrak{M}^r(w)$. Thus we obtain a functor of sites:

$$\rho_{w',w} : \mathfrak{M}^r(w') \longrightarrow \mathfrak{M}^r(w),$$

such that $\rho^*_{w',w}\Omega^\lambda_{\mathfrak{M}^r(w')} \cong \Omega^\lambda_{\mathfrak{M}^r(w)}$. In particular, we have a functor

$$\rho_{qw,w} : \mathfrak{M}^r(qw) \longrightarrow \mathfrak{M}^r(w).$$

Let $\mathcal{A}^r_q(w)$, $\mathcal{A}^r(w)$, $\mathcal{D}$ and $\pi_{\mathcal{D}} : \mathcal{A}^r_q(qw) \longrightarrow \mathcal{A}^r_q(qw)/\mathcal{D}$ be as in Section 3.5.1. Recall that we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^r_q(qw) \arrow[rrrr, "\pi_{\mathcal{D}}"] \arrow[drr] \arrow[dd] & & & & \mathcal{A}^r_q(qw)\big/\mathcal{D} \arrow[dd] \\
& & \mathcal{A}^r(qw) \arrow[urr] \arrow[dd] & & \\
\mathcal{M}^r_q(qw) \arrow[rrrr, equal] \arrow[drr, "\pi_1"'] & & & & \mathcal{M}^r_q(qw) \arrow[dll, "g_{w,qw}\circ\pi_2"] \\
& & \mathcal{M}^r(qw) & &
\end{tikzcd} \tag{6.3}$$

where the left and right squares are Carterian and the square in the back is commutative. Similar as in Section 3.5.1 we obtain a morphism

$$\widetilde{\pi}_{\mathcal{D}} : \mathfrak{p}_2^*\Omega^\lambda_{\mathfrak{M}^r(w)} \longrightarrow \mathfrak{p}_1^*\Omega^\lambda_{\mathfrak{M}^r(qw)}.$$

We define an operator $\widetilde{\mathrm{U}}$ by the following composition:

$$\begin{aligned}
&\mathrm{H}^i\left(\mathfrak{M}^r(qw), \Omega^\lambda_{\mathfrak{M}^r(qw)}\right)\\
&\xrightarrow{\rho^*_{qw,w}} \mathrm{H}^i\left(\mathfrak{M}^r(w), \rho^*_{qw,w}\Omega^\lambda_{\mathfrak{M}^r(qw)}\right)\\
&=\!=\!=\!= \mathrm{H}^i\left(\mathfrak{M}^r(w), \Omega^\lambda_{\mathfrak{M}^r(w)}\right)\\
&\xrightarrow{\mathfrak{p}_2^*} \mathrm{H}^i\left(\mathfrak{M}^r_q(qw), \mathfrak{p}_2^*\Omega^\lambda_{\mathfrak{M}^r(w)}\right)\\
&\xrightarrow{\widetilde{\pi}_{\mathcal{D}}} \mathrm{H}^i\left(\mathfrak{M}^r_q(qw), \mathfrak{p}_1^*\Omega^\lambda_{\mathfrak{M}^r(qw)}\right)\\
&=\!=\!= \mathrm{H}^i\left(\mathfrak{M}^r_q(qw), \mathfrak{p}_{1,*}\mathfrak{p}_1^*\Omega^\lambda_{\mathfrak{M}^r(qw)}\right)\\
&\longrightarrow \mathrm{H}^i\left(\mathfrak{M}^r(qw), \Omega^\lambda_{\mathfrak{M}^r(qw)}\right),
\end{aligned}$$

where the last map is the trace map as described by formula (4.4) in Section 4.3.3, since $\mathfrak{p}_{1,*}$ is exact. Passing to the ind-continuous sheaves and taking the G_r-invariants, we obtain an operator (recall that both $\mathfrak{p}_{1,*}$ and $j_{r,*}$ are exact)

$$\mathrm{U} : \mathrm{H}^i\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right) \longrightarrow \mathrm{H}^i\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right).$$

Moreover, we have the following commutative diagrams:

$$\begin{tikzcd}
\mathrm{H}^0\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right) \arrow[r, "\sim"] \arrow[d, "\mathrm{U}"'] & \mathrm{H}^0\left(M(w), \omega^\lambda_w\right) \hat{\otimes}\, \mathbb{C}_p \arrow[d, "\mathrm{U}"] \\
\mathrm{H}^0\left(\mathfrak{M}(w), \omega^\lambda_{\mathfrak{M}(w)}\right) \arrow[r, "\sim"] & \mathrm{H}^0\left(M(w), \omega^\lambda_w\right) \hat{\otimes}\, \mathbb{C}_p,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}\right) \arrow[r, "\sim"] \arrow[d, "\mathrm{U}"'] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p(-1) \arrow[d, "\mathrm{U}"] \\
\mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}\right) \arrow[r, "\sim"] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p(-1),
\end{tikzcd}$$

where the left vertical maps U are defined as above and the right vertical maps U are as in Section 3.5.1. In other words, all the isomorphisms obtained in Corollary 6.2.1 are invariant under the action of G_L and the U operator.

Similarly, the commutative diagram (6.3) also induces a morphism

$$\widetilde{\pi}_{\mathcal{D}} : \mathfrak{p}_2^*\left(j_r^*\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)\hat{\otimes}B}\right)\right) \longrightarrow \mathfrak{p}_1^*\left(j_r^*\mathcal{D}_{qw}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(qw)\hat{\otimes}B}\right)\right)$$

such that the following diagram

$$\begin{tikzcd}
\mathfrak{p}_2^*\left(j_r^*\mathcal{D}_w^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(w)\hat{\otimes}B}\right)\right) \arrow[r, "\widetilde{\pi}_{\mathcal{D}}"] \arrow[d, "\delta_w^{0,\lambda}"'] & \mathfrak{p}_1^*\left(j_r^*\mathcal{D}_{qw}^{0,\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}^r(qw)\hat{\otimes}B}\right)\right) \arrow[d, "\delta_w^{0,\lambda}"] \\
\mathfrak{p}_2^*\Omega^{\lambda}_{\mathfrak{M}^r(w)} \arrow[r, "\widetilde{\pi}_{\mathcal{D}}"] & \mathfrak{p}_1^*\Omega^{\lambda}_{\mathfrak{M}^r(qw)}
\end{tikzcd}$$

commutes, where $\delta_w^{0,\lambda}$ is the morphism obtained in Section 6.3.2. This implies the map

$$\mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_{\lambda} \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)}\hat{\otimes}B\right)\right) \longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}\right)$$

obtained at the end of the previous section is compatible with the U operator.

Similarly as in Section 3.5.2, we can define other Hecke operators and they are compatible with the morphism obtained in previous section.

Chapter 7

Eichler-Shimura isomorphisms

Recall that in last chapter we obtain a morphism

$$\mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)}\hat{\otimes} B\right) \otimes L\right) \longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}\right).$$

Moreover, by Corollary 6.2.1, we have an isomorphism

$$\mathrm{H}^1\left(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)\right) \cong \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes}_L \mathbb{C}_p .$$

Composing the above we have

$$\Psi : \mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_\lambda \otimes \left(\mathcal{O}_{\mathfrak{M}(H,\pi)}\hat{\otimes} B\right) \otimes L(1)\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right) \hat{\otimes}_L \mathbb{C}_p .$$

In particular, if $\mathfrak{U} \subset \mathcal{W}_r$ is an wide open disk, $B = \Lambda_{\mathfrak{U}}$ and $\lambda_{\mathfrak{U}}$ is the universal weight associated to $\mathfrak{U}$, then we have a morphism

$$\Psi_{\mathfrak{U}} : \mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_{\mathfrak{U}} \otimes \widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)}\hat{\otimes} B_{\mathfrak{U}}(1)\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right) \hat{\otimes}_L \mathbb{C}_p,$$

where $B_{\mathfrak{U}} := \Lambda_{\mathfrak{U}} \otimes L$. If $\lambda \in \mathfrak{U}(L)$ is a weight, we have

$$\Psi_\lambda : \mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_\lambda \otimes \widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)}\hat{\otimes} L(1)\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes}_L \mathbb{C}_p .$$

By construction, the following diagram

$$\begin{tikzcd}
\mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_{\mathfrak{U}} \otimes \widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)} \hat{\otimes} B_{\mathfrak{U}}(1)\right) \arrow[r, "\Psi_{\mathfrak{U}}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right) \hat{\otimes}_L \mathbb{C}_p \arrow[d] \\
\mathrm{H}^1\left(\mathfrak{M}(H,\pi), \mathcal{D}_{\lambda} \otimes \widehat{\mathcal{O}}_{\mathfrak{M}(H,\pi)} \hat{\otimes} L(1)\right) \arrow[r, "\Psi_{\lambda}"] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes}_L \mathbb{C}_p
\end{tikzcd}$$

is commutative, where the vertical maps are induced by the specializations.

The main goal of this chapter is to study the map Ψ_λ using the map $\Psi_{\mathfrak{U}}$. First we figure out what happens when $\lambda = k > 0$ is an integer.

7.1 Classical weights

First we fix some notations for this section. Let $\mathcal{M} := \mathcal{M}(H,\pi)$, $M =: M(H,\pi)$ and $\mathfrak{M} := \mathfrak{M}(H,\pi)$ the Faltings' site associated to the pair $(\mathcal{M}, M)$. Let $\varepsilon : \mathcal{A} \to \mathcal{M}$ be the universal abelian scheme and denote $\underline{\omega} := \left(\varepsilon_* \Omega^1_{\mathcal{A}/\mathcal{M}}\right)_1^{2,1}$. Let $\mathcal{T} := \mathrm{T}_\pi\big((\mathcal{A}[\pi^\infty]_1^{2,1})^\vee\big)$. For any integer $k \geq 0$, consider $\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L$ as in ind-continuous sheaf on $M_{\overline{L}}^{\mathtt{et}}$, it can also be viewed as an ind-continuous sheaf on $\mathfrak{M}$. We have the following proposition.

Proposition 7.1.1. *With the above notations, we have a canonical isomorphism compatible with the actions of G_L and all Hecke operators*

$$\mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1)\right) \cong \left(\mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p\right) \oplus \left(\mathrm{H}^1(M, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1)\right).$$

Proof. We prove the statement by localizing at connected small affine object of $\mathcal{M}^{\mathtt{ket}}$ covering $\mathcal{M}$. Let $\mathcal{U} = (\mathrm{Spf}(R), N)$ be such an object such that $\underline{\omega}$ restricted to U is a free R-module of rank 1. Let A be the corresponding abelian scheme defined over R, $T := \mathrm{T}_\pi\big((A[\pi^\infty]_1^{2,1})^\vee\big)$, $V := \mathrm{Sym}^k(T) \otimes \widehat{\overline{R}}[1/\pi]$ and $\underline{\omega}_R$ the pullback of $\underline{\omega}$ to $\mathcal{U}$. Recall that we have a continuous functor $v : \mathcal{M}^{\mathtt{ket}} \to \mathfrak{M}$ sending $\mathcal{U}$ to $(\mathcal{U}, \mathcal{U}_L)$. The Leray spectral sequence

$$\mathrm{H}^i\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^j\, v_*\left(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L\right)\right) \Longrightarrow \mathrm{H}^{i+j}\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L\right)$$

for $i + j = 1$ degenerates to the exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^1\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L\, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right) \to \\ &\to \mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L\right) \to \mathrm{H}^0\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^1\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right) \to \\ &\to \mathrm{H}^2\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right). \end{aligned}$$

By Lemma 6.2.1, the sheaf $\mathrm{R}^j\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)$ is just the sheaf associated to the following presheaf on $\mathcal{M}^{\mathtt{ket}}$:

$$\mathcal{U} \longmapsto \mathrm{H}^j\left(\Delta, (\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)(\overline{R}, \overline{N})\right),$$

where $\Delta := \mathrm{Gal}\left(\overline{R}L/R_{\overline{L}}\right)$ is a subgroup of $\mathcal{G} := \mathrm{Gal}\left(\overline{R}L/RL\right)$ and the localization

$$\left(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L\right)(\overline{R}, \overline{N}) = \mathrm{Sym}^k(T) \otimes \widehat{\overline{R}}[1/\pi] = V.$$

First we claim that:

$$\mathrm{H}^0(\Delta, V) \cong \underline{\omega}_R^{-k} \otimes \mathbb{C}_p(k),$$

$$\mathrm{H}^1(\Delta, V) \cong \underline{\omega}_R^{k+2} \otimes \mathbb{C}_p(-1).$$

Granted this two claims we deduce:

$$\mathrm{H}^0\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^1\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right) \otimes \mathbb{C}_p \cong \mathrm{H}^0\left(M, \underline{\omega}^{k+2}\right) \otimes \mathbb{C}_p(-1),$$

$$\mathrm{H}^1\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right) \otimes \mathbb{C}_p \cong \mathrm{H}^1\left(M, \underline{\omega}^{-k}\right) \otimes \mathbb{C}_p(k).$$

Moreover we have

$$\mathrm{H}^2\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L)\right) \otimes \mathbb{C}_p \cong \mathrm{H}^2\left(M, \underline{\omega}^{-k}\right) \otimes \mathbb{C}_p(k) = 0$$

since M has dimension 1. Therefore we have an exact sequence of $\mathbb{C}_p$-modules compatible with the actions of G_L and the Hecke operators

$$\begin{tikzcd}
0 \arrow[d] & \\
\mathrm{H}^1\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1))\right) \arrow[d] \arrow[r, equal] & \mathrm{H}^1(M, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1) \\
\mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1)\right) \arrow[d] & \\
\mathrm{H}^0\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^1\, v_*(\mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1))\right) \arrow[d] \arrow[r, equal] & \mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p \\
0. &
\end{tikzcd}$$

Similar to the main result proved in Tate [1967], the above sequence splits canonically and we deduce the proposition.

Now we prove our claims. We start with the following exact Hodge-Tate sequence of $\widehat{\overline{R}}[1/\pi]$-modules with semilinear Δ-actions, associated to A:

$$0 \longrightarrow \underline{\omega}_{A^\vee/R}^{-1} \otimes_R \widehat{\overline{R}}[1/\pi](1) \longrightarrow T \otimes \widehat{\overline{R}}[1/\pi] \xrightarrow{\mathrm{dlog}} \underline{\omega}_{A/R} \otimes_R \widehat{\overline{R}}[1/\pi] \longrightarrow 0.$$

Let e_0, e_1 be an $\widehat{\overline{R}}[1/\pi]$-basis of $T \otimes \widehat{\overline{R}}[1/\pi]$ such that

- e_1 is a $\widehat{\overline{R}}[1/\pi]$-basis of $\underline{\omega}_{A^\vee/R}^{-1}$

  and

- $\mathrm{dlog}(e_0)$ is a basis of $\underline{\omega}_R$, i.e., $\sigma\mathrm{dlog}(e_0) = \mathrm{dlog}(e_0)$ for any $\sigma \in \mathcal{G}$.

This gives us the following filtration of V:

$$0 =: \mathrm{Fil}^{-1}(V) \subseteq \mathrm{Fil}^0(V) \subseteq \mathrm{Fil}^1(V) \subseteq \cdots \subseteq \mathrm{Fil}^{k-1}(V) \subseteq \mathrm{Fil}^k(V) := V,$$

where $\mathrm{Fil}^i(V) := \sum_{n=0}^{i} \widehat{\overline{R}}[1/\pi]e_1^{k-n}e_0^n$, for $i = 0, 1, \ldots, k$. For example,

$$\mathrm{Fil}^0(V) = \widehat{\overline{R}}[1/\pi]e_1^k,$$

$$\mathrm{Fil}^1(V) = \widehat{\overline{R}}[1/\pi]e_1^k + \widehat{\overline{R}}[1/\pi]e_1^{k-1}e_0.$$

We have the following results (refer to Remark 6.2.1):

(i) $\mathrm{H}^0(\Delta, \widehat{\overline{R}}[1/\pi]) = R_{\mathbb{C}_p}$,

(ii) $\mathrm{H}^1(\Delta, \widehat{\overline{R}}[1/\pi]) = \underline{\omega}_R^2 \hat{\otimes} \mathbb{C}_p(-1)$,

where $R_{\mathbb{C}_p}$ represents the completed tensor product $R\hat{\otimes}\mathbb{C}_p$. Using these we have:

$$\begin{aligned}
\mathrm{H}^0(\Delta, \mathrm{Fil}^0(V)) &\cong \mathrm{H}^0(\Delta, \underline{\omega}_R^{-k} \otimes \widehat{\overline{R}}[1/\pi](k)) \\
&\cong \underline{\omega}_R^{-k} \otimes \mathrm{H}^0(\Delta, \widehat{\overline{R}}[1/\pi](k)) \cong \underline{\omega}_R^{-k}\hat{\otimes}\mathbb{C}_p(k), \\
\mathrm{H}^1(\Delta, \mathrm{Fil}^0(V)) &\cong \mathrm{H}^1(\Delta, \underline{\omega}_R^{-k} \otimes \widehat{\overline{R}}[1/\pi](k)) \\
&\cong \underline{\omega}_R^{-k} \otimes \mathrm{H}^1(\Delta, \widehat{\overline{R}}[1/\pi](k)) \cong \underline{\omega}_R^{-k+2}\hat{\otimes}\mathbb{C}_p(k-1).
\end{aligned}$$

Moreover, for any $0 \leq i \leq k-1$, we have

$$\begin{aligned}
\mathrm{H}^0(\Delta, \mathrm{Fil}^{i+1} / \mathrm{Fil}^i) &\cong \mathrm{H}^0(\Delta, \underline{\omega}_R^{2i+2-k} \otimes \widehat{\overline{R}}[1/\pi](k-i-1)) \\
&\cong \underline{\omega}_R^{2i+2-k}\hat{\otimes}\mathbb{C}_p(k-i-1), \\
\mathrm{H}^1(\Delta, \mathrm{Fil}^{i+1} / \mathrm{Fil}^i) &\cong \mathrm{H}^1(\Delta, \underline{\omega}_R^{2i+2-k} \otimes \widehat{\overline{R}}[1/\pi](k-i-1)) \\
&\cong \underline{\omega}_R^{2i+4-k}\hat{\otimes}\mathbb{C}_p(k-i-2).
\end{aligned}$$

The class of extension

$$0 \to \mathrm{Fil}^i / \mathrm{Fil}^{i-1} \to \mathrm{Fil}^{i+1} / \mathrm{Fil}^{i-1} \to \mathrm{Fil}^{i+1} / \mathrm{Fil}^i \to 0$$

in $\mathrm{H}^1\left(\Delta, \underline{\omega}_R^{-2} \otimes_R \widehat{\overline{R}}[1/\pi](1)\right) \cong \underline{\omega}_R^{-2} \otimes_R \mathrm{H}^1(\Delta, \widehat{\overline{R}}[1/\pi](1)) \cong R_{\mathbb{C}_p}$ can be computed from the Kodaira-Spencer class and turns out to be a unit. Then by induction, for any $i = 1, 2, \ldots, k$, we have:

$$\begin{aligned}
\mathrm{H}^0(\Delta, \mathrm{Fil}^i) &= \underline{\omega}_R^{-k} \hat{\otimes} \mathbb{C}_p(k), \\
\mathrm{H}^1(\Delta, \mathrm{Fil}^i) &= \underline{\omega}_R^{-k+2+2i} \hat{\otimes} \mathbb{C}_p(k-1-i).
\end{aligned}$$

In particular,

$$\begin{aligned}
\mathrm{H}^0(\Delta, V) &= \mathrm{H}^0(\Delta, \mathrm{Fil}^k) = \underline{\omega}_R^{-k} \hat{\otimes} \mathbb{C}_p(k), \\
\mathrm{H}^1(\Delta, V) &= \mathrm{H}^1(\Delta, \mathrm{Fil}^k) = \underline{\omega}_R^{k+2} \hat{\otimes} \mathbb{C}_p(-1).
\end{aligned}$$

This proves the claims and the proposition follows. $\square$

Remark 7.1.1. The analogue result for modular curves was proved by Faltings in Faltings [1987]. The above proof follows from the main lines of the arguments in Faltings' paper.

Recall that we have a natural isomorphism

$$\mathrm{H}^1\left(M_{\overline{L}}^{\mathsf{et}}, \mathcal{V}_k(1)\right) \otimes_L \mathbb{C}_p \cong \mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1)\right).$$

Let Φ_k be the composite of the following morphisms

$$\begin{aligned}
& \mathrm{H}^1\left(M_{\overline{L}}^{\mathsf{et}}, \mathcal{V}_k(1)\right) \otimes_L \mathbb{C}_p \\
\xrightarrow{\sim}\; & \mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \otimes \widehat{\mathcal{O}}_{\mathfrak{M}} \hat{\otimes} L(1)\right) \\
\xrightarrow{\sim}\; & \left(\mathrm{H}^1(M, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1)\right) \bigoplus \left(\mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p\right) \\
\longrightarrow\; & \mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p,
\end{aligned}$$

where the last map is the projection to the second factor. Then we have

Proposition 7.1.2. *Let $k \geq 0$ be an integer. Then the following diagram is commutative*

$$\begin{tikzcd}
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_k\right) \otimes_L \mathbb{C}_p(1) \arrow[r, "\Psi_k"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{k+2}\right) \hat{\otimes}_L \mathbb{C}_p \\
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_k(1)\right) \otimes_L \mathbb{C}_p \arrow[r, "\Psi_k"] & \mathrm{H}^0\left(M(H,\pi), \omega^{k+2}\right) \hat{\otimes}_L \mathbb{C}_p, \arrow[u]
\end{tikzcd}$$

where the left vertical map is the one induced by specialization and the right vertical map is the restriction.

Proof. Recall that we have the following commutative diagram (refer to Section 4.4):

$$\begin{tikzcd}
\mathcal{M}^{\mathtt{ket}} \arrow[r, "v_{\mathfrak{M}}"] \arrow[d, "\mu"'] & \mathfrak{M} \arrow[d, "\nu"] \\
\mathcal{M}^{\mathtt{ket}}(w) \arrow[r, "v_{\mathfrak{M}(w)}"'] & \mathfrak{M}(w).
\end{tikzcd} \tag{7.1}$$

Let $\omega^k_{\mathfrak{M}} := v^*_{\mathfrak{M}}\underline{\omega}^k \otimes_{\widehat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}}} \widehat{\mathcal{O}}_{\mathfrak{M}}$, where $\underline{\omega} = \underline{\omega}_{\mathcal{A}/\mathcal{M}}$ with $\mathcal{A} \to \mathcal{M}$ is the universal abelian scheme. Let $\mathcal{T}_w := \nu^*(\mathcal{T})$. We prove the proposition by showing that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{H}^1\left(M^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_k\right) \otimes \mathbb{C}_p(1) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^1\left(M^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_k(1)\right) \otimes \mathbb{C}_p \arrow[d, "\cong"] \\
\mathrm{H}^1\left(\mathfrak{M}, \mathcal{D}_k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}} \otimes L(1)\right) \arrow[r] \arrow[d] & \mathrm{H}^1\left(\mathfrak{M}, \mathcal{V}_k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}} \otimes L(1)\right) \arrow[d, equal] \\
\mathrm{H}^1\left(\mathfrak{M}(w), \nu^*(\mathcal{D}_k) \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}(w)} \otimes L(1)\right) \arrow[d, "\delta^k_w"'] & \mathrm{H}^1\left(\mathfrak{M}, \mathrm{Sym}^k(\mathcal{T}) \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}} \otimes L(1)\right) \arrow[d] \\
\mathrm{H}^1\left(\mathfrak{M}(w), \omega^k_{\mathfrak{M}(w)} \otimes L(1)\right) \arrow[d, equal] & \mathrm{H}^1\left(\mathfrak{M}, \omega^k_{\mathfrak{M}} \otimes L(1)\right) \arrow[l, "\widetilde{\mu}"'] \arrow[d] \\
\mathrm{H}^0\left(M(w), \omega_w^{k+2}\right) \otimes \mathbb{C}_p & \mathrm{H}^0\left(M, \underline{\omega}^{k+2}\right) \otimes \mathbb{C}_p \arrow[l, "\mu^*"']
\end{tikzcd} \tag{7.2}$$

where the two horizontal maps on the top are induced by the specializaion. Moreover, the left and right vertical compositions are just Ψ_k and Φ_k, respectively. We will explain the other maps in the proof.

- The top square is obviously commutative.
- For the square at the bottom, we first explain how to obtain the horizontal map

$$\widetilde{\mu} : \mathrm{H}^1\left(\mathfrak{M}, \omega_{\mathfrak{M}}^k \otimes L(1)\right) \longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega_{\mathfrak{M}(w)}^k \otimes L(1)\right).$$

In fact $\widetilde{\mu}$ is defined by the following compositions:

$$\begin{aligned}
\mathrm{H}^1\left(\mathfrak{M}, \omega_{\mathfrak{M}}^k \otimes L(1)\right) &= \mathrm{H}^1\left(\mathfrak{M}, v_{\mathfrak{M}}^* \underline{\omega}^k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}} \otimes L(1)\right) \\
&\xrightarrow{\nu^*} \mathrm{H}^1\left(\mathfrak{M}(w), \nu^* v_{\mathfrak{M}}^* \underline{\omega}^k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}(w)} \otimes L(1)\right) \\
&\overset{(1)}{=} \mathrm{H}^1\left(\mathfrak{M}(w), v_{\mathfrak{M}(w)}^* \mu^* \underline{\omega}^k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}(w)} \otimes L(1)\right) \\
&= \mathrm{H}^1\left(\mathfrak{M}(w), v_{\mathfrak{M}(w)}^* \omega_w^k \hat{\otimes} \widehat{\mathcal{O}}_{\mathfrak{M}(w)} \otimes L(1)\right) \\
&= \mathrm{H}^1\left(\mathfrak{M}(w), v_{\mathfrak{M}(w)}^* \left(\omega_w^k \hat{\otimes} \mathbb{C}_p(1)\right)\right) \\
&\overset{(2)}{=} \mathrm{H}^1\left(\mathfrak{M}(w), v_{\mathfrak{M}(w)}^* \left(v_{\mathfrak{M}(w),*} \omega_{\mathfrak{M}(w)}^k\right) \hat{\otimes} L(1)\right) \\
&\longrightarrow \mathrm{H}^1\left(\mathfrak{M}(w), \omega_{\mathfrak{M}(w)}^k \hat{\otimes} L(1)\right)
\end{aligned}$$

where equality (1) is obtained from the commutative diagram (7.1) and equality (2) is by Proposition 6.1.1.

Now consider the Leray spectral sequence

$$\mathrm{H}^p\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^q\, v_{\mathfrak{M},*}(\omega_{\mathfrak{M}}^k \otimes L(1))\right) \Longrightarrow \mathrm{H}^{p+q}\left(\mathfrak{M}, \omega_{\mathfrak{M}}^k \otimes L(1)\right).$$

For $p+q=1$, we get an edge map

$$\mathrm{H}^1\left(\mathfrak{M},\omega^k_{\mathfrak{M}}\otimes L(1)\right)\longrightarrow \mathrm{H}^0\left(\mathcal{M}^{\mathsf{ket}},\mathrm{R}^1\,v_{\mathfrak{M},*}(\omega^k_{\mathfrak{M}}\otimes L(1))\right),$$

which is just the right vertical map in the bottom square of diagram 7.2. Recall from Theorem 6.2.1 that we have isomorphisms

$$\begin{aligned}\mathrm{H}^1\left(\mathfrak{M}(w),\omega^k_{\mathfrak{M}(w)}\otimes L(1)\right) &\cong \mathrm{H}^0\left(\mathcal{M}^{\mathsf{ket}}(w),\mathrm{R}^1\,v_{\mathfrak{M}(w),*}\left(\omega^k_{\mathfrak{M}(w)}\otimes L(1)\right)\right)\\ &\cong \mathrm{H}^0\left(M(w),\omega_w^{k+2}\right)\otimes\mathbb{C}_p\,.\end{aligned}$$

Thus we obtain a commutative diagram given by

$$\begin{tikzcd}
\mathrm{H}^1\left(\mathfrak{M}(w),\omega^k_{\mathfrak{M}(w)}\otimes L(1)\right) \arrow[d, equal] & \mathrm{H}^1\left(\mathfrak{M},\omega^k_{\mathfrak{M}}\otimes L(1)\right) \arrow[l, "\widetilde{\mu}"'] \arrow[d] \\
\mathrm{H}^0\left(\mathcal{M}^{\mathsf{ket}}(w),\mathrm{R}^1\,v_{\mathfrak{M}(w),*}\left(\omega^k_{\mathfrak{M}(w)}\otimes L(1)\right)\right) \arrow[d, equal] & \mathrm{H}^0\left(\mathcal{M}^{\mathsf{ket}},\mathrm{R}^1\,v_{\mathfrak{M},*}(\omega^k_{\mathfrak{M}}\otimes L(1))\right) \arrow[l, "\widetilde{\mu}"'] \arrow[d, equal] \\
\mathrm{H}^0\left(M(w),\omega_w^{k+2}\right)\otimes\mathbb{C}_p & \mathrm{H}^0\left(M,\underline{\omega}^{k+2}\right)\otimes\mathbb{C}_p \arrow[l, "\mu^*"']
\end{tikzcd}$$

which implies the commutativity of the bottom square.

- We deduce that the middle square commutes since the following diagram is commutative

$$\begin{tikzcd}
\mathrm{H}^1\left(\mathfrak{M}(w),\nu^*(\mathcal{D}_k)\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\otimes L(1)\right) \arrow[d] & \mathrm{H}^1\left(\mathfrak{M},\mathcal{D}_k\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}}\otimes L(1)\right) \arrow[l, "\nu^*"'] \arrow[d] \\
\mathrm{H}^1\left(\mathfrak{M}(w),\nu^*(\mathcal{V}_k)\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\otimes L(1)\right) \arrow[d, equal] & \mathrm{H}^1\left(\mathfrak{M},\mathcal{V}_k\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}}\otimes L(1)\right) \arrow[l, "\nu^*"'] \arrow[d, equal] \\
\mathrm{H}^1\left(\mathfrak{M}(w),\mathrm{Sym}^k(\mathcal{T}_w)\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}(w)}\otimes L(1)\right) \arrow[d] & \mathrm{H}^1\left(\mathfrak{M},\mathrm{Sym}^k(\mathcal{T})\hat{\otimes}\widehat{\mathcal{O}}_{\mathfrak{M}}\otimes L(1)\right) \arrow[l, "\nu^*"'] \arrow[d] \\
\mathrm{H}^1\left(\mathfrak{M}(w),\omega^k_{\mathfrak{M}(w)}\otimes L(1)\right) & \mathrm{H}^1\left(\mathfrak{M},\omega^k_{\mathfrak{M}}\otimes L(1)\right). \arrow[l, "\widetilde{\mu}"']
\end{tikzcd}$$

Here, both vertical maps in the bottom square are induced by the dlog map. Note that the left vertical composite is just the map δ_w^k as obtained in formula (6.2). The proof of the proposition is completed.

$\square$

We end this section with the following proposition.

Proposition 7.1.3. *Let $\mathfrak{U} \subset \mathcal{W}_r$ be a wide open disk defined over L and $\lambda_{\mathfrak{U}}$ the universal weight associated to $\mathfrak{U}$. Let $\lambda \in \mathfrak{U}(L) \cap \mathbb{Z}$ such that $k \geq 0$. Then the natural diagram*

$$\begin{tikzcd}
\mathrm{H}^1\left(M(H,\pi)_{\bar{L}}^{et}, \mathcal{D}_{\mathfrak{U}}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\mathfrak{U}}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right) \hat{\otimes} \mathbb{C}_p \arrow[d] \\
\mathrm{H}^1\left(M(H,\pi)_{\bar{L}}^{et}, \mathcal{D}_{\lambda}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\lambda}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p \\
\mathrm{H}^1\left(M(H,\pi)_{\bar{L}}^{et}, \mathcal{V}_{\lambda}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Phi_{\lambda}"] & \mathrm{H}^0\left(M(H,\pi), \underline{\omega}^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p \arrow[u]
\end{tikzcd} \tag{7.3}$$

is commutative, where the left and the top right vertical maps are induced by the specializations

$$D_{\mathfrak{U}} \longrightarrow D_{\lambda} \longrightarrow V_{\lambda}.$$

The lower right vertical map is the restriction.

7.2 Main result

7.2.1 Assumptions and notations

To state our main theorem, we first recall some notations and assumptions.

(I) $r \geq 1$ is an integer.

(II) L is a finite field extension over $F_{\mathcal{P}}$ containing an element $\zeta_r \in \mathbb{C}_p$, where $\{\zeta_n\}_{n\geq 1}$ is a fixed sequence of $\mathbb{C}_p$ points of $\mathcal{LT}$ satisfying

- the $\mathcal{O}_{\mathcal{P}}$-order of ζ_n is exactly π^n;
- $\pi\zeta_{n+1} = \zeta_n$ for each $n \geq 1$;
- $\zeta_1 = (-\pi)^{\frac{1}{q-1}}$, where $(-\pi)^{\frac{1}{q-1}}$ is a fixed element in $\mathbb{C}_p$.

(III) $\mathfrak{U} \subset \mathcal{W}_r$ is a wide open disk defined over L, with ring of bounded analytic functions

$$\Lambda_{\mathfrak{U}} = \left\{ f \in \mathcal{O}(\mathfrak{U}) \;\middle|\; |f(x)| \leq 1 \text{ for all } x \in \mathfrak{U} \right\},$$

and universal weight

$$\lambda_{\mathfrak{U}} : \mathcal{O}_{\mathcal{P}}^{\times} \longrightarrow \Lambda_{\mathfrak{U}}^{\times}.$$

(IV) $w > 0$ is a rational number which is adapted to r, i.e.,

$$w < \frac{1}{q^{r-2}(q+1)}.$$

Moreover, if we choose a weight $\lambda \in \mathfrak{U}(L)$, we also assume that

$$w < (q-1)\left(v(s) + r - \frac{e}{p-1}\right),$$

where s is an element in $\mathbb{C}_p$ depending on λ as in Definition 3.4.1, e is the ramification degree of $F_{\mathcal{P}}/\mathbb{Q}_p$ with the assumption that $e \leq p-1$ (Remark 3.4.1) and v is the valuation on $\mathbb{C}_p$ which extends the one on $F_{\mathcal{P}}$, normalized by $v(\pi) = 1$.

(V) $h \in \mathbb{Q}$, $h \geq 0$ is a slope.

Furthermore, we also suppose that

- there exists a weight $\lambda \in \mathfrak{U}(L)$ corresponding to an integer $k \geq 0$ and

$$h < \lambda + 2 - N,$$

where $N = [F_{\mathcal{P}} : \mathbb{Q}_p]$;

- all $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)$, $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)$ and $\mathrm{H}^2\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda}\right)$ have slope $\leq h$ decompositions with respect to the U operator.

Remark 7.2.1. Note that both $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h}$ and $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}$ are finitely generated $B_{\mathfrak{U}}$-modules. Since $B_{\mathfrak{U}}$ is a principal ideal domain, $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h}$ (resp. $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}$) is a direct sum of a finite free $B_{\mathfrak{U}}$-submodule and a finite torsion. We have:

(1) Since $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)$ is an orthonormalizable $B_{\mathfrak{U}}$-module, one can easily prove that the torsion part of $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h}$ is 0. Hence $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h}$ is a finite free $B_{\mathfrak{U}}$-module.

(2) It is not known yet whether $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}$ is a free $B_{\mathfrak{U}}$-module. We denote by $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}_{tor}$ and $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}_{tf}$ the torsion and torsion free part of $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}$, respectively. Then we have an exact sequence

$$0 \longrightarrow \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}_{tor} \longrightarrow \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \longrightarrow$$

$$\longrightarrow \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h}_{tf} \longrightarrow 0,$$

which is G_L and Hecke-equivariant. The above sequence is split as $B_{\mathfrak{U}}$-modules but not as G_L-modules. Since $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h}$ is free, the morphism

$$\Psi_{\mathfrak{U}}^{\leq h} : \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p$$

factors through the morphism

$$\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}_{tf}\hat{\otimes}\,\mathbb{C}_p(1)\longrightarrow \mathrm{H}^0\left(M(w),\omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\le h}\hat{\otimes}\,\mathbb{C}_p$$

which is still denoted by $\Psi^{\le h}_{\mathfrak{U}}$. Moreover, we have the following isomorphism

$$\begin{aligned} & \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}/\pi_\lambda\,\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h} \\ \cong\quad & \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}_{tf}/\pi_\lambda\,\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}_{tf}\end{aligned}$$

for all but finite weights λ in $\mathfrak{U}$, where π_λ is a uniformizer at λ. Hence if we replace $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}$ by $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}_{tf}$, we can also prove our main theorem without assuming that $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}},\mathcal{D}_{\mathfrak{U}}\right)^{\le h}$ is a free $B_{\mathfrak{U}}$-module.

Now we state our main theorem and the proof is left to the following section. Let

$$\chi: G_L := \mathrm{Gal}(\overline{L}/L)\longrightarrow \mathcal{O}_{\mathcal{P}}^\times$$

be the Lubin-Tate character of L and $\chi^{\mathrm{univ}}_{\mathfrak{V}}$ be the character defined by the following composition

$$G_L\xrightarrow{\chi}\mathcal{O}_{\mathcal{P}}^\times\xrightarrow{\lambda_{\mathfrak{V}}}B_{\mathfrak{V}}^\times\longrightarrow\left(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p\right)^\times.$$

Theorem 7.2.1. *There exists a finite subset of weights $Z\subset\mathfrak{U}(\mathbb{C}_p)$ such that*

(a) For each $\lambda\in\mathfrak{U}(L)-Z$, there exists a finite dimensional $\mathbb{C}_p$-vector space $S^{\le h}_\lambda$ endowed with trivial semilinear G_L-action and Hecke operators, such that we have natural G_L and Hecke equivariant isomorphisms

$$\mathrm{H}^1\left(M(H,\pi)^{et}_{\overline{L}},\mathcal{D}_\lambda\right)^{\le h}\hat{\otimes}_L\mathbb{C}_p(1)\cong\left(\mathrm{H}^0\left(M(w),\omega_w^{\lambda+2}\right)^{\le h}\hat{\otimes}_L\mathbb{C}_p\right)\oplus\left(S^{\le h}_\lambda(\lambda+1)\right),$$

where the first projection is $\Psi^{\le h}_\lambda$.

(b) For every wide open disk $\mathfrak{V} \subset \mathfrak{U}$ defined over L satisfying $\mathfrak{V}(\mathbb{C}_p) \cap Z = \emptyset$, there exists a finite free $B_{\mathfrak{V}} \hat{\otimes}_L \mathbb{C}_p$ −module $S_{\mathfrak{V}}^{\leq h}$ endowed with trivial semilinear G_L-action and Hecke operators, for which we have a G_L and Hecke equivariant exact sequence

$$0 \longrightarrow S_{\mathfrak{V}}^{\leq h}\left(\chi \cdot \chi_{\mathfrak{V}}^{\mathrm{univ}}\right) \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \longrightarrow$$

$$\xrightarrow{\Psi_{\mathfrak{V}}^{\leq h}} \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p \longrightarrow 0.$$

Moreover, for any such open disk $\mathfrak{V}$, there exists finite subset $Z' \subset \mathfrak{V}$ with the property that, for any wide open disk $\mathfrak{V}' \subset \mathfrak{V}$ with $\mathfrak{V}'(\mathbb{C}_p) \cap Z' = \emptyset$, we have a natural G_L and Hecke equivariant isomorphism

$$\begin{aligned} & \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et}, \mathcal{D}_{\lambda_{\mathfrak{V}'}}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \\ \cong \quad & \left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}'}+2}\right)^{\leq h} \hat{\otimes}_L \mathbb{C}_p\right) \oplus \left(S_{\mathfrak{V}'}^{\leq h}(\chi \cdot \chi_{\mathfrak{V}'}^{univ})\right), \end{aligned}$$

where the first projection is determined by $\Psi_{\mathfrak{V}}^{\leq h}$.

7.2.2 The proof of the main theorem

We will divide the proof of our main result, Theorem 7.2.1 stated in the previous section, into several steps.

Lemma 7.2.1. *Let $\mathfrak{U}$, $\lambda_{\mathfrak{U}}$, w be as before and $\lambda \in \mathfrak{U}(L) \cap \mathbb{Z}$ satisfying $\lambda > h - 2 + N$. Let $\pi_\lambda \in B_{\mathfrak{U}}$ be a rigid analytic function on $\mathfrak{U}$ which vanishes with order 1 at λ and nowhere else on $\mathfrak{U}$. Then the specialization maps $D_{\mathfrak{U}} \to D_\lambda$ and $\omega_w^{\lambda_{\mathfrak{U}}} \to \omega_w^\lambda$ induce the following exact sequences:*

$$\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et}, \mathcal{D}_{\mathfrak{U}}\right) \xrightarrow{\cdot \pi_\lambda} \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et}, \mathcal{D}_{\mathfrak{U}}\right) \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{et}, \mathcal{D}_\lambda\right) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda}\right) \longrightarrow 0.$$

Proof. We start from the exact sequence:

$$0 \longrightarrow \mathcal{D}_{\mathfrak{U}} \xrightarrow{\cdot \pi_\lambda} \mathcal{D}_{\mathfrak{U}} \longrightarrow \mathcal{D}_\lambda \longrightarrow 0,$$

which induces the following exact sequence of $B_{\mathfrak{U}}$-modules

$$\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right) \xrightarrow{\cdot \pi_\lambda} \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right) \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right)$$

$$\longrightarrow \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right) \xrightarrow{\cdot \pi_\lambda} \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right) \longrightarrow \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right).$$

Then the following sequence induced by the slope decomposition

$$\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} \xrightarrow{\cdot \pi_\lambda} \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right)^{\le h}$$

$$\longrightarrow \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} \xrightarrow{\cdot \pi_\lambda} \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} \longrightarrow \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right)^{\le h},$$

is also exact. Since $\lambda \in \mathfrak{U}(L) \cap \mathbb{Z}$ and $\lambda > h - 2 + N$, by Corollary 5.2.2 we have

$$\mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right)^{\le h} = \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{V}_\lambda\right)^{\le h}.$$

But the latter is equal to 0 since

$$\mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{V}_\lambda\right) = 0$$

by the same argument as in the proof of Proposition 7.1.1. This implies that

$$\pi_\lambda \cdot \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} = \mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h}.$$

By our choice of π_λ, we deduce that

$$\mathrm{H}^2\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\le h} = 0,$$

which gives the first exact sequence in the statement of the lemma.

Moreover, the exact sequence

$$0 \longrightarrow \omega_w^{\lambda_{\mathfrak{U}}} \longrightarrow \omega_w^{\lambda_{\mathfrak{U}}} \longrightarrow \omega_w^{\lambda} \longrightarrow 0.$$

gives another exact sequence of $B_{\mathfrak{U}}$-modules

$$0 \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda}\right) \longrightarrow \mathrm{H}^1\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right).$$

Since $M(w)$ is an affinoid subdomain and $\omega_w^{\lambda_{\mathfrak{U}}}$ is a sheaf of $B_{\mathfrak{U}}$-Banach modules, we can deduce from [Andreatta et al., 2015a, Appendix] that

$$\mathrm{H}^1\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}}\right) = 0.$$

Thus the second exact sequence in the lemma follows. $\square$

Now let r, $\mathfrak{U}$, w and h be as in Section 7.2.1. Let $\lambda \in \mathfrak{U}(L) \cap \mathbb{Z}$, $\lambda \geq 0$. Recall that we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\mathfrak{U}}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right) \hat{\otimes} \mathbb{C}_p \arrow[d] \\
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\lambda}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p \\
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_{\lambda}\right) \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Phi_{\lambda}"] & \mathrm{H}^0\left(M(H,\pi), \underline{\omega}^{\lambda+2}\right) \hat{\otimes} \mathbb{C}_p . \arrow[u]
\end{tikzcd}$$

By our assumptions, both $\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)$ and $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)$ have slope $\leq h$ decompositions. Then the induced diagram

$$\begin{tikzcd}
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\mathfrak{U}}^{\leq h}"] \arrow[d] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p \arrow[d] \\
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Psi_{\lambda}^{\leq h}"] \arrow[d, "\psi"'] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p \\
\mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_{\lambda}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p(1) \arrow[r, "\Phi_{\lambda}^{\leq h}"] & \mathrm{H}^0\left(M(H,\pi), \underline{\omega}^{\lambda+2}\right)^{\leq h} \hat{\otimes} \mathbb{C}_p \arrow[u, "\phi"']
\end{tikzcd} \tag{7.4}$$

is also commutative, where ψ is the specialization map and ϕ is the restriction. Moreover, the surjectivity of Φ_λ implies that the bottom map $\Phi_\lambda^{\leq h}$ in the above diagram is also surjective. We have the following two cases:

(I) $h < \lambda + 2 - N$. Then [Kassaei, 2009, Theorem 5.1] and Proposition 5.2.3 imply that both ψ and ϕ in the above diagram are isomorphisms. Then $\Psi_\lambda^{\leq h}$ is surjective since $\Phi_\lambda^{\leq h}$ is so.

(II) $h \geq \lambda + 2 - N$. Consider the lower rectangle in diagram (7.4). The commutativity of this rectangle implies that the image of $\Psi_\lambda^{\leq h}$ is contained in the set of classical modular forms. In general, $\Psi_\lambda^{\leq h}$ is not surjective.

Now we let $B := B_{\mathfrak{U}} \hat{\otimes}_L \mathbb{C}_p$, we have the following result.

Lemma 7.2.2. *There exits a nonzero element $b \in B$ such that $b \cdot \mathrm{Coker}\left(\Psi_{\mathfrak{U}}^{\leq h}\right) = 0$.*

Proof. Let $\lambda \in \mathfrak{U}(L) \cap \mathbb{Z}$ be an integer such that $\lambda > h - 2 + N$. Note that both $\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1)$ and $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p$ are finite free B-modules. We denote by n and m their ranks as B-modules, respectively. Then we can choose basis for both such that the map $\Psi_{\mathfrak{U}}^{\leq h}$ corresponds to a matrix

$$\Psi_{\mathfrak{U}}^{\leq h} = (b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \in \mathrm{M}_{n \times m}(B).$$

The exact sequences obtained in Lemma 7.2.1 give the following identifications:

$$\begin{aligned} &\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_\lambda\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1) \\ &\quad \cong \left(\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1)\right) \Big/ \pi_\lambda \cdot \left(\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{U}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1)\right), \end{aligned}$$

and

$$\begin{aligned}
&\mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p \\
&\quad \cong \left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\right) \Big/ \pi_\lambda \cdot \left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{U}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\right),
\end{aligned}$$

where π_λ is an element of B which vanishes with order 1 at λ and nowhere else in U as in Lemma 7.2.1.

Hence the map $\Psi_\lambda^{\leq h}$ corresponds to the matrix

$$\Psi_\lambda^{\leq h} = (b_{ij}(\lambda))_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}.$$

Then our assumption on λ implies that $n \geq m$ and the rank of the matrix $\Psi_\lambda^{\leq h}$ is exactly m. This means there exists an $m \times m$-minor of $\Psi_{\mathfrak{U}}^{\leq h} = (b_{ij})$, say A, such that

$$\det\left(A(\lambda)\right) \neq 0.$$

Here, if $A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq m}}$ is a square matrix with entries in B, we denote by $A(\lambda)$ the matrix $(a_{ij}(\lambda))$ with entries in $\mathbb{C}_p$.

Now let $b := \det(A)$. Then b is a nonzero element of B by the above argument. Moreover, we have

$$b \cdot \mathrm{Coker}\left(\Psi_{\mathfrak{U}}^{\leq h}\right) = 0.$$

$\square$

Let Z_1 be the set of zeros of b, where b is as in the above lemma. Let $\mathfrak{V} \subset \mathfrak{U}$ be a connected affinoid subdomain defined over L such that

- $\mathfrak{V}(\mathbb{C}_p) \cap Z_1 = \emptyset$ and
- there exists an integer $\lambda \in \mathfrak{V}(L)$ with the property that $\lambda > h - 2 + [F_{\mathfrak{P}} : \mathbb{Q}_p]$.

Let $T_{\mathfrak{V}}^{\leq h}$ denote the kernel of the map

$$\Psi_{\mathfrak{V}}^{\leq h} : \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p,$$

where $\lambda_{\mathfrak{V}}$ is the restriction of $\lambda_{\mathfrak{U}}$ to $\mathfrak{V}$, which is also the universal character attached to $\mathfrak{V}$. Moreover, $\mathcal{D}_{\mathfrak{V}}$ is just the étale sheaf associated to the distribution $D_{\mathfrak{V}} = D_{\mathfrak{U}}|_{\mathfrak{V}}$. By the above notations, we have an exact sequence of $(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p)$-modules

$$0 \longrightarrow T_{\mathfrak{V}}^{\leq h} \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1) \xrightarrow{\Psi_{\mathfrak{V}}^{\leq h}} \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p \longrightarrow 0,$$

which is split (only as $(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p)$-modules here) since $\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p$ is a finite free $(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p)$-module. Moreover, it follows that $T_{\mathfrak{V}}^{\leq h}$ is a finite projective $(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p)$-module hence is free, since the latter is a principal ideal domain.

Now we let

$$\chi : G_L := \mathrm{Gal}(\overline{L}/L) \longrightarrow \mathcal{O}_{\mathcal{P}}^{\times}$$

be the Lubin-Tate character of L (refer to Remark 6.2.1) and $\chi_{\mathfrak{V}}^{\mathrm{univ}}$ be the character defined by the following composition

$$G_L \xrightarrow{\chi} \mathcal{O}_{\mathcal{P}}^{\times} \xrightarrow{\lambda_{\mathfrak{V}}} B_{\mathfrak{V}}^{\times} \longrightarrow \left(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p\right)^{\times}.$$

Let $S_{\mathfrak{V}}^{\leq h} := T_{\mathfrak{V}}^{\leq h}\left(\chi^{-1}(\chi_{\mathfrak{V}}^{\mathrm{univ}})^{-1}\right)$. This is a free $(B_{\mathfrak{V}}\hat{\otimes}\,\mathbb{C}_p)$-module of rank $l = n - m$ with continuous, semilinear action of G_L. Let $\phi_{\mathfrak{V}}$ be the Sen operator attached to $S_{\mathfrak{V}}^{\leq h}$ and K a finite, Galois extension of L in $\overline{L}$ satisfying

(a) $\widehat{W}_{K_\infty}\left(S_{\mathfrak{V}}^{\leq h}\right) := \left(S_{\mathfrak{V}}^{\leq h}\right)^{H_K}$ is a free $(B_{\mathfrak{V}}\hat{\otimes}_L\widehat{K}_\infty)$-module of rank l. Here H_K is the kernel of the cyclotomic character

$$\chi_{\mathrm{cyc}} : G_K \longrightarrow \mathbb{Z}_p^{\times};$$

(b) there exists a $(B_{\mathfrak{Y}}\hat{\otimes}_L\widehat{K}_\infty)$-basis $\{e_1, e_2, \cdots, e_l\}$ of $\widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right)$ such that

$$W_* := (B_{\mathfrak{Y}} \otimes_L K)e_1 + \cdots (B_{\mathfrak{Y}} \otimes_L K)e_l$$

is stable under $\Gamma_K := G_K/H_K$;

(c) the action of γ, a topological generator of Γ_K, on this basis is given by

$$\gamma(e_i) = \exp\left(\log\left(\chi_{\mathrm{cyc}}(\gamma)\right)\phi_{\mathfrak{Y}}\right)(e_i),$$

for every $1 \leq i \leq l$.

Let $\lambda \in \mathfrak{Y}(L)$ be an integer such that $\lambda > h - 2 + [F_{\mathcal{P}} : \mathbb{Q}_p]$. We have the following exact sequence of finite free $(B_{\mathfrak{Y}}\hat{\otimes}\,\mathbb{C}_p)$-modules, with G_L and Hecke actions

$$\begin{aligned} 0 \quad \longrightarrow \quad & S_{\mathfrak{Y}}^{\leq h} \\ \longrightarrow \quad & \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathrm{et}}, \mathcal{D}_{\mathfrak{Y}}\right)^{\leq h}\hat{\otimes}\,\mathbb{C}_p(1)\left(\chi^{-1}(\chi_{\mathfrak{Y}}^{\mathrm{univ}})^{-1}\right) \\ \longrightarrow \quad & \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{Y}}+2}\right)^{\leq h}\hat{\otimes}\,\mathbb{C}_p\left(\chi^{-1}(\chi_{\mathfrak{Y}}^{\mathrm{univ}})^{-1}\right) \\ \longrightarrow \quad & 0. \end{aligned}$$

Specializing the above exact sequence at λ, i.e., tensoring with L over $B_{\mathfrak{Y}}$ via the map $B_{\mathfrak{Y}} \to L$ sending $f \mapsto f(\lambda)$, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & S_{\mathfrak{Y}}^{\leq h} \arrow[d] & S_{\mathfrak{Y}}^{\leq h} \arrow[d] & S_{\lambda}^{\leq h} \arrow[d] & \\
 & \mathrm{H}^1_{\mathfrak{Y}} \arrow[r] \arrow[d] & \mathrm{H}^1_{\mathfrak{Y}} \arrow[r] \arrow[d] & \mathrm{H}^1_{\lambda} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^0_{w,\lambda_{\mathfrak{Y}}+2} \arrow[r] \arrow[d] & \mathrm{H}^0_{w,\lambda_{\mathfrak{Y}}+2} \arrow[r] \arrow[d] & \mathrm{H}^0_{w,\lambda+2} \arrow[d] & \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where

$$\mathrm{H}^1_{\mathfrak{V}} := \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1)\left(\chi^{-1}(\chi^{\mathrm{univ}}_{\mathfrak{V}})^{-1}\right),$$

$$\mathrm{H}^1_{\lambda} := \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(-\lambda\right),$$

$$\mathrm{H}^0_{w,\lambda_{\mathfrak{V}}+2} := \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(\chi^{-1}(\chi^{\mathrm{univ}}_{\mathfrak{V}})^{-1}\right),$$

$$\mathrm{H}^0_{w,\lambda+2} := \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(-\lambda-1\right),$$

and the twist in the last equation is by the Lubin-Tate character. Then by Snake Lemma, we have an exact sequence

$$S_{\mathfrak{V}}^{\leq h} \longrightarrow S_{\mathfrak{V}}^{\leq h} \longrightarrow S_{\lambda}^{\leq h} \longrightarrow 0,$$

which implies $S_{\lambda}^{\leq h} \cong S_{\mathfrak{V}}^{\leq h}/\pi_\lambda \cdot S_{\mathfrak{V}}^{\leq h}$.

Moreover, since $\lambda > h-2+[F_{\mathfrak{p}}:\mathbb{Q}_p]$, by classicality, we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & S_{\lambda}^{\leq h} \arrow[d] & & \mathrm{H}^1\left(M(H,\pi), \omega^{-\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{D}_{\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(-\lambda\right) \arrow[r, "\sim"] \arrow[d, "\Psi_{\lambda}^{\leq h}"'] & \mathrm{H}^1\left(M(H,\pi)^{\mathtt{et}}_{\overline{L}}, \mathcal{V}_{\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(-\lambda\right) \arrow[r] \arrow[d, "\Phi_{\lambda}^{\leq h}"'] & 0 \\
0 \arrow[r] & \mathrm{H}^0\left(M(w), \omega_w^{\lambda+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\left(-\lambda-1\right) \arrow[r, "\sim"] \arrow[d] & \mathrm{H}^0\left(M(H,\pi), \omega^{\lambda+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(-\lambda-1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0. &
\end{tikzcd}$$

Then by Five Lemma we obtain

$$S_{\lambda}^{\leq h} = \mathrm{Ker}\left(\Psi_{\lambda}^{\leq h}\right) \cong \mathrm{H}^1\left(M(H,\pi), \omega^{-\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p$$

as G_K-modules.

The exact sequence

$$0 \longrightarrow \pi_\lambda \cdot S_{\mathfrak{Y}}^{\leq h} \longrightarrow S_{\mathfrak{Y}}^{\leq h} \longrightarrow S_{\lambda}^{\leq h} \longrightarrow 0$$

induces the following exact sequence

$$0 \longrightarrow \widehat{W}_{K_\infty}\left(\pi_\lambda \cdot S_{\mathfrak{Y}}^{\leq h}\right) \longrightarrow \widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right) \longrightarrow \widehat{W}_{K_\infty}\left(S_{\lambda}^{\leq h}\right) \longrightarrow \mathrm{H}^1(H_K, S_{\mathfrak{Y}}^{\leq h}).$$

Since the extension $\overline{L}/K_\infty$ is almost étale, we have

$$\mathrm{H}^1(H_K, S_{\mathfrak{Y}}^{\leq h}) = 0.$$

Therefore, we have

$$\begin{aligned} \widehat{W}_{K_\infty}\left(S_{\lambda}^{\leq h}\right) &\cong \widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right) \Big/ \widehat{W}_{K_\infty}\left(\pi_\lambda \cdot S_{\mathfrak{Y}}^{\leq h}\right) \\ &= \widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right) \Big/ \left(\pi_\lambda \cdot \widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right)\right), \end{aligned}$$

where the last equality is true since $\pi_\lambda \in B_{\mathfrak{Y}}$.

Now let ϕ_λ be the Sen operator attached to $S_{\lambda}^{\leq h}$. We denote by

$$(d_{ij}) \in \mathrm{M}_{l \times l}\left(B_{\mathfrak{Y}} \otimes_L \widehat{K}_\infty\right)$$

the matrix of $\phi_{\mathfrak{Y}}$ with respect to the $(B_{\mathfrak{Y}} \hat{\otimes}_L \widehat{K}_\infty)$-basis $\{e_1, \cdots, e_l\}$ of $\widehat{W}_{K_\infty}\left(S_{\mathfrak{Y}}^{\leq h}\right)$ as in the assumption. Then the image of $\{e_1, \cdots, e_l\}$ is a basis of $\widehat{W}_{K_\infty}\left(S_{\lambda}^{\leq h}\right)$ of which ϕ_λ has matrix

$$(d_{ij}(\lambda))_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}}.$$

Recall that we have the isomorphism

$$S_{\lambda}^{\leq h} \cong \mathrm{H}^1\left(M(H, \pi), \omega^{-\lambda}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p,$$

thus

$$\widehat{W}_{K_\infty}\left(S_{\lambda}^{\leq h}\right) \cong \mathrm{H}^1\left(M(H, \pi), \omega^{-\lambda}\right)^{\leq h} \hat{\otimes}_L \widehat{K}_\infty.$$

So we have $\phi_\lambda = 0$, which implies that $d_{ij}(\lambda) = 0$ for all $\lambda \in \mathfrak{V}(L) \cap \mathbb{Z}$ with $\lambda > h - 2 + N$, where $n = [F_{\mathcal{P}} : \mathbb{Q}_p]$ (we have infinitely many such λ). Then we can deduce that $\phi_{\mathfrak{V}}(e_i) = 0$ hence γ fixes e_i for all $1 \leq i \leq l$, i.e., $S_{\mathfrak{V}}^{\leq h}$ has trivial semilinear G_K-action.

Finally, by étale descent, $S_{\mathfrak{V}}^{\leq h}$ is a free $(B_{\mathfrak{V}} \hat{\otimes} \mathbb{C}_p)$-module of rank l with trivial G_L-action.

Now we rewrite our exact sequence as follows

$$0 \longrightarrow S_{\mathfrak{V}}^{\leq h}\left(\chi \cdot \chi_{\mathfrak{V}}^{\mathrm{univ}}\right) \longrightarrow \mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1) \longrightarrow \tag{7.5}$$

$$\xrightarrow{\Psi_{\mathfrak{V}}^{\leq h}} \mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p \longrightarrow 0.$$

Let

$$\mathcal{H} := \mathrm{Hom}_{(B_{\mathfrak{V}} \hat{\otimes} \mathbb{C}_p), G_L}\left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p, S_{\mathfrak{V}}^{\leq h}\left(\chi \cdot \chi_{\mathfrak{V}}^{\mathrm{univ}}\right)\right).$$

Then $\mathcal{H}$ is a finite free $(B_{\mathfrak{V}} \hat{\otimes} \mathbb{C}_p)$-module with continuous, semilinear action of G_L. Furthermore, the extension class of the above exact sequence corresponds to a cohomology class in $\mathrm{H}^1(G_L, \mathcal{H})$. Let ϕ denote the Sen operator associated to $\mathcal{H}$. Then by the argument in Sen [1988], the cohomology group $\mathrm{H}^1(G_L, \mathcal{H})$ is killed by $c := \det(\phi) \in B_{\mathfrak{V}}$. Since $c = \det(\phi) \neq 0$, we have a split short exact sequence of G_L-modules

$$0 \longrightarrow \left(S_{\mathfrak{V}}^{\leq h}\left(\chi \cdot \chi_{\mathfrak{V}}^{\mathrm{univ}}\right)\right)_c \longrightarrow \left(\mathrm{H}^1\left(M(H,\pi)_{\overline{L}}^{\mathtt{et}}, \mathcal{D}_{\mathfrak{V}}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p(1)\right)_c \longrightarrow$$

$$\xrightarrow{\Psi_{\mathfrak{V}}^{\leq h}} \left(\mathrm{H}^0\left(M(w), \omega_w^{\lambda_{\mathfrak{V}}+2}\right)^{\leq h} \hat{\otimes}\, \mathbb{C}_p\right)_c \longrightarrow 0.$$

Now let $Z' \subset \mathfrak{V}$ be the set of zeros of c, we obtain our theorem.

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