Geometric Fault Detection and Isolation of Infinite Dimensional Systems

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ABSTRACT

Geometric Fault Detection and Isolation of Infinite Dimensional Systems

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A broad class of dynamical systems from chemical processes to flexible mechanical structures, heat transfer and compression processes in gas turbine engines are represented by a set of partial differential equations (PDE). These systems are known as infinite dimensional (Inf-D) systems. Most of Inf-D systems, including PDEs and time-delayed systems can be represented by a differential equation in an appropriate Hilbert space. These Hilbert spaces are essentially Inf-D vector spaces, and therefore, they are utilized to represent Inf-D dynamical systems. Inf-D systems have been investigated by invoking two schemes, namely approximate and exact methods. Both approaches extend the control theory of ordinary differential equation (ODE) systems to Inf-D systems, however by utilizing two different methodologies. In the former approach, one needs to first approximate the original Inf-D system by an ODE system (e.g. by using finite element or finite difference methods) and then apply the established control theory of ODEs to the approximated model. On the other hand, in the exact approach, one investigates the Inf-D system without using any approximation. In other words, one first represents the system as an Inf-D system and then investigates it in the corresponding Inf-D Hilbert space by extending and generalizing the available results of finite-dimensional (Fin-D) control theory.

It is well-known that one of the challenging issues in control theory is development of algorithms such that the controlled system can maintain the required performance even in presence of faults. In the literature, this property is known as fault tolerant control. The fault detection and isolation (FDI) analysis is the first

step in order to achieve this goal. For Inf-D systems, the currently available results on the FDI problem are quite limited and restricted. This thesis is mainly concerned with the FDI problem of the linear Inf-D systems by using both approximate and exact approaches based on the geometric control theory of Fin-D and Inf-D systems. This thesis addresses this problem by developing a geometric FDI framework for Inf-D systems. Moreover, we implement and demonstrate a methodology for applying our results to mathematical models of a heat transfer and a two-component reaction-diffusion processes.

In this thesis, we first investigate the development of an FDI scheme for discrete-time multi-dimensional (nD) systems that represent approximate models for Inf-D systems. The basic invariant subspaces including unobservable and unobservability subspaces of one-dimensional (1D) systems are extended to nD models. Sufficient conditions for solvability of the FDI problem are provided, where an LMI-based approach is also derived for the observer design. The capability of our proposed FDI methodology is demonstrated through numerical simulation results to an approximation of a hyperbolic partial differential equation system of a heat exchanger that is represented as a two-dimensional (2D) system.

In the second part, an FDI methodology for the Riesz spectral (RS) system is investigated. RS systems represent a large class of parabolic and hyperbolic PDE in Inf-D systems framework. This part is mainly concerned with the equivalence of different types of invariant subspaces as defined for RS systems. Necessary and sufficient conditions for solvability of the FDI problem are developed. Moreover, for a subclass of RS systems, we first provide algorithms (for computing the invariant subspaces) that converge in a finite and known number of steps and then derive the necessary and sufficient conditions for solvability of the FDI problem.

Finally, by generalizing the results that are developed for RS systems necessary and sufficient conditions for solvability of the FDI problem in a general Inf-D system are derived. Particularly, we first address invariant subspaces of Fin-D systems from a new point of view by invoking resolvent operators. This approach enables one to extend the previous Fin-D results to Inf-D systems. Particularly, necessary and sufficient conditions for equivalence of various types of conditioned and controlled invariant subspaces of Inf-D systems are obtained. Duality properties of Inf-D systems are then investigated. By introducing unobservability subspaces for Inf-D systems the FDI problem is formally formulated, and necessary and sufficient conditions for solvability of the FDI problem are provided.

For my mother,

Toomar

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Last but not least, I dedicate this thesis to my family, Rasool, Maryam,

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List of Symbols and Abbreviations

$\langle \cdot, \cdot \rangle$	$\langle x, y \rangle$, inner product of x and y.
$\|\| \cdot \|\|$	$\|\|x\|\|$, norm of x.
$\perp$	$\begin{cases} \mathscr{V}^{\perp}, \text{ orthogonal complement of } \mathscr{V} \\ x \perp y, \, x \text{ is prependicular to } y \end{cases}$.
$\oplus$	$\mathscr{V}_1 \oplus \mathscr{V}_2$, direct sum of $\mathscr{V}_1$ and $\mathscr{V}_2$.
$\overline{\mathscr{V}}$	Closure of $\mathscr{V}$.
$\boxplus \mathscr{V}$	An infinite number of direct sum of $\mathscr{V}$.
$\bigoplus \mathscr{V}$	Largest Banach space contained in $\boxplus \mathscr{V}$.
$\mathcal{A}^*$	Adjoint operator of $\mathcal{A}$.
$\mathcal{A}^{-1}\mathscr{V}$	Inverse image of $\mathscr{V}$ with respect to $\mathcal{A}$.
$\mathbb{C}$	Vector space of complex numbers.
$D(\mathcal{A})$	Domain of $\mathcal{A}$.
$\operatorname{Im} \mathcal{A}$	Image of operator $\mathcal{A}$.
$\ker \mathcal{C}$	kernel of $\mathcal{C}$.
$L_2([a, b], \mathbb{R}^n)$	Set of all square integrable functions $f : [a, b] \to \mathbb{R}^n$.
$\mathcal{L}(\mathcal{X})$	Set of all bounded linear operator defined on $\mathcal{X}$.
$\mathbb{N}$	Set of positive integers.
$\mathbb{R}$	Vector space of real numbers.
$\mathbb{T}_{\mathcal{A}}$	C_0 semigroup generated by the infinitesimal generator $\mathcal{A}$.
$\mathscr{V}_{\overline{\mathrm{R}}}$	An arbitrary $\mathcal{R}(\lambda, \mathcal{A})$-invariant containing $\mathscr{V}$.
$\mathscr{V}_{\underline{\mathrm{R}}}$	An arbitrary $\mathcal{R}(\lambda, \mathcal{A})$-invariant contained in $\mathscr{V}$.
$\mathbb{Z}$	Set of integers.
$< \mathscr{C} \| \mathcal{A} >$	Largest $\mathcal{A}$-invariant subspace contained in $\mathscr{C}$.
ϕ_k	k^{th} (generalized) eigenvector of $\mathcal{A}$.
ψ_k	Corresponding biorthonormal vector to ϕ_k.
$\rho(\mathcal{A})$	Resolvent set of $\mathcal{A}$.
$\rho_\infty(\mathcal{A})$	Largest real interval $[r, \infty) \subseteq \rho(\mathcal{A})$.
$\mathcal{R}(\lambda, \mathcal{A})$	Resolvent operator of $\mathcal{A}$.
$\sigma(\mathcal{A})$	Spectrum of $\mathcal{A}$.

Chapter 1

Introduction

The fault detection and isolation (FDI) problem has attracted a considerable research interest during the past few decades [1–5]. Advancements in the control theory have resulted in development of various robust control algorithms for systems that are subject to disturbances and modeling uncertainties. Consequently, as a result of the introduced robustness of these controllers the task of early fault detection has now become even more challenging, and more advanced FDI methods should be developed and considered. During the past three decades, significant efforts have been made to address control of infinite dimensional (Inf-D) systems [6–9]. However, due to the complexity of Inf-D systems, research on FDI problem of these systems is quite limited and developing an FDI methodology for Inf-D systems is still a very active area of research.

1.1 Fault Detection and Isolation Problem

Nowadays, control algorithms need to be as reliable as possible. For example, consider a gas turbine power plant where one needs a highly accurate and reliable control algorithm to ensure that the generated power has an exact frequency (i.e., 60 Hz). Since shutting down a generator can be costly due to its effect on all power networks [1], another important issue is that one needs to minimize the maintenance time. One of the main measures that defines reliability is performance of control algorithms in presence of faults. In other words, the system can still be operational for a certain set of faults and the maintenance action is not urgent. In other words, one of the challenging issues in control theory is the development of algorithms such that the controlled system can maintain the required performance even in presence of faults. In the literature, this property is known as fault tolerant control (FTC) (refer to [1, 10] and references therein).

FTC algorithms are categorized into passive and active schemes. In the former approach, the corresponding controller is designed such that it is robust to certain set of faults. Whereas, in the active FTC, the controller is reconfigured such that the effects of faults can be rectified as much as possible [1, 10, 11]. For handling faulty scenarios, a passive FTC scheme yields a conservative result due to nature of the design framework. To overcome this drawback, active FTC methods have been proposed in the literature [1, 11].

Specifically, an active FTC approach is the methodology that is mainly concerned with reconfiguring the controllers based on the available fault information [10]. A generic FTC is depicted in Figure 1.1, where $u(t)$ and $y(t)$ denote input and output signals of the plant, respectively. As can be observed, the FDI unit plays a crucial role in an active FTC module and is a cornerstone for active FTC system. Indeed, The FDI analysis is the first step in order to achieve the FTC goal. Moreover, the FDI unit can provide required information for condition-based maintenance that results in a significant maintenance cost reduction [1, 10].

The main goal of an FDI unit is to generate a set of signals, so-called residual signals, such that these signals provide as much information as possible regarding the fault signals [10, 12]. More precisely, by using a residual signal the decision making unit should be able to:

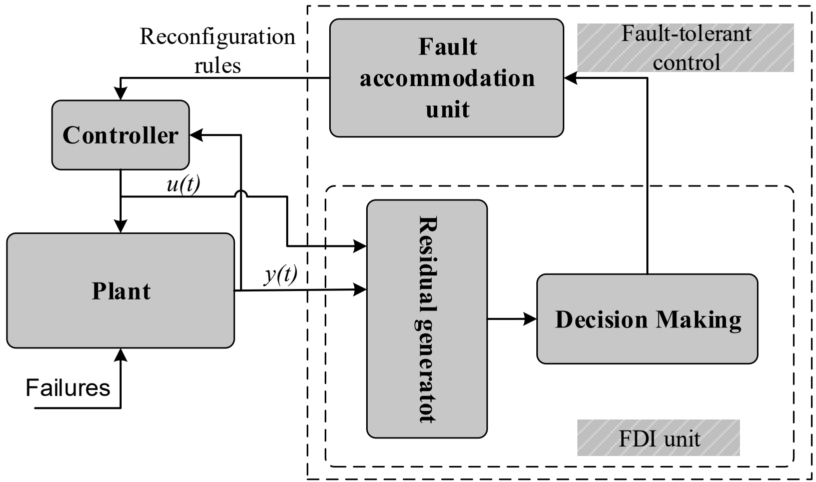


Figure 1.1: General fault-tolerant control methodology.

1. Detect the occurrence of a fault.
2. Determine the location (i.e. which actuator or sensor) the fault has occurred in, which known as the fault isolation.

Therefore, the main part of the FDI problem can be summarized as that of residual generation that is subsequently addressed.

1.1.1 Residual Generation

A residual is a signal that is sensitive to certain set of faults and decoupled from the other inputs of the plant and faults [5, 10] . In this thesis, we derive residuals that are decoupled from all but one fault, and consequently the decision making unit (refer to Figure 1.1) is restricted to a threshold comparison. Figure 1.2 depicts the schematic of the residual generators where the following logic is used in the decision making unit,

$$\text{if } r_i > th_i \Rightarrow f_i \text{ has occured.} \tag{1.1}$$

with th_i is the threshold corresponding to r_i. Thresholds can be determined by utilizing Monte Carlo simulations [13], and this issue is formally addressed in Chapters

3 - 5.

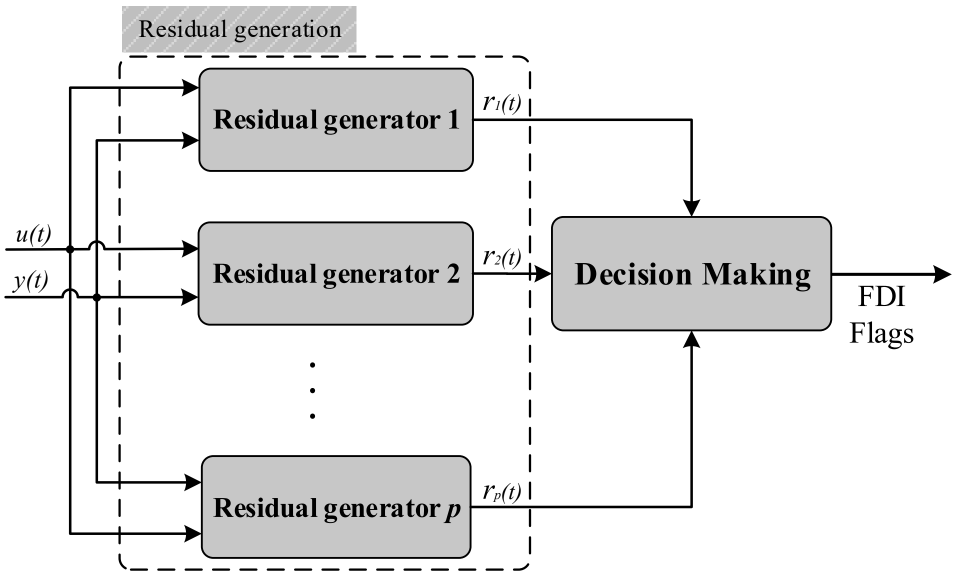


Figure 1.2: General residual generation part, where $u(t)$ and $y(t)$ denote input and output signals of the plant, respectively.

It should be pointed out that one of most prominent issues related to the residual signals is the residual generator realization. For example, a residual generator can be an observer or a parameter estimator. The type of realization identifies the FDI approach (refer to Figure 1.3). However, before reviewing the approaches for FDI, we provide the motivation of the research pursued in this thesis.

1.2 Motivation

There are certain classes of engineering process that cannot be modeled as finite dimensional (Fin-D) systems. For example, heat distribution of a heat exchanger and voltage substations in a distributed transmission system are generally modeled by a set of partial differential equations (PDEs) and time-delay systems, respectively. Indeed, a large class of dynamical systems from the compression process in gas turbine engines to reaction processes in solid-fuel rockets are mathematically represented as Inf-D dynamical systems. A given Inf-D dynamical system is usually modeled by a differential equation in an appropriate Hilbert space [14, Chapter 1], which is an Inf-D vector space.

Although, certain set of Inf-D systems can be approximated by Fin-D systems, the approximation error may result in a significant performance degradation. For example, consider a neutral time-delay system that models a traffic network, where the delay is not negligible and cannot be assumed to be zero. This system cannot be represented by a Fin-D dynamical system that is governed by an ordinary differential equation (ODE) with no delays. Therefore, development of control theory for Inf-D systems is an emerging field of interest and research.

The mathematical control theory of Inf-D systems has seen a considerable progress during the past four decades [6–9]. Particularly, PDE systems have been investigated by using two schemes that are called approximate and exact methods. Both approaches extend the control theory of ODE systems to Inf-D systems, however by invoking two different methodologies. By using the approximation approach, by using finite element or finite difference methods one needs first to approximate the original PDE by an ODE system and then apply the established control theory of ODE systems to the approximated model [15–17]. On the other hand, the exact approach investigates the PDE system without any approximation [14, 18]. In other words, one first represents the PDE system as an Inf-D system and then investigates this system in the corresponding Inf-D Hilbert space by extending the available results of Fin-D control theory. This approach is also applicable to other distributed parameter systems such as time-delayed system (for more detail, refer to [14, chapters 1 and 2]).

In contrast to Fin-D systems, research on the FDI problem for Inf-D systems is quite limited due to the complicated structure of these systems. Recently, some efforts have been made to address the FDI problem for PDEs [19–21]. In this thesis, we address the FDI problem of Inf-D systems, as follows

- How and under what conditions can one detect a fault in the system? In other words, by referring to Figure 1.2 under what conditions one can generate the residual signal r_i such that it is sensitive to f_i?

- Under what conditions can one isolate the detected fault? More precisely, under what conditions one can generate the residual signal r_i that is decoupled from all the faults but f_i (refer to the condition (1.1)).

In order to answer these questions, one needs to derive the necessary and sufficient conditions for the solvability of the FDI problem for Inf-D systems. For Fin-D systems, the geometric FDI approach is one of the main approaches that addresses the solvability of the FDI problem by using observers. The main motivation for this thesis can therefore be summarized in the following question:

- How and under what conditions can one extend the existing geometric theories on the FDI of Fin-D systems to Inf-D ones?

  As reviewed subsequently, the FDI problem of Fin-D systems has extensively been addressed in the literature. Therefore, one approach to tackle the FDI problem of Inf-D systems is to generalize the existence theory to Inf-D systems. This thesis tries to investigate the FDI problem of Inf-D systems by using the available geometric theories on the Fin-D ones as a guide. More precisely, we generalize the results by using two main methodologies; namely the approximate and the exact methods. As stated earlier, in the former one we first approximate the Inf-D system and then apply the FDI theory of Fin-D system to the approximated model with certain modifications (refer to Chapter 3 for more details), whereas in the exact approach we first formulate the Inf-D system as a differential equation in an appropriate Inf-D vector space and then the FDI problem is addressed (refer to Chapters 4 and 5 for more details).

1.3 Literature Review

In this section, we first review the literature on FDI approaches for Fin-D systems followed by a description of the FDI problem of Inf-D systems.

1.3.1 FDI Methods for Fin-D Systems

In the literature, there are various methods that have been developed to tackle the FDI problem of Fin-D systems. These approaches can be categorized into two main schemes known as data driven and model-based methodologies [11, 22, 23]. Figure 1.3 depicts these two schemes.

Data Driven-based Approaches

In the case that the mathematical model of the system is not available or it is very complicated, data driven-based approaches provide the sufficient infrastructure to address the FDI problem [24, 25]. Patton *et al.* proposed a neural network multiple model approach in [26]. The FDI problem was addressed by using feed-forward neural networks in [27]. A dynamic neural network is successfully applied to the gas turbines for performing fault diagnosis in [28, 29]. In [30], a Bayesian neural network was used to optimize the wavelet transform of input-output signals, and then this transformation is used for FDI.

A pattern recognition approach that is based on the fuzzy logic was also applied to the FDI problem in [31]. In [32] a framework for the fault diagnosis problem by using an expert system was developed. Fault diagnosis methods based on the qualitative trend analysis were reviewed in [23]. Moreover, in the literature, the statistical analysis-based methods are also utilized for the FDI purpose. A nonlinear

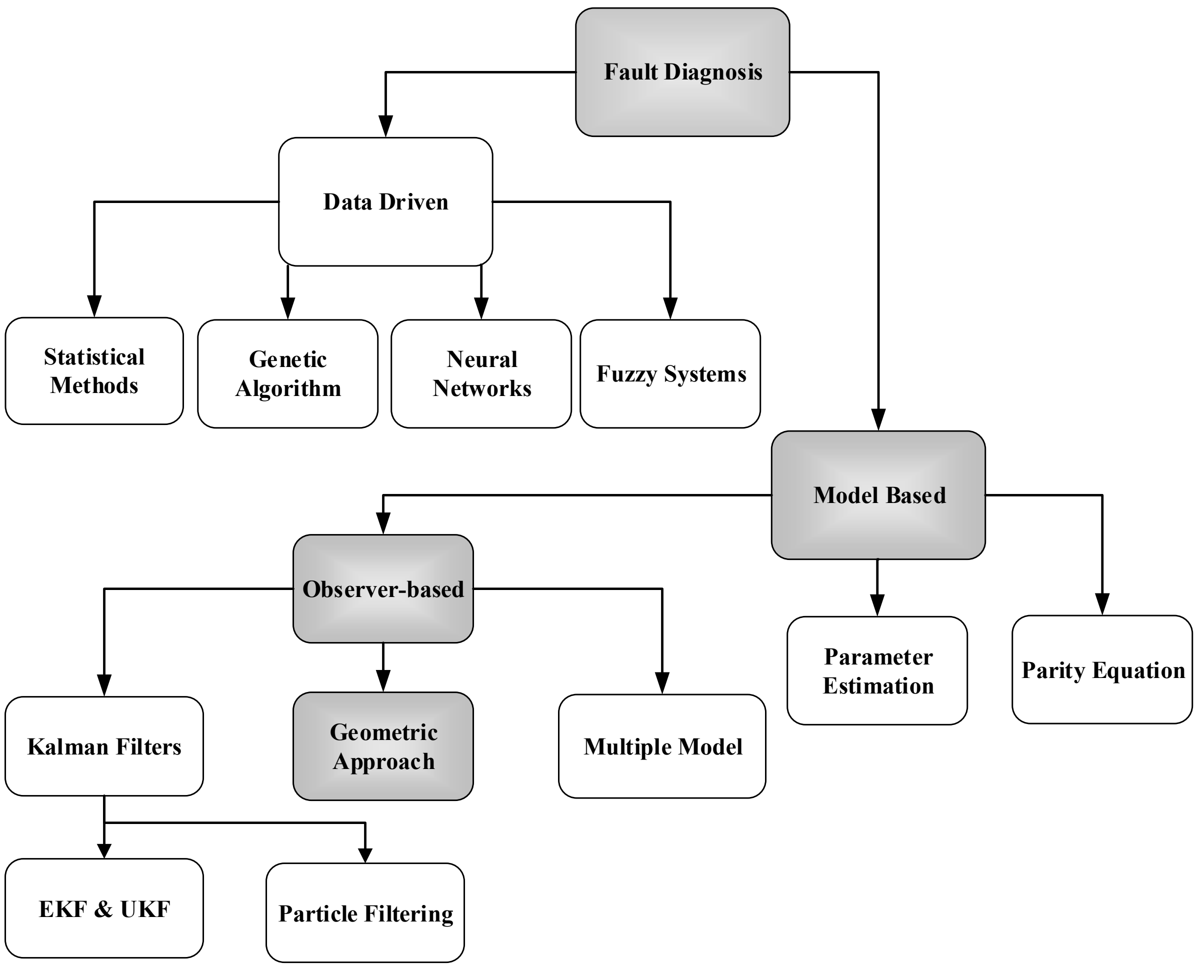


Figure 1.3: The currently available FDI approaches in the literature for Fin-D systems.

principle component analysis (PCA) and a recursive PCA were used in [33] and [34], respectively. Also, independent component analysis was applied to the FDI problem in [35, 36]. In [37, 38], support vector machines (SVM) were utilized for performing health monitoring purposes.

It should be pointed out that since data-driven approaches are model independent, they can be applied to Fin-D and with certain modification to Inf-D systems. For example refer to [39] where a singular value decomposition was utilized for identification purpose that represents the approximated model of an Inf-D system.

Model-based FDI Approaches

In the literature, model-based FDI includes variety of techniques such as particle filters [40], observer-based [41], and parity equations [42]. The parity equation approaches use a set of functions that are so-called parity functions to extract the fault information from the measured input-output data [2, 42, 43]. However, these approaches are sensitive to measurement noise [44]. Observer-based methods that are established tools for model-based fault diagnosis include various approaches such as multiple model [45–47], high-gain observer [48], sliding mode observer [49], and geometric methods [3, 41, 50].

Due to uncertainties in modeling complex systems, a perfect mathematical model is generally not feasible. Neglected dynamics, noise, and disturbances are examples of model uncertainty [51, 52]. Since in model-based approaches the model is utilized for designing detection filters, to minimize the effect of the uncertainties that is decreasing the accuracy of the FDI algorithm and increasing the false alarms, one needs to apply robust FDI algorithms. The FDI of linear systems using robust filters in presence of disturbances were considered in [53–55]. Also, a robust FDI approach for a Lipschitz nonlinear system was provided in [56]. Other important FDI approaches include parameter estimation techniques [2], particle filtering [40,57] and maximum likelihood estimation techniques [58, 59].

Hybrid FDI Approaches

The drawbacks of model-based and data-driven based methods can be addressed by applying a hybrid method of fault diagnosis by integrating the model-based and data-driven based approaches [12]. The hybrid method enables one to detect and isolate faults in presence of different uncertainties due to the modeling errors, parameter variations, unknown external disturbances and measurement noise. In [60], a parity based approach is integrated with a neural network to increase the efficiency of the fault detection. For a nonlinear system, an observer-based approach is modified by using the SVM for the fault isolation purposes. In [61], a hybrid FDI approach was developed, where a data-driven approach is combined with wavelet transformation analysis. Morevoer, various types of hybrid approaches were reviewed in [62].

Geometric FDI Methods

Since this thesis is specifically concerned with geometric approaches for the FDI problem, in this subsection we review the geometric FDI approach. The geometric FDI approach [3] is a model-based method, where necessary and sufficient conditions for solvability of the FDI problem are obtained based on geometric concepts such as invariant subspaces. For the FDI problem of Fin-D systems, the geometric approach developed by Massoumnia [3] has provided a valuable tool for studying the FDI problem not only for basic linear dynamical systems but also for more general cases such as Markovian jump systems [63, 64], time-delay systems [65, 66], linear parameter varying (LPV) systems [67, 68], linear periodic systems [69] and linear impulsive systems [70]. Moreover, the geometric approach has been also extended to affine nonlinear systems in [4, 71]. Furthermore, hybrid geometric FDI approaches for linear and nonlinear systems have been provided in [72] and [73], where a set of residual generators are equipped with a discrete-event based system fault diagnoser to solve the FDI problem.

The geometric approach is based on invariant subspaces, such as unobservable and conditioned invariant subspaces that are formally defined in the next Chapters. These subspaces can fully characterize the behavior of the investigated linear system [74]. The geometric approach also has its application in the control of disturbance decoupling problem [74]. Therefore, development of this framework for Inf-D systems allows us to have not only a novel tool for the FDI purpose of Inf-D systems but also a better understanding of the nature and behavior of these systems.

1.3.2 The FDI Approach for Inf-D Systems

As stated earlier, from the system theory point of view, there are two main approaches to investigate Inf-D systems, namely approximate and exact methods. In approximate approaches, that are applicable to PDE systems, the original PDE is approximated by using a finite element [15, 20, 21] or a finite difference method [75] and then this approximated model is used for designing a controller or FDI unit. However, in exact approaches the system is reformulated as a linear system in an appropriate Inf-D Hilbert space and a controller or FDI unit is designed for this abstract differential equation [14, 76]. In this thesis, we cover both approaches by providing necessary and sufficient conditions for solvability of the FDI problem, in each case.

The FDI Approach Based on the Approximated Model

Two approaches to approximate a PDE system are the finite element (particularly Galerkin) [77] approach and the finite difference method [78]. Finite element-based approaches are applicable to dissipative parabolic PDE systems, for which the eigenspectrum of the spatial differential operators can be partitioned into a finite subset containing all unstable eigenvalues (and a finite subset of stable eigenvalues) and an infinite subset of stable eigenvalues such that the gap between these two sets is sufficiently large. If such a partition exists, a Fin-D ODE could approximate the original PDE [77] which can be employed for designing the FDI filters [15, Assumption 1]. This assumption enables one to apply the singular perturbation theory to approximate the model and derive sufficient condition for solvability of the FDI problem. In [20, 21], it was assumed that the number of actuators (l) is equal to the number of Fin-D states (n). For the situation ($l < n$), the approach that is presented in [20] (Remark 6) is not applicable since the introduced transformation is not invertible. To solve this problem, in [15] we utilized a nonlinear geometric FDI approach as described in [4]. It was shown that the FDI system that is designed based on the approximated Fin-D system can detect almost all the faults injected in the original system. However, since this thesis is mainly concerned with linear Inf-D system, the results of [15] are not presented here.

In this thesis, a finite difference approach is used to approximate the original PDE system. The main reason lies on the following observations and facts:

1. It is well-known that *parabolic* PDE systems can be approximated by ODE representations. These systems can be approximated through application of the finite element methods where sufficient conditions for solvability of the FDI problem can then be derived by using the singular perturbation theory [15, 20, 79]. Unlike parabolic PDE systems, one cannot apply model decomposition, order reduction and singular perturbation theory to hyperbolic PDE systems [80]. Moreover, the order of the resulting approximate Fin-D system can be high. Therefore, the Galerkin method is not applicable to hyperbolic PDE systems for solving the FDI problem.

2. As shown in [75], a *single* hyperbolic PDE system can be approximated by using a two-dimensional (2-D) system that is formally addressed in Section 2.5

and Chapter 3. As we shall see subsequently in Section 2.5 this approximation is also applicable to a *system* of hyperbolic PDEs. Moreover, as shown in [81], parabolic PDE systems can be approximated by three-dimensional (3-D) Fornasini and Marchesini model II (FMII) representation [82–85]. Therefore, multidimensional (n-D) (the generalization of 2-D systems for $n \geq 2$) system approximation can be applied to *both* hyperbolic and parabolic systems.

Moreover, it should be noted that the n-D system framework has other applications in the control field. For example, a class of discrete-time linear repetitive processes can be modeled by n-D systems. These processes play important roles in tracking control and robotics, where the controlled system is required to perform a periodic task with high precision (refer to [86] for more details on repetitive systems). One of the main approaches to control linear repetitive processes is the iterative learning control (ILC) [86]. Since the ILC problem can be formulated as a control problem using n-D system theory [87, 88], n-D systems have been increasingly applied to spatio-temporal and repetitive process control problems in the literature.

There are quite a few results on FDI of 2-D systems in the literature, such as dead-beat based filters [89] and parity equations [90, 91] that utilize the algebraic approaches. As shown in [92] these approaches that are based on polynomial matrices face new challenges for $n \geq 3$. More precisely, due to the complexity of the primeness properties (refer to [92, page 389] for more detail) generalizing the results in [89,91] from 2-D case to n-D systems is not straightforward. In this thesis, we are interested in developing an FDI methodology that is applicable to all n-D systems, $n \geq 2$. Motivated by the above discussion, in this thesis we investigate the FDI problem of n-D systems as the FDI approximate method of Inf-D systems.

2-D systems have been extensively investigated from a system theory point of view [82–85]. Particularly, system theory concepts such as stability [84,93], controllability [94], observability [95], and state reconstruction [96] have been investigated in the literature. However, due to complexity of 2-D systems, unlike one-dimensional (1D) systems, there are various definitions that are introduced for controllability and observability properties. Not surprisingly, the duality between observability and controllability does not hold in 2-D systems.

Recently, the geometric theory of 2-D and 3-D systems has attracted much interest, where basic concepts such as conditioned invariant and controllable subspaces are studied in detail for the FMI model [97,98]. The hybrid 2-D systems have also been investigated from the geometric point of view in [95].

Finally, it must be noted that recently related work has appeared in [99] and [100]. These two papers investigated the FDI problem of 3-D FMII models. Although a geometric FDI methodology is also developed in [99], this thesis is distinct and unique from [99] in *three* main and fundamental perspectives:

1. The approach proposed in [99] is based on the results of [50], whereas our approach is based on the generalization of the results of [50] as reported in [41] (the results in [41] are more general than [50]) for 2-D models.

2. In [99], necessary and sufficient conditions for solvability of the FDI problem were derived for a *subclass of detection filters* where it was assumed that the output map of the detection filters and that of the system are identical. However, in this thesis we consider a *general class of detection filters* for the residual generation and relax this condition.

3. As shown in Section 3.3.2, the observability property of the 3-D model is a fundamental requirement and assumption in [99] (although it is stated in [99] that this assumption was made for simplicity of their presentation). However, our proposed solution *does not* require this condition and assumption, and consequently our approach leads to a less restrictive solution.

FDI Approaches Based on the Exact Model

As we shall see in the next chapter, the systems governed by PDE can be reformulated as an abstract differential equation in an appropriate Hilbert space which is an Inf-D function space. The system theory of Inf-D systems has attracted an increasing attention during the past few decades [7, 8, 14, 101, 102]. Not surprisingly, the control problems of Inf-D systems are more complicated as compared to those of Fin-D systems.

In [7, 8], the optimal control problem of systems governed by PDE (hyperbolic and parabolic) was addressed. The observability and controllability concepts were investigated in [14, 102]. The geometric approaches for Inf-D systems have been addressed for the first time in [18, 101]. Like the Fin-D systems, the geometric approaches in [18, 101, 103] are based on certain invariant subspaces. One of main differences between Fin-D and Inf-D spaces which cause many difficulties is summarized in the following fact [14, 101, 102] that a Fin-D subspace is always closed, whereas an Inf-D subspace can be closed or only dense in a closed subspace. This fact results in a set of open problems in the system theory of linear Inf-D systems [101]. Therefore, geometric approaches on the Inf-D subspaces (such as those used in [101]) need to be investigated with more sophisticated mathematical tools such as topological vector spaces that address the above difficulties. In other words, in a general topological vector space the completeness of a subspace is not a trivial property and there exit subspaces that are not closed [104].

The disturbance decoupling problem has attracted attention in Inf-D systems [103, 105] and has partially been solved. Very recently, the disturbance decoupling of Inf-D systems has been addressed in [106]. However, compared with the Fin-D systems, the currently available results in the literature for the FDI problem of Inf-D systems is very limited [19, 76, 107]. In [19], The FDI problem of positive Inf-D systems was investigated by using the parameter estimation technique.

A cornerstone of the geometric control theory is the invariance properties such as A-invariance, conditioned and controlled invariance. There are various definitions of invariance properties that are equivalent in Fin-D systems, whereas they are not in Inf-D systems. Specifically, due to the complexity of working with unbounded operators, the invariant subspace investigation of Inf-D systems is quite limited and equivalence of different definitions has been shown only for single-input, single-output systems [103, 108]. Also, in [103] a sufficient condition for equivalence of various definitions are provided. Moreover, in [14, 101] by applying the resolvent operators necessary and sufficient conditions for equivalence of various $\mathcal{A}$-invariant definitions, that are addressed subsequently in Chapters 4 and 5, are presented. However, deriving necessary and sufficient conditions for equivalence of various definitions of conditioned and controlled invariant subspaces has been open since middle 1980's [101, 109]. In this thesis, we derive necessary and sufficient conditions for equivalence of various definitions of the above invariant subspaces for a class of multi-input multi-output Inf-D systems.

1.4 General Problem Statement

As stated earlier, one of the fundamental problems that is related to dynamical systems is the FDI problem. The available geometric FDI methods for Fin-D provide useful tools for addressing the FDI problem by taking advantages of invariant subspaces.

The FDI problem of Inf-D systems can be handled by invoking approximate or exact methods. Each method has its own advantages and limitations. On one hand, in the approximate methods the extension of the currently available results for Fin-D system is more straightforward than the exact approach. On the other hand, development of geometric FDI approaches by invoking exact schemes provides a more fundamental and better understanding of the Inf-D systems. For example, as shown in Chapters 4 and 5, we investigate duality of Inf-D systems that enables one to address and investigate the disturbance decoupling problem by using the results of this thesis.

The main objective of this thesis is to investigate the FDI problem of Inf-D systems by using both approximate and exact approaches and to develop new geometric frameworks for Inf-D systems that not only are applicable to the FDI problem, but also can be extended to other fundamental problems, such as the disturbance decoupling problem.

1.4.1 Thesis Objectives

As mentioned earlier, the FDI problem of Inf-D systems can be addressed by using approximate and exact methods. This thesis first provides and develops a geometric framework for the FDI problem of Inf-D systems based on the approximate method (Chapter 3). We then address the FDI problem of Inf-D system by using an exact approach that investigates the FDI problem of Inf-D systems without any approximation (Chapters 4 and 5).

As shown subsequently in Section 2.5, one can approximate a hyperbolic PDE system that is defined on a single spatial coordinate through the finite difference method that results in a 2-D model. By following along the same steps one can show that a hyperbolic PDE defined on an m spatial coordinates can be approximated by a $(m+1)$-D system. Therefore, one can address the FDI problem of a hyperbolic PDE system by using the results of the n-D systems. However, for n-D models [82], the FDI problem is still a challenging task. The geometric analysis of n-D systems are relatively new and for the first time in the literature we address the geometric FDI problem of n-D systems such that it is applicable to any dimension (i.e. $n \geq 2$), as the first objective of this thesis. In other words, the proposed approach can be applied to a large class of PDE systems such as m-D PDE systems (where $m \geq 1$ is the dimension of the spatial coordinates).

We also address the FDI problem of Inf-D systems by using geometric exact methods. This scheme is more general than the n-D methodology and can be applied to a larger class of Inf-D systems. As stated earlier, although there are some results in the literature on the geometric disturbance decoupling problem of Inf-D systems, the geometric FDI problem of these systems has not yet been addressed. In this thesis, we formally formulate the FDI problem of Inf-D systems, and by providing necessary and sufficient conditions for equivalence of various types of invariant subspaces, we investigate the solvability of the FDI problem.

To analyze invariant subspaces of Inf-D systems, we develop and provide two main methodologies that are based on (generalized) eigenvectors and resolvent operators. The former is applicable to Riesz spectral (RS) systems, whereas the latter approach can be applied to a more general class of Inf-D system. Note that a large class of hyperbolic and parabolic PDE systems can be represented and formulated as RS systems in an Inf-D Hilbert space [110].

To summarize, this thesis focuses on development of a geometric FDI framework for Inf-D systems. The main objectives of this thesis are as follows.

1. Develop FDI units that are based on approximated models that are obtained by using the finite difference methods.

2. Geometric analysis of RS systems that are a subclass Inf-D systems and its application to the FDI problem of RS systems

3. Address the FDI problem of a general Inf-D system by developing and utilizing a new geometric framework for Inf-D systems.

1.4.2 Thesis Contributions

The main contributions of this thesis are summarized as follows:

- **Geometric FDI of n-D systems**

  1. By reformulating n-D models as Inf-D systems, the invariance property of an unobservable subspace is investigated as provided in Section 3.2, where an Inf-D unobservable subspace is also introduced. Unlike the work in [99, 100], this result enables one to formally address the solvability of the FDI problem without a restriction on the initial conditions.
  2. Two important Inf-D invariant subspaces, namely, the conditioned invariant and the unobservability, are introduced for the FMII-based n-D models. Although, these subspaces are Inf-D, we provide explicit algorithms that can be invoked to compute these subspaces in a finite and known number of steps.
  3. A novel procedure is developed for designing a detection filter by utilizing the linear matrix inequalities (LMI) technique.
  4. The FDI problem of n-D systems is formulated in terms of the introduced invariant subspaces, and necessary and sufficient conditions for its solvability are derived and formally analyzed by using our proposed LMI-based detection filter.

- **Invariant subspaces of RS systems with their applications to the FDI problem.**

  1. Necessary and sufficient conditions for equivalence of various conditioned-invariant subspaces for RS systems are obtained and analyzed.
  2. By using duality properties, necessary and sufficient conditions for equivalence of various controlled invariant subspaces are provided.

3. An unobservability subspace for RS systems is introduced, and algorithms that converge in a finite number of steps are proposed.

4. By taking advantage of the introduced subspaces, the FDI problem of RS systems is formulated and necessary and sufficient conditions for solvability of the FDI problem are developed and provided.

- **Semigroup invariant concepts and the FDI problem of Inf-D systems**

  1. Necessary and sufficient conditions are obtained for equivalence of conditioned invariant subspaces of Inf-D systems.

  2. Necessary and sufficient conditions are obtained for equivalence of controlled invariant subspaces of Inf-D systems.

  3. The unobservability subspaces of Inf-D systems is addressed.

  4. The FDI problem of Inf-D systems based on the introduced invariant subspaces is formulated and necessary and sufficient conditions for the FDI problem solvability are derived.

1.5 Thesis Organization

In Chapter 2, we briefly review the geometric FDI approach and invariant subspaces of Fin-D systems. Moreover, in this chapter we address Inf-D vector spaces and the system theory of Inf-D systems that is available in [14, 101, 104].

In Chapter 3, we formulate the FDI problem of n-D systems. More precisely, the *preliminary* results including the Inf-D representation, the FDI problem formulation and the n-D Luenberger observers (detection filters) are presented in Section 3.1. The unobservable subspaces of the FMII-based n-D model are introduced in Section 3.2. The geometric property of these subspaces and the invariant concept of n-D models are presented in Section 3.3. In Section 3.4, necessary and sufficient conditions for solvability of the FDI problem are derived and developed. Analytical comparisons between our proposed approach and the available geometric methods in the literature, namely [99] and [100] are also provided in this section. Furthermore, numerical comparisons with both geometric and algebraic methods in [89,91,99,100] are presented in this section. Simulation results for the FDI problem of a heat transfer process in a thermal-fluid system that is expressed as a PDE system are conducted in Section 3.5. Finally, Section 3.6 concludes the chapter.

Chapter 4 focuses on RS systems. In Section 4.1, RS systems are reviewed. The invariant subspaces are developed and analyzed in Section 4.2. In Section 4.3, the FDI problem is first formulated and then its solvability is addressed. A numerical example is provided in Section 4.4 to demonstrate the capability of our proposed strategy. Finally, Section 4.5 provides the conclusions.

Chapter 5 is devoted to the geometric analysis of a general Inf-D system. It is worth nothing that as compared to Chapter 4, in this chapter we consider a more general Inf-D system and provide the results by using the resolvent operators. In Section 5.1, we first review Inf-D systems and our assumption on these systems in this chapter. Then, certain simple but crucial results on geometric theory of Fin-D system that are not available in the literature are presented. These results are based on resolvent operators. The invariant subspaces are investigated in Section 5.2, where we first derive necessary and sufficient conditions for both conditioned and controlled invariant subspaces. The unobservability subspace is also addressed in this section. Section 5.3 is dedicated to the FDI problem of Inf-D systems, where first the FDI problem is precisely formulated and then we derive necessary and sufficient conditions for the FDI problem solvability. Finally, Section 5.4 provides the conclusions.

Chapter 6 concludes the thesis and provides suggestion for future work.

1.6 Notation

The subspaces (Fin-D and Inf-D) are denoted by $\mathscr{A}$, $\mathscr{B}$, $\cdots$. The notations $\overline{\mathscr{V}}$ and $\mathscr{V}^{\perp}$ denote the closure and orthogonal complement of the subspace $\mathscr{V}$, respectively. For a given operator L, the subspace of image of L is denoted by $\mathscr{L}$. The maps between two finite dimensional vector spaces are designated by A, B, $\cdots$. The inverse image of the subspace $\mathscr{V}$ with respect to the operator A is denoted by $A^{-1}\mathscr{V}$. The block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is denoted by $\mathrm{diag}(A, B)$. The real, complex, integer and positive integer numbers are denoted by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$, respectively. The notation $\underline{\mathbb{N}}$ denotes the set $\mathbb{N}\cup\{0\}$. Let $m < n$ and $\mathscr{V} \subseteq \mathbb{R}^m$. The corresponding embedded subspace of $\mathscr{V}$ in $\mathbb{R}^n$ is denoted by $\tilde{\mathscr{V}} \subseteq \mathbb{R}^n$. In other words, $\tilde{\mathscr{V}} = Q\mathscr{V}$, where Q is the embedding operator from $\mathbb{R}^m$ into $\mathbb{R}^n$. In Chapter 3, we deal with special Inf-D subspaces that are defined as follows. The Inf-D subspace $\cdots \boxplus \mathscr{V} \boxplus \mathscr{V} \boxplus \cdots$ (which represents the direct sum of an infinite number of $\mathscr{V}$) is denoted by $\boxplus(\mathscr{V}) \subseteq \mathbb{R}^{\infty}$, where $\mathscr{V} \subseteq \mathbb{R}^m$. Let $\mathbf{x} = (x_j)_{j\in\mathbb{Z}} = (\cdots, x_{-1}^{\mathrm{T}}, x_0^{\mathrm{T}}, x_1^{\mathrm{T}}, \cdots)^{\mathrm{T}} \subseteq \boxplus(\mathscr{V})$ and $|\mathbf{x}|_{\infty} = \sup\limits_{i\in\mathbb{Z}}|x_i|$, where $x_i \in \mathscr{V}$. The vector space $\mathscr{V}_{\infty} = \bigoplus(\mathscr{V})$ is defined by $\{\mathbf{x}|\mathbf{x} \in \boxplus(\mathscr{V}) \text{ and } |\mathbf{x}|_{\infty} < \infty\}$. It can be shown that $\mathscr{V}_{\infty}$ is a Banach (but not necessarily Hilbert) space. Let $i, j \in \mathbb{Z}\cup\{-\infty, \infty\}$ and $j \geq i$. The Inf-D vector $\mathbf{x}_i^j \in \bigoplus(\mathscr{V})$ is expressed as $\mathbf{x}_i^j = [\cdots, 0, 0, x_i^{\mathrm{T}}, \cdots, x_j^{\mathrm{T}}, 0, 0, \cdots]^{\mathrm{T}}$, where $x_\ell \in \mathscr{V}$ for all $i \leq \ell \leq j$, and associated with $\mathbf{x}_{-\infty}^{\infty}$ we simply use $\mathbf{x}$. Consider the real subspace $\mathscr{V} = \overline{\mathrm{span}\{x_i\}_{\ i\in\mathbb{I}}}$ ($\mathbb{I} \subseteq \mathbb{N}$). The corresponding complex subspace $\mathscr{V}_{\mathbb{C}}$ is defined as all vectors z that can be expressed as $z = \sum_{i\in\mathbb{I}} \zeta_i x_i$, where $\zeta_i \in \mathbb{C}$. The notations $\mathcal{A}$, $\mathcal{B}$, $\cdots$ denote the maps between two vector spaces such that at least one of them is Inf-D. Particularly, the notions I and $\mathcal{I}$ denote the identity operators on Fin-D and Inf-D subspaces, respectively. The set of all bounded operators defined on $\mathcal{X}$ are designated by $\mathcal{L}(\mathcal{X})$. The domain of an unbounded operator $\mathcal{A}$ is denoted by $D(\mathcal{A})$. The resolvent set of $\mathcal{A}$ is the set of all λ such that $(\lambda I - \mathcal{A})$ is invertible and $(\lambda\mathcal{I} - \mathcal{A})^{-1}$ is bounded. This set is denoted by $\rho(\mathcal{A})$. Also, the operator $\mathcal{R}(\lambda, \mathcal{A}) = (\lambda\mathcal{I} - \mathcal{A})^{-1}$ ($R(\lambda, A) = (\lambda I - A)^{-1}$) denotes the resolvent operator of $\mathcal{A}$ (A). $\rho_{\infty}(\mathcal{A})$ denotes the largest interval $[r, \infty)$ such that for all $\lambda \in [r, \infty)$, we have $\lambda \in \rho(\mathcal{A})$ (we have the same notations for operator A). $\mathscr{V}_{\overline{\mathrm{R}}}$ and $\mathscr{V}_{\underline{\mathrm{R}}}$ denote arbitrary $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspaces containing and contained in a given subspace $\mathscr{V}$, respectively. The operator of strongly continuous semigroup that is generated by $\mathcal{A}$ is denoted by $\mathbb{T}_{\mathcal{A}}$. The largest invariant subspace with respect to $\mathbb{T}_{\mathcal{A}}$ that is contained in the subspace $\mathscr{V}$ is designated by $< \mathscr{V} | \mathbb{T}_{\mathcal{A}} >$. The other notations are defined within the text of the thesis.

Chapter 2

Background Information

In this chapter, we first review invariant subspaces and the FDI problem of Fin-D systems. The Inf-D vector spaces and Inf-D system theory are then briefly addressed.

2.1 Geometric Analysis of Linear Systems on a Fin-D Hilbert Space

Consider a linear time-invariant state space equation defined on an n-dimensional real Hilbert space. Based on the fact that every n-dimensional real Hilbert space is isometrically isomorphic to $\mathbb{R}^n$ [104], we can represent any linear operator between two Fin-D Hilbert spaces by a matrix. Therefore, without loss of any generality a linear Fin-D dynamical system can be represented as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$. The above equation has a regular (sufficiently smooth and unique) solution for x is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)Bu(s)\mathrm{d}s \tag{2.2}$$

where $\Phi(\cdot)$ is the fundamental matrix which can be computed as

$$\Phi(t) = e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}. \tag{2.3}$$

Jordan Decomposition

In Chapter 4, we define the class of Inf-D RS systems and develop a number of results which lead to necessary and sufficient conditions for FDI. This work requires a generalization of the Jordan canonical form of Fin-D operators. In this subsection, we briefly review the Jordan-form operator and its invariant subspaces.

Definition 2.1. *(General Jourdan Decomposition) [74] For every operator* $A : \mathbb{R}^n \to \mathbb{R}^n$*, there exists a isomorphism* $T : \mathbb{C}^n \to \mathbb{C}^n$ *such that* $J = T^{-1}AT = diag(J_1, \cdots, J_{\ell_1})$*, where* $J_i : \mathbb{C}^{n_i} \to \mathbb{C}^{n_i}$*,* $\sum_{i=1}^{\ell_1} n_i = n$ *and*

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots \\ 0 & \lambda_i & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}, \tag{2.4}$$

where $\lambda_i \in \mathbb{C}$ *is an eigenvalue of* A*.*

Since in the FDI problem we are interested in real systems and real subspaces, the following definition is more suitable for our purpose.

Definition 2.2. *(Real Jordan Decomposition) [74] For every operator* $A : \mathbb{R}^n \to \mathbb{R}^n$*, there exists an isomorphism* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $J = T^{-1}AT = diag(J_1, \cdots, J_\ell)$ *such that* $J_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$*,* $\sum_{i=1}^{\ell} n_i = n$*. The Jordan block* J_i *corresponding to a* real *eigenvalue is defined as per equation* (2.4), *otherwise*

$$J_i = \begin{bmatrix} \Lambda_i & I & 0 & 0 & \cdots \\ 0 & \Lambda_i & I & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \vdots & 0 & \cdots & \Lambda_i & I \\ \vdots & 0 & \cdots & 0 & \Lambda_i \end{bmatrix}, \tag{2.5}$$

where $\Lambda_i = \begin{bmatrix} \mu_{i,1} & \mu_{i,2} \\ -\mu_{i,2} & \mu_{i,1} \end{bmatrix}$*, such that* $\mu_{i,1} \pm j\mu_{i,2}$ *are two* complex *eigenvalues of* A*. Also,* I *is an identity operator defined on* $\mathbb{R}^2$*.*

Invariant subspaces play a central role in the geometric FDI approach. To define invariant subspaces, we need the following notation. Consider a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ and a subspace $\mathscr{V} \subseteq \mathbb{R}^n$. Then, we have

$$A\mathscr{V} = \{y \in \mathbb{R}^n |\ y = Ax,\ x \in \mathscr{V}\} \tag{2.6}$$

Definition 2.3. *A subspace* $\mathscr{V} \subseteq \mathbb{R}^n$ *is called* A*-invariant if* $A\mathscr{V} \subseteq \mathscr{V}$*.*

By considering the Jordan blocks J_i defined by (2.4) and (2.5), it follows that one can decompose $\mathbb{R}^n$ into $\mathbb{R}^n = \tilde{\mathscr{V}}_1 \oplus \cdots \oplus \tilde{\mathscr{V}}_\ell$, where $J|\mathscr{V}_i = J_i$, and $\tilde{\mathscr{V}}$ is the embedded subspace of $\mathscr{V}$ into $\mathbb{R}^n$. In other words, $\tilde{\mathscr{V}} = Q\mathscr{V}$, where Q is the embedding operator from $\mathbb{R}^m$ into $\mathbb{R}^n$, where $m = \dim(\mathscr{V})$. The following lemma shows a relationship between the Jordan decomposition and the J-invariant subspaces.

Lemma 2.4. *[74, Proposition 0.4] The subspace* $\mathscr{V} \subseteq \mathbb{R}^n$ *is* J*-invariant that is* $J\mathscr{V} \subseteq \mathscr{V}$ *if* $\mathscr{V} = \tilde{\mathscr{V}}_1^1 \oplus \cdots \oplus \tilde{\mathscr{V}}_\ell^1$*, such that* $\tilde{\mathscr{V}}_i^1 \subseteq \tilde{\mathscr{V}}_i$ *and* $\mathscr{V}_i^1$ *is* J_i*-invariant for* $i = 1, \ldots, \ell$*.*

It is worth noting that reverse of the above lemma only holds for the operators with distinct eigenvalues. For example, consider $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ that is already in the Jordan canonical form and the subspace $\mathscr{V} = \operatorname{span}\{[1,\ 1,\ 0,\ 0]^{\mathrm{T}}\}$. It follows that $\mathscr{V}$ cannot be decomposed into $\mathscr{V} = \tilde{\mathscr{V}}_1^1 \oplus \cdots \oplus \tilde{\mathscr{V}}_\ell^1$, however it is J-invariant. This problem has its roots in the fact that Lemma 2.4 considers different Jordan blocks without any special concern about the eigenvalues as to whether they are distinct or multiple. In other words, Lemma 2.4 is not a coordinate-free result and the structure of J and the decomposition of $\mathscr{V}$ are based on the coordinates that are defined by the (generalized) eigenvectors. To tackle this problem, one needs to merge all the Jordan blocks corresponding to a given eigenvalue as shown in the next lemma.

Lemma 2.5. *[101] The subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is J-invariant (that is, $J\mathscr{V} \subseteq \mathscr{V}$) if and only if $\mathscr{V} = \tilde{\mathscr{V}}_{\lambda_1}^1 \oplus \cdots \oplus \tilde{\mathscr{V}}_{\lambda_m}^1$, where $\tilde{\mathscr{V}}_{\lambda_i}$ is the corresponding eigenspace to λ_i $\tilde{\mathscr{V}}_{\lambda_i}^1 \subseteq \tilde{\mathscr{V}}_{\lambda_i}$ and $\mathscr{V}_{\lambda_i}^1$ is J_{λ_i}-invariant $i = 1, \cdots, m$, $J_{\lambda_i} = diag(J_{i,1}, \cdots, J_{i,n_i})$, n_i is the algebraic multiplicity of λ_i and $J_{i,k}$ is the Jordan block corresponding to λ_i.*

Fin-D Spectral Projection

According to the above discussion, the subspaces $\tilde{\mathscr{V}}_{\lambda_i}$, $i = 1, \ldots, m$ in Lemma 2.5 are of importance, and consequently it is necessary to formally characterize these subspaces for Inf-D systems. In Fin-D systems, it is obvious that $\tilde{\mathscr{V}}_{\lambda_i} = \operatorname{span}\{\phi_{\lambda_i,k}\}_{k=1}^{n_i}$, where $\phi_{\lambda_i,k}$ are the (generalized) eigenvectors corresponding to λ_i and n_i is the algebraic multiplicity of λ_i. However, to generalize the above results we need to investigate the subspaces $\tilde{\mathscr{V}}_{\lambda_i}$ from a more abstract point of view. To generalize Jordan form to Inf-D systems we need the following projection operator

$$P_{\lambda_i} : \mathbb{C}^n \to \mathbb{C}^n, \quad P_{\lambda_i} z = \frac{1}{2j\pi} \int_{\Gamma_i} (\lambda I - A)^{-1} z \mathrm{d}\lambda, \tag{2.7}$$

where Γ_i is a simple curve in $\mathbb{C}$ surrounding only λ_i. Indeed, it can be shown that

$$P_{\lambda_i} z = \begin{cases} z & z \in \tilde{\mathscr{V}}_{\lambda_i} \\ 0 & \text{Otherwise} \end{cases} \tag{2.8}$$

This is concluded from the Cauchy's integral formula for holomorphic functions ([14, Appendix A]).

By following Definition 2.2, one can construct real subspaces corresponding to a complex eigenvalue λ and its complex conjugate $\overline{\lambda}$. Therefore, we have $\sum_i P_{\lambda_i} = I$ and $P_{\lambda_i}\mathbb{R}^n = \tilde{\mathscr{V}}_{\lambda_i}$. In Chapter 4 by generalizing the above operator for Inf-D systems, we formally define the RS operators.

2.2 Geometric FDI Approach for Finite Dimensional (Fin-D) Systems

In this section, we briefly review the geometric FDI approach for the Fin-D systems that has been developed in [3, 41, 50]. This approach is a cornerstone for this dissertation.

Consider the following linear Fin-D system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \sum_{i=1}^{p} L_i f_i(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2.9}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ denote state, input and output vectors, respectively. Moreover, L_i and f_i denote fault signatures and fault signals, respectively. For example, by setting $p = m$, $f_i(t) \in \mathbb{R}$ and $L = [L_1 \ \cdots \ L_p] = B$, one can model the actuator faults in the system (2.9). In other words, $L_k f_k(t)$ represents the faulty behavior of the k^{th} actuator. For example, by setting $f_k(t) = -0.2u_k(t)$ for all $t \geq t_{\text{f}}$, one can model the permanent fault of 20% loss of effectiveness in the

k^{th} actuator that occurs at $t = t_{\text{f}}$. Also, be setting

$$f_k(t) = \begin{cases} -0.1u_k(t) & t_{\text{f}_1} \leq t \leq t_{\text{f}_2} \\ 0 & \text{Otherwise} \end{cases} \tag{2.10}$$

we model the intermittent fault of 10% loss of effectiveness in the k^{th} actuator that occurs at the interval $[t_{\text{f}_1}, t_{\text{f}_2}]$.

As stated in the previous chapter, the FDI problem solvability is accomplished by generating a set of residual r_i, $i = 1, \ldots, p$ such that each residual is decoupled from all inputs and faults but one fault. In other words, each r_i has the following properties,

- In absence of the fault f_i the residual r_i decays to a neighborhood of zero.
- When the fault f_i occurs, the residual r_i exceeds a predefined threshold.

It should be pointed out that based on the above definition we have one residual for each fault signal. However, one can use a coding approach to use a smaller set of residuals to detect and to isolate the faults (refer to [3, 5] for more details). The detection filters are the realization that is utilized in this thesis for the residual generators. In other words, we use a set of filters such that output of each filter is a residual signal corresponding to one and only one fault.

In the geometric FDI approach for Fin-D systems, the necessary and sufficient conditions for solvability of the FDI problem (generating the residual) have been developed based on invariant subspaces [3]. In particular, the A-invariant (Definition 2.3), conditioned invariant ((C, A)-invariant) and unobservability subspaces play a crucial role in this area. By considering the dynamical system (2.9), these subspaces are defined as follows.

Definition 2.6. *A subspace $\mathscr{W}$ is called conditioned invariant if there exists an output injection map $D : \mathbb{R}^q \to \mathbb{R}^n$ such that $(A + DC)\mathscr{W} \subseteq \mathscr{W}$.*

It can be shown that the subspace $\mathscr{W}$ is conditioned invariant if and only if $A(\mathscr{W} \cap \ker C) \subseteq \mathscr{W}$ [3]. The set of all conditioned invariant subspaces containing a given subspace $\mathscr{L}$ is closed respect to intersection, and consequently this set always holds a minimal subspace in the inclusion sense. The minimal conditioned invariant subspace containing the subspace $\mathscr{L}$ (that is denoted by $\mathscr{W}^*(\mathscr{L})$) is the limiting subspace of the following algorithm,

$$\begin{aligned} \mathscr{W}_0 &= \mathscr{L}, \\ \mathscr{W}_k &= \mathscr{L} + A(\mathscr{W}_{k-1} \cap \ker C),\ k \in \mathbb{N}, \end{aligned} \tag{2.11}$$

where $\mathscr{W}^*(\mathscr{L}) = \mathscr{W}_n$. Another cornerstone of geometric FDI is the unobservability subspace that is defined as follows.

Definition 2.7. *A subspace $\mathcal{S}$ is called unobservability subspace if there exist two maps $D : \mathbb{R}^q \to \mathbb{R}^n$ and $H : \mathbb{R}^q \to \mathbb{R}^{q_0}$ such that $q_0 \leq q$ and $\mathcal{S} =< \ker HC | A + DC >$ (that is largest $A + DC$-invariant contained in $\ker HC$).*

The notation $\underline{\mathcal{S}}(\mathscr{L})$ denotes the family of all unobservability subspaces $\mathcal{S}$ containing the subspace $\mathscr{L}$. As above, it can be shown that the set $\underline{\mathcal{S}}(\mathscr{L})$ always holds a minimal (that is denoted by $\mathcal{S}^*(\mathscr{L})$) [3, Chapter 2, Theorem 18]. The subspace $\mathcal{S}^*(\mathscr{L})$ is given by the following algorithm,

$$\begin{aligned} \mathcal{S}_0 &= \mathbb{R}^n, \\ \mathcal{S}_k &= \mathscr{W}^*(\mathscr{L}) + A^{-1}(\mathcal{S}_{k-1} \cap \ker C),\ k \in \mathbb{N}, \end{aligned} \tag{2.12}$$

where $\mathscr{W}^*(\mathscr{L})$ is the limiting subspace of the algorithm (2.11) and $\mathcal{S}^*(\mathscr{L}) = \mathcal{S}_n$.

The main result on the geometric FDI approach of Fin-D systems has been provided in [3] as follows.

Theorem 2.8. *[3, Chapter 4, Theorem 2] The FDI problem defined above is solvable if and only if there exist the unobservability subspaces*

$$\mathcal{S}_i^* \cap \mathscr{L}_i = 0,\ i = 1, \ldots, p, \tag{2.13}$$

where $\mathscr{L}_i =$ span$\{L_i\}$ *and* $\mathcal{S}_i^*$ *is the smallest unobservability subspace containing* $\mathscr{L} = \sum_{j=1,j\neq i}^{p} \mathscr{L}_j$ *by using algorithm* (2.12).

The prominent feature of the above theorem is that the sufficient part has been shown by a constructive method. Specifically, to show the sufficient part a systematic method to design the filters and residual generators has been provided. The geometric FDI approach for Fin-D systems includes three major steps, namely a signature mapping, factoring out and filter design.

1. *Signature mapping*: In this step, one obtains the maps D and H such that the signatures of all faults but f_i (i.e., $L_j, j \neq i$) are contained in the unobservability subspace defined by D and H (that is $\mathcal{S}^*(\mathscr{L}) =< \ker HC|A + DC >$, $\mathscr{L} = \sum_{j=1,j\neq i}^{p} \mathscr{L}_j$).

2. *Factoring out*: If condition (2.13) is satisfied, the unobservability subspace (that is computed in the previous step) is factored out from the linear system (HC,$A+DC$), where the resulting quotient subsystem is an observable system which is decoupled from all faults but f_i.

3. *Filter Design*: In the filter design step, one designs an observer to decouple the residual (that is output of the observer) from input signal(s) and initial state error. Below, we formulate these steps.

As mentioned above, one needs to factor out unobservability subspaces. Below, we shows the steps. Let $\mathcal{S}_i^*$ be the unobservability subspace containing all fault signatures $L_j, j \neq i$ such that $\mathcal{S}_i^* \cap \mathscr{L}_i = 0$. Also, there exist the maps D_i and H_i are such that $\mathcal{S}_i^* =< \ker H_iC|A + D_iC >$. By definition, $\mathcal{S}_i^*$ is an $(A + D_iC)$-invariant subspace.

Now, consider the the canonical projection map $P_i : \mathbb{R}^n \to \mathbb{R}^n/\mathcal{S}_i^*$ and the following detection filter,

$$\begin{aligned}\dot{\omega}_i(t) &= F_i\omega_i(t) + G_iu(t) - E_iy(t),\\ r_i(t) &= H_iy(t) - M_i\omega_i(t),\end{aligned} \tag{2.14}$$

where $\omega(t) \in \mathbb{R}^o$, $D_\mathrm{f} = D + P_i^{-r}D_oH_i$, D_o is the observer gain, $E_i = P_iD_\mathrm{f}$, $G_i = P_iB$, A_p is an operator such that $A_pP_i = P_i(A + D_iC)$ and $F_i = A_p + D_oM_i$. Also, M_i is the unique solution of $M_iP_i = H_iC$. Note that P_i is a monic operator and hence, the M_i exists and is unique, and $\dim(\mathbb{R}^o) = n - \dim(\mathcal{S}_i^*)$ that is factoring out of $\mathcal{S}_i^*$ - Step 2.

Define $e(t) = P_ix(t) - \omega_i(t)$. It follows that

$$\begin{aligned}\dot{e}(t) =&P_iAx(t) + G_iu(t) + P_iL_if_i(t) - (F_i\omega_i(t) + G_iu(t) - P_iD_iCx(t) - D_oH_iCx(t))\\ =&P_i(A + D_iC)x(t) - F_i\omega_i(t) + D_oH_iCx(t) + P_iL_if_i(t)\\ =&A_pP_ix(t) + D_oM_iP_ix(t) - F_i\omega_i(t) + P_iL_if_i(t) = F_ie(t) + P_iL_if_i(t)\end{aligned} \tag{2.15}$$

Moreover, since $r_i(t) = H_iy(t) - M_i\omega_i(t)$ and $M_iP_i = H_iC$, one obtains $r_i(t) = M_iP_ix(t) - M_i\omega(t) = M_ie(t)$.

Note that based on the fact that $\mathcal{S}_i^*$ is the unobservable subspace, by factoring out this subspace the subsystem (M_i,A_p). The gain D_o is an observer gain for the system (M_i,A_p) that is designed to eliminate the effects of input and initial state error. This is Step 3.

Now, if $f_i(t) \equiv 0$ the residual vanishes to a small neighborhood of origin. However, if $f_i(t) \neq 0$, $r(t)$ exceeds a threshold (that are defined by using Monte Carlo simulation - refer to Chapter 3 for more details). Therefore, the filter (2.14) can detect and isolate the fault f_i.

As can be observed from the above procedure, the unobservability subspace is a central concept in the geometric FDI approach. Therefore, to generalize this approach to a more general class of dynamical systems, defining this subspace is inevitable. For instance, this concept is generalized to Markovian-Jump and Time-Delay systems in [64–66, 111].

2.3 Inf-D Vector Spaces

In this section, we review certain basic mathematical tools that are essential for studying Inf-D systems. These tools can be categorized into two main groups, namely topological (more particularly Hilbert) spaces and theory of semigroups of operators. In the geometric theory of Fin-D systems developed in [74, 112], one does not need to deal with a linear space equipped with a norm function, and a basis that fully characterizes the corresponding space. The main reason lies on the fact that every Fin-D subspaces $\mathscr{V}$ is isomorphic to $\mathbb{R}^{n_\mathrm{v}}$ (i.e. there exists an invertible linear operator $T : \mathscr{V} \to \mathbb{R}^{n_\mathrm{v}}$), where $n_\mathrm{v} = \dim(\mathscr{V})$. Therefore, every Fin-D subspace is closed. However, Inf-D vector subspaces are not necessarily closed and one usually deals with limits. To define a limit in a vector space one needs to define a norm, or in more general sense, a topology on the corresponding vector spaces [104].

This section focuses only on the special class of topological spaces that are induced by an inner product defined on the corresponding spaces. More precisely, we only review the Hilbert spaces in an arbitrary dimension.

However, before going into more detail, we first provide an example which clearly shows that dealing with Inf-D subspaces are more complicated than Fin-D subspaces. Consequently, it leads us to more sophisticated mathematical tools that are provided in this section. Indeed, this example emphasizes the fact that trivial results in Fin-D spaces are not easy to show for Inf-D subspaces and even certain set of them does not hold true.

Example 2.9. *Basis of Inf-D vector spaces:*

Consider an n-dimensional vector space $\mathcal{X}$. It is well-known that every set of n

linearly independent vectors is a basis for $\mathcal{X}$. Now, assume that $\mathcal{X}$ is an Inf-D space and consider $B_1 = \{\phi_1, \phi_2, \cdots\}$ as a set of *infinite* number of linear independent vectors of $\mathcal{X}$. Although cardinality of B_1 is infinite, B_1 is not necessarily a basis for the space $\mathcal{X}$. To see this, let us define a set of linearly independent vectors as $B_2 = \{\phi_2, \phi_4, \cdots, \phi_{2k}, \cdots\}$ which is a subset of B_1. It follows that the number of vectors in B_2 is also infinite, however, it cannot span the space that is constructed by B_1. This lack of completeness causes a set of challenging concepts such as non-closed subspaces, non-complete subspaces, etc.

As emphasized in the above examples, for investigating the Inf-D spaces we need more sophisticated tools and even by using these tools certain results in Fin-D spaces do not hold for Inf-D vector spaces. This section attempts to review certain concepts of linear vector spaces from an abstract point of view that allows one to address both Fin-D and Inf-D dynamical systems by using a unique methodology.

This section is organized as follows. In the next subsection, we investigate the inner product vector spaces from the topological point of view. Hilbert vector spaces are reviewed in Subsection 2.3.2. Basis, dimension, dual spaces and quotient subspaces are briefly addressed in Subsections 2.3.3-2.3.6.

2.3.1 Topological Spaces

In this subsection, we review topological subspaces and focus on the topologies induced by norm. Consider the set $\mathcal{X}$ and a collocation of subsets of $\mathcal{X}$, denoted by $\mathfrak{T}$ such that the elements of $\mathfrak{T}$ (that are subsets of $\mathcal{X}$) satisfy

1. $\mathcal{X}, \varnothing \in \mathfrak{T}$.

2. $V_\alpha \in \mathfrak{T} \Rightarrow \bigcup_{\alpha \in \mathbb{I}} V_\alpha \in \mathfrak{T}$ for arbitrary index set $\mathbb{I}$ (finite, infinite or even uncountable).

3. $V_1, V_2 \in \mathfrak{T} \Rightarrow V_1 \cap V_2 \in \mathfrak{T}$.

V_α and $\mathfrak{T}$ are called open sets and topology, respectively. We denote this topological set as $(\mathcal{X}, \mathfrak{T})$ or simply $\mathcal{X}$ when the topology is known from the context.

Generally, topology is a tool which enables one to define continuity for a map between sets. Consider two topological sets $(\mathcal{X}, \mathfrak{T}_x)$ and $(\mathcal{Y}, \mathfrak{T}_y)$ and the map $F : \mathcal{X} \to \mathcal{Y}$. Then, we say F is continuous at x_0 if and only if for all open sets containing $F(x_0)$, say V_y, $F^{-1}(V_y)$ is an open set of $\mathcal{X}$. This definition is more general than the ε, δ definition that is used usually in the ordinary calculus [113].

As stated earlier, in this section we are only interested in the topology that is defined on a vector space and not on a set. In certain literature [104], the set $\mathcal{X}$ with the topology $\mathfrak{T}$ is called as the topological vector space. However, here we reserve this name for the case when the set $\mathcal{X}$ is a linear vector space that is equipped by a topology. We have:

Definition 2.10. *The vector space $\mathcal{X}$ with the topology $\mathfrak{T}$ is called the topological vector space if the vector addition and scalar product operations defined on $\mathcal{X}$ are continuous with respect to the topology of $\mathcal{X} \times \mathcal{X}$ and $\mathbb{F} \times \mathcal{X}$, respectively, where $\mathbb{F}$ is the scalar field on which the space $\mathcal{X}$ is defined.*

In this thesis, we are mainly concerned with Hilbert spaces defined on $\mathbb{R}$, so we focus on the topologies that are induced by using norm. Note that every inner product space is a normed space with the norm induced from the inner product. Consider the normed vector space $\mathcal{X}$ and the corresponding norm function $||\cdot||$. One can define a topology for this vector space by using the norm as follows. We call a set V as an open set (to define topology) if for every $x_0 \in V$, one can find $r_0 \in \mathbb{R}$ such that $B(x_0, r_0) = \{x \in \mathcal{X} |\ ||x - x_0|| < r_0\} \subseteq V$.

Based on the norm (or generally, open sets and topological structure), one can define the limit as follows. We write $x_1 \to x_2$ when $||x_1 - x_2|| \to 0$ which is well-defined because $||x|| \in \mathbb{R}$ and the limit is well-defined on $\mathbb{R}$. Also, one can define the Cauchy and convergent sequence as usual. Recall that the sequence $\{x_i\}$

is called a Cauchy sequence if for any given $\varepsilon > 0$ one can find a number $N_\varepsilon \in \mathbb{N}$ such that

$$\forall n, m > N_\varepsilon \Rightarrow ||x_n - x_m|| < \varepsilon \tag{2.16}$$

Now, we have following definitions.

Definition 2.11. Open Covers: *The collection of open sets $\{U_\alpha\}$ (finite, countable or uncountable) is an open cover of subspace $\mathscr{V}$ if $\mathscr{V} \subseteq \bigcup_\alpha U_\alpha$.*

Definition 2.12. Completeness: *The topological vector space $\mathcal{X}$ is said to be complete if every Cauchy sequence is convergent.*

Remark 2.13. *As we shall see in the next subsection, in Inf-D subspaces we usually work with the limits of sequences, and the completeness property is crucial to show the existence of limits in the corresponding subspaces.*

2.3.2 Hilbert Spaces

As mentioned earlier, in this dissertation we focus on Hilbert spaces. In this part, we first define Hilbert spaces and then important concepts that relate to our research are provided. In this section, $\mathcal{X}$ denote a Fin-D or Inf-D systems. In other words, all the results are applicable to both Fin-D and Inf-D vector spaces.

Definition 2.14. *Consider the vector space $\mathcal{X}$, the inner product $< \cdot, \cdot >$ and the induced norm that is defined as $|| \cdot || : \mathcal{X} \to \mathbb{R}, \quad ||x|| = \sqrt{< x, x >}$. We call $\mathcal{X}$ a Hilbert space if $\mathcal{X}$ is complete.*

Therefore, the Hilbert space $\mathcal{X}$ is naturally equipped with a topological structure that is induced by a norm.

Example 2.15. *Hilbert vector space:*

Consider the space $L_2([0, 1], \mathbb{R})$ that is the space of all square integrable functions defined on $[0,1]$, and the inner product $< f, g >= \int_0^1 f(x)g(x)\mathrm{d}x$. The induced norm is defined by $||f|| = \sqrt{\int_0^1 f(x)^2\mathrm{d}x}$, and it can be shown that this normed space is complete [104].

2.3.3 Basis

One of the important concepts in vector spaces is the basis. Although, to define a basis the corresponding vector space does not need to be even a normed space, in this section we focus on the Hilbert spaces.

There are various definitions of basis. However, here we review two of these definitions.

Definition 2.16. *Consider the vector space $\mathcal{X}$ and the set of independent vectors $H = \{\phi_i\}_{i\in\mathbb{I}}$, where $\mathbb{I}$ is an index set (not necessarily countable) and $\phi_i \in \mathcal{X}$. The set H is a Hamel basis of $\mathcal{X}$ if every $x \in \mathcal{X}$ can be expressed as a linear combination of a* <u>*finite*</u> *number of ϕ_i.*

In the literature, the Hamel basis is also called an algebraic basis [104]. For the Fin-D vector spaces (dimension is formally defined in the next subsection), every basis is a Hamel basis, and every Hamel basis is a countable basis. However, for Inf-D Banach space (that is a complete vector space equipped with a norm), every Hamel basis is essentially *uncountable* that makes analysis of Inf-D system by using Hamel spaces much more difficult. In other words,

Lemma 2.17. *([104, Problem i.11.2]) Le $\mathcal{X}$ be an Inf-D vector space. Then, every Hamel basis for $\mathcal{X}$ is uncountable.*

Since in Inf-D systems we deal with Hilbert spaces with a countable dimension (that is more structured) one can utilize the following countable basis (that is called as Riesz basis) which has certain nice properties. The following definition formally introduces the Riesz basis.

Definition 2.18. *[14] The set of vectors* $\{\phi_i\}_{i\in\mathbb{I}}$, $\mathbb{I} \subseteq \mathbb{N}$ *is called the Riesz basis for the Hilbert space* $\mathcal{X}$ *if* $\overline{\operatorname{span}\{\phi_i\}}_{\ i\in\mathbb{I}} = \mathcal{X}$.

It can be shown that if $\{\phi_i\}_{i\in\mathbb{I}}$ is a Riesz basis for $\mathcal{X}$, then there exists a set of vectors $\{\psi_i\}_{i\in\mathbb{I}}$ such that $\psi_i \in \mathcal{X}$ and $<\psi_i, \phi_k>= \delta_{ik}$ (δ_{ik} denotes the Dirac delta function), for all $i, k \in \mathbb{I}$ [14, Section 2.3]. In other words, ψ_i's and ϕ_k are biorthonormal vectors [14].

Example 2.19. *Riesz basis:*

Consider the space $L_2([0,1],\mathbb{R})$. It can be shown that $\{\sin(2n\pi x), \cos(2n\pi x)\}_{n\in\underline{\mathbb{N}}}$ is Riesz spectral basis for $L_2([0,1],\mathbb{R})$ (that is a countable basis).

Definition 2.20. *[104] A Hilbert space that admits a countable orthonormal basis is called a separable Hilbert space.*

Since the Riesz bases are equivalent to a countable orthonormal basis, every Hilbert space that has a Riesz basis is separable [104]. The Inf-D systems that are addressed in this thesis are defined on separable Hilbert spaces.

Another difference between Definitions 2.16 and 2.18 does show up in Inf-D vector spaces. By using a Hamel basis, each vector is presented by a finite number of basis vectors. In other word, $x = \sum_{j\in\mathbb{J}} \zeta_j \phi_j$, where $\mathbb{J} \subseteq \mathbb{I}$ is a finite subset. Since $\mathbb{J}$ is finite, the equality $x = \sum_{j\in J} \zeta_j \phi_j$ is well-defined. However, for the Riesz basis (Definition 2.18) in fact we have $x = \lim_{n\to\infty} \sum_{k=1}^{n} \zeta_k \phi_k$, where the limit is well-defined by using the norm defined on the Hilbert space $\mathcal{X}$ and the fact that $\mathcal{X}$ is complete.[1]

2.3.4 Dimension

Generally, the dimension of a vector space is defined based on the basis (as stated above subsequently we deal with Definition 2.18). Based on the basis, we have three

[1]As can be observed, to define a countable basis we need only norm functions (and no inner product). However, for the consistency we provided the Definition 2.18 for Hilbert spaces.

types of Hilbert spaces as follows:

- Fin-D Hilbert spaces - the corresponding basis is a finite set. It includes all the Fin-D spaces.

- Inf-D separable Hilbert spaces - the corresponding basis is an infinite, countable set. For instance $L_2([0,1],\mathbb{R})$.

- Inf-D and uncountable vector spaces - the corresponding basis is an infinite, uncountable set. For example, consider the vector space ℓ^∞ that is the space of all sequences $(\zeta_1, \zeta_2, \cdots)$ such that $\zeta_k \in \mathbb{R},\ k \in \mathbb{N}$ and $\sup_k(|\zeta_k|) < \infty$ (refer to [104, page 57]).

Since the state space of PDE and time-delay systems can be formulated in an appropriate Hilbert space with a countable basis, in this thesis, we are interested on the first two types of Hilbert spaces.

Remark 2.21. *The dimension of a vector space is directly related to its field. For example, consider the plane E^2 (which is all the points in a plane- but it is not still a vector space). Let $\mathbb{R}^2$ and $\mathbb{C}^1$ be the corresponding vector spaces of E^2 defined on $\mathbb{R}$ and $\mathbb{C}$, respectively. It follows that a basis for $\mathbb{R}^2$ is $\{\phi_1, \phi_2\}$, where $\phi_1 = [1,0]^{\mathrm{T}}$ and $\phi_2 = [0,1]^{\mathrm{T}}$, and every $x \in E^2$ can be expressed as $x = \zeta_1\phi_1 + \zeta_2\phi_2$ ($\zeta_i \in \mathbb{R}$ for $i = 1,2$). Hence,* $\dim(\mathbb{R}^2) = 2$. *Now, consider $\mathbb{C}^1$ (that is, E^2 on $\mathbb{C}$). We claim that the dimension of $\mathbb{C}^1$ is one. Towards this end, consider an arbitrary point $x \in E^2$, where $x = (\gamma_1, \gamma_2)$. It is clear, one can represent x as $x = (\gamma_1 + j\gamma_2)$. Let $\phi_1 = 1$ and $\gamma = \gamma_1 + j\gamma_2 \in \mathbb{C}$. Therefore, $\{\phi_1\}$ is a basis for $\mathbb{C}^1$, and* $\dim(\mathbb{C}^1) = 1$.

Now, we are in a position to show the main feature of the Riesz basis.

Consider a Hilbert space with two bases $B_1 = \{\phi_{1,i}\}_{i\in\mathbb{I}}$ and $B_2 = \{\phi_{2,i}\}_{i\in\mathbb{I}}$, $\mathbb{I} \subseteq \mathbb{N}$. We say the bases B_1 and B_2 are equivalent if there exists a topological isomorphism $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ (that is, $\mathcal{T}$ is a linear continuous map with continuous inverse) such that $\phi_{2,i} = \mathcal{T}\phi_{1,i}$. The main features that we are interested in are as follows [104]:

- Every separable Hilbert space has a countable orthogonal basis.
- Every countable basis of a Hilbert space is equivalent to all orthogonal bases.
- A basis of the Hilbert space $\mathcal{X}$ is countable if and only if it is equivalent to a Riesz basis of $\mathcal{X}$.

Therefore, a Riesz basis is an extension of common definition of basis in Fin-D spaces to countable Hilbert spaces. Also, we have the following important theorem.

Theorem 2.22. *[114, Theorem 9] Consider the Hilbert space $\mathcal{X}$ and the set $E = \{\phi_i\}_{i\in\mathbb{I}}$ such that $\overline{\text{span}\{E\}} = \mathcal{X}$. Then, E is a Riesz basis if and only if there exist two positive numbers M_1 and M_2 (independent of n) such that for any $n \in \mathbb{N}$, we have $M_1 \sum_{k=1}^{n} |\alpha_k|^2 \leq || \sum_{k=1}^{n} \alpha_k\phi_k||^2 \leq M_2 \sum_{k=1}^{n} |\alpha_k|^2$, where $|| \cdot ||$ denotes the norm induced from $< \cdot, \cdot >$ and $\alpha_k \in \mathbb{R}$ and $\lim_{n\to\infty} \sum_{k=1}^{n} |\alpha_k|^2 \leq \infty$.*

In the literature, a Riesz basis is defined by using the above theorem (refer to [14, Definition 2.3.1]).

2.3.5 Orthogonal Space

Dual spaces are of special importance in the system theory of Inf-D dynamical systems. For instance, they are essential to address the duality of observability and controllability. Let us first define an orthogonal subspace.

Definition 2.23. *Consider the inner-product vector space $\mathcal{X}$ (not necessarily Fin-D) and the subspace $\mathscr{V}$ (not necessarily closed). Then the orthogonal subspace to $\mathscr{V}$ that is denoted by $\mathscr{V}^\perp$ is defined as*

$$\mathscr{V}^\perp = \{x \in \mathcal{X} | < x, y >= 0, \ \forall y \in \mathscr{V}\} \tag{2.17}$$

This definition is valid for all inner product spaces. However, if $\mathcal{X}$ is a Hilbert space $\mathscr{V}^{\perp}$ is a closed subspace (even if $\mathscr{V}$ is not). One of most important results on Hilbert spaces is the *Riesz Projection Theorem* that is given below.

Theorem 2.24. *[14, Theorem A.3.52] Consider the Hilbert space $\mathcal{X}$ and the* closed *subspace $\mathscr{V} \subseteq \mathcal{X}$. Then each $x \in \mathcal{X}$ can be represented uniquely by $x = v + w$, such that $v \in \mathscr{V}$ and $w \in \mathscr{V}^{\perp}$.*

In the above theorem v is called as the projection of x on $\mathscr{V}$. Also, this theorem is of special interest to us because it shows that we can define quotient subsystems (by factoring out an unobservability subspace).

Remark 2.25. *The "closed" condition in the above theorem is crucial. Indeed, this condition is necessary even for Fin-D subspaces. However, since every Fin-D subspace is closed, the above theorem is valid for all Fin-D subspaces.*

2.3.6 Quotient Subspaces

For Inf-D Hilbert spaces, quotient subspaces are defined as in Fin-D spaces. Quotient subspaces play a key role in the FDI problem. More precisely, by using the quotient subspace we derive a subsystem that is decoupled from all faults but one.

Consider the Hilbert vector space $\mathcal{X}$ and the subspace $\mathscr{M} \subseteq \mathcal{X}$. Then for every $x \in \mathcal{X}$, the element $[x]$ of the quotient subspace $\mathcal{X}/\mathscr{M}$ is defined by the set $\{u|(x-u) \in \mathscr{M}\}$. Now, we have the following result for the Banach spaces.

Theorem 2.26. *[104] Consider the Hilbert space $\mathcal{X}$ and the* closed *subspace $\mathscr{M}$. Then the subspace $\mathcal{X}/\mathscr{M}$ is a complete (Banach) space with respect to the following norm*

$$||[x]||_{\mathcal{X}/\mathscr{M}} = \inf_{u \in [x]} |u| = \inf_{m \in \mathscr{M}} |x - m| \tag{2.18}$$

where $|\cdot|$ is the norm defined on $\mathcal{X}$.

Again, note that one needs the closeness in the above theorem.

Example 2.27. *Quotient Subspace:*

Consider the vector space $L_2([0,1],\mathbb{R})$ and the closed subspace $\mathscr{M} = \overline{\operatorname{span}\{\sin(2n\pi x)_{k\in\mathbb{N}}\}}$. It follows that $L_2([0,1],\mathbb{R})/\mathscr{M}$ is also closed and isomorphic to $\overline{\operatorname{span}\{\cos(2n\pi x)\}_{k\in\mathbb{N}}}$.

2.4 Linear Operators

Given that this thesis is concerned with linear Inf-D systems, in this section we review the linear operators that are defined on Fin-D and Inf-D Hilbert spaces. Generally, one can define two different types of operators on Hilbert spaces as follows:

1. **Bounded Operator:** An operator $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ is a bounded operator, if there exists a positive number $M_0 > 0$ such that $||\mathcal{D}|| < M$, where $|| \cdot ||$ is the operator norm. The bounded operators have the following properties,
   - $D(\mathcal{D}) = \mathcal{X}$, where $D(\mathcal{D})$ denotes the domain of $\mathcal{D}$.
   - A operator $\mathcal{D}$ is bounded if and only if it is continuous (on every point in $\mathcal{X}$).
2. **Unbounded Operator:** An operator that is not bounded, is unbounded.

It should be pointed out that an unbounded operator can only be defined on Inf-D vector space (refer to the following subsection).

Moreover, finite-rank operators are of a special interest that are defined as follows.

Definition 2.28. *The operator $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ is finite-rank if $D(\mathcal{D})$ and/or* $\operatorname{Im}\mathcal{D}$ *is Fin-D.*

Without loss of any generality we can assume $\mathcal{D}$ is onto $\mathcal{Y}$ and $\overline{D(\mathcal{D})} = \mathcal{X}$. Therefore, $\mathcal{D}$ is finite-rank if at least one of the vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is Fin-D.

2.4.1 Unbounded Operators

Since in the Inf-D system theory unbounded operators have a special interest, we review these operators here in more details. As stated earlier, an unbounded operator can only be defined on Inf-D spaces. In other words, we have:

Lemma 2.29. *[104] Consider the operator $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$. Then, $\mathcal{D}$ is unbounded only if* $\dim(\mathcal{X}) = \infty$.

A very important corollary from the above fact can be stated as follows:

Corollary 2.30. *Consider the operator $\mathcal{D} : \mathcal{X} \to \mathcal{Y}$ such that* $\dim(\mathcal{X}) < \infty$. *Then, $\mathcal{D}$ is bounded.*

Remark 2.31. *Note that $\mathcal{D}$ can still be unbounded even if $\mathcal{Y}$ is* Fin-D. *More precisely, finite-rankness is not a sufficient condition for boundedness.*

$\mathcal{A}$-Bounded Operators

A special unbounded operator that is related to another unbounded operator (that is denoted as $\mathcal{A}$) is the $\mathcal{A}$-bounded operator.

Definition 2.32. *[101, Definition II.4] Consider an unbounded operator $\mathcal{A} : \mathcal{X} \to \mathcal{X}$. The operator $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is $\mathcal{A}$-bounded if $D(\mathcal{A}) \subseteq D(\mathcal{F})$ and $\mathcal{F}(\lambda\mathcal{I} - \mathcal{A})^{-1}$ is bounded.*

Note that every bounded operator $\mathcal{F}$ is $\mathcal{A}$-bounded. Moreover, $\mathcal{A}$-boundedness is only defined for the operators that are defined on the same vector space (that is $\mathcal{X}$) as $\mathcal{A}$.

2.4.2 Adjoint Operators

To apply and utilize the duality concept in linear systems, one needs to deal with adjoint operators. In this subsection, we review adjoint operators in more detail.

Definition 2.33. *[14, Definition A.3.57] Consider a* bounded *operator* $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$*. There exists the unique operator* $\mathcal{T}^* : \mathcal{Y} \to \mathcal{X}$ *such that for all* $x \in \mathcal{X}$ *and* $y \in \mathcal{Y}$

$$< \mathcal{T}x, y >=< x, \mathcal{T}^* y > . \tag{2.19}$$

It is worth noting that the left dot product is defined on $\mathcal{Y}$, whereas the right dot product is defined on $\mathcal{X}$. However, since it is clear from the context, we do not use subscript for the dot products. The adjoint operator of an unbounded operator is defined as follows.

Definition 2.34. *[104] Consider an unbounded operator* $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ *such that* $\overline{D(\mathcal{T})} = \mathcal{X}$*. Then, let* $D(\mathcal{T}^*) \subseteq \mathcal{Y}$ *be all* $y \in \mathcal{Y}$ *such that there exists* $x^* \in \mathcal{X}$ *such that* $< \mathcal{T}x, y >=< x, x^* >$ *for all* $x \in D(\mathcal{A})$*. We define* $\mathcal{T}^* : D(\mathcal{T}^*) \to \mathcal{X}$ *and* $\mathcal{T}^* y = x^*$*.*

The following facts are useful once we deal with adjoint operators.

1. If $\mathcal{T}$ is bounded, then $(\mathcal{A} + \mathcal{T})^* = \mathcal{A}^* + \mathcal{T}^*$.

2. Adjoint operator of a finite-rank operator is finite-rank.

3. In general, $(\mathcal{F}^*)^* \neq \mathcal{F}$. For example, consider $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$, where $\dim(\mathcal{Y}) < \infty$. By Definition 2.34, we have $\mathcal{F}^* : \mathcal{Y} \to \mathcal{X}$ and since $\mathcal{Y}$ is Fin-D, by Corollary 2.30 $\mathcal{F}^*$ is bounded. Moreover, given that $\mathcal{F}^*$ is bounded, $(\mathcal{F}^*)^*$ is bounded. Therefore, $(\mathcal{F}^*)^* \neq \mathcal{F}$. However, if $\mathcal{F}$ is bounded, we obtain $(\mathcal{F}^*)^* = \mathcal{F}$.

2.5 Two-Dimensional (2-D) Systems

In this section, we briefly review 2-D systems. The results that are provided in Chapter 3 are applicable to n-D systems, for $n \geq 2$.

There are various models that are adopted in the literature for 2-D systems including the Rosser model [115], the Fornasini-Marichesini model I (FMI) and model II (FMII) [82, 85]. The FMI can be formulated as a Roesser model and the Roesser model is a special case of the FMII model [82]. In this section, we consider and concentrate on the FMII model, and consequently our results are also derived for this general class of 2-D systems.

Consider the following FMII model [85],

$$\begin{aligned}
x(i+1,j+1) &= A_1x(i,j+1) + A_2x(i+1,j) + B_1u(i,j+1) + B_2u(i+1,j) \\
&\quad + \sum_{k=1}^{p} L_k^1 f_k(i,j+1) + \sum_{k=1}^{p} L_k^2 f_k(i+1,j), \\
y(i,j) &= Cx(i,j),\ i,j \in \mathbb{Z},
\end{aligned} \tag{2.20}$$

where $x \in \mathbb{R}^m$, $u \in \mathbb{R}^\ell$, and $y \in \mathbb{R}^q$ denote the state, input and output vectors, respectively. The fault signals and the corresponding fault signatures are designated byf_k, L_k^1 and L_k^2, respectively. Since in this thesis all the introduced invariant subspaces are based on the operators A_1, A_2 and C, we designate the system (2.20) by the triple (C,A_1,A_2).

Remark 2.35. *Note that the system* (2.20) *represents and captures the presence of* both *actuator and component faults. To represent sensor faults, one can augment the sensor dynamics and model the sensor faults as actuator faults in the augmented system (for a complete discussion on this issue refer to [3] - Chapters 3 and 4). Also, it should be pointed out that the fault signal f_k affects the system through two different fault signatures L_k^1 and L_k^2. An alternative fault model could have been expressed according to the following representation,*

$$\begin{aligned}
x(i+1,j+1) =& A_1x(i,j+1) + A_2x(i+1,j) + B_1u(i,j+1) + B_2u(i+1,j) \\
&+ \sum_{k=1}^{p} L_k g_k(i,j), \\
y(i,j) =& Cx(i,j).
\end{aligned} \tag{2.21}$$

Model (2.20) *is more general than the one given by equation* (2.21)*. This is due to* *the fact that by denoting* $f_k(i+1,j) = g_k(i,j)$ *for all* $k = 1, \dots, p$*, one can represent the model* (2.21) *as the model* (2.20).

Let us now consider the Roesser model [115] which is expressed as

$$\begin{aligned}
\begin{bmatrix} r(i+1,j) \\ s(i,j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} r(i,j) \\ s(i,j) \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u(i,j) + \sum_{k=1}^{p} L_k f_k(i,j), \\
y(i,j) &= C \begin{bmatrix} r(i,j) \\ s(i,j) \end{bmatrix},
\end{aligned} \tag{2.22}$$

and where $\begin{bmatrix} r^{\mathrm{T}} & s^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^m$ represents the state, and the variables u, y, f_k and L_k are defined as in equation (2.20). By defining

$$\begin{aligned}
x &= \begin{bmatrix} r \\ s \end{bmatrix}, A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \\
B_1 &= \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ B_{21} \end{bmatrix},
\end{aligned} \tag{2.23}$$

one can formulate the Roesser model (2.22) as in equation (2.20).

2.5.1 The Approximation of Hyperbolic PDE Systems by 2-D Models

Let us first illustrate and demonstrate how one can approximate a general hyperbolic PDE system by using the 2-D models and representation. Consider the following hyperbolic PDE system

$$\frac{\partial \tilde{x}}{\partial t} = \tilde{A}_1 \frac{\partial \tilde{x}}{\partial z} + \tilde{A}_2 \tilde{x} + \tilde{B} u + \sum_{k=1}^{p} \tilde{L}_k \tilde{f}_k, \tag{2.24}$$

where z denotes the spatial coordinate, $\tilde{x}(z,t) \in \mathbb{R}^n$, $u(z,t) \in \mathbb{R}^q$ and $\tilde{f}_k(z,t) \in \mathbb{R}$ denote the state, input and fault signals, respectively. Also, the operators $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{B}$ and $\tilde{L}_k$ are real matrices with appropriate dimensions. Note that every hyperbolic

PDE system with constant coefficient can be represented in the form (2.24) with a diagonalizable $\tilde{A}_1$ [116] (Chapter 1, Detention 1.1.1).

By applying the finite difference method to the system (2.24), one obtains

$$\begin{aligned}\frac{\tilde{x}(i\Delta z,(j+1)\Delta t)-\tilde{x}(i\Delta z,j\Delta t)}{\Delta t} =& \tilde{A}_1\frac{\tilde{x}(i\Delta z,j\Delta t)-\tilde{x}((i-1)\Delta z,j\Delta t)}{\Delta z}+\tilde{A}_2\tilde{x}(i\Delta z,j\Delta t)\\ &+\tilde{B}\tilde{u}(i\Delta z,j\Delta t)+\sum_{k=1}^{p}\tilde{L}_k\tilde{f}_k(i\Delta z,j\Delta t).\end{aligned} \tag{2.25}$$

By setting $x(i,j)=\begin{bmatrix}\tilde{x}((i-1)\Delta z,j\Delta t)\\ \tilde{x}(i\Delta z,j\Delta t)\end{bmatrix}$, we can now write

$$x(i+1,j+1)=A_1x(i,j+1)+A_2x(i+1,j)+B_1u(i+1,j)+\sum_{k=1}^{p}L_kf_k(i+1,j), \tag{2.26}$$

where $u(i+1,j)=\tilde{u}(i,j)$ for all i and j, $A_1=\begin{bmatrix}0 & I\\ 0 & 0\end{bmatrix}$ and

$$A_2=\begin{bmatrix}0 & 0\\ -\frac{\Delta t}{\Delta z}\tilde{A}_1 & (I+\frac{\Delta t}{\Delta z}\tilde{A}_1+\Delta t\tilde{A}_2)\end{bmatrix},\ B_1=\begin{bmatrix}0\\ \tilde{B}\end{bmatrix},\ L_k=\begin{bmatrix}0\\ \tilde{L}_k\end{bmatrix},\ f(i,j)=\tilde{f}(i+1,j).$$

Therefore, the PDE system (2.24) can be approximated by the FMII 2-D model (2.26).

2.6 Semigroups of Operator and Dynamical Systems

In this section, we review some basic concepts of Inf-D dynamical systems.

2.6.1 Linear Systems on an Inf-D Hilbert space

As stated earlier, we mainly focus on systems that are defined on real separable Hilbert vector spaces. Consider the following system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0, \\ y(t) &= \mathcal{C}x(t) \end{aligned} \tag{2.27}$$

where $x(t) \in \mathcal{X}$ and $\mathcal{X}$ is a real separable Hilbert space. Also, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ are the input and output signals. The above equation has a regular (sufficiently smooth and unique) solution for if $u(\cdot) \in L_2((0,\infty), \mathbb{R}^m)$ [14] and the operator $\mathcal{A}$ is an infinitesimal generator of a strongly continuous (C_0) semigroup $\mathbb{T}_{\mathcal{A}}(t)$. Let $\mathcal{L}(\mathcal{X})$ denotes the set of all bounded operators defined on $\mathcal{X}$. A C_0 semigroup $\mathbb{T} : \mathbb{R}^+ \to \mathcal{L}(\mathcal{X})$ is the operator where the following conditions hold [14, Definition 2.1.2]:

- $\mathbb{T}(t+s) = \mathbb{T}(t)\mathbb{T}(s)$ for all $t, s \geq 0$.
- $\mathbb{T}(0) = \mathcal{I}$.
- If $t \to 0^+$, then $||\mathbb{T}(t)x - x|| \to 0$ for all $x \in \mathcal{X}$.

The $\mathbb{T}_{\mathcal{A}}$ is the semigroup that is generated by $\mathcal{A}$ and is related to $\mathcal{A}$ as

$$\mathcal{A}z = \lim_{t \to 0^+} (\mathbb{T}_{\mathcal{A}}(t) - \mathcal{I})z, \; z \in \mathcal{X} \tag{2.28}$$

$D(\mathcal{A})$ is all the $z \in \mathcal{X}$ such that the above limit exists.

The solution x to (2.27) is given by

$$x(t) = \mathbb{T}_{\mathcal{A}}(t)x_0 + \int_0^t \mathbb{T}_{\mathcal{A}}(t-s)\mathcal{B}u(s)ds \tag{2.29}$$

It is worth noting that

- By comparing the solutions (2.2) and (2.29), it follows that $\mathbb{T}_{\mathcal{A}}(t)$ plays the same role as e^{At} in Fin-D systems.

- Due to complexity of Inf-D systems (i.e. unboundedness of $\mathcal{A}$), unlike e^{At} (equation (2.3)), $\mathbb{T}_{\mathcal{A}}$ cannot be computed by using $\mathcal{A}$.
- $\overline{D(\mathcal{A})} = \mathcal{X}$.
- The solution (2.29) involves integral of functions that are Hilbert space valued (note $x \in \mathcal{X}$). This type of integral is known as the Bochner integral that is a generalized version of the Lebesgue integral.

Example 2.36. *Representation of a PDE system as an Inf-D System.*

Consider the following dynamical system that is governed by a parabolic PDE [15]

$$\frac{\partial \tilde{x}}{\partial t} = A_1 \frac{\partial^2 \tilde{x}}{\partial z^2} + A_2 \frac{\partial \tilde{x}}{\partial z} + A_3 \tilde{x} + B(z)u \tag{2.30}$$

where $z \in [0,1]$. Note that in general $z \in [z_1, z_2]$. However it can be easily transformed to $z \in [0,1]$ as in [7], $t \in [0, \infty)$. The input operator $B(z)$ is also defined as

$$B(z) = \mathbf{1}_{z_\alpha, \epsilon} \tag{2.31}$$

in which $\mathbf{1}_{z_\alpha,\epsilon} = \begin{cases} 1 & ;z \in [z_\alpha, z_\alpha + \epsilon] \\ 0 & ;\text{Otherwise} \end{cases}$. The measurements are collected at certain locations as

$$y_i = \int_{z_i}^{z_i+\epsilon} \tilde{x} dz \ , \ i = 1, \cdots, q, z_i \in [0,1] \tag{2.32}$$

Note that the boundary control (that is $B(z) = \{ \begin{smallmatrix} 1 & x=0 \\ 0 & Otherwise \end{smallmatrix})$ and the boundary measurement (that is $y_0 = \tilde{x}(0,t)$) can be approximated by equation (2.31) and (2.32), respectively, where ϵ is sufficiently small.

Moreover, the operators that are defined by (2.31) and (2.32) are finite-rank [14]. The Boundary condition of the system (2.30) is governed by

$$\begin{aligned} C_1 \tilde{x}(t,0) + D_1 \frac{\partial \tilde{x}}{\partial z}(t,0) = R_1, \\ C_2 \tilde{x}(t,1) + D_2 \frac{\partial \tilde{x}}{\partial z}(t,1) = R_2, \end{aligned} \tag{2.33}$$

where C_i, D_i and R_i are real matrices. Also, the initial condition is given by $\tilde{x}(0,z) = \tilde{x}_0$.

Now consider $\mathscr{X} = L_2([0,1],\mathbb{R})$ and

$$\mathcal{A}h = A_1\frac{d^2h}{dt^2} + A_2\frac{dh}{dt} + A_3h \tag{2.34}$$

$$D(\mathcal{A}) = \{h \in \mathscr{X}|h, \frac{dh}{dx} \text{ are absolutely continuous, } h \text{ satisfies the initial condition}\},$$

Consequently, the input and output operators can be defined according to

$$\begin{aligned} &\mathcal{B} : \mathbb{R}^p \to \mathscr{X} \ , \ \mathcal{B}u = B(z)u \\ &\mathcal{C} : \mathscr{X} \to \mathbb{R}^q \ , \ \mathcal{C}x =< \mathbf{1}_{z_i,\epsilon}, x > \end{aligned} \tag{2.35}$$

where $< \cdot, \cdot >$ denotes the inner product defined on $\mathscr{X}$. Therefore, the system (2.30) can be represented as in (2.27).

Compared to the PDE system, expressing a time-delay system as an Inf-D system is more challenging. First, we need the following lemma.

Lemma 2.37. *[14, Lemma 2.1.11] Consider C_0 semigroup $\mathbb{T}_\mathcal{A}$ and its corresponding infinitesimal generator $\mathcal{A}$. Then*

$$\mathcal{R}(\lambda,\mathcal{A})x = \int_0^\infty e^{-\lambda t}\mathbb{T}_\mathcal{A}x\mathrm{d}t, \tag{2.36}$$

where $\mathcal{R}(\lambda,\mathcal{A}) = (\lambda\mathcal{I} - \mathcal{A})^{-1}$ is the resolvent operator of $\mathcal{A}$.

Example 2.38. *Representation of a time-delay system as an Inf-D system.*

Consider the following time-delay system

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_1x(t-h), \\ &y(t) = Cx(t), x(0) = x_0, x(\gamma) = g_0(\gamma), \ -h \le \gamma < 0, \end{aligned} \tag{2.37}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ denote state, input and output, respectively. Also, h is a positive real constant.

For every $x_0 \in \mathbb{R}^n$ and $g(\cdot) \in L_2([-h,0],\mathbb{R})$, the unique solution to the above system is given by [14, Theorem 2.4.1]

$$x(t) = e^{A_0 t}x_0 + \int_0^t e^{A_0(t-s)}A_1 x(s-h)\mathrm{d}s. \tag{2.38}$$

Therefor, by defining $\mathcal{X} = \mathbb{R}^n \oplus L_2([-h,0],\mathbb{R})$, the corresponding semigroup is expressed as [2, 2.4.4]

$$\mathbb{T}(t)\begin{bmatrix} x_0 \\ g(\cdot) \end{bmatrix} = \begin{bmatrix} x(t) \\ x(t+\cdot) \end{bmatrix}, \tag{2.39}$$

where $x(\cdot)$ is the solution (2.38). Now, by using Lemma 2.37, one can write [14, Lemma 2.4.5]

$$\begin{aligned} \mathcal{R}(\lambda,\mathcal{A})\begin{bmatrix} x_0 \\ g_0(\cdot) \end{bmatrix} &= \begin{bmatrix} g_1(0) \\ g_1(\cdot) \end{bmatrix}, \\ g_1(\gamma) &= e^{\lambda\gamma}g_1(0) - \int_0^\gamma e^{\lambda(\gamma-s)}g_0(s)\mathrm{d}s, \\ g_1(0) &= [\Delta(\lambda)]^{-1}\big(x_0 + \int_h^0 e^{-\lambda(s+h)}A_1 g_0(s)\mathrm{d}s\big), \end{aligned} \tag{2.40}$$

where $\Delta(\lambda) = (\lambda I - A_0 - A_1 e^{-h})$. Finally, by using the fact that $(\lambda\mathcal{I} - \mathcal{A})\mathcal{R}(\lambda,\mathcal{A}) = \mathcal{I}$, it can be shown that

$$\begin{aligned} &\mathcal{A}\begin{bmatrix} x \\ g(\cdot) \end{bmatrix} = \begin{bmatrix} A_0 x + A_1 g(-h) \\ \frac{\mathrm{d}g}{\mathrm{d}\gamma}(\cdot) \end{bmatrix}, \\ &D(\mathcal{A}) = \{\begin{bmatrix} x \\ g(\cdot) \end{bmatrix} |g \text{ is absolutely continuous}, \frac{\mathrm{d}g}{\mathrm{d}\gamma} \in L_2([-h,0],\mathbb{R}), g(0) = x\}. \end{aligned} \tag{2.41}$$

Therefor, the time-delay system (2.37) can be represented as an Inf-D system (2.27).

2.7 Summary

In this chapter, we have reviewed linear Fin-D systems and the FDI problem of Fin-D systems. Moreover, Inf-D vector spaces and Inf-D systems have been addressed.

In the following chapters, we use this background information to provide necessary and sufficient conditions for the FDI problem solvability of Inf-D systems.

Chapter 3

Fault Detection and Isolation of Multidimensional Systems

In this chapter, we develop a novel FDI scheme for discrete-time multidimensional (n-D) systems for the first time in the literature. These systems represent as generalization of the Fornasini-Marchesini model II (FMII) two- and three-dimensional (2-D and 3-D) systems. This is accomplished by extending the geometric FDI approach of one-dimensional (1-D) systems to n-D systems. The basic invariant subspaces including unobservable, conditioned invariant and unobservability subspaces of 1-D systems are generalized to n-D models. These extensions have been achieved and facilitated by representing a n-D model as an Inf-D system, and by particularly constructing algorithms that compute these subspaces in a *finite and known* number of steps. By utilizing the introduced subspaces the FDI problem is formulated and necessary and sufficient conditions for its solvability are provided. Sufficient conditions for solvability of the FDI problem for n-D systems using LMI filters are also developed. Moreover, the capabilities and advantages of our proposed approach are demonstrated by performing an analytical comparison with the currently available methods in the literature. Finally, numerical simulations corresponding to an approximation of a hyperbolic PDE system of a heat transfer process, that is mathematically represented as a 2-D model, have also been provided.

3.1 Preliminary Results

In this section, we first review n-D systems and their various representational models. Subsequently, an n-D system is expressed as an Inf-D system that allows one to geometrically analyze the unobservable subspaces (this is to be defined and specified in the next section). The FDI problem of n-D systems is also formulated in this section. Finally, an LMI-based approach is introduced to design an n-D Luenberger observer (also known as a detection filter) for n-D systems.

3.1.1 Discrete-Time n-D Systems

As stated in the previous chapter , the n-D models can be used to represent a large class of problems, such as approximating hyperbolic PDE systems [16, 75], image processing and digital filtering [115] (as 2-D systems) and approximate 2-D parabolic PDE system (as 3-D systems). System theory concepts such as observability, controllability and feedback stabilization have also been investigated in the literature for 2-D systems [16, 82, 85, 95, 98]. However, as emphasized in [92, Section 2] extending the available algebraic methods for $n \geq 3$ deals with certain difficulties. As we shall see, unlike algebraic approaches one can extend the available geometric results of 2-D systems to n-D system.

As stated earlier in Chapter 2, there are various models that are adopted in the literature for 2-D systems including the Roesser model [115], the Fornasini-Marichesini model I (FMI) and FMII [82, 85]. The FMI can be formulated as a Roesser model and the Roesser model is a special case of the FMII model [82]. In this chapter, we consider and concentrate on the FMII model to formulate n-D systems, and consequently our results are also derived for this general class of n-D systems.

To formulate an n-D system we use the following notations. Consider the coordinates $(i_1, \ldots, i_n)$, where $i_k \in \mathbb{Z}$ and the vector $x(i_1, \ldots, i_n) \in \mathbb{R}^m$. We denote σ_k to be a shift operator on the k^{th} coordinate (i.e. $\sigma_k x(i_1, \cdots, i_n) = x(i_1, \cdots, i_k + 1, i_{k+1}, \cdots, i_n)$). Also, let us set $\delta = \Pi_{k=1}^n \sigma_k$ (i.e., $\delta x(i_1, \ldots, i_n) = x(i_1 + 1, \cdots, i_n + 1)$) and $\delta_k = \Pi_{i=1, i\neq k}^n \sigma_k$, that is $\delta_k x(i_1, \cdots, i_n) = x(i_1 + 1, \cdots, i_{k-1} + 1, i_k, i_{k+1} + 1, \cdots, i_n + 1)$.

Consider the following FMII-based n-D model (that is a generalized version of 2-D systems in [85]),

$$\begin{aligned} \delta x(i_1, \ldots, i_n) =& \sum_{k=1}^n A_k \delta_k x(i_1, \ldots, i_n) + \sum_{k=1}^n B_k \delta_k u(i_1, \ldots, i_n) \\ &+ \sum_{j=1}^p \sum_{k=1}^n L_j^k \delta_k f_j(i_1, \ldots, i_n), \\ y(i_1, \ldots, i_n) =& C x(i_1, \ldots, i_n),\ i_k \in \mathbb{Z}, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^m$, $u \in \mathbb{R}^\ell$, and $y \in \mathbb{R}^q$ denote the state, input and output vectors, respectively. The fault signals and the corresponding fault signatures are designated byf_k and $L_i^k : \mathbb{R} \to \mathbb{R}^m$, respectively. Since in this chapter all the introduced invariant subspaces are based on the operators A_k and C, we designate the system (3.1) by the pair (C,A), where $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$.

Remark 3.1. *Note that system* (3.1) *represents and captures the presence of* both *actuator and component faults. To represent sensor faults, one can augment the sensor dynamics and model the sensor faults as actuator faults in the augmented system (for a complete discussion on these issues refer to [3, Chapters 3 and 4]). Moreover, the fault signal f_j affects the system through n different fault signatures L_j^k and $k = 1, \cdots, n$. However, in [99] an alternative fault model is utilized. The* *n-D extension of the fault model in [99] is expressed according to the following representation*

$$\begin{aligned}\delta x(i_1,\ldots,i_n) =& \sum_{k=1}^{n} A_k\delta_k x(i_1,\ldots,i_n) + \sum_{k=1}^{n} B_k\delta_k u(i_1,\ldots,i_n)\\ &+ \sum_{j=1}^{p} L_j g_j(i_1,\ldots,i_n),\\ y(i_1,\ldots,i_n) =& Cx(i_1,\ldots,i_n),\ i_k \in \mathbb{Z},\end{aligned} \tag{3.2}$$

Model (3.1) *is more general than the one given by equation* (3.2). *This is due to the fact that by denoting* $f_j(i_1+1, i_2, \ldots, i_n) = g_j(i_1, i_2, \ldots, i_n)$ *for all* $j = 1, \ldots, p$, *one can represent the model* (3.2) *as in the model* (3.1).

In this work, we will investigate and develop FDI strategies for the model (3.1) such that they are applicable to any $n \geq 2$. It is assumed that A_k in model (3.1) are not necessarily commutative. It should be emphasized that the commutativity of A_k is a strong condition that renders the results in [95] (where for $n = 2$ A_1 and A_2 are assumed to commutate) not applicable to Roesser systems.

3.1.2 Inf-D Representation

In this subsection, we reformulate the n-D model (3.1) as an Inf-D system that will be used to derive the invariance property of unobservable subspaces (for details refer to Subsection 3.2.1).

Consider the fault free system (3.1), that is with $f_j(i_1, \ldots, i_n) \equiv 0$, $j = 1, \ldots, p, i_k \in \mathbb{Z}$, and with zero input (we are mainly interested in the unobservable subspaces and do not need to be concerned with the control inputs in the FDI problem). Let $\mathbf{x}(i) \in \bigoplus(\mathbb{R}^m)$ ($i \in \underline{\mathbb{N}}$) denote an Inf-D vector that is constructed by using all $x(j_1, \ldots, j_n)$, where $j_k \in \mathbb{Z}$, $\sum_{k=1}^{n} j_k = i$ and the position of each $x(j_1, \ldots, j_n)$ in $\mathbf{x}(i)$ is determined by a selected ordering index (refer to the Example 3.1 to observe how one can define an ordering index). Under the above condition and for a given ordering index, the system (3.1) can be represented as,

$$\begin{aligned} \mathbf{x}(i+1) &= \mathcal{A}\mathbf{x}(i), \quad i \in \underline{\mathbb{N}} \\ \mathbf{y}(i) &= \mathcal{C}\mathbf{x}(i), \end{aligned} \tag{3.3}$$

where $\mathbf{x}(i) \in \mathcal{X} = \bigoplus(\mathbb{R}^m)$, $\mathbf{y}(i) \in \bigoplus(\mathbb{R}^q)$ (with the same ordering index that is used to construct $\mathbf{x}(i)$), and $\mathcal{A}$ is an Inf-D block matrix such that each A_k $(k = 1, \cdots, n)$ does appear at every row-block of $\mathcal{A}$ once and only once, with the remaining elements set to zero. Also, we have $\mathcal{C} = \mathrm{diag}(\ldots, C, C, \ldots)$. For example,

$$\mathcal{A} = \begin{bmatrix} & \ddots & \ddots & \ddots & \ddots & & \cdots & \\ \cdots & 0 & A_1 & 0 & A_2 & \cdots & A_n & \cdots \\ \cdots & A_1 & 0 & 0 & A_2 & \cdots & A_n & \cdots \\ \cdots & & \cdots & & \ddots & \ddots & \ddots & \cdots \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} \ddots & & \cdots & & \cdots \\ \cdots & 0 & C & 0 & \cdots \\ \cdots & 0 & 0 & C & \cdots \\ \cdots & & \cdots & & \ddots \end{bmatrix} \tag{3.4}$$

where the position of A_k at each row is determined such that the order that is obtained for $\mathbf{x}(i+1)$ is the same as that of $\mathbf{x}(i)$. For more clarification, consider the following example for $n = 3$.

Example 3.2. Let $n = 3$ and $k \in \mathbb{Z}$. Set

$$\begin{aligned} I_k(i) =& ((k+i, -k, 0), (k+i, -k+1, -1), (k+i, -k-1, 1), \\ & (k+i, -k+2, -2), (k+i, -k-2, 2), \ldots) \\ I(i) =& (\ldots, I_{-1}(i), I_0(i), I_1(i), \ldots) \end{aligned} \tag{3.5}$$

and let $I_k^j(i)$ denote the j^{th} index in $I_k(i)$. For example, $I_k^3(0) = (k, -k-1, 1)$ (the third element in $I_k(0)$). It follows that for each $I_k^j(i)$ in $I_k(i)$, the summation of its element is equal to i (for example for $I_k^3(i)$ we have $(k+i)+(-k-1)+1 = i$). Also, the ordering index for all $\mathbf{x}(i)$ are identical. In other words, the position of $(k+i, -k, 0)$ is the first element of $I_k(i)$ for all $i \in \underline{\mathbb{N}}$. Therefore, without loss of any generality, we show the position of elements by $I_k^j = I_j^k(0)$. By using the above ordering index (that is, $I = I(0)$), we have $\mathbf{x}(0) = (\ldots, x(-1, 1, 0), x(-1, 2, , -1), x(-1, 0, 1), \ldots)$.

Also, $\mathcal{A}$ is constructed such that the order index of $\mathbf{x}(1)$ is the same as that of $\mathbf{x}(0)$. Towards this end, first set $\mathbf{x}(1) = (\ldots, x(1,-1,1), x(1,0,0), x(1,1,,-1), x(1,2,-2), \ldots)$ (the same order as $\mathbf{x}(0)$). Each row-block of $\mathcal{A}$ is individually constructed. For example, the row corresponding to $x(1,0,0)$ (that is the row, $\mathbf{a}^{\mathrm{T}}$ of $\mathcal{A}$ such that $x(1,0,0) = \mathbf{a}^{\mathrm{T}}\mathbf{x}(0)$) has A_1, A_2 and A_3 in the I_0^1, I_1^1 and I_1^2 (by applying the same ordering index that is used in $\mathbf{x}(0)$ for $\mathbf{a}$), respectively (note that $x(1,0,0) = A_1x(0,0,0) + A_2x(1,-1,0) + A_3x(1,0,-1)$). In other words, we have,

$$\mathbf{a}^{\mathrm{T}} = \begin{bmatrix} \cdots & 0 & \underbrace{A_1}_{\text{position } I_0^1} & 0 & \cdots & 0 & \underbrace{A_2}_{\text{position } I_0^1} & \underbrace{A_3}_{\text{position } I_1^2} & 0 & \cdots \end{bmatrix} \tag{3.6}$$

Therefore, the system (3.1) (for $n = 3$, fault free and zero input) can be represented as in equation (3.3).

Various formulations for the initial conditions of the FMII model (3.1) are possible that are based on the separation set introduced in [117]. There are two separation sets that are commonly used in the literature for 2-D systems (refer to [85] and [82]). The generalization of these two formulations to n-D systems are as follows. In the first formulation, the initial conditions are denoted by $\mathbf{x}(0) = (\ldots, x(i_1, i_2, \ldots, i_n)^{\mathrm{T}}, \ldots)^{\mathrm{T}} \in \bigoplus(\mathbb{R}^m)$ (this is compatible with the model (3.3)), where $i_k \in \mathbb{Z}$ and $\sum_{k=1}^{n} i_k = 0$. The second formulation is expressed as $x(0, \cdots, 0, i_k, 0, \cdots, 0) = h_k(i_k)$ for all $k = 1, \ldots, n$, where $h_k(i_k) \in \mathbb{R}^m$ and $i_k \in \mathbb{N}$. This formulation is more compatible with different applications (particularly, in case that the system (3.1) is an approximate model of a PDE system - (refer to [16])). It will be shown subsequently that since we derive our conditions based on an invariant unobservable subspace (this is formally defined in the next section), our proposed methodology is applicable to <u>both</u> initial condition formulations.

As stated in the Notation Section 1.6, it can be shown that $\mathcal{X} = \bigoplus(\mathbb{R}^m)$ (which the vector space for equation (3.3)) is an Inf-D Banach space. The system theory corresponding to Inf-D systems poses a more significantly challenging task than that of the Fin-D system theory (1-D systems) (refer to [101, Chapters I and II]). However, as shown subsequently, for any ordering index the operator $\mathcal{A}$ is bounded (refer to Subsection 2.4 for definition of bounded operators), and consequently, one can readily extend the results of 1-D systems to the system (3.3) [14,101] (for example, refer to [101, Lemma I.3]).

Lemma 3.3. *The operator $\mathcal{A}$, as defined in the Inf-D system* (3.3)*, is bounded.*

Proof. Let $G = n\max_{k=1}^{n}(|A_k|)$, where $|A_k|$ denotes the norm of A_k and $\mathbf{x} = (x_j)_{j\in\mathbb{Z}} \in \mathcal{X}$. For each row-block of $\mathcal{A}$, set the map $m_I^k : \{1,\cdots,n\} \to \mathbb{Z}$ such that $m_I^k(j)$ determines the position of A_j in the k^{th} row-block of $\mathcal{A}$. It follows readily that $|\mathcal{A}\mathbf{x}|_\infty = \sup\limits_{k\in\mathbb{Z}}|\sum_{j=1}^{n} A_j x_{m_I^k(j)}| \leq \sup\limits_{k\in\mathbb{Z}} G\max_{j=1}^{n}(|x_{m_I^k(j)}|) = G\sup\limits_{k\in\mathbb{Z}}|x_k|$. Therefore, $|\mathcal{A}\mathbf{x}|_\infty \leq G|\mathbf{x}|_\infty$. This completes the proof of the lemma. □

Note that the above lemma is independent from the chosen ordering index. The map m_I^k in the above proof is indeed a combination of two maps; first map is from $\{1,\cdots,n\}$ to I (i.e., any ordering index), and second, a map from I to $\mathbb{Z}$. Moreover, The above lemma enables one to formulate the unobservable subspace of the n-D system (3.1) in a geometric framework (for details refer to Section 3.2) based on the operator $\mathcal{A}$ (and consequently, in terms of A_k, where $k = 1,\cdots,n$).

3.1.3 The FDI Problem of n-D FMII Model

In this subsection, we formulate the FDI problem for the n-D system (3.1). Without loss of any generality, it is assumed that the system (3.1) is subject to two faults, and therefore we construct two residuals such that each one is sensitive to only one fault and is decoupled from the other. Our approach can be extended trivially to more than two faults.

More precisely, consider the faulty n-D model (3.1). The solution to the FDI problem of the n-D FMII system can be stated as that of generating two residuals

$r_j(i_1, \dots, i_n), j \in \{1, 2\}$ such that,

$$\forall u, f_2 \text{ and } f_1 = 0 \quad \text{then} \lim_{\sum_{k=1}^{n} i_k \to \infty} r_1(i_1, \dots, i_n) \to 0, \tag{3.7a}$$

$$\text{and if} \quad f_1 \neq 0 \quad \text{then } \exists N_0 \in \mathbb{N} \text{ such that } r_1(i_1, \dots, i_n) > \epsilon_1; \text{ for } \sum_{k=1}^{n} i_k > N_0$$

$$\forall u, f_1 \text{ and } f_2 = 0 \quad \text{then} \lim_{\sum_{k=1}^{n} i_k \to \infty} r_2(i_1, \dots, i_n) \to 0, \tag{3.7b}$$

$$\text{and if} \quad f_2 \neq 0 \quad \text{then } \exists N_0 \in \mathbb{N} \text{ such that } r_2(i_1, \dots, i_n) > \epsilon_2; \text{ for } \sum_{k=1}^{n} i_k > N_0,$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ and $N_0 \in \mathbb{N}$ is a sufficiently large integer (refer to Remark 3.23).

The above residuals are to be constructed by employing the fault detection filters. For the n-D system (3.1), we consider the following FMII-based *fault detection filter*,

$$\begin{aligned} \delta\omega_j(i_1, \dots, i_n) =& \sum_{k=1}^{n} F_k \delta_k \omega_j(i_1, \dots, i_n) + \sum_{k=1}^{n} K_k \delta_k u(i_1, \dots, i_n) \\ &+ \sum_{k=1}^{n} E_k \delta_k y(i_1, \dots, i_n), \\ r_j(i_1, \dots, i_n) =& H_j y(i_1, \dots, i_n) - M_j \omega_j(i_1, \dots, i_n), \end{aligned} \tag{3.8}$$

where $\omega(i_1, \dots, i_n) \in \mathbb{R}^o$ denotes the state of the filter and is used to define the residual signal $r_j(i_1, \dots, i_n)$. The solution to the FDI problem is now reduced to that of selecting the filter gains F_k, K_k, E_k, M_j and H_j corresponding to the filter structure given by (3.8).

Remark 3.4. *The detection filter* (3.8) *can be selected as a full-order* ($H_j = I$) *or as a partial-order* ($\ker H_j \neq 0$) *n-D Luenberger observer. As shown subsequently in Section 3.3, this level of generality allows one to analytically compare our proposed methods with the results reported in [99].*

Remark 3.5. *In this chapter, we investigate the FDI problem by employing two main steps, namely (i) decoupling the faults, and (ii) designing a filter for each fault.*

The first step addresses the existence of a subsystem of (3.1) *such that it is decoupled from f_2 and sensitive to f_1. By the existence of a subsystem, we imply the existence of $n+1$ maps D_k, and H, such that all fault signatures of f_2 (namely L_2^k for all $k = 1, \cdots, n$) are members of the unobservable subspace (defined in the next section) of the system $(H_1C,\ A + D^1C)$ (where $A + DC = \begin{bmatrix} A_1 + D_1^1 C & \cdots & A_n + D_n^1 C \end{bmatrix}$). Moreover, the second step is mainly concerned with the existence of the filter* (3.8) *(i.e., the residual generation) such that the stability of the error dynamics is guaranteed. Indeed, the second step is mainly concerned with a realization of the detection filter (as shown subsequently this could be a Luenberger-based filter). In this chapter, if the first step is solvable for the fault f_1 we say that f_1 is detectable and isolable. We use the same procedure for the fault f_2 (i.e., if there exists a subsystem of* (3.1) *such that it is decoupled from f_1 and sensitive to f_2, we say that f_2 is detectable and isolable). Finally, we will state that there is a solution to the FDI problem if for both fault signals f_1 and f_2 <u>both</u> steps above are solvable.*

3.1.4 LMI-based Observer (Detection Filter) Design

As shown in [118], design of a deadbeat observer requires that one works with polynomial matrices (this is not always a straightforward process). Moreover, polynomiyal matrices face certain difficulties for $n \geq 3$ (refer to [92, Section 2]). In this subsection, we address the design process for the n-D system observer, or the detection filter gains, by using the linear matrix inequalities (LMI) properties. These results will be used subsequently in Section 3.3 to explicitly design an n-D Luenberger detection filter (that can also be formulated as in equation (3.8)) for the purpose of accomplishing the solution to the FDI problem.

In order to show the asymptotic stability of the state estimation error dynamics, one needs to invoke the following stability lemmas. Lemma 3.6 is the generalization of the Proposition 2.1 (and equation (2.7)) in [93], where sufficient conditions for the stability of 2-D systems are provided.

Lemma 3.6. *(Generalization of [93, equation (2.7)]) Let* $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}, A_k \in \mathbb{R}^{m \times m}$ *(refer to the system* (3.1)*). The n-D FMII system* (3.1) *(under the fault free situation) is asymptotically stable if there exist n symmetric positive definite matrices* $R_k \in \mathbb{R}^{m\times m}$, $k = 1, \cdots, n$ *such that*

$$A_c \triangleq A^{\mathrm{T}}(\sum_{k=1}^{n} R_k)A - R < 0, \tag{3.9}$$

where $R = \mathrm{diag}(R_1, \cdots, R_n)$.

Proof. Consider the characteristic polynomial matrix $p(z) = (I - \sum_{k=1}^{n} A_k z_k), z \in \mathbb{C}^n$. By using the Theorem 41 in [119], the system (3.1) is shown to be stable if and only if the equation $\det(p(z)) = 0$ has no zero in the region $\overline{U}^n = \{(z_1, \cdots, z_n)|\ |z_k| \leq 1,\ k = 1, \cdots, n\}$. We show the results by invoking contradiction. Let $A_c < 0$ and there exists $z_0 = (z_1^0, \cdots, z_n^0) \in \overline{U}^n$ and a non-zero $x \in \mathbb{C}^m$ such that $p(z_0)x = 0$. Hence, one can write $x = (\sum_{k=1}^{n} A_k z_k^0)x$ (i.e., $x = Ax_z$, where $x_z = \begin{bmatrix} z_1^0 I, \cdots, & z_n^0 I \end{bmatrix}^{\mathrm{T}} x$), and

$$\begin{aligned} x^*(\sum_{k=1}^{n} R_k)x = x_z^* A^{\mathrm{T}}(\sum_{k=1}^{n} R_k)Ax_z = x_z^*(A_c + R)x_z \\ x_z^* R x_z = x^*(\sum_{k=1}^{n} R_k |z_k^0|^2)x, \end{aligned} \tag{3.10}$$

where superscript $*$ is used as the complex conjugate transpose. Therefore, we obtain $x^*(\sum_{k=1}^{n} R_k)x - x^*(\sum_{k=1}^{n} R_k |z_k^0|^2)x = x_z^* A_c x_z$, i.e., $0 < x^*(\sum_{k=1}^{n} R_k(1 - |z_k^0|^2))x = x_z^* A_c x_z$ (since $|z_k^0| \leq 1$). Consequently, given that A_c is a real matrix there exists $x_r^{\mathrm{T}} A_c x_r > 0$, where x_r is the real part of x_z. This is in contradiction with the assumption $A_c < 0$. This now completes the proof of the lemma. □

Lemma 3.7. *[120, Lemma 3.1] Consider the matrices* $\Phi \in \mathbb{R}^{m\times m}$, $P \in \mathbb{R}^{p\times m}$, $Q \in \mathbb{R}^{q\times m}$ *and the LMI condition* $\Phi + P^{\mathrm{T}}\Lambda^{\mathrm{T}}Q + Q^{\mathrm{T}}\Lambda P < 0$. *Also, define the matrices* W_p *and* W_q *such that the columns of* W_p *and* W_q *are bases of* $\ker P$ *and*

$\ker Q$, *respectively. There exists a matrix* $\Lambda \in \mathbb{R}^{q\times p}$ *satisfying the previous LMI condition if and only if* $W_p^{\mathrm{T}}\Phi W_p < 0$ *and* $W_q^{\mathrm{T}}\Phi W_q < 0$,

Now consider the n-D system (3.1) under the fault free situation and the corresponding state estimation observer that is given by,

$$\begin{aligned}\delta\hat{x}(i_1,\ldots,i_n) =& \sum_{k=1}^{n}(A_k + D_{ok}C)\delta_k\hat{x}(i_1,\ldots,i_n) + \sum_{k=1}^{n}B_k\delta_k u(i_1,\ldots,i_n)\\ &- \sum_{k=1}^{n}D_{ok}\delta_k y(i_1,\ldots,i_n), \qquad (3.11)\\ \hat{y}(i_1,\ldots,i_n) =& C\hat{x}(i_1,\ldots,i_n).\end{aligned}$$

It follows readily that the state estimation error dynamics, as defined by $e(i,j) = x(i_1,\ldots,i_n) - \hat{x}(i_1,\ldots,i_n)$, is governed by,

$$\delta e(i_1,\ldots,i_n) = \sum_{k=1}^{n}(A_k + D_{ok}C)\delta_k e(i_1,\ldots,i_n). \qquad (3.12)$$

The following theorem and corollary provide an LMI-based condition for existence of the state estimation observer gains D_{ok} such that the error dynamics (3.12) is asymptotically stable.

Theorem 3.8. *Consider the n-D system* (3.1) *under the fault free situation. Also, let* $W_{c_n} = \mathrm{diag}(\underbrace{W_c,\cdots,W_c}_{n \text{ times}})$ *where the columns of* $W_c \in \mathbb{R}^{m\times(m-q)}$ *are the basis of* $\ker C$. *There exist* n *maps* $D_{ok} : \mathbb{R}^q \to \mathbb{R}^n$ *and* n *symmetric positive definite matrices* R_k *(with* $G = \begin{bmatrix} A_1 + D_{o1}C & \cdots & A_n + D_{on}C\end{bmatrix}$ *and* R *defined in Lemma 3.6) such that, the LMI* $G^{\mathrm{T}}(\sum_{k=1}^{n} R_k)G - R < 0$ *is satisfied if and only if all* R_k *satisfy the LMI condition* $W_{c_n}^{\mathrm{T}} A_c W_{c_n} < 0$, *where* A_c *is defined in* (3.9).

Proof. Note that without loss of any generality, it is assumed that C is full row rank, and $m > q$ that is equivalent to partial state measurement. Let $A = \begin{bmatrix} A_1 & \cdots & A_n\end{bmatrix}$

and $\Phi = \begin{bmatrix} -(\sum_{k=1}^{n} R_k)^{-1} & A \\ A^{\mathrm{T}} & -R \end{bmatrix}$. By using the Schur complement lemma, we have $W_{c_n}^{\mathrm{T}} A_c W_{c_n} < 0$ if and only if,

$$\begin{bmatrix} -(\sum_{k=1}^{n} R_k)^{-1} & AW_{c_n} \\ W_{c_n}^{\mathrm{T}} A^{\mathrm{T}} & -W_{c_n}^{\mathrm{T}} R W_{c_n} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & W_{c_n}^{\mathrm{T}} \end{bmatrix} \Phi \begin{bmatrix} I & 0 \\ 0 & W_{c_n} \end{bmatrix} < 0. \tag{3.13}$$

Therefore, $\begin{bmatrix} 0_{nm\times m} & I_{nm\times nm} \end{bmatrix} \Phi \begin{bmatrix} 0_{m\times nm} \\ I_{nm\times nm} \end{bmatrix} = -R < 0$, if and only if $R > 0$ (or $R_k > 0$ for all $k = 1, \cdots, n$). By defining $P = \begin{bmatrix} 0_{nq\times m} & C_n \end{bmatrix}$ and $Q = \begin{bmatrix} I_{m\times m} & 0_{m\times nm} \end{bmatrix}$, where $C_n = \mathrm{diag}(\underbrace{C, \cdots, C}_{n \text{ times}})$, and using Lemma 3.7, the LMI condition (3.13) is satisfied if and only if there exits a matrix $\Lambda = \begin{bmatrix} D_{o1} & \cdots & D_{on} \end{bmatrix} \in \mathbb{R}^{m\times nq}$ such that,

$$\begin{aligned} &\Phi + \begin{bmatrix} 0_{m\times nq} \\ C_n^{\mathrm{T}} \end{bmatrix} \Lambda^{\mathrm{T}} \begin{bmatrix} I_{m\times m} & 0_{m\times nm} \end{bmatrix} \\ &+ \begin{bmatrix} I_{m\times m} \\ 0_{nm\times m} \end{bmatrix} \Lambda \begin{bmatrix} 0_{nq\times m} & C_n \end{bmatrix} = \begin{bmatrix} (-\sum_{k=1}^{n} R_k)^{-1} & G \\ G^{\mathrm{T}} & R \end{bmatrix} < 0. \end{aligned} \tag{3.14}$$

Again, by using the Schur complement lemma, we have $G^{\mathrm{T}}(\sum_{k=1}^{n} R_k)G - R < 0$. This completes the proof of the theorem. □

An important corollary to the above theorem and Lemma 3.6 can be stated as follows.

Corollary 3.9. *Consider the n-D system* (3.1) *under the fault free situation and the state estimation observer* (3.11)*. If there are n symmetric positive definite matrices* R_k *satisfying the LMI condition* $W_{c_n}^{\mathrm{T}} A_c W_{c_n} < 0$*, then there exists n maps* D_{ok} *such that the error dynamics* (3.12) *is asymptotically stable.*

Proof. Follow directly from the proofs of Theorem 3.8 and Lemma 3.6, and therefore the details are omitted for sake of brevity. □

Remark 3.10. *Note that by solving the LMI condition $W_{c_n}^{\mathrm{T}} A_c W_{c_n} < 0$, one can obtain symmetric positive definite matrices R_k. Hence, the state estimation observer gains D_{ok} are computed by solving the LMI* (3.14) *(which is an LMI condition in terms of the gains D_{ok}). Therefore, Corollary 3.9 not only provides sufficient conditions for existence of a state estimation observer, but also provides an approach for computing the observer gains D_{ok}.*

The results of this section will now be used subsequently in Section 3.2 to address the unobservable subspace of the system (3.1) as well as to provide sufficient conditions for solvability of the FDI problem, respectively.

3.2 Invariant Subspaces for n-D FMII Models

As described earlier, n-D systems can be represented by Inf-D systems (i.e., the initial condition is a vector of an Inf-D subspace). In this section, we first use the Inf-D representation (3.3) to formally define and construct an unobservable subspace. Next, we define a subspace of the unobservable subspace (this we call as an invariant unobservable subspace) of the n-D system (3.1) that can be represented as an infinite sum of the same Fin-D subspaces. Therefore, one can compute the invariant unobservable subspace (that is, the Inf-D subspace) in a finite number of steps (that is at most equal to m). Also, it is shown that the invariant unobservable subspace enjoys an important geometric property that is crucial for solving the FDI problem.

3.2.1 Unobservable Subspace

The unobservable subspace of the system (3.3) (and consequently of the system (3.1)) is defined as,

$$\mathscr{N}_g \triangleq \bigcap_{i=0}^{\infty} \ker \mathcal{C}\mathcal{A}^i, \tag{3.15}$$

where $\mathcal{A}$ and $\mathcal{C}$ are defined as in (3.3). Note that we define the above unobservable subspace by following along the steps in [103, page 1013], the results in [101, Chapter I], and the fact that the operator $\mathcal{A}$ in (3.3) is bounded (refer to Lemma 3.3).

One of the main difficulties in geometric analysis of Inf-D systems is the convergence of any developed algorithm that involves computation of certain set of subspaces in a finite number of steps. For example, consider the unobservable subspace (3.15). In Fin-D systems, the algorithm for computing the unobservable subspace converges in a finite number of steps [74] (for example, Lemma 5.1). Moreover, one is generally interested in investigating the FMII models in a Fin-D representation (3.1). Motivated by the above, below two important subspaces denoted by $\mathscr{N}_\infty \subseteq \mathscr{N}_g$ and $\mathscr{N}_{s,\infty} \subseteq \mathscr{N}_g$ are introduced. The subspaces $\mathscr{N}_\infty$ and $\mathscr{N}_{s,\infty}$ can be computed in a *finite* number of steps. This also allows one to derive necessary and sufficient conditions for solvability of the FDI problem.

Consider the initial condition $\mathbf{x}(0) = (\dots, 0, x_0, 0, \dots)$ and

$$u(i_1, \dots, i_n) = \begin{cases} u_0 & i_k = 0 \\ 0 & \text{Otherwise} \end{cases}, \tag{3.16}$$

where $u_0 \in \mathbb{R}^\ell$ and $k = 1, \cdots, n$. One can show that the state solution of the model (3.1) under the fault free situation is given by (through generalizing the results of [85] for n-D systems),

$$x(i_1, \dots, i_n) = A^{(i_1,\dots,i_n)} x_0 + A_B^{(i_1,\dots,i_n)} u_0, \tag{3.17}$$

where the matrices $A^{(i_1,\dots,i_n)}$ and $A_B^{(i_1,\dots,i_n)}$ are defined by the following recursive expressions,

$$\begin{aligned} \Delta A^{(i_1,\dots,i_n)} &= \sum_{k=1}^{n} A_k \Delta_k A^{(i_1,\dots,i_n)}, \quad A^{(i_1,\dots,i_n)} = 0 \quad \text{if any } i_k < 0, \\ \Delta A_B^{(i_1,\dots,i_n)} &= \sum_{k=1}^{n} \Delta_k A^{(i_1,\dots,i_n)} B_k, \ A^{(0,\dots,0)} = I, \end{aligned} \tag{3.18}$$

where by following the notation that is used in equation (3.1) we apply the shift operators to $A^{(i_1,\dots,i_n)}$ as

$$\Delta A^{(i_1,\dots,i_n)} = A^{(i_1+1,\dots,i_n+1)},\ \Delta_k A^{(i_1,\dots,i_n)} = A^{(i_1+1,\dots,i_{k-1}+1,i_k,i_{k+1}+1,\dots,i_n+1)} \tag{3.19}$$

For example, in the case $n = 3$, we have $\Delta A^{(i_1,i_2,i_3)} = A^{(i_1+1,i_2+1,i_3+1)}$ and $\Delta_2 A^{(i_1,i_2,i_3)} = A^{(i_1+1,i_2,i_3+1)}$. Based on the solution that is given by (3.17), and considering that $u_0 = 0$, a finite observability matrix (given that its null space is a finite dimensional subspace) can be defined as follows,

$$O \triangleq \left[C^{\mathrm{T}}, (CA_1)^{\mathrm{T}}, (CA_2)^{\mathrm{T}}, \cdots, (CA^{(i_1,\dots,i_n)})^{\mathrm{T}}, \dots\right]^{\mathrm{T}}. \tag{3.20}$$

Let $\mathscr{N} \triangleq \ker\ O = \bigcap_{i_k \geq 0}\left(\ker CA^{(i_1,\dots,i_n)}\right)$. Since $\dim(\mathscr{N}) \leq m < \infty$, we designate $\mathscr{N}$ as the *finite* unobservable subspace of the system (3.1). Also, recall from the generalized n-D Cayley-Hamilton theorem [121] that for all $\sum_{k=1}^{n} i_k \geq m$, one sets $A^{(j_1,\cdots,j_n)} = \sum_{\sum i_k < m} \zeta_{i_1,\dots,i_n} A^{(i_1,\dots,i_n)}$, where $\zeta_{i_1,\dots,i_n}$ are real numbers. Therefore, for all $\sum_{k=1}^{n} j_k \geq m$, we have

$$\bigcap_{\sum_{k=1}^{n} i_k < m} \ker CA^{(i_1,\dots,i_n)} \subseteq \ker CA^{(j_1,\dots,j_n)}. \tag{3.21}$$

Consequently, $\mathscr{N}$ can be computed in a finite number of steps as,

$$\mathscr{N} \triangleq \ker\ O = \bigcap_{i_k \geq 0,\ \sum_{k=1}^{n} i_k < m} \left(\ker CA^{(i_1,\dots,i_n)}\right). \tag{3.22}$$

Now, we consider the following subspace,

$$\mathscr{N}_\infty \triangleq \bigoplus(\mathscr{N}). \tag{3.23}$$

It follows that if $\mathbf{x}_0 = (\dots, x_{-1}, x_0, x_1, \dots) \in \mathscr{N}_\infty$, then $x_j \in \mathscr{N}$ for all $j \in \mathbb{Z}$, and given the zero input assumption one gets $\mathbf{y}(i) = 0$ for all $i \in \underline{\mathbb{N}}$ (in (3.3)). By considering $\mathcal{A}^i$, where $\mathcal{A}$ is defined as in (3.3) and $i \in \underline{\mathbb{N}}$, it can be shown that $\mathscr{N}_\infty \subseteq \mathscr{N}_g$. Also, note that although $\mathscr{N}_\infty$ is an Inf-D subspace, it can be computed in a finite number of steps (one only needs to compute $\mathscr{N}$). However, as explained in [16, 122] the invariance property (this is addressed in the next subsection) of $\mathscr{N}$ is not lucid (even for the case $n = 2$). Therefore, in the following a subspace of $\mathscr{N}$ (that is denoted by $\mathscr{N}_s$) is introduced such that it enjoys this geometric property. To define the subspace $\mathscr{N}_s$ one needs the following notation.

Let us express A^α to denote the sequence of multiplications of A_k, where α is a multi-index parameter that specifies the sequence of the multiplications. For example, for $\alpha = (2, 1, 1, 3, 6)$, we obtain $A^\alpha = A_2 A_1 A_1 A_3 A_6$. The notation $||\alpha||$ denotes the number of all A_k that are involved in the corresponding multiplications (for the above example, we have $||\alpha|| = 5$). Now, consider the following subspace,

$$\mathscr{N}_s \triangleq \bigcap_{||\alpha||<m} \ker CA^\alpha. \tag{3.24}$$

Similar to the above, let us define $\mathscr{N}_{s,\infty} \triangleq \bigoplus(\mathscr{N}_s)$. The following lemma shows that the subspace that is employed in [98–100] as the unobservable (non-observable) subspace is indeed $\mathscr{N}_s$ for the special cases $n = 2, 3$.

Lemma 3.11. *The subspace $\mathscr{N}_s$ (as defined in (3.24)) can be computed in a finite number of steps according to the following algorithm,*

$$\begin{aligned} &\mathscr{V}_0 = \ker C \text{ and } \mathscr{V}_k = (\bigcap_{j=1}^{n} A_j^{-1}\mathscr{V}_{k-1}) \cap \ker C, \\ &\mathscr{V}_m = \mathscr{N}_s. \end{aligned} \tag{3.25}$$

Proof. First, note that $\mathscr{V}_1 = (\bigcap_{j=1}^n A_j^{-1} \ker C) \cap \ker C$. In other words, $\mathscr{V}_1 = (\bigcap_{||\alpha||=1}(A^\alpha)^{-1} \ker C) \cap \ker C$, and $\mathscr{V}_2 = \mathscr{V}_1 \cap (\bigcap_{||\alpha||=2}(A^\alpha)^{-1} \ker C)$. Hence, we obtain $\mathscr{V}_k = \bigcap_{||\alpha||\leq k}(A^\alpha)^{-1} \ker C$. Note that for every pair of operators $C : \mathbb{R}^m \to \mathbb{R}^q$ and $F : \mathbb{R}^m \to \mathbb{R}^m$, one can show that $\ker CF = F^{-1} \ker C$, where $F^{-1} \ker C$ denotes the inverse image of $\ker C$ with respect to F (even if F is non-invertible $F^{-1} \ker C$ is well-defined). Therefore, by setting $F = A^\alpha$ in the above equations it follows that $\mathscr{V}_m = \mathscr{N}_s$. This completes the proof of the lemma. $\square$

3.2.2 A_k-Invariant Subspaces

As stated in Subsection 3.1.2, the n-D system (3.1) can be represented as an Inf-D system (3.3) (with $u \equiv 0$ and $f \equiv 0$). In order to formulate the corresponding Inf-D invariant subspaces one needs the next two definitions.

Definition 3.12. *[101, Definition I.2 for bounded operators] Consider the Inf-D system* (3.3)*, where the operator $\mathcal{A}$ is bounded (according to Lemma 3.3). The closed subspace $\mathscr{V}_\infty \subseteq \mathcal{X} = \bigoplus(\mathbb{R}^m)$ is called $\mathcal{A}$-invariant if $\mathcal{A}\mathscr{V}_\infty \subseteq \mathscr{V}_\infty$.*

Definition 3.13. *A subspace $\mathscr{V} \subset \mathbb{R}^m$ is said to be an $A_{\underline{\mathbf{k}}}$-invariant subspace for the n-D system* (3.1) *if $\sum_{k=1}^{n}(A_k\mathscr{V}) \subseteq \mathscr{V}$, where A_k, $k = 1, \ldots, n$ are the state operators in* (3.1)*.*

Note that $\mathscr{V}$ is $A_{\underline{\mathbf{k}}}$-invariant if and only if it is invariant with respect to <u>all</u> A_k (i.e., $A_k\mathscr{V} \subseteq \mathscr{V}$ for all $k = 1, \ldots, n$). The following theorem provides the connection between the Definitions 3.12 and 3.13.

Theorem 3.14. *Consider the n-D system* (3.1) *and the Inf-D system* (3.3)*. Let $\mathscr{V}_\infty = \bigoplus(\mathscr{V})$, where $\mathscr{V} \subseteq \mathbb{R}^m$. The subspace $\mathscr{V}_\infty$ is $\mathcal{A}$-invariant if and only if $\mathscr{V}$ is $A_{\underline{\mathbf{k}}}$-invariant.*

Proof. First, note that every $\mathbf{x} \in \mathscr{V}_\infty$ can be expressed as $\mathbf{x} = \sum_{j=-\infty}^{\infty} \mathbf{x}_j^j$, where $\mathbf{x}_j^j = (\ldots, 0, 0, x_j^{\mathrm{T}}, 0, 0 \ldots)^{\mathrm{T}} \in \mathscr{V}_\infty$ and $x_j \in \mathscr{V}$. Therefore, one only needs to show the result for $\mathbf{x}_j^j$.

(**If part**): Assume $\mathscr{V}$ is $A_{\underline{\mathbf{k}}}$-invariant. Consider the Inf-D vector $\mathbf{x}_j^j$. It follows that $\mathcal{A}\mathbf{x}_j^j = (\ldots, 0, (A_n x_j)^{\mathrm{T}}\ , 0, \ldots, (A_1 x_j)^{\mathrm{T}},\ 0, 0 \ldots)^{\mathrm{T}}$ (where the position of $A_k x_j$ are determined from the position of $\mathcal{A}_k$ in $\mathcal{A}$. For example, if A_1 is at k^{th} row and j^{th} column, we have $A_1 x_j$ at k^{th} position in the above Inf-D vector). Since $\mathscr{V}$ is $A_{\underline{\mathbf{k}}}$-invariant, it follows that $\mathcal{A}\mathbf{x}_j^j \in \mathscr{V}_\infty$.

(**Only if part**): Let $\mathcal{A}\mathscr{V}_\infty \subseteq \mathscr{V}_\infty$ and $\mathbf{x}_0^0 \in \mathscr{V}_\infty$. Consequently, $x_0 \in \mathscr{V}$. Since

$\mathcal{A}\mathbf{x}_0^0 = (\ldots, 0, (A_n x_0))^{\mathrm{T}}, 0, \ldots, (A_1 x_0)^{\mathrm{T}}, 0, 0, \ldots)^{\mathrm{T}} \in \mathscr{V}_\infty$ (where the position of $A_k x_j$ are determined from the position of $\mathcal{A}_k$ in $\mathcal{A}$. For example, if A_1 is at k^{th} row and j^{th} column, we have $A_1 x_j$ at k^{th} position in the above Inf-D vector), it follows that $A_k x_0 \in \mathscr{V}$ for all $k = 1, \ldots, n$, and consequently $\mathscr{V}$ is $A_{\underline{\mathbf{k}}}$-invariant. This completes the proof of the theorem. $\square$

Consider subspaces $\mathscr{V} \subseteq \mathbb{R}^m$ and $\mathscr{C} \subseteq \mathbb{R}^m$. If $\mathscr{V}$ is the largest $A_{\underline{\mathbf{k}}}$-invariant subspace that is contained in $\mathscr{C}$, we denote $\mathscr{V} =< \mathscr{C} | A_{\underline{\mathbf{k}}} >$. By generalizing the results in [16] (from 2-D to n-D), it follows that $\mathscr{N}_s \subseteq \mathscr{N}$, and $\mathscr{N}_s$ is the largest $A_{\underline{\mathbf{k}}}$-invariant subspace contained in $\ker C$. Therefore, one can write $\mathscr{N}_s =< \ker C | A_{\underline{\mathbf{k}}} >$. Since $\mathscr{N}_s$ is $A_{\underline{\mathbf{k}}}$-invariant, by invoking Theorem 3.14, $\mathscr{N}_{s,\infty}$ is $\mathcal{A}$-invariant. Therefore, if $\mathbf{x}(0) = (\cdots, x_{-1}, x_0, x_1, \cdots) \in \mathscr{N}_{s,\infty}$ (that is, $x_j \in \mathscr{N}_s$ for all $j \in \mathbb{Z}$) and zero input, $\mathbf{x}(i) \in \mathscr{N}_{s,\infty}$ and $\mathbf{y}(i) = 0$ for all $i \in \underline{\mathbb{N}}$ (in equation (3.3)). We designate $\mathscr{N}_{s,\infty}$ as the invariant unobservable subspace.

Remark 3.15. *As stated in Subsection 3.1.1, there are two different types of initial condition formulations. In this chapter, we use the first formulation that is compatible with the Inf-D system* (3.3)*. Recall that the second formulation is expressed as $x(\cdots, 0, i_k, 0, \cdots) = h_k(i_k)$ for all $k = 1, \cdots, n$, where $i_k \in \mathbb{N}$. Now, let $x(\cdots, 0, i_k, 0, \cdots) \in \mathscr{N}_s$. In this case, the $A_{\underline{\mathbf{k}}}$-invariance property of $\mathscr{N}_s$ also ensures that $y(i_1, \ldots, i_n) = 0$. In other words, $\mathscr{N}_{s,\infty}$ is also the invariant unobservable subspace of system* (3.1) *with the second formulation of the initial conditions. Therefore, without loss of any generality, one can apply our proposed approach to* <u>*both*</u> *initial condition formulations as provided in Subsection 3.1.1.*

3.2.3 Conditioned Invariant Subspaces of n-D Systems

Another important subspace in the geometric FDI toolbox is the conditioned invariant (i.e., the (C, A)-invariant) subspace that is defined next. This definition is an extension of the one that has appeared and presented in [123, Definition 3.2].

Definition 3.16. *A subspace* $\mathscr{W}_\infty = \bigoplus(\mathscr{W})$ *(where* $\mathscr{W} \subseteq \mathbb{R}^m$*) is said to be the conditioned invariant subspace for the n-D system* (3.1) *if there exist* n *output injection maps* $D_k : \mathbb{R}^q \to \mathbb{R}^m$ *such that* $(A_k + D_k C)\mathscr{W} \subseteq \mathscr{W}$ *for all* $k = 1, \cdots, n$.

In other words, $\mathscr{W}$ is $[A+DC]_{\underline{\mathbf{k}}}$-invariant (i.e., invariant with respect to $A_k + D_k C$). We designate $\mathscr{W}$ as the finite conditioned invariant subspace (since $\dim(\mathscr{W}) < \infty$) of the n-D system (3.1).

Similar to 1-D systems, one can now state the following result.

Lemma 3.17. *The following statements are equivalents.*

(i) *The subspace* $\mathscr{W}_\infty$ *is conditioned invariant.*

(ii) $\sum_{k=1}^{n}(A_k(\mathscr{W} \cap \ker C)) \subseteq \mathscr{W}$.

(iii) $\mathcal{A}(\mathscr{W}_\infty \cap \ker \mathcal{C}) \subseteq \mathscr{W}_\infty$.

where $\mathscr{W}_\infty = \bigoplus(\mathscr{W})$.

Proof. $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$: By definition, there exist n maps D_k such that $\mathscr{W}$ is $[A + DC]_{\underline{\mathbf{k}}}$-invariant. By utilizing Theorem 3.14, $\mathscr{W}_\infty$ is $\mathcal{A}_d$-invariant, where for example

$$\mathcal{A}_d = \begin{bmatrix} & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \cdots & 0 & A_1 + D_1C & 0 & \cdots & A_n + D_nC & & \cdots \\ \cdots & A_1 + D_1C & 0 & A_2 + D_2C & \cdots & A_n + D_nC & \cdots \\ \cdots & & \cdots & \cdots & \cdots & \ddots & \ddots \end{bmatrix} \tag{3.26}$$

in which the position $A_k + D_kC$ is defined by the position of A_k in $\mathcal{A}$ (i.e., the chosen ordering index). By following along the same lines as in Lemma 3.3, one can show that $\mathcal{A}_d$ is bounded. Consequently, the result of 1-D system is also valid for the Inf-D system (3.3). Hence, we have $\mathcal{A}(\mathscr{W}_\infty \cap \ker \mathcal{C}) \subseteq \mathscr{W}_\infty$ (which shows that $(i) \Leftrightarrow (iii)$). By considering the structure of $\mathcal{A}$ and $\mathcal{C}$ it follows that $\sum_{k=1}^{n}(A_k(\mathscr{W} \cap \ker C)) \subseteq \mathscr{W}$.

$(iii) \Leftrightarrow (i)$: Since $\mathcal{A}$ is bounded, the domain of $\mathcal{A}$ is equal to $\mathcal{X} = \bigoplus(\mathbb{R}^m)$, and therefore, the results of 1-D system are also valid for the Inf-D system (3.3). Therefore, there exists a bounded operator $\mathcal{D}$ such that $\mathscr{W}_\infty$ is $\mathcal{A}+\mathcal{D}\mathcal{C}$-invariant. By considering the structure of $\mathscr{W}_\infty$ and $\ker \mathcal{C}$, it is easy to show that one solution for $\mathcal{D}$ is given by (the position of D_k are determined by the positions of A_k in $\mathcal{A}$)

$$\mathcal{D} = \begin{bmatrix} & \ddots & \ddots & \ddots & \cdots & \cdots & & \\ \cdots & 0 & D_1 & 0 & \cdots & D_n & 0 & \cdots \\ \cdots & D_1 & 0 & 0 & \cdots & D_n & \cdots & \\ \cdots & & \cdots & \cdots & \ddots & \ddots & \ddots & \end{bmatrix} \tag{3.27}$$

Hence, by using Theorem 3.14, the subspace $\mathscr{W}$ is $[A+DC]_{\underline{\mathbf{k}}}$-invariant, and consequently $\mathscr{W}_\infty$ is a conditioned invariant subspace. This completes the proof of the lemma. $\square$

In the geometric FDI approach, one is interested in conditioned invariant subspaces that are containing a given subspace [41]. By following along the same lines as in 1-D systems (refer to [112, Section 4.1.1]), let us define all the conditioned invariant subspaces containing a subspace $\mathscr{L}_\infty = \bigoplus(\mathscr{L})$ ($\mathscr{L} \subseteq \mathbb{R}^m$) as $\mathfrak{Q}(\mathscr{L}) = \{\mathscr{W}_\infty | \ \exists\ D_k\ (A_k + D_kC)\mathscr{W} \subseteq \mathscr{W} \text{ and } \mathscr{W} \supseteq \mathscr{L}, k = 1, \cdots, n\}$. It can be shown that for a given subspace $\mathscr{L}_\infty$ (or $\mathscr{L}$), the set $\mathfrak{Q}(\mathscr{L})$ is closed under intersection, and hence the set $\mathfrak{Q}(\mathscr{L})$ has a minimal member as $\mathscr{W}^*_\infty(\mathscr{L})$. The minimal conditioned invariant subspace containing a given subspace $\mathscr{L}_\infty = \bigoplus(\mathscr{L})$ (that is, $\mathscr{W}^*_\infty(\mathscr{L})$) is obtained by invoking the following non-decreasing algorithm that is provided below,

$$\begin{aligned} \mathscr{W}^0 &= \mathscr{L}, \\ \mathscr{W}^i &= \mathscr{L} + \sum_{k=1}^{n}(A_k(\mathscr{W}^{i-1} \cap \ker C)), \end{aligned} \tag{3.28}$$

$\mathscr{W}^*(\mathscr{L}) = \mathscr{W}^{i_0}, i_0 \leq m$ and $\mathscr{W}^*_\infty(\mathscr{L}) = \bigoplus(\mathscr{W}^*(\mathscr{L}))$.

Note that the above algorithm converges in a <u>finite</u> number of steps (since $\mathscr{W}^i \subseteq \mathbb{R}^m$, the above algorithm converges in maximum m number of steps). Also, let $\mathscr{W}$ be a finite conditioned invariant subspace. The set of all maps $D = \begin{bmatrix} D_1 & \cdots & D_n \end{bmatrix}$ such that $\mathscr{W}$ is $[A + DC]_{\underline{\mathbf{k}}}$-invariant is designated by $\underline{D}(\mathscr{W})$.

3.2.4 Unobservability Subspace of n-D Systems

The unobservability subspace [3, Chapter 4 - Theorem 2] is the cornerstone of geometric FDI approach in 1-D systems. The following definition generalizes and extends this concept to the FMII n-D models.

Definition 3.18. *A subspace $\mathcal{S}_\infty$ is said to be an unobservability subspace for the n-D system* (3.1) *if there are $n+1$ maps D_k and H such that $\mathcal{S} =< \ker HC | [A+DC]_{\underline{\mathbf{k}}} >$ and $\mathcal{S}_\infty = \bigoplus(\mathcal{S})$. We designate $\mathcal{S}$ as the finite unobservability subspace of the n-D system* (3.1).

Note that $\mathcal{S}_\infty$ is also conditioned invariant subspace and an invariant unobservable subspace of the systems (C,A) and (HC, $A+DC$), respectively. For accomplishing the goal of the FDI task, one first computes an unobservability subspace and then obtains the map H [3, Chapter 2 - Theorem 18]. Therefore, it is necessary to compute the unobservability subspace without having any knowledge of H. Let $\mathscr{W}^*$ be the minimal finite conditioned invariant subspace containing $\mathscr{L}$. One can show that the limit of the following algorithm is the smallest unobservability subspace $\mathcal{S}^*(\mathscr{L})$ (and consequently $\mathcal{S}^*_\infty(\mathscr{L})$) that contains a given subspace $\mathscr{L}$.

$$\begin{aligned} &\mathscr{Z}^0 = \mathbb{R}^m \text{ and } \mathscr{Z}^i = \mathscr{W}^*(\mathscr{L}) + \Big(\bigcap_{k=1}^{n} A_k^{-1}\mathscr{Z}^{i-1} \cap \ker C\Big), \\ &\mathcal{S}^* = \mathscr{Z}^m, \end{aligned} \tag{3.29}$$

and $\mathcal{S}^*_\infty(\mathscr{L}) = \bigoplus(\mathcal{S}^*(\mathscr{L}))$. Finally, it is worth noting that $\underline{D}(\mathscr{W}^*(\mathscr{L})) \subseteq \underline{D}(\mathcal{S}^*(\mathscr{L}))$ (since $\mathcal{S}^*(\mathscr{L}) =< \ker C + \mathscr{W}^* | [A + DC]_{\underline{\mathbf{k}}} >$, and consequently $\mathcal{S}^*(\mathscr{L})$ is $[A+DC]_{\underline{\mathbf{k}}}$-invariant).

To summarize, in this section, we first defined the invariance property of the n-D system (3.1) that are Inf-D subspaces of the Inf-D system (3.3). Next, an invariance unobservable subspace (that is generically equivalent to an unobservable subspace) $\mathscr{N}_{s,\infty} = \bigoplus(\mathscr{N}_s)$ was introduced. Moreover, the conditioned and unobservability subspaces that are crucial in determining the solution to the FDI problem have been introduced. By utilizing the above results necessary and sufficient conditions for solvability of the FDI problem are subsequently derived and provided.

3.3 Necessary and Sufficient Conditions for Solvability of the FDI problem

In this section, we first present necessary and sufficient conditions for detectability and isolability of faults. Next, by employing an LMI-based filter sufficient conditions for solvability of the FDI problem are presented.

Consider the faulty FMII model (3.1) (i.e., the system is subjected to two faults f_1 and f_2) and the detection filter (3.8) designed to detect and isolate the fault f_1. By augmenting the detection filter dynamics (3.8) with the faulty n-D model (3.1), one obtains,

$$\begin{aligned}
\delta x_e(i_1,\ldots,i_n) =& \sum_{k=1}^{n} A_k^e \delta_k x_e(i_1,\ldots,i_n) + \sum_{k=1}^{n} B_k^e \delta_k u(i_1,\ldots,i_n) \\
&+ \sum_{k=1}^{n} L_{1,e}^k \delta_k f_1(i_1,\ldots,i_n) + \sum_{k=1}^{n} L_{2,e}^k \delta_k f_2(i_1,\ldots,i_n), \\
r_1(i_1,\ldots,i_n) =& C^e x_e(i_1,\ldots,i_n),
\end{aligned} \tag{3.30}$$

where $x_e = \begin{bmatrix} x^{\mathrm{T}} & \omega_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathscr{X}^e = \mathbb{R}^n \oplus \mathbb{R}^o$ (o refers to the dimension of ω_1), $C^e = \begin{bmatrix} HC & M \end{bmatrix}$ and,

$$
\begin{aligned}
A_k^e &= \begin{bmatrix} A_k & 0 \\ E_k C & F_k \end{bmatrix}, B_k^e = \begin{bmatrix} B_k \\ K_k \end{bmatrix}, \\
L_{i,e}^k &= \begin{bmatrix} (L_i^k)^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}, \ i = 1, 2, \ k = 1, \cdots, n.
\end{aligned} \tag{3.31}
$$

In this section, by considering the invariance unobservable subspace of the above augmented an analytical comparison between our proposed approach and the method developed in [99] is also provided to highlight the strength and capabilities of our proposed methodology when compared to the currently available results in the literature.

3.3.1 Main Results

The following lemma provides an important property for the invariant unobservable subspace $\mathscr{N}_{s,\infty}^e$ (and $\mathscr{N}_s^e$) that is associated with the system (3.30).

Lemma 3.19. *Consider the n-D system* (3.30) *and its invariant unobservable subspace $\mathscr{N}_{s,\infty}^e$. Let Q represent the embedding operator into $\mathscr{X}^e$ (i.e., $Q : \mathbb{R}^m \to \mathscr{X}^e$ and $Qx = \begin{bmatrix} x^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}}$), and $\mathcal{Q} = \mathrm{diag}(\cdots, Q, Q, \cdots)$. Then $\mathcal{Q}^{-1}\mathscr{N}_{s,\infty}^e$ is an unobservability subspace of the n-D system* (3.1)*.*

Proof. First, recall that $\mathscr{N}_{s,\infty}^e = \bigoplus(\mathscr{N}_s^e)$. Note that, $Q^{-1}\mathscr{N}_s^e = \mathscr{S} = \{x | \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathscr{N}_s^e\}$, and assume that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathscr{N}_s^e$. According to the fact that $\mathscr{N}_s^e$ is $A_{\underline{\mathbf{k}}}^e$-invariant, we have $A_k^e \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathscr{N}_s^e$, and if $x \in \ker C$ then $A_k x \in \mathscr{S}$ and it follows that $A_k(\mathscr{S} \cap \ker C) \subseteq \mathscr{S}$ for all $k = 1, \cdots, n$. Therefore, by using Lemma 3.17, $\mathscr{S}_\infty = \mathcal{Q}^{-1}\mathscr{N}_{s,\infty}^e = \bigoplus(\mathscr{S})$ is a conditioned invariant subspace. Moreover, given that $\mathscr{N}_s^e$ is contained in $\ker C^e$, we have $\mathscr{S} \subseteq \ker HC$. Therefore, the subspace $\mathscr{S}$ is a finite conditioned invariant subspace contained in $\ker HC$. Since $\mathscr{N}_s^e$ is the largest $A_{\underline{\mathbf{k}}}^e$-invariant subspace in $\ker C^e$, it follows that $\mathscr{S}$ is the largest $[A + DC]_{\underline{\mathbf{k}}}$ invariant subspace in $\ker HC$ (i.e., $\mathscr{S}$ is a finite unobservability subspace of the n-D system (3.1)). In other words, $\mathscr{S}$ is a finite unobservability subspace of the system (3.1). This completes the proof of the lemma. $\square$

The following theorem provides a single necessary and sufficient condition for detectability and isolability of faults (i.e., the existence of a subsystem such that it is decoupled from all faults but one - refer to Subsection 3.1.3 for more details).

Theorem 3.20. *Consider the n-D system* (3.1) *that is subject to two faults* f_1 *and* f_2*. Also, let* $\mathscr{L}_1^k = \operatorname{span}\{L_1^k\}$ *and* $\mathcal{S}_1^*$ *denote the smallest finite unobservability subspace of the n-D system* (3.1) *containing* $\sum_{k=1}^n \mathscr{L}_2^k$ *(this represents the limit of the algorithm* (3.29)*, where one sets* $\mathscr{L} = \sum_{k=1}^n \mathscr{L}_2^k$ *in the algorithm* (3.28)*).The fault* f_1 *is detectable and isolable if and only if the following condition is satisfied,*

$$(\sum_{k=1}^{n} \mathscr{L}_1^k) \nsubseteq \mathcal{S}_1^* \tag{3.32}$$

Proof. (**If part**): By the definition of $\mathcal{S}_1^*$, there exist $n+1$ maps D_k and H such that $\mathcal{S}_1^* =< \ker H_1C|[A+D^1C]_{\underline{\mathbf{k}}} >$. Since $\sum_{k=1}^n \mathscr{L}_2^k \subseteq \mathcal{S}^*$, it follows that the output of the system $(H_1C,\, A+D^1C)$ is decoupled from f_2. In other words, f_2 has no effect on the output of the system as defined above. Now consider the following detection filter

$$\begin{aligned}
\delta\omega_1(i_1,\ldots,i_n) =& \sum_{k=1}^{n} F_k\delta_k\omega_1(i_1,\ldots,i_n) + \sum_{k=1}^{n} P_1B_k\delta_k u(i_1,\ldots,i_n)\\
&+ \sum_{k=1}^{n} P_1L_1^k\delta_k f_1(i_1,\ldots,i_n), \qquad\qquad (3.33)\\
r_1(i_1,\ldots,i_n) =& M_1\omega_1(i_1,\ldots,i_n),
\end{aligned}$$

where $P_1 : \mathbb{R}^m \to \mathbb{R}^m/\,\mathcal{S}_1^*$ is the canonical projection of $\mathbb{R}^m$ on $\mathbb{R}^m/\mathcal{S}_1^*$ (for simplicity, this map is denoted as the canonical projection of $\mathcal{S}_1^*$). Since $\mathcal{S}_1^*$ is $[A+D^1C]_{\underline{\mathbf{k}}}$-invariant, there exist n maps A_k^p such that $A_k^pP_1 = P_1(A_k+D_k^1C)$ for all $k=1,\cdots,n$. Also, $F_k = A_k^p + D_{ok}M_1$, where H_1 can be obtained from $\ker H_1C = \mathcal{S}_1^* + \ker C$, and

M_1 is the unique solution to $M_1P_1 = H_1C$. Moreover, D_{ok} denote the filter gains. Note that $P_1L_2^k = 0$ (since $\sum_{k=1}^{n} \mathscr{L}_2^k \subseteq \mathcal{S}_1^*$).

Now, by defining $e_1(i_1,\ldots,i_n) = P_1x(i_1,\ldots,i_n) - \omega_1(i_1,\ldots,i_n)$, one obtains

$$\begin{aligned}\delta e_1(i_1,\ldots,i_n) =& \sum_{k=1}^{n} F_k\delta_k e_1(i_1,\ldots,i_n) + \sum_{k=1}^{n} P_1L_1^k\delta_k f_1(i_1,\ldots,i_n),\\ r_1(i_1,\ldots,i_n) =& M_1e_1(i_1,\ldots,i_n),\end{aligned} \tag{3.34}$$

According to the condition in equation (3.32), one has $PL_1^k \neq 0$ for at least one $k \in \{1,\cdots,n\}$. Hence, the residual signal (3.34) is decoupled from the fault f_2, and it follows that the fault f_1 is detectable from the output of the residual signal (3.34). Therefore, the fault f_1 is detectable and isolable in the sense of Remark 3.5.

(**Only if part**): We show the result by contradiction. Assume $\sum_{k=1}^{n} \mathscr{L}_1^k \subseteq \mathcal{S}_1^*$. By using Lemma 3.19, it follows that $\sum_{k=1}^{n} \mathscr{L}_{1,e}^k \subseteq \mathscr{N}_s^e$. As discussed in Subsection 3.2.1, $\mathscr{N}_s^e \subseteq \mathscr{N}^e$, and consequently $\sum_{k=1}^{n} \mathscr{L}_{1,e}^k \subseteq \mathscr{N}^e$. In other words, the fault f_1 is not detectable. This is in contradiction with the assumption. This completes the proof of the theorem. □

Remark 3.21. *It should pointed out that since L_1^k, $k = 1,\ldots,n$ are defined on $\mathbb{R}$ (refer to the system (3.1)), one obtains $\dim(\mathscr{L}_1^k) = 1$. This shows that the condition (3.32) is consistent with the condition (2.13) for the case $n = 1$ that is the 1-D system.*

As stated in Remark 3.5, the FDI problem has two main steps. Theorem 3.20 provides a necessary and sufficient condition for the first step (that is, detectability and isolability of fault f_1). Therefore, condition (3.32) is also necessary for solvability of the FDI problem. For the second step (that is, designing a filter that can detect f_1 and the corresponding error dynamics is asymptotically stable), one needs to design a detection filter. Design of a residual generator to detect and isolate the fault f_1 in the n-D system (3.1) is reduced to that of detecting this fault in the system (3.33) by using an observer. In other words, one needs to determine D_{ok} and consequently F_k in (3.33), such that the error dynamics (3.34) is asymptotically stable. For the special case $n = 2$ one can use deadbeat observers [89, 118]. However, as stated earlier and emphasized in [92], the algebraic approach (that is based on polynomial matrices) faces certain difficulties (for $n \geq 3$). Moreover, as pointed out in [118], design of a deadbeat observer for FMII models is based on polynomial matrices. This method is unfortunately not always numerically or analytically straightforward (even in the case $n = 2$) to develop and therefore, in this work we develop a set of sufficient conditions for solvability of the FDI problem by using an n-D Luenberger observer.

Towards the above end, let $k = 1, \cdots, n$, P_1 is the canonical projection of $\mathcal{S}_1^*$, and A_k^p are defined from (3.33). Also, H_1 and D_k^1 denote the operators that are defined by the smallest unobservability subspace that contains $\sum_{k=1}^n \mathscr{L}_2^k$. The operator P_1^{-r} is the right-inverse of P_1 and D_{ok} are the state estimation observer gains as given by the Corollary 3.9. The residual generator n-D detection filter that is governed by (3.8) is utilized where the filter gains are selected according to,

$$F_k = A_k^p + D_{ok}M_1,\ K_k = P_1 B_k,\ D_k^e = D_k^1 + P_1^{-r} D_{ok} H_1,\ E_k = P_1 D_k^e, \tag{3.35}$$

The next corollary provides sufficient conditions for solvability of the FDI problem by using an n-D Luenberger observer.

Corollary 3.22. *Consider the n-D model* (3.1), *where the condition* (3.32) *is satisfied. Let* $A_p = \begin{bmatrix} A_1^p & \cdots & A_n^p \end{bmatrix}$ *(*A_k^p *are defined in* (3.33)*), and define* W_m *such that the columns of* W_m *are the basis of* $\ker M_1$*. The FDI problem is solvable if there exist n symmetric positive definite matrices* R_k *such that,*

$$W_m^{\mathrm{T}}(\mathrm{diag}(R_1, \ldots, R_n) - A_p^{\mathrm{T}}(\sum_{k=1}^{n} R_k)A_p)W_m < 0. \tag{3.36}$$

Proof. By invoking Theorem 3.20, the fault f_1 is detectable and isolable, and consequently the subsystem (3.33) exists (and is decoupled from all faults but f_1).

Moreover, Corollary 3.9 guarantees the existence of n observer gains D_{ok} that one can construct for the detection filter (3.8), where the operators are defined as in (3.33). Therefore, the state estimation error dynamics can be expressed according to,

$$\begin{aligned} \delta e(i_1, \dots, i_n) &= \sum_{k=1}^{n} F_k \delta_k e(i_1, \dots, i_n) + \sum_{k=1}^{n} P_1 L_1^k \delta_k f_1(i_1, \dots, i_n), \\ r_1(i_1, \dots, i_n) &= M_1 e(i_1, \dots, i_n), \end{aligned} \tag{3.37}$$

where $e(i_1, \dots, i_n) = P_1 x(i_1, \dots, i_n) - \omega_1(i_1, \dots, i_n)$. By considering the LMI condition (3.36) and invoking results from Corollary 3.9, the error dynamics (3.37) is asymptotically stable. If $f_1 \equiv 0$, the residual signal r_1 converges to zero as $\sum_{k=1}^{n} i_k \to \infty$. Otherwise, the residual has a value that is different from zero. Therefore, the condition (3.7a) is satisfied and the FDI problem is solvable. By following along the same lines as those above one can also design another state estimation observer to detect and isolate the fault f_2. Therefore, this completes the proof of the corollary. $\square$

Remark 3.23. *The parameters ϵ_i (refer to equation (3.7)) are determined by using Monte Carlo simulations as follows. When the detection filter gains are obtained, one performs Monte Carlo simulations for the healthy system. Subsequently, one sets $\epsilon_i = Th_i$, where Th_i is the upper bound of r_i in all the simulations. Also, N_0 is dependent to application and selected such that the effects of the initial condition errors are eliminated from r_i.*

Remark 3.24. *It is worth noting that one can directly work with $\mathscr{N}_g$ (as defined in (3.15)) and derive necessary and sufficient conditions by following along the same steps as those that have been proposed in [124]. However, there are two main drawbacks associated with this approach that are as follows:*

1. *The invariant subspaces are <u>not</u> necessarily computed in a finite number of steps.*

2. *By factoring out $\mathscr{N}_g$, the resulting subsystem does not necessarily have a Fin-D representation. For more clarification on n-D realization, refer to the work in [125].*

To summarize, Theorem 3.20 provides a single necessary and sufficient condition for detectability and isolability of the faults (refer to Remark 3.5) that is also necessary for solvability of the FDI problem. Then, a set of sufficient conditions for solvability of the FDI problem by utilizing an n-D Luenberger observer are derived in Corollary 3.22. Note that the provided results are applicable to any n-D system, whereas the algebraic approaches need more investigation for the cases $n \geq 3$.

Table 3.1 summarizes the main results that are developed and presented in this subsection.

Table 3.1: Pseudo-algorithm to detect and isolate the fault f_i in the n-D system (3.1).

1. `Compute the minimal conditioned invariant subspace` $\mathscr{W}_i^*$ `containing all` $\sum_{k=1}^{n} \mathscr{L}_j^k$ `subspaces such that` $j \neq i$ `(by invoking the algorithm` (3.28)`, where` $\mathscr{L} = \sum_{j \neq i} \sum_{k=1}^{n} \mathscr{L}_j^k$`).`
2. `Compute the unobservability subspace` $\mathcal{S}_i^*$ `containing` $\mathscr{L}$ `(by using the algorithm` (3.29)`).`
3. `Compute the operators` D_k^i `such that` $\mathscr{W}_i^*$ `is the minimal conditioned invariant subspace of the n-D system` (3.1)`.`
4. `Find the operator` H_i `such that` $\ker H_i C = \mathcal{S}_i^* + \ker C$`.`
5. `If` $\sum_{k=1}^{n} \mathscr{L}_i^k \nsubseteq \mathcal{S}_i^*$`, then the fault` f_i `is detectable and isolable (refer to Remark 3.5 and Theorem 3.20), and`
6. `If the conditions of Corollary 3.22 are satisfied, there exists an LMI-based observer for detection and isolation of the fault` f_i`. The operator of the detection filter is defined in` (3.35)`.`

   `The output norm of the above detection filter is the residual that satisfies the condition` (3.7)`.`

3.3.2 Comparisons with Other Available Approaches in the Literature

In this subsection, our proposed approach is compared and evaluated with existing geometric methods in the literature [99,100]. We first show that if the FDI problem is solvable by using the approach in the above literature, our approach can also detect and isolate the faults. Furthermore, we provide a numerical example where it is shown that our approach is capable of detecting and isolating a fault, however, the necessary conditions provided in [99] are not satisfied.

The equivalent n-D version of the necessary condition (as derived in [99], Theorems 2 and 3) to detect and isolate two faults can be summarized as follows: The faults f_1 and f_2 in the n-D system (3.1) are detectable and isolable according to [99] if $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$, where $\mathscr{W}_1^*$ and $\mathscr{W}_2^*$ denote the minimal finite conditioned invariant subspaces that contain $\sum_{k=1}^{n} \mathscr{L}_1^k$ and $\sum_{k=1}^{n} \mathscr{L}_2^k$, respectively. It should be pointed out that the observability assumption of $(C,\, A)$ is a fundamental requirement and condition in [99] (although it was stated in [99] that this assumption was made for simplicity of the presentation). The main reason for the above limitation lies on and is due to the fact that the approach in [99] is based on results of [50]. However, as stated in [50] the observability assumption is quite a crucial and critical condition (refer to Section III, Lemma 5, Proposition 6 and Theorem 7 in [50]).

For further illustration and clarification of the above serious concern consider the following 3-D system,

$$
\begin{aligned}
x(i+1,j+1,k+1) = {} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(i,j+1,k+1) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(i+1,j,k+1) \\
& + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(i+1,j+1,k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f_2, \qquad (3.38) \\
y(i,j,k) = {} & \begin{bmatrix} 0 & 1 \end{bmatrix} x(i,j,k).
\end{aligned}
$$

We have $\mathscr{W}_1^* = \mathscr{L}_1 = \ker C$, $\mathscr{W}^* = \mathscr{L}_2$, and consequently $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$. Therefore, the sufficient condition for solvability of the FDI problem under the zero initial condition in [99] (Theorem 2) is satisfied. However, it is easy to verify that $\mathscr{L}_1 \subseteq \mathscr{N}$, and consequently f_1 is <u>not even detectable</u> (in other words, f_1 has no effect on the output signal). It should be pointed out that our proposed methodology does not suffer from the above limitation and restriction.

We are now in a position to state the following theorem.

Theorem 3.25. *Consider the n-D system* (3.1) *and assume that the FDI problem is solvable by using the approach that is proposed in [99]. Then the approach proposed in this work can also detect and isolate the faults in the system* (3.1)*.*

Proof. According to Theorem 3 in [99], $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$, and $\mathscr{W}_1^*$ and $\mathscr{W}_2^*$ are internally/externally stabilizable. Therefore, there exist n maps D_k such that $\mathscr{W}_1^*$ and $\mathscr{W}_2^*$ are *both* $[A+DC]_{\underline{\mathbf{k}}}$-invariant, and the system $(A_1 + D_1C, \cdots, A_n + D_nC)$ is stable (that is, the corresponding n-D system is asymptotically stable). Since $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$, there exists a map H such that $HC\mathscr{W}_1^* = C\mathscr{W}_1^*$ and $HC\mathscr{W}_2^* = 0$ (i.e., $\ker H = C\mathscr{W}_2^*$ and $H|_{C\mathscr{W}_1^*} = I$). Let $\mathscr{N}_s^h$ denote the invariant unobservable subspace of $(HC,\ A+DC)$. It follows that $\mathscr{N}_s^h \cap \mathscr{W}_1^* = 0$. Note that $\mathscr{N}_s^h$ is an unobservability subspace of $(C,\ A)$ containing $\sum_{k=1}^n \mathscr{L}_2^k$, and since $\mathcal{S}_1^*$ is the smallest unobservability subspace containing $\sum_{k=1}^n \mathscr{L}_2^k$, it follows that $\sum_{k=1}^n \mathscr{L}_1^k \nsubseteq \mathcal{S}_1^*$. Also, since $(A_1 + D_1C, \cdots, A_n + D_nC)$ is stable, it can be shown that $(A_1^p, \cdots, A_n^p)$ is also stable, where $P[A_k+D_kC] = A_k^p P$ and P is the canonical projection of $\mathcal{S}_1^*$. Therefore, one can also construct an observer for the quotient system (3.33) to detect and isolate the fault f_1 (by choosing $D_{ok} = 0$ for all $k = 1, \cdots, n$). By following along the same lines, one can detect and isolate the fault f_2. This completes the proof of the theorem. $\square$

Remark 3.26. *Theorem 3.25 shows that our proposed approach can detect and isolate faults that are detectable and isolable by using the geometric method in [99]. However, below an example is provided where this approach fails whereas our proposed approach* can still *detect and isolate the faults.*

Illustrative Example (Limitations of the Method in [99])

Consider the 3-D system (3.1) that is subjected to two faults f_1 and f_2 where,
$A_1 = \begin{bmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 0.5 \end{smallmatrix} & 0.5I_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix}$, $A_2 = 0.5 \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ I_{2\times 2} & I_{2\times 2} \end{bmatrix}$, $A_3 = 0.5I_{4\times 4}$, $L_1^i = L_2^i = 0, i = 2, 3$,

$B_1 = B_2 = B_3 = 0$, $L_1^1 = [0, 0, 0, 1]^{\mathrm{T}}$, $L_2^1 = [0, 0, -1, 1]^{\mathrm{T}}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The necessary condition to detect and isolate the fault f_1 by using the approach in [99] is $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$. Since $\mathscr{L}_1^1, \mathscr{L}_2^1 \not\in \ker C$, by invoking the algorithm (3.28), one obtains $\mathscr{W}_1^* = \mathscr{L}_1$ and $\mathscr{W}_2^* = \mathscr{L}_2$. It follows that, $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = \operatorname{span}\{[0, 1]^{\mathrm{T}}\}$. Therefore, the necessary condition in [99, Theorems 2 and 3] is not satisfied. In other words, the fault f_1 cannot be detected and isolated by using the detection filter (3.8), if one restricts the filter to the case with $M = C$ (or $H = I$), according to the required results in [99].

Finally, it is now shown and demonstrated that one can detect and isolate both faults f_1 and f_2 by using our proposed methodology. Towards this end, by invoking the algorithm (3.29), one can write $\mathcal{S}_1^* = \mathscr{L}_2^1$ (that is, the finite unobservability subspace containing $\mathscr{L}_2^1$) that satisfies the condition (3.32). By considering $D_1 =$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 \end{bmatrix}$ and $D_3 = 0$, $\mathscr{W}_2^*$ is $[A + DC]_{\underline{\mathbf{k}}}$-invariant. Also, since $\ker C + \mathcal{S}_1^* = \ker H_1 C$ and $M_1 P_1 = H_1 C$, one gets $H_1 = [1, 0]$ and $M_1 = [1, 0, 0]$. Hence, the quotient subsystem (3.33) that is only affected by the fault f_1 is given by

$$A_1^p = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2^p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 0.5 \end{bmatrix}, A_3^p = 0.5I, \; B_1 = B_2 = B_3 = 0. \quad (3.39)$$

Since the triple $(A_1^p,\ A_2^p,\ A_3^p)$ is stable (since $\det(I - z_1 A_1 - z_2 A_2 z_3 A_3) \neq 0$ for all $|z_j| \leq 1, j = 1, 2, 3$ - [119, Theorem 41]), by considering $D_{o1} = D_{o2} = 0$, the detection filter for the fault f_1 (as given by equation (3.8)) is obtained according to,

$$\begin{aligned} \omega_1(i_1+1, i_2+1, i_3+1) =& A_1^p \omega_1(i_1, i_2+1, i_3+1) + A_2^p \omega_1(i_1+1, i_2, i_3+1) \\ & + A_3^p \omega_1(i_1+1, i_2+1, i_3), \\ r_1(i,j) =& M_1 \omega_1(i,j) - H_1 y(i,j). \end{aligned} \quad (3.40)$$

By following along the same procedure, one can also design a detection filter to detect and isolate the fault f_2. Therefore, our proposed approach can accomplish the FDI objectives while the approach that is proposed in [99] cannot achieve this goal.

Remark 3.27. *All the conditions for solvability of the FDI problem in the literature (for both 1-D, 2-D and 3-D systems) and also our proposed conditions (for n-D systems) are* generic*, although this fact is not explicitly mentioned. In other words, for every system that satisfies the proposed conditions in the literature (i.e. [89, 91, 99]) or our proposed conditions, the developed methods can detect and isolate almost all the fault signals. For clarification, consider the faulty model* (3.1)*, where* $x \in \mathbb{R}^2$*,* $A_1 = A_2 = 0.4 * I$*,* $B_1^1 = L_1^1 = [1, 1]^{\mathrm{T}}$*,* $B_2^2 = L_2^2 = [0, 1]$ *and* $C = [1, -1]^{\mathrm{T}}$*. Let the initial condition* $\mathbf{x}(0) = 0$ *and* $f_1(i,j) = 1$ *for all* $i + j \geq 0$*. It follows that* $y \equiv 0$*, and* *consequently f_1 is not detectable. However, below we show that sufficient conditions in the literature [89, 91, 99] as well as our proposed conditions are still all satisfied.*

1. *(Conditions in [89] and [91]). It follows that $N(z_1, z_2)PHB(z_1, z_2) = 0$, where $N(z_1, z_2) = \begin{bmatrix} -1 & 0 & 0.4(z_1+z_2)-1 \\ 0 & 1 & 0.4(z_1+z_2)-1 \end{bmatrix}$, and consequently the condition of Theorem 1 in [91] is also satisfied.*

2. *(Conditions in [99]). By following the algorithm* (3.28) *we obtain $\mathscr{W}_1^* = \mathscr{L}_1$ and $\mathscr{W}_2^* = \mathscr{L}_2$. It follows that $C\mathscr{W}_1^* \cap C\mathscr{W}_2^* = 0$, and consequently the condition in [99] is also satisfied.*

3. *(Our proposed Conditions) By following the algorithm* (3.29) *we obtain $\mathcal{S}_1^* = \mathscr{L}_1$ and $\mathcal{S}_2^* = \mathscr{L}_2$. It follows that $\mathcal{S}_1^* \cap \mathscr{L}_2 = 0$, and consequently the condition* (3.32) *is also satisfied.*

To summarize, in this section we have developed and presented a solution to the FDI problem of n-D systems by invoking an Inf-D framework for the first time in the literature and by utilizing invariant subspaces and derived necessary and sufficient conditions for solvability of the problem. It was shown that if the sufficient conditions for solvability of the FDI problem that are provided in [99,100] are satisfied, then our proposed approach can also detect and isolate the faults. However, as shown above there are certain systems that the method in [99] are not applicable to and capable of detecting and isolating faults, whereas our proposed approach can solve this problem successfully.

3.4 Simulation Results

In this section, we apply our proposed FDI methodology to a heat transfer process [75,126]. Figure 3.1 shows a two-line parallel heat transfer system. This process can

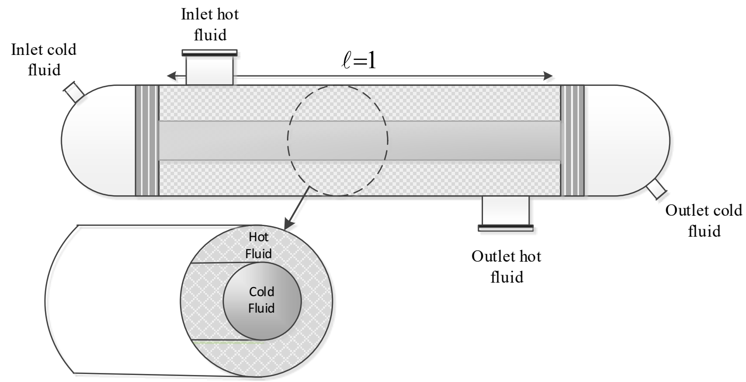


Figure 3.1: The two-line parallel heat transfer process that is considered in this section.

be considered as a model for the heat transfer in a thermal-hydraulic system and heat exchangers.

In this section, we verify the necessary and sufficient conditions that are derived in the previous section, and also design a set of filters to detect and isolate faults under both full and partial state measurement scenarios (this is to be realized by an appropriate selection of the output matrix C).

The heat transfer system is usually subject to two different types of faults, namely the fouling and the leakage [126]. The mathematical model of a typical heat transfer system is governed by the following hyperbolic PDEs,

$$\begin{aligned} \frac{\partial T_f}{\partial t} &= -\alpha_f \frac{\partial T_f}{\partial z} - \beta(T_f - T_g) - f_2(z,t), \\ \frac{\partial T_g}{\partial t} &= -\alpha_g \frac{\partial T_g}{\partial z} - \beta(T_g - T_f) + f_1(z,t) + f_2(z,t), \end{aligned} \tag{3.41}$$

where $z \in [0,1]$, and T_f and T_g denote the temperature of the cold (fuel) and the hot (exhaust gas) sections, respectively. The coefficients α_f and α_g are proportional on the speed of the fluid and the gas, respectively, and the coefficient β is related to the heat transfer coefficient of the wall [126]. Moreover, f_1 and f_2 denote the leakage and the fouling effects, respectively. Finally, it is assumed that the boundary conditions (the inlet temperature) $T_f(t,0)$ and $T_g(t,0)$ are given and only the outer section (i.e. T_g) is subject to the leakage.

For the purpose of conducting simulations, the parameters in equation (3.41) are taken as $\alpha_f = \alpha_g = \beta = 1$. Also, by considering $\Delta z = \Delta t = 0.1$, and following along the same lines as in subsection 2.5.1, one can discretize the system (3.41) as,

$$
\begin{aligned}
x(i+1,j+1) =& \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{bmatrix} x(i,j+1) + \begin{bmatrix} 0_{2\times 2} & 0_{2\times 2} \\ I_{2\times 2} & \begin{smallmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{smallmatrix} \end{bmatrix} x(i+1,j) + L_1^1 f_1(i+1,j) \\
&+ B_2^1 u(i+1,j) + L_2^1 f_2(i+1,j), \\
y(i,j) =& Cx(i,j),
\end{aligned} \tag{3.42}
$$

where $L_1^1 = [0,0,0,1]^{\mathrm{T}}$, $L_2^1 = [0,0,-1,1]^{\mathrm{T}}$ and $x(i,j) = [T_g((i-1)\Delta z, j\Delta t), T_f((i-1)\Delta z, j\Delta t),\ T_g(i\Delta z, j\Delta t), T_f(i\Delta z, j\Delta t)]^{\mathrm{T}}$. Finally, one obtains $B_2 = \begin{bmatrix} 0_{2\times 2} \\ \begin{smallmatrix} 1.1 & -0.1 \\ -0.1 & 1.1 \end{smallmatrix} \end{bmatrix}$, $u(0,j) = [T_f(0,\Delta t), T_g(0,\Delta t)]^{\mathrm{T}}$ and $u(i,j) = u(0,j)$ for all i.

We first assume that both temperatures T_f and T_g are available for measurement along the spatial coordinates at discrete points (i.e., $T_f(i\Delta z, j\Delta t)$ and $T_g(i\Delta z, j\Delta t)$ are available from sensors). Next, we consider the case where only the outer temperature (that is, T_g) is available for measurement. As we shall show subsequently, in the latter case by using the 1-D approximate ODE model (for example, as in [127]), the faults f_1 and f_2 are *not* isolable, whereas by using our proposed n-D FDI methodology one can detect and isolate both faults.

3.4.1 FDI of a Heat Transfer System by Using Full State Measurements

Let us assume that both temperatures (namely, T_f and T_g) are available for measurement. Therefore, one can select the output matrix as $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Detectability and Isolability Conditions

By applying the algorithm (3.28), the minimal finite conditioned invariant subspace containing $\mathscr{L} = \text{span}\{L_2^1\}$ is obtained as $\mathscr{W}_2^* = \mathscr{L}_2^1$, and by applying the algorithm (3.29), one obtains $\mathcal{S}_1^* = \mathscr{W}_2^* = \mathscr{L}_2^1$. It follows that $\mathscr{L}_1^1 \cap \mathcal{S}_1^* = 0$. Therefore, the sufficient condition of Theorem 3.20 is satisfied. By following along the same lines the necessary and sufficient condition for detectability and isolability of the fault f_2 can also be shown to be satisfied.

FDI 2-D Luenberger Filter Design

As stated earlier, we are interested in designing 2-D Luenberger detection filters by using the LMI condition that is proposed in the Subsection 3.1.4. In this part of the chapter, we design a filter for detecting and isolating the fault f_1 (without loss of any generality, by following along the same lines as conducted below one can also design a filter to detect and isolate the fault f_2). The 2-D detection filter must be decoupled from the fault f_2 (refer to the conditions in equation (3.7)). As stated above, the finite unobservability subspaces containing the subspace $\mathscr{L}_2^1$ is obtained by using the algorithm (3.29) and is given by $\mathcal{S}_1^* = \mathscr{W}_2^* = \mathscr{L}_2^1$.

The output injection matrices D_1^1 and D_2^1 for $\mathscr{W}_2^*$ are to be derived such that $\mathscr{W}_2^*$ is $[A + D^1C]_{1,2}$-invariant. Therefore, one can write $(A_1 + D_1^1C)W_2^c = 0$ and $(A_2 + D_2^1C)W_2^c = 0$, where the columns of W_2^c are the basis of $\mathscr{W}_2^* \cap (\mathscr{W}_2^* \cap \ker C)^\perp$. In other words, for the above example, we have $W_2^c = L_2^1$. One solution to D_1 and D_2^1 is $D_1^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}^\mathrm{T}$ and $D_2^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2 & 0.2 \end{bmatrix}^\mathrm{T}$. Also, let $P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (which is the canonical projector of the subspace $\mathcal{S}_1^*$), where P_1 is used in equation (3.33). By using $\ker H_1C = \mathcal{S}_1^* + \ker C$ and $M_1P_1 = H_1C$, we have $H_1 = [1,\ 0,\ 0]$

and the output matrix M_1 becomes $M_1 = [1, 0, 0]$. Hence, the factored out 2-D system is now expressed as,

$$
\begin{aligned}
\omega(i+1,j+1) &= \begin{bmatrix} 0 & 0 & -1.42 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega(i,j+1) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.7 & -0.7 & 0 \end{bmatrix} \omega(i+1,j) \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.7 & 0.7 \end{bmatrix} u(i+1,j) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} f_1(i+1,j), \\
y_p(i,j) &= [1, 0, 0]\omega(i,j),
\end{aligned} \tag{3.43}
$$

where $\omega(i,j) = P_1 x(i,j)$. It is straightforward to show that the positive definite matrices $R_1 = \mathrm{diag}(0.4, 1, 2.133)$ and $R_2 = \mathrm{diag}(0.4, 2.15, 0.86)$ satisfy the inequality $W_{c_d}^{\mathrm{T}} A_C W_{c_d} < 0$. By using Remark 3.10, one can obtain $D_{o1} = 0$ and $D_{o2} = \begin{bmatrix} 0 & 0 & -0.7 \end{bmatrix}^{\mathrm{T}}$. Therefore, the filter to detect and isolate the fault f_1 is given by,

$$
\begin{aligned}
\omega_1(i+1,j+1) &= \begin{bmatrix} 0 & 0 & -1.42 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_1(i,j+1) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.7 & -0.7 & 0 \end{bmatrix} \omega_1(i+1,j) \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.7 & 0.7 \end{bmatrix} u(i+1,j) + D_{o2} P_1 y(i,j+1), \\
r_1(i,j) &= M_1 \omega_1(i,j) - H_1 y(i,j).
\end{aligned}
$$

Designing a filter to detect and isolate the fault f_2 follows along the same lines as those given above for the fault f_1. These details are not included for brevity.

Threshold Computation

Due to presence of input and output noise, disturbances, and uncertainties in the model, the value of the residual $r_k(i,j)$ is not exactly equal to zero under the fault free situation. Therefore, to reduce the number of false alarm flags one needs to apply threshold bands to the residual signals. In this subsection, we present an approach for determining the thresholds that are needed for achieving the FDI task. For 1D systems, there are a number of approaches for computing a threshold, e.g. based on the maximum or the root mean square (RMS) of the residual signal [128]. In this chapter, we use the maximum residual norm. However, one can also apply the RMS approach to 2-D systems.

Consider the FMII 2-D model (3.1) subject to the fault free situation. The threshold th_k is then determined from,

$$th_k = \underset{\ell < N_0}{\mathrm{Max}} \quad |r_k^\ell(i,j)| \quad , \quad \text{for } i,j \le N_1 \tag{3.44}$$

where $|\cdot|$ denotes a norm function (in this chapter we use the norm-2), N_0 is the number of the Monte Carlo simulations (refer to [13]) that are used to determine the threshold, $r_k^\ell(\cdot,\cdot)$ denotes the signal of k^{th} residual in the ℓ^{th} length and N_1 is a sufficiently large number.

By utilizing predefined thresholds that are denoted by $th_k, k = 1,2$, the FDI logic can be summarized as follows,

$$\begin{aligned} &\text{if } r_1 > th_1 \;\Rightarrow\; \text{the fault } f_1 \text{ has occurred.}\\ &\text{if } r_2 > th_2 \;\Rightarrow\; \text{the fault } f_2 \text{ has occurred.}\end{aligned} \tag{3.45}$$

Let us now consider two scenarios. In the first scenario, a single fault f_1 with the severity of 1 occurs at ($i = 5$ and $j \ge 60$). In the second scenario, multiple faults f_1 and f_2 with the severity of 1 occur at ($i = 5$ and $j \ge 50$) and ($i = 5$ and $j \ge 70$), respectively. The residual r_1 for the first scenario is shown in Figure

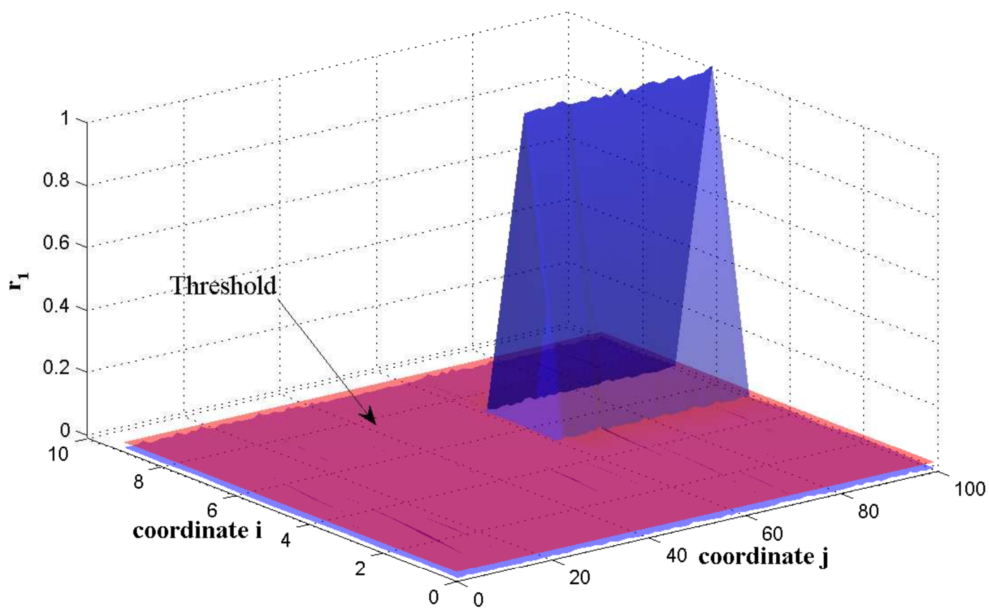


Figure 3.2: The residual signal r_1 for detecting and isolating the fault f_1.

3.2. For the second scenario, the results are shown in Figure 3.3. The thresholds are determined by conducting Monte Carlo simulations [13] corresponding to the healthy 2-D system. As shown in Figures 3.2 and 3.3, our proposed methodology can detect and isolate the faults in both single- and multiple-fault scenarios (according to the FDI logic that is given by equation (3.45)).

3.4.2 FDI of a Heat Transfer System Using Partial State Measurements

In this section, we assume that only the outer temperature (T_g) is available for measurement. This corresponds to a more practical and physically feasible scenario in various applications (sensing the inner temperature requires a more sophisticated hardware). In this case, we set $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (in other words, we measure $T_g(i-1, j)$ and $T_g(i, j)$). In this subsection, we demonstrate and illustrate the capabilities

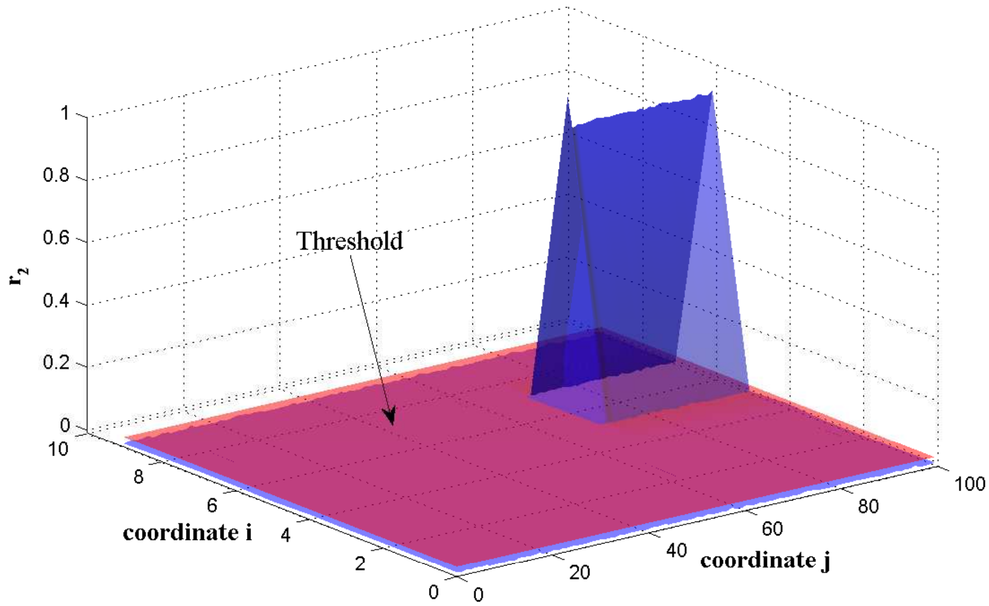


Figure 3.3: The residual signal r_2 for detecting and isolating the fault f_2.

of our proposed FDI approach based on the 2-D system modeling, whereas it is shown that by using a *1D approximation* of the PDE system (3.41) the FDI problem *cannot* be solved.

1-D Approximation of the Heat Transfer System

The hyperbolic PDE system (3.41) can be approximated by applying discretization through the z coordinates as follows. Let ℓ denote the length of the heat transfer system that is discretized into N equal intervals (i.e., $\Delta z = \frac{\ell}{N}$). By using the approximation $\frac{\partial T_g(k\Delta z,t)}{\partial z} = \frac{T_g(k\Delta z,t)-T_g((k-1)\Delta z,t)}{\Delta z}$, one can represent the PDE system (3.41) by the following 1D approximate model,

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + \sum_{k=1}^{N}(L_1^k f_1^k(t) + L_2^k f_2^k(t)), \\
y(t) &= Cx(t),
\end{aligned}
\tag{3.46}
$$

where $x(t) = \begin{bmatrix} T_g(\Delta z,t), T_f(\Delta z,t), \cdots, T_g(N\Delta z,t), T_f(N\Delta z,t) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2N}$, $f_i^k(t) = f_1(k\Delta z,t)$ $(i=1,2)$ and $B = \begin{bmatrix} 1,0,0,0,\cdots,0 \\ 0,1,0,0,\cdots,0 \end{bmatrix}^{\mathrm{T}}$. Also, $L_1^k = [\underbrace{0,\cdots,0}_{(k-1)},-1,1,0,\cdots]^{\mathrm{T}}$, and the fault signatures L_2^k are $2N$-dimensional vectors such that only the k^{th} element is 1 and the rest are zeros. Moreover,

$$A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & \\ A_2 & A_1 & 0 & \cdots & \\ 0 & A_2 & A_1 & \cdots & \\ 0 & 0 & \ddots & \ddots & \ddots \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \end{bmatrix} \tag{3.47}$$

in which $A_1 = \begin{bmatrix} -\frac{1+\Delta z}{\Delta z} & 1 \\ 1 & -\frac{1+\Delta z}{\Delta z} \end{bmatrix}$ and $A_2 = \begin{bmatrix} -\frac{1}{\Delta z} & 0 \\ 0 & -\frac{1}{\Delta z} \end{bmatrix}$. Now, consider the faults f_1^k and f_2^k. It can be shown that $\mathscr{W}_2^* =$ span$\{L_1^k, L_2^k\}$, where $\mathscr{W}_2^*$ is the minimal conditioned invariant subspace containing $\mathscr{L}_2^k$ (from the 1-D system perspective). Consequently, $\mathcal{S}_{1D}^* \cap \mathscr{L}_1^k \neq 0$ ($\mathcal{S}_{1D}^*$ denotes the unobservability subspace containing $\mathscr{L}_2^k$ in the 1D system sense). Consequently, the faults f_1^k and f_2^k **are not** isolable. However, we show below that the faults can be detected and isolated if one approximates the system (3.41) by using the 2-D model representation.

2-D Representation of the Heat Transfer System

Let us set $x_1(i,j) = T_g(i,j) + \frac{\Delta t}{\Delta z} f_1(i,j)$ and $x_2(i,j) = T_f(i,j)$, so that the system (3.41) is approximated by the system (3.42) where all the operators are defined as before except for $L_1^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Note that since only the state T_g is assumed to be available for measurement, we sense $T_g(i-1,j)$ and $T_g(i,j)$. By applying the algorithm (3.29), where $\mathscr{L} = \mathscr{L}_2^1$, one gets $\mathcal{S}_1^* =$ span$\{L_2^1, \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^{\mathrm{T}}\}$. Since, $\mathscr{L}_1^1 \cap \mathcal{S}_1^* = 0$, the fault f_1 is both detectable and isolable (according to Theorem 3.20). It should be noted that by

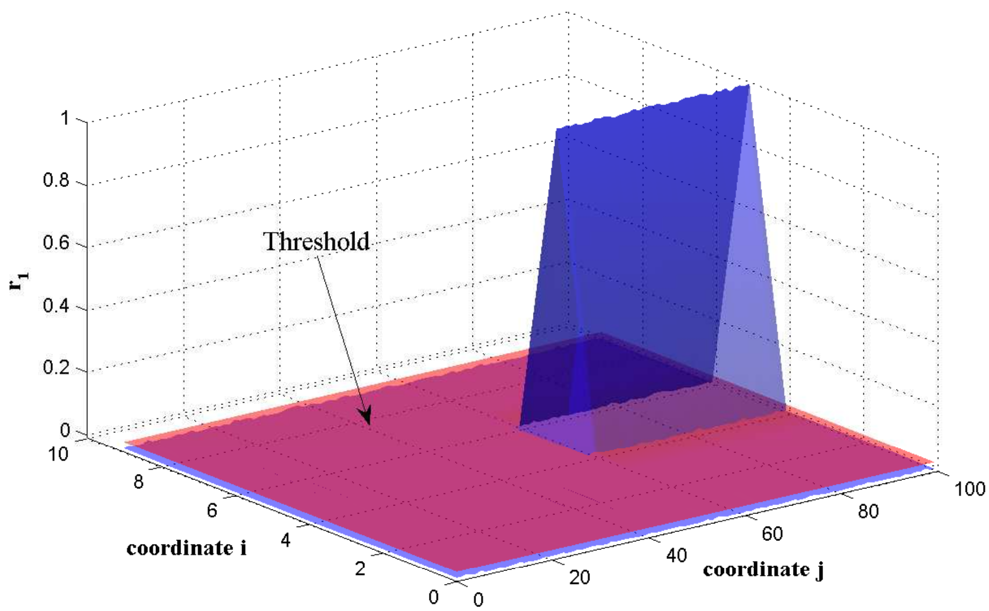


Figure 3.4: The residual signal r_1 for detecting and isolating the fault f_1.

applying the approaches in [89, 91, 99], one can also detect and isolate the fault f_1. By following along the same lines as the ones given earlier one can show that the fault f_2 is also detectable and isolable, where one can design the required detection filters. Figures 3.4 and 3.5 depict the simulation results for the scenarios that presented above. As can be observed from Figures 3.3 and 3.5, it follows that fewer available information (recall that both T_f and T_g are measurable in Figure 3.3, whereas in Figure 3.5, only T_g is measurable) results in a situation where the spatial coordinate of a fault cannot be estimated accurately.

3.5 Summary

The FDI problem for n-D systems represented by the Fornasini-Marchesini model II was investigated In this chapter. In order to derive the necessary and sufficient conditions for solvability of the FDI problem, the notion of the conditioned invariant and unobservability subspace of 1-D systems was generalized to n-D systems by

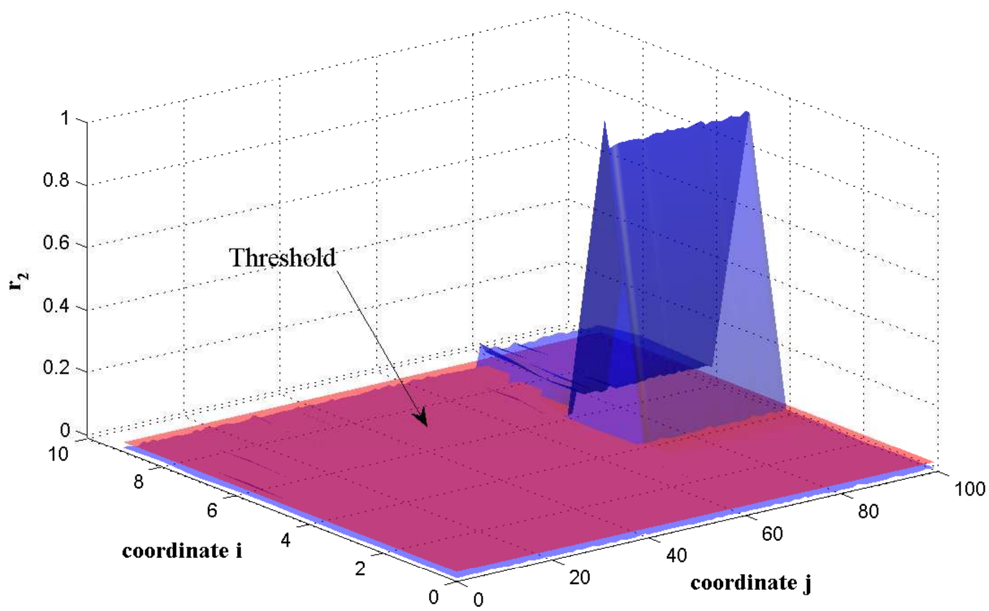


Figure 3.5: The residual signal r_2 for detecting and isolating the fault f_2.

using an Inf-D framework and representation. Moreover, algorithms for computing and constructing these subspaces are introduced and provided that converge in a finite and known number of steps. By applying the LMI approach, sufficient conditions for existence of an asymptotically convergent n-D state estimation observer is derived. Necessary and sufficient conditions for solvability of the FDI problem are also provided. It was shown that although the sufficient conditions for applicability of the currently available geometric results in the literature are also sufficient for our proposed approach to accomplish the FDI goal, however, there are n-D systems where the geometric approaches in the literature are not applicable to detect and isolate the faults, whereas our approach can still achieve the FDI objective and goal. Finally, simulation results are provided for the application of our proposed FDI methodology to a heat transfer process to demonstrate and illustrate the capabilities and advantages of our proposed solution as compared to the alternative 1-D representation and 1-D FDI approaches.

Chapter 4

Invariant Subspaces of Riesz Spectral (RS) Systems with Application to Fault Detection and Isolation

A large class of parabolic PDE systems, such as reaction-diffusion processes can be represented as a RS system in the Inf-D framework. Compared to Fin-D systems, the geometric theory of Inf-D systems to address certain fundamental control problems, such as disturbance decoupling and FDI, is quite limited due to existence of various types of invariant notions and complexity of working with them. Interestingly enough, these invariant concepts are equivalent in Fin-D systems, although they are different in Inf-D representation. In this chapter, first the equivalence of various types of invariant subspaces that are defined for RS systems is investigated. This enables one to define and specify the unobservability subspace for the RS system. Specifically, necessary and sufficient conditions are derived for equivalence of various types of conditioned invariant subspaces. Moreover, by using duality properties the controlled invariant subspaces are investigated and necessary and sufficient conditions of equivalence of various types of controlled invariant subspaces are addressed. It is shown that finite-rank output operator enables one to derive algorithms for computing the invariant subspaces that under certain conditions, and unlike methods in the literature, converge in a finite number of steps. An FDI methodology for RS systems is then developed by using a geometric approach where the FDI problem is formally investigated by invoking the introduced invariant subspaces. Finally, the necessary and sufficient conditions for solvability of the FDI problem are provided.

4.1 RS Systems

In this section, we review some of the basic concepts that are associated with a class of RS systems that will be considered in this chapter. This class of RS is mainly categorized by the state operator $\mathcal{A}$ and the output operator $\mathcal{C}$ as follows.

Consider the following Inf-D system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0, \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathcal{X}$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ denote the state, input and output vectors, respectively, and $\mathcal{X}$ is a real Inf-D separable Hilbert space equipped with the dot-product $< \cdot, \cdot >$. Moreover, we consider the following finite rank output operator

$$\mathcal{C} = \Big[< c_1, \cdot >, < c_2, \cdot >, \cdots, < c_q, \cdot > \Big]^{\mathrm{T}}, \tag{4.2}$$

where $c_i \in \mathcal{X}$.

In this chapter, we assume that the model (4.1) represents a well-posed system that is suitable from practical point of view. This implies that the solution of system (4.1) is continuous with respect to the initial conditions for all $u(t) \in \mathbb{R}^m$ [14]. This assumption is equivalent to stating that $\mathcal{A}$ is a closed infinitesimal generator of a strongly continuous (C_0) semigroup $\mathbb{T}_{\mathcal{A}}(t)$.

The solution of system (4.1) is given by $x(t) = \mathbb{T}_{\mathcal{A}}(t)x_0 + \int_0^t \mathbb{T}_{\mathcal{A}}(t-s)\mathcal{B}u(s)\mathrm{d}s$ [14], where $x_0 \in \mathcal{X}$ denotes the initial condition. The following lemma provides an important feature and property of the Riesz basis (refer to Definition 2.18).

Lemma 4.1. *[14, Lemma 2.3.2-b] Consider the Riesz basis $\{\phi_i\}_{i\in\mathbb{I}}$ of the Hilbert space $\mathcal{X}$ ($\mathbb{I} \subseteq \mathbb{N}$). Then every $z \in \mathcal{X}$ can be* uniquely *represented as $z = \sum_{i\in\mathbb{I}} < z, \psi_i > \phi_i$, where ψ_i is biorthonormal vector to ϕ_i for all $i \in \mathbb{I}$.*

By following the same steps as in Subsection 2.1, one can define the following projection operator for each eigenvalue λ_i of $\mathcal{A}$ [129], namely

$$\mathcal{P}_i : \mathcal{X} \to \mathcal{X}, \;\; \mathcal{P}_i = \frac{1}{2\pi j}\int_{\Gamma_i} (\lambda\mathcal{I} - \mathcal{A})^{-1}\mathrm{d}\lambda, \tag{4.3}$$

where $i \in \mathbb{I}_\lambda$ ($\mathbb{I}_\lambda$ is an index set for $\sigma(\mathcal{A})$), Γ_i is a simple closed curve surrounding only the eigenvalue λ_i. This represents the projection on the subspace of generalized eigenvectors of $\mathcal{A}$ corresponding to λ_i, that is, the subspace spanned by all ϕ_i satisfying $(\lambda_i\mathcal{I} - \mathcal{A})^n\phi_i = 0$, for some positive integer n.

Definition 4.2. *[129] The operator $\mathcal{A}$ is called a regular RS operator, if*

1. *All but finitely many of the eigenvalues (with finite multiplicity) are simple.*

2. *The (generalized) eigenvectors of the operator $\mathcal{A}$, $\{\phi_i\}_{i\in\mathbb{I}}$, form a Riesz basis for $\mathcal{X}$ (but defined on the field $\mathbb{C}$), and consequently, $\sum_{i\in\mathbb{I}_\lambda} \mathcal{P}_i = \mathcal{I}$ (that is an identity operator on $\mathcal{X}$).*

Remark 4.3. *As we shall see subsequently, to derive a necessary condition for solvability of the FDI problem, it is necessary that a bounded perturbation of $\mathcal{A}$ (that is, $\mathcal{A} + \mathcal{D}$ where $\mathcal{D}$ is a bounded operator) is also a regular RS operator. This property holds if $\sum_i \frac{1}{d_i^2} < \infty$, where $d_i = \inf_{\lambda\in\sigma(\mathcal{A})-\{\lambda_i\}} |\lambda - \lambda_i|$ [129, Theorem 1]. Therefore,* *in this chapter it is assumed that the operator $\mathcal{A}$ satisfies the above condition. It should be pointed out that a large class of RS systems, including the discrete RS systems satisfy this condition [109].*

Example 4.4. *Regular RS operator:*

Consider the operators $\mathcal{A}_1 : \mathcal{X}_1 \to \mathcal{X}_1$, $\mathcal{A}_2 : \mathcal{X}_2 \to \mathcal{X}_2$, $\mathcal{A}_1 = \mathrm{diag}(-1, -4, -9, \cdots, -k^2, \cdots)$ and $\mathcal{A}_2 = \mathrm{diag}(-4, -16, -36, \cdots, -(2k)^2, \cdots)$, where $\mathcal{X}_1 = \overline{\{e_i^1\}_{i\in\mathbb{N}}}$ and $\mathcal{X}_2 = \overline{\{e_{2i}^2\}_{i\in\mathbb{N}}}$ (for $k = 1, 2$, e_i^k is an Inf-D vector where all elements are zero except i^{th} element that is one). It follows that $\sigma(\mathcal{A}_1) = \{-1, -4, -9, \cdots\}$, and $\sigma(\mathcal{A}_2) = \{-4, -16, -36, \cdots\}$. Moreover, $\{e_i^1\}_{i\in\mathbb{N}}$ and $\{e_i^2\}_{i\in\mathbb{N}}$ are the eigenvectors of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. It follows that all the eigenvalues are simple and eigenvectors of $\mathcal{A}_1$ and $\mathcal{A}_2$ span $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively. Also, it follows that the condition in Remark 4.3 is satisfied (since $d_i \geq i$ for all i and $\sum_{i\in\mathbb{N}} \frac{1}{i^2} < \infty$). Therefore, the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are regular RS operators.

If the operator $\mathcal{A}$ in system (4.1) is a regular RS operator and the operator $\mathcal{B}$ is bounded and finite rank we designate the system (4.1) as a regular RS system. Moreover, the system (4.1) is well-posed if and only if $\sup\limits_{\lambda\in\sigma(\mathcal{A})} \lambda < \infty$ that is a feasible assumption from the applications point of view [2]. Also, according to Definitions 2.18 and 4.2, one can show that [109]

$$\mathcal{A} = \sum_{i\in\mathbb{I}_\lambda} \lambda_i \sum_{k=1}^{n_i} < \cdot, \psi_{i,k} > \phi_{i,k}, \tag{4.4}$$

where n_i denotes the number of (generalized) eigenvectors corresponding to the eigenvalues λ_i (if λ_i is a distinct eigenvalue then $n_i = 1$, and if λ_i is repeated we have $n_i > 1$). Also, $\phi_{i,k}$ and $\psi_{i,k}$ are the (generalized) eigenvectors and the corresponding biorthonormal vectors of λ_i, respectively.

Given that we are interested in RS systems that are defined on the field $\mathbb{R}$, we need to work with eigenspaces instead of eigenvectors (eigenvalues and eigenvectors in (4.4) can be complex). If an eigenvalue is real, the corresponding eigenspace is equal to $\mathcal{P}_i\mathcal{X}$, where $\mathcal{P}_i$ is the corresponding projection that is defined in (4.3). Let $\lambda = a + jb$ and $\overline{\lambda} = a - jb$ be a pair of complex eigenvalues of $\mathcal{A}$. Since $\mathcal{A}$ is a real operator, it is easy to show that if $\phi = v_1 + jv_2$ is a (generalized) eigenvector corresponding to λ, then $\overline{\phi} = v_1 - jv_2$ is a (generalized) eigenvector corresponding to $\overline{\lambda}$ (the conjugate of λ). The corresponding *real* eigenspace to λ and $\overline{\lambda}$ is constructed by $\text{span}\{v_1^i, v_2^i\}_{i=1}^{n}$, where $v_1^i \pm jv_2^i$ correspond to the (generalized) eigenvectors of $\mathcal{A}$, and n denotes the algebraic multiplicity of λ. We denote the real eigenspace of $\mathcal{A}$ corresponding to λ_i by $\mathscr{P}_i$. It should be pointed out that $\dim(\mathscr{P}_i) = n_i$ and $\dim(\mathscr{P}_i) = 2n_i$ for real and complex eigenvalue λ_i, respectively (where n_i is the algebraic multiplicity of λ_i). Note that Condition 3 in Definition 4.2 implies that $\overline{\sum_{i\in\mathbb{I}_\lambda} \mathscr{P}_i} = \mathcal{X}$ (defined on $\mathbb{R}$). Also, we have $\mathscr{P}_i \subseteq D(\mathcal{A})$ and $\mathcal{A}\mathscr{P}_i \subseteq \mathscr{P}_i$. Moreover, we designate the subspace $\mathscr{E}_i \subseteq \mathscr{P}_i$ as a *sub-eigenspace* if $\mathcal{A}\mathscr{E}_i \subseteq \mathscr{E}_i$.

Remark 4.5. *It is worth noting that the only proper sub-eigenspace of an eigensapce corresponding to a simple eigenvalue is* 0. *In other words, let* $\mathscr{P}$ *be an eigenspace corresponding to a simple eigenvalue* λ. *If* $\mathscr{E} \subseteq \mathscr{P}$ *and* $\mathscr{E} \neq \mathscr{P}$, *then* $\mathcal{A}\mathscr{E} \subseteq \mathscr{E}$ *implies* $\mathscr{E} = 0$.

4.2 Invariant Subspaces

Invariant subspaces play a prominent role in the geometric control theory of dynamical systems [41, 74, 101, 130]. As stated earlier, for the FDI problem one works with three invariant subspaces, namely $\mathcal{A}$-invariant, conditioned invariant, and unobservability subspaces. Also, to investigate the disturbance decoupling problem (refer to [74] for more detail), one deals with controlled invariant and controllability subspaces that are dual to conditioned invariant and unobservability subspaces, respectively [3]. In the literature, $\mathcal{A}$-invariant, conditioned and controlled invariant subspaces have been introduced for Inf-D systems [103, 105, 110, 130]. Due to the complexity of Inf-D systems, various kinds of invariant subspaces are available (although these are all equivalent in Fin-D systems). The necessary and sufficient conditions for equivalence of $\mathcal{A}$-invariant subspaces have been obtained in the literature [14]. However, for equivalence of conditioned invariant subspaces, the results that are available are only limited to sufficient conditions.

In the following subsections, we first review invariant subspaces and provide necessary and sufficient conditions for equivalence of conditioned invariant subspaces for regular RS systems. Then, by using duality properties, the necessary and sufficient conditions for equivalence of controlled invariant subspace are formally shown. Moreover, an unobservability subspace for RS systems is also introduced.

Generally, for Inf-D systems the algorithms that are developed to compute invariant subspaces require an *infinite* number of steps to converge. In this section, it is shown that the finite-rankness of the output operator enables us, for the first time in the literature, to develop an algorithm for computing the conditioned invariant subspace that converges in a *finite* number of steps.

4.2.1 $\mathcal{A}$-Invariant Subspace

There are two different definitions that are related to the $\mathcal{A}$-invariance property. Unlike Fin-D systems, these definitions are not equivalent for Inf-D systems. In this subsection, we review these definitions and investigate various types of unobservable subspaces for the RS system (4.1).

Definition 4.6. *[130]*

1. *The closed subspace* $\mathscr{V} \subseteq \mathcal{X}$ *is called* $\mathcal{A}$*-invariant if* $\mathcal{A}(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V}$.

2. *The closed subspace* $\mathscr{V} \subseteq \mathcal{X}$ *is* $\mathbb{T}_{\mathcal{A}}$*-invariant if* $\mathbb{T}_{\mathcal{A}}(t)\mathscr{V} \subseteq \mathscr{V}$ *for all* $t \in [0, \infty)$, *where* $\mathbb{T}_{\mathcal{A}}$ *denotes the* C_0 *semigroup generated by* $\mathcal{A}$.

In Fin-D systems, items 1) and 2) in the above definition are equivalent, however for Inf-D systems, item 2) is stronger than item 1). In other words, every $\mathbb{T}_{\mathcal{A}}$-invariant subspace is $\mathcal{A}$-invariant, however the reverse is not valid in general [130]. In the geometric control theory of dynamical systems, one needs subspaces that are $\mathbb{T}_{\mathcal{A}}$-invariant. Since dealing with $\mathbb{T}_{\mathcal{A}}$-invariant subspaces is more challenging than $\mathcal{A}$-invariant subspaces, we are interested in cases where they are equivalent. For a general Inf-D system, a sufficient condition to have this equivalence is $\mathscr{V} \subseteq D(\mathcal{A})$ [130], which is quite a restricted and limited condition. However, the following lemma provides necessary and sufficient conditions for $\mathbb{T}_{\mathcal{A}}$-invariance property.

Lemma 4.7. *[14, Lemma 2.5.6] Consider an infinitesimal generator $\mathcal{A}$ (more general than RS operators), and its corresponding $\mathbb{T}_{\mathcal{A}}$ operator and a closed subspace $\mathscr{V}$. Then $\mathscr{V}$ is $\mathbb{T}_{\mathcal{A}}$-invariant if and only if $\mathscr{V}$ is $(\lambda\mathcal{I}-\mathcal{A})^{-1}$-invariant, where $\lambda \in \rho_\infty(\mathcal{A})$.*

Another important result on $\mathbb{T}_{\mathcal{A}}$-invariant subspaces for a regular RS system that is provided in [101, Theorem IV.6] is given next.

Lemma 4.8. *[101] Consider the Inf-D system (4.1), where $\mathcal{A}$ is a regular spectral operator and the $\mathcal{A}$-invariant subspace is denoted by $\mathscr{V}$. Then $\mathscr{V}$ is $\mathbb{T}_{\mathcal{A}}$-invariant if and only if $\mathscr{V} = \overline{\operatorname{span}\{\mathscr{D}_i\}_{\,i\in\mathbb{I}_\lambda}}$, where $\mathbb{I}_\lambda \subseteq \mathbb{N}$ and $\mathscr{D}_i \subseteq \mathcal{P}_i\mathcal{X}$, is $\mathcal{A}$-invariant.*

Note that the above lemma is a generalization of Lemma 2.5 to Inf-D systems. As stated in the preceding section, the eigenvalues (and the corresponding eigenvectors) of $\mathcal{A}$ may be complex, and Lemmas 4.7 and 4.8 are provided for complex subspaces. However, for the geometric control approach one needs to work with real subspaces. The following corollary provides the necessary and sufficient conditions for equivalence of Definition 4.6, items 1) and 2) for the regular RS system and real subspaces.

Corollary 4.9. *Consider the regular RS system (4.1) and the $\mathcal{A}$-invariant subspace $\mathscr{V}$. The real subspace $\mathscr{V}$ is $\mathbb{T}_{\mathcal{A}}$-invariant if and only if $\mathscr{V} = \overline{\operatorname{span}\{\mathscr{E}_i\}_{\,i\in\mathbb{I}}}$ for a given* *index set* $\mathbb{I} \subseteq \mathbb{N}$*, where* $\mathscr{E}_i$ *denote the sub-eigenspaces of* $\mathcal{A}$*.*

Proof. Let $\phi^k = v_1^k + jv_2^k, k = 1, \cdots, n_i$ denote the corresponding (generalized) eigenvectors for the eigenvalue $\lambda_i = \gamma_1 + j\gamma_2$ of $\mathcal{A}$, where n_i denotes the algebraic multiplicity of λ_i, and γ_ℓ and v_ℓ^k (for $\ell = 1, 2$) are real numbers and vectors, respectively. Since $\mathcal{A}$ is a regular RS operator, it follows that the eigenspace corresponding to λ_i (and its conjugate) is equal to $\operatorname{span}\{v_1^k, v_2^k\}_{k=1}^{n_i}$.
(**If part**): Let $\mathscr{V} = \overline{\operatorname{span}\{\mathscr{E}_i\}}_{i\in\mathbb{I}}$. The corresponding complex subspace of $\mathscr{V}$ $\mathscr{V}_\mathbb{C}$ is defined as all vectors z that can be expressed as $z = \sum_{i\in\mathbb{I}} \zeta_i x_i$ for $\zeta_i \in \mathbb{C}$ and $x_i \in \mathscr{V}$. Consequently, one obtains $\mathscr{V}_\mathbb{C} = \overline{\operatorname{span}\{\mathscr{D}_i\}}_{i\in\mathbb{I}}$, where $\mathscr{D}_i$ (and its conjugate) is the corresponding complex subspace to $\mathscr{E}_i$. Consequently, $\mathscr{V}_\mathbb{C}$ is $\mathcal{A}$-invariant. By Lemma 4.8, $\mathscr{V}_\mathbb{C}$ is $\mathbb{T}_\mathcal{A}$-invariant. Hence, $\mathbb{T}_\mathcal{A}(t)(v_1 + jv_2) \in \mathscr{V}_\mathbb{C}$, for all $v_1 + jv_2 \in \mathscr{V}_\mathbb{C}$ and $t \geq 0$. Since $\mathcal{A}$ and $\mathbb{T}_\mathcal{A}$ are real, by referring to the definition of $\mathscr{V}_\mathbb{C}$ we have $v_1, v_2 \in \mathscr{V}$ and $\mathbb{T}_\mathcal{A}(t)v_1, \mathbb{T}_\mathcal{A}(t)v_2 \in \mathscr{V}$ for all $t \geq 0$. Therefore, $\mathbb{T}_\mathcal{A}(t)\mathscr{E}_i \subseteq \mathscr{E}_i$ implying that $\mathscr{V}$ is $\mathbb{T}_\mathcal{A}$-invariant.
(**Only if part**): Let $\mathscr{V}$ be $\mathbb{T}_\mathcal{A}$-invariant. The corresponding complex subspace $\mathscr{V}_\mathbb{C}$ is also $\mathbb{T}_\mathcal{A}$-invariant. Again, by using Lemma 4.8, $\mathscr{V}_\mathbb{C} = \overline{\operatorname{span}\{\phi_i\}}_{i\in\mathbb{I}}$. Therefore, $\mathscr{V} = \overline{\operatorname{span}\{\mathscr{E}_i\}}_{i\in\mathbb{I}}$. This completes the proof of the corollary. □

As stated earlier, we are mainly concerned with two important invariant subspaces of Inf-D systems as discussed below. We denote the largest $\mathcal{A}$- and $\mathbb{T}_\mathcal{A}$-invariant subspaces that are contained in $\mathscr{C}$ by $< \mathscr{C}|\mathcal{A} >$ and $< \mathscr{C}|\mathbb{T}_\mathcal{A} >$, respectively. The $\mathcal{A}$-unobservable subspace of system (4.1) is defined by $\mathscr{N}_\mathcal{A} =< \ker \mathcal{C}|\mathcal{A} >= \bigcap_{n\in\mathbb{N}} \ker \mathcal{C}\mathcal{A}^n$. Also, the unobservable subspace of system (4.1) is defined by $\mathscr{N} =< \ker \mathcal{C}|\mathbb{T}_\mathcal{A} >= \bigcap_{t\geq 0} \ker \mathcal{C}\mathbb{T}_\mathcal{A}(t)$ [103]. Note that $\mathscr{N}_\mathcal{A} \subseteq D(\mathcal{A}^n)$ for all $n \in \mathbb{N}$ and is not necessarily $\mathbb{T}_\mathcal{A}$-invariant. However, as shown subsequently, using this subspace one can develop an algorithm to compute the conditioned invariant subspaces in a *finite* number of steps. These subspaces will be used in Section 4.2.3 to introduce the unobservability subspace of RS systems, where the following corollary plays a crucial role.

Corollary 4.10. *Consider the RS system* (4.1)*, where $\mathcal{A}$ is a regular RS operator with a bounded output operator $\mathcal{C}$. The unobservable subspace $\mathscr{N}$ is the largest subspace contained in* $\ker \mathcal{C}$ *that can be expressed as* $\overline{\operatorname{span}\{\mathscr{E}_i\}_{i\in\mathbb{I}}}$*, where $\mathscr{E}_i$ are sub-eigenspaces of $\mathcal{A}$ and $\mathbb{I} \subseteq \mathbb{N}$.*

Proof. As stated above, $\mathscr{N}$ is $\mathbb{T}_{\mathcal{A}}$-invariant, and consequently by using Corollary 4.9, $\mathscr{N} = \overline{\operatorname{span}\{\mathscr{E}_i\}_{i\in\mathbb{I}}}$. Moreover, since $\mathscr{N}$ is the largest $\mathbb{T}_{\mathcal{A}}$-invariant that is contained in $\ker \mathcal{C}$ [103], the result now follows. □

4.2.2 Conditioned Invariant Subspaces

In this subsection, the conditioned invariant subspaces of system (4.1) are defined and characterized. Not surprisingly, various definitions that are all equivalent in Fin-D systems are available for conditioned invariant subspaces of Inf-D systems that are *not* equivalent [103]. This subsection mainly concentrates on deriving necessary and sufficient conditions where these definitions are shown to be equivalent. Let us first define the notion of conditioned invariant subspace.

Definition 4.11. *[103]*

1. *The closed subspace $\mathscr{W} \subseteq \mathcal{X}$ is designated as ($\mathcal{C}$,$\mathcal{A}$)-invariant if $\mathcal{A}(\mathscr{W} \cap D(\mathcal{A}) \cap \ker \mathcal{C}) \subseteq \mathscr{W}$.*

2. *The closed subspace $\mathscr{W} \subseteq \mathcal{X}$ is feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant if there exists a bounded operator $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ such that $\mathscr{W}$ is invariant with respect to $(\mathcal{A} + \mathcal{D}\mathcal{C})$, as per Definition 4.6, item 1).*

3. *The closed subspace $\mathscr{W} \subseteq \mathcal{X}$ is $\mathbb{T}$-conditioned invariant if there exists a bounded operator $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ such that (i) the operator $(\mathcal{A} + \mathcal{D}\mathcal{C})$ is the infinitesimal*

generator of a C_0-semigroup $\mathbb{T}_{\mathcal{A}+\mathcal{DC}}$; and (ii) $\mathscr{W}$ is invariant with respect to $\mathbb{T}_{\mathcal{A}+\mathcal{DC}}$, as per Definition 4.6, item 2).

It should be pointed out that in the literature $\mathbb{T}$-conditioned invariant is also called $\mathbb{T}(\mathcal{C},\mathcal{A})$-invariant [103]. It can be shown that Definition 4.11, item 3) $\Rightarrow$ item 2) $\Rightarrow$ item 1) [103]. A sufficient condition for equivalence of the above definitions is developed in [103].

Lemma 4.12. *[103] A given $(\mathcal{C},\mathcal{A})$-invariant subspace $\mathscr{W}$ is $\mathbb{T}$-conditioned invariant, if $\mathcal{C}\mathscr{W}$ is closed and $\mathscr{W} \subseteq D(\mathcal{A})$.*

In this subsection, we show that Definition 4.11, item 1) and item 2) are equivalent for the system (4.1), when the finite rank output operator is represented by (4.2) (even if $\mathscr{W} \not\subset D(\mathcal{A})$). Moreover, we derive the necessary and sufficient conditions for $\mathbb{T}$-conditioned invariance. These results enable us to subsequently derive the necessary and sufficient conditions for solvability of the FDI problem. Towards this end, we first need the following lemma.

Lemma 4.13. *Consider the closed subspace $\mathscr{V} = \overline{\operatorname{span}\{x_i\}_{i\in\mathbb{I}}}$, where $x_i \in \mathcal{X}$ (and not necessarily orthogonal) and $\mathbb{I} \subseteq \mathbb{N}$. Then*

$$\mathscr{V} = \overline{\mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}} = \mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}, \tag{4.5}$$

where $\mathscr{V}_{\text{f}} = \operatorname{span}\{x_i\}_{i\in\mathbb{J}}$, $\mathscr{V}_{\text{inf}} = \overline{\operatorname{span}\{x_i\}_{i\in\mathbb{I}-\mathbb{J}}}$ and $\mathbb{J}$ is a finite subset of $\mathbb{I}$.

Proof. It follows readily that $\operatorname{span}\{x_i\}_{i\in\mathbb{I}-\mathbb{J}} + \mathscr{V}_{\text{f}}$ is dense in $\mathscr{V}$. Hence, the subspace $\mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}$ is also dense in $\mathscr{V}$. Furthermore, since $\mathscr{V}_{\text{f}}$ is a Fin-D subspace, it is a closed subspace. Therefore, by using the Proposition 1.7.17 in [131] (which states that the sum of two closed subspaces is also closed if at least one of them is Fin-D), it follows that $\mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}$ is closed. Since, $\mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}$ is closed and dense in $\mathscr{V}$, we have $\mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}} = \mathscr{V}$. This completes the proof of the lemma. $\square$

The following lemma shows the equivalence of ($\mathcal{C}$,$\mathcal{A}$)- and feedback ($\mathcal{C}$,$\mathcal{A}$)-invariance properties for a general Inf-D system provided that the output operator is a finite rank operator (as considered to be satisfied by (4.2) in this chapter).

Lemma 4.14. *Consider the Inf-D system* (4.1)*, where* $\mathcal{A}$ *is the infinitesimal generator of a* C_0 *semigroup (more general than the regular RS systems) and the finite rank output operator is given by* (4.2)*. Let* $\mathscr{W} \subseteq \mathcal{X}$ *be a closed subspace such that* $\overline{D(\mathcal{A}) \cap \mathscr{W}} = \mathscr{W}$*. The subspace* $\mathscr{W}$ *is (*$\mathcal{C}$*,*$\mathcal{A}$*)-invariant if and only if it is feedback (*$\mathcal{C}$*,*$\mathcal{A}$*)-invariant.*

Proof. As pointed out earlier, every feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace is ($\mathcal{C}$,$\mathcal{A}$)-invariant. Therefore, we only show the converse. By definition, we have $\mathcal{A}(\mathscr{W} \cap \ker \mathcal{C} \cap D(\mathcal{A})) \subseteq \mathscr{W}$. Since $\overline{\mathscr{W} \cap D(\mathcal{A})} = \mathscr{W}$, and $\mathscr{W}$ is separable ($\mathscr{W}$ is a closed subspace of the separable Hilbert space $\mathcal{X}$), there exists a basis $\{w_i\}_{i\in\mathbb{I}}$ for $\mathscr{W}$ such that $w_i \in D(\mathcal{A})$. Let us rearrange the basis $\{w_i\}_{i\in\mathbb{I}}$ such that the first n_{f} vectors construct the Fin-D subspace $\mathscr{W}_{\mathrm{f}} = \operatorname{span}\{w_i\}_{i=1}^{n_{\mathrm{f}}} \subset D(\mathcal{A})$, where $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{C} = 0$ and $n_{\mathrm{f}} = \dim(\mathscr{W} \cap (\mathscr{W} \cap \ker \mathcal{C})^{\perp})$. From (4.2) (i.e., the finite rankness of $\mathcal{C}$) and the fact that $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{C} = 0$, it follows that $\dim(\mathscr{W}_{\mathrm{f}}) = n_{\mathrm{f}} \leq q < \infty$. Note that if $n_{\mathrm{f}} = 0$ it implies that $\mathscr{W} \subseteq \ker \mathcal{C}$ and therefore $\mathscr{W}$ is $\mathcal{A}$-invariant and by setting $\mathcal{D} = 0$ it is also feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant. Now, without loss of any generality we assume that $w_i \in \ker \mathcal{C}$ for all $i > n_{\mathrm{f}}$ (if $w_i \notin \ker \mathcal{C}$, one can remove the projection of w_i on $\mathscr{W}_{\mathrm{f}}$ and calls it as $w_i^n \in \ker \mathcal{C}$. Since $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A})$, it follows that $w_i^n \in D(\mathcal{A})$). Also, given that $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$, and by using Lemma 4.13, we obtain $\mathscr{W} = \mathscr{W}_{\mathrm{inf}} + \mathscr{W}_{\mathrm{f}}$, where $\mathscr{W}_{\mathrm{inf}} = \mathscr{W} \cap \ker \mathcal{C} = \overline{\operatorname{span}\{w_i\}}_{i>n_{\mathrm{f}}}$.

Now, we show how one can construct a bounded operator $\mathcal{D}$ such that $(\mathcal{A} + \mathcal{D}\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subset \mathscr{W}$. Let $\mathcal{A}w_i = x_i \in \mathcal{X}$, $i = 1, \cdots, n_{\mathrm{f}}$. Below, we construct $\mathcal{D}$ such that $\mathcal{D}\mathcal{C}[w_1, \cdots, w_{n_{\mathrm{f}}}] = -[x_1, \cdots, x_{n_{\mathrm{f}}}]$. Given that $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{C} = 0$, $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$, and $\mathcal{C}$ is a bounded operator, it follows that $\mathcal{C}$ is an invertible operator from $\mathscr{W}_{\mathrm{f}}$ onto

$\mathscr{Y} = \mathcal{C}\mathscr{W}_{\mathrm{f}} \subseteq \mathbb{R}^q$. In other words, $C_w = \mathcal{C}|_{\mathscr{W}_{\mathrm{f}}} : \mathscr{W}_{\mathrm{f}} \to \mathscr{Y}$ is a bijective map. Therefore, $C_w = \mathcal{C}[w_1, \cdots, w_{n_{\mathrm{f}}}]$ is a monic matrix (i.e., $\ker C_w = 0$), and consequently always there is a solution for $D_w : \mathscr{Y} \to \mathcal{X}_{\mathrm{f}}$, such that $D_w C_w = -[x_1, \cdots, x_{n_{\mathrm{f}}}]$, where $\mathcal{X}_{\mathrm{f}} = \operatorname{span}\{x_i\}_{i=1}^{n_{\mathrm{f}}}$. A solution to $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ is an extension of D_w as $\mathcal{D}y = \mathcal{Q}D_w y_1$, where $y \in \mathbb{R}^q$, $y = y_1 + y_2$, $y_1 \in \mathscr{Y}$, $y_2 \in \mathscr{Y}^{\perp}$ and $\mathcal{Q}$ is the embedding operator from $\mathcal{X}_{\mathrm{f}}$ to $\mathcal{X}$. Since $\mathscr{Y}$ is Fin-D, it follows that $\mathcal{D}$ is bounded. Now, set $x \in (\mathscr{W} \cap D(\mathcal{A}))$. Since $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A})$, one can write $x = x_{\mathrm{inf}} + x_{\mathrm{f}}$, where $x_{\mathrm{inf}} \in (\mathscr{W}_{\mathrm{inf}} \cap D(\mathcal{A}))$ and $x_{\mathrm{f}} \in \mathscr{W}_{\mathrm{f}}$. Given that $\mathscr{W}$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant, it follows that $(\mathcal{A} + \mathcal{D}\mathcal{C})x_{\mathrm{inf}} = \mathcal{A}x_{\mathrm{inf}} \in \mathscr{W}$, and by definition $\mathcal{D}$, we obtain $(\mathcal{A} + \mathcal{D}\mathcal{C})x_{\mathrm{f}} = 0$. Therefore, $(\mathcal{A} + \mathcal{D}\mathcal{C})x \in \mathscr{W}$, and consequently $\mathscr{W}$ is a feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace. This completes the proof of the lemma. □

As shown in [101] the $\mathbb{T}$-conditioned invariance and ($\mathcal{C}$,$\mathcal{A}$)-invariance are not generally equivalent. Moreover, if $\mathcal{C}$ is not a finite rank the feedback ($\mathcal{C}$,$\mathcal{A}$)-invariance and ($\mathcal{C}$,$\mathcal{A}$)-invariance are not equivalent [101,103]. However, Lemma 4.14 shows the equivalence between feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant and ($\mathcal{C}$,$\mathcal{A}$)-invariant in the sense of Definition 2, if the output operator $\mathcal{C}$ is finite rank.

The following lemma shows that the $\mathbb{T}$-conditioned invariance is an independent property from the bounded operator $\mathcal{D}$. This result allows us to derive the necessary and sufficient conditions for the $\mathbb{T}$-conditioned invariance.

Lemma 4.15. *Consider a $\mathbb{T}$-conditioned invariant subspace $\mathscr{W}$ such that $\mathbb{T}_{\mathcal{A}+\mathcal{D}_1\mathcal{C}}\mathscr{W} \subseteq \mathscr{W}$, and consider a bounded operator $\mathcal{D}_2$ such that $(\mathcal{A} + \mathcal{D}_2\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$. Then $\mathbb{T}_{\mathcal{A}+\mathcal{D}_2\mathcal{C}}\mathscr{W} \subseteq \mathscr{W}$.*

Proof. By invoking Lemma 4.7, we have $(\lambda\mathcal{I} - (\mathcal{A} + \mathcal{D}_1\mathcal{C}))^{-1}\mathscr{W} \subseteq \mathscr{W}$, for all $\lambda \in \rho_\infty(\mathcal{A} + \mathcal{D}_1\mathcal{C})$. Let us set $\lambda \in \rho_\infty(\mathcal{A} + \mathcal{D}_1\mathcal{C}) \cap \rho_\infty(\mathcal{A} + \mathcal{D}_2\mathcal{C})$ (by using the Hille-Yosida theorem ([14, Theorem 2.1.12], where it is shown that for every infinitesimal generator $\mathcal{A}$ there exists a real number $r \in \mathbb{R}$ such that $[r, \infty) \subset \rho_\infty(\mathcal{A})$ we have the set $\rho_\infty(\mathcal{A}+\mathcal{D}_1\mathcal{C}) \cap \rho_\infty(\mathcal{A}+\mathcal{D}_2\mathcal{C})$ non-empty). Based on Lemma 4.7, we need to show that $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}\mathscr{W} \subseteq \mathscr{W}$. First, let $\mathscr{W}_c = \{y|y \in \mathscr{W}\ ;\ (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}y \in \mathscr{W}_\mathrm{f}\}$, where $\mathscr{W}_\mathrm{f} \subset D(\mathcal{A})$ is defined as in the proof of Lemma 4.14 and $\mathscr{W}_\infty = \{y|y \in \mathscr{W}\ ;\ (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}y \in \mathscr{W} \cap \ker\mathcal{C}\}$. Since $\mathscr{W} = \mathscr{W}_\mathrm{f} + \mathscr{W} \cap \ker\mathcal{C}$, $\mathscr{W}$ is $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}$-invariant and $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}$ is bounded and bijective, it follows that $\mathscr{W} = \mathscr{W}_c + \mathscr{W}_\infty$. Let $y \in \mathscr{W}_\infty$ and $x = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}y$. Given that $x \in \ker\mathcal{C}$, it follows

$$y = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))x = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))x = (\lambda\mathcal{I}-\mathcal{A})x. \tag{4.6}$$

Since $\mathscr{W}$ is $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}$-invariant, one obtains $x \in \mathscr{W}$, and consequently we have $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}y = x \in \mathscr{W}$.

Next, by following the steps, below, we show that if $y \in \mathscr{W}_c$ then $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}y \in \mathscr{W}$.

1. Let $\{w_i\}_{i=1}^{n_\mathrm{f}}$ be a basis of $\mathscr{W}_\mathrm{f}$ and set $z_i = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))w_i \in \mathscr{W}$ for $i = 1, \cdots, n_\mathrm{f}$ (as $(\mathcal{A}+\mathcal{D}_2\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$). Since $\mathscr{W} = \mathscr{W}_c + \mathscr{W}_\infty$ one can write $z_i = z_c^i + z_\infty^i$, where $z_c^i \in \mathscr{W}_c$ and $z_\infty^i \in \mathscr{W}_\infty$.

2. We show that z_c^i are linearly independent. Towards this end, assume z_c^i are linearly dependent and therefore we obtain $\sum_{i=1}^{n_\mathrm{f}} \zeta_i z_c^i = 0$, where $\zeta_i \in \mathbb{R}$ for $i = 1, \cdots, n_\mathrm{f}$. Hence, one can write $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))w = z_\infty$, where $w = \sum_{i=1}^{n_\mathrm{f}} \zeta_i w_i \neq 0$ (since w_i are basis), and $z_\infty = \sum_{i=1}^{n_\mathrm{f}} \zeta_i z_\infty^i \in \mathscr{W}_\infty$. Consequently, since $w = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}z_\infty$, and since $z_\infty \in \mathscr{W}_\infty$, we obtain $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1}z_\infty \in \ker\mathcal{C}$, and consequently $w = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_1\mathcal{C}))^{-1} = (\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}z_\infty \in \ker\mathcal{C}$. Hence, we have $w \in \ker\mathcal{C}$ that is in contradiction with the fact $w \in \mathscr{W}_\mathrm{f}$ (recall $\mathscr{W}_\mathrm{f} \cap \ker\mathcal{C} = 0$). Therefore, z_c^i are linearly independent. Since the resolvent operators are bijective and $\mathscr{W}_c$ is Fin-D, we obtain $\dim(\mathscr{W}_c) = \dim(\mathscr{W}_\mathrm{f}) = n_\mathrm{f}$, and consequently $\{z_c^i\}_{i=1}^{n_\mathrm{f}}$ is a basis of $\mathscr{W}_c$.

3. We show that $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}z_c^i\in\mathscr{W}$. Set $w_\infty^i=(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}z_\infty^i$, z_∞^i are defined as above. As shown above in (4.6), we have $w_\infty^i\in\mathscr{W}$. Since $w_i\in\mathscr{W}_{\mathrm{f}}\subseteq\mathscr{W}$ it follows that $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}z_c^i=w_i-w_\infty^i\in\mathscr{W}$. Given that $\operatorname{span}\{z_c^i\}_{i=1}^{n_{\mathrm{f}}}$ is a basis of $\mathscr{W}_c$, we obtain $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}\mathscr{W}_c\subseteq\mathscr{W}$.

Finally, for every $y\in\mathscr{W}$ one can write $y=y_c+y_\infty$, where $y_c\in\mathscr{W}_c$ and $y_\infty\in\mathscr{W}_\infty$. As we have shown above $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}\mathscr{W}_\infty\subseteq\mathscr{W}$ and $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}\mathscr{W}_c\subseteq\mathscr{W}$. Therefore, $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}y\in\mathscr{W}$, and consequently $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_2\mathcal{C}))^{-1}\mathscr{W}\subseteq\mathscr{W}$. This completes the proof of the lemma. □

A bounded operator $\mathcal{D}$ is called a *friend* of the $\mathbb{T}$-conditioned subspace $\mathscr{W}$ if $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}\mathscr{W}\subseteq\mathscr{W}$. The set of all friend operators of $\mathscr{W}$ is denoted by $\underline{\mathcal{D}}(\mathscr{W})$. Let $\mathcal{D}\in\underline{\mathcal{D}}(\mathscr{W})$ and consider a bounded operator $\mathcal{D}_0$. As in Fin-D systems [3, page 31], it follows (by using the above lemma) that a sufficient condition for $\mathcal{D}_0$ to be a friend of $\mathscr{W}$ is $(\mathcal{D}-\mathcal{D}_0)\mathcal{C}\mathscr{W}\subseteq\mathscr{W}$.

Remark 4.16. *It worth noting that in the proofs of Lemmas 4.14 and 4.15, it is not necessary $\mathcal{A}$ to be regular RS operator. Indeed, we showed the results for every infinitesimal generator operator. Therefore, we also use these lemmas in the next chapter.*

Below, we provide the main results of this subsection leading to the necessary and sufficient conditions for the $\mathbb{T}$-conditioned invariance of regular RS systems.

Theorem 4.17. *Consider the regular RS system* (4.1) *such that the operator $\mathcal{C}$ is defined according to* (4.2). *The ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace $\mathscr{W}$ is an $\mathbb{T}$-conditioned invariant subspace if*

$$\mathscr{W}=\mathscr{W}_\phi+\mathscr{W}_{\mathrm{f}}, \tag{4.7}$$

and $\overline{D(\mathcal{A})\cap\mathscr{W}}=\mathscr{W}$, where $\dim(\mathscr{W}_{\mathrm{f}})<\infty$ and $\mathscr{W}_\phi$ is the largest subspace contained in $\mathscr{W}$ that can be expressed as

$$\mathscr{W}_\phi=\overline{\operatorname{span}\{\mathscr{E}_i\}_{i\in\mathbb{I}}}, \tag{4.8}$$

in which $\mathscr{E}_i$ is the sub-eigenspace of $\mathcal{A}$ and $\mathbb{I} \subseteq \mathbb{N}$. Moreover, if $\mathscr{W} \cap (\mathscr{W} \cap \ker \mathcal{C})^\perp$ is a sum of sub-eigenspaces of $\mathcal{A} + \mathcal{D}\mathcal{C}$, then the condition (4.7) *is also necessary.*

Proof. (**If part**): Let $\mathscr{W} = \mathscr{W}_\phi + \mathscr{W}_\mathrm{f}$. We show that $\mathscr{W}$ can be spanned by eigenspaces of $\mathcal{A} + \mathcal{D}\mathcal{C}$, for a bounded $\mathcal{D}$ (and therefore by Corollary 4.9, $\mathscr{W}$ is $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant). By using Lemma 4.15 we need to show the result for only one $\mathcal{D} \in \underline{\mathcal{D}}(\mathscr{W})$. Without loss of any generality, we assume $\mathscr{W}_\phi \cap \mathscr{W}_\mathrm{f} = 0$ (if $\mathscr{W}_1 = \mathscr{W}_\phi \cap \mathscr{W}_\mathrm{f} \neq 0$, redefine $\mathscr{W}_\mathrm{f} = \mathscr{W}_\mathrm{f}/\mathscr{W}_1$).

First, we show that one can assume $\mathscr{W}_\mathrm{f} \subset D(\mathcal{A})$ without loss of any generality. Since $\mathscr{W}_\phi$ is $\mathbb{T}_\mathcal{A}$-invariant, it follows that $\overline{\mathscr{W}_\phi \cap D(\mathcal{A})} = \mathscr{W}_\phi$ [101]. Also, as an assumption we have $\overline{\mathscr{W} \cap D(\mathcal{A})} = \mathscr{W}$. If $\mathscr{W}_\phi$ is Fin-D, $\mathscr{W}$ is Fin-D and hence $\mathscr{W}_\mathrm{f} \subseteq \mathscr{W} \subset D(\mathcal{A})$. Let, $\mathscr{W}_\phi$ be Inf-D. By following along the same steps in the proof of Lemma 4.14, we define the basis $\{w_i\}_{i=1}^\infty$ of $\mathscr{W}$ such that $w_i \in D(\mathcal{A})$ for all $i \in \mathbb{N}$ and $\{w_i\}_{i=n_\mathrm{f}+1}^\infty$ is a basis for $\mathscr{W}_\phi$, where $n_\mathrm{f} = \dim(\mathscr{W}_\mathrm{f})$ (since $\overline{\mathscr{W}_\phi \cap D(\mathcal{A})} = \mathscr{W}_\phi$ the existence of the basis $\{w_i\}_{i=n_\mathrm{f}+1}^\infty$ with $w \in D(\mathcal{A})$ is guaranteed). Set $\mathscr{W}_\mathrm{ff} = \mathrm{span}\{w_i\}_{i=1}^{n_\mathrm{f}} \subset D(\mathcal{A})$, and it follows that $\mathscr{W} = \mathscr{W}_\phi + \mathscr{W}_\mathrm{ff}$. Therefore, without loss of any generality, we assume $\mathscr{W}_\mathrm{f} = \mathscr{W}_\mathrm{ff} \subset D(\mathcal{A})$.

Next, to show the result we construct the bounded operator $\mathcal{D}$ such that (i) $(\mathcal{A} + \mathcal{D}\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$, and (ii) $\mathcal{D}\mathcal{C}\mathscr{W}_\phi = 0$. Define $\mathscr{W}_\mathrm{fpc} = \mathscr{W}_\mathrm{f} \cap (\mathscr{W}_\mathrm{f} \cap \ker \mathcal{C})^\perp$ and $\mathscr{W}_\mathrm{fc} = \{w | w \in \mathscr{W}_\mathrm{fpc}\ ,\ \mathcal{C}w \neq \mathcal{C}w_\phi, \forall w_\phi \in \mathscr{W}_\phi\}$. In other words, $\mathscr{W}_\mathrm{fc}$ is the largest subspace in $\mathscr{W}_\mathrm{fpc}$ such that $\mathscr{W}_\mathrm{fc} \cap \ker \mathcal{C} = 0$ and $\mathcal{C}\mathscr{W}_\mathrm{fc} \cap \mathcal{C}\mathscr{W}_\phi = 0$. Moreover, by definition of $\mathscr{W}_\mathrm{fpc}$, we obtain $\ker \mathcal{C} + \mathscr{W}_\mathrm{f}/\mathscr{W}_\mathrm{fc} = \ker \mathcal{C} + \mathscr{W}_\mathrm{fpc}/\mathscr{W}_\mathrm{fc}$. Since $\mathscr{W}_\mathrm{f} \subset D(\mathcal{A})$, we have $\mathscr{W}_\mathrm{fc} \subset D(\mathcal{A})$. Now, consider the operator H_f such that $\ker H_\mathrm{f}\mathcal{C} = \ker \mathcal{C} + \mathscr{W}_\phi + \mathscr{W}_\mathrm{f}/\mathscr{W}_\mathrm{fc} = \ker \mathcal{C} + \mathscr{W}_\phi + \mathscr{W}_\mathrm{fpc}/\mathscr{W}_\mathrm{fc}$ and define $\mathcal{C}_1 = H_\mathrm{f}\mathcal{C}$ (since $\ker \mathcal{C} \subseteq \ker \mathcal{C}_1$, there always exists a solution for H_f). First, we show that $\mathscr{W}$ is also $(\mathcal{C}_1,\mathcal{A})$-invariant subspace in two steps as follows.

1. Let $w \in \mathscr{W}_\mathrm{fpc}/\mathscr{W}_\mathrm{fc}$. We prove that $\mathcal{A}w \in \mathscr{W}$ (if $\mathscr{W}_\mathrm{fpc} = \mathscr{W}_\mathrm{fc}$, we have $w = 0$ and we skip this step). Since $\mathscr{W}_\mathrm{fpc} \subset \mathscr{W}_\mathrm{f}$, $\mathscr{W}_\mathrm{fc} \subset \mathscr{W}_\mathrm{f}$ and $\mathscr{W}_\mathrm{f} \subset D(\mathcal{A})$, it

follows that $w \in D(\mathcal{A})$. By definition of $\mathscr{W}_{\text{fc}}$, there exists a $w_\phi \in \mathscr{W}_\phi$ such that $\mathcal{C}w = \mathcal{C}w_\phi \neq 0$. Next, we show that $w_\phi \in D(\mathcal{A})$. Let $\mathscr{W}_\phi^{\text{p}} \subset \mathscr{W}_\phi$ be the subspace such that $\mathcal{C}\mathscr{W}_\phi^{\text{p}} = \mathcal{C}(\mathscr{W}_{\text{fpc}}/\mathscr{W}_{\text{fc}})$ and $\dim(\mathscr{W}_\phi^{\text{p}}) = \dim(\mathscr{W}_{\text{fpc}}/\mathscr{W}_{\text{fc}})$. Also, let $\{w_\phi^i\}_{i=1}^\infty$ be a basis of $\mathscr{W}_\phi$ such that $w_\phi^i \in D(\mathcal{A})$ (since $\overline{\mathscr{W}_\phi \cap D(\mathcal{A})} = \mathscr{W}_\phi$, this basis exists). By following along the same steps in Lemma 4.14, we can assume w_ϕ^i be such that $w_\phi^i \in \mathscr{W}_\phi^p$ for all $i \leq n_\phi$ and $w_\phi^i \in \mathscr{W}_\phi/\mathscr{W}_\phi^p$ for $i > n_\phi$. Therefore, since $\mathcal{C}$ on $\mathscr{W}_{\text{fpc}}/\mathscr{W}_{\text{fc}}$ is bijective, one can find $w_\phi \in \operatorname{span}\{w_\phi^i\}_{i=1}^{n_\phi}$ such that $\mathcal{C}w = \mathcal{C}w_\phi$ and since $w_\phi^i \in D(\mathcal{A})$, it follows that $w_\phi \in D(\mathcal{A})$. Now, set $w_c = (w - w_\phi) \in \mathscr{W} \cap \ker \mathcal{C} \cap D(\mathcal{A})$. Since $\mathcal{A}w_\phi \in \mathscr{W}$ (recall $\mathscr{W}_\phi$ is $\mathcal{A}$-invariant), and $\mathcal{A}(\mathscr{W} \cap \ker \mathcal{C} \cap D(\mathcal{A})) \subseteq \mathscr{W}$, it follows that $\mathcal{A}w \in \mathscr{W}$.

2. By considering the subspace $\mathscr{W}_\phi^{\text{p}}$, we decompose $\mathscr{W}_\phi$ as $\mathscr{W}_\phi = \mathscr{W}_\phi^{\text{p}} + \mathscr{W}_\phi^{\text{c}} + \mathscr{W}_\phi \cap \ker \mathcal{C}$, where $\mathscr{W}_\phi^{\text{c}} \cap \mathscr{W}_\phi^{\text{p}} = 0$ and $\mathscr{W}_\phi^{\text{c}} \cap \ker \mathcal{C} = 0$. As above, we can assume $\mathscr{W}_\phi^c \subset D(\mathcal{A})$ (i.e. there exists a subspace $\mathscr{W}_\phi^c \subset D(\mathcal{A})$ that satisfies the above conditions). By definition of H_{f}, it follows that $\ker H_{\text{f}}\mathcal{C} = \ker \mathcal{C} + (\mathscr{W}_{\text{fpc}}/\mathscr{W}_{\text{fc}}) + \mathscr{W}_\phi^{\text{p}} + \mathscr{W}_\phi^{\text{c}}$. Let $w \in (\mathscr{W} \cap \ker \mathcal{C}_1 \cap D(\mathcal{A}))$. It follows that $w = w_{\text{p}} + w_\phi + w_\infty$, where $w_{\text{p}} \in \mathscr{W}_{\text{fpc}}/\mathscr{W}_{\text{fc}} \subset D(\mathcal{A})$, $w_\phi \in (\mathscr{W}_\phi^{\text{p}} + \mathscr{W}_\phi^{\text{c}}) \subset D(\mathcal{A})$ and $w_\infty \in \mathscr{W} \cap \ker \mathcal{C}$. Since $w, w_{\text{p}}, w_\phi \in D(\mathcal{A})$, it follows that $w_\infty \in D(\mathcal{A})$. As shown above, $\mathcal{A}w_p \in \mathscr{W}$, $\mathcal{A}w_\phi \in \mathscr{W}_\phi \subseteq \mathscr{W}$ (since $\mathscr{W}_\phi$ is $\mathcal{A}$-invariant) and also $\mathcal{A}w_\infty \in \mathscr{W}$ (recall that $\mathscr{W}$ is $(\mathcal{C},\mathcal{A})$-invariant). Therefore, $\mathcal{A}w \in \mathscr{W}$ and consequently $\mathcal{A}(\mathscr{W} \cap \ker \mathcal{C}_1 \cap D(\mathcal{A})) \subseteq \mathscr{W}$.

Second, by following along the same steps in Lemma 4.14, we construct $\mathcal{D}_{\text{f}}$ such that $(\mathcal{A} + \mathcal{D}_{\text{f}}\mathcal{C}_1)(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$. By setting $\mathcal{D} = \mathcal{D}_{\text{f}}H_{\text{f}}$, one can write $(\mathcal{A} + \mathcal{D}\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$.

It should be pointed out that since $\mathscr{W}_\phi \subseteq \ker H_{\text{f}}\mathcal{C}$ (refer to definition of H_{f}), we obtain $\mathscr{W}_\phi \subseteq \ker \mathcal{C}_1$, and therefore, we have $\mathcal{D}\mathcal{C}\mathscr{W}_\phi = \mathcal{D}_{\text{f}}\mathcal{C}_1\mathscr{W}_\phi = 0$. Consequently, it follows that every sub-eigenspace $\mathscr{E}_i \subset \mathscr{W}_\phi$ is also the sub-eigenspaces of the operator $\mathcal{A} + \mathcal{D}\mathcal{C}$. Therefore, $(\lambda\mathcal{I} - (\mathcal{A} + \mathcal{D}\mathcal{C}))^{-1}\mathscr{W}_\phi \subseteq \mathscr{W}_\phi$. Moreover, recall that

$\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A})$ and also the operator $\mathcal{D}_{\mathrm{f}}$ is defined such that $(\mathcal{A}+\mathcal{D}_{\mathrm{f}}\mathcal{C}_1)\mathscr{W}_{\mathrm{f}} \subseteq \mathscr{W}_{\mathrm{f}}$ (refer to the proof of Lemma 4.14). Therefore, by using Lemmas 4.7 and 4.12, we obtain $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}_{\mathrm{f}}\mathcal{C}_1))^{-1}\mathscr{W}_{\mathrm{f}} \subseteq \mathscr{W}_{\mathrm{f}}$, and consequently $(\lambda\mathcal{I}-(\mathcal{A}+\mathcal{D}\mathcal{C}))^{-1}\mathscr{W}_{\mathrm{f}} \subseteq \mathscr{W}_{\mathrm{f}}$. Finally, by using Lemma 4.7 and Corollary 4.9, it follows that $\mathscr{W}_{\mathrm{f}}$ is also a summation of sub-eigenspaces of $(\mathcal{A}+\mathcal{D}\mathcal{C})$. Therefore, $\mathscr{W}$ is spanned by the sub-eigenspaces of $(\mathcal{A}+\mathcal{D}\mathcal{C})$, and again by using Corollary 4.9, $\mathscr{W}$ is $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant, that is $\mathbb{T}$-conditioned invariant.

(**Only if part**): Consider $\mathscr{W}$ is $\mathbb{T}$-conditioned invariant. By Definition 4.11, item 3), there exists a bounded operator $\mathcal{D}$ such that $\mathscr{W}$ is $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant (and also $(\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant) and $\mathscr{W} = \overline{\mathrm{span}\{\mathscr{E}_i^D\}_{\ i\in\mathbb{I}_D}}$, where $\mathscr{E}_i^D$ are sub-eigenspaces of $(\mathcal{A}+\mathcal{D}\mathcal{C})$. As in the first part of the proof, first we construct a bounded operator $\mathcal{D}$ such that (i) $(\mathcal{A}+\mathcal{D}\mathcal{C})(\mathscr{W}\cap D(\mathcal{A})) \subseteq \mathscr{W}$, and (ii) $\mathcal{D}\mathcal{C}\mathscr{W}_{\phi} = 0$, where $\mathscr{W}_{\phi}$ is the largest $\mathbb{T}_{\mathcal{A}}$-invariant contained in $\mathscr{W}$. Consequently, we have $\mathscr{W} = \overline{\mathscr{W}_{\phi}+\mathscr{W}_{\mathrm{f}}}$, and we then show that $\mathscr{W}_{\mathrm{f}}$ is Fin-D.

Let $\mathcal{D}$ be a bounded operator such that $\mathcal{D} = \mathcal{D}_{\mathrm{f}}H_{\mathrm{f}}$, where $\mathscr{W}_{\phi}$ is the largest $\mathbb{T}_{\mathcal{A}}$-invariant contained in $\mathscr{W}$ (that is expressed as (4.8)) and $\ker H_{\mathrm{f}}\mathcal{C} = \mathscr{W}_{\phi}$. Moreover, $\mathcal{D}_{\mathrm{f}}$ is defined by following along the same lines as in the proof of Lemma 4.14. By using the fact that $\mathcal{D}\mathcal{C}\mathscr{W}_{\phi} = 0$, it follows that $\mathscr{W}_{\phi} = \overline{\mathrm{span}\{\mathscr{E}_j\}_{\ j\in\mathbb{I}}}$, where $\mathbb{I}$ is an index set such that for each $j \in \mathbb{I}$ there exists an $i \in \mathbb{I}_D$ (recall $\mathscr{W} = \overline{\mathrm{span}\{\mathscr{E}_i^D\}_{\ i\in\mathbb{I}_D}}$) such that $\mathscr{E}_j = \mathscr{E}_i^D \subseteq (\mathscr{W} \cap \ker H_{\mathrm{f}}\mathcal{C})$.

Now, set $\mathscr{W} = \overline{\mathscr{W}_{\phi}+\mathscr{W}_{\mathrm{f}}}$, where $\mathscr{W}_{\mathrm{f}} \cap \mathscr{W}_{\phi} = 0$. We show that $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$ by contradiction. Since $\mathscr{W}$ and $\mathscr{W}_{\phi}$ are the summations of sub-eigenspace of $(\mathcal{A}+\mathcal{D}\mathcal{C})$, it follows that $\mathscr{W}_{\mathrm{f}}$ does so. Assume $\dim(\mathscr{W}_{\mathrm{f}}) = \infty$ and consider the subspace $\mathscr{W}_{\mathrm{fc}} \subset (\mathscr{W}_{\mathrm{f}} \cap (\mathscr{W}_{\mathrm{f}} \cap \ker H_{\mathrm{f}}\mathcal{C})^{\perp})$. By the assumption stated in the theorem $\mathscr{W}_{\mathrm{fc}} \subset D(\mathcal{A})$. Also, $\mathscr{W}_{\mathrm{f}} = \mathscr{W}_{\mathrm{fc}} + \mathscr{W}_{\mathrm{f}} \cap \ker H_{\mathrm{f}}\mathcal{C}$ (as above since $H_{\mathrm{f}}\mathcal{C}$ is finite rank, by following along the same lines in the proof of Lemma 4.14, existence of this subspace is guaranteed). Again, by the above assumption, $\mathscr{W}_{\mathrm{fc}}$ is a sub-eigenspaces of $(\mathcal{A}+\mathcal{D}\mathcal{C})$. Since $\mathscr{W}_{\rm f}$ is a summation of sub-eigenspaces of $\mathcal{A}+\mathcal{D}\mathcal{C}$, we obtain $\mathscr{W}_{\rm f}\cap\ker H_{\rm f}\mathcal{C} = \overline{\text{span}\{\mathscr{E}_i^D\}_{\ i\in\mathbb{I}_{\rm f}}} + \mathscr{W}_{\rm ff}$, where $\mathbb{I}_{\rm f}\subseteq\mathbb{I}_D$, and $\mathscr{W}_{\rm ff}+\mathscr{W}_{\rm fc}$ is also a sub-eigenspace of $(\mathcal{A}+\mathcal{D}\mathcal{C})$ (note that it is possible to have $\mathscr{W}_{\rm ff}=0$). Since $\mathcal{A}+\mathcal{D}\mathcal{C}$ is a regular RS operator (refer to Remarks 4.3 and 4.5), it is necessary $\dim(\mathscr{W}_{\rm ff})<\infty$. Hence, since $\mathscr{W}_{\rm f}$ is Inf-D, we obtain $\mathbb{I}_{\rm f}\neq\emptyset$. However, this is in contradiction with the definition of $\mathscr{W}_\phi$ (that is the largest subspace in the form (4.8)), and consequently $\mathscr{W}_{\rm f}$ is a Fin-D subspace, and $\mathscr{W}=\mathscr{W}_\phi+\mathscr{W}_{\rm f}$ (refer to Lemma 4.13). This completes the proof of the theorem. □

Remark 4.18. *Theorem 4.17 shows that every $\mathbb{T}$-conditioned invariant subspace is constructed by sum of the subspace $\mathscr{W}_\phi$, that is $\mathbb{T}_{\mathcal{A}}$-invariant (and possibly Inf-D), and the Fin-D subspace $\mathscr{W}_{\rm f}$ such that $\mathscr{W}_{\rm f}\subseteq D(\mathcal{A})$ and $\mathscr{W}_{\rm f}\cap\mathscr{W}_\phi=0$. Given that $\mathscr{W}$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant and $\mathscr{W}_\phi$ is $\mathcal{A}$ invariant, it follows that $\mathscr{W}_{\rm f}$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant. Hence, by using Lemma 4.12, $\mathscr{W}_{\rm f}$ is $\mathbb{T}$-conditioned invariant.*

For design of our subsequent FDI scheme, we need to obtain the smallest $\mathbb{T}$-condition invariant subspace (in the inclusion sense) containing a given subspace. The following lemma allows us to show that this smallest subspace always exists.

Lemma 4.19. *The set of $\mathbb{T}$-conditioned invariant subspaces containing a given Fin-D subspace $\mathscr{L}$ and satisfying the conditions of Theorem 4.17 is closed with respect to the intersection.*

Proof. Consider the $\mathbb{T}$-conditioned invariant subspaces $\mathscr{W}_1$ and $\mathscr{W}_2$ containing $\mathscr{L}$. Hence, $\mathcal{A}(\mathscr{W}_1\cap\ker\mathcal{C}\cap D(\mathcal{A}))\subseteq\mathscr{W}_1$ and $\mathcal{A}(\mathscr{W}_2\cap\ker\mathcal{C}\cap D(\mathcal{A}))\subseteq\mathscr{W}_2$, and consequently $\mathcal{A}(\mathscr{W}_1\cap\mathscr{W}_2\cap\ker\mathcal{C}\cap D(\mathcal{A}))\subseteq\mathscr{W}_1\cap\mathscr{W}_2$. Also, given that $\mathscr{W}_1$ and $\mathscr{W}_2$ are closed, so does the subspace $\mathscr{W}_1\cap\mathscr{W}_2$. Therefore, $\mathscr{W}_1\cap\mathscr{W}_2$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant. Moreover $\mathscr{W}_1\cap\mathscr{W}_2\cap D(\mathcal{A})$ is dense in $\mathscr{W}_1\cap\mathscr{W}_2$. Consequently, $\mathscr{W}_1\cap\mathscr{W}_2$ is feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant (refer to Lemma 4.14).

By invoking Theorem 4.17, let $\mathscr{W}_1=\mathscr{W}_{\phi_1}+\mathscr{W}_{{\rm f}_1}$, $\mathscr{W}_2=\mathscr{W}_{\phi_2}+\mathscr{W}_{{\rm f}_2}$ with $\mathscr{W}_{\phi_k}=\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_k}}$, $k=1,2$, where we have $\mathscr{W}_k=\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_k}}+\mathscr{W}_{{\rm f}_k}$, for $k=1,2$ ($\mathscr{W}_{{\rm f}_k}\subset$

$D(\mathcal{A})$ denote two Fin-D subspaces-refer to Remark 4.18). Now, we show that $\mathscr{W}_1 \cap \mathscr{W}_2$ can be represented by $\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_3}} + \mathscr{W}_{\text{f}_3}$. Let $x \in \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_1}} \cap \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_2}}$. Therefore, x can be expressed as

$$x = \sum_i \zeta_i^1 \phi_i^1 = \sum_i \zeta_i^2 \phi_i^2, \tag{4.9}$$

where ϕ_i^1 and ϕ_i^2 are the generalized eigenvectors that span the subspaces $\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_1}}$ and $\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_2}}$, respectively. Since $\mathcal{A}$ is a regular RS operator (i.e. only finitely many eigenvalues are multiple), therefore all but finitely many of eigenspaces and corresponding sub-eigenspace are equivalent. In other words, there are finitely many (generalized) eigenvector corresponding to the same eigenvalue, and there are infinite eigenvectors for distinct eigenvalues (Also, refer to Remark 4.5). By using Lemma 4.1 (i.e., a unique representation of x), the fact that the (generalized) eigenvectors are independent, it follows that $\overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_1}} \cap \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_2}} = \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_3}} + \mathscr{W}_{\text{f}_3}$, where $\mathscr{W}_{\text{f}_3} \subset D(\mathcal{A})$ (since $\mathscr{E}_i \subset D(\mathcal{A})$) is a Fin-D subspace. Finally, given that $\mathscr{W}_{\text{f}_1} \subset D(\mathcal{A})$ and $\mathscr{W}_{\text{f}_2} \subset D(\mathcal{A})$ are Fin-D subspaces, by contradiction we show that $\mathscr{W}_1 \cap \mathscr{W}_2 = \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_3}} + \mathscr{W}_{\text{f}_4}$, where $\mathscr{W}_{f_4} \subset D(\mathcal{A})$ is a Fin-D subspace. Let $\mathscr{W}_1 \cap \mathscr{W}_2 = \overline{\mathscr{V} + \mathscr{V}_{\text{f}}}$, where $\mathscr{V}_{\text{f}}$ is Inf-D and $\mathscr{V} \cap \mathscr{V}_{\text{f}} = 0$. Also, consider $\overline{\{v_i\}_{i=1}^{\infty}}$ be a basis for $\mathscr{V}_{\text{f}}$. It follows, we can express every v_i as $v_i = v_{\phi_1}^i + v_{\text{f}1}^i = v_{\phi_2}^i + v_{\text{f}2}^i$, where $v_{\phi_k}^i \in \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_k}}$ and $v_{\text{f}_k}^i \in \mathscr{W}_{\text{f}_k}$ for $k = 1, 2$ and $i \in \mathbb{N}$. Since $\mathscr{W}_{\text{f}_1}$ and $\mathscr{W}_{\text{f}_2}$ are Fin-D it follows that we can assume the basis $\{v_i\}_{i=1}^{\infty}$ be such that for a sufficiently large number $n_0 \geq \dim(\mathscr{W}_{\text{f}1}) + \dim(\mathscr{W}_{\text{f}2})$ we have $v_{\text{f}_k}^i = 0$ for $k = 1, 2$ and $i > n_0$ (consider the fact that only for finite number n_0 of vectors we have $v_{\text{f}_k}^i$ are linearly independent, therefore we can remove the projection of v_i $(i > n_0)$ from $\mathscr{W}_{\text{f}1} + \mathscr{W}_{\text{f}2}$ to make sure that $v_{\text{f}k}^i = 0$ for both $k = 1, 2$). It follows that $v_{\phi_k}^i \in \mathscr{V}$ for $i > n_0$ that is in contradiction with the fact that $\mathscr{V} \cap \mathscr{V}_{\text{f}} = 0$ (since in this case we have $v_{\phi_1}^i = v_{\phi_2}^i = v_i \in \mathscr{V}$). Therefore, $\mathscr{V}_{\text{f}}$ is Fin-D, and by setting $\mathscr{W}_{\text{f}_4} = \mathscr{W}_{\text{f}_3} + \mathscr{V}_{\text{f}}$ we have $\mathscr{W}_1 \cap \mathscr{W}_2 = \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}_3}} + \mathscr{W}_{\text{f}_4}$.

Hence, by invoking Theorem 4.17, it follows that $\mathscr{W}_1 \cap \mathscr{W}_2$ is a $\mathbb{T}$-conditioned invariant subspace. This completes the proof of the lemma. $\square$

Remark 4.20. *Set $\mathbb{I}_3^1$ be all indices i such that $i \in \mathbb{I}_1 \cap \mathbb{I}_2$ and $\mathscr{E}_i$ be one-dimensional (1-D) (or two-dimensional (2-D) in the complex case) eigenspace corresponding a simple eigenvalue. Consider two eigenspaces $\mathscr{E}_i \subset \mathscr{W}_1$ and $\mathscr{E}_j \subset \mathscr{W}_2$ corresponding to simple eigenvalues. Since they are corresponding to* simple *eigenvalues, we have only two possibilities that are $\mathscr{E}_i \cap \mathscr{E}_j = 0$ and $\mathscr{E}_i = \mathscr{E}_j$ (also, refer to Remark 2). Therefore, $\mathbb{I}_3^1$ is related to common eigenspaces (for simple eigenvalues) contained in both $\mathscr{W}_1$ and $\mathscr{W}_2$.*

However, this does not hold when we deal with multiple eigenvalues (even in Fin-D systems). To see this, refer to the following example.

Example 4.21. *Set $A = I_{2\times 2}$, $\mathscr{E}_1 = e_1$, $\mathscr{E}_2 = [1,1]^{\mathrm{T}}$, $\mathscr{E}_3 = e_1, e_2$. It follows that $\mathscr{E}_1 \cap \mathscr{E}_2 = 0$ and $\mathscr{E}_2 \cap \mathscr{E}_3 = \mathscr{E}_2$ and $\mathscr{E}_2 \neq \mathscr{E}_3$. However, all these subspaces are A-invariant.*

Now, let $\mathscr{E}_i \subset \mathscr{W}_1$ and $\mathscr{E}_j \in \mathscr{W}_2$ be two sub-eigenspaces corresponding to the same multiple eigenvalue λ. Set $\mathscr{E}_i \cap \mathscr{E}_j = \mathscr{E}_k + \mathscr{V}$, where $\mathscr{E}_k$ is a sub-eigenspace corresponding to λ and $\mathscr{V}$ is not a sub-eigenspace. In general, we have following possibilities (refer to the example below)

1. *$\mathscr{E}_k = 0$.*

2. *$\mathscr{E}_k \neq 0$. Therefore, we need to add $\mathscr{E}_k$ to $\overline{\mathrm{span}\{\mathscr{E}_i\}}_{\, i\in\mathbb{I}_3^1}$ and then construct $\mathbb{I}_3$.*

Therefore, generally $\mathbb{I}_3$ cannot be specified only based on $\mathbb{I}_3^1$. Also, we do not *use any specific characteristic of $\mathbb{I}_3$ to show Lemma 8. The key point is that $\mathscr{W}_{\mathrm{f}_3}$ is Fin-D.*

As shown in [103], the smallest $\mathbb{T}$-conditioned invariant subspace containing $\mathscr{L}$ may not exists for a general Inf-D operator $\mathcal{A}$. However, the fact all but only finitely many eigenvalues of $\mathcal{A}$ are simple play a crucial role in the above proof to assure that $\mathscr{W}_{\mathrm{f}_3} \subset D(\mathcal{A})$.

We are now in a position to introduce an algorithm for computing the smallest $\mathbb{T}$-conditioned invariant subspace containing a given subspace. The algorithm for computing the smallest $(\mathcal{C},\mathcal{A})$-invariant subspace containing a given subspace $\mathscr{L}$ is given by [103], namely

$$\mathscr{W}^0 = \mathscr{L}, \;\; \mathscr{W}^k = \overline{\mathscr{L} + \mathcal{A}(\mathscr{W}^{k-1} \cap \ker \mathcal{C} \cap D(\mathcal{A}))}. \tag{4.10}$$

As pointed out in [103], the limit of the above algorithm may be a non-closed subspace, and consequently, it is not conditioned invariant in the sense of Definition 4.11. Below, we now provide an algorithm that computes the minimal $\mathbb{T}$-conditioned invariant subspace in a finite number of steps provided that the subspace $\mathscr{N}_{\mathcal{A}} = \bigcap_{n\in\mathbb{N}} \ker \mathcal{C}\mathcal{A}^n$, which denotes the $\mathcal{A}$-unobservable subspace of the system (4.1), is known.

Theorem 4.22. *Consider the RS system* (4.1) *and a given Fin-D subspace* $\mathscr{L} \subset D(\mathcal{A})$ *and* $\mathscr{L} \cap \ker \mathcal{C} \subset D(\mathcal{A}^\infty)$, *where* $D(\mathcal{A}^\infty) = \bigcap_{k=1}^{\infty} D(\mathcal{A}^k)$ *that is decomposed to disjoint subspaces* $\mathscr{L} = \mathscr{L}_{\mathscr{N}^\perp} + \mathscr{L}_{\mathscr{N}}$, *such that* $\mathscr{L}_{\mathscr{N}^\perp} \cap \mathscr{N}_{\mathcal{A}} = 0$ *and* $\mathscr{L}_{\mathscr{N}} = \mathscr{L} \cap \mathscr{N}_{\mathcal{A}}$. *The smallest* $\mathbb{T}$*-conditioned invariant subspace that satisfies the condition* (4.11) *and containing* $\mathscr{L}$ *(denoted by* $\mathscr{W}^*$*) is given by* $\mathscr{W}^*(\mathscr{L}) = \mathscr{W}_\ell + \mathscr{Z}^*$, *where* $\mathscr{Z}^*$ *is the limiting subspace of the following algorithm*

$$\mathscr{Z}_0 = \mathscr{L}_{\mathscr{N}^\perp}, \;\; \mathscr{Z}_k = \mathscr{L}_{\mathscr{N}^\perp} + \mathcal{A}(\mathscr{Z}_{k-1} \cap \ker \mathcal{C} \cap D(\mathcal{A})), \tag{4.11}$$

and $\mathscr{W}_\ell = \overline{\text{span}\{\mathscr{E}_i\}_{\,i\in\mathbb{J}}}$ *denotes the smallest subspace in the form of* (4.8) *(sum of the sub-eigenspaces of* $\mathcal{A}$*) such that* $\mathscr{L}_{\mathscr{N}} \subseteq \mathscr{W}_\ell$. *Moreover, the above algorithm converges in a finite number of steps.*

Proof. First, we show that this algorithm converges in a finite number of steps by contradiction. Assume that there exists at least a vector $x \in \mathscr{L}_{\mathscr{N}^\perp} \cap D(\mathcal{A}^\infty)$ such that $\mathcal{A}^n x \subseteq \ker \mathcal{C}$ and $\mathcal{A}^n x$ are independent vectors of all n. Otherwise, there is a n_0 such that $\mathcal{A}^{n_0} x \notin \ker \mathcal{C}$ for all $x \in \mathscr{L}_{\mathscr{N}^\perp}$. Therefore, $(\mathscr{Z}_{n_0+1} \cap \ker \mathcal{C} \cap D(\mathcal{A})) =$

$(\mathscr{Z}_{n_0} \cap \ker \mathcal{C} \cap D(\mathcal{A}))$, and consequently we obtain $\mathscr{Z}_{n_0+2} = \mathscr{Z}_{n_0+1}$. In other words, the above algorithm converges in a finite number of steps. Since $\ker \mathcal{C}$ is a closed subspace $\mathcal{A}^n x \in \ker \mathcal{C}$ for all $n \in \mathbb{N}$, and consequently $x \in \mathscr{N}_{\mathcal{A}}$, which is in contradiction with the fact that $\mathscr{L}_{\mathscr{N}^\perp} \cap \mathscr{N}_{\mathcal{A}} = 0$. Therefore, there exists a $k \in \mathbb{N}$ such that $\mathscr{Z}^* = \mathscr{Z}_k$. Moreover, since $\mathscr{L} \cap \ker \mathcal{C} \subset D(\mathcal{A}^\infty)$, it follows that $\mathscr{Z}^* \subset D(\mathcal{A})$.

Second, since $\mathscr{L}$ is Fin-D it follows that $\dim(\mathscr{Z}^*) < \infty$. By considering the definition of $\mathscr{W}_\ell$, we obtain $\overline{\mathscr{W}^* \cap D(\mathcal{A})} = \mathscr{W}^*$, and by using Theorem 4.17, it follows that $\mathscr{W}^*(\mathscr{L})$ is a $\mathbb{T}$-conditioned invariant subspace.

Finally, we show that $\mathscr{W}^*(\mathscr{L})$ is the smallest $\mathbb{T}$-conditioned invariant subspace. Consider a $\mathbb{T}$-conditioned invariant subspace $\mathscr{W}$ such that $\mathscr{L} \subseteq \mathscr{W}$. Given that $\mathscr{W}$ is $\mathbb{T}$-conditioned invariant and $\mathcal{A} + \mathcal{D}\mathcal{C}$ is a regular RS operator (Remark 4.3), $\mathscr{W} = \overline{\text{span}\{\mathscr{E}_i^D\}_{i \in \mathbb{I}}}$, where $\mathbb{I} \subseteq \mathbb{N}$ and $\mathscr{E}_i^D$ is a sub-eigenspace of $\mathcal{A} + \mathcal{D}\mathcal{C}$. Next, we show that $(\mathscr{W}_\ell + \mathscr{L}) \subseteq \mathscr{W}$. Towards this end, let $\mathcal{D}$ be the injection operator that is defined in the proof of Theorem 4.17, where $\mathscr{W} = \mathscr{W}_\phi + \mathscr{W}_{\mathrm{f}}$ and $\mathcal{D}\mathcal{C}\mathscr{W}_\phi = 0$. Also, as above we can assume that there is no sub-eigenspace $\mathscr{E}$ of $\mathcal{A}$ such that $\mathscr{E} \subset \mathscr{W}_{\mathrm{f}}$ (i.e. $\mathscr{W}_\phi$ is the largest subspace in the form (4.8) contained in $\mathscr{W}$). Since $\mathscr{L}_{\mathscr{N}} \subset D(\mathcal{A}^\infty)$ and $\mathscr{L}_{\mathscr{N}} \subseteq \mathscr{N}_{\mathcal{A}}$ it follows that $(\lambda\mathcal{I} - \mathcal{A})^k \mathscr{L}_{\mathscr{N}} = (\lambda\mathcal{I} - (\mathcal{A} + \mathcal{D}\mathcal{C}))^k \mathscr{L}_{\mathscr{N}} \subset \ker \mathcal{C}$ for all $k \in \mathbb{N}$. Therefore, $\mathscr{L}_{\mathscr{N}} \subseteq \mathscr{W}_\phi$. Otherwise, if $\mathscr{L}_{\mathscr{N}} \cap \mathscr{W}_{\mathrm{f}} \neq 0$, there exists $x \in \mathscr{L}_{\mathscr{N}} \cap \mathscr{W}_{\mathrm{f}}$ such that $(\lambda\mathcal{I} - \mathcal{A})^k x \in \ker \mathcal{C} \cap \mathscr{W}_{\mathrm{f}}$ for all $k \in \mathbb{N}$ (recall $\mathscr{W}_{\mathrm{f}}$ is $(\mathcal{C},\mathcal{A})$-invariant). Since, $\mathscr{W}_{\mathrm{f}}$ is Fin-D, it follows that there exists a sub-eigenspace contained in $\mathscr{W}_{\mathrm{f}}$ that is in contradiction with definition of $\mathscr{W}_{\mathrm{f}}$. Since $\mathscr{W}_\ell$ is the smallest subspace in the form of (4.8) such that $\mathscr{L}_{\mathscr{N}} \subseteq \mathscr{W}_\ell$, it follows that $\mathscr{W}_\ell \subset \mathscr{W}_\phi$. Furthermore, as we assume $\mathscr{L} \in \mathscr{W}$, we obtain $(\mathscr{W}_\ell + \mathscr{L}) \subseteq \mathscr{W}$. Now, since the algorithm is increasing and starts from $\mathscr{L}_{\mathscr{N}^\perp} \subseteq \mathscr{L} \subseteq \mathscr{W}$, we obtain $\mathscr{Z}_k \subseteq \mathscr{W}$ and consequently $\mathscr{W}^* \subseteq \mathscr{W}$. It follows that $\mathscr{W}^*$ is the smallest $\mathbb{T}$-conditioned invariant subspace containing $\mathscr{L}$. This completes the proof of the lemma. $\square$

It should be pointed out that one can compute $\mathscr{W}_\ell$ as follows.

1. Let $\mathcal{X}_{\text{inf}} = \overline{\text{span}\{\phi_i\}_{\ i\in\mathbb{J}_s}}$ and $\mathcal{X}_{\text{f}} = \text{span}\{\phi_j\}_{\ i\in\mathbb{J}_m}$, where $\mathbb{J}_s$ and $\mathbb{J}_m$ are the index sets for simple and multiple eigenvalues, respectively. Also, ϕ_i correspond to the simple eigenvalues of $\mathcal{A}$ and ϕ_j are the (generalized) eigenvectors that correspond to the multiple eigenvalues (note that $\dim(\mathcal{X}_{\text{f}}) < \infty$).

2. Compute, $\mathscr{W}_\ell^m$, the smallest sub-eigenspace in $\mathcal{X}_{\text{f}}$ containing $\mathcal{P}_{\text{f}}\mathscr{L}_{\mathcal{N}}$, where $\mathcal{P}_{\text{f}}$ is the projection from $\mathcal{X}$ on $\mathcal{X}_{\text{f}}$. It follows that $\mathscr{W}_\ell^m = \text{span}\{\phi_k\}_{\ k\in\mathbb{I}_m}$, where $\mathbb{I}_m \subseteq \mathbb{J}_m$, and therefore $\dim(\mathscr{W}_\ell^m) < \infty$.

3. Let $\mathscr{W}_\ell^s = \overline{\text{span}\{\phi_k\}_{\ k\in\mathbb{I}_s}}$, where $\mathbb{I}_s \subseteq \mathbb{J}_s$ and ϕ_k does appear in the representation of at least one member of $\mathscr{L}_{\mathcal{N}}$ (refer to Lemma 4.1).

4. Set $\mathscr{W}_\ell = \mathscr{W}_\ell^s + \mathscr{W}_\ell^m$.

Example 4.23. *Computing $\mathscr{W}_\ell$:*

Consider the following regular RS system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{L}f(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{4.12}$$

where,

$$\mathcal{A} = \begin{bmatrix} J_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & J_2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & \cdots \\ & & \ddots & & \ddots & & \ddots & \end{bmatrix}, \quad J_1 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \tag{4.13}$$

Also, $\mathcal{A} : \mathcal{X} \to \mathcal{X}$, $\mathcal{L} = \sum_{k=0}^{\infty} e^{-(2k+1)^2}\phi_{2k+1}$. where ϕ_k are generalized eigenvetors of $\mathcal{A}$. Moreover, $\mathcal{C} =< c_i, \cdot >$ where $c_i = \sum_{k=1}^{\infty} \frac{1}{4k^2}\phi_{2k}$. Also, $\mathcal{X} = \ell_2$ is the Hilbert space that induced from $\mathbb{R}^\infty$ (that is all $x = [x_0, x_1, \cdots]^{\text{T}} \in \mathbb{R}^\infty$ such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$). Note that for computing ϕ_k we do not need to run an algorithm infinite number of times and we have them as $\phi_k = [0,\ 0,\ \cdots,1,\ 0,\ 0,\ \cdots]^{\mathrm{T}}$, where 1 is in the k^{th} position.

It is follows that $\mathscr{L} = \text{ span}\{\mathcal{L}\} \subset \mathscr{N}_{\mathcal{A}}$ and $\mathcal{L} \in D(\mathcal{A}^{\infty})$ (since $\sum_{k=1}^{\infty}((2k+1)^n e^{-(2k+1)^2}) < \infty$ for any n and hence $\mathcal{A}^n\mathcal{L} \in \mathcal{X}$ for any n). $\mathscr{W}_\ell$ is constructed as follows.

By using the above discussion, we have $\phi_1, \phi_3 \in \mathscr{W}_\ell$ (since span$\{\phi_1\}$ and span$\{\phi_3\}$ are J_{-1}-and J_{-2}-invariant, respectively). Also, since other eigenvalues are simple we have $\mathscr{W}_\ell = \overline{\text{span}\{\phi_{2k-1}\}_{\ k\in\mathbb{N}}}$. Note that in this case since $\mathscr{L} \subseteq \mathscr{N}_{\mathcal{A}}$, it follows that in the algorithm (4.11) one obtains $\mathscr{L}_{\mathscr{N}^{\perp}} = 0$, and consequently $\mathscr{Z}^* = 0$. therefore, we have $\mathscr{W}^*(\mathscr{L}) = \mathscr{W}_\ell$.

Remark 4.24. *We compute $\mathscr{W}_\ell$ in two steps. As stated above, we assume that we have complete knowledge of the eigenvalues and eigenvectors of $\mathcal{A}$ and we can construct the given subspace (i.e. $\mathscr{V}_\ell$) in the terms of these eigenvectors. Hence, it is not necessary to examine one by one the eigenvetrors to construct $\mathscr{W}_\ell$.*

4.2.3 Unobservability Subspace

In the geometric FDI approach, one needs to work with another invariant subspace known as the unobservability subspace. In this subsection, we first provide two definitions for this subspace, and then develop an algorithm to construct it computationally.

Definition 4.25.

1. *The subspace $\mathcal{S}$ is called an $\mathcal{A}$-unobservability subspace for the RS system* (4.1)*, if there exist two bounded operators $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ and $H : \mathbb{R}^q \to \mathbb{R}^{q_h}$, where $q_h \leq q$, such that $\mathcal{S}$ is the largest $\mathcal{A}+\mathcal{D}\mathcal{C}$-invariant subspace contained in* $\ker HC$ *(i.e., $\mathcal{S} =< \ker H\mathcal{C} | \mathcal{A}+\mathcal{D}\mathcal{C} >$).*

2. *The subspace $\mathcal{S}$ is called a unobservability subspace for the RS system* (4.1)*, if there exist two bounded operators $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ and $H : \mathbb{R}^q \to \mathbb{R}^{q_h}$, where $q_h \leq q$, such that $\mathcal{S}$ is the largest $\mathbb{T}_{\mathcal{A}+\mathcal{DC}}$-invariant subspace contained in* $\ker HC$ *(i.e., $\mathcal{S} =< \ker HC|\mathbb{T}_{\mathcal{A}+\mathcal{DC}} >$).*

Remark 4.26. *It follows that the $\mathcal{A}$- and unobservability subspaces are the $\mathcal{A}$- and unobservable subspaces of the pair ($\mathcal{HC}$,$\mathcal{A}+\mathcal{DC}$), respectively. Also, by definition $\mathcal{A}$- and unobservability subspaces are also feedback ($\mathcal{C}$,$\mathcal{A}$)- and $\mathbb{T}$-conditioned invariant, respectively.*

Computing the Unobservability Subspace: As stated earlier, for the FDI problem one is interested in computing the smallest unobservability subspace containing a given subspace. By following along the same lines as in Lemma 4.19, and the fact that $\mathcal{A} + \mathcal{DC}$ is a regular operator, and finally by invoking Remark 4.26, one can show that the set of all unobservablity subspaces containing a given subspace always admits a minimal in the inclusion sense. In the Fin-D case, the unobservability subspace computing algorithm involves the inverse of the state dynamic operator (i.e., the operator A) [3, equation 2.61]. However, for Inf-D systems, the inverse image of $\mathcal{A}$ is not convenient to deal with (if $0 \notin \rho(\mathcal{A})$). To overcome this difficulty, one can compute the unobservability subspace by using its dual subspace which is the controllability subspace. Therefore, one needs to compute the adjoint operators of $\mathcal{A}$ and $\mathcal{C}$ as was pointed out in [124].

The above method in [124] uses a non-decreasing algorithm that converges in a countable number of steps. However, since the algorithm is non-decreasing, the limiting subspace is not necessarily closed. The following theorem provides an approach to compute the smallest unobservability subspace containing a given Fin-D subspace $\mathscr{L} \subseteq D(\mathcal{A})$.

Theorem 4.27. *Consider model* (4.1) *which is assumed to be a regular RS system and a given Fin-D subspace $\mathscr{L} \subset D(\mathcal{A})$. Let $\mathscr{W}^*$ denotes the smallest $\mathbb{T}$-conditioned* *invariant subspace containing $\mathscr{L}$, where $\mathscr{W}^* = \mathscr{W}_\phi^* + \mathscr{W}_{\rm f}^*$ (by Theorem 4.17), $\mathscr{W}_\phi^*$ is the largest subspace contained in $\mathscr{W}^*$ in the form* (4.8) *and $\mathscr{W}_{\rm f}^* \subset D(\mathcal{A})$ is a Fin-D subspace. The smallest unobservability subspace containing $\mathscr{L}$ (denoted by $\mathcal{S}^*$) is given by*

$$\mathcal{S}^* = \overline{\mathscr{W}_\phi^* + \mathscr{N}} + \mathscr{W}_{\phi,\rm f}^*, \tag{4.14}$$

in which $\mathscr{N}$ is the unobservable subspace of ($\mathcal{C}$,$\mathcal{A}$), $\mathscr{W}_{\phi,\rm f}^*$ is the largest subspace in the form of $\overline{\text{span}\{\mathscr{E}_i^D\}_{\ i\in\mathbb{I}_D}}$ such that $\mathscr{W}_{\phi,\rm f}^*$ contains $\mathscr{W}_{\rm f}^*$ and is contained in $\overline{\mathscr{W}^* + \ker\mathcal{C}}$. Also, $\mathscr{E}_i^D$ are sub-eigenspaces of $(\mathcal{A} + \mathcal{D}\mathcal{C})$.

Proof. Let us first show that $\mathcal{S}^*$ is $\mathbb{T}$-conditioned invariant subspace. Since $\mathscr{N}$ is $\mathbb{T}_\mathcal{A}$-invariant, we obtain $\mathscr{N} = \overline{\text{span}\{\mathscr{E}_i\}_{\ i\in\mathbb{I}}}$, where $\mathscr{E}_i$ are sub-eigenspaces of $\mathcal{A}$ (by using Corollary 4.9). Let $\mathcal{D} \in \underline{D}(\mathscr{W}^*)$ that is constructed in Theorem 4.17 (i.e., $\mathcal{D}\mathcal{C}\mathscr{W}_\phi = 0$ and $(\lambda\mathcal{I} - (\mathcal{A} + \mathcal{D}\mathcal{C}))^{-1}\mathscr{W}^* \subseteq \mathscr{W}^*$). Since $\mathscr{N} \subseteq \ker\mathcal{C}$, as shown above (in the proof of Theorem 4.17) $\mathscr{E}_i$ are also sub-eigenspace of $(\mathcal{A} + \mathcal{D}\mathcal{C})$. Also, by definition, $\mathscr{W}_{\phi,\rm f}^*$ is a summation of sub-eigenspaces of $(\mathcal{A} + \mathcal{D}\mathcal{C})$. Therefore, $\mathcal{S}^*$ is a summation of sub-eigenspaces of $(\mathcal{A} + \mathcal{D}\mathcal{C})$ and by Corollary 4.9, it follows that $\mathcal{S}^*$ is $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant (i.e. $\mathbb{T}$-conditioned invariant).

Second, let H be a map such that $\ker H\mathcal{C} = \overline{\mathscr{W}^* + \ker\mathcal{C}}$ (one choice is $H : \mathbb{R}^q \to \mathbb{R}^{q_h}$, where $\ker H = \mathscr{W}^* \cap (\mathscr{W}^* \cap \ker\mathcal{C})^\perp$). Since $\mathscr{W}_{\phi,\rm f}^* \subseteq \overline{\mathscr{W}^* + \ker\mathcal{C}}$, and $\mathscr{W}_{\rm f}^* \subseteq \mathscr{W}_{\phi,\rm f}^*$, it follows that $\overline{\mathscr{W}_\phi^* + \ker\mathcal{C}} + \mathscr{W}_{\phi,\rm f}^* = \overline{\mathscr{W}^* + \ker\mathcal{C}}$. Also, given that $\mathscr{N} \subseteq \ker\mathcal{C}$, we obtain $\overline{\mathscr{W}^* + \ker\mathcal{C}} = \overline{\mathcal{S}^* + \ker\mathcal{C}}$, and consequently, we have $\mathcal{S}^* \subseteq \ker H\mathcal{C}$.

Third, we show $\mathcal{S}^*$ is an unobservable subspace of the system $(H\mathcal{C}, \mathcal{A} + \mathcal{D}\mathcal{C})$. As shown above $\mathcal{S}^* = \overline{\text{span}\{\mathscr{E}_i^D\}_{\ i\in\mathbb{I}}}$, where $\mathscr{E}_i^D$ is a sub-eigenspace of $\mathcal{A} + \mathcal{D}\mathcal{C}$. Next, it is shown that $\mathcal{S}^*$ contains all sub-eigenspaces of $(\mathcal{A} + \mathcal{D}\mathcal{C})$ that are contained in $\ker H\mathcal{C}$. Let $\mathscr{E}_0^D$ be a given sub-eigenspace of $\mathcal{A} + \mathcal{D}\mathcal{C}$, such that $\mathscr{E}_0^D \subseteq \ker H\mathcal{C}$. If $\mathscr{E}_0^D \nsubseteq \ker\mathcal{C}$, since $\mathscr{W}_\phi^* + \mathscr{W}_{\phi,\rm f}^*$ contains all sub-eigenspaces that may not contained in $\ker\mathcal{C}$ (recall the definition of H and $\mathscr{W}_{\phi,\rm f}^*$) but in $\ker H\mathcal{C}$, we obtain $\mathscr{E}_0^D \subseteq (\mathscr{W}_\phi^* +$

$\mathscr{W}^*_{\phi,\mathrm{f}}) \subseteq \mathcal{S}^*$. Now, assume $\mathscr{E}_0^D \subseteq \ker \mathcal{C}$. It follows that $(\lambda\mathcal{I} - (\mathcal{A}+\mathcal{D}\mathcal{C}))^{-1}\mathscr{E}_0^D = (\lambda\mathcal{I} - \mathcal{A})^{-1}\mathscr{E}_0^D \subseteq \ker \mathcal{C}$, and consequently, $\mathscr{E}_0^D \subseteq \mathscr{N} \subseteq \mathcal{S}^*$. Hence, $\mathcal{S}^*$ is the largest subspace contained in $\ker H\mathcal{C}$ that is spanned by the sub-eigenspace of $\mathcal{A}+\mathcal{D}\mathcal{C}$ (i.e. every sub-eigenspace in $\ker H\mathcal{C}$ is contained in $\mathcal{S}^*$). Therefore, $\mathcal{S}^*$ is the unobservable subspace of the pair $(H\mathcal{C},\mathcal{A}+\mathcal{D}\mathcal{C})$.

Finally, we show $\mathcal{S}^*$ is the smallest unobservability subspace containing $\mathscr{L}$. Let, $\mathcal{S}$ be another unobservability subspace containing $\mathscr{L}$. Since $\mathcal{S}$ is $\mathbb{T}$-conditioned invariant containing $\mathscr{L}$, it follows that $\mathscr{W}^* \subseteq \mathcal{S}$ ($\mathscr{W}^*$ is the smallest $\mathbb{T}$-conditioned invariant containing $\mathscr{L}$). Now, let H_1 such that $\ker H_1\mathcal{C} = \overline{\mathcal{S} + \ker \mathcal{C}}$. Since $\mathcal{S}^* \subseteq \overline{\mathscr{W}^* + \ker \mathcal{C}}$, it follows $\mathcal{S}^* \subseteq \ker H_1\mathcal{C}$. Also, given that $\mathcal{S}$ is the largest $\mathbb{T}$-conditioned invariant in $\ker H_1\mathcal{C}$, by using Theorem 4.17 $\mathcal{S}$ is the largest subspace in form (4.7) that is contained in $\ker H_1\mathcal{C}$. Since $\mathcal{S}^*$ is also expressed in the form (4.7) (as $\mathcal{S}^*$ is also $\mathbb{T}$-conditioned invariant) it follows that $\mathcal{S}^* \subseteq \mathcal{S}$. This completes the proof of the theorem. $\square$

It should be pointed out that since $\mathscr{W}^*_{\mathrm{f}}$ is Fin-D and the operator $\mathcal{A}+\mathcal{D}\mathcal{C}$ is regular RS, we obtain $\mathscr{W}^*_{\phi,\mathrm{f}}$ is Fin-D. Therefore, one can compute $\mathscr{W}^*_{\phi,\mathrm{f}}$ based on sub-eigenspaces of $\mathcal{A}+\mathcal{D}\mathcal{C}$ (i.e., for every the sub-eigenspace $\mathscr{E}_0^D$ of $\mathcal{A}+\mathcal{D}\mathcal{C}$ that (i) $\mathscr{E}_0^D$ is contained in $\overline{\mathscr{W}^* + \ker \mathcal{C}}$, (ii) $\mathscr{E}_0^D \not\subseteq \overline{\mathscr{W}^*_{\phi} + \mathscr{N}}$, and (iii) $\mathscr{E}_0^D \not\perp \mathscr{W}^*_{\mathrm{f}}$, we have $\mathscr{E}_0^D \subseteq \mathscr{W}^*_{\phi,\mathrm{f}}$).

As an example, refer to Subsection 4.4, where we provide a numerical example.

4.2.4 Controlled Invariant Subspaces and the Duality Property

As stated above, for addressing the FDI problem one needs to construct the conditioned invariant subspace. However, for the disturbance decoupling problem the controlled invariant subspaces (that are dual to the conditioned invariant subspaces) are needed. For sake of completeness of the chapter, in this subsection we review controlled invariant subspaces of the RS system (4.1), where necessary and sufficient conditions for controlled invariance are provided. We address the controlled invariant subspaces by using the duality property. Similar to conditioned invariant subspaces, there are three types of controlled invariant subspaces. These are discussed further below.

Definition 4.28. *[103] Consider the closed subspace $\mathscr{V} \subseteq \mathcal{X}$ and $\mathscr{B} = \operatorname{Im} \mathcal{B}$, where $\mathcal{B}$ is defined by the system* (4.1)*. Then,*

1. *$\mathscr{V}$ is called ($\mathcal{A}$,$\mathcal{B}$)-invariant if $\mathcal{A}(\mathscr{V} \cap D(\mathcal{A})) \subseteq \overline{\mathscr{V} + \mathscr{B}} = \mathscr{V} + \mathscr{B}$ (since $\dim(\mathscr{B}) < \infty$).*

2. *$\mathscr{V}$ is called feedback ($\mathcal{A}$,$\mathcal{B}$)-invariant if there exists a bounded operator $\mathcal{F}$: $\mathcal{X} \to \mathbb{R}^m$ such that $(\mathcal{A} + \mathcal{B}\mathcal{F})(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V}$.*

3. *$\mathscr{V}$ is called $\mathbb{T}$-controlled invariant if there exists a bounded operator $\mathcal{F} : \mathcal{X} \to \mathbb{R}^m$ such that (i) the operator $\mathcal{A} + \mathcal{B}\mathcal{F}$ is the infinitesimal generator of a C_0-semigroup $\mathbb{T}_{\mathcal{A}+\mathcal{B}\mathcal{F}}$; and (ii) $\mathscr{V}$ is invariant with respect to $\mathbb{T}_{\mathcal{A}+\mathcal{B}\mathcal{F}}$ as per Definition 4.6, item 2). In the literature, it is also called closed feedback invariant [101] and $\mathbb{T}(\mathcal{A}, \mathcal{B})$-invariant [103].*

From the above, it follows that Definition 4.28, item 3) $\Rightarrow$ item 2) $\Rightarrow$ item 1) [103]. In this subsection, we are interested in developing and addressing the necessary and sufficient conditions for equivalence of the above definitions. In [103], the duality between the Definitions 4.11 and 4.28 was shown by using the following lemmas (the superscript $*$ is used for adjoint operators).

Lemma 4.29. *[103, Lemma 5.2] Consider the system* (4.1)*, where $\mathcal{A}$ is an infinitesimal generator of the C_0 semigroup $\mathbb{T}_{\mathcal{A}}$ (more general than the regular RS operator) and the operator $\mathcal{C}$ is bounded (but not necessarily finite rank), and two subspaces $\mathscr{S}_1$ and $\mathscr{S}_2$. We have*

1. $(\mathscr{S}_1 + \mathscr{S}_2)^\perp = \overline{\mathscr{S}_1^\perp + \mathscr{S}_2^\perp}$.

2. $(\ker \mathcal{C})^\perp = \overline{\operatorname{Im} \mathcal{C}^*}$.

3. *If* $\mathbb{T}_{\mathcal{A}} \mathscr{S}_1 \subseteq \mathscr{S}_2$, *then* $\mathbb{T}_{\mathcal{A}^*} \mathscr{S}_2^\perp \subseteq \mathscr{S}_1^\perp$.

4. *If* $\mathcal{A}(\mathscr{S}_1 \cap D(\mathcal{A})) \subseteq \mathscr{S}_2$, *then* $\mathcal{A}^*(\mathscr{S}_2^\perp \cap D(\mathcal{A}^*)) \subseteq (\mathscr{S}_1 \cap D(\mathcal{A}))^\perp$.

By using Lemma 4.29, item 3) the following result can be obtained.

Lemma 4.30. *[103] Consider the Inf-D system* (4.1). *The subspace $\mathscr{V}$ is $\mathbb{T}$-controlled invariant if and only if $\mathscr{V}^\perp$ is $\mathbb{T}$-conditioned invariant with respect to ($\mathcal{B}^*$,$\mathcal{A}^*$).*

However, as can be observed from Lemma 4.29 (item 4), the dual equivalence of Lemma 4.14 will not be straightforward to show by using the duality property. The following lemma now directly provides our proposed result.

Lemma 4.31. *Consider the regular RS system* (4.1) *and the closed subspace $\mathscr{V}$ such that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$. The feedback ($\mathcal{A}$,$\mathcal{B}$)-invariance property is equivalent to the ($\mathcal{A}$,$\mathcal{B}$)-invariance property.*

Proof. It is sufficient to show that ($\mathcal{A}$,$\mathcal{B}$)-invariance $\Rightarrow$ feedback ($\mathcal{A}$,$\mathcal{B}$)-invariance. Let $\mathscr{V}$ be ($\mathcal{A}$,$\mathcal{B}$)-invariant. Since $D(\mathcal{A})$ is dense in $\mathscr{V}$, one can construct the basis $\{v_i\}_{i\in\mathbb{I}}$ (where $\mathbb{I} \in \mathbb{N}$) such that $v_i \in D(\mathcal{A})$. Since $\mathcal{B}$ is finite rank, we have $\mathscr{V} = \mathscr{V}_{\text{inf}} + \mathscr{V}_{\text{f}}$, such that $\mathcal{A}(\mathscr{V}_{\text{inf}} \cap D(\mathcal{A})) \subseteq \mathscr{V}$, $\mathscr{V}_{\text{f}} \subset D(\mathcal{A})$ and $\mathcal{A}v_i$' are linearly independent for all $i = 1, \cdots, n_f$, where without loss of any generality we assume that $\mathscr{V}_{\text{f}} = \operatorname{span}\{v_i\}_{i=1}^{n_{\text{f}}}$ and $\mathcal{A}\mathscr{V}_{\text{f}} \subseteq \mathscr{B}$ (by following along the same steps in Lemma 4.14). Therefore, there exist u_i such that $\mathcal{A}v_i = -\mathcal{B}u_i$ for all $i = 1, \cdots, n_{\text{f}}$. Let us now define F such that $F[v_1, \cdots, v_{n_{\text{f}}}] = [u_1, \cdots, u_{n_{\text{f}}}]$ (note since $\ker[v_1, \cdots, v_{n_{\text{f}}}] = 0$, F always exists), and let $\mathcal{F}$ be the extension of F to $\mathcal{X}$. In other words, for all $x \in \mathcal{X}$, we have $\mathcal{F}x = Fx_v$, where $x = x_{v^\perp} + x_v$, $x_v \in \mathscr{V}_{\text{f}}$ and $x_{v^\perp} \perp \mathscr{V}_{\text{f}}$. It follows that

$||\mathcal{F}|| = ||F|| < \infty$ (i.e., $\mathcal{F}$ is bounded) and $(\mathcal{A}+\mathcal{B}\mathcal{F})(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V}$. Therefore, $\mathscr{V}$ is feedback ($\mathcal{A}$,$\mathcal{B}$)-invariant. This completes the proof of the lemma. $\square$

Remark 4.32. *In [101] feedback ($\mathcal{A}$,$\mathcal{B}$)-invariant is defined as follows. The subspace $\mathscr{V}$ is feedback ($\mathcal{A}$,$\mathcal{B}$)-invariant if there exists an $\mathcal{A}$-bounded (refer to Section 2.4) state feedback (instead of bounded state feedback as in Definition 4.6) $\mathcal{F}$, such that $(\mathcal{A}+\mathcal{B}\mathcal{F})(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V}$. By this definition, in [101, Theorem II.26], it is shown that ($\mathcal{A}$,$\mathcal{B}$)-invariant and feedback ($\mathcal{A}$,$\mathcal{B}$)-invariant are equivalent. However, Lemma 4.31 shows the result (by also using an extra condition that is $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$) when we restrict the feedback to the bounded operators (i.e. as per Definition 4.6). Note that this result cannot be concluded from Lemma II.25 and Theorem II.26 in [101].*

However, we are interested in deriving a direct necessary and sufficient condition for the $\mathbb{T}$-controlled invariance property. By taking advantage of the duality property, the following theorem now provides the necessary and sufficient conditions for the $\mathbb{T}$-controlled invariance property.

Theorem 4.33. *Consider the regular RS system* (4.1) *and the closed subspace $\mathscr{V}$ such that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$ and $\mathcal{A}(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V} + \operatorname{Im}\mathcal{B}$. Then, $\mathscr{V}$ is $\mathbb{T}$-controlled invariant if $\mathscr{V}$ can be represented as $\mathscr{V} = \mathscr{V}_\phi \cap \mathscr{V}_{\mathrm{f}}^{\perp}$, where $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$ ($\mathcal{A}^*$ denotes adjoint operator $\mathcal{A}$) is a Fin-D subspace and $\mathscr{V}_\phi$ is the smallest subspace containing $\mathscr{V}$ that can be expressed as*

$$\mathscr{V}_\phi = \overline{\operatorname{span}\{\mathscr{E}_i\}}_{\, i\in\mathbb{I}}, \tag{4.15}$$

in which $\mathscr{E}_i$ are the sub-eigenspaces of $\mathcal{A}$ and $\mathbb{I} \subseteq \mathbb{N}$. Moreover, if $\mathscr{V} + (\mathscr{V} + \operatorname{Im}\mathcal{B})^{\perp}$ is a sum of sub-eigenspaces of $\mathcal{A}+\mathcal{D}\mathcal{C}$, then the condition (4.15) *is also necessary.*

Proof. (**If part**): Let $\mathscr{V} = \mathscr{V}_\phi \cap \mathscr{V}_{\mathrm{f}}^{\perp}$. It follows that $\mathscr{W}_\psi = \mathscr{V}_\phi^{\perp}$ can be expressed as $\mathscr{W}_\psi = \overline{\operatorname{span}\{\mathscr{E}_i^*\}}_{\, i\in\mathbb{I}}$, where $\mathscr{E}_i^*$ are sub-eigenspaces of $\mathcal{A}^*$ (since $\mathscr{W}_\psi$ is $\mathbb{T}_{\mathcal{A}^*}$). Given that $\mathscr{V}_{\mathrm{f}} \subseteq D(\mathcal{A}^*)$, $\dim(\mathscr{V}_{\mathrm{f}}) < \infty$ and $\overline{\mathscr{W}_\psi \cap D(\mathcal{A}^*)} = \mathscr{W}_\psi$ (since it is $\mathbb{T}_{\mathcal{A}^*}$-invariant), it follows that $\overline{\mathscr{V}^{\perp} \cap D(\mathcal{A}^*)} = \mathscr{V}^{\perp}$. Also, By invoking Lemma 4.29 (item 4)) and the fact that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$, we have (note that $\dim(\operatorname{Im}\mathcal{B}) < \infty$ and consequently $\operatorname{Im}\mathcal{B} = \overline{\operatorname{Im}\mathcal{B}}$)

$$\mathcal{A}^*(\mathscr{V}^\perp \cap (\operatorname{Im}\mathcal{B})^\perp \cap D(\mathcal{A}^*)) \subseteq \mathscr{V}^\perp. \tag{4.16}$$

Hence, $\mathscr{V}^\perp$ is a ($\mathcal{B}^*$,$\mathcal{A}^*$)-invariant subspace. By using Theorem 4.17, it follows that $\mathscr{V}^\perp$ is $\mathbb{T}$-conditioned invariant with respect to ($\mathcal{B}^*$,$\mathcal{A}^*$), and consequently, by using Lemma 4.30 it follows that $\mathscr{V}$ is $\mathbb{T}$-controlled invariant.

(**Only if part**): Let $\mathscr{V}$ be $\mathbb{T}$-controlled invariant. By invoking Lemma 4.30, it follows that $\mathscr{V}^\perp$ is $\mathbb{T}$-conditioned invariant. Therefore, from Theorem 4.17 it follows that $\mathscr{V}^\perp = \mathscr{W}_\psi + \mathscr{W}_{\mathrm{f}}$, with $\mathscr{W}_\psi$ defined as above and $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$. Also, since $D(\mathcal{A}^*)$ is densely defined on $\mathscr{V}^\perp$ (by Lemma 4.30, we obtain $\mathscr{V}^\perp$ is $\mathbb{T}_{\mathcal{A}^*}$-invariant, and consequently $\overline{\mathscr{V}^\perp \cap D(\mathcal{A}^*)} = \mathscr{V}^\perp$) and $\mathscr{W}_\psi$ (since it is $\mathbb{T}_{\mathcal{A}^*}$-invariant), it follows $\mathscr{W}_f \subset D(\mathcal{A}^*)$. Hence, $\mathscr{V} = \mathscr{V}_\phi \cap (\mathscr{W}_{\mathrm{f}})^\perp$, where $\mathscr{V}_\phi = \mathscr{W}_\psi^\perp = \overline{\operatorname{span}\{\mathscr{E}_i\}_{\,i\in\mathbb{I}}}$ and $\mathscr{W}_f \subset D(\mathcal{A}^*)$. This completes the proof of the theorem. □

Remark 4.34. *Below, we emphasize that Theorem 4.33 is compatible with the currently available results in the literature. In the literature, there are following main results corresponding to* $\mathbb{T}$*-controlled invariant subspaces.*

1. *As shown in [105, Theorem 3.1] and [132, Theorem 2.2] the necessary condition for* $\mathbb{T}$*-controlled invariant is* $\overline{\mathscr{V}^\perp \cap D(\mathcal{A}^*)} = \mathscr{V}^\perp$*. Since in* $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$*, this result is compatible with Theorem 4.33 (only if part).*

2. *In [133] its is shown that for single-input single-output (SISO) systems if* $c \in D(\mathcal{A}^*)$ *and* $< c, b > \neq 0$*, then the subspace* $\ker\mathcal{C}$ *is* $\mathbb{T}$*-controlled invariant, where* $\mathcal{C} =< c, \cdot >$*, and the corresponding bounded feedback gain is given by* $\mathcal{F} = -\frac{<\mathcal{A}^* c, \cdot>}{<c,b>}$*. Now, we show that this result and Theorem 4.33 coincide. Since* $\mathcal{X} = \ker\mathcal{C} + \operatorname{Im}\mathcal{B}$*,* $\mathscr{V} = \ker\mathcal{C}$ *is (*$\mathcal{A}$*,*$\mathcal{B}$*)-invariant and consequently feedback (*$\mathcal{A}$*,*$\mathcal{B}$*)-invariant (by using Lemma 4.31). Moreover,* $\mathscr{V} = \mathcal{X} \cap (\operatorname{Im}\mathcal{C}^*)^\perp$

(note that $\operatorname{Im}\mathcal{C}^* = \operatorname{span}\{c\}$*) and hence by Theorem 4.33 (as* $\mathcal{X}$ *is summation of all sub-eigenspaces of* $\mathcal{A}$ *and* $c \in D(\mathcal{A}^*)$*, one can set* $\mathscr{V}_\phi = \mathcal{X}$ *and* $\mathscr{V}_{\mathrm{f}} = \operatorname{span}\{c\}$*),* $\mathscr{V}$ *is* $\mathbb{T}$*-controlled invariant. In other words, sufficient conditions of Theorem 4.33 are also compatible with the result in [133] (for SISO systems).*

Moreover, note that $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$ *is a crucial condition. As above, consider a SISO system and the subspace* $\mathscr{V} = \mathcal{X} \cap (\operatorname{Im}\mathcal{C}^*)^\perp$*, and assume that* $c \notin D(\mathcal{A}^*)$*, and consequently the feedback introduced in [133] (i.e.* $\mathcal{F} = -\frac{<\mathcal{A}^*c,\cdot>}{<c,b>}$*) is not bounded. In fact* $\mathscr{V}$ *is not* $\mathbb{T}$*-invariant (since it does not satisfies the necessary condition in [105, Theorem 3.1]). It should be pointed out that although one can still construct another the bounded feedback* $\mathcal{F}$ *that is derived in the proof of Lemma 4.31 and consequently* $\mathscr{V}$ *is feedback (*$\mathcal{A}$*,*$\mathcal{B}$*)-invariant, however, even with this bounded feedback,* $\mathscr{V}$ *is not* $\mathbb{T}$*-controlled invariant (since* $\mathscr{V}$ *does not satisfies the necessary conditions).*

4.3 Fault Detection and Isolation (FDI) Problem

In this section, we first formulate the FDI problem for the RS system (4.1) and then the methodology that was developed in the previous section is utilized to derive and provide necessary and sufficient conditions for solvability (formally defined in Remark 4.35) of the FDI problem.

4.3.1 The FDI Problem Statement

Consider the following regular RS system

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \sum_{i=1}^{p} \mathcal{L}_i f_i(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{4.17}$$

where $\mathcal{L}_i$ and f_i denote the fault signatures and signals, respectively. The other variables and operators are defined as in the model (4.1). The FDI problem is specified in terms of generating a set of residual signals, denoted by $r_i(t)$, $i = 1, \cdots, p$ such that each residual signal $r_i(t)$ is decoupled from the external input and all the faults, except one fault $f_i(t)$. In other words, the residual signal $r_i(t)$ satisfies the following conditions for all $u(t)$ and f_j $(j \neq i)$

$$\text{if } f_i = 0 \ \Rightarrow r_i \to 0 \text{ (stability and decoupling condition)}, \tag{4.18a}$$

$$\text{if } f_i \neq 0 \ \Rightarrow r_i \neq 0. \tag{4.18b}$$

The residual signal $r_i(t)$ is to be generated from the following detection filter

$$\begin{aligned} \dot{\omega}_i(t) &= \mathcal{A}_o \omega_i(t) + \mathcal{B}_o u(t) + \mathcal{E}_i y(t), \\ r_i(t) &= H_i y(t) - \mathcal{M}_i \omega_i(t), \end{aligned} \tag{4.19}$$

where $\omega_i \in \mathcal{X}_o^i$, $\mathcal{X}_o^i$ is a separable Hilbert space (Fin-D or Inf-D), and $\mathcal{A}_o$ is a regular RS operator. The operators $\mathcal{B}_o$, $\mathcal{E}_i$, $\mathcal{M}_i$ and H_i are closed operators with appropriate domains and codomains (for example, $\mathcal{A}_o : \mathcal{X}_o^i \to \mathcal{X}_o^i$ and $\mathcal{E} : \mathbb{R}^q \to \mathcal{X}_o^i$). In this work we investigate, develop, and derive conditions for constructing the detection filter (4.19) by utilizing invariant subspaces such that the condition (4.18) is satisfied.

Remark 4.35. *Design of the detection filter* (4.19) *involves satisfying two main requirements:*

1. *The residual signal $r_i(t)$ should be decoupled from all faults except $f_i(t)$.*

2. *The corresponding filter error dynamics should be stable.*

If the first requirement is satisfied, we say that the fault f_i is detectable and isolable*. However, the FDI problem is said to be* solvable *if* both *requirements are simultaneously satisfied.*

In the next subsection, we derive necessary and sufficient conditions for solvability of the FDI problem for the RS system (4.17).

4.3.2 Necessary and Sufficient Conditions

As stated above, the FDI problem can be cast as that of designing detection filters having the structure (4.19) such that each detection filter output is decoupled from all faults but one.

By augmenting the RS system (4.17) and the detection filter (4.19), one can write

$$\begin{aligned}\dot{x}^e(t) &= \mathcal{A}^e x^e(t) + \mathcal{B}^e u(t) + \sum_{i=1}^{p} \mathcal{L}_i^e f_i(t),\\ r_i(t) &= \mathcal{C}^e x^e(t),\end{aligned} \tag{4.20}$$

where $x^e(t) = \begin{bmatrix} x(t) \\ \omega_i(t) \end{bmatrix} \in \mathcal{X}^e = \mathcal{X} \oplus \mathcal{X}_o^i$, $\mathcal{C}^e = \begin{bmatrix} H_i\mathcal{C} & -M_i \end{bmatrix}$ and

$$\mathcal{A}^e = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{E}_i\mathcal{C} & \mathcal{A}_o \end{bmatrix}, \ \mathcal{B}^e = \begin{bmatrix} \mathcal{B} \\ \mathcal{B}_o \end{bmatrix}, \ \mathcal{L}_i^e = \begin{bmatrix} \mathcal{L}_i \\ 0 \end{bmatrix}. \tag{4.21}$$

First, let us present the following important lemma.

Lemma 4.36. *Assume that the operators $\mathcal{A}_{11} : \mathcal{X}_1 \to \mathcal{X}_1$ and $\mathcal{A}_{22} : \mathcal{X}_2 \to \mathcal{X}_2$ are infinitesimal generators of two C_0 semigroups $\mathbb{T}_{\mathcal{A}_{11}}$ and $\mathbb{T}_{\mathcal{A}_{22}}$, respectively. Let the operator $\mathcal{A}_{21} : \mathcal{X}_1 \to \mathcal{X}_2$ be bounded. Then*

(a) $\mathcal{A}_e = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$ *is infinitesimal generator of the following C_0 semigroup in $\mathcal{X}_e = \mathcal{X}_1 \oplus \mathcal{X}_2$*

$$\mathbb{T}_{\mathcal{A}} = \begin{bmatrix} \mathbb{T}_{\mathcal{A}_{11}} & 0 \\ \mathbb{T}_{\mathcal{A}_{21}} & \mathbb{T}_{\mathcal{A}_{22}} \end{bmatrix}, \ \mathbb{T}_{\mathcal{A}_{21}}(t)x = \int_0^t \mathbb{T}_{\mathcal{A}_{22}}(t-s)\mathcal{A}_{21}\mathbb{T}_{\mathcal{A}_{11}} x ds.$$

(b) Moreover, if $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are regular RS operators with finitely many multiple eigenvalues and only finitely many common eigenvalues, then $\mathcal{A}_e$ is also a regular RS operator.

Proof. (a) This follows from the Proposition 4.7 in [130].

(b) We first show that the operator $\mathcal{A}_d = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ 0 & \mathcal{A}_{22} \end{bmatrix}$ is a regular RS with a finitely many multiple eigenvalues. It can be shown that λ is an eigenvalue of $\mathcal{A}_d$ if and only if λ is an eigenvalue of $\mathcal{A}_{11}$ or $\mathcal{A}_{22}$. Hence, $\mathcal{A}_d$ is an operator with finitely many multiple eigenvalues. Moreover, each generalized eigenvector of $\mathcal{A}_d$ can be expressed as $\begin{bmatrix} \phi_1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ \phi_2 \end{bmatrix}$, where ϕ_1 and ϕ_2 are the generalized eigenvectors of $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$, respectively. It follows that $\mathscr{P}_i^d = \mathcal{Q}\mathscr{P}_i$ (where $\mathscr{P}_i$ is the eigenspace of the operator $\mathcal{A}_{11}$ and $\mathcal{Q}$ is an embedding operator such that $\mathcal{Q} : \mathcal{X}_1 \to \mathcal{X}_e$ and $\mathcal{Q}x = \begin{bmatrix} x \\ 0 \end{bmatrix}$) is an eigenspace of $\mathcal{A}^e$. Furthermore, the same result holds for the eigenspaces of $\mathcal{A}_{22}$. Hence, it can be shown that the condition (3) in Definition 4.2 is satisfied. Finally, we show the inequality that is defined in Remark 4.3 holds. If $\lambda_i \in \sigma(\mathcal{A}_{11}) \cap \sigma(\mathcal{A}_{22})$, we select $d_i = \min(\inf_{\lambda \in \sigma(\mathcal{A}_{11}) - \lambda_i} |\lambda - \lambda_i|, \inf_{\lambda \in \sigma(\mathcal{A}_{22})} |\lambda - \lambda_i|)$. Since the number of common eigenvalues of $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ is finite, it follows that the inequality in Remark 4.3 is satisfied, and consequently $\mathcal{A}_d$ is a regular RS with a finitely many multiple eigenvalues. Given that the operator $\begin{bmatrix} 0 & 0 \\ \mathcal{A}_{21} & 0 \end{bmatrix}$ is bounded (with a bound equal to the bound of $\mathcal{A}_{21}$) and by invoking Remark 4.3, it follows that the operator $\mathcal{A}_e$ is a regular RS operator. This completes the proof of the lemma. $\square$

Note that $\mathcal{A}_o$ in (4.19) is assumed to be a regular RS operator and the operator $\mathcal{E}$ (and consequently $\mathcal{EC}$) is a bounded operator if $\mathcal{A}_o$ and $\mathcal{A}$ have only finitely many common eigenvalues. Hence, by using Lemma 4.36, it follows that $\mathcal{A}^e$, as per equation (4.21), is an infinitesimal generator of a C_0 semigroup, and also a regular RS operator. Next, we need to establish an important relationship between the unobservable subspace of the system (4.20) and the unobservability subspace of the system (4.17) as shown in the following lemma.

Lemma 4.37. *Consider the augmented system* (4.20) *and let* $\mathscr{N}^e = < \ker \mathcal{C}^e | \mathbb{T}_{\mathcal{A}^e} >$. *Then,* $\mathcal{Q}^{-1}\mathscr{N}^e$ *is an unobservability subspace of system* (4.17)*, where* $\mathcal{Q}$ *is the embedding operator.*

Proof. Let $\mathcal{S} = \mathcal{Q}^{-1}\mathscr{N}^e$, where $\mathcal{Q}$ is the embedding operator as defined above. We first show that $\mathcal{S}$ is a ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace of the system (4.17) (that is, $\mathcal{A}(\mathcal{S} \cap \ker \mathcal{C} \cap D(\mathcal{A})) \subseteq \mathcal{S}$). Let us show that $\overline{\mathcal{S} \cap D(\mathcal{A})} = \mathcal{S}$. Since $\mathscr{N}^e$ is $\mathbb{T}_{\mathcal{A}^e}$-invariant, we have $\overline{\mathscr{N}^e \cap D(\mathcal{A}^e)} = \mathscr{N}^e$. Assume $\overline{\mathcal{S} \cap D(\mathcal{A})} \neq \mathcal{S}$, and consequently there exits $x \in \mathcal{S}$ and a neighborhood $B \ni x$ such that $B \cap D(\mathcal{A}) = 0$. It follows that $\mathcal{Q}B \cap D(\mathcal{A}^e) = 0$ (note that $\mathcal{Q}x = \begin{bmatrix} x \\ 0 \end{bmatrix}$) that is in the contradiction with the fact that $\overline{\mathscr{N}^e \cap D(\mathcal{A}^e)} = \mathscr{N}^e$. Let $x \in (\mathcal{S} \cap \ker \mathcal{C} \cap D(\mathcal{A}))$. Now, since $\mathscr{N}$ is $\mathcal{A}^e$-invariant, one can write $\mathcal{A}^e \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}x \\ 0 \end{bmatrix} \in \mathscr{N}^e$. Hence, $\mathcal{A}x \in \mathcal{S}$, and $\mathcal{S}$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant, and consequently feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace (according to Lemma 4.14).

Now, we show that $\mathcal{S}$ satisfies the conditions in Theorem 4.17. Since $\mathscr{N}^e$ is $\mathbb{T}_{\mathcal{A}^e}$-invariant and $\mathcal{A}^e$ is a regular RS operator, from Corollary 4.10 we have $\mathscr{N}^e = \overline{\text{span}\{\mathscr{E}_i^e\}}_{\ i\in\mathbb{I}}$, where $\mathscr{E}_i^e$ are the sub-eigenspaces of $\mathcal{A}^e$. There are three possibilities for a sub-eigenspace of $\mathcal{A}^e$ as follows

1. $\mathscr{E}_i^e = \begin{bmatrix} \mathscr{E}_i \\ 0 \end{bmatrix}$, where $\mathscr{E}_i$ is a sub-eigenspace of $\mathcal{A}$.

2. $\mathscr{E}_i^e = \begin{bmatrix} 0 \\ \mathscr{E}_i^o \end{bmatrix}$, where $\mathscr{E}_i^o$ is sub-eigenspace of $\mathcal{A}_o$.

3. $\mathscr{E}_i^e = \begin{bmatrix} \mathscr{E}_i \\ \mathscr{E}_o \end{bmatrix}$, such that $\mathscr{E}_i$ and $\mathscr{E}_o$ are *not* sub-eigenspaces of $\mathcal{A}$ and $\mathcal{A}_o$ (this sub-eigenspace is corresponded to common eigenvalues of $\mathcal{A}$ and $\mathcal{A}_o$).

Let $\mathcal{S}_\phi$ denotes the largest subspace in the form $\mathcal{S}_\phi = \overline{\text{span}\{\mathscr{E}_i\}}_{\ i\in\mathbb{I}}$ such that $\mathscr{E}_i$ is a sub-eigenspace of $\mathcal{A}$ that is contained in $\ker H\mathcal{C}$. It follows that $\mathcal{S}_\phi \subseteq \mathcal{S}$ and

$\mathcal{S} = \overline{\mathcal{S}_\phi + \mathcal{S}_\mathrm{f}}$, where $\mathcal{S}_\mathrm{f}$ is a summation of sub-eigenspaces in the form item 3). Since there are only finitely many common eigenvalues of $\mathcal{A}$ and $\mathcal{A}_o$, it follows that $\mathcal{S}_\mathrm{f}$ is Fin-D. Therefore, $\mathcal{S}$ satisfies the condition of Theorem 4.17, and consequently $\mathcal{S}$ is $\mathbb{T}$-conditioned invariant.

Finally, given that $\mathcal{S} \subseteq \ker H_i\mathcal{C}$ and $\mathscr{N}^e$ is the largest $\mathbb{T}_{\mathcal{A}^e}$-invariant subspace in $\ker \mathcal{C}$, it follows that $\mathcal{S}$ is the largest $\mathbb{T}$-conditioned invariant subspace contained in $\ker H_i\mathcal{C}$ (i.e., $\mathcal{S}$ is an unobservability subspace of the RS system (4.17)). This completes the proof of the lemma. $\square$

To clarify of existence the subspace $\mathcal{S}_\mathrm{f}$ in the above proof, consider the following Fin-D example.

Example 4.38.

Assume

$$A^e = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{4.22}$$

Also, let $L_1 = [0,1,0,0]^\mathrm{T}$ and $L_2 == [1,1,1,0]^\mathrm{T}$. It follows that $L_2 = A^eL_1$ and $A^eL_2 = 2L_2 - L_1$. Therefore, $\mathscr{E} =$ span$\{L_1, L_2\}$ is sub-eigenspace of A^e (corresponding to $\lambda = 1$). However, $Q^{-1}\mathscr{E} = \mathscr{L}_1$ is not a sub-eigenspace of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This is the reason with consider $\mathcal{S}_\mathrm{f}$ in the proof of the above Lemma.

In order to provide sufficient conditions for solvability of the FDI problem, one also needs to show that the error dynamics corresponding to the designed fault detection observer is stable. The following theorem provides necessary and sufficient conditions for stability of a general Inf-D system.

Lemma 4.39. *[14] Consider the Inf-D system $\dot{e}(t) = \mathcal{A}'_e e(t)$, such that $\mathcal{A}'_e$ is an infinitesimal generator of a C_0 semigroup. This system is exponentially stable if and* *only if there exists a positive definite and bounded operator* $\mathcal{P}_e : \mathcal{X} \to \mathcal{X}$ *such that*

$$< \mathcal{A'}_e z, \mathcal{P}_e z > + < \mathcal{P}_e z, \mathcal{A'}_e z >= - < z, z > . \tag{4.23}$$

We are now in the position to derive the solvability necessary and sufficient conditions for the FDI problem corresponding to the RS system (4.17).

Theorem 4.40. *Consider the regular RS system* (4.17)*. The FDI problem has a solution only if*

$$\mathcal{S}_i^* \cap \mathscr{L}_i = 0, \tag{4.24}$$

where $\mathcal{S}_i^* =< \ker H_i \mathcal{C} | \mathcal{A} + \mathcal{D}_i \mathcal{C} >$ *is the smallest unobservability subspace containing* $\mathscr{L}_j$*, where* $j = 1, \cdots, p$ *and* $j \neq i$*, and* $\mathscr{L}_i = \operatorname{span}\{\mathcal{L}_i\}$ *. On the other hand, if the above condition is satisfied and there exist two maps* $\mathcal{D}_o$ *and* $\mathcal{P}_e$ *such that* $(\mathcal{A}_p + \mathcal{D}_o \mathcal{M}_i)$ *and* $\mathcal{P}_e$ *satisfy the condition* (4.23)*, then the FDI problem is solvable where* $\mathcal{A}_p = (\mathcal{A} + \mathcal{D}_i \mathcal{C})|_{\mathcal{X}/\mathcal{S}_i^*}$ *(i.e.,* $\mathcal{A}_p$ *is the operator induced by* $\mathcal{A} + \mathcal{D}_i \mathcal{C}$ *on the factor space* $\mathcal{X}/\mathcal{S}_i^*$*), and* $\mathcal{M}_i$ *is the solution to* $\mathcal{M}_i \mathcal{P}_i = H_i \mathcal{C}$*, where* $\mathcal{P}_i$ *is the canonical projection from* $\mathcal{X}$ *on* $\mathcal{X}/\mathcal{S}_i^*$*.*

Proof. (**Only if part**): We consider the system (4.17) that is subject to two faults f_1 and f_2. Assume that the detection filter (4.19) is designed such that the residual (that is, the output of the filter) is decoupled from the fault f_2 but requires to be sensitive to the fault f_1. By considering the augmented system (4.20), it is necessary that $\mathscr{L}_2^e = \operatorname{span}\{\mathcal{L}_2^e\} \subseteq \mathscr{N}^e$, ($\mathcal{L}_2^e$ is defined in (4.21)) where $\mathscr{N}^e$ is the unobservable subspace of (4.20), and by using Lemma 4.37, the subspace $\mathcal{S} = \mathcal{Q}^{-1}\mathscr{N}^e$ is an unobservability subspace of the pair ($\mathcal{C}$,$\mathcal{A}$) containing $Q^{-1}\mathscr{L}_2^e = \mathscr{L}_2$. Moreover, in order to detect the fault f_1 (which can be an arbitrary function of time), it is necessary that $\mathscr{N}^e \cap \mathscr{L}_1^e = 0$. Hence, $\mathcal{S} \cap \mathscr{L}_1 = 0$. Since $\mathcal{S}_1^*$ is the minimal unobservability subspace containing $\mathscr{L}_2$ (i.e., $\mathcal{S}_1^* \subseteq \mathcal{S}$), the necessary condition for satisfying the above condition is $\mathcal{S}_1^* \cap \mathscr{L}_1 = 0$.

(**If part**): Assume that $\mathcal{S}_1^* \cap \mathscr{L}_1 = 0$, and let $\mathcal{D}_1$ and H_1 be defined according to $\mathcal{S}_1^*$

(refer to the Definition 4.25). By definition, $\mathcal{L}_2 \subseteq \mathcal{S}_1^*$ where $\mathcal{S}_1^*$ is the unobservable subspace of the system $(H_1\mathcal{C}, \mathcal{A}+\mathcal{D}_1\mathcal{C})$. In other words, $\mathcal{S}_1^* =< \ker H_1\mathcal{C}|\mathcal{A}+\mathcal{D}_1\mathcal{C} >$.

Now consider the canonical projection $\mathcal{P}_1 : \mathcal{X} \to \mathcal{X}/\mathcal{S}_1^*$ and the following detection filter

$$\begin{aligned}\dot{\omega}_1(t) =& \mathcal{F}_1\omega_1(t) + \mathcal{G}_1 u(t) - \mathcal{E}_1 y(t)\\ r_1(t) =& \mathcal{M}_1\omega_1(t) - H_1 y(t)\end{aligned} \tag{4.25}$$

where $\mathcal{F}_1 = \mathcal{A}_p + \mathcal{D}_o\mathcal{M}_1$, $\mathcal{G} = \mathcal{P}_1\mathcal{B}$ and $\mathcal{E}_1 = \mathcal{D}_1 + \mathcal{P}_1^{-r}\mathcal{D}_o H_1$. By defining the error $e(t) = \mathcal{P}_1 x(t) - \omega_1(t)$ and following along the same steps as in Section 2.2, one can obtain

$$\begin{aligned}\dot{e}(t) &= \mathcal{F}_1 e(t) + \mathcal{P}_1\mathcal{L}_1 f_1(t),\\ r_1(t) &= \mathcal{M}_1 e(t).\end{aligned} \tag{4.26}$$

By invoking Lemma 4.39, it follows that the error dynamics (4.26) is exponentially stable. Therefore, if $f_1 \equiv 0$ (for any value of f_2) then $r_1(t) \to 0$. Otherwise, $||r_1(t)|| \neq 0$. This completes the proof of the theorem. □

Remark 4.41. *Note that the FDI problem was solved by designing a fault detection filter to estimate x_1. However, unlike the Fin-D case, the condition $\mathscr{N} = 0$ (the unobservable subspace) is not sufficient for the existence of an observer for a general Inf-D system [130]. Therefore, the condition* (4.24) *is not sufficient for solvability of the FDI problem, and one needs the extra condition that is stated in Theorem 4.40.*

Remark 4.42.
1. *The condition that $\mathcal{A}$ and $\mathcal{A}_o$ must have only finite many common eigenvalues is only needed for "only if" part of the above proof. Note that this condition is necessary to show that $\mathcal{Q}^{-1}\mathscr{N}^e$ is an unobservability subspace.*
2. *For "if part", we do not need this condition, since*
   (a) *We do not deal with the augmented system.*

(b) *Directly by using Lemma 14, we assume that* $\mathcal{A}_p+\mathcal{D}_o\mathcal{M}_1$ *is asymptotically stable (i.e. the error dynamics of the detection filter is asymptotically stable).*

4.3.3 Solvability of the FDI Problem Under Two Special Cases

In this subsection, we investigate two special cases, where the condition (4.24) provides a *single* necessary and sufficient condition for solvability of the FDI problem.

<u>**Case 1**</u>

The following theorem provides a necessary and sufficient condition for solvability of the FDI problem when the number of positive eigenvalues of the quotient subsystem is finite.

Theorem 4.43. *Consider the faulty system* (4.17) *with* $\mathcal{C}$ *specified as in equation* (4.2)*, and let the operator* $(\mathcal{A}+\mathcal{D}_1\mathcal{C})$ *has only finite number of positive eigenvalues and the operator* $\mathcal{A}_p = (\mathcal{A}+\mathcal{D}_1\mathcal{C})|_{\mathcal{X}/\mathcal{E}^+}$ *is asymptotically stable, where* $\mathcal{E}^+$ *is the sum of eigenspaces corresponding to the positive eigenvalues. The FDI problem is solvable if and only if the condition* (4.24) *holds.*

Proof. (**if part**): Consider the detection filter (4.25). As stated above, the observer gain $\mathcal{D}_o$ is designed such that the operator $\mathcal{A}_p+\mathcal{D}_o\mathcal{M}_1$ is asymptotically stable.

Given that the unobservable subspace of system ($\mathcal{M}_1$, $\mathcal{A}_p$) is zero (since it is obtained by factoring out $\mathcal{S}_1^*$), the pair Fin-D (M_1^+, A_p^+) (that are induced from $\mathcal{M}_1$ and $\mathcal{A}_p$ on $\mathcal{X}_1^+$) is observable. Therefore, there exists an operator $D_o : \mathbb{R}^{q_h} \to \mathcal{X}_1^+$ such that all the eigenvalues of $A_p^+ + D_oM_1^+$ are negative. By invoking the asymptotic stability of $\mathcal{A}_p^-$, and considering $\mathcal{D}_o$ as the extension of D_o, one can show that the error dynamics (4.26) is asymptotically stable. By following along the same lines as in the proof of Theorem 4.40, it follows that the FDI problem is solvable.

(**only if part**): This follows from the results that are stated in Theorem 4.40. This completes the proof of the theorem. $\square$

Case 2

In this case, the system (4.17) is specified according to the operator given by equation (4.2), however c_i are governed and restricted to

$$c_i = \sum_{i=1}^{n_c} \zeta_{i,j}\psi_j. \tag{4.27}$$

In other words, c_i vectors lie on a finite dimensional subspace of $\mathcal{X}$. Since $<\phi_i, \psi_j >= \delta_{ij}$, it follows that $\mathcal{C}\phi_i = 0$ for all $i > n_c$. Therefore, $\mathrm{span}\{\phi_i\}_{i=n_c+1}^{\infty} \subseteq \ker \mathcal{C}$, and consequently, $\ker \mathcal{C} = \overline{\mathscr{C}_f^0 \oplus \mathrm{span}\{\phi_i\}_{i=n_c+1}^{\infty}}$, where $\mathscr{C}_f^0 \subseteq \mathrm{span}\{\{\phi_j\}_{j=1}^{n_c}\}$. By invoking Lemma 4.13 and the fact that $\dim(\mathscr{C}_f^0) < \infty$, we have $\ker \mathcal{C} = \mathscr{C}_f^0 \oplus \overline{\mathrm{span}\{\phi_i\}_{i=n_c+1}^{\infty}}$. Since every $\overline{\{\phi_i\}_{i=n_c+1}^{\infty}} \subseteq \ker \mathcal{C}$ is also $\mathbb{T}_{\mathcal{A}+\mathcal{DC}}$-invariant and contained in $\ker H\mathcal{C}$, it follows that the unobservability subspace $\mathcal{S}$ containing a given subspace $\mathscr{L}$ necessarily contains the Inf-D subspace $\overline{\{\phi_i\}_{i=n_c+1}^{\infty}}$. Therefore, the factored out quotient subsystem ($\mathcal{M}_1$, $\mathcal{A}_p$) is Fin-D and one can provide necessary and sufficient conditions for solvability of the FDI problem. The following theorem summarizes this result.

Theorem 4.44. *Consider the faulty system* (4.17) *that is assumed to be an RS system and specified according to the output operator* (4.27)*. The FDI problem is solvable if and only if $\mathcal{S}_i^* \cap \mathscr{L}_i = 0$, where $\mathcal{S}_i^*$ is the smallest unobservability subspace containing $\mathscr{L}_j$, $j = 1, \cdots, p$ and $j \neq i$.*

Proof. (**if part**): Since $\mathcal{X}/\mathcal{S}_1^*$ is a Fin-D vector space and the system (M, A_{11}) (where $A_{11} = (\mathcal{A} + \mathcal{DC})|_{\mathcal{S}_1^*}$ and $M\mathcal{P} = H\mathcal{C}$) is observable and Fin-D. Therefore, there always exists the operator $\mathcal{D}_o$ such that the observer (4.19) can both detect and isolate the fault f_i. Given that the detection filter is Fin-D, the stability of the error dynamics is gauranteed by the observability of the system $(M,\, A_{11})$.

(**only if part**): This follows from the results that are stated in Theorem 4.40.

This completes the proof of the theorem. $\square$

4.3.4 Summary of Results

In this section, the FDI problem was formulated by invoking invariant subspaces that were introduced and developed in Section 4.2. We first derived in Theorem 4.40 necessary and sufficient conditions for solvability of the FDI problem. Moreover, it was shown that for two special classes of regular RS systems there exists a *single* necessary and sufficient condition (that is, the condition (4.24)) for solvability of the FDI problem. Table 4.1 summarizes and provides a pseudo-code and procedure for detecting and isolating faults in the RS system (4.17).

Remark 4.45. *As illustrated above, the main difficulty in deriving a single necessary and sufficient condition for solvability of the FDI problem for a general RS system has its roots in the relationship between the condition* $\mathscr{N} = 0$ *and the existence of a bounded observer gain* $\mathcal{D}$ *such that the corresponding error dynamics is exponentially stable. Another possible approach that one can investigate and pursue is through a frequency-based approach that was originally developed in [101] to investigate the disturbance decoupling problem. This approach deals with the Hautus test, and as shown in [134] the Hautus test does also involve certain new challenges for Inf-D systems. Specifically, there exist certain Inf-D systems that pass the Hautus test, however they are not observable. Notwithstanding the above, the investigation of utilization of a frequency-based approach for tackling the FDI problem and its relationship with invariant subspaces introduced in this chapter is beyond the scope of this thesis, therefore we suggest this line of research as part of our future work.*

Table 4.1: Pseudo-algorithm for detecting and isolating the fault f_i in the regular RS system (4.17).

1. Compute the minimal conditioned invariant subspace $\mathscr{W}^*$ containing all $\mathcal{L}_k$ subspaces such that $k \neq i$ (by using the algorithm (4.11) where $\mathscr{L} = \sum_{j \neq i} \mathcal{L}_j$).
2. Compute the unobservability subspace $\mathcal{S}_i^*$ containing $\sum_{j \neq i} \mathcal{L}_j^1$ (by using the algorithm (4.14)).
3. Compute the operator $\mathcal{D}_i$ such that $\mathcal{D}_i \in \underline{D}(\mathscr{W}^*)$.
4. Find the operator H_i such that $\ker H_i \mathcal{C} = \overline{\mathscr{W}^* + \ker \mathcal{C}} = \overline{\mathcal{S}_i^* + \ker C}$.
5. If $\mathcal{S}_i^* \cap \mathcal{L}_i = 0$, then the necessary condition for solvability of the FDI problem is satisfied. Moreover, if one of the following conditions are satisfied, the FDI problem is solvable. In other words, one can design a detection filter according to the structure provided in (4.19) to detect and isolate f_i,
   - If there exists a bounded operator $\mathcal{D}_o$ such that the conditions of Theorem 4.40 are satisfied, or
   - The operator $\mathcal{A}_p = (\mathcal{A} + \mathcal{D}_i \mathcal{C})|_{\mathcal{X}/\mathcal{S}_i^*}$ has finite number of positive eigenvalues, or
   - If $\dim(\mathcal{X}/\mathcal{S}_i^*) < \infty$.

   The operators in the detection filter (4.19) are defined as follows. Let $\mathcal{P}_i$ be the canonical projection of $\mathcal{S}_i^*$, then $\mathcal{A}_o = (\mathcal{A} + \mathcal{D}_i \mathcal{C})|_{\mathcal{X}/\mathcal{S}_i^*} + \mathcal{D}_o \mathcal{M}_i$, $\mathcal{B}_o = \mathcal{P}_i \mathcal{B}$, $\mathcal{M}_i \mathcal{P}_i = H_i \mathcal{C}$, $\mathcal{E} = \mathcal{D}_o H_i$ and $\mathcal{D}_o$ is selected such that $\mathcal{A}_o$ satisfies the condition of Lemma 4.39. Moreover, the output of the detection filter (i.e., $r_i(t)$) is the residual that satisfies the condition (4.18).

Finally, to add further clarification and information we have provided in Figure 4.1 a schematic summarizing and depicting the relationships among the various lemmas, theorems and corollaries that are presented and developed in this chapter.

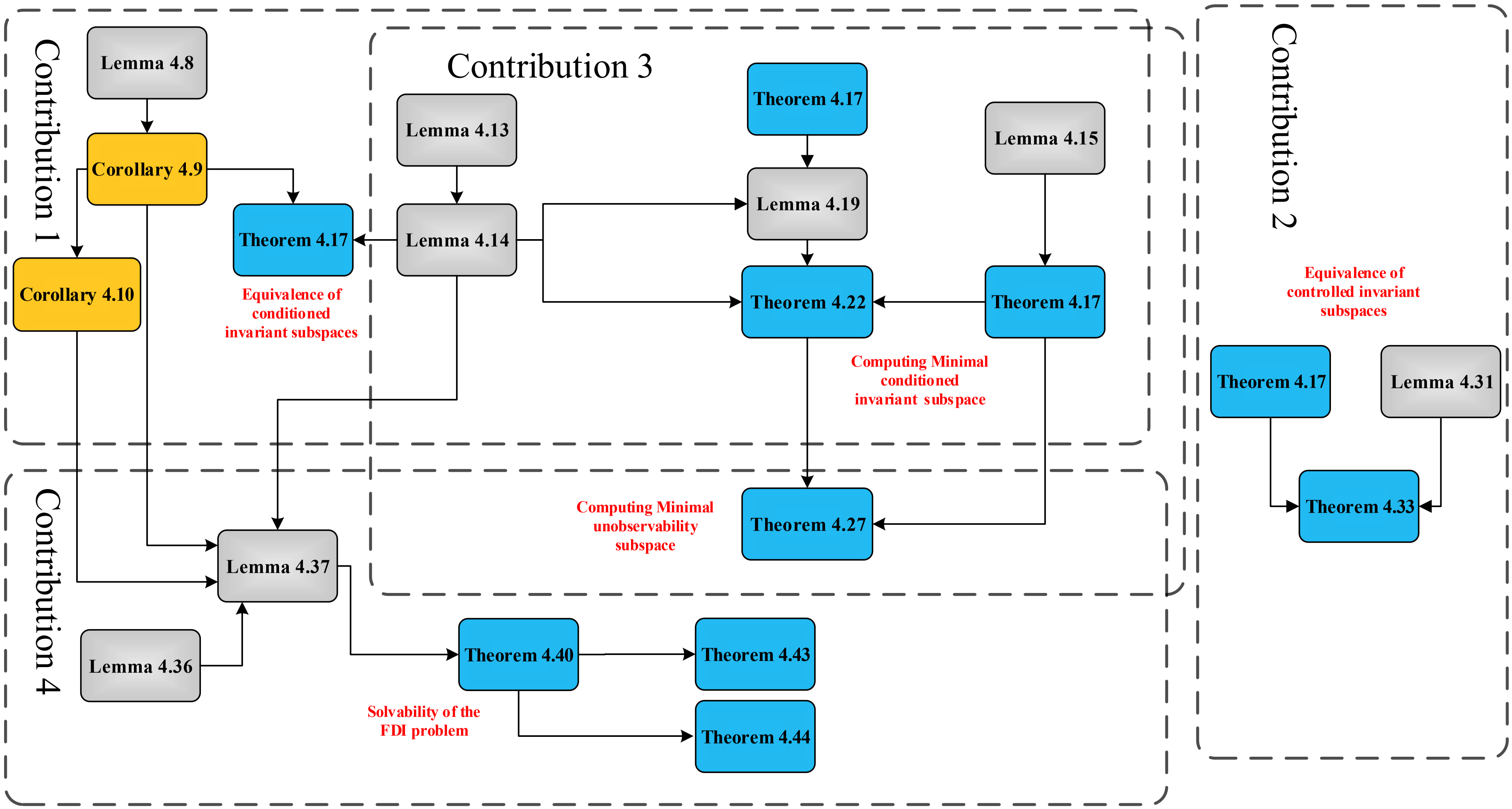


Figure 4.1: The flowchart diagram depicting the relationships among lemmas, theorems and corollaries that are developed and presented in this chapter.

4.4 Numerical Example

In this section, we provide a numerical example to demonstrate the applicability of our proposed approach. We consider a PDE system that represents a linearized approximation to the model that corresponds to a large class of chemical processes, such as the two-component reaction-diffusion process (for more detail refer to [79]).

Consider the following parabolic PDE system.

$$\begin{aligned}
\begin{bmatrix} \frac{\partial \tilde{x}_1(t,z)}{\partial t} \\ \frac{\partial \tilde{x}_2(t,z)}{\partial t} \end{bmatrix} &= \begin{bmatrix} \frac{\partial^2}{\partial z^2} & 0.1 \\ 0.1 & \frac{\partial^2}{\partial z^2} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t,z) \\ \tilde{x}_2(t,z) \end{bmatrix} + b_1(z)\tilde{u}_1(t,z) + b_2(z)\tilde{u}_2(t,z) + L_1(z)\tilde{f}_1(t,z) \\
&\quad + L_2(z)\tilde{f}_2(t,z) + \begin{bmatrix} \nu_1(t,z) \\ \nu_2(t,z) \end{bmatrix}, \\
y_1(t) &= \int_0^\pi c_1(z)\tilde{x}(t,z)\mathrm{d}z + w_1(t,z), \tilde{x}_i(t,0) = 0,\ i = 1,2, \qquad (4.28) \\
y_2(t) &= \int_0^\pi c_2(z)\tilde{x}(t,z)\mathrm{d}z + w_2(t), \tilde{x}_i(t,\pi) = 0,\ i = 1,2,
\end{aligned}$$

where $\tilde{x}(t,z) = [\tilde{x}_1(t,z), \tilde{x}_2(t,z)]^{\mathrm{T}} \in \mathbb{R}^2$ and $\tilde{u}_i(t,z) \in \mathbb{R}$ denote the state and input, respectively. Also, $z \in [0,\pi]$ denotes the spatial coordinate, and $c_i \in L_2([0,\pi],\mathbb{R})$, where $L_2([0,\pi],\mathbb{R})$ denotes the space of all square integrable functions over $[0,\pi]$. Also ν_i and w_i $(i = 1,2)$ denote the process and measurement noise that are assumed to be normal distributions with 0.5 and 0.2 variances, respectively. Note that the boundary conditions assure that the temperatures at $z = 0$ and $z = \pi$ are fixed. Here for simplicity of presentation we assume $\tilde{x}_i(t,0) = 0$ and $\tilde{x}_i(t,\pi) = 0$, $i = 1,2$. Moreover, the faults f_1 and f_2 represent malfunctions in the heat jackets (that are modeled by invoking the input vectors b_1 and b_2).

The system (4.28) can be expressed in the representation of (4.17) (by neglecting the disturbances and noise signals ν_i and w_i) by utilizing the spectral operator $\mathcal{A} = \begin{bmatrix} \frac{\partial^2}{\partial z^2} & 0.1 \\ 0.1 & \frac{\partial^2}{\partial z^2} \end{bmatrix}$, where the domain of $\mathcal{A}$ is defined by [14, Chapter 1]:

$$D(\mathcal{A}) = \{x \in \mathbb{L}_2([0,\pi]) \mid x, \frac{\mathrm{d}x}{\mathrm{d}z} \text{ are absolutely continuous}\}.$$

By solving the corresponding Sturm-Liouville problem [135], the eigenvalues of $\mathcal{A}$ are obtained as $\lambda_k^1 = 0.1 - k^2, \lambda_k^2 = -0.1 - k^2, k \in \mathbb{N}$, and the corresponding eigenfunctions are given by $\phi_k^1 = \sqrt{\frac{2}{\pi}}[\sin(kz), \sin(kz)]^{\mathrm{T}}$ and $\phi_k^2 = \sqrt{\frac{2}{\pi}}[\sin(kz), -\sin(kz)]^{\mathrm{T}}$. Note that ϕ_k^1 and ϕ_k^2 are bi-orthonormal. In other words, $\phi_k^1 \perp \phi_j^2$, for all $j \in \mathbb{N}$ and $\phi_k^1 \perp \phi_j^1$, for $j \neq k$. Therefore, $\psi_k^i = \phi_k^i$, $i = 1, 2$ and $k \in \mathbb{N}$. Consider the system (4.28), where

$$c_1(z) = \begin{cases} [1,1]^{\mathrm{T}} & 0 \leq z \leq \pi/4 \\ 0 & ;\ \text{Otherwise} \end{cases}, \quad c_2(z) = \begin{cases} [1,-1]^{\mathrm{T}} & 3\pi/4 \leq z \leq \pi \\ 0 & ;\ \text{Otherwise} \end{cases} \tag{4.29}$$

Note that from practical point of view the structure of c_1 and c_2 is determined by characteristics of sensors.

Let us assume $b_i(z) = \sum_{k=5}^{\infty} \zeta_k^i \phi_k^i$, where $\zeta_k^1 = [\frac{1}{k}\ , \frac{1}{k}]^{\mathrm{T}}$, and $\zeta_k^2 = [\frac{1}{k^2}\ , -\frac{1}{k^2}]^{\mathrm{T}}$ for $k \geq 5$. Moreover, let $L_i(z) = b_i(z)$ $i = 1, 2$ (for all $z \in [0, \pi]$) represent actuator faults. Finally, let $\mathcal{C} = [< c_1, \cdot >,\ < c_2, \cdot >]^{\mathrm{T}}$, with c_1 and c_2 given above. As observed below the condition for the Case 1 stated in Section 4.3.3 does hold.

In the following, a detection filter is designed for detecting and isolating the fault f_1. Since $\mathscr{L}_2 = \text{span}\{L_2\} \in D(\mathcal{A})$ and $\mathscr{L}_2 \notin \ker \mathcal{C}$, we obtain $\mathscr{Z}^* = \mathscr{Z}_1 = \mathscr{L}_2$ from the algorithm (4.11). Hence, one can write $\mathscr{W}_\ell = 0$ (since $\mathscr{L}_{\mathscr{N}} = 0$). Therefore, $\mathscr{W}^* = \mathscr{L}_2$. By setting $\mathscr{W}_{\phi,\mathrm{f}}^* = \mathscr{W}_{\mathrm{f}}^*$ and since $c_1 \perp \phi_k^2$ for all $k \in \mathbb{N}$, $0 \in \rho_\infty(\mathcal{A})$, we have $\mathscr{N} + \mathscr{L}_2 = \overline{\text{span}\{\phi_k^2\}\ _{k\in\mathbb{N}}}$ (i.e., the unobservable subspace of system (4.28) with only one input $y = c_1 x$). Given that $\mathscr{W}^* = \mathscr{L}_2$, we obtain that $\mathcal{S}_1^* = \overline{\text{span}\{\phi_k^2\}\ _{k\in\mathbb{N}}}$. It follows that $\mathscr{L}_1 \cap \mathcal{S}_1^* = 0$, and a solution for the corresponding maps $\mathcal{D}_1$ and H_1 is given by $\mathcal{D}_1 = 0$ and $H_1 = [1, 0]$. The factored out subsystem can therefore be specified by using the canonical projection on $\mathcal{S}_1^*$, that is $\mathcal{P}_1 : \mathcal{X} \to \mathcal{X}/\mathcal{S}_1^*$, as follows

$$\begin{aligned} \dot{\omega}_1(t) &= \mathcal{A}_p \omega(t) + \mathcal{P}_1 \mathcal{B} u(t) + \mathcal{P}_1 \mathcal{L}_1 f_1(t), \\ y_\omega(t) &= \mathcal{M}_1 \omega_1(t), \end{aligned} \tag{4.30}$$

where $\omega_1 \in \mathcal{X}/\mathcal{S}_1^*$, $u = [u_1,\ u_2]^{\mathrm{T}}$, $y_\omega = H_1 y$, $\mathcal{A}_p$ and $\mathcal{M}$ are solutions to the equations

$\mathcal{A}_p\mathcal{P}_1 = \mathcal{P}_1\mathcal{A}$ and $\mathcal{M}\mathcal{P} = H\mathcal{C}$, respectively, and are given by

$$\mathcal{A}_p = \frac{\partial^2}{\partial z^2} + 0.1, \quad \mathcal{M}_1\omega_1 =< c_2, \omega_1 > . \tag{4.31}$$

By solving the corresponding Sturm-Liouville problem [135], the eigenvalue of $\mathcal{A}_p$ is given as $\lambda_k^1 = 0.1 - k^2$, $i \in \mathbb{N}$. The eigenvalue of $\mathcal{A}_p$ is Since all the eigenvalues of $\mathcal{A}_p$ are negative (the condition for Case 1 in Subsection 4.3.3), by using Theorem 4.43 a detection filter is therefore specified according to

$$\begin{aligned} \dot{\omega}_1(t) &= \mathcal{A}_o\omega_1(t) + \mathcal{P}_1\mathcal{B}u(t), \\ r_1(t) &= H_1y(t) - \mathcal{M}_1\omega_1(t), \end{aligned} \tag{4.32}$$

where $\mathcal{A}_o = \mathcal{A}_p$. In other words, the detection filter to detect and isolate the fault f_1 is given by

$$\frac{\partial\tilde{\omega}_1(t,z)}{\partial t} = \frac{\partial^2\tilde{\omega}_1(t,z)}{\partial z^2} + 0.1\tilde{\omega}_1(t,z) + b_{11}(z)\tilde{u}_1(t,z) + b_{21}(z)\tilde{u}_2(t,z) \tag{4.33}$$

where $\tilde{\omega}_1(t,z) \in \mathbb{R}$ is the corresponding function to $\omega_1(t) \in \mathcal{X}$, $[b_{11}(z), b_{21}(z)]^{\mathrm{T}} = \mathcal{P}_1[b_1(z), b_2(z)]^{\mathrm{T}}$. The error dynamics corresponding to the above detection filter (i.e., $e(t) = \mathcal{P}_1x(t) - \omega_1(t)$) is given by $\dot{e}(t) = \mathcal{A}_pe(t) + \mathcal{P}_1\mathcal{L}_1f_1(t)$. Therefore, if $f_1 = 0$, the error converges to the origin exponentially. Otherwise, $e \neq 0$. The above residual (i.e, r_1) corresponding to the fault f_1 is also decoupled from f_2. By following the same steps as above, one can design a detection filter to detect and isolate the fault f_2. These details are therefore not included. We no have $\mathcal{S}_2^* = \overline{\ \mathrm{span}\{\phi_k^1\}\ _{k\in\mathbb{N}}}$, and the filter to detect and isolate f_2 is given by

$$\frac{\partial\tilde{\omega}_1(t,z)}{\partial t} = \frac{\partial^2\tilde{\omega}_1(t,z)}{\partial z^2} - 0.1\tilde{\omega}_1(t,z) + b_{12}(z)\tilde{u}_1(t,z) + b_{22}(z)\tilde{u}_2(t,z), \tag{4.34}$$

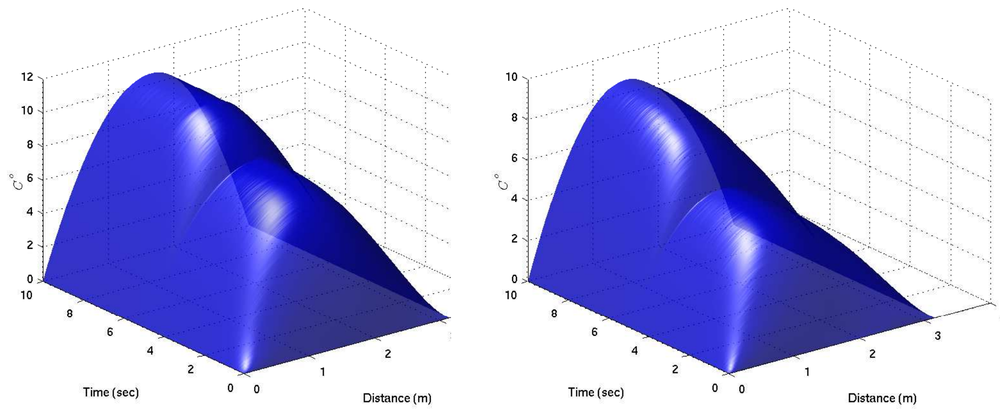
where $[b_{11}(z), b_{21}(z)]^{\mathrm{T}} = \mathcal{P}_2[b_1(z), b_2(z)]^{\mathrm{T}}$, and $\mathcal{P}_2 : \mathcal{X} \to \mathcal{X}/\mathcal{S}_2^*$ is the canonical projection on $\mathcal{S}_2^*$.

For the purpose of simulations, we consider a scenario where the fault f_1 with a severity of 2 occurs at $t = 5$ *sec* and the fault f_2 with a severity of -1 occurs at $t = 7$ *sec*. Figure 4.2 shows the states of the system (4.28) (namely, $\tilde{x}_1$ and $\tilde{x}_2$ with disturbances and noise signals ν_i and w_i included in the simulations), and Figure 4.3 depicts the residuals r_1 and r_2. It clearly follows that r_i is only sensitive to the fault f_i, $i = 1, 2$. Note that the thresholds are computed based on running 70 Monte Carlo simulations for the *healthy system* with the thresholds selected as the maximum residual signals r_1 and r_2 during the entire simulation runtime. The selected thresholds are $th_1 = 0.09$ and $th_2 = 0.064$, corresponding to the residual signals r_1 and r_2, respectively. The faults f_1 and f_2 are detected at $t = 5.051$ *sec* and $t = 7.31$ *sec*, respectively. Table 4.2 shows the detection times corresponding to various severity fault cases that are simulated. This table clearly shows the impact of the fault severity levels on the detection times. In other words, the lower the fault severity, the longer the detection time delay. Moreover, the *minimum* detectable fault severities associated with f_1 and f_2 are determined to be 0.05 and 0.15, respectively.

Remark 4.46. *The conducted simulations are performed by using the finite-element methods that are not based on the eigenvalues of $\mathcal{A}$. More specifically, we use the "*`pdepe`*" function in MATLAB to generate the data and simulate the filters. In other words, we design the filters based on the eigenvalues and eigenvectors and perform the simulations by using a different approach. Therefore, the provided results emphasize that the proposed method can detect and isolate faults in the actual PDE system.*

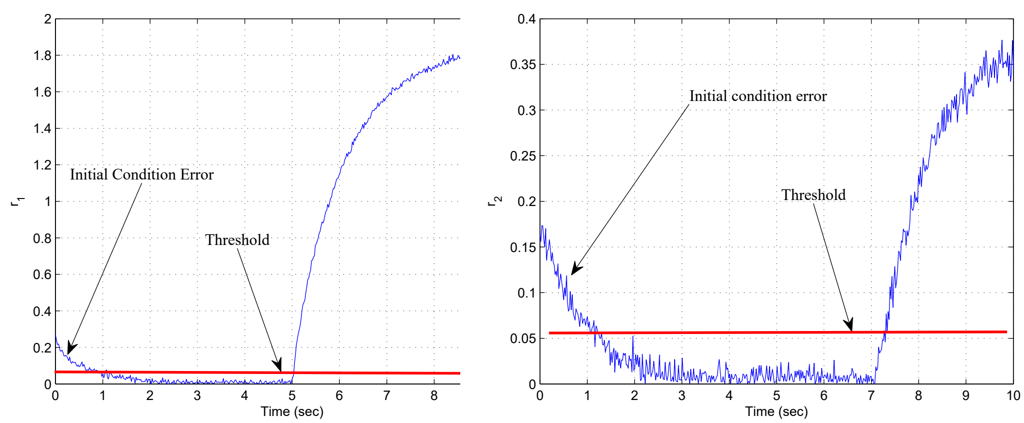
Remark 4.47. *When compared with approximate approaches that are developed in [15, 17] and [20] two main issues are worth pointing out:*

1. *The approximation of the system* (4.17) *is based on only the operator $\mathcal{A}$. As stated in [20], system* (4.17) *was approximated by using the first two to four eigenvalues. However, since the fault signatures (namely, L_1 and L_2) in the above example have no effect on the eigenspaces of the first five eigenvalues,*



(a) The state $\tilde{x}_1$. (b) The state $\tilde{x}_2$.

Figure 4.2: The states of the system (4.28). The faults f_1 and f_2 occur at $t = 5$ *sec* and $t = 7$ *sec* with severities of 2 and -1, respectively.



(a) The residual signal r_1 for detecting and isolating the fault f_1. (b) The residual signal r_2 for detecting and isolating the fault f_2.

Figure 4.3: The residual signals for detecting and isolating the faults f_1 and f_2. The faults occur at $t = 5$ *sec* and $t = 7$ *sec* with severities of 2 and -1, respectively.

Table 4.2: Detection time delay of the faults f_1 and f_2 corresponding to various severities.

Severity \ Fault	f_1 (sec)	f_2 (sec)
$f_1 = 2$, $f_2 = -1$	0.051	0.31
$f_1 = 0.5$, $f_2 = 0.5$	0.21	0.555
$f_1 = 0.09$, $f_2 = 0.2$	1.18	1.04
$f_1 = 0.05$, $f_2 = 0.15$	4.7	1.34

the faults f_1 *and* f_2 would not have been detectable *by using the approaches in [15, 17] and [20].*

2. *In the references [15, 17] and [20], it is necessary that the Inf-D system has eigenvalues that are far in the left-half plane, that result in extremely fast transient times (refer to Assumption 1 in [15]), whereas our proposed approach in this chapter does not suffer from this restriction and limitation.*

4.5 Summary

In this chapter, geometric characteristics associated with the regular Riesz spectral (RS) systems are investigated and new properties are introduced, specified, and developed. Specifically, various types of invariant subspaces such as the $\mathcal{A}$- and $\mathbb{T}$-conditioned invariant and $\mathbb{T}$-unobservability subspaces are developed and analyzed. Moreover, necessary and sufficient conditions for equivalence of various conditioned invariant subspaces are also provided. Under certain conditions, the algorithms corresponding to computing invariant subspaces are shown to indeed converge in a finite number of steps. Finally, we formulate and introduce the problem of fault detection and isolation (FDI) of RS systems, for the first time in the literature, in terms of invariant subspaces. For regular RS systems, we have developed and presented necessary and sufficient conditions for solvability of the FDI problem.

Chapter 5

Fault Detection and Isolation of Inf-D Systems by Using Semigroup Invariant Subspaces

In this chapter we focus on derivation of necessary and sufficient conditions for equivalence of invariant subspaces of Inf-D systems with their applications on the FDI problem. The Inf-D system that is considered in this chapter is more general than RS systems (which were addressed in Chapter 4).

As stated earlier, due to complexity of unbounded operators various definitions of invariant subspaces are introduced, where these subspaces are equivalent in Fin-D systems and inequivalent in Inf-D systems. In this chapter, we first address invariant subspaces of Fin-D systems from a new point of view by invoking resolvent operators. This approach enables one to extend the results to Inf-D systems. Particularly, we derive necessary and sufficient conditions for equivalence of various types of conditioned and controlled invariant subspaces of Inf-D systems. Duality properties of Inf-D systems are then investigated. Finally, by introducing unobservability subspaces for Inf-D systems we precisely formulate the FDI problem, and necessary and sufficient conditions for solvability of the FDI problem are provided.

5.1 Inf-D Systems

Consider the following Inf-D system.

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{5.1}$$

where $x(t) \in \mathcal{X}$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ are the state, input and output vectors, respectively. $\mathcal{X}$ is a real separable Hilbert space equipped by the dot-product $< \cdot, \cdot >$. As in Chapter 4, we consider the following input and output operators,

$$\mathcal{B} : \mathbb{R}^m \to \mathcal{X},\ \mathcal{B}u = \sum_{i=1}^{m} u_i b_i, \tag{5.2}$$

where $u = \begin{bmatrix} u_1, \cdots, u_m \end{bmatrix}^{\mathrm{T}}$ and $b_i \in \mathcal{X}$, $i = 1, \cdots, m$, and

$$\mathcal{C} = \begin{bmatrix} < c_1, \cdot >, < c_2, \cdot >, \cdots, < c_q, \cdot > \end{bmatrix}^{\mathrm{T}} \tag{5.3}$$

in which $c_i \in \mathcal{X}$, $i = 1, \cdots, q$. It follows that the operators $\mathcal{B}$ and $\mathcal{C}$ are bounded and finite rank. However, unlike Chapter 4, in this chapter we do not restrict $\mathcal{A}$ to be a regular RS operator. More precisely, we consider the unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{X}$ that is closed and is the infinitesimal generator of a strongly continuous (C_0) semigroup $\mathbb{T}_{\mathcal{A}}(t)$ [14]. Therefore, system (5.1) is more general than the RS system (4.1).

5.2 Invariant Subspaces of Fin-D Systems

As mentioned earlier, one of main problems in geometric theory of Inf-D systems is to characterize conditioned and controlled invariant subspaces. As we observed in Chapter 4, due to inherent complexities of unbounded operators. For example, consider domain of unbounded operators. Unlike bounded ones, domain of an unbounded operator is not $\mathcal{X}$ and indeed $\overline{D(\mathcal{A})} = \mathcal{X}$), extension of invariant concepts from Fin-D to Inf-D systems faces certain challenges. In this subsection, we review invariant subspaces of Fin-D systems from a new perspective and point of view that is not available in the literature. Specifically, we provide below two important lemmas on conditioned and controlled invariant subspaces that can be generalized and extended (unlike the available results in geometric theory of Fin-D systems in [74, 112]) to Inf-D systems.

Consider the following Fin-D system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{5.4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^q$ denote the state, input and output vectors, respectively. Based on [74], a subspace $\mathscr{V}$ is A-invariant if $A\mathscr{V} \subseteq \mathscr{V}$, and the A-invariance property is equivalent to e^{At}-invariance. In other words, $e^{At}\mathscr{V} \subseteq \mathscr{V}$ if and only if $A\mathscr{V} \subseteq \mathscr{V}$. However, as discussed in Chapter 4, these subspaces are not necessarily equivalent in Inf-D systems.Therefore, one cannot formally and fully characterize the Inf-D system (5.1) by only addressing the $\mathcal{A}$-invariant subspaces [103]. The main reason for this fact is unboundedness property of $\mathcal{A}$. However, the resolvent operator of an unbounded operator is always a bounded operator. Hence, in this section we investigate the invariant subspaces of Fin-D systems by using the resolvent operator and then generalize and extend these results to Inf-D systems in the next section.

The following lemma describes the relationship between the $R(\lambda, A)$-invariance and the A-invariance concepts.

Lemma 5.1. *Consider the system* (5.4) *and the A-invariant subspace $\mathscr{V}$. Then for all $\lambda \in \rho_\infty(A)$ we have $R(\lambda, A)\mathscr{V} = \mathscr{V}$.*

Proof. Let $\{v_1, \cdots, v_k\}$ denote a basis of $\mathscr{V}$ and $\lambda \in \rho_\infty(A)$. Since $(\lambda I - A)$ is invertible, $\{(\lambda I - A)v_1, \cdots, (\lambda I - A)v_k\}$ are independent. By the A-invariance property of $\mathscr{V}$, it follows that $(\lambda I - A)v_i \in \mathscr{V}$ for all $i \in \{1, \cdots, k\}$. Therefore, $\{(\lambda I-A)v_1, \cdots, (\lambda I-A)v_k\}$ is a basis of $\mathscr{V}$, and consequently $(\lambda I-A)\mathscr{V} = \mathscr{V}$. Also, since $(\lambda I - A)$ is invertible, $R(\lambda, A)$ is well-defined, so that we have $R(\lambda, A)\mathscr{V} = \mathscr{V}$. This completes the proof of the lemma. $\square$

As pointed out in [14, 101], $R(\lambda, A)\mathscr{V} \subseteq \mathscr{V}$ is equivalent to $A\mathscr{V} \subseteq \mathscr{V}$ (since $D(A) = \mathbb{R}^n$). However, the above lemma shows that the condition $R(\lambda, A)\mathscr{V} = \mathscr{V}$ is not stronger than $A\mathscr{V} \subseteq \mathscr{V}$, and they are indeed equivalent. The following lemma provides an important perspective on the unobservable subspace (that is $\mathscr{N} = \bigcap_{k=0}^{n-1} \ker CA^k$) for the system (5.4).

Lemma 5.2. *Consider the Fin-D system* (5.4) *and its unobservable subspace* $\mathscr{N} = \bigcap_{k=0}^{n-1} \ker CA^k$. *The subspace* $\mathscr{N}$ *can be computed as*

$$\mathscr{N} = \bigcap_{k=0}^{n-1} \ker CR(\lambda, A)^k \tag{5.5}$$

Proof. It is well-known that $\mathscr{N}$ is largest A-invariant and consequently largest e^{At}-invariant for all $t \geq 0$ contained in $\ker C$ [74]. Also, as stated above every A-invariant is also $R(\lambda, A)$-invariant. Now, we show that $\mathscr{N}_{\rm r} = \bigcap_{k=0}^{n-1} \ker CR(\lambda, A)^k$ is the largest $R(\lambda, A)$-invariant contained in $\ker C$ and therefore, $\mathscr{N}_{\rm r} = \mathscr{N}$.

First, we show $\mathscr{N}_{\rm r}$ is $R(\lambda, A)$-invariant. It is lucid that $\mathscr{N}_{\rm r} \subseteq \ker C$. Let $x \in \mathscr{N}_{\rm r}$. Since $x \in \ker CR(\lambda, A)^k$ and $x \in \ker CR(\lambda, A)^{k+1}$ for $k = 0, \cdots, n-2$, we obtain $R(\lambda, A)x \in \bigcap_{k=0}^{n-2} \ker CR(\lambda, A)^k$. Also, by using Cayley-Hamilton theorem for $R(\lambda, A)$ (note that $R(\lambda, A)$ is a matrix with the same dimension as A), it follows that $R(\lambda, A)x \in \ker CR(\lambda, A)^{n-1}$, and consequently $R(\lambda, A)x \in \mathscr{N}_{\rm r}$. Therefore, $\mathscr{N}_{\rm r}$ is $R(\lambda, A)$-invariant. Now, we show $\mathscr{N}_{\rm r}$ is the largest $R(\lambda, A)$-invariant subspace contained in $\ker C$. Let $\mathscr{V} \subseteq \ker C$ is a $R(\lambda, A)$-invariant, and set $x \in \mathscr{V}$. It follows that $x \in \ker C$ and since $\mathscr{V}$ is $R(\lambda, A)$-invariant, we obtain $R(\lambda, A)^k x \in$

$\mathscr{V} \subseteq \ker C$, for $k \in \mathbb{N}$. Hence, $x \in \ker CR(\lambda, A)^k$ for all $k = 0, \cdots, n-1$, and consequently $x \in \mathscr{N}_{\mathrm{r}}$ (i.e., $\mathscr{V} \subseteq \mathscr{N}_{\mathrm{r}}$). Therefore, $\mathscr{N}_{\mathrm{r}}$ is the largest $R(\lambda, A)$-invariant (and consequently, A-invariant) subspace contained in $\ker C$, and consequently $\mathscr{N}_{\mathrm{r}} = \mathscr{N}$. This completes the proof of the lemma. □

Next, let us review the conditioned and controlled invariant subspaces of the system (5.4).

Definition 5.3. *[74]*

1. *The subspace $\mathscr{W} \subseteq \mathbb{R}^n$ is called a conditioned invariant subspace if $A(\mathscr{W} \cap \ker C) \subseteq \mathscr{W}$.*

2. *The subspace $\mathscr{V} \subseteq \mathbb{R}^n$ is called a controlled invariant subspace if $A\mathscr{V} \subseteq \mathscr{B} + \mathscr{V}$.*

The following lemma provides the main available result in the literature on conditioned and controlled invariant subspaces of Fin-D systems.

Lemma 5.4. *[74] Consider the system* (5.4) *and the conditioned (controlled) invariant subspace $\mathscr{W}$ ($\mathscr{V}$). Then, there exists a map $D : \mathbb{R}^q \to \mathbb{R}^n$ ($F : \mathbb{R}^n \to \mathbb{R}^m$) such that $\mathscr{W}$ ($\mathscr{V}$) is invariant with respect to $\mathrm{e}^{(A+DC)t}$ ($\mathrm{e}^{(A+BF)t}$).*

As shown in the next section, the above lemma
emphdoes not hold for Inf-D systems. Therefore, we subsequently use the resolvent operator instead of A to address the conditioned and controlled invariant subspaces of Fin-D systems. This new point of view will subsequently enable us to formally address the conditioned and controlled invariant subspaces for the Inf-D systems.

Lemma 5.5. *Consider system* (5.4)*, and the subspace $\mathscr{W} \subseteq \mathbb{R}^n$. The subspace $\mathscr{W}$ is conditioned invariant if and only if for any $\lambda \in \rho_\infty(A)$, we have $R(\lambda, A)\mathscr{W} \cap \ker C = \mathscr{W} \cap \ker C$.*

Proof. **(if part):** Let $\mathscr{W} \bigcap \ker C = R(\lambda, A)\mathscr{W} \bigcap \ker C$. Therefore, we have $\mathscr{W} \bigcap \ker C \subseteq R(\lambda, A)\mathscr{W} \bigcap \ker C \subseteq R(\lambda, A)\mathscr{W}$. Since $R(\lambda, A)$ is invertible it follows that $(\lambda I - A)(\mathscr{W} \bigcap \ker C) \subseteq \mathscr{W}$. Now, let $x \in \mathscr{W} \bigcap \ker C$. Hence, $\lambda x - Ax \in \mathscr{W}$, and consequently $Ax \in \mathscr{W}$. In other words, $A(\mathscr{W} \bigcap \ker C) \subseteq \mathscr{W}$. Therefore, $\mathscr{W}$ is conditioned invariant.

(only if part): Let $\mathscr{W}$ be a conditioned invariant subspace and an arbitrary $\lambda \in \rho_\infty(A)$. Since $A(\mathscr{W} \bigcap \ker C) \subseteq \mathscr{W}$, we obtain $(\lambda I - A)(\mathscr{W} \bigcap \ker C) \subseteq \mathscr{W}$, and consequently $(\mathscr{W} \bigcap \ker C) \subseteq R(\lambda, A)\mathscr{W}$. Also, since $(\mathscr{W} \bigcap \ker C) \subseteq \ker C$, it follows that $(\mathscr{W} \bigcap \ker C) \subseteq R(\lambda, A)\mathscr{W} \bigcap \ker C$.

We now show that $R(\lambda, A)\mathscr{W} \bigcap \ker C \subseteq (\mathscr{W} \bigcap \ker C)$. Let $x \in R(\lambda, A)\mathscr{W} \bigcap \ker C$. it follows that there exists a $y \in \mathscr{W}$ such that $R(\lambda, A)y = x$ or $(\lambda I - A)x = y$. Since $x \in \ker C$, we obtain $(\lambda I - A - DC)x = (\lambda I - A)x = y$ and therefore $x \in R(\lambda, A + DC)\mathscr{W}$. Since $\mathscr{W}$ is conditioned invariant, by using Lemma 5.1, we have $R(\lambda, A + DC)\mathscr{W} = \mathscr{W}$, and hence $x \in \mathscr{W}$. It follows that $x \in \mathscr{W} \cap \ker C$ which completes the proof. $\square$

Remark 5.6. *It should be pointed out that the condition $R(\lambda, A)\mathscr{W} \bigcap \ker C \subseteq \mathscr{W} \bigcap \ker C$ is not sufficient for conditioned invariance. For example, consider the system* (5.4) *with $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $C = [1,\ -1]^{\mathrm{T}}$. Also, let $\mathscr{W} = \operatorname{span}\{[1,\ 1]^{\mathrm{T}}\}$. It follows that $0 = R(\lambda, A)\mathscr{W} \bigcap \ker C \subseteq (\mathscr{W} \cap \ker C) = \mathscr{W}$, however $\mathscr{W}$ is not conditioned invariant (since $A(\mathscr{W} \bigcap \ker C) = \operatorname{span}\{[2,\ 1]^{\mathrm{T}}\} \not\subseteq \mathscr{W}$).*

By following along the same lines as above, one can derive the following lemma.

Lemma 5.7. *Consider the system* (5.4)*. Then the subspace $\mathscr{V}$ is controlled invariant if and only if for any $\lambda \in \rho_\infty(A)$, we have $R(\lambda, A)(\mathscr{V} + \mathscr{B}) = \mathscr{V} + R(\lambda, A)\mathscr{B}$, where $\mathscr{B} = \operatorname{Im} B$.*

Proof. We show the result by using duality property that holds for Fin-D systems [74].

(if part): Let $R(\lambda, A)(\mathscr{V} + \mathscr{B}) = \mathscr{V} + R(\lambda, A)\mathscr{B}$ and $\mathscr{W} = \mathscr{V}^{\perp}$. Since $\ker B^{\mathrm{T}} = \mathscr{B}^{\perp}$ and $(F^{-1}\mathscr{R})^{\perp} = F^{\mathrm{T}}\mathscr{R}^{\perp}$ for any $\mathscr{R} \in \mathbb{R}^n$ and operator $\mathscr{F} : \mathbb{R}^n \to \mathbb{R}^n$ [136], it follows that $(\lambda I - A^{\mathrm{T}})(\mathscr{W} \cap \ker B^{\mathrm{T}}) = \mathscr{W} \cap (\lambda I - A^{\mathrm{T}}) \ker B^{\mathrm{T}}$, and consequently, $R(\lambda, A^{\mathrm{T}})\mathscr{W} \cap \ker B^{\mathrm{T}} = \mathscr{W} \cap \ker B^{\mathrm{T}}$ (note that $\rho_\infty(A^{\mathrm{T}}) = \rho_\infty(A)$). By using Lemma 5.5, $\mathscr{W}$ is a conditioned invariant subspace of the pair $(B^{\mathrm{T}}, A^{\mathrm{T}})$. Therefore, by duality $\mathscr{V} = \mathscr{W}^{\perp}$ is a controlled invariant subspace for the pair (A, B).

(only if part): By following along the same steps as above one can show this part. This completes the proof of the lemma. $\square$

5.3 Invariant Subspaces for Inf-D Systems

As stated earlier in Chapter 4, invariant subspaces play a prominent role in geometric control theory that includes studies in FDI and disturbance decoupling problems [41, 50, 74, 112]. In the FDI problem, one needs three types of invariant subspaces, namely $\mathbb{T}_{\mathcal{A}}$-invariant, conditioned invariant, and unobservability subspaces. For the disturbance decoupling problem, the controlled invariant subspace is necessary. In the literature, $\mathbb{T}_{\mathcal{A}}$-invariant, conditioned and controlled invariant subspaces (that are subsequently defined) have been introduced for Inf-D systems [103, 105, 110, 130]. The necessary and sufficient condition for equivalence of $\mathcal{A}$-invariance and $\mathbb{T}_{\mathcal{A}}$-invariance has been addressed in the literature. However, the conditioned and controlled invariant subspaces have been partially addressed. More specifically, the necessary and sufficient conditions for equivalence of various types of these subspace is still an open problem. In this section, we first review invariant concepts of Inf-D systems and develop two important results that are related to the conditioned and controlled invariant subspaces. Then, an unobservability subspaces of Inf-D systems is addressed.

5.3.1 $\mathcal{A}$-Invariant Subspace

As stated in Chapter 4, there are two different definitions that are related to the $\mathbb{T}_{\mathcal{A}}$-invariance property, and unlike the Fin-D case, these definitions are not equivalent for Inf-D systems. We repeat the definitions as follows,

Definition 5.8. *[130]*

1. *The closed subspace $\mathscr{V} \subseteq \mathcal{X}$ is called $\mathcal{A}$-invariant if $\mathcal{A}(\mathscr{V} \cap D(\mathcal{A})) \subseteq \mathscr{V}$.*

2. *The closed subspace $\mathscr{V} \subseteq \mathcal{X}$ is $\mathbb{T}_{\mathcal{A}}$-invariant if $\mathbb{T}_{\mathcal{A}}(t)\mathscr{V} \subseteq \mathscr{V}$ for all $t \in [0, \infty)$, where $\mathbb{T}_{\mathcal{A}}$ is the C_0 semigroup generated by $\mathcal{A}$.*

In the geometric approach of the FDI problem one needs the subspaces that are $\mathbb{T}_{\mathcal{A}}$-invariant. Since dealing with $\mathbb{T}_{\mathcal{A}}$-invariant subspaces are more challenging than $\mathcal{A}$-invariant subspaces, we are interested in cases where the Definition 5.8 item 1) and 2) are equivalent. The necessary and sufficient condition for $\mathbb{T}_{\mathcal{A}}$-invariance is provided in the literature that is presented as follows.

Lemma 5.9. *[101, Lemma I.4] Consider the C_0 semigroup $\mathbb{T}_{\mathcal{A}}$ and its infinitesimal generator $\mathcal{A}$. Let $\mathscr{V}$ be a closed subspace. Then the following statements are equivalent.*

1. *$\mathscr{V}$ is $\mathbb{T}_{\mathcal{A}}$-invariant.*

2. *$\mathscr{V}$ is $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant for a $\lambda_1 \in \rho_\infty(\mathcal{A})$ (for definition of $\rho_\infty(\mathcal{A})$ refer to the Notation Section 1.6).*

3. *$\mathscr{V}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant for all $\lambda \in \rho_\infty(\mathcal{A})$.*

4. *The range of $(\lambda\mathcal{I} - \mathcal{A})$ restricted to $\mathscr{V}$ is $\mathscr{V}$ (that is, $(\lambda\mathcal{I} - \mathcal{A})(\mathscr{V} \cap D(\mathcal{A})) = \mathscr{V}$).*

As in the previous chapter, a subspace of particular $\mathbb{T}_{\mathcal{A}}$-invariant of interest that we are mainly concerned in this chapter is the unobservable subspace. The unobservable subspace of the system (5.1) is defined as $\mathscr{N} =< \ker \mathcal{C} | \mathbb{T}_{\mathcal{A}} >= \bigcap_{t\geq 0} \ker \mathcal{C}\mathbb{T}_{\mathcal{A}}(t)$ [103] that is the largest $\mathbb{T}_{\mathcal{A}}$-invariant that is contained in $\ker \mathcal{C}$ [103]. By extending Lemma 5.2, the following lemma shows an important property for $\mathscr{N}$.

Lemma 5.10. *The unobservable subspace can be computed as*

$$\mathscr{N} = \bigcap_{k=0}^{\infty} \ker \mathcal{C}\mathcal{R}(\lambda, \mathcal{A})^k \tag{5.6}$$

Proof. Since $\mathscr{N}$ is the largest $\mathbb{T}$-invariant contained in $\ker \mathcal{C}$, it is also the largest $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspace in $\ker \mathcal{C}$. Set $\mathscr{N}_{\mathrm{r}} = \bigcap_{k=0}^{\infty} \ker \mathcal{C}\mathcal{R}(\lambda, \mathcal{A})^k$. As in Lemma 5.2, we show that $\mathscr{N}_{\mathrm{r}}$ is the largest $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspace contained in $\ker \mathcal{C}$ and consequently $\mathscr{N} = \mathscr{N}_{\mathrm{r}}$.

First, we show that $\mathscr{N}_{\mathrm{r}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant. It is lucid that $\mathscr{N}_{\mathrm{r}} \subseteq \ker \mathcal{C}$. Also, since $\mathcal{C}$ and $\mathcal{R}(\lambda, \mathcal{A})^k$ (and consequently $\mathcal{C}\mathcal{R}(\lambda, \mathcal{A})^k$) are bounded, the null space of $\ker \mathcal{C}\mathcal{R}(\lambda, \mathcal{A})^k$ is closed for all $k \in \mathbb{N}$. Therefore, $\mathscr{N}_{\mathrm{r}}$ is a closed subspace. Let $x \in \mathscr{N}_{\mathrm{r}}$ by following along the same steps in the proof of Lemma 5.2, one can show that $\mathcal{R}(\lambda, \mathcal{A})x \in \ker \mathcal{C}\mathcal{R}(\lambda, \mathcal{A})^k$ for all $k \in \underline{\mathbb{N}}$, and consequently it follows that $\mathcal{R}(\lambda, \mathcal{A})x \in \mathscr{N}_{\mathrm{r}}$. Therefore, $\mathscr{N}_{\mathrm{r}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant.

Next, we show that $\mathscr{N}_{\mathrm{r}}$ is the largest $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspace contained in $\ker \mathcal{C}$. Let $\mathscr{V} \subseteq \ker \mathcal{C}$ be a $\mathcal{R}(\lambda, \mathcal{A})$ invariant subspace and set $x \in \mathscr{V}$. It follows that $\mathcal{R}(\lambda, \mathcal{A})^k x \in \mathscr{V} \subseteq \ker \mathcal{C}$ for all $k \in \underline{\mathbb{N}}$, and consequently $x \in \mathscr{N}_{\mathrm{r}}$. Therefore, $\mathscr{V} \subseteq \mathscr{N}_{\mathrm{r}}$ and $\mathscr{N}_{\mathrm{r}}$ is the largest $\mathcal{R}(\lambda, \mathcal{A})$ in variant subspace contained in $\ker \mathcal{C}$. This completes the proof of the lemma. $\square$

We use the unobservable subspace $\mathscr{N}$ to introduce the unobservability subspaces for Inf-D systems.

As emphasized in Chapter 4, there are three different definitions for conditioned invariant subspaces (Definition 4.11). In order to derive necessary and sufficient conditions for equivalence of these definitions, we need the following lemma.

Lemma 5.11. *Consider the infinitesimal generator $\mathcal{A}$, $\lambda \in \rho_\infty(\mathcal{A})$ and the $\mathcal{A}$-invariant subspace $\mathscr{V}$ such that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$. If there exists a Fin-D $\mathcal{A}$-invariant subspace $\mathscr{V}_{\mathrm{f}}$ such that $\mathscr{V}_{\overline{\mathrm{R}}} = \mathscr{V} + \mathscr{V}_{\mathrm{f}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant, then $\mathscr{V}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant.*

Proof. As Lemma 4.17, we first show that one can assume $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A})$. Given that $V_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant [101], we have $\overline{\mathscr{V}_{\overline{\mathrm{R}}} \cap D(\mathcal{A})} = \mathscr{V}_{\overline{\mathrm{R}}}$. Also, as an assumption we have $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$. If $V_{\overline{\mathrm{R}}}$ is Fin-D, $\mathscr{V}$ is Fin-D and hence $\mathscr{V}_{\mathrm{f}} \subseteq \mathscr{V} \subset D(\mathcal{A})$. Let, $V_{\overline{\mathrm{R}}}$ be Inf-D. By following along the same steps in the proof of Lemma 4.14, we define the basis $\{v_i\}_{i=1}^{\infty}$ of $\mathscr{V}$ such that $v_i \in D(\mathcal{A})$ for all $i \in \mathbb{N}$ and $\{v_i\}_{i=n_{\mathrm{f}}+1}^{\infty}$ is a basis for $V_{\overline{\mathrm{R}}}$, where $n_{\mathrm{f}} = \dim(\mathscr{V}_{\mathrm{f}})$. Set $\mathscr{V}_{\mathrm{ff}} =$ span$\{w_i\}_{i=1}^{n_{\mathrm{f}}} \subset D(\mathcal{A})$, and it follows that $\mathscr{V} = V_{\overline{\mathrm{R}}} + \mathscr{V}_{\mathrm{ff}}$. Therefore, without loss of any generality we assume $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A})$.

Next, by following along the same steps as in Lemma 5.1, we show $(\lambda\mathcal{I} - \mathcal{A})\mathscr{V}_{\mathrm{f}} = \mathscr{V}_{\mathrm{f}}$. Let $\{v_1, \cdots, v_{n_{\mathrm{f}}}\}$ denote a basis of $\mathscr{V}_{\mathrm{f}}$, where $v_k \in D(\mathcal{A}), k = 1, \ldots, n_{\mathrm{f}}$ and $\lambda \in \rho_\infty(A)$. Since $(\lambda\mathcal{I} - \mathcal{A})$ is invertible, $\{(\lambda\mathcal{I} - \mathcal{A})v_1, \ldots, (\lambda\mathcal{I} - \mathcal{A})v_{-} - - - n_{\mathrm{f}}\}$ are independent. By the $\mathcal{A}$-invariance property of $\mathscr{V}$ and $\mathcal{R}(\lambda, \mathcal{A})$-invariant property $V_{\overline{\mathrm{R}}}$, it follows that $\mathscr{V}_{\mathrm{f}}$ is $\mathcal{A}$-invariant, and $(\lambda\mathcal{I} - \mathcal{A})v_k \in \mathscr{V}_{\mathrm{f}}$ for all $k \in \{1, \cdots, n_{\mathrm{f}}$. Therefore, $\{(\lambda\mathcal{I} - \mathcal{A})v_1, \cdots, (\lambda\mathcal{I} - \mathcal{A})v_k\}$ is a basis of $\mathscr{V}_{\mathrm{f}}$, and consequently $(\lambda\mathcal{I} - \mathcal{A})\mathscr{V} = \mathscr{V}$.

Now, we show the result by contradiction. Assume that $\mathscr{V}$ is not $\mathcal{R}(\lambda, \mathcal{A})$-invariant. Therefore, there exists $\mathscr{U}_{\mathrm{f}} \subset \mathscr{V}$ such that $\mathscr{V}_{\mathrm{u,f}} = \mathcal{R}(\lambda, \mathcal{A})\mathscr{U}_{\mathrm{f}}$ (i.e. $(\lambda\mathcal{I} - \mathcal{A})\mathscr{V}_{\mathrm{u,f}} = \mathscr{U}_{\mathrm{f}}$) and $\mathscr{V}_{\mathrm{u,f}} \cap \mathscr{V} = 0$. Now, let $\lambda_1 = \lambda + \zeta$, where $\zeta > 0$. By definition of $\rho_\infty(\mathcal{A})$, it follows that $\lambda_1 \in \rho_\infty(\mathcal{A})$ and $(\lambda_1\mathcal{I} - \mathcal{A})\mathscr{V}_{\mathrm{u,f}} \cap \mathscr{V} = 0$ (given that $v_u \in \mathscr{V}_{\mathrm{u,f}}$, we obtains $(\lambda_1\mathcal{I} - \mathcal{A})v_u = (\lambda_1\mathcal{I} - \mathcal{A})v_u + \zeta v_u$, $(\lambda_1\mathcal{I} - \mathcal{A})v_u \in \mathscr{V}$, and consequently $(\lambda_1\mathcal{I} - \mathcal{A})\mathscr{V}_{\mathrm{u,f}} \cap \mathscr{V} = 0)$. Moreover, $(\lambda_1\mathcal{I} - \mathcal{A})(\mathscr{V} \cap D(\mathcal{A})) \subset \mathscr{V}$, and given that

$(\lambda_1\mathcal{I}-\mathcal{A})(\mathscr{V}_{\overline{\mathrm{R}}}\cap D(\mathcal{A}))=\mathscr{V}_{\overline{\mathrm{R}}}$ (by using Lemma 5.9) and $\mathcal{R}(\lambda_1,\mathcal{A})\mathscr{V}_{\mathrm{f}}=\mathscr{V}_{\mathrm{f}}$, we obtain $(\lambda_1\mathcal{I}-\mathcal{A})(\mathscr{V}\cap D(\mathcal{A}))=\mathscr{V}$ (otherwise, there exists $v\in\mathscr{V}\subset\mathscr{V}_{\overline{\mathrm{R}}}$ such that $\mathcal{R}(\lambda_1,\mathcal{A})v\notin\mathscr{V}_{\overline{\mathrm{R}}}$), and by Lemma 5.9, item 4), $\mathscr{V}$ is $\mathcal{R}(\lambda_1,\mathcal{A})$-invariant that is in contradiction with the above assumption. This completes the proof of the lemma. □

Remark 5.12. *In the above proof, the finite dimensionality of $\mathscr{V}_{\mathrm{f}}$ plays a crucial role as explained, below.*

1. *Since $\dim(\mathscr{V}_{\mathrm{f}})<\infty$, we obtain $\mathcal{R}(\lambda,\mathcal{A})\mathscr{V}_{\mathrm{f}}=\mathscr{V}_{\mathrm{f}}$.*

2. *If $\dim(\mathscr{V}_{\mathrm{f}})=\infty$, then $\overline{\mathscr{V}+\mathscr{V}_{\mathrm{f}}}\neq\mathscr{V}+\mathscr{V}_{\mathrm{f}}$ does not hold in general.*

5.3.2 Conditioned Invariant Subspaces

In this subsection, the conditioned invariant subspaces of the system (5.1) are characterized. As in Chapter 4, we have the following definitions.

Definition 5.13. *[103]*

1. *The closed subspace $\mathscr{W}$ is called ($\mathcal{C}$,$\mathcal{A}$)-invariant if $\mathcal{A}(\mathscr{W}\cap D(\mathcal{A})\cap\ker\mathcal{C})\subseteq\mathscr{W}$.*

2. *The closed subspace $\mathscr{W}$ is feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant if there exists a bounded operator $\mathcal{D}:\mathbb{R}^q\to\mathcal{H}$ such that $\mathscr{W}$ is invariant with respect to $(\mathcal{A}+\mathcal{D}\mathcal{C})$ as per Definition 5.8.*

3. *The closed subspace $\mathscr{W}$ is $\mathbb{T}$-conditioned invariant if there exists a bounded operator $\mathcal{D}:\mathbb{R}^q\to\mathcal{H}$ such that (i) the operator $(\mathcal{A}+\mathcal{D}\mathcal{C})$ is the infinitesimal generator of a C_0-semigroup $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$; and (ii) $\mathscr{W}$ is $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant.*

Example 5.14. *Definitions 5.13 are not equivalent.*

Let us now consider the following dynamical system

$$\begin{aligned}\frac{\partial\tilde{x}}{\partial t}&=\frac{\partial^2\tilde{x}}{\partial z^2},\\ y(t)&=\int_0^{0.5}\tilde{x}(t,z)\mathrm{d}z,\end{aligned}\tag{5.7}$$

where $z \in [0,1]$ and $\tilde{x}(t,0) = \tilde{x}(t,1) = 0$. Let $\mathscr{W}$ denote the subspace of all functions satisfying the boundary conditions and are equal to zero almost everywhere in $(0.5, 1]$. By following the same steps as in [101, Example I.6], it can be shown that $\mathscr{W}$ is

emphnot $\mathbb{T}_{\mathcal{A}}$-invariant. However, by setting $\mathcal{D} = 0$, $\mathscr{W}$ is feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant and invoking Lemma 4.15, it follows that $\mathscr{W}$ is not $\mathbb{T}$-conditioned invariant). Therefore, the $\mathbb{T}$-conditioned invariance property is more stronger than feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant.

For developing a solution to the FDI of Inf-D systems, we are interested in cases in which the Definition 5.13, items 1) to 3) coincide. To the best of our knowledge, there is no necessary and sufficient condition for this equivalence in the literature. It should be pointed out that the results in [108] are applicable to

emphonly single input single output systems. Also, the results of Chapter 4 are mainly applied to regular RS systems.

Motivated by the above example, and following Lemmas 5.1 and 5.5, we provide the main result of this section in theorem, below.

Theorem 5.15. *Consider the system* (5.1) *and the bounded finite rank output operator* $\mathcal{C}$ *(*$\mathrm{rank}(\mathcal{C}) = q$*). Also, let* $\mathscr{W}$ *denote a (*$\mathcal{C}$*,*$\mathcal{A}$*)-invariant subspace such that* $\overline{D(\mathcal{A}) \cap \mathscr{W}} = \mathscr{W}$.

1. $\mathscr{W}$ *is* $\mathbb{T}$*-conditioned invariant.*

2. *There exists a* $\lambda_1 \in \rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{DC})$ *such that* $\mathcal{R}(\lambda_1, \mathcal{A})\mathscr{W} \cap \ker \mathcal{C} \subseteq \mathscr{W} \cap \ker \mathcal{C}$ *and*

   (a) *The subspace* $\mathscr{W}$ *can be represented as* $\mathscr{W} = \mathscr{W}_{\underline{\mathrm{R}}} + \mathscr{W}_{\mathrm{f}}$ *such that* $\mathscr{W}_{\underline{\mathrm{R}}}$ *is a* $\mathcal{R}(\lambda, \mathcal{A})$*-invariant subspace contained in* $\mathscr{W}$*, and* $\mathscr{W}_{\mathrm{f}}$ *is a Fin-D subspace, or*

(b) *There exists a* $\mathscr{W}_{\mathrm{f}} \subseteq D(\mathcal{A})$ *such that (i)* $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{C} = 0$*, (ii)* $\underline{D}(\mathscr{W}) \cap \underline{D}(\mathscr{W}_{\mathrm{f}}) \neq \emptyset$*, and (iii)* $\mathscr{W}_{\overline{\mathrm{R}}} = \mathscr{W} + \mathscr{W}_{\mathrm{f}}$*, where* $\mathscr{W}_{\overline{\mathrm{R}}}$ *is a* $\mathcal{R}(\lambda, \mathcal{A})$*-invariant containing* $\mathscr{W}$.

3. *For every* $\lambda \in \rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{D}\mathcal{C})$*, we have* $\mathcal{R}(\lambda, \mathcal{A})\mathscr{W} \cap \ker \mathcal{C} \subseteq \mathscr{W} \cap \ker \mathcal{C}$ *and*

   (a) *The subspace* $\mathscr{W}$ *can be represented as* $\mathscr{W} = \mathscr{W}_{\underline{\mathrm{R}}} + \mathscr{W}_{\mathrm{f}}$ *such that* $\mathscr{W}_{\mathrm{f}}$ *is a Fin-D subspace, or*

   (b) *There exists a Fin-D* $\mathscr{W}_{\mathrm{f}}$ *such that(i)* $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{C} = 0$*, (ii)* $\underline{D}(\mathscr{W}) \cap \underline{D}(\mathscr{W}_{\mathrm{f}}) \neq \emptyset$*, and* $\mathscr{W}_{\overline{\mathrm{R}}} = \mathscr{W} + \mathscr{W}_{\mathrm{f}}$.

Proof. Since $\mathscr{W}$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant, by using Lemma 4.14 there exists a bounded operator $\mathcal{D}$ such that $(\mathcal{A} + \mathcal{D}\mathcal{C})(\mathscr{W} \cap D(\mathcal{A})) \subseteq \mathscr{W}$ and the operator $\mathcal{A} + \mathcal{D}\mathcal{C}$ is infinitesimal generator of the C_0 semigroup $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$. By using Theorem 2.1.12 in [14] (the Hille-Yosida theorem), where it is shown that for every infinitesimal generator $\mathcal{A}$ there exists a finite real number $r \in \mathbb{R}$ such that $[r, \infty) \subset \rho_\infty(\mathcal{A})$), it follows that $(\rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{D}\mathcal{C}))$ is not empty.

(**1** $\Rightarrow$ **2 and 3**): Assume $\mathscr{W}$ is $\mathbb{T}$-conditioned invariant, and consider the bounded operator $\mathcal{D}$ such that $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}\mathscr{W} \subseteq \mathscr{W}$. Let $\lambda_1 \in \rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{D}\mathcal{C})$. By using Lemma 5.9, one obtains $\mathcal{R}(\lambda_1, \mathcal{A} + \mathcal{D}\mathcal{C})\mathscr{W} \subseteq \mathscr{W}$. Let $x \in \mathcal{R}(\lambda_1, \mathcal{A})\mathscr{W} \cap \ker \mathcal{C}$ and $y = (\lambda\mathcal{I} - \mathcal{A})x \in \mathscr{W}$. Since $x \in \ker \mathcal{C}$ one can write $y = (\lambda_1\mathcal{I} - \mathcal{A} + \mathcal{D}\mathcal{C})x = (\lambda_1\mathcal{I} - \mathcal{A})x$. and consequently since $\mathscr{W}$ is $\mathcal{R}(\lambda_1, \mathcal{A} + \mathcal{D}\mathcal{C})$-invariant we have $x = (\lambda_1\mathcal{I} - \mathcal{A})^{-1}y \in \mathscr{W}$. In other words, $\mathcal{R}(\lambda_1, \mathcal{A})\mathscr{W} \cap \ker \mathcal{C} \subseteq \mathscr{W}$.

By invoking contradiction we show that one of the conditions 2-a) or 2-b) is a necessary condition. Assume that both conditions fail, and consider the smallest subspace $\mathscr{W}_{\mathrm{f}}$ such that $\mathscr{W}_{\overline{\mathrm{R}}} = \overline{\mathscr{W} + \mathscr{W}_{\mathrm{f}}}$ is $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant containing $\mathscr{W}$ and $\dim(\mathscr{W}_{\mathrm{f}}) = \infty$. Note that $\mathscr{W}_{\mathrm{f}}$ does not contain any $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant subspace (since $\mathscr{W}_{\mathrm{f}}$ is the smallest subspace to construct $W_{\overline{\mathrm{R}}}$). Given that $\mathcal{C}$ is finite rank and

$\mathcal{R}(\lambda_1,\mathcal{A})\mathscr{W}\cap\ker\mathcal{C}\subsetneq\mathscr{W}$, it follows that $\mathscr{W}_1=\mathcal{R}(\lambda_1,\mathcal{A})\mathscr{W}\cap(\mathcal{R}(\lambda_1,\mathcal{A})\mathscr{W}\cap\mathscr{W})^{\perp}\subset D(\mathcal{A})$ is Fin-D and $\mathscr{W}_1\cap\ker\mathcal{C}=0$. Moreover, $\mathscr{W}_1\subset\mathscr{W}_{\mathrm{f}}$, and since $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda_1,\mathcal{A})$-invariant, and conditions 2-a) and 2-b) are not satisfied there are only two possibilities as follows:

1. There exists a $k_0\geq 0$ such that for every $n\geq k_0$ we have $\mathcal{R}(\lambda_1,\mathcal{A})^n\mathscr{W}_1\cap\mathscr{W}=0$. Note that if the condition 2-b) is satisfied, this condition fails. For example, assume condition 2-b) is satisfied (i.e., there exists a Fin-D $\mathscr{W}_{\mathrm{f},2}\subseteq\mathscr{W}_1$ satisfying condition 2-b) and consider the Inf-D subspace $\mathscr{W}_c\subset\mathscr{W}$ such that $\mathscr{W}_c+\mathscr{W}_{\mathrm{f},2}$ is $\mathcal{R}(\lambda,\mathcal{A})$-invariant. Therefore, for k_0 it is possible to have n and $y\in\mathscr{W}_{\mathrm{f},2}$ such that $\mathcal{R}(\lambda_1,\mathcal{A})^n y\in\mathscr{W}\subset\mathscr{W}$.

   By selecting $\mathscr{W}_{\mathrm{f,R}}=\operatorname{span}\{\mathcal{R}(\lambda_1,\mathcal{A})^n y\}_{n=k_0}^{\infty}$ and $y\in\mathscr{W}_1$, it follows that $\mathscr{W}_{\mathrm{f,R}}\subsetneq\mathscr{W}_{\mathrm{f}}$ and $\mathscr{W}_{\mathrm{f,R}}$ is $\mathcal{R}(\lambda_1,\mathcal{A})$-invariant that is in contradiction with the definition of $\mathscr{W}_{\mathrm{f}}$.

2. For every $k\geq 1$, there exists an $n\geq k$ and $y_k\in\mathscr{W}_1$ such that $x_k=\mathcal{R}(\lambda_1,\mathcal{A})^n y\in\mathscr{W}$. If $x_k\in\ker\mathcal{C}$, it follows that $(\lambda_1\mathcal{I}-\mathcal{A})x_k\notin\mathscr{W}$ that is in contradiction with the $(\mathcal{C}$,$\mathcal{A})$-invariance assumption of $\mathscr{W}$. Hence, let $x_k\notin\ker\mathcal{C}$, and given that $\mathcal{C}$ is finite rank, it follows that there exist k_0 sufficiently large such that $\operatorname{span}\{x_k\}_{k=1}^{k_0}$ constructs a $\mathcal{R}(\lambda_1,\mathcal{A})$-invariant subspace, and consequently $\mathscr{W}+\operatorname{span}\{x_k\}_{k=1}^{k_0}$ is also $\mathcal{R}(\lambda_1,\mathcal{A})$-invariant that is again in contradiction with definition of $\mathscr{W}_{\mathrm{f}}$.

Hence, it is necessary that the condition 2-b) is satisfied (that is in contradiction with the assumption) or to have $\dim(\mathscr{W}_{\mathrm{f}})<\infty$ that is equivalent to the condition 2-a). Therefore, at least one of the conditions 2-a) or 2-b) is satisfied.

Finally, given that the above analysis holds for any $\lambda_1\in\rho_\infty(\mathcal{A})\cap\rho_\infty(\mathcal{A}+\mathcal{D}\mathcal{C})$, it follows that the condition 1) also implies conditions 3).

(**3** $\Rightarrow$ **1**): Let $y\in\mathscr{W}$ such that $x=\mathcal{R}(\lambda,\mathcal{A})^{-1}y\in\ker\mathcal{C}$. It follows that $y=(\lambda\mathcal{I}-\mathcal{A}-\mathcal{D}\mathcal{C})x=(\lambda\mathcal{I}-\mathcal{A})x$ for any bounded operator $\mathcal{D}$ and $\lambda\in\rho_\infty(\mathcal{A})\cap$

$\rho_\infty(\mathcal{A}+\mathcal{D}\mathcal{C})$. Consequently, we obtain $\mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})y \in \mathscr{W}$ (since $\mathcal{R}(\lambda,\mathcal{A})\mathscr{W} \cap \ker\mathcal{C} \subseteq \mathscr{W}$).

First, assume that condition 3-a) is satisfied and let $\mathscr{W} = \mathscr{W}_{\underline{\mathrm{R}}} + \mathscr{W}_{\mathrm{f}}$. Without loss of any generality we assume $\mathscr{W}_{\underline{\mathrm{R}}} \cap \mathscr{W}_{\mathrm{f}} = 0$. Since $\overline{\mathscr{W} \cap D(\mathcal{A})} = \mathscr{W}$, $\overline{\mathscr{W}_{\underline{\mathrm{R}}} \cap D(\mathcal{A})} = \mathscr{W}_{\underline{\mathrm{R}}}$ and $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$, without loss of any generality we assume $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A})$. Also, given that $\mathscr{W}$ is $(\mathcal{C}$,$\mathcal{A})$-invariant and $\mathscr{W}_{\underline{\mathrm{R}}}$ is $\mathcal{A}$-invariant, we obtain $\mathscr{W}_{\mathrm{f}}$ is $(\mathcal{C}$,$\mathcal{A})$-invariant, and consequently it is $\mathbb{T}$-conditioned invariant (by using Lemma 4.12). Therefore, there exists an operator $\mathcal{D}$ such that $\mathcal{D}\mathcal{C}\mathscr{W}_{\underline{\mathrm{R}}} = 0$ and $(\mathcal{A}+\mathcal{D}\mathcal{C})\mathscr{W}_{\mathrm{f}} \subseteq \mathscr{W}_{\mathrm{f}}$ (since $\mathscr{W}_{\underline{\mathrm{R}}} \cap \mathscr{W}_{\mathrm{f}} = 0$ and $\mathscr{W}_{\mathrm{f}}$ is $\mathbb{T}$-conditioned invariant, by following along the same steps as in Theorem 4.17, the operator $\mathcal{D}$ always exists). Now, let $y = y_{\mathrm{r}} + y_{\mathrm{f}} \in \mathscr{W}$ and $x = \mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})y$, where $y_{\mathrm{r}} \in \mathscr{W}_{\underline{\mathrm{R}}}$ and $y_{\mathrm{f}} \in \mathscr{W}_{\mathrm{f}}$. Also, set $x = x_{\mathrm{r}} + x_{\mathrm{f}}$, where $x_{\mathrm{r}} = \mathcal{R}(\lambda,\mathcal{A})y_{\mathrm{r}}$ and $x_{\mathrm{f}} = \mathcal{R}(\lambda,\mathcal{A})y_{\mathrm{f}} \in \mathscr{W}$. It follows that $x_{\mathrm{r}} \in \mathscr{W}_{\underline{\mathrm{R}}}$ and $(\lambda\mathcal{I}-\mathcal{A})x_{\mathrm{r}} = (\lambda\mathcal{I}-\mathcal{A}+\mathcal{D}\mathcal{C})x_{\mathrm{r}} = \mathcal{R}(\lambda,\mathcal{A})y_{\mathrm{r}}$, and consequently $x \in \mathscr{W}$. In other words, $\mathscr{W}$ is $\mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant, and hence it is $\mathbb{T}$-conditioned invariant.

Now, assume that the condition 3-b) is satisfied and let $\mathscr{W}_{\overline{\mathrm{R}}} = \mathscr{W} + \mathscr{W}_{\mathrm{f}}$, where $\dim(\mathscr{W}_{\mathrm{f}}) < \infty$ and $\mathscr{W}_{\mathrm{f}}$ is the smallest subspace to ensure $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda,\mathcal{A})$-invariant and $\lambda \in \rho_\infty(\mathcal{A})$. As above, without loss of any generality we assume $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A})$, and $\mathscr{W} \cap \mathscr{W}_{\mathrm{f}} = 0$. We first show that $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})$. Since $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{A}$-invariant, it can be shown that $\operatorname{Im}\mathcal{D}\mathcal{C} \subset \mathscr{W}_{\overline{\mathrm{R}}}$, and also by definition we have $(\lambda\mathcal{I}-\mathcal{A})(\mathscr{W}_{\overline{\mathrm{R}}} \cap D(\mathcal{A})) \subseteq \mathscr{W}_{\overline{\mathrm{R}}}$. Now, set $x = \mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})y$ for an arbitrary $y \in \mathscr{W}_{\overline{\mathrm{R}}}$. It follows that $z = (\lambda\mathcal{I}-\mathcal{A})x = y + \mathcal{D}\mathcal{C}x \in \mathscr{W}_{\overline{\mathrm{R}}}$. Given that $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda,\mathcal{A})$-invariant, it follows that $x \in \mathscr{W}_{\overline{\mathrm{R}}}$. Therefore, $\mathscr{W}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant. Finally, since $\mathscr{W}$ is $(\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant, $\mathscr{W}_{\mathrm{f}}$ is Fin-D and $(\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant, by using Lemma 5.11 we obtain $\mathscr{W}$ is $\mathcal{R}(\lambda,\mathcal{A}+\mathcal{D}\mathcal{C})$-invariant. This completes the proof of the theorem. $\square$

Remark 5.16. *By invoking Corollary 4.9 in Chapter 4, every $\mathbb{T}_{\mathcal{A}}$-invariant subspace of RS systems can be expressed as a sum of sub-eigenspaces (as per* (4.8)*). Therefore, for RS systems conditions 2-a) can be represented as $\mathscr{W} = \mathscr{W}_\phi + \mathscr{W}_{\mathrm{f}}$.*

As emphasized earlier, for design of an FDI scheme, one needs to determine the smallest unobservability subspace (in the inclusion sense) containing a given subspace. For computing the smallest unobservability subspace one needs first to compute the minimal conditioned invariant subspace [3]. However, as shown in [105] the smallest $\mathbb{T}$-conditioned invariant may not always exist. In [103], the following algorithm is proposed for computing the minimal ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace (and emphnot $\mathbb{T}$-conditioned invariant) containing a given subspace $\mathscr{L}$. Specifically,

$$\text{Set } \mathscr{W}^0 = \mathscr{L}, \ \ \mathscr{W}^k = \overline{\mathscr{L} + \mathcal{A}(\mathscr{W}^{k-1} \cap \ker \mathcal{C} \cap D(\mathcal{A}))}. \tag{5.8}$$

However, as pointed out in [103], given that the above algorithm is non-decreasing, its limiting subspace may not be closed, and consequently it is not conditioned invariant in the sense of Definition 5.13, item 1)-item 3). Note that the above algorithm may not converge to the minimal $\mathbb{T}$-conditioned invariant even if such a subspace exists.

5.3.3 Controlled Invariant Subspaces

In this subsection, we address the controlled invariant subspaces and investigate the duality property between conditioned and controlled invariant subspaces.

Consider system (5.1) and the finite rank input operator (5.2). Corresponding to a conditioned invariant subspace, we have three different controlled invariant subspaces as follows:

Definition 5.17. *[103]*

1. *The closed subspace $\mathscr{V} \subseteq \mathcal{X}$ is called ($\mathcal{A}$,$\mathcal{B}$)-invariant if $\mathcal{A}(\mathscr{V} \cap D(\mathcal{A})) \subseteq \overline{\mathscr{V} + \operatorname{Im} \mathcal{B}}$.*

2. *The closed subspace $\mathscr{V} \subseteq \mathcal{X}$ is feedback-($\mathcal{A}$,$\mathcal{B}$)-invariant if there exists an operator $\mathcal{F} : \mathcal{X} \to \mathbb{R}^m$ such that $\mathscr{V}$ is invariant with respect to $(\mathcal{A} + \mathcal{B}\mathcal{F})$ as per Definition 5.8.*

3. *The closed subspace $\mathscr{V} \subseteq \mathcal{X}$ is $\mathbb{T}$-controlled invariant subspace if there exists an operator $\mathcal{F} : \mathcal{X} \to \mathbb{R}^m$ such that (i) the operator $(\mathcal{A} + \mathcal{B}\mathcal{F})$ is the infinitesimal generator of a C_0-semigroup $\mathbb{T}_{\mathcal{A}+\mathcal{B}\mathcal{F}}$; and (ii) $\mathscr{V}$ is invariant with respect to $\mathbb{T}_{\mathcal{A}+\mathcal{B}\mathcal{F}}$.*

Note that since $\mathcal{B}$ is finite rank ($\dim(\operatorname{Im}\mathcal{B}) < \infty$), using Lemma 4.13 we have $\overline{\mathscr{V} + \operatorname{Im}\mathcal{B}} = \mathscr{V} + \operatorname{Im}\mathcal{B}$. Also, as mentioned before we consider the controlled invariant subspaces that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$. As $\mathbb{T}$-conditioned invariant subspaces, for the $\mathbb{T}$-controlled invariant subspace, the set of all state feedbacks $\mathcal{F}$ such that $\mathbb{T}_{\mathcal{A}+\mathcal{F}\mathcal{B}}\mathscr{V} \subseteq \mathscr{V}$ (i.e., $\mathcal{F}$ is friend operator of $\mathscr{V}$) is denoted by $\underline{F}(\mathscr{V})$.

Now, we provide an important result on the controlled invariant subspaces. In fact, this theorem is dual to that of Theorem 5.15.

Theorem 5.18. *Consider the system* (5.1) *and the $(\mathcal{A},\mathcal{B})$-invariant subspace $\mathscr{V}$ such that $\overline{\mathscr{V} \cap D(\mathcal{A})} = \mathscr{V}$. Then the following statements are equivalent.*

1. *$\mathscr{V}$ is $\mathbb{T}$-controlled invariant.*

2. *There exists a $\lambda_1 \in \rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{B}\mathcal{F})$ such that $\mathcal{R}(\lambda_1, \mathcal{A})\mathscr{V} \subseteq \mathcal{R}(\lambda_1, \mathcal{A})\mathscr{B} + \mathscr{V}$ and*

   (a) *The subspace $\mathscr{V}$ can be represented as $\mathscr{V} = \mathscr{V}_{\overline{\mathrm{R}}} \cap \mathscr{V}_{\mathrm{f}}^{\perp}$ such that $\mathscr{V}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant and is defined in Section 1.6 and $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$, or*

   (b) *There exists a Fin-D $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$ such that (i) $\mathscr{V}_{\mathrm{f}}^{\perp} + \mathscr{B} = \mathcal{X}$, (ii) $\mathscr{V}_{\mathrm{f}}^{\perp}$ is $\mathbb{T}$-controlled invariant with $\underline{F}(\mathscr{V}_{\mathrm{f}}^{\perp}) \cap \underline{F}(\mathscr{V}) \neq 0$, and $\mathscr{V}_{\underline{\mathrm{R}}} = \mathscr{V} \cap \mathscr{V}_{\mathrm{f}}^{\perp}$, where $\mathscr{V}_{\underline{\mathrm{R}}}$ $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant is defined in Section 1.6.*

3. *For every $\lambda \in \rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A} + \mathcal{B}\mathcal{F})$, we have $\mathcal{R}(\lambda, \mathcal{A})\mathscr{V} \subseteq \mathcal{R}(\lambda, \mathcal{A})\mathscr{B} + \mathscr{V}$ and*

   (a) *The subspace $\mathscr{V}$ can be represented as $\mathscr{V} = \mathscr{V}_{\overline{\mathrm{R}}} \cap \mathscr{V}_{\mathrm{f}}^{\perp}$ such that $\mathscr{V}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant and is defined in Section 1.6 and $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$, or*

(b) *There exists a Fin-D* $\mathscr{V}_{\mathrm{f}} \subset D(\mathcal{A}^*)$ *such that (i)* $\mathscr{V}_{\mathrm{f}}^{\perp} + \mathscr{B} = \mathcal{X}$, *(ii)* $\mathscr{V}_{\mathrm{f}}^{\perp}$ *is* $\mathbb{T}$*-controlled invariant with* $\underline{F}(\mathscr{V}_{\mathrm{f}}^{\perp}) \cap \underline{F}(\mathscr{V}) \neq 0$ *and* $\mathscr{V}_{\underline{\mathrm{R}}} = \mathscr{V} \cap \mathscr{V}_{\mathrm{f}}^{\perp}$, *where* $\mathscr{V}_{\underline{\mathrm{R}}}$ $\mathcal{R}(\lambda, \mathcal{A})$*-invariant is defined in Section 1.6.*

where $\mathscr{B}$ *is the range of the input operator* $\mathcal{B}$ *(i.e.* $\mathscr{B} = \operatorname{Im} \mathcal{B}$*).*

Proof. We show the results by using the duality property.

(**1** $\Rightarrow$ **2 and 3**): Since $\mathscr{V}$ is $\mathbb{T}$-controlled invariant, it follows that $\mathscr{W} = \mathscr{V}^{\perp}$ is $\mathbb{T}$-conditioned invariant with respect to the operator $(\mathcal{A}^* + \mathcal{F}^*\mathcal{B}^*)$ (by using Lemma 4.29). From Theorem 5.15 it follows that $\mathcal{R}(\lambda, \mathcal{A}^*)\mathscr{W} \cap \ker \mathcal{B}^* \subseteq \mathscr{W}$. Hence, we obtain $\mathscr{V} \subseteq ((\lambda\mathcal{I} - \mathcal{A})(\mathscr{V} \cap D(\mathcal{A})) + \mathscr{B})$. Therefore, $\mathcal{R}(\lambda, \mathcal{A})\mathscr{V} \subseteq \mathscr{V} + \mathcal{R}(\lambda, \mathcal{A})\mathscr{B}$.

Now, we show that the conditions 2-a) and 2-b) are dual to the conditions 2-a) and 2-b) in the Theorem 5.15, respectively. By using Theorem 5.15, there are two cases as follows:

1. Let $\mathscr{W} = \mathscr{W}_{\underline{\mathrm{R}}} + \mathscr{W}_{\mathrm{f}}$ such that $\mathscr{W}_{\underline{\mathrm{R}}}$ is defined in Theorem 5.15 and $\mathscr{W}_{\mathrm{f}}$ is a Fin-D subspace. Since $D(\mathcal{A}^*)$ is densely defined in the both subspaces $\mathscr{W}$ and $W_{\underline{\mathrm{R}}}$, we obtain $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A}^*)$. Given that $\mathscr{W}_{\underline{\mathrm{R}}}$ is $\mathbb{T}_{\mathcal{A}^*+\mathcal{F}^*\mathcal{B}^*}$-invariant, the subspace $\mathscr{V}_{\overline{\mathrm{R}}} = (\mathscr{W}_{\underline{\mathrm{R}}})^{\perp}$ is $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant. Moreover, set $\mathscr{V}_{\mathrm{inf}} = (\mathscr{W}_{\mathrm{f}})^{\perp}$. Consequently, $\mathscr{V} = \mathscr{V}_{\overline{\mathrm{R}}} \cap \mathscr{W}_{\mathrm{f}}^{\perp}$.

2. Assume that there exists a Fin-D $\mathscr{W}_{\mathrm{f}}$ such that $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{B}^* = 0$, and $\mathscr{W}_{\overline{\mathrm{R}}} = \mathscr{W} + \mathscr{W}_{\mathrm{f}}$, where $\mathscr{W}_{\overline{\mathrm{R}}}$ is defined in Theorem 5.15 for $\mathcal{A}^*$. As above, $\mathscr{V}_{\underline{\mathrm{R}}} = (\mathscr{W}_{\overline{\mathrm{R}}})^{\perp}$ is $\mathcal{R}(\lambda_1, \mathcal{A})$-invariant, and $\mathscr{W}_{\mathrm{f}} \subset D(\mathcal{A}^*)$. Moreover, since $\mathscr{W}_{\mathrm{f}} \cap \ker \mathcal{B}^* = 0$, we obtain $\mathscr{W}_{\mathrm{f}}^{\perp} + \mathscr{B} = \mathcal{X}$, and $\mathscr{W}_{\mathrm{f}}^{\perp}$ is $\mathbb{T}$-controlled invariant with $\underline{F}(\mathscr{W}_{f}^{\perp}) \cap \underline{F}(\mathscr{V}) \neq \emptyset$ (since $\mathscr{W}_{\mathrm{f}}$ is $\mathbb{T}$-conditioned invariant with $\underline{D}(\mathscr{W}_{f}) \cap \underline{D}(\mathscr{W})$). Finally, it follows that $\mathscr{V}_{\underline{\mathrm{R}}} = \mathscr{V} \cap \mathscr{W}_{\mathrm{f}}^{\perp}$.

Since all the above derivations hold for every $\lambda_1 \in \rho_{\infty}(\mathcal{A}) \cap \rho_{\infty}(\mathcal{A} + \mathcal{B}\mathcal{F})$, we also obtain the conditions 3-a) and 3-b).

($\mathbf{3} \Rightarrow \mathbf{1}$): By following along the same steps (i.e. by applying the duality property), one can show this part. This completes the proof of the theorem. $\square$

In the next section, we use the above theorem to address unobservability subspaces.

5.3.4 Unobservability Subspace

In the geometric FDI approach, one needs another invariant subspace, namely the unobservability subspace. In this section, we characterize this subspace.

Definition 5.19. *Unobservability subspace*

1. *The subspace $\mathcal{S}$ is called an $\mathcal{A}$-unobservability subspace for the system* (5.1)*, if there exist two bounded operators $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ and $H : \mathbb{R}^q \to \mathbb{R}^{q_h}$ where $q_h \leq q$ such that $\mathcal{S}$ is the largest $(\mathcal{A} + \mathcal{D}\mathcal{C})$-invariant subspace contained in* $\ker H\mathcal{C}$*. We denote the largest $\mathcal{A}$-invariant subspace contained in $\mathscr{C}$ by $<\mathscr{C}|\mathcal{A}>$, and hence we have $\mathcal{S} = <\ker H\mathcal{C}|\mathcal{A} + \mathcal{D}\mathcal{C}>$.*

2. *The subspace $\mathcal{S}$ is called an $\mathbb{T}$-unobservability subspace for the system* (5.1)*, if there exist two bounded operators $\mathcal{D} : \mathbb{R}^q \to \mathcal{X}$ and $H : \mathbb{R}^q \to \mathbb{R}^{q_h}$ where $q_h \leq q$ such that $\mathcal{S}$ is the largest $\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}$-invariant subspace contained in* $\ker H\mathcal{C}$*, and hence we have $\mathcal{S} = <\ker H\mathcal{C}|\mathbb{T}_{\mathcal{A}+\mathcal{D}\mathcal{C}}>$.*

It follows that the $\mathbb{T}$-unobservability subspace $\mathcal{S}$ is the unobservable subspace of the system ($\mathcal{H}\mathcal{C}$,$\mathcal{A} + \mathcal{D}\mathcal{C}$). Also, every $\mathbb{T}$-unobservability subspace is a $\mathbb{T}$-conditioned invariant subspace.

Remark 5.20. *As conditioned invariant there is no algorithm that compute the smallest conditioned invariant. However, if the following algorithm converges in a finite number of steps the limiting subspace is the smallest unobservability subspace* *containing a Fin-D subspace $\mathscr{L} \subset D(\mathcal{A})$ [103].*

$$\mathcal{S}^0 = \mathcal{X},\ \mathcal{S}^k = \overline{\mathscr{W}^* + \mathcal{R}(\lambda, \mathcal{A})\mathcal{S}^{k-1} \cap \ker \mathcal{C}}, \tag{5.9}$$

where $\mathscr{W}^*$ is the smallest $\mathbb{T}$-conditioned invariant subspace containing $\mathscr{L}$.

5.3.5 Summary

In this section, we have first reviewed $\mathbb{T}_{\mathcal{A}}$-invariant subspaces. Necessary and sufficient conditions for equivalence of conditioned invariant subspaces were derived and obtained. Also, we reviewed the duality and controlled invariant subspaces. Moreover, we have introduced the unobservability subspace of Inf-D systems.

5.4 Fault Detection and Isolationof Inf-D Systems

In this section, we first formulate the FDI problem and then by using the methodology that is developed in the previous section, we provide a necessary and sufficient conditions for solvability of this problem.

5.4.1 The FDI Problem Statement

Consider the following faulty Inf-D systems

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \sum_{i=1}^{p} \mathcal{L}_i f_i(t), \\ y(t) &= \mathcal{C}x(t), \end{aligned} \tag{5.10}$$

where f_i and the bounded operator $\mathcal{L}_i$ are fault signals and signatures, respectively. The other operators are defined as in (5.1), (5.2) and (5.3). The FDI problem is now specified as that of generating a set of residual signals, $r_i(t)$, $i = 1, \cdots, p$ such that each residual signal $r_i(t)$ is decoupled from the inputs and all faults, but one fault $f_i(t)$. In other words, the residual signal $r_i(t)$ satisfies the following conditions

$$\forall u, f_j, \text{ if } f_i = 0 \ \Rightarrow r_i \to 0, \tag{5.11a}$$

$$\forall u, f_j, \text{ if } f_i \neq 0 \ \Rightarrow r_i > \epsilon, \ \epsilon > 0. \tag{5.11b}$$

In other words, condition (5.11a) ensures that r_i is decoupled from all faults but f_i and condition (5.11b) guarantees that r_i is sensitive to f_i. Moreover, the realization of the residual signal $r_i(t)$ is accomplished by using the following fault detection filter

$$\begin{aligned} \dot{\omega}_i(t) &= \mathcal{A}_o\omega_i(t) + \mathcal{B}_o u(t) + \mathcal{E}_i\mathcal{C}x(t), \\ r_i(t) &= H_i y(t) - \mathcal{M}_i\omega_i(t), \end{aligned} \tag{5.12}$$

where $\omega_i(t) \in \mathcal{X}_o^i$ and $\mathcal{X}_o^i$ is a separable Hilbert space (Fin-D or Inf-D). The operators $\mathcal{A}_o$, $\mathcal{B}_o$, and $\mathcal{E}_i$ are closed with appropriate domains and codomains. For example, $\mathcal{A}_o : \mathcal{X}_o^i \to \mathcal{X}_o^i$ and $\mathcal{E}_i : \mathbb{R}^q \to \mathcal{X}_o^i$. However, unlike Chapter 4, in this chapter the operators $\mathcal{A}$ and $\mathcal{A}_o$ are not necessarily RS operators. The specific characteristics of the filter operators and parameters will be designed and determined subsequently.

Remark 5.21. *As in Chapters 3 and 4, the problem of detection and isolation of a fault f_i involves two main steps as follows: First, (i) derive a subsystem that is decoupled from all faults but f_i (this is denoted as the decoupling problem), and second (ii) design a detection filter (as per equation* (5.12)*) to detect and isolate the fault f_i. If both steps are solvable, we then say that the FDI problem is solvable.*

5.4.2 Necessary and Sufficient Conditions for Solvability of the FDI Problem

In this subsection, by using the methodology that was developed above, necessary and sufficient conditions for solvability of the FDI problem are provided. As stated in Subsection 5.4.1, the FDI problem can be cast as that of designing detection filters as per equation (5.12) such that each residual is decoupled from all faults but one.

By augmenting the system (5.1) and the detection filter (5.12), one can write

$$
\begin{aligned}
\dot{x}^e(t) &= \mathcal{A}^e x^e(t) + \mathcal{B}^e u(t) + \sum_{i=1}^{p} \mathcal{L}_i^e f_i(t), \\
r_i(t) &= \mathcal{C}^e x^e(t),
\end{aligned}
\tag{5.13}
$$

where $x^e(t) = \begin{bmatrix} x(t) \\ \omega_i(t) \end{bmatrix} \in \mathcal{X}^e = \mathcal{X} \oplus \mathcal{X}_o^i$, $\mathcal{C}^e = \begin{bmatrix} H_i\mathcal{C} & -\mathcal{M}_i \end{bmatrix}$ and

$$
\mathcal{A}^e = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{E}_i\mathcal{C} & \mathcal{A}_o \end{bmatrix}, \; \mathcal{B}^e = \begin{bmatrix} \mathcal{B} \\ \mathcal{B}_o \end{bmatrix}, \; \mathcal{L}_i^e = \begin{bmatrix} \mathcal{L}_i \\ 0 \end{bmatrix}. \tag{5.14}
$$

Given that $\mathcal{A}_o$ in equation (5.12) is an infinitesimal generator of a C_0 semigroup and $\mathcal{E}_i$, and consequently $\mathcal{E}_i\mathcal{C}$ are bounded operators, by using Lemma 4.36 - item 1) the operator $\mathcal{A}^e$ is an infinitesimal generator of a C_0 semigroup. The following lemma characterizes the resolvent operator of $\mathcal{A}^e$.

Lemma 5.22. *Consider the operator* $\mathcal{A}^e = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$, *where* $\mathcal{A}_{11}$ $\mathcal{A}_{22}$ *are infinitesimal generator and* $\mathcal{A}_{21}$ *is a bounded operator. If* $\lambda \in (\rho_\infty(\mathcal{A}_{11}) \cap \rho_\infty(\mathcal{A}_{22}))$, *then* $\lambda \in \mathcal{R}(\lambda, \mathcal{A}^e)$ *and* $\mathcal{R}(\lambda, \mathcal{A}^e) = \begin{bmatrix} \mathcal{R}(\lambda, \mathcal{A}_{11}) & 0 \\ -\mathcal{R}(\lambda, \mathcal{A}_{22})\mathcal{A}_{21}\mathcal{R}(\lambda, \mathcal{A}_{11}) & \mathcal{R}(\lambda, \mathcal{A}_{22}) \end{bmatrix}$.

Proof. Let $\lambda \in (\rho_\infty(\mathcal{A}_{11}) \cap \rho_\infty(\mathcal{A}_{22}))$. Therefore, $\mathcal{R}(\lambda, \mathcal{A}_{11})$ and $\mathcal{R}(\lambda, \mathcal{A}_{22})$ are well-defined and bounded. Now, let $\mathcal{R} = \begin{bmatrix} \mathcal{R}(\lambda, \mathcal{A}_{11}) & 0 \\ -\mathcal{R}(\lambda, \mathcal{A}_{22})\mathcal{A}_{21}\mathcal{R}(\lambda, \mathcal{A}_{11}) & \mathcal{R}(\lambda, \mathcal{A}_{22}) \end{bmatrix}$. It follows that $(\lambda\mathcal{I} - \mathcal{A}^e)\mathcal{R} = \mathcal{I}$. Also, given that $\mathcal{R}(\lambda, \mathcal{A}_{11})$ and $\mathcal{R}(\lambda, \mathcal{A}_{22})$ are bounded, it follows that $\mathcal{R}$ is bounded. Hence, $\mathcal{R}(\lambda, \mathcal{A}^e) = \mathcal{R}$ and $\lambda \in \rho_\infty(\mathcal{A}^e)$. This completes the proof of the lemma. □

The following lemma shows the relationship between the unobservable subspace of the system (5.13) and the unobservability subspace of the system (5.1). This property will be used to derive the necessary conditions for the FDI problem solvability.

Lemma 5.23. *Consider the system* (5.13) *and let* $\mathscr{N}^e = <\ker \mathcal{C}^e | \mathbb{T}_{\mathcal{A}^e}>$. *Then,* $\mathcal{Q}^{-1}\mathscr{N}^e$ *is an unobservability subspace of the system* (5.1), *where* $\mathcal{Q}$ *is the embedding operator given by* $\mathcal{Q} : \mathcal{X} \to \mathcal{X}^e$ *and* $\mathcal{Q}x = \begin{bmatrix} x \\ 0 \end{bmatrix}$.

Proof. Since $\mathscr{N}^e$ is $\mathbb{T}_{\mathcal{A}^e}$-invariant, it is also $\mathcal{A}^e$-invariant [103]. Also, by using Lemma 5.9 we have $\mathcal{R}(\lambda, \mathcal{A}^e)\mathscr{N}^e \subseteq \mathscr{N}^e$, where $\lambda \in \rho_\infty(\mathcal{A}^e)$. Let $\mathcal{S} = \mathcal{Q}^{-1}\mathscr{N}^e$, and assume that $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathscr{N}^e$ and $x \in D(\mathcal{A}) \cap \ker \mathcal{C}$. It follows that $\mathcal{A}x \in \mathcal{S}$ (or $\mathcal{A}(\mathcal{S} \cap \ker \mathcal{C} \cap D(\mathcal{A})) \subseteq \mathcal{S}$). Since $\overline{\mathscr{N}^e \cap D(\mathcal{A})} = \mathscr{N}^e$, by following along the same steps as in Lemma 4.37, we obtain $\overline{\mathcal{S} \cap D(\mathcal{A})} = \mathcal{S}$. Therefore, by using Lemma 4.14, it follows that $\mathcal{S}$ is a feedback ($\mathcal{C}$,$\mathcal{A}$)-invariant subspace.

Now consider $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathscr{N}^e$, such that $x \in \mathcal{R}(\lambda, \mathcal{A})\mathcal{S} \cap \ker \mathcal{C}$, where $\lambda \in (\rho_\infty(\mathcal{A}) \cap \rho_\infty(\mathcal{A}_o))$. By using Lemma 5.22, one can write $\lambda \in \rho_\infty(\mathcal{A}^e)$, and (refer to Lemma 5.22)

$$\mathcal{R}(\lambda, \mathcal{A}^e) = \begin{bmatrix} \mathcal{R}(\lambda, \mathcal{A}) & 0 \\ -\mathcal{R}(\lambda, \mathcal{A}_o)\mathcal{E}\mathcal{C}\mathcal{R}(\lambda, \mathcal{A}) & \mathcal{R}(\lambda, \mathcal{A}_o) \end{bmatrix}. \tag{5.15}$$

Therefore, by using the fact that $\mathscr{N}^e$ is $\mathcal{R}(\lambda, \mathcal{A}^e)$-invariant one can write $\mathcal{R}(\lambda, \mathcal{A}^e) \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{R}(\lambda, \mathcal{A})x \\ -\mathcal{R}(\lambda, \mathcal{A}_o)\mathcal{E}\mathcal{C}\mathcal{R}(\lambda, \mathcal{A})x \end{bmatrix} \in \mathscr{N}^e$. Invoking the assumption $\mathcal{R}(\lambda, \mathcal{A})x \in \ker \mathcal{C}$, we have $\mathcal{R}(\lambda, \mathcal{A})x \in \mathcal{S} \subseteq \ker H\mathcal{C}$. In other words, $\mathcal{R}(\lambda, \mathcal{A})\mathcal{S} \cap \ker \mathcal{C} \subseteq \mathcal{S} \cap \ker \mathcal{C}$.

Finally, by invoking contradiction, we show that one of the conditions 2-a) or 2-b) in Theorem 5.15 is satisfied. Hence, let us now assume that both conditions fail and $\mathcal{S}_{\overline{\mathrm{R}}} = \overline{\mathcal{S} + \mathcal{S}_{\mathrm{f}}}$, where $\dim(\mathcal{S}_{\mathrm{f}}) = \infty$, $\mathcal{S}_{\mathrm{f}}$ contains no $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspace, and $\mathcal{S}_{\overline{\mathrm{R}}}$ is $\mathcal{R}(\lambda, \mathcal{A})$-invariant subspace. Since both conditions 2-a) and 2-b) fail and $\mathcal{R}(\lambda, \mathcal{A})\mathcal{S} \cap \ker \mathcal{C} \subseteq \mathcal{S}$, it follows that there is a subspace $\mathcal{S}_{\mathrm{r}} \subset \mathcal{S}$ such that $\mathcal{S}_{\mathrm{r}} \cap (\ker H\mathcal{C})^{\perp} \neq 0$, and for all $n \in \mathbb{N}$, we have $\mathcal{R}(\lambda, \mathcal{A})^n \mathcal{S}_{\mathrm{r}} \nsubseteq \mathcal{S}$. Again, since $\mathcal{R}(\lambda, \mathcal{A})\mathcal{S} \cap \ker \mathcal{C} \subseteq \mathcal{S}$, there exists a k_0 and $y \in \mathcal{S}_{\mathrm{r}}$ such that $\mathcal{S}_{k_0} = \operatorname{span}\{\mathcal{R}(\lambda, \mathcal{A})^n y\}_{n=k_0}^{\infty}$ and $\mathcal{R}(\lambda, \mathcal{A})\mathcal{S}_{k_0} \subseteq \mathcal{S}_{k_0}$ that is in contradiction with the definition of $\mathcal{S}_{\mathrm{f}}$. In other words, $\mathcal{S}$ satisfies at least one of the conditions 2-a) and 2-b), and consequently it is $\mathbb{T}$-conditioned invariant of the system (5.1). Moreover, since $\mathscr{N}^e$ is the largest $\mathbb{T}_{\mathcal{A}^e}$-invariant subspace in $\ker \begin{bmatrix} H_i\mathcal{C} & \mathcal{M}_i \end{bmatrix}$, the subspace $\mathcal{S}$ is the largest $\mathbb{T}$-conditioned invariant subspace that is contained in $\ker H_i\mathcal{C}$. Therefore, $\mathcal{S}$ is an unobservability subspace of the system (5.1). This completes the proof of the lemma. $\square$

To provide sufficient conditions, one needs to show that the error dynamics of the designed fault detection observer is stable. As in Chapter 4, we use Lemma 4.39 for this purpose. Now, we are in the position to provide the necessary and sufficient conditions for solvability of the FDI problem.

Theorem 5.24. *Consider the system* (5.10)*. The FDI problem is solvable only if*

$$\mathcal{S}_i^* \cap \mathscr{L}_i = 0, \tag{5.16}$$

where $\mathcal{S}_i^* =< \ker H_i\mathcal{C} | \mathcal{A} + \mathcal{D}_i\mathcal{C} >$ *is the smallest unobservability subspace containing all* $\mathscr{L}_j = \operatorname{span}\{\mathcal{L}_j\}$ *, where* $j = 1, \cdots, p$ *and* $j \neq i$*, and* $\mathscr{L}_i = \operatorname{span}\{\mathcal{L}_i\}$ *. Moreover, if there exist two maps* $\mathcal{E}$ *and* $\mathcal{P}_e$ *such that* $(\mathcal{A}_p + \mathcal{D}_o H_i\mathcal{C})$ *and* $\mathcal{P}_e$ *satisfy the condition* (4.23)*, then the FDI problem is solvable, where* $\mathcal{A}_p = (\mathcal{A} + \mathcal{D}_i\mathcal{C})|_{\mathcal{X}/\mathcal{S}_i^*}$ *is the induced operator of* $\mathcal{A} + \mathcal{D}_i\mathcal{C}$ *on the factor space* $\mathcal{X}/\mathcal{S}^*$ *and* $\mathcal{D}_i \in \underline{D}(\mathcal{S}_i^*)$*.*

Proof. Without loss of generality, we consider the system (5.10) that is subject to two faults f_1 and f_2. However, the results are applicable to any number of faults.
(**Necessary Part**): Assume that the detection filter (5.12) is designed such that the residual (the output of the filter) is decoupled from f_2 but it is sensitive to f_1.

By considering the augmented system (5.13), it is necessary that $\mathscr{L}_2^e \subseteq \mathscr{N}^e$, ($\mathscr{L}_i^e$ is defined in equation (5.14)), where $\mathscr{N}^e$ is the unobservable subspace of system (5.13), and by using Lemma 5.23, the subspace $\mathcal{S} = \mathcal{Q}^{-1}\mathscr{N}^e$ is an unobservability subspace of the pair ($\mathcal{C}$,$\mathcal{A}$) that contains $Q^{-1}\mathscr{L}_2^e = \mathscr{L}_2$ (i.e., $\mathscr{L}_2 \subseteq \mathcal{S}$). By detecting the fault f_1 (that is an arbitrary function of time), it implies that $\mathscr{N}^e \cap \mathscr{L}_1^e = 0$, and consequently $\mathcal{S} \cap \mathscr{L}_1 = 0$. Since $\mathcal{S}_1^*$ is the minimal unobservability subspace containing $\mathscr{L}_2$ (i.e., $\mathcal{S}_1^* \subseteq \mathcal{S}$), the necessary condition for satisfying the above condition is $\mathcal{S}_1^* \cap \mathscr{L}_1 = 0$.

(**Sufficient Part**): If the condition (5.16) is satisfied for f_1, one can write $\mathcal{X} = \overline{\mathcal{X}/\mathcal{S}_1^* \oplus \mathcal{S}_1^*}$, and $\mathcal{S}_1^*$ is $\mathbb{T}_{A+DC}$-invariant. As in previous chapters, consider the canonical projection $\mathcal{P}_1 : \mathcal{X} \to \mathcal{X}/\mathcal{S}_1^*$ and the following detection filter

$$\begin{aligned} \dot{\omega}_1(t) =& \mathcal{F}_1\omega_1(t) + \mathcal{G}_1 u(t) - \mathcal{E}_1 y(t) \\ r_1(t) =& \mathcal{M}_1\omega_1(t) - H_1 y(t) \end{aligned} \tag{5.17}$$

where $\mathcal{F}_1 = \mathcal{A}_p + \mathcal{D}_o\mathcal{M}_1$, $\mathcal{G} = \mathcal{P}_1\mathcal{B}$ and $\mathcal{E}_1 = \mathcal{D}_1 + \mathcal{P}_1^{-r}\mathcal{D}_o H_1$. By following along the same steps in Section 2.2 one obtains the error dynamics associated with the signal $e(t) = \mathcal{P}_1 x(t) - \omega_1(t)$ is given by

$$\dot{e}(t) = \mathcal{F}_1 e(t) + \mathcal{L}_{11} f_1(t). \tag{5.18}$$

By using results in Lemma 4.39, the error dynamics (5.18) can be made to be exponentially stable. Therefore, if $f_1 = 0$ then $e(t) \to 0$. Otherwise $||e(t)|| > \epsilon$, $\epsilon > 0$, where $e(t) \in \mathbb{R}^{q_h}$ and $q_h = \dim(\operatorname{Im} H)$. By setting $r_1(t) = \mathcal{M}e(t)$, the conditions in equation (5.11) are satisfied and the FDI problem is rendered solvable. This completes the proof of the theorem. □

As developed in Chapter 4, the following corollaries present two special cases, where the condition (5.16) is a single necessary and sufficient condition for solvability of the FDI problem (5.11).

Corollary 5.25. *Consider the unobservability subspace $\mathcal{S}^*$ and the system ($\mathcal{M}_1$, $\mathcal{A}_p$), where the operator $\mathcal{A}_p$ is exponentially stable (by satisfying the condition in Lemma 4.39). The condition* (5.16) *is necessary and sufficient for solvability of the FDI problem* (5.11).

Proof. The necessary condition follows from Theorem 5.24. Hence, we show the sufficient part. As illustrated above, the detection and isolation of the fault f_i is restricted to design of an observer for the quotient subsystem $(\mathcal{M}_1, \mathcal{A}_p)$ (that is to obtain $\mathcal{D}_o$). However, given that $\mathcal{A}_p$ is exponentially stable, by defining $\mathcal{D}_o = 0$, we obtain the following detection filter

$$\begin{aligned}\dot{\omega}_1(t) &= \mathcal{A}_p\omega_1(t) + \mathcal{G}_1 u(t),\\ r_1(t) &= H_1 y(t) - \mathcal{M}_1\omega_1(t),\end{aligned} \tag{5.19}$$

where $\mathcal{M}_1$ is the solution of $\mathcal{M}_1\mathcal{P}_1 = H_1\mathcal{C}$. It follows that the error dynamics (that is $e(t) = \mathcal{P}_1 x(t) - \omega_1(t)$) is given by $\dot{e}(t) = \mathcal{A}_p e(t)$, which is exponentially stable. Therefore, if $f_1(t) = 0$, it follows that $r_1(t) \to 0$, and if $f_1(t) \neq 0$ one obtains $||r_1(t)|| > \epsilon$, $\epsilon > 0$. This completes the proof of the corollary. □

Corollary 5.26. *Consider the system* (5.10) *and the unobservability subspace $\mathcal{S}^*$ such that $\mathcal{X}/\mathcal{S}^*$ is Fin-D. The condition* (5.16) *is necessary and sufficient for solvability of the FDI problem* (5.11).

Proof. The necessary condition follows from the results in Theorem 5.24. Moreover, by following along the same steps as in Theorem 5.24 one obtains the quotient subsystem (M_1, A_p) such that it is decoupled from all fault but one. Moreover, since this subsystem is a Fin-D system with $\mathscr{N} = 0$, it follows that it is observable and there exists an observer gain such that $A_p + D_o M_1$ is Hurwitz. By following along the same steps as in the Corollary 5.25 the sufficient result follows. This completes the proof of the corollary. □

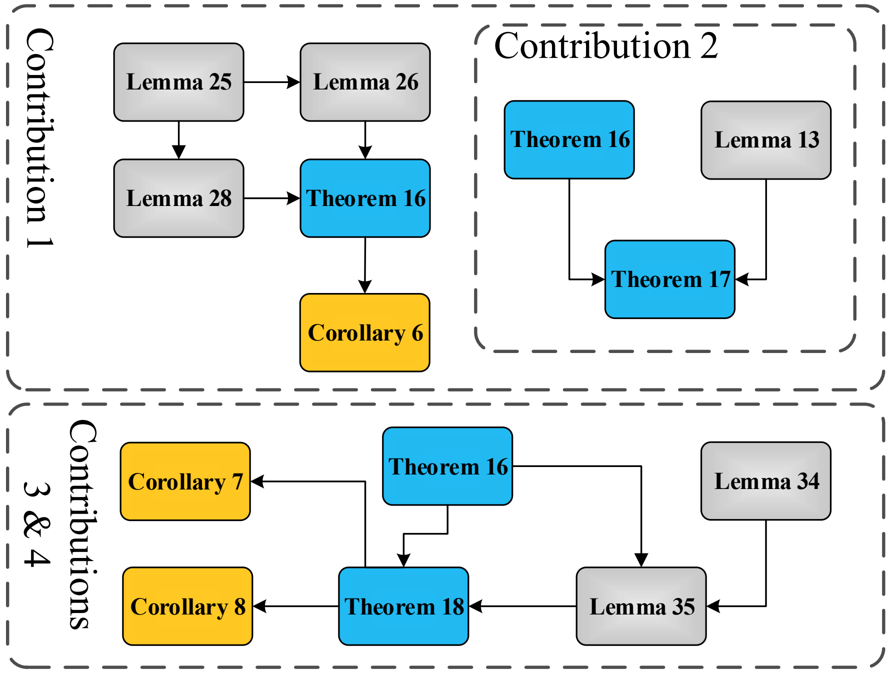


Figure 5.1: The diagram showing the relationships among lemmas, theorems and corollaries that are developed and presented in this chapter. For definition of the contributions refer to Subsection 1.4.2.

5.4.3 Summary

In this section, we have formulated the FDI problem and by utilizing the geometric theory that was developed in the preceding section, we have derived, for the first time in the literature, necessary and sufficient conditions for solvability of the FDI problem for Inf-D systems. For more details and clarification Figure 5.1 provides a schematic that depicts the relationship among the various lemmas, theorems and corollaries that are developed in this chapter. The contribution of this chapter has been summarized in Subsection 1.4.2.

Moreover, Table 5.1 summarizes the FDI scheme that was developed in this section.

Table 5.1: A Pseudo-algorithm for detecting and isolating the fault f_i for the Inf-D system (5.10).

1. Obtain the smallest conditioned invariant subspace $\mathscr{W}^*$ containing all $\mathcal{L}_k$ subspaces such that $k \neq i$.
2. Obtain the unobservability subspace $\mathcal{S}_i^*$ containing $\sum_{j \neq i} \mathcal{L}_j^1$.
3. Compute the operator $\mathcal{D}_i$ such that $\mathcal{D}_i \in \underline{\mathcal{D}}(\mathscr{W}^*)$.
4. Find the operator H such that $\ker H_i = \mathcal{C}\mathcal{S}_i^*$.
5. If $\mathcal{S}_i^* \cap \mathcal{L}_i = 0$, then the necessary condition for solvability of the FDI problem is satisfied. Moreover, if one of the following conditions are satisfied, the FDI problem is solvable. In other words, one can design a detection filter according to the structure provided in (5.12) to detect and isolate the fault f_i,
   - If there exists a bounded operator $\mathcal{D}_o$ such that the conditions of Theorem 5.24 are satisfied, or
   - The condition of the Corollary 5.25 is satisfied.
   - If $\dim(\mathcal{X}/\mathcal{S}_i^*) < \infty$ (i.e. the condition of the Corollary 5.26 is satisfied).

   The operators in the detection filter (5.12) are defined as follows. Let $\mathcal{P}_i$ be the canonical projection of $\mathcal{S}_i^*$, then $\mathcal{A}_o = (\mathcal{A} + \mathcal{D}_i\mathcal{C})|_{\mathcal{X}/\mathcal{S}_i^*}$, $\mathcal{B}_o = \mathcal{P}_i\mathcal{B}$, $\mathcal{M}_i\mathcal{P}_i = H_i\mathcal{C}$, $\mathcal{E} = \mathcal{D}_o H_i$ and $\mathcal{D}_o$ is selected such that $(\mathcal{A}_p + \mathcal{D}_o\mathcal{M}_i)$ satisfies the condition of Lemma 4.39. Moreover, the output of the detection filter (i.e., $r_i(t)$) is the residual that satisfies the conditions in equation (5.11).

5.5 Numerical Example

In this section, we provide a numerical example to emphasize the applicability of our proposed approach in this chapter. Consider the following Inf-D system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{L}_1 f_1(t) + \mathcal{L}_2 f_2(t) + \nu(t),$$
$$y(t) = \mathcal{C}x(t) + w(t), \tag{5.20}$$

where ν_i and w_i ($i = 1, 2$) denote the process and measurement noise that are assumed to be normal distributions with 0.1 and 0.2 variances, respectively. Also,

$$
\begin{aligned}
&J_k = \begin{bmatrix} -k & 1 & 0 & 0 \\ 0 & -k & 1 & 0 \\ 0 & 0 & -k & 1 \\ 0 & 0 & 0 & -k \end{bmatrix}, \mathcal{A} = \mathrm{diag}(J_1, J_2, \cdots), \\
&b_{1,1} = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; b_{2,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \end{bmatrix}, \\
&b_k = \begin{bmatrix} \frac{1}{k^3} & \frac{1}{k^3} & \frac{1}{k^3} & \frac{1}{k^3} \end{bmatrix}, b^1 = \begin{bmatrix} b_{1,1} & b_3 & b_4 & \cdots \end{bmatrix}^{\mathrm{T}}, \; b^2 = \begin{bmatrix} b_{2,1} & b_3 & b_4 & \cdots \end{bmatrix}^{\mathrm{T}}, \\
&c_{1,1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \; c_{21} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \\
&c_k = \begin{bmatrix} \frac{1}{k^3} & 0 & 0 & 0 \end{bmatrix}, c^1 = \begin{bmatrix} c_{11} & c_3 & c_4 & \cdots \end{bmatrix}, \; c^2 = \begin{bmatrix} c_{2,1} & c_3 & c_4 & \cdots \end{bmatrix}, \mathcal{C} = \begin{bmatrix} c^1 \\ c^2 \end{bmatrix}, \\
&\mathcal{B} = \begin{bmatrix} b^1 & b^2 \end{bmatrix}, \mathcal{L}_1 = b^1, \mathcal{L}_2 = b^2.
\end{aligned}
\tag{5.21}
$$

It should be pointed out that the Inf-D system (5.20) is not a regular RS system (since the number of multiple eigenvalues in $\mathcal{A}$ is infinite), and consequently the approach in Chapter 4 is not applicable to this system.

In the following, a detection filter is designed for detecting and isolating the fault f_1. Since $\mathscr{L}_2 = \mathrm{span}\{L_2\} \in D(\mathcal{A})$ and $\mathscr{L}_2 \notin \ker \mathcal{C}$, we obtain $\mathscr{W}^* = \mathscr{L}_2$ ($\mathscr{L}_2 \in D(\mathcal{A})$ is ($\mathcal{C}$,$\mathcal{A}$)-invariant and therefore by Lemma 4.12 $\mathscr{L}_2$ is $\mathbb{T}$-conditioned invariant). Given that $\mathscr{W}^* = \mathscr{L}_2$, we obtain that $\mathcal{S}_1^* = \mathscr{W}^* + \mathrm{span}\{b_{\mathrm{s}}\}$ (by using Remark 5.20), where

$$
\begin{aligned}
&b_{\mathrm{s},1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, b_{\mathrm{s,k}} = -\frac{1}{k^2} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \\
&b_{\mathrm{s}} = \begin{bmatrix} b_{\mathrm{s},1,2} & b_{\mathrm{s},3} & b_{\mathrm{s},4} & \cdots \end{bmatrix}.
\end{aligned}
\tag{5.22}
$$

It follows that $\mathscr{L}_1 \cap \mathcal{S}_1^* = 0$, and a solution for the corresponding maps $\mathcal{D}$ and H is given by $D_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$ and $H_1 = [-1, 1]$.

Since all the eigenvalues of $\mathcal{A}_p$ are negative, by using Corollary 5.25 a detection filter is therefore specified according to

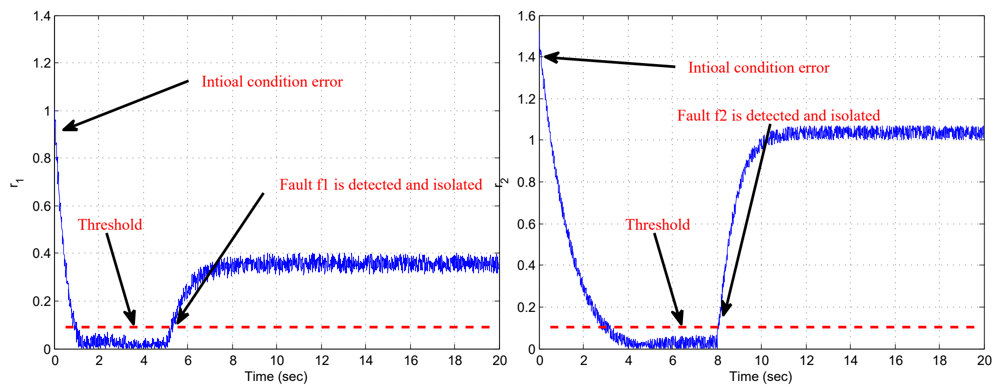
$$\begin{aligned}\dot{\omega}_1(t) &= \mathcal{A}_o\omega_1(t) + \mathcal{P}_1\mathcal{B}u(t),\\ r_1(t) &= H_1y(t) - \mathcal{M}_1\omega_1(t),\end{aligned} \tag{5.23}$$

where $\mathcal{A}_o = \mathcal{A}_p$. The error dynamics corresponding to the above detection filter (i.e., $e(t) = \mathcal{P}_1x(t) - \omega_1(t)$) is given by $\dot{e}(t) = \mathcal{A}_pe(t) + \mathcal{P}_1\mathcal{L}_1f_1(t)$. Therefore, if $f_1 = 0$, the error converges to origin exponentially. Otherwise, $e \neq 0$. The above residual (i.e, r_1) corresponding to the fault f_1 is also decoupled from f_2. By following along the same steps as above, one can design a detection filter to detect and isolate the fault f_2.

For the purpose of simulations, we consider a scenario where the fault f_1 with a severity of 1 occurs at $t = 5$ *sec* and the fault f_2 with a severity of 1 occurs at $t = 8$ *sec*. Figure 4.3 depicts the residuals r_1 and r_2. It clearly follows that r_i is only sensitive to the fault f_i, $i = 1, 2$. Note that the thresholds are computed based on running 100 Monte Carlo simulations for the healthy system with the thresholds selected as the maximum residual signals r_1 and r_2 during the entire simulation runtime. The selected thresholds are $th_1 = 0.1$ and $th_2 = 0.15$, corresponding to the residual signals r_1 and r_2, respectively. The faults f_1 and f_2 are detected at $t = 5.51$ *sec* and $t = 8.25$ *sec*, respectively. Table 4.2 shows the detection times corresponding to various severity fault cases that are simulated. This table clearly shows the impact of the fault severity levels on the detection times. In other words, the lower the fault severity, the longer the detection time delay. Moreover, the minimum detectable fault severities associated with f_1 and f_2 are determined to be 0.22 and 0.17, respectively.

Table 5.2: Detection time delay of the faults f_1 and f_2 corresponding to various severities.

Severity / Fault	f_1 (sec)	f_2 (sec)
$f_1 = 3$, $f_2 = 3$	0.2	0.15
$f_1 = 1$, $f_2 = 1$	0.51	0.25
$f_1 = 0.35$, $f_2 = 0.2$	1.02	1.07
$f_1 = 0.22$, $f_2 = 0.17$	4.2	1.35



(a) The residual signal r_1 for detecting and isolating the fault f_1.

(b) The residual signal r_2 for detecting and isolating the fault f_2.

Figure 5.2: The residual signals for detecting and isolating the faults f_1 and f_2. The faults occur at $t = 5$ sec and $t = 8$ sec with severities of 1 for both faults.

5.6 Conclusions

In this chapter, the semigroup invariant subspace of infinite dimensional (inf-D) systems were addressed. Particularly, for the first time in the literature, we have provided necessary and sufficient conditions for equivalence of various types of conditioned invariant subspaces. These represent extensions of new geometric perspectives of finite dimensional dynamical systems that were provided in this work and generalized to Inf-D systems. By utilizing duality, controlled invariant subspaces were addressed. The unobservability subspaces of Inf-D systems were then developed and provided. Finally, by utilizing the developed geometric methodology the fault detection and isolation (FDI) problem of Inf-D systems was first formally formulated and then the necessary and sufficient conditions for the FDI problem solvability were presented.

Chapter 6

Conclusions and Future Directions of Research

This dissertation was mainly concerned with the fault detection and isolation (FDI) of infinite dimensional (Inf-D) systems by using a geometric approach. We developed FDI algorithms by using both approximate and exact methods, where in the former approach the original Inf-D system is first approximated and then the currently available results for Fin-D systems are applied (with certain modifications) to the approximate model. In the exact approach, we dealt with Inf-D systems without any approximation, where extension of the Fin-D systems to Inf-D systems is more challenging than the approximate method. For example, compare the results in Chapter 3 with those in Chapters 4 and 5. Below, we provide the thesis summary based on the results that were provided in Chapters 3 to 5.

6.1 FDI of Multi-Dimensional Systems

As shown in Chapter 3, a set of Inf-D systems (including hyperbolic PDE systems) can be approximated by a multidimensional (n-D) model. In Chapter 3, the FDI problem for discrete-time n-D systems was addressed, where we first generalized the invariant subspaces of one-dimensional (1-D) systems to n-D systems by using an Inf-D representation. Sufficient conditions for existence of an asymptotically convergent n-D state estimation observer were derived, where an LMI-based approach was utilized to show the stability of the error dynamics. Finally, we provided necessary and sufficient conditions for solvability of the FDI problem by using our proposed methodology. It should be pointed out that although the sufficient conditions for applicability of the currently available geometric results in the literature are also sufficient for our proposed approach to accomplish the FDI goal, however, there are n-D systems where the geometric approaches in the literature are not applicable to detect and isolate the faults, whereas our approach can still accomplish the FDI mission.

The future directions for research can be summarized as follows:

- It is well-known that the disturbance decoupling (DD) and the FDI problems are highly related to each other. Therefore, one can applied the proposed approaches in Chapter 3 to the DD problem of n-D systems.

- As mentioned in Chapter 3, we formulate n-D systems as Inf-D systems defined on $\mathcal{X} = \bigoplus(\mathbb{R}^m)$, where $\mathcal{X}$ is the largest Banach vector space contained in $\mathbb{R}^\infty$. Although $\mathbb{R}^\infty$ is not a Banach space, but it can be shown that it is topological vector space (refer to Chapter 2). Therefore, one may extend the results of Chapter 3 to Inf-D systems that are defined on topological vector spaces.

6.2 Invariant Subspaces of Riesz Spectral Systems

In Chapter 4, we first reviewed the available geometric control theory results on Riesz Spectral (RS) systems and then invariant subspaces of RS systems (with bounded input and output operators) were formally introduced. Specifically, necessary and sufficient conditions for equivalence of various conditioned invariant subspaces were provided. Moreover, by using the developed geometric machinery, necessary and sufficient conditions for solvability of the RS system FDI problem were derived.

The future directions of research is this area can be summarized below.

- Development the DD controller for the RS system, by using the duality property and our proposed approach in Chapter 4.
- Generalize the results for the regular RS systems, where the output injection operator (for example $\mathcal{D}$ in the Definition 4.11) can be unbounded.

6.3 Fault Detection and Isolation of Infinite Dimensional Systems

Chapter 5 considered a more general class of Inf-D systems than those that were addressed in Chapters 3 and 4. In this chapter, we first reviewed the invariant subspaces of Fin-D system from a new point of view such that it enables one to investigate the invariant subspaces of Inf-D systems by using resolvent operators.However, as stated in Chapter 5, the computing algorithms the conditioned and controlled invariant are still open problem. It was shown that for accomplishing the FDI objectives one needs semigroup invariant subspaces. Necessary and sufficient conditions for equivalence of invariant subspaces were provided. The FDI problem of Inf-D systems was formulated based on the invariant concepts. Finally, by using the developed generic tools we derived necessary and sufficient conditions for solvability of the Inf-D systems FDI problem.

The future directions for research are provided below.

- Development of computing algorithms for conditioned and controlled invariant subspaces.
- Development of a DD controller for Inf-D systems.

- In this thesis, we considered static output injector operators, whereas one can consider dynamic feedback that results in dynamic feedback invariant subspaces. Dynamic feedbacks have been investigated for Fin-D systems. However, dynamic feedback invariant subspaces have not been addressed for Inf-D systems. Therefore, this direction could open a new door to geometric control theory of Inf-D systems.

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