EXPLORING NARRATIVE INQUIRY IN AN INTRODUCTION TO MATHEMATICAL THINKING COURSE

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ABSTRACT

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This thesis presents a study of the mathematical behaviour of students in a first year undergraduate course entitled *Introduction to Mathematical Thinking*. Previous data collected from a design experiment carried out in the same course (see Hardy et al. 2013) proved insufficient to discuss/characterize the mathematical behaviour that emerged in the classroom. I suggest that a new generation of research in this area needs to address the *lived experiences* of students as they are learning to think mathematically. This thesis is motivated by two research goals: (1) to construct a rich characterization of mathematical behaviours, and (2) to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in institutional education settings (in this case an undergraduate classroom).

The educational philosophy of John Dewey, who claims that all education comes about through experience, is central to the theoretical perspective of this research. Drawing on previous characterizations, a model of mathematical behaviour is proposed and used as a tool for characterizing students' mathematical behaviours. In order to foreground individual experience, I use Clandinin and Connolly's (2000) methodological framework to conduct a *narrative inquiry*. This methodology allows me to construct meaningful narrative depictions of students' mathematical behaviour, and to explore the significance of these experiences within the continuity and wholeness of their individual narratives. The findings from this research provide rich characterizations of some elements of mathematical behaviour, and offer insight into my own experiences using narrative inquiry methodology. Implications for teaching and future research are discussed.

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CONTENTS

PROLOGUE

> *People live stories, and in the telling of these stories, reaffirm them, modify them, and create new ones. Stories lived and told educate the self and others.*
> *(Clandinin & Connolly, 2000, p. xxvi)*

> *Try to love the questions themselves as if they were locked rooms or books written in a very foreign language. Don't search for the answers, which could not be given to you now, because you would not be able to live them. And the point is, to live everything. Live the questions now. Perhaps then, someday far in the future, you will gradually, without even noticing it, live your way into an answer. (Rilke, 1984, p. 34)*

Nadia and I are sitting in her office on the ninth floor. Her back is to the window, and facing her, I am looking out across the Montreal skyline. It is near the end of the winter term, April 2014. I am just finishing my first year in the Masters of Teaching Mathematics program, in a course-based program. Nadia was my professor for one of the courses I have just finished, on research methods in Mathematics Education. I have been toying with the idea of switching into a thesis-based program for the past couple months.

"I am thinking of doing a thesis," I tell her.

"Good," she replies, in her simple, direct tone, "you should."

Nadia never uses more words than she needs to. Although I am beginning to get used to her straightforward way of speaking, I still find it hard to read her expressions, which makes me nervous. I babble to fill the space.

"And, I was thinking, well, I've really liked coming to your course this semester and working with the students, and, uh, I was wondering about maybe doing a project with you in MAST 217?"

"Good. I will be teaching 217 again next winter, so we can arrange for you to be the TA for that course. You can do your thesis project then if you like."

The course that Nadia teaches, MAST 217, is called *Introduction to Mathematical Thinking;* a required course for undergraduate students taking a Major in Mathematics and Statistics. I am drawn to the work she is doing in this course because it resonates with an idea that I have been reflecting on since a few years previously, when I read an essay by Paul Lockhart. At some point during my teaching degree, I came across *A Mathematician's Lament*, a short, impassioned indictment of the traditional model of mathematics education:

> By concentrating on *what*, and leaving out *why*, mathematics is reduced to an empty shell. The art is not in the "truth" but in the explanation, the argument. It is the argument itself which gives the truth its context, and determines what is really being said and meant. Mathematics is *the art of explanation*. If you deny students the opportunity to engage in this activity— to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs— you deny them mathematics itself. So no, I'm not complaining about the presence of facts and formulas in our mathematics classes, I'm complaining about the lack of *mathematics* in our mathematics classes. (Lockhart, 2009, p. 29)

Lockhart's main criticism is that students are, in general, not being given enough time to formulate their own ideas. Worse, they are often never given opportunities to ask questions, and as a result very rarely understand the motivation behind the facts and formulas they are using. In *Introduction to Mathematical Thinking*, Nadia had made it expressly her goal to offer opportunities for students to experience the *real mathematics* that Lockhart is talking about.

I don’t mention Lockhart, or any of this to Nadia, though, not at this first meeting.

We continue our discussion for some time, and agree to set up a reading list for me to look at over the summer. As a starting point, Nadia assigns me several readings by different authors who have tried to characterize ‘mathematical thinking’. I am to reflect on these and consider possible research questions. I need to be prepared to start collecting data by the winter, so I have some time to think about and plan a project.

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In the fall, we are meeting again to discuss my project. I have spent the summer creating a list of what different authors have identified as essential features of ‘mathematical thinking’ or ‘mathematical behaviour.’ I feel I have learned some interesting things, but still have no clear vision or idea for a research project.

“I think your research should be about trying to characterize these behaviours, and trying to capture them as we observe them in the course,” says Nadia.

“So how will that work?” I ask, “Will I take this list and use it as a kind of observation guide, like a checklist, or rubric, and then, um, observe the students in class and make note of which behaviours I see, that kind of thing?”

“That will be the topic you will research this semester as your reading course: methodology. I have an idea for something which you might be interested to try,” she tells me.

“Okay…”

“I have been thinking about trying to find a methodological approach that uses stories or narratives,” she tells me, “I know of examples in other fields, like anthropology and psychology, where researchers use stories and story-telling as a tool for research. I don’t know of any examples in education, but they may exist. I want you to try and find out more about these methods and think about how we could apply it in 217.”

“Woah, okay. That sounds like it could be interesting. How would it work as research?” I ask.

Nadia takes a moment to pause and think before speaking. “In 217, I have this experience where the mathematical thinking takes place in these fleeting moments, and it is not something I am able to capture in my research. But I am still convinced it is there. We have discussed already how many of the research methods used in math education are trying to imitate scientific methods - trying to capture the ‘truth’ about an experience. There is this apartness that researchers try to have, as if they are objective recorders of the events they describe. But they are not.”

She pauses again, and I wait for her to continue. “There is this feeling that if what you are doing is not generalizable and reproducible, in the sense that is expected of quantitative research, then it is worthless; that individual, subjective experiences do not have any relevance. But for writers of fiction, this is exactly what they do. Tell a story of particular individuals in a particular place and time, and that story does have a certain generalizable quality, since so many different people can read it and find meaning there.”

“Sort of like how the reader can see themselves in the story and they imagine what it would be like to be that person?” I ask. Nadia nods, so I continue with my thought, “I remember a talk I went to once, about storytelling traditions in Native culture. The woman spoke about how a story offered something different to each person who heard it. Like, you heard the lesson you needed to hear at that moment in your life. And that the same story could mean, for the same person, different things at different times, too."

"Yes," says Nadia, "that is part of what makes it a good story."

Nadia's idea is that if we could write stories about our experiences as teachers and researchers, that this might be an effective and interesting way to communicate our research findings. I am excited about the idea of doing something different, something creative and original with my thesis research, while simultaneously daunted by the prospect. It is a much more challenging project than I had originally envisioned. We talk for a while longer about stories, the ones we have enjoyed, how different narrative forms could be interesting to experiment with. She makes some recommendations for where I should start looking to get ideas. I leave her office feeling thrilled.

Several weeks later, I come across the work of Jean Clandinin and Michael Connolly and a research method they call *narrative inquiry.* It is a methodology they have used for their research in education, and reading it, I have the feeling that they are concerned with the same issues Nadia and I had been discussing, about truth, about capturing experience. I share what I am reading with Nadia, and she agrees. And so, I set about researching narrative inquiry and planning my project for the following semester.

CHAPTER 1: Introduction

Overview of the Research

In this thesis, I weave together the threads of many different stories. Stories of people who are students, teachers, researchers, and my own story as someone occupying all three of these roles. This research begins with my own story, my belief that teaching mathematics is about providing opportunities for students to *experience* mathematical practices and ideas. It begins with Nadia's story as well, and her previous experiences teaching and researching in the course where this research was conducted, and the questions that her previous research raised for her with respect to research methodology.

The two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in our classroom interactions with students. The research described in this thesis was carried during a one-semester course entitled MAST 217: *Introduction to Mathematical Thinking*[1], which for several years previous had been the site of an ongoing design experiment conducted by Nadia Hardy, the course instructor. It is a first-year undergraduate course that is compulsory for students in the Major in Mathematics and Statistics program. These students have some background in college calculus and linear algebra and, for the most part, have not taken courses that require proving.

Nadia's goal in the course was that students learn to behave as mathematicians do, to learn to think mathematically; e.g., exploring mathematical problems, proposing and testing conjectures, developing proofs or solutions, and explaining their ideas. As part of her ongoing research, she had worked with research assistants in a (partial) characterization of mathematical behaviours. As mentioned above, one of the main goals of this thesis was to work on such characterization, which I did prior to engaging in the second goal of the thesis. However, once the narrative inquiry started (see below), the characterization of

[1] Throughout this thesis, I use both the title of the course (*Introduction to Mathematical Thinking)* and the course code (MAST 217) to refer to the course in which this research was conducted

mathematical behaviours was constantly challenged and revised – the process only interrupted by the need of writing of this thesis.

Integral to Nadia's teaching design was her belief that all education comes about through experience (Dewey, 1938), and so to achieve her goal, she set out to design situations which provided students with opportunities to *behave mathematically,* to experience mathematics the way a mathematician does.

The second main goal of this thesis was to explore a methodology that could allow us to discuss and construct accounts of students' experiences behaving mathematically. Narrative inquiry presented itself to me as an appropriate methodological tool for this. As described by Clandinin and Connolly (2000) this methodology provides an appropriate framework for thinking about and discussing the lived experiences of our students, as well as our own experiences. Instead of a disembodied recorder of someone else's experience, narrative inquiry positions the researcher as an integral element of the phenomena they set out to explore. Narrative inquiry offers "a way for someone both to lead a life and to reflect on it, thereby combining living with self-criticism and growth" (Clandinin & Connolly, 2000, p. 82).

I was joined by two other researchers in this inquiry, Nadia and Genevieve, both of whom contributed narrative data to this thesis. Nadia is my supervisor for this thesis and was also the instructor of the course in which this inquiry took place. Genevieve is a doctoral student who was doing a research internship with Nadia. She was interested in exploring different methodologies, and agreed to join us for this project. All three of us participated as teachers in every class, and wrote extensive field notes and reflections about our experiences.

We met weekly to discuss our writing and our experiences, collaborating in what Clandinin and Connelly (2000) refer to as the "living, telling, reliving, and retelling" (p. 70) of stories. Over the course of the semester, we generated a considerable volume of narrative data. I also conducted hour-long interviews with ten of the students from the course, and wrote several pages of notes about each. It was then my task to bring together all these narrative threads and look for patterns, tensions and themes.

The outcomes of working on the two goals of this thesis are a model of mathematical behaviour and the 'results' which take the form of a series of short narratives about some of the students we met in this course, together with an analysis of the features of mathematical behaviour which (I believe) are visible in the stories. These narratives aim to capture, in as much detail as possible, some of the fleeting moments where we felt a student was behaving mathematically in this course.

Structure of the Thesis

The main results of this thesis are the storied accounts of students' experiences presented in chapter 7. However, the students' stories are not the only ones which must be told; there is also the story of this thesis, as a research endeavour and personal growth. Clandinin and Connolly (2000) emphasize that a narrative inquiry should acknowledge "the centrality of the researcher's own experience - the researcher's own livings, tellings, retellings, and relivings" (p. 70). My own experience as the researcher, as the eyes, ears and voice that narrates these stories, is also an important part of this thesis. At different points throughout the thesis, I include brief snippets of dialogue between Nadia and myself, as a way of including our story. These conversations make visible some of the difficulties I encountered and questions I reflected on throughout this research, and point to the fact that this work is situated in the midst of an ongoing conversation, and ongoing story.

The story of this thesis will be told in eight chapters, including this first chapter, which serves to introduce the research. Chapter 2 provides a review of some of the literature pertaining to transition-to-proof courses. The goal of this chapter is to situate this thesis within the current context of mathematics education research, and to describe how Nadia's experience with traditional inquiry plotlines triggered the need to explore a different methodological approach.

The theoretical framework of this research will be the focus of chapter 3, the main focus being Dewey's educational theory, which I use to conceptualize the teaching and learning of mathematics, while also borrowing some concepts from Ostrom's (2005) Institutional Analysis and Development (IAD) framework to allow me to discuss teaching and learning in institutional educational settings.

In chapter 4, I present my model of *mathematical behaviour*, which arises as an extension of the model of mathematical thinking proposed by Hardy et al. (2013) in their research in the same course. I provide a review of literature that has sought to characterize mathematical behaviour in order to generate a list of features of mathematical behaviour. I also describe how this model is used as a tool in this research.

Chapter 5 focuses on my reasons for using narrative inquiry as a methodological framework, focusing in particular on the epistemological issues that motivate this choice.

This chapter also includes an overview of this methodological framework, as described in the relevant literature.

Chapter 6 outlines the particular details of how this research was carried out, and how the narrative data that was collected during this inquiry was iteratively re-written and re-storied until it took its final form, which are the stories that are presented in chapter 7. The series of short narratives in chapter 7 are the ‘results’ of this research. Finally, in chapter 8 I summarize my results, providing further discussion and analysis of the stories presented in chapter 7, including an explanation of how these stories allow me to more richly characterize mathematical behaviour. This chapter also includes a series of reflections on narrative inquiry, the methodology that I am exploring in this thesis. Finally, I discuss some of the questions which remain open for future research.

Overview of Results

As stated above, the two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in our classroom interactions with students. The first of these two goals is addressed in chapter 4, where I present a model of mathematical behaviour. The second is addressed by constructing storied accounts of students’ experiences that appear in chapter 7. In these storied accounts, some of the different features of mathematical behaviour (described in chapter 4) are discernible. In chapter 8, I provide a discussion of each of those features, reflecting on the significant differences and nuances that are visible in the accounts. The result is a rich description of the ways in which students engaged in mathematical behaviour in this course.

The second goal of this thesis is also addressed in chapter 8, where I reflect on some of the advantage and difficulties involved with using narrative inquiry as my methodological framework. Using this methodology, I was able to construct accounts of the mathematical behaviour that emerged in the classroom. Narrative inquiry allowed me, and my fellow researchers, to generate rich remembrances of our experiences, to reflect on these experiences, and to learn from them in ways that were meaningful to our development as teachers and researchers.

CHAPTER 2: Review of Literature on Transition to Proof Courses

As mentioned in the previous chapter, the research in this thesis was conducted in a course called *Introduction to Mathematical Thinking,* a first-year undergraduate course that is compulsory for students in the Major in Mathematics and Statistics program. Nadia Hardy, the instructor of the course, is also a mathematics education researcher, and the course was already a site for her ongoing research. Since she had begun teaching it five years previously, she had been carrying out a design experiment, modifying the structure of the course and her teaching approach. This design experiment is briefly described in Hardy et al. (2013).

When Nadia took over from retiring colleagues, she gleaned the didactic organization of the course from the notes and assessment documents (assignments, midterms and final exams). She assumed that the institutional approach consisted of what is commonly known as a *transition to proof* course. A course organized around the teaching of basic propositional logic, proof structure, and proving techniques. Nadia had the impression that the course was meant to be taught by giving lectures presenting proofs in algebra and arithmetic to exemplify different proving techniques. Nadia found teaching the course using this approach unsatisfying for a number of reasons, and this motivated her to begin her design experiment research. Nadia's goal was to design a course where students would be expected to develop their mathematical thinking skills, which certainly involves writing proofs, but not only that. Over several iterations of her design experiment, the course evolved away from what is traditionally considered a transition to proof course, as the focus expanded to include other aspects of mathematical practices besides just proof-related activities, including problem solving, problem posing, conjecturing and validating, reading and understanding mathematical definitions. In order to clarify how this thesis is situated within the domain of mathematics education research, this chapter presents a review of some of the literature concerned with transition to proof courses, and discusses how it relates to Nadia's redesign of the course, and to the goals of this thesis.

The main purpose of this chapter is to explain how **what** we are aiming to study is different from that which the research on transition to proof courses reviewed in this chapter has sought to characterize. I use the term 'mathematical behaviour' (to be more thoroughly defined in chapter 4) to describe something quite different from the proof schemes, proof construction, and student strategies that are discussed in other research. In the final section of this chapter, I will argue that the research tools that have been developed and used for investigating transition to proof courses are inadequate to the task of capturing the behaviour I wish to discuss in this thesis. Thus, in the first three sections of this chapter I will address some of the literature pertaining to transition to proof courses, with a particular focus on the methodological tools that have been used.

In the area of undergraduate mathematics, researchers have only recently turned their attention to transition to proof courses, and have raised many questions about the nature of proofs, how students learn to prove, and how best to teach the mathematical practice of proving. As the object of study has been refined, so have the research tools and methods that are used to study the teaching and learning of proofs. Within the research around transition to proof courses, several concerns about the learning and teaching of mathematical reasoning are discussed. I will begin by addressing those studies which have sought to identify, characterize, and evaluate students' proof construction processes. Next, I shall consider some studies that investigate the different strategies and processes that mathematicians use when they construct or validate proofs. Finally, I will concern myself with what has been said about the teaching design of transition to proof courses, in particular studies that advocate the effectiveness of an inquiry-based teaching approach.

The chapter will then conclude with a discussion of how the literature presented in the first three sections relates to Nadia's redesign of the course, and to the goals of this thesis.

2.1 Proof Construction and Proof Schemes

Within the domain of undergraduate mathematics education research, a significant body of research has been devoted to investigating and characterizing what students are doing and thinking as they construct proofs. Many studies have focused on the difficulties students encounter when constructing proofs (Moore, 1994; Selden & Selden, 2003). These studies aim to identify students' difficulties with the purpose of suggesting teaching strategies or interventions that will address these difficulties.

More recently, researchers have begun to move beyond identifying and categorizing students' difficulties to characterizing students' proof schemes (Grundmeier et al., 2012; Harel & Sowder, 1998), addressing the cognitive processes and reasoning involved in students' proof construction (Alcock & Weber, 2010; Mejia-Ramos et al., 2012; Mills, 2010; Selden et al., 2014), and creating a richer picture of how students comprehend and evaluate mathematical proofs (Brandt & Rimmasch, 2012; Roy et al., 2010; Selden & Selden, 2014; Weber, 2008).

Although they used different terminology, I contend that these studies have a shared goal, that is characterizing what students are doing and thinking as they (successfully or unsuccessfully) construct proofs. In terms of methodology, most of these studies analyze students' written proof attempts, and some use task-based interviews wherein the researcher observes a student engaging in a proving task (selected by the researcher) and asks questions about their work. This analysis of student work generally takes place after the student has completed, or nearly completed a transition to proof course. As such, the researchers often use the results of their studies to make claims about, or suggest possible changes to, the teaching approach used in transition to proof courses. In what follows, I will consider several of the studies mentioned above in more detail.

In their study of students' proof construction in an inquiry-based introductory proofs course, Grundmeier et al. (2012) categorize student proofs based on their use of different *proof schemes* (as defined by Harel and Sowder, 1998). Harel and Sowder (1998) define proof scheme as "what constitutes ascertaining and persuading for that person" (p. 244) when they construct a proof, and propose three categories to describe these schemes: external conviction, empirical, and analytical. The external conviction proof scheme classifies student proofs that depend on outside sources and reflect little independent thought. For the empirical proof scheme, all cases of the problem may not be addressed or be sufficiently justified. Finally, the analytical proof scheme classifies proofs that validate conjectures by means of logical deductions (Harel & Sowder, 1998). Note that only the analytic proof scheme represents a convincing or rigorous mathematical proof.

Grundmeier et al. (2012) collected the written proof attempts of 70 students from a transition to proof course and classified them according to which proof scheme they used. For three out of the five questions they looked at, over 95% of the student work was classified as having used an analytical proof scheme. "From this," they write, "it is logical to conclude that the classroom environment had a positive effect on students and prepared them to attempt formal proofs on their own" (Grundmeier et al., 2012, p. 227).

Other researchers have also categorized the techniques that students use to convince themselves and others of mathematical statements. In one of many studies centred on students' proof techniques, Alcock and Weber (2010) identify two approaches to proving: the *syntactic* approach and the *referential* or *semantic* approach. The syntactic approach is characterized by the use of formalism, precise mathematical language, and logical inference.

Individuals using this approach may focus entirely on logical syntax, without thinking about other representations of mathematical concepts. On the other hand, when using a referential approach to proving, the act of proving a statement is understood as forming a convincing explanation for why mathematical concepts have the properties ascribed to them by that statement. Such an explanation, "deals not with logical symbols themselves, but with meaningful representations of the mathematical concepts to which the symbols relate" (Alcock & Weber, 2010, p. 102).

Alcock and Weber (2010) conducted structured, task-based interviews with students who had nearly completed a transition to proof course. The interviews included both proof production and proof reading tasks, which students were asked to complete, and to discuss their steps with the interviewer. Their findings showed that although effective mathematical reasoning generally requires the use of both referential and syntactic strategies, many students exhibit a tendency to use either one or the other preferentially. They conclude by suggesting that in class settings, "students with both syntactic and referential preferences might benefit from instruction that explicitly draws attention to links between formal statements and proofs and their referent objects and relationships, at a detailed, step-by-step level" (Alcock & Weber, 2010, p.120).

As a final example, I will turn to Selden, Benkhalti, and Selden's (2014) work analyzing students' proof construction. Drawing on terms and concepts from cognitive psychology, Selden et al. (2014) view the process of proof construction as "a *sequence of mental* (e.g., "unpacking" the meaning of the conclusion in inner speech) *or physical* (e.g., drawing a diagram) *actions*" (p. 247). These actions derive from a person's *inner interpretations* of situations in a partly completed proof construction, which are not directly observable, but can be inferred, they claim, from the students actions. In their perspective, proving actions appear to be the result of the enactment of small, linked, automated situation-action pairs that they have termed *behavioral schemas* (Selden, McKee, & Selden, 2010).

Selden et al.'s (2014) perspective includes drawing inferences from students' written proof attempts about students' sometimes automated links between situations and mental, as well as physical, actions. In their study, they collected written proofs from the final examinations of students in a transition to proof course. Their observations allow them to claim that some actions are beneficial for proof construction and should be initiated or encouraged, while other actions can be detrimental and should be eliminated or discouraged. Elsewhere, they have identified three useful actions in proof construction: exploring, reworking an argument in the case of a suspected error or wrong direction, and validating a completed proof (Selden & Selden, 2013). They conclude by suggesting possible teaching interventions which might alleviate student difficulties, and encourage these useful proving actions (Selden et al., 2014).

Recent researches have further refined the study of students in transition to proofs courses by selecting particular proving actions, and investigating how students engage in that action when constructing proofs. Individual studies have focused on how students use examples when constructing proofs (Hanush, 2015; Holguìn, 2015), how students make use of mathematical definitions (Holguìn, 2015) and how students use mathematical language in their written proofs (Lew & Mejia-Ramos, 2015). Other researchers have focused on how students construct particular types of mathematical proofs, such as inductive proofs (Harel, 2002; Harel & Brown, 2008), indirect proofs (Brown, 2012), or existence proofs (Roh & Lee, 2015). Still other studies have looked at how students complete other tasks related to proving, such as students' ability to validate a written proof (Brandt & Rimmasch, 2012; Selden & Selden, 2014).

Throughout this considerable body of research on students in transition to proofs courses, the central questions are: how do students construct proofs, what are they thinking and doing as they construct proofs, what difficulties/errors do they encounter when writing proofs, and how do they decide, when reading a proof, if it is correct. The theoretical frameworks that many authors use have their roots in cognitive psychology, as do their research methods. Many studies analyze student work (written proofs) or conduct task-based interviews. The theoretical perspectives that researchers take in their studies justify inferring students' cognitive processes from their actions and their responses to interview questions.

Although some studies are interested only in characterizing student thinking and/or cognitive behaviour, many use the same models and research methods as a means of evaluating the teaching approach. In Grundmeier et al. (2012), for example, the fact that most students from a particular transition to proofs course were classified as having used analytic proof scheme is taken as evidence that the teaching approach had a positive effect on those students. There are many other studies which evaluate the 'success' of a teaching approach by measuring the ability of students to perform given tasks at a particular level (Brandt & Rimmasch, 2012; Eubank et al., 2013; Grundmeier et al., 2012; Selden & Selden, 2014).

Based on what is taken as evidence of a successful teaching approach, one might reasonably assume that the pedagogical goal of the transition to proof course is for students to be able to comprehend, construct, and validate proofs. This is important, because as I will discuss later in this chapter, Nadia's goals as the instructor of MAST 217 are slightly different from the ones that are evident in most studies on transition to proofs course, and that is what motivates the research in this thesis.

All of the above studies conclude by offering pedagogical recommendations based on their observations. For example, having observed and described some difficulties students encounter when constructing proofs, Selden et al. (2014) suggest some possible teaching interventions that might alleviate these difficulties. These recommendations include explicitly teaching students to write *proof frameworks*, and engaging students in small group discussion about *operable definitions*. They also suggest that future research might gauge the effectiveness of such interventions by, "interview[ing] students towards the end of the course, asking them to construct a few relatively easy proofs, and observe them to see if they wrote proof frameworks. One might also analyze examination proofs for difficulties that might be traceable to not having written a proof framework" (p. 257).

In order to make pedagogical recommendations, many researchers compare student behaviour to the ways that mathematicians construct and validate proofs. For example, Alcock and Weber (2010) recommend that instructors explicitly draw attention to links between formal statements and the objects and relationships they refer to, in order to help students develop both referential and syntactic strategies. This recommendation is based on evidence from an earlier study by Alcock (2008) that "a successful mathematician will be able to use both [types of strategies], switching from one to the other in response to the perceived demands of the situation" (cited in Alcock & Weber, 2010, p.118).

Pedagogical recommendations often hinge upon this idea that mathematicians' practices represent an 'expert' way of constructing proofs. This idea has generated a considerable body of research investigating mathematicians' proving behaviour. In the next section, I will review some of the literature which explores how mathematicians construct, comprehend, and validate proofs.

2.2 Mathematicians as Source of Expert Knowledge

An important branch of research related to transition to proof courses is those studies which investigate the proving behaviour of mathematicians. These studies are generally of two different types. On the one hand, there are those that investigate mathematicians' views on proof construction in order to better understand their pedagogical practices as instructors of transition to proof courses (Lai & Weber, 2010; Lockwood & Weber, 2015). On the other hand, there are studies which situate the proving or problem solving activity of mathematicians as a model of expert behaviour (Burton, 2001; Samkoff et al., 2011; Samkoff et al., 2102; Savic, 2012; Schoenfeld, 1987; Selden & Selden, 2013). Such research is based on the assumption that a goal of instruction in advanced mathematics courses is to lead students to reason like mathematicians with respect to proof (a position endorsed by Harel

& Sowder, 2007), and that we can improve our understanding of mathematical practice by carefully observing mathematicians engaged in mathematical tasks. In the context of this thesis, I am more interested in discussing researches of this second type, and these will be used in chapter four to develop a model for mathematical behaviour.

In this section, I will review some of the literature discussing researches of this second type, that is, how mathematicians comprehend, construct, and validate proofs. I will focus my discussion on how mathematicians' practices have been characterized by these studies, and highlight some of the pedagogical suggestions made by researchers. The research tools that are used for these investigations are very similar to those which have been used to investigate students' proof construction; researchers observe mathematicians as they engage in a proving task and aim to categorize their behaviours. Some of the traits which characterize mathematicians' proving behaviour are: an ability to generate multiple ideas for solving or proving (Schoenfeld, 1987, Selden & Selden, 2013), the ability to efficiently manage their time and resources (Schoenfeld, 1987), and persistence in the face of challenges or impasses (Savic, 2012; Selden & Selden, 2013).

Schoenfeld (1987) videotaped students and mathematicians as they attempted to solve problems, and then analyzed their attempts according to how much time they spent engaged in six different stages of problem solving: reading, analyzing, exploring, planning, implementing, and verifying. Whereas the students tended to get stuck on their first idea and go off on a "wild goose chase," the mathematician considered many different approaches, and was "as ruthless in testing and rejecting ideas as he was ingenious in generating them" (p. 194). Schoenfeld (1987) concludes that the key difference in the expert's success on a problem and the student's failure was not a lack of knowledge in the subject matter, but an inability to effectively make use of what they did know: "With the efficient use of self-monitoring and self-regulation [the mathematician] solved a problem that many students – who knew a lot more geometry than he did – failed to solve" (p. 195).

The ability to generate many possible paths for a proof and then test those ideas is an important feature of mathematicians' proof construction. Selden and Selden (2013) observed a mathematician as he worked on a single proof over several hours. The mathematician spent most of his time exploring the problem, trying several different approaches and even putting the problem aside for some time before arriving at a proof with which he was satisfied. He demonstrated *persistence* when faced with difficulties in the proof process*;* there is no evidence that the mathematician "thought there was anything wrong with having gone in all of those unhelpful directions or with having thought that some theorems were false, that he later discovered were true" (p. 307).

The ability to be persistent in the face of difficulty when working on a proof seems to be an important trait for a mathematician. Savic (2012) focuses particularly on what mathematicians do when they have a proving impasse, as well as the incubation and insight that often allows them to continue. Observing mathematicians as they engaged in proving, he found several impasse recovery actions that were directly related to the proof that they were working on, such as using prior knowledge from their own research, using a (mental) database of proving techniques, or generating examples and/or counterexamples. Most of the mathematicians also engaged in impasse recovery actions that were unrelated to the proof at hand, such as doing other mathematics, walking around, going to lunch, doing other tasks, or sleeping on it (Savic, 2012).

Selden and Selden (2013) suggest that in order to be persistent when constructing a proof, a student needs to believe that they can personally benefit from persistence or exploration. This is called a *self-efficacy* belief (in the sense of Bandura, 1994), which is best developed through, perhaps numerous, successes in similar actions. They suggest a teaching approach which allows students to experience success early on in the course, in order to help build the self-efficacy and persistence needed for more challenging situations.

As a slightly different example, instead of looking at proof construction, Samkoff, Weber and Mejia-Ramos (2012) asked mathematicians and undergraduate students to read and summarize mathematical proofs that they read in order to investigate which ideas they consider important in a proof. They found that when summarizing proofs, mathematicians highlighted the overarching goals of the proof; the ideas they found novel or unfamiliar; theorems and facts used in the proof; and encapsulations of a series of inferences as an application of a general method. The undergraduate students, on the other hand, did not mention the goals, important theorems or facts, or the methods in their summaries, but often mentioned the type of proof (such as direct, indirect, contrapositive). Samkoff et al. (2012) suggest that the students are not reading proofs the same way that mathematicians do, and they recommend that instructors encourage students to reflect on the proofs that they have written by providing opportunities to revise and summarize proofs in their courses.

Weber (2008) stated that, "investigations into the practices of professional mathematicians should have a strong influence on what is taught in mathematics classrooms" (p. 451). The studies discussed above seek to characterize the practices of mathematicians and offer pedagogical suggestions that they believe will encourage students to develop similar practices. However, few address the difficulties of fostering these practices within the classroom settings. In their study on students' ability to validate proofs, Selden & Selden (2014) stress that it is "not enough for professors to model validation, […] and for students to observe those validations, as our participants did, rather students also need to *practice* validation" (p. 240, my emphasis).

I agree that it is essential for students to *experience* using mathematical strategies and practices if they are to develop those practices for themselves. However, there are inherent challenges in trying to expose students to the practices of mathematicians, since these are sometimes directly at odds with the institutional norms and practices of undergraduate mathematics classrooms (Burton, 2001; Lampert, 1990; Schoenfeld, 1987; Smith & Hungwe, 1998). I discuss these challenges in more detail in Chapter 3 of this thesis, using Ostrom's (2005) framework for Institutional Analysis and Development (IAD) to help illustrate and conceptualize the role that institutional constraints play in Nadia's course design and teaching strategies.

Many mathematics education researchers have argued that the traditional lecture-based approach may not encourage the development of deep problem solving techniques that students will need to utilize in higher-level classes (Burton, 2001; Grundmeier et al., 2012; Sierpinska, 2007; Smith & Hungwe, 1998). An increasingly popular method for the teaching of undergraduate mathematics courses, and in particular transition to proof courses, is what has been called an *inquiry-based learning* approach. This teaching approach is characterized by a desire to engage undergraduates in learning new mathematics by exploring mathematical problems, proposing and testing conjectures, developing proofs or solutions, and explaining their ideas. In the following section, I consider studies which are more explicitly directed at evaluating the 'effectiveness' of this teaching method.

2.3 Transition to Proofs Courses and Teaching Design

Current mathematics undergraduate teaching typically consists of in-class lectures followed by students completing homework outside of class. From large introductory courses to smaller advanced courses, instructors and professors tend to follow the transmission model of teaching, and there is little pressure and certainly no requirement to do otherwise. Sierpinska (2007) explains:

> This may not be the most effective method of teaching but it is the least costly one in terms of intellectual and emotional effort. Professors are usually more interested in a neat organization and smooth presentation of the mathematical content than in knowing what and how students in their class think about it. Indeed, they may not want to know, for fear of losing morale. It is more pleasant to live in the illusion that students think exactly the way we think ourselves. (p. 6)

She further contends that this approach to teaching fosters and maintains a tendency among students to rely on teachers for the validity of their solutions, and does not foster independent problem solving skills like self-regulation (Schoenfeld, 1987).

Recently, many mathematics educators have argued that lecture-based teaching may not encourage the development of essential proving and problem solving techniques (Burton, 2001; Grundmeier et al., 2012; Sierpinska, 2007; Smith & Hungwe, 1998). Burton (2001) argues that in contrast to how mathematicians describe coming to know mathematics, undergraduates frequently encounter mathematical knowledge without the exciting experiences of making personal, sociocultural and aesthetic connections through their own style of learning. Proofs, theorems and the accumulated results of hundreds of years of mathematical study are presented in carefully crafted, perfectly polished forms. Smith and Hungwe (1998) maintain that the cycle of conjecture and verification, so essential for research activity in mathematics, is often absent from the narrative that students are presented with in undergraduate courses.

Many researchers in undergraduate mathematics education advocate that an inquiry-based teaching approach is more effective than traditional lecture-based approach, in particular when it comes to transition to proofs courses. Inquiry-based learning refers to those teaching approaches which place a "responsibility on students by encouraging students to discover mathematics for themselves rather than simply relying on the authority of an instructor" (Eubank et al., 2013, p. 205). By engaging in peer collaboration, peer critique, and student presentations, students "learn new mathematics through engagement in genuine argumentation" and come to "see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity" (Rasmussen & Kwon, 2007, p. 190). Such approaches are supported by current socio-constructivist views of learning that "emphasize individual constructions and ways of thinking and learning developed in social interactions in classrooms" (Hassi et al., 2011, p.73).

As she began her design experiment, Nadia shared the views espoused by these researchers, and felt the need to move away from a traditional lecture-based approach to teaching introductory proofs courses and towards an approach where students are required to construct their own proofs, discuss them, and validate them together. Overall, she found this teaching approach more satisfying than the lecture-based course design, but after several iterations of the design experiment, she remained unsatisfied with the methodology available for demonstrating the success she felt she was having with her new teaching approach.

Recent studies deduce that inquiry-based learning students experience mathematics in a way that deepens their comprehension of abstract ideas that are essential for proving (Boaler, 1998; Grundmeier et al., 2012; Hassi et al. 2011; Rasmussen & Kwon, 2007). Studies which validate this claim often use before and after methods (analysis of student work, interviews, and task-based interviews) for testing students' abilities to perform certain proving tasks, i.e. the same methods as discussed in the first section of this chapter. Some studies only do an analysis of students' abilities at the end of an inquiry-based transition to proofs course, and infer from their ability to perform tasks whether the course was effective.

There is an assumption in all of these studies that if students are able to successfully comprehend, construct or validate proofs at a certain level by the end of a transition to proofs course, then the teaching method was effective. In the first few iterations of her design experiment in this course, Nadia followed similar methodological practices as those used in the researches reviewed above. She collected students' written work, including exams, assignments, and online discussion posts, and looked for evidence of different types of mathematical thinking. She came to realize, however, that these written productions were only what was left over after an experience had taken place, like signposts that pointed to what students had actually said and done. They were not sufficient to talk about the behaviour that she saw emerging in the classroom.

In the next section, I will discuss Nadia's design experiment in more detail, and how her work has inspired the central questions which this thesis will investigate. In particular, I will examine how the questions posed in this thesis build upon the research reviewed above, and also where they diverge.

2.4 MAST 217: *Introduction to Mathematical Thinking*

As mentioned in the introduction to this chapter, Nadia initially taught her course using the institution's normal, lecture-based, approach. Students were presented with results and techniques, which in this case looked like a toolbox from which students were expected to pick the right tool to prove this or that. This approach, however, felt unsatisfying in at least two ways. For one, it felt unsatisfying when considering students' learning: by the end of the course most students, "although able to write simple proofs, didn't seem to understand why these were necessary or why they were actually *proofs* of anything" (Hardy et al., 2013, p. 280). It seemed quite likely that one of the problems with the teaching approach was that, as Mason (1989) remarked, "the more explicitly the teacher indicates the behaviour which would arise from understanding, the more likely students are to be able to produce that behaviour without generating it from understanding"[2] (quoted in Mason, 1994, p. 178). Furthermore, it was quite unclear (to her) that the fact that students could write some simple proofs meant that they could read, understand and write more difficult proofs or proofs in domains of mathematics other than elementary algebra and arithmetic.

In addition, she considered the fact that the course was required for students in the Major in Mathematics program, but not the Honours or Specialization programs. These students, because they are undertaking a less mathematically intensive degree, are not likely to construct many proofs in their program, and thus have less of a need for developing formal, rigorous, proof-writing skills. She wanted to focus more on their ability to make arguments, pose problems and ask good (mathematical) questions.

Her design experiment was motivated by an unsatisfactory teaching experience and her (constructivist) view that people learn best through experience. Instead of a course where students were presented with proofs and proof techniques, she wanted to make a shift to a course where students would be expected to actively develop their own mathematical thinking skills. Nadia saw an opportunity in the name of the course (*Introduction to Mathematical Thinking*) to build up a *new* course, one about *thinking mathematically* – which certainly requires knowing how to write proofs, but it is not only about that. The emphasis of her teaching approach was to be on providing opportunities for students to engage in thinking mathematically. The design experiment's main goal was to provoke a shift from a

[2] See also Brousseau's (1997) discussion of the tensions inherent in the didactic contract: the more clearly the teacher indicates the behaviour being sought, the more easier it is for the students to display the behaviour without generating it from themselves (from understanding).

course *about mathematical thinking* to a course where students could (consciously) *experience thinking mathematically*.

It could be said that the course evolved away from what is traditionally considered a transition-to-proofs course, as the focus expanded to include other aspects of mathematical practices besides just proof-related activities, including problem solving, problem posing, conjecturing and validating, reading and understanding mathematical definitions. Her goal was similar to that of Schoenfeld (1994), who aimed to create a "microcosm of selected aspects of mathematical practice and culture" within their classroom (p. 66).

The teaching approach she adopted fits the definitions/description of an inquiry-based learning approach and involved having students work in small groups on carefully selected open-end problems. In order to assess the effectiveness of her design, she used methodological tools very similar to those used by other researchers inquiring into transition to proof courses. She gathered data in the form of students' written productions, from assignments and quizzes, as well as from the on-line discussion forum they were expected to participate in every week. She created a model of what would be considered evidence of mathematical thinking (based on the work of Sierpinska et al. 2002 and Schoenfeld, 1987) and analyzed the students' work accordingly. After several iterations of the design experiment, Nadia felt unsatisfied with her ability to describe the mathematical behaviour of her students. She felt that there were events taking place in the classroom where students were thinking mathematically, but that these were fleeting moments; just short snippets of conversations, which she did not have a way to meaningfully capture. Socio-constructivist perspectives have influenced a shift in the teaching approaches (i.e. towards inquiry-based learning) but not, as yet, in the methods being used to conduct research in these courses. The research in this thesis seeks to bridge that 'gap' by exploring a different methodological approach.

In this thesis, I propose a study that seeks to provide a richer characterization of students' mathematical behaviours *as they emerge* in classroom settings, and I explore a methodological approach that will (hopefully) allow us to discuss and construct accounts of these behaviours. I will now briefly discuss some of the theoretical issues which will be explored throughout this thesis, in order to clarify the methodological shift that needs to happen. A full treatment of my theoretical perspective appears in chapter three.

Teachers and Learners

Many researchers discuss "inquiry-based learning" as a good approach to teaching, and I contend that this label is a misnomer. The term is used to refer to the actions a teacher can perform in the hopes that their students will take the opportunity to engage in inquiry-based learning. Therefore, perhaps a more accurate label would be "inquiry-based teaching." This difference in labelling highlights an important way in which this thesis departs from the trajectory of previous research. The methodology used in the studies I have discussed in this chapter assumes that the ability to perform a particular proving task (such as constructing or validating) is evidence of *learning*. From this perspective, an instructor's goal for a transition to proofs course is to ensure that students can construct, comprehend and validate proofs.

As mentioned in chapter one, the two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in the classroom. Nadia's course design has shifted the main pedagogical goal, from getting students to produce a particular type of activity (proof construction and/or proof validation), to providing opportunities for students to engage in thinking mathematically. The goals of this thesis are motivated by a desire to investigate/study whether she is providing these opportunities. An important part of the theoretical perspective I take with respect to teaching and learning is that *the only way to know that we have presented an opportunity is if a student takes it*. There is a difference between a student taking an opportunity to engage in mathematical thinking in that moment, in the classroom, and that same student actually *learning* a mathematical practice. The latter being something that likely takes much more time.

This thesis explores whether or not students engage in mathematical behaviour in the classroom, and seeks to characterize what these behaviours look like. The question of whether they *learn* this behaviour, in the sense that they will perform it every time they are faced with situations of a certain type, is different. Dewey (1933) understood the importance of experience as the grounding for all learning, and turned this into an educational principle:

> Since all learning is something the pupil has to do himself and for himself, the initiative lies with the learner. The teacher is a guide and director; he steers the boat, but the energy that propels it must come from those who are learning. The more a teacher is aware of the past experiences of students, of their hopes, desires, chief interests, the better will he understand the forces at work that need to be directed and utilized for the formation of reflective habits. (Dewey, 1933, p. 36)

All we can aim for as educators is to expose students to the ways of doing of mathematicians, and to engage them in trying out these behaviours themselves. This is what constitutes teaching. The rest, that which constitutes learning, where the student reflects on the experience and learns to use that behaviour again, is not something a teacher can control/manipulate in the same way. Thus, assessing the 'effectiveness' of our teaching practice means assessing if students are taking the opportunities presented in the course. In the places where we hope they will behave mathematically, how do they behave? Do we observe them using mathematical practices in our class?

The research tools used in previous research have proven inadequate to the task of capturing this behaviour in the moment, as it emerges in the classroom setting. In chapter five I present an argument for why I feel narrative inquiry is an appropriate choice for this research, not only because it allows me to construct accounts of aspects of students' mathematical behaviour, but also because it meshes with my own theoretical and epistemological positions, which I will outline in the next chapter.

CHAPTER 3: Theoretical Framework

> *Educating people mathematically consists of much more than teaching them some mathematics… It requires a fundamental awareness of the values which underlie mathematics and a recognition of the complexity of educating children about those values. It is not enough to teach them mathematics; we need also to educate them about mathematics, to educate them through mathematics, and to educate them with mathematics. (Bishop, 1991, page 3)*

This chapter provides an overview of my theoretical framework for this research, that is, my epistemological and ontological positions with respect to mathematics and learning, as well as the language and concepts that I have used to conceptualize and discuss the research in this thesis. I began this research with several questions about teaching and learning mathematics. These questions were based in my experiences working with Nadia in her course and my understanding of her goals as a teacher in that course. My questions were also based in my understanding of what it means to behave mathematically, and how people, especially undergraduate students, learn to behave mathematically; ideas which for me are grounded in the two theories I will present in this chapter.

The process of developing the theoretical framework presented in this chapter was an interaction between my own epistemological and ontological positions, which I held prior to engaging in this research, and my theoretical choices. In my reading of different theories, I found terminology that allowed me to clarify my own positions, and provided me with the language to discuss the research and results presented in this thesis. In order to reflect this process, the first section of this chapter provides a brief discussion of my own epistemological position on mathematical knowledge. In the second and third sections, I provide an overview of two theories that I will be using to frame my research. The first is the educational theory of John Dewey, which I outline in section 2 of this chapter. The second is the framework for Institutional Analysis and Development (IAD) developed by Ostrom (2005), which I outline in section 3.

Together, these two theories form my theoretical framework, that is, they delineate my understanding of how mathematics is learned and taught in institutional educational settings. The chapter concludes with a discussion of how I have used this theoretical framework to conceptualize the research in this thesis. It is important to note that Dewey's theory provides the main theoretical framework for this thesis. Dewey's work provides the language for discussing my initial research questions, is evident in Nadia's course design, and is central to my entire methodological framework. The IAD framework is less central to this research, but still plays an important role in how I pose my questions at the beginning of this thesis, and in how I formulate my model of mathematical behaviour in chapter four.

3.1 Mathematics as an Institutional Product

It is my perspective that mathematics is not objective and absolute, but rather is socially and collectively constructed by those who claim membership to the mathematical community (namely, mathematicians). The institutional culture of mathematics rests on hundreds of implicit assumptions and beliefs, which may be so integrated into a mathematician's thinking and functioning that they are not consciously aware of them operating. Entry into this culture is often necessary in order to understand and appreciate mathematics and mathematical ways of thinking (Schoenfeld, 1987, p. 214).

Mathematical claims arise from a "deep enculturation into mathematical ways of thinking, resting on a conception of mathematics that assumes regularity, connectedness and the legitimacy of using analogy to discover (or construct) new types of objects. These claims would not make sense to a reader who did not share this enculturation" (Morgan 2001, quoted in Burton, 2004, p. 71). The institutional culture of mathematics is identifiable and teachable and includes not only learning to speak an acceptable mathematical discourse, but also the feelings associated with that learning (Burton, 2001). Understandings of what counts as mathematics – more specifically what counts as interesting, meaningful and significant mathematics, as well as sufficient justification and explanation – are all *norms of the mathematical institution*. Norms refer to understandings or interpretations that become normative or taken-as-shared by a group. In the third section of this chapter, I will elaborate on how my conception of norms, as well as what I mean by 'the mathematical institution' are ideas grounded in Ostrom's (2005) Institutional Analysis and Development (IAD) framework.

My perspective on mathematics as an institutional product has an important implication for how I conceptualize teaching and learning mathematics. If mathematics is a human activity and a social practice, then teaching mathematics involves more than the transmission of facts and formula. Students should also be enculturated into a discipline that views certain relationships and ways of thinking as valid and meaningful, while others are not.

Thus, from my perspective, the only way to learn mathematics is to experience this institutional culture, that is, to experience doing mathematics and thinking/behaving mathematically. One of Nadia's goals as the teacher of *Introduction to Mathematical Thinking* is to provide opportunities for students to *experience* mathematics the way mathematicians do. Similar to an immersion program for learning languages, her goal is to immerse students in mathematical experiences in the hopes of enculturating them into the practices of doing mathematics.

Educational theorist John Dewey wrote extensively about the connection between education and experience, claiming that all genuine education comes about through experience (Dewey, 1938). Because it echoes and clarifies my own perspective, I have used Dewey's philosophy of education and experience as the theoretical framework for this thesis. In the following section, I provide an overview of his philosophy.

3.2 Dewey's Philosophy of Education and Experience

The history of educational theory is marked by an opposition between the idea that education is development from within and that it is formation from without. Rejecting this debate as a false dichotomy, John Dewey (1938) proposed an educational theory based on his carefully developed philosophy of experience. Although Dewey wrote about education in the context of school, where the two main participants are teachers and children, I believe his theories can be extended to any situation involving an educator and learner, an expert and a novice.

Dewey's educational theory rests on two fundamental principles: that *all genuine education comes about through experience*, and that *all human experience is ultimately social* (Dewey, 1938). These two ideas are the philosophical foundation of this thesis, as well as my understanding of what learning and education mean. Dewey asserts that experience is the source of all education, although he cautions that experience and education are not synonymous, as "not all experiences are equally educative" (Dewey, 1938, p. 25). In order to better elucidate the relationship between experience and what he calls *genuine education*, Dewey proposes a theory of experience based in two criteria: *continuity* and *interaction.*

Continuity

Continuity refers to the notion that "every experience both takes up something from those [experiences] which have gone before it, and modifies in some way the quality of those which come after" (Dewey, 1938, p. 35). Experiences are not discrete points, but can be conceived as vectors; having direction in the sense that they are 'coming from' previous experiences and 'heading towards' future experiences. Every experience influences (for better or for worse) the formation of intellectual and emotional attitudes, which in turn will play a part in determining the quality of future experiences.

The notion of continuity of experience allows Dewey to differentiate between experiences that are educative and those which are not. Every experience is a moving force, and this means that "its value can be judged only on the ground of what it moves toward and into" (Dewey, 1938, p. 38). Any experience which arrests or distorts the growth of future experiences, which engenders a lack of sensitivity, curiosity and responsiveness, which restricts the possibility of having richer experiences in the future, is said to be *mis-educative*. For Dewey, the central problem educators face is cultivating experiences which "live fruitfully and creatively in subsequent experiences" (Dewey, 1938, p. 28).

Interaction

For Dewey, experience is not something that goes on purely inside the mind of the individual, but neither is it exclusively determined by external factors. Every experience is an *interaction* between the individual and their environment. The *environment* (Dewey also uses the term *situation)* refers to whatever conditions, be they physical, social, cultural, or institutional, which interact with an individual's internal personal needs, desires, capacities, and purposes in order to create an experience (Dewey, 1938, p. 44). Both the personal and the social are always present. Interaction makes it possible to conceptualize an individual's learning, while simultaneously recognizing that said learning is taking place in a classroom, in a school, in a wider social context.

Role of the Educator

Given his understanding of human experience, Dewey advocates that there is no "best way" to educate all people, since both the external factors and the student as an individual are equally important in an educational experience. What takes place in the classroom is the dual product of the student's previous experiences and the *environment*. The environment refers to the *objective conditions* in which the individual is having the experience. Some of these conditions lie to some extent within the possibility of regulation by the educator. For Dewey, "objective conditions" include:

> what is done by the educator and the way in which it is done, not only words spoken but the tone of voice in which they are spoken. It includes equipment, books, apparatus, toys, games played. It includes the materials with which an individual interacts, and, most important of all, the total *social* set-up of the situations in which a person is engaged (1938, p. 45).

The role of the educator is thus to organize, as much as possible, the conditions of the learner's experiences towards those which they believe will be most educative. The greater maturity of experience which (should) belong to the educator as an expert, puts them in a position to evaluate the direction of an experience in a way in which the individual new to the experience cannot (Dewey, 1938). Educators must be alert to the attitudes and habits that are being created and be able to judge which of these are most conducive to learning. Further, educators must have a sympathetic understanding of individuals as individuals, in order to be able to facilitate connections with the learner's previous experiences (Dewey, 1938, p. 45). Dewey concedes that the necessity for these characteristics in an educator makes a system of education based upon living experience a difficult thing to conduct and follow (Dewey, 1938, p. 39).

If an experience cannot be put into continuity with the student's previous experiences, it may fail to be educative for that student. In the previous section, I claimed that the only way to learn mathematics is to experience doing mathematics and thinking/behaving mathematically. However, Dewey's theory of experience maintains that simply having an experience does not guarantee that the experience will be educative. It is often necessary to reflect on the experience and identify why and how a particular action or behaviour led to success. Then the learner may see the value of that behaviour in mathematical situations, which will increase the likelihood of it being engaged in a future situation.

It follows that from this perspective, learning mathematics is enabled and constrained not only by an individual learner's needs, capacities and previous experiences, but also by the social and cultural fact that this learning, today, takes place in educational *institutions* and within their particular *institutional practices.* In the following section of this chapter, I will draw on the IAD framework to define and describe the various institutions and institutional practices that are relevant to this research context, and conceptualize how individuals behave as subjects of these institutions.

3.3 Institutional Analysis and Development

In the existing literature in mathematics education, the term "institution" has mostly been used in a very broad sense, referring to any kind of formal or informal structure that shapes the social and cultural activities of a group. The Institutional Analysis and Development (IAD) framework provides precise language for discussing institutions in terms of the structures that emerge through repeated interactions.

Ostrom (2005) defines *institutions* as any rule-structured organization within which humans repeatedly interact to achieve certain goals. Individuals interacting within institutions face choices regarding the actions and strategies they take, leading to consequences for themselves and others (Ostrom, 2005, p. 3). The mechanisms that regulate these interactions are categorized into rules, norms and strategies. *Rules* are explicit regulations that are established by a recognized authority and contain sanctions against those who break them. *Norms* are guidelines for "how things are usually done." They are part of the generally accepted moral fabric of the community, and are learned through practice and experience. *Strategies* are individual plans of actions for achieving a goal or accomplishing a task and are learned through practice or instruction. (Hardy, 2009, p. 30)

It is important to note that, in most cases, institutions will be nested within larger institutions; what is a whole system on one level is part of a larger structure on another level. For example, a Mathematics Department is an institution within a larger institution which could be the Faculty of Sciences, itself a sub-institution within a university (Hardy, 2009).

In the context of this research, I am primarily interested in discussing how institutions influence behaviour (rather than how these institutions are formed or how they evolve). Within a particular situation, individuals construct implicit or spontaneous models for acting based on their beliefs about the opportunities and constraints of the situation (Hardy, 2009). In what follows, I will discuss two institutions that are involved in this study, as well as the models for acting in each institution that the participants of that institution may have constructed. I will then situate these two institutions within the particular context of this research.

Although there are many different institutions that coexist in any undergraduate class, there are two institutions that are of primary interest to me in this study, which I will call the *mathematical institution* and the *undergraduate-mathematics-program* (UMP) *institution.* The mathematical institution refers to mathematics as a research field, while the UMP institution refers to the particular expectations and regulations in place at the university with respect to how undergraduate mathematics courses are conducted, including the number of course hours, a marking scheme, or the physical space of classrooms.

The **Mathematical Institution** is regulated by:

<u>Rules</u>

- A set of mathematical rules and laws. For example, mathematical arguments must be logically consistent, according to a set of logic rules.
- A set of mathematical facts, axioms, and theorems.

<u>Norms</u>

- A normalized way of presenting and communicating the results of mathematical work, including syntax, notation, terminology, etc.
- A set of mathematical norms, taken-as-shared practices and ways of doing. For example, what counts as a significant, interesting, or elegant result.

The primary participants of the mathematical institution are mathematicians. As they work within this institution, there are many strategies that mathematicians may use. In Chapter 4, I present a model of mathematical behaviour, which seeks to characterize some, though not all, of these mathematical strategies.

The **Undergraduate Mathematics Program (UMP)** institution is regulated by:

Rules

- University regulations and guidelines, including the academic calendar, courses which may have prerequisites, and assessment constraints.
- Class sizes, classroom layout, and number of required course hours.
- Course outlines and course content are often fixed by the department or a course examiner.

Norms

- Professor lectures, students take notes. In the mathematics class, the professor presents a theorem/definition, then a proof/explanation, then examples. Rinse repeat.
- Weekly assignments will contain 'problems' similar to the exercises done in class. These problems will have clear, unambiguous solutions.

When considering the regulatory mechanisms of the UMP institution, it is important to distinguish between rules, which are given by the university mandates, and norms, which are general expectations for what a university mathematics class should be like. There are many different participants in this institution, including students, professors, administrators, and teaching assistants, however, in this thesis I am primarily interested in discussing the positions and actions of students.

In this research, I am looking at a group of students in a particular undergraduate mathematics course entitled *Introduction to Mathematical Thinking*. This course is mandatory for all students taking a major in Mathematics and Statistics, and, according to the course outline, the aim of the course is to introduce students to the practices/strategies of mathematicians:

> Most math courses present the results of mathematicians' work; what we call mathematical results. In this course, we will focus on how mathematicians arrive at these results. In particular, we will consider the processes of conjecturing, solving, and proving. (Hardy, 2015, para. 4)

In other words, the aim of the course is to provide students with opportunities to develop a set of behaviours/strategies that we believe characterize the *mathematical institution.*

Both Dewey's philosophy of experience and Ostrom's framework for IAD are apparent in Nadia's course design, and are part of the starting assumptions from which I pose my research questions. In the following section, I will discuss how these two theories come together to form the theoretical framework for this thesis, and will outline on how they are applied throughout this thesis.

3.4 Theoretical Framework

The two theories I have presented in this chapter come together to form my theoretical framework, in the sense that they frame how I believe mathematics is learned and taught in institutional educational settings. In this section, I present a discussion of how this framework is used to conceptualize the research in this thesis.

Of the two theories discussed in this chapter, the IAD framework is used primarily in the posing of my research problems, and as a basis for formulating my model of mathematical behaviour, although Dewey's theory of experience influenced these elements as well.

Dewey's theory of experience is used throughout this thesis: it informs my understanding of how mathematics is learned and taught, is a key element in Nadia's teaching philosophy, is the foundation of the narrative inquiry research methodology I have used, and provides the language for analyzing and discussing my results.

As discussed in chapter 2, my research goals were motivated by Nadia's previous research in MAST 217. In chapter 4, I will provide the definition of *mathematical behaviour* that I use in this thesis. In the context of this thesis, mathematical behaviour means behaving like a mathematician, in other words, engaging the strategies of the mathematical institution, abiding by its rules and norms. I will draw on different sources in mathematics education literature to characterize these strategies/behaviour and formulate the model of mathematical behaviour which I use in this thesis.

One of the Nadia's goals as the teacher and course designer for MAST 217 is to provide opportunities for students to *experience* mathematics the way mathematicians do. She aims to provide students with experiences which she believes will be *educative* (in the sense of Dewey), which in this case means experiences which will allow students to develop their mathematical behaviours.

Thinking in terms of institutional practices played an important role in the course design for MAST 217. Nadia sought to create a model of the mathematical institution *within* the UMP institution, so that students might experience, and hopefully integrate into their thinking/functioning, the norms, rules and strategies of the mathematical institution. For example, because collaboration is an important strategy within the mathematical community (Burton, 2001; Schoenfeld, 1987), Nadia had students work in groups and encouraged them to discuss their solutions and share ideas. Rather than following the theorem-proof-examples-exercises format, a common teaching strategy in the UMP institution, she regularly presented students with open-ended problems that had no clear or definite solution (Hardy et al., 2013).

There are, however, some clear challenges associated with trying to foster within one institution (UMP) the behaviours characteristic of another institution (those of the mathematical institution). As course instructor and designer, Nadia was required to adhere to the rules (but not necessarily the norms) of the UMP institution, while at the same time seeking to promote/foster the norms of the mathematical institution. Often the rules and the expected norms of an undergraduate mathematics classroom were directly at odds with the norms and strategies of the mathematical institution she was trying to foster.

Mathematics as a research field is very different from the mathematics students typically experience in high school and undergraduate courses. Mathematics as a research discipline is a creative endeavour that is driven by curiosity and the resultant pleasure when something is resolved (Burton, 1999). In contrast to how mathematicians in Burton (2001; 2004) describe coming to know mathematics, undergraduates frequently encounter mathematical knowledge without the exciting experiences of making personal, sociocultural and aesthetic connections through their own style of learning (Burton, 2001). Proofs, theorems and the accumulated results of hundreds of years of mathematical study are presented in carefully crafted, perfectly polished forms. The cycle of conjecture and verification, so essential for research activity in mathematics, is often absent from the way that students encounter mathematics in undergraduate courses (Smith & Hungwe, 1998).

Reflecting Dewey's philosophy, Nadia's course design is based on her belief that it is essential for students to *experience* using the strategies and practices of the mathematical institution, if they are to recognize that these strategies are useful and to develop mathematical behaviours.

Dewey's philosophy of experience is also at the heart of the methodology used in this thesis, that of *narrative inquiry*. As I will discuss in Chapter 5, narrative inquirers adopt Dewey's philosophy of experience for the basis of their understanding of experience as a narrative phenomenon. Clandinin (2006) argues that human beings structure their experiences according to personal and social narratives, not only in the stories they tell retrospectively, but in the lived immediacy of an experience. Reflecting Dewey's two criteria for experience (continuity and interaction), Clandinin and Connolly (2000) maintain that the stories which structure an individual's experience are the result of a convergence of social influences on that person's inner life, social influences of their environment, and their own personal history. Narrative inquiry is a research methodology that engages the researcher in the living and telling of these stories. As I progressed in this research, I grew to understand that narrative inquiry is more than just a methodological tool, it is a way of living in an inquiry, which shapes the experiences you will have and the significance you will draw from those experiences.

Narrative inquiry, with its foundation in Dewey's philosophy of experience, is used to frame and justify the methodological choices in this thesis, including the design of the project (chapter 6), the methods for gathering and sorting data (chapters 6 and 7), and the categories for analyzing and discussing results (chapters 7 and 8). As I will discuss further in later chapters, using Dewey's philosophy as a framework meant that I considered each individual in the course as being in the process of living their own life story. Thinking narratively about experience made me more aware of how a student's individual needs, capacities, and previous experiences inform and determine the nature of the experiences they will have in the course.

CHAPTER 4: Model of Mathematical Behaviour

In section 4.2, I present a model of mathematical behaviour. This model arises as an extension of the model of mathematical thinking used by Hardy et al. (2013) in their design experiment, which is based on the work of Sierpinska et al. (2002), and Schoenfeld (1987). For this reason, and because the narrative data I collected for this thesis was based on my experiences as a teaching assistant/instructor in one of the iterations of that design experiment, section 4.1 of this chapter provides an account of their research, with the main goal of contextualizing mine. Thus, in section 4.1, I give an overview of their research, including their model of mathematical thinking, and in section 4.2, I extend their model to a model of *mathematical behaviour*. Drawing on insights from Burton's (2001) work on the epistemological practices of mathematicians, I include beliefs about mathematics (Burton, 2004; Schoenfeld, 1987), feelings associated with doing mathematics (Burton,2004; Mason et al., 2010), and attitudes towards learning mathematics (Mason et al., 2010; Selden and Selden, 2013). Section 4.2 concludes with a discussion of how this model will be used in this thesis.

4.1 *Introduction to Mathematical Thinking*: A Design Experiment

This thesis, like all narrative inquiry, is situated within an ongoing story. Nadia Hardy, the instructor of the course in which the narrative study of this thesis took place, is also a mathematics education researcher, and the course was already a site for her ongoing research. Since she had begun teaching the course five years previously, she had been carrying out a design experiment, modifying the structure of the course and her teaching approach. My research was strongly influenced by her work in a number of ways: I created a model of mathematical behaviour by extending the model of mathematical thinking used in her design experiment, and I conducted my narrative research study in the class where this design experiment was being implemented (one of the iterations of the experiment). Additionally, Nadia, as instructor of the course, contributed narrative data to this research. Her voice, thoughts and experiences as both a teacher and a researcher are woven into these stories.

Despite the name of the course (Introduction to Mathematical Thinking), the institutional approach had been that of what is commonly known as a *transition to proofs* course, as discussed in chapter 2. Nadia saw an opportunity in the name of the course to build up a *new* course, one about thinking mathematically – which certainly requires knowing how to write proofs, but it is not only about that. To set up the design experiment, there was a need for a model of mathematical thinking.

4.1.1 Model of Mathematical Thinking

Hardy et al. (2013) give an account of the ongoing design experiment for the teaching of the undergraduate course *Introduction to Mathematical Thinking,* wherein the course was redesigned with the goal of providing a learning environment where students could develop mathematical thinking. They propose a model of mathematical thinking which results from combining Sierpinska et al.'s (2002) model of theoretical thinking with aspects of metacognition (Schoenfeld, 1987).

In Sierpinska et al.'s (2002) model, theoretical thinking is described as the opposite of practical thinking, and is characterized as reflective thinking, systemic thinking, and analytic thinking. Reflective thinking is thinking for the sake of thinking. It is not about achieving a specific practical goal, but instead is geared towards speculation, reflection and further investigation of the task at hand. The reflective thinker is curious and able to apply the meaning of concepts freely, adapting and changing them as new problems arise. Systemic thinking is concerned with the internal validity and coherence of systems of concepts. Sierpinska et al. (2002) identify three subcategories of systemic thinking: hypothetical, proving, and definitional. The systemic thinker is able to understand and accept that meaning is found in the definition of concepts (definitional); is aware of the conditional character of statements (proving); and aims to consider all logically possible cases, while being sensitive to implicit assumptions and ambiguities (hypothetical). Lastly, analytic thinking is concerned with symbolic notation and the structure and logic of mathematical language. The analytic thinker is sensitive to linguistic concerns of notation and terminology, as well as the distance between symbol and object.

Theoretical thinking develops against practical thinking, and although both are required for the study of mathematics, Sierpinska et al. (2002) suggest that many undergraduate students fail to develop theoretical thinking skills because it is not required for success in their courses.

Schoenfeld (1987) highlights *self-regulation*, an aspect of metacognition, as having direct implications for mathematics education. Self-regulation refers to how well someone keeps track of what they are doing, and how well they manage their time and resources when working on a mathematical problem. The self-regulated problem solver reads the problem carefully, tries different strategies, and always takes time to re-assess whether their current course of action is leading somewhere fruitful. Effective and efficient self-regulation was shown to be more important for success in problem solving than particular knowledge in the subject matter: "With the efficient use of self-monitoring and self-regulation, [the mathematician] solved a problem that many students – who knew a lot more geometry than he did – failed to solve" (Schoenfeld, 1987, p. 195).

According to Hardy et al.'s (2013) model of mathematical thinking an individual thinks mathematically when they show evidence of theoretical thinking (in the sense of Sierpinska et al., 2002) or self-regulation (in the sense of Schoenfeld, 1987). Nadia's goal in designing the course was (and is) to provide students with opportunities to develop and engage in theoretical thinking and to develop and practice self-regulation. In the language proposed in chapter 3, it can be said that Nadia's goal was to provide students with opportunities to experience some of the rules, norms and strategies that regulate the behaviour of participants of the mathematics institution (the mathematicians).

4.1.2 Course Design

If the goal of the course is for students to think mathematically, what should the course look like? What is the best way to provide opportunities for students to experience theoretical thinking and make use of mathematical strategies? These were the questions that guided Nadia and her two research/teaching assistants as they began their design experiment.

Design experiment methodology (Brown, 1992) is modeled on the iterative procedures of design sciences such as aeronautics or robotics. Based on the criteria of their study, researchers engineer and design new educational environments and simultaneously conduct experimental studies on those designs. After each implementation of the design, the researcher assesses its effectiveness and may or may not make changes to the design before implementing it again.

Based on this methodology, Nadia set out to drastically change the traditional undergraduate mathematics classroom into what Brown (1992) calls an *intentional learning environment,* where students are encouraged to engage in self-reflective learning and critical inquiry:

> They act as researchers responsible to some extent for defining their own expertise. Teachers' roles also change dramatically in that they are expected to serve as active role models of learning and as responsive guides to students' discovery processes. (Brown, 1992, pp. 149-150)

There were two 75 minute classes per week where students would work in small groups (4-5 students) on 1 or 2 problems from a set of activities designed to engage them in mathematical thinking. The problems were usually easy enough so that students could mathematize them and make conjectures about their solutions but difficult enough so that they couldn't easily (or at all) construct a mathematical proof; problems for which they couldn't provide a definite answer. They included activities on the notion of truth, definitions, axioms; and reading and deconstructing proofs. Hardy et al. (2013) describe the goals of the activities:

> The ultimate goals of these activities were to engage students in situations where we believe mathematical strategies, reflective thinking and systemic thinking are particularly useful, and in oral and written argumentations in which we believe systemic thinking and analytic thinking strongly come into play. (p. 281)

As students were discussing the problems, the instructor and teaching assistant(s) would circulate in the room, listen to what a group was discussing and then challenge their approach with "what if…" type of questions. Concepts and techniques that the students were not familiar with were explained on an as-needed basis.

An online forum was created to be an extension of the classroom work space where students could continue to discuss the problems from class and share proofs, solutions, and ideas outside of the class time. Making at least one *meaningful* posting on the forum per week was a mandatory component of the course (in that grades were given for doing so). The criteria for being meaningful were established through practice, and were intended to encourage students to seek new ideas and approaches. They could initiate a thread or respond to peers' or teachers' postings, and postings had to be unique; they were asked not to repeat questions or comments that others had posted, which required students to read the forum before posting.

The previous edition of the course notes (Hillel & Byers, 2010) contained many *examples of proofs*; the carefully crafted propositions and polished arguments which one finds in mathematics textbooks. Such texts are not a faithful representation of the (often messy) thought processes that created those results in the first place. With this in mind, new course notes were written which attempted to make visible the cycle of guessing and inquiry that goes into constructing a proof. The main goal was to create a document which would help students increase their intuition for how to approach and deal with mathematical problems. The writing style of the new notes was more conversational than a standard math textbook, but still contained definitions and examples intended to complement the activities done in class.

In small groups, students were assigned to each section of the notes and were responsible for reading and presenting that section to the rest of the class. Nadia's goal in having these presentations was for students to experience reading and understanding new mathematical topics on their own without the support of a teacher or lecture.

Approximately every 3 weeks the students had a quiz (4 in total) and there was a final exam at the end of the session – in every case, the students were given time to discuss the assessment problems in groups before engaging in writing individual answers.

From the first class, students engaged actively in the in-class activities and seemed to enjoy the experience of doing mathematics in this way. Many groups developed a dynamic where those who were stronger in math became leaders, explaining their ideas to the others in the group, who still participated actively by asking "why…" and "what if..." questions. Almost all students contributed to the forum every week, often posting more frequently than was required. Discussions usually centred on the activities from class, although sometimes students would post about new topics. They participated in long back-and-forth discussion threads and shared pertinent resources they found online.

Although students were not assigned to fixed groups, many tended to sit in roughly the same places each class, and developed relationships with their peers. Some groups would develop stronger group dynamics/bonds than others, likely depending on the personalities of the students in the group. Although most students sat with the same people each class, there were some who remained quite nomadic, and would sit at different tables each class and there were a few who would mostly work alone.

Overall, after this first iteration of the design experiment, and based on an analysis of the collected data (see Hardy et al., 2013), Nadia felt satisfied that the design provided students with "opportunities to engage in mathematical thinking; something that doesn't seem to be a commonality in first-year undergraduate mathematics courses – and students did take advantage of these" (p. 283). There were still, however, many aspects of the course which required improvement, and she continued to make adjustments to the course design which were motivated by both didactic concerns and institutional constraints.

Didactic changes were those concerned with making sure the activities were as useful and meaningful as possible. For example, Nadia decided to eliminate the student presentations of sections of the booklet (course notes), as they often felt disconnected from the activities being done in class, and replaced them with weekly assignments based on the readings. She also changed the order of the activities to try and make the presentation of the material more coherent and logical, something which she continues to adjust each time she teaches the course, never fully satisfied. Nadia writes in one of her narratives:

> "It looked to me that the order of the activities was a little weird. Now I feel it flows better. I guess the problem is whether to formalize first or to work first. And it seems I have been alternating between the two […] It is just about not being able to convince anyone, not even myself, of what is best."

Another didactic change was to include new activities to ensure that opportunities were provided for student to engage in all aspects of mathematical thinking. The data analysis performed after the first iteration showed that of the different aspects of mathematical thinking in the model, students engaged in analytic thinking the least. The researchers conjectured that if students were not engaging in analytic thinking, it was because they did not see it as useful for working on the problems in class. They were able to understand the problems they were working on, and make themselves understood by each other, without using formal notation and structure. In the hopes of providing more opportunities for students to use and develop their analytic thinking, new activities were added which were mostly about reading and analyzing the structure of statements and (sometimes incorrect) proofs.

Institutional constraints such as time limitations, class size, and the physical space of the classroom also affected decisions about teaching methods. Nadia has usually been able to teach this course in a room where the desks could be re-arranged for group work, and class sizes have been under 40 students. These two details are important to make the teaching design work. One of the questions about this particular design is whether and how it can be adapted to larger class sizes.

One of the problems with this approach was Nadia's sense that, mostly due to time constraints, some students couldn't explore a variety of approaches when dealing with a task, nor could they reach an understanding entirely on their own. Furthermore, the course had been structured to emphasize experiential learning, but there was no class time devoted to reflecting on those experiences, formalizing them, or situating them within the context of mathematics and the goals of the course. For example, a student could be trying to explain or prove a statement, and they notice that if they assume the statement is false they arrive at a contradiction. The student may think to themselves: "If it is not possible for the statement to be false, then I know it is true" and provide a clear argument as to where the contradiction arises, all without recognizing that they are using the technique of *proof by contradiction*. This would be a lack of institutionalization: although the student is able to think mathematically about the problem, they do not recognize (formalize or generalize) the underlying structure of their mathematical argument. The way that Nadia decided to address these issues was by two means:

- sometimes step in and provide guidance, encouraging the students to reflect on what they had been doing, in order to assist the students in creating an educational meaning from the experience; and
- by setting class time aside for *institutionalizing* knowledge (in the sense of Chevallard, 1999).

Essentially, the difficulty Nadia faced was that a succession of experiences does not necessarily add up to an experience of that succession. The experience of doing/encountering something many times is not usually sufficient if not accompanied by some reflection on the experience that allows the students to identify key moments, ideas, and actions in their experiences. Additionally, the students need to be aware (or need to be made aware) of how the experiences they are having are connected to mathematical concepts and/or elements of mathematical thinking.

In order to address this concern, and to provide more opportunities for knowledge to be institutionalized, Nadia adjusted her teaching method to include some brief mini-lectures at either the beginning or end of class. She would use these to explain directly how the activities of that class were related to concepts found in the course notes, give definitions of important new terms and concepts, explain the structure of a statement or proof, or provide examples. However, the main goal of these (interactive) mini-lectures was to highlight the systemic nature of mathematics; how the ideas, concepts, techniques, and strategies being discussed, are connected or related to one another; to illustrate the type of reflection needed to transform the succession of experiences into the experience of such succession

My research for this thesis begins in the midst of this ongoing design experiment. I first learned about Nadia's work on *Introduction to Mathematical Thinking* when she mentioned it during a course I was taking with her in my first year of the master's program. She invited her Mathematics Education students to attend the class, which I did several times that semester, observing and sometimes helping out as a teaching assistant. Many of the questions and goals of her research were ones that interested me, and she agreed to take me on as a teaching assistant for the following iteration of the course, and to participate in and supervise the research for this thesis.

For this thesis, I have chosen to extend the model of mathematical thinking used in Nadia's design research to a model of *mathematical behaviour.* In the following section, I outline the reasons for this decision, and describe the elements of this new model.

4.2 A Model of Mathematical Behaviour

In this section, I will present the model of *mathematical behaviour* I use in this thesis, which extends upon the model of mathematical thinking proposed by Hardy et al. (2013). In the first subsection, I begin with some reasons why I decided to add to their model, along with a discussion of how I arrived at using the term "mathematical behaviour" instead of "mathematical thinking." Next, I define what I mean by "mathematical behaviour" and present the different elements of my model in the form of a glossary, drawing on relevant literature as well as on my own experiences during the research. In the second subsection, I expand on how mathematical behaviour is observed in the classroom.

4.2.1 Defining/Describing Mathematical Behaviour

Developing the theoretical framework for this thesis was, for me, a struggle of language, of trying to find the right words. I am by no means the first researcher to encounter this difficulty, as Pirie and Kieran (1994) write:

> When seeking to provide labels for new conceptions one is faced with the dilemma of either choosing existent words which one can hope already convey

> some of the desired meaning or creating new terminology and then attempting to invest in it associations and connotations that will carry the new ideas to the reader. (p. 166)

In this section I will outline why I decided to extend the model of mathematical thinking proposed in Hardy et al. (2013), and discuss how this led me to adopting the term *mathematical behaviour* instead of 'mathematical thinking'. I will do my best to provide a description of what the term mathematical behaviour means in the context of this thesis.

The model of mathematical thinking used by Hardy et al. (2013) focused primarily on the types of thinking associated with working on and solving mathematical problems. In fact, the term *mathematical thinking*, particularly how it is employed in mathematics education research, is a term that is associated mostly with how one thinks about mathematical problems (Selden & Selden, 2005). To me, this term did not convey the totality of what it means to experience doing mathematics the way a mathematician does. The view of problem solving as an operational definition of thinking mathematically is, I feel, too narrow. I take the term *thinking mathematically* to mean, in general, *thinking like a mathematician;* it is an attitude, an approach to the world. Thinking like a mathematician means not only a way of approaching problems, but the feelings associated with doing mathematics, as well as a set of beliefs about the nature of mathematics in general. Learning to think mathematically is a social act of apprenticeship, wherein students are being acculturated to view certain relationships and ways of thinking as valid, meaningful and interesting, while others are not.

Schoenfeld (1987) writes:

> I remember discussing with some colleagues, early in our careers, what it was like to be a mathematician. Despite obvious individual differences, we had all

> developed what might be called *the mathematician's point of view* – a certain way of thinking about mathematics, of its value and how it is done. What we had picked up was more than a set of skills; it was a way of viewing the world and our work. We came to realize that we had undergone a process of acculturation, in which we had become members of, and accepted the values of, a particular community. As the result of a protracted apprenticeship, we had become mathematicians in a deep sense (p. 213).

Language is ever an imperfect tool. Words can only point us towards ideas, and never seem to completely capture what we mean. The meaning I am searching for lies in the union of many different terms from the literature. I have chosen to use the term *mathematical behaviour* because I feel it comes closest to what I want to describe. For me, mathematical behaviour refers to the broad scope of activities, actions, feelings, and beliefs shared by (most) members of the mathematical community.

There is a wealth of literature in mathematics education that has sought to characterize mathematical behaviour. I drew on many of these sources to begin generating a comprehensive list of mathematical behaviours, which I used to guide my observations. Drawing on the literature I read, I want the term *mathematical behaviour* to encompass all of the following perspectives on mathematical activity:

a) *thinking mathematically* in problem solving situations, as described in Mason et al. (2010), including theoretical thinking (in the sense of Sierpinska et al., 2002), and metacognition (in the sense of Schoenfeld, 1987);

b) displaying/utilizing *mathematical habits of mind* (in the sense of Cuoco et al., 1996) - mental tools needed to use, understand, and create mathematics;

c) abiding by the set of norms, rules and strategies which constitute the *regulatory mechanisms* of the mathematical institution (in the sense of Ostrom, 2005)

d) having a certain *mathematical world view* – beliefs about the nature of mathematics and the way it is done (Burton, 2001; Burton, 2004; Schoenfeld, 1987)

In what follows, I provide a list of elements of mathematical behaviour, drawn from the literature and from personal experiences. Throughout the course of this research, elements were changed or added based on the experiences/discussions of our research group. The list is non-hierarchical (that is, the order does not intend to convey that any one element is more important or 'more mathematical' than another), and is by no means exhaustive. Following the list, in section 4.2.2, I provide an explanation for how this list of mathematical behaviour is used as a tool for researching students' mathematical behaviour in this thesis.

In this thesis, I say that a student *behaves mathematically*, if in their actions I can see evidence of any one of the elements in the following list.

Reflective Thinking: The reflective thinker engages in speculation, curiosity, reflection and further investigation of the task at hand. They are able to apply the meaning of concepts freely, adapting and changing them as new problems arise. A reflective thinker is not geared towards a particular goal: they enjoy thinking for the sake of thinking. (Sierpinska et al., 2002)

Systemic Thinking: A systemic thinker is someone who is concerned with the internal validity and coherence of systems of concepts. They are able to understand that meaning is found in the definition of concepts and aware of the conditional character of statements. A systemic thinker thinks hypothetically and considers all logically possible cases; they are sensitive to ambiguities and assumptions. (Sierpinska et al., 2002)

Analytic Thinking: An analytic thinker is someone who is sensitive to the linguistic concerns of notation and terminology, and the structure and logic of mathematical language. They understand the distance between symbol and object. (Sierpinska et al, 2002)

Self-Regulation: Schoenfeld (1985) describes self-regulation as the thinker's ability to (a) make sure that they understand what a problem is all about before hastily attempting a solution; (b) plan; (c) monitor, or keep track of how well things are going during a solution; and (d) allocate resources, or decide what to do, and for how long, as they work on the problem. Good self-regulation skills are what differentiate an 'expert' problem solver from a novice.

Intuition: Mathematicians describe intuition as 'seeing connections', 'a light which switches on', or 'knowing the possible or even likely' (Burton, 1999). Intuition is an ability to see or guess the way forward on a problem, or a sense that two areas of mathematics may be connected, without being able to show it rigorously. Sometimes described as not-straightforward, non-linear thinking, intuition gives an idea or direction, which must be tried and tested. Intuition is frequently regarded by mathematicians as something quite essential to doing of mathematics. Most mathematicians cite knowledge and experience as the source of their own intuition. (Burton, 1999)

Conjecturing and Justification: Mason et al. (2010) highlight conjecturing and justifying as essential mathematical skills for problem solving. Conjecturing is the process of sensing or guessing that something may be true and investigating its truth. For example, if a problem solver has generated some examples and thinks they see a pattern emerging, then they might *conjecture* that this pattern will be true for all cases, and seek to justify this claim. What counts as a justification in mathematics is decided by the community. In contemporary mathematics, acceptable justification has to abide by the rules of mathematical logic and logical inference.

Deal with Frustration: A mathematician interviewed in Burton (2004) expressed that, "the natural condition of doing research [in mathematics] is to be stuck, most of the time, on most of the things you are doing" (p. 59). Mathematicians are able to manage their frustration when stuck on a problem, ideally viewing being stuck as "an honourable and positive state from which much can be learned" (Mason et al., 2010, p. 45). When 'stuck', mathematicians will choose to take a break from the problem, go for a walk, work on something else and then return to it later, or discuss it with a colleague (Burton, 2004; Selden & Selden, 2013).

Persistence: Mathematical problems take time and patience to solve. Mathematicians understand that it may be, and often is, necessary to go in many unhelpful directions before arriving at a solution (Selden & Selden, 2013). A persistent problem solver is someone who seeks to generate as many ideas for solving a problem as possible, will try things without knowing ahead of time if they will work, and is not discouraged when an approach proves unsuccessful (Schoenfeld, 1987).

Self-Efficacy: Selden and Selden (2013) claim that in order to tackle difficult problems which require persistence, the problem solver needs self-efficacy. Self-efficacy is the belief in one's own ability to succeed in a particular situation. I often equate self-efficacy with having a particular *attitude towards learning*: a belief that one's ability can be improved by practice and reflection. Individuals with strong self-efficacy:

> (a) view challenging problems as tasks to be mastered; (b) develop a deeper interest in the activities in which they participate; (c) form a stronger sense of commitment to their interests and activities; and (d) recover quickly from setbacks and disappointments. In contrast, people with a weak sense of self-efficacy: (a) avoid challenging tasks; (b) believe that difficult tasks and situations are beyond their capabilities; (c) focus on personal failings and negative outcomes; (d) quickly lose confidence in personal abilities. (Bandura, 1994, quoted in Selden & Selden, 2013, p. 305).

Reflecting on one's own thinking: In their discussion of how students can learn to think mathematically, Mason et al. (2010) observe that although we learn from experience, experience alone is not enough. One cannot learn from experience unless they reflect on their own thinking after the task, recalling and identifying the key moments and key ideas in their resolution process (Mason et al., 2010).

Collaboration: Contrary to the pervasive cultural image of the isolated, lonely mathematician, many mathematicians work collaboratively or co-operatively. They speak of many different reasons for collaborating. Many find collaborating more enjoyable than working alone for the simple reason that they like discussing their ideas with others. Collaborating also provides many advantages over working alone: mathematicians benefit from the experience of their peers and colleagues whose skills may somewhat complement (rather than duplicate) their own. Sharing workloads allows them to work more quickly and collaborating often increases the quantity and quality of ideas generated. (Burton, 2001; Schoenfeld, 1987)

Consider multiple points of view: Mathematicians will try to consider a problem or task through multiple lenses, employing different techniques, etc. until they find a method which will work for the particular task. They will often compare their methods and solutions with others and determine whether one is better, according to certain shared criteria such as simplicity or elegance. They also see value in having multiple different solutions or solving methods for a given problem.

Skepticism: "Ask mathematicians what makes their discipline different from others, and many will say that mathematicians prove things; they will say that the standard for truth in mathematics is just higher than anywhere else; they will say that mathematicians simply are not convinced of a fact, in spite of what would seem like overwhelming evidence to people in other (even scientific) disciplines, unless the fact comes with a proof" (Cuoco et al., 1996, pp. 386 – 387).

Obsessive Behaviour: I had some difficulty naming this particular feature of mathematical behaviour. The label 'obsessive behaviour' refers to the tendency of mathematicians to work tirelessly on a problem, to the point where their minds often continue working on a question, even when they believe they have put it aside. Mathematicians frequently share experiences of being asleep, or engaged in some other task, when they are suddenly struck with an insight into the problem they have been working on. This 'obsessive behaviour' may at first glance seem similar to persistence, so it is important to distinguish between the two. Obsessive behaviour has an unconscious element to it: the brain keeps working on a problem after one has moved on to other tasks.

Personal and Emotional Connectedness: Many mathematicians do mathematics for the very good reason that they enjoy it. The cognitive and affective aspects of doing mathematics are indivisible. Mathematicians often have a personal/emotional connection to their work, and their mathematical activity is driven by curiosity and the resultant pleasure when something was resolved, or the strong feeling of frustration when they reached a dead end (Burton, 2004; Mason et al., 1981).

Mathematical Beliefs: Schoenfeld (1985) points out that the performance of any intellectual task always takes place within the context of one's perspective regarding that task. Beliefs about the nature of mathematics are forged by prior experiences, and will determine what information and techniques the problem solver considers to be relevant and appropriate to the problem at hand. Through their experiences with mathematics at school, students develop a tacit understanding about "what mathematics is all about;" they develop their own "mathematical world view," even if they are not consciously aware of holding those beliefs (Schoenfeld, 1987). These beliefs will shape how one engages in the task at hand.

4.2.2 Observing Mathematical Behaviour

In one of our early conversations, I met with Nadia to discuss possible topics for my thesis. I asked her about her goals as a teacher in this course.[3]

[3] Dialogues between Nadia and myself such as the one I present here appear in several places throughout this thesis. It is important to note that these dialogues are not transcripts of conversations that took place, but rather re-constructions, which I have created based on field texts and my own remembering.

Nadia: As a teacher, my main goal is to provide students with opportunities to experience mathematics the way that mathematicians do; and for me the only way of knowing if an opportunity has been given is if a student takes it.

Erin: What do you mean by that? Couldn't you say that you provided the opportunity either way, whether the student takes it or not?

Nadia: Perhaps opportunity is not the right word… but what I mean is that as a teacher this is how I judge if my teaching is 'successful'. If I don't observe mathematical behaviour in my classroom, what does it mean? It is not that students are not reflective thinkers or persistent, it is not that they don't self-regulate, or that they are not creative or engaged. It is that I haven't offered them enough or 'good' situations to enact or practice these (perhaps, dormant) skills. I want my class to be a space where these skills cease to be potential and become actual. Only then, students will recognize them and value them.

Erin: Okay, I understand what you mean. Seeing students behave mathematically in the classroom is a way of measuring or knowing that the teaching method is achieving the goal you intended.

Nadia: Right, but often the moments when I see a student behaving mathematically are fleeting. It is there and then it is gone. There is this in-the-moment quality to their behaviour which research data, at least it seems to me at this point, fails to capture. This is why I think it could be interesting for you to explore something different in terms of methodology. I want to find some way to create accounts of what I see and experience.

Erin: Sounds like quite a challenge…

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In this section, I want to explain how the model I have developed was used in this research. A more traditional research plotline would have had me test the model by first operationalizing it into a set of *observable features*, and then trying to detect those features in student behaviour. However, the question Nadia raised in our discussions was not one that could be answered using such a method. She had the sense that the behaviours were there but she could not say why; she did not know what the observable features of those behaviours were. So rather than coming up with a set of observable features to be then looked for, the (methodological) goal was to try to make accounts of the moments in which we believe students are behaving mathematically, and perhaps draw some conclusions about what it was that we observed in those moments that made us feel that way.

Thus, in the previous section, I provided general descriptions of each element of this model, but refrained from using these descriptions to construct a rubric for observation. In this thesis, I wanted to be able to construct records of the moments that seemed significant or interesting, and to create space to reflect on these moments in all their particularity and complexity, only afterwards thinking about how they might be connected to the elements of the model. Following the narrative inquiry framework, which I will describe in more detail in the following chapter, we constructed narrative data after each class by writing accounts of the moments that seemed significant to us. Moments when we felt that the teaching was in some way successful or when we felt the students were behaving mathematically, without necessarily trying to label those behaviours in that moment. In reflecting on those moments and behaviours, we would ask ourselves why we felt they were examples of mathematical behaviours, trying to formulate the basis on which we imagined we found meaning or understanding in those experiences.

The two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in the classroom. Because I was interested in inquiring into and understanding the lived immediacy of our students, experiences of mathematical behaviour, I needed to find a research methodology that was appropriate to that goal. Narrative inquiry, as described by Clandinin and Connolly (2000) provides an appropriate framework for thinking about and discussing the lived experiences of ourselves and those of our students. In the following chapter, I will provide a detailed description of this methodology, as well as further justification for its use in this thesis.

CHAPTER 5: Methodological Framework

The purpose of this chapter is to provide justifications for why narrative inquiry is an appropriate choice of methodology for my research goals, and to provide an overview of this research methodology, which was conceived by Clandinin and Connolly (1990; 2000), and which has guided the design and implementation of my research.

When Nadia and I first discussed this thesis project, she asked if I would be interested in exploring a methodology based on story telling. She told me about her previous research in the course and why she was not satisfied with the methods for constructing and reporting data that she had been using up until now (see chapter 2, pp. 30 – 32). We met many times to discuss what a methodology based in storytelling might look like, while I searched for methodologies in research fields other than mathematics education, that may help us in achieving our goals. When I came across Clandinin and Connolly's (2000) book, it was as if a light switched on.

In the first section of this chapter, I explain how my decision to use narrative inquiry as a methodological framework for this thesis was motivated by both my own epistemological perspective on research activity, as well as my shared interest with Nadia in storytelling and narrative as a potential research methodology. In the second section of this chapter, I review the work of Clandinin and Connolly (1990; 2000) in order to outline and define the terms and elements of narrative inquiry research methods.

5.1 Ontological and Epistemological Justifications: Why Narrative Inquiry?

> *But if our interest as researchers is lived experience - that is, lives and how they are lived - how did our research conversations become focused on the measurement of student responses? How did education come to be seen as something that could be measured in this way?* (Clandinin & Connolly, 2000, pg. xxii)

In this section I will provide my own epistemological and ontological position with respect to research activity. My epistemological position on research activity is what led me to narrative inquiry methodology, and for me justifies why narrative inquiry provides an appropriate framework for thinking about and discussing the lived experiences of individual learners. In order to contextualize my position, I will briefly outline some of the history of mathematics education research, with a particular focus on the epistemological character of research data. A dialogue between Nadia and I highlights some of our reasons for seeking out a research methodology that was different from those more established in mathematics education research. The section concludes with a discussion of how the epistemological stance required to do narrative inquiry, as described by Clandinin and Connolly (2000), is compatible with my own, and is therefore an appropriate choice for this research.

Education as a research discipline emerged in North America at the beginning of the 20th century with the establishment of university schools and departments of education, their goal being to institute a "science of education." Research was heavily influenced by the era's "increasingly common faith in the value of deriving generalizations from empirical data and its widespread disdain for knowledge based on logic or speculation" (Lagemann, 1997, p. 6). Early research in mathematics education was also heavily influenced by psychology, and looked to the natural sciences for models and methods (Kilpatrick, 2014). Quantitative measurement became the mark of validity in education research, and research in mathematics education was no exception. Phenomena were taken to be fixed and measurable, and researchers employed tests and statistical devices to measure the achievement of students and the costs of instruction in a desire to determine the most efficient practices (Lagemann, 1997).

In the latter half of the twentieth century, educational psychologists were gradually less concerned with emulating the natural sciences, and began to develop their own techniques for studying learning, and researchers in mathematics education followed suit (Kilpatrick, 2014). Researchers began investigating the development of concepts in young children using observation and interview. As mathematics education researchers began to study the learning and thinking of children, they increasingly recognized that laboratory studies present a restricted view of those processes, since children do most of their learning in the school classroom, in interactions with other children and with their teachers. Thus, many mathematics education researchers began conducting their studies in classrooms, observing both teachers and learners in interaction (Kilpatrick, 2014).

Kilpatrick (2014) summarizes an important shift in the focus of mathematics education research, away from the simplistic stimulus-response model it inherited from the natural sciences, towards a more complex social-cultural perspective:

> In a number of studies conducted in the first half of the twentieth century, components of teaching or characteristics of teachers were linked to learners' performance in an effort to understand what might constitute effective teaching. Researchers eventually moved from such simple "process-product" models to more sophisticated efforts that attempted to capture more of the complexity of the teaching-learning process, including the knowledge and beliefs of the participants as well as their activities during instruction (p. 270).

A different, less quantitative, research plotline began to emerge in mathematics education, as researchers began to study the teaching and learning of mathematics as a complex social-cultural phenomenon. Researchers studied how discourse is structured in mathematics classes, how norms for learning and doing mathematics are established in classrooms, and how teachers and students build relationships (Kilpatrick, 2014). With this shift in focus came a shift in methodological practices; qualitative research, or at least research which used qualitative sources of data, became increasingly established.

Barwell (2009) writes that mathematics education research, in its current incarnation, is primarily a discursive process; it involves the production and interpretation of various kinds of spoken and written texts, such as interviews, classroom observations, transcripts, or journal articles. But although the sources of data in mathematics education are no longer strictly quantitative, the established inquiry tradition still views analyzing said data as an endeavour to give an accurate, credible, confirmable account (Barwell, 2009; Maheux & Proulx, 2014). Barwell (2009) points out that although most researchers would agree it is not possible to simply read off what learners are thinking from what they say or do, in most research, such data (interviews, observations of classroom interaction, written tests, etc.) are nevertheless treated as relatively transparent, with interpretation of data being dealt with as a technical issue (e.g. through triangulation). So, although researchers have moved away from the 'scientific' model of inquiry, there is still a tendency in mathematics education research to treat data and observation as relatively objective. Researchers working on students' conceptions, for example, often position themselves as if said conceptions existed independently of the observer and the act of observing (Proulx & Maheux, 2014).

As Barwell (2009) summarizes:

> For me, there is a danger that 'observing' and 'viewing' have an air of 'apartness'. The [student] is doing the thinking whilst the researcher is merely observing. But what does it entail, this observing? Surely, if learners' mathematical thinking is seen as a discursive process, then so too must *researchers'* thinking. Observing or viewing are cognitive processes and, as such, subject to the same general theoretical ideas as the cognitive processes involved in mathematical thinking. Doing research in mathematics education, then, is as much a discursive process as doing mathematics (Barwell, 2009, p. 256).

I share the concerns raised by these authors. As a researcher, I take the epistemological position that an observer/researcher does not describe what is being observed, but constructs their own account of their perceptions. Any representation, no matter how faithful it tries to be, involves interpretation and selective emphasis of one's own experience. In mathematics education research, data is *constructed* by the researcher, based on their own interpretations and experience. The goal of doing research and analyzing data is not about providing accurate, objective accounts of that experience: it is, in Kieran and Simmt's (2009) words, about 'bringing forth a world of significance'.

Following Dewey (1938), I maintain that the study of education is the study of experience, a study of humans and their relations with themselves and their environments. If we are interested in inquiring into and understanding lived human experiences, then the tools we use should be ones that are commensurate with that goal. Clandinin and Connolly (2000) argue that, since it is a complex and dynamic human experience, empirical methods are wholly inappropriate for studying education. To attempt to quantify experience is to strip it of its richness and expression. Or, as anthropologist Clifford Geertz (1973) writes, "it is not in our interest to bleach human behaviour of the very properties that interest us before we begin to examine it" (p. 17).

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Nadia: I can remember a few years ago there was this great conversation that I had with a student. It was so interesting, and seemed so rich, one of those great moments where the student had said something that was exactly what I, as a teacher, was hoping they would say. I can't remember now what it was, but I was excited about it, so I went and shared it with a colleague and re-told her the little story of my conversation. She told me, "You need to find a way to capture that in your research; those fleeting moments." In the first stages of my design experiment in this course, I collected only written products, the traces left on the page, and I came to realize they were not sufficient to talk about behaviour.

Erin: I remember when we first met to discuss this project; you asked if I would be interested in exploring a methodology that used storytelling.

Nadia: Yes, because I knew we needed to find a way to capture these conversations and other fleeting moments that take place in the class. I knew that the things I really wanted to discuss were happening, but I was not capturing them. I know that some researchers use video or audio recordings in their research, but this is not what I want to do.

Erin: I feel like having video cameras would be distracting and disruptive to the students, to know they were being recorded. I've always felt this would change the way students interact with us and vice versa.

Nadia: More than that, though, I feel video recording would be missing the point of what I want to capture. A video recording or transcript doesn't convey what I thought or felt or experienced in that moment, only the dry details of who said what. I want to be able to convey my own experience of that conversation, why it was significant to me, which is different than recording precisely what was said.

Erin: That reminds me of a part in Clandinin and Connolly's work where they write about how tape recorders and videotape tend to be overused in education research: "it is the fear that somehow experience will be lost that drives researchers to try to record and tape all of experience. What we fail to acknowledge clearly enough is that all field texts are constructed representations of experience" (2000, p. 106).

Nadia: Yes, that's exactly it. The very act of recording events comes from this epistemological stance where you believe you can objectively capture an experience. And for me, from my epistemological perspective, an experience as I remember it is real and accurate in a way a transcript is not.

Erin: In a sense, there is no one true version of events, only interpretations.

Nadia: I was interested in stories as a research method, because the truth is in the telling. A story is good if it conveys something, if it creates an experience for the reader to live in. Stories are not concerned with accurately recording the details, but about communicating what an experience was like.

Erin: In good stories, the author writes about an individual or group of individuals in such a way that the reader will see some aspects of themselves reflected in the tale. Author Stephen Mitchell wrote that a good story is one that can "mirror back to us our condition, changing as we change, clarifying as we become more clear" (Mitchell, 1984, *xvi*). A story is an invitation to the reader to walk along with the author, to try and see what they see.

Nadia: I was inspired to start thinking about storytelling in research by an article in FLM, where they quoted Alice Munro, and talked about how her stories had this unique appeal because they depict little slices of life. A quote from that article really stuck with me: "In a field at times neurotically obsessed with the general and the generalizable, it is important to be reminded (and constantly so) that the particular, the specific, the local and the unexpected also have their place in our field" (Pimm & Sinclair, 2014, p. 5). I wanted to try and stay with the complexity of my experiences and to convey them in my research.

Erin: Stories were appealing to me because it seemed like a good way to investigate my own experiences, and also a good way to share those experiences in a way that might be meaningful for others. It seems to me that if we want to be able to meaningfully inform teaching practice (both our own practice, and that of our readers), then it makes sense to do and discuss research in a way that reflects that practice.

Nadia: All these things made me think there might be some way to make storytelling a main feature of our research method. I was so pleased when you found the work of Clandinin and Connolly and we started reading about narrative inquiry. They express many things that I feel I had been circling around for some time.

Erin: Yeah, I was excited when I found their book, because it connected so well with the conversations we had been having about epistemology and research methods. I read it quickly, and by the time I finished I knew two things. First, that I wanted to try and do a narrative inquiry for my research, and second, that I had no idea how to do a narrative inquiry.

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Narrative inquiry provides a robust framework for thinking about and discussing the lived experiences of ourselves and those of our students. Building on Dewey's notion of continuity, Clandinin and Rosiek (2007) put forward the idea that:

> inquiry, [narrative or otherwise,] is not a search 'behind the veil' of appearances that ends in the identification of an unchanging transcendent reality. Instead, inquiry is an act within a stream of experience that generates new relations that then become a part of future experiences. (p. 41)

The epistemology of narrative inquiry is transactional, not transcendent. The regulative ideal is not to generate an exclusively faithful representation of some reality independent of the knower, but rather to generate a new relation between oneself as inquirer-researcher and one's environment. Narrative inquiry, both as research and as personal experience, consists of trying to formulate the basis on which one imagines one has found meaning or understanding in an experience.

Clandinin and Rosiek (2007) note that in other social sciences and education research traditions, the stories people live and tell are often treated as reflections of important social realities, but not as realities themselves. Narrative inquirers reject this philosophy, which impoverishes experience as a source of knowledge. Inspired by Dewey's ontology of experience, they argue that our experience is "all we ultimately have in which to ground our understanding. And that is all we need" (p. 41). Sharing this ontological view, Bruner (2004) argues that a life as led is inseparable from a life as told; events are not "how they were" but rather how they were interpreted, told and retold.

One might be tempted to view this as a threat to the 'objective status' of research knowledge, but it is an empty one. Geertz (1973) reminds us that, "where an interpretation comes from does not determine where it can be impelled to go" (p. 23). A narrative inquiry presents a subjective interpretation of a particular experience in a particular place at a particular time. The telling of such individual stories, in all their complexity and particularity, has general relevance because "they present the sociological mind with bodied stuff on which to feed" (p. 23). As Brown (1981) remarked, "One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children" (cited in Pimm & Sinclair, 2014, p. 5).

In this thesis, the research findings I present are stories of students behaving (and learning to behave) mathematically. The purpose of these stories is to share these experiences, to invite the reader to share in our experiences, and rethink and reimagine their own experiences and the ways they relate to others:

> Our hope is to create research texts that allow audiences to engage in resonant remembering as they lay their experiences alongside the inquiry experiences, to wonder alongside participants and researchers who were part of the inquiry (Clandinin, 2013, p. 51).

If we imagine research as the study of experience then narrative inquiry is a way of understanding experience. Instead of being generalizable, the results of a narrative inquiry are *appropriable.*

In this section, I have sought to clarify how my decision to use narrative inquiry as a methodological framework for this thesis was motivated by both my own epistemological perspective on research activity, as well as my shared interest with Nadia in storytelling and narrative as a potential research methodology. In the following section, I will review some of the main literature pertaining to this methodology, in order to outline and define the terms and elements of narrative inquiry.

5.2 Methodological Framework: What is Narrative Inquiry?

Clandinin and Connolly (2000) do not offer an exact formula or prescribed way of doing narrative inquiry, instead proposing general guidelines for designing and engaging in narrative inquiry. Drawing on their work, this section will provide a summary of narrative inquiry as a research methodology.

Clandinin (2013) is careful to point out that narrative inquiry as a methodology is distinguished from other forms of narrative research and narrative analysis, as well as from other forms of qualitative research, by a particular ontological view of experience. To use narrative inquiry methodology is to adopt a view of human experience as a *storied phenomenon;* people by nature lead storied lives, and tell stories of those lives. These lived and told stories, both personal and social, are one of the ways in which we fill our lives with meaning. According to Clandinin and Connolly (2000), since human experience is storied in nature, narratives are the best way of representing and understanding our experience. In a narrative inquiry, the term *narrative* describes not only the quality and nature of the phenomenon we hope to study (experience), but it also describes the method of how it should be studied.

Narrative as Phenomenon

Narrative inquiry is "the study of experience as story" and is therefore "first and foremost a way of thinking about experience" (Clandinin and Rosiek, 2007, p. 38). In Chapter 3, I have given an overview of Dewey's philosophy of experience which is centered on two criteria for experience - continuity and interaction. Narrative inquirers adopt these criteria to form the basis of their understanding of experience as a narrative phenomenon.

Human beings both live and tell stories about their living, and these stories are how we interpret our experience in the world and construct personal meaning. Connolly and Clandinin (2006) drew on Dewey's concept of *continuity* when they wrote that "[people] shape their lives by stories of who they and others are and as they interpret their past in terms of these stories" (p. 42). The narrative structures that have characterized our prior experiences will have an effect on present experience. Human beings structure their experiences according to personal narratives, as well as larger cultural, institutional, political, and social narratives, not only in the stories they tell retrospectively, but in the lived immediacy of that experience (Bruner, 2004).

Dewey's second criterion for experience, *interaction*, also provides helpful insight into the understanding of experience as a storied phenomenon. It is not just our own personal stories that will be significant to our experiences. In addition to being an examination of personal experience, narrative inquiry is also "an exploration of the social, cultural and institutional narratives within which individual's experiences are constituted, shaped, expressed and enacted" (Clandinin & Rosiek, 2007, p. 42). Life stories are the results of a convergence of social influences on a person's inner life, social influences of their environment, and their own personal history.

If one views human experience as a storied phenomenon, Clandinin and Connolly (2000) argue that stories must be the best tool for representing and understanding experiences.

Narrative as Methodology

Narrative inquirers begin their inquiries either by engaging with participants through telling stories, or by coming alongside participants in the living of stories. Whether they begin by living or telling stories, they are always entering the field in the midst of ongoing narratives. Participants' stories, inquirers' stories, as well as social, cultural, and institutional stories, are all in progress as the inquiry begins, and are not completed once the inquiry ends (Clandinin & Connolly, 2000). Narrative inquirers enter the field with a particular set of research puzzles, understanding that these questions are not fixed, but must be free to develop throughout the inquiry.

Throughout their time in the field, narrative inquirers collect and generate *field texts* to document their experience, and these form the 'data' of the narrative inquiry. A field text is any piece of writing or artifact reflective of the experiences of researchers and participants. Field texts can have many forms: autobiographical writing, journal writing, letters, conversations, character sketches, interviews, photographs, student work, or poems. Narrative inquirers use the term *field text* rather than *data* to convey that these texts are experiential, intersubjective (not objective) accounts which reveal only "those aspects of the experience that the relationship [between researchers and participants] allows" (Clandinin, 2013, p. 46).

In order to reflect a Deweyan understanding of experience, and to assist narrative inquirers in creating richly detailed field texts, Clandinin and Connolly (2000) suggest that the field texts should be composed along three major dimensions: temporality, sociality, and space.

Temporality (Continuity): All experience is shaped and informed by previous experiences, and has the potential to determine the scope and nature of future experiences. When inquiring narratively into an event, one must consider its past and future: what previous events and experiences are shaping the way inquirers and participants are experiencing the present event? How are these present and past experiences likely to affect future experience?

Sociality (Interaction): Sociality refers to the interactions between internal and external aspects of experience. The goal of attending to sociality is to consider what wider narratives (of family life, culture, institutions, etc.) the participant is relating to, and what role these narratives may be playing in shaping their present experience. For example, one prevalent sociocultural narrative about mathematics is that you are either 'a math person,' who 'gets it' or you are not, and that one's mathematical ability is to some extent fixed and determined. We could ask ourselves whether this narrative appears to be present in the participant's experience.

Place (Situation): Attends to specific concrete physical and topological boundaries to the space in which the experience is taking place. How does this environment shape the experiences that are possible within it? What expectations does the environment create? From my perspective, the environment also includes institutional practices, norms and strategies (discussed in chapter 3).

At some point, the inquirer moves away from the close, intensive contact with participants in the field to begin to work with the field texts and begin to shape them into research texts. In narrative inquiry, there is no linear path from data gathering to data analysis to publishing research findings. Moving from field texts to research texts is a complex, iterative, and often tension-filled process (Clandinin, 2013).

Field texts are created in the field, in the midst of the relational phenomenon under study, and represent fragments of our lived experiences. Over time, layering the many field texts alongside each other, narrative threads, patterns, and themes become visible. By revisiting and re-reading what is often a considerable volume of field texts, the inquirer's task is to discover and construct meaning across an individual's experience.

The transition from field texts to research texts is rarely a linear path, nor is it self-evident. Given the quantity and variety of field texts that are generally created, the research potential is vast and rich. Often, in re-searching and re-storying the field texts, new questions and interpretations will emerge. Narrative inquirers frequently engage in writing a variety of different *interim research texts* – partial texts that are open, allowing both researchers and participants opportunities to "further co-compose storied interpretations and to negotiate the multiplicity of possible interpretations" (Clandinin, 2013, p. 47). Interim research texts seek to make sense of multiple and diverse field texts, and often take the form of narrative retellings of the experiences as they relate to the evolving research problems.

Narrative inquirers cannot seek to separate themselves from the inquiry but rather need to find ways to inquire into participants' experiences, their own experiences, as well as the co-constructed experiences developed through their relationships with one another. Through this iterative process of re-living and re-writing stories, attending closely to the three-dimensional narrative inquiry space, eventually researchers move toward final research texts. Final research texts do not provide answers, since narrative inquiry does not begin with questions. What we present as the product of our inquiries are stories that seek to capture and situate these experiences; stories of particular people in a particular time and place.

The two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in the classroom. In the language used by Dewey, we could say that we are trying to understand *experiences* (our student's and our own). Following the framework of narrative inquiry, I take the position that these experiences are storied in nature. My goal is to construct an account of students' experiences through the personal stories they are living, the storied landscapes in which they are living (institutional, cultural, etc.), and my own stories of the experiences I shared in living alongside them.

Narrative inquiry, as described by Clandinin and Connolly (2000) provides a framework for thinking about and discussing the lived experiences of ourselves and those of our students. In this chapter, I have provided a general description of this methodology. These are the principle ideas and guidelines which constitute the methodological framework of this thesis - they are the implicated in every methodological decision I made throughout this research. In the following chapter, I will provide a detailed account of how I engaged in this narrative inquiry.

CHAPTER 6: Methodology

In this chapter, I provide a description of the research design, data collection, data analysis and reporting methods used in this thesis project. It is important to note that although I used the framework of narrative inquiry as a guiding principle, no two narrative inquiries are conducted in the same way. The details of how relationships are negotiated in the field, how field texts are constructed, reconstructed and analyzed, and how research texts are created, are unique to each researcher, setting, and group of participants.

In each section of this chapter, I focus on a particular phase of this narrative inquiry. In section 6.1 I provide some background information on the three members of our research group, and outline our initial strategies and goals as we began this research. The two main goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to discuss and construct accounts of these behaviours as they emerge in the classroom. With these goals in mind, and our familiarity with the course we would be entering (see Section 4.1), the research group came up with a plan for entering and working in the field, and a set of guidelines for how we would compose field texts, which are also provided in section 6.1.

In section 6.2, I describe the experiences of the research group living in the field and composing field texts. All the members of our research group were new to narrative inquiry, and it took us a bit of time to find our stride when it came to composing field texts. We met weekly to discuss the field texts we had written and situate them with respect to the central questions of the inquiry. As the inquiry progressed, five different narrative topics emerged and grew in our writing and discussion, and I expand on each of them in this section.

In section 6.3, I describe the two ways that student voices were included in this research: through a short questionnaire given to all students in the course, and through interviews I conducted with ten participants. I provide details about the interviews I conducted with students, as well as the interim research texts I composed beforehand, and continued to co-compose with the students during the interviews. I also describe how I began to compile, re-read, and re-write these texts to prepare for writing final research texts.

Lastly, in section 6.4, I discuss how I moved from these field texts and interim texts towards the final research texts which appear in Chapter 7 (Results) of this thesis. The chapter concludes with a brief summary.

6.1 Inquiry Design

> *One of the methodological principles we were taught in quantitative analysis courses was to specify hypotheses to be tested in research. It does not work like that in narrative inquiry. The purposes, and what one is exploring and finds puzzling, change as the research progresses* (Clandinin & Connolly, 2000, p. 73).

The class was held twice weekly, and the members of our research group agreed that we would plan to always arrive 10-15 minutes early so we would have time to rearrange the desks in the classroom into small groups for four to five students. Nadia planned to run the course in much the same way as she had the previous semester, with the majority of the class time devoted to problem solving and discussion in small groups. The design of the course and our style of intervention followed the model Nadia had been using previously in her design experiment (see section 4.1)

As students worked on the problems, the three instructors (myself, Nadia and Genevieve) circulated among the groups and asked them to explain their reasoning or their thinking, why they were doing something, or why they thought a particular method was correct/incorrect. We explained content and gave hints as needed, but avoided providing solutions.

Aside from participating in class we had other responsibilities as instructors. Nadia had taught the same course several times, and as course instructor her role was to prepare the problems for the students to work on and discuss in class, manage and reply to student posting on the online forums, manage the course website, and prepare and mark all assessments (four quizzes and one final exam).

In my role as teaching assistant, it was my responsibility to help manage the online forums and to mark all weekly assignments, providing feedback that would help students improve their ability to express their mathematical thinking,

Genevieve is a doctoral student who was participating in this course as part of a research internship in Mathematics Education. She participated in class discussions and in composing field texts, but had no administrative duties.

Clandinin and Connolly (2000) write that field texts must be regularly and routinely kept, and as richly detailed as possible. As a research group, we made a commitment to write a reflective narrative after each class. The goal of these narratives was to give an account of and to reflect upon our experiences in class that day and our interactions with students. Some guiding questions we considered were: What did students say or do while engaging in the tasks that were part of the day's lesson? Were there any moments where we felt students were behaving mathematically? What makes us feel they were or were not? What do we know about that particular student that may be influencing their behaviour? What kinds of *mathematical behaviours* did you observe in yourself and in the students? What aspects of the task seemed to prompt students to behave mathematically, and why might that be? The goal of these reflective writings was to record, as much as we could, students and their behaviour, everything we noticed.

Student behaviour was not the only thing we planned to write about in these narratives. As narrative inquirers, we knew it was essential to situate ourselves within our writing by reflecting on how we experienced the events of each class. Some guiding questions we considered were: What were you feeling? Why do you think you found certain moments more salient or interesting than others? Why were you drawn to reflect on the experiences of particular students and not others? Were you yourself behaving mathematically at any point? What effect did that have on your interactions with others? If the events of the day called forward any of your own past experiences (as a student, as a teacher, or even in a book or movie you once saw), write about these as well.

In addition to our individual writing tasks, we agreed to meet once per week. We would each read the narratives written by the other members of the research group and then convene to re-tell the stories of the past week, and write additional field notes about our shared and different experiences. The meetings would also be a space to reflect on how the inquiry was progressing, how our puzzles and questions were evolving, and whether we wanted to make any changes to our initial inquiry strategy.

Having a well thought out plan is important, but equally important in a narrative inquiry is to recognize that the plan will likely need to change or shift as the inquiry progresses. In our planning stages, the research group considered several potential sources of additional field texts such as interviews, student questionnaires, or inviting students to compose narratives with us. We knew that at some point in the inquiry we were going to want to include the voices of the students in the stories we were telling, but as we had not yet met the students, we agreed it made sense to wait until a later date to decide how best to do this.

6.2 Composing Field Texts

> *Because field texts are our way of talking about what passes for data in narrative inquiry and because data tend to carry with them the idea of objective interpretation, it is important to note how imbued field texts are with interpretation….We deliberately select some aspects that turn up in field texts. Other aspects, less consciously and deliberately selected, also show up in field texts* (Clandinin & Connolly, 2000, p. 93).

Although we spent a great deal of time thinking about the course and planning and preparing for this research study, the first day of classes still managed to sneak up on me. Suddenly there we were, thrown right into the midst. As I sat down to write a narrative account of that first day I remember being overwhelmed with details, something which is a normal experience for beginning narrative inquirers (Clandinin and Connolly, 2000).

It was very helpful to share our narratives with each other. There were many things I had noticed but not written into my narrative, which either Nadia or Genevieve mentioned.

Having multiple versions of the same day's events provided a richer picture of the experience and allowed us to capture more details. It was clear from our narratives of that first week that Nadia and I had both written narratives from our perspective as teachers, whereas Genevieve had been more successful at tapping into a researcher-angle. Nadia and I both wrote about how the class went, how well we felt it was received by the students, how well our goals as educators appeared to be realized. Genevieve, on the other hand, wrote more about the students themselves, what they were doing and how they were thinking about problems. She wrote about the tendency for students to immediately search for a formula instead of seeking to understand the question as a whole.

Over the course of the semester, the field texts we wrote tended to be descriptive and shaped around particular events. Our weekly writings had a record-keeping quality to them, a retelling of events that occurred and conversations that stuck out as particularly interesting. The weekly meetings gave us a chance to reflect on our writings, to position the events and our reflections within the larger scope of our inquiry and ask questions of meaning and significance.

After about a month, there were five narrative topics that emerged in our field texts: (1) stories about individual students; (2) stories about groups of students as individual entities; (3) reflections on the class as a whole; (4) reflections on ourselves as individuals and as a research group; and (5) reflections on methodology and the goals of our research/inquiry. We had not imposed these five directions at the start of our inquiry, but we found they arose as we lived out the inquiry and composed field texts. Some further detail on each dimension seems appropriate here.

(1) Stories about individual students

Stories about individual students made up the majority of what the research group wrote about in our field texts. We tended to write about moments where students said or did something that we considered to be evidence of mathematical behaviour according to our model. We wrote about students who suddenly had a stroke of insight, an ‘aha!’ moment, and about students who surprised us in their way of thinking about or explaining a problem. We also wrote about students who were having difficulties: those who did not seem able to “buy into” the course, those for whom this course was not ‘working’ and who seemed to be struggling, and those who seemed unwilling to put in the work for this course.

Stories about individual students were usually told from the first person point of view, and went a bit further than recounting simply what was said and done. Occasionally, we would write short bits of remembered dialogue with a student. Usually we would include some reflection on why that particular moment or conversation was interesting/striking, as well as our own interpretations and wonderings.

At our weekly meetings, we would discuss individual students, and over time we got to know them better. As our understanding of their personalities developed and became sharper, we tended to tell stories that brought these characteristics more into focus. Once we identified that a student possessed a particular characteristic or exhibited a given mathematical behaviour, we were more inclined to notice it and write about it in our future field texts. For example, there was one student, Micah[4], who we all agreed was good at playing the skeptic, at always needing a convincing argument. As the semester went on, we

[4] All names are fictional, to preserve the identities of student participants who consented to be part of this study.

wrote more stories that featured and highlighted his skepticism. Another student, Hannah, was often stubborn, and the more we discussed her stubbornness in our group meetings, the more we came to label her behaviour as 'stubborn' or 'overly-persistent' in our writing.

(2) Stories about groups of students as individual entities

Some of the stories we wrote were about our interactions with entire groups as opposed to individual students. We would ask questions and interact with the whole group, drawing ideas forward and asking students to explain themselves to each other. A student would give a possible solution, or ask one of us if their work was correct and we would throw it out to the rest of the group, asking "what does everyone think." We wrote accounts of these moments when we were acting like facilitators to student conversations.

Although students were not assigned to particular groups, they often sat in roughly the same places each class. As time went by and students spent more time working with the same people, some (not all) of the groups developed their own identities. Jonathan's group, for example, developed strong collaborative behaviour when working on problems. They worked together as a team; conjectures were made by members of the group, and tested by others. Everyone participated in the discussions, and made sure that each person understood the problem before moving on to something else. The research group always had the impression that everyone in this group was enjoying working on the problems at hand. Micah's group is a different example. Their group was much larger than the other groups in the class (9 people, as compared to 4 or 5)[5], and so it was rare they were all discussing

[5] This larger group formed on its own. Although the instructor asked students to form groups of 1 to 5 participants, due to the arrangement of the desks and the classroom space, two groups quickly merged into one larger group. This larger group continued to work all together for the rest of the semester.

problems together, but instead worked in smaller subgroups, and then compared their work afterwards.

(3) Reflections on the class as a whole

These reflections usually took the form of discussing how the activity was received by the students, and other general notes about class that day. We attended to the physical space of the classroom as well as to our sense of the prevailing mood. Did students seem anxious (as they did on the day of the first quiz)? Were they excited and engaged in the activity? There were some days where all three of us agreed that students seemed to enjoy the problems and were engaged, and other days when many seemed less interested in the task at hand.

We also wrote about the particular problems we were working on that day and how they were received by the students. Because neither Genevieve nor I had seen the problems before they were given to the students in class, we would write about what we felt the intended purpose of the activity was and how effective it was at engaging students in what we believed it was meant to. Nadia would also write about her perception of the effectiveness of the activities, but from a different perspective since she was the one who had chosen them with the intention of provoking particular types of mathematical behaviour.

There were some classes where the majority of students seemed to be encountering the same difficulty or using the same incorrect approach. Sometimes it was because the activity in question was designed so that they were meant to encounter this difficulty and grapple with it, other times it was not foreseen. If most of the students in the class had the same response to a problem, whether it was an approach to solving it or a difficulty they encountered, this was noted and reflected in our narratives, and often helped to lend context to the accounts of individual behaviour.

(4) Reflections on ourselves as individuals and as a research group

> *When narrative inquirers are in the field, they are never there as disembodied recorders of someone else's experience. They too are having an experience, the experience of the inquiry that entails the experience they set out to explore* (Clandinin & Connolly, 2000, pg. 81).

The phrase *experiencing an experience* is a reminder that narrative inquiry is aimed at understanding and making meaning out of lived experiences. It is important to recognize that although we wrote primarily about students and events in the class, our own experiences were still the central story. All of the field texts we wrote contained our own interpretation of the events we were describing, as well as reflections on how we were positioning ourselves in the classroom and in this research. Part of our work as narrative inquirers was to examine our own behaviour and assumptions, to ask why we had interpreted events in a particular way, and whether other interpretations were possible.

Throughout the inquiry, each member of our research group took on many different roles and engaged in different relationship dynamics with each other and with the students in the class – teaching assistant, researcher, instructor, student/academic supervisor, friends, research colleagues. In life, we switch easily between our many different roles without really noticing, but one of our tasks as narrative inquirers was to pay attention to which of these positions we were adopting at any given moment both in the lived immediacy of our experiences, and in our written reflections on those experiences.

We wrote about our past experiences whenever we were reminded of them by something which happened in class. Past experiences always shape/influence how we experience and interpret things in the present. We sometimes told our own remembered stories of when we were students, comparing our past experiences to the ones our students were having in the present. We also wrote about previous experiences as teachers, Nadia in particular because she had taught this course before. There were certain student behaviours or difficulties which were familiar, which we felt we had seen or experienced in other teaching situations, so we wrote texts which wove these past and present experiences together.

We reflected on our interactions with students, wondering if we had really understood what they were saying/thinking, or had been clearly understood by them. When she spent time lecturing to the whole class, Nadia wrote about her feeling of being less able to tell if the students understood what she was saying, as compared to when she spoke with them one-on-one. Genevieve and I both experienced moments where we felt we had said the 'wrong' thing to a student or group of students, that we directed them too strongly to follow a particular way of thinking, rather than letting them explore their own ideas first. We reflected on why we had done or said particular things and speculated as to how these actions may influence the students' experiences.

We wrote about our own state of mind on each day, and how that may have had an influence on our interactions with students. If I was running late, for example, or feeling stressed in another aspect of my life, this was certain to affect how I experienced the class on that day. We tried to keep track of internal factors such as our moods, as well as external factors such as really terrible or beautiful weather, which could influence our students' behaviour or our own.

Finally, we also wrote about ourselves as researchers, examining our own research activities. In one of her field texts, for example, Nadia wrote about how Genevieve's questions help keep her on her toes, help her to refocus on the aims of this inquiry, and raise critical questions for her about the amount of detail included in her field texts. In her reflection, Nadia appreciated Genevieve's criticism, while at the same time wondering if Genevieve was comfortable criticizing her. Through this type of reflection, Nadia made herself (as a researcher) visible in her writing.

(5) Reflections on methodology and research questions.

The discussions we had in our weekly meetings had the goal of bringing meaning and significance to our field texts of that week, and of trying to make connections between our experiences and the model of mathematical behaviour. We examined each other's writing and through our discussions we generated questions which usually became topics of focus for the following week. In one meeting, for example, Nadia pointed out that all three of us had been using the terms *weak* and *strong* to describe student behaviour, but that it was possible we meant different things by them. We subsequently agreed to make characterizing strong and weak students the focus of our inquiry for the next week. In our field texts, when we wrote about particular students, we tried to specify the reasons why we felt they were weak or strong, looking for (and trying to articulate) the general criteria by which we were making such a determination.

In another instance, Genevieve wrote in her field text that she would like to spend more time sitting with a single group during a class, in order to observe and understand in more detail how they interact with each other and with the problems. When she was circulating, she had the sense of only ever seeing snippets and portions of student activity, and wanted an opportunity to observe students engaging in problems from start to finish. She also suggested that, in order to bring more specificity to the categories of our model, we might try to begin each class with a particular behaviour from our model in mind and then try to observe it in the students and describe it in our writing. We selected which students we wanted to observe more closely based on who we felt exemplified the behaviour we wanted to write about. For the following four classes, Genevieve and I sat with a single group for the entire class, while Nadia continued to circulate as before.

Engaging in weekly dialogues around our field texts gave us the opportunity to create connections between our experiences and to continually re-examine our purposes. We would sometimes decide to make changes to how we were engaging in the inquiry space, adjusting the way we positioned ourselves in the classroom, both mentally and physically. Adopting these different positions allowed different aspects of mathematical behaviour to become visible to us.

I wrote about our weekly meetings in my narratives; what questions we discussed, what important ideas came up, and how these were likely to be incorporated into the inquiry going forward. Genevieve and Nadia also occasionally wrote reflections about the meetings, but not every week. These pieces of writing are of a different character than the other field texts I wrote during the semester, since they are written with a bit more distance from the events themselves (that is, student behaviour and being in the classroom), and draw connections between our different field texts.

The reflections I wrote about our meetings are actually what Clandinin and Connolly (2000) would call *interim research texts:* texts which seek to make sense of multiple field texts, to make connections and give meaning to multiple experiences as they relate to the evolving research problems.

6.3 Adding Student Voices

> *The dialogue around interim research texts can lead the inquirer back for more intensive work with the participants if more field texts are needed to be able to compose research texts that researcher and participants see as authentic and compelling* (Clandinin, 2013, p. 47).

How we interpret and are interpreted by others will shape the stories we tell about them. Over the course of the semester, as we developed relationships with the students and discussed those experiences with one another in the research group, we started to label students with characteristics. Early on, we identified that Micah is often skeptic, and so stories featuring him being skeptical got told and written about more often in our weekly field texts. We created, in our discussions and in our texts, a skeptical character whom we called Micah.

As we got to know individual students better and better, we became curious to know more about how they perceive the course and their own behaviour. What do they think the aims are? What do they remember from class as important moments? What do they believe it means to *think mathematically?*

The research group discussed these questions during one of our meetings and decided that we needed to have students contribute to our growing volume of field text. There were two times where we had students engage in composing field texts: once as a reflection assignment within the course (all students were expected to participate), and a second time through an interview (only some students were selected to participate). The reflection assignment was given in the 8th week of the course, and the interviews were conducted during the last week of the course.

(1) Reflection Assignment

With the goal of getting a better sense of how students experience their own mathematics and mathematical behaviour, we gave the following five questions as one of their weekly assignments:

1. What is mathematics (in your opinion)? What is a mathematician?

2. What are your strengths and weaknesses when it comes to mathematics? Which aspects of doing math do you enjoy/dislike?

3. Think of the other students in your group/class. Which ones do you think are strong (or weak) in doing math and why?

4. What is the purpose of this course? What do you feel you are supposed to be learning? What are you actually learning?

5. What parts of this course (class discussions, forum, assignments, and quizzes) do you find the most helpful/interesting/challenging/boring? Why?

Only a few students did not complete the assignment, but it was very interesting to read the responses of those students who took the time to answer thoughtfully. In the final drafting of my research texts, I did not make much use of the responses we gathered. Mostly this exercise served to help us understand the students' perspectives a bit better and to identify which students were more able to reflect on their own mathematical behaviour and might be willing to engage in further conversations about it. When it came time to select students for the interviews, we considered, among other factors, those whose answers to the reflection assignment had been most interesting to us.

(2) Student Interviews

As the semester drew to a close, I was envisioning the final research product of this thesis as a set of short stories (vignettes) organized around the different mathematical behaviours in the model. My goal was to be able to illustrate the different behaviours through these brief accounts of students in the class. I had a vision of creating a set of fictional characters based on some of the students in the class, with each character representing and acting out one of the behaviours in the model. However, I felt I needed more details about the individual students in order to be able to create rich characters.

Working with Nadia and Genevieve, we chose 10 students from the class to interview. We chose students who we felt had shown evidence of the different behaviours over the course of the semester. Each student had displayed some combination of the different behaviours in the model, and together the ten chosen for interviewing covered all the aspects I planned to write about in my final research texts. Because we felt it would make for more interesting and revealing interviews, our choice was also based on which students I felt I had developed the best rapport with over the course of the semester.

I prepared for the interviews by re-reading all of the narratives that Nadia, Genevieve and I had written over the course of the semester. For each student I was planning to interview, I would re-read every storied account that we had written about them, as well as their responses to the reflection assignment (see above). After getting a good sense of what a student's experience/trajectory through the course had been, I composed a short character sketch. I tried to create a short synopsis of their time in the class based on what I had come to know about them through both their work and my experiences working on and discussing mathematical problems with them.

These documents were interim research texts; drawing meaning and detail from the field texts we had written, I wrote storied accounts of particular students, told entirely from my perceptions. Because the field texts and these interim texts I had written arose "from the relational experience that is co-constructed between researcher and participants, they must return to that experience for validation" (Clandinin & Murphy, 2009, pg. 600). Thus, the goal of the interview was to have conversations with the students about their experiences in the classroom and to share with them parts of the character sketches I had written. Together we would re-tell these stories and co-compose new field texts.

Interviews were conducted in an unused office in the mathematics department, were audio recorded, and each lasted between 1 and 1.5 hours. Although I had a set of questions prepared for each interview, I allowed the conversation to be flexible, and tried to let students tell their own stories in their own words as much as possible. In some cases, students were particularly engaged in discussing previous math courses, or their family history, and so we would talk about that. In one case a student wanted to discuss a proof he had written for a problem we had been working on together in class.

For the most part, the things students told me about themselves resonated with the perception I had of them, with the notable exception of Micah. We discussed what they enjoyed or disliked about learning math in general and about the way they were learning in MAST 217 in particular. In all cases, I gained a better understanding of who each student was as a person, their character traits, and how their previous experiences may have shaped their experiences in our course.

In the week after completing all 10 interviews, I listened to the recordings and made a set of notes (field texts) about the experience. For each student I interviewed, I wrote a brief account of my own experience/impressions of the interview, and then made note of all the information/details I had learned about each person. Wherever possible, I tried to connect what they had said to some aspects of the mathematical behaviour model I had constructed (chapter 4). I grouped the different discussion points together according to categories of the model, and included my own reflections and commentary about the discussion.

These reflections were more interim research texts, and although at the time I was still not certain how they would be incorporated into the final research texts, it was helpful to be able to create a richer picture of these individual students, of their past present and possible future selves. Experiences are the stories people live, and in a narrative inquiry, the researcher seeks to understand and give meaning to experiences by taking into account all of the different stories a person is living. The interviews and the reflections I wrote about them were essential to understanding more about each individual student, and were the last interim research texts I wrote before moving towards final research texts.

6.4 From Field Texts to Research Text

> *There is no clear path to follow that works for every inquiry… The doubt and uncertainty are lived out in endless false starts. As we begin to write interim and final research texts, we may try out one kind of research text and find it does not capture the meanings we had in mind, find it lifeless and lacking in the spirit we wish to portray, find that research participants do not feel the text captures their experience, or find the research text to be inappropriate to our intended audience. We try out other kinds and continually compose texts until we find ones that work for us and for our purposes* (Clandinin & Connolly, 2000, pp. 134-135).

After the interviews were completed came the difficult task of moving towards a research text. Clandinin and Connolly (2000) write that the transition from field texts to research texts is always challenging and time-consuming. In any narrative inquiry of reasonable scope, the volume of field texts that are produced can be overwhelming, but before coming to the question of what to do with all these field texts, one needs to know what there is. I re-read all of the field texts we had written, reviewed the course notes, looked at student submissions to the forum and their assignments, and re-visited my notes and comments on the interviews I had conducted.

I had some idea that I wanted to write short stories, possibly small vignettes organized around particular aspects of mathematical behaviour. My first attempt was to write stories about particular days. I tried writing narrative accounts of the class time, what problems we worked on and what students said and did, followed by a fictionalized conversation between the three members of the research group discussing the behaviours we had seen that day. I wanted to be able to present the lived experience of class time alongside some texts which would expand on the meaning and significance we saw in those experiences and situate them within the model of mathematical behaviour.

But these texts did not feel quite right as final research texts. The texts were too scattered, and moved from one idea to another without any smooth, unifying theme or transition. They were very similar in their format and style to our field notes, and I found that they were too descriptive, that there were too many distracting details which acted like noise, obscuring instead of clarifying the moments I wanted to focus on and discuss. Clandinin and Connolly (2000) write that it is normal to have 'false starts' when writing research texts, that it may take a few attempts in order to arrive at a research text that hopefully allows/enables the reader to find in the stories of these experiences the same meaning that I have found.

At our weekly research meetings, our conversations invariably centered on individual students. We would discuss what they had done or said, how they interacted with others in their groups, what we knew about their background and how this could be shaping their experience. As I was reviewing all the field texts we had written, I realized I had already constructed interim research texts in the form of writing about these discussions, and in my reflections about the student interviews. Since so much of the reflection we had done during the semester seemed to be organized around individual students, I decided to try writing stories in this way. I focused on each of the students I had interviewed and wrote short vignettes about them, trying to capture moments when they did or said something that struck me (or another member of the research group) as being mathematical.

This approach fit better. By focusing on the individual students - who are they, why are they doing what they are doing - I was able to give meaning and social significance to their actions and to the stories about them. I knew from the outset of this project that my model was only a general description of mathematical behaviours and that the particular ways these behaviours were enacted and lived out by students were always unique, individual. This format seemed better suited to highlighting these individual differences.

In composing my final research text, I wrote stories about 10 students, 9 of whom I had interviewed, and one who I had not. I added stories about Donnie, who I had not interviewed, because we spent so much time talking about him and because he represents one of the biggest problems/difficulties in this course. We felt there was something about his case that was worth considering more deeply. I chose not to interview Donnie because he is an example of a lack of mathematical behaviour (or, at least, he seems to lack *the right kind* of mathematical behaviour - more on this in chapter 8). At the time when I was setting up my interviews, we chose students based on who we felt exhibited mathematical behaviours, so Donnie was not chosen. Reflecting on it now, I feel it was right not to interview him, since it might have stressed him out or made him uncomfortable. It is one thing to ask students about how they think and behave when they are having success, but I could not imagine having a conversation with Donnie where I asked him, "Why do you think you did not succeed in this course?" It is important to note that my stories about Donnie are positioned differently than my stories about other students, since I did not have the opportunity to discuss his behaviour with him in an interview. My stories about Donnie are constructed exclusively from our interpretations of his behaviour.

I also decided not to include stories about one of the students who I did interview, Jessica. It is harder to articulate my reasoning for this decision; I think I felt like there was something missing there, in terms of individual mathematical behaviour. For whatever reason it was not as interesting to me, I did not feel that her story added anything to my understanding of student behaviour. We had chosen to interview her because Genevieve grew close with her and often sat with their group, and so she was someone who regularly appeared as a character in our stories. But when we met for an interview, I could not seem to reflect with her on her own behaviour in quite the same way as I had with the other students.

After writing this set of short vignettes that featured the different students behaving mathematically in their particular ways, there still seemed to be a piece of the puzzle missing. I felt these stories had accurately captured the behaviours I wanted to portray, but still needed some vulgarization of what I felt these stories represented. I needed to convey to the reader why these stories were chosen, what I saw in them, before I could embark on any discussions or conclusions.

One of my reasons for using narrative inquiry in this research was to foreground the lived immediacy of experiences. In traditional research formats, data is often presented separately from data analysis. When reading data analysis, the reader often finds they must flip back to a previous section to recall the facts or events that are being referred to. I wanted to present my results in a way that would make it possible for the reader to follow my thinking, and experience the connections I was making right alongside me.

I tried several formats, each with its own benefits and limitations, before eventually deciding to present the stories as they appear in the following chapter, with comments embedded throughout to highlight the mathematical behaviour I observed in students.

Summary

In this chapter, I have provided a description of the design and implementation of this narrative inquiry. The two main goals of this thesis were to construct a characterization of mathematical behaviours and to explore a methodological approach that allows us to create accounts of these behaviours as they emerge in the classroom. To address these goals I, along with my research group, designed and carried out a narrative inquiry in the undergraduate course *Introduction to Mathematical Thinking.*

Each member of the research group attended every class and wrote narrative reflections (field texts) about their experience in class that day. We met once a week to discuss the themes and questions that came up in these field texts and reflect on how they were connected to our larger research questions. As the semester progressed, we continued to refine our characterizations of mathematical behaviours through our writing and reflection. In addition to discussing theoretical questions, we also spoke in our meetings about individual students. We would discuss our interactions with them, who we felt was strong or weak and why, who seemed well-suited to the course and who did not. This interest in understanding individual students led to interviews, the purpose of which were to inquire into “the continuity and wholeness” of each student’s experiences in the classroom” (Clandinin and Connolly, 2000, p. 17).

The final products of this inquiry are the stories presented in the next chapter. Each set of stories focuses on a particular student from the course and tries to capture their experiences with behaving (and learning to behave) mathematically. The purpose of these stories is to invite the reader to share in the experiences we reflected on as a research group, and to perhaps rethink and reimagine their own experiences as a result.

CHAPTER 7: Results

> *To restore the human subject at the centre … we must deepen a case history to a narrative or tale; only then do we have a 'who' as well as a 'what'.*
> *- Oliver Sacks, 1985, viii*

My presentation of results involves ten sets of short narratives, with each set focused on the experiences of a particular student in the course. The ultimate goal of the final research texts presented in this chapter is twofold: to communicate the nature of students' experiences as I perceive them in my own experience and in the narratives written by the research group; and to highlight the elements of mathematical behaviour that become visible (to me, and to the research group) through the telling of these stories. In order to construct these tales of student's experiences, I drew on my own perceptions and experiences of the events, as well as the narratives the research group wrote about the students. I also drew on the insights I gained through my interviews with the students about their experiences, and their own descriptions of their experiences.

The goal of the course was to offer opportunities for students to engage in mathematical behaviour. From our perspective, the only way we, as instructors, can know if we gave an opportunity is if a student takes it (see section 2.4, p. 33). In presenting these storied accounts of students' behaviour, I wish to make clear what I, both as an instructor and as a researcher, take as evidence that a student has taken an opportunity. My central purpose in using narrative inquiry in this research is to foreground the lived immediacy of experiences. In the storied accounts I present in this chapter, I wanted to be able to point to particular moments, particular behaviours, and say, "this is what I noticed, here." Thus, the stories I present in this chapter contain comments and notes embedded in the text of the story. These are meant to encourage the reader to notice what I noticed, and to (hopefully) construct a similar meaning from the stories as I have. The result is a layered text which presents, at the same time, stories of students engaging in mathematical behaviour, and my own experiences and perceptions of those students as a teacher and a researcher.

Some stories revolve around a particular problem or activity that students were working on in class, and where this is the case I include (in blue) the activity that was given at the start of that class. There are three different types of comments which appear in the narratives presented in this chapter:

Comments in the text (yellow) are intended to:

- Highlight/draw reader attention to specific examples of what I take as evidence that a student is behaving (or not behaving) mathematically. Where possible, I label the features of mathematical behaviour according to the categories/terminology presented in chapter four. These labels are in **bold** text.
- Explore and examine how a given behaviour or action is (or may be) connected to the student's personality and their previous experiences.

Other comments in the text (green) provide:

- Comments on my own actions in the stories which reveal what my intentions and/or feelings were in a particular moment.
- Comments on particular moments that raised questions and/or were important sites of discussion within the research group.

Comments that appear following a problem (within a box with a white background) that was posed in class provide:

- Explanations of the intended purpose of the activities that Nadia had in mind when selecting them, that is, the opportunity she imagines/believes she is presenting to students by giving them these problems to work on.

For the most part, the comments in a given set of stories focus on the actions of the student they are about, however it is occasionally possible to see evidence of other students behaving mathematically as they interact with the main character in the story. On a couple occasions, comments about one student appear in the section of stories dedicated to another student

7.1 Storied Accounts of Student Experience

HANNAH

Activity 20

1. The barber is a man in town who shaves all those, and only those, men in town who do not shave themselves. Who shaves the barber?

2. Let us call a set "abnormal" if it is a member of itself, and "normal" otherwise. For example, take the set of all squares in the plane. That set is not itself a square in the plane, and therefore is not a member of the set of all squares in the plane. So it is "normal". On the other hand, if we take the complementary set that contains all non-squares in the plane, that set is itself not a square in the plane and so should be one of its own members as it is a non-square in the plane. It is "abnormal". Now we consider the set of all normal sets, R. Is R normal or abnormal?

3. Suppose that every public library has to compile a catalogue of all its books. Since the catalogue is itself one of the library's books, some librarians include it in the catalogue for completeness; while others leave it out as it being one of the library's books is self-evident. Now imagine that all these catalogues are sent to the national library. Some of them include themselves in their listings, others do not. The national librarian compiles two master catalogues—one of all the catalogues that list themselves, and one of all those that don't. The question is: should these catalogues list themselves?

These activities introduce students to the notion of paradoxes. Do they understand how the paradox arises? Are they able to recognize the similarities between these three questions?

As soon as Nadia puts this activity up, I know already that Hannah will be thrilled. Given her passion for logic and philosophy, Hannah will certainly have seen this (and Russell's paradox) in a previous course, and I have no doubt she will be pleased that it is our topic today. She always enjoys an opportunity to discuss (and debate) with her peers, and I can hear her dominating the conversation in their group right from the beginning, confidently explaining how the paradox arises.

I am glad she is confident, but the others at her table are not saying much, and I do not have the sense they fully understand her rapid explanations. When Hannah feels she understands something well, she wants to explain it to others. Some of her peers find this very helpful, but others feel she has a tendency to steamroll the conversation.

"An interesting question was asked at another table," I interject, trying to open up the conversation to the group, "as to whether the barber question is a math problem."

"It's a logic problem," Hannah responds in her brusque, decisive manner.

"Okay, sure, but I asked if it is a math problem," I say, looking towards the others in the group.

Hannah answers immediately that math is just calculations and manipulating numbers and symbols. Everything else, including problem solving, deduction, thinking, is logic. "To me, this course is not a math course, it's a Logic course," she replies.

I am quite surprised when she says this, as is Genevieve who has now joined our conversation. Hannah explains that this class is more like her previous logic courses than her previous math courses (and is therefore, logically, a logic course). Her previous experiences have led her to believe that mathematics is just about finding the right answer, doing calculations over and over. I ask her if that is the job of a mathematician: to find the right answers and solve similar problems over and over again. She says, "I don't know. What does a mathematician even do?"

I remember reading in one of Nadia's narratives from earlier in the semester that Hannah and Judy had spent several hours trying to answer this question during one of their study sessions, searching and reading articles online. Nadia writes:

I sat on the floor and we talked. They said that they reflected on what it means to be a historian (someone who studies history but not necessarily someone who makes history); is a mathematician someone who knows or studies math or is he or she required to do and create math (prove a new theorem)? I mentioned that someone who studies and knows a lot of math and tries to prove new theorems may never succeed. I wonder (ask them) whether we should consider that person a mathematician or not. They also said they reflected on what it means to be a poet. Someone can be a specialist in poetry but that doesn't make him or her a poet. But what makes you a poet (I asked)? Who gets to decide that your lines are a poem and therefore you are a poet? Who gets to decide that your efforts in proving something in math grant you the title of mathematician? If you prove a theorem in complete isolation and don't share it with others, are you a mathematician –equivalently (they said), are you a poet?

> *Hannah is a* ***reflective thinker****, in the sense that she enjoys "thinking for the sake of thinking." She has a* ***questioning attitude,*** *and often thinks critically about whatever task is at hand. Her default mode is to ask questions; in this case, she is asking questions about the nature of mathematical activity and mathematical thought.*
>
> *It is also interesting that the questions Hannah raised in this conversation were ones that we had asked ourselves many times over the course of this research. What makes someone a member of the mathematical institution? Do we consider ourselves mathematicians? Is there more than one way to do mathematics?*

*****

Hannah is a philosophy student and is everything I imagine a philosopher should be – clever, frenetic, and intense, with a mind that thinks in categories and is persistent to the point of stubbornness. As soon as she has an idea, she will begin talking quickly to the others at her table. Sometimes they understand what she is saying and they can all work together on the problem, other times they are lost in the flurry of her words, in which case she will usually ignore them and work on her own. "It's difficult for me to just drop something when I've started it," she tells me, "I just get absorbed and need to completely exhaust the possibility. When I get stubborn, it is because I have a firm grip on the ideas I'm dealing with, and when you have a firm grip, you really don't want to let go."

To me, it seems that stubbornness is both strength and weakness for Hannah. She considers her persistence a great asset, even though it can take her some time to realize if her approach is not working. Her tendency to focus on her first idea or instinct about a problem can sometimes make it hard for her to understand other approaches.

*Hannah is very **persistent**. Her desire to work on something until she has exhausted it as a possibility is very mathematician-like. But she sometimes lacks the flexibility/**self-regulation** skills to change directions when an approach is not working.*

"Sometimes it gets frustrating when everyone at the table understands something [and I don't], and I think it's because I learn a little bit differently, their explanations are just not sufficient for me to get it. And that can get really frustrating. Like, when Akara explains things to me I just never get it. There's not been one time where she tells me something and I understand it, but the thing is that she gets things right. So yeah, I get impatient and frustrated, but I really hold it in because it's not her fault."

The lack of communication doesn't seem to go both ways: Hannah feels like she can communicate her ideas to Akara, and that Akara can understand her, just not the other way around.

*Hannah is a **collaborator** in the sense that she enjoys discussing her ideas with others in the class, and likely benefits from stating them aloud. She enjoys explaining her thinking to others, often taking on a **peer tutoring** role, but she has difficulty understanding how other people think. When Hannah is involved, collaboration seems to occur in only one direction: she is able to offer her insights to others who often benefit from them, but she is rarely able to benefit from the experiences and skills of her peers.*

*****

Activity 8

Prove the following statements are true:

1. $(\exists\, x \in \mathbb{R})\ x \notin \mathbb{Q}$
2. $(\exists\, A \subseteq \mathbb{R})$ A is infinite and A^c is infinite

Prove that the following statements are false:

1. $(\forall n \in \mathbb{N})$ n is divisible by 2
2. $(\forall n \in \mathbb{N})$ n is divisible by 2 or by 3
3. $(\forall n \in \mathbb{N})(\exists\, m \in \mathbb{N})\ m^2 = n$
4. $(\forall m, n \in \mathbb{N})$ if m + n is even, then m and n are both even

This activity, done early in the semester, is meant to introduce students to logical quantifiers and their symbolic representations (∀ and ∃). *It is also one of the first activities where students are asked to* ***prove*** *something, and is meant to engage them in the question of what counts as sufficient evidence that a statement is true or false.*

Hannah is familiar with statements like these from her previous courses in logic, and is able to quickly explain them to Judy, who seems to understand quite readily. Each is a statement that can be proven (or disproven) by finding a single example (or counterexample), and they are able to complete the task quite quickly. From a nearby table, Nadia overhears Judy saying, "Okay, so, if question 4 is reversed, then it is true. If m and n are even, then m + n is also even."

Once they are finished with the task at hand, both Judy and Hannah are being ***reflective thinkers*** *when they continue to explore the problem, investigate other possibilities, and ask themselves more questions.*

Intrigued by their discussion, Nadia walks over to join them and asks if this new statement can be proven by example. They both agree that it can't, so Nadia asks them if they can prove it, which they both think about for a moment.

"Well, if both m and n are even, then they are multiples of 2, so their sum will also be a multiple of two," Hannah says, sketching out a quick proof.

*****

Outside of class, Hannah works as a math tutor. She feels she offers something different than most math tutors because she takes what she calls a "philosophical approach." For her this means putting a focus on conceptual understanding and explaining why things work, rather than simply telling students how to get answers.

We once had an interesting conversation about understanding why things work versus understanding how to do a technique: She feels she cannot teach something unless she understands it in depth (understands why). As a tutor, she sometimes tries to teach by explaining the definition (of, say, a derivative) and then showing students how to find it. However, this often does not work for students, so she will teach the mechanical technique first and once they have an idea of the how-to, she will explain the related mathematical concepts in order to "show them what they have been doing."

> *The question of what to teach first - theory or practice - is a perennial dilemma in mathematics education; one which we discussed many times in our research group. It is interesting that this has come up for Hannah in her experiences as a tutor, and that she has developed her own teaching strategy based on how she learns best. Several of the students I interviewed have jobs as math tutors (Jonathan, Judy, Hannah), and hearing them explain how they teach math to their students revealed some interesting aspects of how they approach learning.*

She teaches this way because it is how she learned many concepts in math, but she describes this as an 'unpleasant' learning process: "Nobody likes to do mechanical work and not understand what's going on behind the scenes but, for me at least, it's necessary, and then when I'm ready, I can learn what's actually going on." Understanding the foundations and being able to explain the "why" are very important to her. She has a hard time trying to learn a concept by herself (using, say, a textbook), because she will become fixated on understanding the conceptual before she has mastered the practical. She will sometimes try to follow this path, to learn a concept by beginning with the definition and moving towards practical applications, but she has learned that she needs to work in the other direction.

> *Hannah **reflects on her own thinking**, and tries to identify what aspects of her previous learning experiences are helpful and which are not. She understands the way she learns best, and seems to be able to recognize her own strengths and weaknesses.*

*****

It is near the end of the semester, and students have been told that only the best quiz grade will be counted, so for the last quiz, Nadia offers the students a choice. Those who wish to take the quiz may do so, and those who are satisfied with their marks up to this point can work on other, slightly more challenging questions.

Hannah has done very well on the quizzes, but Judy, whom she has sat with and worked with all semester, has not gotten a mark above 80% and wants to do better. Hannah offers to help her and another student at their table with their quizzes. Nadia suggests that Alan, a student from one of the other tables, should join them as well, since everyone else at his table has chosen not to write the quiz. Soon, Hannah is carefully explaining the questions to these three other students, even though she herself is not planning to write the quiz. When the time comes for them to stop discussing and start writing their answers, Hannah wishes them good luck and then moves to another table where they are discussing the other problems.

"I decided to do it to help them out," she says, "I wanted to help Judy and the others to get a better mark, if I could. In a situation like that, where I feel relied upon, I work a lot better, and I actually become smarter than I normally am. When I feel needed, it's a really good thing."

> ***Collaboration*** *is a part of the mathematical culture and although it is something of a one-way street in her case, it is clear that Hannah enjoys working and discussing with others. Hannah feels she benefits from working with her peers – that it gives her a sense of purpose and increases her ability to focus. And it is clear that many of her peers benefit from working with her, as well.*

Judy ended up getting 100% on that quiz, and both of the other students did better than they had up to that point. Hannah feels proud that she was able to help her friend do well. Judy's success is also her success.

AKARA

Akara is very good at being able to come up with a variety of ideas to tackle a problem and choose which one is most likely to work. She can't really explain how she knows, but most of the time she can guess the right way to solve a problem, and she has learned to trust this intuition. When she has a 'feeling' that it is working or going somewhere, she will not give up. If she does not have that feeling, she will look for other ways. It is difficult for her to put into words where the feeling comes from. "I can't explain it," she says, "I don't know how to say it, but sometimes I just know if it is working or not."

This is from a conversation Akara and I had in our interview. Genevieve, Nadia and I had all noticed that Akara has a strong ***intuition –*** *an ability to guess what is possible or likely to work for a given problem. In the way she describes her approach to problem solving, she makes reference to this intuition as well as to her ability to* ***self-regulate.*** *If something is not working or does not seem to be leading in a fruitful direction, she will re-evaluate her strategy.*

Akara knows that she is a persistent person; she likes to take on challenges. She can be stubborn, but only when she feels she has the right idea.

*****

Activity 10

Prove, by contrapositive, the statement: ($\forall$n $\in \mathbb{N}$) if n^2 is even then n is even.
Now try to prove it by direct proof.

In the activity for today, we ask students to attempt to prove a statement by contrapositive and by direct proof. The main point of the activity is for them to see how, in this case, the contrapositive proof was much easier to construct.

Akara had understood that this was the objective, but is still curious to know if the direct proof is possible. I tell her that it can be done, but it is not as straightforward as the proof by contrapositive. She decides to try the direct proof, and quickly becomes fixated on trying to write one. Although the others in her group move on to the next activity, she continues to work on the direct proof for the rest of the class. At one point, I give her a hint that she might need to use the uniqueness of prime factorization.

> *Akara demonstrated remarkable **persistence** when working on this problem. Once she knew that a solution was possible, she would not be satisfied until she figured it out.*

Akara hangs back after class, wanting to discuss how she thought she could write a proof for n^2 even → n even. She has not written out the whole proof, just a sketch of her ideas. It was slightly different than how I would have imagined the proof, but as far as I could tell her ideas made sense.

"Yes, I think this works," I say, and then add, "but there are details you would need to fill in to make it a complete, proper proof. But it looks like you have the right ideas."

"I got it?!? Really?" she exclaims, the words bursting out of her. She had been grappling with this proof for most of the class, and she is excited, thankful, and pleased with herself to have found a way forward.

> *Akara's reaction is an example of what we have called **personal and emotional connectedness**; her mathematical activity is driven by curiosity and the resultant pleasure when something was resolved. I could tell she was **personally invested** in the problem because of the **persistence** she showed when working on it, and the pleasure that radiated from her when she was successful.*

*****

At the beginning of the semester, Akara was enrolled in the Major in Mathematics and Statistics program, and at Nadia's suggestion/encouragement she made an appointment to see an advisor about switching to the specialization[6]. At this meeting, she also asked about adding a minor in Philosophy. She enjoyed the philosophy course she had taken while studying in California, and could see the similarities between philosophy and how she thinks about math. After talking with Hannah, who is studying in both math and philosophy, she was interested in exploring this possibility.

She was very discouraged when the advisor replied, "Why would you want to minor in Philosophy, it doesn't help, you should minor in Economics."

He told her that a degree in economics is better for finding a job, and asked her what she wants to do after she graduates. She told him that she really loves math and might like to go all the way to doing a PhD in Mathematics. He replied, "Oh no, don't set your goal too high. We don't know how far you can go so just try to finish a Bachelor first, and maybe you can get a job after."

I felt enraged when I heard this story, convinced that this older white male academic would have responded very differently if a male student told him they were interested in one day

[6] This is something that Nadia often does when she instructs this course. Sometimes students are enrolled in the Major because they do not know about the Specialization program. Every semester she has taught the course, she has recruited several (very good) students for the Specialization. She has a sense that it is easier to see and recognize a student's potential in this course, a course where they get to do mathematics, as opposed to a 'traditional' approach, where they are only mimicking math techniques

pursuing a PhD. I ask her, in our interview, what she thought of this advice. She replies, "Maybe it's because I'm an immigrant, the way I speak English, he thinks I can't do it."

Nadia ran into Akara shortly after her discouraging meeting with the advisor and promptly gave Akara the exact opposite advice, telling her to take as much math as she was interested in. After discussing it with her husband, she decided to continue working towards what she wants. She told me, "I want to find myself and find out what I want. I want to study what I am passionate for."

> *Akara enjoys math, describes it as something she is passionate about. Like many mathematicians, she feels a* ***personal connection*** *to mathematics.*

*****

Akara received perfect marks on her first two quizzes, and Nadia had told the students that only the best of their four quizzes would count toward their final grade. As far as her grade in the course was concerned, the 7.6/10 she just received on her third quiz would not be counted.

After class, however, she stays back to ask Nadia to explain why she had lost marks on the quiz and how she should do it differently in the future. She had trouble with the questions, which required her to unpack mathematical statements. Nadia tells her that although she seemed to have the right idea and understand the meaning, the structure of what she had written was incorrect. Nadia had given Akara 7.6 on the quiz because she had already gotten 10 on her previous quizzes. She explains that if Akara had not yet gotten a 10, Nadia might have given her higher marks because the understanding was there, just poorly communicated. Akara replies, “You had to give me the 7 so that I would learn my mistake. If you gave me 10 on 10, I would not have come to ask you how to do it better.”

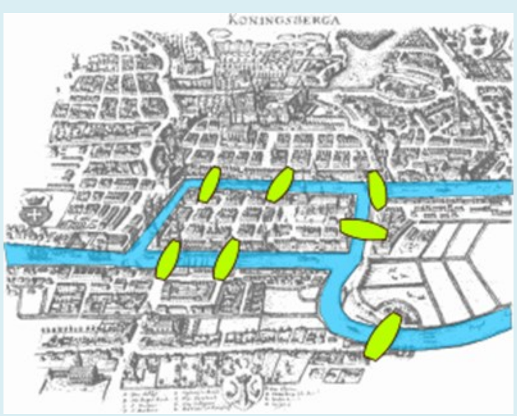
*This is an example of Akara’s **self-efficacy**. It is clear that she is not discouraged at having made a mistake, but rather sees it as an opportunity to learn. This is not the only time she showed this attitude towards learning; she would often come by my office after receiving feedback on an assignment to get clarification on how she could improve her work. She was rarely concerned about the mark she had received, but was instead focused on understanding how she could improve.*

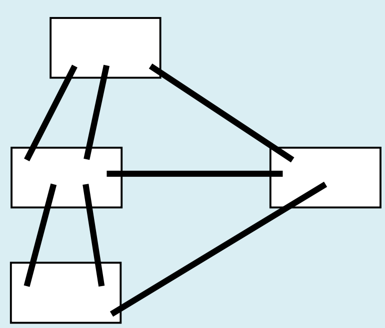
*****

Today, we are working on the Bridges of Königsberg activity:

Activity 18: Bridges of Königsberg

The city of Königsberg (now known as Kaliningrad) has four pieces of land separated by a river. There are seven bridges connecting the land as shown in the picture on the left. Another way of representing this would be the figure on the right: the rectangles represent the pieces of land and the lines represent the bridges.





Is it possible to do a walk through the city so that every bridge is crossed exactly once?

(Your answer should not be just "yes" or "no", the goal of the activity is to discuss "why yes" or "why not" and how the answer would change if ...)

A mathematical problem rarely has a yes or no answer; explanation and justification are always required. This activity invites students to make and test conjectures, and to learn to justify their claims. It also invites students to pose their own problems, asking how the question would change if bridges were added/removed, and whether there is a general rule.

Akara notices almost right away that it will not be possible to do a walk, although at first she is not able to explain why not. She asks me, "is it because some are odd and some are even?"

Her intuition tells her that the solution has to do with the parity of the vertices, although she does not use this term. She explains to me why she thinks that there are too many vertices with an odd number of bridges for a path to exist. It is a good explanation, and together she and Pierre-Luc are able to work out a correct approach.

Akara is left with the difficult task of trying to convince Hannah, who often has difficulty accepting the explanations and justifications of others, and will relentlessly question everything she is told. It is not clear what drives this questioning – skepticism, a desire to understand, or perhaps she feels threatened when someone else is correct (or 'more' correct than she is). In any case, Hannah is not convinced of something unless it comes with a proof she understands

Pierre-Luc doesn't seem to care if Hannah agrees with his ideas; once he is convinced and understands, he is not interested in arguing or discussing. But Akara does care, she can't let it go and will keep trying to explain her thinking. She hasn't really spoken much English in the past five years, and it can be frustrating to not have the words for what she is thinking. Despite the frustration and difficulty, she does not like to give up on trying to explain things. It matters to her to have someone understand her work and agree with her.

> *Akara and Pierre-Luc have a **collaborative** relationship. Despite their language barriers, they are able to communicate well together and explain their ideas and help each other understand better.*
>
> *Akara's relationship with Hannah is quite different. Akara is often in the position of trying to convince Hannah, of trying to **justify** her conjectures or ideas. For Akara, it seems to really matter that she can justify her proof to the others at her table. Collaborating is a way of validating her work.*

As often happens, she calls Nadia over to review her work and help her convince Hannah.

JUDY

Judy has something of a chip on her shoulder when it comes to math teachers. She has had some negative experiences that have clearly left their mark on her; it is one of the reasons why she wants to be a math teacher herself. She is currently taking a major in Mathematics and Statistics and a minor in Education.

She finds many of her university math professors arrogant and unhelpful. She was particularly displeased when, in her Linear Algebra class last semester, over half of her class failed the midterm, including herself, despite having studied with her study group for many hours. She was upset because she worked hard, wrote long answers for each problem on the test, and felt too many marks were taken away for small mistakes. She felt it unfair that the professor expected the students to give perfect answers; if half the class fails, she feels it is the fault of the teacher, not the students. She ended up failing the course and is re-taking it this semester with the same professor. (She rolls her eyes when she tells me this, “Of course, it had to be with him again!”)

Talking about her experience in the Linear Algebra course, she says that she spent so much time studying and understanding the material, that she actually grew to really enjoy it, but still did not get a good grade. She contrasts this with her Education courses, where the more she enjoyed the material and put in effort, the better her marks tended to be. She doesn’t like when her grades do not reflect her effort.

Math was never particularly hard for her, but it is not her passion. She is passionate about working with youth, mentoring them and getting to know them. Particularly in high school, when students are experiencing a lot of personal change, she believes it is important for teachers to be skilled in more than just content knowledge. There are many reasons why a student may be having trouble in high school math, so she wants to be the kind of teacher who can make personal connections with students and help them with any type of setback. She feels she knows enough math content now to teach at the high school level, and so she doesn't enjoy the courses she is taking right now because she doesn't really see how they are helping her work towards her goal (aside from the fact that she needs the credits).

She freely admits to not being very highly motivated in MAST 217 - she is only taking it because it is required for her degree. "I just kind of go through the courses, and like learn what I got to learn and then get out. As fast as possible." In the classes she took for her minor in Education, she was really interested in the material, and she worked much harder.

> *Interviewing Judy gave me some helpful insight into her behaviour during the semester. Judy had the potential to be a very good student in this course. Her reasoning about problems and ability to explain her thinking always impressed me, and I often wondered why she was not more engaged in the course. Her explanation from our interview gave me the answer: because she didn't enjoy it. She had her own very good reasons for not having a* ***personal connection*** *to the mathematics we were doing. Her goal was to get the credit and move on.*

*****

She didn't like Linear Algebra because she finds the concepts too abstract: "Subspaces and dimensions, how am I supposed to use this in real life situations?" she asks, "Unless I'm going to be, like, a mathematician as my job, and I'm going to make an equation or whatever… But other than that, it's not valuable for me to know this information, what's valuable to me is to be a teacher."

She much prefers the problems in her Calculus course, because they are mostly mechanical exercises, and she gets to work with equations and formulas. She finds this work easy, and is more likely to do her homework carefully and completely.

Her strategy for doing well in math has always been to do a lot of practice problems so she learns the techniques which will show up on the exam. She always thought that this was the way to do well in math, and that it would be this way for all the math courses she would ever take. In her recent courses, this idea has been challenged; she now recognizes that there is an aspect of doing and learning math which requires one to understand why something works, and has only recently come to appreciate that this matters. She likes MAST 217 because it teaches her how to teach herself math, how to ask herself why questions and how to understand.

> *Judy describes her experiences in previous math course and how these informed her* ***beliefs about mathematics*** *and what doing mathematics is all about. For her, math is about applying certain rules and procedures.*

*****

"I've always been into group work," Judy says. She finds it motivating to work with others because she doesn't like being the weakest in any group (whether a math study group or her soccer team). She has to push herself to work harder, put more effort in, and means that she has to stay engaged during class time. She says that if this course had a more traditional lecture-style format, she would, "definitely have checked-out a while ago."

*Judy enjoys **collaboration** because it is more interesting to her than working alone, and it keeps her motivated. When her interest in a question is flagging, having others around to discuss with is what will keep her going.*

In class discussions, Judy says, "I just ask questions all the time," and laughs nervously. Judy benefits from group work in the course because she is not the kind of person who enjoys doing abstract problems for their own sake, but she sits at a table with others who do. Unlike when she is working alone at home, when she is doing problems in class she will enjoy it more because her classmates are very engaged. She will be motivated to participate because, she says, "I don't want to be the one who is continually asking questions and making everyone else explain it to me. I want to be able to have something valuable to put on the table."

*****

After the first class when they sat together, Hannah and Judy learned they both lived in the same part of town so Judy offered Hannah a lift home, and they became friends. Judy describes herself as a person who makes friends easily. She will meet a new group of people each semester and form study groups with them.

Judy and Hannah are an interesting study in opposites. When it comes to solving problems, Judy is relaxed and Hannah is intense. Hannah will get very invested in the problem, and will not move on to something else until she understands every single line of the proof. Judy thinks Hannah sometimes gets too deep into a problem, to the point where she gets tunnel-vision and can't see other options, which to Judy will appear obvious. On the other hand, Hannah will often be quicker to find a possible solution or point of entry than Judy. They both benefit from working with each other

> *Judy and Hannah have a good* ***collaborative*** *relationship because their ways of thinking complement each other well.*

When they work together on the homework, they do not solve the problems together, but instead will take time to do their solutions separately and compare solutions afterwards, explain them to each other, help each other if they get stuck.

STEPHANIE

Stephanie usually does not enjoy working in groups. She has had too many group projects where other members of the group did not do their share of the work until after the deadline had passed, and has learned that working with others can be frustrating and unsatisfying. In this course, however, it is not so bad. The fact that the problems are more open-ended and there is no mark associated with the work done in class has meant there was less pressure involved and she was able to benefit from working with her peers.

Stephanie is not tied to any group in particular, and is quite happy to sit with whomever. She can't really name any others in the class, and has little to say about the experience of working with them. I try to ask her a couple times about others in the class, but she speaks very vaguely about her experiences working collaboratively:

"I sit with whomever, no one in particular. Most of the people I work with were pleasant to work with; I never had anyone I didn't get along with."

For Stephanie, the most important thing is understanding the problems, collaborating is secondary.

> *Although collaboration is a mathematical behaviour, not all mathematicians are collaborators, and not all mathematicians collaborate in the same way. Stephanie participates much more in the on-line discussions than in those that take place in class. One of Nadia's goals in designing the course was to provide students with the space they need to practice mathematical behaviour. Although the course has a "group work" format, some students do work alone or prefer the forum space for communication or debating, rather than the classroom space.*

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Stephanie: "I have a better time expressing myself through writing, rather than talking. Because, talking for me, it's like, I stumble over words and what not, but writing down things is more … I can spell out my thoughts better and organize them better, so I find I have a better time writing than speaking."

Stephanie's written work is easily the best in the class. When doing her assignments, she will imagine that she is explaining the questions to someone else, and will read over her carefully written steps once she is done, imagining she is a person who does not know how to do the problem to see if she would be able to understand it. Her favourite part of this course has been the focus on how to write mathematically, how to use different mathematical symbols, and how to write a proper proof.

> *Stephanie is an* ***analytic thinker.*** *She is sensitive to the linguistic concerns of notation and terminology, and to the structure and logic of mathematical language. Stephanie sees the value in notation, because it allows her to be precise. She is one of the few students in the course who uses notation and symbols flawlessly in her written work.*

She often preferred discussing things on the forum, where it was easier to organize and express her ideas, as opposed to in class. Annoyed that many students seem to just dump their proofs on the forum and not read what others had posted, Stephanie was consistently someone who read other people's posts and proofs and responded to them. I asked her about this once and she said, "I just want to help people out, mostly." She would often post several times in a week, answering people's questions or pointing out mistakes in their work. Her responses were always very well thought out and clearly worded. In response to a question a classmate posted about relations, Stephanie wrote the following explanation of the definition:

Forum Week 4
by Stephanie Chan

Well, since we're talking about relations here, an ordered pair ((0,2) for example) is an element of a relation. Yes, an ordered pair is defined as having two elements (a,b) in which a belongs to A and b belongs to B but the ordered pair itself is an element in a relation.

Let me explain it in more detail:

A Cartesian product of A x B for example is defined as:

A x B = { (a,b) | a∈A, b∈B}

Basically, it is the set of all the ordered pairs (a,b) such that a is an element of set A and b is an element of set B. That means a given ordered pair (a,b) belongs to the Cartesian product A x B if it meets those conditions (a∈A, b∈B). In other words, **the ordered pair is an element of A x B.**

A relation is a **subset** of a Cartesian product. Since it is a subset, that means a relation has at least some of the elements of the Cartesian product. Since elements of a Cartesian product (such as A x B) are ordered pairs, not the numbers inside of these ordered pairs, **every element of a relation is an ordered pair** (thus (0,2) for example is one element, not two).

I hope that clears that up for you.

Stephanie is very good at being able to read a definition and understand it. She appreciates that the meaning of an object is given by its definition, one of the traits of a ***systemic thinker****. In this forum post, she carefully explains the definition of a relation for her classmates, pointing out where she believes they may have misunderstood the written definition.*

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Stephanie likes math because she is good at it. Going through a set of problems and realizing she can figure them out gives her satisfaction and confidence. She feels that her ability to come up with an idea (know what to do on a problem) comes quite naturally. When she is stuck on a problem, she will try as many different ways as she can think of before giving up.

Stephanie is very ***persistent*** *when solving problems. She tries to generate as many ideas as possible, and is not discouraged if her initial idea or approach is unsuccessful. Unlike with some of the other students I interviewed, I do not feel I have a clear picture of where this persistence comes from in Stephanie's case.*

She has always had an affinity for math, and has always liked, and been good at, solving mathematical problems. For her, this is an innate ability, she can't explain (or has never tried to) where it comes from and why she is good at it. She does recognize that she has an ability that goes beyond the simple application of formulas (what is normally taught in math class). She is very good at reading a question and knowing which concepts are likely to apply or can be used to answer it.

> *Stephanie usually has good* ***intuition*** *about problems, something that seems to come from her previous experiences. When faced with a new problem, she will try to identify elements of the problem that are familiar, or may be connected to something she has done before.*

She says that the non-procedural questions, which require critical thinking, are more challenging and therefore more enjoyable for her. "It's a fun kind of challenge," she says. She enjoys questions that are more mechanical too, but these are satisfying in a different way. She feels this course gave her more opportunities than previous (more traditional) math courses to use her creativity and critical thinking when solving problems.

JAVIER

Early in the semester, we are working on some exercises involving Peano's axioms:

Activity 12

Consider the set of real numbers over which we define a binary operation called + (addition). It is a binary operation because it acts on two elements of the given set. We state the following axioms:

0. For every real numbers a and b, $a + b$ is a real number.
1. Associative law for +: $a + (b + c) = (a + b) + c$ for any real numbers a, b and c.
2. Commutative law for +: $a + b = b + a$ for any real numbers a and b.
3. Existence of neutral element for +: there's a number, denoted by 0, such that for any real number a, $a + 0 = a$.
4. Existence of additive inverse: for every real number a, there exists a real number a' such that $a + a' = 0$. a' is denoted $-a$.
5. For every real number a, $a = a$.
6. For every real numbers a and b, if $a = b$, then $b = a$.
7. For every real numbers a, b and c, if $a = b$ and $b = c$, then $a = c$.
8. For every real numbers a, b and c, $a = b$ if and only if $a + c = b + c$.

(This means that two statements are true: "if a = b, then a + c = b + c" and "if a + c = b + c, then a = b", this last statement is called the cancellation property.)

Using only the axioms above, prove the following theorems:
Theorem 1: 0 + 0 = 0

Theorem 2: For every real numbers a and b, there exists a real number c such that $a + c = b$. In symbols: $(\forall a,b \in R)\ (\exists c \in R)\ a + c = b$

Theorem 3: For every real number a, if $a + a = a$, then $a = 0$.
In symbols: $(\forall a \in R)\ a + a = a \Rightarrow a = 0$

Theorem 4: For every real number a, $a = -(-a)$.
In symbols: $(\forall a \in R)\ a = -(-a)$

The activity is asking students to ***think axiomatically*** *about arithmetic, a topic with which they are already very familiar. I claim that this activity offers an opportunity for students to engage in* ***systemic thinking.*** *It asks them to be sensitive to the internal coherence of an axiomatic system, to examine the difference between familiar and formal definitions of mathematical objects, and to be sensitive to ambiguities and assumptions.*

The group sitting at the front right of the room (consisting of Javier, Jonathan, Scott, Marco, Viviane, Joseph and Shannon) spent the first few minutes working independently and are now comparing and discussing their answers. They ask me to take a look at the proofs they have written. I spot a couple of instances where they have done a step without explicitly stating the axioms they have used, so I point out what is missing. In one case they have "subtracted" a variable from both sides of an equation, so I point out to them that this is not defined by the given axioms. I am impressed that everyone seems to readily understand what is missing and they start to work on adding the missing steps.

> *What I mean by this is that the students were willing to work at being* ***rigorous****. I would point to a step and ask, "How do you know you can do this part?" or tell them, "You need more than one axiom to do this," and the students would understand what was needed to make their work more complete.*

They all seem to be having lots of fun! More than any other table (where some are confused, frustrated, lost), this group seems to be enjoying themselves, discussing and working together. They do not get the answer right away, but no one seems frustrated or worried about that.

At one point, Javier asks me if he can assume that the additive inverse of *a* is unique, which is a great question! Just like a mathematician to wonder and ask "Is it unique?" I ask him if the uniqueness of –*a* is guaranteed by any of the axioms, and he says no. I ask him if he thinks the inverse is unique, and he says yes, so I reply, "Then it should be possible to prove it. Why not try?"

By asking this question about uniqueness, Javier shows his potential for ***systemic thinking*** *by being sensitive to ambiguities and assumptions in his own work. He asks a mathematical question, and is able to recognize that although he believes his statement is true, it is an assumption that has not been justified or proven.*

He smiled at this, sighed slightly, and said, "Somehow I knew you were going to say that."

By asking Javier these questions, I am trying to ***model*** *mathematical behaviour in our interaction. I am asking the type of questions which I would like students to develop the habit of asking themselves when they are working on problems. If you have a conjecture or a guess about something, ask yourself if you can prove it, and then try to!*

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Javier is doing his degree in Biochemistry and works as a Research Assistant in a Neuroeconomics lab. The main idea of this field of research is that economic markets react and behave like human beings. The research he is working on studies the neural basis of decision-making, reward, and motivation, and how these can then be applied to economic situations.

In our interview, I ask him why he thinks some students prefer to work on easy questions, while others like challenging ones. Javier tells me that he sees this as a neuroeconomical question: "For some people it is more rewarding to do something that they know how to do. They feel better about doing something and feeling capable, because the reward is more immediate. But when you delay a little bit for the same reward, it is not as satisfying." This is something he calls 'delay discounting': for some people it is more rewarding to accumulate many small rewards in a short time than to work longer for a larger reward. If someone is working on a problem and they do not know if what they are doing is going to work out (in other words, they do not know if the reward is coming), certain people will be motivated by the fact that if they do succeed, the reward will be bigger.

Javier says that both of these ways of working apply to him (most people will fall somewhere along a spectrum between the two). He is able to work hard on tougher problems if he feels like his effort is leading somewhere. He needs to feel what he is doing is moving him forward in some way, even if not in the right direction.

Javier had some really interesting insights into motivation and learning, and often ***reflects on his own thinking****. I am sure that his research interest in the subject has a lot to do with this. He notices and is aware of his own habits of mind, what motivates him, and what causes him difficulty.*

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Javier often reflects on his own motivation and learning. It seems likely that because this is a question he is investigating in his work as a research assistant, he often thinks about his own behaviour in such terms. Javier is not motivated by getting high marks, since he believes that grades do not always correlate to a good understanding. His focus is on learning and understanding to a point where he is satisfied.

He tells me, “If I am in an exam and I don't know the answer, but I can probably bullshit my way through, I will just not write anything; because that's not the purpose of the course. It's not my goal to confuse professors into giving me a better grade: I want to actually understand things. It doesn't matter how much you score in a test where you just guessed all the answers.”

*This is indicative of Javier’s **attitude towards learning.** He is a student who is not really motivated by marks, but by his own curiosity and the pleasure he gets from understanding things. He views challenges as tasks to be mastered and believes that his success is determined by his efforts, both indications of his strong **self-efficacy.***

Similarly, he says that in this course he will not bother to post on the forum just to get the marks, but only if he feels he has something worthwhile to contribute.

In his courses, Javier focuses on understanding the general principles, and is bored by facts and names, which he often has to memorize in his Biology and Chemistry classes. When studying for an exam (Calculus, for example) he will look at the problem, and if he can grasp the general idea of how it is done he will stop there. He does not like to practice the actual solving of problems, because he finds it boring to repeat things he has already understood. Often, this lack of practice will cause him to make many small errors on the exam and get a poor result.

He was excited to take MAST 217 from the very first day because the focus was put on the bigger picture, on developing general understanding, getting the right ideas but not necessarily the right answer. The course was a good fit for him, because he wanted to learn this way.

"I'm blessed, too, because my parents don't really care about grades. What's important to them is that we are learning." Both his parents are actuarial mathematicians. His father comes from a very poor background in Mexico City. He was born in a slum and managed to work his way to doing a Master's here in Montreal. Javier speaks of his father's accomplishments with pride and respect. His father worked hard to get where he is, and was driven to achieve his goals by his own passion for learning. So both his parents have instilled these values in him from a young age: that the most important things are to be curious and to work hard.

*Javier's strong **self-efficacy** is something he learned at home, from his parents. The structure of the course gave him an opportunity to use this skill.*

*****

Activity 22

Prove the following using induction:

1. $(\forall n \in N)\ 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

2. $(\forall n \in N, n \geq 5)\ (n+1)! > 2^{n+3}$

3. $(\forall n \in N, n \geq 5)\ 2^n > n^2$

4. $(\forall n \in N)\ 8 \mid 5^n + 2(3^{n-1}) + 1$

5. Consider the Fibonacci sequence and prove that:
   a. For all *n*, f_{3n} is even
   b. For all *n*, f_{4n} is divisible by 3

6. For all n , $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \geq \sqrt{n}$

This activity was used to introduce students to induction proofs. The first problem was accompanied by a complete proof, with an explanation for how this method of proving works. The rest of the problems were left for the students to work on in class.

Javier really struggled with writing proofs by induction. Nadia had done the first proof on the board, explaining the steps and giving a general explanation of how induction works and why. The rest of the group moved on to the third proof and then the fourth, but Javier was still stuck working on the second. In particular, he was having a hard time assuming the inductive hypothesis. I discussed it with him several times, showing that we are trying to prove an implication of the form P(k) implies P(k+1), and so we must begin by assuming P(k) is true. But he could not understand, and kept asking, "How do I know that's true, though?"

At one point, ever-so-slightly frustrated because no one else at the table seemed bothered by it, Javier asked, "But how can I assume it's true for k?" and Jonathan replied, "You just have to assume it, you just have to take the plunge." His answer did little to allay Javier's concern.

> *Here, Javier struggles with being able to understand that meaning is found in the definition of the concept, whereas Jonathan seems to be able to do this more easily. In Javier's case, one element of his aptitude for* ***systemic thinking*** *(his sensitivity to assumptions) seems to be actually hindering another element of systemic thinking – understanding that meaning is found in definitions.*

What struck me as particularly interesting was that Javier was very aware of what kind of difficulty he was having. At one point when I offered him help with the particular problem (I think I suggested multiplying both sides of the inequality by 2), he said, "It's not that I can't figure out this question; I don't understand how this method works, what the process is and why I am able to even do that…"

> *Javier did not want to be given a hint or told how to answer the problem. His concern was not about being able to answer the question, but being able to understand the structure of the proof, so that he could apply it to any question.*

This is not the first time that Javier has had this difficulty. He tells me that he often has a hard time with proving implication statements of the form "P implies Q" because he doesn't like to assume the hypothesis is true.

"Logically, I understand why it works, you are just ignoring the case when P is false. I understand how it works as a structure. But practically, when I am doing it, I just forget and don't want to assume that part is true unless I understand why and can show it."

> *When working on the activity with Peano's axioms, Javier demonstrated that he is very sensitive to assumptions; he will not use a fact or statement unless he knows for certain that it is true. What was an asset in one situation, however, appears to be causing him some difficulty here. I was really amazed with how well he **identified and articulated** this difficulty, once again demonstrating his ability to **reflect on his own learning.***

He is often aware of when this is happening and will say to himself, "Oh, you're doing that same mistake again." When he realizes, he will try to change his strategy to correct for it, but often seems to arrive back at the same impasse; "I don't know why that happens!"

*****

Javier will often withdraw from participating in the group discussions and work on his own. He needs to write things down and figure it out on his own before he can share. Because the others in the group work a bit faster than him, by the time he is writing down his thoughts, they may be already discussing the solution or moving on to the next problem. Javier prefers to work alone until he is stuck, and only then will he discuss what he is doing with the others to see if they can give him an idea of what to try next.

> *Many students (Stephanie, Judy, Hannah, Jonathan, Scott) expressed that they need to write down their thoughts before they can discuss them.*

Genevieve and I often discussed that we, too, found it hard to discuss problems without first trying to write out our ideas. I have a feeling that understanding through writing (needing to write it down to understand) is probably a very common trait among mathematicians.

Javier has a very short attention span, and finds that if he is working with other people he will always end up talking about something else. When overhearing conversations it is difficult to concentrate, so the only way he can really work on the problems is to shut the others out. So unless he needs help to understand, he will mostly work on his own.

He prefers to work alone in most areas of his life. "It gets messy working with other people," he says, and mentions some previous experiences where working in a group was not just about sharing ideas and deciding which idea is better, but about people wanting to prove that they are smarter than everyone else. He doesn't like working with people who aren't able to be wrong.

"If I sensed that kind of antagonism," he tells me, "I wouldn't participate at all." In this course, he has been glad to have such a positive group dynamic and is not afraid to share his ideas. For him, it is better to have a wrong idea and share it and learn what the flaw is, than to continue working with a wrong idea, thinking it is correct.

Javier enjoys working with Scott and Jonathan because they always have at least one idea for how to approach a problem and are both pretty good at explaining their ideas to the others (something which Javier finds very difficult). Javier really admires their ability to not remain stuck or fixated in one way of thinking about a problem, but to shift their approach and try something else. He has enjoyed working with them because he gets to observe their more intuitive problem solving in action, and he feels it is helping him understand what he needs to do to develop this skill.

"Like anything, it is a skill and you can learn it. It's just a matter of practice."

*This is an example of Javier's **self-efficacy**: he views difficult tasks as challenges that can be mastered, and believes his ability is something that can be improved with practice and effort. He is reflecting on his own thinking, recognizing that Scott and Jonathan have good self-regulation (although he does not use this term), and that this is one area in which he would like to improve.*

*****

In our interview, I ask Javier about his persistence when it comes to math problems, and he tells me, "it's just stubbornness. I'm a very pride-driven person. I feel like if I cannot solve something and somebody else can, I am doing something wrong and I should change my way of thinking in order to solve it. Because I know there is an answer, and if I cannot solve it, I feel bad about myself, so I think it's not persistence, it's just stubbornness"

He describes his feelings about mathematics as a love-hate relationship: "I hate it when I'm doing it and I can't get it, that frustrating feeling. But once you get it, it can be the greatest feeling in the world, when you see where you were wrong."

*A sentiment that I am sure many mathematicians would understand and recognize. Javier is **personally/emotionally invested** in the mathematics he is doing. The fact that he associates a 'great feeling' with being stuck and then seeing where he was wrong shows that he has a strong **self-efficacy**; he sees his mistakes as important sites of learning.*

SCOTT

Scott has done rock climbing since he was in high school. There are two types of rock climbing competitions – one is a straightforward race to see who can reach the top fastest, and another contest where the wall is really hard and climbers compete to see how many tries it takes them to be able to complete it. Scott prefers the second type of competition because it is more of a problem-solving competition; it is about finding the best route to take. Since going quickly is not the main goal, each person will spend time studying the wall before they get started, making a plan. He feels there is definitely a connection between his interest in mathematics and his love of rock climbing; climbing is "like problem solving with your whole body."

Scott knows the importance of having a plan when attacking problems, and this skill transfers to other areas of his life. When working on math problems, he often seems to be thinking a few steps ahead, and asking himself where his current approach may be leading.

> *Scott regularly displays good **self-regulation** when working on problems. There was one time I was sitting with their group and I noticed that he acted as a good self-regulator, not just for himself, but for the whole group. Because he was slightly faster at setting up proofs, doing calculations, whatever, he would provide hints/directions to the others, such as "Seems like it gets pretty messy to try and prove by contradiction, maybe we should try something else."*

Scott: *"I find with most problems there is like, one part of the problem which I have always called the crux - it's a rock climbing term, I used to rock climb a lot, and the crux is the most difficult part of your climb. As soon as you get past that, you're in the clear. And I always find* *that most math problems have at least one, and that's the real meat of the problem, and I get tunnel-vision on that part of the problem and trying to get through it, and sometimes the way I figure out for doing it may disregard or contradict the initial assumptions I made earlier. It's like I forget about what came before because I'm focused on the crux."*

> *Scott is talking with me about how he* ***responds to being stuck.*** *He has a fairly positive relationship with being stuck, and feels that the part of a problem where you get stuck or challenged is the most important part. He views these challenges as learning opportunities and does not seem to become too easily frustrated by them.*
>
> *He also describes his experience of sometimes getting "tunnel-vision" when working on a problem - focusing too much on one particular detail or element, and failing to consider the problem as a whole.*

*****

Scott likes challenging problems. This year is the first year that he has looked at proof-based math. His Linear Algebra course forced him to really work at understanding how proofs are constructed, and this has given him a strong foundation to draw on in MAST 217.

He has always preferred more theoretical proof- based questions. If an assignment has 10 exercises and one tough question, he will ignore the easy ones and focus on the challenging one for the whole day. He says he prefers these more challenging problems because he feels like he is, "actually doing math instead of just being a calculator." He prefers to work on one very tough problem, or on a set of different problems which explore a topic in different ways, than to do a set of exercises where you practice a technique 10 times.

*Scott's statement reveals something about his **beliefs about mathematics** and what doing mathematics is all about. Scott recognizes that doing mathematics is about tackling problems that are challenging, that one doesn't necessarily know the answer to. For him, performing mechanical calculations is not 'doing math.'*

*****

Scott is explaining to me his proof of one of the problems we did in class. I remember that class very well. I had been sitting with their group for the entire class working on some proofs about certain patterns in the Fibonacci sequence.

Activity 23

Consider the Fibonacci sequence:
$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2}$ for all n ≥ 3

Prove that:

a. For all $n \in \mathbb{N}$, F_{3n} is even
b. For all $n \in \mathbb{N}$, F_{4n} is divisible by 3
c. For all $k, n \in \mathbb{N}$, F_{kn} is divisible by F_k

This activity is about conjecturing and justifying. It is also about reading mathematical symbols. Students will likely need to try some examples to get a sense of the patterns and what the question is asking.

The questions were all ones I had not seen before either, and it was fun to be working on them with this group. Jonathan had been explaining a pattern he noticed: every term in the sequence will divide certain subsequent terms, for example 8 is the sixth term, and every sixth term is divisible by 8. As he is explaining the pattern, I realize this is exactly what the last theorem on the board is stating. Jonathan noticed the pattern without realizing it is the same as the statement we are asked to prove! I remember that I was very excited when I saw the connection, and explained it to the others. Class time was over at that point, so there was no time to work on a proof.

This moment stands out for me because I was sitting at the same table for the whole class, working on problems I didn't know how to solve with the students. I was doing what they were doing, living their experiences with them, sharing in the excitement of realizing a connection. It was fun!

Just over a week later, Scott approaches me after class to share his proof with me. He tells me it had been something of an obsession, that he had trouble doing his work for other classes until he had figured out this question.

"I'm positive it's true, because of the pattern of the sequence, but I thought I could show it is always true by induction, but it doesn't seem to be working. But the pattern is there, I can see it, I just don't know how to fully explain it."

He has not written a full proof, but has sketched out his idea of how it might be proven. It is an extension of the method he used to prove the first two problems, and seems like a good strategy.

"I made this, and I was convinced that I had proven it, but then I realized that there was a difference between even and odd terms, and I had to go back and do a second part. It gave me a lot of trouble…. And this second part is even less well thought out and sketchy, but it works out in a similar way."

He says that he did the whole thing and then later, while riding a Greyhound bus home for the weekend, he had a moment where it just sort of struck him that he had not considered the cases of even and odd k separately. He says this happens sometimes, as if some part of his brain is still working on the problem even when he is no longer consciously focused on it, even when he feels it is solved and correct.

*I wrote about Scott's experience with this particular problem because I feel it shows him behaving mathematically in a few different ways and is a good example of how he engages in mathematical problem solving. He has a **conjecture**, based on a pattern that he sees in the terms, which he is trying to prove. He does not seem concerned that his first attempts do not work. Like many mathematicians, he **deals with being stuck** by putting the problem aside for a bit and returning to it later. He is **persistent**, and continues to work on the problem over several days, displaying the **obsessive behaviour** of a mathematician.*

Even though he has not found a solution or a proof, he is proud of his work and feels he has accomplished something.

"Sometimes I will have this stroke of insight to myself, and I will think I have fully understood a problem, and then as soon as I try to explain it to someone, I will realize that I've missed, like, this huge gaping hole, right from the start that is really obvious."

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"I've been coming to appreciate how there is an element of creativity to mathematics in doing proofs. I like the feeling of having to come up with an idea and see if it works."

Scott once told me he enjoys doing "math for math's sake." He is doing the specialization program in Pure and Applied Mathematics, which he chose because he has always been more interested in the theoretical side of things and wants to be around peers who share his enthusiasm. He has always been passionate about math, he even told me that when he was a kid, he wanted to be a mathematician (and also and astronaut and a firefighter).

He has really enjoyed this course, in particular because of the group atmosphere. He likes that the time is given in class to 'wander' through a problem, play with it, and try things out, without as much pressure to get an answer quickly. He has always been a person who likes to work in groups, to be able to bounce ideas off other people. He feels that discussing his ideas makes the experience of learning much richer.

*From the way Scott describes himself and his relationship to doing mathematics, I claim he is a **reflective thinker**. He is curious and genuinely enjoys thinking for the sake of thinking.*

He enjoyed the course so much, that he wanted to continue to participate after it was done, and volunteered his time to help as a Teaching Assistant when Nadia taught the course the following semester. He told me that the learning experience in Nadia's course was unique in his experience of university mathematics courses, and the opportunity to discuss mathematics problems was something he greatly valued.

Scott really enjoys mathematics, and enjoys it even more when he is able to ***collaborate*** *and discuss with others. The fact that he wanted to continue to participate in the collaborative aspect of this course even after he was done, shows just how much this means to him.*

MARCO

For the first few weeks of the course, Marco sits with a few different groups before eventually finding one he likes working with.

On one of those days, he sits with Hannah's group. The task for the day is about logical implication. Mentioning that she has already seen this in her previous courses in logic, Hannah tells the others she knows what to do and quickly writes out the answers. Marco, confused and not quite following her explanation, asks her to explain what she is doing, but she simply replies, "trust me, I've seen this before."

Marco didn't understand what she had done so he took a step back and worked it out on his own for a while. Eventually he came up with an answer which was different from Hannah's.

He later told me about his frustration with this experience:

"It turned out that what she had written was kind of irrelevant and overcomplicated the problem. And I didn't really like that because it almost came off as arrogant, but also because I think the point of the activity was to *do* the activity and actually think about it; you know, the whole thought process of it, and she kind of ruined it for me. She took away my learning experience."

The next class, he sat with Jonathan's group.

Marco does not want to be given the answer to a problem; he is focused on ***understanding why*** *and on being able to explain what he is doing. To me, this shows his* ***attitude towards learning****. His purpose in a math class is not just to give answers, but also to understand the answers. He would rather work hard to understand something difficult, than to have someone else do the work for him, evidence of his strong* ***self-efficacy.***

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Marco: *"I know Nadia says this class is about mathematical communication, and I didn't realize how not-good I was at that until taking this class. And it was weird to me to not be good at something relating to math. I'm definitely glad I took this course; I think I've gotten better, not perfect, still room for improvement. But now I kind of know more what it is supposed to be like."*

Marco was uneasy at the beginning of this course, especially after the first quiz when he did not do very well. The others at his table had all done fairly well so he had to ask himself: what are they doing that I'm not? He was aware that he was one of the weaker members of his group, and that is precisely why he decided to stay at that table for the rest of the term. It pushed him to try and do things differently and keep improving.

I feel that Marco has strong ***self-efficacy*** *and is very* ***persistent.*** *Nadia once mentioned that Marco reminds her of other students she has known; students she categorizes as a particular* <u>*type*</u>*:*

> *"weak students (possibly in the sense of lacking math background) who are very determined, self-conscious, have*

clear goals, and are willing to do what it takes to achieve them (not necessarily that they will achieve them)"

Marco is someone who regularly reflects on his own learning - how he learns best and what he needs to be doing in order to make sure he understands concepts, etc. He feels that this is something he did more particularly in this course because it was the type of class which encourages this type of reflection.

Nadia's goal in designing this course was to create space and situations for students to enact/practice mathematical behaviour. Marco is someone who has a strong ability to self-reflect, likely a skill he possessed before the course began. The fact that we can observe him reflecting on his own thinking and learning is ***because*** *the course design creates a space that encourages him to do this.*

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Marco spent the rest of the semester sitting with Jonathan, Scott, Shannon, Javier, Viviane, and Joseph. He liked this group because of the friendly and supportive group dynamic.

Scott asks good questions and gives good suggestions. Marco finds that Scott is very *modest* about his strengths: even when Scott knows what to do he will always say "I think it might be…" or "Maybe we could…" as opposed to saying "It's this, this is the answer". Marco really appreciates that Scott is open to suggestion and invites the opinions of others: there is always the shared understanding that any idea is subject to revision or could be wrong. For Marco, not being open to the ideas and opinions of others is missing an opportunity to learn and improve, and he is glad to be with a group that is able to share ideas well.

Marco also appreciates that Jonathan takes the time to explain things patiently and carefully when Marco is having trouble understanding, without overcomplicating things. The group works fairly methodically through the questions and time is taken to ensure that everyone understands what is going on. They have a good dynamic where anyone can ask a question or ask someone to repeat themselves if they didn't understand, and no one will rush ahead to finish the problem, or make you feel stupid for asking.

> *I keep coming back to descriptions of this group and their supportive dynamic. Nadia told me that it was unique in her experience of teaching this course, and strongly represents the* ***community of learners*** *(in the sense of Brown, 1992) that she wants to cultivate through the design of the course. So I write about it often because it was exciting and interesting to witness. There was something there that just worked.*

Marco is aware that he is often the person in the group who understands more slowly, and will often ask questions or ask to have something explained again.

*****

Near the end of the semester, Nadia, Genevieve and I decide to try something new and ask all the students to sit with a new group with at least two other people they have not sat with before. We are working on induction problems.

We met before class (for our regularly scheduled weekly meeting), and as we discussed individual students and their strengths and weaknesses, we began speculating about whether it might be helpful/interesting/fruitful for certain students who normally sat at different tables to work together. In particular we were wondering about whether Micah and Scott would benefit from working together.
We got so excited about the idea that we wanted to try it! We came up with a plan to mix up the seating. It seemed like a great idea, and there were some interesting interactions on that day, such as Marco's experience.
I think some students felt it was odd to be asked to switch tables for just one class. Micah told me later that he didn't understand why we wanted them to do it, and that it had bothered him. Many of the groups were quite fixed, stable, and some students, like Hannah, did not like having their established routine changed.

Marco is sitting with an entirely new group of people. He tries to start a discussion about a challenging induction problem, but others at the table were not able to follow what he is doing, so he has to explain it to them carefully. For the first time in this course, Marco finds himself in the position of leading the group discussion.

"The fact that I had to explain what I was doing had me go back and change some things up and re-think it for myself. I think that in the group I usually am in I tend to listen more and talk less, but on that day I did more talking because they weren't really talking. Being able to talk through my work and explaining what I was trying to do helped me understand what was going on and what I needed to do."

Marco was able to solve the problem on his own, and felt an enormous sense of accomplishment after this class.

Nadia: *I've seen students like Marco before; they need a ground to make their jump to becoming mathematical thinkers/to behave mathematically. This simple experiment of shuffling was his ground and that makes (to me the teacher) the whole experiment worthwhile.*

JONATHAN

"Is it possible to re-submit my assignment?" Jonathan asks me one day at the beginning of class. The assignments had been due the previous day, and had to be submitted online before midnight.

"I submitted it last night, but then this morning as I was driving to school I suddenly realized I made a mistake on one of the proofs - I need to consider two cases, and I only did one. It just hit me all of a sudden; I almost had to pull over."

*We have called this **obsessive behaviour:** the mind keeps working on a problem, even when you aren't looking at it anymore; often taking a break can lead to a breakthrough.*

He is holding his assignment in his hand. I recognize his neat, careful handwriting from previous weeks. His assignments are always very thorough and well-written.

"I re-did the proof earlier today. If I scan this one and send it to you will you mark it instead?"

Jonathan is something of a perfectionist; whatever he does, he wants to do it well. He focuses on getting all the details correct, and genuinely enjoys working on problems. To me, the fact that he wants to correct his previous work shows he is invested in the problems.

*One of the traits that marks a student with strong **self-efficacy** is that they develop a deeper interest and form a stronger sense of commitment to the activities they work on. I definitely have the sense that Jonathan is strongly committed to mathematical problem solving.*

This type of investment and persistence is a behaviour I want to encourage, so I tell him to send me his corrected proof before the end of the day.

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Jonathan works as a math tutor for a private tutoring center, and he is insistent that students learn the basics (e.g., algebraic manipulations) because he sees this as a necessary skill for being able to do the 'real' math down the road. "If you read a lot of books, you have a good vocabulary," he explains, "Same with math, if you didn't know what you were doing in high school algebra it will be harder in this class."

*Like Hannah, Jonathan also works as a math tutor, an experience which has helped him develop his skills for explaining mathematical concepts to others. It is interesting to note that Jonathan, like Hannah, often took on a **peer-tutoring** role in his group; the others in Jonathan's group spoke with praise of his ability to explain his ideas for a problem in a way that was easy for them to understand.*

One of the reasons he puts a strong emphasis on mastering technical skills is because he believes this is one of the main reasons he is able to do well in the courses he is currently taking. Jonathan often finds that others in his group may understand the structure of the question but make mistakes on their manipulations and can't work out the solution.

"Because I think my elementary algebra is really good, I don't make any mistakes, so like, I don't get stuck unless I'm *really* stuck," he says. If he is having difficulty with a problem, it is almost always because he is using the wrong approach, not because of a calculation or algebra error.

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It was our last quiz on Thursday, and Nadia gave students the option of writing a quiz to hand in, or, if they were already happy with their grades from previous quizzes, there were some more challenging problems involving strong induction to work on.

Activity 24

Recall the Fibonacci sequence:

$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2}$ for all n ≥ 3

Prove:

1. $(\forall n \geq 2)\ \phi^{n-2} \leq F_n \leq \phi^{n-1}$

Where $\phi = \frac{1+\sqrt{5}}{2}$

Hint 1: You need strong induction.
Hint 2: You need to use that $\phi + 1 = \phi^2$ (if you do use it, prove it!)

- What is exactly the base case?
- What is exactly the inductive hypothesis?

2. $(\forall n \geq 1)\ F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\overline{\phi^n}$

This activity was given as a challenge problem for those students who chose not to write the final quiz; it is the only activity involving strong induction that we did in the course.

Jonathan is working with some of his usual group (Shannon, Marco, Viviane) and a couple other students (Steven and David) on these questions. He tried the first problem and got stuck, so he decided to start over and try something else, but then got stuck again. He asked the others at the table, but it seemed they were all stuck in the same place, so they needed some help. In an effort to help them understand that they could use more than one case of the induction hypotheses to prove the inductive step, I wrote down the two inequalities*:*

$\phi^{k-4} \leq F_{k-2} \leq \phi^{k-3}$ and $\phi^{k-3} \leq F_{k-1} \leq \phi^{k-2}$

Jonathan immediately saw that the two equations should be added together. He said it seemed obvious. I asked him why it was obvious and he said, "Some things you just know, I guess it is a kind of intuition in a way, for me the first thing to do would be to add them together. I know other people might be like, 'how did you do that?' but I don't know, I think it just comes with experience in math, if you have enough experience doing these things you get an idea of what is most likely going to work, how to get the terms you need."

*Jonathan uses the word '**intuition**' to describe his ability to sense what is likely to work on a given problem. He explains what this intuition feels like, and that he feels it comes from having lots of experience.*

But previous experience alone (encountering something many times) is not sufficient to build this intuition, unless accompanied by some reflection on the experience. And not every type of reflection will work. Jonathan reflects in a way that allows him to make meaning out of the experience; he tries to identify what worked and why, and that helps him remember/recall/have that experience come to mind in a future *situation. His **reflection on his own thinking** is what makes his experiences 'educative' (in the sense of Dewey, 1938).*

Once he has understood the first problem, Jonathan finds that the second problem is easy – the structure of the proof is essentially the same. He quickly writes out a solution to the second problem and then turns to Steven, who is sitting next to him, to offer help. Jonathan was able to quickly see how the two problems were connected, but Steven could not. I think this is an important strength for Jonathan – his ability to understand both the content and the structure of a proof, and his ability to transfer this understanding to another similar situation: "Once I see something once, I know what to do."

*More evidence that the experience alone is not what makes it possible to develop intuition. All the students at the table saw the same thing, and once Jonathan explained the similarity to them, they saw it and understood it. However, only Jonathan showed evidence of **reflecting on his own behaviour** in the first problem in order to be able to apply the same technique to the second problem.*

*****

The group that Jonathan usually sits with functions as an entity. More than any other group in the class, they work together and collaborate and help one another. They often overhear the table next to them, where Hannah always appears to be controlling and dominating the conversation and they joke among themselves about how they would not like to be there, that it seems too intense. They have a sense of teamwork, and they take pride in their ability to work together.

Jonathan is not someone who will argue with others very much, but will become defensive when he knows he is correct and someone is trying to tell him he is wrong. If he thinks he is right, he needs to be shown and convinced otherwise before he will back down from his position.

On the day when we asked everyone to sit with a new group, Jonathan found he missed the conviviality of his usual group. Instead of discussing ideas, he felt as though he was arguing with the others in the group in order to convince them of his idea. "I didn't want to argue, but somehow there we were. It didn't need to be like that, and it was kind of upsetting," he said later.

Jonathan had come up with a proof, and everyone had a feeling that it was incorrect, but no one could explain why. Eventually, Stephanie pointed out that the negation of an existential statement is a universal statement (this somehow explained why his proof was not correct), but Jonathan could not remember ever having heard that before, so he asked her for an example or to explain why that was the case, and she couldn't. "It's just the rule," she told him. There was a tension in their exchange, and Jonathan didn't like it; it made him feel instantly defensive.

*****

Jonathan wants to be a physician, and it is clear that he has a knack for understanding and working with people. Jonathan is the most group-oriented of anyone I interviewed, and seems to have a very good sense of other people's strengths and weaknesses. In our interview, he spoke about the other students in his group, and I found his descriptions very insightful:

Jonathan knows that he and Scott are the strongest students in their group, in the sense that they usually have an idea of what to do when faced with a problem. They both recognize the importance of holding back from telling the others what to do and allowing time for everyone to think through the problem. Jonathan says it is important that everyone has a chance to learn and understand at their own pace. He is very invested in keeping the group working as a team, always making sure those who are having trouble understanding are not left out, and creating space for everyone to participate.

When the problems had a visual component, Jonathan tells me, Javier seemed to be all over them, but once the course moved into other topics, it seemed like he had a harder time getting it. Jonathan pinpoints the exact same difficulty that Javier described to me, saying, "Javier gets it, but he overcomplicates things and confuses himself in the details. When that happens, he doesn't really talk, he just sits there trying to work it out. I try to ask him what he thinks and keep him included in the discussion."

Jonathan believes that Marco came to join their group because their dynamic was inviting and encouraging. He says Marco struggled at first to keep up, but now: "I don't know if you've noticed, but he seems to be getting the concepts really well. He has put a lot of effort in, you can tell."

*Jonathan is definitely a **collaborator** - he is the lynch pin of the discussions that happen at his usual table, and always makes an effort to ensure that everyone has an opportunity to contribute or ask a question.*

I think the group work aspect of this course really suited Jonathan because it drew on one of his natural strengths – working with others.

MICAH

It is early in the semester and we are working on an activity about proving statements involving existential and universal quantifiers. The statements are written symbolically, so part of the task is for students to learn to read the statements, as well as introducing them to the question of what makes a convincing or sufficient proof. Each of these cases only requires a single example to prove (in the case of existential statements) or disprove (in the case of universal statement).

Activity 8

Prove the following statements are true:

1. $(\exists\, x \in \mathbb{R})\ x \notin \mathbb{Q}$
2. $(\exists\, A \subseteq \mathbb{R})$ A is infinite and A^c is infinite

Prove that the following statements are false:

1. $(\forall n \in \mathbb{N})$ n is divisible by 2
2. $(\forall n \in \mathbb{N})$ n is divisible by 2 or by 3
3. $(\forall n \in \mathbb{N})(\exists\, m \in \mathbb{N})\ m^2 = n$
4. $(\forall m, n \in \mathbb{N})$ if m + n is even, then m and n are both even

*This activity, done early in the semester, serves as an introduction/refresher on mathematical notation and symbols. It is intended to provide opportunities for students to engage in **analytic thinking***

At the back table, where Micah sits, they are having quite a bit of difficulty unpacking the mathematical statements. Only Leona seems to be comfortable with reading and using symbols, it is clear she has seen them before. The others are struggling. It is always interesting to see how students react to new, unfamiliar mathematics. Many in this group, like Donnie or Mariola, are paralyzed when they look at a problem and do not immediately know what to do. They give up easily and wait for a teacher to come by and help them.

Micah is also baffled by the new symbols, but pushes everyone in the group to begin discussing. He notices that Leona knows the symbols and so he asks her to explain them to him. Soon they are all discussing. They repeat the meaning of each symbol for those in the group who are still learning them. When Genevieve comes by their table, Micah is arguing against the rest of the group, trying to explain why only a single example is needed to prove a certain statement. His explanation is correct, but not everyone in the group is convinced.

*Micah is highly **collaborative**, he is someone who learns better when he is able to discuss things with peers. He sees his peers as resources to help him learn and understand.*

Micah, lacking confidence, is not sure if he can defend his point, so he looks to Genevieve for back-up and she confirms he has the right idea, so he continues trying to explain his thinking to the others with her help.

*****

Micah once told me that having group discussions was his favourite part of this course. For him, talking through questions has always been the best way to learn.

After finishing high school, Micah attended Yeshiva, a Jewish institution that focuses on the study of traditional religious texts, primarily the Talmud and Torah, for four years. The program of study was about understanding fundamental concepts, about breaking things down to their most basic elements and then building them back up. Micah told me that he finds it odd that the study areas at Concordia are so silent. At Yeshiva, the study hall would be cacophonous, everyone discussing and debating simultaneously. Students were paired up with a study partner for each topic and would spend hours at a time discussing and debating the texts they had read. These study pairs are a traditional approach to Talmudic study in which each student is responsible for analyzing the text, organizing his thoughts into logical arguments, explaining his reasoning to his partner, hearing out his partner's reasoning, and questioning and sharpening each other's ideas

> *Micah's previous learning experiences, the classroom norms and expectations he was used to, were an important element in how he engaged in the course. In particular, he was very used to* ***discussion and dialogue*** *as a means for learning. Some students in the class found the group discussion element foreign, new, a challenging transition. For Micah it is the way he is used to learning. I think this matters a great deal in understanding his (mathematical) behaviour in the class.*

This style of learning is something he brings with him to this course, and is clearly visible in the way he engages in MAST 217. His learning style is characterized by a desire to understand things in their depth and to make connections between different ideas. He takes pleasure in discussing his ideas, and seems to care that he can express what he is thinking to others and be understood.

Micah wants there to be an explanation for everything. He is very persistent in the sense that he will continue asking questions (even the same question) until he arrives at a satisfying answer. He must understand something before he can move on. He is a skeptical student who is not satisfied with accepting something is correct just because someone (even a math professor) tells him it is.

*****

Genevieve is discussing a question that came up on the forum with Micah. Another student (Nisha) had posted a slightly different proof for the theorem "*If* $a + a = a$, *then* $a = 0$". Instead of starting with the hypothesis $a + a = a$ and transforming it into the thesis $a =0$, she had started with the true statement $a + 0 = a$, and used the given axioms as well as the hypothesis of the statement to show that $a = 0$. Her proof was less conventional in the form of argument, but still completely correct.

Micah was confused by the proof, and in particular by Nisha's use of the axiom $a + 0 = a$ combined with the hypothesis of the theorem. They exchanged messages back and forth a couple times, but although she knew her proof was correct, Nisha did not understand Micah's confusion. She was satisfied that the proof was correct because she had shown it to Nadia in class.

> *Micah is paying close attention to the structure of the argument and questioning its validity. To me, this is an example of* ***systemic thinking*** *- being sensitive to logical consistency. Micah is very detail-oriented; he wants to make sure every part of the proof is correct. When he comes across something he does not understand, he questions it, and will continue to ask questions until he understands.*

As part of the discussion on the issue, Genevieve tries to explain to Micah that there is a difference between using the hypothesis of the theorem in the proof and using the thesis (the result one is trying to prove) in the proof.

"So it's okay to replace $a + a$ with a anywhere in the proof, since it's part of what we are assuming at the beginning?" he asks.

"Yes, but you can't use $a = 0$, since this is what you are trying to prove," Genevieve replies.

"Okay, but something is still bothering me," Micah continued, "Suppose I start with $a + a = a$ and I can make a substitution to get $a = a$, then isn't this a contradiction? Since if $a = a$, it can be any value. It could be, say, $a = 1$, and then we would have a does not equal 0. See what I'm saying?"

"Hmmm," said Genevieve, "but this is not a contradiction. If $a = a$, it does not necessarily mean that a does not equal 0. And showing $a = a$ is not the same as showing that $a = 1$"

"What would a contradiction look like, then?"

"You would have to start with your hypothesis, or with some other true statement, and then using the axioms show that $a = 1$. Then you would know it does not equal zero*."*

> *This conversation is very typical of the kind of interactions we had with Micah. Micah can be very* ***skeptical****; when he is confronted with a proof that does not make sense to him, it takes a lot for him to be convinced. He is very* ***persistent*** *in asking questions until he understands. In this instance, his reasoning is (mathematically) incorrect, but he is still behaving mathematically in his desire to understand the structure of the argument, and to consider other possible situations.*

Genevieve struggled to explain to him something that, from her own experience, seemed quite natural and obvious to her. She was certain no one had ever taken the time to explain the underlying logical structure of proofs to her, certainly not this carefully. She also realized that as a student she had never been as curious as Micah; she did not ask the kinds of questions he was asking her. In her case, it was only by reading and doing many proofs that she developed her instinct for when something is correct or not.

*****

Micah: *"Everything we know is kind of stored away in certain blocks or structures; you just need to find the connection somewhere."*

Micah is very structured and his weakness is sometimes his lack of flexibility.

At the end of class one day, Micah asks me whether there are any set of steps or method that you should always do when writing a proof. I tell him it is different for different proofs, some methods are going to work better than others. I mention that we will be looking at proof by contradiction and proof by contraposition - methods that can offer alternate ways of entering into a proof when the direct approach proves difficult. There is no hard and fast rule I can give him that will tell him how to prove any statement. I tell him that only by reading and doing lots of proofs will he get better at knowing what might work.

As the semester progresses, we look at these different proof structures, and Micah is able to develop his own systematic way of determining what kind of proof he is reading, or what may be a helpful way of proving different types of statements. Categorizing and organizing in this way comes quite naturally to him. "We only have a certain set of tools, so something has to work," he says, "It will fit into one of the structures, so you just have to figure out which one it is."

> *Micah has a very organized mind, with an ability to recognize/identify* ***structures****. This is another form of systemic thinking, different from instances where he is understanding/using/working within a system of mathematical concepts/ideas; in this case he is creating his own system for analyzing proofs and problems.*

*****

Donnie, Janelle and Mariola are working on the Bridges of Königsberg, and they are stuck. They have created a second graph where they took one of the bridges away and found a path, so I asked them why this change made it possible, what did they think was different, but they had no ideas.

Micah interrupted our discussion to ask a question about odd and even vertices. I asked him to explain what he was thinking, and he gave a good explanation: he spoke in terms of comings and goings and concluded that any vertex with odd parity would have to be the one you start at, or one you finish on.

Like Akara, Micah identifies the important variables/elements in the problem quite quickly. He is beginning to develop a better ***intuition*** *for mathematical problems.*

It was a very clear explanation, but when I looked back to Mariola, Donnie, and Janelle, it was clear they had not been paying attention. I asked them if they understood his explanation, and they said they weren't listening. I was very annoyed by this, especially because it was not the first time that the group was not listening when Micah was giving an explanation.

Micah had joined in the conversation I was having with them in order to offer his help. He seemed annoyed that he had to repeat himself, and that they appeared to make no effort to understand what was being said, even as he was explaining a second time. When I came by a bit later, they had not progressed any further with the problem. At this point, I am finding it increasingly frustrating to work with this group because it often feels they are not giving a full effort and if the answer or explanation is not provided by a teacher, they just seem to stall out.

At the beginning of the semester, their group would often be the most engaged in discussion, usually spurred by Micah who would ask questions and encourage others to explain their reasoning. Towards the middle of the semester, the engagement really dropped off in this group.

> *Micah was really the driving force behind the discussion and* ***collaboration*** *in this group. Once he was not willing to keep the discussion going by asking questions and demanding explanations, it diminished noticeably in this group.*

I had the chance to ask Micah about this and he told me that he got lazy as the semester progressed. He has always been the type of person who will do as much work as is needed to succeed, but no more. At the beginning of the semester many of the terms and concepts were new to him and he did not understand how to read or prove statements, so he worked hard and engaged lots. As the semester progressed, he began to feel more confident about proofs. If he read a statement and knew more or less how to prove it, he was less interested in actually doing it. He was not the only one in the group who slackened their effort, and once he felt the others were not as committed to listening or working as hard, he became less willing to put effort into explaining.

> *For Micah, discussion is a way to engage with something he doesn't know, it is the best way to explore a topic and learn new ideas. As the semester progressed, Micah felt more confident and comfortable with the material and as a result less interested in discussing it. Micah likes to engage in* ***discussion as inquiry****, but is less interested in discussing something he understands well.*

DONNIE

> ***Activity 16***
>
> Theorem: For any integer n, if n^2 is a multiple of 3, then n is a multiple of 3.
>
> Proof 1: If n is not a multiple of 3, then n = 3k + 1 for some integer k. We get then that $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Therefore, n^2 is not a multiple of 3. QED

Proof 2: Let n^2 be a multiple of 3. Then $n^2 = 3q$ for some integer q. By uniqueness of prime factorization, 3 is in the prime factorization of n^2 . Suppose 3 is not in the prime factorization of n and let $p_1...p_s$ be the prime decomposition of n; then $n^2 = p_1p_1...p_sp_s$ but $3 \neq p_i$ for all i = 1, ..., s. Contradiction. QED

Proof 3: Let n be an integer such that $n^2 = 3q$ for some integer q. Then nn=3q which implies that 3|n; therefore, n is a multiple of 3. QED

Proof 4: Let n be an integer such that n^2 is a multiple of 3. Then $n^2 = (3q)^2$ for some integer q. So $n^2 = 9q^2 = (3q)(3q)$. Therefore, n is a multiple of 3. QED

Proof 5: Let n be a multiple of 3, then n = 3k for some integer k. Then $n^2 = (3k)^2 = 9k^2 = 3(3k^2)$ so n^2 is a multiple of 3. QED

Proof 6: If n is not a multiple of 3, then n = 3k + 1 or n = 3k + 2 for some integer k. We get then that $n^2 = 9k^2 + 6k + 1$ or $n^2 = 9k^2 + 12k + 4$. In either case, n^2 is not a multiple of 3. QED

Questions:
Are these proofs? Why? Why not? What proving technique is being used? Rewrite the proofs “step” by “step” and explain every “step” (why it’s correct or why it is incorrect).

When I return to the back table, Donnie and Leona are reading through the proofs, and are having difficulty deciding which proof method is being used in each instance. Donnie asks me to explain again the difference between contrapositive and contradiction. This is something Donnie has been having trouble with all semester; it is not the first time he has asked me to explain these definitions to him. I am not sure why he has so much difficulty remembering or internalizing the definitions, but he seems to need them explained again every time they come up.

In the second proof, Donnie tells me, “It is a proof by contradiction,” and when I ask him why, he replies, “because it has the word contradiction right there, so you know it has to be.” He does not really understand how the contradiction arises, or why it was a proof by contradiction. It is true that the way the proof is written, it is hard to pull this information out.

As I am reminding him that the negation of P -> Q is to have P and not Q, Leona interjects, “Yeah, I still don’t get that! Why is the negation like this?”

I re-explain the truth table for implication and show them where it comes from. As I am writing down the truth table, both Leona and Donnie get confused about why the implication statement is true whenever P is false. I know that this is not the point of today’s exercise, so I remind them that they are right to find this non-intuitive, that it is a rule of mathematical logic, not human intuition.

They really seem to struggle with this aspect of mathematical thinking – understanding that the meaning of an object is given by its definition. Donnie in particular is always flipping back through his notes because he cannot remember any of the rules.

Donnie is a very persistent student. Even though he is having difficulty in the course, he continues to work hard, and really wants to understand how everything works. Although others at his table sometimes “slack off” during class time, Donnie is almost always focused on understanding the problems.

Reflecting on my experiences in this course, one of the open questions, for me, is why Donnie had such difficulty in the course. He is a very hard-working and committed student; I can tell he is giving his full effort and genuinely wants to understand (not just apply a rule or formula by rote), but he does not made much progress over the course of the semester. Why not?

*Donnie does have a **questioning** attitude and wants to understand the material. He is not satisfied with simply knowing what to do and, like Micah, he will ask questions. But for some reason, he does not seem to get to the point where he understands. Although he is **persistent**, it seems like he is **not reflective**, and so does not benefit from his experiences - they are not (mathematically) educative.*

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On the day of the last quiz, Nadia gives students the option of writing a quiz to hand in, or if they were already happy with their grades from previous quizzes, there were some other problems to work on. Some students seem quite stressed out, mostly those doing the quiz and who have not yet received a passing grade on any of their quizzes. Donnie is quite frantic trying to get all the answers before it was time to write his quiz. I realize that Donnie's weakness is that he works too quickly on problems. I have this feeling whenever I work with him (it is particularly noticeable today because of the added stress of the quiz) that he is in such a rush to get the answers and have an explanation that he does not take any time to pause and think about what he is saying.

Having a series of experiences does not guarantee those experiences will be educative. Donnie is an example of a student who works hard, does all the activities in class, but for some reason does not seem to be internalizing. We would occasionally observe him exhibiting features of mathematical behaviour in one situation, but in a similar situation a week later, he might be unable to recall what had led him to success previously.

His method is to quickly write down his first ideas of how to do something, and then ask one of the teachers if it is correct before even reading it over to himself. Often he has made several mistakes that are easy to spot, but he has not taken the time to look at them.

He asks me questions about what the theorem of induction means, and how induction works, and becomes confused about the structure of the statement. When working with him, I can never tell if he does not understand something or if he is just speaking carelessly, quickly, and so ends up saying something which is incorrect. I do not have the sense that he takes time to reflect/formulate his thoughts before asking questions, and it seems that he does not think carefully about the meaning of the words he is using.

CHAPTER 8: Discussion

> *Abstract theoretical categories might be uppermost prior to research, but participants, and one's relationship to them, are key by the time the research text is to be written. The researcher learns that people are never only (nor even a close approximation to) any particular set of isolated theoretical notions, categories, or terms. They are people in all their complexity. They are people living storied lives on storied landscapes*
>
> *(Clandinin & Connolly, 2000, p. 145)*

Nadia’s main goal, as a teacher for the course, is to provide students with opportunities where they could learn/develop/practice/engage in mathematical behaviours – that is, opportunities where they can experience mathematics the way that mathematicians do. From our perspective, the only way of knowing that such an opportunity was given is if we see the student behaving mathematically in response to the situation. In the first iterations of her design experiment, Nadia and her colleagues collected and analyzed the student’s written productions (quizzes, assignments, online forum posts). These data were insufficient to discuss/characterize the mathematical behaviour that emerged in the classroom, since they could not capture conversations that took place, interactions between students, their discussions and questions.

This thesis seeks to address this ‘methodological’ difficulty. The two principal goals of this thesis are to construct a characterization of mathematical behaviours and to explore a methodological approach that may allow us to talk about, and report on, these behaviours as they emerge in the classroom. The results of this thesis take the form of the stories presented in the previous chapter: stories of students behaving mathematically in a particular time and place, an account of our shared experiences. The purpose of this chapter is to discuss the mathematical behaviour that is visible in these experiences (as ‘reported’ in the stories in chapter 7) and place them within my characterization of mathematical behaviour (presented in chapter 4). This chapter also discusses the exploration of narrative inquiry methodology in the context of mathematics education research. I outline some of the advantages and limitations I experienced using narrative inquiry in this research, and discuss some possibilities for future implementation of this methodological approach.

The first section of this chapter is devoted to responding to the first goal mentioned above: providing a characterization of mathematical behaviours. In Chapter 4, I generated a non-exhaustive list of features of mathematical behaviours, drawn from the literature, as a starting point for the characterization of mathematical behaviour I wanted to develop in this thesis. Although every feature in that list was in evidence in the classroom at some point, I have chosen, in this chapter, to provide further discussion on each of those features which seemed most significant to me and to the research group. By significant, I mean those instances where the experiences reported in these stories allowed us to gain a richer understanding of mathematical behaviour, where we felt there was something we learned. In many cases, although students were engaging in the same type of mathematical behaviour, they did so very differently, and so by comparing the behaviour of different students, I seek to create a "thicker" description of that feature.

In the second section of this chapter, I address the second goal of this thesis, which was to explore a methodological approach that may allow us to talk about, and report on, these mathematical behaviours as they emerge in the classroom. Overall, I feel that the narrative inquiry methodology allowed me to report on these behaviours, and provided a methodological framework for understanding and interpreting what I experienced. I will discuss what seems to have worked and what did not, what this methodology and framework allowed me to see, and what it did not, and make some comments about future research possibilities.

In the third and final section of this chapter I reflect on how these mathematical behaviours may be fostered and developed in institutional educational environments, referring in particular to what insights the research in this thesis can offer to instructors like Nadia, in her continued quest to engage students in mathematical behaviours. Insights and discussion around how to engage in the teaching of this course are a by-product of this research. It was not something I set out looking for, but the experiences we shared led to some potentially useful conclusions, so I mention them here.

8.1 Mathematical Behaviour

In this section I discuss the mathematical behaviour that is visible to me in the stories presented in the previous chapter. For each element of mathematical behaviour in my model (as outlined in chapter 4), I provide a brief discussion of the examples of that behaviour, in some cases drawing direct comparisons between two or more students to highlight individual variances. This is not to suggest that these examples were the only instances when I observed any particular behaviour during the semester, only that these are strong examples, and that they offer significant, meaningful insight into how these behaviours emerged in the classroom.

After treating the different elements of the model, I discuss some general observations/conclusions about the mathematical behaviour of students in this course. The section concludes with a discussion of some of the questions that were raised by this research, which remain open and are potential avenues for future study.

<u>Collaboration</u> **(see p. 68)**

Many students enjoyed that this course offered them opportunities to discuss and collaborate on mathematical problems, something which few other undergraduate courses in mathematics offer. However, not all students are collaborators. It is important to note that it is not Nadia's goal, as a teacher, to have everyone participate in the group discussions. Rather, her goal is to provide students with the space they need to practice mathematical behaviour. Although the course has a "group work" format, some students do work alone, or prefer to use the forum for communication or debating rather than the classroom space. Javier, for example, enjoyed the camaraderie of his group, but once it was time to work on math problems he would withdraw and work on his own (see pp. 151 - 152 of this thesis). Another example is the case of Stephanie, who would engage in group discussions in class because she felt that was what was expected of her, but preferred to contribute to on-line exchanges (p. 140).

Many students enjoyed collaborating with their groups. For Micah, discussion is a way to engage with something he does not fully understand, and is the best way to explore a topic and learn new concepts. This is a practice he was exposed to quite a lot in his time studying at Yeshiva, and as a result, Micah likes to collaborate and discuss as a form of inquiry; he is less interested in discussing something he understands well. At the beginning of the semester, most of the concepts we were discussing were new to him, and he engaged actively in discussions, often leading his group and encouraging the others to contribute/participate. As the semester progressed, Micah felt more confident and comfortable with the material and as a result became less interested in discussing it. On the occasions where he understood a problem and the others did not, one of the teachers might try and encourage him to explain it to them and help them out, but this was not something he enjoyed doing or engaged in readily. As he explained it to me, he prefers to discuss with those who can help him advance his own understanding. Micah genuinely enjoys discussion and collaborating, but only when he feels it is helping him learn more. Otherwise, he does not see the point. (pp. 179 - 180)

Hannah, by contrast, likes to assume a peer-tutoring role and explain things to others. For her, the group discussion is a space where those who understand a given problem should explain it to those who do not, and she prefers to be the one who is explaining. She is less comfortable with having others in the group explain things to her, and often has difficulty understanding interpretations of a problem which are different than her own (pp. 121 – 122). Like Micah, she genuinely enjoys discussion and collaboration as a way of understanding and learning, but she is less able to consider multiple points of view.

Jonathan, like Hannah, enjoys taking on a peer-tutoring role in his group and explaining his thinking and his solutions to the others at his table. He is very able to explain his ideas in a way that others can follow and understand. Unlike Hannah, however, Jonathan is more receptive to having things explained to him and is more flexible when it comes to considering multiple points of view. Jonathan is more comfortable than Hannah with having his own ideas/suggestions questioned by others in the group, and sees value in having multiple interpretations of the problem. He actively encourages the other members of his group to explain their thinking and asks them questions.

Akara, on the other hand, sees the group discussion as a space for validating her work and ideas. She listens to the explanations that others provide and asks questions. When she thinks she understands something or has a solution, it really matters to her that she can explain it to another person in her group and have them understand and agree with her (p.

133). I believe this desire stems equally from wanting to make sure she is explaining using the right words since she is still not completely confident in her language skills, and from her sense that a result needs to be validated by her peers in order to have value.

Reflective Thinking (see p. 65)

The narratives in chapter 7, and my analysis of them, provide examples of students engaging in reflective thinking. For example, Scott told me in our interview that he enjoys doing "math for math's sake" (p. 159). He gets satisfaction from thinking about and working on problems, trying to come up with different ideas and approaches, and looking for patterns. He liked that the structure of this course gave him time and space to 'wander' through a problem, play with it, and try things out, without needing to find an answer right away. His favourite type of problem is one where he encounters something which challenges his intuitive understandings of a concept and forces him to shift his way of thinking. In these cases, Scott is not focused on any practical end; the purpose of such thinking is further thinking, inquiry, and speculation.

I feel Hannah, with her background in logic and philosophy, was certainly exposed to reflective thinking before she began this class. She is someone who enjoys thinking for the sake of thinking, and so it is no surprise that she showed a proclivity for reflective thinking when working on the problems in this course. Sierpinska et al. (2002) characterize a reflective thinker as someone who

> considers him or herself, to a certain extent, an "owner" or a participant in the construction of the meaning of the concept. He or she feels entitled to apply it

> freely and to change it, adapt it to new problems. This implies the belief that concepts are human creations (p. 17).

Hannah regularly questioned categories and definitions, and reflected on what counts as mathematics and what makes someone a mathematician. She understands that the definition of "mathematics" and "mathematician" are human creations, are not necessarily fixed, and thus she feels entitled and able to question them. The analogy of comparing mathematicians to the question of what makes someone a poet is a great example of reflective thinking (p. 120).

However, there are notable differences between how Hannah and Scott engage in reflective thinking. Possibly because he has more previous experience studying mathematics, Scott is more comfortable with the malleability of mathematical concepts. He enjoys having ideas about a problem or concept challenged, and takes pleasure in having to shift his way of thinking to adapt to new situations. Hannah is also able to shift her thinking, but seems to enjoy it less. She likes well-defined categories, and is less flexible than Scott.

<u>Systemic Thinking</u> (see p. 65)

Although there were many examples of students engaging in systemic thinking, I feel that Micah was one of the 'most' systemic thinkers of all the students we met in this course. Micah is very sensitive to logical consistency and structure, and has a fine eye for details and structure, something which I feel relates to his previous learning experiences. Studying at Yeshiva trained him to look carefully at the details, be sensitive to contradictions, and question the logic and structure of arguments. This exposure seems to transfer well, for him, to the study of mathematics. In particular, when he first encountered proofs and proof-writing in this course, he examined the proofs he read very carefully, and would not move on to something new until he understood the meaning of every step.

Another example of a student engaging in systemic thinking is Stephanie, who demonstrated a very strong ability to understand that the meaning of a mathematical object is contained in its definition. She is very good at articulating the nuances of these axiomatic definitions, and their accompanying notation, in her assignments and in her forum posts (pp. 140 – 141). Often, she would post a response to another student clarifying the meaning of a term, or correcting an inappropriate use of notation or terminology.

Although Stephanie and Micah are both engaging in systemic thinking, it is worth noting that they are not doing so in the same way. Sierpinska et al. (2002) identify three subcategories of systemic thinking: hypothetical, proving, and definitional. Stephanie's systemic thinking is very definitional; she recognizes that the meaning of a mathematical concept is stabilized by its definition. Micah, on the other hand, exhibits aspects of systemic thinking which fall more into the subcategories of hypothetical and proving. He is deeply and genuinely concerned with the internal consistency of arguments and uncovering implicit assumptions. This is not to suggest that Micah never engaged in definitional thinking, or that Stephanie was not able to think hypothetically, but that each exhibited certain aspects of systemic thinking more strongly than others.

Javier, on the other hand, is an example of a student who engages readily in hypothetical thinking, can read, understand and write proofs, but struggles with definitional thinking. In one of our earliest classes, where we worked on problems that used Peano's axioms, he showed a good sensitivity to assumptions and logical coherence, both in the way he approached the problems and in the type of questions he asked about them. But once we got to working on induction, he could not wrap his head around the axiomatic nature of a definition (in particular the definition of an implication statement), which is another aspect of systemic thinking. It is a problem he had throughout the course, although it became particularly problematic when he tried to understand proofs by induction. He would repeatedly get stuck whenever he was attempting a direct proof on an implication statement because he felt he could not assume the hypothesis was true (pp. 149 – 151). He understood the structure of these proofs, and how the implication statement is defined, but for whatever reason whenever he was actually doing the problem, he would invoke one aspect of his systemic thinking (sensitivity to assumptions) and not the other (understanding that meaning is found in definitions). It created this cognitive obstacle for him, which he was very aware of, but still caused him difficulty.

Analytic Thinking (see p. 66)

With the notable exception of Stephanie, the narrative data I gathered and the subsequent research texts I wrote contain very few examples of students engaging in analytic thinking. Stephanie's analytic thinking is evidenced by her written assignments and forum posts. For the most part, the students we saw in this course were not sensitive to linguistic or meta-linguistic concerns when they were working on mathematics. They are often careless with the words they use and do not see the importance/value/necessity of using precise language. Stephanie would frequently post clarifications on the on-line forum for students who were misusing/abusing notation (p. 140 – 141). The part of this course that she appreciated the most and felt was the most beneficial to her, was what she learned about writing mathematics. She was never taught syntax, notation, and how to write a mathematical argument in her high school math courses, and was glad of to have an opportunity to improve these skills because she prefers communicating in written forms.

To me, the fact that we did not observe many students engaging in analytic thinking indicates that there is something still imperfect about the opportunities for analytic thinking offered in the course. During class times, students are more likely to be engaged in thinking and discussing than in writing. The only times students are really asked to write is when we ask them to hand something in, such as an assignment or a quiz. Nadia wrote in one of her narratives: "it is important for them to develop a sense of whether what they write is correct or not, regardless of whether they know how to write it correctly." The students' behaviour may be evidence that they are given fewer situations in which analytic thinking is a necessity.

Persistence **(see p. 67)**

During the inquiry, we saw many examples of students being persistent. For example, Hannah is very persistent when working on mathematical problems; once she starts working on a problem, she will stick with it until she figures out a solution. She sees this persistence as an important strength, and it often serves her well. She does not give up easily on problems, but sometimes she can be persistent to the point of stubbornness. She will have an initial idea about how to do a problem, and even when other ideas or approaches are suggested by her group she will continue trying her first one. She says this is because she has a hard time understanding others and she can't make room for other ways of thinking until she is certain she has exhausted her own.

Hannah's persistence, the way she becomes fixated on problems, is a mathematical behaviour, but it can also be a weakness for her. One of her main difficulties is that she lacks the ability to self-regulate: she cannot put an approach aside when it is not working and try something different.

Scott is also a persistent problem solver, like Hannah, but he has a better ability to self-regulate, to change strategies when the one he is using does not seem to be working. Scott is more comfortable with trying many approaches, without knowing in advance which will be most useful, and is not discouraged when an approach is not successful. He will leave a question and come back to it, and on several occasions continued working on a problem over several weeks.

Micah is persistent about asking questions, something which we sometimes referred to as being *skeptical*. He will not believe something just because a teacher or a classmate told it to him. He will question a statement or a proof very persistently until he finds an explanation or justification with which he is satisfied. I feel this persistence was evident when he was having difficulty understanding Nisha's proof, and asked her several questions about it on the forum over the week-end (p. 176). Nisha, for her part, was satisfied that Nadia had validated her work, but having a teacher stamp-of-approval did not satisfy Micah, because he still did not understand the explanation. At our next class, he approached Genevieve to discuss his confusion, and the three of us ended us discussing it for some time. Once Micah has asked a question, he will persist until he finds an answer.

Self-Regulation (see p. 66)

The narratives presented in chapter 7 contain several examples of students using self-regulation when solving problems. Self-regulation seems like an important mathematical behaviour to have paired with others. Hannah, for example, is a very persistent problem solver, but since she does not self-regulate, her persistence can become an obstacle.

Scott often uses self-regulation, and this is part of what made him such an effective problem solver. In our interview, he compared solving mathematical problems to rock climbing puzzles and spoke of the importance of planning. When he is solving problems, Scott is usually thinking a few steps ahead of whatever he is working on, thinking about whether it is likely to lead to a resolution or not. There was even one day when I was sitting with Scott's group and noticed he was letting others know what strategies he had tried and why they had not worked and warning them of difficult or tricky algebraic manipulations that were coming up (p. 154). He was essentially being a self-regulator, but for the entire group.

<u>Obsessive Behaviour</u> (see p. 69)

The main difference between obsessive behaviour and persistence is that obsessive behaviour has an unconscious element to it; the brain keeps working on a problem after one has moved on to other tasks. For example, Jonathan had a sudden realization while driving to school that the solution he submitted in his homework assignment the previous evening was incorrect because he had forgotten to consider separate cases (p. 165). Scott was on a bus headed home for the long weekend when he suddenly realized that the problem he was working on might be made simpler by considering odd and even cases separately (p. 158). Both are examples of this type of behaviour.

It seems likely that obsessive behaviour goes hand-in-hand with personal/emotional connectedness (being personally and emotionally invested in the problems at hand). Both Scott and Jonathan demonstrate that they are invested in the problems, and that they have taken the task as something more than a school-task to be completed for evaluation. Mathematicians don't work on problems because they were asked to, but because the problems have some significance and are in some way important, because there is a desire to know the answer that drives their work.

<u>Personal/Emotional Connectedness</u> (see p. 69)

One example of personal/emotional connectedness is the 'obsessive behaviour' that we saw in Jonathan and Scott, as discussed in the previous section. Other evidence of a personal connection could be that a student experiences frustration when having difficulty or a sense of pride/pleasure when something is resolved. Seeing Akara's face break into a smile when she was able to construct a difficult proof for the first time, it was clear she was emotionally connected to the work she had done (p. 128). Javier wrote that he has a love-hate relationship with mathematics. Working on a problem can make him feel very frustrated if he cannot solve it, but he also describes feeling great satisfaction and pleasure when he is able to see what had been missing (p. 153).

Self- Efficacy (see pp. 67 – 68)

Self-efficacy and persistence are related behaviours; if a student has strong self-efficacy, they are more likely to be persistent when solving problems (Selden & Selden, 2013). Problem solving situations are not the only sites where students in the course exhibited self-efficacy. Several students exhibited self-efficacy in their general attitudes towards learning, and their ability to recover from setbacks.

Marco is an example of a student who showed strong self-efficacy throughout the course. He did poorly on his first quiz, and when we met in our interview, he told me that this was a bit of a surprise because he was used to doing well in math courses. He recovered quickly from this setback. Realizing that there were some new concepts in this course and that he was not understanding or communicating at the level that was expected, he set to work on improving. The fact that he was having difficulty was motivation for him to work harder. He was particularly hard working and determined.

Likewise, Akara had some difficulty at the beginning of the course. With a stronger mathematics background than Marco, her difficulties were less about understanding the problems, and more in communicating her answers in acceptable mathematical discourse. When she did not do well on her first assignment, she requested a meeting with me. She was not upset or perturbed by the low mark, but wanted to know what she could improve for next time. She truly seems to see her mistakes as an opportunity to learn, and is strongly committed to understanding.

For Javier, self-efficacy is evident in his general attitude towards learning and in particular towards having difficulty and making mistakes. In our interview, I was asking him about solving problems and he said, "I hate it when I can't get it, that frustrating feeling. But once you get it, it can be the greatest feeling in the world, when you see where you were wrong" (p. 153). The fact that he associates the 'greatest feeling' with seeing where he was wrong rather than where he was right is an indication of his strong self-efficacy. Javier views challenging problems as tasks that can be mastered, and sees mistakes as important sites for learning.

Reflecting on one's own thinking (see p. 68)

This element of mathematical behaviour is deeply connected to the theoretical framework of this thesis. The central goals of this thesis are concerned with observing and characterizing mathematical behaviour as it emerges in the classroom. The fact that we observe a mathematical behaviour only confirms that the situation we have designed offered that student an opportunity to behave that way, but does not guarantee that the experience will be *educative* (in the sense of Dewey, 1939). This takes something else. I believe/claim/suspect that reflecting on your own thinking and behaviour in the way that Mason et al. (2010) describe is an essential factor in determining whether an experience is mathematically educative or not.

According to Dewey's theory of education and experience (chapter 3), just having an experience is not sufficient for that the experience to be educative. Reflecting on one's experiences, identifying one's own behaviours and ways of acting/engaging in the problem are essential actions for learning. Reflecting on one's own learning can be helpful/effective in many different ways, and seems to be an important complement to developing other mathematical behaviours. Consider Jonathan, as an example; reflecting on his own previous experiences is what has helped him develop a good intuition about problem solving. Not every kind of reflection will do; it requires a particular way of assimilating that experience into his existing experiences. He looks for commonalities, patterns, and ways in which the situation might be generalizable. Not every student approaches learning situations from this vantage point, looking for what they can learn from that situation.

Reflecting on one's own learning is not only important for problem solving situations. I see evidence of this type of thinking in Marco's explanation for why he chose not to sit with Hannah's group any more. He was reflecting on his own learning, and felt that the experience of working with Hannah's group was not providing him with an educative experience. One could imagine a different student who would be glad to have someone like Hannah at their table - someone who would do the work and get the answers for them. But Marco wanted to be able to understand on his own and he felt that working with Hannah in a dynamic where she explains everything to him was not going to be as educative (p. 160).

I suspect that a lack of this feature of mathematical behaviour is one of the reasons Donnie had so much difficulty in this course. He was very persistent and hardworking, but often did not seem to be reflecting on what he had learned in a way that would make his experiences mathematically educative.

Donnie's difficulty in the course raises some interesting questions about mathematical behaviour, which could be avenues for future research. It seems that, considering all the features of mathematical behaviour listed in the model, there may be some subsets of those features which are more *effective* than others. By effective, I mean that they might enable a student to have a more educative experience. Some features of mathematical behaviour seem to reinforce one another, such as self-efficacy and persistence, since individuals with strong self-efficacy are more likely to be persistent. In other cases, it seems likely that certain features of mathematical behaviour must both be in evidence in order to enable mathematically educative experiences. In Hannah's case, persistence which is not accompanied by an ability to self-regulate can become an obstacle to making progress on a mathematical problem, and in Donnie's case, persistence that is not accompanied with reflection proves inadequate for developing understanding. Further study would be required to better understand these relationships.

In this section, I have highlighted the significant features of mathematical behaviour which were visible to me in this inquiry. In cases where there were many examples of students with a particular feature of mathematical behaviour, I have sought to delineate the nuances between them, to create a richer characterization of mathematical behaviour. In the above discussion, I was able to say more about certain behaviours than others in my discussion and in the stories. This is part of the nature of a narrative inquiry, and should be noted that this does not suggest that the model for mathematical behaviour needs to be changed, only that the nature of the situation/experience means that I am able to say more about some behaviours than others. It makes sense, for example, that I have lots to discuss on the subject of collaboration, as this was something that I saw students engaging in (in one way or another) every day - the nature of the class work space was such that they were always talking to one another. How a student reflects on their own thinking was less visible to me in our day-to-day interaction, and was something I tried to delve into in our interviews. As a result, I feel I have a richer picture of the collaborative behaviour of individual students than their reflective behaviour or their beliefs about mathematics. This does not mean that students were engaging in collaborative behaviour more than in any other, only that it was more visible in our shared experiences.

In other words, I have discussed only those features of mathematical behaviour that the methodology allowed me to see and understand. In the following section, I will elaborate on the affordances of narrative inquiry methodology, however, it seems important for me to remark at this point on what I perceive as an important limitation of this thesis in terms of characterizing mathematical behaviour. That is, that not all the behaviour that I describe in my model (chapter 4) and in the storied accounts presented in chapter 7, is specifically mathematical in nature. Some of the features of mathematical behaviour in the model are not specific to mathematics and mathematicians, and are in fact characteristics common to many disciplines. Collaboration, persistence, self-efficacy, or being personally connected to one's work, are examples of features that, although present in many mathematicians, may not be in and of themselves specifically or inherently mathematical. This raises the question: is it possible for students in this course to be persistent, but perhaps not mathematically persistent? In discussing Akara's persistence, for example, what is it that makes it mathematical persistence, besides the fact that she is working on a mathematical task? Or is the context of the task enough? In the case of Jonathan's collaboration it seems as though collaboration is certainly an essential part of who he is as a person, but is there anything specifically mathematical about his collaboration in the course? If there is, I feel the nuance/detail of the specifically mathematical nature of some features/behaviours may be incompletely captured in this research.

The storied accounts I present in chapter 7 certainly focus much more on social behaviors, social relations and affect than specifically mathematical behaviors. It is not clear if this is something that is inherent in the methodology I have chosen to explore, or a result of the fact that my fellow researchers and I were new to narrative inquiry and perhaps need to become more skilled in using this methodology in order to reveal this type of behaviour. In the following section, as part of my discussion of the limitations of narrative inquiry in this thesis, I will offer some suggestions for how this type of research might be conducted in the future to better address this difficulty.

8.2 Methodological Affordances

> *We need to be wakeful about what we are doing as narrative inquirers, so we can continue to learn what it means to do narrative inquiry. (Clandinin & Connolly, 2000, p. 184)*

Erin: So, now that we are nearing the end of this inquiry, it seems a good time to reflect on what we have been able to do with narrative inquiry. In what ways does this methodology respond to our needs as researchers and as teachers?

Nadia: Well, I feel like it certainly gave us space to talk about and make a record of how the students were behaving in the classroom, which was one of our goals. I enjoyed writing narratives, although it was sometimes very difficult to make time, and there were some weeks where I did not write as much as I could have or would have wanted to. It was good to have the weekly meetings as a space to share things that may not have made it into my narratives.

Erin: I was thinking about that recently too; there was definitely more that we could have written down, more field texts which would have been interesting to generate and reflect on. I think the meetings were really fruitful, but I was not always diligent about taking notes or writing reflections about what we discussed. It might have been a good idea to have all three of us writing reflections about our discussions in the meeting as part of our weekly writing commitment.

Nadia: True, there are many other ways we could have done things. It could have been interesting to have students do more writing as well, maybe a journal-writing activity. And this is something I may try in the future. What is great about this experience is that it gave us a first idea of how this methodology can work, where we might be able to take it in the future.

Erin: During this inquiry, I was constantly dogged by a feeling that time was passing by too quickly, and that there was not enough time to create field texts about all of my experiences. I still feel that there are things we missed; more we could have looked at.

Nadia: It is important to remember, too, that all three of us had many other responsibilities, other academic commitments, family lives; there was a limit to how much of ourselves and our time we were able to commit to this project. But those limits are part of the experience, as well. As researchers, we are not machines who record everything; we are people who are living an experience (amidst many other experiences) and trying to make sense of it.

Erin: You're right, of course. When we create field texts we are constructing a representation of our experience, so naturally other representations or constructions will always be possible, and narrative inquiry isn't intended to arrive at final conclusions of any kind. But all the same, it is hard to be completely comfortable with the sense of incompleteness.

Nadia: I know it is hard, especially as you are finishing writing your thesis, and there is this temptation or maybe a desire to seek answers to the questions you began your research with. Remember that the purpose of this research is not to draw final conclusions, but to offer an account of your experience so that a reader can slip into it, try it on. You can't try and answer every question; that is not the point.

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One of the main goals of this thesis was to explore a methodological approach that would allow us to talk about, and report on, mathematical behaviours as they emerge in the classroom. This is the section where I mean to discuss whether using narrative inquiry as a methodological framework was 'successful' with respect to this goal. But before I reflect on this question of the 'success' of the methodology I have used, I feel I must first offer some discussion about what it means for this methodological approach to be successful.

When I say, for example, that the goal of this thesis was to explore a methodological approach that would allow *us* to talk about, and report on, mathematical behaviours as they emerge in the classroom, it is important to clarify that '*us*' refers only to the members of our research group, chiefly Nadia and myself. In his plenary address to PME in 1994, John Mason said,

> I see working on education not in terms of an edifice of knowledge, adding new theorems to old, but rather as a journey of self-discovery and development in which what others have learned has to be re-experienced by each traveler, re-learned, re-integrated, and re-expressed in each generation (p. 177).

He went on to say that, "the most important effect of educational research is on the researcher themselves. What a researcher finds out most about is themselves" (p. 179). Using narrative inquiry allowed us to capture and discuss our experiences of the behaviours we saw emerging in the classroom, while at the same time changing and shaping the nature of those experiences. The field texts we constructed, the discussions we had, allowed us to create, for ourselves, rich remembrances of our own experiences, to reflect on these experiences, and to learn from them in ways that were meaningful to our own development as teachers and as researchers. In this section, I will discuss some of the important reflections that this research triggered for us.

When considering whether the methodology was 'successful', it is enough, for us, that this inquiry triggered important reflections, for both Nadia and I, on our own practices as researchers and as teachers. Nevertheless, as discussed in chapter 5, it is my hope that the results of this thesis will be *appropriable* for other researchers and teachers. That is, that the stories and discussion I have presented as the result of this research will provide opportunities for readers to engage in *resonant remembering* (Clandinin, 2013) as they lay their experiences alongside ours, and to rethink and reimagine their own experiences and the ways they relate to others. As Peshkin (1985) writes:

> When I disclose what I have seen, my results invite other researchers to look where I did and see what I saw. My ideas are candidates for others to entertain, not necessarily as truth, let alone Truth, but as positions about the nature and meaning of a phenomenon that may fit their sensibility and shape their thinking about their own inquiries (quoted in Clandinin & Connolly, 1990, p. 8)

To create a reconstruction of my own experiences that will have meaning and value to other researchers and teachers is part of my intention for this thesis, but I cannot comment (yet) on the success of this. To answer that question would require a different dimension of inquiry. I feel this would be an interesting question to explore: who are the readers (aside from researchers and participants involved) for whom this narrative inquiry has meaning?

Thus, the main focus of the discussion in this section is on how this inquiry allowed me to develop as a researcher and a teacher. The first subsection discusses the writing process, and my experience with something called 'writing as inquiry', and how writing this narrative inquiry increased my ability to reflect as a researcher. In the second subsection, I present a dialogue between Nadia and myself about what we feel we have gained as teachers and as researchers from this narrative inquiry. In particular, we reflect on what 'living and thinking narratively' meant to us, and the insights it afforded us in this research. In the third section, I discuss the limitations of this methodology as a tool for capturing specifically mathematical behaviour, and propose some strategies for addressing these difficulties in future research. The section concludes with a discussion of how this thesis has sought to address some of the methodological issues raised in chapters 2 and 5.

Writing as Inquiry

The idea of writing as inquiry is something I came across while reading Carolyn Ellis' book, *The Ethnographic I* (2004). It became an important feature of my writing process, often helping me get started again when I reached a block. Writing as inquiry means removing any constraint of style and using writing to explore the questions one does not yet know how to answer or even to write about. Writing as inquiry allowed me to give myself freedom to explore ideas, writing styles, and the many possible stories that were happening, looking for those that reveal some deeper meaning. As Richardson (1997) writes, "I write because I want to find something out. I write in order to learn something that I didn't know before I wrote it. … Writing is a method of knowing" (quoted in Ellis, 2004, p. 171).

Writing field texts about our daily experiences in the classroom was a form of inquiry. We would write about what had happened and propose possible explanations, possible connections to previous experiences. Often, especially as we were beginning our narrative inquiry, these connections and explanations occurred to me only at the moment of writing. The act of constructing the field text gave a certain shape to my experiences. I saw connections and understood meanings in a way that would not otherwise have been accessible to me. My experience with writing was similar to what author Alison Bechdel describes: "it is by writing, by stepping back from the real thing to look at it, that we are most present" (Bechdel, 2012, p. 242).

Clandinin and Connolly (2000) write that one of the roles of field texts is to serve as memory signposts, allowing the researcher to return to tap into the basis of their experiences, to return to what they thought in that moment (p.143). But field texts are not only signposts, since they have the ability to transform memories as well. As I read and re-read a field text, returning to the memory of that experience, I continued to re-shape the experience in light of what has transpired since. Writing a narrative inquiry is a recursive process of re-writing and re-telling stories.

The writing process itself has been long and full of challenges. Writing field texts was difficult in the sense that it required diligence, but easy in the sense that I was writing an account of events that had just transpired, so details were fresh and (for the most part) easily recalled. Moving towards research text was much more challenging. There is no one moment where I stopped gathering data and began to analyze it. Analysis had been taking place already, as we constructed field texts, and my task in writing a research text was to gather all those strands and try to weave them together into something that could stand on its own. It required me to try to hold a sense of the whole in mind, while undertaking to compose individual parts, and there was a constant back and forth between reading (and re-reading) field texts and writing new interim research texts. In chapter 6, I mentioned some of the 'false starts' I had in trying to move towards final research texts. I would write a story or a section which felt like a final research text, and then return to re-read field texts related to that event and find myself re-interpreting the story again. Writing as inquiry gave me the freedom to try different forms and styles of writing, as well as different voices. I even experimented with writing brief accounts from the point of view of individual students, trying to imagine how their voice would come across in an account of an event. In the end, these texts are not included in this thesis, since I felt they lacked a certain authenticity. Nevertheless, the act of writing them was a way of learning something about each student, a way of exploring what I knew (and did not know) about them.

All this recursive, back-and-forth writing activity leaves me with the feeling that the 'final' research texts I have presented in this thesis are anything but. Clandinin and Connolly (2000) write about the importance of maintaining a sense of 'work in progress', even in final research texts:

> A written document appears to stand still; the narrative appears finished. It has been written, characters' lives constructed, social histories recorded and meanings expressed for all to see. Yet, those engaged in narrative inquiry know that the written document, the research text, like life, is a continual unfolding. (p.166)

The stories I have written are like a photograph: they capture only one particular moment. They may hint at what came before, what may come afterwards, but they can still only show one small slice of an experience which is still unfolding.

Thinking Narratively

Erin: As a teacher, one of the most important things I learned from this research was about trying to recognize where students are coming from. Some students arrive at the beginning of the course with some features of mathematical behaviour already integrated into their functioning (in some way or another), others are willing/interested in developing these behaviours. Still others are not engaged, not inclined to work on changing their thinking habits, or are only seeking to pass the class. It helped me to begin to think narratively about the students.

Nadia: You often talk about 'thinking narratively' - this is a phrase that Clandinin and Connolly use as well. What does it mean to you?

Erin: Thinking narratively meant that I considered each individual in the course as being in the process of living their own life story. People, at any point in time, are in a process of personal change and from an educational point of view, it is important to be able to narrate the person in terms of the process (Clandinin and Connolly, 2000, p. 30). So when you are thinking about a particular student, you consider more than just what they are doing in that particular moment. You try to imagine how their actions fit within the stream of their experiences.

Nadia: Because understanding more about individual students makes it easier, as educators, to construct situations where they can behave mathematically and see the value of those skills in relation to their own goals.

Erin: Yes, and it helped me to understand their behaviour in a more nuanced way. For example, both Micah and Hannah are very reflective individuals, who are constantly asking questions and wanting to understand deeply whatever subject they are engaged with. In understanding Hannah's mathematical behaviour, it *matters* that she is a philosophy student. Likewise, it matters that Micah attended Yeshiva for four years. These previous experiences are part of how they structure their present experiences. It gave me a better sense of how complex experiences are, something which Dewey was trying to capture, something that is hard to put into general, abstract terms. But in these particular examples, I can see it.

Nadia: We are not at a place where we can make general statements about student experiences, I agree. This research is a beginning, and any pattern we see is tentative, one possible explanation of many.

Erin: What about you, what did thinking narratively look like for you?

Nadia: This methodology we have used, writing narratives and reflecting on them, it has helped me to become aware of certain types of students. I have worked at constructing characters, fictional characters, which represent these different types. We talked about doing something like this for your thesis, I think.

Erin: Oh yeah, we talked about creating one character for each behaviour in the model, and then describing how these imaginary characters would behave or interact. But it became clear to me that this was not the right approach, because students have these complex combinations of behaviours. In the end I decided to stick with representing individual students and narrating their stories.

Nadia: Yes, and I think that was the right decision. But the character types I am thinking of are slightly different. I have taught this course for several years, and there are some students who I meet who are the same as students I have met before. Not exactly the same, obviously, but there is some important way in which they are like a recurrence of the same character. Thinking of students as characters in our stories helped me clarify these student archetypes. Now when I teach the course, I recognize those characters again.

Erin: You mean like when you told me that Marco reminded you of a student you had in a previous semester?

Nadia: Yes, Marco is one of those types of students. I saw him the other day, by the way. Not Marco, but the other student who reminds me of Marco. I ran into him in the hallway, and we talked for a bit. He is in the actuarial science program now. That program is very hard to get into, so he has clearly had to work hard, I was impressed.

Erin: Yeah, no kidding!

Nadia: But he was like Marco, hard-working and very determined, so it is no surprise that he has reached his goal, I think. They are students who are behind the rest of the class at the start of the semester, perhaps having less mathematical background, and work tirelessly and patiently to catch up to everyone else.

Another example is E______ and X_______, who were both in the course last semester, they were the same character as Akara, from our narrative inquiry. Like variations on the same theme, expressions of the same type of student: mature, hardworking, dedicated, patient students who listen to everything that is going on and really absorb it, and who are interested in and excited about mathematical ideas. I have started writing little sketches of these different types of students and I can already tell it will help me in the future.

Erin: Help you how? As a teacher or as a researcher?

Nadia: Both. Becoming aware of these types of students, and starting to pin down who they are, has triggered important reflections for me as a teacher. We talked a lot about how

> there are some students for whom this teaching method seems to work, and others for whom it does not. Now, as I continue to develop the course, I can think about these different types of students and reflect on how to address their different learning needs.
>
> Understanding these types of students has also been important to me as a researcher. I can use them as a means to think about research. It allows me to be curious about the different types of students, to pose questions about their learning that can be investigated further.

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It is important to understand that writing a narrative inquiry is both a research methodology and a personal experience for the researcher. At first, the act of constructing field texts shaped my experiences, but only as I was writing about them and reflecting on them after the fact. Writing about my experiences encouraged me to consider the different possible meanings of those experiences. As the semester progressed, constructing field texts began to shape my experiences as I was living them; this is what Clandinin and Connolly mean by 'living narratively':

> I believe that the ways of telling and the ways of conceptualizing that go with them become so habitual that they finally become recipes for structuring experience itself, for laying down routes into memory, for not only guiding the life narrative up to the present, but directing it into the future.
> (Bruner, 2004, p. 708)

This is something I only really became aware of the following semester, when I was a TA for Nadia once again in the same course, entering the class only as a teacher, not as a researcher, not as a narrative inquirer. Without the regular reflection and re-telling of my daily experiences, I was no longer living and thinking narratively about my experiences, and so I found that they were not as rich, deep, or meaningful. I did not spend as much time trying to get to know students, to understand them; I wasn't looking to generate meaning from our interactions in the same way. I realized that narrative inquiry is more than just a methodological tool, it is a way of living in an inquiry, which shapes the experiences you will have and the significance you will draw from those experiences.

Mason (1994) discusses the importance to researchers of developing and strengthening an inner observer (Mason et al., 2010; Schoenfeld, 1987) who can inform practice in the moment by awakening the functioning self to alternative possibilities. He defines *spection* (as opposed to retrospection) as being awake in the moment, noticing and responding freshly and creatively in the instant, catching oneself before embarking on habitual behaviour (Mason, 1994). Clandinin and Connolly use the term 'wakefulness' to describe a similar type of alertness and awareness, which is necessary for carrying out a narrative inquiry. Because of the amount of reflection (retrospection) we were doing in this inquiry, we became more aware of our actions as being part of the larger narrative, as having potential meaning and significance in both our own stories and those of our students, both in our reflections and *in the very moment of living*. And this is important. The methodology shaped my experiences, my awareness changed in the actual living and doing. To borrow Mason's terms: by engaging in lots of retrospection, I began to develop a capacity for spection.

Limitations of Narrative Inquiry and Suggestion for Future Use

As mentioned in the section 8.1, one of the shortfalls of this research is a certain inability to capture enough specifically mathematical aspects of student behaviour. It is not clear to me whether this is a function of my own interest and perspective as a researcher, or whether it is inherent in the structure of narrative inquiry. The storied accounts I present in chapter 7 focus primarily on social behaviors, social relations and the students' (often very strong) engagement with the course. Itnis certainly true that I was personally more drawn to telling these aspects of the story, as they seemed to me to be more generally relevant and interesting. I attempted to include more details in my accounts of the mathematical problems students were working on, and what steps they were taking. These accounts, however, did not seem as fruitful for reflection or making connections between different experiences.

When one considers that narrative inquiry is structured along three dimensions of inquiry - personal, social, and place – perhaps it is inevitable that the products of this type of research will have such a strong focus on the personal and social aspects of students' behaviour. Perhaps to use narrative inquiry effectively in the context of mathematics education, one might consider adding a fourth dimension to the inquiry space; a dimension that would reflect some specifically mathematical aspect of the experiences one is trying to capture. Developing and implementing this dimension seems to me a fruitful and important avenue for future exploration of narrative inquiry methodology in a mathematics education context.

In chapter 2, I described how socio-constructivist perspectives have influenced a shift in the teaching approaches being implemented in transition to proof courses (i.e. inquiry-based learning) but not, as yet, in the methods being used to conduct research in these courses. While teaching approaches have been influenced by current views of learning that "emphasize individual constructions and ways of thinking and learning developed in social interactions in classrooms" (Hassi et al., 2011, p.73), the research methods employed remain focused on the measurement of student responses. In chapter 5, I discussed the history of this 'scientific' model of inquiry in mathematics education research, and, following Dewey's theory, described how such empirical methods can feel quite inappropriate for studying such a complex human experience as education. In an effort to address what I perceive as a 'methodological gap', this thesis explored the use of a narrative inquiry framework for researching students' mathematical behaviour.

The choice of methodology was motivated by both a desire to move away from research methods oriented towards providing accurate, objective accounts of experience, and a need to capture student behaviours which emerge and exist but momentarily, in the classroom. I feel that narrative inquiry satisfied both of these requirements. Writing field texts and meeting regularly with my fellow researchers allowed me to construct accounts of the significant moments that transpired in the classroom, moments that might otherwise have been forgotten. Constructing narrative data allowed me to focus on the meaning and significance of experiences, rather than on their 'objective status' as research data. By considering the individual students, and their prior experiences, I was able to construct a rich, detailed characterization of their mathematical behaviour.

8.3 Further Questions on Learning and Teaching

In this section, I will discuss some of the questions that were raised in this inquiry which remain unanswered, and suggest possible directions for future study. The goal of Nadia's original design experiment was to investigate how mathematical behaviours can be fostered in institutional educational settings. She designed the course with the goal of providing opportunities for students to engage in mathematical behaviour. As previously discussed (chapter 3), from our theoretical perspective, the only way to know that such an opportunity was provided is if we observe a student behaving mathematically in that situation. Thus, the research in this thesis was conceived to investigate, capture and discuss what these behaviours look like as they emerge, as a way of evaluating whether we, as educators are succeeding in our pedagogical goals.

Based on the experiences I had engaging in this thesis, I can claim that Nadia's course design does offer opportunities and situations which provoke and encourage students to engage in mathematical behaviour. What this thesis does not speak to is whether or not this teaching method is effective in the sense that students learn things and retain things. This is a fundamental aspect of the theoretical framework I have used in my research; if we only observe the students while they are having these experiences, *we cannot say anything about learning*. According to Dewey, the educative value of an experience is determined by how it lives on in future experiences, something upon which I cannot meaningfully comment in this thesis.

There is anecdotal evidence which suggests that many students find the course helpful in their future mathematics studies. Almost all of the students I interviewed said that the course was different than their previous experiences with mathematics, that it had been meaningful and enriching for them. Nadia has had several experiences of meeting former students she taught in the course, and having them tell her the course was important for them, that they learned things in this course which they have found important/useful/meaningful in the courses they have taken since, that they felt it was their first 'real' math course, that it shifted their perceptions about what it means to do mathematics. So there is reason to believe that the experiences some students have in the course are *educative.*

Regardless of this anecdotal data, whether these classroom experiences are educative for the students, and whether they are actually learning to behave mathematically, are not questions this research can answer. Any discussion of learning would require a long-term study, since we would need to investigate how the experiences student have in the course influence their future experiences. Dewey's educational theory provides a framework for conceptualizing learning which could be used to structure such an investigation.

It would be interesting to explore the question of whether or not the experiences students have of behaving mathematically in MAST 217 are educative, whether they are experiences from which the students feel they learned something, and to try to qualify those experiences of learning. Gathering together former students of the course and interviewing them or having them construct field texts of some kind that reflect on what they feel they learned in the course, whether or not they (and in what ways) have found that learning useful and valuable. It would be interesting to see what mathematical behaviours from our model (or other ones besides) are seen by former students of the course to be important.

Although investigating the teaching design was not the main goal of this inquiry it was nevertheless a large part of our reflection and discussion - how Nadia might continue to change the teaching design, concerns around assessment, and discussing our feelings about when we felt we 'got it right' in our interventions with a student. These reflections were ever-present throughout this inquiry, so that although this thesis focuses on the mathematical behaviour of students, there is collateral learning/research taking place on teaching design. It could be interesting to formulate an inquiry based around teaching practices in this course, and there are several interesting reflections about this already present in the field texts we constructed for this research, which could serve as a good jumping off point.

Summary

In this final chapter, I have discussed how this thesis addresses its two main goals. The first was to construct a characterization of some of the features of mathematical behaviour. I constructed storied accounts of students' experiences, in which some of the different features of mathematical behaviour were visible. In this chapter, I have provided discussion of each of those features, reflecting on the significant differences and nuances of behaviour that I saw. The result is a rich description of the ways in which students engaged in mathematical behaviour in this course, which is significant because it can help us (as teachers and as researchers) to better understand which of those opportunities we believe we are providing are actually being taken as such by students.

The second goal of this thesis was to explore a methodological approach that may allow us to talk about, and report on, features of mathematical behaviour as they emerge in the classroom. Using narrative inquiry as a methodological framework allowed me to generate rich remembrances of my own experiences, to reflect on these experiences, and to learn from them in ways that were meaningful to my development as a teacher and researcher. I was able to construct accounts of the mathematical behaviour that emerged in the classroom.

EPILOGUE

> *My aim is to extend my awareness, increase my sensitivities to others, and to do this by telling stories, and by evoking stories from others. My position is easily stated: I cannot change others, but I can change myself. (Mason, 1994, p. 177)*

The end is not the end. Although I find myself here on the last few pages, and there is a certain sense of completion (finally, done!), this thesis is an important catalyst for how I will continue to grow as an educator. Thus, I try in these last pages, which I do not see as a conclusion, to look back on how I told my narratives of inquiry in this thesis, and what I have learned. This work is only a snapshot, one frame of the moving picture of experience. An attempt to capture and render something that cannot ever be wholly captured and rendered. What I have learned most of all is to appreciate the sheer *complexity* of any experience.

It is my hope that the conversations between Nadia and I, which surface throughout the thesis, help point to the fact that this work is situated in the midst of an ongoing conversation, and ongoing story. It is a story which began for both Nadia and I some time ago, before we met. My work with my former mentor Edison instilled me with a desire to investigate mathematical thinking, and Nadia's previous research conducting a design experiment led her to the point of being interested in exploring a new methodology. We constructed and negotiated this research together, at the intersection of our two perspectives and previous experiences. The themes which I have begun to explore and articulate in this thesis are ones which I will continue to reflect on and develop in my teaching practice.

Many times in this work, when discussing students and their learning, I mention my belief that a certain meaningful reflection is essential in order for an experience to be educative. In teaching the course and living alongside students, I tried to offer what I could to help them have experiences which I hoped would be valuable in their future experiences. Nevertheless, as Clandinin and Connolly (2000) would remind me, I too was having an experience. The work of this thesis has been my attempt to make my own experiences in MAST 217 as educative for myself as possible. Engaging in this narrative inquiry meant I had to learn how to notice, how to be wakeful, and to reflect constantly on my own actions, thinking, and beliefs. Which questions do I want to ask? Why those, and not others? According to Mason (1994), developing this type of awareness is an essential action for becoming a better teacher.

The educative value of an experience cannot be determined in the moment when one is having that experience. The educative value of an experience is determined by how it lives "fruitfully and creatively in subsequent experiences" (Dewey, 1938, p. 28). I look forward to continuing to develop the awarenesses and sensitivities to others, which I feel have begun to emerge for me through this research. I believe that the experience of writing this thesis is one that will live fruitfully and creatively in many of my future endeavours as an educator.

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