Bad reduction of Hilbert modular varieties with parahoric level structure

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A Thesis
In the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the For the Degree of
Doctor of Philosophy (Mathematics) at
Concordia University
Montreal, Canada

April 2017

CONCORDIA UNIVERSITY
School of Graduate Studies

This is to certify that the thesis was prepared

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Entitled: Bad reduction of Hilbert modular varieties with parahoric level structure

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Doctor of Philosophy (Mathematics)

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Date 22 April 2017

ABSTRACT

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Concordia University, 2017

Abelian varieties can be thought of as a higher dimensional analogue to elliptic curves. Over fields, they are defined as complete algebraic varieties with a compatible group structure. One of the most fertile fields in arithmetic geometry is concerned with the study of abelian varieties in prime characteristic. The fundamental reason why this area has become so central is that the many interesting phenomena arising in positive characteristic provide us with very powerful geometric tools. For instance, several results in characteristic zero can be derived studying the reduction to positive characteristic.

A very fruitful approach for describing such phenomena is to look at several abelian varieties at once. Roughly speaking, the abelian varieties of a given dimension are seen as points of a space, namely a *moduli space*. In general, in order for these spaces to have *nice* geometric properties, we will expect the abelian varieties to have additional structure, such as, for example, some data on the N-torsion (kernel of the multiplication by N map, for some positive integer N), called a structure of level N. The fundamental example in this context is that of modular curves that is, spaces whose points parametrize (isomorphism classes of) elliptic curves with some level N structure. The work of the author focuses on a generalization of modular curves, that is, Hilbert modular varieties, parametrizing abelian varieties with polarization, N-level structure and real multiplication, that is, an action by a suitable finite field extension of $\mathbb{Q}$.

The most immediate questions arising from the study of such spaces are purely of geometric nature. The geometry of Hilbert modular varieties is well understood in every characteristic p, as long as p does not divide the level structure. When p divides the level structure however, things get complicated. In this thesis we provide a description of Hilbert modular varieties in the case when p divides the level N (case of *parahoric* level). In particular, we obtain an understanding of these spaces by giving a full description of the deformation of abelian varieties and by defining suitable *stratifications* depending on the p-torsion of abelian varieties.

Contents

CHAPTER 1

Introduction

The main object of study of this thesis is the geometry of moduli spaces of Abelian schemes with extra structure. In particular we fix a totally real field L of degree g over $\mathbb{Q}$ and study schemes whose points correspond to isomorphism classes of Abelian schemes with multiplication by the ring of integers $\mathcal{O}_L$ of L and some level structure, called Hilbert modular varieties. More precisely in this study we concentrate on Hilbert modular varieties $\mathcal{M}$ over $\overline{\mathbb{F}}_p$ with Iwahori at p level structure. Namely, we use the theory of local models to compute completed local rings at all geometric points of $\mathcal{M}$ and the theory of relative displays to describe certain natural stratifications of this moduli space. The study of the geometry of Hilbert modular varieties $\mathcal{M}$ in positive characteristic depends strongly on the ramification properties of p in $\mathcal{O}_L$.

The geometry of Hilbert modular varieties in positive characteristic p is well understood in the case where the level structure is relatively prime to p by work of Rapoport [Rap78], Deligne-Pappas [DP94], Oort [Oor99], Goren-Oort [GO00], Goren [Gor01], Andreatta-Goren [AG03, AG04]. Goren-Kassaei [GK12] considered the case with Iwahori level structure at p when p is unramified in $\mathcal{O}_L$.

In this thesis we first describe the geometry of Hilbert modular varieties in positive characteristic p with Iwahori level structure at p unramified in $\mathcal{O}_L$ with methods different from those used by Goren-Kassaei. Moreover we are also able to apply such methods to Hilbert modular varieties where p is ramified in $\mathcal{O}_L$. In this case our results are completely new.

As mentioned above one of the main ingredients is the computation of the completed local rings of such a Hilbert modular variety $\mathcal{M}$ at all its geometric points by means of the theory of local models. This general approach was first introduced by Deligne-Pappas [DP94] and the results are due to Stamm [Sta97] in the unramified case and to Pappas [Pap95] in the ramified case. The theory of Grothendieck-Messing describes the deformations of an Abelian variety over $\overline{\mathbb{F}}_p$ in terms of the deformations of the Hodge filtration of its first de Rham cohomology group. This allows us to identify the completed local rings of geometric points in $\mathcal{M}$ with completed local rings at geometric points of a suitable Grassmannian, which in certain cases can be explicitly computed.

Our main contribution is the definition and the study of stratifications of our Hilbert modular varieties in characteristic p with Iwahori at p level structure. In other words we write our moduli space as a disjoint union of locally closed subschemes (strata) by choosing suitable invariants of Abelian varieties with real multiplication depending on the structure of their p-torsion. We moreover study the geometric properties of such strata by means of the theory of relative displays (these are defined in Section (3), Chapter (2)).

We next describe in deeper detail the structure of the thesis.

Chapter (2) contains fundamental definitions in the theory of group schemes and abelian varieties. In particular we present the Dieudonné theory for finite group schemes in positive characteristic with an emphasis on the p-torsion of abelian varieties. Finally we introduce the theory of displays as presented by Zink [Zin02] and their relation to the deformation theory of abelian varieties.

In Chapter (3) we present our main results in the case of Hilbert modular varieties $\mathcal{M}$ with Iwahori level structure at p of dimension 1, that is, modular curves with $\Gamma_0(p)$-level structure. The geometry in positive characteristic p of such modular varieties is already well understood and the results we present are those already obtained by Deligne-Rapoport [DR75], Katz-Mazur [KM85] and others. Our methods consist in the description of the deformation theory of geometric points of $\mathcal{M}$ in terms of relative displays, both in the equi characteristic and mixed characteristic cases. To a geometric $\overline{\mathbb{F}}_p$-point x of $\mathcal{M}$ we associate a morphism of displays $\mathcal{P}^{(\sigma)} \to \mathcal{P}$ over $\overline{\mathbb{F}}_p$. We describe a criterion for a morphism of displays $\mathcal{P}_1 \to \mathcal{P}_2$ to be universal with respect to the deformation of $\mathcal{P}^{(\sigma)} \to \mathcal{P}$ and provide an explicit candidate. This approach will be generalized and used in the following chapters.

Chapter (4) contains the results related to the geometry of Hilbert modular varieties with Iwahori level structure at p when p is inert in $\mathcal{O}_L$. As already stated, we recover results of Stamm [Sta97] and Goren-Kassaei [GK12] by means of relative displays. Hopefully such methods lead in a more straightforward way to results related to the description of the functor forgetting the Iwahori p-level structure on $\mathcal{M}$ (Key Lemma in [GK12]) and the theory of canonical subgroups, also in the case when p is ramified in $\mathcal{O}_L$, which has not been studied yet.

In Chapter (5) we describe the geometry of Hilbert modular varieties with Iwahori level structure at p in the case where p is ramified in $\mathcal{O}_L$. By developing the local model in this case, following Pappas [Pap95] we compute explicitly the completed local rings at any geometric point in terms of suitable invariants related to the p-torsion of the corresponding abelian variety; see Section (4.1). In terms of the above mentioned invariants we define stratifications for the space whose strata are non-singular and locally irreducible. See Theorem (5.4.1) and Theorem (5.4.2) for details. Such geometric properties of the strata and their dimension are obtained through the deformation theory of displays. We finally show that the closure of each stratum consists of the union of other strata (Theorem (7.2.1)).

We believe that the methods used in this thesis can be generalized to a wider class of Shimura varieties of PEL type over $\overline{\mathbb{F}}_p$ with Iwahori level structure at p, especially in the case of bad reduction, which has not been studied yet.

CHAPTER 2

Preliminaries

1. Group schemes and Dieudonné modules

DEFINITION 1.0.1. *Fix a base scheme S. We say that $G \to S$ is an S-*group scheme *if it has a group structure as an object in the category Sch/S. Explicitly, this means that there are S-maps*

$$m : G \times_S G \to G \text{ multiplication},$$
$$i : G \to G \text{ inverse},$$
$$e : S \to G \text{ unit},$$

satisfying the group axioms. We say that a group scheme is commutative *if the commutativity is satisfied by these maps.*

In other words, a group scheme over S corresponds to a contravariant functor from the category of schemes over S to the category of groups.

Basic examples. Let R be a commutative ring with unity.

1. The *additive group scheme* $\mathbb{G}_a/R = (\mathbb{A}^1_R, +)$, corresponding to the additive group structure underlying the affine line.

2. The *multiplicative group scheme* $\mathbb{G}_m/R = (\mathbb{A}^1_R - \{0\}, \times)$, corresponding to the multiplicative group structure underlying the affine line without the origin.

3. The constant group scheme $(\mathbb{Z}/p\mathbb{Z})_R$ over R.

REMARK 1. *We will always use commutative groups schemes and therefore all the results will refer to these (even though some of them might be true in general).*

1.1. Affine group schemes.

Let $S = \mathrm{Spec}(R)$ be an affine base scheme.

DEFINITION 1.1.1. *An S-group scheme G which is affine as a scheme is called an* affine group scheme.

Assume that $G = \mathrm{Spec}(A)$ is an affine group scheme. Note that the describing the group operations m, i and e defined in Definition (1.0.1) is equivalent to describing R-algebra morphisms

$$m^*\colon A \to A \otimes_R \otimes A \text{ co-multiplication},$$
$$i^*\colon A \to A \text{ co-inverse}$$
$$e^*\colon A \to R \text{ co-unit},$$

satisfying various identities corresponding to the group axioms for m, i and e.

Examples. 1. The *p-th roots of unity* $\mu_p = \{x \in \mathbb{G}_m \mid x^p = 1\} \subset \mathbb{G}_m$. It is a group scheme over R, but we will omit this in the notation. In other words μ_p is the kernel of the Frobenius morphism in $\mathbb{G}_m$. Note that as a scheme μ_p is identified

$$\mu_p \simeq \mathrm{Spec}(R[T]/(T^p - 1))$$

and the group operation is defined by the morphism of R-algebras

$$\begin{array}{ccc} R[T]/(T^p-1) & \to & R[T]/(T^p-1)\otimes R[T]/(T^p-1) \\ T & \mapsto & T\otimes T \end{array}$$

2. Assume that R is an $\mathbb{F}_p$-algebra. The *p-th roots of zero* $\alpha_p = \{x\in\mathbb{G}_a \mid x^p = 0\}\subset\mathbb{G}_a$. As schemes we have the identification

$$\alpha_p \simeq \mathrm{Spec}(R[T]/(T^p))$$

and the group operation is defined by the morphism of R-algebras

$$\begin{array}{ccc} R[T]/(T^p) & \to & R[T]/(T^p)\otimes R[T]/(T^p) \\ T & \mapsto & (T\otimes 1)+(1\otimes T) \end{array}$$

Note that in particular μ_p and α_p are isomorphic as schemes, but they are not as group schemes, as their group operations are different.

Definition 1.1.2 (Cartier dual). *Let G be a commutative group. We define its dual as the group of characters*

$$G^* := \mathrm{Hom}_{GrSch/S}(G,\mathbb{G}_m).$$

Examples. Let the base scheme $S = \mathrm{Spec}(k)$ with k of positive characteristic.
1. The Cartier dual of μ_p is $\mathbb{Z}/p\mathbb{Z}$.
2. The Cartier dual of α_p is α_p itself.

Definition 1.1.3. *A* finite flat group scheme *is a commutative group scheme $f: G\to S$, such that the structural morphism is finite and flat and such that $f_*(\mathcal{O}_G)$ is a locally free $\mathcal{O}_S$-module of locally constant rank $r>0$ (note that if S is noetherian this condition is always verified).*

1.2. The Dieudonné theory. Let k be a perfect field of positive characteristic p. Let $W(k)$ be the ring of Witt vectors and denote by ${}^F\colon W(k)\to W(k)$ the induced Frobenius. See the Appendix for details on $W(k)$.

Definition 1.2.1 (Dieudonné modules). *A Dieudonné module D is a module over $W(k)$ equipped with two maps $F\colon D\to D$ and $V\colon D\to D$ satisfying the following identities:*

$$F(wx) = {}^F wF(x), \qquad V(wx) = {}^{F^{-1}}wV(x), \qquad w\in W(k), x\in D$$
$$F\circ V = p, \qquad V\circ F = p.$$

Theorem 1.2.2 (Dieudonné). *There is a contravariant equivalence of categories between the category of finite flat commutative group schemes over k of p-power rank and the category of Dieudonné modules over $W(k)$ of finite length.*

Given a finite commutative group scheme G over k, we denote by $\mathbb{D}(G)$ the corresponding Dieudonné module. If G has rank p^n, then $\mathbb{D}(G)$ has length n.

Definition 1.2.3 (p-divisible groups). *A p-divisible group G over a base scheme of height h is an inductive system (G_n,ι_n), $n\geq 0$, where for every n*

(1) *G_n is a finite flat group scheme over S of order p^{nh},*
(2) *the sequence*

$$0\to G_n \xrightarrow{\iota_n} G_{n+1} \xrightarrow{\times p^n} G_{n+1}$$

is exact.

THEOREM 1.2.4. *Let k be a perfect field of positive characteristic $p > 0$. There is a contravariant equivalence of categories between the category of p-divisible groups over k and the category of finite free Dieudonné modules over $W(k)$:*

$$\begin{array}{ccc} \{p\text{-divisible groups over } k\} & \to & \{\text{Dieudonné modules over } W(k)\} \\ G = (G_n, \iota_n) & \mapsto & \varprojlim_n \mathbb{D}(G_n). \end{array}$$

2. Abelian varieties

We will give here a *sketchy* introduction to abelian varieties. For proofs and a more detailed theory we refer to Mumford's book [Mum08].

DEFINITION 2.0.1 (Abelian variety). *An abelian variety A over a field k is a projective variety with compatible group structure, that is, a projective variety over k, together with k-morphisms*

$$\begin{aligned} m&: A \times A \to A, \\ i&: A \to A, \\ e&: A \to k, \end{aligned}$$

satisfying the group axioms.

One can prove that the group law defined above is commutative. We will write the group operation additively.

There is an analogue to abelian varieties over any base scheme.

DEFINITION 2.0.2 (Abelian scheme). *An* abelian scheme *over a base scheme S of relative dimension g is a proper, smooth group scheme over S whose geometric fibers are connected and of dimension g.*

In other words an abelian scheme over S is a family of abelian varieties parametrized by S. An abelian scheme over a field k is an abelian variety as defined in Definition (2.0.1).

Given an abelian scheme A over a base S we define the *multiplication by n* map for any non-zero integer n. For $n \neq 0$ it is a surjective map and its kernel $A[n]$ is a group scheme of rank n^{2g}, where g denotes the relative dimension of A/S.

DEFINITION 2.0.3 (Isogenies). *An* isogeny *of abelian schemes is a surjective homomorphism whose kernel is a finite flat group scheme. The* degree *of an isogeny is the degree of its kernel as a group scheme.*

Example: The multiplication by n map for an integer $n > 0$ is an isogeny.

Let p be a prime number and let A be an abelian scheme over S. The group schemes $A[p], A[p^2], A[p^3], \dots, A[p^n], \dots$ form an inductive system $A[p^\infty] = (A[p^n], A[p^n] \xrightarrow{\times p} A[p^{n+1}])$. It is a p-divisible group called the p-divisible group associated to the abelian variety A.

To an abelian scheme A over a base scheme S one associates a *dual abelian scheme* $A^\vee$ over S. The construction of the dual abelian scheme $A^\vee$ is obtained by means of a line bundle assosciated to A, the *Poincaré bundle* $\mathscr{P}_A$.

Definition 2.0.4 (Polarizations). *Given A an abelian scheme over a base S, a homomorphism $\lambda\colon A \to A^\vee$ is called a* polarization *if it a symmetric isogeny such that the line bundle* $(\mathrm{id}, \lambda)^*(\mathscr{P})$ *is ample.*

2.1. Abelian varieties over fields of positive characteristic. In this section we are going assume k to be a perfect field of positive characteristic $p > 0$.

Definition 2.1.1. *An abelian variety A over k of dimension g is said to be* ordinary *if*

$$|(A[p](\bar{k}))| = p^g.$$

In particular, an elliptic curve E over k is said to be supersingular *if it is not ordinary. In this case we have that*

$$|(E[p](\bar{k}))| = \{0\}.$$

Let $\sigma\colon k \to k$ be the absolute Frobenius $x \to x^p$. On an abelian variety A over k of dimension g we may define relative Frobenius and Verschiebung morphisms

$$\mathrm{Fr}_A\colon A \to A^{(p)}, \qquad \mathrm{Ver}_A\colon A^{(p)} \to A,$$

where $A^{(p)} \simeq A \times_{\mathrm{Spec}(k),\sigma} \mathrm{Spec}(k)$. They satisfy the identities

$$\mathrm{Ver}_A \circ \mathrm{Fr}_A = [p]_A, \qquad \mathrm{Fr}_A \circ \mathrm{Ver}_A = [p]_{A^{(p)}}.$$

It follows therefore that

$$\mathrm{Ker}(\mathrm{Fr}_A) \subset A[p], \qquad \mathrm{Ker}(\mathrm{Ver}_A) \subset A^{(p)}[p].$$

The relative Frobenius and the Verschiebung morphisms

$$\mathrm{Fr}_A\colon A \to A^{(p)}, \qquad \mathrm{Ver}_A\colon A^{(p)} \to A$$

are isogenies of degree p^g.

We have the following functorial in A commutative diagram with exact rows providing a differential interpretation of the Dieudonné modules associated to $A[p], \mathrm{Ker}(\mathrm{Fr}_A), \mathrm{Ker}(\mathrm{Ver}_A)$:

$$\begin{tikzcd}
0 \arrow[r] & H^0(A, \Omega_{A/k}) \arrow[r] \arrow[d, equal] & H^1_{dR}(A/k) \arrow[r] \arrow[d, equal] & H^1(A, \mathcal{O}_A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A)) \otimes_{k,\sigma^{-1}} k \arrow[r] & \mathbb{D}(A[p]) \arrow[r] & \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A)) \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

Here $H^1_{dR}(A/k)$ denotes the first de Rham cohomology group of A over k and $\Omega_{A/k}$ is the sheaf of invariant differentials of A. The result was established by Oda [Oda69]; see Goren-Kassaei [GK12, 2.2.1] for details.

On an abelian variety A over k there is a perfect bilinear canonical pairing

$$e_N\colon A[N] \times A^\vee[N] \to \mu_N$$

called the *Weil pairing*. If the abelian variety is endowed with a polarization $\lambda\colon A \to A^\vee$ then the Weil pairing induces a bilinear pairing

$$\begin{array}{rcl} e_N^\lambda\colon A[N] \times A[N]\colon & \to & \mu_N \\ (x,y) & \mapsto & e_N(x, \lambda(y)) \end{array}$$

Also the first de Rham cohomology group $H^1_{dR}(A/k)$ is endowed with a perfect alternating pairing

$$H^1_{dR}(A/k) \times H^1_{dR}(A/k) \to \mathcal{O}_L \otimes k,$$

see [DP94, 1.5] for details.

2.2. Deformations of abelian varieties. Let Art_k denote the category of local Artinian k-algebras with residue field k, whose objects are pairs (R,ϕ) where R is a local Artinian k-algebra with maximal ideal $\mathfrak{m}_R$ and ϕ is the unique k-isomorphism $\phi\colon R/\mathfrak{m}_R \xrightarrow{\simeq} k$.

DEFINITION 2.2.1. *Let A be an abelian variety over k. The deformation functor*

$$\mathrm{Def}(A/k)\colon \mathrm{Art}_k \to \mathrm{Sets}$$

associating to $(R,\phi) \in \mathrm{Art}_k$, the set $\mathrm{Def}(A/k)(R,\phi)$ of isomorphism classes of pairs $(\widetilde{A}/R,\psi)$, where $\widetilde{A}/R$ is an abelian scheme over $\mathrm{Spec}(R)$ and ψ is a k-isomorphism

$$\psi\colon \widetilde{A} \times_{R,\phi} k \xrightarrow{\simeq} A.$$

Analogously, given a p-divisible group G over k we define the deformation functor

$$\mathrm{Def}(G/k)\colon \mathrm{Art}_k \to \mathrm{Sets}$$

associating to (R,ϕ) the isomorphism classes of pairs $(\widetilde{G}/R,\psi)$, where $\widetilde{G}/R$ is a p-divisible group over $\mathrm{Spec}(R)$ and $\psi\colon \widetilde{G} \times_{\mathrm{Spec}(k),\phi} \mathrm{Spec}(k) \simeq G$.

The following result states that it is equivalent to deform an abelian variety A over k or its associated p-divisible group $A[p^\infty]$.

THEOREM 2.2.2 (Serre-Tate). *Let A be an abelian variety over k and let $A[p^\infty]$ be the associated p-divisible group. The deformation functors $\mathrm{Def}(A/k)$ and $\mathrm{Def}(A[p^\infty]/k)$ are canonically isomorphic.*

PROOF. For Drinfeld's proof, see [Kat81]. □

2.3. Abelian varieties with real multiplication. Let k be an algebraically closed field of positive characteristic $p > 0$. Denote by L a totally real number field of degree g:

$$\begin{tikzcd} L \arrow[r, dash] \arrow[d, dash, "g"'] & \mathcal{O}_L \arrow[d, dash] \\ \mathbb{Q} \arrow[r, dash] & \mathbb{Z} \end{tikzcd}$$

Let S be a scheme over k. An *RM-abelian scheme* (or *Hilbert-Blumenthal abelian scheme*) over S is a pair $(A/S,\iota)$, where

- A is an abelian scheme over S of relative dimension g,
- $\iota\colon \mathcal{O}_L \to \mathrm{End}_S(A)$ is a ring homomorphism ($\mathcal{O}_L$*-action*); such ι gives A/S the structure of an $\mathcal{O}_L$-module scheme.

Note that duality induces an $\mathcal{O}_L$-action $\iota^\vee\colon \mathcal{O}_L \to \mathrm{End}_S(A^\vee)$ and hence the pair $(A^\vee,\iota^\vee)$ is an RM-abelian scheme as well.

We say that a p-divisible group G over a finitely generated k-algebra R has real multiplication by $\mathcal{O}_L$, if $\mathcal{O}_L$ acts on G as endomorphisms and if $\mathbb{D}(G \otimes_k k^{\mathrm{alg}})$ is a free $\mathcal{O}_L \otimes_{\mathbb{Z}} W(k^{\mathrm{alg}})$-module of rank 2.

The moduli problem we are about to define carries a choice of polarization respecting the real multiplication. More precisely, given an abelian scheme A over S we consider the $\mathcal{O}_L$-module $M(A)$ of $\mathcal{O}_L$-linear symmetric morphisms from A to its dual $A^\vee$, that is, the $\mathcal{O}_L$-module with elements $\lambda \in \mathrm{Hom}_S(A,A^\vee)$, such that $\lambda \circ \iota = \iota^\vee \circ \lambda$ and such that $\lambda = \lambda^\vee$.

More generally, following Rapoport we may view the $\mathcal{O}_L$-module $M(A)$ as a sheaf on the étale site of S

$$\begin{array}{rcc} Sch/S & \longrightarrow & Ab \\ (T \to S) & \longmapsto & \{\lambda \in \mathrm{Hom}_T(A \times_S T, A^\vee \times_S T) \mid \mathcal{O}_L\text{-linear, symmetric}\} \end{array}$$

By [Rap78, 1.17] this sheaf is locally constant whose values are projective $\mathcal{O}_L$-modules of rank 1, equipped with positivity notions corresponding to the $\mathcal{O}_L$-polarizations of (A,ι). We call the pair $(M(A), M(A)^+)$ the *étale polarization module* of (A,ι).

Recall that given an abelian variety over a perfect field k of positive characteristic and of dimension g, the $W(k)$-module $H^1_{cris}(A/W(k))$ is free of rank $2g$. If A has a real multiplication action by a totally real field L, then $H^1_{cris}(A/W(k))$ inherits a structure of $\mathcal{O}_L \otimes W(k)$-module.

LEMMA 2.3.1 (Rapoport, Lemma 1.3). *Let (A,ι) be a real multiplication abelian variety over a perfect field k of positive characteristic $p > 0$. The first crystalline cohomology group $H^1_{cris}(A/W(k)$ is a free $\mathcal{O}_L \otimes W(k)$-module of rank 2.*

Note that by reducing modulo p we obtain that the first de Rham cohomology group $H^1_{dR}(A/k)$ is a free $\mathcal{O}_L \otimes k$-module of rank 2. More generally, both Ver_A and Fr_A commute with the $\mathcal{O}_L$-action, that is $\mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A))$ and $\mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A))$ have structures of $\mathcal{O}_L \otimes k$-modules and the diagram (2.1) is a diagram of $\mathcal{O}_L \otimes k$-modules as well.

3. Dieudonné displays

The theory of displays is one of the main tools to study local deformation theory of abelian varieties. It was introduced by D. Mumford and P. Norman and was generalized to a much wider class of objects by Thomas Zink. The notion we will be using is in particular that of *Dieudonné displays* [Zin02, Zin01a, Lau14]. The fundamental result is that, under relatively mild restrictions, Dieudonné displays are equivalent to p-divisible groups. Under Serre-Tate's theorem it is therefore possible to study deformation theory of abelian varieties through Dieudonné displays. Here is a list of results about Dieudonné displays that we are going to use; the main references are [Zin01a] and [Lau14].

3.0.1. *Admissible rings.* Let p be a prime number. The following definitions are due to Lau [Lau14, Definition 1.1, Definition 1.2].

DEFINITION 3.0.1 (Admissible rings). *A ring R is called* admissible *if its nilradical $\mathcal{N}_R$ is bounded nilpotent and if the quotient $R_{\mathrm{red}} = R/\mathcal{N}_R$ is a perfect ring of characteristic p. An* admissible local *ring is an admissible ring such that R_{red} is a field.*

REMARK 2. *Artinian local rings with perfect residue field are admissible local rings.*

DEFINITION 3.0.2 (Admissible topological rings). *An* admissible topological *ring is a complete and separated topological ring R with linear topology such that the ideal $\mathcal{N}_R$ of topologically nilpotent elements is open, the ring $R_{\mathrm{red}} = R/\mathcal{N}_R$ is perfect of characteristic p, and for each open ideal N of R contained in $\mathcal{N}_R$ the quotient $\mathcal{N}_R/N$ is bounded nilpotent. Thus R is the projective limit of the admissible rings R/N.*

REMARK 3. *Complete local rings with perfect residue field are admissible topological rings.*

3.0.2. *The Zink ring.* Given an admissible ring R, the exact sequence

$$0 \to W(\mathcal{N}_R) \to W(R) \to W(R_{\mathrm{red}}) \to 0$$

has a unique section $s: W(R_{\mathrm{red}}) \to W(R)$, which is ^{F}-equivariant. See [Lau14, Lemma 1.4] for details. Denote by $\widehat{W}(\mathcal{N}_R)$ the subring of $W(\mathcal{N}_R)$ whose elements have only finitely many non-zero components. Define the Zink ring as the direct sum

$$\mathbb{W}(R) \coloneqq \widehat{W}(\mathcal{N}_R) \oplus W(R_{\mathrm{red}}).$$

It is an ^{F}-stable subring of $W(R)$.

Lemma 3.0.3. *If $p \geq 3$, the Zink ring $\mathbb{W}(R)$ is ^{V}-stable and there is an exact sequence*

$$0 \to \mathbb{W}(R) \xrightarrow{V} \mathbb{W}(R) \xrightarrow{w_0} R \to 0.$$

An analogous result holds also when $p = 2$ if we substitute the Verschiebung morphism ^{V} with ${}^{\mathbb{V}}x = {}^V(u_0 x)$, where u_0 denotes the image of ${}^{V^{-1}}(2 - [2]) \in W(\mathbb{Z}_p)$ in $W(R)$.

Proof. See [Lau14, Lemma 1.6, 1.7]. The result is due to Zink [Zin02, Lemma 2] when $p \geq 3$. □

Denote by $\mathbb{I}(R)$ the kernel of $w_0: \mathbb{W}(R) \to R$. It is the image of $W(R)$ through ^{V} (${}^{\mathbb{V}}$ in the case of $p = 2$). Then, as for regular Witt vectors, we have that

$$ {}^F\mathbb{I}(R) \subseteq p\mathbb{W}(R); \tag{3.1}$$

see Lemma (0.1.1), Appendix, for details.

3.0.3. *Case of topological admissible rings.* For an admissible topological ring R take

$$\mathbb{W}(R) = \varprojlim_N \mathbb{W}(R/N),$$

where the limit is taken over all open ideals of R contained in $\mathcal{N}_R$. Lemma (3.0.3) holds also for admissible topological rings.

3.1. Fundamental definitions.

Definition 3.1.1 (Dieudonné Displays). *A* Dieudonné display $\mathcal{P}$ *over an admissible (topological) ring R is a quadruple $\mathcal{P} = (P, Q, F, V^{-1})$, where*

- *P is a finitely generated projective $\mathbb{W}(R)$-module,*
- *Q is a $\mathbb{W}(R)$-submodule of P,*
- *$F: P \to P$ and $V^{-1}: Q \to P$ are ^{F}-linear maps,*

such that

1. *$\mathbb{I}(R)P \subset Q \subset P$ and P/Q is a direct summand of $P/\mathbb{I}(R)P$ (as a $\mathbb{W}(R)$-module),*
2. *$V^{-1}: Q \to P$ is an ^{F}-linear epimorphism,*
3. *the following relation holds*

$$V^{-1}({}^V w \cdot x) = w \cdot Fx, \qquad \forall x \in P, w \in W(R).$$

Remark 4. *In the notation of Definition (3.1.1), for $y \in Q$ we always have the relation*

$$Fy = p \cdot V^{-1} y.$$

Indeed $Fy = V^{-1}({}^V 1 y) = pV^{-1}y$ by ^{F}-linearity.

Example When $R = k$ is a perfect field, the category of displays is equivalent to the category of Dieudonné modules over k. Recall that a Dieudonné module M is a finitely generated module over $W(k)$ equipped with two operators F and V (see Section (1.2) for details). Since V is injective, there is an inverse operator $V^{-1}: VM \to M$. Since trivially $\mathbb{W}(k) = W(k)$, one obtains a Dieudonné display (M, VM, F, V^{-1}).

Definition 3.1.2. *A morphism of Dieudonné displays*

$$f: \mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1}) \to \mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$$

is a morphism of $\mathbb{W}(R)$*-modules* $f: P_1 \to P_2$ *respecting filtrations and maps, that is,*

$$f(Q_1) \subseteq Q_2, \qquad fF_1 = F_2 f, \qquad fV_1^{-1} = V_2^{-1} f.$$

The category of Dieudonné displays is an additive category.

For a Dieudonné display $\mathcal{P} = (P, Q, F, V^{-1})$, there exists a *normal decomposition* of P, that is a direct sum decomposition

$$P = L \oplus T,$$

as a $\mathbb{W}(R)$-module, such that the equality $Q = L \oplus \mathbb{I}(R)T$ holds.
Giving a Dieudonné display $\mathcal{P}$ corresponds to giving a normal decomposition of a $\mathbb{W}(R)$-module $P = L \oplus T$, and a semi-linear isomorphism

$$V^{-1} \oplus F: L \oplus T \longrightarrow P.$$

By localizing on R, we may assume the modules L and T to be free. By choosing bases for L and T, one may represent $V^{-1} \oplus F$ through an invertible matrix, the *displaying matrix*, that encodes all the information about the Dieudonné display.

We will be particularly interested in the reduction modulo p of Dieudonné displays. Given a Dieudonné display $\mathcal{P}$ denote

$$D_{\mathcal{P}} = P/\mathbb{I}(R)P, \qquad H_{\mathcal{P}} = Q/\mathbb{I}(R)P;$$

we call $H_{\mathcal{P}} \subset D_{\mathcal{P}}$ the *Hodge filtration* of $\mathcal{P}$; it is a locally direct summand. It follows easily from the definition that $F(Q) = pP$, hence the reduction modulo $\mathbb{I}(R)$ is such that $\mathrm{Ker}(\overline{F}) = Q/\mathbb{I}(R)P$.

In what follows we will need Dieudonné displays equipped with an action of the ring $\mathcal{O}_L$.

Definition 3.1.3 (RM Dieudonné displays). *A* real multiplication (RM) Dieudonné display $\mathcal{P}$ *over an admissible (topological) ring* R *is a quintuple* $(P, Q, F, V^{-1}, \langle\ ,\ \rangle)$ *such that*

(1) P *is a projective* $\mathcal{O}_L \otimes \mathbb{W}(R)$*-module of rank* 2,
(2) Q *is a projective* $\mathcal{O}_L \otimes \mathbb{W}(R)$*-submodule of* P *such that* $\mathbb{I}(R)P \subset Q \subset P$ *and such that, as* R*-modules, the quotient* P/Q *is a direct summand of the module* $P/\mathbb{I}(R)P$,
(3) F *and* V^{-1} *are the usual morphisms of modules of displays,*
(4) $\langle\ ,\ \rangle$ *is an* $\mathcal{O}_L \otimes \mathbb{W}(R)$*-bilinear map:* $P \times P \to \mathfrak{D}_L^{-1} \otimes \mathbb{W}(R)$ *such that*

$$^V\langle V^{-1}(x), V^{-1}(y)\rangle = \langle x, y\rangle, \qquad \forall x, y \in Q,$$

(here $\mathfrak{D}_L$ *denotes the different ideal of the number field* L*).*

3.2. Relation to p-divisible groups. In [Lau14], for every admissible (topological) ring R a functor

$$\Phi_R\colon \{p\text{-divisible groups over } R\} \to \{\text{Dieudonné displays over } R\}$$

is defined and the following theorem is proved.

THEOREM 3.2.1 (Lau). *For any admissible (topological) ring R the functor Φ_R is an equivalence of exact categories.*

For details see [Lau14, Corollary 5.4, Corollary 5.5]. Such a result is proven in [Zin01a] in the case when p is odd.

We may extend this result to RM Dieudonné displays.

THEOREM 3.2.2. *If R is an admissible (topological), the category of RM Dieudonné displays over R is equivalent to the category of polarized RM p-divisible groups over R.*

3.3. Deformation of Dieudonné displays. Consider a homomorphism $\psi\colon S \to R$ of admissible (topological) rings and a Dieudonné display $\mathcal{P} = (P, Q, F, V^{-1})$ over S. We may base change $\mathcal{P}$ to R and obtain a Dieudonné display $\mathcal{P}_R$ over R by putting

- $P_R = P \otimes_{\mathbb{W}(S)} \mathbb{W}(R)$,
- $Q_R = \mathrm{Ker}(P \otimes_{\mathbb{W}(S)} \mathbb{W}(R) \to P/Q \otimes_S R)$,
- F_R such that $F_R(\lambda \otimes x) = {}^F\lambda \otimes F(x)$,
- V_R^{-1} as the only ^{F}-linear operator satisfying the necessary relations for a Dieudonné display.

DEFINITION 3.3.1 (Lifts). *Consider a Dieudonné display $\mathcal{P}_0$ over an admissible (topological) ring R and let $\psi\colon S \longrightarrow R$ be a homomorphism of admissible (topological) rings. A* deformation *or* lift *of $\mathcal{P}_0$ to S is a pair $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is a Dieudonné display over S and ϕ is an isomorphism of Dieudonné displays $\mathcal{P}_R \simeq \mathcal{P}_0$.*

DEFINITION 3.3.2 (The deformation functor). *Let $\mathcal{P}_0$ be a Dieudonné display over a perfect field k of characteristic $p > 0$. The deformation functor of $\mathcal{P}_0$ is defined as*

$$\begin{array}{rcc} \mathrm{Def}(\mathcal{P}_0/k)\colon \mathrm{Art}_k & \to & \mathrm{Sets} \\ S & \mapsto & \{\textit{isomorphism classes of } (\mathcal{P}, \phi)\} \end{array}$$

where $(\mathcal{P}, \phi)$ is a lift of $\mathcal{P}_0$.

The following result is a consequence of Theorem (3.2.2).

COROLLARY 3.3.3. *Let $\mathcal{P}_0$ be a Dieudonné display over k and let G_0 be the associated p-divisible group as in Theorem* (3.2.2). *Then the deformation functors* $\mathrm{Def}(\mathcal{P}_0/k)$ *and* $\mathrm{Def}(G_0/k)$ *are canonically isomorphic.*

3.4. The crystalline nature of Dieudonné displays. Let A be a commutative ring and $I \triangleleft A$ and ideal.

DEFINITION 3.4.1 (Divided powers). *By* divided powers on I *we mean a collection of maps $\{\gamma_i : I \to A\}_{i \geq 0}$ satisfying the following properties:*

(1) *for $x \in I$ we have $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\forall i \geq 1$, $\gamma_i(x) \in I$,*
(2) *if $x, y \in I$ then $\gamma_k(x+y) = \Sigma_{i+j=k}\gamma_i(x)\gamma_j(y)$,*
(3) *for $\lambda \in A$, $\gamma_i(\lambda x) = \lambda^i \gamma_i(x)$,*
(4) *for $x \in I$, $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$,*

(5) $\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$.

We call (I,γ) *a P.D. ideal,* (A,I,γ) *a P.D. ring and* γ *a P.D. structure on* I.

DEFINITION 3.4.2. *Let* (A,I,γ) *and* (B,J,δ) *be P.D. rings. A* morphism *of PD rings* $f:(A,I,\gamma)\to(B,J,\delta)$ *(or* PD-morphism *is a ring homomorphism* $f: A\to B$ *such that* $f(I)\subseteq J$ *and such that* $f(\gamma_n(x)) = \delta_n(f(x))$, $\forall x\in I, n\in\mathcal{N}$. *We say that* J *is a* sub PD ideal *of* I *if* $\gamma(x)\in J$ *for every* $x\in J$.

Let $S\to R$ be a surjective PD morphism of admissible (topological) rings, and let $\mathcal{P} = (P,Q,F,V^{-1})$ be a Dieudonné display over R and $\widetilde{\mathcal{P}}_S = (P_S,Q_S,F_S,V_S^{-1})$ be a Dieudonné display over S lifting $\mathcal{P}$. Denote by $\widehat{Q}_S$ the inverse image of Q through the natural projection $P_S\to P$. Note that V_S^{-1} extends uniquely to $\widehat{Q}_S$. Denote by $\widehat{\mathcal{P}}$ the quadruple $(P,\widehat{Q},F,V^{-1})$.

THEOREM 3.4.3 (Zink). *Let* $S\to R$ *be a surjective PD-morphism of admissible rings and let* $f:\mathcal{P}_1 = (P_1,Q_1,F_1,V_1^{-1},\langle\ ,\ \rangle_1)\to\mathcal{P}_2 = (P_2,Q_2,F_2,V_2^{-1},\langle\ ,\ \rangle_2)$ *be a morphism of RM Dieudonné displays over R. Let* $\mathcal{P}_{1,S} = (P_{1,S},Q_{1,S},F_{1,S},V_{1,S}^{-1},\langle\ ,\ \rangle_{1,S})$ *and* $\mathcal{P}_{2,S} = (P_{2,S},Q_{2,S},F_{2,S},V_{2,S}^{-1},\langle\ ,\ \rangle_{2,S})$ *be two Deidonné displays over* S *lifting* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *respectively. There exists a unique morphism*

$$\widehat{\mathcal{P}}_{1,S}\xrightarrow{f_S}\widehat{\mathcal{P}}_{2,S}$$

over S *lifting* $f:\mathcal{P}_1\to\mathcal{P}_2$ *and commuting with the* $\mathcal{O}_L$*-action.*

PROOF. See [Zin01a, Theorem 3]. □

The result extends to admissible topological rings. We will in fact use the result for such objects.

REMARK 5. *If* $\mathcal{P} = \mathcal{P}_1 = \mathcal{P}_2$ *and* $f:\mathcal{P}\to\mathcal{P}$ *is the identity morphism, and if* $\mathcal{P}_{1,S}$ *and* $\mathcal{P}_{2,S}$ *are two RM Dieudonné displays over* S *lifting* $\mathcal{P}$, *then the map* $f_S:\widehat{\mathcal{P}}_{1,S}\to\widehat{\mathcal{P}}_{2,S}$ *obtained by Theorem* (3.4.3) *is an isomorphism.*

REMARK 6. *The identification* $P_1 := P(\mathrm{Spec}(R)\subset\mathrm{Spec}(S))$ *defines a sheaf on the crystalline site over* R. *By Theorem* (3.4.3) P *is a crystal. The result implies also that* $D_{\mathcal{P}_1}$ *and* $D_{\mathcal{P}_2}$ *are canonically isomorphic and hence also the sheaf* $D(\mathrm{Spec}(R)\subset\mathrm{Spec}(S)) = D_{\mathcal{P}_1}$ *is a crystal on the crystalline site over* R, *namely the* Dieudonné crystal.

Given an abelian variety A with real multiplication, and $\mathcal{P}_A$ the Dieudonné display associated to the p-divisible group $A[p^\infty]$ by Theorem (3.2.2), we have that $D_{\mathcal{P}} = \mathbb{D}(A[p^\infty])$. In fact, the crystalline theory gives a canonical, explicit way to associate displays to abelian varieties. The following result is due to Langer-Zink, see [LZ04, Section 3.4] for details.

THEOREM 3.4.4 (Langer-Zink). *Given a perfect field* k *of positive characteristic* p *and an abelian variety* A *over* k, *there is a unique Dieudonné display* $\mathcal{P}_A = (P_A,Q_A,F_A,V_A^{-1})$ *such that*

- $P_A = H^1_{cris}(A/W(k))$ *and* F_A *is the canonical crystalline Frobenius on* P_A,
- $Q_A = \mathrm{Ker}(H^1_{cris}(A/W(k))\to H^1(A,\mathcal{O}_A))$ *(under the composition of the canonical maps* $H^1_{cris}(A/W(k))\to H^1_{dR}(A/R)\to H^1(A,\mathcal{O}_A)$*),*
- *the arrow* $A\to\mathcal{P}_A$ *defines a functor which commutes with base change.*

Note that this makes sense since $\mathbb{W}(k) = W(k)$. Let us explain this construction concretely. Given an abelian variety over k, the first crystalline cohomology group is identified with the Dieudonné module

$$\mathbb{D}(A) = \mathbb{D}(A)_{(W(k)/k)} \simeq H^1_{cris}(A/W(k)).$$

It has natural ^{F} – *linear* morphisms F_A and V_A. The map V_A^{-1} is defined as the map $V_A\mathbb{D}(A) \to \mathbb{D}(A)$. Note that indeed $Q = V_A(\mathbb{D}(A))$.

Note moreover that the Hodge filtration of the display $\mathcal{P}_A$ is given by the Hodge filtration of the de Rham cohomology

$$H^0(A, \Omega^1_{A/k}) \subseteq H^1_{dR}(A/k).$$

Moreover, $\mathrm{Ker}(\overline{F}) = H^0(A, \Omega^1_{A/k})$.

Let us understand how the functor works on isogenies.

LEMMA 3.4.5. *Given an isogeny $f: A \to B$ of abelian varieties over k, there is a morphism*

$$\mathcal{P}_B = (P_B, Q_B, F_B, V_B^{-1}) \to \mathcal{P}_A = (P_A, Q_A, F_A, V_A^{-1})$$

between the associated Dieudonné displays.

PROOF. Consider an isogeny of abelian varieties

$$f: A \to B$$

over k and the Dieudonné displays $\mathcal{P}_A$ and $\mathcal{P}_B$ associated to A and B respectively, following Langer-Zink. We are going to use the description in terms of Dieudonné theory. By functoriality, there exists a morphism of Dieudonné modules

$$\mathbb{D}(f): \mathbb{D}(B) \to \mathbb{D}(A).$$

It respects the ^{F}-linear morphisms F_A, F_B and V_A and V_B by construction. It moreover respects the filtrations Q_B and Q_A, that is,

$$\mathbb{D}(f)(Q_B) \subset Q_A,$$

or equivalently, $\mathbb{D}(f)(V_B(\mathbb{D}(B))) \subset V_A(\mathbb{D}(A))$. Indeed $\mathbb{D}(f)(V_B(\mathbb{D}(B)) = (V_A \circ \mathbb{D}(f))(\mathbb{D}(B)) \subset V_A(\mathbb{D}(A))$. We need finally to show that

$$\mathbb{D}(f) \circ V_B^{-1} = V_A^{-1} \circ \mathbb{D}(f).$$

Take $b \in \mathbb{D}(B)$; on the one hand, trivially by definition of V_B^{-1}

$$\mathbb{D}(f) \circ V_B^{-1}(V_B(b)) = \mathbb{D}(f)(b),$$

on the other hand, by definition of V_A^{-1} we have

$$(V_A^{-1} \circ \mathbb{D}(f))(V_B(b)) = V_A^{-1} \circ (V_A \circ \mathbb{D}(f))(b) = \mathbb{D}(f)(b),$$

hence the conclusion. □

If the abelian variety has real multiplication, then the associated display is an RM display.

LEMMA 3.4.6 ([Rap78], 1.3). *Let $\mathcal{P} = (P, Q, F, V^{-1})$ be the display associated to a real multiplication abelian variety A over k after Theorem* (3.4.4). *Then P is a free $\mathcal{O}_L \otimes W(k)$-module of rank* 2.

Remark that by the construction of $\mathcal{P}$, we may identify $Q/I(k)P$ with $H^0(A,\Omega^1_A)$ and P/Q with $H^1(A,\mathcal{O}_A)$ *as* $\mathcal{O}_L\otimes k$*-modules*:

$$\begin{tikzcd}
0 \arrow[r] & H^0(A,\Omega_{A/k}) \arrow[r] \arrow[d, equal] & H^1_{dR}(A/k) \arrow[r] \arrow[d, equal] & H^1(A,\mathcal{O}_A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_{\mathcal{P}_A} \arrow[r] & D_{\mathcal{P}_A} \arrow[r] \arrow[d, equal] & P_A/Q_A \arrow[r] & 0 \\
& & \mathbb{D}(A[p^\infty]) & &
\end{tikzcd}$$

The above diagram allows us to find an explicit basis for P. Consider for instance an $\mathcal{O}_L\otimes k$-basis $\overline{\eta}_0$ for $H^0(A,\Omega^1_A)$ and the dual $\mathcal{O}_L\otimes k$-basis $\overline{\eta}_1$ for $H^1(A,\mathcal{O}_A)$. We choose an $\mathcal{O}_L\otimes W(k)$-basis $\{\eta_0,\eta_1\}$ for P, with $\eta_0\in Q$ and $\eta_1\in P$, reducing modulo $I(k)P$ to $\overline{\eta}_0$ and $\overline{\eta}_1$ respectively. By a slight abuse of notation, in what follows we will not make the distinction between η_i and $\overline{\eta}_i$.

4. Hilbert modular varieties

Let k be a perfect field of characteristic $p>0$ and let $N\geq 5$ be an integer coprime with p. Denote by

$$\mathcal{M}=\mathcal{M}(k,\mathcal{O}_L,\mu_N)$$

the moduli space over k parametrized by quadruples

$$\underline{A}=(A/S,\iota,\lambda,\alpha),$$

where

(1) A is an abelian scheme over a k-scheme S of relative dimension g,
(2) $\iota\colon\mathcal{O}_L\to\mathrm{End}_S(A)$ is a ring homomorphism,
(3) λ is a polarization respecting the $\mathcal{O}_L$-action, that is, an $\mathcal{O}_L$-linear isomorphism on the étale site over S

$$\lambda\colon(M(A),M(A)^+)\xrightarrow{\sim}(\mathfrak{I},\mathfrak{I}^+),$$

where $(\mathfrak{I},\mathfrak{I}^+)$ is a representative in $[\mathrm{CL}^+(\mathrm{L})]$, identifying $M(A)^+$ with $\mathfrak{I}^+$.
(4) α is an $\mathcal{O}_L$-linear injective homomorphism

$$\mu_{N,S}\otimes_{\mathbb{Z}}D_L^{-1}\hookrightarrow A,$$

where

$$(\mu_{N,S}\otimes_Z D_L^{-1})(T)=\mu_{N,S}(T)\otimes_{\mathbb{Z}}D_L^{-1},$$

for any S-scheme T.

We will assume moreover the quadruple to satisfy the *Deligne-Pappas* condition (see [DP94]), that is, the natural morphism

$$A\otimes_{\mathcal{O}_L}M(A)\longrightarrow A^\vee \tag{DP}$$

is an isomorphism. This is equivalent to requiring that for any prime ℓ, there exists, étale locally on S, a symmetric $\mathcal{O}_L$-linear polarization

$$\lambda_\ell\colon A\to A^\vee$$

of degree prime to ℓ (see [AG03, Proposition 3.1] for details).

If the characteristic p of k is unramified in L, then (DP) is equivalent to the *Rapoport condition* (R) [Rap78], that is, $\mathrm{Lie}(A)$ is a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$-module of rank 1. See for example [Gor02, Lemma 5.5]. Note that in general, (DP) always implies (R).

CHAPTER 3

Modular curve

1. Description of the moduli space

Let k be an algebraically closed field of characteristic $p > 0$. Let $N \geq 5$ be an integer such that $(p, N) = 1$. Denote by $X_0(p, N)$ the modular curve over k with $\Gamma_0(p) \cap \Gamma_1(N)$-level structure. It is parametrizing triples $\underline{E} = (E, \alpha, H)$ where

- E/S is an elliptic curve over a k-scheme S,
- α is a rigid $\Gamma_1(N)$-level structure $\mu_{N,S} \hookrightarrow E$,
- $H \subseteq E[p]$ is a subgroup scheme of $E[p]$ of rank p, isotropic with respect to the Weil pairing $E[p] \times E[p] \to \mu_{p,S}$.

In what follows we will denote by $\underline{E}$ an elliptic curve, together with the datum of a $\Gamma_0(N)$-level structure.

Lemma 1.0.1. *The space $X_0(p, N)$ defined above corresponds to the moduli space parametrized by p-isogenies $\underline{E}_1 \xrightarrow{f} \underline{E}_2$ of degree p between elliptic curves E_1 and E_2 with $\Gamma_0(N)$-level structure.*

Proof. Let (E, μ_N, H) be an S-point in $X_0(p, N)$. We obtain a p-isogeny $f\colon \underline{E} \to \underline{E}'$ as follows. Let $E' = E/H$ and let $f\colon E \to E'$ be the natural isogeny. Denote finally by μ'_N the natural level structure induced by μ_N on the quotient.

On the other hand, consider a p-isogeny $\underline{E}_1 \xrightarrow{f} \underline{E}_2$ of degree p between elliptic curves with $\Gamma_0(N)$-level structure and let $H = \mathrm{Ker}(f)$. All we need to show is that H is isotropic with respect to the Weil pairing $E[p] \times E^\vee[p] \to \mu_p$, that is, that given any $\gamma \in M(E)$, the composition $(f^t)^\vee \circ \gamma(H) = 0$. This is proved in [GK12, Lemma 2.1.2] and explained carefully in Lemma (2.0.4), Chapter (4). □

From now on, given a point $f\colon \underline{E}_1 \to \underline{E}_2$ in $X_0(p, N)$, we will denote by f^t the unique isogeny $f^t\colon \underline{E}_2 \to \underline{E}_1$ such that $f^t \circ f = [p]$ and $f^t \circ f = [p]$.

1.1. A concrete description of the geometric k-points of $X_0(p, N)$. Let $\underline{E} = (E, \mu_N, H) \in X_0(p, N)(k)$ be a geometric point. By the Oort-Tate classification of p-group schemes of rank p over algebraically closed fields [OF70, Lemma 1] we have the following possibilities for H

(1) $H = \mu_p$,
(2) $H = \mathbb{Z}/p\mathbb{Z}$,
(3) $H = \alpha_p$.

We have a description of the p-torsion of an elliptic curve E over k:

- If E/k is ordinary, we have that
$$E[p] \simeq \mu_p \oplus \mathbb{Z}/p\mathbb{Z},$$
and $\mu_p = \mathrm{Ker}(\mathrm{Fr}_E)$, and $\mathbb{Z}/p\mathbb{Z} = \mathrm{Ker}(\mathrm{Ver}_E)$.

- If E/k is supersingular, we have that $E[p]$ fits into the non-split exact sequence

$$0 \to \alpha_p \to E[p] \to \alpha_p \to 0.$$

In the equivalent description of the moduli space in terms of isogenies we have the following types of k-points:

(1) the point $(\underline{E}, \mu_p)$ corresponds to the Frobenius morphism of an ordinary elliptic curve

$$\mathrm{Fr}_E \colon E \to E^{(\sigma)},$$

(2) the point $(\underline{E}, \mathbb{Z}/p\mathbb{Z})$ corresponds to the Verschiebung morphism of an ordinary elliptic curve

$$\mathrm{Ver} \colon E \to E^{(1/\sigma)},$$

(3) the point $(\underline{E}, \alpha_p)$ corresponds to the Frobenius morphism of a supersingular elliptic curve

$$\mathrm{Fr} \colon E \to E^{(\sigma)};$$

note that it corresponds to the Verschiebung morphism as well.

1.2. From geometry to linear algebra. Consider an elliptic curve E over k. Recall that the p-torsion $E[p]$ is a group scheme of rank p^2 and hence the Dieudonné module $\mathbb{D}(E[p])$ is a k-vector space of dimension 2. We have therefore by Oda's identification (2.1), Chapter (2), that the de Rham cohomology group $H^1_{dR}(E/k) \simeq \mathbb{D}(E[p])$ is a k-vector space of dimension 2. Moreover, the Hodge filtration $H^0(E, \Omega^1_{E/k}) \subseteq H^1_{dR}(E/k)$ is a sub-k-vector space of dimension 1.

Let now $(\underline{E}_1 \xrightarrow{f} \underline{E}_2) \in X_0(p, N)(k)$ be a geometric point. By means of the Dieudonné functor it induces a morphism

$$\mathbb{D}(f) \colon H^1_{dR}(E_2/k) \to H^1_{dR}(E_1/k)$$

respecting the Hodge filtrations, that is, such that

$$\mathbb{D}(f)(H^0(E_2, \Omega^1_{E_2/k})) \subseteq H^0(E_1, \Omega^1_{E_1/k}). \tag{1.1}$$

Lemma 1.2.1. *Let $x = (\underline{E}_1 \xrightarrow{f} \underline{E}_2) \in X_0(p, N)$ be a geometric k-point. There exist k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ for $H^1_{dR}(E_1/k)$ and $H^1_{dR}(E_2/k)$ respectively such that the induced morphism $\mathbb{D}(f) \colon H^1_{dR}(E_2/k) \to H^1_{dR}(E_1/k)$ is represented by the matrix*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proof. Note that the conditions $f \circ f^t = [p]$ and $f^t \circ f = [p]$ induce the conditions

$$\mathbb{D}(f^t) \circ \mathbb{D}(f) = [p]_{H^1_{dR}(E_2/k)}, \qquad \mathbb{D}(f) \circ \mathbb{D}(f^t) = [p]_{H^1_{dR}(E_1/k)},$$

that is, in terms of matrices

$$\mathbb{D}(f) \circ \mathbb{D}(f^t) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{D}(f^t) \circ \mathbb{D}(f) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1.2}$$

Therefore both the images and the kernels of $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ have dimension 1. We may hence choose k-bases

$$H^1_{dR}(E_1/k) \simeq ke_1 \oplus ke_2, \qquad H^1_{dR}(E_2/k) \simeq k\eta_1 \oplus k\eta_2,$$

such that

$$\mathbb{D}(f)(\eta_2) = e_1, \qquad \mathbb{D}(f^t)(e_2) = \eta_1.$$

By (1.2), we obtain that

$$\mathbb{D}(f)\circ\mathbb{D}(f^t)(e_2) = \mathbb{D}(f)(\eta_1) = pe_2, \qquad \mathbb{D}(f^t)\circ\mathbb{D}(f)(\eta_2) = \mathbb{D}(f^t)(e_1) = p\eta_2,$$

hence the conclusion. □

Denote by $\mathcal{N}_p$ the Grassmann variety of pairs (W_1, W_2) where W_1 and W_2 are k-sub-vector spaces of k^2 of dimension 1 such that, given the morphism

$$k\oplus k \xrightarrow{h=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} k\oplus k,$$

$h(W_1)\subseteq W_2$ and $h(W_2)\subseteq W_1$.

By Lemma (1.2.1) and by (1.1), to a geometric point $(x = \underline{E}_1 \xrightarrow{f} \underline{E}_2) \in X_0(p,N)(k)$ we may associate a point (W_1, W_2) of $\mathcal{N}_p$ by taking

$$W_1 = H^0(E_2, \Omega^1_{E_2/k}), \qquad W_2 = H^0(E_1, \Omega^1_{E_1/k}). \tag{1.3}$$

Proposition 1.2.2. *Given a geometric k-point $(\underline{E}_1 \xrightarrow{f} \underline{E}_2) \in X_0(p,N)(k)$ and the associated point (W_1, W_2) as in (1.3), we have an identification*

$$\widehat{\mathcal{O}}_{X_0(p,N)_{/k},x} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_p,(W_1,W_2)}.$$

Proof. See [DP94, dJ93, GK12] for complete proofs. □

Recall that by Remark (3), Chapter (2), the ring $\widehat{\mathcal{O}}_{X_0(p,N),x}$ is an admissbile topological ring.

We shall analyse the local deformation theory by translating the data of the geometric points in terms of the bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ as in Lemma (1.2.1).

2. Supersingular elliptic curves

Consider a geometric supersingular point $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)}) \in X_0(p,N)(k)$.

2.1. The local model. We are going to compute $\widehat{\mathcal{O}}_{X_0(p,N)_{/k},x}$ in terms of the Grassmannian $\mathcal{N}_p$ after Proposition (1.2.2). Take hence k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ for $H^1_{dR}(E/k)$ and $H^1_{dR}(E^{(\sigma)}/k)$ as in Lemma(1.2.1). Recall that the induced morphism

$$\mathbb{D}(\mathrm{Fr}_E)\colon H^1_{dR}(E^{(\sigma)}/k) \to H^1_{dR}(E/k)$$

is represented by the matrix

$$\mathbb{D}(\mathrm{Fr}_E) = \begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

This makes sense since the Frobenius morphism of a supersingular elliptic curve is nilpotent. Note that the Hodge filtrations are identified with the kernel of $\mathbb{D}(\mathrm{Fr}_E)$ and hence we have that

$$W_1 = H^0(E^{(\sigma)}, \Omega^1_{E^{(\sigma)}/k}) = \langle\eta_1\rangle, \qquad W_2 = H^0(E, \Omega^1_{E/k}) = \langle e_1\rangle.$$

A deformation $\widetilde{W}_1$ (resp. $\widetilde{W}_2$) of W_1 (resp. W_2) to an artinian k-algebra S is uniquely described by a vector $\langle\eta_1 + X\eta_2\rangle$ (resp, $\langle e_1 + Ye_2\rangle$), with $X, Y \in \mathfrak{m}_S$. The condition $h(\widetilde{W}_1) \subset \widetilde{W}_2$

and $h(\widetilde{W_2}) \subset \widetilde{W_1}$, gives the condition $X \cdot Y = 0$, therefore the deformation of (W_1, W_2) is pro-represented by the ring

$$R = \widehat{\mathcal{O}}_{\mathcal{N}_p,(W_1,W_2)} = k[\![X,Y]\!]/(XY)$$

and the universal object of the Grassmannian local model is

$$W_1^{\mathrm{un}} = \langle \eta_1 + X\eta_2 \rangle \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)} W_2^{\mathrm{un}} = \langle e_1 + Ye_2 \rangle \subset Re_1 \oplus Re_2. \tag{2.1}$$

2.2. Equi-characteristic deformation. Following Langer-Zink (see Theorem (3.4.4), Chapter (2)), we associate Dieudonné displays $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}^{(\sigma)} = (P^{(\sigma)}, Q^{(\sigma)}, F^{(\sigma)}, (V^{-1})^{(\sigma)})$ over k to E and $E^{(\sigma)}$ respectively. We have

$$P := \mathbb{D}(E)_{(W(k)/k)} \simeq H^1_{cris}(E/W(k)),$$
$$Q := \mathrm{Ker}(P \to \mathrm{Lie}(E)),$$

and since the construction commutes with base change, we obtain

$$P^{(\sigma)} := \mathbb{D}(E^{(\sigma)})_{(W(k)/k)} \simeq \sigma^*(P)$$
$$Q^{(\sigma)} := \mathrm{Ker}(P^{(\sigma)} \to \mathrm{Lie}(E^{(\sigma)})).$$

Recall that the first crystalline cohomology group is a free $W(k)$-module of rank 2. We have the identifications

$$H^1_{cris}(E/W(k))/(p) \simeq H^1_{dR}(E/k), \qquad H^1_{cris}(E^{(\sigma)}/W(k))/(p) \simeq H^1_{dR}(E^{(\sigma)}/k).$$

Lemma 2.2.1. *We may lift the k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ for $H^1_{dR}(E/k)$ and $H^1_{dR}(E^{(\sigma)}/k)$ obtained in Lemma (1.2.1) to $W(k)$-bases of $H^1_{cris}(E/W(k))$ and $H^1_{cris}(E^{(\sigma)}/W(k))$ such that the induced morphism*

$$\mathbb{D}(\mathrm{Fr}_E)\colon H^1_{cris}(E^{(\sigma)}/W(k)) \to H^1_{cris}(E/W(k))$$

is represented by the matrix

$$\mathbb{D}(\mathrm{Fr}_E) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

By abuse of notation we will denote them $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ as well.

Proof. Recall that by construction (see Lemma(1.2.1)), the bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ we have

$$\mathbb{D}(\mathrm{Fr}_E)(\eta_2) = e_1, \qquad \mathbb{D}(\mathrm{Ver}_E)(e_2) = \eta_1.$$

Lift hence e_1 to $\tilde{e}_1$ and η_2 to $\tilde{\eta}_2$ over $W(k)$. Set now

$$\tilde{e}_1 = \mathbb{D}(\mathrm{Fr}_E)(\tilde{\eta}_2), \qquad \tilde{\eta}_1 = \mathbb{D}(\mathrm{Ver}_E)(e_2). \tag{2.2}$$

Clearly $\{\tilde{e}_1, tildee_2\}$ and $\{\tilde{\eta}_1, \tilde{\eta}_2\}$ reduce to $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ modulo p. Since $W(k)$ is a DVR with residue field k, we may apply the Nakayama lemma and conclude that $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{\eta}_1, \tilde{\eta}_2\}$ are $W(k)$-bases of $H^1_{cris}(E/W(k))$ and $H^1_{cris}(E^{(\sigma)}/W(k))$ respectively. Note that the conditions

$$\mathbb{D}(\mathrm{Fr}_E) \circ \mathbb{D}(\mathrm{Ver}_E) = [p]_{H^1_{cris}(E/W(k))}, \qquad \mathbb{D}(\mathrm{Ver}_E) \circ \mathbb{D}(\mathrm{Fr}_E) = [p]_{H^1_{cris}(E^{(\sigma)}/W(k))}$$

hold, that is in particular

$$\mathbb{D}(\mathrm{Fr}_E) \circ \mathbb{D}(\mathrm{Ver}_E)(e_2) = p\eta_2, \qquad \mathbb{D}(\mathrm{Ver}_E) \circ \mathbb{D}(\mathrm{Fr}_E)(\eta_2) = pe_2.$$

Combining this with (2.2) we get

$$\mathbb{D}(\mathrm{Fr}_E)(\eta_1) = pe_2, \qquad \mathbb{D}(\mathrm{Ver}_E)(e_1) = p\eta_2,$$

hence the conclusion. □

Write

$$\begin{aligned} P^{(\sigma)} &\simeq W(k)\eta_1 \oplus W(k)\eta_2, \qquad & P &\simeq W(k)e_1 \oplus W(k)e_2, \\ Q^{(\sigma)} &\simeq W(k)\eta_1 oplus (p)\eta_2, \qquad & Q &\simeq W(k)e_1 \oplus (p)e_2; \end{aligned}$$

the induced map of displays $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$ is represented by the matrix

$$\mathbb{D}(\mathrm{Fr}) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Since $\mathbb{D}(\mathrm{Fr})$ is the linearization of F and $\mathcal{P}^{(\sigma)}$ is the Frobenius base change of $\mathcal{P}$ we obtain also that the ^{F}-linear maps $F\colon P \to P$ and $F^{(\sigma)}\colon P^{(\sigma)} \to P^{(\sigma)}$ are represented by

$$F^{(\sigma)} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix},$$

respectively in the bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$.

Our goal is to define a universal object

$$\mathcal{P}_1^{\mathrm{un}} \xrightarrow{\alpha^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}, \tag{2.3}$$

over a complete Noetherian k-algebra R for the deformation of

$$\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$$

over k. This means that given any deformation $\widetilde{\mathcal{P}}^{(\sigma)} \xrightarrow{\widetilde{\mathbb{D}(\mathrm{Fr}_E)}} \widetilde{\mathcal{P}}$ of $\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$ to an Artinian k-algebra S, there exists a unique morphism $\rho\colon R \to S$ such that

$$(\mathcal{P}_1^{\mathrm{un}} \xrightarrow{\alpha^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}) \otimes_{R,\rho} S \simeq \widetilde{\mathcal{P}}^{(\sigma)} \xrightarrow{\widetilde{\mathbb{D}(\mathrm{Fr}_E)}} \widetilde{\mathcal{P}}.$$

By the Serre-Tate Theorem (Theorem (2.2.2), Chapter (2)) and by the equivalence between p-divisible groups and Dieudonné displays over admissible topological rings (Theorem (3.2.2), Chapter (2)) such a universal object exists, and by the theory of local models it exists over $R = k[\![X, Y]\!]/(XY)$.

2.3. Strategy for the construction of a universal object. The idea is to define Dieudonné displays $\mathcal{P}_1$ and $\mathcal{P}_2$ over R, lifting $\mathcal{P}^{(\sigma)}$ and $\mathcal{P}$ respectively, and a map between them

$$\mathcal{P}_1 \xrightarrow{\alpha} \mathcal{P}_2,$$

reducing to α modulo $\mathfrak{m}_R = (X, Y)$, such that crystalline theory and display theory *compare well*. By this, we mean that the Hodge filtrations of the two Dieudonné displays should correspond to the universal object given by the local model described in (2.1) and that the isomorphisms of quadruples

(2.4)

$$\begin{tikzcd} \widehat{\mathcal{P}}^{(\sigma)} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\approx"] & \widehat{\mathcal{P}}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) & \widehat{\mathcal{P}}_2 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\approx"] & \widehat{\mathcal{P}} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \\ \mathcal{P}^{(\sigma)} \arrow[r, "\mathrm{id}"] & \mathcal{P}^{(\sigma)} & \mathcal{P} \arrow[r, "\mathrm{id}"] & \mathcal{P}. \end{tikzcd}$$

obtained by the crystalline theory (see Theorem (3.4.3), Chapter (2)) identify the Hodge filtrations. The quadruples $\widehat{\mathcal{P}}^{(\sigma)}$ and $\widehat{\mathcal{P}}$ are taken with respect to the identity on k. If these conditions are satisfied, then we are able to adapt Theorem 5.6.2, by Andreatta-Goren ([AG04]) to the $\Gamma_0(p)$-case.

Define two quadruples

$$\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1}) \text{ and } \mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$$

over R, as follows. Let $P_1 \coloneqq \mathbb{W}(R)\eta_1 \oplus \mathbb{W}(R)\eta_2$, $P_2 \coloneqq \mathbb{W}(R)e_1 \oplus \mathbb{W}(R)e_2$ and Q_1 (resp. Q_2) the inverse image of W_1^{un} (resp. W_2^{un}) via the projection $P_1 = \mathbb{W}(R)\eta_1 \oplus \mathbb{W}(R)\eta_2 \to R\eta_1 \oplus R\eta_2$ (resp. $P_2 = \mathbb{W}(R)e_1 \oplus \mathbb{W}(R)e_2 \to Re_1 \oplus Re_2$), namely

$$Q_1 \simeq \mathbb{I}(R)\eta_1 \oplus \mathbb{I}(R)\eta_2 + \mathbb{W}(R)(\eta_1 + \hat{X}\eta_2), \qquad Q_2 \simeq \mathbb{I}(R)e_1 \oplus \mathbb{I}(R)e_2 + \mathbb{W}(R)(e_1 + \hat{Y}e_2).$$

We want to construct semi-linear maps $F_i \colon P_i \to P_i$, such that $\overline{F}_i = F_i \pmod{\mathbb{I}(R)}$ has kernel equal to W_i^{un}, for $i = 1, 2$. This property is satisfied by the maps F_i represented by the matrices

$$F_1 = \begin{pmatrix} -{}^F\hat{X} & 1 \\ p & -{}^F\hat{Y} \end{pmatrix}, \qquad F_2 = \begin{pmatrix} -{}^F\hat{Y} & 1 \\ p & -{}^F\hat{X} \end{pmatrix},$$

in the bases $\{\eta_1, \eta_2\}$ and $\{e_1, e_2\}$. Note that the coefficients of the matrices of F_1 and F_2 do belong to $\mathbb{W}(R)$. Indeed, both X and Y belong to $\mathfrak{m}_R$ and their Teichmüller lifts have only one non-zero compoment and hence belong to $\widehat{W}(\mathfrak{m}_R) \subset \mathbb{W}(R)$. Since $\mathbb{W}(R)$ is ^{F}-equivariant, we have that ${}^F\widehat{X}$ and ${}^F\widehat{Y}$ belong to $\mathbb{W}(R)$.
Define $V_i^{-1} \colon Q_i \to P_1$ as the restriction to Q_i of F_i/p.

Theorem 2.3.1. *The quadruples $\mathcal{P}_1$ and $\mathcal{P}_2$ are Dieudonné displays over R. They reduce to $\mathcal{P}^{(\sigma)}$ and $\mathcal{P}$ modulo $\mathfrak{m}_R$ respectively and their Hodge filtrations are $W_1^{\mathrm{un}} \subset R\eta_1 \oplus R\eta_2$ and $W_2^{\mathrm{un}} \subset Re_1 \oplus Re_2$.*

Proof. We will prove this for $i = 1$, the case for $i = 2$ being essentially identical. The only part that is left to prove is that V_1^{-1} is well defined over Q_1 and that is is an ^{F}-linear epimorphism.

Recall that we set $V_1^{-1} \coloneqq \frac{F_1}{p}|_{Q_1}$. We claim that $F_1(Q_1) \subseteq pP_1$. An element $x \in Q_1 \simeq \mathbb{I}(R)(\eta_1 \oplus \eta_2) + \mathbb{W}(R)(\eta_1 + \hat{X}\eta_2)$ can be written as a vector $\begin{pmatrix} \theta_1 + w \\ \theta_2 + w\hat{X} \end{pmatrix}$, where $\theta_1, \theta_2 \in \mathbb{I}(R)$ and $w \in \mathbb{W}(R)$. Therefore $F_1 x = \begin{pmatrix} -{}^F\hat{X} & 1 \\ p & -{}^F\hat{Y} \end{pmatrix} \begin{pmatrix} \theta_1 + w \\ \theta_2 + w\hat{X} \end{pmatrix} = \begin{pmatrix} -{}^F\hat{X}{}^F\theta_1 + {}^F\theta_2 \\ p^F\theta_1 + p^F w - {}^F\hat{Y}{}^F\theta_2 \end{pmatrix} \in {}^F\mathbb{I}(R)(\eta_1 \oplus \eta_2) + p\mathbb{W}(R)(\eta_1 + \hat{X}\eta_2)$. By (3.1), Chapter (2), we have that ${}^F\mathbb{I}(R) \subseteq p\mathbb{W}(R)$, hence we conclude that V_1^{-1} is well-defined on Q_1.

We have left to show that V_1^{-1} is an ^{F}-linear epimorphism, that is, that its linearization

$$(V_1^{-1})^\sharp \colon \mathbb{W}(R) \otimes_{\mathbb{W}(R), F} Q_1 \to P_1$$

is an epimorphism. By [Zin01a], see Section (3), Chapter (2), this equals to showing that, for a suitable decomposition $P = L \oplus T$ such that $Q = L \oplus \mathbb{I}(R)T$, the morphism

$$V^{-1} \oplus F \colon L \oplus T \longrightarrow P$$

is an ^{F}-linear isomorphism. Take hence

$$L_1 = \langle \eta_1' \rangle, \qquad T_1 = \langle \eta_2' \rangle,$$

with $\eta_1' = \eta_1 + \hat{X}\eta_2$ and $\eta_2' = \eta_2$. If we show that the *displaying* matrix associated to $V^{-1} \oplus F$ is invertible, we are done. Remark that, from the basis $\{\eta_1', \eta_2'\}$ to the basis $\{\eta_1, \eta_2\}$, the ^{F}-linear map F is represented by $\begin{pmatrix} 0 & 1 \\ p & -{}^F Y \end{pmatrix}$, that is, the ^{F}-linear map $V^{-1} \oplus F \colon L \oplus T \longrightarrow P$ is represented by

$$\begin{pmatrix} 0 & 1 \\ 1 & -{}^F Y \end{pmatrix},$$

which is invertible, hence the conclusion.

□

THEOREM 2.3.2. *For $\mathcal{P}_1$ and $\mathcal{P}_2$ constructed as above, there exists a unique map of Dieudonné displays $\alpha \colon \mathcal{P}_1 \to \mathcal{P}_2$ lifting $\mathbb{D}(f) \colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$ to R. Moreover α is universal in the sense of* (2.3).

PROOF. Consider $\alpha \colon P_1 \to \mathcal{P}_2$ represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

CLAIM. *The map α is a morphism between the Dieudonné displays $\mathcal{P}_1$ and $\mathcal{P}_2$ lifting the morphism $\mathbb{D}(\mathrm{Fr}_E) \colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$.*

In order to prove the claim we need to show that the following conditions hold

$$\alpha(Q_1) \subseteq Q_2, \tag{2.5}$$

$$\alpha \circ F_1 = F_2 \circ \alpha^{(\sigma)}, \qquad \alpha \circ V_1^{-1} = V_2^{-1} \circ \alpha^{(\sigma)}. \tag{2.6}$$

Let us first show (2.5). Recall that

$$Q_1 = \mathbb{I}(R)\eta_1 \oplus \mathbb{I}(R)\eta_2 + \mathbb{W}(R)(\eta_1 + \hat{X}\eta_2), \qquad Q_2 \mathbb{I}(R)e_1 \oplus \mathbb{I}(R)e_2 + \mathbb{W}(R)(e_1 + \hat{Y}e_2).$$

A vector in Q_1 is therefore of the form $\theta\eta_1 + \zeta\eta_2 + w(\eta_1 + \hat{x}\eta_2)$ with $\theta, \zeta \in \mathbb{I}(R)$ and $w \in \mathbb{W}(R)$. Hence $\alpha\begin{pmatrix} \theta + w \\ \zeta + w\hat{X} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\begin{pmatrix} \theta + w \\ \zeta + w\hat{X} \end{pmatrix} = \begin{pmatrix} \zeta + w\hat{X} \\ p\theta + pw \end{pmatrix} = \begin{pmatrix} \zeta + w\hat{X} \\ p\theta + \hat{Y}\hat{X}w \end{pmatrix} =: q$. Note that $q = \zeta e_1 + p\theta e_2 + \hat{X}w(e_1 + \hat{Y}) \in Q_2$ since $\zeta, p\theta \in \mathbb{I}(R)$ and $\hat{X}w \in \mathbb{W}(R)$. Condition (2.6) is proved by direct computation:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha} \underbrace{\begin{pmatrix} -{}^F\hat{X} & 1 \\ p & -{}^F\hat{Y} \end{pmatrix}}_{F_1} = \begin{pmatrix} p & -{}^F\hat{Y} \\ -p{}^F\hat{X} & p \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} -{}^F\hat{Y} & 1 \\ p & -{}^F\hat{X} \end{pmatrix}}_{F_2} \underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha^{(\sigma)}} = \begin{pmatrix} p & -{}^F\hat{Y} \\ -p{}^F\hat{X} & p \end{pmatrix}.$$

The commutativity of α with V_1^{-1} and V_2^{-1} follows, since V_1^{-1} and V_2^{-1} were defined from F_1 and F_2. Recall that $\mathfrak{m}_R = (X, Y)$. We can see easily that the morphism of displays

$$\alpha \colon \mathcal{P}_1 \to \mathcal{P}_2$$

reduces to the morphism of Dieudonné displays

$$\mathbb{D}(\mathrm{Fr}_E) \colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$$

modulo $\mathfrak{m}_R$.

We have left to show that $\alpha: \mathcal{P}_1 \to \mathcal{P}_2$ is universal with respect to the deformation of $\mathbb{D}(f): \mathcal{P}^{(\sigma)} \to \mathcal{P}$. Note first that the isomorphisms in (2.4) are both the identity and that $\alpha \pmod{\mathfrak{m}_R^2} = \mathbb{D}(f) \otimes \mathbb{W}(R/\mathfrak{m}_R^2)$. Therefore the natural trivializations $\rho: D_{\mathcal{P}_1} \xrightarrow{\sim} R\eta_1 \oplus R\eta_2$ and $\tau: D_{\mathcal{P}_2} \xrightarrow{\sim} Re_1 \oplus Re_2$ are horizontal modulo $\mathfrak{m}_R^2$ and such that $\rho(H_{\mathcal{P}_1}) = W_1^{\mathrm{un}}$ and $\tau(H_{\mathcal{P}_2}) = W_2^{\mathrm{un}}$ and the reduction $\overline{\alpha}$ to R of α is equal to h.

Consider hence $\mathcal{P}_1^{\mathrm{un}} \xrightarrow{\alpha^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}$ as in (2.3). By universality, there exists a unique homomorphism $\xi: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ such that

$$\xi^*(\mathcal{P}_1^{\mathrm{un}} \xrightarrow{\alpha^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}) \simeq (\mathcal{P}_1 \xrightarrow{\alpha} \mathcal{P}_2).$$

Claim. *The morphism $\xi: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is an isomorphism.*

Recall the universal object for the Grassmannian moduli problem

By universality of

$$(W_1^{\mathrm{un}} \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} W_2^{\mathrm{un}} \subset Re_1 \oplus Re_2)$$

with respect to the deformation of $(W_1, W_2) \in \mathcal{N}$, we there exist two unique morphisms

$$\psi_1: \mathrm{Spec}(R) \to \mathrm{Spec}(R), \qquad \psi_2: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$$

such that

$$\psi_1^*(W_1^{\mathrm{un}} \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} W_2^{\mathrm{un}} \subset Re_1 \oplus Re_2) \simeq (H_{\mathcal{P}_1^{\mathrm{un}}} \subset D_{\mathcal{P}_1^{\mathrm{un}}} \xrightarrow{\overline{\alpha}^{\mathrm{un}}} H_{\mathcal{P}_2^{\mathrm{un}}} \subset D_{\mathcal{P}_2^{\mathrm{un}}}) \tag{2.7}$$

and

$$\psi_2^*(W_1^{\mathrm{un}} \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} W_2^{\mathrm{un}} \subset Re_1 \oplus Re_2) \simeq (H_{\mathcal{P}_1} \subset D_{\mathcal{P}_1} \xrightarrow{\overline{\alpha}} H_{\mathcal{P}_2} \subset D_{\mathcal{P}_2}). \tag{2.8}$$

Note that we may choose trivializations

$$\begin{tikzcd} D_{\mathcal{P}_1^{\mathrm{un}}} \arrow[r, "\simeq"] \arrow[d, "\alpha^{\mathrm{un}}"'] & R^2 \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)"] \\ D_{\mathcal{P}_2^{\mathrm{un}}} \arrow[r, "\simeq"] & R^2 \end{tikzcd}$$

such that the isomorphism in (2.7) is horizontal modulo $\mathfrak{m}_R^2$ with respect to the Gauss-Manin connection. We want to show now that also the isomorphism in (2.8) is horizontal modulo $\mathfrak{m}_R^2$ by construction of $\alpha: \mathcal{P}_1 \to \mathcal{P}_2$. We are in fact going to show that the isomorphism (2.8) is the identity modulo $\mathfrak{m}_R^2$. Note that the isomorphisms of quadruples in (2.4)

(2.9)

$$\widehat{\mathcal{P}}^{(\sigma)} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\sim} \widehat{\mathcal{P}}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R), \qquad \widehat{\mathcal{P}} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\sim} \widehat{\mathcal{P}}_2 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2),$$

lifting the identity morphisms on $\mathcal{P}^{(\sigma)}$ and $\mathcal{P}$ are actually the identity. Indeed, the $\mathbb{W}(R)$-bases $\{\eta_1, \eta_2\}$ and $\{e_1, e_2\}$ are lifts of the bases $\{\eta_1, \eta_2\}$ and $\{e_1, e_2\}$ describing the local model. Moreover by construction

$$F_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = F^{(\sigma)} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2), \qquad F_2 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = F \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2).$$

From this we obtain that modulo $\mathfrak{m}_R^2$ the Hodge filtrations $(H_{\mathcal{P}_1}, H_{\mathcal{P}_2})$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ are identified with the universal object $(W_1^{\mathrm{un}}, W_2^{\mathrm{un}})$ describing the local model. Now, by Theorem (3.4.3) the isomorphisms in (2.9) are unique and they are therefore the identity. Note that also the morphism of displays $\alpha\colon \mathcal{P}_1 \to \mathcal{P}_2$ lifting $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$ is crystalline in nature. Indeed, by Theorem (3.4.3) we obtain that modulo $\mathfrak{m}_R^2$ the morphism of quadruples obtained from α is unique:

$$\begin{tikzcd} \widehat{\mathcal{P}}^{(\sigma)} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\simeq"] & \widehat{\mathcal{P}}_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\overline{\alpha}", "\exists!"'] & \widehat{\mathcal{P}}_2 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\simeq"] & \widehat{\mathcal{P}} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \\ \mathcal{P}^{(\sigma)} \arrow[r, "\mathrm{id}"] & \mathcal{P}^{(\sigma)} \arrow[r, "\mathbb{D}(\mathrm{Fr}_E)"] & \mathcal{P} \arrow[r, "\mathrm{id}"] & \mathcal{P}. \end{tikzcd}$$

We obtain therefore that the isomorphism (2.8) is horizontal with respect to the Gauss-Manin connection.

Note that there is a commutative diagram $\psi_1 \circ \phi = \psi_2$. By the theory of local models by Deligne and Pappas, also ψ_1 is an isomorphism on tangent spaces. Therefore ϕ is an isomorphism on tangent spaces. We conclude that ϕ is an isomorphism by a commutative algebra argument identical to the one given in [AG04, Theorem 5.6.2]. Namely, since ϕ is an isomorphism on tangent spaces, we have that for any n the induced morphism $\mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$ is surjective. Let $\mathrm{Gr}(R) = \oplus_n \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$ be the graded ring associated to R. the map ϕ induces a map $\mathrm{Gr}(\phi^\sharp)\colon \mathrm{Gr}(R) \to \mathrm{Gr}(R)$ which is surjective on each graded piece and hence by dimension considerations $\mathrm{Gr}(\phi^\sharp)$ is an isomorphism. By [AM69, Lemma 10.23], $\varphi^\sharp$ is an isomorphism as well. ☐

3. Ordinary elliptic curves

Consider first a geometric point $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)}) \in X_0(p,N)^{\mu_p} \subseteq X_0(p,N)$.

3.1. The local model and deformation theory for $X_0(p,N)^{\mu_p}$. We would like the induced morphism

$$\mathbb{D}(\mathrm{Fr}_E)\colon H^1_{dR}(E^{(\sigma)}/k) \to H^1_{dR}(E/k)$$

to reflect the geometric properties of the Frobenius morphism. Since the elliptic curve is ordinary, we know that the Frobenius morphism is idempotent.

LEMMA 3.1.1. *Let $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)}) \in X_0(p,N)(k)$ be the Frobenius morphism of an elliptic curve. There exist k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ for $H^1_{dR}(E/k)$ and $H^1_{dR}(E^{(\sigma)}/k)$ respectively such that the induced morphism $\mathbb{D}(\mathrm{Fr}_E)\colon H^1_{dR}(E^{(\sigma)}/k) \to H^1_{dR}(E/k)$ is represented by*

$$\mathbb{D}(\mathrm{Fr}_E) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

PROOF. We can easily obtain such bases from the bases obtained in Lemma (1.2.1), where the matrix of $\mathbb{D}(\mathrm{Fr}_E)$ is represented by the matrix $\mathbb{D}(\mathrm{Fr}_E) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Keep for instance $\{e_1, e_2\}$ as it is and invert the order of the basis $\{\eta_1, \eta_2\}$. ☐

By taking into account the point

$$W_1 \subset k\eta_1 \oplus k\eta_2 \xrightarrow{h} W_2 \subset ke_1 \oplus ke_2$$

in the Grassmannian $\mathcal{N}_p$ associated to $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)})$, we have that h is represented by $h = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $W_1 = \langle \eta_1 \rangle$ and $W_2 = \langle e_1 \rangle$. If we take a general deformation $(\widetilde{W}_1, \widetilde{W}_2)$ to an artinian k-algebra S of the point (W_1, W_2), that is, $\widetilde{W}_1 = \langle \eta_1 + T\eta_2 \rangle$ and $\widetilde{W}_2 = \langle e_1 + Ze_2 \rangle$ with

$T, Z \in \mathfrak{m}_S$, from the conditions $h(\widetilde{W}_1) \subseteq \widetilde{W}_2$ and $h^t(\widetilde{W}_2) \subseteq \widetilde{W}_1$ we obtain the condition $T = 0$. It follows that the universal object for the deformation of (W_1, W_2) is

$$W_1^{\text{un}} = \langle \eta_1 \rangle \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)} W_2^{\text{un}} = \langle e_1 + Ze_2 \rangle \subset e_1 R \oplus e_2 R,$$

where $R \simeq k[\![Z]\!]$.

Again, to E and $E^{(\sigma)}$ respectively, through crystalline theory (see Theorem (3.4.4), Chapter (2)) we associate Dieudonné displays $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}^{(\sigma)} = (P^{(\sigma)}, Q^{(\sigma)}, F^{(\sigma)}, (V^{-1})^{(\sigma)})$ over k. By a reasoning identical in substance to the one proving Lemma (2.2.1) we lift the bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ to $W(k)$-bases for $H^1_{cris}(A/W(k))$ and $H^1_{cris}(B/W(k))$. We may therefore write

$$\begin{aligned} P^{(\sigma)} &= W(k)\eta_1 \oplus W(k)\eta_2, \qquad & P &= W(k)e_1 \oplus W(k)e_2, \\ Q^{(\sigma)} &= W(k)\eta_1 + (p)\eta_2, \qquad & Q &= W(k)e_1 + (p)e_2. \end{aligned}$$

In these bases the induced map of displays $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$ is represented by

$$\mathbb{D}(\mathrm{Fr}_E) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $\mathbb{D}(\mathrm{Fr}_E)$ is the linearization of the semi-linear map $F\colon P \to P$ and that $F^{(\sigma)}\colon P^{(\sigma)} \to P^{(\sigma)}$ is its ^{F}-linear base change. Hence F and $F^{(\sigma)}$ are represented by the matrix

$$F = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = F^{(\sigma)}.$$

Define quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ over R, with $P_1 \coloneqq P_0^{(\sigma)} \otimes \mathbb{W}(R)$ and $P_2 \coloneqq P \otimes \mathbb{W}(R)$, and Q_1 (resp. Q_2) the pre-image of W_1^{un} (resp. W_2^{un}) through the projection $P_1 \to H^1_{dR}(B/k) \otimes_k R$ (resp. $P_2 \to H^1_{dR}(A/k) \otimes_k R$). In terms of the bases $\{\eta_1, \eta_2\}$ and $\{e_1, e_2\}$ we have

$$\begin{aligned} P_1 &= \mathbb{W}(R)\eta_1 \oplus \mathbb{W}(R)\eta_2, \qquad & P_2 &= \mathbb{W}(R)e_1 \oplus \mathbb{W}(R)e_2, \\ Q_1 &= \mathbb{W}(R)\eta_1 + \mathbb{I}(R)\eta_2, \qquad & Q_2 &= \mathbb{I}(R)(e_1 \oplus e_2) + \mathbb{W}(R)(e_1 + Ze_2). \end{aligned}$$

Define $F_i\colon P \to P$ as the semi-linear maps that in the bases $\{\eta_1, \eta_2\}$ and $\{e_1, e_2\}$ are represented by

$$F_1 = \begin{pmatrix} p & 0 \\ -p{}^F\hat{Z} & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} p & 0 \\ -{}^F\hat{Z} & 1 \end{pmatrix}.$$

Note that, by an argument identical to the supersingular case the coefficients of these matrices belong to $\mathbb{W}(R)$. Finally, define $V_i^{-1}\colon Q_i \to P_i$ as the restriction to Q_i of F_i/p. These are Dieudonné displays over R. The argument for proving this is identical to that proving Theorem (2.3.1). In particular we provide a normal decomposition for $\mathcal{P}_1$ and $\mathcal{P}_2$, namely take:

$$\begin{aligned} L_1 &= \langle \eta_1 \rangle, \qquad & T_1 &= \langle \eta_2 \rangle, \\ L_2 &= \langle e_1 + \hat{Z}e_2 \rangle, \qquad & T_2 &= \langle e_2 \rangle. \end{aligned}$$

Note that the map $\alpha\colon P_1 \to P_2$ represented by

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies the relation

$$\alpha \cdot F_1 = F_2 \cdot \alpha^{(\sigma)}.$$

Hence, it is a map between $\mathcal{P}_1$ and $\mathcal{P}_2$, lifting $\mathbb{D}(\mathrm{Fr})\colon \mathcal{P} \to \mathcal{P}^{(\sigma)}$. Therefore, by Theorem (2.3.2),

$$\mathcal{P}_1 \xrightarrow{\alpha} \mathcal{P}_2$$

is universal for the deformation of $\mathcal{P}_0^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}_0$.

3.2. The local model and deformation theory for $X_0(p,N)^{\mathbb{Z}/p\mathbb{Z}}$. The construction is analogous for a geometric k-point $z = (E \xrightarrow{\mathrm{Ver}_E} E^{(1/\sigma)}) \in X_0(p,N)^{\mathbb{Z}_p}$.

LEMMA 3.2.1. *Let $x = (\underline{E} \xrightarrow{\mathrm{Ver}_E} \underline{E}^{(1/\sigma)}) \in X_0(p,N)^{\mathbb{Z}/p\mathbb{Z}}$ be a geometric k-point. There exist k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ for $H^1_{dR}(E/k)$ and $H^1_{dR}(E^{(1/\sigma)}/k)$ respectively such that the induced morphism $\mathbb{D}(\mathrm{Ver}_E)\colon H^1_{dR}(E^{(1/\sigma)}/k) \to H^1_{dR}(E/k)$ is represented by the matrix*

$$\mathbb{D}(\mathrm{Ver}_E) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

PROOF. Such bases are obtained from the bases described in Lemma (1.2.1) by inverting the order of the basis $\{e_1, e_2\}$. □

The point $(W_1, W_2) \in \mathcal{N}_p$ associated to $x = (\underline{E} \xrightarrow{\mathrm{Ver}_E} \underline{E}^{(1/\sigma)})$ is such that

$$W_1 = \langle \eta_1 \rangle, \qquad W_2 = \langle e_1 \rangle$$

and the morphism h is represented by the matrix $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. A general deformation $(\widetilde{W_1}, \widetilde{W_2})$ to an artinian k-algebra S of the corresponding point (W_1, W_2) is given by $\widetilde{W_1} = \langle \eta_1 + Z\eta_2 \rangle$ and $\widetilde{W_2} = \langle e_1 + Te_2 \rangle$. The conditions $h(\widetilde{W_1}) \subset \widetilde{W_2}, h^t(\widetilde{W_2}) \subset \widetilde{W_1}$ give the condition $T = 0$. The universal object

$$W_1^{\mathrm{un}} = \langle \eta_1 + Z\eta_2 \rangle \subset R\eta_1 \oplus R\eta_2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} W_2^{\mathrm{un}} = \langle e_1 \rangle \subset Re_1 \oplus Re_2$$

is defined over $R := k[\![Z]\!]$.

To x we associate now a map of Dieudonné displays $\mathbb{D}(\mathrm{Ver}_E)\colon \mathcal{P}^{(1/\sigma)} = (P^{(1/\sigma)}, Q^{(1/\sigma)}, F^{(1/\sigma)}, (V^{-1})^{(1/\sigma)}) \to \mathcal{P} = (P, Q, F, V^{-1})$. We lift the k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ to $W(k)$-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$. Analogously to the argument in Lemma (2.2.1) we have

$$P^{(1/\sigma)} = W(k)\eta_1 \oplus W(k)\eta_2, \qquad P = W(k)e_1 \oplus W(k)e_2$$

and in the same bases

$$\mathbb{D}(\mathrm{Ver}_E) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

The matrix of $\mathbb{D}(\mathrm{Ver}_E)$ represents the Verschiebung morphism. The semi-linear maps $F\colon P \to P$ and $F^{(1/\sigma)}\colon P^{(1/\sigma)} \to P^{(1/\sigma)}$ are represented by

$$F = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = F^{(1/\sigma)}.$$

The strategy to determine a universal object over R is identical to the one used above. A choice for a universal object is $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1}) \xrightarrow{\alpha} \mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ over R with

$$F_1 = \begin{pmatrix} p & 0 \\ -{}^F\hat{Z} & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} p & 0 \\ -p^F\hat{Z} & 1 \end{pmatrix},$$

and the map α represented by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

4. The deformation theory in mixed characteristic

Consider the modular curve $X_0(p,N)$ over $W(k)$ parametrizing p-isogenies of elliptic curves $\underline{E}_1 \xrightarrow{f} \underline{E}_2$, where $\underline{E}_1$ and $\underline{E}_2$ are elliptic curves over an $W(k)$-schemes S.

An elliptic curve E over a $W(k)$-scheme S, the first de Rham cohomology group $H^1_{dR}(E/S)$ is a locally free $\mathcal{O}_S$-module of rank 2.

Lemma 4.0.1. *Let $(S, \mathfrak{m}_S)$ be an Artinian local $W(k)$-algebra with residue field k. Consider a point $x = (\underline{E}_1 \xrightarrow{f} \underline{E}_2) \in X_0(p,N)(\mathrm{Spec}(S))$ for a $W(k)$-scheme $\mathrm{Spec}(S)$. On a small enough neighbourhood of x there exists trivializations*

$$H^1_{dR}(E_1/S) \xrightarrow{\simeq} Se_1 \oplus Se_2, \qquad H^1_{dR}(E_2/S) \xrightarrow{\simeq} S\eta_1 \oplus S\eta_2,$$

such that the induced morphism

$$\mathbb{D}(f) \colon H^1_{dR}(E_1/S) \to H^1_{dR}(E_2/S)$$

is represented by the matrix

$$\mathbb{D}(f) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Proof. By the considerations preceding the statement of the Lemma, in a small enough neighbourhood of x, the S-modules $H^1_{dR}(E_1/S)$ and $H^1_{dR}(E_2/S)$ are free of rank 2. By Lemma (1.2.1), given the reduction $\overline{x}$ modulo $\mathfrak{m}_S$ of x there exist trivializations

$$H^1_{dR}(\overline{E}_1/k) \xrightarrow{\simeq} ke_1 \oplus ke_2, \qquad H^1_{dR}(\overline{E}_2/k) \xrightarrow{\simeq} k\eta_1 \oplus k\eta_2,$$

such that the induced morphism

$$\mathbb{D}(\overline{f}) \colon H^1_{dR}(\overline{E}_1/k) \to H^1_{dR}(\overline{E}_2/k)$$

is represented by the matrix

$$\mathbb{D}(\overline{f}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{D}(\overline{f^t}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

By Nakayama's Lemma, we may lift the k-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ to S-bases for $H^1_{dR}(E_1/S)$ and $H^1_{dR}(E_2/S)$ respectively. Note that given $f^t \colon B \to A$ the unique p-isogeny satisfying the relations $f^t \circ f = [p]_{E_1}$ and $f \circ f^t = [p]_{E_2}$ we obtain relations

$$\mathbb{D}(f) \circ \mathbb{D}(f^t) = [p]_{H^1_{dR}(E_1/S)}, \qquad \mathbb{D}(f^t) \circ \mathbb{D}(f) = [p]_{H^1_{dR}(E_2/S)}.$$

In the S-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$ this translates into the relations

$$\mathbb{D}(f) \circ \mathbb{D}(f^t) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \qquad \mathbb{D}(f^t) \circ \mathbb{D}(f) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix},$$

which combined with (4.1) gives us

$$(\mathbb{D}(f)\circ\mathbb{D}(f^t))(e_2)=\mathbb{D}(f)(\eta_1)=pe_2,\qquad (\mathbb{D}(f^t)\circ\mathbb{D}(f))(\eta_2)=\mathbb{D}(f^t)(e_1)=p\eta_2,$$

from which we obtain that

$$\mathbb{D}(f)=\begin{pmatrix}0&1\\p&0\end{pmatrix}.$$

□

As in the equi-characteristic case, to a geometric point $x=\underline{E}_1\to\underline{E}_2\in X_0(p,N)(k)$ we will associate a point of a suitable Grassmann variety and construct the local model. Denote by $\mathcal{N}_p$ the Grassmann variety of pairs (W_1,W_2), where W_1 and W_2 are free k-submodules of k^2, such that given the k-linear map

$$k^2\xrightarrow{h=\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)}k^2$$

we have the inclusions $h(W_1)\subseteq W_2$ and $h^t(W_2)\subseteq W_1$. By applying crystalline theory we obtain an isomorphism of completed local rings

$$\widehat{\mathcal{O}}_{X_0(p,N),x}\simeq\widehat{\mathcal{O}}_{\mathcal{N}_p,(W_1,W_2)}.$$

Let us therefore understand the deformation theory of points (W_1,W_2) in order to understand the local geometry of $X_0(p,N)/W(k)$. As in the equi-characteristic case, a full description of completed local rings of $X_0(p,N)$ was provided by Deligne and Rapoport in their Antwerp paper [DR75].

NOTE 1. *Let (W_1,W_2) be a k-point in $\mathcal{N}_p$. We have the following possibilities*

(1) $W_1=\mathrm{Ker}(h)=W_2$, *that is*

$$W_1=\langle\eta_1\rangle,\qquad W_2=\langle e_1\rangle.$$

Given generic deformations

$$\widetilde{W_1}=\langle\eta_1+X\eta_2\rangle,\qquad \widetilde{W_2}=\langle e_1+Ye_2\rangle,$$

the hypotheses

$$h(\widetilde{W_1})\subseteq\widetilde{W_2},\qquad h(\widetilde{W_2})\subseteq\widetilde{W_1},$$

we get the condition $X\cdot Y=p$. The deformation ring is therefore $W(k)[\![X,Y]\!]/(XY-p)$.

(2) $W_1=\mathrm{Ker}(h)$ *and* $W_2\neq\mathrm{Ker}(h)$, *in terms of bases*

$$W_1=\langle\eta_1\rangle,\qquad W_2=\langle e_2\rangle.$$

Given generic deformations

$$\widetilde{W_1}=\langle\eta_1+X\eta_2\rangle,\qquad \widetilde{W_2}=\langle e_2+Ye_1\rangle,$$

the hypotheses

$$h(\widetilde{W_1})\subseteq\widetilde{W_2},\qquad h_i(\widetilde{W_2})\subseteq\widetilde{W_1},$$

we get the condition $X=0$. The deformation ring is therefore $W(k)[\![Y]\!]$.

(3) $W_1 \neq \mathrm{Ker}(h)$ *and* $W_2 = \mathrm{Ker}(h)$*, that is*

$$W_1 = \langle \eta_2 \rangle, \qquad W_2 = \langle e_1 \rangle.$$

Given generic deformations

$$\widetilde{W_1} = \langle \eta_2 + X\eta_1 \rangle, \qquad \widetilde{W_2} = \langle e_1 + Ye_2 \rangle,$$

the hypotheses

$$h(\widetilde{W_1}) \subseteq \widetilde{W_2}, \qquad h(\widetilde{W_2}) \subseteq \widetilde{W_1}$$

provide us with the condition $Y = 0$*. The deformation ring is therefore* $W(k)[\![X]\!]$*.*

Lemma 4.0.2. *Given a geometric point* $\underline{E}_1 \xrightarrow{f} \underline{E}_2 \in X_0(p,N)(k)$ *there exist trivializations*

$$H^1_{dR}(A/k) \simeq ke_1 \oplus ke_2, \qquad H^1_{dR}(B/k) \simeq k\eta_1 \oplus ke_2, \tag{4.2}$$

such that the Hodge filtrations are represented by

$$H^0(A, \Omega^1_{A/k}) = \langle e_1 \rangle, \qquad H^0(B, \Omega^1_{B/k}) = \langle \eta_1 \rangle,$$

and such that

$$\mathbb{D}(f) = \begin{cases} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } x \text{ is the Frob of a ssing ell curve },\\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } x \text{ is the Frob of an ord ell curve,}\\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } x \text{ is the Ver of an ord ell curve .} \end{cases}$$

See Lemmas (1.2.1), (3.1.1) and (3.2.1) for details.

4.1. The case of supersingular points. Let $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)}) \in X_0(p,N)$ be a supersingular k-point. The universal object for the deformation of the associated point $(W_1, W_2) \in \mathcal{N}_p$ is

$$W_1^{\mathrm{un}} = \langle \eta_1 + X\eta_2 \rangle \subset S\eta_1 \oplus S\eta_2 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)} W_2^{\mathrm{un}} = \langle e_1 + Ye_2 \rangle \subset Se_1 \oplus Se_2,$$

where $S = W(k)[\![X,Y]\!]/(XY - p)$.

The morphism of Dieudonné displays

$$\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$$

over k associated to the point x was already defined in Section (2.2). By Lemma (2.2.1), the Dieudonné displays $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}^{(\sigma)} = (P^{(\sigma)}, Q^{(\sigma)}, F^{(\sigma)}, (V^{-1})^{(\sigma)})$ were defined as follows in the $W(k)$-bases $\{e_1, e_2\}$ and $\{\eta_1, \eta_2\}$:

$$\begin{aligned} P^{(\sigma)} &\simeq W(k)\eta_1 \oplus W(k)\eta_2, & P &\simeq W(k)e_1 \oplus W(k)e_2,\\ Q^{(\sigma)} &\simeq W(k)\eta_1 + (p)\eta_2, & Q &\simeq W(k)e_1 \oplus (p)e_2,\\ F^{(\sigma)} &= \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, & F &= \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix},\\ (V^{-1})^{(\sigma)} &= \frac{F^{(\sigma)}}{p}|_{Q^{(\sigma)}}, & (V^{-1}) &= \frac{F}{p}|_Q, \end{aligned}$$

and the induced morphism of Dieudonné displays $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$ is represented by the matrix

$$\mathbb{D}(\mathrm{Fr}) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

THEOREM 4.1.1. *Consider the quadruples* $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ *and* $\mathcal{P}_2 = (P_2, Q_2, F_2 V_2^{-1})$ *over* $S = W(k)[\![X,Y]\!]/(XY - p)$ *defined as follows:*

$$\begin{aligned} P_1 &= \mathbb{W}(S)\eta_1 \oplus \mathbb{W}(S)\eta_2, & P_2 &= \mathbb{W}(S)e_1 \oplus \mathbb{W}(S)e_2,\\ Q_1 &= \mathbb{I}(S)(\eta_1 \oplus \eta_2) + \mathbb{W}(S)(\eta_1 + \hat{X}\eta_2), & Q_2 &\simeq \mathbb{I}(S)(e_1, \oplus e_2) + \mathbb{W}(S)(e_1 + \hat{Y}e_2),\\ F_1 &= \begin{pmatrix} -^F\hat{X} & 1 \\ p & -^F\hat{Y} \end{pmatrix}, & F_2 &= \begin{pmatrix} -^F\hat{Y} & 1 \\ p & -^F\hat{X} \end{pmatrix},\\ V_1^{-1} &= \frac{F_1}{p}|_{Q_1}, & V_2^{-1} &= \frac{F_2}{p}|_{Q_2}. \end{aligned}$$

Both $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are Dieudonné displays over* S *and the morphism* $\alpha\colon P_1 \to P_2$ *described by the matrix*

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

is a morphism of Dieudonné displays

$$\alpha\colon \mathcal{P}_1 \to \mathcal{P}_2.$$

Moreover, it is a universal object for the deformation of $\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$*, that is, for every deformation* $\widetilde{\mathcal{P}}_1 \xrightarrow{\widetilde{\alpha}} \widetilde{\mathcal{P}}_2$ *of* $\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$ *to an Artinian* $W(k)$*-algebra* $(A, \mathfrak{m}_A)$ *there exists a unique morphism* $\psi\colon S \to A$ *such that*

$$(\mathcal{P}_1 \xrightarrow{\alpha} \mathcal{P}_2) \otimes_{S,\psi} A \simeq \widetilde{\mathcal{P}}_1 \xrightarrow{\widetilde{\alpha}} \widetilde{\mathcal{P}}_2.$$

PROOF. The argument for showing that $\mathcal{P}_1$ and $\mathcal{P}_2$ are Dieudonné displays over S is analogous to the proof of Theorem (2.3.1). An element $x \in Q_1 \simeq \mathbb{I}(S)(\eta_1 \oplus \eta_2) + \mathbb{W}(S)(\eta_1 + \hat{X}\eta_2)$ can be written as a vector $\begin{pmatrix} \theta_1 + w \\ \theta_2 + w\hat{X} \end{pmatrix}$, where $\theta_1, \theta_2 \in \mathbb{I}(S)$ and $w \in \mathbb{W}(S)$. Hence $F_1 x = \begin{pmatrix} -^F\hat{X} & 1 \\ p & -^F\hat{Y} \end{pmatrix} \begin{pmatrix} \theta_1 + w \\ \theta_2 + w\hat{X} \end{pmatrix} = \begin{pmatrix} -^F\hat{X}^F\theta_1 + {}^F\theta_2 \\ p^F\theta_1 + p^Fw - {}^F\hat{Y}^F\theta_2 - {}^F\hat{Y}^F\hat{X}^Fw \end{pmatrix} \in {}^F\mathbb{I}(S)(\eta_1 \oplus \eta_2) + p\mathbb{W}(S)(\eta_1 + \hat{X}\eta_2)$. As already seen in the equi-characteristic case, this suffices to conclude that V_1^{-1} is well-defined on Q_1. Showing that $V_1^{-1}\colon Q_1 \to \mathcal{P}_1$ is an ^{F}-linear epimorphism is identical to the proof of Theorem (2.3.1).

Observe that $\alpha\colon P_1 \to P_2$ defined in the statement is a morphism of Dieudonné displays. Let us show that the conditions

$$\alpha(Q_1) \subseteq Q_2, \tag{4.3}$$

$$\alpha \circ F_1 = F_2 \circ \alpha^{(\sigma)}, \qquad \alpha \circ V_1^{-1} = V_2^{-1} \circ \alpha^{(\sigma)} \tag{4.4}$$

hold. Let us first show (4.3). A vector in Q_1 has the form $\begin{pmatrix} \theta+w \\ \zeta+w\hat{X} \end{pmatrix}$, with $\theta, \zeta \in I(S)$ and $w \in W(S)$. Hence $\alpha\begin{pmatrix} \theta+w \\ \zeta+w\hat{X} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\begin{pmatrix} \theta+w \\ \zeta+w\hat{X} \end{pmatrix} = \begin{pmatrix} \zeta+w\hat{X} \\ p\theta+pw \end{pmatrix} \in Q_2$, since $\hat{Y}\begin{pmatrix} \zeta+w\hat{X} \\ p\theta+pw \end{pmatrix} = \begin{pmatrix} \hat{Y}\zeta+\hat{X}\hat{Y}w \\ p\hat{Y}\theta+p\hat{Y}w \end{pmatrix} \in Q_2$. Condition (4.4) is proved by direct computation:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha} \underbrace{\begin{pmatrix} -^F\hat{X} & 1 \\ p & -^F\hat{Y} \end{pmatrix}}_{F_1} = \begin{pmatrix} p & -^F\hat{Y} \\ -p^F\hat{X} & p \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} -^F\hat{Y} & 1 \\ p & -^F\hat{X} \end{pmatrix}}_{F_2} \underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha^{(\sigma)}} = \begin{pmatrix} p & -^F\hat{Y} \\ -p^F\hat{X} & p \end{pmatrix}.$$

The commutativity of α with V_1^{-1} and V_2^{-1} follows, given that V_1^{-1} and V_2^{-1} were defined from F_1 and F_2.

Let us now prove universality.

Recall that we denoted $R = k[\![X,Y]\!]$. Both S and R are Artinian rings with maximal ideals $\mathfrak{m}_S = (X,Y)S$ and $\mathfrak{m}_R = (X,Y)R$ and residue field $S/\mathfrak{m}_S \simeq k \simeq R/\mathfrak{m}_R$. Note that there is a surjection

$$S \twoheadrightarrow S/(p) \simeq R.$$

We observe finally that

$$\begin{aligned} S/\mathfrak{m}_S^2 &= (W(k)[\![X,Y]\!]/(XY-p))/(X,Y)^2 \simeq (W(k)[\![X,Y]\!]/(XY-p))/(X^2,Y^2,XY) \\ &\simeq (W(k)[\![X,Y]\!]/(XY-p))/(X^2,Y^2,p) \simeq (k[\![X,Y]\!]/(XY))/(X^2,Y^2) \simeq R/\mathfrak{m}_R^2. \end{aligned}$$

□

4.2. The case of ordinary points. Let $x = (\underline{E} \xrightarrow{\mathrm{Fr}_E} \underline{E}^{(\sigma)}) \in X_0(p,N)$ be a geometric ordinary point. By Note (1) and Lemma (4.0.2), the universal object for the associated Grassmannian moduli problem is

$$W_1^{\mathrm{un}} = \langle e_1 \rangle \subset S\eta_1 \oplus S\eta_2 \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)} W_2^{\mathrm{un}} = \langle e_1 + Ye_2 \rangle,$$

where $S = W(k)[\![Y]\!]$. The morphism of Dieudonné displays $\mathcal{P}^{(\sigma)} \xrightarrow{\mathbb{D}(\mathrm{Fr}_E)} \mathcal{P}$ over k was already defined in (3):

$$\begin{aligned} P^{(\sigma)} &= W(k)\eta_1 \oplus W(k)\eta_2, & P &= W(k)e_1 \oplus W(k)e_2, \\ Q^{(\sigma)} &= W(k)\eta_1 + (p)\eta_2, & Q &= W(k)e_1 + (p)e_2, \\ F^{(\sigma)} &= \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, & F &= \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\ (V^{-1})^{(\sigma)} &= \frac{F^{(\sigma)}}{p}|_{Q^{(\sigma)}}, & V^{-1} &= \frac{F}{p}|_Q, \end{aligned}$$

and the morphism $\mathbb{D}(\mathrm{Fr}_E)$ is represented by the matrix

$$\mathbb{D}(\mathrm{Fr}_E) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

THEOREM 4.2.1. *The quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ defined by*

$$\begin{aligned} P_1 &= \mathbb{W}(S)\eta_1 \oplus \mathbb{W}(S)\eta_2, & P_2 &= \mathbb{W}(S)e_1 \oplus \mathbb{W}(S)e_2, \\ Q_1 &= \mathbb{W}(S)\eta_1 + \mathbb{I}(S)\eta_2, & Q_2 &= \mathbb{I}(S)(e_1 \oplus e_2) + \mathbb{W}(S)(e_1 + Ye_2), \\ F_1 &= \begin{pmatrix} p & 0 \\ -p^F\hat{Y} & 1 \end{pmatrix}, & F_2 &= \begin{pmatrix} p & 0 \\ -^F\hat{Y} & 1 \end{pmatrix}, \\ V_1^{-1} &= \frac{F_1}{p}|_{Q_1}, & V_2^{-1} &= \frac{F_2}{p}|_{Q_2}, \end{aligned}$$

are Dieudonné displays over S and the morphism $\alpha\colon P_1 \to P_2$ represented by the matrix

$$\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

is a morphism of Dieudonné displays

$$\alpha\colon \mathcal{P}_1 \to \mathcal{P}_2$$

over S which is universal with respect to the deformation of $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$.

Consider now $x = (\underline{E} \xrightarrow{\mathrm{Ver}_E} \underline{E}^{(1/\sigma)}) \in X_0(p,N)$ a geometric ordinary point. By Note (1) and Lemma (4.0.2), the universal object for the associated Grassmannian moduli problem is

$$W_1^{\mathrm{un}} = \langle e_1 + Xe_2 \rangle \subset S\eta_1 \oplus S\eta_2 \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)} W_2^{\mathrm{un}} = \langle e_1 \rangle,$$

where $S = W(k)[\![X]\!]$. The morphism of Dieudonné displays $\mathbb{D}(\mathrm{Ver}_E)\colon \mathcal{P}^{(1/\sigma)} \to \mathcal{P}$ associated to $x = (\underline{E} \xrightarrow{\mathrm{Ver}_E} \underline{E}^{(1/\sigma)})$ was already defined in Section (3):

$$P^{(1/\sigma)} = W(k)\eta_1 \oplus W(k)\eta_2, \qquad P = W(k)e_1 \oplus W(k)e_2,$$
$$Q^{(1/\sigma)} = W(k)\eta_1 + (p)\eta_2, \qquad Q = W(k)e_1 + (p)e_2,$$
$$F^{(1/\sigma)} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix},$$
$$(V^{-1})^{(1/\sigma)} = \frac{F^{(1/\sigma)}}{p}|_{Q^{(1/\sigma)}}, \qquad V^{-1} = \frac{F}{p}|_Q,$$

and in the same bases

$$\mathbb{D}(\mathrm{Ver}) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

THEOREM 4.2.2. *The quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ defined by*

$$P_1 = \mathbb{W}(S)\eta_1 \oplus \mathbb{W}(S)\eta_2, \qquad P_2 = \mathbb{W}(S)e_1 \oplus \mathbb{W}(S)e_2,$$
$$Q_1 = \mathbb{I}(S)(\eta_1 \oplus \eta_2) + \mathbb{W}(S)(\eta_1 + X\eta_2), \qquad Q_2 = \mathbb{W}(S)e_1 + \mathbb{I}(S)e_2,$$
$$F_1 = \begin{pmatrix} p & 0 \\ -p^F\hat{X} & 1 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} p & 0 \\ -^F\hat{X} & 1 \end{pmatrix},$$
$$V_1^{-1} = \frac{F_1}{p}|_{Q_1}, \qquad V_2^{-1} = \frac{F_2}{p}|_{Q_2},$$

are displays over S and the morphism $\alpha\colon P_1 \to P_2$ represented by the matrix

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

is a morphism of Dieudonné displays

$$\alpha\colon \mathcal{P}_1 \to \mathcal{P}_2$$

over S which is universal with respect to the deformation of $\mathbb{D}(\mathrm{Fr}_E)\colon \mathcal{P}^{(\sigma)} \to \mathcal{P}$.

CHAPTER 4

Inert Spaces

1. Unramified primes in totally real extensions

Fix a prime p. Consider a totally real extension L of $\mathbb{Q}$ of degree g and denote by $\mathcal{O}_L$ its ring of integers. Assume the prime p to be inert in $\mathcal{O}_L$:

$$\begin{tikzcd} L \arrow[r, dash] \arrow[d, dash, "g"'] & \mathcal{O}_L \arrow[d, dash] & \mathfrak{p} = p\mathcal{O}_L \\ \mathbb{Q} \arrow[r, dash] & \mathbb{Z} & p \end{tikzcd}$$

Let $k = \overline{k}$ be a field of characteristic p such that there exists an embedding

$$\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_{p^g} \hookrightarrow k.$$

Denote by $\sigma : k \to k$ the Frobenius on k.

Note that if there exists an embedding of $\mathbb{F}_{\mathfrak{p}}$ into k, there are precisely g

$$\beta_j : \mathbb{F}_{\mathfrak{p}} \hookrightarrow k,$$

and we may order them so that $\beta_{j+1} = \sigma \circ \beta_j$; it follows therefore that if $\rho : \mathbb{F}_{\mathfrak{p}} \hookrightarrow k$ is an embedding, also $\sigma \circ \rho$ is. Note moreover that we may lift them at the level of Witt vectors

$$\hat{\beta}_j : W(\mathbb{F}_{\mathfrak{p}}) \to W(k),$$

(in what follows we shall abuse notation and denote $\hat{\beta} = \beta$). Again ${}^F \circ \beta_j = \beta_{j+1}$.

One has the following trivial identities

$$\mathcal{O}_L \otimes \mathbb{F}_p \simeq \mathbb{F}_{\mathfrak{p}}, \qquad \mathcal{O}_L \otimes \mathbb{Z}_p \simeq W(\mathbb{F}_{\mathfrak{p}}). \tag{1.1}$$

Remark that the embeddings β_j induce an isomorphism

$$\mathbb{F}_{\mathfrak{p}} \otimes_{\mathbb{F}_p} k \simeq \bigoplus_{j=1}^{g} k, \tag{1.2}$$

and hence on Witt vectors we have:

$$W(\mathbb{F}_{\mathfrak{p}}) \otimes_{\mathbb{Z}_p} W(k) \simeq \bigoplus_{j=1}^{g} W(k). \tag{1.3}$$

By combining this with (1.1), we obtain the following identifications

$$\mathcal{O}_L \otimes k \simeq \bigoplus_{j=1}^{g} k, \qquad \mathcal{O}_L \otimes W(k) \simeq \bigoplus_{j=1}^{g} W(k). \tag{1.4}$$

We may generalize these considerations to the unramified case. Indeed, for any ideal $\mathfrak{p} \triangleleft \mathcal{O}_L$ dividing $p\mathcal{O}_L$, we may consider the residue field $\mathbb{F}_{\mathfrak{p}}$ of degree $f_{\mathfrak{p}}$ over $\mathbb{F}_p$ and a perfect field k containing all residue fields for any $\mathfrak{p}$ dividing p. We consider hence the set of embeddings from $\mathcal{O}_L$ into $W(k)$

$$\mathbb{B} = \mathrm{Emb}(\mathcal{O}_L, W(k)).$$

It follows that the decomposition obtained in (1.3) generalizes to the following:

$$\mathcal{O}_L \otimes_{\mathbb{Z}} W(k) = \bigoplus_{\beta\in\mathbb{B}} W(k)_\beta.$$

Given a subset $\theta \subseteq \mathbb{B}$, we will use the following notation

$$\ell(\theta) = \{\sigma^{-1} \circ \beta : \beta \in \theta\}, \qquad r(\theta) = \{\sigma \circ \beta : \beta \in \theta\}.$$

2. Hilbert modular varieties in unramified characteristic

Consider $\mathcal{M}$ the *Hilbert modular variety* over k of dimension g, as defined in Section (1).

LEMMA 2.0.1. *When p is unramified in $\mathcal{O}_L$, the Deligne-Pappas condition (DP) is equivalent to the Rapoport condition (R). In other words, given $\underline{A}/S \in \mathcal{M}$, the $\mathcal{O}_L \otimes \mathcal{O}_S$-module* $\mathrm{Lie}(A)$ *is free of rank* 1.

PROOF. See Lemma [Gor02, Lemma 5.5]. □

The geometry of $\mathcal{M}$ has been studied extensively by Rapoport [Rap78], Goren and Oort [GO00, Gor01] in the unramfied case. Let us summarize briefly some of the results that were obtained. Goren and Oort [GO00] define the notion of *type*, an invariant attached to the geometric k-points of $\mathcal{M}$.

DEFINITION 2.0.2. *Given a geometric point $\underline{A} \in \mathcal{M}(k)$, we define the* type of $\underline{A}$ *as the subset*

$$\tau(A) = \{\beta \in \mathbb{B} \mid \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}) \cap \mathrm{Ker}(\mathrm{Ver}))_\beta\} \subseteq \mathbb{B}.$$

In their paper, they describe moreover the moduli space $\mathcal{M}$ as a disjoint union of locally closed subspaces.

DEFINITION 2.0.3. *For every subset $\tau \subseteq \mathbb{B}$ denote by W_τ the subset of $\mathcal{M}$ whose geometric points $\underline{A}$ are such that $\tau \subseteq \tau(A)$.*

Here is a list of the most meaningful results

- The W_τ's are closed, locally irreducible.
- Given $\tau \subseteq \mathbb{B}$, the subscheme W_τ has generic points of type τ and it has dimension $g - |\tau|$.
- Given two subsets $\sigma, \tau \subseteq \mathbb{B}$, one has $W_\sigma \cap W_\tau = W_{\sigma\cup\tau}$.

Denote by $\mathcal{M}_p$ the Hilbert modular variety over k of dimension g and $\Gamma_0(p)$-level structure, that is, the fine moduli space of quadruples $(A/S, \iota, \lambda, \mu_N)$ as above, together with a finite flat $\mathcal{O}_L$-subgroup scheme H of $A_0[p]$ of rank p^g, isotropic with respect to the λ-Weil pairing.

LEMMA 2.0.4. *The moduli space $\mathcal{M}_p$ can be seen equivalently as the moduli space with points $t = (\underline{A} \xrightarrow{f} \underline{B})$ where $\underline{A} = (A/S, \iota_A, \lambda_A, (\mu_N)_A)$ and $\underline{B} = (B/S, \iota_B, \lambda_B, (\mu_N)_B)$ are quadruples as the ones parametrizing $\mathcal{M}$ and $f: \underline{A} \to \underline{B}$ is an $\mathcal{O}_L$-isogeny killed by p and of degree p^g, such that $f^*M(B) = pM(A)$. Denote moreover by f^t the unique isogeny $f^t: \underline{B} \to \underline{A}$ such that*

$$f \circ f^t = [p]_B, \qquad f^t \circ f = [p]_A.$$

Proof. The proof of this result is given by Goren-Kassaei [GK12, Lemma 2.1.2]). We want to define an abelian variety B, with real multiplication by $\mathcal{O}_L$, an $\mathcal{O}_L$-polarization and a $\Gamma_{00}(N)$-level structure. Take the quotient $B = A/H$: the real multiplication ι_A and level structure $(\mu_N)_A$ on A induce a real multiplication ι_B and $(\mu_N)_B$ on B. Finally, we wish to define a polarization λ_B on B. Define the map

$$\begin{array}{rcl} f^*: M(B) & \to & M(A) \\ \delta & \mapsto & f^\vee \circ \delta \circ f. \end{array}$$

We want to describe a polarization on B. Put

$$\begin{array}{rcl} \lambda_B: M(B) & \xrightarrow{\simeq} & \mathfrak{I} \\ \delta & \mapsto & \frac{1}{p}\lambda_A \circ f^*(\delta) \end{array}$$

We need to show that this makes sense, namely that, as in the statement,

$$f^*(M(B)) = pM(A).$$

Consider the isogeny $A \xrightarrow{f} A/H$ and the unique isogeny $A/H \xrightarrow{f^t} A$ such that

$$f \circ f^t = [p]_{A/H}, \qquad f^t \circ f = [p]_A.$$

We have the following exact sequences

$$0 \longrightarrow H \longrightarrow A \xrightarrow{f} A/H \longrightarrow 0 \tag{2.1}$$

$$0 \longrightarrow A[T]/H \longrightarrow A/H \xrightarrow{f^t} A \longrightarrow 0 \tag{2.2}$$

By dualizing (2.2) we obtain

$$0 \longleftarrow (A[T]/H)^\vee \xleftarrow{(f^t)^\vee} A^\vee \longleftarrow (A/H)^\vee \longleftarrow 0\ .$$

By definition $H \subseteq A[T]$ is isotropic with respect to the γ-Weil pairing $A[p] \times A^\vee[p] \to \mu_p$ for any $\gamma \in M(A)$. Fix such a $\gamma \in M(A)$. From isotropy it follows in particular that $\gamma(H) \subseteq (A[T]/H)^\vee$ and we obtain hence a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "\gamma"'] & A/H \arrow[r] \arrow[d, "i\gamma"] & 0 \\ 0 \arrow[r] & (A[T]/H)^\vee \arrow[r] & A^\vee \arrow[r, "(f^t)^\vee"] & (A/H)^\vee \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

inducing the map $i\gamma: A/H \to (A/H)^\vee$ satisfying the relation

$$(f^t)^\vee \circ \gamma = i\gamma \circ f.$$

This defines an $\mathcal{O}_L$-linear morphism

$$i: M(A) \to M(B);$$

it is injective as $\deg(i\gamma) = \deg(\gamma)$. Note moreover that $i\gamma$ fits in the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & A/H \arrow[d, "i\gamma"] \\ A^\vee & (A/H)^\vee. \arrow[l, "f^\vee"] \end{tikzcd}$$

We have therefore

$$f^*(i\gamma) := f^\vee \circ i\gamma \circ f = f^\vee \circ (f^t)^\vee \circ \gamma = (f^t \circ f)^\vee \circ \gamma = p\gamma.$$

This proves in particular that

$$pM(A) \subseteq f^* M(B).$$

We want now to prove the opposite inclusion. Consider once again the exact sequence (2.2)

$$0 \longrightarrow A[T]/H \longrightarrow A/H \xrightarrow{f^t} A \longrightarrow 0.$$

Note that $A[T]/H$ is isotropic with respect to the pairing $A[p] \times A^\vee[p] \to \mu_p$. It is enough to show that it is isotropic with respect to the $i\gamma$-Weil paring, where $\gamma \in M(A)$, which is equivalent to the equality $i\gamma(A[T]/H) = H^\vee$. We have

$$i\gamma(A[T]/H) = i\gamma \circ f(A[T]) = (f^t)^\vee \circ \gamma(A[T]) = (f^t)^\vee(A^\vee[T]) = A^\vee/(A[T]/H)^\vee = H^\vee,$$

which proves the isotropy of $A[T]/H$. This means in particular that for any $\delta \in M(B)$, we have $\delta(A[T]/H) = H^\vee$. By dualizing the exact sequence (2.1) we obtain therefore a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A[T]/H \arrow[r] \arrow[d] & A/H \arrow[r, "f^t"] \arrow[d, "\delta"'] & A \arrow[r] \arrow[d, "j\delta"] & 0 \\ 0 \arrow[r] & H^\vee \arrow[r] & (A/H)^\vee \arrow[r, "f^\vee"] & A^\vee \arrow[r] & 0 \end{tikzcd}$$

which defines an $\mathcal{O}_L$-linear map

$$j\colon M(B) \to M(A)$$

satisfying the relations

$$f^\vee \circ \delta = j\delta \circ f^t$$

for any given $\delta \in M(B)$. Showing that $f^*(M(B)) \subseteq pM(A)$ is equivalent to showing that for $\delta \in M(B)$ we have

$$f^*\delta = pj\delta. \tag{2.4}$$

We claim that j satisfies the relation

$$(f^t)^* \circ j\delta = p\delta,$$

where $(f^t)^* \circ j\delta := (f^t)^\vee \circ j\delta \circ f^t$ for any $\delta \in M(B)$. Indeed

$$(f^t)^* \circ j\delta = (f^t)^\vee \circ j\delta \circ f^t = (f^t)^\vee \circ f^\vee \circ \delta = (f \circ f^t)^\vee \circ \delta = p\delta.$$

In order to show the equality in (2.4), it is enough to show that $(f^t)^* pj\delta = (f^t)^* f^*\delta$. We have that on the one hand

$$(f^t)^* pj\delta = p(f^t)^* k\delta = p^2\delta,$$

while

$$p^*\delta = p^2\delta,$$

which concludes the proof of the equality

$$f^*M(B) = M(A).$$

Note finally that the quadruple $\underline{B} = (B = A/H, \iota_B, \lambda_B, (\mu_N)_B)$ satisfies the Deligne-Pappas condition. Indeed, by [AG03, Proposition 3.1] it is enough to show that there exists an element $\delta \in M(B)$ of degree prime to ℓ for every prime ℓ different from p. Since $\underline{A}$ satisfies the Deligne-Pappas condition, there exists $\gamma \in M(A)$ of degree prime to ℓ for every $\ell \neq p$. Since $\deg(\gamma) = \deg(i\gamma)$, take $\delta = i\gamma$.

In order to show the other direction, given $f\colon \underline{A} \to \underline{B}$ an $\mathcal{O}_L$-isogeny killed by p and of degree p^g such that $f^*M(B) = pM(A)$, we just need to show that $H = \mathrm{Ker}(f)$ is isotropic with respect to the Weil-pairing. This is equivalent to showing that for any $\gamma \in M(A)$, we have that $\gamma(H) \subseteq (A[p]/H)^\vee$, in other words that the composition $(f^t)^\vee \circ \gamma(H) = 0$. Note that $(f^t)^\vee \circ \gamma \circ f^t = (f^t)^*\gamma = \delta p = \delta \circ f \circ f^t$ for some $\delta \in M(B)$. Therefore we get that $(f^t)^\vee \circ \gamma(H) = \delta \circ f(H) = 0$ since $H = \mathrm{Ker}(f)$. □

Denote by

$$\begin{array}{ccc} \mathcal{M}_p & \xrightarrow{\pi} & \mathcal{M} \\ (\underline{A}, H) & \longmapsto & \underline{A} \end{array}$$

the natural forgetful functor.

3. Discrete invariants

We have observed in the considerations following Lemma (2.3.1), Chapter (2), that if A has a real multiplication action $\iota\colon \mathcal{O}_L \hookrightarrow A$, then $\mathbb{D}(A[p])$, $\mathbb{D}(\mathrm{Fr}_A)$ and $\mathbb{D}(\mathrm{Ver}_A)$ are $\mathcal{O}_L \otimes k$-modules. By Section (1) we have therefore the following decomposition

$$\mathbb{D}(A[p]) = \bigoplus_{i=1}^{g} \mathbb{D}(A[p])_i.$$

Note moreover that $\mathbb{D}(\mathrm{Fr}_A)(\mathbb{D}(A[p])_i) \subseteq \mathbb{D}(A[p])_{i+1}$ and that $\mathbb{D}(\mathrm{Ver}_A)(\mathbb{D}(A[p])_{i+1}) \subseteq \mathbb{D}(A[p])_i$. Finally, we also have the decompositions

$$\mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A)) = \bigoplus_{i=1}^{g} \mathbb{D}(\mathrm{Ker}(\mathrm{Fr}_A))_i, \qquad \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A)) = \bigoplus_{i=1}^{g} \mathbb{D}(\mathrm{Ker}(\mathrm{Ver}_A))_i.$$

By (2.1), the same decompositions hold for $H^1_{dR}(A/k)$, $H^0(A, \Omega^1_{A/k})$ and $\mathrm{Lie}(A)$.

The isogeny f of a given a k-point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$ induces homomorphisms

$$\bigoplus_{i=1}^{g} \mathrm{Lie}(f)_i\colon \bigoplus_{i=1}^{g} \mathrm{Lie}(\underline{A})_i \longrightarrow \bigoplus_{i=1}^{g} \mathrm{Lie}(\underline{B})_i,$$
$$\bigoplus_{i=1}^{g} \mathrm{Lie}(f^t)_i\colon \bigoplus_{i=1}^{g} \mathrm{Lie}(\underline{B})_i \longrightarrow \bigoplus_{i=1}^{g} \mathrm{Lie}(\underline{A})_i.$$

The following notions were introduced by Goren and Kassaei in [GK12]. Consider a point $(\underline{A}, H) = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$. We define the following parameters:

$$\varphi(\underline{A}, H) = \varphi(f) = \{i \in \{1, \dots, g\} \mid \mathrm{Lie}(f)_{i-1} = 0\},$$
$$\eta(\underline{A}, H) = \eta(f) = \{i \in \{1, \dots, g\} \mid \mathrm{Lie}(f^t)_i = 0\},$$
$$I(\underline{A}, H) = \ell(f) = \ell(\varphi(f)) \cap \eta(f) = \{i \in \{1, \dots, g\} \mid \mathrm{Lie}(f)_i \cap \mathrm{Lie}(f^t)_i = 0\}.$$

Given $\varphi, \eta \subseteq \mathbb{B}$, we say that the pair (φ, η) is *admissible* if $\ell(\varphi^c) \subseteq \eta$, or equivalently, if $r(\eta^c) \subseteq \varphi$. There are 3^g admissible pairs and the discrete invariants $(\varphi(\underline{A}, H), \eta(\underline{A}, H))$ of a geometric k-point $(\underline{A}, H) \in \mathcal{M}_p$ are an admissible pair.

We will rely on the description of these parameters given by Goren and Kassaei in [GK12, Lemma 2.3.3], that we recall shortly here:

Lemma 3.0.1 (Goren-Kassaei). *Consider a geometric point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$. We have the following descrption:*

(1) $\beta \in \tau(A) \Leftrightarrow \mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_i = \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_i$,
(2) $i \in \varphi(f) \Leftrightarrow \mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_i = \mathrm{Im}(\mathbb{D}(f))_i$,
(3) $i \in \eta(f) \Leftrightarrow \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_i = \mathrm{Im}(\mathbb{D}(f))_i$.

4. The local model of the Hilbert modular variety

4.1. The points of $\mathcal{M}_p$ in terms of linear algebra. Let $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$ be a geometric k-point. By applying the contravariant Dieudonné theory to the p-torsion subgroups of both A and B, we obtain two k-vector spaces of dimension $2g$ and induced k-morphisms

$$\mathbb{D}(f)\colon \mathbb{D}(B[p]) \to \mathbb{D}(A[p]), \qquad \mathbb{D}(f^t)\colon \mathbb{D}(A[p]) \to \mathbb{D}(B[p]).$$

Recall the identification with de Rham cohomology as in Oda [Oda69]:

$$\mathbb{D}(A[p]) \simeq H^1_{dR}(A/k), \qquad \mathbb{D}(B[p]) \simeq H^1_{dR}(B/k);$$

By [Rap78, Lemma 1.3] they are free $\mathcal{O}_L \otimes k$-modules of rank 2.

Lemma 4.1.1. *There exist $\mathcal{O}_L \otimes k$-bases $\{a, b\}$ and $\{\alpha, \beta\}$ of $H^1_{dR}(A/k)$ and $H^1_{dR}(B/k)$ respectively, such that the induced morphism*

$$\mathbb{D}(f)\colon H^1_{dR}(B/k) \to H^1_{dR}(A/k)$$

is represented by the matrix

$$\mathbb{D}(f) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Proof. The conditions $f \circ f^t = [p]_B$ and $f \circ f^t = [p]_A$ induce the conditions

$$\mathbb{D}(f^t) \circ \mathbb{D}(f) = [p]_{\mathbb{D}(B[p])}, \qquad \mathbb{D}(f) \circ \mathbb{D}(f^t) = [p]_{\mathbb{D}(A[p]}. \tag{4.1}$$

Hence the compositions are represented by zero matrices

$$\mathbb{D}(f^t) \circ \mathbb{D}(f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{D}(f) \circ \mathbb{D}(f^t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We deduce therefore that both $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ have kernel and image of rank 1. Choose hence $\mathcal{O}_L \otimes k$-bases $\{\alpha, \beta\}$ and $\{a, b\}$ for $\mathbb{D}(A[p])$ and $\mathbb{D}(B[p])$ respectively, such that

$$\begin{aligned} \mathbb{D}(f)(\beta) &= a, & \mathbb{D}(f)(\alpha) &= 0, \\ \mathbb{D}(f^t)(a) &= 0, & \mathbb{D}(f^t)(b) &= \alpha, \end{aligned}$$

hence the conclusion. □

Consider the Hodge filtrations

$$H^0(A, \Omega^1_{A/k})) \subseteq H^1_{dR}(A/k), \qquad H^0(B, \Omega^1_{B/k})) \subseteq H^1_{dR}(B/k).$$

Being p unramified, the Rapoport condition (R) (see Section (1), Chapter (2)) holds, that is $H^0(A, \Omega^1_{A/k}))$ and $H^0(B, \Omega^1_{B/k}))$ are free of rank 1 over $\mathcal{O}_L \otimes k$. Finally, the morphisms $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ respect the Hodge filtrations, that is

$$\mathbb{D}(f)(H^0(B, \Omega^1_{B/k})) \subseteq H^0(A, \Omega^1_{A/k})), \qquad \mathbb{D}(f^t)(H^0(A, \Omega^1_{A/k})) \subseteq H^0(B, \Omega^1_{B/k})). \tag{4.2}$$

4.2. The associated Grassmannian. We will sketch the local description of the Hilbert modular variety through the theory of local models. It allows to identify the completed local ring at a geometric point of $\mathcal{M}_p$ with some universal deformation ring R. The idea is to associate to each k-point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$ an element of a certain Grassmann variety $\mathcal{N}_p$, or, more precisely, to construct a morphism from a Zariski-open neighborhood of t to $\mathcal{N}_p$, and compare the completed local rings of the two moduli spaces through crystalline theory.

We summarize here some ideas described in [GK12, Section 2.4]. Let $\mathcal{N}_p = \mathcal{N}_p(k, \mathcal{O}_L \otimes k, p)$ be the Grassmann variety of pairs (W_1, W_2) of free $\mathcal{O}_L \otimes k$-submodules of $(\mathcal{O}_L \otimes k)^2$ of rank 1, such that given the $\mathcal{O}_L \otimes k$-map

$$\begin{array}{ccc} (\mathcal{O}_L \otimes k)^2 & \xrightarrow{h} & (\mathcal{O}_L \otimes k)^2 \\ (x, y) & \longrightarrow & (y, 0) \end{array}$$

we have $h(W_1) \subset W_2$ and $h^t(W_2) \subset W_1$.

By Lemma (4.1.1) and condition (4.2) and the following considerations, to a point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$ we may associate a point $(W_1, W_2) \in \mathcal{N}_p$ by putting $h = \mathbb{D}(f)$ and $(W_1, W_2) = (H^0(B, \Omega^1_{B/k}), H^0(A, \Omega^1_{A/k}))$. By applying the crystalline theory by Grothendieck and Messing we obtain an isomorphism of the completed local rings of the two varieties

$$\widehat{\mathcal{O}}_{\mathcal{M}_p, t} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_p, (W_1, W_2)};$$

(See Deligne-Pappas ([DP94]) and Goren-Kassaei ([GK12]) for a proof). We sketch here the way this is done.

We can actually refine the description of the local model. Recall that for p inert, see Section (1), there exist isomorphisms

$$\mathcal{O}_L \otimes W(k) \simeq \bigoplus_{i=1}^{g} W(k)$$

and

$$\mathcal{O}_L \otimes k \simeq \bigoplus_{i=1}^{g} k.$$

We obtain therefore the map $h : (\mathcal{O}_L \otimes k)^2 \to (\mathcal{O}_L \otimes k)^2$ as a direct sum

$$h = \bigoplus_{i=1}^{g} h_i : \bigoplus_{i=1}^{g} k^2 \to \bigoplus_{i=1}^{g} k^2,$$

where each h_i is a k-linear map represented by a 2×2 matrix $h_i = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. We may therefore consider the indices $i \in \{1, \dots, g\}$ one at the time and reduce to studying the Grassmann variety $\mathcal{N}_p(i)$ of pairs $(W_1(i), W_2(i))$ of one-dimensional subspaces of k^2 satisfying the compatibility conditions

$$h_i(W_1(i)) \subseteq W_2(i), \qquad h_i(W_2(i)) \subseteq W_1(i). \tag{4.3}$$

Let us analyse its deformations. Consider bases $\{\alpha_i, \beta_i\}$ and $\{a_i, b_i\}$ for the two-dimensional vector spaces

$$\alpha_i k \oplus \beta_i k \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} a_i k \oplus b_i k.$$

Note 2. *We have the following possibilities*

(1) $W_1(i) \neq \operatorname{Ker}(h_i)$ *and* $W_2(i) = \operatorname{Ker}(h_i)$, *that is*

$$W_1(i) = \langle \beta_i \rangle, \qquad W_2(i) = \langle a_i \rangle.$$

A generic deformation $(\widetilde{W_1}(i), \widetilde{W_2}(i))$ *to an Artinian* k*-algebra* $(S, \mathfrak{m}_S)$ *is described, in the bases* $\{\alpha_i, \beta_i\}$ *and* $\{a_i, b_i\}$, *by*

$$\widetilde{W_1}(i) = \langle \beta_i + X_i \alpha_i \rangle, \qquad \widetilde{W_2}(i) = \langle a_i + Y_i b_i \rangle,$$

for $X_i, Y_i \in \mathfrak{m}_S$. *The hypotheses*

$$h_i(\widetilde{W_1}(i)) \subseteq \widetilde{W_2}(i), \qquad h_i(\widetilde{W_2}(i)) \subseteq \widetilde{W_1}(i),$$

translate into the conditions

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} X_i \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} 1 \\ Y_i \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ Y_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} Y_i \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} X_i \\ 1 \end{smallmatrix}\right),$$

which are satisfied when $Y_i = 0$. *The universal deformation ring is therefore* $k[\![X_i]\!]$.

(2) $W_1(i) = \operatorname{Ker}(h_i)$ *and* $W_2(i) \neq \operatorname{Ker}(h_i)$, *in terms of the bases* $\{\alpha_i, \beta_i\}$ *and* $\{a_i, b_i\}$ *give*

$$W_1(i) = \langle \alpha_i \rangle, \qquad W_2(i) = \langle b_i \rangle.$$

A generic deformation $(\widetilde{W_1}(i), \widetilde{W_2}(i))$ *of* (W_1, W_2) *to an Artinian* k*-algebra* $(S, \mathfrak{m}_S)$ *is described by*

$$\widetilde{W_1}(i) = \langle \alpha_i + X_i \beta_i \rangle, \qquad \widetilde{W_2}(i) = \langle b_i + Y_i a_i \rangle,$$

with $X_i, Y_i \in \mathfrak{m}_S$. *The hypotheses*

$$h_i(\widetilde{W_1}(i)) \subseteq \widetilde{W_2}(i), \qquad h_i(\widetilde{W_2}(i)) \subseteq \widetilde{W_1}(i),$$

translate into

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ X_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} X_i \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} Y_i \\ 1 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} Y_i \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} 1 \\ X_i \end{smallmatrix}\right),$$

from which we get the condition $X_i = 0$. *The universal deformation ring is therefore* $k[\![Y_i]\!]$.

(3) $W_1(i) = \mathrm{Ker}(h_i) = W_2(i)$, *that is*

$$W_1(i) = \langle \alpha_i \rangle, \qquad W_2(i) = \langle a_i \rangle.$$

A generic deformation $(\widetilde{W_1}(i), \widetilde{W_2}(i))$ *to an Artinian* k*-algebra* $(S, \mathfrak{m}_S)$ *is of type*

$$\widetilde{W_1}(i) = \langle \alpha_i + X_i \beta_i \rangle, \qquad \widetilde{W_2}(i) = \langle a_i + Y_i b_i \rangle,$$

with $X_i, Y_i \in \mathfrak{m}_S$. *The hypotheses*

$$h_i(\widetilde{W_1}(i)) \subseteq \widetilde{W_2}(i), \qquad h_i(\widetilde{W_2}(i)) \subseteq \widetilde{W_1}(i),$$

give the conditions

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ X_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} X_i \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} 1 \\ Y_i \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ Y_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} Y_i \\ 0 \end{smallmatrix}\right) \subseteq \left(\begin{smallmatrix} 1 \\ X_i \end{smallmatrix}\right),$$

from which we obtain the condition $X_i \cdot Y_i = 0$. *The universal deformation ring is therefore* $k[\![X_i, Y_i]\!]/(X_i Y_i)$.

LEMMA 4.2.1. *Consider a point* $(W_1, W_2) \in \mathcal{N}_p$ *associated to a geometric point* $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$, *that is* $W_1 = H^0(B, \Omega^1_{B/k})$ *and* $W_2 = H^0(A, \Omega^1_{A/k})$, *and for* $i \in \{1, \dots g\}$, $W_1(i) = H^0(B, \Omega^1_{B/k})(i)$ *and* $W_2(i) = H^0(A, \Omega^1_{A/k})(i)$. *In terms of the* discrete invariants *of* t *(see Section* (3)*), the condition* $W_2(i) = \mathrm{Ker}(h_i) = \mathrm{Ker}(\mathbb{D}(f^t))_i$ *is the condition* $i \in \eta(f)$, *while the condition* $W_1(i) = \mathrm{Ker}(h_i) = \mathrm{Ker}(\mathbb{D}(f))_i$ *is the condition* $\sigma \circ i \in \varphi(f)$.

PROOF. Note that the condition $\mathbb{D}(f^t)_i \circ \mathbb{D}(f)_i = 0$ implies that $\mathrm{Ker}(\mathbb{D}(f^t))_i \subseteq \mathrm{Im}(\mathbb{D}(f))_i$ and by dimension reasons we get the equality

$$\mathrm{Ker}(\mathbb{D}(f^t))_i = \mathrm{Im}(\mathbb{D}(f))_i.$$

By an identical reasoning, we have that

$$\mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_i = \mathrm{Ker}(\mathbb{D}(\mathrm{Ker}_A))_i.$$

Hence $W_2(i) = \mathrm{Ker}(\mathbb{D}(\mathrm{Fr}_A))_i = \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_i$ and the condition $W_2(i) = \mathrm{Ker}(\mathbb{D}(f^t))_i$ equals to the condition $W_2(i) = \mathrm{Im}(\mathbb{D}(\mathrm{Ver}_A))_i = \mathrm{Im}(\mathbb{D}(f))_i$. We conclude hence by Lemma (3.0.1) that $W_2(i) = \mathrm{Ker}(\mathbb{D}(f^t))_i$ if and only if $i \in \eta(f)$.

Recall that the notation $i \in \ell(\varphi)$ means that $i + 1 \in \varphi$. By definition $1 + i \in \varphi$ if and only if the map in Lie algebras $\mathrm{Lie}(f)_i \colon \mathrm{Lie}(A)_i \to \mathrm{Lie}(B)_i$ is the zero map, which equals to its dual

$$\begin{tikzcd}
\mathrm{Lie}(B)^*_i \arrow[r, "\mathrm{Lie}(f)^*_i"] \arrow[d, equal] & \mathrm{Lie}(A)^*_i \arrow[d, equal] \\
H^0(B, \Omega^1_{B/k})(i) \arrow[r, "\mathbb{D}(f)_{H^0(B,\Omega^1_{B/k})(i)}"] \arrow[d, equal] & H^0(A, \Omega^1_{A/k})(i) \arrow[d, equal] \\
W_1(i) \arrow[r, "h_i"] & W_2(i)
\end{tikzcd}$$

being the zero map. Hence the condition $W_1(i) = \mathrm{Ker}(h_i) = \mathrm{Ker}(\mathbb{D}(f))_i$ equals to the condition $i + 1 \in \varphi$. $\square$

Put hence $I = I(f), \varphi = \varphi(f), \eta = \eta(f)$. Putting together Note (2) and Lemma (4.2.1) we recover [Sta97] (cf. [GK12, Theorem 2.4.1])

$$R := \widehat{\mathcal{O}}_{\mathcal{M}_p, t} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_p, (W_1, W_2)} \simeq k[\![\{X_i, i \in \ell(\varphi)\}, \{Y_i, i \in \eta\}]\!]/(\{X_i Y_i, i \in I\}).$$

Note that as a completed local ring, R is an admissible topological ring.

We obtain moreover a description of the universal object for the deformation of $(W_1, W_2) \in \mathcal{N}_p$:

$$W_1^{\mathrm{un}} = \oplus_{i=1}^{g} \widetilde{W_1}(i) \subset \oplus_{i=1}^{g} R\alpha_i + R\beta_i \xrightarrow{h=\oplus_{i=1}^{g} h_i} W_2^{\mathrm{un}} = \oplus_{i=1}^{g} \widetilde{W_2}(i) \subset \oplus_{i=1}^{g} Ra_i + Rb_i, \tag{4.4}$$

where $\widetilde{W_1}(i)$ and $\widetilde{W_2}(i)$ were described in Note (2).

There exist σ-linear morphisms

$$F_A : H^1_{dR}(A/k) \to H^1_{dR}(A/k), \qquad F_B : H^1_{dR}(B/k) \to H^1_{dR}(B/k)$$

defined as follows. We are going to show the construction for A, as the one for B is similar. Let

$$\mathscr{F}_A : H^1_{dR}(A^{(p)}/k) \simeq H^1_{dR}(A/k) \otimes_{k,\sigma} k \to H^1_{dR}(A/k)$$

be the linear morphism induced by the relative Frobenius $\mathrm{Fr}_A : A \to A^{(p)}$. Define

$$F_A(\alpha x) := \mathscr{F}_A(x \otimes 1), \qquad x \in H^1_{dR}(A/k).$$

Note that it is σ-linear: indeed, for $\alpha \in k$ and $x \in H^1_{dR}(A/k)$ we have $F_A(\alpha x) := \mathscr{F}_A(\alpha x \otimes 1) = \mathscr{F}_A(x \otimes \sigma(\alpha)) = \sigma(\alpha) F_A(x)$. Moreover, since $\mathscr{F}_A$ commutes with the $\mathcal{O}_L$-action, also F_A does.

The map F_A decomposes into g maps

$$F_A(i) : H^1_{dR}(A/k)(i) \to H^1_{dR}(A/k)(i+1).$$

Indeed, we may identify

$$H^1_{dR}(A/k)(i) = \beta_i(H^1_{dR}(A/k))$$

with $\theta_i = \sum_j \lambda_j \otimes x_j \in \mathcal{O}_L \otimes k$ and hence

$$F_A(H^1_{dR}(A)(i)) = F_A(\theta_i(H^1_{dR}(A/k)).$$

Then for $h \in H^1_{dR}(A/k)$, $F_A(\theta_i h) = F_A(\sum_j \lambda_j \otimes xh) = \sum_j \lambda_j F_A(x_j h) = \sum_j \lambda_j \otimes \sigma(x_j) F_A(h) \in H^1_{dR}(A/k)(i+1)$.

We wish to represent the maps $F_A(i)$ and $F_B(i)$ in terms of the bases $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$ for $i \in \{1, 2, \dots, g\}$.

Lemma 4.2.2. *Fix an index $i \in \{1, \dots, g\}$. The induced semi-linear morphisms*

$$F_A(i) : ka_i \oplus kb_i \to ka_{i+1} \oplus kb_{i+1}, \qquad F_B(i) : k\alpha_i \oplus k\beta_i \to k\alpha_{i+1} \oplus k\beta_{i+1}$$

are represented by the matrices

$$F_A(i) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{when } i \in \eta - I, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{when } i \in \ell(\varphi) - I, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{when } i \in I. \end{cases} \qquad F_B(i) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{when } i \in \eta - I, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{when } i \in \ell(\varphi) - I, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{when } i \in I. \end{cases}$$

Proof. As a general observation we have that the semi-linear Frobenius morphisms

$$F_A : H^1_{dR}(A/k) \to H^1_{dR}(A/k), \qquad F_B : H^1_{dR}(B/k) \to H^1_{dR}(B/k)$$

commute with the morphism

$$D(f) : H^1_{dR}(B/k) \to H^1_{dR}(A/k),$$

that is, by semi-linearity, for every index $i \in \{1, \dots g\}$, the following relation is satisfied:

$$h_{i+1} \circ F_B(i) = F_A(i) \circ h_i^{(\sigma)}; \tag{4.5}$$

(here we use the convention that the index $g + 1 = 1=$.

- Case when $i \in \eta - I$: By Note (2), point (1) we have

$$W_1(i) = \langle \beta_i \rangle, \qquad W_2(i) = \langle a_i \rangle,$$

and hence we have the following possibilities

$$F_B(i) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \text{ or} \\ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right). \end{cases} \qquad F_A(i) = \begin{cases} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \text{ or} \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right). \end{cases}$$

We need now to impose condition (4.5):

$$\text{Option 1 for } F_B(i) : \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{h_{i+1}} \underbrace{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}_{F_B(i)} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \qquad \text{Option 2 for } F_B(i) : \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{h_{i+1}} \underbrace{\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)}_{F_B(i)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

$$\text{Option 1 for } F_A(i) : \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{F_A(i)} \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{h_i^{(\sigma)}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \qquad \text{Option 2 for } F_A(i) : \underbrace{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)}_{F_A(i)} \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{h_i^{(\sigma)}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

From this we see that

$$F_B(i) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, the condition $i \notin \ell(\varphi)$ tells us by Lemma (3.0.1) that

$$\operatorname{Im}(\mathbb{D}(\mathrm{Fr}_A))_{i+1} \neq \operatorname{Im}(\mathbb{D}(f))_{i+1} = \langle a_{i+1} \rangle,$$

from which we conclude that

$$F_A(i) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Case when $i \in \ell(\varphi) - I$: By Note (2), point (2):

$$W_1(i) = \langle \alpha_i \rangle, \qquad W_2(i) = \langle b_i \rangle.$$

Moreover

$$\operatorname{Im}(\mathbb{D}(\mathrm{Fr}_A))_{i+1} = \operatorname{Im}(\mathbb{D}(f))_{i+1} = \langle a_{i+1} \rangle.$$

We have therefore that

$$F_A(i) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$\underbrace{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}_{F_A(i)} \underbrace{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}_{h_i^{(\sigma)}} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

On the other hand we have a priori the following options for $F_B(i)$:

$$F_B(i) = \begin{cases} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \text{ or} \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{cases}$$

and the following relations:

Option 1 for $F_B(i)$: $\underbrace{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}_{h_{i+1}}\underbrace{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}_{F_B(i)} = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, Option 2 for $F_B(i)$: $\underbrace{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)}_{h_{i+1}}\underbrace{\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)}_{F_B(i)} = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$.

From condition (4.5) we get that

$$F_B(i) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

- Case when $i \in I = \ell(\varphi) \cap \eta$: By Note (2), point (3):

$$W_1(i) = \langle \alpha_i \rangle, \qquad W_2(i) = \langle a_i \rangle.$$

Moreover

$$\mathrm{Im}(\mathbb{D}(\mathrm{Fr}_A))_{i+1} = \mathrm{Im}(\mathbb{D}(f))_{i+1} = \langle a_{i+1} \rangle.$$

We have therefore that

$$F_A(i) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

For $F_B(i)$ the same conditions as in the previous point hold, from which we obtain that

$$F_B = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

□

5. Dieudonné Displays on Hilbert modular varieties

By taking advantage of the decomposition in 2×2 blocks of the local model of the Hilbert modular variety, we wish to use the arguments of Theorem (2.3.1) and Theorem (2.3.2), Chapter (3), to describe the deformation theory of $\mathcal{M}_p$ through Dieudonné displays.

Consider a k-point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$. Let

$$\{a_1, \dots, a_g, b_1, \dots, b_g\}, \qquad \{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\} \tag{5.1}$$

be k-bases for $H^1_{dR}(A/k)$ and $H^1_{dR}(B/k)$ respectively so that given the corresponding point $(W_1, W_2) \in \mathcal{N}_p$, for every $i \in \{1, \dots, g\}$ the point

$$H^0(B, \Omega^1_{B/k})(i) = W_1(i) \subset k\alpha_i \oplus k\beta_i \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} H^0(A, \Omega^1_{A/k}) = W_2(i) \subset ka_i \oplus kb_i$$

in $\mathcal{N}_p(i)$ can be described as in Note (2), Section (4.2). For instance we will take trivializations

$$H^1_{dR}(A/k)(i) \simeq ka_i \oplus kb_i, \qquad H^1_{dR}(B/k)(i) \simeq k\alpha_i \oplus k\beta_i. \tag{5.2}$$

By the results outlined in Section (3), Chapter (2), to the point $t = (\underline{A} \xrightarrow{f} \underline{B})$ we associate a morphism of RM Dieudonné displays

$$\mathbb{D}(f)\colon \mathcal{P}_B = (P_B, Q_B, F_B, (V_B)^{-1}) \to \mathcal{P}_A = (P_A, Q_A, F_A, (V_A)^{-1})$$

over k. Following Theorem (3.4.4), Chapter (2), we will put

$$P_A = H^1_{cris}(A/W(k)), \qquad P_B = H^1_{cris}(B/W(k)),$$
$$Q_A = \mathrm{Ker}(H^1_{cris}(A/W(k)) \to \mathrm{Lie}(A)), \qquad \mathrm{Ker}(H^1_{cris}(B/W(k)) \to \mathrm{Lie}(B));$$

they are free $\mathcal{O}_L \otimes W(k)$-modules of rank 2. Recall the isomorphism

$$\mathcal{O}_L \otimes W(k) \simeq \bigoplus_{i=1}^{g} W(k)$$

obtained in Section (1). We wish to describe the Dieudonné displays $\mathcal{P}_A$ and $\mathcal{P}_B$ in terms of 2×2 blocks, that is

$$P_A = \bigoplus_{i=1}^{g} P_A(i) = \bigoplus_{i=1}^{g} H^1_{cris}(A/W(k))(i), \qquad P_B = \bigoplus_{i=1}^{g} P_B(i) = \bigoplus_{i=1}^{g} H^1_{cris}(B/W(k))(i),$$
$$Q_A(i) = \mathrm{Ker}(P_A(i) \to H^1(A,\mathcal{O}_A)(i), \qquad Q_B(i) = \mathrm{Ker}(P_B(i) \to H^1(B,\mathcal{O}_B)(i)),$$

where

$$Q_A = \bigoplus_{i=1}^{g} Q_A(i), \qquad Q_B = \bigoplus_{i=1}^{g} Q_B(i),$$

and morphisms

$$F_A(i)\colon P_A(i) \to P_A(i+1) \qquad F_B(i)\colon P_B(i) \to P_B(i+1).$$

Also the morphism of Dieudonné dislplays $\mathbb{D}(f)\colon \mathcal{P}_B \to \mathcal{P}_A$ admits a decomposition

$$\bigoplus_{i=1}^{g} P_B(i) \xrightarrow{\oplus_{i=1}^{g}\mathbb{D}(f)(i)} \bigoplus_{i=1}^{g} P_A(i).$$

Note that trivially $H^1_{cris}(A/W(k))(i)/(p) \simeq H^1_{dR}(A/k)(i)$ and the same holds for $H^1_{cris}(B/W(k))(i)$.

Let $\{a,b\}$ and $\{\alpha,\beta\}$ be $\mathcal{O}_L \otimes k$-bases for $H^1_{cris}(A/W(k)) = \oplus_{i=1}^{g} H^1_{cris}(A/W(k))(i)$ and $H^1_{cris}(B/W(k)) = \oplus_{i=1}^{g} H^1_{cris}(B/W(k))(i)$ respectively. Then the by applying the idempotents $\theta_i \in \mathcal{O}_L \otimes W(k)$ we obtain bases $\{a_i,b_i\}$ and $\{\alpha_i,\beta_i\}$ for $H^1_{cris}(A/W(k))(i)$ and $H^1_{cris}(B/W(k)$ for every $i \in \{1,\dots,g\}$.

Lemma 5.0.1 (Shape of morphisms). *There exists $W(k)$-bases $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ and $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ for $H^1_{cris}(A/W(k))$ and $H^1_{cris}(B/W(k))$, lifting the k-bases $\{a_1,\dots,a_g,b_1,\dots,b_g\}$ and $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ as in (5.1), such that*

$$M^{\{\alpha_i,\beta_i\}}_{\{a_i,b_i\}}(\mathbb{D}(f)(i)) = \begin{pmatrix} 0 & 1 \\ p & 1 \end{pmatrix},$$

and such that when $i \in \{1,\dots,g-1\}$

$$M^{\{a_i,b_i\}}_{\{a_{i+1},b_{i+1}\}}(F_A(i)) = \begin{cases} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } i \in \eta - I, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } i \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } i \in I, \end{cases} \quad M^{\{\alpha_i,\beta_i\}}_{\{\alpha_{i+1},\beta_{i+1}\}}(F_B(i)) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } i \in \eta - I, \\ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } i \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } i \in I, \end{cases}$$

and

$$M^{\{a_g,b_g\}}_{\{a_1,b_1\}}(F_A(g)) = \begin{cases} \left(\begin{smallmatrix} p(1+a) & b \\ p\beta & 1+\alpha \end{smallmatrix}\right) & \text{when } g \in \eta - I, \\ \left(\begin{smallmatrix} 1+a & b \\ p\beta & p(1+\alpha) \end{smallmatrix}\right) & \text{when } g \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} a & 1+b \\ p(1+\beta) & \alpha \end{smallmatrix}\right) & \text{when } g \in I, \end{cases}$$

$$M^{\{\alpha_g,\beta_g\}}_{\{\alpha_1,\beta_1\}}(F_B(g)) = \begin{cases} \left(\begin{smallmatrix} 1+\alpha & \beta \\ pb & p(1+a) \end{smallmatrix}\right) & \text{when } g \in \eta - I, \\ \left(\begin{smallmatrix} p(1+\alpha) & \beta \\ pb & 1+\alpha \end{smallmatrix}\right) & \text{when } g \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} \alpha & 1+\beta \\ p(1+b) & a \end{smallmatrix}\right) & \text{when } g \in I, \end{cases}$$

with $\alpha,\beta,a,b,\in W(k)$ and $p \mid \alpha, p \mid \beta, p \mid a, p \mid b$.

In order to give the proof, we need some linear algebra results.

Lemma 5.0.2. *Let* $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$ *be a geometric point. Then*

(1) *For every* $i \in \{1, \ldots, g\}$ *there exist* $\mathcal{O}_L \otimes W(k)$*-bases* $\{a_i, b_i\}$ *and* $\{a_{i+1}, b_{i+1}\}$ *for* $H^1_{cris}(A/W(k))(i)$ *and* $H^1_{cris}(A/W(k))(i+1)$ *respectively, lifting the* k*-bases* $\{a_i, b_i\}$ *and* $\{a_{i+1}, b_{i+1}\}$ *as in* (5.1)*, such that*

$$\det(M^{\{a_i,b_i\}}_{\{a_{i+1},b_{i+1}\}}(F_A(i))) = \pm p$$

where $M^{\{a_i,b_i\}}_{\{a_{i+1},b_{i+1}\}}(F_A(i))$ *denotes the matrix representing* $F_A(i)$ *in the bases* $\{a_i, b_i\}$ *and* $\{a_{i+1}, b_{i+1}\}$*.*

(2) *For every* $i \in \{1, \ldots, g\}$ *there exist* $\mathcal{O}_L \otimes W(k)$*-bases* $\{\alpha_i, \beta_i\}$ *and* $\{\alpha_{i+1}, \beta_{i+1}\}$ *for* $H^1_{cris}(B/W(k))(i)$ *and* $H^1_{cris}(B/W(k))(i+1)$ *respectively, lifting the* k*-bases* $\{\alpha_i, \beta_i\}$ *and* $\{\alpha_{i+1}, \beta_{i+1}\}$ *as in* (5.1)*, such that*

$$\det(M^{\{\alpha_i,\beta_i\}}_{\{\alpha_{i+1},\beta_{i+1}\}}(F_B(i))) = \pm p,$$

where $M^{\{\alpha_i,\beta_i\}}_{\{\alpha_{i+1},\beta_{i+1}\}}(F_B(i))$ *denotes the matrix representing* $F_B(i)$ *in the bases* $\{\alpha_i, \beta_i\}$ *and* $\{\alpha_{i+1}, \beta_{i+1}\}$*.*

(3) *For every* $i \in \{1, \ldots, g\}$ *there exist* $\mathcal{O}_L \otimes W(k)$*-bases* $\{a_i, b_i\}$ *and* $\{\alpha_i, \beta_i\}$ *for* $H^1_{cris}(A/W(k))(i)$ *and* $H^1_{cris}(B/W(k))(i)$ *respectively, lifting* $\{a_i, b_i\}$ *and* $\{\alpha_i, \beta_i\}$ *as in* (5.2)*, such that*

$$\det(M^{\{\alpha_i,\beta_i\}}_{\{a_i,b_i\}}(\mathbb{D}(f)(i))) = -p,$$

where $M^{\{\alpha_i,\beta_i\}}_{\{a_i,b_i\}}(\mathbb{D}(f)(i))$ *denotes the matrix representing* $\mathbb{D}(f)(i)$ *in the bases* $\{\alpha_i, \beta_i\}$ *and* $\{a_i, b_i\}$*.*

Proof. (1) Note that for every i, the σ^{-1}-linear map $V_A(i+1)\colon H^1_{cris}(A/W(k))(i+1) \to H^1_{cris}(A/W(k))(i)$ is such that

$$V_A(i+1) \circ F_A(i) = [p]_{H^1_{cris}(A/W(k))(i)}, \qquad F_A(i) \circ V_A(i+1) = [p]_{H^1_{cris}(A/W(k))(i+1)}.$$

The multiplication by p map has determinant p^2 and given that both $F_A(i)$ and $V_A(i)$ are not invertible, there exist bases $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$ such that

$$\det(M^{\{a_i,b_i\}}_{\{a_{i+1},b_{i+1}\}}(F_A(i))) = \pm p.$$

(2) The proof is identical to the proof of point (1).

(3) Recall that there exists a morphism $f^t\colon B \to A$ such that $f^t \circ f = [p]$. Now apply the proof of point (1).

□

Lemma 5.0.3. *Let* U *and* V *be free modules of rank* 2 *over* $W(k)$ *and denote by* $\phi\colon U \to V$ *a semi-linear map between them. Assume that there exist* $W(k)$*-bases* $\{e_1, e_2\}$ *and* $\{\eta_1, \eta_2\}$ *for* U *and* V *respectively such that*

$$\det(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi)) = -p.$$

Then

(1) *given any* $W(k)$*-basis* $\{e_1', e_2'\}$ *of* U *then there exists an* $W(k)$*-basis* $\{\eta_1', \eta_2'\}$ *for* V *such that*

$$\det(M^{\{e_1',e_2'\}}_{\{\eta_1',\eta_2'\}}(\phi)) = -p;$$

(2) *given any $W(k)$-basis $\{\eta_1', \eta_2'\}$ of V then there exists an $W(k)$-basis $\{e_1', e_2'\}$ of U such that*
$$\det(M^{\{e_1',e_2'\}}_{\{\eta_1',\eta_2'\}}(\phi)) = -p.$$

PROOF. (1) Let $S \in \mathrm{Gl}_2(R)$ be the base change matrix such that $S\{e_1, e_2\} = \{e_1', e_2'\}$. Take hence the $W(k)$-basis $\{\eta_1', \eta_2'\}$ of V defined as $\{\eta_1', \eta_2'\} := (S^{-1})^{(\sigma)}\{\eta_1, \eta_2\}$. Then
$$M^{\{e_1',e_2'\}}_{\{\eta_1',\eta_2'\}}(\phi) = (S^{-1})^{(\sigma)}(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi))S^{(\sigma)}$$
and hence
$$\det(M^{\{e_1',e_2'\}}_{\{\eta_1',\eta_2'\}}(\phi)) = \det(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi)).$$
(2) The proof is similar to the proof of point (1).

□

LEMMA 5.0.4. *Let U and V be free modules of rank 2 over $W(k)$ and denote by $\overline{U}$ and $\overline{V}$ their reduction modulo p. Let $\phi: U \to V$ be a semi-linear map such that there exist $W(k)$-bases $\{e_1, e_2\}$ abd $\{\eta_1, \eta_2\}$ such that*
$$\det(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi)) = -p \tag{5.3}$$
$$M^{\{\overline{e}_1,\overline{e}_2\}}_{\{\overline{\eta}_1,\overline{\eta}_2\}}(\phi) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.4}$$

Then

(1) *there exists a $W(k)$-basis $\{e_1', e_2'\}$ such that*
$$M^{\{e_1',e_2'\}}_{\{\eta_1,\eta_2\}}(\phi) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix};$$
(2) *there exists a $W(k)$-basis $\{\eta_1', \eta_2'\}$ such that*
$$M^{\{e_1,e_2\}}_{\{\eta_1',\eta_2'\}}(\phi) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

PROOF. (1) By (5.4) we know that
$$M(\phi)^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}} = \begin{pmatrix} a' & 1+b' \\ c' & d' \end{pmatrix}$$
with $p \mid a', p \mid b', p \mid c', p \mid d'$. From (5.3) we deduce that $c' = p(1 + p\alpha)$ with $\alpha \in W(k)$. Indeed $\det(M(\phi)^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}) = a'd' - (a + b')c' = a'd' - c' - b'c' = -p$ which tells us that $c' \equiv -p \pmod{p^2}$ and hence the conclusion. We get therefore that
$$M(\phi)^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}} = \begin{pmatrix} a' & 1+b' \\ p(1+p\alpha) & d' \end{pmatrix}.$$
Consider now the $W(k)$-basis
$$\{e_1'', e_2''\} = \{\frac{1}{(1+p\alpha)^{(\sigma^{-1})}}, (1+p\alpha)^{(\sigma^{-1})}\}.$$
Then
$$M(\phi)^{\{e_1'',e_2''\}}_{\{\eta_1,\eta_2\}} = \begin{pmatrix} a & 1+b \\ p & d \end{pmatrix},$$

with $p \mid a, p \mid b, p \mid d$. Note that we chose the basis so that

$$\det(M(\phi)^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}) = \det(M(\phi)^{\{e''_1,e''_2\}}_{\{\eta_1,\eta_2\}}) = -p = -p(1+b) + ad.$$

From the last equality we obtain that $pb = ad$ and hence that $b = \frac{ad}{p}$ (note that this fraction makes sense since by hypothesis $p^2 \mid ad$. Therefore

$$M(\phi)^{\{e''_1,e''_2\}}_{\{\eta_1,\eta_2\}} = \begin{pmatrix} a & 1 + \frac{ad}{p} \\ p & d \end{pmatrix}.$$

By multiplying on the right by the base change matrix

$$\begin{pmatrix} 1 + \frac{a^{(\sigma^{-1})}d^{(\sigma^{-1})}}{p} & -\frac{d^{(\sigma^{-1})}}{p} \\ -a^{(\sigma^{-1})} & 1 \end{pmatrix},$$

we obtain the matrix

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

(2) The proof is similar to the one of point (1).

□

LEMMA 5.0.5. *Let U and V be free modules of rank 2 over $W(k)$ and denote by $\overline{U}$ and $\overline{V}$ their reduction modulo p. Let $\phi: U \to V$ be a semi-linear map such that there exist $W(k)$-bases $\{e_1, e_2\}$ abd $\{\eta_1, \eta_2\}$ such that*

$$\det(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi)) = p \tag{5.5}$$

$$M^{\{\bar{e}_1,\bar{e}_2\}}_{\{\bar{\eta}_1,\bar{\eta}_2\}}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.6}$$

Then

(1) *there exists an $W(k)$-basis $\{e'_1, e'_2\}$ such that*

$$M^{\{e'_1,e'_2\}}_{\{\eta_1,\eta_2\}}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix};$$

(2) *there exists a $W(k)$-basis $\{\eta'_1, \eta'_2\}$ such that*

$$M^{\{e_1,e_2\}}_{\{\eta'_1,\eta'_2\}}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}.$$

PROOF. The proof is analogous to the proof of Lemma (5.0.4). □

LEMMA 5.0.6. *Let U and V be free modules of rank 2 over a $W(k)$ and denote by $\overline{U}$ and $\overline{V}$ their reduction modulo p. Let $\phi: U \to V$ be a semi-linear map such that there exist $W(k)$-bases $\{e_1, e_2\}$ abd $\{\eta_1, \eta_2\}$ such that*

$$\det(M^{\{e_1,e_2\}}_{\{\eta_1,\eta_2\}}(\phi)) = p \tag{5.7}$$

$$M^{\{\bar{e}_1,\bar{e}_2\}}_{\{\bar{\eta}_1,\bar{\eta}_2\}}(\phi) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.8}$$

Then

(1) *there exists a $W(k)$-basis $\{e'_1, e'_2\}$ such that*

$$M^{\{e'_1,e'_2\}}_{\{\eta_1,\eta_2\}}(\phi) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix};$$

(2) *there exists a $W(k)$-basis $\{\eta'_1, \eta'_2\}$ such that*

$$M^{\{e_1,e_2\}}_{\{\eta'_1,\eta'_2\}}(\phi) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

PROOF. The proof is analogous to the proof of Lemma (5.0.4). □

PROOF OF LEMMA (5.0.1). For every i, by Nakayama's lemma we may find bases $W(k)$-bases $\{\gamma_i, \delta_i\}$ and $\{c_i, d_i\}$ of $H^1_{cris}(B/W(k))(i)$ and $H^1_{cris}(A/W(k))(i)$ respectively such that

$$M^{\{\overline{\gamma}_i,\overline{\delta}_i\}}_{\{\overline{c}_i,\overline{d}_i\}}(\mathbb{D}(\overline{f})(i) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Moreover, by Lemma (5.0.2), point (3) we may assume that

$$\det(M^{\{\gamma_i,\delta_i\}}_{\{c_i,d_i\}}(\mathbb{D}(f)(i))) = -p.$$

Analogously, by Nakayama's lemma there exist $W(k)$-bases $\{\epsilon_1, \phi_i\}$ and $\{e_i, f_i\}$ of $H^1_{cris}(B/W(k))(i)$ and $H^1_{cris}(A/W(k))(i)$ such that

$$M^{\{\overline{e}_i,\overline{f}_i\}}_{\{\overline{e}_{i+1},\overline{f}_{i+1}\}}(\overline{F}_A(i)) = \begin{cases} \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } i \in \eta - I, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } i \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } i \in I, \end{cases} \quad M^{\{\overline{\epsilon}_i,\overline{\phi}_i\}}_{\{\overline{\epsilon}_{i+1},\overline{\phi}_{i+1}\}}(\overline{F}_B(i)) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } i \in \eta - I, \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } i \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \text{when } i \in I. \end{cases}$$

and by (5.0.2), points (1) and (2) we may assume that

$$\det(M^{\{e_i,f_i\}}_{\{e_{i+1},f_{i+1}\}}(F_A(i))) = \begin{cases} p & \text{when } i \notin I, \\ -p & \text{when } i \in I, \end{cases} \qquad \det(M^{\{\epsilon_i,\phi_i\}}_{\{\epsilon_{i+1},\phi_{i+1}\}}(\overline{F}_B(i))) = \begin{cases} p & \text{when } i \notin I, \\ -p & \text{when } i \in I. \end{cases}$$

We have the following commutative diagrams

(5.9)
$$\begin{tikzcd} H^1_{cris}(B/W(k))(1) \arrow[r, "F_B(1)"] \arrow[d, "\mathbb{D}(f)(1)"] & H^1_{cris}(B/W(k))(2) \arrow[r] \arrow[d, "\mathbb{D}(f)(2)"] & \dots \arrow[d] \\ H^1_{cris}(A/W(k))(1) \arrow[r, "F_A(1)"] & H^1_{cris}(A/W(k))(2) \arrow[r] & \dots \end{tikzcd}$$

$$\begin{tikzcd} \dots \arrow[r] \arrow[d] & H^1_{cris}(B/W(k))(g-1) \arrow[r, "F_B(g-1)"] \arrow[d, "\mathbb{D}(f)(g-1)"] & H^1_{cris}(B/W(k))(g) \arrow[d, "\mathbb{D}(f)(g)"] \\ \dots \arrow[r] & H^1_{cris}(A/W(k))(g-1) \arrow[r, "F_A(g-1)"] & H^1_{cris}(A/W(k))(g) \end{tikzcd}$$

Each square satisfies the relation

(5.10)
$$F_A(i) \circ \mathbb{D}(f)(i)^{(\sigma)} = \mathbb{D}(f)(i+1) \circ F_B(i).$$

Consider the first diagram on the left:

$$\begin{tikzcd} H^1_{cris}(B/W(k))(1) \arrow[r, "F_B(1)"] \arrow[d, "\mathbb{D}(f)(1)"] & H^1_{cris}(B/W(k))(2) \arrow[d, "\mathbb{D}(f)(2)"] \\ H^1_{cris}(A/W(k))(1) \arrow[r, "F_A(1)"] & H^1_{cris}(A/W(k))(2) \end{tikzcd}$$

By Lemma (5.0.4), there exist $W(k)$-bases $\{\alpha_1, \beta_1\}$ and $\{a_1, b_2\}$ of $H^1_{cris}(B/W(k))(1)$ and $H^1_{cris}(A/W(k))(1)$ such that

$$M^{\{\alpha_1,\beta_1\}}_{\{a_1,b_1\}}(\mathbb{D}(f)(i)) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Now, by Lemmas (5.0.4), (5.0.5) and (5.0.6) point (2) there exist $W(k)$-bases $\{\alpha_2, \beta_2\}$ and $\{a_2, b_2\}$ of $H^1_{cris}(B/W(k))(2)$ and $H^1_{cris}(A/W(k))(2)$ such that

$$M^{\{a_1,b_1\}}_{\{a_2,b_2\}}(F_A(1)) = \begin{cases} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } 1 \in \eta - I, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } 1 \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } 1 \in I, \end{cases} \quad M^{\{\alpha_1,\beta_1\}}_{\{\alpha_2,\beta_2\}}(F_B(1)) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } 1 \in \eta - I, \\ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } 1 \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } 1 \in I, \end{cases}$$

We have hence fixed all the bases in the diagram:

$$\begin{tikzcd} \overbrace{H^1_{cris}(B/W(k))(1)}^{\{\alpha_1,\beta_1\}} \arrow[r, "F_B(1)"] \arrow[d, "\mathbb{D}(f)(1)"] & \overbrace{H^1_{cris}(B/W(k))(2)}^{\{\alpha_2,\beta_2\}} \arrow[d, "\mathbb{D}(f)(2)"] \\ \underbrace{H^1_{cris}(A/W(k))(1)}_{\{a_1,b_1\}} \arrow[r, "F_A(1)"] & \underbrace{H^1_{cris}(A/W(k))(2)}_{\{a_2,b_2\}} \end{tikzcd}$$

We claim that

$$M^{\{\alpha_2,\beta_2\}}_{\{a_2,b_2\}}(\mathbb{D}(f)(2)) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Recall that by (5.10) the relation

$$\mathbb{F}_A(1) \circ D(f)(1)^{(\sigma)} = \mathbb{D}(f)(2) \circ F_B(1)$$

must be satisfied. For $1 \in I$ this automatically proves the claim. For $i \in \eta - I$, we have

$$\underbrace{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}_{F_A(1)} \circ \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbb{D}(f)(2)} = \underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\mathbb{D}(f)(1)^{(\sigma)}} \circ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}_{F_B(1)},$$

$$\begin{pmatrix} pa & pb \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix},$$

from which we easily see that $a = d = 0$, $b = 1$ and $c = p$. The proof in the case when $i \in \varphi - I$ is analogous.

Passing to the second square of diagram

$$\begin{tikzcd}
\overbrace{H^1_{cris}(B/W(k))(2)}^{\{\alpha_2,\beta_2\}} \arrow[r, "F_B(2)"] \arrow[d, "\mathbb{D}(f)(2)"] & H^1_{cris}(B/W(k))(3) \arrow[d, "\mathbb{D}(f)(3)"] \\
\underbrace{H^1_{cris}(A/W(k))(2)}_{\{a_2,b_2\}} \arrow[r, "F_A(2)"] & H^1_{cris}(A/W(k))(3)
\end{tikzcd}$$

of (5.9), by Lemma (5.0.4), (5.0.5) and (5.0.6) point (2) we take $W(k)$-bases $\{\alpha_3,\beta_3\}$ and $\{a_3,b_3\}$ for $H^1_{cris}(B/W(k))(3)$ and $H^1_{cris}(A/W(k))(3)$ such that

$$M^{\{a_2,b_2\}}_{\{a_3,b_3\}}(F_A(2)) = \begin{cases} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } 2 \in \eta - I, \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } 2 \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } 2 \in I, \end{cases} \quad M^{\{\alpha_2,\beta_2\}}_{\{\alpha_3,\beta_3\}}(F_B(2)) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) & \text{when } 2 \in \eta - I, \\ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{when } 2 \in \ell(\varphi) - I, \\ \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right) & \text{when } 2 \in I, \end{cases}$$

Again, by imposing (5.10) we obtain that

$$M^{\{\alpha_3,\beta_3\}}_{\{a_3,b_3\}}((\mathbb{D}(f)(3)) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

We go on with this approach and we arrive to the diagram

$$\begin{tikzcd}
\overbrace{H^1_{cris}(B/W(k))(g-1)}^{\{\alpha_{g-1},\beta_{g-1}\}} \arrow[r, "F_B(g-1)"] \arrow[d, "\mathbb{D}(f)(g-1)"] & \overbrace{H^1_{cris}(B/W(k))(g)}^{\{\alpha_g,\beta_g\}} \arrow[r, "F_B(g)"] \arrow[d, "\mathbb{D}(f)(g)"] & \overbrace{H^1_{cris}(B/W(k))(1)}^{\{\alpha_{g-1},\beta_{g-1}\}} \arrow[d, "\mathbb{D}(f)(1)"] \\
\underbrace{H^1_{cris}(A/W(k))(g-1)}_{\{a_{g-1},b_{g-1}\}} \arrow[r, "F_A(g-1)"] & \underbrace{H^1_{cris}(A/W(k))(g)}_{\{a_g,b_g\}} \arrow[r, "F_B(g)"] & \underbrace{H^1_{cris}(A/W(k))(1)}_{\{a_1,b_1\}}
\end{tikzcd}$$

where all the bases are set and where

$$M^{\{\alpha_g,\beta_g\}}_{\{a_g,b_g\}}(\mathbb{D}(f)(g)) = M^{\{\alpha_1,\beta_1\}}_{\{a_1,b_g\}}(\mathbb{D}(f)(g)) = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix},$$

and where only $M^{\{\alpha_g,\beta_g\}}_{\{\alpha_1,\beta_1\}}(F_B(g))$ and $M^{\{a_g,b_g\}}_{\{a_1,b_1\}}(F_B(g))$ have to be determined. We will show in detail the case $g \in I$, the other cases being similar. In full generality we know that

$$M^{\{\alpha_g,\beta_g\}}_{\{\alpha_1,\beta_1\}}(F_B(g)) = \begin{pmatrix} \alpha & 1+\beta \\ \gamma & \delta \end{pmatrix}, \qquad M^{\{a_g,b_g\}}_{\{a_1,b_1\}}(F_A(g)) = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix},$$

with $p \mid \alpha, p \mid \beta, p \mid \gamma, p \mid \delta, p \mid a, p \mid b, p \mid c, p \mid d$. By (5.10) we get the condition

$$\begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\begin{pmatrix} \alpha & 1+\beta \\ \gamma & \delta \end{pmatrix}$$

$$\begin{pmatrix} p(1+b) & a \\ pd & c \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ p\alpha & p(1+\beta) \end{pmatrix},$$

from which we get

$$\gamma = p(1+b), \delta = a, d = \alpha, c = p(1+\beta)$$

and hence that

$$M^{\{\alpha_g,\beta_g\}}_{\{\alpha_1,\beta_1\}}(F_B(g)) = \begin{pmatrix} \alpha & 1+\beta \\ p(1+b) & a \end{pmatrix}, \qquad M^{\{a_g,b_g\}}_{\{a_1,b_1\}}(F_A(g)) = \begin{pmatrix} a & 1+b \\ p(1+\beta) & \alpha \end{pmatrix},$$

with $p \mid \alpha, p \mid \beta, p \mid a, p \mid b$. □

In the bases obtained in Lemma (5.0.1) we have

$$P_A = \bigoplus_{i=1}^{g} P_A(i) = \bigoplus_{i=1}^{g}(W(k)a_i \oplus W(k)b_i), \qquad P_B = \bigoplus_{i=1}^{g} P_B(i) = \bigoplus_{i=1}^{g}(W(k)\alpha_i \oplus W(k)\beta_i),$$

$$Q_A = \bigoplus_{i=1}^{g} Q_A(i), \qquad Q_B = \bigoplus_{i=1}^{g} Q_B(i).$$

In particular by Note (2), Section (4.2), we have

$$\begin{aligned} &Q_A(i) = W(k)a_i \oplus p\cdot W(k)b_i,\ Q_B(i) = p\cdot W(k)\alpha_i \oplus W(k)\beta_i && \text{when } i \in \eta - I,\\ &Q_A(i) = p\cdot W(k)a_i \oplus W(k)b_i,\ Q_B(i) = W(k)\alpha_i \oplus p\cdot W(k)\beta_i && \text{when } i \in \ell(\varphi) - I,\\ &Q_A(i) = W(k)a_i \oplus p\cdot W(k)b_i,\ Q_B(i) = W(k)\alpha_i \oplus p\cdot W(k)\beta_i && \text{when } i \in I. \end{aligned}$$

Note that reducing modulo (p) the objects introduced above, we recover an object of the Grassmann variety as in Section (4.2):

$$\bigoplus_{i=1}^{g} W_1(i) \subset \bigoplus_{i=1}^{g}(k\alpha_i \oplus k\beta_i) \xrightarrow{\oplus h_i} \bigoplus_{i=1}^{g} W_2(i) \subset \bigoplus_{i=1}^{g}(ka_i \oplus kb_i).$$

Finally, by semi-linearity of the Frobenius morphisms, we have that, in the bases $\{\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g\}$ and $\{a_1,\dots,a_g,b_1,\dots,b_g\}$, F_A, F_B and $\mathbb{D}(f)$ are represented by the matrices

$$F_A = \begin{pmatrix} 0 & 0 & \dots & 0 & F_A(g) \\ F_A(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_A(g-1) & 0 \end{pmatrix}, \ F_B = \begin{pmatrix} 0 & 0 & \dots & 0 & F_B(g) \\ F_B(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_B(g-1) & 0 \end{pmatrix}.$$

$$\mathbb{D}(f) = \begin{pmatrix} \mathbb{D}(f)(1) & 0 & \dots & \dots & 0 \\ 0 & \mathbb{D}(f)(2) & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \dots & 0 & \mathbb{D}(f)(g-1) & 0 \\ 0 & \dots & \dots & 0 & \mathbb{D}(f)(g) \end{pmatrix}.$$

Remark that by construction, for every index i we have that

$$\mathbb{D}(f)(i+1) \circ F_B(i) = F_A(i) \circ \mathbb{D}(f)(i)^{(\sigma)}.$$

Therefore we have that

$$\mathbb{D}(f) \circ F_B = F_A \circ \mathbb{D}(f)^{(\sigma)}.$$

We expect this since

$$\mathbb{D}(f) \colon \mathcal{P}_B \longrightarrow \mathcal{P}_A$$

is a morphism of Dieudonné displays. We would like to write a universal object

$$\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1}) \xrightarrow{\alpha} \mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1}),$$

over $R = \widehat{\mathcal{O}}_{\mathcal{M},t} \simeq k[\![X_i, Y_i : i \in I]\!][\![Z_i : i \in I^c]\!]/\{(X_i Y_i)\}_{i \in I}$ for the deformation of

$$\mathbb{D}(f) : \mathcal{P}_B \to \mathcal{P}_A.$$

We will follow a strategy analogous to the one outlined in (2.3), Chapter (3).

Recall the description (4.4) of the universal object for the deformation of the point $(W_1, W_2) \in \mathcal{N}_p$ associated to $t = \underline{A} \xrightarrow{f} \underline{B}$:

$$W_1^{\mathrm{un}} = \bigoplus_{i=1}^{g} \widetilde{W_1}(i) \subset \bigoplus_{i=1}^{g} R\alpha_i + R\beta_i \xrightarrow{h = \oplus_{i=1}^{g} h_i} W_2^{\mathrm{un}} = \bigoplus_{i=1}^{g} \widetilde{W_2}(i) \subset \oplus_{i=1}^{g} Ra_i + Rb_i.$$

Define

$$P_1 := P_B \otimes_{W(k)} \mathbb{W}(R), \qquad P_2 := P_A \otimes_{W(k)} \mathbb{W}(R),$$

and let Q_1 (resp. Q_2) be the pre-image of W_1^{un} (resp. W_2^{un}) through the projection $P_1 \to H^1_{dR}(B/k) \otimes_k R$ (resp. $P_2 \to H^1_{dR}(A/k) \otimes_k R$). Note that there is an identification

$$\mathcal{O}_L \otimes \mathbb{W}(R) \simeq \bigoplus_{i=1}^{g} \mathbb{W}(R).$$

(This is true since $\mathcal{O}_L \otimes_{W(k)} \mathbb{W}(R) \simeq (\mathcal{O}_L \otimes W(k)) \otimes_{W(k)} \mathbb{W}(R) \simeq (\oplus_{i=1}^{g} W(k)) \otimes_{W(k)} \mathbb{W}(R)$.) The $\mathcal{O}_L \otimes \mathbb{W}(R)$-modules P_1, P_2, Q_1, Q_2 inherit decompositions from the $\mathcal{O}_L \otimes W(k)$-modules P_B and P_A:

$$P_1 = \bigoplus_{i=1}^{g} P_1(i), \qquad P_2 = \bigoplus_{i=1}^{g} P_2(i), \qquad Q_1 = \bigoplus_{i=1}^{g} Q_1(i), \qquad Q_2 = \bigoplus_{i=1}^{g} Q_2(i).$$

In particular, in the bases $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ and $\{a_1, \dots, a_g, b_1, \dots, b_g\}$:

$$\begin{aligned} P_1 &= \bigoplus_{i=1}^{g} P_1(i) = \bigoplus_{i=1}^{g} (W(k)\alpha_i \oplus W(k)\beta_i) \otimes_k \mathbb{W}(R) = \bigoplus_{i=1}^{g} (\mathbb{W}(R)\alpha_i \oplus \mathbb{W}(R)\beta_i), \\ P_2 &= \bigoplus_{i=1}^{g} P_2(i) = \bigoplus_{i=1}^{g} (W(k)a_i \oplus W(k)b_i) \otimes_k \mathbb{W}(R) = \bigoplus_{i=1}^{g} (\mathbb{W}(R)a_i \oplus \mathbb{W}(R)b_i). \end{aligned}$$

and

$$\begin{array}{rr} Q_1(i) = \mathbb{I}(R)(\alpha_i \oplus \beta_i) + \mathbb{W}(R)(\alpha_i + \hat{X}_i\beta_i), & Q_2(i) = \mathbb{I}(R)(a_i \oplus b_i) + \mathbb{W}(R)(a_i + \hat{Y}_i b_i), \quad i \in I, \\ Q_1(i) = \mathbb{I}(R)\beta_i \oplus \mathbb{W}(R)\alpha_i, & Q_2(i) = \mathbb{I}(R)(a_i \oplus b_i) + \mathbb{W}(R)(b_i + \hat{Y}_i a_i), \quad i \in \ell(\varphi) - I, \\ Q_1(i) = \mathbb{I}(R)(\alpha_i \oplus \beta_i) + \mathbb{W}(R)(\alpha_i + \hat{X}_i\beta_i), & Q_2(i) = \mathbb{I}(R)a_i \oplus \mathbb{W}(R)b_i, \quad i \in \eta. \end{array}$$

We will define the semi-linear maps $F_1 : P_1 \to P_1$ and $F_2 : P_2 \to P_2$ in terms of their decompositions

$$\bigoplus_{i=1}^{g} F_1(i) : \bigoplus_{i=1}^{g} P_1(i) \to P_1(i+1), \qquad \bigoplus_{i=1}^{g} F_2(i) : \bigoplus_{i=1}^{g} P_2(i) \to P_2(i+1).$$

Following is the description of the matrices associated to

$$F_1(i) : R\alpha_i \oplus R\beta_i \to R\alpha_{i+1} \oplus R\beta_{i+1}, \qquad F_2(i) : Ra_i \oplus Rb_i \to Ra_{i+1} \oplus Rb_{i+1}.$$

For $i \in \{1, \dots, g-1\}$ take

$$F_1(i) = \begin{cases} \begin{pmatrix} -^F\hat{X}_i & 1 \\ p & -^F\hat{Y}_i \end{pmatrix} & \text{if } i \in I, \\ \begin{pmatrix} p & 0 \\ -p^F\hat{Y}_i & 1 \end{pmatrix} & \text{if } i \in \ell(\varphi) - I, \\ \begin{pmatrix} 1+ & -^F\hat{X}_i \\ 0 & p \end{pmatrix} & \text{if } i \in \eta - I, \end{cases} \qquad F_2(i) = \begin{cases} \begin{pmatrix} -^F\hat{Y}_i & 1 \\ p & -^F\hat{X}_i \end{pmatrix} & \text{if } i \in I, \\ \begin{pmatrix} 1 & -^F\hat{Y}_i \\ 0 & p \end{pmatrix} & \text{if } i \in \ell(\varphi) - I, \\ \begin{pmatrix} p & 0 \\ -p^F\hat{X}_i & 1 \end{pmatrix} & \text{if } i \in \eta - I, \end{cases}$$

and for $i = g$,

$$F_1(g) = \begin{cases} \begin{pmatrix} \alpha - {}^F\hat{X}_g & 1+\beta \\ p(1+b) & a - {}^F\hat{Y}_g \end{pmatrix} & \text{if } g \in I, \\ \begin{pmatrix} p(1+\alpha) & \beta \\ p(b - {}^F\hat{Y}_g) & 1+a \end{pmatrix} & \text{if } g \in \ell(\varphi) - I, \\ \begin{pmatrix} 1+\alpha & \beta - {}^F\hat{X}_g \\ pb & p(1+a) \end{pmatrix} & \text{if } g \in \eta - I, \end{cases} \qquad F_2(g) = \begin{cases} \begin{pmatrix} a - {}^F\hat{Y}_g & 1+b \\ p(1+\beta) & \alpha - {}^F\hat{X}_g \end{pmatrix} & \text{if } g \in I, \\ \begin{pmatrix} 1+a & -{}^F\hat{Y}_g + b \\ p\beta & p(1+\alpha) \end{pmatrix} & \text{if } g \in \ell(\varphi) - I, \\ \begin{pmatrix} p(1+a) & b \\ p(\beta - {}^F\hat{X}_g) & 1+\alpha \end{pmatrix} & \text{if } g \in \eta - I, \end{cases}$$

By an argument identical to the one outlined in the case of the modular curve, Chapter (3), the coefficients of the matrices belong to $\mathbb{W}(R)$. Note that by construction, the reductions

$$\overline{F}_1(i) = F_1(i) \bmod \mathbb{I}(R), \qquad \overline{F}_2(i) = F_2(i) \bmod \mathbb{I}(R)$$

have respectively kernel $\widetilde{W}_1(i)$ and $\widetilde{W}_2(i)$ as in Note (2). Hence, in the bases $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ and $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ the semi-linear morphisms are represented by the matrices

$$F_2 = \begin{pmatrix} 0 & 0 & \dots & 0 & F_2(g) \\ F_2(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_2(g-1) & 0 \end{pmatrix}, \ F_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & F_1(g) \\ F_1(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_1(g-1) & 0 \end{pmatrix}.$$

By construction the kernels of $\overline{F}_1$ modulo $\mathbb{I}(R)$ and $\overline{F}_2$ modulo $I(R)$ are W_1^{un} and W_2^{un} as in (4.4).

For $i = 1, 2$, define

$$V_i^{-1} \colon Q_i \to P_i$$

as the restriction to Q_i of F_i/p.

Theorem 5.0.7. *The semi-linear maps $V_1^{-1} \colon Q_1 \to P_1$ and $V_2^{-1} \colon Q_2 \to P_2$ are well-defined and the quadruples $\mathcal{P}_1$ and $\mathcal{P}_2$ are Dieudonné displays over R. They reduce to $\mathcal{P}_B$ and $\mathcal{P}_A$ modulo $\mathfrak{m}_R$ respectively and their Hodge filtrations are $W_1^{\mathrm{un}} \subset \oplus_{i=1}^g R\alpha_i \oplus R\beta_i$ and $W_2^{\mathrm{un}} \subset \oplus_{i=1}^g Ra_i \oplus Rb_i$.*

Proof. The argument is a generalization of the proof of Theorem (2.3.1), Chapter (3). We will prove this for $\mathcal{P}_1$, the case for $\mathcal{P}_2$ being essentially identical. They only part that is left to prove is that V_1^{-1} is well defined over Q_1 and that it is an ^{F}-linear epimorphism.

We have set $V_1^{-1} \coloneqq \frac{F_1}{p}|_{Q_1}$. Showing that V_1^{-1} is well-defined is equivalent to showing that $F_1(Q_1) \subseteq pP_1$. Note that by definition V_1^{-1} admits a decomposition

$$\bigoplus_{i=1}^{g} V_1^{-1}(i) \colon \bigoplus_{i=1}^{g} Q_1(i) \to \bigoplus_{i=1}^{g} P_1(i+1),$$

where for each i, $V_1^{-1}(i)$ is defined on $Q_1(i)$ as the restriction of $F_1(i)/p$. Showing our claim equals to showing that for every $i \in \{1, \dots, g\}$ we have that $F_1(i)(Q_1(i)) \subseteq pP_1(i+1)$. We will show this extensively in the case $i \in I$. An element $x \in Q_1(i) \simeq \mathbb{I}(R)(\alpha_i \oplus \beta_i) + \mathbb{W}(R)(\alpha_i + \hat{X}_i\beta_i)$ can be written as a vector $\begin{pmatrix} \theta + w \\ \zeta + w\hat{X}_i \end{pmatrix}$, where $\theta, \zeta \in \mathbb{I}(R)$ and $w \in \mathbb{W}(R)$. Therefore, when $i \in \{1, \dots, g-1\}$, we have $F_1(i)(x) = \begin{pmatrix} -{}^F\hat{X}_i & 1 \\ p & -{}^F\hat{Y}_i \end{pmatrix} \begin{pmatrix} \theta + w \\ \zeta + \hat{X}_i w \end{pmatrix} = \begin{pmatrix} -{}^F\hat{X}_i{}^F\theta + {}^F\zeta \\ p^F\theta + p^F w - {}^F\hat{Y}_i{}^F\zeta \end{pmatrix} \in {}^F\mathbb{I}(R)(\alpha_i \oplus \beta_i) + p\mathbb{W}(R)(\alpha_i + \hat{X}_i\beta_i)$, and the result follows also for $i = g$ as $a, b, \alpha, \beta \in p\mathbb{W}(R)$.

In order to prove the claim we need therefore to show that $F_1(i)(\mathbb{I}(R)(\alpha_i \oplus \beta_i) \subseteq pP_1(i)$, that is, that ${}^F\mathbb{I}(R) \subseteq p\mathbb{W}(R)$. This was shown in (3.1), Chapter (2), hence V_1^{-1} is well-defined on Q_1.

We have left to show that V_1^{-1} is an ^{F}-linear epimorphism, that is, that its linearization

$$(V_1^{-1})^\sharp\colon \mathbb{W}(R) \otimes_{\mathbb{W}(R),F} Q_1 \to P_1$$

is an epimorphism (see Appendix for details). This equals to showing that for a suitable decomposition $P_1 = L_1 \oplus T_1$ such that $Q_1 = L_1 \oplus \mathbb{I}(R)T_1$, the morphism

$$V_1^{-1} \oplus F_1\colon L_1 \oplus T_1 \longrightarrow P_1$$

is an ^{F}-linear isomorphism. Note that after localizing R, we may assume L_1 and T_1 to be free. We take

$$L_1 = \langle \lambda_1, \dots, \lambda_g \rangle, \qquad T_1 = \langle \tau_1, \dots, \tau_g \rangle,$$

with

$$\lambda_i = \begin{cases} \alpha_i + \hat{X}_i\beta_i & \text{if } i \in I \\ \alpha_i & \text{if } i \in \ell(\varphi) - I \\ \beta_i + \hat{X}_i\alpha_i & \text{if } i \in \eta - I \end{cases} \qquad \tau_i = \begin{cases} \beta_i & \text{if } i \in I \\ \beta_i & \text{if } i \in \ell(\varphi) - I \\ \alpha_i & \text{if } i \in \eta - I \end{cases} \tag{5.11}$$

If we show that the *displaying* matrix associated to $V_1^{-1} \oplus F_1$ is invertible, we are done. Remark that, from the basis $\{\lambda_i, \tau_i\}_i$ to the basis $\{\alpha_i, \beta_i\}_i$, the ^{F}-linear map F_1 is represented by $F_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & F_1(g) \\ F_1(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & F_1(g-1) & 0 \end{pmatrix}$ where

$$F_1(i) = \begin{cases} \begin{pmatrix} -{}^F\hat{X}_i & 1 \\ p & -{}^F\hat{Y}_i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ {}^F\hat{X}_1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & -{}^F\hat{Y}_i \end{pmatrix} & \text{if } i \in I \\ \begin{pmatrix} p & 0 \\ -p{}^F\hat{Y}_i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & 0 \\ -p{}^F\hat{Y}_i & 1 \end{pmatrix} & \text{if } i \in \ell(\varphi) - I \\ \begin{pmatrix} 1 & -{}^F\hat{X}_i \\ 0 & p \end{pmatrix}\begin{pmatrix} {}^F\hat{X}_i & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} & \text{if } i \in \eta - I \end{cases}$$

Hence the morphism $V_1^{-1} \oplus F_1\colon L_1 \oplus T_1 \to P_1$ is represented by the matrix

$$F_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & (V_1^{-1} \oplus F_1)(g) \\ (V_1^{-1} \oplus F_1)(1) & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (V_1^{-1} \oplus F_1)(g-1) & 0 \end{pmatrix},$$

where

$$F_1(i) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & -{}^F\hat{Y}_i \end{pmatrix} & \text{if } i \in I \\ \begin{pmatrix} 1 & 0 \\ -{}^F\hat{Y}_i & 1 \end{pmatrix} & \text{if } i \in \ell(\varphi) - I \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } i \in \eta - I \end{cases}$$

They are all invertible, hence the conclusion. $\square$

Take now $\alpha\colon P_1 \to P_2$ as the map represented by

$$\alpha = \begin{pmatrix} \alpha(1) & 0 & \dots & \dots & 0 \\ 0 & \alpha(2) & 0 & \dots & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \dots & 0 & \alpha(g-1) & 0 \\ 0 & \dots & \dots & 0 & \alpha(g) \end{pmatrix}$$

in the bases $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ and $\{a_1, \dots, a_g, b_1, \dots, b_g\}$, with $\alpha_i = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ for every i.

Lemma 5.0.8. *The map $\alpha\colon P_1 \to P_2$ is a morphism of Dieudonné displays*

$$\alpha\colon \mathcal{P}_1 \to \mathcal{P}_2.$$

Proof. In order to show that it is a morphism of Dieudonné displays we need to show that it respects the filtrations

$$\alpha(Q_1) \subseteq Q_2, \tag{5.12}$$

and that it commutes with the semi-linear maps

$$\alpha \circ F_1 = F_2 \circ \alpha^{(\sigma)}, \qquad \alpha \circ V_1^{-1} = V_2^{-1} \circ \alpha^{(\sigma)}. \tag{5.13}$$

For proving (5.12) it is enough to show that for every $i \in \{1, \dots, g\}$ the inclusion

$$\alpha(i)(Q_1(i)) \subseteq Q_2(i)$$

holds. We will show this in the case $i \in I$. Recall that in this case

$$Q_1(i) = \mathbb{I}(R)(\alpha_i \oplus \beta_i) + \mathbb{W}(R)(\alpha_i + \hat{X}_i\beta_i),\ Q_2(i) = \mathbb{I}(R)(a_i \oplus b_i) + \mathbb{W}(R)(a_i + \hat{Y}_i b_i).$$

An generic element of $Q_1(i)$ is a vector $\left(\begin{smallmatrix} \theta+w \\ \zeta+w\hat{X}_i \end{smallmatrix}\right)$ with $\theta, \zeta \in \mathbb{I}(R)$ and $w \in \mathbb{W}(R)$ and $\mathbb{D}(f)\left(\begin{smallmatrix} \theta+w \\ \zeta+w\hat{X}_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \theta+w \\ \zeta+w\hat{X}_i \end{smallmatrix}\right) = \left(\begin{smallmatrix} \zeta+w\hat{X}_i \\ p\theta+pw \end{smallmatrix}\right) \in Q_2(i)$, since $\hat{Y}_i\left(\begin{smallmatrix} \zeta+w\hat{X}_i \\ p\theta+pw \end{smallmatrix}\right) = \left(\begin{smallmatrix} \zeta\hat{Y}_i \\ p\theta\hat{Y}_i+pw\hat{Y}_i \end{smallmatrix}\right) \in Q_2(i)$. The cases when $\imath \in \ell(\varphi) - I, i \in \eta - I$ are similar.

Showing condition (5.13) is equivalent to showing that

$$\alpha(i+1) \circ F_1(i) = F_2(i) \circ \alpha(i)^{(\sigma)}. \tag{5.14}$$

Note that the commutativity with V_1^{-1} and V_2^{-1} follows given that these maps were defined starting from F_1 and F_2. We will show (5.14) for $i \in I$, the other cases being similar. It comes from direct computations:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha(i+1)} \underbrace{\begin{pmatrix} -^F\hat{X}_i & 1 \\ p & -^F\hat{Y}_i \end{pmatrix}}_{F_1(i)} = \begin{pmatrix} p & -^F\hat{Y}_i \\ -p^F\hat{X}_i & p \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} -^F\hat{Y}_i & 1 \\ p & -^F\hat{X}_i \end{pmatrix}}_{F_2(i)} \underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha(i)^{(\sigma)}} = \begin{pmatrix} p & -^F\hat{Y}_i \\ -p^F\hat{X}_i & p \end{pmatrix}.$$

when $i \in \{1, \dots, g-1\}$ and for $i = g$:

$$\underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha(1)} \underbrace{\begin{pmatrix} \alpha-^F\hat{X}_g & 1+\beta \\ p(1+b) & a-^F\hat{Y}_g \end{pmatrix}}_{F_1(g)} = \begin{pmatrix} p(1+b) & a-^F\hat{Y}_g \\ p(\alpha-^F\hat{X}_g) & p(1+\beta) \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} a-^F\hat{Y}_g & 1+b \\ p(1+\beta) & \alpha-^F\hat{X}_g \end{pmatrix}}_{F_2(g)} \underbrace{\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}}_{\alpha(g)^{(\sigma)}} = \begin{pmatrix} p(1+b) & a-^F\hat{Y}_g \\ p(\alpha-^F\hat{X}_g) & p(1+\beta) \end{pmatrix}.$$

Hence $\alpha: \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of displays.

□

Theorem 5.0.9. *The morphism of displays*

$$\alpha: \mathcal{P}_1 \to \mathcal{P}_2$$

is universal with respect to the deformation of $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$.

Proof. By Theorems (3.2.2) and (2.2.2), Chapter (2), we know that a universal object

$$\alpha^{\mathrm{un}}: \mathcal{P}_1^{\mathrm{un}} \to \mathcal{P}_2^{\mathrm{un}}$$

exists over $R = \widehat{\mathcal{O}}_{\mathcal{M}_p,t}$. By universality there exists a unique morphism $\phi: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ such that

$$\phi^*(\mathcal{P}_1^{\mathrm{un}} \xrightarrow{\alpha^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}) \simeq \mathcal{P}_1 \xrightarrow{\alpha} P_2.$$

The argument for showing that ϕ is an isomorphism is identical to the proof of Theorem (2.3.2), Chapter (3). Here is a sketch in the current case. By the theory of local models there exist morphisms

$$\psi_1: \mathrm{Spec}(R) \to \mathrm{Spec}(R), \qquad \psi_2: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$$

such that

$$\psi_1 * (W_1^{\mathrm{un}} \subset \oplus_{i=1}^g R\alpha_i + R\beta_i \xrightarrow{h=\oplus_{i=1}^g h_i} W_2^{\mathrm{un}} \subset \oplus_{i=1}^g Ra_i + Rb_i) \simeq (H_{\mathcal{P}_1^{\mathrm{un}}} \subset D_{\mathcal{P}_1^{\mathrm{un}}} \xrightarrow{\overline{\alpha^{\mathrm{un}}}} H_{\mathcal{P}_2^{\mathrm{un}}} \subset D_{\mathcal{P}_2^{\mathrm{un}}}),$$

$$\psi_2 * (W_1^{\mathrm{un}} \subset \oplus_{i=1}^g R\alpha_i + R\beta_i \xrightarrow{h=\oplus_{i=1}^g h_i} W_2^{\mathrm{un}} \subset \oplus_{i=1}^g Ra_i + Rb_i) \simeq (H_{\mathcal{P}_1} \subset D_{\mathcal{P}_1} \xrightarrow{\overline{\alpha}} H_{\mathcal{P}_2} \subset D_{\mathcal{P}_2}).$$

The first isomorphism is horizontal modulo $\mathfrak{m}_R^2$. By Theorem (3.4.3), Chapter (2), we obtain the following picture

$$\widehat{\mathcal{P}_B} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\simeq} \widehat{\mathcal{P}_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow[\exists!]{\alpha\otimes\mathbb{W}(R/\mathfrak{m}_R^2)} \widehat{\mathcal{P}_2} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\simeq} \widehat{\mathcal{P}_A} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2)$$

$$\mathcal{P}_B \xrightarrow{\mathrm{id}} \mathcal{P}_B \xrightarrow{\mathbb{D}(f)} \mathcal{P}_A \xrightarrow{\mathrm{id}} \mathcal{P}_A.$$

Note that the Hodge filtrations $(H_{\mathcal{P}_1}, H_{\mathcal{P}_2})$ are identified in this diagram with the universal object $(W_1^{\mathrm{un}}, W_2^{\mathrm{un}})$ for the deformation of the Grassmann variety $\mathcal{N}_p$. Hence the induced isomorphism

$$\psi_2 * (W_1^{\mathrm{un}} \subset \oplus_{i=1}^g R\alpha_i + R\beta_i \xrightarrow{h=\oplus_{i=1}^g h_i} W_2^{\mathrm{un}} \subset \oplus_{i=1}^g Ra_i + Rb_i) \simeq (H_{\mathcal{P}_1} \subset D_{\mathcal{P}_1} \xrightarrow{\overline{\alpha}} H_{\mathcal{P}_2} \subset D_{\mathcal{P}_2}).$$

is horizontal modulo $\mathfrak{m}_R^2$.

Hence both ψ_1 and ψ_2 are isomorphisms on tangent spaces. By the factorization

$$\psi_1 \circ \phi = \psi_2$$

we get that also ϕ is an isomorphism on tangent spaces. By a commutative algebra argument we get that ϕ is an isomorphism (see the proof of Theorem (2.3.2), Chapter (3), for details).

□

6. Strata in the inert case

6.1. Recalling some notation introduced in the first part. The following definitions are due to Goren and Kassaei ([GK12]). Given an admissible pair (φ,η) as in (3), we are going to study those subsets of $\mathcal{M}_p$ whose geometric points have invariants (φ,η). The disjoint union of these sets is equal to the space $\mathcal{M}_p$ itself.

DEFINITION 6.1.1. *Let $\varphi,\eta\in\mathbb{B}$. Denote by U_φ the subset of $\mathcal{M}_p$ with geometric points $t=(\underline{A}\xrightarrow{f}\underline{B})\in\mathcal{M}_p(k)$ such that $\varphi(x)=\varphi$ and denote by V_η the subset whose geometric points $t=(\underline{A}\xrightarrow{f}\underline{B})\in\mathcal{M}_p(k)$ are such that $\eta(x)=\eta$.*

Denote by U_φ^+ the subset whose closed points $x\in\mathcal{M}_p(k)$ are such that $\varphi\subseteq\varphi(x)$, and by V_η^+ the subset whose closed points $x\in\mathcal{M}_p(k)$ are such that $\eta\subseteq\eta(x)$.

The following relations hold

$$U_\varphi^+=\bigcap_{\beta\in\varphi}U_{\{\beta\}}^+,\qquad V_\eta^+=\bigcap_{\beta\in\eta}V_{\{\beta\}}^+,\tag{6.1}$$

$$U_\varphi=U_\varphi^+-\bigcup_{\varphi\subsetneq\varphi'}U_{\varphi'}^+,\qquad V_\eta=V_\eta^+-\bigcup_{\eta\subsetneq\eta'}V_{\eta'}^+.\tag{6.2}$$

LEMMA 6.1.2. *Given $\varphi,\eta\in\mathbb{B}$, the sets U_φ^+ and V_η^+ are closed and the sets U_φ and V_η are locally closed.*

PROOF. Given that for $\beta\in\mathbb{B}$, the sets $U_{\{\beta\}}^+$ and $V_{\{\beta\}}^+$ are both the degeneracy loci of morphisms of line bundles, namely of

$$\mathrm{Lie}(f)_{\sigma\circ\beta}\colon\mathrm{Lie}(\underline{A}^{\mathrm{un}})_{\sigma\circ\beta}\to\mathrm{Lie}(\underline{B}^{\mathrm{un}})_{\sigma\circ\beta},\qquad \mathrm{Lie}(f^t)_\beta\colon\mathrm{Lie}(\underline{B}^{\mathrm{un}})_\beta\to\mathrm{Lie}(\underline{A}^{\mathrm{un}})_\beta$$

respectively, it follows that both U_β^+ and V_β^+ are closed. By (6.1), for $\varphi,\eta\in\mathbb{B}$ the sets U_φ^+ and V_η^+ are closed and by (6.2) the sets U_φ and V_η are locally closed. □

DEFINITION 6.1.3. *Let (φ,η) be an admissible pairs. Denote by $W_{(\varphi,\eta)}$ the subset of $\mathcal{M}_p$ whose geometric points have invariants (φ,η) and by $Z_{(\varphi,\eta)}$ subset of $\mathcal{M}_p$ whose geometric points $x=(\underline{A}\xrightarrow{f}\underline{B})\in\mathcal{M}_p(k)$ are such that $\varphi\subseteq\varphi(x)$ and $\eta\subseteq\eta(x)$.*

Note that

$$W_{(\varphi,\eta)}=U_\varphi\cap V_\eta,\qquad Z_{(\varphi,\eta)}=U_\varphi^+\cap V_\eta^+$$

and that

$$Z_{(\varphi,\eta)}=\bigcup_{(\varphi',\eta')\geq(\varphi,\eta)}W_{(\varphi',\eta')}.$$

We see therefore that the $Z_{(\varphi,\eta)}$'s are closed and that the $W_{(\varphi,\eta)}$ are locally closed.

Remark the points $x=(\underline{A}\xrightarrow{\mathrm{Fr}_A}\underline{B})$ where A is superspecial have invariants $(\varphi(x),\eta(x))=(\mathbb{B},\mathbb{B})$, that is, such points belong to any set $Z_{(\varphi,\eta)}$. It follows that for any admissible pair (φ,η) the set $Z_{(\varphi,\eta)}$ is not empty. This implies that also every set $W_{(\varphi,\eta)}$ is non-empty, since

$$Z_{(\varphi,\eta)}-W_{(\varphi,\eta)}=\bigcup_{(\varphi,\eta)\subset(\varphi',\eta')}Z_{(\varphi',\eta')}.\tag{6.3}$$

THEOREM 6.1.4 (Strata of $\mathcal{M}_p$). *Let (φ,η) be an admissible pair and let $I=\ell(\varphi)\cap\eta$.*

1. *The closed subset $Z_{(\varphi,\eta)}$ is non-empty, non-singular and of dimension*
$$\dim(Z_{(\varphi,\eta)})=2g-(|\varphi|+|\eta|).$$
2. *The closure of $W_{(\varphi,\eta)}$ is $Z_{(\varphi,\eta)}$ and the $W_{(\varphi,\eta)}$'s are a stratification of the space $\mathcal{M}_p$.*

7. Proof of Theorem (6.1.4)

Already for the moduli space $\mathcal{M}$ Goren and Oort used displays as a tool for giving a local description of the strata (see [GO00, Theorem 2.3.4]). We show here that this can be done also in the case with $\Gamma_0(p)$-level.We are going to describe a given stratum *around a given geometric point* $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p$. We recover the results obtained by Goren and Kassaei [GK12] with this new method.

By (6.1), it is enough to understand the local geometry of $U^+_{\{\beta\}}$ and $V^+_{\{\beta\}}$ for a given $\beta \in \mathbb{B}$.

Proposition 7.0.1. *Let $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$ be a geometric point with parameters $\varphi(x) = \varphi$ and $\eta(x) = \eta$ and let $\beta \in \mathbb{B}$.*

(1) *Assume $x \in U^+_{\{\beta\}}$; then locally around x, $U^+_{\{\beta\}}$ is equal to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{M}_p,x})$ if $\beta \notin r(I)$, while $U^+_{\{\beta\}}$ is given by the vanishing of the variable $X_{\sigma^{-1}\circ\beta}$ if $\beta \in r(I)$.*
(2) *Assume $x \in V^+_{\{\beta\}}$; then locally around x, $V^+_{\{\beta\}}$ is equal to $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{M}_p,x})$ if $\beta \notin I$, while $V^+_{\{\beta\}}$ is given by the vanishing of the variable Y_β if $\beta \in I$.*

Proof. Recall that by Lemma (4.2.1) the condition $\beta \in \varphi$ is equivalent to the condition

$$\mathrm{Ker}(\mathbb{D}(\overline{\mathrm{Fr}}_B(\sigma^{-1}\circ\beta))) = \mathrm{Ker}(\mathbb{D}(f)(\sigma^{-1}\circ\beta)), \tag{7.1}$$

while the condition $\beta \in \eta$ is equivalent to the condition

$$\mathrm{Ker}(\mathbb{D}(\overline{\mathrm{Fr}}_A(\beta))) = \mathrm{Ker}(\mathbb{D}(f^t)(\beta)). \tag{7.2}$$

(1) Note that by hypothesis $\beta \in \varphi$. Recall that $i \in r(I)$ if and only if $\sigma^{-1}\circ\beta \in I$, that is $\sigma^{-1}\circ\beta \in \eta$. By Lemma (??) when $\sigma^{-1}\circ\beta \in I$, the matrix of Frobenius F_1 of $\mathcal{P}_1$ is

$$F_1(\sigma^{-1}\circ\beta) = \begin{pmatrix} -{}^F\hat{X}_{\sigma^{-1}\circ\beta} & 1 \\ p & {}^F\hat{Y}_{\sigma^{-1}\circ\beta} \end{pmatrix},$$

while when $\sigma^{-1}\circ\beta \in \ell(\varphi) - I$, F_1 is

$$F_1(\sigma^{-1}\circ\beta) = \begin{pmatrix} p & 0 \\ -p{}^F\hat{Y}_{\sigma^{-1}\circ\beta} & 1 \end{pmatrix}.$$

Now, let us impose condition (7.1). Note that $\mathrm{Ker}(D(f)(\sigma^{-1}\circ\beta)) = \binom{1}{0}$. When $\beta \in r(I)$, we have that $\mathrm{Ker}(\overline{F}_1(\sigma^{-1}\circ\beta)) = \binom{1}{0}$ if and only if $X_{\sigma^{-1}\circ\beta} = 0$, while if $\beta \notin r(I)$ we have that $\mathrm{Ker}(\overline{F}_1(\sigma^{-1}\circ\beta)) = \binom{1}{0}$ without any additional condition.
(2) Since $x \in V^+_{\{\beta\}}$, we have that $\beta \in \eta$. When $\beta \in I$, the matrix of Frobenius of $\mathcal{P}_2$ is such that

$$F_2(\beta) = \begin{pmatrix} -{}^F\hat{Y}_\beta & 1 \\ p & {}^F\hat{X}_\beta \end{pmatrix},$$

while if $\beta \in \eta - I$, F_2 is such that

$$\begin{pmatrix} p & 0 \\ -p^F\hat{X}_\beta & 1 \end{pmatrix}.$$

Note that $\mathrm{Ker}(D(f^t)(\beta)) = \binom{1}{0}$. When $i \in I$, condition (7.2) is satisfied when $\mathrm{Ker}(\overline{F}_2(\beta)) = \binom{1}{0}$, that is, when $Y_\beta = 0$, while when $i \notin I$, condition (7.2) is automatically satisfied. Note that this holds also when $\beta = g$, since we are dealing with the reduction modulo p. □

COROLLARY 7.0.2. *Let $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$ with parameters (φ, η). Locally around x, U^+_φ is isomorphic to the closed subscheme defined by the ideal*

$$\langle \{X_\beta\} | \beta \in I \rangle,$$

while V^+_η is isomorphic to the closed subscheme defined by the ideal

$$\langle \{Y_\beta\} | \beta \in I \rangle.$$

The closed set $Z_{(\varphi,\eta)}$ is locally isomorphic to the closed subscheme described to the ideal

$$\langle \{X_\beta\}, \{Y_\beta\}, \beta \in I \rangle.$$

PROOF OF THEOREM (6.1.4), POINT (1). Recall that the local model (Section (**??**)) for $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_p(k)$ with parameters (φ, η) gives

$$\widehat{\mathcal{O}}_{\mathcal{M}_p,x} = k[\![\{X_\beta, \beta \in \ell(\varphi)\}, \{Y_\beta, \beta \in \eta\}]\!]/(\{X_\beta Y_\beta, \beta \in I\}),$$

and by Corollary (7.0.2)

$$\widehat{O}_{Z_{(\varphi,\eta)},x} \simeq \widehat{O}_{\mathcal{M}_p,x} / \langle \{X_\beta\}, \{Y_\beta\}, \beta \in I \rangle \simeq k[\![\{X_\beta, \beta \in \ell(\varphi) - I\}, \{Y_\beta, \beta \in \eta - I\}]\!].$$

From this we obtain that $\dim(Z_{(\varphi,\eta)}) = 2g - (|\varphi| + |\eta|)$ and that the $Z_{(\varphi,\eta)}$'s are pure-dimensional and non-singular. □

PROOF OF THEOREM (6.1.4), POINT (2). We have already observed that $W_{(\varphi,\eta)}$ is non-empty for any admissible pair (φ, η). Since $Z_{(\varphi,\eta)}$ contains $W_{(\varphi,\eta)}$ and since $Z_{(\varphi,\eta)}$ is closed, it contains also the closure $\overline{W_{(\varphi,\eta)}}$. By Corollary (7.0.2) we see that also $W_{(\varphi,\eta)}$ is pure of dimension $2g - \varphi - \eta$ and hence by dimension considerations we see that $\overline{W_{(\varphi,\eta)}} = Z_{(\varphi,\eta)}$. □

8. An example

CHAPTER 5

Totally ramified spaces

1. Hilbert modular varieties in the totally ramified case

Let $k = \overline{k}$ be a field of characteristic $p > 0$ and let L be a totally real number field of degree g. Assume moreover that the prime p is totally ramified, that is

$$p\mathcal{O}_L = \mathfrak{p}^g,$$

where $\mathfrak{p}$ is a prime ideal in the ring of integers $\mathcal{O}_L$ of L. We have that

$$\mathcal{O}_L \otimes W(k) \simeq W(k)[T]/(E(T)),$$

where $E(T)$ is an Eisenstein polynomial over $W(k)$ of degree g, and that

$$\mathcal{O}_L \otimes k \simeq k[T]/(T^g).$$

The moduli space $\mathcal{M}$ was studied extensively by Rapoport [Rap78], Deligne-Pappas [DP94] and Andreatta-Goren [AG03, AG04]. For $N \geq 4$ it is a scheme of dimension g. Most of the results by Deligne-Pappas and Andreatta-Goren hold on the definition of suitable invariants attached to the geometric k-points of the moduli space $\mathcal{M}$.

We define the *singularity index* of a given a geometric point $\underline{A} \in \mathcal{M}(k)$ as the integer $j = j(\underline{A})$ such that $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr})$ but $A[T^{j+1}] \nsubseteq \mathrm{Ker}(\mathrm{Fr})$. Note that $A[T^j]$ is a subgroup scheme of $A[p]$ of rank p^{2j}, and hence from the condition $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr})$ we obtain that $0 \leq 2j \leq g$.

The second fundamental invariant is the *slope* of a k-point $\underline{A} \in \mathcal{M}_{\mathfrak{p}}$, that is, the integer $n = n(\underline{A}) = a(\underline{A}) - j(\underline{A})$ where $a(\underline{A})$ denotes the a-number of A.

Lemma 1.0.1. *If $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr})$, then $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr}) \cap \mathrm{Ker}(\mathrm{Ver})$.*

Proof. I believe that it is enough to point out that $A[T]$ and hence all the subgroups $A[T^j]$ are self-dual, that is, if they are killed by Frobenius, then they are also killed by the Verschiebung. However, we may also give a more down to earth proof, involving the perfect $\mathcal{O}_L$-pairing.

Suppose that $x \in A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr})$. In order to show that $x \in \mathrm{Ker}(\mathrm{Ver})$, since the Weil pairing induced by the polarization is perfect, it is enough to show that $\langle u, Vx\rangle = 0$ for every u. But remark that $\langle u, Vx\rangle^\sigma = \langle Fu, x\rangle = \langle Fu, T^{g-j}x'\rangle$ for some x'. The last equality is true since $x \in A[T^j]$. since there is an exact sequence

$$0 \to A[T^{g-j}] \to A[p] \xrightarrow{\times T^{g-j}} A[T^j] \to 0.$$

Moreover the pairing is self-adjoint, that is it commutes with the $\mathcal{O}_L$-action, hence $\langle Fu, T^{g-j}x'\rangle = \langle FT^{g-j}u, x'\rangle = 0$, hence the conclusion. $\square$

The lemma tells us in particular that if $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr})$ then $A[T^j] \subseteq \alpha_p(A) = \mathrm{Ker}(\mathrm{Fr}) \cap \mathrm{Ker}(\mathrm{Ver})$. Since $\alpha_p(A)$ has rank $p^{a(A)}$ by definition, and $A[T^j]$ has rank p^{2j}, we conclude that $a \geq 2j$. Hence the slope tells us more or less how much "bigger" is $\alpha_p(A)$ with respect to $A[T^j]$; in particular $n \geq j$.

Example: The Hilbert modular surface.
Let $g = 2$. We may classify the points of the moduli space $\mathcal{M}$ according to their p-torsion of the underlying abelian varieties. In general, given $\underline{A} \in \mathcal{M}(k)$, the p-group scheme $A[p]$ has rank p^4 and we have the following possibilities:

- A is ordinary: it has a-number 0 and hence it has singularity index 0. One could also point out that by self-duality $A[T] \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ and hence $A[T] \notin \mathrm{Ker}(\mathrm{Fr}) = \mu_p^2$.
- A is supersingular, not superspecial: it has a-number 2 and $A[T]$ is isogenous to M^2 where M denotes the p-torsion of a supersingular elliptic curve. We have therefore that $A[T] \notin \mathrm{Ker}(\mathrm{Fr})$ and hence the singularity index of $\underline{A}$ is 0.
- A is supersingular, superspecial: it has a-number 2 and $A[T] \simeq \alpha_p^2$. It follows that $A[T] = \mathrm{Ker}(\mathrm{Fr})$ and hence the singularity index of $\underline{A}$ is 1.

2. The space with $\Gamma_0(\mathfrak{p})$-level structure

Denote by $\mathcal{M}_p = \mathcal{M}(k, \mathcal{O}_L, \mu_N, \Gamma_0(p))$ the Hilbert modular variety over k of dimension g and $\Gamma_0(p)$-level structure, that is, the fine moduli space parametrizing quadruples $(A/S, \iota, \lambda, \mu_N)$ as above, together with the choice of a finite flat $\mathcal{O}_L$-subgroup scheme H of $A[p]$ of rank p^g, isotropic with respect to the λ-Weil pairing.

Considering the moduli space $\mathcal{M}$ defined above gives rise to several moduli spaces parametrizing isogenies in the ramified case. Together with $\mathcal{M}_p$, we may consider the Hilbert modular variety $\mathcal{M}_{\mathfrak{p}} = \mathcal{M}(k, \mathcal{O}_L, \mu_N, \Gamma_0(\mathfrak{p}))$ over k with $\Gamma_0(\mathfrak{p})$-level structure. It parametrizes pairs $(\underline{A}, H)$, where H is an $\mathcal{O}_L$-invariant flat sub-group scheme of $A[T]$ of rank p, isotropic with respect to the Weil pairing.

Lemma 2.0.1 (Equivalency of moduli spaces). *The moduli space $\mathcal{M}_{\mathfrak{p}}$ can be seen equivalently as the moduli space of points of type $(\underline{A} \xrightarrow{f} \underline{B})$, where f is an $\mathcal{O}_L$-isogeny killed by p and of degree p, such that $f^*M(B) = \mathfrak{p} \cdot M(A)$. We denote by $f^t\colon \underline{B} \to \underline{A}$ the unique isogeny such that $f^t \circ f = [\mathfrak{p}]_A$ and $f \circ f^t = [\mathfrak{p}]_B$.*

Proof. We want to define a quadruple $\underline{B} = (B, \iota_B, \lambda_B, (\mu_N)_B)$. Take hence $B = A/H$: the real multiplication ι_A and level structure $(\mu_N)_A$ on A induce a real multiplication ι_B and $(\mu_N)_B$ on the quotient. Finally, we wish to define a polarization λ_B on B. Define the map

$$\begin{array}{rcl} f^*\colon M(B) & \to & M(A) \\ \delta & \mapsto & f^\vee \circ \delta \circ f. \end{array}$$

We want to describe a polarization on B. Put

$$\begin{array}{rcl} \lambda_B\colon M(B) & \xrightarrow{\simeq} & \mathfrak{I} \\ \delta & \mapsto & \frac{1}{\mathfrak{p}}\lambda_A \circ f^*(\delta) \end{array}$$

We need to show that this makes sense, namely that, as in the statement,

$$f^*(M(B)) = \mathfrak{p}M(A).$$

Consider the isogeny $A \xrightarrow{f} A/H$ and the unique isogeny $A/H \xrightarrow{f^t} A$ such that

$$f \circ f^t = [\mathfrak{p}]_{A/H}, \qquad f^t \circ f = [\mathfrak{p}]_A.$$

We have the following exact sequences

$$0 \longrightarrow H \longrightarrow A \xrightarrow{f} A/H \longrightarrow 0 \tag{2.1}$$

$$0 \longrightarrow A[T]/H \longrightarrow A/H \xrightarrow{f^t} A \longrightarrow 0 \tag{2.2}$$

By dualizing (2.2) we obtain

$$0 \longleftarrow (A[T]/H)^\vee \xleftarrow{(f^t)^\vee} A^\vee \longleftarrow (A/H)^\vee \longleftarrow 0\ .$$

By definition $H \subseteq A[T]$ is isotropic with respect to the Weil pairing that is, with respect to the γ-Weil pairing $A[p]\times A^\vee[p] \to \mu_p$ for any $\gamma \in M(A)$. Fix such a $\gamma \in M(A)$. From isotropy it follows in particular that $\gamma(H) \subseteq (A[T]/H)^\vee$ and we obtain hence a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "\gamma"'] & A/H \arrow[r] \arrow[d, "i\gamma"] & 0 \\ 0 \arrow[r] & (A[T]/H)^\vee \arrow[r] & A^\vee \arrow[r, "(f^t)^\vee"] & (A/H)^\vee \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

inducing the map $i\gamma\colon A/H \to (A/H)^\vee$ satisfying the relation

$$(f^t)^\vee \circ \gamma = i\gamma \circ f.$$

This defines an $\mathcal{O}_L$-linear morphism

$$i\colon M(A) \to M(B);$$

it is injective as $\deg(i\gamma) = \deg(\gamma)$. Note moreover that $i\gamma$ fits in the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\gamma"'] & A/H \arrow[d, "i\gamma"] \\ A^\vee & (A/H)^\vee. \arrow[l, "f^\vee"] \end{tikzcd}$$

We have therefore

$$f^*(i\gamma) := f^\vee \circ i\gamma \circ f = f^\vee \circ (f^t)^\vee \circ \gamma = (f^t \circ f)^\vee \circ \gamma = \mathfrak{p}\gamma.$$

This proves in particular that

$$\mathfrak{p}M(A) \subseteq f^* M(B).$$

We want now to prove the opposite inclusion. Consider once again the exact sequence (2.2)

$$0 \longrightarrow A[T]/H \longrightarrow A/H \xrightarrow{f^t} A \longrightarrow 0.$$

Note that $A[T]/H$ is isotropic with respect to the Weil pairing $A[p] \times A^\vee[p] \to \mu_p$. It is enough to show that it is isotropic with respect to the $i\gamma$-Weil paring, where $\gamma \in M(A)$, which is equivalent to the equality $i\gamma(A[T]/H) = H^\vee$. We have

$$i\gamma(A[T]/H) = i\gamma \circ f(A[T]) = (f^t)^\vee \circ \gamma(A[T]) = (f^t)^\vee(A^\vee[T]) = A^\vee/(A[T]/H)^\vee = H^\vee,$$

which proves the isotropy of $A[T]/H$. This means in particular that for any $\delta \in M(B)$, we have $\delta(A[T]/H) = H^\vee$. By dualizing the exact sequence (2.1) we obtain therefore a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A[T]/H \arrow[r] \arrow[d] & A/H \arrow[r, "f^t"] \arrow[d, "\delta"'] & A \arrow[r] \arrow[d, "j\delta"] & 0 \\ 0 \arrow[r] & H^\vee \arrow[r] & (A/H)^\vee \arrow[r, "f^\vee"] & A^\vee \arrow[r] & 0 \end{tikzcd}$$

which defines an $\mathcal{O}_L$-linear map

$$j\colon M(B) \to M(A)$$

satisfying the relations

$$f^\vee \circ \delta = j\delta \circ f^t$$

for any given $\delta \in M(B)$. Showing that $f^*(M(B)) \subseteq \mathfrak{p}M(A)$ is equivalent to showing that for $\delta \in M(B)$ we have

$$f^*\delta = \mathfrak{p}j\delta. \tag{2.4}$$

We claim that j satisfies the relation

$$(f^t)^* \circ j\delta = \mathfrak{p}\delta,$$

where $(f^t)^* \circ j\delta := (f^t)^\vee \circ j\delta \circ f^t$ for any $\delta \in M(B)$. Indeed

$$(f^t)^* \circ j\delta = (f^t)^\vee \circ j\delta \circ f^t = (f^t)^\vee \circ f^\vee \circ \delta = (f \circ f^t)^\vee \circ \delta = \mathfrak{p}\delta.$$

In order to show the equality in (2.4), it is enough to show that $(f^t)^*\mathfrak{p}j\delta = (f^t)^*f^*\delta$. We have that on the one hand

$$(f^t)^*\mathfrak{p}j\delta = \mathfrak{p}(f^t)^*k\delta = \mathfrak{p}^2\delta,$$

while

$$\mathfrak{p}^*\delta = \mathfrak{p}^2\delta,$$

which concludes the proof of the equality

$$f^*M(B) = M(A).$$

Note finally that the quadruple $\underline{B} = (B = A/H, \iota_B, \lambda_B, (\mu_N)_B)$ satisfies the Deligne-Pappas condition. Indeed, by [AG03, Proposition 3.1] it is enough to show that there exists an element $\delta \in M(B)$ of degree prime to ℓ for every prime ℓ different from p. Since $\underline{A}$ satisfies the Deligne-Pappas condition, there exists $\gamma \in M(A)$ of degree prime to ℓ for every $\ell \neq p$. Since $\deg(\gamma) = \deg(i\gamma)$, take $\delta = i\gamma$.

□

From the proof of Lemma (2.0.1) we see that given a point $(\underline{A}, H) = (\underline{A} \xrightarrow{f} \underline{B})$, there are several related points:

$$\begin{tikzcd} (\underline{A}, H) = (\underline{A} \xrightarrow{f} \underline{B}) \arrow[rr, "\omega"] \arrow[d] \arrow[drr, "\mathrm{inv}"] & & (\underline{B} = A/H, A[T]/H) = (\underline{B} \xrightarrow{f^t} \underline{A}) \arrow[d] \\ (\underline{A}^\vee, (A[T]/H)^\vee) = (\underline{A}^\vee \xrightarrow{(f^t)^\vee} \underline{B}^\vee) \arrow[rr, "\omega"'] & & (\underline{B}^\vee, H^\vee) = (\underline{B} \xrightarrow{f^\vee} A^\vee) \end{tikzcd} \tag{2.5}$$

Both ω and inv are involutions on the space $\mathcal{M}_\mathfrak{p}$ and ω goes under the name of *Atkin-Lehner involution.*

Moreover, the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\mathfrak{p}}$ are related by means of a forgetful functor

$$\begin{array}{rcl} \pi\colon \mathcal{M}_{\mathfrak{p}} & \longrightarrow & \mathcal{M} \\ (\underline{A}, H) & \longmapsto & \underline{A} \end{array}$$

Example: The fibers of π on the modular surface.
Let $\underline{A} \in \mathcal{M}(k)$ be a geometric point. We may understand the fiber $\pi^{-1}(\underline{A})$ by looking at the p-torsion $A[p]$.

- A is ordinary: $A[T] \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ and we have therefore two choices for $H \subseteq A[T]$ of rank p:

$$H = \mu_p$$
$$H = \mathbb{Z}/p\mathbb{Z}.$$

- A has étale part of order p: $A[T] \simeq E[p]$ hence we have one choice for H:

$$H = \alpha_p.$$

- A supersingular, not superspecial. Then there is only one subgroup $H \subseteq A[T]$ of order p and it is isomorphic to α_p.
- A supersingular, superspecial: $A[T] \simeq \alpha_p^2$ and hence H is parametrized by $\mathbb{P}^1$; this is explained in [AG03, Prop. 8.7].

We can look at the forgetful functor in terms of the interpretation of the points of $\mathcal{M}_{\mathfrak{p}}$ as p-isogenies of degree p. We will use the following notation:

$$\begin{array}{rcl} \mathcal{M}_{\mathfrak{p}} & \xrightarrow{\pi_1} & \mathcal{M} \\ (\underline{A} \xrightarrow{f} \underline{B}) & \longmapsto & \underline{A} \end{array} \qquad \begin{array}{rcl} \mathcal{M}_{\mathfrak{p}} & \xrightarrow{\pi_2} & \mathcal{M} \\ (\underline{A} \xrightarrow{f} \underline{B}) & \longmapsto & \underline{B} \end{array}$$

3. The local model in the totally ramified case

The results obtained in this section were obtained by Pappas, see in particular [Pap95, Lemma 4.3.1, Lemma 4.3.2].

3.1. From geometry to linear algebra. Given a geometric point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$, we may consider the first de Rham cohomology groups $H^1_{dR}(A/k)$ and $H^1_{dR}(B/k)$ of the associated abelian varieties A and B. Recall that by (2.1), Chapter (2), they are related to the p-torsion subgroups by means of Dieudonné theory

$$H^1_{dR}(A/k) \simeq \mathbb{D}(A[p]), \qquad H^1_{dR}(B/k) \simeq \mathbb{D}(B[p]),$$

and they are k-vector spaces of dimension $2g$. Recall moreover that for a RM-abelian scheme A over k, the first de Rham cohomology group is a free $\mathcal{O}_L \otimes_{\mathbb{Z}} k$-module of rank 2 (see Lemma (3.4.6), Chapter (2), for details. We may as well consider the Hodge filtrations

$$H^0(A, \Omega^1_{A/k}) \subseteq H^1_{dR}(A/k), \qquad H^0(B, \Omega^1_{B/k}) \subseteq H^1_{dR}(B/k);$$

as k-vector spaces they have dimension g. Note that p being ramified, Rapoport's condition (R) is not equivalent to Deligne-Pappas' condition (DP). The global differentials $H^0(A, \Omega^1_{A/k})$ and $H^0(B, \Omega^1_{B/k})$ are in general not free over $\mathcal{O}_L \otimes k$.

By functoriality the isogenies $f\colon A \to B$ and $f^t\colon B \to A$ induce morphisms

$$\mathbb{D}(f)\colon H^1_{dR}(B/k) \to H^1_{dR}(A/k), \qquad \mathbb{D}(f^t)\colon H^1_{dR}(A/k) \to H^1_{dR}(B/k)$$

such that

$$\mathbb{D}(f)\circ\mathbb{D}(f^t) = [T]_{H^1_{dR}(A/k)}, \qquad \mathbb{D}(f^t)\circ\mathbb{D}(f) = [T]_{H^1_{dR}(B/k)},$$

and that preserve the Hodge filtrations, that is

$$\mathbb{D}(f)(H^0(B,\Omega^1_{B/k})) \subseteq H^0(A,\Omega^1_{A/k}), \qquad \mathbb{D}(f^t)(H^0(A,\Omega^1_{A/k})) \subseteq H^0(B,\Omega^1_{B/k}).$$

LEMMA 3.1.1. *There exist $\mathcal{O}_L\otimes k$-bases $\{e_1,e_2\}$ and $\{e'_1,e'_2\}$ for $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$ respectively, such that the morphisms $\mathbb{D}(f)\colon H^1_{dR}(B/k)\to H^1_{dR}(A/k)$ and $\mathbb{D}(f)\colon H^1_{dR}(A/k)\to H^1_{dR}(B/k)$ are represented by the matrices*

$$\mathbb{D}(f) = \begin{pmatrix}1 & 0\\ 0 & T\end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix}T & 0\\ 0 & 1\end{pmatrix}.$$

PROOF. We shall use extensively the identifications

$$\mathbb{D}(A[p]) \simeq H^1_{dR}(A/k), \qquad \mathbb{D}(B[p]) \simeq H^1_{dR}(B/k).$$

Consider the inclusion $A[T]\subseteq A[p]$ (resp. $B[T]\subseteq B[p]$). After applying the contravariant Dieudonné functor there are surjections

$$\mathbb{D}(A[p]) \twoheadrightarrow \mathbb{D}(A[T]), \qquad \mathbb{D}(B[p]) \twoheadrightarrow \mathbb{D}(B[T]),$$

the surjectivity coming from the identifications

$$\mathbb{D}(A[T]) \simeq \mathbb{D}(A[p])/T\mathbb{D}(A[p]), \qquad \mathbb{D}(B[T]) \simeq \mathbb{D}(B[p])/T\mathbb{D}(B[p]).$$

Note that $\mathbb{D}(B[T])$ and $\mathbb{D}(A[T])$ are k-vector spaces of dimension 2. Moreover, the morphisms $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ respect $\mathbb{D}(A[T])$ and $\mathbb{D}(B[T])$, that is

$$\mathbb{D}(f)(\mathbb{D}(B[T]) \subseteq \mathbb{D}(A[T]), \qquad \mathbb{D}(f^t)(\mathbb{D}(A[T])) \subseteq \mathbb{D}(B[T]).$$

We may therefore consider the commutative diagram

$$\begin{tikzcd}
\mathbb{D}(B[p]) \arrow[r, "\mathbb{D}(f)"] \arrow[d, two heads] & \mathbb{D}(A[p]) \arrow[d, two heads] \\
\mathbb{D}(B[T]) \arrow[r, "\mathbb{D}(f)"] & \mathbb{D}(A[T])
\end{tikzcd}$$

We want to construct suitable k-bases for $\mathbb{D}(B[T])$ and $\mathbb{D}(A[T])$. From the condition

$$\mathbb{D}(f^t)\circ\mathbb{D}(f) = [T]_{\mathbb{D}(B[p])} \tag{3.1}$$

we obtain that the morphism $\mathbb{D}(f^t)\circ\mathbb{D}(f)\colon\mathbb{D}(B[T])\to\mathbb{D}(A[T])$ is always represented by the zero matrix

$$\mathbb{D}(f^t)\circ\mathbb{D}(f) = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Hence $\mathrm{Ker}(\mathbb{D}(f))$ has k-dimension 1 in $\mathbb{D}(B[T])$ and $\mathrm{Im}(\mathbb{D}(f))$ has k-dimension 1 in $\mathbb{D}(A[T])$ (and similarly for $\mathbb{D}(f^t)$). Choose hence bases $\{e_1,e_2\}$ and $\{e'_1,e'_2\}$ for $\mathbb{D}(B[T])$ and $\mathbb{D}(A[T])$ respectively such that

$$e'_1 = \mathbb{D}(f)(e_1), \qquad e_2 = \mathbb{D}(f^t)(e'_2). \tag{3.2}$$

In these two bases $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ are represented by

$$\mathbb{D}(f) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Recall that $\mathbb{D}(B[p])$ and $\mathbb{D}(A[p])$ are free of rank two over $k[T]/(T^g)$, which is a local algebra with residue field k. Lift now e_1 to $\tilde{e}_1 \in (\mathcal{O}_L \otimes k)^2$ and e_2' to $\tilde{e}_2' \in (\mathcal{O}_L \otimes k)^2$. Define now

$$\tilde{e}_1' = \mathbb{D}(f)(\tilde{e}_1), \qquad \tilde{e}_2 = \mathbb{D}(f^t)(\tilde{e}_2').$$

Note that $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{e}_1', \tilde{e}_2'\}$ reduce to $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ modulo T and hence by Nakayama's Lemma they are $\mathcal{O}_L \otimes k$-bases for $\mathbb{D}(B[p])$ and $\mathbb{D}(A[p])$ respectively. By abuse of notation we will denote by $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ the $\mathcal{O}_L \otimes k$-bases as well. From condition (3.1) we obtain in particular that

$$\mathbb{D}(f^t) \circ \mathbb{D}(f)(e_1) = Te_1, \qquad \mathbb{D}(f) \circ \mathbb{D}(f^t)(e_2') = Te_2'.$$

Combining this with (3.2) we obtain that

$$\mathbb{D}(f^t)(e_1') = Te_1, \qquad \mathbb{D}(f)(e_2) = Te_2',$$

which concludes the proof. □

Corollary 3.1.2. *In the bases obtained in Lemma* (3.1.1)*, the Hodge filtrations* $H^0(B, \Omega^1_{B/k}) \subseteq H^1_{dR}(B/k)$ *and* $H^0(A, \Omega^1_{A/k}) \subseteq H^1_{dR}(A/k)$ *are represented by direct sums*

$$H^0(B, \Omega^1_{B/k}) = T^a e_1 \oplus T^b e_2, \quad H^0(A, \Omega^1_{A/k}) = T^{a'} e_1' \oplus T^{b'} e_2',$$

with $a + b = g = a' + b' = g$*. Moreover, the integers* a, b, a', b' *satisfy the following relations*

Case 1: $a = a'$ *and* $b = b'$,

Case 2: $a = a' + 1$ *and* $b = b' - 1$.

Proof. The Hodge filtration, as a k-vector space, has dimension p^g, we conclude hence by the elementary divisors theorem.

Recall moreover that the maps $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ respect the Hodge filtrations, that is

$$\mathbb{D}(f)(H^0(B, \Omega^1_{B/k})) \subseteq H^0(A, \Omega^1_{A/k}), \qquad \mathbb{D}(f^t)(H^0(A, \Omega^1_{A/k})) \subseteq H^0(B, \Omega^1_{B/k}),$$

hence in the bases obtained in Lemma (3.1.1)

$$\mathbb{D}(f)(T^a e_1 \oplus T^b e_2) \subseteq T^{a'} e_1' \oplus T^{b'} e_2', \qquad \mathbb{D}(f^t)(T^{a'} e_1' \oplus T^{b'} e_2') \subseteq T^a e_1 \oplus T^b e_2.$$

In particular, since

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix},$$

one gets relations

$$\begin{aligned}
\mathbb{D}(f)(T^a e_1) = T^a e_1' \subseteq T^{a'} e_1' \oplus T^{b'} e_2' \qquad &\Rightarrow \qquad a \geq a', \\
\mathbb{D}(f)(T^b e_2) = T^{b+1} e_2' \subseteq T^{a'} e_1' \oplus T^{b'} e_2' \qquad &\Rightarrow \qquad b + 1 \geq b', \\
\mathbb{D}(f^t)(T^{a'} e_1') = T^{a'+1} e_1 \subseteq T^a e_1 \oplus T^b e_2 \qquad &\Rightarrow \qquad a' + 1 \geq a, \\
\mathbb{D}(f^t)(T^{b'} e_2') = T^{b'} e_2 \subseteq T^a e_1 \oplus T^b e_2 \qquad &\Rightarrow \qquad b' \geq b.
\end{aligned}$$

That is, we have $a' \leq a \leq a' + 1$ and $b \leq b' \leq b + 1$. Given the conditions $a + b = g$ and $a' + b' = g$ we get the possibilities

(1) $a = a'$ and $b = b'$,
(2) $a = a' + 1$ and $b = b' - 1$,

which concludes the proof. □

Remark 7. *With the aim of relating the relations obtained in Corollary* (3.1.2) *to the geometry of the moduli space* $\mathcal{M}_{\mathfrak{p}}$, *let us add a few comments. Let* $j = \min(a,b)$ *and* $j' = \min(a',b')$. *We will moreover denote* $i = g - j$ *and* $i' = g - j'$. *We wish to have bases such that the Hodge filtrations are represented by*

$$H^0(A,\Omega^1_{A/k}) = T^{i'}e'_1 \oplus T^{j'}e'_2, \qquad H^0(B,\Omega^1_{B/k}) = T^{i}e_1 \oplus T^{j}e_2,$$

given that Andreatta-Goren use this convention in [AG03, AG04] when describing the related moduli space $\mathcal{M}$. *We have the following four cases*

Case 1: $j = b = j' = b'$. *Then without any effort we obtain the desired bases with* $\mathbb{D}(f)$ *represented by*

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

Case 2: $j = a = j' = a'$. *We obtain the desired bases by inverting the order of the bases obtained in Lemma* (3.1.1). *The map* $\mathbb{D}(f)$ *is represented by*

$$\mathbb{D}(f) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 3: $j = a$ *and* $a' = a - 1 = j'$. *We invert* e_1 *with* e_2 *and* e'_1 *with* w'_2. *The map* $\mathbb{D}(f)$ *is represented by*

$$\mathbb{D}(f) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 4: $j' = b'$ *and* $j = b = b' - 1$. *We have to change nothing and*

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

Such a translation of the data attached to geometric points in terms of linear algebra allows us to give a description of the local geometry of $\mathcal{M}_{\mathfrak{p}}$ by means of the *theory of local models*.

Define the Grassmann variety $\mathcal{N}_{\mathfrak{p}}$ of pairs of $\mathcal{O}_L \otimes k$-subspaces (F,F') of $(k[T]/(T^g))^2$ of k-dimension g, which are isotropic (i.e. equal to their own orthogonal) with respect to the standard $k[T]/(T^g)$-linear alternating form and such that, given the map

$$\begin{array}{rcl} u : (k[T]/(T^g))^2 & \longrightarrow & (k[T]/(T^g))^2 \\ (x,y) & \longmapsto & (x,Ty) \end{array}$$

we have that $u(F) \subset F'$ and $u^t(F') \subset F$.

As in Lemma (3.1.1), by choosing suitable $k[T]/(T^g)$-bases $\{e_1,e_2\}$ and $\{e'_1,e'_2\}$ for $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$, we may associate an element $(F,F') \in \mathcal{N}_{\mathfrak{p}}$ to a k-point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}$. Take namely $(F,F') = (H^0(B,\Omega^1_{B/k}), H^0(A,\Omega^1_{A/k}))$ and $u = \mathbb{D}(f)$, $u^t = \mathbb{D}(f^t)$. Moreover, by Corollary (3.1.2) $F \subseteq (k[T]/(T^g))^2$ and $F' \subseteq (k[T]/(T^g))^2$ are represented by

$$F = T^a e_1 \oplus T^b e_2, \qquad F' = T^{a'}e'_1 \oplus T^{b'}e'_2,$$

with $a + b = g$ and $a' + b' = g$.

The reasoning above leads to the construction of a functor

$$
\begin{array}{ccc}
U \subset \mathcal{M}_{\mathfrak{p}} & \longrightarrow & \mathcal{N}_{\mathfrak{p}} \\
(\underline{A} \xrightarrow{f} \underline{B}) & \longmapsto & (F, F')
\end{array}
\tag{3.3}
$$

where U denotes an open Zariski neighbourhood $\mathcal{M}_{\mathfrak{p}}$ containing $t = (\underline{A} \xrightarrow{f} \underline{B})$. By crystalline theory we may relate the local geometry of $\mathcal{M}_{\mathfrak{p}}$ and $\mathcal{N}_{\mathfrak{p}}$ (see [DP94] and [dJ93]) for detail.

Proposition 3.1.3. *The morphism defined in* (3.1.3) *is étale, that is, there is an isomorphism of completed local rings*

$$\widehat{\mathcal{O}}_{\mathcal{N}_{\mathfrak{p}},(F,F')} \simeq \widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},t}.$$

In order to understand the completed local ring of the moduli space $\mathcal{M}_{\mathfrak{p}}$ at a geometric k-point it is therefore enough to compute the completed local ring at the corresponding point of the Grassmannian $\mathcal{N}_{\mathfrak{p}}$.

Note 3. *As pointed out in Remark* (7)*, given a geometric k-point $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}$ we wish to represented the $k[T]/(T^g)$-submodules $H^0(B, \Omega^1_{B/k})$ and $H^0(A, \Omega^1_{A/k})$ with $k[T]/(T^g)$-bases $\{e_1, e_2\}$ and $\{e'_1 e'_2\}$ such that*

$$H^0(B, \Omega^1_{B/k}) = T^i e_1 \oplus T^j e_2, \qquad H^0(A, \Omega^1_{A/k}) = T^{i'} e'_1 \oplus T^{j'} e'_2,$$

with $j \le i$ and $j' \le i'$. This corresponds to the RM-abelian schemes $\underline{A}$ and $\underline{B}$ having singularity indices j' and j respectively. From now on we shall analyse the two cases:

1. *$j(\underline{A}) = j(\underline{B})$, taking implicitly into account Case 1 (as in Remark* (7)*). Given the associated point $(F, F') \in \mathcal{N}_{\mathfrak{p}}$, the subspace F is generated by $(f, u^t(f'))$ and F' is generated by $(u(f), f')$ with $f = T^i e_1$ and $f' = T^{j'} e'_2$. Moreover*
$$u = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \qquad u^t = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$
By applying Atkin-Lehner to Case 1 we obtain Case 2, hence slightly different from the computational point of view, but identical in essence;
2. *$j(\underline{A}) = j(\underline{B}) - 1$, taking into account Case 3. Given the associated point $(F, F') \in \mathcal{N}_{\mathfrak{p}}$, the subspace F is generated by $(f, u^t(f'))$ and F' is generated by $(u(f), f')$ with $f = T^i e_1$ and $f' = T^{j'} e'_2$. Moreover*
$$u = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}, \qquad u^t = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$
Case 4 is the Atkin-Lehner image of Case 3.

From these considerations one sees moreover that the singularity indices of $\underline{A}$ and $\underline{B}$ are related. A full description of the relation between the singularity indices of $\underline{A}$ and $\underline{B}$ can by found in [AG04, Table 8.1].

4. The completed local ring at a geometric point

By Proposition (3.1.3), in order to compute the completed local ring $\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},t}$ at a geometric k-point $t = (\underline{A} \xrightarrow{f} \underline{B})$ it is enough to compute the completed local ring of the Grassmann variety $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathfrak{p}},(F,F')}$ at the associated point (F, F'). We are going to divide the computation into the cases

(1) $j(\underline{A}) = j(\underline{B})$,
(2) $j(\underline{A}) = j(\underline{B}) - 1$.

4.1. The local model at $t = (\underline{A} \to \underline{B})$ when $j(\underline{A}) = j(\underline{B})$. Put $j = j(\underline{B}) = j(\underline{A})$ and denote by (F, F') the point in $\mathcal{N}_\mathfrak{p}$ associated through the morphism defined in (3.3). In the bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ constructed in Note (3), F and F' are generated by $(f, u^t(f')) = (T^i e_1, T^j e_2)$ and $(u(f), u^t(f')) = (T^i e'_1, T^j e'_2)$, where $i = g - j$, and the maps u and u^t are represented by

$$u = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \qquad u^t = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the $\mathcal{O}_L$-deformations of $(F, F') \in \mathcal{N}_\mathfrak{p}$

$$\tilde{f} = T^i e_1 + a e_1 + b e_2, \qquad \tilde{f}' = T^j e'_2 + c e'_1 + d e'_2,$$
$$u^t(\tilde{f}') = T^j e_2 + cT e_1 + d e_2, \qquad u(\tilde{f}) = T^i e'_1 + a e'_1 + bT e'_2,$$
$$a = \sum_{s=0}^{i-1} a_s T^s,\ \ b = \sum_{s=0}^{j-1} b_s T^s,\ \ c = \sum_{s=0}^{i-1} c_s T^s,\ \ d = \sum_{s=0}^{j-1} d_s T^s.$$

By isotropy of F and F'

$$u(f) \wedge f' = 0 = f \wedge u^t(f'),$$

we obtain the condition

$$aT^j + dT^i + ad - bcT = 0.$$

We obtain therefore that the completed local ring at $t = (\underline{A} \to \underline{B})$ is

$$\widehat{\mathcal{O}}_{\mathcal{M}_\mathfrak{p}, t} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_\mathfrak{p}, (F,F')} = k[T]/(T^g)[\![a_0, \dots, a_{i-1}, b_0, \dots, b_{j-1}, c_0, \dots, c_{i-1}, d_0, \dots, d_{j-1}]\!]$$
$$/(aT^j + dT^i + ad - bcT).$$

Example: Hilbert modular surface, $g = 2$.

- Ordinary points: $t = (\underline{A} \to \underline{B})$ with
$$j(\underline{A}) = j(\underline{B}) = 0.$$
Both F and F' in the associated pair in $\mathcal{N}_\mathfrak{p}$ are free and we have
$$a = a_0 + a_1 T, \qquad b = 0, \qquad c = c_0 + c_1 T, \qquad d = 0,$$
and from the condition
$$aT^0 + dT^2 + ad - bcT = a = 0$$
we obtain that $a_0 = 0$ and $a_1 = 0$. Therefore the completed local ring at t is isomorphic to
$$k[\![c_0, c_1]\!].$$
- Superspecial points: $t = (\underline{A} \to \underline{B})$ such that
$$j(\underline{A}) = 1 = j(\underline{B}).$$
It follows that in the notation above
$$a = a_0, b = b_0, c = c_0, d = d_0,$$
and hence the condition
$$aT + dT + ad - bcT = 0$$

we get relations $a_0 d_0 = 0$ and $d_0 + a_0 - b_0 c_0$ and hence we have that the completed local ring is isomorphic to

$$k[\![a_0, b_0, c_0]\!]/(a_0 b_0 c_0 - a_0^2).$$

Example: Case when $g = 3$.

- Ordinary points: $t = (\underline{A} \to \underline{B})$ with

$$j(\underline{A}) = 0 = j(\underline{B}).$$

The completed local ring at t is isomorphic to

$$k[\![c_0, c_1, c_2]\!].$$

In fact, it is easy to see that for any dimension g, the completed local ring at an ordinary point is isomorphic to

$$k[\![c_0, c_1, \ldots, c_{g-1}]\!].$$

- Points $t = (\underline{A} \to \underline{B})$ such that

$$j(\underline{A}) = 1 = j(\underline{B}).$$

We have that

$$a = a_0 + a_1 T, \qquad b = b_0, \qquad c = c_0 + c_1 T, \qquad d = d_0.$$

The condition

$$aT + dT^2 + ad - bcT$$

gives the relations $-a_1 + b_0 c_1 = d_0$, $a_0 d_0 = 0$ and $a_0 = b_0 c_0 - a_1 d_0$. We conclude therefore that the completed local ring at t is isomorphic to

$$k[\![a_1, b_0, c_0, c_1]\!]/((b_0 c_1 - a_1)(b_0 c_0 - a_1 b_0 c_1 - a_1^2)).$$

4.2. The local model at $t = (\underline{A} \to \underline{B})$ when $j(\underline{A}) = j(\underline{B}) - 1$. Denote by $j = j(\underline{B})$ and $j' = j(\underline{A}) = j - 1$. We will use the notation $i = g - j$ and $i' = g - j'$. By Note (3) there exist bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ such that $(f, u^t(f')) = (T^i e_1, T^j e_2)$ and $(u(f'), f') = (T^{i'} e_1', T^{j'} e_2')$ generate F and F' respectively and in such bases

$$u = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}, \qquad {}^t u = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

The $\mathcal{O}_L$-deformations of (F, F') in $\mathcal{N}_{\mathfrak{p}}$ are

$$\tilde{f} = T^i e_1 + a e_1 + b e_2, \qquad \tilde{f}' = T^{j'} e_2' + c e_1' + d e_2',$$
$$u^t(\tilde{f}') = T^j e_2 + c e_1 + dT e_2, \qquad u(\tilde{f}) = T^{i'} e_1' + aT e_1' + b e_2',$$

with

$$a = \sum_{s=0}^{i-1} a_s T^s,\ b = \sum_{s=0}^{j-1} b_s T^s,\ c = \sum_{s=0}^{i'-1} c_s T^s,\ d = \sum_{s=0}^{j'-1} d_s T^s.$$

From the isotropy condition of F and F' we obtain that the completed local ring of $\mathcal{M}_{\mathfrak{p}}$ at t is isomorphic to

$$\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}}, t} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_{\mathfrak{p}}, (F, F')} = k[T]/(T^g)[\![a_0, \ldots, a_{i-1}, b_0, \ldots, b_{j-1}, c_0, \ldots, c_{i'-1}, d_0, \ldots, d_{j'-1}]\!]$$
$$/(dT^{i'} + aT^j + adT - bc).$$

Example: Hilbert modular surface, $g = 2$. Consider $t = (\underline{A} \to \underline{B})$ such that $j(\underline{B}) = 1$ and $j(\underline{A}) = 0$. In this case

$$a = a_0, \qquad b = b_0, \qquad c = c_0 + c_1 T, \qquad d_0,$$

and hence the condition

$$aT - bc = 0$$

gives the relations $a_0 = b_0 c_1$ and $b_0 c_0 = 0$. The completed local ring at t is therefore isomorphic to

$$k[\![a_0, b_0, c_0, c_1]\!]/(a_0 - b_0 c_1, b_0 c_0).$$

This tells us that locally around t there are two irreducible components.

Example: Case of dimension $g = 3$. Consider $t = (\underline{A} \to \underline{B})$ with $j' = j(\underline{A}) = 0$ and $j = j(\underline{B}) = 1$. We have

$$a = a_0 + a_1 T, \qquad b = b_0, \qquad c = c_0 + c_1 T + c_2 T, \qquad d_0 = 0.$$

The condition

$$aT - bc = 0$$

yields to relations $b_0 c_0 = 0$, $a_0 = b_0 c_1$ and $a_1 = b_0 c_2$ and therefore we get that the completed local ring at t is isomorphic to

$$k[\![a_0, a_1, b_0, c_0, c_1, c_2]\!]/(b_0 c_0, a_0 - b_0 c_1, a_1 - b_0 c_2).$$

Hence locally around t there are three irreducible components.

In order to understand the local structure of points of Case 2 and Case 4 as in Remark (7), we have the following result.

LEMMA 4.2.1. *Let $\omega \colon \mathcal{M}_{\mathfrak{p}} \to \mathcal{M}_{\mathfrak{p}}$ be the Atkin-Lehner involution. Given a geometric point $x \in \mathcal{M}_{\mathfrak{p}}(k)$ we have that*

$$\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}}, \omega(x)} = \widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}}, x}.$$

PROOF. Let $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ be the isogeny associated to x. Its image $\omega(x)$ is represented by $(\underline{B} \xrightarrow{f^t} \underline{A}) \in \mathcal{M}_{\mathfrak{p}}(k)$. In terms of the associated points of the Grassmannian $\mathcal{N}_{\mathfrak{p}}$, the Atkin-Lehner automorphism ω maps $(F, F') = (H^0(B, \Omega^1_{B/k}), H^0(A, \Omega^1_{A/k}))$ to $(F', F) = (H^0(A, \Omega^1_{A/k}), H^0(B, \Omega^1_{B/k}))$. Hence in terms of deformations ω maps $(\widetilde{F}, \widetilde{F'})$ to $(\widetilde{F'}, \widetilde{F})$, with $\widetilde{F}$ generated by $(\tilde{f}, u^t(\tilde{f}'))$ and $\widetilde{F'}$ generated by $(u(\tilde{f}), \tilde{f}')$. Since the automorphy conditions

$$u(\tilde{f}) \wedge \tilde{f}' = 0 \text{ and } \tilde{f} \wedge {}^t u(\tilde{f}') = 0$$

are equivalent, we conclude that the completed local rings are the same. □

5. Stratifications

A stratification is a description of a moduli space as a disjoint union of locally closed subsets. Strata are defined by taking into account invariants of the geometric points on the space.

Deligne-Pappas [DP94] and Andreatta-Goren [AG03, AG04] described stratifications on the Hilbert modular variety $\mathcal{M}$ in positive ramified characteristic by looking at invariants arising from the p-torsion of the abelian varieties associated to the geometric points of $\mathcal{M}$. Let us recall some of the results they obtained.

In Section (1), Chapter (2), we defined the *singularity index* of a geometric point $\underline{A} \in \mathcal{M}(k)$ as the integer $j = j(\underline{A})$ such that $A[T^j] \subseteq \mathrm{Ker}(\mathrm{Fr}_A)$, but $A[T^{j+1}] \notin \mathrm{Ker}(\mathrm{Fr}_A)$. Equivalently, it can be defined by looking at the Hodge filtration of the de Rham cohomology, as follows: there exists a $k[T]/(T^g)$-basis $\{\eta_1, \eta_2\}$ of $H^1_{dR}(A/k)$ such that

$$H^0(A, \Omega^1_{A/k}) = T^{g-j(A)}\eta_1 \oplus T^j \eta_2,$$

with i, j integers such that $i + j = g$ and $j \leq i$.

Denote by S_j the subset of $\mathcal{M}$ whose geometric points $\underline{A}$ have singularity index $j(\underline{A}) \geq j$. The S_j's are closed subsets and

$$\mathcal{M} = \bigcup_j S_j.$$

Deligne and Pappas proved in particular that the S_j's are pure of dimension $g - 2j$ and that for every positive j, $S_j - S_{j+1}$ is non-singular.

With the aim of relating such strata to the Newton polygon, Andreatta and Goren [AG03, AG04] define a finer stratification by considering a second invariant of a geometric point $\underline{A} \in \mathcal{M}(k)$, the slope $n(A)$, that is, the difference between the a-number and the singularity index of A:

$$n(A) = a(A) - j(A).$$

As shown in Lemma (1.0.1), the singularity index and the slope of A satisfy the following inequality

$$0 \leq j(A) \leq n(A) \leq i(A) = g - j(A).$$

It will be convenient to set the following notation introduced in [AG03]

$$J = \{(j, n) | 0 \leq j \leq n \leq g - j; j, n \in \mathbb{Z}\}.$$

Given a pair $(j, n) \in J$, denote by $W_{(j,n)}$ the locally closed subset S_j whose geometric points are the points $\underline{A} \in \mathcal{M}(k)$ such that $j(A) = j$ and $n(A) = n$. Andreatta and Goren obtain most of their results by studying carefully the two projections

$$\begin{array}{ccccccc} \mathcal{M}_{\mathfrak{p}} & \xrightarrow{\pi_1} & \mathcal{M}, & \qquad & \mathcal{M}_{\mathfrak{p}} & \xrightarrow{\pi_2} & \mathcal{M} \\ (\underline{A} \xrightarrow{f} \underline{B}) & \longmapsto & \underline{A} & & (\underline{A} \xrightarrow{f} \underline{B}) & \longmapsto & \underline{B} \end{array}$$

The following statement provides the most meaningful properties of the $W_{(j,n)}$'s. See [AG03, Theorem 10.1] for details.

THEOREM 5.0.1. *The $W_{(j,n)}$'s are non-empty and non-singular and* $\dim(W_{(j,n)}) = g - (j + n)$. *Moreover, every irreducible component of $\mathcal{M}$ contains a point of $W_{(j,n)}$. The $W_{(j,n)}$'s determine a stratification of the space $\mathcal{M}$.*

The results are obtained by studying the local model of $\mathcal{M}$ and by looking at the strata *close to* a geometric point.

5.1. The singularity stratification of $\mathcal{M}_{\mathfrak{p}}$. As already seen in Section (3) while studying the local model of $\mathcal{M}_{\mathfrak{p}}$, given a geometric point $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}$ we may consider the singularity indices of both $\underline{A}$ and $\underline{B}$ and define the pair $(j(A), j(B))$. By Note (3), in terms of linear algebra this corresponds to the existence of $k[T]/(T^g)$-bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ for $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$ respectively such that

$$H^0(B, \Omega^1_{B/k}) = T^{g-j(B)}e_1 \oplus T^{j(B)}, \qquad H^0(A, \Omega^1_{A/k}) = T^{g-j(A)}e'_1 \oplus T^{j(A)}e'_2.$$

Recall that there are relations between $j(A)$ and $j(B)$, namely $j(B) \in \{j(A), j(A)+1, j(A)-1\}$.

As an analogue to the work done by Deligne and Pappas [DP94], we define the singularity stratification in the case with $\Gamma_0(\mathfrak{p})$-level structure.

DEFINITION 5.1.1 (Singularity stratification). *Denote by $S_{(j,j')}$ the subsets of $\mathcal{M}_{\mathfrak{p}}$ whose geometric points are the points $(\underline{A} \xrightarrow{f} \underline{B})$ in $\mathcal{M}_{\mathfrak{p}}(k)$ such that $j(A) \geq j$ and $j(B) \geq j'$.*

The relation to the singularity stratification of $\mathcal{M}$ is straightforward:

$$\pi_1(S_{(j,j')}) = S_j, \qquad \pi_2(S_{(j,j')}) = S_{j'}.$$

Given that the strata S_j are closed, we have that the pre-images are closed. It follows therefore that, for any pair (j', j), the set

$$S_{(j',j)} = \pi_1^{-1}(S_{j'}) \cap \pi_2^{-1}(S_j)$$

is closed.

Denote by $S^0_{(j,j')}$ the subset of $S_{(j,j')}$ whose geometric points $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ have singularity indices $j(\underline{A}) = j'$ and $j(\underline{B}) = j$. Note that the following description holds:

$$S_{(j,j')} = \coprod_{(\tilde{j},\tilde{j}') \geq (j,j')} S^0_{(\tilde{j},\tilde{j}')}.$$

5.2. The singularity-slope stratification. Let us now define a finer stratification by taking into account, together with the singularity indices, the a-numbers of the associated abelian varieties. Namely, given a geometric point $\underline{A} \xrightarrow{f} \underline{B} \in \mathcal{M}_{\mathfrak{p}}(k)$, consider the slopes $n(\underline{A})$ and $n(\underline{B})$ of $\underline{A}$ and $\underline{B}$ respectively. This leads to considering the quadruple $(j(\underline{A}), j(\underline{B}), n(\underline{A}), n(\underline{B}))$. A full description between the pair $(j(\underline{A}), n(\underline{A}))$ and the pair $(j(\underline{B}), n(\underline{B}))$ was provided in [AG03, Table 8.1]. From this derives naturally the definition of a function [AG03, Definition 8.8]

$$\Lambda \colon 2^J \longrightarrow 2^J,$$

such that given $(j,n) \in J$, we have that $(j', n') \in \Lambda(j,n)$ if and only if there exists a geometric point $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}$ such that $(j,n) = (j(B), n(B))$ and $(j', n') = (j(A), n(A))$. It makes therefore sense to consider quadruples (j, n, j', n') where $(j,n) \in J$ and $(j', n') \in \Lambda(j,n)$. By abuse of notation, given $0 \leq j \leq [g/2]$ we will write $j' \in \Lambda(\{j\})$ if and only if there exists a geometric point $(\underline{A} \xrightarrow{f} \underline{B})$ such that $j(\underline{A}) = j'$ and $j(\underline{B}) = j$.

DEFINITION 5.2.1 (Singularity-slope stratification). *Given a quadruple (j, n, j', n') where $(j,n) \in J$ and $(j', n') \in \Lambda(j,n)$, we define the subset $W_{(j,n,j',n')}$ of $S_{(j,j')}$ whose geometric points are the points $(\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ such that $j(A) = j$, $n(A) = n$, $j(B) = j'$ and $n(B) = n'$.*

There is an identification

$$W_{(j,n,j',n')} = \pi_1^{-1}(W_{(j,n)}) \cap \pi_2^{-1}(W_{(j',n')}),$$

from which we obtain that the $W_{(j,n,j',n')}$'s are non-empty.

Let us give some low-dimensional examples.

$$\begin{array}{c}
\underline{g=1} \\
(0,1,0,1) \\
(0,0,0,0)
\end{array}
\qquad
\begin{array}{cc}
 & \underline{g=2} \\
 & (1,1,0,2) \\
(1,1,1,1) & (1,1,0,1) \\
 & (0,0,0,0)
\end{array}
\qquad
\begin{array}{cc}
 & \underline{g=3} \\
 & (1,2,0,3) \\
(1,2,1,2) & (1,2,0,2) \\
(1,1,1,1) & (1,1,0,1) \\
 & (0,0,0,0)
\end{array}$$

$$\begin{array}{ccc}
 & & \underline{g=4} \\
 & & (1,3,0,4) \\
 & (2,2,1,3) & (1,3,0,3) \\
(2,2,2,2) & (2,2,1,2) & (1,2,0,2) \\
 & (1,1,1,1) & (1,1,0,1) \\
 & & (0,0,0,0)
\end{array}
\qquad
\begin{array}{ccccc}
 & & & & \underline{g=8} \\
 & & & & (1,7,0,8) \\
 & & & (2,6,1,7) & (1,7,0,7) \\
 & & (3,5,2,6) & (2,6,1,6) & (1,6,0,6) \\
 & (4,4,3,5) & (3,5,2,5) & (2,5,1,5) & (1,5,0,5) \\
(4,4,4,4) & (4,4,3,4) & (3,4,2,4) & (2,4,1,4) & (1,4,0,4) \\
 & (3,3,3,3) & (3,3,2,3) & (2,3,1,3) & (1,3,0,3) \\
 & & (2,2,2,2) & (2,2,1,2) & (1,2,0,2) \\
 & & & (1,1,1,1) & (1,1,0,1) \\
 & & & & (0,0,0,0)
\end{array}$$

In full generality, at least when g is even, we have:

$$\begin{array}{cccccccc}
 & & & & & & & (1,g-1,0,g) \\
 & & & & & & (2,g-2,1,g-1) & (1,g-1,0,g-1) \\
 & & & & & \dots & (2,g-2,1,g-2) & (1,g-2,0,g-2) \\
 & & & & \dots & \dots & \dots & \dots \\
 & (g/2,g/2,g/2-1,g/2+1) & \dots & & & & \dots & (1,g/2+1,0,g/2+1) \\
(g/2,g/2,g/2,g/2) & (g/2,g/2,g/2-1,g/2) & & \dots & \dots & & (2,g/2,1,g/2) & (1,g/2,0,g/2) \\
 & & \dots & & \dots & & \dots & \dots \\
 & & & & \dots & \dots & \dots & (1,3,0,3) \\
 & & & & & (2,2,2,2) & (2,2,1,2) & (1,2,0,2) \\
 & & & & & & (1,1,1,1) & (1,1,0,1) \\
 & & & & & & & (0,0,0,0)
\end{array}$$

5.3. The Atkin-Lehner automorphism and relations with the moduli space $\mathcal{M}_p$.

PROPOSITION 5.3.1. *The Atkin-Lehner involution* $\omega(\underline{A} \xrightarrow{f} \underline{B}) = (\underline{B} \xrightarrow{f^t} \underline{A})$ *acts on the stratifications of* $\mathcal{M}_{\mathfrak{p}}$ *by*

$$\omega(S_{(j,j')}) = S_{(j',j)}, \qquad \omega(W_{(j,n,j',n')}) = W_{(j',n',j,n)}.$$

The involution $\mathrm{inv}(\underline{B} \xrightarrow{f^\vee} \underline{A})$ *acts on the stratifications of* $\mathcal{M}_{\mathfrak{p}}$ *by*

$$\mathrm{inv}(S_{(j,j')}) = S_{(j',j)}, \qquad \mathrm{inv}(W_{(j,n,j',n')}) = W_{(j',n',j,n)}.$$

PROOF. Recall the diagram constructed in (2.5):

$$\begin{tikzcd}
(\underline{A}, H) = (\underline{A} \xrightarrow{f} \underline{B}) \arrow[r, "\omega"] \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{inv}"] & (\underline{B} = A/H, A[T]/H) = (\underline{B} \xrightarrow{f^t} \underline{A}) \arrow[d, "\alpha"] \\
(\underline{A}^\vee, (A[T]/H)^\vee) = (\underline{A}^\vee \xrightarrow{(f^t)^\vee} \underline{B}^\vee) \arrow[r, "\omega"'] & (\underline{B}^\vee, H^\vee) = (\underline{B} \xrightarrow{f^\vee} A^\vee)
\end{tikzcd}$$

By [AG03, Lemma 8.5], given $\underline{A} \in W_{(j,n)}$, the point associated to the dual $\underline{A}^\vee$ belongs to $W_{(j,n)}$. From this, it follows that $\alpha(W_{(j,n,j',n')}) = W_{(j,n.j',n')}$. On the other and, by construction $w(W_{(j,n,j',n')}) = W_{(j',n',j,n)}$. Since $\mathrm{inv} = \alpha \circ \omega$, hence the conclusion. We have the following picture:

$$\begin{tikzcd}
W_{(j,n,j',n')} \arrow[r, "\omega"] \arrow[d, "\alpha"'] \arrow[dr, "\mathrm{inv}"] & W_{(j',n',j,n)} \arrow[d, "\alpha"] \\
W_{(j,n,j',n')} \arrow[r, "\omega"'] & W_{(j',n',j,n)}
\end{tikzcd}$$

□

COROLLARY 5.3.2. *Assume that* $(j,n),(j',n') \in J$*. The pair* $(j',n') \in \Lambda(j,n)$ *if and only if* $(j,n) \in \Lambda(j',n')$.

PROOF. The original result can be found in [AG03, Corollary 8.11]. We give here a different proof. Let $(j,n),(j',n') \in J$ and assume $(j',n') \in \Lambda(j,n)$. By Theorem (5.0.1), there exists a geometric point $\underline{A} \in \mathcal{M}$ such that $(j(A),n(A)) = (j,n)$. The hypothesis $(j',n') \in \Lambda(j,n)$ equals to saying that there exists a geometric point $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \pi_1^{-1}(\underline{A}) \subset \mathcal{M}_p$ such that $(j(A),n(A),j(B),n(B)) = (j,n,j',n')$. The point $\underline{B} = \pi_2(x) \in \mathcal{M}$ has parameters $(j(B),n(B)) = (j',n')$. In order to show that $(j,n) \in \Lambda(j'n')$ we need to show that there exists a geometric point $y \in \pi_1^{-1} \subset \mathcal{M}_p$ with parameters (j',n',j,n). Take for instance the Atkin-Lehner image $y = \omega(x) = (\underline{B} \xrightarrow{f^t} \underline{A})$. By Proposition (5.3.1) it has parameters (j',n',j,n). The other direction of the statement is obtained by switching (j,n) with $(j'n')$. This shows in particular the stratification in $\mathcal{M}$ respects the relation

$$\pi_2(\pi_1^{-1}(W_{(j,n)}) = W_{\Lambda(j,n)};$$

the original statement can be found in [AG03, Proposition 8.10]. □

5.4. Main statements.

THEOREM 5.4.1 (On the singularity stratification). *Let (j, j') be a pair of integers with $0 \leq j \leq [g/2]$ and $j' \in \Lambda(\{j\})$.*

(1) *The $S^0_{(j,j')}$'s are non-empty, non-singular, locally irreducible of dimension $g - j' - j$.*
(2) *The closed set $S_{(j,j')}$ has dimension $g - j' - j$ and its generic points have singularity indices (j, j'). In particular the $S_{(j,j')}$ are a stratification of the space $\mathcal{M}_{\mathfrak{p}}$.*

THEOREM 5.4.2 (On the singularity-slope stratification). *Let $(j, n) \in J$ and $(j', n') \in \Lambda(j, n)$.*

(1) *The locally closed set $W_{(j,n,j',n')}$ is non-singular, locally irreducible of dimension $g - j' - n'$.*
(2) *The $W_{(j,n,j',n')}$'s are a stratification of the space $\mathcal{M}_{\mathfrak{p}}$.*

6. Dieudonné Displays

In order to proceed to the proof of the main statements, see Section (7), we need to develop a theory of relative Dieudonné displays in the ramified case.

We are going to relate a geometric point $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ to an object in characteristic 0 by means of Dieudonné theory. More precisely, following the results by Langer-Zink (3.4.4), Chapter (2), we may associate RM Dieudonné displays $\mathcal{P}_A = (P_A, Q_A, F_A, V_A^{-1})$ and $\mathcal{P}_B = (P_B, Q_B, F_B, V_B^{-1})$ to $\underline{A}$ and $\underline{B}$ respectively as follows. Put

$$P_A = H^1_{cris}(A/W(k)), \qquad P_B = H^1_{cris}(B/W(k)), \tag{6.1}$$

$$Q_A = \mathrm{Ker}(H^1_{cris}(A/W(k)) \to \mathrm{Lie}(A)), \qquad Q_B = \mathrm{Ker}(H^1_{cris}(B/W(k)) \to \mathrm{Lie}(B)). \tag{6.2}$$

By functoriality, the p-isogeny $f: \underline{A} \to \underline{B}$ induces morphisms of Dieudonné displays

$$\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A, \qquad \mathbb{D}(f^t): \mathcal{P}_A \to \mathcal{P}_B, \tag{6.3}$$

and the conditions

$$f \circ f^t = [\mathfrak{p}]_B, \qquad f^t \circ f = [\mathfrak{p}]_A$$

induce the conditiosn

$$\mathbb{D}(f) \circ \mathbb{D}(f^t) = [\mathfrak{p}]_{H^1_{cris}(A/W(k))}, \qquad \mathbb{D}(f^t) \circ \mathbb{D}(f) = [\mathfrak{p}]_{H^1_{cris}(B/W(k))}. \tag{6.4}$$

Moreover, by [Rap78, Lemma 1.3], $H^1_{cris}(A/W(k))$ and $H^1_{cris}(B/W(k))$ are free $\mathcal{O}_L \otimes W(k)$-modules of rank 2 and their reduction modulo (p) is identified with de Rham cohomology groups:

$$H^1_{cris}(A/W(k))/(p) \simeq H^1_{dR}(A/k), \qquad H^1_{cris}(B/W(k))/(p) \simeq H^1_{dR}(B/k).$$

We would like to describe $\mathcal{P}_A$ and $\mathcal{P}_B$ in terms of suitable $\mathcal{O}_L \otimes W(k)$-bases.

LEMMA 6.0.1. *We may lift the $\mathcal{O}_L \otimes k$-bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ of $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$ as in Remark (7), Case 1, to $\mathcal{O}_L \otimes W(k)$-bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ of $H^1_{cris}(B/W(k))$ and $H^1_{cris}(A/W(k))$. In these bases, the morphisms $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ are represented by*

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We will use an argument analogous to the proof of Lemma (3.1.1). The fundamental observation is that there is an identification

$$H^1_{cris}(B/W(k))/TH^1_{cris}(B/W(k)) \simeq H^1_{dR}(B/k)/TH^1_{dR}(B/k) \simeq \mathbb{D}(B[T]),$$

and similarly

$$H^1_{cris}(A/W(k))/TH^1_{cris}(A/W(k)) \simeq H^1_{dR}(A/k)/TH^1_{dR}(A/k) \simeq \mathbb{D}(A[T]).$$

As already seen in the proof of Lemma (3.1.1), both $\mathbb{D}(B[T])$ and $\mathbb{D}(A[T])$ are k-vector spaces of rank 2 and are respected by the morphism $\mathbb{D}(f)$. In fact, there is a diagram

$$\begin{tikzcd}
H^1_{cris}(B/W(k)) \arrow[r, "\mathbb{D}(f)"] \arrow[d] & H^1_{cris}(A/W(k)) \arrow[d] \\
H^1_{dR}(B/k) \arrow[r, "\mathbb{D}(f)"] \arrow[d] & H^1_{dR}(A/k) \arrow[d] \\
\mathbb{D}(B[T]) \arrow[r, "\mathbb{D}(f)"] & \mathbb{D}(A[T])
\end{tikzcd}$$

where all the squares are commutative. Take, as in Lemma (3.1.1), k-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ for $\mathbb{D}(B[T])$ and $\mathbb{D}(A[T])$ respectively such that $\mathbb{D}(f)$ and $\mathbb{D}(f^t)$ are represented by the matrices

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular note that

$$\mathbb{D}(f)(e_1) = e_1', \qquad \mathbb{D}(f^t)(e_2') = e_2.$$

Lift hence e_1 and e_2' to $\tilde{e}_1$ and $\tilde{e}_2'$ in $(\mathcal{O}_L \otimes W(k))^2$ respectively. Set

$$\tilde{e}_1' = \mathbb{D}(f)(\tilde{e}_1), \qquad \tilde{e}_2 = \mathbb{D}(f^t)(\tilde{e}_2'). \tag{6.5}$$

Note that $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{e}_1', \tilde{e}_2'\}$ reduce to the k-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ respectively and hence by the Nakayama lemma they are $\mathcal{O}_L \otimes W(k)$ bases for $H^1_{cris}(B/W(k))$ and $H^1_{cris}(A/W(k))$ respectively. We will forget the tilde and denote them by $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ by abuse of notation. From (6.4) we obtain

$$\mathbb{D}(f^t) \circ \mathbb{D}(f)(e_1) = Te_1, \qquad \mathbb{D}(f) \circ \mathbb{D}(f^t)(e_2') = Te_2',$$

and combining this with (6.5) we get that

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \qquad \mathbb{D}(f^t) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

□

In what follows we need to make the distinction between the case when $j(\underline{A}) = j(\underline{B})$ and $j(\underline{A}) \neq j(\underline{B})$.

6.1. Case when $j = j(\underline{A}) = j(\underline{B})$. Using the bases obtained in Lemma (6.0.1), take

$$P_B = (\mathcal{O}_L \otimes W(k))e_1 \oplus (\mathcal{O}_L \otimes W(k))e_2, \qquad P_A = (\mathcal{O}_L \otimes W(k))e_1' \oplus (\mathcal{O}_L \otimes W(k))e_2'.$$

COROLLARY 6.1.1. *In the bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ we have that*

$$Q_B = (\mathcal{O}_L \otimes W(k)) \cdot (T^i e_1 \oplus T^j e_2),$$
$$Q_A = (\mathcal{O}_L \otimes W(k)) \cdot (T^{i'} e_1' \oplus T^{j'} e_2').$$

PROOF. Note that by Corollary (3.1.2) we have that

$$H^0(B, \Omega^1_{B/k}) = T^i e_1 \oplus T^j e_2, \qquad H^0(A, \Omega^1_{A/k}) = T^i e_1' \oplus T^j e_2',$$

and by Remark (7) we may assume $j \leq i$. □

In what follows, we will be taking into account Case 1 from Remark (7). The morphism of displays $\mathbb{D}(f) \colon \mathcal{P}_B \to \mathcal{P}_A$ is represented by the matrix

$$\mathbb{D}(f) = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

Note that Case 2 from Remark (7) is obtained by taking the Atkin-Lehner involution of elements of Case 1. The matrices representing the ^{F}-linear matrices $F_A \colon P_A \to P_A$ and $F_B \colon P_B \to P_B$ satisfy the two following conditions

$$\text{(6.6)} \qquad \mathrm{Ker}(\overline{F}_A) = H^0(A, \Omega^1_{A/k}), \qquad \mathrm{Ker}(\overline{F}_B) = H^0(B, \Omega^1_{B/k}),$$

$$\text{(6.7)} \qquad \mathbb{D}(f) \circ F_B = F_A \circ \mathbb{D}(f)^\sigma.$$

Condition (6.6) tells us that F_B are represented by matrices of type

$$F_B = \begin{pmatrix} T^j g_{1,1} & T^i g_{1,2} \\ T^j g_{2,1} & T^i g_{2,2} \end{pmatrix}, \qquad F_A = \begin{pmatrix} T^j \tilde{g}_{1,1} & T^i \tilde{g}_{1,2} \\ T^j \tilde{g}_{2,1} & T^i \tilde{g}_{2,2} \end{pmatrix},$$

where $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}, \tilde{g}_{1,1}, \tilde{g}_{1,2}, \tilde{g}_{2,1}, \tilde{g}_{2,2} \in W(k)[T]/(E(T))$. Moreover, (6.7) translates into

$$\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} T^j g_{1,1} & T^i g_{1,2} \\ T^j g_{2,1} & T^i g_{2,2} \end{pmatrix} = \begin{pmatrix} T^j \tilde{g}_{1,1} & T^i \tilde{g}_{1,2} \\ T^j \tilde{g}_{2,1} & T^i \tilde{g}_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix},$$

which gives the conditions

$$\tilde{g}_{1,1} = g_{1,1}, \qquad g_{1,2} = T\tilde{g}_{1,2}, \qquad \tilde{g}_{2,1} = T g_{2,1}, \qquad \tilde{g}_{2,2} = g_{2,2}.$$

From this we obtain that F_B and F_A are represented by

$$F_B = \begin{pmatrix} T^j g_{1,1} & T^{i+1} g_{1,2} \\ T^j g_{2,1} & T^i g_{2,2} \end{pmatrix}, \qquad F_A = \begin{pmatrix} T^j g_{1,1} & T^i g_{1,2} \\ T^{j+1} g_{2,1} & T^i g_{2,2} \end{pmatrix},$$

with $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2} \in W(k)[T]/(E(T))$.

Our goal is to construct a universal object

$$\text{(6.8)} \qquad \mathcal{P}_1^{\mathrm{un}} = (P_1^{\mathrm{un}}, Q_1^{\mathrm{un}}, F_1^{\mathrm{un}}, (V_1^{-1})^{\mathrm{un}}) \xrightarrow{u^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}} = (P_2^{\mathrm{un}}, Q_2^{\mathrm{un}}, F_2^{\mathrm{un}}, (V_2^{-1})^{\mathrm{un}})$$

over R with respect to the equi-characteristic deformation of

$$\mathbb{D}(f) \colon \mathcal{P}_B \to \mathcal{P}_A.$$

Namely, $\mathcal{P}_1^{\mathrm{un}} \xrightarrow{u^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}$ is such that for every deformation $\tilde{u}\colon \widetilde{\mathcal{P}_B} \to \widetilde{\mathcal{P}_A}$ to an Artinian k-algebra S, there exists a unique morphism $\tau\colon R \to S$ such that

$$(\mathcal{P}_1^{\mathrm{un}} \xrightarrow{u^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}) \otimes_{W(R),\tau} W(S) \simeq (\widetilde{\mathcal{P}_B} \xrightarrow{\tilde{u}} \widetilde{\mathcal{P}_A}).$$

By Theorem (2.2.2) and Theorem (3.2.2), Chapter (2), such an object exists and it is defined over the ring

$$R = \widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},t} = k[T]/(T^g)[\![a_0,\dots,a_{i-1},b_0,\dots,b_{j-1},c_0,\dots,c_{i-1},d_0,\dots,d_{j-1}]\!] \\ /(aT^j + dT^i + ad - bcT)..$$

Note that R is an admissible topological ring.

Let $t = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ and denote $j = j(\underline{B})$. Assume first that $j(\underline{A}) = j(\underline{B})$.

Consider hence the universal object pro-representing the local model associated to $t = (\underline{A} \to \underline{B})$ described in section (4.1)

$$\tilde{F} \subset (R[T]/(T^g)e_1 \oplus R[T]/(T^g)e_2) \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & T\end{smallmatrix}\right)} \tilde{F}' \subset (R[T]/(T^g)e_1' \oplus R[T]/(T^g)e_2'),$$

with $\tilde{F}$ generated by $\tilde{f} = T^i e_1 + ae_1 + be_2$ and $u^t(\tilde{f}') = T^j e_2 + de_2 + cTe_1$ and $\tilde{F}'$ spanned by $\tilde{f}' = T^j e_2' + ce_1' + de_2'$ and $u(\tilde{f}) = T^i e_1' + ae_1' + bTe_2'$.

We are going to use the following result:

Lemma 6.1.2. *The element*

$$w = 1 + T^{-g}({}^F\hat{a}T^j + {}^F\hat{d}T^i + {}^F\hat{a}{}^F\hat{d} - {}^F\hat{b}{}^F\hat{c}T)$$

is a unit in $(\mathcal{O}_L \otimes W(k)) \otimes_{W(k)} W(R) \simeq \mathcal{O}_L \otimes W(R)$.

Proof. The strategy for the proof is provided in [AG04, Lemma 5.5.1, Lemma 5.5.2]. The tensor $\mathcal{O}_L \otimes W(R)$ has a notion of Frobenius ${}^F\colon \mathcal{O}_L \otimes W(R) \to \mathcal{O}_L \otimes W(R)$ and Verschiebung ${}^V\colon \mathcal{O}_L \otimes W(R) \to \mathcal{O}_L \otimes W(R)$ morphisms. Namely, for $\lambda \otimes x \in \mathcal{O}_L \otimes W(R)$

$${}^F(\lambda \otimes x) = \lambda \otimes {}^F x, \qquad {}^V(\lambda \otimes x) = \lambda \otimes {}^V x.$$

Since the ring R is reduced we have the identification

$${}^{FV}(\mathcal{O}_L \otimes W(R)) = {}^{VF}(\mathcal{O}_L \otimes W(R)) = p(\mathcal{O}_L \otimes W(R)).$$

Recall the exact sequence

$$0 \to {}^V(\mathcal{O}_L \otimes W(R)) \to \mathcal{O}_L \otimes W(R) \to \mathcal{O}_L \otimes R \to 0.$$

Clearly $\hat{a}T^j + \hat{d}T^i + \hat{a}\hat{d} - \hat{b}\hat{c}T \in {}^V(\mathcal{O}_L \otimes W(R))$ and hence $\kappa = {}^F\hat{a}T^j + {}^F\hat{d}T^i + {}^F\hat{a}{}^F\hat{d} - {}^F\hat{b}{}^F\hat{c}T \in p(\mathcal{O}_L \otimes W(R))$. Note that $T^{-g}\kappa = up^{-1}\kappa$ for some unit u; we conclude therefore that

$$w = 1 + T^{-g}\kappa \in \mathcal{O}_L \otimes W(\mathfrak{m}_R) \subset \mathcal{O}_L \otimes W(R).$$

We have left to show that w is a unit in $\mathcal{O}_L \otimes W(R)$. Note that $\kappa \in \mathcal{O}_L \otimes W(\mathfrak{m}_R)$. Indeed $\mathfrak{m}_R = (a_0,\dots,a_{i-1},b_0,\dots,b_{j-1},c_0,\dots,c_{i-1},d_0,\dots,d_{j-1})$ and trivially $W(\mathfrak{m}_R) = \{(a_0,a_1,a_2,\dots)|a_i \in \mathfrak{m}_R, \forall i\}$. Hence

$$w \in 1 + (\mathcal{O}_L \otimes W(\mathfrak{m}_R)).$$

Claim.

$$1 + (\mathcal{O}_L \otimes W(\mathfrak{m}_R)) \subset (\mathcal{O}_L \otimes W(R))^\times.$$

In what follows we may assume $\mathcal{O}_L = \mathbb{Z}$. Indeed, define a norm operator N on $\mathcal{O}_L \otimes W(R)$ by taking the norm as a module over $W(R)$. Then $x \in \mathcal{O}_L \otimes W(R)$ is a unit if and only if $N(x)$ is a unit. Consider hence $x = 1 + \lambda \in W(R)$, such that $\lambda \in W(\mathfrak{m}_R)$. Note that the reduction $\overline{1+\lambda} \in R$ is a unit, since $\overline{\lambda} \in \mathfrak{m}_R$. Hence there exists $y, s \in W(R)$ such that

$$yx = 1 - s, \qquad \overline{s} = 0 \in R.$$

Note that in particular $s = (0, s_1, s_2, s_3, \dots) \in {}^V W(R) = I(R)$, hence the n-th power $s^n \equiv 0$ in $W(R)/I(R)^n W(R)$. By [Zin02, Proposition 3] the ring $W(R)$ is $I(R)$-adically complete and separable. From this we get that the geometric series $z = \sum_n s^n \in W(R) = \varprojlim_n W(R)/I(R)^n W(R)$ and

$$zyx = (\sum_n s^n) yx = (\frac{1}{1-s}) 1 - s = 1,$$

hence zy is the inverse of x, which proves the Claim and concludes the proof of the Lemma. $\square$

Corollary 6.1.3. *In fact w belongs to $(\mathcal{O}_L \otimes \mathbb{W}(R))^\times$.*

Proof. Note first that the elements ${}^F\hat{a}_0, {}^F\hat{a}_1, \dots {}^F\widehat{a_{i-1}}, {}^F\hat{b}_0, {}^F\hat{b}_1, \dots, {}^F\widehat{b_{j-1}}, {}^F\hat{c}_0, {}^F\hat{c}_1, \dots, {}^F\widehat{c_{i-1}}, {}^F\hat{d}_0, {}^F\hat{d}_1, \dots$ belong to $\mathbb{W}(R)$. Indeed the elemnts $a_0, \dots, a_{i-1}, b_0, \dots,\ b_{j-1}, c_0, \dots, c_{i-1}, d_0, \dots, d_{j-1}$ all belong to the maximal ideal $\mathfrak{m}_R$ and their Teichmüller lifts belong to $\widehat{W}(\mathfrak{m}_R) \subset \mathbb{W}(R)$. Since $\mathbb{W}(R)$ is ^{F}-equivariant, we conclude.

Hence by the proof of the Lemma $w \in \mathcal{O}_L \otimes \widehat{W}(\mathfrak{m}_R) \subset (\mathcal{O}_L \otimes \mathbb{W}(R))^\times$, since $\mathbb{W}(R)$ is a complete local ring. $\square$

Define the quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ so that

$$P_1 \coloneqq P_B \otimes_{W(k)} \mathbb{W}(R) = (\mathcal{O}_L \otimes \mathbb{W}(R)) e_1 \oplus (\mathcal{O}_L \otimes \mathbb{W}(R)) e_2,$$
$$P_2 \coloneqq P_A \otimes_{W(k)} \mathbb{W}(R) = (\mathcal{O}_L \otimes \mathbb{W}(R)) e_1' \oplus (\mathcal{O}_L \otimes \mathbb{W}(R)) e_2'$$

and the $\mathcal{O}_L \otimes \mathbb{W}(R)$-submodule Q_1 (resp. Q_2) is the pre-image of $\tilde{F}$ (resp. $\tilde{F}'$) through the natural map $P_1 \to (R[T]/(T^g))^2$ (resp. $P_2 \to (R[T]/(T^g))^2$):

$$Q_1 = \mathbb{I}(R)(e_1 \oplus e_2) + (\mathcal{O}_L \otimes \mathbb{W}(R)) \widehat{\tilde{F}}, \qquad Q_2 = \mathbb{I}(R)(e_1' \oplus e_2') + (\mathcal{O}_L \otimes \mathbb{W}(R)) \widehat{\tilde{F}}',$$

where $\widehat{\tilde{F}}$ is generated by $\widehat{\tilde{f}} = T^i e_1 + \hat{a} e_1 + \hat{b} e_2$ and $u^t(\widehat{\tilde{f}}') = T^j e_2 + \hat{d} e_2 + \hat{c} T e_1$ and $\widehat{\tilde{F}}'$ is generated by $\widehat{\tilde{f}}' = T^j e_2' + \hat{c} e_1' + \hat{d} e_2'$ and $u(\widehat{\tilde{f}}) = T^i e_1' + \hat{a} e_1' + \hat{b} T e_2'$. Define F_1 and F_2 as the semi-linear maps represented, with respect to the bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$, by the matrices

$$F_1 = w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T^{i+1} g_{1,2} - {}^F\hat{c} T g_{1,1} + {}^F\hat{a} T g_{1,2} \\ T^j g_{2,1} + {}^F\hat{d} g_{2,1} - {}^F\hat{b} g_{2,2} & T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2} \end{pmatrix},$$
$$F_2 = w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T^i g_{1,2} - {}^F\hat{c} g_{1,1} + {}^F\hat{a} g_{1,2} \\ T^{j+1} g_{2,1} + {}^F\hat{d} T g_{2,1} - {}^F\hat{b} T g_{2,2} & T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2} \end{pmatrix}.$$

Theorem 6.1.4. *Take $V_i^{-1}\colon Q_i \to P_i$ defined as F_i/p. The quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ are Dieudonné displays over R. They reduce to $\mathcal{P}_B$ and $\mathcal{P}_A$ mod $\mathfrak{m}_R$ respectively and their Hodge filtrations are $\tilde{F} \subset (R[T]/(T^g) e_1 \oplus R[T]/(T^g) e_2)$ and $\tilde{F}' \subset (R[T]/(T^g) e_1' \oplus R[T]/(T^g) e_2')$.*

PROOF. We are going to show first that $F_i(Q_i) \subseteq pP_i$, that is, V_i^{-1} is well defined. We show here the argument for $i = 1$, the case for $i = 2$ being identical. Note that $F_1(\mathbb{I}(R)e_1) \subset pP_1$ and $F_1(\mathbb{I}(R)e_2) \subset pP_1$. This just follows from ^{F}-linearity of F_1: indeed, since R is reduced, we have that ${}^F\mathbb{I}(R) = {}^{FV}\mathbb{W}(R) = p\mathbb{W}(R)$. One can see from direct computations that the image through F_1 of the generators $T^i e_1 + \hat{a}e_1 + \hat{b}e_2$ and $T^j e_2 + \hat{d}e_2 + \hat{c}Te_1$ lies in pP_1:

$$
\begin{aligned}
F_1 \cdot \widehat{\widetilde{f}} &= w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}g_{1,1} - {}^F\hat{b}Tg_{1,2} & T^{i+1}g_{1,2} - {}^F\hat{c}Tg_{1,1} + {}^F\hat{a}Tg_{1,2} \\ T^j g_{2,1} + {}^F\hat{d}g_{2,1} - {}^F\hat{b}g_{2,2} & T^i g_{2,2} - {}^F\hat{c}Tg_{2,1} + {}^F\hat{a}g_{2,2} \end{pmatrix} \begin{pmatrix} T^i + \hat{a} \\ \hat{b} \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} T^g g_{1,1} + T^{j\,F}\hat{a}g_{1,1} + T^{i\,F}\hat{d}g_{1,1} + {}^F\hat{a}^F\hat{d}g_{1,1} - T^{i+1\,F}\hat{b}g_{1,2} - T^F\hat{a}^F\hat{b}g_{1,2} + T^{i+1\,F}\hat{b}g_{1,2} - T^F\hat{b}^F\hat{c}g_{1,1} + T^F\hat{a}^F\hat{b}g_{1,2} \\ T^g g_{2,1} + T^{j\,F}\hat{a}g_{2,1} + T^{i\,F}\hat{d}g_{2,1} + {}^F\hat{a}^F\hat{d}g_{2,1} - {}^F\hat{b}T^i g_{2,2} - {}^F\hat{a}^F\hat{b}g_{2,2} + T^{i\,F}\hat{b}g_{2,2} - T^F\hat{b}^F\hat{c}g_{2,1} + {}^F\hat{a}^F\hat{b}g_{2,2} \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} g_{1,1}(T^g + T^{j\,F}\hat{a} + T^{i\,F}\hat{d} + {}^F\hat{a}^F\hat{d} - T^F\hat{c}^F\hat{b}) \\ g_{2,1}(T^g + T^{j\,F}\hat{a} + T^{i\,F}\hat{d} + {}^F\hat{a}^F\hat{d} - T^F\hat{c}^F\hat{b}) \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} T^g g_{1,1} w \\ T^g g_{2,1} w \end{pmatrix} \\
&= T^g \begin{pmatrix} g_{1,1} \\ g_{2,1} \end{pmatrix} \in T^g P_1,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
F_1 \cdot u^t(\widehat{\widetilde{f}}') &= w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}g_{1,1} - {}^F\hat{b}Tg_{1,2} & T^{i+1}g_{1,2} - {}^F\hat{c}Tg_{1,1} + {}^F\hat{a}Tg_{1,2} \\ T^j g_{2,1} + {}^F\hat{d}g_{2,1} - {}^F\hat{b}g_{2,2} & T^i g_{2,2} - {}^F\hat{c}Tg_{2,1} + {}^F\hat{a}g_{2,2} \end{pmatrix} \begin{pmatrix} \hat{c}T \\ T^j + \hat{d} \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} T^{j+1}g_{1,1}{}^F\hat{c} + {}^F\hat{c}^F\hat{d}Tg_{1,1} - {}^F\hat{b}^F\hat{c}T^2 g_{1,2} + T^{g+1}g_{1,2} + T^{i+1\,F}\hat{d}g_{1,2} - {}^F\hat{c}T^{j+1}g_{1,1} - {}^F\hat{c}^F\hat{d}Tg_{1,1} + {}^F\hat{a}T^{j+1}g_{1,2} + {}^F\hat{a}^F\hat{d}Tg_{1,2} \\ T^{j+1\,F}\hat{c}g_{2,1} + {}^F\hat{c}^F\hat{d}Tg_{2,1} - {}^F\hat{c}^F\hat{b}Tg_{2,2} + T^g g_{2,2} + {}^F T^i g_{2,2} - {}^F\hat{c}T^{j+1}g_{2,1} - {}^F\hat{c}^F\hat{d}Tg_{2,1} + {}^F\hat{a}T^j g_{2,2} + {}^F\hat{a}^F\hat{d}g_{2,2} \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} -{}^F\hat{b}^F\hat{c}T^2 g_{1,2} + T^{g+1}g_{1,2} + T^{i+1\,F}\hat{d}g_{1,2} + {}^F\hat{a}T^{j+1}g_{1,2} + {}^F\hat{a}^F\hat{d}Tg_{1,2} \\ -{}^F\hat{c}^F\hat{b}Tg_{2,2} + T^g g_{2,2} + {}^F T^i g_{2,2} + {}^F\hat{a}T^j g_{2,2} + {}^F\hat{a}^F\hat{d}g_{2,2} \end{pmatrix} \\
&= w^{-1} \begin{pmatrix} T^{g+1}g_{1,2}w \\ T^g g_{2,2}w \end{pmatrix} \\
&= T^g \begin{pmatrix} Tg_{1,2} \\ g_{2,2} \end{pmatrix} \in T^g P_1.
\end{aligned}
$$

We need to show now that the image $V_i^{-1}(Q_i)$ generates P_i. Again, we will do this only for $i = 1$. By Section (2.3.1), Chapter (2), it is enough to show that there exist a suitable decomposition $P_1 = L_1 \oplus T_1$ such that

$$V_1^{-1} \oplus F_1 \colon L_1 \oplus T_1 \xrightarrow{\simeq} P_1$$

is an ^{F}-linear isomorphism. Remark that F_1 is the composition of the extension of the ^{F}-linear base change of F_B to P_1 after θ_1, where

$$\theta_1 = w^{-1} \cdot \begin{pmatrix} 1 + \hat{d}T^{-j} & -\hat{c}T(T^{-j}) \\ -\hat{b}T^{-i} & 1 + \hat{a}T^{-i} \end{pmatrix}.$$

Consider the Dieudonné display $\mathcal{P}'_B \coloneqq \mathcal{P}_B \otimes_{W(k)} W(R)$ over R obtained from base change. Being $\mathcal{P}'_B$ a display, there exists a decomposition $P'_B = L'_B \oplus T'_B$ and an ^{F}-linear isomorphism

$$\psi'_B \coloneqq (V_B^{-1})' \oplus F'_B \colon L'_B \oplus T'_B \xrightarrow{\simeq} P'_B.$$

Given the identification

$$F'_B(x) = p(V_B^{-1})'(x), \qquad x \in Q,$$

we have that

$$\psi'_B \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) = F'_B. \tag{6.9}$$

Note that by construction $P'_B = P_1$ and that the semi-linear Frobenius F'_B is the ^{F}-linear base change of F_B to R. Consider

$$\tilde{\theta}_1 \coloneqq \left(\begin{smallmatrix} T^g & 0 \\ 0 & 1 \end{smallmatrix}\right) \circ \theta_1 \circ \left(\begin{smallmatrix} 1 & 0 \\ 0 & T^{-g} \end{smallmatrix}\right).$$

It is an isomorphism, since $\det(\tilde{\theta}_1) = \det(\theta_1)$. Consider hence

$$L_1 := \tilde{\theta}_1^{-1}(L'_B), \qquad T_1 := \tilde{\theta}_1^{-1}(T'_B).$$

This leads to the diagram

$$\begin{tikzcd}
 & L'_B \oplus T'_B \arrow[r, "\psi'_B", "\simeq"'] \arrow[d, equal] & P_1 \\
 & P_1 & \\
P_1 \arrow[r, equal] & L_1 \oplus T_1 \arrow[u, "\theta_1", "\simeq"'] \arrow[uur, "\simeq"] &
\end{tikzcd}$$

defining an ^{F}-linear isomorphism

$$\psi\colon L_1 \oplus T_1 \xrightarrow{\simeq} P_1.$$

The following diagram describes the various relations involved:

$$\begin{tikzcd}
P'_B \arrow[r, "\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[rr, bend left, "F'_B"] & L'_B \oplus T'_B \arrow[r, "\psi'_B"] & P_1 \\
P_1 \arrow[u, "\theta_1"] \arrow[r, "\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] \arrow[ur, phantom, "\circlearrowleft"] & L_1 \oplus T_1 \arrow[u, "\tilde{\theta}_1"] \arrow[ur, "\psi"'] \arrow[u, phantom, shift right=4ex, "\circlearrowleft"] &
\end{tikzcd}$$

We have therefore

$$F_1 = F'_B \circ \theta_1 = \psi'_B \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \circ \theta_1 = \psi'_B \circ \tilde{\theta}_1 \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) = \psi \circ \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

which concludes the proof. □

Consider the $\mathcal{O}_L \otimes \mathbb{W}(R)$-linear morphism $u\colon P_1 \to P_2$ represented by the matrix

$$u = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$$

in the bases $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$. Note that

$$u(Q_1) \subseteq Q_2.$$

Recall indeed that $Q_1 = \mathbb{I}(R)(e_1 \oplus e_2) + (\mathcal{O}_L \otimes \mathbb{W}(R))\widehat{\widetilde{F}}$ and $Q_2 = \mathbb{I}(R)(e'_1 \oplus e'_2) + (\mathcal{O}_L \otimes \mathbb{W}(R))\widehat{\widetilde{F}}'$, where $\widehat{\widetilde{F}}$ is generated by $\widehat{\widetilde{f}} = T^i e_1 + \hat{a} e_1 + \hat{b} e_2$ and $u^t(\widehat{\widetilde{f}}') = T^j e_2 + \hat{d} e_2 + \hat{c} T e_1$ and $\widehat{\widetilde{F}}'$ is generated by $u(\widehat{\widetilde{f}}) = T^j e'_2 + \hat{c} e'_1 + \hat{d} e'_2$ and $\widehat{\widetilde{f}}' = T^i e'_1 + \hat{a} e'_1 + \hat{b} T e'_2$. Then we see that $u(\widehat{\widetilde{f}}) \in \widehat{\widetilde{F}}'$ by the construction of the basis and that $u(u^t(\widehat{\widetilde{f}}')) = T\widehat{\widetilde{f}}' \in \widehat{\widetilde{F}}'$.

Moreover it commutes with the semi-linear maps F_1 and F_2 (and hence by definition the semi-linear-maps V_1^{-1} and V_2^{-1}), that is $u \circ F_1 = F_2 \circ u^\sigma$:

$$\begin{aligned}
u \circ F_1 &= w^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T^{i+1} g_{1,2} - {}^F\hat{c} T g_{1,1} + {}^F\hat{a} T g_{1,2} \\ T^j g_{2,1} + {}^F\hat{d} g_{2,1} - {}^F\hat{b} g_{2,2} & T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2} \end{pmatrix} \\
&= w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T^{i+1} g_{1,2} - {}^F\hat{c} T g_{1,1} + {}^F\hat{a} T g_{1,2} \\ T(T^j g_{2,1} + {}^F\hat{d} g_{2,1} - {}^F\hat{b} g_{2,2}) & T(T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2}) \end{pmatrix}, \\
F_2 \circ u^\sigma &= w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T^i g_{1,2} - {}^F\hat{c} g_{1,1} + {}^F\hat{a} g_{1,2} \\ T^{j+1} g_{2,1} + {}^F\hat{d} T g_{2,1} - {}^F\hat{b} T g_{2,2} & T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix} \\
&= w^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d} g_{1,1} - {}^F\hat{b} T g_{1,2} & T(T^i g_{1,2} - {}^F\hat{c} g_{1,1} + {}^F\hat{a} g_{1,2}) \\ T^{j+1} g_{2,1} + {}^F\hat{d} T g_{2,1} - {}^F\hat{b} T g_{2,2} & T(T^i g_{2,2} - {}^F\hat{c} T g_{2,1} + {}^F\hat{a} g_{2,2}) \end{pmatrix}.
\end{aligned}$$

It follows hence that u is a morphism of Dieudonné displays

$$u: \mathcal{P}_1 \to \mathcal{P}_2.$$

THEOREM 6.1.5. *The morphism of Dieudonné displays $u: \mathcal{P}_1 \to \mathcal{P}_2$ over R is universal with respect to the deformation of $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$.*

PROOF. The argument uses extensively the crystalline nature of Dieudonné displays. Let us recall the main consequences of Theorem (3.4.3), Chapter (2). Both P_B and P_A are crystals and the morphism $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$ induces a morphism of crystals. Analogously, by reducing modulo p, both $D_{\mathcal{P}_B}$ and $D_{\mathcal{P}_A}$ are contravariant Dieudonné crystals and the morphism of Dieudonné displays $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$ induces a morphism of crystals $D_B \to D_A$.

Note that the morphism Dieudonné of displays $u: \mathcal{P}_1 \to \mathcal{P}_2$ reduces to $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$ modulo $\mathfrak{m}_R$.

Let $u^{\mathrm{un}}: \mathcal{P}_1^{\mathrm{un}} = (P_1^{\mathrm{un}}, Q_1^{\mathrm{un}}, F_1^{\mathrm{un}}, (V_1^{-1})^{\mathrm{un}}) \to \mathcal{P}_2^{\mathrm{un}} = (P_2^{\mathrm{un}}, Q_2^{\mathrm{un}}, F_2^{\mathrm{un}}, (V_2^{-1})^{\mathrm{un}})$ be a universal object for the deformation of $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$ as in (6.8). There exists therefore a morphism of artinian k-algebras $\phi: \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ such that

$$\phi^*(\mathcal{P}_1^{\mathrm{un}} \xrightarrow{u^{\mathrm{un}}} \mathcal{P}_2^{\mathrm{un}}) \simeq (\mathcal{P}_1 \xrightarrow{u} \mathcal{P}_2).$$

CLAIM. *The morphism ϕ is an isomorphism.*

Observe that $\mathcal{P}_1^{\mathrm{un}}$ and $\mathcal{P}_2^{\mathrm{un}}$ are RM Dieudonné displays over R, that is P_1^{un} and P_2^{un} are projective $\mathcal{O}_L \otimes \mathbb{W}(R)$-modules of rank 2. Hence the quotients $D_{\mathcal{P}_1^{\mathrm{un}}} = P_1^{\mathrm{un}}/\mathbb{I}(R)P_1^{\mathrm{un}}$ and $P_2^{\mathrm{un}}/\mathbb{I}(R)P_2^{\mathrm{un}}$ are projective $\mathcal{O}_L \otimes R$-modules of rank 2 and since $\mathcal{O}_L \otimes R \simeq R[T]/(T^g)$ is a local ring they are free. Note morover that the reduction $\overline{u^{\mathrm{un}}}$ of the morphism of Dieudonné displays $u^{\mathrm{un}}: \mathcal{P}_1^{\mathrm{un}} \to \mathcal{P}_2^{\mathrm{un}}$ respects the Hodge filtrations $H_{\mathcal{P}_1^{\mathrm{un}}} = Q_1^{\mathrm{un}}/\mathbb{I}(R)P_1^{\mathrm{un}}$ and $H_{\mathcal{P}_2^{\mathrm{un}}} = Q_2^{\mathrm{un}}/\mathbb{I}(R)P_2^{\mathrm{un}}$, that is

$$\overline{u^{\mathrm{un}}}(H_{\mathcal{P}_1^{\mathrm{un}}}) \subseteq H_{\mathcal{P}_2^{\mathrm{un}}}.$$

By the theory of local models (see Section (4.1) for details) there exists a morphism on $\mathrm{Spec}(R)$, respecting the closed subscheme $\mathrm{Spec}(R/\mathfrak{m}_R^2)$

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "\psi_1"] & \mathrm{Spec}(R) \\
\mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[r] \arrow[u, hook] & \mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[u, hook]
\end{tikzcd}$$

such that

$$\begin{tikzcd} \psi_1^*(\tilde{F} \subset (R[T]/(T^g))^2 \xrightarrow{u} \tilde{F}' \subset (R[T]/(T^g))^2) \arrow[d, "\simeq"] \\ (H_{\mathcal{P}_1^{\mathrm{un}}} \subset D_{\mathcal{P}_1^{\mathrm{un}}} \xrightarrow{\overline{u^{\mathrm{un}}}} H_{\mathcal{P}_2^{\mathrm{un}}} \subset D_{\mathcal{P}_2^{\mathrm{un}}}) \end{tikzcd} \tag{6.10}$$

In particular, we may find trivializations

$$\begin{tikzcd} D_{\mathcal{P}_1^{\mathrm{un}}} \arrow[r, "\simeq"] \arrow[d, "\overline{u^{\mathrm{un}}}"'] & (R[T]/(T^g))^2 \arrow[d, "u"] \\ D_{\mathcal{P}_2^{\mathrm{un}}} \arrow[r, "\simeq"] & (R[T]/(T^g))^2 \end{tikzcd}$$

such that on $R/\mathfrak{m}_R^2$

$$\psi_1^*(R[T]/(T^g)e_1 \oplus R[T]/(T^g)e_2 \xrightarrow{u} R[T]/(T^g)e_1' \oplus R[T]/(T^g)e_2') \xrightarrow{\simeq} (D_{\mathcal{P}_1^{\mathrm{un}}} \xrightarrow{\overline{u^{\mathrm{un}}}} D_{\mathcal{P}_2^{\mathrm{un}}})$$

induced by (6.10) is horizontal with respect to the Gauss-Manin connection.

By the theory of local model there exists as well a morphism

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[r, "\psi_2"] & \mathrm{Spec}(R) \\ \mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[r] \arrow[u, hook] & \mathrm{Spec}(R/\mathfrak{m}_R^2) \arrow[u, hook] \end{tikzcd}$$

such that

$$\begin{tikzcd} \psi_2^*(\tilde{F} \subset (R[T]/(T^g))^2 \xrightarrow{u} \tilde{F}' \subset (R[T]/(T^g))^2) \arrow[d, "\simeq"] \\ (H_{\mathcal{P}_1} \subset D_{\mathcal{P}_1} \xrightarrow{\overline{u}} H_{\mathcal{P}_2} \subset D_{\mathcal{P}_2}) \end{tikzcd} \tag{6.11}$$

We claim that by construction of $u: \mathcal{P}_1 \to \mathcal{P}_2$, not only the isomorphism in (6.11) is horizontal modulo $\mathfrak{m}_R^2$ with respect to the Gauss-Manin connection, it is actually the identity.

Let us describe in detail the picture provided by Theorem (3.4.3), Chapter (2), in this case. The identity map $\mathrm{id}: \mathcal{P}_B \to \mathcal{P}_B$ lifts to a unique isomorphism of quadruples

$$\widehat{\mathcal{P}_B} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\simeq} \widehat{\mathcal{P}_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2), \tag{6.12}$$

the same holding for $\mathcal{P}_A$ and its lift $\mathcal{P}_2$:

$$\widehat{\mathcal{P}_A} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \xrightarrow{\simeq} \widehat{\mathcal{P}_2} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2). \tag{6.13}$$

By construction of $\mathcal{P}_1$ and $\mathcal{P}_2$, the isomorphisms in (6.12) and (6.13) are the identity. Indeed, the $\mathcal{O}_L \otimes \mathbb{W}(R)$-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ describing P_1 and P_2 are obtained by lifting the $\mathcal{O}_L \otimes W(k)$-bases describing P_B and P_A; remark that in fact such bases are obtained from the description of the local model itself! Note that by construction

$$F_1 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = F_B \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2), \qquad F_2 \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = F_A \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2),$$

indeed

$$\underbrace{w^{-1}\cdot\begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}g_{1,1} - {}^F\hat{b}Tg_{1,2} & T^{i+1}g_{1,2} - {}^F\hat{c}Tg_{1,1} + {}^F\hat{a}Tg_{1,2} \\ T^j g_{2,1} + {}^F\hat{d}g_{2,1} - {}^F\hat{b}g_{2,2} & T^i g_{2,2} - {}^F\hat{c}Tg_{2,1} + {}^F\hat{a}g_{2,2}\end{pmatrix}}_{F_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = \underbrace{\begin{pmatrix} T^j g_{1,1} & T^{i+1}g_{1,2} \\ T^j g_{2,1} & T^i g_{2,2}\end{pmatrix}}_{F_B} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2),$$

$$\underbrace{w^{-1}\cdot\begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}g_{1,1} - {}^F\hat{b}Tg_{1,2} & T^{i}g_{1,2} - {}^F\hat{c}g_{1,1} + {}^F\hat{a}g_{1,2} \\ T^{j+1} g_{2,1} + {}^F\hat{d}Tg_{2,1} - {}^F\hat{b}Tg_{2,2} & T^i g_{2,2} - {}^F\hat{c}Tg_{2,1} + {}^F\hat{a}g_{2,2}\end{pmatrix}}_{F_2} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) = \underbrace{\begin{pmatrix} T^j g_{1,1} & T^{i}g_{1,2} \\ T^{j+1} g_{2,1} & T^i g_{2,2}\end{pmatrix}}_{F_A} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2).$$

Since $H_{\mathcal{P}_1} = \mathrm{Ker}(\overline{F}_1)$ and $H_{\mathcal{P}_2} = \mathrm{Ker}(\overline{F}_2)$, also the Hodge filtrations $(H_{\mathcal{P}_1}, H_{\mathcal{P}_2})$ of the two displays are identified with the universal object $(\tilde{F}, \tilde{F}')$ of the Grassmannian moduli problem modulo $\mathfrak{m}_R^2$. By uniqueness of the isomorphisms (6.12) and (6.13), we deduce that the identity is precisely the crystalline morphism provided by Zink's theory.

Note that this is compatible with the morphisms of display $\mathbb{D}(f)\colon \mathcal{P}_B \to \mathcal{P}_A$ and its lift $u\colon \mathcal{P}_1 \to \mathcal{P}_2$. Namely, by Theorem (3.4.3), Chapter (2), we get the following picture:
(6.14)

$$\begin{tikzcd} \widehat{\mathcal{P}_B} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\simeq"] & \widehat{\mathcal{P}_1} \otimes_{\mathbb{W}(R)} \mathbb{W}(R/\mathfrak{m}_R^2) \arrow[r, "\overline{u}", "\exists!"'] & \widehat{\mathcal{P}_2} \otimes_{\mathbb{W}(R)} W(R/\mathfrak{m}_R^2) \arrow[r, "\simeq"] & \widehat{\mathcal{P}_A} \otimes_{W(k)} \mathbb{W}(R/\mathfrak{m}_R^2) \\ \mathcal{P}_B \arrow[r, "\mathrm{id}"] & \mathcal{P}_B \arrow[r, "\mathbb{D}(f)"] & \mathcal{P}_A \arrow[r, "\mathrm{id}"] & \mathcal{P}_A. \end{tikzcd}$$

Therefore, trivially the induced isomorphism on $R/\mathfrak{m}_R^2$

$$\psi_2^*(R[T]/(T^g)e_1 \oplus R[T]/(T^g)e_2 \xrightarrow{u} R[T]/(T^g)e_1' \oplus R[T]/(T^g)e_2') \xrightarrow{\simeq} (D_{\mathcal{P}_1} \xrightarrow{\overline{u}} D_{\mathcal{P}_2})$$

is horizontal with respect to the Gauss-Manin connection.

The morphisms ϕ, ψ_1 and ψ_2 are all canonical and there is a factorization

$$\psi_1 \circ \phi = \psi_2.$$

Both ψ_1 and ψ_2 are isomorphisms on tangent spaces, and hence also ϕ is an isomorphism on tangent spaces. From this it follows that for any n the induced morphism $\mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \to \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$ is surjective. Let $\mathrm{Gr}(R) = \oplus_n \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$ be the graded ring associated to R. the map ϕ induces a map $\mathrm{Gr}(\phi^\sharp)\colon \mathrm{Gr}(R) \to \mathrm{Gr}(R)$ which is surjective on each graded piece and hence by dimension considerations $\mathrm{Gr}(\phi^\sharp)$ is an isomorphism. By [AM69, Lemma 10.23] $\varphi^\sharp$ is an isomorphism as well, which proves the Claim.

□

6.2. Case when $j' = j(\underline{A}) = j(\underline{B})-1$. Following Langer-Zink (Theorem (3.4.4), Chapter (2)) we associate Dieudonné displays $\mathcal{P}_A = (P_A, Q_A, F_A, V_A^{-1})$ and $\mathcal{P}_B = (P_B, Q_B, F_B, V_B^{-1})$ to $\underline{A}$ and $\underline{B}$ respectively, and the isogeny f translates into a morphism of Dieudonné displays

$$\mathbb{D}(f)\colon \mathcal{P}_B \to \mathcal{P}_A.$$

We shall use ideas identical to the ones exposed in detail in the case when $j(\underline{A}) = j(\underline{B})$, Section (6.1). From now on, we will assume that $f\colon \underline{A} \to \underline{B}$ falls into Case 3 from Remark (7). Recall hence that by Note (3), there exist $k[T]/(T^g)$-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ for $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$ such that the induced morphism $\mathbb{D}(f)\colon H^1_{dR}(B/k) \to H^1_{dR}(A/k)$ is represented by the matrix

$$\mathbb{D}(f) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix},$$

and such that the Hodge filtrations are described by

$$H^0(B,\Omega^1_{B/k}) = T^i e_1 \oplus T^j e_2, \qquad H^0(A,\Omega^1_{A/k}) = T^{i'} e_1' \oplus T^{j'} e_2'$$

with $j \le i$ and $j' \le i'$. Case 4 (Remark (7)) can be obtained simply through Atkin-Lehner. We will comment briefly on this at the end of this section.

Lemma 6.2.1. *There exist $\mathcal{O}_L \otimes W(k)$-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ of $H^1_{cris}(B/W(k))$ and $H^1_{cris}(A/W(k))$ lifting the $\mathcal{O}_L \otimes k$-bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ of $H^1_{dR}(B/k)$ and $H^1_{dR}(A/k)$ as in Note (3). In such bases, the induced morphism $\mathbb{D}(f)\colon H^1_{cris}(B/W(k)) \to H^1_{cris}(A/W(k))$ is represented by the matrix*

$$\mathbb{D}(f) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Note that Lemma (6.0.1) holds also in this case. In order to obtain this description of the Hodge filtrations and of $\mathbb{D}(f)$ we just need to switch e_1 with e_2 and e_1' with e_2' as in Remark (7). □

We may hence represent the Dieudonné displays $\mathcal{P}_B$ and $\mathcal{P}_A$ in terms of $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ as in Lemma (6.2.1). Take

$$\begin{aligned} P_B &= W(k)[T]/(E(T))e_1 \oplus W(k)[T]/(E(T))e_2, \\ P_A &= W(k)[T]/(E(T))e_1' \oplus W(k)[T]/(E(T))e_2', \\ Q_B &= W(k)[T]/(E(T)) \cdot (T^i e_1 \oplus T^j e_2), \\ Q_A &= W(k)[T]/(E(T)) \cdot (T^{i'} e_1' \oplus T^{j'} e_2'), \end{aligned}$$

and that the morphism of Dieudonné displays $\mathbb{D}(f)\colon \mathcal{P}_B \to \mathcal{P}_A$ is represented by

$$\mathbb{D}(f) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

By imposing conditions (6.6) and (6.7) in Section (6.1) we obtain a description of the semilinear maps $F_B\colon P_B \to \mathcal{P}_B$ and $F_A\colon P_A \to P_A$:

$$F_B = \begin{pmatrix} T^j g_{1,1} & T^i g_{1,2} \\ T^j g_{2,1} & T^{i+1} g_{2,2} \end{pmatrix}, \qquad F_A = \begin{pmatrix} T^{j'+1} g_{1,1} & T^{i'} g_{1,2} \\ T^{j'} g_{2,1} & T^{i'} g_{2,2} \end{pmatrix} = \begin{pmatrix} T^j g_{1,1} & T^{i+1} g_{1,2} \\ T^{j-1} g_{2,1} & T^{i+1} g_{2,2} \end{pmatrix},$$

with $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2} \in W(k)[T]/(E(T))$.

We wish to construct an object

$$\mathcal{P}_1 \xrightarrow{u} \mathcal{P}_2$$

over

$$\begin{aligned} R := \widehat{\mathcal{O}}_{\mathcal{M}_\mathfrak{p},t} = k[\![&a_0, \dots, a_{i-1}, b_0, \dots, b_{j-1}, c_0, \dots, c_{i'-1}, d_0, \dots, d_{j'-1}]\!] \\ &/(dT^{i'} + aT^j + adT - bc), \end{aligned}$$

universal with respect to the deformation of $\mathbb{D}(f)\colon \mathcal{P}_B \to \mathcal{P}_A$.

Recall the universal object pro-representing the local model associated to the k-point $t = (\underline{A} \xrightarrow{f} \underline{B})$ described in section (4.2):

$$\tilde{F} \subset (R[T]/(T^g)e_1 \oplus R[T]/(T^g)e_2) \xrightarrow{\left(\begin{smallmatrix} T & 0 \\ 0 & 1 \end{smallmatrix}\right)} \tilde{F}' \subset (R[T]/(T^g)e_1' \oplus R[T]/(T^g)e_2'),$$

with $\tilde{F}$ generated by $f = T^i e_1 + ae_1 + be_2$ and $u^t(f') = T^j e_2 + ce_1 + dTe_2$ and $\tilde{F}'$ generated by $f' = T^{j'} e'_2 + ce'_1 + de'_2$ and $u(f) = T^{i'} e'_1 + aTe'_1 + be'_2$.

Define hence $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ as quadruples over R with

$$\begin{aligned} P_1 &:= P_B \otimes_{W(k)} \mathbb{W}(R) = (\mathcal{O}_L \otimes \mathbb{W}(R))e_1 \oplus (\mathcal{O}_L \otimes \mathbb{W}(R))e_2, \\ P_2 &:= P_A \otimes_{W(k)} \mathbb{W}(R) = (\mathcal{O}_L \otimes \mathbb{W}(R))e'_1 \oplus (\mathcal{O}_L \otimes \mathbb{W}(R))e'_2, \end{aligned}$$

Q_1 (resp. Q_2) the pre-image of $\tilde{F}$ (resp. $\tilde{F}'$) through the natural map $P_1 \to (R[T]/(T^g))^2$:

$$Q_1 = \mathbb{I}(R)(e_1 \oplus e_2) + (\mathcal{O}_L \otimes \mathbb{W}(R))\widehat{\tilde{F}}, \qquad Q_2 = \mathbb{I}(R)(e'_1 \oplus e'_2) + (\mathcal{O}_L \otimes \mathbb{W}(R))\widehat{\tilde{F}}',$$

where $\widehat{\tilde{F}}$ is generated by $\widehat{\tilde{f}} = T^i e_1 + \hat{a}e_1 + \hat{b}e_2$ and $u^t(\widehat{\tilde{f}}') = T^j e_2 + \hat{c}e_1 + \hat{d}Te_2$ and $\widehat{\tilde{F}}'$ is generated by $u(\widehat{\tilde{f}}) = T^{i'} e'_1 + \hat{a}Te'_1 + \hat{b}e'_2$ and $\widehat{\tilde{f}}' = T^{j'} e'_2 + \hat{c}e'_1 + \hat{d}e'_2$.

Take the semi-linear maps $F_i: P_i \to P_i$ as the arrow represented by

$$\begin{aligned} F_1 &= z^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}Tg_{1,1} - {}^F\hat{b}g_{1,2} & T^i g_{1,2} + {}^F\hat{a}g_{1,2} - {}^F\hat{c}g_{1,1} \\ T^j g_{2,1} + {}^F\hat{d}Tg_{2,1} - {}^F\hat{b}Tg_{2,2} & T^{i+1} g_{2,2} + {}^F\hat{a}Tg_{2,2} - {}^F\hat{c}g_{2,1} \end{pmatrix}, \\ F_2 &= z^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}Tg_{1,1} - {}^F\hat{b}g_{1,2} & T^{i+1} g_{1,2} + {}^F\hat{a}Tg_{1,2} - {}^F\hat{c}Tg_{1,1} \\ T^{j-1} g_{2,1} + {}^F\hat{d}g_{2,1} - {}^F\hat{b}g_{2,2} & T^{i+1} g_{2,2} + {}^F\hat{a}Tg_{2,2} - {}^F\hat{c}g_{2,1} \end{pmatrix}, \end{aligned}$$

where

$$z := 1 + T^{-g\,F}(\hat{a}T^j + \hat{d}T^{i+1} + \hat{a}\hat{d}T - \hat{b}\hat{c});$$

(it makes sense to write z^{-1}, exactly as in Lemma (6.1.2)). Finally, for $i = 1, 2$, define $V_i^{-1}: Q_i \to P_i$ as $V_i^{-1} := F_i/p$.

THEOREM 6.2.2. *The maps $V_i: Q_i \to P_i$ are well defined and the quadruples $\mathcal{P}_1 = (P_1, Q_1, F_1, V_1^{-1})$ and $\mathcal{P}_2 = (P_2, Q_2, F_2, V_2^{-1})$ are Dieudonné displays over R.*

PROOF. We will show that $\mathcal{P}_1$ is a Dieudonné display. The proof for $\mathcal{P}_2$ is identical. Showing that V_1^{-1} is well defined, that is that $F_1(Q_1) \subseteq pP_1$ is a matter of computations. As already seen in the proof of Theorem (6.1.4), $F_1(\mathbb{I}(R)(e_1 \oplus e_2)) \subseteq P_1$. It is therefore enough to understand the image through F_1 of the generators $\widehat{\tilde{F}}$ and $u^t(\widehat{\tilde{f}}')$ of $\widehat{\tilde{F}}$.

$$\begin{aligned} F_1 \cdot \tilde{f} &= z^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}Tg_{1,1} - {}^F\hat{b}g_{1,2} & T^i g_{1,2} - {}^F\hat{c}g_{1,1} + {}^F\hat{a}g_{1,2} \\ T^j g_{2,1} + {}^F\hat{d}Tg_{2,1} - {}^F\hat{b}Tg_{2,2} & T^{i+1} g_{2,2} - {}^F\hat{c}g_{2,1} + {}^F\hat{a}Tg_{2,2} \end{pmatrix} \begin{pmatrix} T^i + \hat{a} \\ \hat{b} \end{pmatrix} \\ &= z^{-1} \begin{pmatrix} T^g g_{1,1} + T^{j\,F}\hat{a}g_{1,1} + T^{i+1\,F}\hat{d}g_{1,1} + {}^F\hat{a}^F\hat{d}Tg_{1,1} - T^{i\,F}\hat{b}g_{1,2} - {}^F\hat{a}^F\hat{b}g_{1,2} + T^{i\,F}\hat{b}g_{1,2} - {}^F\hat{b}^F\hat{c}g_{1,1} + {}^F\hat{a}^F\hat{b}g_{1,2} \\ T^g g_{2,1} + T^{j\,F}\hat{a}g_{2,1} + T^{i+1\,F}\hat{d}g_{2,1} + T^F\hat{a}^F\hat{d}g_{2,1} - {}^F\hat{b}T^{i+1}g_{2,2} - T^F\hat{a}^F\hat{b}g_{2,2} + T^{i+1\,F}\hat{b}g_{2,2} - {}^F\hat{b}^F\hat{c}g_{2,1} + T^F\hat{a}^F\hat{b}g_{2,2} \end{pmatrix} \\ &= z^{-1} \begin{pmatrix} g_{1,1}(T^g + T^{j\,F}\hat{a} + T^{i'\,F}\hat{d} + T^F\hat{a}^F\hat{d} - {}^F\hat{c}^F\hat{b}) \\ g_{2,1}(T^g + T^{j\,F}\hat{a} + T^{i'\,F}\hat{d} + T^F\hat{a}^F\hat{d} - {}^F\hat{c}^F\hat{b}) \end{pmatrix} \\ &= z^{-1} \begin{pmatrix} T^g g_{1,1} z \\ T^g g_{2,1} z \end{pmatrix} \\ &= T^g \begin{pmatrix} g_{1,1} \\ g_{2,1} \end{pmatrix} \in T^g P_1. \end{aligned}$$

And similarly

$$
\begin{aligned}
F_1 \cdot u^t(\widehat{\hat{f}'}) &= z^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}T g_{1,1} - {}^F\hat{b} g_{1,2} & T^i g_{1,2} - {}^F\hat{c} g_{1,1} + {}^F\hat{a} g_{1,2} \\ T^j g_{2,1} + {}^F\hat{d}T g_{2,1} - {}^F\hat{b}T g_{2,2} & T^{i+1} g_{2,2} - {}^F\hat{c} g_{2,1} + {}^F\hat{a}T g_{2,2} \end{pmatrix} \begin{pmatrix} \hat{c} \\ T^j + \hat{d}T \end{pmatrix} \\
&= z^{-1} \cdot \begin{pmatrix} T^{j\,F}\hat{c} g_{1,1} + {}^F\hat{d}^F\hat{c}T g_{1,1} - {}^F\hat{b}^F\hat{c} g_{1,2} + T^g g_{1,2} + {}^F\hat{d}T^{i+1} g_{1,2} + {}^F\hat{a}T^j g_{1,2} + {}^F\hat{a}^F\hat{d}T g_{1,2} - {}^F\hat{c}T^j g_{1,1} - {}^F\hat{c}^F\hat{d}T g_{1,1} \\ {}^F\hat{c}T^j g_{2,1} + {}^F\hat{c}^F\hat{d}T g_{2,1} - {}^F\hat{b}^F\hat{c}T g_{2,2} + T^{g+1} g_{2,2} + {}^F\hat{d}T^{i+2} g_{2,2} + {}^F\hat{a}T^{j+1} g_{2,2} + {}^F\hat{a}^F\hat{d}T^2 g_{2,2} - {}^F\hat{c}T^j g_{2,1} - {}^F\hat{c}^F\hat{d}T g_{2,1} \end{pmatrix} \\
&= z^{-1} \cdot \begin{pmatrix} -{}^F\hat{b}^F\hat{c} g_{1,2} + T^g g_{1,2} + {}^F\hat{d}T^{i+1} g_{1,2} + {}^F\hat{a}T^j g_{1,2} + {}^F\hat{a}^F\hat{d}T g_{1,2} \\ {}^F\hat{b}^F\hat{c}T g_{2,2} + T^{g+1} g_{2,2} + {}^F\hat{d}T^{i+2} g_{2,2} + {}^F\hat{a}T^{j+1} g_{2,2} + {}^F\hat{a}^F\hat{d}T^2 g_{2,2} \end{pmatrix} \\
&= z^{-1} \begin{pmatrix} T^g g_{1,2} z \\ T^{g+1} g_{2,2} z \end{pmatrix} \\
&= T^g \begin{pmatrix} g_{1,2} \\ T g_{2,2} \end{pmatrix} \in T^g P_1.
\end{aligned}
$$

Note that F_1 is the composition of the extension of the ^{F}-linear base change of F_B to P_1 after

$$\theta_1 = z^{-1} \cdot \begin{pmatrix} 1 + \hat{d}T^{-j'} & -\hat{c}T^{-j} \\ -\hat{b}T^{-i} & 1 + \hat{a}T^{-i} \end{pmatrix}.$$

The proof for showing that V^{-1} is a semi-linear epimorphism is identical to the proof of Theorem (6.1.4). □

Denote by u the morphism

$$u \colon P_1 \to P_2$$

represented, in the bases $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ by the matrix

$$u = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

By construction of the Q_1 and Q_2 it verifies the inclusion

$$u(Q_1) \subseteq Q_2,$$

and it verifies

$$u \circ F_1 = F_2 \circ u^\sigma.$$

Indeed

$$
\begin{aligned}
u \circ F_1 &= z^{-1} \cdot \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}T g_{1,1} - {}^F\hat{b} g_{1,2} & T^i g_{1,2} + {}^F\hat{a} g_{1,2} - {}^F\hat{c} g_{1,1} \\ T^j g_{2,1} + {}^F\hat{d}T g_{2,1} - {}^F\hat{b}T g_{2,2} & T^{i+1} g_{2,2} + {}^F\hat{a}T g_{2,2} - {}^F\hat{c} g_{2,1} \end{pmatrix} \\
&= z^{-1} \cdot \begin{pmatrix} T(T^j g_{1,1} + {}^F\hat{d}T g_{1,1} - {}^F\hat{b} g_{1,2}) & T(T^i g_{1,2} + {}^F\hat{a} g_{1,2} - {}^F\hat{c} g_{1,1}) \\ T^j g_{2,1} + {}^F\hat{d}T g_{2,1} - {}^F\hat{b}T g_{2,2} & T^{i+1} g_{2,2} + {}^F\hat{a}T g_{2,2} - {}^F\hat{c} g_{2,1} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
F_2 \circ u^{(\sigma)} &= z^{-1} \cdot \begin{pmatrix} T^j g_{1,1} + {}^F\hat{d}T g_{1,1} - {}^F\hat{b} g_{1,2} & T^{i+1} g_{1,2} + {}^F\hat{a}T g_{1,2} - {}^F\hat{c}T g_{1,1} \\ T^{j-1} g_{2,1} + {}^F\hat{d} g_{2,1} - {}^F\hat{b} g_{2,2} & T^{i+1} g_{2,2} + {}^F\hat{a}T g_{2,2} - {}^F\hat{c} g_{2,1} \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \\
&= z^{-1} \cdot \begin{pmatrix} T(T^j g_{1,1} + {}^F\hat{d}T g_{1,1} - {}^F\hat{b} g_{1,2}) & T^{i+1} g_{1,2} + {}^F\hat{a}T g_{1,2} - {}^F\hat{c}T g_{1,1} \\ T(T^{j-1} g_{2,1} + {}^F\hat{d} g_{2,1} - {}^F\hat{b} g_{2,2}) & T^{i+1} g_{2,2} + {}^F\hat{a}T g_{2,2} - {}^F\hat{c} g_{2,1} \end{pmatrix}
\end{aligned}
$$

Therefore $u \colon P_1 \to P_2$ is a morphism of Dieudonné displays

$$u \colon \mathcal{P}_1 \to \mathcal{P}_2.$$

THEOREM 6.2.3. *The morhism of Dieudonné displays $u \colon \mathcal{P}_1 \to \mathcal{P}_2$ over R is universal with respect to the deformation of $u \colon \mathcal{P}_B \to \mathcal{P}_A$.*

PROOF. By construction $\mathcal{P}_1$ and $\mathcal{P}_2$ reduce to $\mathcal{P}_B$ and $\mathcal{P}_A$ modulo $\mathfrak{m}_R$ respectively. Moreover the Hodge filtrations are

$$(H_{\mathcal{P}_1} \subset D_{\mathcal{P}_1}) = (\tilde{F} \subset (R[T]/(T^g)e_1 \oplus R[T]/(T^g)e_2)),$$
$$(H_{\mathcal{P}_2} \subset D_{\mathcal{P}_2}) = (\tilde{F}' \subset (R[T]/(T^g)e_1' \oplus R[T]/(T^g)e_2')).$$

The morphism $u: \mathcal{P}_1 \to \mathcal{P}_2$ lifts $\mathbb{D}(f): \mathcal{P}_B \to \mathcal{P}_A$ and by the crystalline nature of Dieudonné displays, it is the unique lift of $\mathbb{D}(f)$ to a morphism between $\mathcal{P}_1$ and $\mathcal{P}_2$.

The proof for the universality is identical to the proof of Theorem (6.1.5). □

6.3. Dieudonné Displays in normal forms. By [AG03, Proposition 4.10] we may write Dieudonné displays in a *normal form.* For $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ with $j(\underline{A}) = j(\underline{B}) = j$, there exists a $W(k)[T]/(E(T))$-basis $\{e_1, e_2\}$ (resp. $\{e_1', e_2'\}$) such that the Dieudonné display $\mathcal{P}_B$ (resp. $\mathcal{P}_A$) associated to $\underline{B}$ (resp. $\underline{A}$) has semi-linear Frobenius represented by the matrix

$$F_B = \begin{pmatrix} T^n & gT^{i+1} \\ T^j & 0 \end{pmatrix},$$

with $g \in W(k)[T]/(E(T))$ and

$$F_A = \begin{pmatrix} T^{n'} & \tilde{g}T^{i'} \\ T^{j'} & 0 \end{pmatrix},$$

with $\tilde{g} \in W(k)[T]/(E(T)))$. Similarly, in the case $j' = j(\underline{A}) \neq j(\underline{B}) = j$, the semi-linear Frobenius morphisms have normal forms

$$F_B = \begin{pmatrix} T^n & gT^{i} \\ T^j & 0 \end{pmatrix}, \qquad F_A = \begin{pmatrix} T^{n'} & \tilde{g}T^{i'} \\ T^{j'} & 0 \end{pmatrix};$$

note that this description does not hold in the case of superspecial points.

We also get an analogue to [AG03, Lemma 4.12].

LEMMA 6.3.1. *Let $\overline{F}_B$ and $\overline{F}_A$ denote the reductions modulo p of F_B and F_A respectively. The square $\overline{F}_B^2$ (resp. $\overline{F}_A^2$) is zero on $D_{\mathcal{P}_B}$ (resp. $D_{\mathcal{P}_A}$) if an only if $n = i = g - j$ (resp. $n' = i' = g - j'$).*

PROOF. By a direct computation we see that for ℓ a positive integer we have

$$\mathrm{rank}_{H_{\mathcal{P}_B}}(\overline{F}_B^{\ell}) = g - j - \min(i, \ell n), \qquad \mathrm{rank}_{H_{\mathcal{P}_A}}(\overline{F}_A^{\ell}) = g - j' - \min(i', \ell n'),$$

hence the conclusion. □

6.4. Remarks on ramification. Note that it is enough to consider the totally ramified case. Indeed, suppose more in general that

$$p\mathcal{O}_L = \mathfrak{p}^e,$$

with $f = [\mathcal{O}_L/\mathfrak{p} : \mathbb{F}_p]$. Then the f embeddings $\mathcal{O}_{L,\mathfrak{p}}^{\hat{ur}} \to W(k)$ induce an isomorphism

$$\mathcal{O}_L \otimes W(k) \simeq \bigoplus_{i=1}^{f} (W(k)[T]/(E_i(T))),$$

where the E_i's are polynomials with coefficients in $W(k)$ and degree e. We have moreover that

$$\mathcal{O}_L \otimes k \simeq \bigoplus_{i=1}^{f} (k[T]/(T^e)).$$

We may therefore decompose the local model further, but the methods would not change (see [AG04, Section 5.4] for details).

7. Proof of the main statements

In order to describe deeper geometric properties of the strata, we need to extract information provided by the description of the local deformation theory. In particular, the explicit description of the variation of Frobenius, by means of the theory of Dieudonné displays (Section (6)), will allow us to understand the local structure of the $W_{(j,n,j',n')}$'s.

As a consequence to the construction of the local model obtained in Section (4), we obtain local equations for the $S_{(j,j')}$'s:

Proposition 7.0.1. *Consider a geometric point $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ with singularity indices $j(\underline{A}) = j'$ and $j(\underline{B}) = j$. In the notation of Section (4), the deformations to $S_{(\tilde{j},\tilde{j}')}$, with $\tilde{j} \leq j$ and $\tilde{j}' \leq j'$ are described locally by the closed subscheme defined by the ideal*

$$\begin{aligned} &\langle a_k, b_k, c_k, d_k \mid 0 \leq k \leq \tilde{j} - 1 \rangle, && \text{when } \tilde{j} = \tilde{j}', \\ &\langle a_k, b_k, c_{k'}, d_k \mid 0 \leq k \leq \tilde{j} - 1, 0 \leq k' \leq \tilde{j}' - 1 \rangle, && \text{when } j = j' \text{ and } \tilde{j} = \tilde{j}' + 1, \\ &\langle a_k, b_k, c_k, d_{k'} \mid 0 \leq k \leq \tilde{j} - 1, 0 \leq k' \leq \tilde{j}' - 1 \rangle, && \text{when } j = j' + 1 \text{ and } \tilde{j} = \tilde{j}' + 1. \end{aligned}$$

Proof. Let $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ be a geometric point with $j(\underline{B}) = j$ and $j(\underline{A}) = j'$. Assume $\tilde{j} \leq j, \tilde{j}' \leq j'$, which implies that $x \in S_{(\tilde{j},\tilde{j}')}$. The completed local ring $\widehat{\mathcal{O}}_{S_{\tilde{j},\tilde{j}'},x}$ is a quotient of $\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},x}$. Indeed the closed immersion

$$S_{(\tilde{j},\tilde{j}')} \hookrightarrow \mathcal{M}_{\mathfrak{p}}$$

induces the surjection

$$\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},x} \twoheadrightarrow \widehat{\mathcal{O}}_{S_{(\tilde{j},\tilde{j}')}}.$$

In Section (4) we have determined the universal ring $\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},x} \simeq \widehat{\mathcal{O}}_{\mathcal{N}_{\mathfrak{p}},(W_1,W_2)}$. The universal object for the Grassmannian local model changes according to whether $j = j'$ or $j \neq j'$. Let us analyse in detail the two possibilities.

- $j = j'$: we know that the completed local ring $\widehat{\mathcal{O}}_{\mathcal{M}_{\mathfrak{p}},x}$ is isomorphic to $\widehat{\mathcal{O}}_{\mathcal{N}_{\mathfrak{p}},(F,F')}$ and the universal object for the Grassmannian local model is

$$\tilde{F}: \begin{cases} \tilde{f} = T^i e_1 + a e_1 + b e_2 \\ u^t(\tilde{f}') = T^j e_2 + c T e_1 + d e_2 \end{cases}, \qquad \tilde{F}': \begin{cases} u(\tilde{f}) = T^i e_1' + a e_1' + b T e_2' \\ \tilde{f}' = T^j e_2' + c e_1' + d e_2' \end{cases}$$

with

$$a = \sum_{s=0}^{i-1} a_s T^s, \qquad b = \sum_{s=0}^{j-1} b_s T^s, \qquad c = \sum_{s=0}^{i-1} c_s T^s, \qquad d = \sum_{s=0}^{j-1} d_s T^s.$$

The deformations of $(\tilde{F}, \tilde{F}')$ to $S_{(\tilde{j},\tilde{j}')}$ are parametrized by

$$\begin{cases} T^i e_1 + \tilde{a} e_1 + \tilde{b} e_2 \\ T^j e_2 + \tilde{c} T e_1 + \tilde{d} e_2 \end{cases}, \qquad \begin{cases} T^i e_1' + \tilde{a} e_1' + \tilde{b} T e_2' \\ T^j e_2' + \tilde{c} e_1' + \tilde{d} e_2' \end{cases},$$

where

$$\tilde{a} = \sum_{s=\tilde{j}}^{i-1} a_s T^s, \qquad \tilde{b} = \sum_{s=\tilde{j}}^{j-1} b_s T^s, \qquad \tilde{c} = \sum_{s=\tilde{j}}^{i-1} c_s T^s, \qquad \tilde{d} = \sum_{s=\tilde{j}}^{j-1} d_s T^s \qquad \text{if } \tilde{j} = \tilde{j}', \tag{7.1}$$

$$\tilde{a} = \sum_{s=\tilde{j}}^{i-1} a_s T^s, \qquad \tilde{b} = \sum_{s=\tilde{j}}^{j-1} b_s T^s, \qquad \tilde{c} = \sum_{s=\tilde{j}'}^{i-1} c_s T^s, \qquad \tilde{d} = \sum_{s=\tilde{j}'}^{j-1} d_s T^s \qquad \text{if } \tilde{j} = \tilde{j}' + 1. \tag{7.2}$$

The description of the deformation in (7.1) concludes the proof in the case $\tilde{j} = \tilde{j}'$. On the other hand, looking at the deformation of $\tilde{F}$ in the case $\tilde{j} = \tilde{j}' + 1$ we see that $\tilde{d}_{\tilde{j}} = 0$.

- $j = j' + 1$: The universal object for the deformation of the local model is in this case

$$\tilde{F} = \begin{cases} \tilde{f} = T^i e_1 + a e_1 + b e_2 \\ u^t(\tilde{f}') = T^j e_2 + c e_1 + d T e_2 \end{cases}, \qquad \tilde{F}' = \begin{cases} u(\tilde{f}) = T^{i'} e_1' + a T e_1' + b e_2' \\ \tilde{f}' = T^{j'} e_2' + c e_1' + d e_2' \end{cases},$$

with

$$a = \sum_{s=0}^{i-1} a_s T^s, \qquad b = \sum_{s=0}^{j-1} b_s T^s, \qquad c = \sum_{s=0}^{i'-1} c_s T^s, \qquad d = \sum_{s=0}^{j'-1} d_s T^s.$$

The deformations of $(\tilde{F}, \tilde{F}')$ to $S_{(\tilde{j},\tilde{j}')}$ are now parametrized by

$$\begin{cases} T^i e_1 + \tilde{a} e_1 + \tilde{b} e_2 \\ T^j e_2 + \tilde{c} e_1 + \tilde{d} T e_2 \end{cases}, \qquad \begin{cases} T^{i'} e_1' + \tilde{a} T e_1' + \tilde{b} e_2' \\ T^{j'} e_2' + \tilde{c} e_1' + \tilde{d} e_2' \end{cases}, \tag{7.3}$$

with

$$\tilde{a} = \sum_{s=\tilde{j}}^{i-1} a_s T^s, \qquad \tilde{b} = \sum_{s=\tilde{j}}^{j-1} b_s T^s, \qquad \tilde{c} = \sum_{s=\tilde{j}}^{i'-1} c_s T^s, \qquad \tilde{d} = \sum_{s=\tilde{j}}^{j'-1} d_s T^s, \qquad \text{if } \tilde{j} = \tilde{j}', \tag{7.4}$$

$$\tilde{a} = \sum_{s=\tilde{j}}^{i-1} a_s T^s, \qquad \tilde{b} = \sum_{s=\tilde{j}}^{j-1} b_s T^s, \qquad \tilde{c} = \sum_{s=\tilde{j}'}^{i'-1} c_s T^s, \qquad \tilde{d} = \sum_{s=\tilde{j}'}^{j'-1} d_s T^s, \qquad \text{if } \tilde{j} = j' \tilde{+} 1. \tag{7.5}$$

The description in (7.4) concludes the proof in the case $\tilde{j} = \tilde{j}'$. In order to conclude the proof of the case when $\tilde{j} = \tilde{j}' + 1$ we just need to look at the deformation of $\tilde{F}$ and conclude that $\tilde{c}_{\tilde{j}'} = 0$.

□

COROLLARY 7.0.2. *The $S^0_{(j,j')}$'s are locally closed. Given a geometric point $x = (\underline{A} \xrightarrow{f} \underline{B}) \in \mathcal{M}_{\mathfrak{p}}(k)$ of singularity indices (j, j'), the formal neighborhood of $S^0_{(j,j')}$ at x is isomorphic to*

$$k[\![c_j, c_{j+1}, \ldots, c_{g-j'-1}]\!].$$

In particular, $S^0_{(j,j')}$ is non-singular of dimension $g - j' - j$.

PROOF. We just need to apply Proposition (7.0.1) with $\tilde{j} = j$ and $\tilde{j}' = j'$. □

COROLLARY 7.0.3. *The stratum $S_{(j,j')}$ has dimension $g - j' - j$ and its generic points have singularity indices (j, j').*

COROLLARY 7.0.4. *The points $\underline{A} \xrightarrow{f} \underline{B}$ with singularity index $(j(\underline{A}), j(\underline{B})) = (0, 0)$ are dense in $\mathcal{M}_{\mathfrak{p}}$. In particular the Hilbert modular variety with $\Gamma_0(\mathfrak{p})$-level structure $\mathcal{M}_{\mathfrak{p}}$ has dimension g.*

Note that there are the following relations between the $S^0_{(j,j')}$'s and the singularity-slope subsets:

- When $j \notin \{0, [g/2]\}$:

$$S^0_{(j,j-1)} = W_{(j,j,j-1,j)} \sqcup W_{(j,j+1,j-1,j+1)} \sqcup W_{(j,j+2,j-1,j+2)} \sqcup \cdots \sqcup W_{(j,g-j,j-1,g-j)} \sqcup W_{(j,g-j,j-1,g-j+1)}.$$

-

$$S^0_{(0,1)} = W_{(0,1,1,1)} \sqcup W_{(0,2,1,2)} \sqcup \cdots \sqcup W_{(0,g-1,1,g-1)} \sqcup W_{(0,g,1,g-1)}.$$

- When g is even:

$$S^0_{(g/2,g/2-1)} = W_{(g/2,g/2,g/2-1,g/2)} \sqcup W_{(g/2,g/2,g/2-1,g/2+1)}.$$

- When g is odd:

$$S^0_{((g-1)/2,(g-1)/2-1)} = W_{((g-1)/2,(g-1)/2,(g-1)/2-1,(g-1)/2)} \sqcup W_{((g-1)/2,(g-1)/2,(g-1)/2-1,(g-1)/2+1)}.$$

- We always have

$$S^0_{(j,j)} = W_{(j,j,j,j)}.$$

By Lemma (6.3.1) we obtain a local description of the $W_{(j,n,j',n')}$ as a consequence of the construction of universal Dieudonné displays obtained in Section (6).

LEMMA 7.0.5. *Fix a geometric point $x = (\underline{A} \xrightarrow{f} \underline{B}) \in W_{(j,n,j',n')}$. For $(\tilde{j}', \tilde{j}) \leq (j', j)$, locally around x, the deformations to $W_{(\tilde{j}',\tilde{n}',\tilde{j},\tilde{n})}$ are parametrized by the closed subscheme of deformations to $S_{(\tilde{j}',\tilde{j})}$ intersected with the closed subscheme given by relations $T^{\tilde{j}'+\tilde{n}'}|F_2^2$ and $T^{\tilde{j}+\tilde{n}}|F_1^2$.*

PROOF. By taking Dieudonné displays in their normal form as in Section (6.3) we see through direct computations that

$$T^{j+n}|\overline{F}^2_B, \qquad T^{j'+n'}|\overline{F}^2_A,$$

and $T^{j+n+1} \nmid F_B^2, T^{j'+n'} \nmid \overline{F}^2_A$. For instance, when $j(\underline{A}) = j(\underline{B})$, the Dieudonné displays in normal form are

$$F_B\begin{pmatrix} T^n & gT^{i+1} \\ T^j & 0 \end{pmatrix}, \qquad F_A\begin{pmatrix} T^{n'} & \tilde{g}T^i \\ T^j & 0 \end{pmatrix}, \qquad g, \tilde{g} \in W(k)[T]/(E(T)),$$

(note that for instance $n = n'$ when $j = j'$). Hence

$$\overline{F}^2_B = \begin{pmatrix} T^{2n} & 0 \\ T^{j+n} & 0 \end{pmatrix}, \qquad \overline{F}^2_A = \begin{pmatrix} T^{2n} & 0 \\ T^{j+n} & 0 \end{pmatrix}.$$

The computations for the case $j = j'$ are similar. See (6.3.1) for details. □

COROLLARY 7.0.6. *Given a geometric point of parameters (j, n, j', n'), the formal neighbourhood of $W_{(j,n,j',n')}$ at x is isomorphic to*

$$k[\![c_n, c_{n+1}, \ldots, c_{g-j'-1}]\!].$$

In particular, $W_{(j,n,j',n')}$ is locally irreducible, non-singular of dimension $g - j' - n'$. Finally, the generic points of $S_{(j,j')}$ have parameters (j, j, j', j).

PROOF. The results follows from Corollary (7.0.2) when $j = j'$. Indeed in this case $n = n'$ and $W_{(j,j,j,j)} = S^0_{(j,j)}$. Let us therefore show the result in the case $j = j' + 1$. This comes from a direct application of Lemma (7.0.5). As $W_{(j,n,j',n')} \subset S^0_{(j,j')}$, by Corollary (7.0.2) we may simplify F_1 and F_2. We get namely

$$F_1 = \begin{pmatrix} T^n & T^i g - {}^F\hat{c} \\ T^j & -{}^F\hat{c} \end{pmatrix}, \qquad F_2 = \begin{pmatrix} T^{n'} & T^{i+1}\tilde{g} - T^F\hat{c} \\ T^{j-1} & -{}^F\hat{c} \end{pmatrix},$$

with $c = \sum_{s=j}^{g-j'-1} c_s T^s$. For convenience, let us denote $\tau(\sum_{s=j}^{g-j'-1} c_s T^s)$ the Teichmüller lift of $\sum_{s=j}^{g-j'-1} c_s T^s$. The condition $T^{j+n}|F_1^2$ is the condition

$$T^{j+n}|\begin{pmatrix} T^{2n} + T^g g - T^{j\,F}\tau(c_jT^j + c_{j+1}T^{j+1} + \dots c_{g-j}T^{g-j}) & T^{n+i}g - T^{n\,F^2}\hat{c} - T^i g^{F^2}\hat{c} + {}^F\tau(c_jT^j + \dots c_{g-j}T^{g-j})^{F^2}\tau(c_jT^j + \dots c_{g-j}T^{g-j}) \\ T^{j+n} + T^{j\,F}\tau(c_jT^j + \dots c_{g-j}T^{g-j}) & T^g g - T^{j\,F^2}\hat{c} + {}^F\tau(c_jT^j + \dots c_{g-j}T^{g-j})^{F^2}\tau(c_jT^j + \dots c_{g-j}T^{g-j}) \end{pmatrix},$$

from which we get

$$c_j = c_{j+1} = \dots = c_{n-1} = 0.$$

On the other hand, the condition $T^{j-1+n}|F_2^2$ translates into

$$T^{j-1+n'}|\begin{pmatrix} T^{2n'} + T^g\tilde{g} - T^{j-1\,F}T\tau(c_jT^j + c_{j+1}T^{j+1} + \dots c_{g-j}T^{g-j}) & T^{n'+i+1}\tilde{g} - T^{n'\,F^2}\hat{c} - T^{i+1}\tilde{g}^F\hat{c} + T^F\hat{c}^{F^2}\tau(c_jT^j + \dots c_{g-j}) \\ T^{j-1+n'} + T^{j-1\,F}\tau(c_jT^j + \dots c_{g-j}T^{g-j}) & T^g\tilde{g} - T^{j-1}T^{F^2}\hat{c} + T^F\hat{c}^{F^2}\tau(c_jT^j + \dots c_{g-j}T^{g-j}), \end{pmatrix}$$

from which we obtain the conditions

$$c_j = c_{j+1} = \dots c_{n'-1} = 0.$$

Since $n \leq n'$ we conclude the proof. □

REMARK 8. *Note that n can be different from n' only on superspecial points.*

DEFINITION 7.0.7 (a-number strata). *Let $0 \leq a, a' \leq g$. Denote by $Z^0_{(a,a')}$ the locally closed subset of $\mathcal{M}_{\mathfrak{p}}$ whose geometric points $\underline{A} \xrightarrow{f} \underline{B} \in \mathcal{M}_{\mathfrak{p}}(k)$ are such that $a(\underline{A}) = a$ and $a(\underline{B}) = a'$.*

PROPOSITION 7.0.8. *The $Z_{(a,a')}$'s are non-empty.*

PROOF. This follows from the definition of slope n as the difference of the a-number and the singularity index. In particular, we have the relations

$$Z_{(a,a')} = \coprod_{j+n=a,j'+n'=a'} W_{(j,n,j',n')}. \tag{7.6}$$

Since the $W_{(j,n,j',n')}$'s are non-empty, the $Z_{(a,a')}$'s are non-empty. □

In particular the following relations occur.

- <u>if g is even:</u>

  $0 \leq a = a' \leq g$ with $a = a'$ even, or $0 < a = a' + 1 \leq g$ with a even and a'odd.
- <u>if g is odd:</u>

$0 \leq a = a' \leq g - 1$ with $a = a'$ even, or $a = a' = g$, or $0 < a = a' + 1 \leq g$ with a odd and a' even.

We always have the following identification:

$$W_{(j,n,j',n')} = S^0_{(j,j')} \cap Z^0_{(j+n,j'+n')}.$$

PROPOSITION 7.0.9. *The set $Z_{(a,a')}$ has dimension $g - a'$, when $a' < g$ and $g - 1$ when $a' = g$.*

PROOF. The assertion follows from Corollary (7.0.6) and from (7.6). □

7.1. Some examples in low dimension.

7.1.1. *Case when g=1.* There are the following strata:

$$\underline{g=1}$$

$$\begin{array}{lll}
(0,1,0,1) & S^0_{(0,0)} = \mathcal{M}^R_{\mathfrak{p}} = \mathcal{M}_{\mathfrak{p}} & \dim(W_{(0,1,0,1)}) = 0 \\[2em]
(0,0,0,0) & & \dim(W_{(0,0,0,0)}) = 1
\end{array}$$

7.1.2. *Case when g=2.* Here is the stratification of the space:

$$\underline{g=2}$$

$$\begin{array}{ccc}
 & \boxed{\begin{array}{c}(1,1,0,2)\\[2em](1,1,0,1)\end{array}} & \begin{array}{c} S^0_{(1,0)} = W_{(1,1,0,2)} \sqcup W_{(1,1,0,1)} \\[2em] S^0_{(1,1)} = W_{(1,1,1,1)} \end{array} \\
\boxed{(1,1,1,1)} & & \\[1em]
 & \boxed{(0,0,0,0)} & \mathcal{M}^{\mathrm{ord}}_{\mathfrak{p}} = S^0_{(0,0)} = W_{(0,0,0,0)}
\end{array}$$

The various strata have the following dimensions:

$$\dim(S^0_{(0,0)}) = 2, \qquad \dim(S^0_{(1,0)}) = 1, \qquad \dim(S^0_{(1,1)}) = 0$$
$$\dim(W_{(1,1,0,1)}) = 1, \qquad \dim(W_{(1,1,0,2)}) = 0$$

7.1.3. *Case when g=3.* There are the following singularity-slope strata:

$$\underline{g=3}$$

$$\begin{array}{cccc}
 & & \boxed{\begin{array}{c}(1,2,0,3)\\[2em](1,2,0,2)\\[2em](1,1,0,1)\end{array}} & S^0_{(1,0)} = W_{(1,1,0,1)} \sqcup W_{(1,2,0,2)} \sqcup W_{(1,2,0,3)} \\
S^0_{(1,1)} = W_{(1,2,1,2)} \sqcup W_{(1,1,1,1)} & \boxed{\begin{array}{c}(1,2,1,2)\\[2em](1,1,1,1)\end{array}} & & \\[1em]
 & & \boxed{(0,0,0,0)} & S^0_{(0,0)} = W_{(0,0,0,0)}
\end{array}$$

The strata have the following dimensions:

$$\dim(S^0_{(0,0)}) = 3, \qquad \dim(S^0_{(1,0)}) = 2, \qquad \dim(S^0_{(1,1)}) = 1,$$
$$\dim(W_{(1,1,0,1)}) = 2, \qquad \dim(W_{(1,1,1,1)}) = 1, \qquad \dim(W_{(1,2,0,2)}) = 1,$$
$$\dim(W_{(1,2,1,2)}) = 0, \qquad \dim(W_{(1,2,0,3)}) = 0.$$

Consider $x = (\underline{A} \xrightarrow{f} \underline{B})$ with $j(\underline{A}) = 1 = j(\underline{B})$. After the local model

$$\widehat{\mathcal{O}}_{\mathfrak{M}_{\mathfrak{p}},x} \simeq k[\![a_1, b_0, c_0, c_1]\!]/((b_0c_1 - a_1)(b_0c_0 - a_1b_0c_1 - a_1^2)).$$

According to Lemma (7.0.1) we obtain the following local description for the stratum $S_{(1,0)}$:

$$S_{(1,0)} \simeq_{loc} \mathrm{Spf}(k[\![c_1]\!]).$$

7.2. The closure of the $W_{(j,n,j',n')}$. Let us now show that the closure of a locally closed subset $W_{(j,n,j',n')}$ is a union of sets of type $W_{(\tilde{j},\tilde{n},\tilde{j}',\tilde{n}')}$, that is, the $W_{(j,n,j',n')}$ are a stratification of the space $\mathcal{M}_{\mathfrak{p}}$.

With the notation introduced in Section (5), we have

$$\pi_1^{-1}(W_{(j,n)}) = W_{(j,n,\Lambda(j,n))} = \bigcup_{(j',n')\in\Lambda(j,n)} W_{(j,n,j',n')},$$

that is, the fiber of a stratum in $\mathcal{M}$ is a union of strata in $\mathcal{M}_{\mathfrak{p}}$. In what follows we are going to use extensively the notation

$$W_{(j,n,\Lambda(j,n))} = \bigcup_{(j',n')\in\Lambda(j,n)} W_{(j,n,j',n')}.$$

THEOREM 7.2.1. *The $W_{(j,n,j',n')}$'s are a stratification of the moduli space $\mathcal{M}_{\mathfrak{p}}$. In particular, given $(j,n)\in J$ and $(j',n')\in\Lambda(j,n)$, we have*

$$\overline{W_{(j,n,j',n')}} = \Big(\bigcup_{(\tilde{j},\tilde{n})\in\Delta(j,n)} W_{(\tilde{j},\tilde{n},\Lambda(\tilde{j},\tilde{n}))}\Big) \cap \Big(\bigcup_{(\tilde{j}',\tilde{n}')\in\Delta(j',n')} W_{(\Lambda(\tilde{j}',\tilde{n}'),\tilde{j}',\tilde{n}')}\Big),$$

where Δ was defined by Andreatta and Goren [AG03, Theorem 8.14] as the unique function $\Delta\colon 2^J \longrightarrow 2^J$ characterized by the following relations for $0\le j\le g/2$:

$$\Delta(j,j) = \{(j'n')\in J : j\le j'\}, \qquad \Delta(j,g-j) = \{(j,g-j)\}, \qquad \Delta(j-1,n) = \Lambda(\Delta(j,n)).$$

PROOF. Assume first $j,j'>0$. Recall that

$$W_{(j,n,j',n')} = \pi_1^{-1}(W_{(j,n)}) \cap \pi_2^{-1}(W_{(j',n')}).$$

By [AG03, Lemma 8.4] both π_1 and π_2 are proper. From this, it follows that

$$\pi_1(\overline{W_{(j,n,j',n')}}) = \overline{\pi_1(W_{(j,n,j',n')})} = \overline{W_{(j,n)}},$$
$$\pi_2(\overline{W_{(j,n,j',n')}}) = \overline{\pi_2(W_{(j,n,j',n')})} = \overline{W_{(j',n')}},$$

from which we get that

$$\pi_1^{-1}(\overline{W_{(j,n)}}) \bigcap \pi_2^{-1}(\overline{W_{(j',n')}}) \subseteq \overline{W_{(j,n,j',n')}}.$$

Note moreover that by [AG03, Proposition 8.7] we have that

$$\pi_1^{-1}(\overline{W_{(j,n)}}) = \overline{\pi_1^{-1}(W_{(j,n)})}, \qquad \pi_2^{-1}(\overline{W_{(j',n')}}) = \overline{\pi_2^{-1}(W_{(j',n')})}$$

and hence

$$\begin{aligned}\overline{W_{(j,n,j',n')}} &= \overline{\pi_1^{-1}(W_{(j,n)}) \cap \pi_2^{-1}(W_{(j',n')})}\\ &\subseteq \overline{\pi_1^{-1}(W_{(j,n)})} \bigcap \overline{\pi_2^{-1}(W_{(j',n')})}\\ &= \pi_1^{-1}(\overline{W_{(j,n)}}) \bigcap \pi_2^{-1}(\overline{W_{(j',n')}}).\end{aligned}$$

We obtain hence the equality

$$\overline{W_{(j,n,j',n')}} = \pi_1^{-1}(\overline{W_{(j,n)}}) \bigcap \pi_2^{-1}(\overline{W_{(j',n')}}).$$

Andreatta and Goren described the closure of a stratum $W_{(j,n)}$ in $\mathcal{M}$ as follows:

$$\overline{W_{(j,n)}} = W_{\Delta(j,n)} = \bigcup_{(\tilde{j},\tilde{n})\in\Delta(j,n)} W_{(\tilde{j},\tilde{n})},$$

see [AG03, Theorem 8.14] for details. Therefore we have that the closure of the fiber is the union of strata:

$$\overline{\pi_1^{-1}(W_{(j,n)})} = \pi_1^{-1}(W_{\Delta(j,n)}) = \bigcup_{(\tilde{j},\tilde{n})\in\Delta(j,n)} W_{(\tilde{j},\tilde{n},\Lambda(\tilde{j},\tilde{n}))}.$$

Assume now that $j = 0$ and $j' > 0$. It follows that $j' = 1$, that is, we want to understand the closure of $W_{(0,n,1,n')}$. Observe that

$$\begin{aligned}\overline{W_{(0,n,1,n')}} &= \overline{\pi_1^{-1}(W_{(0,n)} \cap \pi_2^{-1}(W_{(1,n')})} \\ &= \overline{W_{(0,n,1,n')}} \cap \overline{\pi_2^{-1}(W_{(1,n')})} \\ &= \overline{\pi_2^{-1}(W_{(1,n')})},\end{aligned}$$

the last equality being true since obviously $W_{(0,n,1,n')} \subseteq \pi_2^{-1}(W_{(1,n')})$. Therefore we recover the previous case. □

From deformation theory we have seen that

$$\overline{W_{(j,j,j,j)}} = S_{(j,j)}.$$

In particular,

$$\overline{W_{(0,0,0,0)}} = S_{(0,0)} = \mathcal{M}_{\mathfrak{p}}.$$

Here is a description of the closure relations in low dimension.

$\underline{g = 1}$

$$\begin{tikzcd}
(0,1,0,1) \\
(0,0,0,0) \arrow[u]
\end{tikzcd}$$

$\underline{g = 2}$

$$\begin{tikzcd}
& (1,1,0,2) \\
(1,1,1,1) & (1,1,0,1) \arrow[l] \arrow[u] \\
& (0,0,0,0) \arrow[u]
\end{tikzcd}$$

$\underline{g = 3}$

$$\begin{tikzcd}
& (1,2,0,3) \\
(1,2,1,2) & (1,2,0,2) \arrow[l] \arrow[u] \\
(1,1,1,1) \arrow[u] & (1,1,0,1) \arrow[l] \arrow[u] \\
& (0,0,0,0) \arrow[u]
\end{tikzcd}$$

$\underline{g = 4}$

$$\begin{tikzcd}
& & (1,3,0,4) \\
& (2,2,1,3) & (1,3,0,3) \arrow[l] \arrow[u] \\
(2,2,2,2) & (2,2,1,2) \arrow[l] \arrow[u] & (1,2,0,2) \arrow[l] \arrow[u] \\
& (1,1,1,1) \arrow[u] & (1,1,0,1) \arrow[l] \arrow[u] \\
& & (0,0,0,0) \arrow[u]
\end{tikzcd}$$

$\underline{g = 8}$

$$\begin{tikzcd}
& & & & (1,7,0,8) \\
& & & (2,6,1,7) & (1,7,0,7) \arrow[l] \arrow[u] \\
& & (3,5,2,6) & (2,6,1,6) \arrow[l] \arrow[u] & (1,6,0,6) \arrow[l] \arrow[u] \arrow[dll] \\
& (4,4,3,5) & (3,5,2,5) \arrow[l] \arrow[u] & (2,5,1,5) \arrow[l] \arrow[u] \arrow[dll] & (1,5,0,5) \arrow[l] \arrow[u] \arrow[dll] \\
(4,4,4,4) & (4,4,3,4) \arrow[l] \arrow[u] & (3,4,2,4) \arrow[l] \arrow[u] & (2,4,1,4) \arrow[l] \arrow[u] & (1,4,0,4) \arrow[l] \arrow[u] \\
& (3,3,3,3) \arrow[u] & (3,3,2,3) \arrow[l] \arrow[u] & (2,3,1,3) \arrow[l] \arrow[u] & (1,3,0,3) \arrow[l] \arrow[u] \\
& & (2,2,2,2) \arrow[u] & (2,2,1,2) \arrow[l] \arrow[u] & (1,2,0,2) \arrow[l] \arrow[u] \\
& & & (1,1,1,1) \arrow[u] & (1,1,0,1) \arrow[l] \arrow[u] \\
& & & & (0,0,0,0) \arrow[u]
\end{tikzcd}$$

7.3. Proofs of the Main statements.

Proof of Theorem (5.4.1). (1): From the description of the $S^0_{(j,j')}$'s as a disjoint union of $W_{(j,n,j',n')}$'s we obtain that the $S^0_{(j,j')}$'s and the $S_{(j,j')}$'s are non-empty since the $W_{(j,n,j',n')}$'s are non-empty. By Corollary (7.0.2) we get that the $S^0_{(j,j')}$'s are non-singular, locally irreducible of dimension $g-j-j'$.
(2): This comes from Corollary (7.0.3). $\square$

Proof of Theorem (5.4.2). (1) This is a consequence of Corollary (7.0.6). (2) In Section (7.2) a description of the closure of a set $W_{(j,n,j',n')}$ is provided. In particular $\overline{W_{(j,n,j',n')}}$ is described as the union of other strata, from which we get that the $W_{(j,n,j',n')}$'s are a stratification of $\mathcal{M}_{\mathfrak{p}}$. $\square$

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APPENDIX A

Witt vectors

For a full introduction on Witt vectors, see [Ser79, Zin02, FJM].

In what follows, p is a fixed prime. Let R be a commutative ring of characteristic p. We set
$$W(R) = R^{\mathbb{N}^+}$$
as sets. We would like to give $W(R)$ a ring structure such that the relation between R and $W(R)$ generalizes the relation between $\mathbb{F}_p$, the field with p elements, and $\mathbb{Z}_p$, the ring of the p-adic integers.

We define $W_n(X) = \sum_{i=0}^n p^i X_i^{p^{n-i}}$ the *Witt polynomials* associated to the sequence $X = (X_0, \dots, X_n, \dots)$.

Denote by
$$\mathbb{Z}[\underline{X}, \underline{Y}] = \mathbb{Z}[X_0, \dots, X_n, \dots, Y_0, \dots, Y_n, \dots].$$
One can prove that for any polynomial $\Phi \in \mathbb{Z}[\underline{X}, \underline{Y}]$ there exists a unique sequence $\{\Phi_n\}_n$ of polynomials in $\mathbb{Z}[\underline{X}, \underline{Y}]$ such that
$$\Phi(W_n(X), W_n(Y)) = (\Phi_0(\underline{X}, \underline{Y}))^{p^n} + p(\Phi_1(\underline{X}, \underline{Y}))^{p^{n-1}} + \dots + p^n(\Phi_n(\underline{X}, \underline{Y})). \tag{0.1}$$
In fact one has
$$\Phi_n \in \mathbb{Z}[X_0, X_1, \dots, X_n, Y_0, Y_1, \dots, Y_n].$$
In the above notation, if $\Phi = X + Y$, we denote by Σ_i the associated Φ_i; if $\Phi = X \cdot Y$, we denote by Π_i the associated Φ_i. Remark that both $\Sigma_i, \Pi_i \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$.

For two elements $a = (a_0, \dots, a_n, \dots), b = b_0, \dots, b_n, \dots) \in W(R)$, we define their sum and multiplication respectively
$$a + b = (\Sigma_0(a_0, b_0), \Sigma_1(a_0, a_1, b_0, b_1), \dots), \qquad a \cdot b = (\Pi_0(a_0, b_0), \Pi_1(a_0, a_1, b_0, b_1), \dots).$$
From (0.1) we may deduce explicitly some of these polynomials. We have namely:
$$\Sigma_0(X_0, Y_0) = X_0 + Y_0, \qquad \Pi_0(X_0, Y_0) = X_0 Y_0,$$
$$\Sigma_1(X_0, X_1, Y_0, Y_1) = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X_0^i Y_0^{p-i},\ \Pi_1(X_0, X_1, Y_0, Y_1) = X_0^p Y_1 + X_1 Y_0^p + p X_1 Y_1.$$
The expressions become rather complicated as n grows. However we can easily see that for $n \geq 1$, the equality
$$W_n(X)W_n(Y) = (\Pi_0(X_0, Y_0))^{p^n} + p(\Pi_1(X_0, X_1, Y_0, Y_1))^{p^{n-1}} + \dots + p^n(\Pi_n(X_0, \dots, X_n, Y_0, \dots, Y_n)),$$
can be written as
$$\cancel{X_0^{p^n} Y_0^{p^n}} + X_0^{p^n} \sum_{i=1}^n p^i Y_i^{p^{n-i}} + Y_0^{p^n} \sum_{i=1}^n p^i X_i^{p^{n-i}} + \sum_{i=1}^n p^i X_i^{p^{n-i}} \sum_{i=1}^n p^i Y_i^{p^{n-i}}$$
$$= \cancel{X_0^{p^n} Y_0^{p^n}} + p(X_0^p Y_1 + X_1 Y_0^p + p X_1 Y_1)^{p^{n-1}} + \dots + p^n(\Pi_n(X_0, \dots, X_n, Y_0, \dots, Y_n)).$$

From this we obtain that

$$(a_0, 0, \dots, 0, \dots) \cdot (b_0, 0, \dots, 0, \dots) = (a_0 b_0, 0, \dots, 0, \dots).$$

It is *not* true in general that $(a_0, 0, \dots, 0, \dots) + (b_0, 0, \dots, 0, \dots) = (a_0 + b_0, 0, \dots, 0, \dots)$.

The map

$$\begin{array}{ccc} W(R) & \xrightarrow{\rho} & R^{\mathbb{N}^+} \\ \underline{a} = (a_0, a_1, \dots) & \longmapsto & (W_0(\underline{a}), W_1(\underline{a}), \dots) \end{array}$$

is a ring homomorphism. It is an isomorphism if p is invertible in R.

Example. *One has* $W(\mathbb{F}_p) = \mathbb{Z}_p$.

0.1. Maps related to Witt vectors. The Witt polynomials induce ring homomorphisms

$$\begin{array}{ccc} W(R) & \xrightarrow{w_n} & R \\ (x_0, \dots, x_n, \dots) & \longmapsto & x_0^{p^n} + p x_1^{p^{n-1}} + \dots + p^n x_n \end{array}$$

We denote by $I(R)$ the kernel of w_0.

We define moreover the *Verschiebung* map (or *shift*) as

$$\begin{array}{ccc} {}^V\colon W(R) & \to & W(R) \\ (x_0, \dots, x_n, \dots) & \longmapsto & (0, x_0, \dots, x_n, \dots) \end{array}$$

Denote $I(R) = {}^V W(R)$.

Given an element of R, this can be seen as embedded in the Witt vectors $W(R)$ through the *Teichmüller* map

$$\begin{array}{ccc} R & \longrightarrow & W(R) \\ x & \longmapsto & \widehat{x} = (x, 0, \dots, 0, \dots) \end{array}$$

Suppose now that R has positive characteristic p. Then the Frobenius homomorphism

$$\begin{array}{ccc} \sigma\colon R & \longrightarrow & R \\ x & \longmapsto & x^p \end{array}$$

induces a ring homomorphism on the Witt vectors

$$\begin{array}{ccc} {}^F\colon W(R) & \to & W(R) \\ (x_0, x_1, \dots) & \longmapsto & (x_0^p, x_1^p, \dots) \end{array}$$

It is called the *Frobenius map* on the Witt vectors.

The following relations between Frobenius and Verschiebung are satisfied:

$$^{FV} = p, \qquad {}^V({}^F x y) = x\,{}^V y, \qquad x, y \in W(R).$$

Lemma 0.1.1. *We have that*

$$^F I(R) \subseteq pW(R).$$

Proof. Since $I(R) = {}^V W(R)$, for a given an element $\eta \in I(R)$, there exists an element $\theta \in W(R)$ such that ${}^V\theta = \eta$. Therefore ${}^F\eta = {}^{FV}\theta = p\theta$, hence the conclusion. □

The Witt vectors construction is functorial. Given a ring homomorphism $\phi\colon A \to B$, we obtain a ring homomorphism

$$\begin{array}{cccc} W(\phi)\colon & W(A) & \longrightarrow & W(B) \\ & (a_0, a_i, \dots) & \longmapsto & (\phi(a_0), \phi(a_1), \dots) \end{array}$$

0.2. The Witt vectors of a perfect field. Let k be a perfect field of positive characteristic p. Then $W(k)$ is a discrete valuation ring with maximal ideal generated by p and fraction field of characteristic 0. More precisely, the element p in $W(k)$ is the vector $(0,1,0,\dots)$ and the maximal ideal is $\{(0,x_1,x_2,\dots)\}$. In this case ^{F} and ^{V} commute with each other and $I(k)=pW(k)$. The Witt vectors construction is universal in this case.

THEOREM 0.2.1. *Let R be a complete noetherian local ring with perfect residue field k of characteristic p. Then there is a uinque canonical map $\phi\colon W(k)\to R$ such that the following diagram commutes*

$$\begin{tikzcd} W(k) \arrow[r, "\phi"] \arrow[d, "/pW(k)"'] & R \arrow[dl] \\ k & \end{tikzcd}$$

PROOF. See [FJM, Theorem 0.40] □

0.3. On ^{F}-linearity. Let R be a ring of positive characteristic, and let P_1 and P_2 be two modules over $W(R)$. An *^{F}-linear homomorphism* $\phi\colon P_1\to P_2$ is a homomorphism of abelian groups between P_1 and P_2 such that $\phi(wm)={}^Fw\phi(m)$, for $m\in P_1$ and $w\in W(R)$. Consider $P_1^{(\sigma)}:=W(R)\otimes_{W(R),F}P_1$, it has the following structure of $W(R)$-module:

$$w_1(w_2\otimes x)=(w_1w_2)\otimes x,\qquad w_2\otimes w_1x={}^Fw_1w_2\otimes x,\qquad w_1,w_2\in W(R),x\in P_1.$$

Then we consider the *linearization* of ϕ

$$\begin{array}{rccc}\phi^\sharp\colon & P_1^{(\sigma)} & \longrightarrow & P_2\\ & w_2\otimes x & \longmapsto & w_2\phi(x)\end{array}$$

Following Zink Zink, we will call ϕ an *^{F}-linear epimorphism* (resp. *isomorphism*) if $\phi^\sharp$ is an epimorphism (resp. isomorphism).

Acknowledgements

I would like to thank my supervisors for their guidance and help during my PhD studies.

I am very grateful to Fabrizio Andreatta and Eyal Goren for suggesting the topic of this thesis and for all the time and assistance they provided. I benefitted immensely from their knowledge on the subject and from their numerous explanations. I also want to thank them for being so understanding when I needed to take some months off the PhD.

I am grateful to Adrian Iovita for teaching me a lot of the mathematics that I know and for giving me his help, his time and his encouragement when I most needed it.

I am grateful to Bas Edixhoven for having been so welcoming during my time in Leiden and for his help during the writing phase of my thesis. He read my thesis several times and gave me many useful comments, his kindness arriving to even meeting me on Sundays so that our meetings wouldn't conflict with my daily job!

A PhD in three different countries comes with a great deal of bureaucracy. I want to thank my supervisors for all the non-scientific work they had to do. I am also very grateful to Marie-France Leclere and Jose Garrido for all of their help with this.

I want to thank the ALGANT consortium, Concordia University, McGill University and the University of Milan for the financial support that made this PhD possible.

During these years I benefitted from the company of several people that made my days always interesting; the fear of turning this page into a soap opera script forces me to mention only few of them.

Eleonora, Dalia, Oscar, Rodrigo, Manuela, Sandro, Mariolys and all the Montreal gang, these years would never have been nearly as enriching without your friendship.

I am grateful to Dimitris, for his company and for being such an inspiring example of a researcher.

I want to thank Giulio and Yan for the nice conversations and for enduring me when I was struggling with motivation.

Thank you Mr Braver, for being such a good lawyer, translator, couple therapist, mechanic, yoga teacher and finally, such a good friend.

Thank you, my girl.

Sono immensamente grata alla mia famiglia per tutto l'appoggio e la comprensione. Voglio ringraziare Marta, Marco e Maria perché è importante avere un porto quieto e sicuro. Voglio ringraziare Ilaria, per il suo affetto e per la sua energia e mio padre per la sua discreta presenza e per avermi sempre lasciato libera nelle mie scelte. La tenacia, la dedizione e la passione per la conoscenza che ho messo in questi anni di studio sono valori che mi ha trasmesso mia madre. Questa tesi è dedicata alla sua memoria.