# A Praxeological Model of Future Elementary Teachers' Envisioned Practice of Teaching Geometric Transformations

Ildiko Pelczer

A Thesis

in

the Individualized Program (INDI)

Presented in Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy (Individualized Program) at

Concordia University

Montreal, Quebec, Canada

July, 2017

**CONCORDIA UNIVERSITY**

**SCHOOL OF GRADUATE STUDIES**

This is to certify that the thesis prepared

By: Ildiko Pelczer

Entitled: A Praxeological Model of Future Elementary Teachers' Envisioned Practice of Teaching Geometric Transformations

and submitted in partial fulfillment of the requirements for the degree of

Doctor Of Philosophy (Individualized Program)

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

______________________________ Chair
Dr. Nadia Hardy

______________________________ External Examiner
Dr. Viktor Freiman

______________________________ External to Program
Dr. Charles Reiss

______________________________ Examiner
Dr. Helena Osana

______________________________ Examiner
Dr. Vivek Venkatesh

______________________________ Thesis Supervisor
Dr. Anna Sierpinska

Approved by ______________________________
Dr. Sandra Chang-Kredl, Graduate Program Director

Monday August 28, 2017 ______________________________
Dr. Paula Wood-Adams, Dean
School of Graduate Studies

# Abstract


## A praxeological model of future elementary teachers' envisioned practice of teaching geometric transformations



**Ildiko Pelczer, Ph. D.**

**Concordia University, 2017**



Research on future teachers' geometrical knowledge mainly focused on comparing their knowledge against a reference body of knowledge.  Consequently, these studies were categorized into deficit (what is lacking) and affordance (what knowledge enables) approaches. Yet, research literature is very scarce on what future teachers' geometry knowledge *is*: what is its content and structure.

In research, under the term of personal epistemology are regrouped studies on people's perceptions on the nature of knowledge and knowing, in particular those of future or in-service teachers. The main research tool of these studies is a closed item questionnaire with questions usually targeting four aspects of knowledge: certainty, simplicity, source, and justification of knowledge. A shortcoming of this approach is in the difference between professed and enacted epistemologies: the questionnaires measure professed epistemologies, yet these can differ from what is manifested in a spontaneous manner, for example during classroom instruction.  A different line of research, pertinent to this thesis, is on teachers' conceptions of teaching and learning. Several studies suggest that teachers' conceptions rely on their epistemic beliefs. Both aspects are comprised in the term of *teachers' epistemology*, introduced by Brousseau (1987). In brief, teachers' epistemology comprises aspects of *what*, *why* and *how* to teach.

The aim of the thesis is to construct a praxeological model of future elementary teacher's envisioned practice of teaching geometric transformations. These praxeologies comprise the tasks future teachers would use in their classroom to teach the topic, as well the techniques they envision students should learn. Along with these elements, we describe the theory underlying the techniques. The theory is the expression of future teachers' geometric knowledge. In addition, by analyzing future teachers' lesson plans, we identify manifestations of their spontaneous, enacted epistemology. By extrapolating from the detailed findings about their praxeologies, a broader characterization of future teachers' epistemology is proposed.

The thesis discusses the potential implications of future teachers' praxeologies for students' learning of mathematics and proposes some recommendations for the preparation of teachers to teach mathematics.

# Acknowledgement

For Anna – with gratitude: for she taught me words to express my thoughts.

# Table of Contents

# List of Figures

# List of Tables

# 1 INTRODUCTION

*"We can find no scar,*
*But internal difference –*
*Where the Meanings, are – "*
EMILY DICKINSON: *There's a certain Slant of light*

## 1.1 THE RESEARCH QUESTION

Every mathematics teacher has their own vision of the nature of mathematical knowledge, of what is worth teaching and how: this vision has sometimes been called the teacher's epistemology (Brousseau, 1997). This epistemology changes with experience of teaching and the institution in which the profession is exercised. Young teachers enter the profession and, according to a popular myth, their lofty ideals are crushed by the sad everyday reality of the common classroom; the changes are for the worse. In this thesis, I challenge this myth and ask the question: What are the future teachers' ideals and are they indeed worth saving from change? I focus on the future teachers' ideals of teaching mathematics, more precisely – geometry.

My data are drawn from observations and written productions of a group of future teachers enrolled in a math methods course of an elementary teacher preparation program in a large, urban North American university. The written productions consist mainly of the future teachers' lesson plans on the topic of geometric transformations (translations, reflections, rotations). I assumed that these lesson plans embodied the future teachers' vision of an ideal teaching of the topic, and thus reflected their personal epistemology: what they think they should be teaching, why and how. The lesson plans described their envisioned classroom interactions, the tasks they wanted to propose their students and why, the expected solutions to these, as well as expected students' mistakes and how they would react to these, among other things, and thus were a rich source of information about the future teachers' epistemology.

I also assumed that the future teachers' epistemology is a product of their school experience with mathematics as students, their interpretations of current official school programs, and, to some extent, the ideas they were exposed to in the pedagogical and content courses in the teacher preparation program they were enrolled in, including the math methods course in which my observations took place. I also noted that the future teachers' interpretations of school programs were influenced by how they seemed to be realized in textbook activities and internet resources for teachers.

The teachers' epistemology is not explicit in the statements they make; it has to be constructed by digging deep into the meanings that can be gleaned from their discourse in each and every element of their lesson plans. It takes a theoretical framework to organize all those meanings into some cogent system. I found that Chevallard's Anthropological Theory of the Didactic (ATD) (1999) and especially the notion of praxeology were quite useful for this purpose. The result is a *model* (in the sense of mental construction, theory) of the future teachers' envisaged practice of teaching geometric transformations. I refer to it as "praxeological model" because it is couched in terms of Chevallard's notion of praxeology, a theory of practice as composed of tasks, techniques to accomplish them, methods to justify the techniques and theories to justify the methods. ATD builds on the fundamental view of human practice as directed towards solving problems in a discursive environment that happens in a historical context. By the central role it attributes to practice and the supporting discourse, ATD was my choice as the main part of the conceptual framework for my research.

## 1.2 THE BROADER CONTEXT OF THE RESEARCH

Research on future teachers' subject matter knowledge is motivated by the fact that it has proven crucial for students' learning and achievement (e.g. Hill, Rowan, & Ball (2005); Morris, Hiebert, & Spitzer (2009); Harris & Sass (2011)). Of the different areas of mathematics, geometry and measurement are among the frequently studied areas for subject matter knowledge. While the study of geometry is considered an important means to develop students' skills of visualization, perspective, problem solving, conjecturing, deductive reasoning, etc. (Jones, 2002), the time allotted to geometry in curricula was reduced and the amount of content to be taught also significantly diminished (Whiteley (1999); Jones (2002); Žilková, Gunčaga, & Kopáčová (2015)). With less geometry instruction received by the future teachers' during their schooling, the question raised by researchers focused on how much and how well future teachers know geometry and to what extent they are prepared to teach geometry.

It must be remarked that not only the extent of geometry curricula changed in time, but also the fundamental position on how to introduce and discuss geometry. Felix Klein proposed a reconceptualization of the nature of geometry in his 1872 address at the Erlangen University. In this new approach, a geometry is defined by a group of transformations and the properties of the given geometric space invariant under these transformations. His address represented the opportunity to link developments in algebra, mainly related to group theory, with geometry. The beginning of the twentieth century witnessed a widespread initiative to change the teaching of synthetic geometry, or Euclidean geometry (Fujita, 2001), leading to a more practical approach and gradual building of theoretical knowledge. However, another change was triggered almost simultaneously, when several European countries (for example, for the case of France, see Bkouche, 2009) modified their curriculum so as to include plane transformations. The trend to introduce geometry through transformations gained popularity also due to mathematicians who engaged in writing books based on this approach or developed curriculum (Herbst, Fujita, Halverscheid, & Weiss, 2017). In practical terms, textbooks often do not present a uniform view on the nature of geometry and, in consequence, the place of plane transformations in the curriculum is not straightforward. For example, Quebec curriculum foresees, at secondary level, to study the geometric properties invariant under transformations, yet the congruency of triangles is introduced through the three fundamental cases of congruency (and not through the existence of an isometry).

One can hypothesize at least two consequences of the above outlined situation (diminishing geometry content and unclear positioning on the nature of geometry): a) future teachers' content knowledge of geometry will be reduced; and b) the place and role of particular geometrical concepts in the construction of other geometrical concepts will be unclear.

### 1.2.1 Research on teachers' knowledge of geometry

Based on a review of 26 articles of peer-reviewed research on future-teachers' geometry and measurement knowledge published up to 2012, Browning et al. (2014), summarized the findings along two main lines: a) future teachers' overall conceptions in geometry and measurement are weak and limited; b) future teachers rely mainly on memorized procedures. The studies concur that future teachers' understanding of geometry is underdeveloped and relies on limited concept images which, as a consequence, cannot support reasoning and the formation of connections among concepts. As reported, future teachers' sources of definitions are visual examinations of shapes; they rely on procedural knowledge and, in general, lack conceptual understanding of geometric concepts.

Research published after 2012 reports similar conclusions: low level of geometry knowledge in terms of concepts and relations among concepts, but also reasoning. Future teachers seem to rely on prototypical representations of certain concepts instead of their definitions, lack factual knowledge and do not recognize contradictions in their justifications (Couto & Vale, 2014). Similarly, they face difficulties when in need to define figures (Gomes, Ribeiro, Pinto, & Martins, 2013).

On the topic of transformations, Harper (2003) reports an improvement of future teachers' knowledge of transformations after activities where they were using dynamic geometric software. Although, at the beginning, future teachers demonstrated a partial understanding of geometrical plane transformations, after working with the software, they significantly improved their knowledge of definitions and properties of transformations. The initial observations referred to preservice teachers' difficulties with a) correctly identifying the parameters of transformation; b) foreseeing the results of a transformation applied to a complex figure, and c) conceptualize transformations as operations that require the specification of inputs and parameters.

A case study on the development of future teachers' conceptualizations of translation, Yanik and Flores (2009), revealed successive phases of interpretations of transformation as: a) undefined motions of a single object; b) defined motions of a single object, and c) defined motions of all points on the plane. In a follow-up study, Yanik (2011) reported the future teachers' difficulty to recognize, execute and represent translations. The study reports that such conceptualizations of translations and of vectors later proved to prevent future teachers even to accept the role of vector in the definition of translation – a result that is of interest for my research.

The usually employed research tools were: questionnaires or specific tasks aimed at eliciting a certain type of reasoning and use of concepts. Some tasks aimed at identifying future teachers' geometrical thinking level in order to categorize their thinking on different geometrical topics or to reveal the conceptions future teachers had about a given topic. Questionnaires were developed for some of the studies, with items on a Likert scale; others had multiple choice questions related to concepts taught at a certain school level.

To summarize, it can be said that most of the studies compare future teachers geometrical knowledge with a body of knowledge considered as reference, usually scholarly geometry. In this sense, the questionnaires represent normative knowledge and the studies uncover future teachers' lack of knowledge. As for the studies on geometrical transformations, the focus was on unveiling future teachers' conceptualizations of these concepts. In this sense, they sought to identify what future teachers' *think* transformations are. While these are not normative evaluations, they were still studied in isolation, without an interest in describing the future teachers' epistemology.

### 1.2.2 Research on "personal epistemology"

Another line of research relevant to my topic is about epistemology, more precisely, what researchers named as *personal epistemology.* The construct refers to an individual's perspective on the characteristics of knowledge and the nature of knowing (Hofer & Pintrich, 1997). Personal epistemology is different from the scholarly epistemologies, yet it can still be seen as organized along similar issues: origin of knowledge, sufficient justification and dealing with validity claims. In the questionnaire proposed by Hofer & Pintrich these aspects are assessed along four continuous dimensions. The dimensions are: certainty, simplicity, source, and justification of knowledge. The dimensions range from objectivist to relativist. The objectivist epistemology is characterized by the following features: source of knowledge is external, be it the observable world or entities considered authorities (such as teachers, textbooks) and thus, certain. In addition, those adhering to an objectivist epistemology would consider knowledge as simple – which can be translated into self-evident truths (nothing to prove) or single correct answers. On the other end of the spectrum is the relativist position where knowledge is a subjective construction, thus uncertain. Those in relativist position also consider knowledge as being complex with multiple justifiable knowledge claims (Roth & Weinstock, 2013).

Interest in personal epistemology of teachers is motivated by the possible links that exists between teachers' conceptions of learning and teaching and personal epistemology. The expression 'teachers' conceptions' is used to refer to teachers' classroom practices. In its turn, research showed a link between teachers' conceptions of teaching and student learning (Trigwell, Prosser, & Waterhouse, 1999).

In a follow-up research, using a statistical technique, confirmatory factor analysis, Chan and Elliott (2004) showed a causal relation between epistemological beliefs about content knowledge and conceptions on teaching and learning. Their result is aligned with previous research that suggested that teachers' conceptions are belief driven (Richardson (1996); Samuel & Ogunkola (2015)). For example, Jones and Carter (2007) argued that teachers' personal epistemologies may serve as reference points for "constructing and evaluating their own teaching practices" (p. 1072).

In my research, in addition to conceptions about the nature of knowledge and knowing, one of the objectives is to unveil also the content to be taught (*what*), as well the reasons (*why*) for teaching that content as perceived by future teachers. Brousseau's (1997) concept of *teacher's epistemology* captures all these three aspects. Brousseau argues that in the decision-making process inherent to teaching situations, teachers routinely rely, in an explicit or tacit way, on methods and beliefs concerning the organization and acquisition of mathematical knowledge. These epistemological beliefs are often rooted in teachers' experiences as students (Hewson & Hewson, 1988) and, then, are revised and further constructed during their career. The importance of identifying them can be seen in the way in which they define students' potential actions and justify teachers' decisions.

### 1.2.3 Teachers' epistemology; focus on knowers rather than on knowledge

Despite considerable amount of research on the themes of future teachers' knowledge (or rather lack of it), epistemic beliefs and conceptions of teaching and learning, there is a paucity of studies on what future teachers *do* know and *how* they know it: *what* is the underlying epistemology. Karl Maton, an educational sociologist, in his book *Knowledge and knowers,* called this phenomenon "knowledge blindness" (2014, p. 4), and remarked:

> *Over recent decades, the theory of learning offered by constructivism has become propagated as a theory of everything, including teaching, curriculum, and research. Different knowledge practices have thereby been reduced to a logic of learning, based on the belief that 'the more basic phenomenon is learning' (Lave and Wenger 1991:92). From this perspective, what is being learned is of little significance. Accordingly, research typically focuses on generic processes of learning and sidelines differences between forms of knowledge being learned. (Maton, 2014b, p. 4)*

My interest in teachers' epistemology can be explained by the fact that questions such as "What is knowledge? How to describe it? How do we decide some knowledge is valid?" are among the most basic ones to ask by anyone engaged in teaching,  besides being among the earliest asked and still debated by philosophers.

The question of knowledge is not restrained to philosophy – each domain of science, whether social sciences or natural sciences – has to deal with the issue of knowledge. For the purpose of my thesis, the interest is to look at knowledge as defined and studied in social sciences to which educational research pertains. An interesting observation is that in many domains, there is an explicit effort to separate knowing from knowledge.

Maton (2014b) argues that a social realist approach can overcome the 'knowledge blindness' phenomena - or the 'subjectivist doxa' in educational research -, where studies focus only on people's relation to knowledge, in other words, on the process of knowing, and tend to treat knowledge as "having no inner structures with properties, powers, tendencies of their own, as if all forms of knowledge are identical, homogeneous and neutral" (p.2).

In my research, I am interested not so much in teachers' relation to knowledge, their "knowing", but in the concrete content and structure of their mathematical knowledge for teaching a particular topic in geometry. I follow Maton's call for "taking knowledge seriously" by looking at what future teachers' knowledge is in itself, how it can be described and what affordances and limitations this knowledge creates in the long run. Future teachers' knowledge, of mathematics and its teaching, creates a **sort** of mathematics – and while this is not **the** mathematics one can find in academic mathematical texts, we use the same word for lack of a better suited one.

Social realism underlines the importance of focusing on the process of defining knowledges in particular social and historical contexts, study their forms and their effects. From this perspective, educational fields are seen as comprising both structures of knowledge practice and actors. As stated by Maton, social realist account considers that "knowledge practices are both emergent from and irreducible to their context of production – the forms taken by knowledge in practice in turn shape those contexts."

The above standpoint resonates with Chevallard's (1999) view on knowledge, according to which knowledge is a product of human construction and its function and place depends on historical and social contexts. The core idea of ATD is *didactic transposition* as a social construct, a process through which knowledge is transformed into knowledge to be taught. The continuous transformation of knowledge, and its clear distinction from knowing, is what makes it irreducible to the one in the context of production.

In ATD, mathematics is conceived as a human activity that studies types of problems and, as any human activity, co-exists with a discursive environment (Chevallard, 1999). In practical terms, knowledge is seen as praxeology comprising both tasks, techniques as ways of performing the tasks and the discourse that validates or justifies the ways of dealing with the task. The distinction between knowledge and knowing is made clearer in French by using distinct terms: 'savoir' refers to knowledge, product of research recognized by a certain community, while 'connaissance' refers to personal knowledge that fits well the verb 'knowing'.

The stance I take on knowledge is in line with the one proposed by Chevallard (Wozniak, Bosch, & Artaud, 2012): knowledge is "…before all a discourse making possible to justify, produce, make comprehensible techniques", where techniques are the ways to perform certain tasks. In other words, knowledge is identified through the kinds of tasks proposed, and solved.

Thus, in my research, future teachers' knowledge will be identified through commonalities found in tasks proposed by them. The existence of these commonalities is essential for talking about knowledge, in contrast with personal processes of knowing, as it supports the emergence of a shared knowledge base and feeds from it. Yet, a proper analysis of this knowledge requires situating it in a wider context.

Educational policies, institutional practices and organizational structures in education all intertwine in generating a (mathematical) culture: an underlying characteristic feature of any educational system. In their article, Adam & Chigeza (2014), state as a premise that ways of knowing mathematics (epistemologies) are instantiated in ways of teaching (pedagogies) and vice-versa, and they are reciprocally related to cultural values. Thus, what we recognize as recurrent tasks, approaches, ideas in future teachers are, in fact, expressions and creators of a mathematical culture – of its epistemology and pedagogy.

The need to attend to the symbiosis between future teachers' knowledge and the overall educational environment, understood here as the set of educational goals, values, and fundamental principles, was also advanced by Petrou & Goulding (2011) in their review of different models of teacher's knowledge. The following citation pertinently resumes their viewpoint to which I also adhere:

> *"One of the common features of the different models of teacher knowledge discussed here is the largely individualistic assumption which underpins them. Despite the acknowledgement of context, the focus tends to be on the knowledge that an individual teacher brings to a course of teacher education and then into the classroom. This can result in a deficit view of the individual teacher, who at worst needs remediating and at best developing, rather than seeing teacher knowledge as a product of the educational system in which she is located. We cannot assume that the frameworks discussed here are universal. Even if there are some commonalities, there may be great differences in emphasis in various cultural contexts and different priorities for research and development. Switching attention to the system would mean paying more attention to the prior mathematical experiences of teachers and to the resources available to teachers for their own use." (p.23)*

### 1.2.4 Models of teacher knowledge in mathematics education

In mathematics education, research on student learning gradually expanded to include studies relating students' learning with teachers' knowledge. In a review of these studies, Park (2012) underlines the variety of approaches in studying the above relationship. Review studies are interesting since they put in perspective the evolution of views on each of the involved variables, as here we have: teacher knowledge, teaching as teacher-student interaction and students' learning. There are several conceptualizations of teachers' mathematical knowledge, as there are several ways of measuring students' learning. Early studies used quantitative elements to capture teachers' knowledge, such as number of mathematics courses taken at college level (for a review, see Hill et al. (2008)). These studies were not conclusive on the relationship; however, they pointed towards the need to perform a finer analysis of teachers' knowledge and, eventually, led to the development of Shulman's model (1986) of teacher knowledge. The model proposed by Shulman placed into focus the inherent complexity of teaching as a profession and sparked a fury of research on the topic.

Petrou and Golding (2011) analyzed four currently used frameworks for conceptualizing teacher knowledge. I shall briefly present them in the following.

Shulman's (1986) research focused on the teaching process with the purpose of identifying types of knowledge needed by teacher in deciding about the teaching content, reaction to errors or misunderstandings. He proposed seven categories of teacher knowledge: general pedagogical knowledge; knowledge of learners' characteristics; knowledge of educational context; knowledge of educational purposes and values; content knowledge; curriculum knowledge; pedagogical content knowledge.

The first four categories are considered as general knowledge, while the last three refer to dimensions of teachers' content knowledge. Content knowledge, called by Shulman as Subject Matter Knowledge (SMK), refers to two aspects: knowledge of the subject and knowledge about its structure. In case of mathematics, SMK contains knowledge of mathematical content and overall organization of mathematics as theory. Schwab (1978) used the terms 'substantive' and 'syntactic' knowledge to designate the two components of SMK. Substantive knowledge consists of knowledge of theories, models and concepts, while syntactic knowledge refers to processes by which a theory is generated. For the case of mathematics, Ball (1991) used the terms "knowledge of mathematics" and "knowledge about mathematics", respectively.

The second category, curriculum knowledge, has two further sub-components. One refers to the knowledge of available teaching materials as curriculum and textbooks (especially in countries where educational policies are de-centralized); while the second refers to the knowledge of ordering the topics in time and ways of presenting them at different levels of schooling. Schulman uses the *term lateral curriculum knowledge* for the first one and, *vertical curriculum knowledge* for the second.

The third category of teacher's content knowledge refers to a mix of content and pedagogy, unique to the teaching profession. This knowledge has been coined by Shulman as Pedagogical Content Knowledge (PCK). It refers to content specific representations, examples and applications to illustrate concepts and strategies to deal with students' difficulties. In mid-nineties, (Meredith, 1995) critiqued this interpretation of PCK for it focusing only on teacher-directed instruction without taking into account students' understanding and personal approaches to the subject matter. Another critique raised by

Meredith referred to the fact that PCK doesn't seem to account for the influences of teachers' beliefs and knowledge on its development.

Other researchers raised concerns about the unclear distinction between SMK and PCK (Ball et al., 2008), but also the clear-cut separation of the knowledge components that ignores interactions between knowledge types (Hashweh, 2005) and does not account for the growth one obtains from the experience of teaching (Fennema & Franke, 1992).

In their article, Fennema and Franke (1992) critiqued Shulman's model for presenting the knowledge for teaching as a static entity. They proposed a model that accounts for the development of this knowledge through experience and refined the components initially included into the Shulman model. In this model teacher knowledge is conceptualized as it occurs in the context of the classroom and it postulates that mathematical knowledge for teaching is constituted of knowledge of the content, knowledge of pedagogy, knowledge of students' cognition and teachers' beliefs. In this view, teaching practices and classroom behavior of the teacher is defined by the three knowledge components that are coordinated by the teacher's belief. It is suggested that central to classroom interaction is the context specific knowledge that emerges from the combination of these elements. Similarly, this could account for variations in the knowledge used or manifested in certain contexts.

The Knowledge Quartet (Rowland, 2005), as theoretical framework, aims at categorizing classroom situations, where mathematical knowledge "surfaces in teaching", by focusing on the relation between SMK and PCK.  The Knowledge Quartet consists of four dimensions: Foundation, Transformation, Connection and Contingency. The Foundation category contains teachers' knowledge and understanding of pedagogy together with the beliefs they hold about the subject and its teaching.  Transformation includes representations and examples used by the teacher, along with explanations. Connection comprises the links made between different units: lessons, lessons parts or mathematical ideas. Contingency refers to teachers' readiness to respond to students' questions, to follow through a student reasoning.

In comparison with the previous models, the Knowledge Quartet allows explaining, through the inclusion of the Foundation component, how trainees can develop different PCK.

The different models complement the initial conceptualization of Shulman, by extending, clarifying and operationalizing the different components. Although some researchers later questioned the distinction between SMK and PCK, it seems clear that reducing subject matter knowledge to pedagogical one would bring a loss of depth in teaching. In this idea, Petrou and Goulding remark that "the unpacking and deepening of SMK can be seen as part of the process of transformation required for robust PCK to be developed" (p.20).

A recurrent critique of the models refers to the broad claims about what teachers' should know. Often this knowledge is specified as normative ("teachers should know…"), without an empirical base (Ball, Thames, & Phelps, 2008). In their study on the link between teachers' knowledge and quality of instruction, Hill et al. (2008) categorize previous studies as "deficit" and "affordance" approaches. The former focus on identifying links between the lack of mathematical understanding of a teacher and his teaching, while the second focus on illustrating how solid mathematical understanding creates more opportunities for student learning and advancement. Albeit these studies provided fine grained insight into links between teachers' mathematical knowledge and instruction, the authors argue that they were too particular for provide generalizable results – thus to inform the content of certain teacher knowledge categories. Nevertheless, these studies were used as springboard for introducing certain content matter into teacher training courses.

Ponte and Chapman (2008) offer a broad review not only of preservice teachers' mathematical knowledge, but also of the mathematical knowledge for teaching. However, the knowledge elements in the reviewed studies are formulated (again) in broad terms, as, for example: beliefs about the means and purposes of mathematics teaching, nature of tasks for functions, working with different representations (for example, fractions), algebraic and graphical representations of functions, instructional explanations, etc. Generic descriptions are not enough to clearly understand what constitutes the teachers' knowledge, how can it be described – and without such detailed specification, it is hard to imagine how that knowledge could be developed, for example, in teacher training courses.

In order to address the issue of concreteness, (Ball, Thames, & Phelps, 2008) proposed a practice based approach for developing a teacher knowledge model. They argue: "Because it seemed obvious that teachers need to know the topics and procedures that they teach—primes, equivalent fractions, functions, translations and rotations, factoring, and so on—we decided to focus on *how* teachers need to know that content." (p. 395, emphasis mine). Furthermore, they concentrated their analysis around two identification issues, that: 1) of recurrent tasks and problems; 2) of mathematical knowledge, skill and sensibilities for dealing with these.

The result of this work was a 'Practice-Based Framework of Teachers' Mathematical Knowledge for Teaching' (MKT) that refines Shulman's initial categories. The Subject Matter Knowledge category is split into three components. *Common content knowledge* (CCK) refers to mathematical knowledge and skills that are general, used in any setting. *Specialized content knowledge* (SCK) is the knowledge that is used in classroom settings and is needed by teachers in order to teach effectively. *Horizon knowledge* includes teachers' awareness of how the mathematical topics are laid out across the curriculum. Shulman's PCK is also split into three subcategories: *Knowledge of content and students (KCS)* is 'knowledge that combines knowledge about students and knowing about mathematics'. *Knowledge of content and teaching (KCT)* is 'knowledge that combines knowledge about mathematics and knowledge about teaching' (Ball et al., 2008, p. 401). And lastly, there is the category of *Knowledge of Content and Curriculum (KCC)*.

Even though the model tries to better delimitate the different knowledge components teachers' knowledge entails, its content is not explicit when reading the description of categories. Concrete examples of what this knowledge consists of are found scattered in different articles. In the review of Liping Ma's book (Ma, 1999), *Knowing and teaching elementary mathematics*, the author (Howe, 1999) gives an example of a situation where the teacher would need to use specialized content knowledge in order to attend to the student's "theory". The situation is presented below:

> *4) Suppose you have been studying perimeter and area and a student comes to you excited by a new "theory": area increases with perimeter. As justification the student provides the example of a 4 x 4 square changing to a 4 x 8 rectangle: perimeter increases from 16 to 24, while area increases from 16 to 32. How would you respond to this student?*

It is assumed that the teacher should propose a *task* so that while working on the task, the student would realize the limitation of his theory. The emphasis is on *task* – the model suggests that SCK consist of tasks, as the guiding questions in its development suggests.

At this point, an articulation with Chevallard's ATD (Chevallard, 1999) is possible. Knowledge is what is manifested in solving tasks – both in MKT and ATD. Thus, it is possible to characterize future teachers' knowledge in terms of tasks they envision performing along with the discourse they attach to them.

## 1.3 THE FOCUS OF THE RESEARCH

As mentioned, in my research, I aim at identifying future teachers' epistemology for teaching geometric transformations and, thus, I ask *what* exactly they think is the knowledge to be taught about this topic.

More precisely, I propose a praxeological model of future elementary teachers' envisioned practice of teaching geometric transformations, focusing on what future teachers consider should be taught in school concerning the topic of transformations and how. In so doing, the future teachers' beliefs about the nature of mathematics will also be in view and I will highlight how these beliefs articulate with their declared and enacted didactic and pedagogic principles. The research goes beyond studies reporting the shortcomings of future teachers' mathematical knowledge, and also avoids reducing the description of their geometry to a global, qualitative way (e.g., as "intuitive"). It is my hope that it can bring a better insight into their epistemology than what we can derive from closed item questionnaires.

## 1.4 OUTLINE OF THE THESIS

The thesis is organized as follows.

The second chapter presents the conceptual framework used in the thesis. The main part of the conceptual framework is the Anthropological Theory of the Didactic (Chevallard, 1999). In this chapter, I will argue for the pertinence of the choice of the ATD framework given the nature of my research.

The third chapter describes the methodology of my research as qualitative: interpretive and analytical. The method consists in an *interpretation* of the discourse in written documents produced by a group of participants in a certain social and institutional context, in the aim of building a model of the *meanings* of this discourse. The social and institutional context is important in this type of research; it is described in the chapter in much detail. The participants and the documents used in the research are also described in detail.

Chapter four contains the results of the research – the praxeological model of the future teachers' envisaged practice of teaching geometric transformations – together with the analyses that led to its construction and justifications of the interpretations. The praxeological model is a system of "punctual praxeologies" related to three types of tasks: to perform a transformation, to identify a transformation and to identify the parameters of a given transformation. The presentation of each praxeology is followed by a discussion of its limitations and potentialities, in terms of opportunities for children's learning mathematics that a lesson based on this praxeology could offer.

In chapter five, extrapolating from the praxeological model constructed in chapter 4, I discuss the epistemology of future teachers: the WHAT, WHY and HOW to teach about geometry, and, more generally, mathematics.

Chapter six presents some conclusions and recommendations from the study and outlines a few potential future avenues of research.

The thesis closes with a list of references.

# 2 CONCEPTUAL FRAMEWORK

In this chapter, I will present the conceptual framework underlying my research. The research focused on a group of future elementary teachers' proposals of geometric activities for the classroom. Assuming that the purpose of these activities was to help children learn some knowledge or develop some knowing, our goal in this research was to identify the mathematical content of this knowledge/knowing. Given that this knowledge/knowing would manifest itself in expected children's actions in response to mathematical tasks, it made sense for us to take the epistemological stance of the Anthropological Theory of the Didactic (Chevallard, 1999) whereby knowledge/knowing is viewed as a certain practice, characterized by types of tasks, techniques for accomplishing them and a discourse allowing those immersed in the practice to justify the techniques and teach them to others. The Anthropological Theory of the Didactic (ATD) constitutes the most fundamental part of our conceptual framework, and it will be presented in the first part of the chapter. This theory was useful in capturing the future teachers' envisioned work in their role of teachers and, consequently, as participants in a particular institution, the primary school system in Quebec.

Another part of our conceptual framework is the van Hiele model of geometric thinking (van Hiele P. , 1959). Van Hiele levels of geometric thinking were useful in characterizing the geometric thinking that the future teachers aimed at in their planned activities. The model presents a trajectory of evolution of a new learner's geometrical thinking, something we assumed that future teachers will have to facilitate in their students as teachers. Moreover, the geometry instruction offered to the future teachers in the methods course they followed was structured around the van Hiele theory. This instruction will be described in Chapter 3, as it was the context in which the data for the research had been collected.

The last part of our conceptual framework is the Geometrical Paradigms theory proposed by Houdement and Kuzniak (2003). The explicit purpose of Houdement and Kuzniak's model was to capture the work of adult learner, future teachers in their case, already familiar with geometry. Their model is to account for the choices future teachers make when working on particular tasks themselves or with children. From this point of view, their proposal is close to the concept of praxeology introduced by Chevallard (1999); however the granularity is significantly finer, since it refers to the task level. The Geometrical Paradigms theory is useful to explain the heterogeneity in future teachers' approaches to tasks – their level of geometrical thinking, as defined by the van Hiele model, is one of the many factors that influence these approaches.

## 2.1 ATD

### 2.1.1 Outline of the elements of Anthropological Theory of the Didactic that will be used in this thesis

The ATD postulates a certain view of human activity, in general, not only mathematical, that implies a specific view of knowledge.

Human activity is directed towards solving the tasks humans face or have imposed on them. Along with the activity of doing or accomplishing the tasks, a discursive environment is being developed, providing elements that give reasons for the specific way a given task is handled thus allowing it to be explicit and communicable to others (Chevallard, 1999).

In line with this view, knowledge is a result of a human construction and, in consequence, its function and place vary accordingly with social context and time (Wozniak, Bosch, & Artaud, 2012). Such view of knowledge asks for methods of study to consider the particular contexts in which the knowledge is (re-)produced. In this sense, ATD postulates an institutional view of mathematics. The word "institution" is understood in the theory as standing for "any social or cultural practice that takes place in an institution" (Artigue, 2002).

Consequently, mathematics, as human activity, is seen as dependent on the social and cultural environment where it is developed and practiced. An implication for research in mathematics education is to look at practices as being defined and sustained by an institutional environment, rather than being independent objects.

### 2.1.2 ATD as an organizing framework for this study

In our study, the mathematical content of the future teachers' lesson plans is analyzed from the perspective of the Anthropological Theory of Didactics (Chevallard, 1999). The theory proposes an epistemological model of mathematics and has at its core the view of mathematics as a purposeful human activity – a practice as any other. ATD proposes to model this activity in the form of a *praxeology*. The general structure of a praxeology can be described as consisting of two blocks: the *know-how* (or the practical block) and the *know-why* (the theoretical block).

The practical block concerns the *types of tasks* (denoted by T) that are to be solved or accomplished and the *techniques* (denoted by $\tau$) used to solve them. The word *technique* is employed here in a general sense as referring to any way of solving a certain type of tasks. It can be an algorithm or "rule of thumb", a way of doing things that is communicated by showing how it works on an example. The fundamental assumption of ATD is that any task that has to be done repetitively will call for some technique, some way of dealing with it. When a way of doing things is generalized and described in decontextualized and depersonalized terms, it is no longer a technique but a method – a "technology" as it is often called in English translations of the praxeology model – and it belongs to the theoretical block.

The theoretical block concerns the reasons, explanations why a technique is considered as being adequate for solving a certain type of task. A first layer of explanations (denoted by $\theta$) called "technologie" in French and often translated as "*technology"* in English, although "method" would be closer to the intended meaning, concerns the reasons and justifications as for why a given technique works. In the context of mathematical praxeology, this could refer to rules, properties and theorems that are justifying the technique, from the point of view of mathematical appropriateness. Yet, rules and theorems are valid only within a system, a mathematical theory; therefore, a second layer of explanation is needed, one that can justify the rules used in the technological discourse. This layer is called *theory* (denoted by $\Theta$) and its purpose is to make explicit the assumptions at the basis of a method. In the analysis of a mathematical praxeology, the theoretical level should refer back to definitions and to the processes of deriving properties and theorems. The validity of techniques, ultimately, is rooted in the constructions present at theoretical level.

One can find example of research focusing on teachers' practice in a more global way, namely, considering teaching as a complex activity where all sort of constraints are present at every level of decision taking (on student-teacher, teacher-teacher or teacher-institution), that opted for ATD as conceptual framework. Such choice is due to the fact that ATD captures praxeologies at different levels of granularity and, thus, offers the opportunity to examine their correlations across different levels (punctual, local or institutional). Examples of studies using ATD as framework include those focused on perceptions of institutional constraints by students (Hardy, 2009), and studies on the impact of such constraints on teachers' practice (Barbe, Bosch, Espinoza & Gascon, 2005).

The ATD will be used to identify praxeologies among the FT. The process requires identifying types of tasks, the techniques applied to solve them and the method and theory underlying these techniques.

## 2.2 OTHER THEORIES CONTRIBUTING TO DESCRIBING FUTURE ELEMENTARY TEACHERS' PRAXEOLOGIES

In the following, I briefly present two commonly used theories in the study of geometry teaching and learning. The first one, the van Hiele model, describes a progression of more and more abstract levels of geometrical thinking. The second model is the Geometrical Paradigms model proposed by Houdement and Kuzniak (2003). In mathematics education, we can find also theories of psychological roots (based on Gestalt principles or Piaget's theory) or cognitive one (for example, (Duval, 1995)), yet those theories do not put the epistemology of geometry at their center. For this reason, I limit to the above mentioned two models.

Each model will be presented briefly, by describing its elements and the relations that exist between them. I will conclude with a short comparison between the two and then explain their relevance to my research.

### 2.2.1 The van Hiele model of geometrical thinking

The van Hiele model (1959) is a generic description of the stages of a theory development. At the foundation of the model lies the assumption that geometry is rooted in observation and mental structuring of the physical space but, in the end, it becomes an abstract mathematical theory. In consequence, although some objects of this theory look and are named the same as objects from everyday life, their nature is very different and the nature of questions we ask about them is different from questions we commonly pose in everyday settings. It also assumes that the role of school education is to acquaint students with the theory. In consequence, school education should help students to gradually shift their attention from…

…identifying and naming observable individual shapes in the environment ("Visualization")

… to the less directly observable properties of those shapes, ("Analysis")

… then to logical relations among these properties, ("Informal deduction")

… and further to deductive systems of such properties,("Deduction")

… to, ultimately, systems of such deductive systems ("Rigor"). (Figure 1)

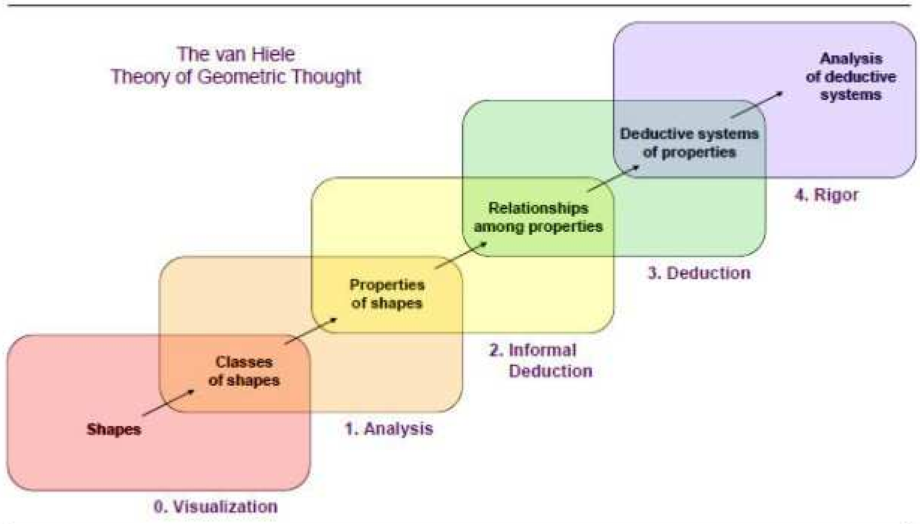



*Figure 1. A summary of the van Hiele levels (Rynhart, 2012)*

The above image is complemented by the following description of the levels.

At level 0, *visualization*, students recognize figures holistically, often by comparing with a prototypical image. They will not pay attention to individual features of geometrical object and their reasoning (and justification) is based on perception. However, as result of this understanding (product of this level), they start to identify properties and these properties become subject of analysis at the next level.

At level 1, *analysis*, students are aware of the properties of an object: they can recognize and name them. However, they are not yet able to see the relationships between the properties, thus; for example, they can't differentiate between sufficient and necessary conditions for a shape to be of a certain type. Exactly the relationships between properties will be the subject of the next level.

At level 2, *abstraction*, students perceive relations between shapes and properties. These elements allow them to produce useful definitions and perform informal reasoning. Classification are understood, however the importance of formal deduction not. Formal deduction becomes the subject of the next level.

At level 3, *deduction*, students understand the importance of the formal system in their reasoning: they recognize the role of axioms, definition and are able to produce a proof.

The last level, *rigor*, makes reference to the understanding of a formal system as a mathematical entity. As such they grasp the idea that different geometries are different interpretations of a coherent, sound formal system and, in consequence, Euclidean geometry is one of possible geometries.

If we were to replace the geometry specific terms in the above descriptions by "objects" (for example, in the description of level 2, *abstraction,* we would replace "shapes" by objects) we would conclude that the levels describe the sequential process of theory formation. In this sense, van Hiele's model is a model of theory formation where the levels describe general, in the sense of domain-independent, processes of knowledge construction and advancement in abstraction through a sequence of exploration and reasoning. The following characteristics confer to the model this possibility (based on (Usiskin, 1982)):

a) Sequential: progression through levels is sequential in nature.
b) Advancement: transition from one level to other is based on instruction (method and content) and is not age-related maturity.
c) Intrinsic and extrinsic: the basic objects from one level become object of study at the next level.
d) Linguistic: each level is characterized by its own language and set of symbols. For example, a relationship might be revisited once the students progresses to a different level. The typical example offered for understanding how this change happens is at follows. Initially, a square is a square: there is no other name available for it. Yet, with the progression to a higher level, a student can consider the square as a rectangle or parallelogram; in other words, the student acknowledges that the set of squares is a subset of larger sets (Crowley, 1987). Another, common situation would be the case where justification is given in different terms, not yet attained by the student: for example, by employing theorems that the student doesn't know yet.
e) Mismatch: if there is a mismatch between the levels a student is in and the instructional material developed for him or her, the student will be in impossibility to follow the thought processes employed in teaching the prepared material.

Among the above characteristics, two arise as being central to the van Hiele model and need further detailing: the role of language and the transition from a level to the next one. Specifying the modality to progress from level to level is essential if the model is to be used in teaching and learning of geometry. Yet, this comes down to the fundamental position one has about how learning happens. Contrary to Piaget's view, for whom learning and growth are consequences of biological maturation, van Hiele states the central role of instruction in learning. In the words of Pegg (1998): "he saw development in terms of students' confrontation with the cultural environment, their own exploration, and their reaction to a guided learning process." (p. 112)

In consequence, it is the method and organization of the instruction, content and materials that must be properly prepared. The model proposes 5 phases through which, by sequentially progressing, one can facilitate the transition to an upper level. For teachers, these phases serve as guidelines, for the nature and ordering of tasks. These phases are: inquiry, directed orientation, explication, free orientation and integration.

During the inquiry phase, the teacher engages students in discussions about the objects of study at this level. Through observations and questioning, the teacher identifies students' previous knowledge on the topic and can define the direction study can progress. At this point, the teacher will also introduce level specific vocabulary (Crowley, 1987). Directed orientation serves the purpose of identifying structures specific to this level and, often, it relies on short tasks designed to elicit particular responses. The third phase is mainly played out among the students, who share their observations and views about the structures they explored earlier. The free orientation phase gives students an opportunity to come up with new solutions to open-ended tasks by using their knowledge and creativity. The last phase aims at integrating all previous experiences into a network of objects and relations.

Since the content and instruction are central elements for transition between levels, a particular attention is needed to teach in accordance with the level. A common situation is that of content level reduction, where a lower level activity is proposed instead of one in line with the level. One example, mentioned in (Crowley, 1987) would be using two or three cases of angles measured in triangles to deduce that the sum of angles in a triangle is 180 degrees. In such case, students would not have the opportunity for observing regularities and generating conjectures, both fundamental for gathering rich experiences that underlie the integration phase. Thus, teachers must pay a special attention to offer tasks corresponding to the level.

Further didactic guidance can be derived from the linguistic characteristic of the model. As described earlier, in the model each level has a specific language and symbols.

The language use refers as much to students' way of explaining their ideas, their use of terminology as it refers to teachers' language use. By formulating their ideas and using, or not, specific terms, students express their understanding of the geometric objects and the relations between them. In consequence, teachers construct an image of the students' understanding through the answers received to the questioning concerning the student's reasoning. It is not uncommon for students to give a correct answer, yet the justification for that answer not to be at the expected level of geometrical thought. Hence, the teacher must inquire about the student's thought process.

Similarly, teachers will use different discourse depending on the level their students are. While at *Analysis* level they employ words as "always", "sometimes", etc., in an effort to enlarge the sets for which certain statement holds, at level of *Informal deduction* they should use "if...then" type of statements in order to facilitate synthesis of cases. At level *Deduction*, they should introduce and use the words characteristically used in theory formation, such as axiom, definition, etc.

Since language use is related to levels, the mismatch between the teacher's and the student's language use can lead to failure of communication, since the two are not acting in the same space. A typical example would be where the teacher explains or justifies a result by using a result with which the students are not acquainted with.

Mathematics educators, interested in operationalizing the van Hiele model, proposed specific tasks and descriptors for each level (for example, Wallrabenstein (1973); Burger & Shaughnessy (1986), Fuys, Geddes, & Tischler (1988)) to act as an instrument for measuring a child's level of geometrical thinking. The core element in the evaluation was the student's verbalization of his or her work and reasoning. Considerable amount of research used these instruments to evaluate students and categorize them by levels. However, more recently, some educators revisited the van Hiele model and questioned some of its elements.

For example, Papademetri-Kachrimani (2012) highlighted that the centrality of language in the van Hiele model and the categorization of children into levels might be more part of a research culture than an intrinsic feature of the model.

As Papademetri-Kachrimani argues, the initial claim of van Hiele (1959) that "thinking without words is not thinking" is later revisited in his article (van Hiele P. , 1999), where he underlines the role of intuitive knowledge in the construction of rational knowledge. On its turn, the intuitive approach builds on recognition of a structure and, van Hiele also claims, this recognition doesn't require language. This might explain the real difficulty teachers face when they interpret children's explanations: there might be a deeper understanding than one that is perceived through the employed words when answering targeted questions.

A second aspect questioned by the above author is the typical interpretation of the model as a hierarchical one. Results from studies (ex. Gutiérrez, Jaime, & Fortuny (1991); Lehrer, Jenkins, & Osana (1998); Clements, Battista, & Sarama (2001)) showed that students develop several levels at once, and also, they can be at different levels on different topics. In other words, progression of geometrical thinking is not a linear or sequential process: a student might be more advanced on some topics than others and develop simultaneously several levels.

Papademetri-Kachrimani (2012) suggests that these situations are better aligned with the model if we understand the levels as ways of thinking, that rely on different epistemologies, instead of seeing them as a linear sequence of subordinate, increasingly more complex thinking. From this viewpoint, multiple mathematics are possible, with different ways of constructing knowledge and knowing.

In my personal opinion, we face two different issues here. The van Hiele model as a model of theoretical thinking (thus, more general than geometrical thinking) is hierarchical in the sense that theory formation follows this increasingly abstract process of analysis-synthesis, where inherent objects from one level become objects of study at the next level. It is a process characteristic of theory formation, identifiable in the work of mathematicians who during decades worked on the systematization of mathematical knowledge. One must keep in mind that the initial model was proposed before the rise of the multitude of learning and teaching theories we have today, in a context where Euclidean geometry was still very much part of the curriculum. In a certain sense, the van Hiele model is a-temporal since it concerns the development of a theory. When the theory is applied to a particular domain to model the development of thinking of a child in that domain, it is possible to have multiple levels simultaneously present for different topics. In other words, we are looking at the model at different level of granularity.

On the other hand, the model as applied to assess children's geometrical thinking need not be sequential and hierarchical, allowing the co-existence of multiple kinds of mathematics. Then, it remains to be seen in which way these 'different mathematics' align, on one hand, with the officially promoted mathematics (through official texts such as curriculum and textbooks) and, on the other, with the discipline of mathematics as established by the community of mathematicians. This remains, however, a subject for another type of research; the model itself is not concerned with the institutional aspects of learning of geometry: global and local goals of education, constraints, curriculum organization, etc. It is one way to progress towards deductive, synthetic geometry starting from tactile and visual experience with the surrounding space.

At the same time, the model did serve as a guide to the development of curricular description of the geometry to be learnt. No coincidence here; first, as already mentioned, the van Hiele model was developed almost 50 years ago when the dominant view of geometry at school was the Euclidean geometry. Second, as remarked in (Clements & Battista, 1992), school geometry refers usually to scholarly geometry, approaches differing only by the choice of the "entry-point" into geometry (whether transformational, coordinate, vector) and/or the modalities of transitioning towards deductive reasoning.

For example, the Progression of Learning in Elementary School (MELS) - which is one of the main official documents in Quebec concerning specification of content to be taught - specifies about plane figures the following sequence of study: *compare…, identify…, describe…, classify*…which are the main type of activities one is engaged with on visualization and analysis level. Figure 2 contains the specifications for plane figures in the above mentioned document.

| C. Plane figures | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1. Compares and constructs figures made with closed curved lines or closed straight lines | → | ★ | | | | |
| 2. Identifies plane figures (square, rectangle, triangle, rhombus and circle) | → | ★ | | | | |
| 3. Describes plane figures (square, rectangle, triangle and rhombus) | → | ★ | | | | |
| **Vocabulary**<br>Straight line, closed straight line, curved line<br>Plane figure, side<br>Square, circle, rectangle, triangle, rhombus | → | ★ | | | | |
| 4. Describes convex and nonconvex polygons | | | → | ★ | | |
| 5. Identifies and constructs parallel lines and perpendicular lines | | | → | ★ | | |
| 6. Describes quadrilaterals (e.g. parallel segments, perpendicular segments, right angles, acute angles, obtuse angles) | | | → | ★ | | |
| 7. Classifies quadrilaterals | | | → | ★ | | |
| **Vocabulary**<br>Quadrilateral, parallelogram, trapezoid, polygon<br>Convex polygon, nonconvex polygon, segment<br>*Is parallel to . . . ; is perpendicular to . . .*<br>**Symbols**<br>//, ⊥ | | | → | ★ | | |
| 8. Describes triangles: scalene triangles, right triangles, isosceles triangles, equilateral triangles | | | | | → | ★ |
| 9. Classifies triangles | | | | | → | ★ |
| 10. Describes circles | | | | | → | ★ |
| **Vocabulary**<br>Equilateral triangles, isosceles triangle, right triangle, scalene triangle<br>Circle, central angle, diameter, radius, circumference | | | | | → | ★ |

*Figure 2. Excerpt from the progression of Progression of Learning in Elementary School on "Plane figures"*

The van Hiele model offers the tools to assess students' geometric thinking level of particular topics. As outlined before, there can be considerable variation among the levels on different topics, and, even on one topic, the level might not be clearly cut. We could say that the van Hiele model situates student's knowledge of a topic in relation with the Euclidean geometry as a mathematical theory characterized by the rigor of deductive reasoning.

### 2.2.2 Geometrical Paradigms

To capture the changes of the objectives of geometry teaching throughout schooling, Houdement and Kuzniak (2003) proposed the notion of Geometrical Paradigms. Thus, Geometrical Paradigms model focuses on the institutional organization of geometrical knowledge to be taught at different levels.

Each of these paradigms specifies the nature of the geometrical objects (ontology), the employable methods and the validation mode (epistemology).

*Geometry I* (Natural Geometry) is closely linked to reality; deduction can be done by means of experimentation or arguments carried out on concrete, physical objects. In this geometry, model and reality are employed in an interchangeable way. Problems in this geometry concern practical aspects of the environment, for example, fitting square tiles onto a rectangular floor. A solution is validated by comparing expected and obtained results (Berthelot & Salin, 1994).

*Geometry II* (Natural Axiomatic Geometry) relies on the existence of a system of axioms and, to be valid, a proof must be constructed based on the axioms of the system. The objects are abstractions from the physical objects and are considered only by their defining properties. For example, a square table becomes a "square", understood as four segments with identical lengths and a right angle defined by two adjacent sides. In this sense, we deal with models that approach reality. The nature of questions we want to answer is different from the ones posed in Geometry I. We aim to solve problems about abstract objects: deciding on their defining conditions; deducing their properties from the chosen definitions; discovering logical connections between the properties of different objects; verifying the truth of conjectures and statements about these objects (i.e., proving); making sure there are no contradictions among the chosen definitions and the conclusions deduced from them. In consequence, solving a practical problem in this workspace means to construct a model of the problem, where the irrelevant details of the concrete context are ignored. Once the problem is solved, the results are interpreted in terms of the physical context. In this sense, knowledge of Natural Axiomatic Geometry is useful in many professions. The classical example for this kind of geometry is the Euclidean geometry.

In Geometry III, Formal Axiomatic Geometry, the system of axioms is not based on any sensible reality and proof and validation can be achieved only through the axioms. In other words, it is the system that defines reality. As an example, Euclidean geometry would be one of the possible interpretations of this Formal Axiomatic Geometry, and geometry on the sphere would be another. That is to say, the term "straight line" would refer to an abstract object that has some specified characteristics without necessarily resembling the straight line as we know it from Euclidean geometry.

To summarize and bring further details about the paradigms, we reproduce in Table 1 as given in Houdement & Kuzniak (2003).

*Table 1. Comparison of characteristics of the three geometrical paradigms (Houdement & Kuzniak, 2003)*

| | **Geometry I (Natural Geometry)** | **Geometry II Natural Axiomatic Geometry** | **Geometry III (Formal Axiomatic Geometry)** |
|---|---|---|---|
| Intuition | Sensible, linked to the perception, enriched by the experiment | Linked to figures | Internal to mathematics |
| Experience | Linked to the measurable space | Linked to schemas of the reality | Logical |
| Deduction | Close to the reality and linked to experiment | Demonstration based on axioms | Demonstration on a complete system of axioms |
| Kinds of spaces | Intuitive and physical space | Physical and Geometrical space | Abstract Euclidean space |
| Status of drawing | Object of study and validation | Support of reasoning and "figural concept" | Schema of a theoretical object, heuristic tool |
| Privileged aspect | Self-evidence and construction | Properties and demonstration | Demonstration and links between the objects. Structure. |

Houdement & Kuzniak (2003) suggested that the main problem of teaching geometry originates in a conflict between the students' and the teacher's working spaces. Curriculum specifications may require from the teacher to work in a paradigm to which students do not have access yet. It may also happen that the teacher, because of his or her geometrical knowledge level or to facilitate students' work, prefers working in Geometry I, even if Geometry II is required (Girnat, 2009).

While the van Hiele model describes the advancement of geometrical thinking of a novice learner, new to the topic of geometry, Houdement and Kuzniak's proposal of Geometrical Paradigms was thought for adult learners, future teachers, who already learnt geometry and have, presumably, mastered the material. In this sense, the purpose of the paradigms is to capture the future teachers' choice of a paradigm when dealing with particular tasks and not the level of their geometrical thinking.

A second element of comparison is the homogeneity of the theory underlying a given reasoning. Houdement and Kuzniak argue that reasoning, if qualified based on the levels in the van Hiele model, can be considered as involving several levels at once, yet in geometrical paradigms proposed by them, theories on which the paradigms are based are homogeneous. By this, the authors meant that a given reasoning uses the tools available in one of the paradigm and, thus, it is being completely contained in it. As already mentioned, research failed to confirm a clear cut separation of the levels in the van Hiele model in an individual's geometric thinking, instead showing development on multiple levels simultaneously. These results brought into question one of the features of the model: the linguistic unity. Therefore, it may be more difficult to create a clear view on a student's work, distinguish between choice and limitation of their thinking. Houdement and Kuzniak (*idem*) argue that each paradigm has its own theory and techniques, resembling more what Chevallard (1999) coined as praxeology. For the purpose of understanding teacher's work in the classroom, the authors argued for the theory of geometrical paradigms as a frame of analysis.

In my research the focus is on future teachers' envisioned practice of geometrical transformations. Although future teachers are adult learners in the sense used by Houdement and Kuzniak, they prepared the lesson plans about teaching a topic to novice learners. From this point of view, it is important to identify the potential of a task to advance the geometrical thinking of the students and to do such identification, the van Hiele model can serve as a guideline. Therefore, the methods course in which the future teachers participated was conducted from the point of view of the goals of teaching geometry underlying the van Hiele theory of geometric thinking.  Yet, as already mentioned, the analysis of the lesson plans is done through the lens of ATD with the purpose of identifying the underlying praxeologies of geometric transformations.

The teaching of geometry in the methods course will be described in the next chapter.

### 2.2.3 Theoretical Thinking model

In this research, theoretical thinking is considered as thinking defined by three characteristics: thinking that is *reflective, systemic* and *analytic* (Sierpinska, Bobos, & Pruncut, 2011). The reflective feature refers to one's ability to adopt a critical attitude towards the product of thinking. Thus, a reflective thinker will continue thinking back on the problem even after having solved it, with the purpose to identify alternative solutions, explore the generality of the solution, to search for links among previously seen situations. Systemic thinking is thinking in terms of a system of concepts and not individual cases or events. Three inter-related aspects can be distinguished here (Bobos and Sierpinska, 2017): a) meanings of concepts are established by definitions (*systemic-definitional thinking*); b) the validity of a statement is derived within the system (*systemic-based on proofs*); and c) the validity of statements is confined inside the system, that is to say, one is aware of the dependence of the statements truth value on the set of initial assumptions (*systemic-hypothetical*).

A person exhibiting systemic-based proving will question the conditions under which certain statements were made and will employ in reasoning only elements from the system (definitions of other concepts, theorems valid in the system). The systemic-hypothetical aspect expresses an epistemological position: the constructed knowledge is not absolute, but relative to the basic assumptions accepted as valid. It also assumes that a person is aware and familiar with the rules of the logic that was adopted in the system. In this sense, the individual will be concerned with the consistency of the system.

*Analytic thinking* can be formulated along two aspects: linguistic and meta-linguistic sensibility. The first one emphasizes awareness of the distance between concepts and their symbolic representation, while also being acquainted with a specialized terminology. Metalinguistic sensibility is also along two lines, as awareness of a) notational conventions and b) structure and logic of mathematical language.

The TT model has been employed in research focusing on students' thinking, as for example in Challita (2013) or Zachariades, Christou, & Pitta-Pantazi (2013).

Theoretical thinking is a basis for awareness of the reasons of why we do things the way we do. Future teachers should be in position to own their knowledge, thus to think at theoretical level. From this point of view, noting the presence or absence of the three components described above is informative about their epistemology.

# 3 METHODOLOGY

In this chapter, I present the methodology used in the research. Given that the aim of the research is the construction of a praxeological model (Chevallard, 1999) of future teachers' envisioned practice of teaching geometric transformations to elementary school children, based on an analysis of their lesson plans, the research is analytical and qualitative in nature. It is analytical in that its object of study is the mathematical meanings of the activities proposed in the lessons plans. It is qualitative in that it attempts to explain those meanings by the future teachers' experiences and sense-making practices in a given social context. From this point of view, there is compatibility between the fundamental position adopted by Chevallard (1999) in the development of the Anthropological Theory of the Didactic (ATD) and the fundamental assumption of qualitative, interpretive approach in social sciences, since both consider human experience as context-bounded.

Hammersley and Atkinson (2007) argue that qualitative research can provide descriptions of a culture, since researchers must start from observations and narratives of participants acting and being in a given context, and, at the same time, be familiar with the "world" in which participants live. The description of a culture encompasses also descriptions of the values and ideologies of the participants (Flick, 2014); an aspect that is captured by the theoretical block of the ATD.

Consequently, the qualitative approach requires not only to collect data directly relevant to the participants' practice that we want to describe (lesson plans in our case), but also to describe precisely the context in which these data have been created (in our case: the pre-service mathematics methods course in which the lesson plans were produced) and the larger influences that are present in the given context (textbooks, internet resources, etc., that the participants used to create their lesson plans).

Therefore, I'll proceed by describing the participants in the study – a group of pre-service elementary teachers enrolled in a math methods course – and the main data source used in the study: the lessons plans written as part of a final assignment in the course, called "The Problem Book". For this purpose, a separate section will contain the detailed description of the specifications future teachers received from their instructor about writing the problem book. Another section will describe the part of the methods course that was devoted to geometry. Additional sources of information, available to future teachers and probably significant to the preparation of the problem book will be presented in the last part of the chapter.

## 3.1 PARTICIPANTS

Participants in the study is a class of 32 university students, future teachers, enrolled in a "Teaching Mathematics II" course, a part of an undergraduate program leading to elementary school teacher certification in the Canadian province of Quebec. All students in the class signed a consent form agreeing to participate in the study. They agreed to be observed in class (without audio- or video-recording) and have their written productions in the course analyzed and results of the analysis published provided the authorship of the productions, when reproduced in publications, is kept anonymous.

I – the author of this thesis – was both a teaching assistant in the course, marking assignments, responding to students' questions, and a researcher, present in all classroom sessions (3 hours and 15 minutes altogether weekly, with two meetings per week, for 13 weeks), observing classes and taking notes about students' interventions particularly during geometric activities. The instructor in the course was the supervisor of this thesis.

There are three mathematics methods courses in the teacher education program the participants were enrolled in. The Mathematics Teaching I course focuses on whole numbers and problem solving involving only these numbers; Teaching Mathematics II deals with fractions and geometry, while Teaching Mathematics III is about proportionality and statistics.

As a final assignment in the particular Teaching Mathematics II course (hereafter called "the methods course"), students were asked to produce a "Problem Book", addressed to teachers, containing descriptions of 12 activities for elementary school, about fractions, ratio and proportion, percents and geometry. It is these descriptions that I called, earlier, the "lesson plans". All thirty two participants submitted the Problem Book, the main source of data for the current research.

In the next section, the course – which constitutes the context of the data – is described in some detail.

## 3.2 Research procedures

The analytic approach to the data can be summarized in the following procedure. Each sentence from the lesson plan was analyzed for its mathematical meaning and this meaning was categorized as source for one of the four components of the ATD: task, technique, method or theory. Sentences were contrasted with the already categorized one, thus constantly probing the similarity of their mathematical meanings with already established categories.

Given that the categories of meanings originated from mathematics, and were not defined by myself, the research methodology differs from a grounded theory approach. Neither was the categorization a result of a triangulation method: as mentioned, the already categorized data served as source for contrasting meanings for subsequently analyzed sentences. In other words, the evidence for a claim resided in the data and not in the opinion of independent raters.

For illustration, I briefly outline here an example of an application of this procedure. Consider the following excerpt of a lesson plan for a given translation task:

> Assumptions about students' previous knowledge:
>
> *Students will already have basic knowledge of the definition of a translation in geometrical transformation terms.*
>
> *Definition of translation: According to Geometry & Spatial Sense: Grades 4-6 (2008), "a translation can be described as a transformation that slides every point of a shape the same distance in the same direction. During a translation the orientation of the shape does not change and the image is congruent to the original shape. A translation can occur in any direction."*
>
> Instructional objectives of the activity:

*Through this activity, I expect students to become familiar with the use of a grid. For instance, each square on the grid represents a single unit.*

First, one can observe that the future teacher makes reference to the "basic knowledge of definition of a translation", and then presents this definition (which is not tagged as definition in the original source cited by the future teacher). The first part of the "definition" (*"a translation can be described as a transformation that slides every point of a shape the same distance in the same direction")* makes reference to "every point of a shape" and these are "slided" by "same distance" and "same direction". These statements suggest a certain interpretation of the translation, as transformation, but also about the nature of the object to be translated. In this sense, it provides information about the *nature* of objects in this geometry, namely, that they are defined by *points*.

The second phrase refers to the preservation of the orientation and size of the shape; thus, it refers to the object as a *compact shape*. This represents a different understanding of the shapes which are no longer made of *points*, but exist as compact objects.

Each of these two phrases suggests a certain way of performing translation (technique in terms of ATD) and these techniques are different, since the object to which they apply are different: the first is applied to a *point*, while the second is applies to a *shape*. While informative about the techniques, the phrases also tell us about the *nature* of geometric objects: as made of points, or as compact shapes. In consequence, two different techniques and underlying method / theory are delimitated.

The phrase in the *Instructional objectives* section hints to the underlying geometry: a "grid geometry" where the unit is the "square" – a bi-dimensional compact object.

Subsequent analyses of the mathematical meaning of other phrases in the corpus of data seek similarities and differences with these established categories in order to clarify whether new categories should be defined. Coherence between the consequences of attributing a meaning to a certain category and the meanings identifiable in other parts of the lesson plan is the guiding element in this classification. The analysis of lesson plans, thus the research methodology, will be presented in detail in chapter 4.

## 3.3 THE CONTENT OF THE METHODS COURSE

The description of the course, given to students at the beginning of the term, announced that it will focus mainly on the development of the first three of the twelve professional competencies, as formulated in the Quebec Educational Program (Martinet, Raymond, & Gauthier, 2001):

Competency #1: The teacher perceives and communicates mathematics as a cultural heritage as well as a means for understanding and solving problems in today's world.

• Competency #2: The teacher communicates mathematics clearly, consistently using correct mathematical terminology in the accepted meanings.

• Competency #3: To teacher develops teaching/learning situations that are appropriate to the students concerned and the mathematical content with a view to developing the competencies targeted in the programs of study of mathematics in elementary school.

The Problem Book assignment asked students to engage most explicitly with the third competency, while expecting them to apply competencies 1 and 2 in the design and description of the "teaching/ learning situations" (activities, problems). The Problem Book was a culmination of the whole course where each homework assignment required students to invent a problem or an activity, and also conduct a 40-minute "workshop" where, during the first 20 minutes, they would simulate a designed activity with their peers in the role of elementary school children, and then conduct a discussion on the lesson with their peers in the role of fellow teachers. Students could include, in their Problem Books, revised versions of their activities invented during the course.

As mentioned, the mathematical content of the course included fractions, ratio, proportion, percents and geometry. The arithmetic concepts were dealt with in the context of problems of measurement of magnitudes and geometry focused on relations within and among classes of figures and on transformations of the plane (reflections, translations and rotations only).

There were 6 homework assignments administered weekly in the first 6 weeks of the course. The next 5 weeks were devoted to the workshops and whole class reflection on them. Students had to submit Reports on their workshops, containing the initial conception of the activity, a critical appraisal of the activity in view of the received feedback and a revised version of the activity. The last two weeks of the course were devoted to preparing students to write the Problem Books, with students presenting samples of their lessons plans and feedback from their peers and the instructor of the course. More details follow.

The homework assignments in the first six weeks of the course consisted of 4 or 5 tasks: three or four questions to answer / problems to solve or pose; the last task asked students to imagine and describe a classroom activity on a particular topic discussed in class. This requirement was central to the course, given the importance accorded to the competence 3, mentioned above.

The posed problems and activities were evaluated from the point of view of mathematical correctness, relatedness to the situation given as initial context, and for practical correctness and relevancy. On some occasions, the future teachers were asked to pose a problem that would be harder or easier than a given one. In such situations future teachers were required to explain the reasons why they considered the posed problems as easier/harder.

Homework assignments were corrected by the two teaching assistants in the course. Correction was based on a grading scheme developed by the course instructor. In a week's delay, future teachers received back their graded homework with detailed feedback. A special attention was given to the evaluation and comments on lesson plan development / activity planning and problem posing. At the end of the semester, future teachers could decide if they wanted to bring improvements to the homework assignments, by attending to the comments.

During the five weeks of workshops, each future teacher animated a workshop on a topic studied in class. Students were split into three groups and three students ran the workshop simultaneously; one presenter with the group of fellow future teachers acting as students. The teacher assistants and the course instructor were one in each group and took notes during the workshop. The workshop material was developed by future teachers and after the workshop, in a week's delay, they had to write a full report about the workshop. The purpose of the report was to position future teachers in the role of reflective practitioner (Colton & Sparks-Langer, 1993). Future teachers received detailed feedback on their workshop reports from the course instructor.

Towards the end of the course, future teachers were presented with the Problem Book assignment. The final submission date was a week after the last class; however, future teachers were aware from the beginning of the course of this assignment. Given the central role of the Problem Book in the present study, in the next section I will present in detail this data source.

## 3.4 THE PROBLEM BOOK

The main source of data for my research is the Problem Book: a set of lesson plans prepared by future teachers and addressed to fellow elementary teachers. Future teachers received a template for the Problem Book containing the organizational structure. Each element of the Problem Book contained a brief description of what should contain.

The main sections in the Problem Book were the following: a) A word to teachers b) three chapters containing four activities each; c) Reflection and, lastly, d) a list of references.

### 3.4.1 The expected content of the introduction to the Problem Book, titled "A word to teachers"

In the section "A word to teachers", the Problem Book template had further specification on the content of this part. These details were given as short descriptions of what the main idea of the paragraph should be. In this introductory section, for example, the Problem Book template contained the following ideas:

a) This Problem Book is addressed to elementary school teachers in....
b) The intent of this book is…
c) The book is organized as follows….
d) How can this book be used in the teachers' practice? …

The requirement to complete the Problem Book under a pre-defined structure gives the possibility to perform comparisons among future teachers' answers and derive the beliefs they hold about teaching. This type of information is essential for the qualitative research, since it gives insight into the "personal" (as opposed to the "institutional") context.

### 3.4.2 Lesson plan template

The introduction was followed by three chapters containing four lesson plans each. The internal organization of these chapters is identical. The theme of the categories was not predefined, but the number of chapters, three, was thought to be in line with the categories of topics explored during the course (fractions in context, ratio and proportion, and geometry). The structure of each chapter was in five parts: "Introduction" and 4 detailed lesson plans on the chosen topic.

The "Introduction" described the core ideas that should be addressed in order to facilitate seeing the connections among the follow-up activities. These were reminders of the subject to be treated:

*a) There will be four activities in this chapter…*
*b) The focus of the chapter is…*
*c) All the activities [have what in common?].*
*d) The activities differ by…*
*e) Here is how the activities are related to the curriculum….*

Asking the future teachers to address these aspects was intended to bring them to reflect on teaching, to see continuity among teaching activities as they progress towards an instructional goal. At the same time, the ideas expressed by future teachers were helpful for the researcher to understand their views on the role individual activities play in teaching.

The lesson plan template given to future teachers was the most extensive description in the Problem Book. The elaboration of a lesson plan is the most frequent task of a teacher and future teachers are often expected to learn how to write one. The main goal of the detailed specification was to underline the many aspects future teachers must consider when developing a lesson plan. I will go over in detail on every element of the lesson plan. The overall structure was the following:

a) Activity Title
b) Assumptions about the students' previous knowledge and/or level of thinking
c) Instructional objectives of the activity
d) Means used to achieve the instructional objectives
e) Tasks for students
f) Expected solutions
g) Means of checking if the instructional objectives were achieved

The description of the lesson plan starts with the title of activity. There were no indications on what a title should comprise and, as it turned out, most future teachers considered this as an occasion to give a "fun", "engaging" and catchy title that would capture the attention of their students. Under this part of the lesson plan, future teachers were also asked to specify the target audience ("Grade range"), the mathematical topic in focus ("Mathematical topics") and the required materials ("Materials").

The section "Assumptions about the students' previous knowledge and/or level of thinking" contained two indications in the form of sentences to complete: "It is assumed that students have been taught… and demonstrated… " The main purpose was to bring to the future teachers' attention the assumption that progression in learning builds on previous experience and there is a certain dependence among the topics and skills or skill levels. In addition, this section was expected to incite the future teachers to reflect on and analyse what is necessary, in terms of skills and knowledge, for accomplishing the task proposed in their activity. In a way, this section is the very first element where future teachers must think as teachers, and not as students solving a problem given by someone else.

In the "Instructional objectives of the activity", the template gave future teachers a definition of this concept and an example of categorisation of these objectives.

The detailed template had the goal to guide future teachers through the development of lesson plans. Although, officially, this was the first occasion future teachers were required to submit lesson plans and, thus, understandably they were novice in elaborating them, future teachers also could get inspired from the textbooks associated with the course and use any kind of resource they could have access to. Similarly, although never stated explicitly, the teaching carried out in the classroom with them as students, was also a way to model teaching and an opportunity for them to study the internal coherence of the lesson succession. In what measure classroom experience was a source of learning *about teaching* is not clear; the personal impression I had from being observer of the classes was that future teachers were too engaged in being students and learning the mathematics to distance themselves from it and reflect on the experience as teachers.

Future teachers were given the following definition of instructional objective of an activity: "concise, explicit statements that describe what exactly you expect students to learn and the skills you hope they will acquire" by participating in the activity (CIRTL, 2012). The specifications in this section also included Bloom's (1956) categorisation of instructional objectives, along with the suggestion that, for the sake of focus, to consider only cognitive instructional objectives pertinent to mathematics. A set of concrete examples of instructional objectives was given in the template, as follows:

> *"Through this activity, I expect students to …*
>
> *[e.g.,*
> *… overcome a misconception about…*
> *… reason with the concept of… at a higher cognitive level (e.g., conceptually and not only procedurally)*
> *… discover the property… / the concept… / a relationship between…*
> *… understand the concept of …*
> *… consolidate an understanding of the concept of…*
> *… practice the skill of…*
> *… apply the knowledge of… in solving real-life problems…*
> *….]"*

The follow-up section in the lesson plan, "Means used to achieve the instructional objectives" required to describe how the characteristics of the proposed tasks would help in achieving the objectives. In other words, future teachers had to detail the elements of the task that will direct the student to act or think in such a way that the instructional objective is achieved. In order to facilitate the understanding of these ideas, future teachers were presented with concrete examples during one of the classes towards the end of the course. These examples were elements from the workshops presented by future teachers. The following are some of the presented situations[1].

*I. Instructional objective: students will overcome a misconception related to scalene triangles.*
- *Example of a way to overcome the idea that all scalene triangles are obtuse.*
- *The idea may appear if all the examples of scalene triangles that children are shown happen to be obtuse triangles.*

*II. Instructional objective: students will reason about fractions at a higher conceptual level*

[1] These are taken from the slides presented during Lab 10.

- *Example of a way to "force" conceptual thinking about fractions by choosing the numerical values in a problem so as to make the procedural approach to a problem very tedious.*

III. *Instructional objective: students will overcome a misconception about fractions*
   - *Suppose the set {1,2,3,4,5,6) is my "whole".*
   - *I say that the set {1,2,6} is a fraction.*
   - *No?*
   - *But it's a part of the whole. So surely it must be a fraction?*
   - *[Let the children formulate their conjectures. They might eventually come to the conclusion that taking the phrase "a fraction is a part of the whole" too literally is misleading. ]*

IV. *Instructional objective: students will understand the concept of fractions in the context of quantities*
   *The means to achieve this objective are many and varied:*
   - *Brief and concise definition of the word "fraction"*
   - *Explanation of the meaning of the expression "This quantity is* $\frac{a}{b}$ *of that other quantity"*
     - *in general terms*
     - *and on examples of use covering a range of possibilities (fractions less than one, and greater than one; fractions of the same quantity and fractions of different quantities, etc.)*
   - *Graphical representations*
   - *Physical representations*

Future teachers were also asked to come up with ideas and propose the "means" as in the following case:

V. *Instructional objective: Students will discover a relationship between the number of sides in a polygon and the sum of its interior angles.*
   - *A volunteer to give an example of the means to achieve this objective?*

The next section of the lesson plan is "Tasks for students". The indications given to students for this section were:

*"At the beginning of the activity, students are asked the following questions, to make sure that…*

*Next, students are given these tasks….*

*The tasks are written in a handout, included at the end of this section. The handouts are accompanied by the following materials (drawings or photos are included at the end of this section)."*

Once again, the purpose of the indents was to draw the attention of future teachers to the need of articulating the task with an introduction that assures all students are ready to take on the task. Also, it makes a distinction between the handout given to students that contains the task to be solved (as students) and the task (as future teachers). Ideally, it is expected that future teachers solve the tasks from the handout, in the role of students, so that they can verify that all elements are understandable and well-formulated. Switching between the role of teacher and student is necessary for ensuring that no tacit elements and interpretations of the future teachers are taken into account when solving. This phase proves essential also for the next section in the lesson plan, "Expected solutions".

In this section, it was expected that future teachers specify what they consider as the most probable answer given the hypothesis already specified in the beginning of the lesson plan. The template specified the following:

*"Since I want the students to…, I expect that, ideally, they will answer the question… as follows….*

*But I also know that some students are likely to… . In this case, this is what I plan to do…."*

The expected solution illustrates future teachers' view of "ideal" solution. For the analysis I am interested in, this part is valuable since it permits to compare it with the instructional objectives, so to better understand what future teachers *mean* when using some of the proposed formulations for the objectives. It also gives an insight into what future teachers mean by *solution*; in particular, if they see the difference between answer and solution. In addition to the "correct" solution, future teachers were directed to mention potentially "wrong" solutions and, briefly, explain their origin. Often, a wrong solution is a consequence of misconceptions or partial understanding of procedures or concepts. Although future teachers didn't have a course on common misconceptions of children in elementary mathematics, quite a few of them reported extensive tutoring experience and they often justified their choices of tasks or follow-up questions based on that experience. In this sense, their descriptions are a rich source of information about the beliefs they have on students' learning, difficulties, but also, on the nature of mathematics to be taught.

The last section in the lesson plan is about "Means of checking if the instructional objectives were achieved". It was suggested to specify how the learning would be verified:

*"At the end of the activity, students are given the following "test" questions, to probe if they have indeed learned what I wanted them to learn."*

Once again, the future teachers' elaboration on ways of checking the learning is also a source of knowledge on their views about *what* has to be learned, independently from what they specified as the instructional goal of the activity. It is also telling about their conception of what a formative / summative evaluation is. All the elements in the lesson plans are influenced by their experience in school as students, and also experience in tutoring, impressions from visits to schools, etc. For this reason they are valuable sources of knowledge about their beliefs and attitudes and, of course, mathematical content knowledge.

### 3.4.3 Reflections

The closing section in the Problem Book is called "Reflections". It was conceived as the place for some closing thoughts about the process of creating the activities and, also, the place for concise enumeration of advice on what to do and what not do in the classroom. The introductory part invited future teachers to reflect in their Problem Book by asking about the sources of inspiration and the guiding process in the development of the activities. The second half of this closing chapter, asked for lists of "Do's and Don't's" in the classroom.

In the "Do's" it was suggested to think along the terms "Make sure… " or "Test the activity on yourself and at least one another person, preferably an elementary school student of the target age or grade." In the "Don'ts" no special indications were given; however it was expected to have concrete advice on what not to do in a classroom as a teacher.

At the very end it was expected to have the “References” section completed with the sources used in the development of the book.

How the Problem Book is useful for the purpose of this research, namely for the identification of future teachers’ mathematical praxeologies? As mentioned above, the methodology relies on an interpretive study which requires having information about what future teachers are considering to do in their future practice, but also about the context in which they live and act. Among the numerous trends in mathematics education, the journal writing gained some interest in the past decade as a way to promote reflection, whether on student’s, teacher’s or teacher educator’s side (Crespo (2003); Cervello Rogers (2014); Ching-shu Shen (2015)) and way to get insight into the thinking and belief these actors hold (Liljedahl (2005); Kenney, Shoffner, & Norris (2013)).

Writing the Problem Book is not the same as writing a journal on a regular basis; however, the process of reflection is present while writing about what the activities should be and justifying their place in the teaching sequence. In addition, in the sections where future teachers had the liberty to express their ideas (as advice for fellow teachers, or even in the sections where they must justify their choices) are sources from where to infer about their beliefs on teaching, learning and mathematics and even their attitudes towards mathematics. In the description of the Problem Book’s structure, I mentioned in which way its sections can act as such sources. By identifying recurrent ideas and common themes, one can draw a group portrait of the future teachers concerning their beliefs and attitudes.

For identifying the mathematical praxeologies related to plane transformations, the main source were the lessons plans on this topic. The mathematical content is most visible in the task specification, the handout described in the lesson plan. In addition to the content, in order to construct the underlying technology/method and theory (as constituent parts of the theoretical block of the praxeology in ATD), it is necessary to look into all sections of the lesson plan, but with special attention into the parts concerning “Expected solutions” and “Means of checking…”

Notes taken during classroom observations were additional sources of information. Workshop activities and the reports written *a posteriori* were also very useful. They were especially rich in conveying future teachers’ thinking, since they gave the opportunity to compare the initial and revised versions of the activities, but also gain insight into the future teachers’ reflection about the way their workshop unfolded and what they considered to be the necessary improvements.

Instruction during the course constituted the major content information for the future teachers; therefore, it is important to describe its content. In the following, I describe the part of the course instruction concerning geometry, particularly the instruction on transformations.

## 3.5 COURSE INSTRUCTION CONCERNING GEOMETRY

In this section, some details of the geometric content of the methods course, ways of presenting it and engaging the future teachers with it will be offered. I chose details that convey the instructor’s efforts to bring the future teachers to perceive geometry as a theoretical mathematical knowledge and reason about geometric tasks at, at least the Analysis level in the sense of the van Hiele level. I also describe in some detail the instructor’s interventions and activities devoted to geometric transformations. These details can explain some of the future teachers' choices in writing their lesson plans on transformations that became the object of a praxeological analysis in my research for this thesis.

### 3.5.1 The first lessons on geometry in the methods course

Geometrical content appeared in various places in the course. For example, similarity of rectangles was discussed in the context of proportionality; the notion of area of a circle appeared in the context of a problem of comparing fractions of pizzas with different diameters; the ratio of the circumference of a circle to its diameter was measured in the context of distinguishing between rational and irrational numbers, etc. There was, however, a sequence of classroom sessions totalling 7 hours and 45 minutes (3 labs of 1h15 mn each and 2 so-called "lectures" of 2 hours each) that were explicitly devoted to geometry.

The sequence on geometry started with *an activity of sorting shapes*, borrowed from van de Walle & Lovin's textbook for teachers of grades 5-8 (Walle & Lovin, 2006, p. 188). Students worked first in small groups and then representatives of the groups gave brief accounts of their work to the whole class. The presentations revealed that the future teachers function mainly on the van Hiele Visualization level with some arguments presenting characteristics of the Analysis level.

This common experience then served as a reference and source of examples for *a lecture-like presentation of the van Hiele model of geometric thinking*. The students – future teachers – were presented with descriptions of these levels, examples of answers to the same question from students' at different levels of geometrical thinking and tasks that would facilitate the transition from one level to the next. The instructor's aim was to bring the future teachers to reason at, hopefully, the Informal Deduction level or at least at the Analysis level, and to have some idea of the Deduction and Rigor levels.

As part of the *presentation of thinking at the Deduction level,* future teachers were introduced to Euclid's "Elements" and referred to an online version of T.L. Heath's translation with commentaries[2]. An important aspect highlighted during this presentation was the difference between geometry as empirical science (which existed long before Euclid), and formalized theory, as presented in Euclid's books.

In relation to the level of *Rigor*, future teachers were informed about the existence of different geometries, obtained by modifications of the sets of axioms used as fundamental elements in Euclidean geometry. Elementary teachers are not expected to teach or even mention other geometries during their practice, yet the instructor believed that it is important for them to be aware of their existence. The presentation seemed to have left an impression on the future teachers; one of them even crocheted a model of a hyperbolic plane, based on the work of Daina Taimina (presented during the lecture, see Figure 3).

[2] http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

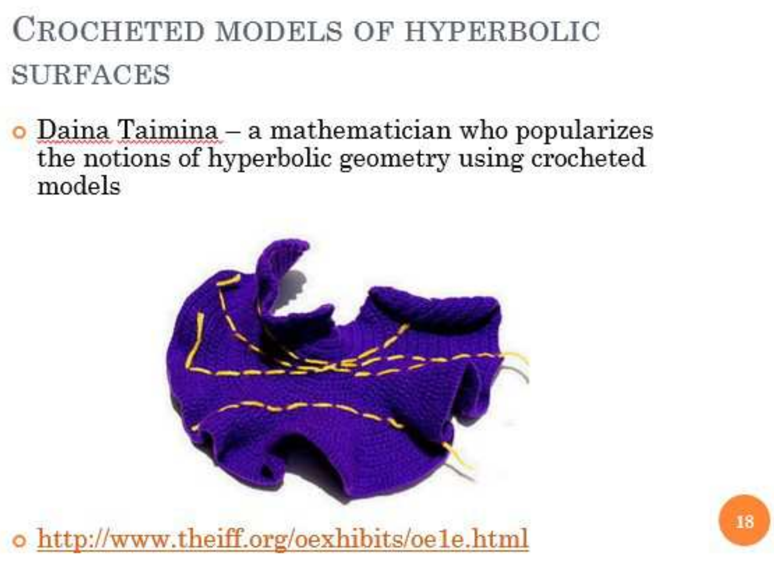



*Figure 3. A slide presented during Lecture 6 of the methods course, illustrating the notion of a straight line in a hyperbolic plane.*

The presentation proceeded by sequentially building the knowledge required for accomplishing the tasks planned for an activity to be done in small groups. (The first task was to construct, by paper folding, a square with a given line segment as its side and to identify the geometric properties used in the construction. More details about the tasks will follow.) The “sequential” aspect was considered important for acquainting the future teachers with the nature of a mathematical theory construction, namely, building on previously introduced definitions and proved results.

First, the definition of *plane angle* was introduced, as presented in Book I of the “Elements” where a plane angle is defined as an inclination between two lines (Definition I.8). Next, the notion of right angle was defined, together with perpendicularity (Definition I.10), as a particular relation between two straight lines: “When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.” The future teachers, used to defining the right angle as a “90 degrees angle”, found this surprising or strange. Definitions I.11 (of obtuse angle) and I.12 (of acute angle) further illustrated the sequential and cumulative character of the Euclidean geometric theory, since these definitions build on the definition of the right angle: the acute and obtuse angles are defined by comparison with the right angle.

Second, definitions of rectilinear figures (triangles, quadrilaterals, in particular case) were introduced, followed by the categorisation of triangles as acute-angled, right-angled and obtuse-angled ones.

Third, the definitions of commonly known quadrilaterals were given. This was followed by a comparison of these formal definitions with the definitions given in primary school.

After this “lecture”, interspersed with questions or remarks from the class, the future teachers were presented with the following task, titled “Activity 6.1 Understanding the concept of square at the Analysis level”. The instructions were given on prepared slides as follows:

*Work in groups of 4.*

*Each group is given one sheet of paper on which a line segment not parallel to any of the sides of the paper is drawn.*

*[A square with a segment not parallel to any of the sides is drawn on the slide]*

*The TASK:*

*(a) To make a square with a given side, from a given sheet of paper by folding the paper (no rulers, no scissors).*

*(b) To formulate, in writing, mathematical questions that arise in accomplishing task (a).*

*(c) To list geometric properties that have been used to produce the square.*

*ROLES OF MEMBERS OF THE GROUP:*

*ALL: Everybody makes suggestions on how to fold the paper and justifies their suggestions, until some agreement is reached.*

*QUESTIONS PERSON: One person takes note of the mathematical questions that arise in the discussion.*

*PROPERTIES PERSON: Another person lists geometric properties mentioned in the discussion.*

*CONSTRUCTION PERSON: After an agreement is reached on what to do, one person folds the paper.*

*ALL: Next, the group discusses if the shape obtained is indeed a square.*

*QUESTIONS PERSON: Again, notes are taken of any mathematical questions that arise in this discussion and geometric properties that are mentioned.*

*ALL: Finally, the group agrees on the properties that have been necessary in the construction.*

*PROPERTIES PERSON: The "Properties person" marks those necessary properties on the list of all properties mentioned in the discussion.*

The task was intended as a context where future teachers had the opportunity to realize that the definitions and properties provided earlier are answers to questions naturally arising during a construction process (e.g., How to construct a perpendicular line to a given one? How can we be sure if a line constructed in such and such way is indeed perpendicular to the given one?).

A second activity given to future teachers was to decide whether a given quadrilateral is a square. Once again, the purpose of the activity was to highlight the different types of argumentation and their links with the geometrical level of thinking according to van Hiele. In this sense, the discussion was intended to contribute to future teachers' understanding of the nature of argumentation expected in different grades, in a school environment.

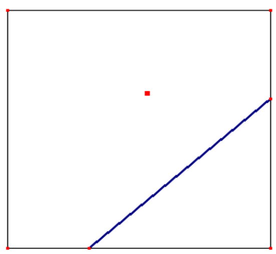
The third activity was given on a slide as follows:

*TO CONSTRUCT A LINE PARALLEL TO GIVEN LINE THROUGH A GIVEN POINT*

*Everybody gets one piece of paper with a line and a point drawn in on it.*

*TASK: To construct a straight line through a given point parallel to a given line by paper folding.*

*What geometric properties have you used in the construction?*



A special characteristic of this task is that, for solving it, one must (at least implicitly) rely on the notion of perpendicular lines conveyed in Definition I.10 of Euclid's "Elements". The folding of a line over itself creates two equal angles, thus, according to this definition, the angles are right angles. From this point of view, the task offers a rich experience: on the one hand, it clearly highlights the importance of definitions (here, that of right angle and perpendicular lines) and, on the other, it draws the future teachers' attention to the role of the environment ("milieu" in the sense of Brousseau (1986)), in which the task is to be accomplished. The environment enforces a selective use of knowledge. It determines the tools (information, instruments) available for solving the problem. In this particular task, the environment was made of a sheet of paper, the possibility of folding it but not cutting or gluing, and the definitions and theorems of the "Elements". Constructing a perpendicular through a given point on a given line with compass and straightedge would have been a very different task, from the point of view of knowledge required for solving it.

As previously mentioned, all these activities were designed with the purpose of highlighting the importance of definitions and properties in solving certain geometry problems, the role of the context in the mobilization of knowledge and in asking new questions or posing problems.

The lessons and activities described so far occupied the lab session in week 5 (Lab 5) and the lecture session in week 6 (Lecture 6).

Further geometric activities were given in the following lectures and labs with the same purpose to highlight the use of properties and definitions in solving tasks. These tasks included: an Analysis level proof of the sum of angles in triangle and the construction of tangram pieces only by folding a square piece of paper. The activity with tangram pieces allows simultaneously to treat some topics and ideas in mathematics: fractions (by searching for relations among the areas of the pieces), types of polygons (the characteristics of the pieces), properties of the polygons (especially, those useful in constructing them), but also fundamental ideas such as the idea of preservation of the area in rearrangement. From this point of view, the tangram activity serves as an example of a *rich* task, in the sense educators use the word. Cambridge University's NRich team (University of Cambridge, 2017) provides a selection of descriptions, based on a variety of official sources, of a *rich task.* A brief, concise formulation of a rich task / context would be as a task with "a range of characteristics that together offer different opportunities to meet the different needs of learners at different times."[3]

The next section contains some details about instruction related to geometric transformations.

### 3.5.2 Lessons on geometric transformations

Lab 6 (in week 6) was devoted to the introduction of transformations of the plane. The first transformation was the reflection about a line. Its definition was preceded by a definition of distance from a point to a line (Figure 4) and this definition was preceded by a recall of the definition of perpendicular lines. The line in the figure accompanying the definition of distance was purposefully drawn obliquely so as not to reinforce the common confusion between the vertical direction and perpendicularity.

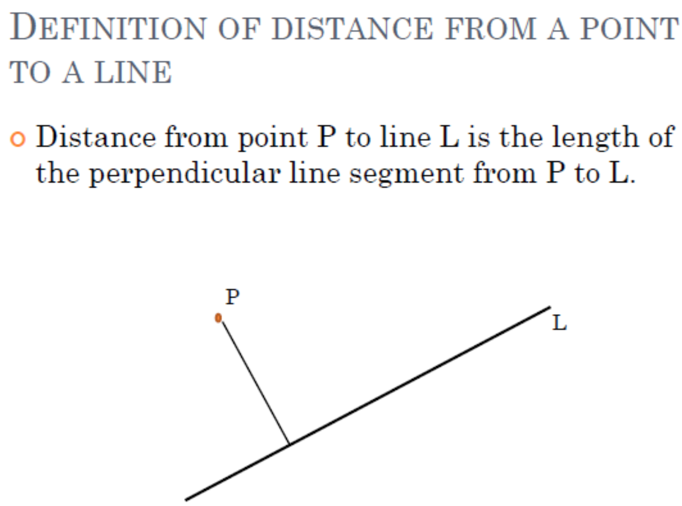



*Figure 4. The slide shown in Lab 6 with the definition of the distance from a point to a straight line.*

Next, the instructor engaged the students in attempting to characterize the reflection about a line. Various metaphors (flip), gestural representations, and properties were proposed. The instructor then proposed to distinguish the definition from the properties and to express the definition in terms of the relation between an arbitrary point of the plane and its image under a reflection about a give line. The definition was partly said in words and partly written on the board, and illustrated with drawings. The properties were also expressed that way. All that was said and drawn was then summarized on a slide (Figure 5).

[3] http://nrich.maths.org/5662

## Properties of a Reflection about a line

- **Definition:**
- Reflection about a line L is a transformation of the plane such that
- (a) if point P' is the reflection of point P about L, then the line through P and P' is perpendicular to L, and
- (b) the distance from P' to the line L is the same as the distance of P to L.
- **Properties**
- If F and F' are figures that are reflections of each other about L, then F and F' are congruent: they can be superposed on each other.
- The line L is a line of symmetry of the union of figures F and F'.
- If L is a line of symmetry of a figure, and points P and P' are symmetric relative to this line (reflections of each other), then the line PP' is perpendicular to L.
- In a reflection about a line L, if P' is the reflection of P and O is any point on L, then the line L bisect the angle POP'.
- Conversely, in a reflection about a line L, for a point P' to be the reflection of a point P, the line L must bisect the angle POP' for any point O on L.

*Figure 5. Summary slide institutionalizing the definition of a reflection about a line and its properties.*

Note that two of the properties in Figure 5 speak about the axis of reflection as being the line of symmetry of a certain figure (the union of the initial figure and its image under the reflection). We will return to this fact later.

The first activity following this discussion was asking the future teachers to draw the reflection of a polygon given on an isometric dot paper. (Figure 6)

- Use isometric dot paper to draw the reflection of a shape about a line.

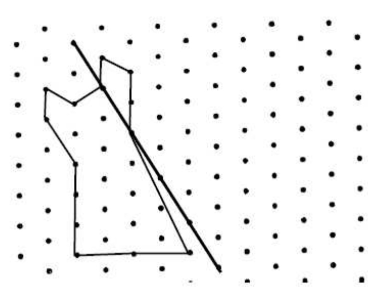


*Figure 6. Activity to be done in small groups in Lab 6.*

Future teachers received the task on paper and were expected to perform the reflection. There are several observations to make about the above task; these remarks are essential for our analysis of future teachers' task proposals in Chapter 4. The paper is an isometric paper, meaning that the distance between two neighboring dots is the same (basically, these are vertices of equilateral triangles). Although very particular as paper, it is not so commonly used as the square dot paper or square grid paper. Certainly, it allows working in a special way, by considering the particularity of distances.

Next, observe the axis of reflection: although it passes through the grid points, it is not in standard position in relation with the frame of the paper (horizontal or vertical). Observe the shape: it is not a regular polygon, but a concave polygon. Lastly, the axis of reflection cuts through the shape. The purpose is to understand that (1) no particular position is required for the axis of reflection and (2) points on the reflection line are mapped onto themselves.

The shapes and the context for the task (the "milieu") were carefully chosen so as to allow future teachers seeing certain properties of the transformation or the generality of the definition (the definition can be applied to any shape). Similarly, the task can be performed by constructing *with ruler* the distance of each point from the reflection axis (and constructing the reflection of each vertex) or *without ruler*, by using properties of equilateral triangles. For all the mentioned reasons, the task permits seeing the general in the particular – once this is performed, future students should be able to perform reflection on any kind of shape. From this point of view, the task is not a randomly selected one, but one *purposefully crafted* by the course instructor in order to illustrate certain ideas. In a methods course, the instructor aims not only to convey the mathematics, but also the ways in which this mathematics becomes available to students. The question remains again, as mentioned already, in what measure this intent is recognized and brought into awareness by future teachers.

The next activity done in this lab referred to the notion of line of symmetry, mentioned already in one of the slides (Figure 5). It was asking students to identify the lines of symmetry of a number of plane figures (square, rectangle, parallelogram, kite, isosceles triangle, equilateral triangle, circle, line segment, infinite straight line).Treating the subject of line of symmetry at this point had two purposes. On the one hand, the intent was to make a clear distinction between reflection (a transformation) and symmetry (a property of a figure). On the other hand, it represented the occasion to talk about another way of looking and categorizing geometrical objects. Thus, the next activity required to specify for a set of quadrilaterals certain properties, including the number of lines of symmetry.

The lab ended with a question for the future teachers to think about: How would you define a rotation about a given point by a given angle? The question was accompanied by a reproduction of a task from Walle & Lovin (2006, p. 212).

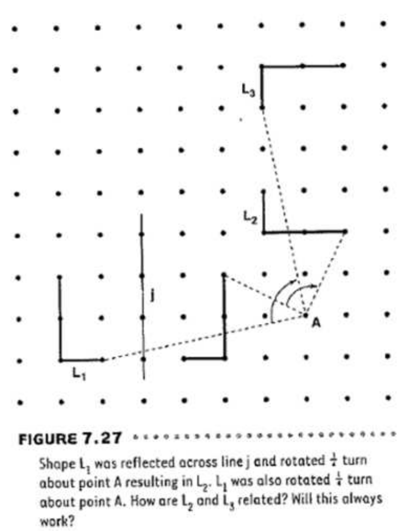



*Figure 7. A task from (Walle & Lovin, 2006).*

The example shown brings up an interesting question for future teachers to think about, namely about the preservation of the initial relation between shapes (one being a reflection of the other) through rotation around the same point and with the same angle. It also provides an example of rotation which is on square grid paper, with a figure on grid points and by an angle of quarter turn – thus, quite standard and very particular configuration – yet the center of rotation is not on the figure. In consequence, the attention is drawn to the importance of the rotation center – without specifying the center of rotation, we can't perform a rotation. Also, it underlines that the definition does not contain any restriction on the center of rotation.

The instructor returned to the topic of geometric transformations in Lab 7. The lab started with a discussion of an activity about a tiling on which the future teachers worked in Lecture 7. (Figure 8) Part (d) of the activity asked to identify the geometric transformations the tiling was constructed with.  In this section we will give an account of the instructor's interventions related to this part of the discussion only.

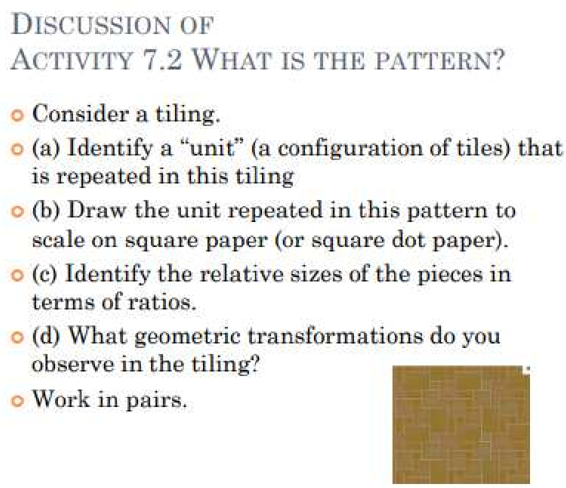



*Figure 8. The first slide in Lab 7:*
*recall of an activity worked on in Lecture 7 and invitation to discuss it.*

Tiling activities were chosen in the course because they are "rich": they combine knowledge about angles with knowledge about polygons and plane transformations.

In connection with the question (d), the instructor proposed the following problem: "What transformation could be applied to the dark blue-green polygon to move it into the position of the light blue-green polygon?" where the polygons were identified repetitive parts of the studied tiling.  (Figure 9)

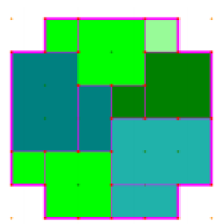


*Figure 9. Fragment of the studied tiling analyzed into polygons.*

To equip the future teachers with knowledge allowing them to solve the problem at the level of Analysis, the instructor recalled the formal definition of reflection and introduced formal definitions of rotation and translation. (Figure 10)

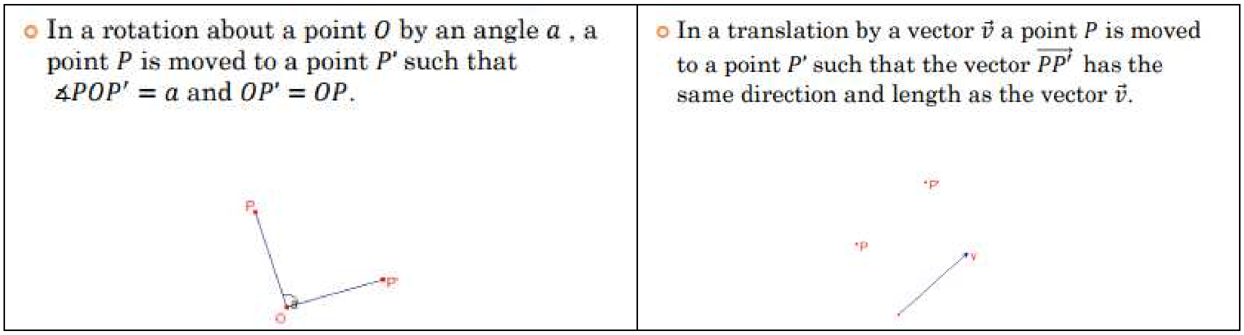



*Figure 10. Definitions of rotation and translation as given in Lab 7.*

For the purposes of the course, the concept of vector was described as a directed line segment. Thus, a single line segment $AB$ gives rise to two vectors, with opposite orientations, but the same direction and length: $\overrightarrow{AB}$ and $\overrightarrow{BA}$ . The first example of rotation given after the definition was closely related to question (d): it involved the polygon representing the "blue-green polygon". (Figure 11) It was a rotation about one of the vertices of the polygon, which may have impressed the future teachers to the point of believing that the center of rotation must be part of the figure to rotation – a belief which appeared in the activities they designed for their Problem Books later. There were several examples of rotations further in the course with the center of rotation elsewhere – one of them being a solution to question (d) – but this did not appear to uproot the belief.

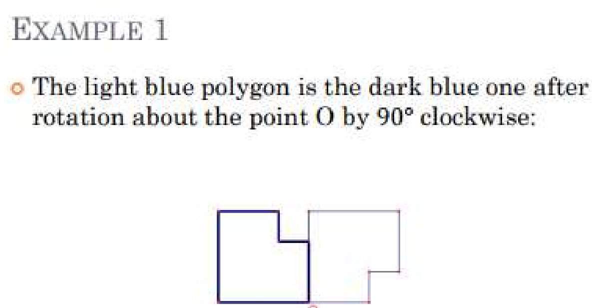



*Figure 11. The first example of rotation given in the course.*

In their responses to question (d), some future teachers proposed a sequence of two transformations: a rotation by 90 degrees clockwise about the bottom right corner of the piece followed by a vertical translation downwards by two "units" (the piece being seen as made of 8 such units – a 3 by 3 square with one unit removed in the upper right corner). But there is a rotation that directly transforms the dark blue-green polygon into the light-green polygon; its center is even at a grid point and it is easily found (without using constructions with ruler and compass) as the intersection of perpendicular bisectors of BB' and EE', using the grid. (Figure 12)

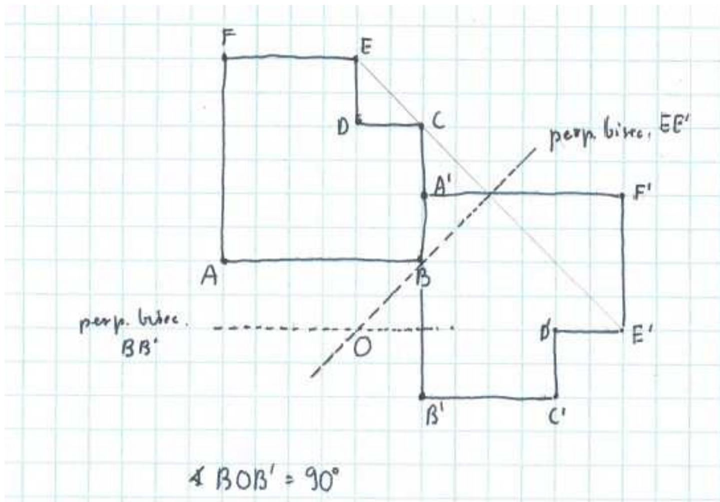
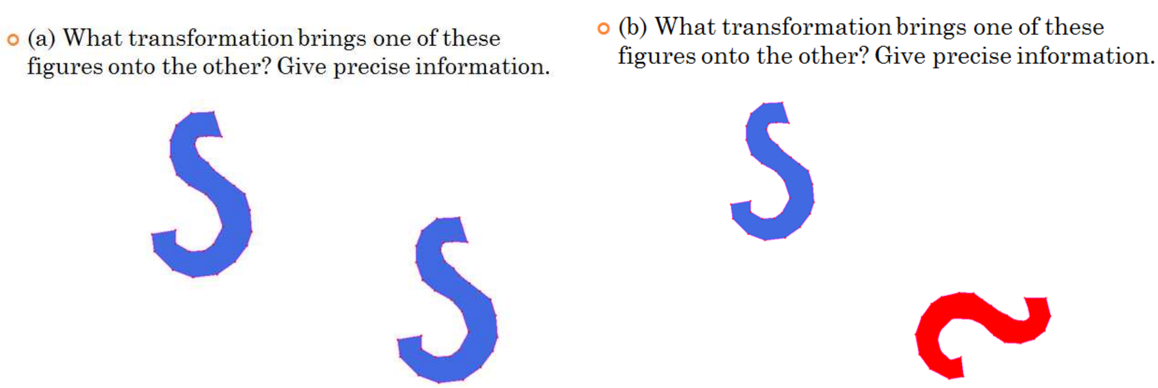



*Figure 12. Construction of the center of rotation mapping ABCDEF onto A'B'C'D'E'F' as the intersection of the perpendicular bisectors of BB' and EE', using only the affordances of the grid.*

After having observed several limited conceptions of transformations in the first round of workshops run by the future teachers, in the Lab 8 session, the instructor decided to propose an activity where the future teachers were given the task to identify the transformation performed on an initial figure. These tasks are presented in Figure 13.



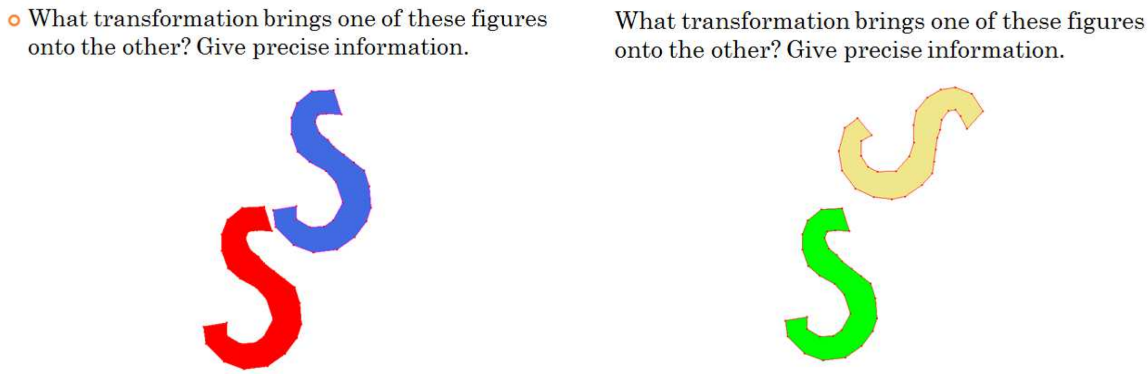



*Figure 13. "What is the transformation?" tasks given in Lab 8.*

While in the first three cases, future teachers were able, first, to visually identify the transformation and, then identify the parameters of the transformation, the fourth context caused quite a difficulty among them. At the visual level, it looks like a rotation; however, it is not clear how the shape has been rotated, if at all. The task gave rise to a discussion about how to decide in this situation, what is a way to deal with such a problem. The intention of the activity, thus the reason for choosing such a configuration, was to raise these questions. They are deeply connected to the nature of argument in mathematics, simply put, to its epistemology. In situations where the rotation is "obvious" (I mean here, that one can easily perform it mentally), students do not see "the point" in trying to produce mathematical arguments. Yet in this case, it is not easy to see or be sure that one shape could be the result of a single rotation of the other.  Thus, the situation requests to use the definition for deciding and, for using the definition, future teachers must find a center of rotation and an angle of rotation. Finding the center requires to reason backwards: from the effects of the rotation to its parameters. The task has been performed during the lab session and the instructor and the teacher assistants made sure all future teachers understood the procedure of finding a candidate for the center of rotation and verifying that it indeed is one. In other words, each future teacher experienced at least one activity where they did find the center of rotation and angle of rotation in a very general configuration.

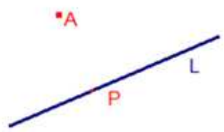
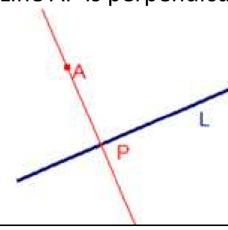
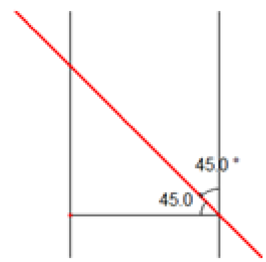
### 3.5.3 Instructor's explicit interventions aimed at issues observed during workshops or classroom activities

I will give a couple of examples of such interventions. The interventions were meant to bring improvements to the creation of activities and draw attention to imprecisions. They were related to mathematical and pedagogical aspects to improve. Here, I will focus only on interventions concerning the mathematical aspects to improve. The following aspects were brought up for discussion: a) the precision of the mathematical language;   b) accuracy of the mathematical concepts; c) justification of the mathematical statements.

The first example is related to the future teachers' responses to the activity of constructing a square with a given side by paper folding. The presentation of mathematical aspects to improve was done under the format: *Initial formulation; Justification of "incorrectness"* and *Correct formulation*[4] (see Table 2).

[4] Information taken from the course slides.

*Table 2. Examples of mathematical aspects to improve*

<table>
<tr><th>Initial formulation</th><th>Justification</th><th>Correct formulation</th></tr>
<tr><td>Point A is perpendicular on line L at P.<br>
</td><td>Perpendicularity is a relation between two lines, not a point and a line.</td><td>Line AP is perpendicular to line L.<br>
</td></tr>
<tr><td>The 45° angle intersects the sides of the square.</td><td>An angle can’t intersect a line. An angle refers to a relation between two lines: an angle is the measure of inclination between two lines. We can speak of an angle between two lines.</td><td>In a square, the bisector of an angle at the vertex intersects the opposite side of the square.<br>
</td></tr>
<tr><td>The divisions are proportionate on all sides of the figure.</td><td>“Proportionate”, in ordinary language, means “in due proportion”; there is judgment on the quality of proportion. But in mathematics, we do not judge the quality of proportion, so we just use the adjective “proportional”.</td><td>The divisions are proportional on all sides of the figure.</td></tr>
</table>

The second example is an intervention in Lab 8 after the first round of workshops in which future teachers were observed to use three-dimensional representations to convey the meaning of transformations of the plane. *Accuracy of mathematical concept* was at stake here. So the instructor made several remarks on the correct use of a concept: in the right context, with the features corresponding to it. One of the future teachers used physical mirrors to help students visualize the notion of reflection, did not give a definition of the reflection of the plane and expected students to be able to perform reflections of plane figures on paper based on this visualization alone. Another future teacher did the same thing with translation, representing it by motion in step dancing. Also the problem of the *justification of mathematical statements* was brought up in the intervention, highlighting the role of definitions in the construction of a mathematical justification. It was stressed that at the level of *Informal reasoning* students are expected to reason in the frame of a theory, thus organizing mathematical statements into a logical sequence.

The instructor also made some concrete *points* about transformations aimed at clearing up certain misunderstandings about them. In the case of reflection the differences between 2d and 3d reflections were highlighted (“Reflection in 2d is relative to a line: “reflection about a line”, while “reflection in 3d is relative to a plane: “reflection through a plane””). As for translation, the clarification concerned the relation between translation and motion in general (“Not all motion is translation”) and the particularity of translation (“Translation is relative to a vector: “translation by a vector” and “A vector is defined by its direction and its length”). For rotation, the difference between 2d and 3d rotation was discussed as difference between *plane* and *spatial* transformation (“Rotation in 2d is relative to a point (“center of the rotation”) and an angle (“angle of rotation”): “rotation by an angle around a point” while “rotation in 3d is relative to a line (“axis of rotation”) and an angle”.

In this chapter, the methodology for the research has been described. The most detailed part consists of the description of the Problem Book, since this serves as main source of data. The geometry part of the methods course future teachers participated in is also described in some detail, since it constitutes the knowledge imparted in lectures, and as such, it serves as reference knowledge when analyzing the problems posed by future teachers. These elements prove critical when analyzing future teachers’ problems proposed in the Problem Book. The result of this analysis is a set of praxeologies and it will be presented in the next chapter.

# 4 A PRAXEOLOGICAL MODEL OF FUTURE ELEMENTARY TEACHERS' ENVISIONED PRACTICE OF TEACHING GEOMETRIC TRANSFORMATIONS

*What is the "educational value" of a school subject? Probably this – that the method treating problems in this subject becomes so much the student's own that he also transfers it to other areas of his thinking.*
TATJANA EHRENFEST-AFANASSJEWA (1931)

This chapter contains an analysis of a group of future elementary teachers' proposals of geometric activities for the classroom. The analysis is structured by the model of mathematical activity proposed in the Anthropological Theory of the Didactic (ATD, already described in Chapter 2) – mathematical praxeology. It is a group portrait that is attempted here: not individual teachers' beliefs about or vision of school geometry or even a statistical distribution of such beliefs or visions in the group of future teachers, but a model of school geometry that transpires from the collection of geometric activities proposed by the group taken as a whole. In this research, the role of the collection of activities is similar to that of a cultural artefact in the work of an anthropologist.

We assume that the goal of the activities described by the future teachers (in the format of lesson plans) was to help children learn some knowledge ("apprendre certains savoirs") or develop some knowing ("développer certaines connaissances") that they believed were part of school geometry. Our goal in this research is to identify the mathematical content of this knowledge/knowing. We want to describe it not in general epistemological or cognitive terms (such as, "intuitive", "practical", "inconsistent", "procedural", etc.) but as a particular practice, with its typical *tasks*, characteristic *techniques* used to accomplish them, *methods* explaining and justifying the choices of techniques, and *theories* giving meaning to the methods, techniques and tasks. In ATD, a description of knowledge/knowing in such terms is called a "praxeology". Types of tasks and techniques of solving them are called "the practical block" or the "know-how" block of the praxeology; methods and theories are called "the theoretical block" or the "logos" or, again, the "know-why" block. A remark must be made about the meaning of the word *methods* in the context of ATD. In French, the language in which the ATD was first formulated, the theoretical block consists of "technologie" and "théorie". In French, based on Larousse[5], the word "technologie" is defined as "Théorie générale des techniques", that is *theory of techniques*. This meaning is in line with the etymology of the word, where "techno" refers to *technique* and "logos" which is *study of a domain.* In English translations of the ATD, the word "technology" has been used as a mirror translation of "technologie". However, the meaning of "technology", as per Merriam-Webster[6] dictionary, is "the practical application of knowledge". In conclusion, although the word sounds similarly in the two languages, the meaning in the two languages differs. Consequently, we decided to adopt the word *methods* as a translation of "technologie" and we will use it from now on.

In research framed in the ATD, it has become customary to denote the types of tasks by the letter T (sometimes with a subscript, if more than one type of tasks are discussed: $T_1$, $T_2$, etc.). Techniques are

[5] http://larousse.fr/dictionnaires/francais/technologie/76961?q=technologie#76059
[6] https://www.merriam-webster.com/dictionary/technology

denoted by the Greek letter $\tau$ (lower case tau), also with subscripts if necessary; methods – by the letter $\theta$ (small theta), and theories – by the letter $\Theta$ (capital theta). The practical block is denoted by the letter $\Pi$ (capital pi); so $\Pi = [T, \tau]$. The theoretical block is denoted by $\Lambda$ (capital lambda): $\Lambda = [\theta, \Theta]$. Thus, in a formal way, a praxeology can be represented as a system of four layers of knowledge/knowing [T, $\tau$, $\theta$, $\Theta$].

Praxeologies can be described at different levels of granularity. At the base level, *punctual* praxeologies are considered. These are constructed to deal with a single, particular type of task. The next level, *local praxeology*, integrates several punctual praxeologies that use the same methods. The third, and last, level refers to *regional praxeology* that integrates several local praxeologies all built around the same theory.

In the following, I shall present praxeologies identified in future teachers' lesson plans concerning the topic of geometric transformations in the plane. The presented praxeologies are aggregated and integrated by nature, since they are abstracted from common elements found in several lesson plans rather than (necessarily) in one lesson plan of a single future teacher. In this sense, they are models of a group-shared vision of future practice, a group portrait as said above. The data for this analysis consist of individual lesson plans, proposed by future teachers as part of their work in the methods course. Given the nature of the source material, at least two points must be underlined. First, the lesson plans were conceived for individual topics; therefore they represent punctual praxeologies. We can consider all these punctual praxeologies as constituting a sort of "praxeology pool" produced by the particular group of future teachers.

Second, since the available data are in the form of written lesson plans and no individual discussions with the future teachers concerning their beliefs and thinking were available, the underlying theoretical block of the praxeology is constructed based on my own interpretation of the different elements specified in the lesson plan (based on responses to the items in the template for the Problem Book): especially, means of achieving the instructional objectives, justification of why the means used are conducive to this achievement, means of checking if the instructional objectives have been achieved. In addition, the sections on expected solutions of the lesson plans were a rich source of information, since they referred to the mistakes future teachers imagined students could commit. Most importantly, every section contains the arguments they bring for the potential occurrence of such mistakes and, also, the ways they would react to these. From this point of view, the elements of the *know-why* block are expression not only of future teachers subject matter knowledge, but they also carry, at least implicitly, messages about what they think mathematics is (its subject and purpose) and what constitutes a valid argument in a given situation.

In this chapter, I will describe the main types of tasks on plane transformations as identified in the future teachers' lesson plans and identify the praxeologies around these tasks. The praxeologies are specified in terms of the ATD framework. The presentation of each praxeology is followed by a discussion of its affordances and limitations for the future mathematics learning of schoolchildren. Limitations are presented in terms of tasks impossible to solve in the frame of the given praxeology. Future teachers' didactical choices will be commented on as part of a general discussion on the praxeologies. At the end of the chapter, I will summarize the findings concerning the geometry of future teachers.

We consider the collection of these tasks as an expression of the future teachers' beliefs regarding the content to be learned about geometric transformations: these are the major *types* of tasks that future teachers consider that elementary students should be able to perform. The tasks are an expression of what a child must be able to do in order to be considered as knowledgeable about the topic.

## 4.1 Types of tasks related to the topic of plane transformation in Future Teachers' lesson plans

A template for the Problem Book given to the class of future teachers proposed a title, "A Problem Book for the use of Elementary School Teachers. Twelve Activities on Fractions, Ratio, Proportion, and Informal Geometry", but the future teachers were free to choose their own titles. One student gave his Problem Book the title: "A Problem Book for the use of Elementary School Teachers. Twelve Activities on measurements, comparisons, conversions, fractions, percentages, rate, incrementing, speed, distance, time and probabilities" (note the absence of geometry). Another added "using the theme of the Muppet Show" to the title proposed in the template.

In the template, the proposed structure of the book was: 3 chapters with 4 "activities" each, with one chapter on geometry, one on fractions and one – on proportions, rates and percent. The future teachers were, however, free to integrate these topics in more complex activities and structure the book into chapters along different organizing principles, provided they had 12 activities in all. Most (30 out of 32) did choose to have a separate chapter on geometry with 3 or 4 activities, so that there were 117 activities on geometry in total.

Twenty one (21) of these tasks were about geometric transformations (translations, rotations, and/or reflections) and it is on these tasks that the present praxeological analysis will focus. We chose to focus our analysis on this area, because knowledge of geometric transformations is built on knowledge of many more elementary geometric concepts (line, point, angle, distance, parallel lines, perpendicular lines, polygons, types of polygons and their properties, etc.) and processes (e.g., reasoning based on definitions). Therefore, an analysis of the future teachers' views of school knowledge about transformations promises to provide a good insight into their vision of school geometry as a whole.

The 21 tasks on geometric transformations can be grouped into three main types, according to whether the transformation was to be performed or something about the transformation had to be identified: the transformation as a whole or a parameter of the transformation.

**TASKS TYPE T1**. To <u>perform</u> a given transformation (translation, reflection, rotation) on a given figure. (11 tasks)

**TASKS TYPE T2**. To <u>identify the transformation</u>(s) that have(s) been applied to a given figure. (9 tasks)

**TASKS TYPE T3**. To <u>identify a parameter</u> of a transformation (vector of a translation, axis of reflection, angle of rotation, center of rotation). (1 task)

These main types could be further differentiated according to the techniques the future teachers expected children to use in accomplishing the tasks. For example, all tasks of type T1 asked to perform a transformation; however there are slight variations as to the specific context and tools available to perform them. We list the identified subtypes of all three main types below. In Table 3, we describe the subtypes of the categories. The associated illustration consists of tasks from the future teachers' problem books. The description of the subtype is naturally broader than the example taken from the Problem Book, given that the task was considered in its generality. However, the concrete example helps in understanding what the task *actually* is. For example, although the sub-category doesn't specify that the polygonal figure is convex, in most tasks from the future teachers' problem books, the polygons were convex. Examples of subtypes of types T2 and T3 are in Table 4 and Table 5, respectively.

In the description of the subtypes of tasks and the related praxeologies, I will use the term "**grid paper**" to refer to any paper with a pattern of lines or dots at various intervals. So a grid paper can be covered with crossing lines or it can be covered with dots only. Lined grid paper is usually called "**graph paper**". The lines can form little squares (I will call this paper "**square grid paper**") or little rectangles ("**rectangular grid paper**"). Although there are graph papers with oblique lines, here, I will only have in mind the horizontal-vertical pattern of lines. A grid paper with dots only will be called "**dot paper**". The dots can be in horizontal-vertical configuration or some other configuration (as in, e.g., isometric dot paper, already mentioned in Chapter 3).[7] In the future teachers' lesson plans, if dot paper was used it was only the one with horizontal-vertical configuration with four neighboring dots forming the vertices of a square. So in this chapter, "dot paper" will only refer to "square dot paper". I will also use the expression "grid points of a grid paper", referring to either the intersection of lines on the lined grid paper or to the dots on the dot paper.

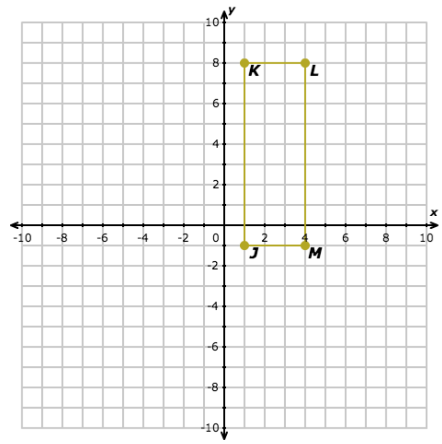
### 4.1.1 "PERFORM TASKS": Subtypes of type T1 tasks

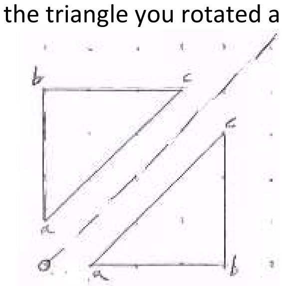
- **T1.1** Perform a translation of a polygon with vertices on the grid points of a grid paper (square or rectangular). The vector of the translation is implicit or, when specified, the vector's head and tail are on the grid points as well.
- **T1.2** Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection *not in a standard direction* (not horizontal and not vertical) but still passing through the grid points and such that its angle of inclination to the horizontal direction is a multiple of a half of the right angle.
- **T1.3** Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection *in a standard direction* (horizontal or vertical) and lying on a grid line.
- **T1.4** Perform a reflection of a polygon with *vertices not on grid points* of a rectangular grid paper; line of reflection is in standard direction and lies on a grid line.
- **T1.5** Perform a rotation of a polygon with (some) vertices not on the grid points of a rectangular grid; the center of rotation is in a grid point and not on the polygon. Rotation is done with an angle that is a quarter or half of a full angle.
- **T1.6** Perform a rotation of a polygon with vertices on the grid points of a grid paper; the center of rotation lies on a line that is an extension of one of the sides of the polygon; the angle of rotation is a multiple of the right angle.

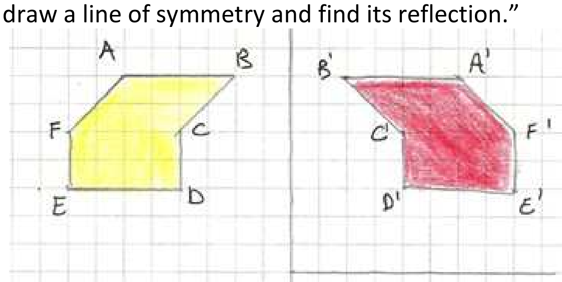
[7] Examples of various grid papers can be seen at http://highered.mheducation.com/sites/0072532947/student_view0/grid_and_dot_paper.html

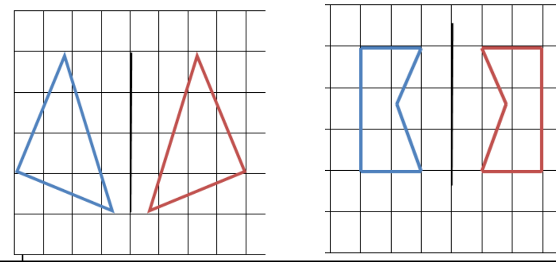
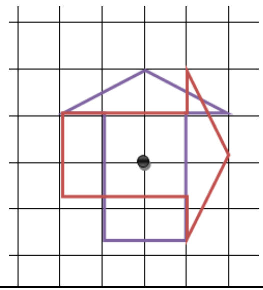
*Table 3. Examples of subtypes of PERFORM TASKS*

<table>
<tr><th>Description of a subtype of T1, the “PERFORM TASKS”</th><th>Illustration of the subtype with a task taken from a future teacher’s problem book.</th></tr>
<tr><td>T1.1 Perform a translation of a polygon with vertices on the grid points of a grid paper (square or rectangular). The vector of the translation is implicit or, when specified, the vector’s head and tail are on the grid points as well.</td><td>“Given the rectangle JKLM, graph the image after a translation of 9 units to the left.”


</td></tr>
<tr><td>T1.2 Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection not in a standard direction (not horizontal and not vertical) but still passing through grid points and such that its angle of inclination to the horizontal direction is a multiple of a half of the right angle.</td><td>“After the rotation, draw a broken line extending diagonally from the center point to the end of the page in the top right corner. This will be our axis of symmetry for making a reflection about a line. Draw a reflection of the triangle you rotated along this line.”

</td></tr>
<tr><td>T1.3 Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection in a standard direction (horizontal or vertical) and on a grid line.</td><td>“Trace the shape you have chosen on you[r] graph paper. Using this shape, draw a line of symmetry and find its reflection.”

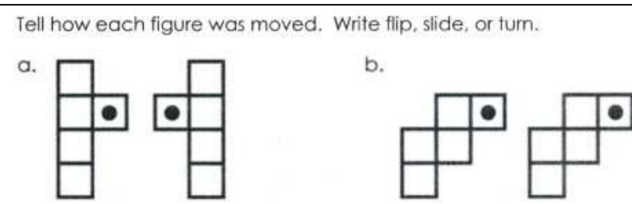
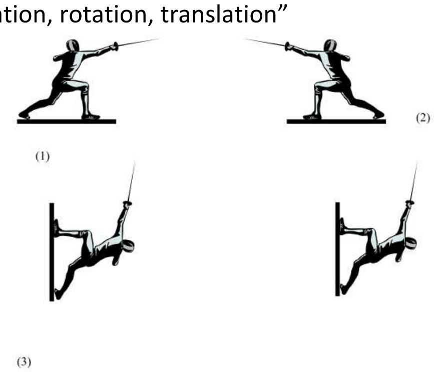

</td></tr>
</table>

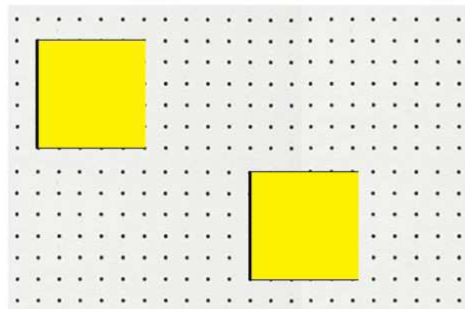
| | |
|---|---|
| **T1.4** Perform a reflection of a polygon with (some) vertices not on the grid points of a rectangular grid paper; line of reflection is in standard direction and lies on a grid line. | “Draw the reflection image of each shape.”  |
| **T1.5** Perform a rotation of a polygon with (some) vertices not on grid points of a rectangular grid; the center of rotation is in grid point and not on the polygon. Rotation is done with an angle that is quarter or half of a full angle. | “Draw the rotation image of each shape (Use the turning point):<br>- $\frac{1}{4}$ turn clockwise”  |
| **T1.6** Perform a rotation of a polygon with vertices on the grid points of a square grid paper; the center of rotation lies on a line that is an extension of one of the sides of the polygon; the angle of rotation is a multiple of a right angle. | “Ask the students to draw a regular polygon on their graph paper (…)<br>- Demonstrate to the students how to rotate their regular polygon<br>- On the projector show the students how to rotate their regular polygon by indicating what angle they will be rotating by as well as in which direction.<br>- Once the students have rotated their regular polygons have them color the image, using a different color, of the regular polygon that resulted after the rotation.” |

### 4.1.2 “IDENTIFY TRANSFORMATIONS TASKS”: Subtypes of type T2 tasks

- **T2.1** Identify the single transformation performed on polygon P. Initial and final figures are drawn on blank paper.
- **T2.2** Identify the single transformation performed on polygon P. Initial and final figures are drawn on square grid paper and vertices are on the grid.
- **T2.3** Identify the single transformation performed on polygon P made of identical squares, one of which is marked with a dot. Initial and final figures are drawn on blank paper.
- **T2.4** Identify the transformations performed on a figure and their order given the initial, the final and some intermediary positions.
- **T2.5** Identify two transformations performed on polygon P and their order given the initial and the final position. Intermediary position is not shown. Initial and final figure are drawn on square grid paper and vertices are on grid points.

*Table 4. Examples of subtypes of IDENTIFY TRANSFORMATIONS TASKS*

| **Description of subtype of T2, the "IDENTIFY TRANSFORMATIONS TASKS"** | **Illustration with a task taken from a future teacher's problem book** |
|---|---|
| **T2.1** Identify the single transformation performed on polygon P. Initial and final figures are drawn on blank paper. | "Professor Peter is trying to plan out his classroom. He draws out a diagram of the layout of the classroom depicting his desks. He then moves his desk around until he reaches a configuration that he likes. Comparing the new (B) and old (A) location of his desk, what type of transformation has occurred?"   |
| **T2.2** Identify the single transformation performed on polygon P. Initial and final figures are drawn on square grid paper and vertices are on the grid. | "Identify what transformation had been done by looking at the same figure in two places on a grid (for example, they should be able to tell that a square has been translated horizontally)." |
| **T2.3** Identify the single transformation performed on polygon P made of identical squares, one of which is marked with a dot. Initial and final figures are drawn on white paper. |   |
| **T2.4** Identify the transformations performed on a figure and their order given the initial, the final and some intermediary positions. | "What series of transformations were used to move the man in picture 1 to the man in picture 4?<br>a) Reflection, reflection, translation<br>b) Rotation, translation, reflection<br>c) Reflection, rotation, translation<br>d) Rotation, rotation, translation"   |

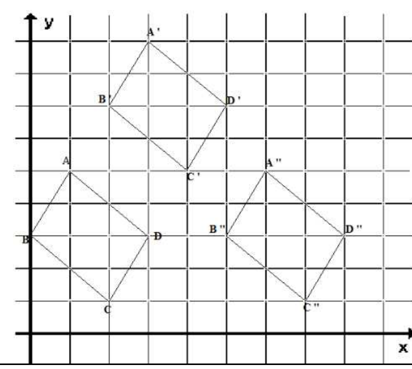
| | |
|---|---|
| **T2.5** Identify two transformations performed on polygon P and their order given the initial and the final position. Intermediary position is not shown. Initial and final figures are drawn on square grid paper and vertices are on grid points. | "For this series of transformations, the student's partner might initially guess that it was a reflection, but if they were to count the dots or draw a line of reflection, then the students would realize that there was another step in between. Students may figure out that the square has been translated to the right by 10 spaces and then was reflected vertically over a line of reflection."  |

### 4.1.3 "IDENTIFY PARAMETER TASKS": Subtype of type T3 tasks

There was only one task of type T3 in the future teachers' problem books.

- **Task type T3.1** Identify the vector of a translation of a polygon P with vertices on square grid points.

*Table 5. Example of an IDENTIFY PARAMETER TASK*

| **Description of subtype of T3, the "IDENTIFY PARAMETER OF TRANSFORMATION TASKS"** | **Illustration with a task taken from a future teacher's problem book** |
|---|---|
| **T3.1** Identify the vector of a translation of a polygon P with vertices on square grid points. | "How many units did parallelogram ABCD move on the grid to become A'B'C'D'?"  |

### 4.1.4 Summary

The tree diagram in Figure 14 summarizes the types of tasks about geometric transformations proposed by the group of future teachers.

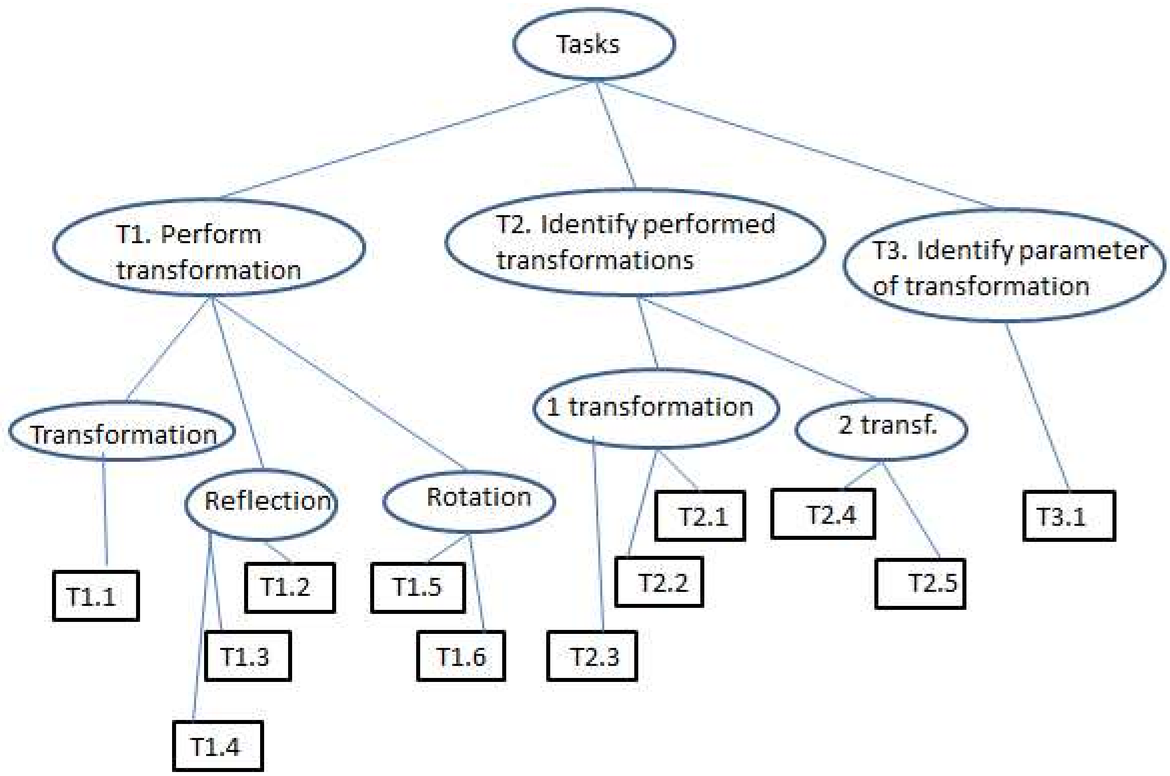



*Figure 14. Types of tasks about geometric transformations identified in future teachers' problem books*

In this section (4.1), I presented a categorization of the tasks about geometric transformations in plane into three types. Inside of each category, a further breakdown of the types of tasks was done based on the type of transformation and / or the particularities of the context (e.g., blank paper, grid paper, etc.) specified for the task. In the following section (4.2), I will describe the praxeologies (i.e., I will specify the techniques, methods and theories) built around each type of tasks.

## 4.2 PUNCTUAL MATHEMATICAL PRAXEOLOGIES RELATED TO "PERFORM TASKS"

In this section, I will present praxeologies related to each type of tasks. Consequently, three sections will follow. The section 4.2.1 is on praxeologies around the PERFORM TASKS, section 4.2.2 – on those related to IDENTIFY TRANSFORMATION TASKS, and section 4.2.3 – on praxeologies related to IDENTIFY PARAMETER TASKS.

The sections on the praxeological models are structured into four parts:

- Description of the type of tasks

- Description of the techniques for accomplishing the tasks, built mainly on future teachers' descriptions of expected solutions

- Description of method(s) and theories used to justify the techniques; this part contains a thorough explanation of the sources (emblematic cases and examples from the problem books) of my construction of this description that served as source for them

- Discussion of the limitations of the praxeology in terms of tasks that cannot be solved, or proposed in the frame of the underlying theory. When more than one praxeology is identified for a certain type of task, this part will also contain their comparison in terms of obstacles and affordances created by each of them.

A very brief closing paragraph will summarize the main characteristics of the praxeology before passing to the next section.

The first praxeological model presented is related to the Perform Translation tasks. For this type of tasks, two techniques have been identified, along with two methods and the corresponding theories. Given the differences between the underlying theories, we can differentiate here between two praxeologies, each associated with a technique, a method and a theory.

### 4.2.1 Praxeologies related to PERFORM TRANSLATION tasks

In this section, we describe praxeologies related to the type of task T1.1. We recode this task now as an element of a whole praxeology, using the following conventions:

- The first two letters refer to the nature of the task. For example, PT will stand for *performing translation*;
- Underscore;
- The next letter refers to the status of the element of the praxeology: if it is a task – the letter T will be used; if a technique - $\tau$, if a method - $\theta$, if a theory - $\Theta$;
- The last symbol of the code is a number signaling a sequential order.

**Type of task** PT_T*: Perform a translation of a polygon with vertices on the grid points of a grid paper (square or rectangular). The vector of the translation is implicit or, when specified, the vector's head and tail are on the grid points as well.*

As mentioned earlier, the techniques of the praxeologies are based on my interpretations of sections of the lesson plans (for example, interpretation of elements specified in expected solution, the expected students' mistakes, assumed previous knowledge, etc.). None of the elements of the praxeology (except the task type) is explicitly described in the lesson plans, therefore variants of techniques, technological discourse and underlying theoretical elements can be identified by studying all elements of the lesson plans. For this category of tasks, two techniques were identified, as specified below.

**Techniques**

PT_$\tau$1*. Construct the image by transformation of each vertex and connect the vertices.*

PT_$\tau$2*. A two-steps technique:*

*a) For one arbitrary vertex, find its image by counting the required units in the required direction.*

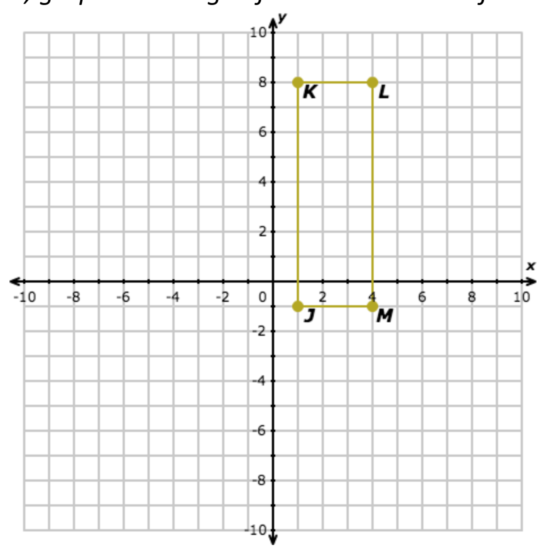
*b) From the position of the new vertex, redraw an identical figure respecting the initial orientation of the figure on the grid.*

In continuation, in discussing the methods and theory, I will discuss two representative examples from future teachers' problem books.

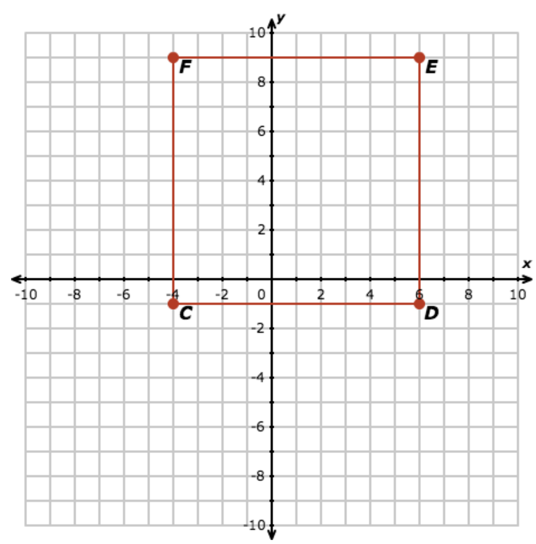
**Methods / Theory**

We start with an example of this type of task, from a future teacher's lesson plan:

*Translate the following figures:*
*1) Given the rectangle JKLM, graph the image after a translation of 9 units to the left*

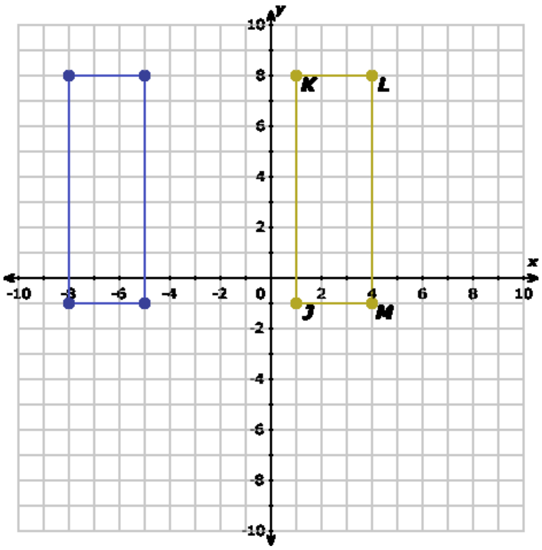



*2) Given the square CFED, graph the image after a translation of 1 unit down*

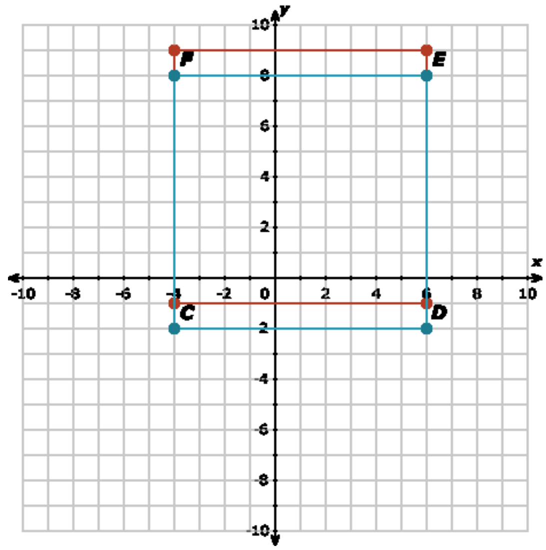



Our interpretation of the intended meaning of the task and its description as part of a praxeology was informed by the expected solution specified by the future teacher, given below:

*The students are expected to answer the following question as follows:*




[Expected response to 1)]




[Expected response to 2)]

What methods and theory could justify the validity of the expected solutions?

The future teacher who proposed the task specified the definition that her students in class should know and we can suppose that this is the definition to which she refers in the construction of the expected solution. In the **Assumptions about the students' previous knowledge and/or level of thinking** section of the lesson plan, she mentioned:

> *Students will already have basic knowledge of the definition of a translation in geometrical transformation terms.*
>
> *Definition of translation: According to Geometry & Spatial Sense: Grades 4-6 (2008)*[8]*, "a translation can be described as a transformation that slides every point of a shape the same distance in the same direction. During a translation the orientation of the shape does not change and the image is congruent to the original shape. A translation can occur in any direction."*

This text – legitimized by being formulated in a published book – is the official knowledge to rely on for both the teacher and the students. Observe that what the future teacher labelled as one single definition contains, in fact, theoretical elements with different status: a part that can be seen as a definition (first sentence) and a part that describes properties that can be deduced from this definition (second sentence). However, there is no explicit separation between the two parts. The distinct nature of these statements has not been recognized by the future teacher and instead, the whole description was treated as "definition." It has to be pointed out that this text is not named "definition" in the reference from which it was taken.

The first part of the "definition" could justify the use of the first technique presented (PT_$\tau$1*)*; the second part could support the use of the second technique (PT_$\tau$2)*.*

---

[8] Geometry and Spatial Sense: Grades 4-6. (2008). Retrieved March 20, 2013 from http://www.eworkshop.on.ca/edu/resources/guides/Guide_Geometry_Spatial_Sense_456.pdf

In addition, in the **Instructional objectives of the activity** section of her lesson plan, the future teacher specified that "*Through this activity, I expect students to become familiar with the use of a grid. For instance, each square on the grid represents a single unit.*"

First, observe that the reference to the "unit" is contextual – its meaning depends on the particular context in which the word is used. In the case of the given task, a "unit" for the future teacher is the minimal "square" of the grid and it will depend on the task's context whether the square is meant to represent a unit of distance or of area. In a transformation task, the future teacher expects her students to understand that she refers to length or distance, even if she says "nine squares to the left", and a square is not a one-dimensional figure. On the other hand, in a measurement of area problem, she would expect children to understand the square as the unit of measurement of a surface. This way, it is the side of the square or the area of the square that are intended, depending on the context. For students in a classroom this ambiguity could be one of the sources of confusion between the concepts of area and perimeter. It does not help to understand the notion of the vector of translation, either.

Further confusions could arise from the use of the grid's "square" as unit of length. There is a possibility of a belief that distances between any two vertices of the square, not necessarily two adjacent ones, represent the same "unit". This would make the diagonal the same length as the side. Thus, in a task of finding the perimeter of a figure drawn on a square grid, where the figure has sides that are not along the grid lines, students might become tempted to still count those sides as having "unit" length.

The above discussion considered the *nature* of the "unit". Another aspect in which we analyze the "unit" is its *role* in performing the transformation. Along with the grid, the "unit" is seen as essential element for understanding the transformations. The following is another excerpt from the **Instructional objectives of the activity** section of a lesson plan: *"I also expect them to consolidate their understanding of these transformations with the concept of units so that they understand the relationship between the transformation and the given unit (such as each point of triangle ABC' being at an equal distance from the axis of symmetry as triangle ABC)."*
The crucial role of the unit arises from the grid geometry and the particular technique to perform transformations: "counting units". Transformations rely on the concept of distance, which, in turn, is based on the existence of some unit as a core element of a metric. Yet, there is no dependence relationship between transformation and unit – a transformation can be performed in any space where a metric is defined. What we witness here is the reduction of the transformation to a procedure of counting "units" on a grid. "*While showing the transformation of an original shape such as the translation of triangle ABC to ABC' prime, the students will have to respect the unit represented, which is one square on the graph paper*." More arguments and examples about the "counting of squares" will be presented in the section about praxeology of reflection.

Future teachers thus appear to consider the grid to be a central element of knowledge about transformations.

The development of a mathematical vocabulary / language is also considered by future teachers as an important aspect to attend to during instruction.

For example, in one of the lesson plans, another future teacher listed among the instructional goals of the lesson the development of a language to describe translation ("*Through this activity students will discover the ways in which you describe a translation on a grid.*") This would be achieved through a class brainstorming activity followed by institutionalization by the teacher: "*The students will be presented* *with the terminology for describing translations after having brainstormed ideas of what the terminology may be*." Later on, she specifies: "*Once the students understand that a translation has occurred ask them what the terms might be to describe the directions of the moves? Come to the understanding that the terminology used is left, right, up and down.*"

The comment implies the use of what, following Vergnaud, we might call a "theorem-in-act" or "théorème en acte" (Vergnaud, 1983), namely that "there are only two orientations on the grid" and four directions: up and down, left and right. Observe that the future teacher capitalizes on these "direction" words, attributing a very special meaning to them, distinct and disjoint from everyday usage, and trying to make them "officially" a part of a mathematical vocabulary.

Thus, the core elements to teach about translations, as a particular case of transformations, is the nature of grid-geometry with a special attention given to the "unit", implicitly understood as the minimal "square" of the grid. The lesson plans suggest that, in order to perform the transformations, one must identify the "unit", understand what "left, right" in this context means and have a certain visual association with each transformation ("slide" for translation, "flip" for reflection and "turn" for rotation). As a consequence, the teacher must be clear about how these elements work.

By considering the grid as an object of teaching and institutionalizing the directions of up and down, left and right as the directions together with the unit – the future teacher treats the grid as inherently containing these directions and unit, in an absolute manner (there is a "well-defined" left and there is a "well-defined" right, etc.). In this approach, the grid becomes a mathematical object to be taught and not only a particular and temporary support used in initiating children to the topic of geometrical transformations. The process through which support material (didactical aid, representation) becomes an object of instruction was identified by Brousseau (1998) as the meta-cognitive shift ("glissement méta-didactique"). What we witness here is an example of such process.

Let us further look at another detail of the expected solution (Figure 15). There are two figures on the grid.

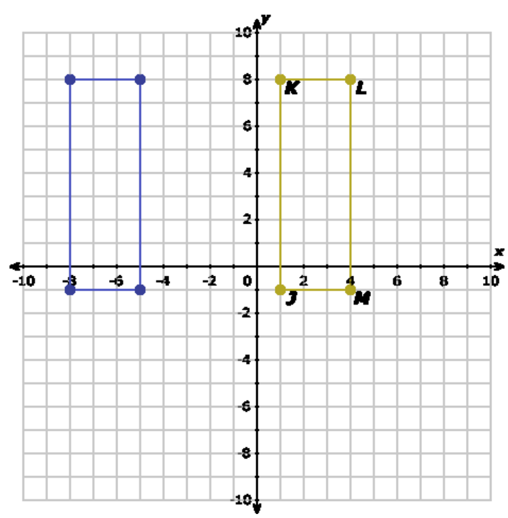



*Figure 15. Expected solution to a* Perform translation *task.*

The translated figure contains no labels; therefore, the correspondence between vertices of initial and translated figure is not clear. The solution could be thus better justified by the second sentence of the given "definition": the translated figure is identical to the original one and it is in the same orientation.

By not specifying vertices, the figure acts as a compact shape that can be "moved" with one gesture. This suggests that the expected technique was (PT_$\tau$2) rather than the other (PT_$\tau$1*)*.

The method, then, can be formulated as follows:

PT_$\theta$1.

> *Distance on a grid is calculated by counting "units" (the "squares").*
>
> *There are only two directions on the grid: vertical and horizontal.*
>
> *There are four orientations on the grid: Down, Up, Right and Left.*
>
> *A translated polygon has the same shape and orientation as the original one.*
>
> *The unit is deduced contextually.*
>
> *A translation is specified by two displacements on the standard orientations or standard directions.*

As mentioned earlier, the theory underlying this method relies on the "definition of translation" used by the future teacher and on the consideration of the grid as the underlying structure.

PT_$\Theta$1*.*

> *"Definition" presented earlier: a translation can be described as a transformation that slides every point of a shape the same distance in the same direction and orientation. During a translation the orientation of the shape does not change and the image is congruent to the original shape. A translation can occur in any direction.*
>
> *The grid and the geometry defined on it: the only points that exist are the grid points; segments connect two points along the grid-lines; distance can be measured vertically or horizontally by number of "units".*

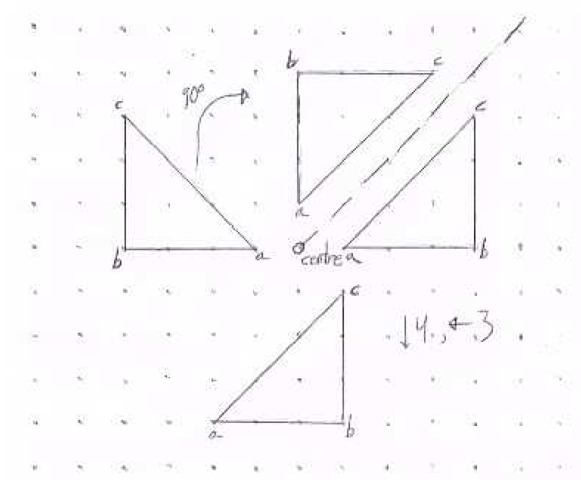
The expected solution illustrates what the future teacher would consider as a valid solution or as one that demonstrates understanding of the concepts involved. The underlying theory is most likely to be communicated in the classroom, in an explicit manner. In the end, this is what will be transmitted from the curriculum as she understands it.

In the above praxeology, the implicit material support for performing the task is a lined grid paper (graph paper). Dot paper is the implicit support for the other one of the two identified praxeologies.

I will first present an example of a future teacher's task that served as source for the identification of the second technique.

This example of translation task, presented below, was part of a longer task in which the three transformations were studied in sequence. Students are presented with a triangle that, after performing several transformations on it, would create a windmill-like figure. Here, we present the part referring to translation:

> *For the final wing of our windmill, translate the triangle you just reflected by a vector of four points down, and three points to the left. Draw this translated triangle.*




In the following, I will argue that for this Perform Translation task, the technique expected by the future teacher to be used by the students was PT_τ1, presented earlier. I rephrase it here for the current task:

PT_τ1*:*

> a) *Identify the new position for each of the three vertices by counting the required number of points in the required direction.*
>
> *b) Connect the new points to obtain the triangle.*

Observe that the counting of "units" happens differently, since the unit is defined differently. The nature of the grid (as points and not a network of parallel lines) makes the unit to be the *distance* between two neighboring points along the same direction. In consequence, this technique is fundamentally different from PT_τ1 in the sense that figures are defined by vertices (and not by a global position on the grid) and the "unit" is one-dimensional. An important consequence of this technique is that the congruence of the image and initial figure is deduced and not given *a priori* by the nature of the underlying geometry. In a certain sense the geometry of the dot paper is "vertices geometry", while the geometry of the lined grid paper can be understood as "rigid geometry", a geometry of rigid shapes. We will refer to the former as "dot geometry" and to the latter as "lined grid geometry" (or just "grid geometry", for the sake of brevity).

The justification of PT_τ2 seems to be a collage of theoretical elements imparted during methods class instruction along with practical elements from personal experience.

In the above example, the task description specifies to translate the triangle "*by a vector of four points down, and three points to the left".* The reference to the "*vector"* is a reminiscence of class instruction where the following definition was given for translation:

> *Translation: In a translation by a vector $v$ a point $P$ is moved to a point $P'$ such that the vector $PP'$ has the same direction and length as the vector $v$. (Course slides, Lab 7)*

Although the concept of vector has been briefly explained in the course and characterized by direction and magnitude, in the context of dot paper, the vector is specified by two displacements, each along a standard direction.

Given that most students' memories of performing translation tasks are in the context of grid or dot paper, as the preferred medium in primary school, we can observe here an adaptation of the concept of vector to the descriptions that dot or grid paper allow – namely, in terms of "dots", "points" or "units" counted in standard directions. A vector, then, is seen as a pair of numbers representing displacements along standard directions and the direction of a vector is given by the words designating standard orientations ("Up", "Down", "Left", "Right").

A second element to mention is the labeling of the vertices. In the **Assumptions about the students' previous knowledge and/or level of thinking** section of the lesson plan, for this task, the future teacher specified: *"Students should be able to use letters to indicate the vertices of a shape, and they must be able to keep the notation consistent."* The idea is once again reinforced in the **Instructional objectives of the activity** section "*Students will understand the importance of consistently identifying points on a shape undergoing transformation*."

The insistence on these elements illustrates a vague reference to the definition given in class, which refers to points only, not full figures. In this context, labelling is seen as an expression of having applied the definition of translation, instead of relying only on visual similarity of shapes. This element is even more important in cases where the symmetries of a figure would not allow a clear identification of the correctness of a transformation.

As for the second part of the technique, connecting translated vertices to obtain the sides of the translated triangle, no theoretical element is referenced as justification. Yet, an explanation of the lack of such justification might come from the way in which figures are defined on the grid.

It seems that in this "dot geometry" figures are described by their vertices. That is to say, the only points in the figure are the vertices. If this is accepted as basic assumption, then the definition has been applied correctly for performing translation: each vertex has been translated and then, these vertices were connected. The only "lines" in this geometry seem to be those in standard directions (horizontal, vertical) and those at 45 degrees to the horizontal direction. These have the peculiarity of going through "most" of the points. We shall return to this in the description of reflection.

Synthesizing, the methodological discourse underlying the technique can be formulated as:

PT_$\theta$2:

*A vector on a grid can be described by vertical and horizontal distance between its head and tail.*

*Translating a point is the same as to identify the head of a vector when the tail and vertical and horizontal displacements are given.*

The "theory" of this praxeology is made of the definition of translation, given in the methods course, and the principles of what we will call "dot geometry":

PT_$\Theta$2:

*Definition of translation: In a translation by a vector $v$ a point $P$ is moved to a point $P'$ such that the vector $PP'$ has the same direction and length as the vector $v$. (Course slides, Lab 7)*

*The dot geometry.*

The dot geometry, just as the lined grid geometry, limits the figures to those that have their vertices on the grid. The only vectors that exist are the ones that can be specified by the vertical and horizontal displacements, measured in whole numbers of dots to the left or to the right, and up or down. Since in this case, and contrary to the previous praxeology, figures are defined as sets of vertices, the correctness of performing a translation of a figure relies on the explicit labeling of the translated vertices and the connection of corresponding ones.  In other words, one must know *how to* connect the vertices and for this purpose the correspondence between vertices is essential.

For the purpose of a synthesis, I list the basic elements of the geometries in the Table 6.

*Table 6. Basic elements of the dot and grid geometry*

| **Elements** | **Grid geometry** | **Dot geometry** |
|---|---|---|
| Plane | Regular square grid | Regularly spaced dots |
| Point | Grid points | Dots |
| Line | Grid lines: vertical and horizontal | Lines passing through the dots (horizontal, vertical, at 45 degrees with the horizontal direction) |
| Segment | Portion of “line” defined by two “points”. | Portion of a “line” defined by two “points”. |
| Angle | Right angle and multiples of it | 1/8 of full angle as smallest angle and multiples of it |
| Figure | A portion of the “plane” defined by its position on the grid. It is a rigid figure that can be described by its global position. | A collection of vertices, connected in certain order – although a segment connecting two vertices is not referred to as side or edge of the figure in transformation tasks. |
| Direction and orientation | Standard cardinal directions (horizontal, vertical) and orientation on these directions is described by the following terms: Right and Left, Up and Down. | Standard cardinal directions (horizontal, vertical) and orientation on these directions is described by the following terms: Right, Left, Up and Down. |
| Metric | “Unit” square | Distance between two adjacent dots in standard directions |
| Instrument | Straightedge for connecting points | Straightedge |

We can remark that distance between two points aligned on a horizontal or vertical line is calculated as number of “units” between two points on horizontal or vertical line. The “unit” allows referencing any point relative to a given one by specifying the displacements along standard directions. For this reason, in transformation tasks, reference to sides of figures is not mandatory.

In dot geometry, figures, in general, are defined by vertices connected in a certain order, the kind of figures that “make sense” depend on the circumstances in which they are used. For example, a triangle (as defined by three connected vertices) is a valid figure in this geometry for a transformation task, yet it is not a valid figure for tasks concerning the perimeter of the triangle (or even area if the decomposition of the triangle is not visually immediate). In grid geometry, the figure is always aligned with grid lines; therefore it is possible to calculate the perimeter and area of any figure.

Next, I highlight the limitations of these praxeologies.

**Limitations of the PT praxeologies**

The very first observation is that figures not having any of their vertices on grid points do not exist in the grid geometry as the ones not aligned with grid lines do not exists in the grid geometry. As for translation, given a valid figure in this geometry, it would not be possible to translate to a point that is not on the grid, since such points simply do not exist.

In case of the grid geometry, the manner in which the grid has been introduced and used restricts the types of tasks that can be performed and, in consequence, will also limit the understanding of translation as a transformation. Moreover, the limitation created by the task, when repeatedly used in teaching, impedes future revisions of the translation because of its strong emphasis on standard directions and typical, rigid shapes. The contextual treatment of the “unit” concept, namely the blurred line between 2 dimensional and 1 dimensional units, will, most probably, create misconceptions. These misconceptions may have a negative influence on the distinction and learning of certain measures, as area and perimeter, for example. By considering the figures as compact, rigid shapes, the future teacher reinforces the typical metaphors used in context of transformations: slide, flip, and turn. These metaphors reduce the understanding of transformations to the visualization level (van Hiele P. , 1959), since they promote a certain (mental) image of the outcome of the process and hide details about how it is to be performed. The appeal to metaphors is yet another manifestation of meta-cognitive shift since it replaces the study of the transformation by an image reducing the concept to a certain interpretation of it.

I became acutely aware of the importance of the grid in the future teachers’ geometry when, while teaching a math methods course in another university, I saw quite a few of them reject a correct solution to a problem just because the figure in the solution was not aligned with the grid. More precisely, the problem was to draw a rectangle with the same perimeter as the given one (Figure 16).

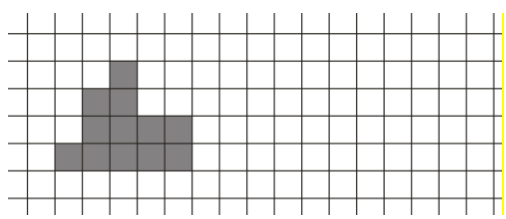


*Figure 16. The figure from the problem: “Draw a rectangle with the same perimeter as the shaded figure.”*

The future teachers were asked to assess a solution which showed a square with the side of length 4.5 “units” long, where the unit is the length of the side of the grid’s smallest square. (Figure 17)

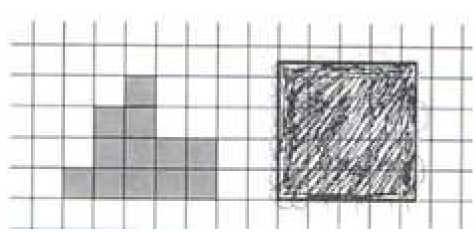


*Figure 17. A solution to the problem in the previous Figure, given to future teachers to assess.*

This is a correct solution since a square is a rectangle and the perimeter of this square is 18 ($4 \times 4.5$) such units. Yet some future teachers claimed that the solution was incorrect not just because, according to them, a square is not a rectangle, but also because the figure “does not respect the grid”, or “goes beyond the grid, which is the unit of measure”; some even claimed that the perimeter of the square was 16, as if treating the bits not aligned with the grid as mere errors of inaccuracy of the drawing.

In conclusion, even the basic elements of this geometry are fine-tuned to the particularities of the usage context. Yet, this is not perceived by future teachers as an issue, since they consider tasks separately from each other (thus, perimeter related tasks are not related to translation related tasks), just as textbooks tend to organize teaching material into units with no visible links between them.

**Comparison of the two praxeologies**

Although similar at the task level, the two praxeologies presented in this section (4.2.1) rely on very different theories and each, in its turn, is different from the mathematical theory of translations. It can be argued that in each case, the validity of solutions relies on a theory, but an individual one with very loose connections to the referent mathematical theory.

In the case of the first praxeology, defined by grid geometry, translation was reinterpreted as being applied to figures seen as compact, solid constructions. Under such circumstances, there is no need to label corresponding vertices, because the whole figure moves as one single entity. The grid serves as a means to “see”, by counting, the congruence of the figures and as a medium for performing the “slide”. As for the level of geometrical knowledge required by the task, it can be concluded that the task remains at visualization level. The validity of solution is judged by perception and consists of finding a correspondence between the two figures by “mentally sliding one onto the other”.

The second praxeology (related to the dot paper) relies on a more complex theory, which is, however, not uniform in the nature of its elements. It is a personal mixture of bits of mathematical theory corresponding to analysis level and practical shortcuts afforded by the presence of the dot plane. As mentioned, in this geometry one can deduce the congruency of initial and translated image from the procedure that was performed on each vertex.

Primary school programs do not prescribe presenting children with the reference mathematical theory underlying transformations. Yet, in those primary years, teachers need to prepare the foundations for the construction of such theory. Consequently, the nature of the arguments teachers use to justify the validity or non-validity of a solution, the “definitions” which they plan to rely on in their teaching and the nature of arguments they accept as being valid are critical elements in creating such foundations. In addition, these arguments hint to the nature of mathematics as a theoretical construction: they convey ideas about what it means to define, to prove, to justify, etc. which are essential components for the epistemology of the discipline (mathematics). Praxeologies differ in the underlying discourse, but these differences present different affordances for future learning. From this point of view, the second praxeology presents more potential for learning, despite its limitations.

By insisting on the importance of correspondence between the vertices, better conditions are created for accommodating, later on, the mathematical definition of translation in generic context. It is not as much the concept of translation that must be re-constructed and integrated, but the concept of vector. The critical features of the translation concept are highlighted and once the concept of vector is reworked, the student will not have major difficulties in solving the task without reference to grid paper and, hopefully, to construct the corresponding mathematical theory. This technique is also more in line with the construction process of synthetic geometry: translation is introduced on points and, from this definition we progress towards deducing the properties of translation (such as preservation of shape, for example).

The common element between the two praxeologies is the presence of grid/dot as context for the task. The option for grid is determined by didactical considerations; it is an invented didactical object that allows performing these transformations (namely, translation and reflection) even when students have not been acquainted with the concept of distance of a point to line.

In the dot-geometry, congruent figures can be constructed by "measuring" distances**.** Given the particular way of calculating distance, a vector's length will be described by two numbers in case of dot-plane and a translation not in standard direction will be considered as consisting of two "basic" translations in standard directions. The prevalence of points and that of standard directions forces a modified angle concept. An angle can only be a multiple of quarter turns (or $1/8^{th}$ turns) as defined by horizontal and vertical lines (and oblique for dot-geometry). We shall come back to this when we discuss rotation.

Therefore, a translation, rather than being specified by a vector of arbitrary direction and length, is seen as a composition of two translations in standard directions. As mentioned before, thinking in terms of grid geometry does not impede a rigorous study of translations; however it creates limitations in the study of reflection and rotation.

### 4.2.2 Praxeology related to PERFORM REFLECTION tasks

The overall structure of this section is as described in the introduction to section 4.2. First, I will formulate a general form of the PERFORM REFLECTION tasks. This will be followed by the presentation of the technique identified based on future teachers lesson plans, then the method/theory part of the praxeology, along with a detailed discussion of examples from the lesson plans at the origin of my claims in the model. Next, I will discuss the limitations of the praxeology.

Given that, in primary school, programs expect students to learn to perform translations and reflection only (not rotation), I include at the end of this section a discussion about future teachers' didactical choices focusing both on translation and reflection. Potential, long term impact of this praxeology is discussed by examining solutions to translation and reflection tasks of future teachers.

**Type of task** PR_T*:* Perform reflection (PR) of a polygon on grid paper

This general formulation includes task types 1.2, 1.3 and 1.4 from the initial categorization. As a reminder, these were:

- **T1.2** Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection *not in a standard direction* (not horizontal and not vertical) but still passing through the grid points and such that its angle of inclination to the horizontal direction is a multiple of a half of the right angle.
- **T1.3** Perform a reflection of a polygon with vertices on the grid points of a square grid paper, with the line of reflection *in a standard direction* (horizontal or vertical) and lying on a grid line.

- **T1.4** Perform a reflection of a polygon with *vertices not on grid points* of a rectangular grid paper; line of reflection is in standard direction and lies on a grid line.

**Technique**

PR_τ1. *(Counting squares) Perform the following steps*:

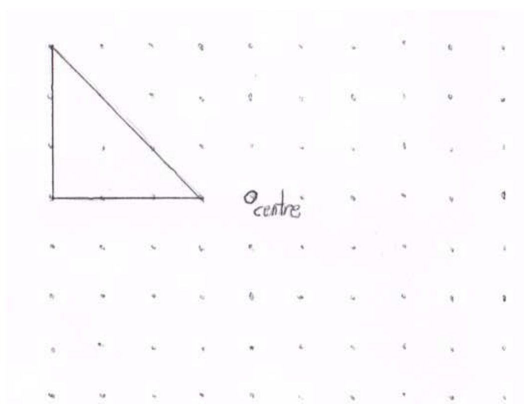
*a) Identify image of vertex by "symmetry";*

*b) Draw the figure "by symmetry" – that is, draw a figure such that the two figures together form a symmetrical shape. Count "units" till the reflection axis and then count the same number of units on the other side of it.*

**Methods / theory**

To illustrate the process my construction of a model of the technique and bootstrap the discussion about the method that legitimizes or justifies it, the following lesson plan is presented as a representative case for this type of tasks (part of this task has been presented in section 4.1.1):

*Here is a triangle on a point grid with its position relative to the center point:*




*We are going to use what we know about transformations to make a windmill that revolves around the center. We are going to do this by rotating, reflecting, and translating the triangle we see on this grid.*

*1. Start by choosing a point near the middle of your point grid page. Circle it and make a note that this is your center point.*

*2. Now draw the triangle you see in the picture above at the same distance from the center point. Use a letter to indicate the vertices of the triangle so that you can follow them through the transformations. This is the first of our windmill wings.*

*3. Rotate the triangle clockwise around the center point by an angle of 90°. Remember to use your letter markings to keep track of the vertices!*

*4. After the rotation, draw a broken line extending diagonally from the center point to the end of the page in the top right corner. This will be our axis of symmetry for making a reflection about a line. Draw a reflection of the triangle you rotated along this line.*

*5. For the final wing of our windmill, translate the triangle you just reflected by a vector of four points down, and three points to the left. Draw this translated triangle.*

*And now you're done!*

Given that the reflection must be done on a figure that is the result of a previous transformation, I will illustrate the figure that must be reflected if the rotation has been performed correctly. The use of a figure that is the result of a previous transformation presents a particularity of this (complex) task, yet for the reflection part, I will assume that the triangle is given.

The reflection line must be created by the student, following the instructions given in the section:

*After the rotation, draw a broken line extending diagonally from the center point to the end of the page in the top right corner. This will be our axis of symmetry for making a reflection about a line. Draw a reflection of the triangle you rotated along this line.*

Although the creation of the reflection line is an important element in the task, at this point I will not discuss it, but focus only on performing the reflection based on the desired configuration. As such, the desired configuration consists of an isosceles right angled triangle and a line of reflection parallel to one of the sides. (Figure 18)

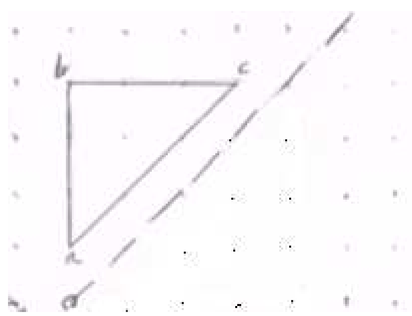



*Figure 18. The task is to reflect the triangle abc about the dotted line.*

Two elements can be interpreted as indicators of the technique: first, the instructions insist on correct labeling of the vertices and, second, they refer to (erroneously) to the line of reflection as axis of symmetry. The expected solution is shown in Figure 19.

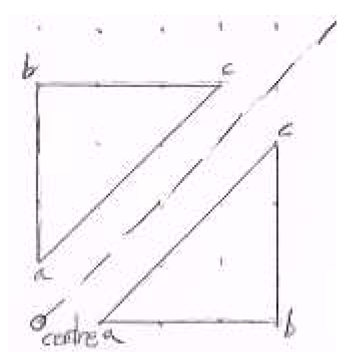



*Figure 19. Expected solution to the problem in the previous figure.*

The importance accorded to vertices may have two sources: on one hand, it comes from the way in which figures are defined in this particular dot geometry and, on other hand, from the notational convention used for transformations imparted during lectures. Labeling corresponding vertices could look as an application of the definition; however, the lack of details does not allow us to assert that this is the case. We need to look further.

The following definition and properties were given during lectures (this definition has been already presented in the Methodology chapter; I include it here for ease of reference):

> ***Definition.*** *Reflection about a line L is a transformation of the plane such that*
> *a) If point P' is the reflection of point P about L, then the line through P and P' is perpendicular to L, and*
> *b) The distance from P' to the line L is the same as the distance of P to L.*
> ***Properties. (…)***
> *The line L is a line of symmetry of the union of figures F and F'.*

The fact that the terms "symmetry" and "reflection" are used interchangeably by the future teachers it suggests that the property rather than the definition was used as a foundation for the technique. It is reasonable to think that reflection is operationalized as the transformation done in a way that the resulting configuration is symmetrical. This is the property of the transformation, a consequence of correctly performing the procedure; however, based on the language use of future teachers, we conclude that it replaces the procedure from the definition. Such amalgamation between reflection and symmetry is fairly common among future teachers. As a further example on this issue, let us consider the following excerpt from a lesson plan.

In a task proposed for the study of symmetry, we can find the following description in the section **Materials** of the lesson plan of the same future teacher: *"A small translucent 'mirror' that can stand on the worksheet (to check reflections for symmetry)"*. In this task, in the section **Means to achieve instructional objectives** he further elaborates: "*When they are asked to consider the pattern of shapes as a whole, it will challenge the students to see how two-dimensional symmetry is not a property of a given shape; it is simply a demonstration of an equal relationship existing between the opposing sides of a given line."*

The last statement suggests that symmetry is not a property of a figure, but a relation between certain segments and this relation exists after some segments have been constructed. As such the reference is, actually, to reflection since one must perform something. Yet, in the task for studying reflection, the terminology is "symmetry axis": *"This will be our axis of symmetry for making a reflection about a line."* All this indicates equivalence in the meaning of reflection and symmetry (thus using them as synonyms). A consequence is the interchangeability in methods used for performing and verifying reflection: thus, we can perform reflection so that the final configuration looks symmetrical (independently on how we defined reflection), and we can verify the correctness of the procedure by using a mirror (which is a method belonging to symmetry).

The difference might be subtle to perceive, but it is essential. If reflection is seen as symmetry, it opens up to experimentation in the sense that "performing a reflection" becomes "finding (potentially by trial-error) the position of a copy of the figure such that the compound figure (a union of the two) presents a symmetry with respect the line specified". Yet, in mathematical terms, performing a reflection means to follow an exact procedure to construct the elements a figure and congruency will follow, from the way the procedure was defined.

Symmetry is a property of a figure: a figure can have it or not. Reflection is a transformation, a mapping defined on a geometric space (a plane in our case): we act on the plane and transform it. Figures are mapped on other figures. Symmetry is static, while reflection is dynamic. Yet, neither translation, nor reflection are seen by future teachers as transformations of the *plane,* since initial and transformed figures cohabit a static underlying grid. Identifying reflection with symmetry eliminates the idea of "motion" - the accent is not anymore on how we proceed, but on the final result (no matter how we proceeded).

Textbooks often represent the initial and resulting figures in the same plane, on the same drawing. This is a didactic choice, so as to facilitate students' work in "verifying" the reflection. At some point, the verification of correctly performing reflection has been equated with verification of the final configuration's symmetry and not with the correspondence of the applied procedure with the mathematical definition. The question of "was the reflection correctly performed?" is replaced by "can we, visually, decide that the configuration we have is symmetrical?" Usually, we would deal with the issue of symmetry of the overall configuration at the (informal) deduction level, based on congruency of triangles, and prove it as a consequence of the definition of reflection; yet, this element of geometrical knowledge is not available to primary school students and so they can only deal with the problem at the visualization level. In this view, the reflection is correctly performed, and so, the concept is grasped if, by folding along the axis, the figures overlap. This could explain the prevalence of considering reflection and symmetry as synonymous.

An unforeseen consequence of the above equivalence manifests itself in the case of certain configurations. When a figure is symmetrical, by folding along the axis of symmetry, one "half" overlaps with the other "half". In other words, these "halves" are in separate half-planes. Transfer of this idea to reflection this suggests that the line of reflection should not intersect the figure. Thus, one can expect students to have difficulties when this is not the case.

In the following, another example of a task is presented where the same equivalence between reflection and symmetry is assumed.

> *Trace the shape you have chosen on your graph paper. Using this shape, draw a line of symmetry and find its reflection. Repeat this three times so that you have a total of 4 shapes on your graph paper.*
>
> *Your first and original shape should be colored in yellow, your first reflection in red, your second reflection in blue and your third reflection in green.*
>
> *Remember to:*
> *1. Identify your line of symmetry with a letter (p for example).*
> *2. Do not forget to identify all vertices of your shape with letter: use the same one as the original shape, only adding prime (A') on every new shape.*

In spite of the insistence on illustrating the correspondence of vertices by labeling, the procedure relies on the concept of symmetry. That is to say, the "reflection" has been performed correctly if the resulting configuration is symmetrical. Hidden in the coloring of resulting figures is the idea that reflection axis should not intersect the original figure. Once more, for didactical purposes, reflection axis should be separated from the figure so that the students can color and label, in a visually clear way, the resulting figure. While such concern for visual clarity can be justified in the introductory phases of the concept, it quickly becomes a source of a limited comprehension that leads to difficulties in performing reflection. I l mention this issue later in the section, in discussing limitations of the praxeology.

The technique can be recognized in the following excerpt from a lesson plan:

> *First, trace a shape. Second trace a line of symmetry. Third, from every vertices, count the number of squares until the line of symmetry, count the same number of squares on the other side of the line and dot your vertices. Repeat this step for every vertices of your polygon. Fourth, place your Mira mirror on you line of symmetry to see if you have correctly traced your reflection. Do not forget to identify every vertices on your polygon using prime, i.e. A will become A'.*

This completes the description of the PR_τ1 by giving the details on *how* to identify the new vertices. The verification of the correctness of construction relies on the use of the Mira mirror. Although this is different from visual perception (which happens without instrument), it still relies on the idea of symmetry and leaves the task, again, at visualization level in the van Hiele model.

The justification behind the technique ("counting squares") seen in the above excerpt, is given in the description on how the future teacher would start the class.

> *"At the beginning of the activity, the teacher will ask students question on reflections: what can they tell the teacher about reflection…"*
> *"Next, with what the children have provided as explanations to the teacher, the teacher should build on the children's knowledge, now using appropriate mathematical language: what the line of symmetry represents (compared to the mirror line), every point is at the same distance from the line of symmetry, the size of the reflected shape is conserved."*

Students don't seem to be provided with a formal definition, but with the practical description "*every point is at the same distance from the line of symmetry, the size of the reflected shape is conserved"*.

Once again, we observe that the description bundles together elements from the formal definition with logical consequences of this definition. The central element in performing reflection became the "counting of squares"; and this is also reflected in the imagined potential mistakes students might commit. In the words of this future teacher:

> *"The most common mistakes that children will make are to misinterpret the distance between the vertices and the line of symmetry. In this case, simply reminding them that they need to count adequately should a way of assuring their correct answers."*

The prevalence of the procedure can be perceived here: if there is an error, it must be corrected on the procedural level by directly telling them what to do to correct it, and not investigate how the concept of reflection has been understood.
In conclusion, the method and theory can be formulated as follows.

**Method**

PR_θ1. *Reflection axis is along grid-lines or, at most in diagonal, in case of dot-plane;*

*Distance from reflection axis is the number of "units" from a point till the axis;*

The underlying **theory** can be summarized as follows:

PR_Θ1.

*Reflection and symmetry are synonyms for the same concept*

*Grid or dot geometry.*

Thus, there are two core elements to the concept of reflection as identified from future teachers' lesson plans, and, therefore, to be transmitted in the classroom: the "unit" and "counting units". There are some shortcomings to this reduced view, as illustrated in the following.

**Limitations of the PR praxeology**

First, the reference to distance of a point from reflection axis remains implicit. Although future teachers think of "counting" as a way to measure the distance, this may not be clear to their future students and the question remains to what extent they will understand it as distance. Without reformulating the "counting" as "establishing distance" students will not be able to later generalize the concept and will face difficulties in performing the task outside of the grid paper context.

Second, the reference to the unit as "squares on the grid", as often seen in these lesson plans, induces further confusion.

The limitations of the grid-geometry are quite clear in the case of reflection. Since distance is defined as "number of units", in practical terms one must be able to "count those units". In turn, this forces the lines of reflection (or, as they say, "symmetry") to be aligned with the grid or with the dots. Otherwise one cannot identify the distance to the reflection line by directly counting the "units". The only possible tasks are those involving lines along the grid-lines and figures with vertices aligned with grid-points.

Therefore, it is not possible to perform a task such as in Figure 20. In fact, the line as given in the figure doesn't exist in this geometry since it does not pass through the grid points.

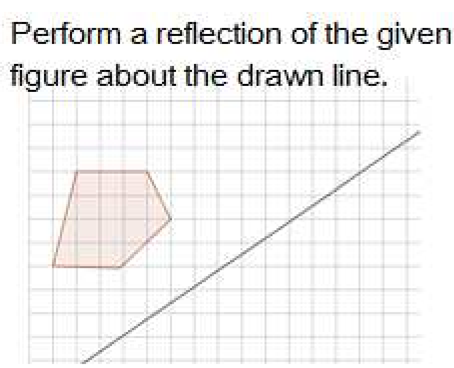


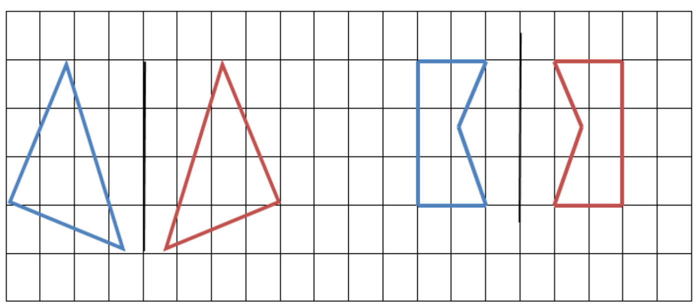

*Figure 20. Example of a task that is outside the grid geometry praxeology.*

The relative position between the reflection line and the figure does not allow the "counting of squares" approach, thus this task cannot be performed under the restrictions of the grid-geometry.

At this point, I briefly return to the construction of the line of reflection as presented in the task at the beginning of the description of the Perform Reflection praxeology. The task instructed the students to "*draw a broken line extending diagonally from the center point to the end of the page in the top right corner".* In this case, *"diagonally"* means "obliquely, through grid points"; in other words, at a 45 degree in relation to a horizontal axis. The term "diagonally" is loosely used, without its referent "diagonal". At the same time, it is interesting to observe the concrete, physical nature of this grid-geometry (support): the reference is made to the "*end of the page in top right corner*". The discourse is highly contextual and, in consequence, hinders generalizations and development of abstract meanings. If all goes well, the line of reflection will be in such a position that will allow performing the reflection with the "counting of squares". However, a different paper format or an earlier mistake can lead to a line as in the task just above. In this case, the technique promoted by the future teacher cannot be applied. The question remains whether, in class, the future teacher would be able to build on such situation and arrive at a revision of the concept of reflection.

With the advent of free software for geometry, future teachers could also envision performing tasks in these environments. None of them mentioned this idea, yet in a single case software was used to design the task. The following is the task as formulated by a future teacher:

"*Draw the reflection image of each shape.*


"

(Figures in red – at the right – are the expected answers).

Since the future teacher gave no details about the procedure to follow, it is not clear how the task should be solved. Her solution is done by the software with which the drawing was created, thus we cannot deduce anything about her geometry. Given the fact that the drawing was done by software, she could have thought of an exploration task where students should identify the software's "way to proceed" when identifying a figure and doing the reflection. Such exploration could lead to an interesting analysis of how mathematical concepts are implemented in software applications. However, the task proposed by the future teacher was not intended in this direction. We can consider the task as an expression of the "idea" of the sort of task she would give: polygonal figure on grid paper with vertices in grid-points.
As outlined at the beginning of the section, I will close the discussion on the praxeology of performing reflection with a short section on future teachers' didactical choices.

**Comment on didactical choices**

At last, I shall mention the potential impact of didactical choices in both cases: reflection and translation. It is known (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006; Vinner, 2011) that the number and variety of examples seen influence the understanding of a concept. In case of transformations, the variety of contexts (grid or blank paper, complexity of the figure to be transformed, position of the figure on the support, etc.) will have an impact on the meaning associated with *transformation*. From the analysis of future teachers' tasks, we saw that the line of reflection has been defined as a line that does not intersect the figure to be reflected. The didactical choice is justified by the fact that future teachers want to focus on the steps to follow for performing the transformation, along with transmitting conventional notation and, also, to develop a "visual feel" of how the particular transformation should look. Yet, students who consistently receive tasks with this constraint can develop a belief that, in any other case, reflection is not possible.

For example, a task such as presented in Figure 21 can be perceived as not well-defined, since the axis passes through the figure. Some students might even refuse to try and solve it.

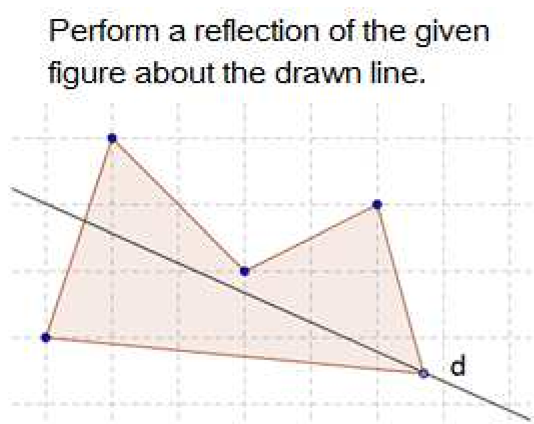



*Figure 21. Another example of a task that is outside the grid geometry praxeology.*

Or, if considered as a legitimate task, students may have difficulties in performing it, since they lack practice with tasks with such features. Later on in this section, some examples are shown that support this claim.

However, repeated use of the same context (square grid paper) can be source of difficulties not only for performing a reflection, but also in deciding whether such transformation has taken place. In other words, it can become problematic on what to focus if the "units" are not there to decide about distance. Among the tasks proposed in the problem books, however, none required to decide about a certain configuration if it is the result of a reflection. Yet, this type of task is important in order to identify the elements of reflection students focus on, and should be offered, once again, on a variety of supports and instruments available.

In case of translation, the frequent didactical choice is a vector aligned with gridlines horizontally or vertically, so that students can easily count the "units". When this is not the case, the vector is often specified as a pair of displacements represented by small numbers and making it sure that the resulting figure would not overlap with the original one. The possibility of visually distinguishing between original and image figure is a decision made with the purpose of reinforcing the visual grasp of the transformation. As for the direction of the vector, this is usually oriented to the right and upwards – maybe as to follow the direction of writing. The overabundance of such cases leads to difficulties with tasks with different specifications. For example, a task such as shown in Figure 22 can cause difficulties.

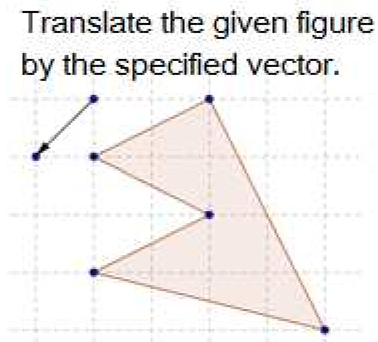



*Figure 22. Example of a Perform Translation task where the translated figure overlaps with the original one.*

Although reflection (and, more generally, transformation) is revisited in high school, it is not clear to what extent the concept is restructured and linked with the previous study of it, in elementary school. It must be said that in grade 7 ("secondaire 1"), transformations are supposed to be performed with ruler and compass; in other words, future teachers should be familiar with other techniques than "counting units". A detailed analysis of how transformations are *actually* taught in later schooling is beyond the scope of the current study, yet a brief overview of topics studied in geometry during secondary school years suggests that ruler-and-compass techniques cannot be completely be understood by students given the lack of theoretical elements needed to justify their validity (for example, congruency of triangles and circles, central angles are studied later). A consequence of this situation is a partial understanding of the techniques, partly owing to the "learning by heart" approach students are somehow forced to take. In this sense, we can suppose that there is a limited acceptance and familiarity of the standard ruler-compass methods for performing transformations.

Furthermore, even when these procedures have been taught, again, to future teachers in their methods course, results indicate that they still have difficulty in assimilating them and accommodating with previous experience, although the theoretical geometrical knowledge should be available to them at this point. Our observations suggest that for many future teachers, their preference for grid-geometry and prototypical tasks might not be *only* a didactical *choice*, but one imposed by their mathematical content knowledge. The geometry of translations and reflections *is* grid-geometry. At the same time, we see that revisiting this knowledge has limited impact.

In conclusion, in case of reflection, the future teachers' grid-geometry, combined with a limited variety of contexts, presents a serious obstacle for proper learning of the concept of reflection. Yet, one can imagine other types of grids, like isometric grids, that would, at least in part, allow for a more diverse formulation of tasks. The prevalence of square grid is owed to its omnipresence in textbooks and online materials. One can say that out of the sheer frequency of these types of tasks in official and online materials, reflections (transformations) on grid-geometry became a new, though artificial and not necessarily useful, learning topic in school.

### 4.2.3 Praxeology related to PERFORM ROTATION tasks

In this section two praxeologies are presented, linked to the types T1.5 and T1.6 of tasks. As in previous two sections, I start by presenting/recalling the types of tasks. This is followed by a formulation of the techniques, the methods and the theories of each of the praxeologies. An explanation of the sources of the models in the future teachers' lesson plans will accompany the formulation. Two representative examples will be included. The section will end with a commentary on the limitations of the praxeologies.

**Type of task** PRo_T*: Perform a rotation of a figure on grid paper*.
The task corresponds to the task types T1.5 and T1.6 from the initial categorization. As a reminder, these were:

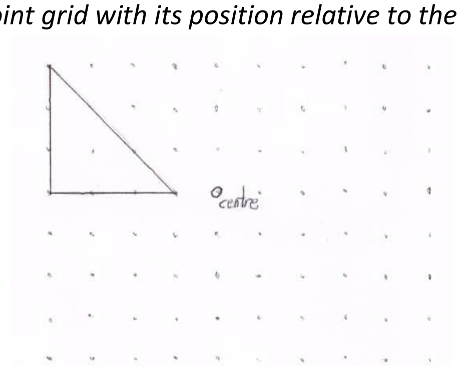
- **T1.5** Perform a rotation of a polygon with (some) vertices not on the grid points of a rectangular grid; the center of rotation is in a grid point and not on the polygon. Rotation is done with an angle that is a quarter or half of a full angle.
- **T1.6** Perform a rotation of a polygon with vertices on the grid points of a square grid paper; the center of rotation lies on a line that is an extension of one of the sides of the polygon; the angle of rotation is a multiple of the right angle.

**Techniques**
PRo__τ1.
*Rotate each vertex and connect their images, respecting the order of the vertices in the original figure.*
PRo_ τ2.
*Rotate one vertex and draw the rest of the vertices by identifying their relative position with respect to the initial vertex.*

**Methods/Theory**

A task to perform a rotation would normally specify the figure to be rotated, and the center and the angle of rotation. The following task is the first part of an already quoted task about "windmills".

*"Here is a triangle on a point grid with its position relative to the center point:*



*We are going to use what we know about transformations to make a windmill that revolves around the center. We are going to do this by rotating, reflecting, and translating the triangle we see on this grid.*

*1. Start by choosing a point near the middle of your point grid page. Circle it and make a note that this is your center point.*

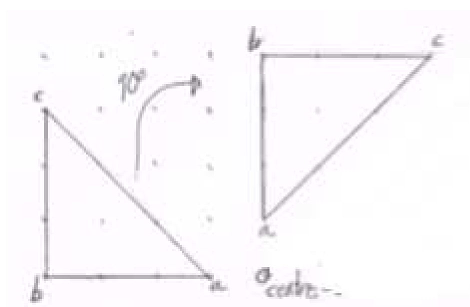
*2. Now draw the triangle you see in the picture above at the same distance from the center point. Use a letter to indicate the vertices of the triangle so that you can follow them through the transformations. This is the first of our windmill wings.*

*3. Rotate the triangle clockwise around the center point by an angle of 90°. Remember to use your letter markings to keep track of the vertices!"*

As in case of translation, the focus placed on labelling the vertices and keeping track of their new positions by labelling their images accordingly, suggests the first technique. The definition given by the instructor in the course (as presented in Chapter 3) was the following:

*In a rotation about a point O by an angle a, a point P is moved to a point P' such that <POP'=a and OP'=OP.*

The definition can be applied in this geometry (thus, applied to valid figures in this geometry and valid angles). The following image shows the future teacher's expected solution; however, there is no indication how this solution is to be obtained. Thus, I will illustrate how both techniques can be applied in this case.




Expected solution for the rotation tasks

According to PRo_τ1, the problem can be solved this way:

First, we identify the direction in which point a will move ("a quarter-turn") and then we count the distance from the center in this direction. In this case, we "count one dot" in vertical direction. Point *b* on the initial figure is 4 "dots" to the left from the center; therefore its rotated image is 4 "dots" "up" from the center. Point *c* is 3 "dots" "up" and 4 "dots" to the "left" from the center, thus its rotated image will be 3 "dots" right and 4 "dots" "up" from the center. In this technique, all points were considered in their relative position to the center of rotation expressed as a pair of displacements.

Since the distance between any two grid points can be described by a pair of values (horizontal and vertical displacements), the technique PRo_τ2 can be applied as follows.

> Start with a vertex "aligned" with the center. For example, one can start again with point $a$ and construct its image. However, the remaining vertices can be constructed by their relative position to this vertex or any other vertex. That is, the image of vertex $b$ can be constructed by "moving" 3 "dots" "up" from the image of vertex $a$; then, the image of vertex $c$ can be constructed as "moving" 3 "dots" to the "right" from the image of $b$.

Given, however, that the instructions to the task (step 3) asked students to label corresponding vertices, it is likely that the technique the future teacher had in mind was PRo_τ1, similarly as in the case of translation.

The underlying method and theory will be then the same as in the case of translation. The first technique is supported by the following theoretical block:

PRo_θ1.

*Figures are defined by their vertices.*

*Rotation is a "turn" made of one or more quarter turns*

PRo_Θ1.

*Dot geometry.*

The second technique relies on grid geometry.

PRo_θ2.

*Figures are compact, defined by their position on grid.*

*Rotation is a "turn" made of one or more quarter turns.*

PRo_Θ2.

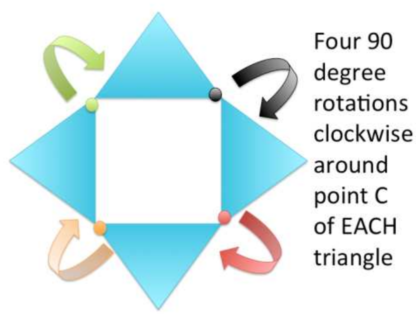
*Grid geometry.*

In grid geometry, the figure moves as whole; therefore, for drawing the image after rotation it is enough to rotate one vertex, and then draw the remaining ones in relation with the one rotated. This is possible, because, in this geometry, the distance between two points can be specified as an ordered pair of directed numbers. As a consequence, the position of each vertex of a figure can be determined from an initial one. This is why the technique PRo_τ2 can be applied. The two geometries have already been compared, but it is worth coming back to the differences between them, one more time. One difference is quite fundamental, from the epistemic point of view. By treating figures as rigid, the grid geometry assumes that the figure will conserve its "shape" (as relative distances between vertices); the dot geometry makes no such assumption. In dot geometry, the conservation of shape has to be proved. Obviously, this has consequences for the nature of errors one can commit and the ways they can be corrected.

If one "knows" (as in grid geometry) that rotation preserves the shape of the figure, an error in performing the rotation will, most likely, concern the rotation of the first vertex. It is also very likely that the correctness of rotation will be judged at perceptual level with the main focus on the shape and not on defining elements of the rotation (center, angle of rotation, for example). If the defining elements of rotation are sidestepped, the task is kept at the visualization level in the sense of the van Hiele model, and the only arguments for correct execution of the task, available to students, are of the type “it looks so since the shape is the same” or cutting out the shapes and comparing…. The metaphors (“turned, but congruent figure”) associated with the concept are so dominant that the defining elements of rotation are ignored even though they are mentioned. The second technique, PRo_τ2, seems to facilitate the retention of the metaphor for rotation, and this can replace or stand for knowledge long after schooling is over.

The following is an example in line with the perception of rotation as a “turn of a rigid figure”. It comes from a lesson plan of a different future teacher. The image in the quote below was planned to be used as an illustration of the concept of rotation at the beginning of the class.

> *As practice for the activity, the teacher will ask the class to draw, on a piece of graph paper, triangle ABC. She will then ask the students to draw a 90---degree rotation clockwise around point C. This new triangle, which can be called ABC' will be the new point of reference for another 90 degree rotation clockwise around the new point C or C’, etc. (see picture below). This will give a pattern, which is often found in a mosaic.*




Although the triangle in the image does not contain labels, by being considered (by the future teacher) as an illustration of the procedure described in the text, one can suppose that the reference triangle is one of the four triangles in the image and the center of rotation is a vertex of the triangle. Based on the mathematical definition of rotation (in the particular case of rotation by 90 degrees around a vertex belonging to the figure), the above configuration cannot be obtained by the specified rotation. Following the instructions, the configuration obtained could be something like the shape in Figure 23.

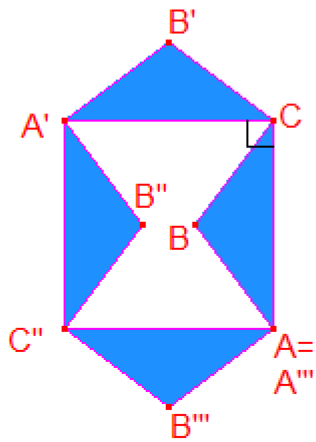



*Figure 23. Image obtained from triangle ABC by performing a rotation around point C by 90 degrees clockwise, then a rotation of the image A'B'C around point A' by the same angle, and finally a rotation of triangle A'B"C" around point C" by the same angle.*

This illustrates the point made above: instead of a reference to the mathematical definition, rotation is treated at the perceptual level; something that has to do with turning a shape around in the plane. In the given example, the future teacher specifies a 90 degree angle of rotation, however what is considered angle of rotation does not correspond to the mathematical definition of it.

It is possible that, in case this example is brought to the classroom, a student might object that this image cannot be obtained from a single triangle done by the specified rotation. The follow-up given to such (though, imagined only) statement is in the hands of the teacher. Yet, if the teacher's mathematical knowledge of the topic is not strong and clear enough to even understand the student's comment, then there is no way to follow up on it.

In fact, instead of using the above task as an illustration of the rotation concept, the teacher could take the configuration to the class and set the students a different task that would test their understanding of this transformation. She could ask:

> *Can the following configuration be obtained by one or several rotations when starting from one of the triangles?*
> *A. Justify your answer.*
> *In case you answered yes to the original question: B1. Write a series of instructions on the rotations to be performed so as to obtain the given configuration when starting from a triangle. Give your instructions to a friend and ask him or her to perform it. Compare their results with your expectations. If needed, revise your instructions.*
> *In case you answered "No" to the original question: B2. Modify the configuration to one that you consider possible to obtain by rotation(s) of a triangle. Explain, by comparing the two configurations, why the latter can be obtained by rotation(s).*

In this variant of the task, the first part may be answered by visual references, complemented by gestures. However, requesting a series of instructions to be written imposes the need of precise elements for performing the rotation. First, it asks the student to identify the center of rotation and the angle of rotation, and then to describe the construction of the center of rotation in precise terms. The section referring to the writing of instructions brings into focus the defining elements of rotation. Even if this was not clear in the beginning, most likely the feedback from students trying to follow the instructions would bring into this issue to the fore.

The alternative section (in case of No answer) is intended to facilitate a better understanding of the student's conception of rotation.
The above task serves as an illustration of the idea, in which I believe, that tasks are not good or bad in themselves; all depends on how they are played out in the classroom. Yet, this requires the teacher to be sufficiently mathematically experienced to realize (often on the spot) the flaws of an original formulation and be able to discuss these and reformulate them.

The potential of a task in creating a substantial learning opportunity depends on the mathematical knowledge of the teacher since this knowledge orients the teacher towards interpretations of the given context. A solid mathematical content knowledge allows aligning contextual particularities with general ideas we want to get across and facilitates an orientation towards abstract meanings. Such are the hopes expressed in the concept of "mathematical horizon" in the Mathematical Knowledge for Teaching model (Ball & Bass, 2009).

**Limitations of the PRo praxeologies**

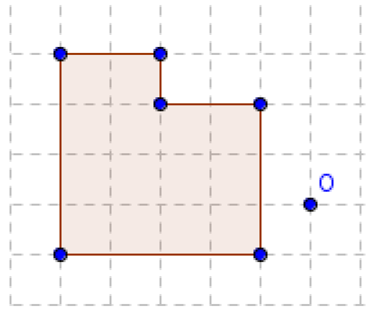
The differences in the fundamental assumptions of the two geometries, and thus, of the techniques are not easy to see, since it requires systemic thinking. Such theoretical distinctions are not only absent from the future teachers' geometric horizon, but they make no sense at all from their point of view. As seen also in case of translation and reflection, future teachers often rely on descriptions that amalgamate definitions and their consequences, properties and theorems, thus blurring, or even annihilating the distinctions of the epistemic status of the different theoretical elements. We cite an excerpt from a future teacher's lesson plan used as a "definition" for rotation.

> *A rotation is a type of transformation that moves an object around a given point. The point around which a rotation occurs is called the center of rotation. Any point on an original object and the corresponding point on its rotated image will be exactly the same distance from the center of rotation. The size or shape of the rotated image does not change from the original. (Dorling, 2010)*[9]

Although in all tasks proposed by the future teachers the center of rotation center was a vertex of the figure to rotate or it was chosen so that the line passing through the center and the vertex was a line in this grid geometry, this is not a restriction arising from the geometry they consider. Defining the relative position of two points by means of the horizontal and vertical displacements creates the possibility to have the rotation center not aligned with any of the figure's sides (rotation angle still being a multiple of "quarter-turns"). The following task illustrates these conditions.

*Rotate the given figure by a quarter turn, clockwise, around the point O*.

[9] Dorling Kindersley Limited. (2010). *Help Your Kids With Math.* New York: DK Publishing.




How could one still solve this task in the grid geometry (thus, with PRo_$\tau$2)? A simple solution would be to use a “tool”, one that can be constructed in this geometry. Conform to the technique; we only need to rotate one vertex. One can think of a rectangle with the center of rotation O and one vertex of the figure as two of its non-adjacent vertices. One can then rotate this rectangle around O by a quarter turn. This gives the position of the image of one of the vertices of the original figure. The remaining vertices are then drawn with reference to the rotated one.(Figure 24)

Defining the “tool” | Rotating a reference vertex | The original and the rotated figures

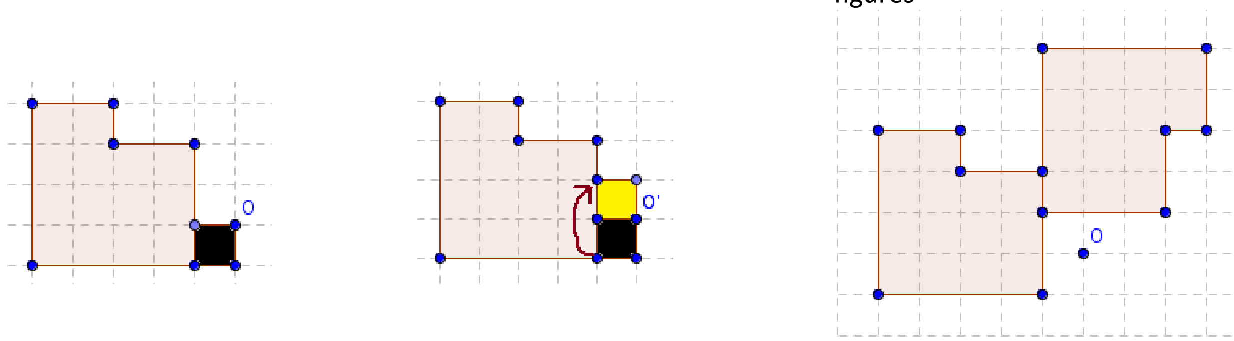



*Figure 24. Solution of a PERFORM ROTATION task with the technique PRo_$\tau$2*

Both geometries could also be the starting point of some interesting explorations - for example, about angles. As presented earlier, the concept of angle is reduced to the particular case of multiples of one-eighth turns. (I will consider this case since it is more general than the quarter turns in grid geometry). However, the possible lines might be extended to include any lines passing through two grid-points and an angle could be defined as being (a measure of) the inclination between two such lines intersecting in a grid-point (Figure 1).

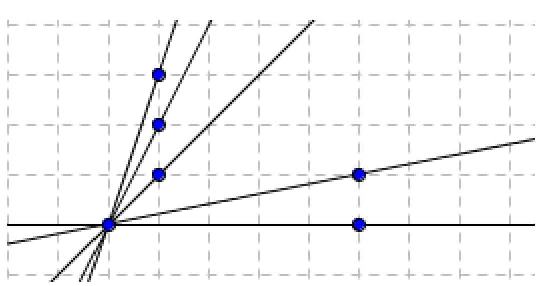


*Figure 25. Examples of angles under the new definition*

One could go on and ask the question: How can I compare these angles, if at all? What does it mean that two angles are equal? Is there a way to “measure” angles? What “unit of measure” could we define for measuring angles? (in analogy with the concept of *degree*).

The use of these explorations can be seen in their potential to engage students in “meta-level” thinking. Specifically, by making explicit the elements of a system (what is defined in the system, what objects exist), the relations between objects (for example, what does it mean for two line segments, angles, etc., to be congruent), the actions one can take (for creating new objects, for example, for creating polygons) and measures that can be associated with objects, the focus shifts toward the construction of a theory, which also immediately appears as one among many other possible theories. One could become aware of the limitations of certain definitions, the arbitrariness – in sense of conventions – of some aspects considered in the theory, among other aspects. In this sense, activities of such type could sustain the development of theoretical thinking.

If one engages in this type of questioning, one has a chance to realize how the choice of definitions of certain elements of the theory limits the operations that can be performed and one could propose ways to overcome these limitations. For example, even if we extend the concept of angle to a measure of inclination between two lines crossing each other at a grid-point, rotations of figures in this geometry are limited. For illustration, consider the rotation represented in Figure 26: it is a counterclockwise rotation of the triangle ABC with the center of rotation at point O and a given angle of rotation, with vertex labeled D. The rotated triangle is labeled A’B’C’.

The result of the rotation is not a figure in this geometry, however, because the vertices of the triangle A’B’C’ are not grid-points. Still, this “failure” could be the starting point of a different generalization, and so on.

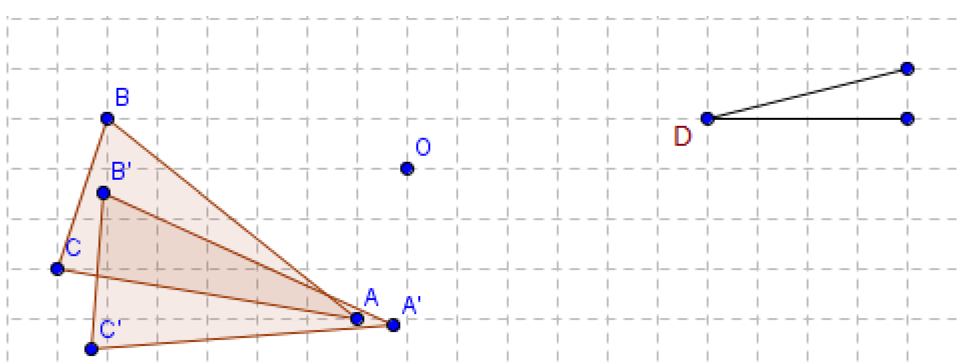



*Figure 26. Counterclockwise rotation of triangle ABC around point O by the given angle with vertex D.*

Of course, one could have chosen the position of the center of rotation O so as the angle XOX’ (where X is A, or B, or C ) is equal to the given angle of rotation. Or, as an additional instrument the teacher could allow the use of a “cardboard angle”, that is, a piece of cardboard having the same angle measure as the given angle of rotation.

One can propose interesting and rich activities involving rotations on grid-geometry, too. It is necessary, however, to help future teachers open their mathematical (and didactic) horizons to such possibilities. How to achieve this is an important and challenging question for instructors of mathematics methods courses.

As argued before, these geometries are the context for teaching transformations in such a way that it is not necessary to introduce the concept of distance from a point to a line. Thus, they are meant to “simplify” the concept of distance. However, as a consequence, the concept of distance disappeared and all that remained was the technique of “counting units”. In other words, the technique is the central object of teaching, not theory: geometry or the concept of transformation. The intention of future teachers is then to transmit the technique, to give their students a procedure to perform transformations rather than focus on teaching transformations and see what techniques are available to perform it in different contexts (with different geometries, maybe) and with different tools. The tasks proposed in the problem books are in line with the purpose of focusing on the technique; this can explain the limited variety of rotation tasks on grid paper.

In section 4.2 the praxeological models related to the first category of tasks, PERFORM TASKS were presented. These tasks are central for the teaching of transformations, hence the interest in an attentive analysis. In each sub-section, related each to one type of transformation, we gave a detailed account on how the praxeological model was identified. The geometries at the heart of these praxeologies represent the future teachers’ view of the geometry to be taught. Given the central role of these geometries, they were discussed repeatedly, each subsection bringing further specifications about them.

Each section also contained a discussion about the limitations of these praxeologies. At the same time, in this latter discussion we also included some examples of tasks that could assist a deeper level of understanding of the transformation and of geometry, in general.

The following section is about the praxeological models identified for the tasks from the category IDENTIFY TRANSFORMATION TASKS. Along with PERFORM TASKS, these tasks are central in evaluating students’ learning of transformations.

## 4.3 Punctual mathematical praxeologies related to “identify transformation tasks”

The tasks of identifying performed transformation(s) are set in a variety of contexts in future teachers’ lesson plans. We recall the tasks types that were identified:

- **T2.1** Identify the single transformation performed on polygon P. Initial and final figures are drawn on blank paper.
- **T2.2** Identify the single transformation performed on polygon P. Initial and final figures are drawn on square grid paper and vertices are on the grid.
- **T2.3** Identify the single transformation performed on polygon P made of identical squares, one of which is marked with a dot. Initial and final figures are drawn on blank paper.
- **T2.4** Identify the transformations performed on a figure and their order given the initial, the final and some intermediary positions.
- **T2.5** Identify two transformations performed on polygon P and their order given the initial and the final positions. Intermediary position is not shown. Initial and final figure are drawn on square grid paper and vertices are on grid points.

In the following, I will describe two praxeologies. The first one is related to identifying single transformation or multiple ones, but with intermediary positions shown. The second is related to identifying a sequence of transformations where intermediary positions are not shown.

The structure of the section is as follows. There are two parts, each devoted to one of the two praxeologies identified. In each part, at the beginning, the task and the technique associated with the task are presented. This is followed by the method/theory section where I will present representative examples on which the elements of the praxeological model have been built. This presentation is followed by a discussion of the nature of future teachers' geometrical knowledge.

### 4.3.1 The first praxeology related to Identify transformation tasks

**Type of task** IT_T1. Identify a single transformation where intermediate figures are given.

The tasks from categories T2.1 till T2.4 are included here.

**Technique** IT_τ1*:*

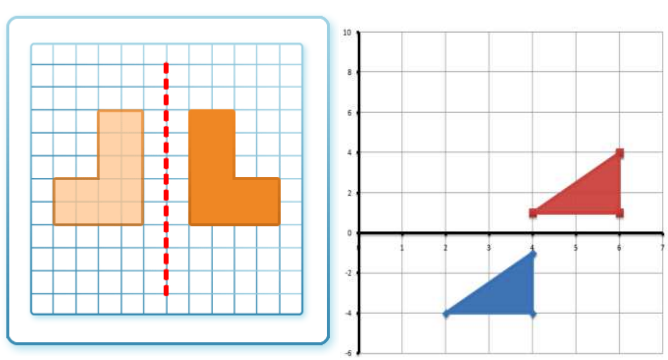
*Compare images on perceptual level and apply one of the following:*
*a) If the resulting figure has the same orientation, then the transformation is a translation.*
*b) If there is a line of symmetry in the final configuration, then the transformation is a reflection.*
*c) If the resulting figure is turned, then the transformation is a rotation.*

**Method/Theory**

The tasks involving identification of a transformation in the problem books were often borrowed from books and websites, rather than originally designed by the future teachers. Therefore, the tasks also illustrate how this type of task is conceived in some popular media. We give examples for each sub-type of tasks as they appeared in the future teachers' lesson plans before discussing them.

Example 1. (for T2.2**)**

*Name the transformation and justify.*



(Source: google images[10])

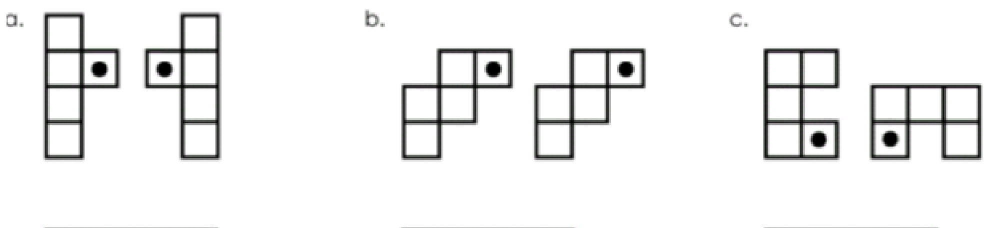
Example 2. (for T2.1)

*Professor Peter is trying to plan out his classroom. He draws out a diagram of the layout of the classroom depicting his desks. He then moves his desk around until he reaches a configuration that he likes. Comparing the new (B) and old (A) location of his desk, what type of transformation has occurred?*

”

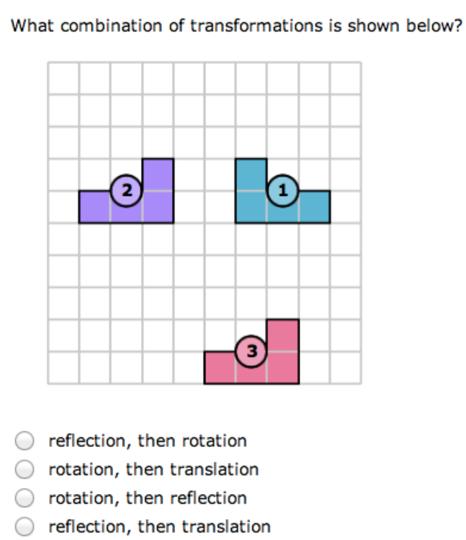
Example 3. (for T2.3)

*Tell how each figure was moved. Write translations, rotations or reflections.*



”

Example 4. (for T2.4)

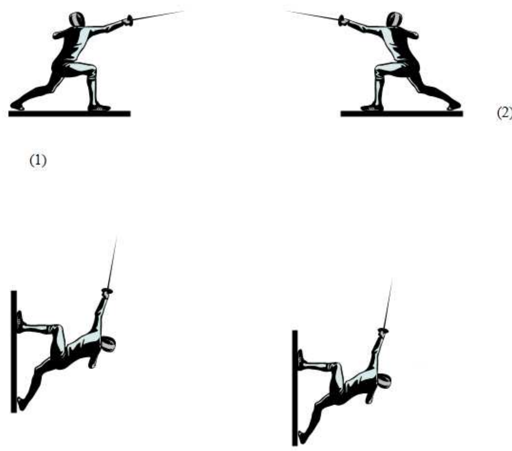



Example 5. (for T2.4)

[10] This is the information about the origin of the images given in the future teacher lesson plan.

*What series of transformations were used to move the man in picture 1 to the man in picture 4?*

*a) Reflection, reflection, translation*
*b) Rotation, translation, reflection*
*c) Reflection, rotation, translation*
*d) Rotation, rotation, translation*




As can be seen from the instructions in the examples, the student must *name* the transformation – that is, the expected answer is a *name.* Students are not asked to justify the name. The focus being on naming, references to the defining elements of the transformations are ignored and are, instead, visual; recognition is favored. In this sense, the tasks remain at the visualization level.

If such tasks are given with the intention of assessing students' knowledge of the concept of transformation, then this implies that for the future teachers to "know" a transformation means to recognize it visually once it has been performed. Students may end up grasping the message and come to see no need of paying attention to the defining elements (parameters) of the transformation. This would only reinforce the belief they would have built already based on the Perform tasks.

The method / theory on which the technique relies, resides in the descriptions future teachers use in "defining" the transformations. These "definitions" are descriptive passages that do not distinguish between the different theoretical status statements can have; definitions and properties that logically follow from them as well as metaphors and analogies can be brought together under the same title of "definition". Some examples of the "definitions" proposed in the Problem Books are given below.

IT_θ1 / IT_Θ1.

> *A translation is a type of transformation. It moves an object to a new position. The translated object is called an image and it is the exact same size and shape as the original object. (Dorling, 2010)*

*(…) a reflection can be identified if a "2-D shape and its image are congruent" and if "a 2-D shape and its image are of opposite orientation" (Newfoundland Department of Education, 2009, p. 102)*[11].

*Reflection: every point is the same distance from the axis of symmetry; the reflection has the same size as the original image. It is a flip over a line. (mathisfun.com, 2012)*

*A rotation is a transformation that is performed by 'spinning' the object around a fixed point known as the center of rotation (turning point). You can rotate your object at any degree measure, but 90° and 180° are two of the most common. Also, rotations are done counterclockwise.*[12]

In the following, I will comment on the potential consequences of using the tasks as envisaged by the future teachers.

**Comments on the praxeology**

In the section **Expected solution**, future teachers often make references to these "definitions" when referring to perception as source of validity. This becomes visible not only through the citations of these, but also in the language they use in the justification of the answer. As illustration, consider the following excerpt, part of the **Expected solution** section for the task presented as example 5.

*"The reason option c would be the answer is because it is expected that students would recognize that the second image is a direct reflection of the first image, and that the third image is a rotation from the second image as it has been rotated 90 degrees to the right, and the final image is a direct translation of the third image." (My emphasis).*

The "recognition" process relies on a standard image associated with each *verb* used to express the transformation ("flip", "turn", and "slide") as if these were the definitional meanings of the *mathematical* concepts reflection, rotation and translation. The mental image of a transformation, or "concept image" in the terminology of Tall & Vinner (1981), has its source in the everyday meanings of the verbs and not in the detailed examples of performed transformations, thoroughly analyzed from the *mathematical* point of view and distilled into definitions. The everyday *verb* together with the *image* is elevated to the status of "definition" through the short paragraphs collecting properties and preceded by the word *definition*. As such, this process of deriving a "definition" has nothing to do with *mathematical* definitions – its purpose is to "institutionalize" an everyday interpretation into an official piece of knowledge and not to give a minimal list of properties that define a concept, and distinguish it clearly from other concepts. As a result, knowledge encapsulated in the "definition" may stay isolated from all other knowledge: it can remain vague and no consistency with other concepts is expected. In practical terms, this also means that a transformation is clearly identifiable by perception and there is no need to identify its parameters: translations are interesting only as translations, as not being reflections, for example. From the point of view of the Theoretical Thinking model, this aspect is related to the

[11] Department of Education, Province of Newfoundland. (2009). *Mathematics Grade 5 Interim Edition Curriculum Guide.* Retrieved from http://www.ed.gov.nl.ca/edu/k12/curriculum/guides/mathematics/gr5_math_guide.pdf

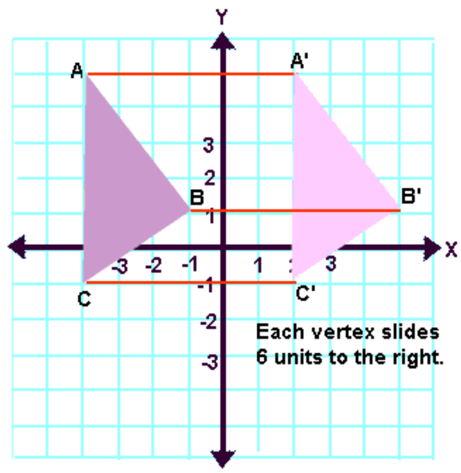
[12] http://www.gradeamathhelp.com/transformation-geometry.html

feature *systemic-definitional thinking*. It can be said that future teachers' thinking does not have this feature.

The difference between mathematical and everyday definition can be stated as a difference between *extracted* and *stipulated* definition (Edwards and Ward, 2008), where "*extracted* definitions tell about usage; while *stipulated* ones create usage, create concepts by decree" (*idem*, p. 224). The difference between these types of definitions is often not recognized by future teachers. However, knowing about the stipulated nature of a definition is related to another feature of systemic thinking, *systemic-hypothetical*. This feature expresses awareness of the hypothetical character of knowledge, meaning that knowledge is valid in the frame of a set of assumptions in which it was constructed.

Future teachers reinforce the visual association suggested by *verbs* by using visuals. These were not always together with the "definitions" they cite, but obtained from internet resources. Consider the following example used by a future teacher to introduce the concept of translation.

> *A translation is a type of transformation. It moves an object to a new position. The translated object is called an image and it is the exact same size and shape as the original object. (Dorling, 2010)*

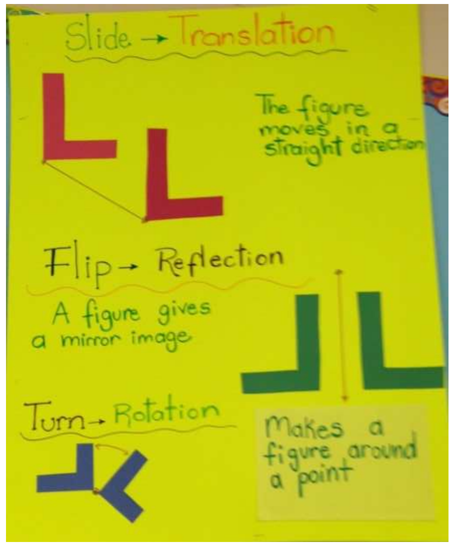



"

The image contains a caption that makes reference to units and direction (performed on grid paper), however it is not clear how these elements correspond to the description to which they supposedly refer. It is likely that the segments connecting corresponding vertices serve more as a reference of the trace of a movement than to the defining parameter of translation, namely the vector of translation. Although translation is "defined" as a transformation, the specific elements that distinguish it from other transformations are not specified. The paragraph applies equally well to rotation and reflection (both will preserve shape and size and both can "move" an object into new position. The specific elements are supposed to be deduced from the illustration, reinforcing a certain image associated with translation.

In some other situations, the future teacher creates such visuals in order to illustrate the "definition". In this case, we have a direct access to the geometry of the future teacher. Consider the following example where the future teacher proposed to use the visual to introduce transformations to the class.

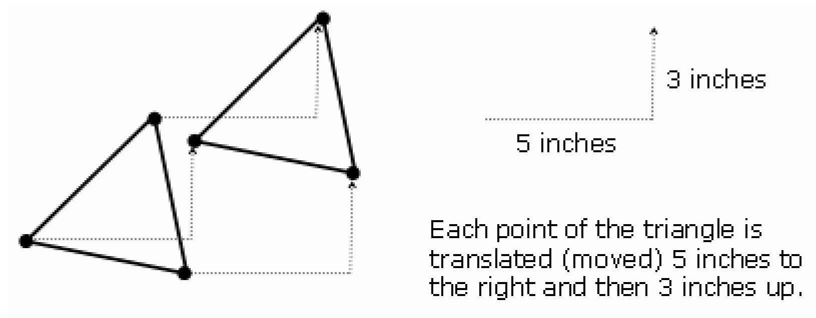
*Students will be explained the three types of transformations and shown with a pictoral demonstration what occurs to each shape when it is rotated, reflected, translated.*




Although formal mathematical definitions have been provided during the methods course, future teachers seem to rely more on their own understandings when it comes to create lesson plans where to introduce or apply these concepts. Similarly as in the case of rotation discussed in section 4.2.3, the metaphoric language (“slide → translation; flip→ reflection; turn→ rotation”) relies on a perceptual, or, on occasion, gestural, element. While the role of gestures and perception in the initial phases of learning is known to be beneficial (as creating a concept image), problems arise if these early understandings are not revised so to introduce a mathematical definition to them. On the above visual, however, one can identify the intent to make elements of the transformations visible. Yet, the image illustrates a particular case of rotation (with the center on the figure. Moreover, the angle of rotation is wrongly identified.

As mentioned, online resources are often found in future teachers lesson plans, whether for providing “definitions” or as sources for their handouts and tasks. The pertinence of such resources is not, at least not explicitly, discussed and it l might well be that the decision to employ them relies, again, on some sort of trust in the authority of published material. For example, the following “definition” is used by one future teacher in a review of translation.

*The most basic transformation is the translation. The formal definition of a translation is "every point of the pre-image is moved the same distance in the same direction to form the image. Example:*



"

One might ask why, in the example given, "same direction, same distance" is illustrated by a pair of displacements (5 inches, 3 inch), even when there is no explicit reference to a grid. The source of this information is a webpage dedicated to, what is called, *transformational geometry.* At the center of this geometry are transformations of the plane, in contrast with the Euclidean approach where the central elements are straightedge and compass constructions. Thus, the epistemology of transformational geometry is different from the classical, Euclidean one. The cited webpage, in addition, presents information with the goal to quickly switch towards descriptions of such transformations in a Cartesian plane. The image is followed by a formalization of the translation as $T(x, y) = (x + 5, y + 3)$. This is the ultimate reason why the caption contains reference to the pair of displacements. As a consequence, the information provided on the webpage (or in other sources) must be interpreted along with its underlying epistemology. Yet, in the case of a future teacher's lesson plan, the context in which the cited definition has been given was ignored and not even acknowledged as an aspect to take into consideration before using it.

Besides the image serving as a paradigmatic case for the concept, gestures are often associated with transformations – transformations are "acted out". Some of the visuals already suggest the movement – for example by using arrows; yet, some future teachers make an explicit effort to "feel the concept in one's body". The following examples are presented as instance of such approach (emphasis is mine).

> *But I also know that some students are likely to confuse reflection with translation and vise-versa. In this case, we can practice the transformations <u>by acting them out in class</u>. Having <u>the students physically motion the transformations</u> might help them to remember the actions.*

From the particular phrasing used by the future teacher, one can conclude the importance attributed by the future teacher to gestures in the learning of a concept. As mentioned before, a gesture can support the intuitive understanding of a concept, yet remaining at the level of gestures will create obstacles to the transition from visual level to analysis level of geometric thinking.

Another future teacher also specifies:

> *Through the use of physical movement, manipulative and a practice exercise, students will familiarize themselves and learn about rotations, reflections and translations.*
>
> *At the beginning of the activity, students are asked the following questions; to physically do a rotation, a translation and a reflection. By using their bodies they will be instructed to keep one foot anchored and make a half turn, then to imagine a reflection line in front on them and reflect*

*themselves onto the other side of it then to slide their feet toward in one movement. The notion of congruence will be explained and how these transformations are moving which is why the shapes remain the same but are in a different space.*

Everyday gestures are not what plane transformations are in mathematics. A human body is three-dimensional, therefore reflection about a line has no sense – although the idea may sound attractive and might be quite amusing and could even give the impression to students that they understood what the transformation is about – in fact, it is a counter-productive approach. In everyday interactions, fuzzy approach is acceptable; meanings are often highly contextual and we learn to navigate by completing information with circumstantial one. Yet, mathematics deals with abstract objects brought into existence through definitions, and thinking and reasoning about them happens in the frame of a system, with elements linked by deduction. For this reason, acting out the transformations cannot bring students closer to understanding what transformations are about. One can understand the *intentions* of the future teachers: many of them try to find ways to show that mathematics requires no special thought, it is just a sort of extension of everyday thinking and deals with everyday objects ("*math is all around us*", as they state). Math is all around us, but it is invisible to the naked eye; it is not in things but in relations among them and relations can only be thought: constructed mentally, postulated, defined, deduced. This thinking is rather different from everyday thinking. It is pertinent to cite Vinner here:

> *In mathematics, however, we form concepts by means of definitions and verify conjectures by mathematical proofs. Thus, mathematics imposes on students certain ways of thinking, which are counterintuitive and not spontaneous. In other words, mathematical thinking requires a kind of inhibition from the learners. (Vinner, 2011)*

These beliefs of future teachers shed light on the epistemology of the mathematics they envision teaching: mathematics is knowledge where justification is mainly perceptual, concepts are contextual and situations should be dealt in individual manner, without looking for coherence among the way we handle them.

We can conclude by highlighting two aspects of future teachers' theory from the above cases.

First, such situations bring up the issue of future teachers' meta-level knowledge, for example, the one concerning the selection of an illustrative example for a certain context / concept or the selection of "definition".

Second, on subject matter level, the situation brings forth the question of future teachers' understanding of the concept of transformation and the existence of misconceptions related to the concept.

Hundreds of websites offer free resources for teachers, particularly worksheets. The worksheets come with answers, too, which is practical. The worksheet partly reproduced in Figure 27 was a popular choice between future teachers (also mentioned as task type T2.3). As it can be observed, the header of the worksheet contains a "reminder" on what these terms "mean" and, one can suppose, it was given as to serve as support for students' reasoning. Student can *look* at these "visual shortcuts" and, mainly by analogy, decide which name best fits each of the situations in the worksheet. In this sense, the task remains at the visualization level.

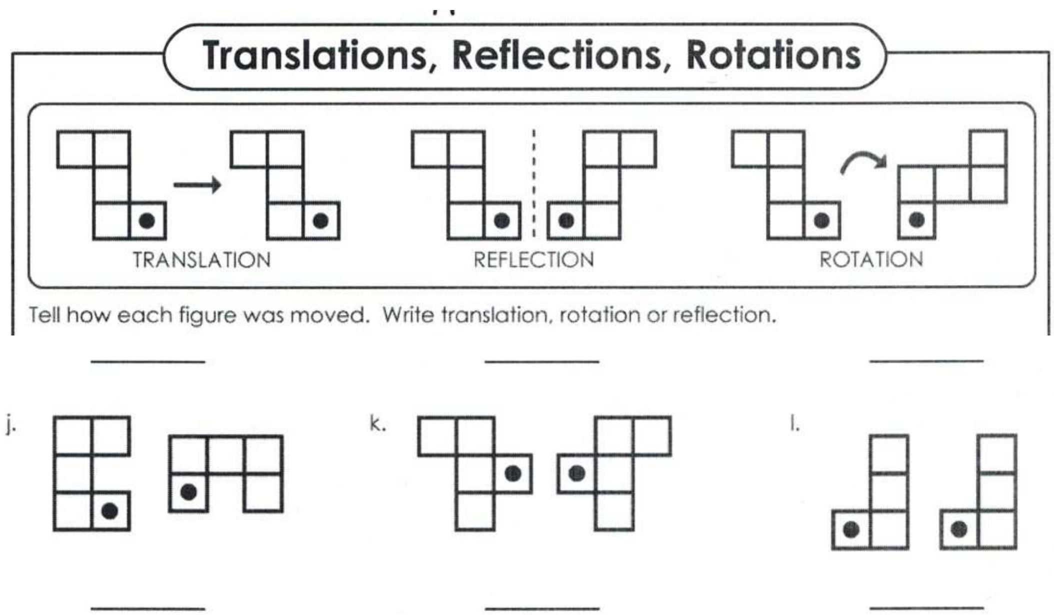



*Figure 27. A worksheet on transformations from the internet[13], popular among future teachers.*

The reminder hints to some element to be identified (at least in case of reflection); however, is not clear from the drawing how the line of reflection should be situated in relation with the two figures. One can observe that nothing more than a name is expected since there is very little space for the answer.

Yet, if the future teachers would decide to ask students to clearly identify the elements in the definition of each transformation, the task could be well adapted for practicing recognition of transformation at informal deduction level. Interestingly, the configuration presented in part *j* of the task in Figure 27 requires further clarification of the rotation concept in the sense that the reminder is not enough to conclude the name of the transformation. Given that the two figures are not aligned on an (imaginary) horizontal line, this situation is not "obviously" a rotation as per the reminder. In a classroom environment it can easily happen that students will still answer "rotation" by exclusion of the other two options (already "occupied" by images *k* and *l*) rather than by finding arguments to justify that it is rotation. We do not know what would have happened in a classroom if the future teacher arrives with this activity. We can only hope that encountering this example (as in *j*), she would have the tools to clarify the definition of rotation, go on and be able to identify the center of rotation and angle for this figure. Furthermore, we can only hope that seeing the distinct nature of this particular example (as one that asks for identification of elements of the transformation), future teacher can become inspired to use this as source idea for developing their own material in the same vein.

In conclusion, it can be said that the tasks about identification of performed transformation do not focus, for justifying, on the mathematical definition of the transformation, but on a standard image associated with a descriptive passage. As such these tasks remain mainly at visualization level.

[13] https://www.superteacherworksheets.com/geometry/translation-rotation-reflection-1_TZQTQ.pdf?up=1466611200

### 4.3.2 The second praxeology related to Identify transformation tasks

I will next look at the techniques used by future teachers in case of tasks where the students were asked to identify the order of several transformations that were performed, however without having information about the intermediary figures. This corresponds to task type T2.7, as outlined at the beginning of the chapter.

**Task type** IT_T2**.** Identify the order of the two transformations performed on figure F where only initial and last position is shown. Transformations are illustrated on graph paper.

**Technique**

IT_τ2.

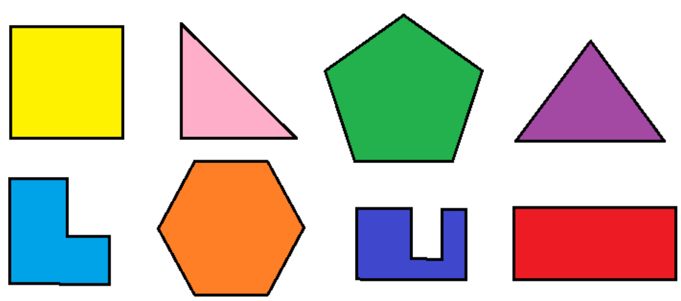
> *Compare images by perception and decide on one of the following:*
>
> *a) If the figure is in the same final orientation as the initial one, then try to identify an axis of reflection and a translation or vice versa;*
>
> *b) If not, then try to identify (visually, expressed by units) a translation and a rotation or translation and reflection, depending on the orientation of the figure.*

**Method/Theory**

The following is an example of such task. It serves as base for further discussion on techniques and method (emphasis is in the original formulation).

Example:

> *To begin this activity, students should find a partner and arrange themselves in pairs.*



> *Next, the teacher would hand out the materials required for this activity. Every student would receive three pieces of dotted paper, as well as a small plastic bag that holds paper cut-outs of different shapes. These shapes may include the following, but the teacher may adjust these shapes based on the needs of the students.*
>
> *Students would each take some time to work individually and develop a figure using the shapes that they had been given. They may choose one figure and trace it onto their dotted paper, or they may choose to combine two or more figures and trace that shape onto their dotted paper. The students would then be expected to perform two transformations on that figure; they may choose whatever they'd like to do, but they must*

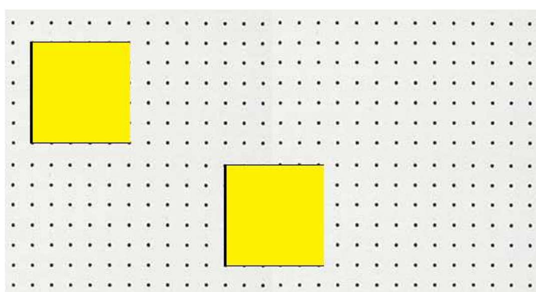
*transform the figure twice (a reflection and then a rotation, for example). The teacher must be sure to tell the students that once they have performed the two transformations, they must <u>erase</u> the first transformation so that only the final product is left (the figure that has undergone the two transformations).*

*Once the students have completed the drawing of their figures as well as the two transformations, they would be expected to switch their papers with their partners so that they will each have a paper with a figure that has been transformed twice. The students would then need to try and figure out what transformations had been done to that figure to have gotten it from the initial drawing to the final product.*

The task setup is long; however, a part of it refers to the identification of two transformations that have been performed sequentially on an initial figure, yet the intermediary position of the figure was deleted. In the following, the proposed solution is presented and, based on it, I will discuss the method and theory.

The technique has been identified from the **Expected solution** section of the lesson plan where the future teacher gives two examples of tasks that students in the classroom could set up. Here, I present the concrete task and solution briefly in order to highlight the origin of the technique.

*(…) some examples of what their work might look like may include the following:*



*For this series of transformations, the student's partner might initially guess that it was a reflection, but if they were to count the dots or draw a line of reflection, then the students would realize that there was another step in between. Students may figure out that the square has been translated to the right by 10 spaces and then was reflected vertically over a line of reflection.*

Note: One may wonder if the square shape in the expected solution represents a geometric square (a figure with 4 lines of symmetry) or a graphical object with different thicknesses of borders (and thus with less symmetry). Given, however, that this shape is one of the shapes in the future teacher's description of the task and the square shape in this description has no such asymmetry, we conclude that the intended object is a geometric square.

As can be observed from the description, the future teacher relies on visual elements and techniques mentioned in the description of praxeologies for performing transformations ("counting units") to decide about the nature of transformation performed on the figure.

Thinking about the possible solutions for this task, one quickly arrives at the conclusion that the solution is not unique. If we allow ourselves to beyond the strict dot geometry and include vectors not passing through grid points, there is a possibility of obtaining the image with only one translation. If one wants to stay within dot geometry then one could (always) argue that two sequentially done translations by vectors in standard directions will do. If the future teacher insisted on identifying two *different* transformations, one can still have, for example, a translation followed by a reflection (both in standard directions), besides the reflection followed by a translation expected by the future teacher.

It seems from the future teacher's comment that the orientation of the figure plays a role in looking for translation and reflection, since the future teacher doesn't even bring up rotation as potential transformation. For this reason, comparing the orientations of the initial and the final figures seems to be a central criterion for identifying the performed transformations.

The fact that the option of one single reflection has been eliminated by the future teacher suggests that in spite of using dot paper, her thinking is more aligned with grid geometry than dot geometry: she does not see the possibility of having an oblique line of reflection. There is no possibility in her mind of a single translation either, since translation can happen only along the standard directions (horizontal, vertical). Thus, there must be another transformation (another translation, or a reflection) to the figure. We have seen that, on occasion, translation is perceived as composed of two, more "basic" translations along the standard directions; however, in this case, the future teacher considered only one component. On the other hand, she measures the distance more like in the dot geometry by counting the spaces between the dots along standard directions. From this point of view, the reason why she could not conceive the transformation to be a single reflection is that, in dot geometry, we cannot find a line at the same number of units from the two figures. Yet, it is interesting to observe that the future teacher relies on an axis of reflection that is not defined by grid-points (thus, a line that should not exist in the geometry). If we follow the solution proposed by the future teacher, the line passes between two rows of dots. In this sense, it is a modified version of dot geometry: it is extended to allow this particular type of line, since it is useful for solving the task. At the same time, by stating that the transformation could not be *only* a translation, the future teacher might introduce / reinforce misconceptions about transformations.

In conclusion, the underlying method and theory seems to be an extended version of dot geometry, along with the visual versions of "definitions" of transformations.

IT_θ2 / IT _Θ2

> *"Dot-geometry+": dot geometry extended to allow lines not through the grid points.*
>
> *"Definitions" of concepts along with standard images associated with them.*

**Comments on the praxeology**

In the preceding example, the future teacher expected a solution consisting of two different transformations. From the description, it was also understood that a unique solution was expected. This expectation is not singular; on several occasions, we found tasks that could be solved in different ways, yet future teachers thought of one single possible solution. In the case of geometry, this was quite common for the "mystery definition" (Walle & Lovin, 2006, p. 197) type of tasks. In these tasks, the students are asked to find a property common to all figures listed as examples and missing from all those listed as non-examples and, based on the identified property, to select figures that satisfy the property.

The failure to recognize the non-unique nature of a solution suggests a difficulty of distancing oneself from the concrete context of the task (and also from its creation if it was created by the future teacher and not borrowed from external sources), and exploring the task from a more general perspective. Such distancing would require asking questions such as: What are the defining elements of the context? Under what assumptions (conventions, available knowledge) are we solving the problem? What are the possible interpretations of the different elements of the task? etc. The ability to distance oneself from the tasks at the time of analysis is related to systemic thinking, as defined in the Theoretical Thinking model. But theoretical thinking requires reflection, so an attitude of inquiry: is there any other way to get to this answer? Easier, harder or maybe just different in terms of elements involved? The questioning attitude is linked to reflective thinking, as defined in the Theoretical Thinking model. It would help future teachers gain more depth in thinking about the tasks they propose and be more prepared for classroom interactions. I will illustrate this point with more discussion of the task in the previous example.

Even in dot geometry as theoretical support, there are several solutions to this task in the sense that there are pairs of transformations that would transform the first figure into the second one. We already mentioned some above. Yet, I include here two new solutions.

The first solution, shown in Figure 28, uses reflection about a line $d$ which is oblique but passes through grid points, followed by a translation by a vector $v$. The line $d$ and the vector exist in dot geometry and the transformations are valid in this geometry. Given the particular position of the line of reflection, the distance from a point to the line can be expressed uniquely, as the number of "units" (here, spaces between dots in standard direction) till the line.

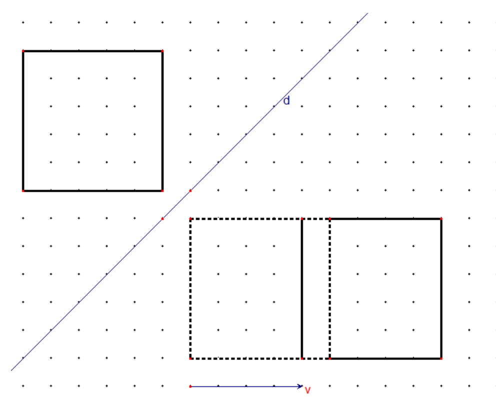



*Figure 28. A possible solution of the task of finding two transformations whose combination maps the upper thick square onto the lower thick square:*
*reflection about line d, followed by translation by vector v.*

Another solution, included in Figure 29, consists of two consecutive rotations, half turns, once around point O and, then, around point O'. Both rotations are well defined in the dot geometry.

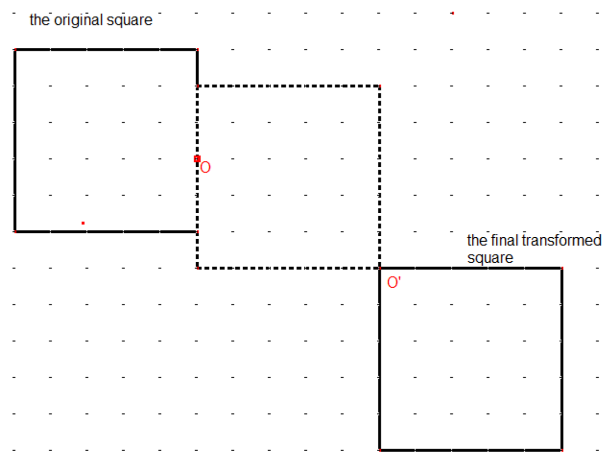



*Figure 29. Solution of the same task as in previous figure, by two consecutive rotations by 180 degrees, around point O and then around point O'.*

If the constraint of dot geometry is removed, then there are many other solutions (rotation(s), reflection(s), and translation(s)).

Given the multitude of solutions, the task could be a good context to raise the question about *relations* between different transformations. For example, if we understand that one single translation can be the answer to the task, seeing the solution by two reflections, raises the question "Is it always possible to replace a translation by two reflections with a suitably defined line of reflection?". Similarly, the solution given as two half-turn rotations can lead to the very same question.

If we go back to the solution specified as reflection followed by translation, we can imagine someone proposing a translation followed by a reflection instead (with the same parameters). We could ask if changing the order of transformations when combining them always leads to the same result (in more mathematical terms, whether the composition of transformations is commutative). With a question about the *nature* and *generality* of some observed relations, the discussion can move to a higher level – about the theory behind – and students can reach for a more profound understanding of the *relation* between these transformations even if they studied them only separately. The progression from studying individual mathematical objects towards the study of relations among them with the finality to reorganize the initial knowledge at a higher level of abstraction is how mathematical theory develops.

The passage from focusing on the individual study of mathematical objects / concepts to focusing on the relations that exists between them, if done repeatedly, with different concepts, could allow students an insight into the nature of mathematics as a discipline and about mathematical thinking: constantly widening, generalizing, and reorganizing at ever higher level of abstraction.

The situation described above talks about the affordances multiple solutions can entail. Once more, even the limitations of dot geometry can lead to very deep mathematical questions. The teacher's role is to sustain such conversation and direct it towards discussing relevant issues. The real issue is if the future teachers are ready to grasp the opportunity. What it does it take to be able to do so? Or teach someone to do so?

On the teacher side, this requires subject-matter knowledge, but at more advanced level than the one that must be taught. And, it also requires theoretical thinking. Yet, it seems from our data, that future teachers have weak subject matter knowledge, at least in the domain of geometry. Their preference for "definitions" – consisting, mainly, of metaphors and prototypical images associated to concepts – is not only a watering down of solid, consistent mathematical knowledge in the name of didactics, but it is their personal content knowledge.

Although during instruction, they have been provided with formal definitions and properties, they prefer to employ their previous experience and memories when they plan to carry out classroom activities. These two aspects, experience and memory of own school years, seem to act as a filter to the taught material; so, that formal mathematical definitions are retained in the measure they reinforce the descriptive, sometimes gestural and image focused "definitions" they possess. As for experience, they often refer to tutoring experiences where their students are faced with tasks just as they propose and, in case of which, verb based metaphors come handy. By having made the student "see and feel" what transformation is about, they reinforce their belief that teaching transformations it is about relaying these verb laced images to students. And since there is no requirement for internal consistency among the "definitions", contextual interpretations will resolve possible conflicts. This might also explain why future teachers continue to make mistakes, but do not see mistakes as motivation for revision and growth.

The distinction between "I can't find a solution" and "There is no solution", or between "I can see only one solution" and "There is a unique solution" is more present in geometry than in other domains of (school) mathematics, especially at primary level. Future teachers need the content knowledge in order to decide about the pertinence of a question or answer in a task. Although this is the case in general, it seems that in geometry this aspect is more pressing than in problem solving in arithmetic, for example. Typology of problems related to topics in arithmetic are known and future teachers may feel more at ease, and less prone to errors, when proposing problems for those topics. However, in geometry, the assumptions underlying certain types of tasks (like, "mystery definition" or "identify transformation") are not so evident. The future teachers will propose a problem that seems to be the same kind as the ones they been taught or saw in textbooks, yet in its essence it will be very different.

As in previous cases, it remains an open question of what could happened if the problem such as in the example discussed above is proposed in class. We can hypothesize that children would find multiple solutions and, then, it is the teacher's role to build on those multiple solutions and lead a classroom conversation that extends beyond the procedure of performing transformations. As shown, repeatedly, there are still many opportunities even within the constraints of the geometries they act in. However, if the future teacher doesn't have a solid content knowledge that is paired up with theoretical thinking, such conversation may be hard to conduct so that new knowledge or understanding emerges from it. Without the discussion of the theoretical validity of the proposed solutions, the teacher's polite acknowledgment of the children's many different solutions – a behavior frequently observed in the workshops during the course – may lead children to believe that, in mathematics, "anything goes".

## 4.4 PUNCTUAL MATHEMATICAL PRAXEOLOGIES RELATED TO "IDENTIFY PARAMETERS TASKS"

In this section, I will analyze a praxeology related to the task of parameter identification of a given transformation. While it is true that identification of the parameters of transformation can be seen also as part of a task where the transformation must be identified, in this case the task is proposed with the only purpose to identify the parameter. In addition, the tasks future teachers proposed for identifying the transformation required only the name of it, ignoring the importance of parameters. At the level of concept understanding such task may prove to be instrumental, since it explicitly focuses on a defining element of a transformation and therefore the task cannot remain at a visual level.

**Type of task** IP_T. *Identify the vector of translation of a polygon P with vertices on square grid points*.

**Technique**

IP_τ1.

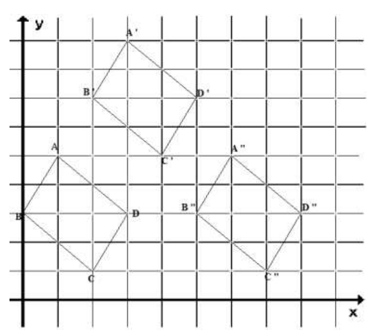
> *Identify the transformation visually.*
>
> *For translation: Count the number of units in standard directions between two points labeled with the same letter or what looks like corresponding vertices.*
>
> *For rotation: Identify the quarter turns between points with corresponding letters or what looks like corresponding vertices.*
>
> *For reflection: Identify the line of symmetry between points with corresponding letters or what looks like corresponding vertices.*

Although the type of task we identified among the proposed ones referred only to translation, the technique above is extended to tasks involving rotations and reflections, too.

**Method/Theory**

Only one task was proposed by future teachers in this category. It was specified as follows:

***Translation***




1. *How many units did parallelogram ▱ABCD move on the grid to become ▱A'B'C'D'?*
2. *How many units did parallelogram ▱A'B'C'D' move on the grid to become ▱A''B''C''D''?*
3. *How many translations would it take for ▱A''B''C''D'' to translate onto ▱ABCD?*

The answer the future teacher expected to the first question was, “4 units Up and 2 units Left”. Thus, the technique identified from the section **Expected answer** is: count the number of units in both standard directions between two corresponding points. However, we can extrapolate from here to what might have been the technique were we having a reflection, or rotation. Once again, the grid / dot geometry is at the basis of the technique, with its standard directions and special “units”.

IP_$\theta$1 / IP_$\Theta$ 1.

*Grid geometry;*
*Translation is described by two movements, each in standard direction;*
*Rotation is a “turn” by some number of quarter angles;*
*Reflection is a “flip” about a line;*
*Labeling convention (transformed vertex has the same letter, but indexed).*

Although *the* task included already labeled figures, in formulating the technique (a *model*, not just a description, of the envisaged practice) I included also a more general situation, where correspondence is not through labeling, but some recognized relation between the vertices. For example, one can imagine a scalene triangle and its transformed image where, even without labels, one would be able to identify corresponding vertices.

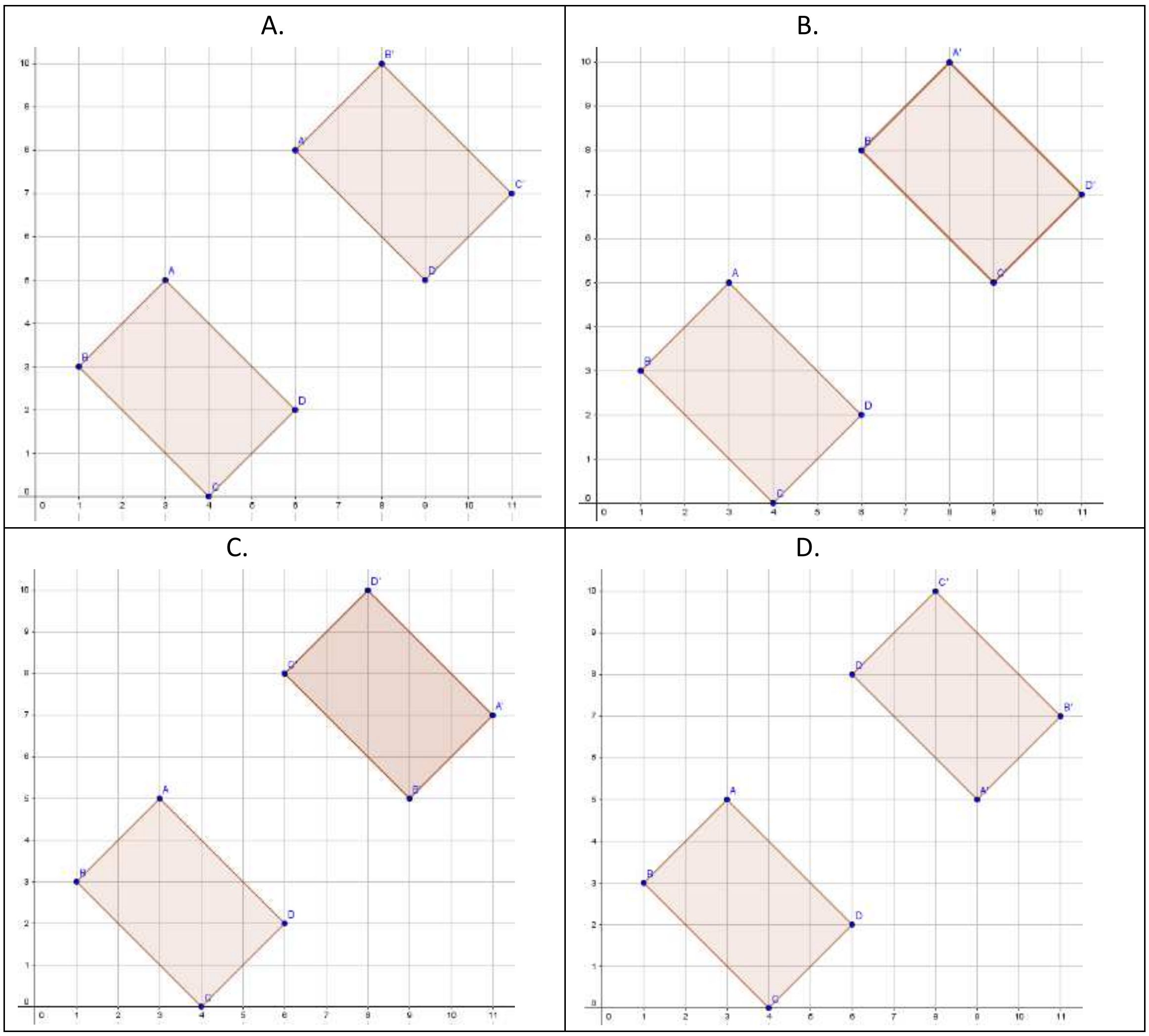
**Comment on praxeology IP**

The task has a title, “translation”, and this title establishes the context of the task. Although the parallelogram ABCD can be transformed in many ways into the parallelogram A’B’C’D’, here the title wants to tell students that they should focus only on translation. At the same time, it tells them, at least by virtue of *didactical contract* (Brousseau, 1997) , that there has been a translation and, as a consequence, students do not need to verify if this is the case.

Even if we limit ourselves to translations, there is no unique answer to the question. The fact that the future teacher believes in the existence of a unique answer to the question illustrates their interpretation of translation on a grid. The vector of translation is expressed as a single pair of relative displacements together with directions. The “displacement should be the shortest possible” is a tacit requirement for specifying translation vectors as pairs of numbers. As argued before, such view is a consequence of the grid geometry that is underlying future teachers understanding of geometry to be taught.

The task relies on the convention of labeling corresponding vertices with the same letter. As discussed earlier, focusing on transformation of vertices allows the direct application of the definition, since the definition refers to transformations of points. Insisting on the definition permits the study of the relation of these three transformations and isometries, identification of invariants and conservation of properties. The convention for labeling corresponding vertices must be clearly transmitted in the classroom if we are to make later reference to it. Such convention might be useful for setting tasks of identification of transformations so as to include the need to identify the parameters, too. By providing visually identical settings (except labeling, of course) and the labeling convention, students are put in the situation to identify the transformation and its parameters by going beyond perception. The task of identification of transformation, initially at perceptual level, becomes a task requiring analysis level only if it explicitly requires the identification of the parameters and a decision about whether a single transformation has occurred. An example of such task could be the following:

> *The following images contain two figures each. For each case, verify if one of the figures is the image of the other by a <u>single</u> transformation if it is assumed that corresponding vertices are labeled similarly, using the apostrophes as distinction (Thus, A would be in correspondence with A', etc.). If yes, specify the transformation. Justify by identifying the parameter(s) of the transformation.*

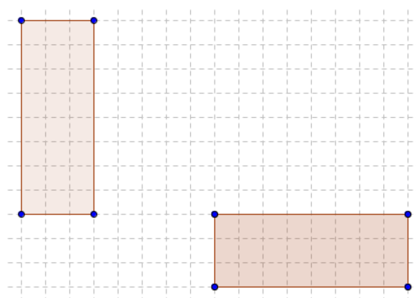



Although the geometries (grid / dot) limit the kind of situations we can define, it is still possible to design rich tasks that would help clarifying conceptual differences between the transformations.

Tasks for students can also be outside the convention of notation. The task from the previous category ("Identify transformation") can be easily be transformed into tasks about parameters of transformations, if we specify the transformation. And this can also be rich enough to lead to interesting questions, even if we remain in the constraints of dot geometry. Consider the following task.

*The following image contains two congruent rectangles: rectangle R’ was obtained from rectangle R through a single rotation.*

A. *Identify center of rotation and angle of rotation.*
B. *Is there a unique solution? Justify your answer.*
C. *If the solution is not unique:*
- *Explore if there is a relation between the multiple solutions and what might justify the relation;*
- *Formulate a conjecture and verify it in a new configuration.*



The task has two solutions (Figure 30). One is a rotation around center O with a quarter turn in clockwise direction, while the second is a counterclockwise half-turn with center O’.

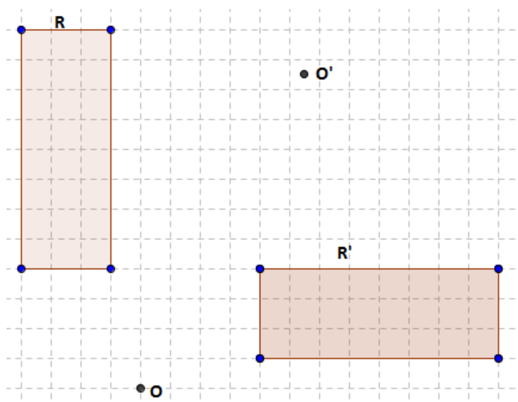



*Figure 30. Two rotations can be identified to transform rectangle R into rectangle R’.*

If students identified the corresponding vertices through each of the transformations, they would be able to understand why there are two solutions: the rectangle has a rotational symmetry in relation to its center (intersection of its diagonals). Working further on this idea, if the task is done in the class, the teacher could conduct a discussion about polygons displaying such property and lead students to formulate a conjecture about rotational symmetry. The activity can give a real opportunity for investigations into transformations.

Relying on convention, without explicitly introducing them to students, is at the origin of a different type of error: confusing the necessary with the arbitrary in mathematics. As Hewitt (Hewitt, 1999) outlines in his article, words, symbols, notation and conventions are arbitrary, in the sense that they are not epistemologically determined (they do not follow logically from previous assumptions and established knowledge), but rather were agreed upon in a community. One must be informed about these in order to become acquainted with them; one cannot deduce them from other knowledge. Convention, like the one concerning the notation of corresponding vertices, must be introduced by the teacher. The categories "necessary" and "arbitrary" are directly linked to, defined by, the epistemology of the discipline. To say that a situation is necessary requires a certain organization of the knowledge: there should be some hierarchical structuring with clear distinction among the nature of its constituting elements. From this point of view, there must be a clear understanding of the differences in the theoretical status of a definition, property, theorem, axiom, etc. and this can be contrasted with the role and status these have in a formal theory. Thus, by examining the future teachers' views on the two categories, we can say something about their epistemology and so, about their mathematics.

There is nothing epistemologically necessary about labeling the corresponding vertices of a transformed polygon. Notation is not part of the theory of transformations. Yet, future teachers seem to attribute great importance to these conventions. As previously shown, in some tasks they insist on correct labeling, since this is considered as expression of understanding how the transformation applies. The risk is in replacing (informal) deductive reasoning about a situation with visual arguments based on perceived superficial elements.

For example, in the following task, the future teacher explicitly asks to pay attention to this aspect:

> *Trace the shape you have chosen on your graph paper. Using this shape, draw a line of symmetry and find its reflection. Repeat this three times so that you have a total of 4 shapes on your graph paper.*
>
> *Remember to:*
> 1. *Identify your line of symmetry with a letter (p for example).*
> 2. *Do not forget to identify all vertices of your shape with letter: use the same one as the original shape, only adding prime (A') on every new shape.*

Similarly, in a task presented earlier, another future teacher mentions as **Instructional objective***:* *"*Students will understand the importance of consistently identifying points on a shape undergoing transformation."

While knowing about conventions is important, it remains questionable if respecting them is in itself an expression of understanding transformations. At the same time, there are significant differences among conventions: some are essential for communication in mathematics and those who wish to belong to the mathematical community must know and respect them. We could bring up here the notational conventions in drawing 3D figures in perspective, as 2D figures, or the conventions used in geometry to mark congruent angles, segments etc.

The examples illustrate the complicated relation that FT have with (some) conventions, and in a broader sense, with necessary and arbitrary in mathematics. Awareness of what is necessary and arbitrary is linked to the systemic-definitional aspect of systemic thinking, part of the Theoretical Thinking model. I will revisit this topic in the Discussion chapter.

## 4.5 Summary of the mathematical praxeologies related to geometric transformations

In this chapter, I constructed a praxeological model of future teachers' envisaged practice of teaching geometric transformations, based on an analysis of a group of future teachers' lesson plans. The model contains three sets of punctual mathematical praxeologies, each related to a different type of tasks. In this section, I summarize the praxeologies in several tables (Table 7; Table 8; Table 9; Table 10), defining them by type of tasks, technique, method and theory. This juxtaposition of the definitions helps in seeing the similarities and differences among the praxeologies and also gives an overview of the epistemic nature of the future teachers' knowledge of geometry, quite different from the view of mathematics underlying the Theoretical Thinking model.

The recurrent element in the tasks proposed by the future teachers was the grid (lined or dot) as context and frame for teaching transformations. The grid defined new geometries and this is what future teachers consider as the geometry to be taught, at least in relation with transformations.

As in any geometry, basic elements must be introduced. In dot geometry, points are dots, while lines are horizontal or vertical, but also in a 1/8$^{th}$ turn from the horizontal direction ("diagonal line"). Segments are defined by points on the same line. Angle is the relations between lines, thus angle measures are multiples of 1.8$^{th}$ turn. Two points are aligned if there is a line to which they both belong. The geometry has an inherent unit, defined by the distance on standard direction between two points. Distance between two points is defined as the relative displacement in standard directions, expressed in units. A figure is defined by its vertices. Transformations, thus, apply to points. A vector is defined by a pair of displacements in standard directions of its head and tail. Translations by a vector, as a consequence, are performed as a succession of two simple translations in standard directions with the number of units indicated by the vector. The line of reflection must be a line in this geometry. Reflection is performed by counting the units from a vertex to the reflection axis and creating a point at the same distance on opposite side of the line of reflection. Rotation can be performed with multiples of 1/8$^{th}$ turns with the center on grid point.

The square grid geometry is very similar to dot geometry; however figures are defined as compact parts of the plane that can be described by their global position on the grid. The "unit" is bi-dimensional, is the square of the grid. In consequence, transformations are performed differently: one vertex is transformed and, then, the figure redrawn by considering the relative position of the other vertices from the initial one.

In dot geometry, given that transformations are applied on vertices, one can deduce the preservation of shape. In square grid geometry, preservation of shape is the precondition for performing transformation.

As such, the square grid geometry can be envisioned as geometry of rigid shapes; while the dot geometry is the geometry of figures defined by vertices. Many of the objects are similar in these two, yet the difference in how a figure is defined has an impact on what must be proved, thus on the construction of the theory. At the level of theory, this means that mathematical definitions of transformations are replaced with images or descriptions of how a transformation should look.

The geometry future teachers envision to teach is different from Euclidean geometry; it is much simpler, but also more restrictive. Yet, as I repeatedly tried to show by proposing additional activities, in itself is not an impediment for growing mathematically; however, it would require from future teachers a solid content knowledge so to design rich activities that go beyond tasks at visualization level.

*Table 7. Praxeology of PERFORM TRANSLATION tasks*

| **Type of task** | **Technique** | **Method** | **Theory** |
|---|---|---|---|
| PT_T.<br>Perform a translation of a polygon with vertices on the grid points of a grid paper (square or rectangular). The vector of the translation is implicit or, when specified, the vector's head and tail are on the grid points as well. | PT_τ1.<br>Construct the image by transformation of each vertex and connect the vertices.<br>PT_τ2.<br>A two-steps technique:<br>a) For one arbitrary vertex, find its image by counting the required units in the required direction.<br>b) From the position of the new vertex, redraw an identical figure respecting the initial orientation of the figure on the grid. | PT_θ1.<br>• Distance on a grid is calculated by counting "units" (the "squares").<br>• There are only two orientations on the grid: vertical and horizontal.<br>• There are four directions on the grid: Down, Up, Right and Left.<br>• A translated polygon has the same shape and orientation as the original one.<br>• The unit is deduced contextually.<br>• A translation is specified by two displacements on the standard orientations or standard directions.<br>PT_θ2:<br>• A vector on a grid can be described by vertical and horizontal distance between its head and tail.<br>• Translating a point is the same as to identify the tail of a vector when the head and vertical and horizontal displacements are given. | PT_Θ1.<br>• "Definition" presented earlier:" a translation can be described as a transformation that slides every point of a shape the same distance in the same direction. During a translation the orientation of the shape does not change and the image is congruent to the original shape. A translation can occur in any direction."<br>• The grid and the geometry defined on it: the only points that exist are the grid points; segments connect two points along the grid-lines; distance can be measured vertically or horizontally by number of "units".<br>PT_Θ2:<br>• Definition of translation: In a translation by a vector $v$ a point $P$ is moved to a point $P'$ such that the vector $PP'$ has the same direction and length as the vector $v$.<br>• The dot-geometry. |

*Table 8. Praxeology of PERFORM REFLECTION and PERFORM ROTATION tasks*

| **Task** | **Techniques** | **Method** | **Theory** |
|---|---|---|---|
| PR_T.<br>Perform reflection (PR) of a polygon on graph paper (with or without coordinate system) | PR_τ1. (Counting squares)<br>Perform the following steps:<br>a) Identify image of vertex by "symmetry";<br>b) Draw figure "by symmetry" – that is, draw a figure such that the two figures together form a symmetrical shape. Count "units" till the reflection axis and then count the same number on the other side of it. | PR_θ1.<br>• Reflection axis is along grid-lines or, at most in diagonal, in case of dot-plane;<br>• Distance from reflection axis is the number of "units" from a point till the axis; | PR_Θ1.<br>• Reflection and symmetry are synonyms for the same concept;<br>• Grid or dot geometry. |
| PRo_T.<br>Perform a rotation of a figure on grid paper. | PRo_τ1.<br>Rotate each vertex and connect them, respecting the order.<br>PRo_ τ2.<br>Rotate one vertex and draw the rest of vertices by identifying their relative position with the initial vertex. | PRo_θ1.<br>• Figures are defined by their vertices.<br>• Rotation is a "turn" made of one or more quarter turns | PRo_Θ1.<br>Grid or dot geometry. |

*Table 9. Praxeology of IDENTIFY TRANSFORMATION tasks*

| **Task** | **Techniques** | **Method / Theory** |
|---|---|---|
| IT_T1.<br>Identify single transformation where intermediate figures are given. | IT_τ1**:**<br>Compare images on perceptual level and apply one of the following:<br>a) If the resulting figure has the same orientation, then the transformation is a translation;<br>b) If there is a line of symmetry in the final configuration, then the transformation is a reflection;<br>c) If the resulting figure is "turned", then the transformation is a rotation. | IT_θ1 / IT_Θ1.<br>"A translation is a type of transformation. It moves an object to a new position. The translated object is called an image and it is the exact same size and shape as the original object. (Dorling, 2010)"<br>"(…) a reflection can be identified if a "2-D shape and its image are congruent" and if "a 2-D shape and its image are of opposite orientation" (Newfoundland Department of Education, 2009, p. 102)[14]."<br>"Reflection: every point is the same distance from the axis of symmetry; the reflection has the same size as the original image. It is a flip over a line. (mathisfun.com, 2012) "<br><br>"A rotation is a transformation that is performed by 'spinning' the object around a fixed point known as the center of rotation (turning point). You can rotate your object at any degree measure, but 90° and 180° are two of the most common. Also, rotations are done counterclockwise."[15] |
| IT_T2**.**<br>Identify the order of the two transformations performed on figure F where only initial and last position is shown. Transformations are illustrated on grid paper. | IT_τ2.<br>Compare images by perception and decide on one of the following:<br>a) If the figure is in the same final orientation as the initial one, then try to identify an axis of reflection and a translation or vice versa;<br>b) If not, then try to identify (visually, expressed by units) a translation and a rotation or translation and reflection, depending on the orientation of the figure. | IT_θ2 / IT _Θ2<br>"Dot-geometry+": dot geometry adapted to allow lines not through the grid points.<br>"Definitions" of concepts along with standard images associated with them. |

[14] Department of Education, Province of Newfoundland. (2009). *Mathematics Grade 5 Interim Edition Curriculum Guide.* Retrieved from http://www.ed.gov.nl.ca/edu/k12/curriculum/guides/mathematics/gr5_math_guide.pdf

[15] http://www.gradeamathhelp.com/transformation-geometry.html

*Table 10. Praxeology of IDENTIFY PARAMETER tasks*

| **Task** | **Techniques** | **Method / Theory** |
|---|---|---|
| IP_T.<br>Identify the vector of a translation of a polygon P with vertices in grid points. | IP_τ1.<br>Identify, by perception, the transformation.<br>For translation: Count the number of units in standard directions between two points labeled with the same letter or what might be like corresponding vertices.<br>For rotation: Identify the quarter turns between corresponding letters or what might be like corresponding vertices.<br>For reflection: Identify axis of "symmetry" between corresponding letters or what might be like corresponding vertices. | IP_θ1 / IP_Θ 1.<br>• Grid geometry;<br>• Translation is described by two movements, each in standard direction;<br>• Rotation is a turn by some number of quarter angles;<br>• Reflection is flip over a line;<br>• Labeling convention (transformed vertex has the same letter, but indexed); |

# 5 DISCUSSION OF FUTURE TEACHERS' EPISTEMOLOGY

*Prospective elementary teachers do not come to teacher education feeling unprepared for teaching.*
FEIMAN-NEMSER et al., 1987

In this chapter, I will extrapolate on the ideas observed before in dot and grid geometries in order to generate a characterization of future teachers' epistemology. The analysis will focus on extracting elements that integrate into a more or less coherent set of views on content (WHAT to teach), reasons (WHY to teach it) and means for teaching (HOW to teach it).

Future teachers' view on *what* to teach is hard to separate from their views on *how* to teach it and *why* to teach it. All three aspects (what, how and why) are taken into account in the notion of *teacher's epistemology* proposed by Brousseau (1997). The concept of *personal epistemology* (Hofer & Pintrich, 1997) which has been conceptualized as an individual's view on the nature of knowledge and knowing has been shown to influence teachers' conceptions on teaching and learning, that is the *how*. In the following, I will focus on future teachers' epistemology, in the sense of Brousseau.

The chapter is organized into two main sections. The first focuses on WHAT to teach and WHY to teach it, and has two subsections. The first subsection is about the topics or concepts future teachers deem important to teach. The second subsection is about mathematical reasoning and how the future teachers understand teaching it. The second section is about future teachers' ideas about HOW to teach.

Thus, in section 5.1.1, I will present the main topics future teachers consider important for teaching and the place of transformations in this content. I will also speak about the complex relationships that exist between the content to teach and reasons and ways of teaching it.

In section 5.1.2, on reasoning, I will look at the mathematical thinking that the lessons or activities proposed by the future teachers engage from two points of view: a) the van Hiele levels of geometrical thinking, and b) the features of the Theoretical Thinking model. Here, I will also show examples where there was a discrepancy between the declared and actual thinking required by the task.

In the section 5.2, about future teachers' conceptions of HOW to teach, the main aspects considered in the analysis refer to teachers' view on the organization of the teaching: ranging from physical organization of space to ideas about assessment.

## 5.1 WHAT AND WHY TO TEACH ABOUT GEOMETRY

### 5.1.1 Which topics and why?

Assuming that future teachers' view on what to teach about geometry is influenced by official curriculum documents, it is presumed that it contains not only a list of topics but also descriptions what geometry is. In Quebec, the section on mathematics of the curriculum starts with a presentation of the discipline[16] (MELS, 2008, p. 140), which states that mathematics plays an important role not only in science and technology but also in everyday life. An echo of this view is found in the **Introduction** sections to the chapter of geometry in the future teachers' problem books.

> *"Geometry is everywhere we look; from the cars we drive and the roads and bridges they travel on to the food we eat and the buildings where the food is sold. Geometry surrounds us…"*
>
> *"Geometry is not just about angles and plane shapes. It is very broad and has many applications."*
>
> *"Geometry is important because it is found in our everyday lives."*
>
> *"By virtue of the nature of geometry, all of the activities below use manipulatives and images."*
>
> *"Geometry is about shapes, compasses, protractors and therefore, measuring angles."*

There is a shared view that geometry is in the environment, "around us". Therefore, we can learn geometry by studying what is around us, and using manipulatives. This view could be associated with natural geometry (Houdement & Kuzniak, 2003), where the "source of validation is the sensitive" (idem, p. 4) and reasoning acts on material objects through perception and instrument. While it is true that the environment is one possible source of inquiry in mathematics, and much of the learning in early years is about spatial orientation, visualization, etc., it is expected that based on this interaction with the immediate, a process of abstraction will be triggered. As stated in the official curriculum,

> *Mathematics involves abstraction. Although it is always to the teacher's advantage to refer to real-world objects and situations, he/she must nevertheless set out to examine, in the abstract, relationships between the objects or between the elements of a given situation.*[17] *(MELS, 2008, p. 140)*

In this abstract geometry, the nature of questions we ask is not the same as of questions about everyday life, nor are the objects of study everyday objects. The objects of geometry are abstract entities that exist by definition – we do not have a square in the real environment. Blurring the difference between physical reality and the abstract world of geometry limits the knowledge construction about both. In practical reality we deal with particular contexts, and often we use those particularities to perform tasks. In geometry, we want to arrive at abstract and general meanings. On the epistemic level, not separating physical reality from the abstract world of geometry changes the nature of arguments that validate or justify a claim. In case of future teachers, the "everyday-ness" of geometry often manifests itself in language: the kind of words that are used, the way they are used and the meanings that are associated with them. This aspect is related also to theoretical thinking, and I will return to it later, in section 5.2.

In the last quote from future teachers' problem books, geometry is declared to be about "shapes, instruments and measurement". Equating geometry with measurement is quite frequent among future

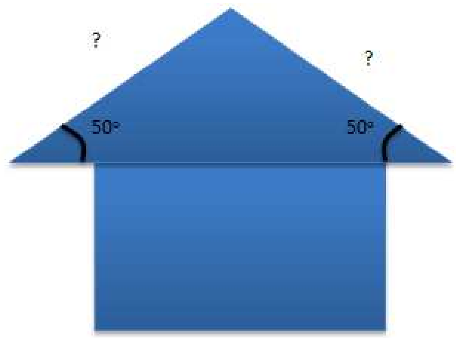
---

[16] http://www1.education.gouv.qc.ca/sections/programmeFormation/primaire/pdf/educprg2001bw/educprg2001bw.pdf

[17] The same link as in previous footnote.

teachers; often the confusion is revealed in language use. They would say “I measured the area with the formula” or “I calculated with the ruler”. But the epistemic value of information derived from measurement is different from information that is deduced in a theoretical system. One example, from a future teacher’s lesson plan, is given next. In the **Instructional objectives** it is stated: “*Through this activity, I expect students to understand the concept of how to calculate the length of the sides of a triangle*”. Then, the task is specified as follows:

> *Principal Simpson is getting a new roof for the school. The base of the roof is 20 ft long and two of the angels* (sic!) *are 50°. The principal would like you to draw the plans for his roof. For the diagram replace 20ft with 20cm. What is the length of the two top sides of the roof as presented below.*




Next, in the expected solution, the steps to solve the problem are outlined as follows:

> *The ideal expected solution is the following:*
> - *Students will measure out a 20 cm line.*
> - *Using a protractor they will measure the two bottom angels to measure 50°.*
> - *Students will then draw the lines until they intersect at the top point.*
> - *Students will then measure the length of both sides.*
> - *Students will find that both sides are 15.25cm long.*

The word *calculate* is used with the meaning of *measure* in this case; the distinction between the two is not recognized. I will return to this example later when discussing the language employed by future teachers.

Table 11 lists topics considered important to be taught in primary school in geometry. The list was compiled from the recurrent topics in the lesson plans from the geometry chapter in future teachers’ problem books.

*Table 11. Geometry topics considered important to be taught in primary school*

| **Things to know** | **Things to be able to do** |
|---|---|
| “Points, lines, planes, and axis of symmetry” | “differentiate shapes by properties” |
| “Acute angles, obtuse angles, as well as right angles” | “translate polygons, rotate polygons, and reflect polygons” |
| “terminology”<br><br>“precise geometric terms students are assumed to | “classify triangles” |

| | |
|---|---|
| have learned previously” | |
| “symmetry” | “measure area and perimeter”; “calculate area and perimeter” |
| “tessellations” | “use formulas that they will learn (area), or asked to produce a frieze using reflection” |
| “basic concepts of geometry such as angle measurements and geometric shapes” | “calculate interior angles of polygons” |
| “Pythagorean theorem” | “labeling the correct parts to a shape” |

The topics are closely related to what is prescribed by the provincial curriculum. This could be expected, given that the problem book template provided by the instructor of the methods course suggested consulting the official curriculum materials.

Given the prevalent view of geometry as being “all around us”, when justifying the reasons for teaching those topics of geometry, I was expecting to see arguments related to this view. However, the argument was, mostly: “*It is important for students to learn about triangles because it is required by the Quebec Education Program*”.

What is the place of transformations in the curriculum, as seen by future teachers? Why should students know about them? Why should we study transformations? The arguments future teachers bring are very similar to the one quoted above: we study transformations, because this topic is prescribed by the curriculum. Some quotes from future teachers about this issue are:

> *It is important for students to learn about transformations because it is required by the Quebec Education Program. Students are required to learn about reflection, rotation, and translations (QEP, 2012, p. 153). Students will be required to identify as well as produce examples of reflections, and translations.*
>
> *Other activities are related to the Quebec curriculum (…) an activity also involves effectuating translations of geometric figures using graph paper (QEP, p.152)*

At this point, at least two observations are in order. First, it seems future teachers have no particular opinion about the reasons for studying transformations. A possible explanation of such state of affairs could be the limited horizon knowledge (Ball, Thames, & Phelps, 2008). Horizon knowledge is considered as part of subject matter knowledge and refers to "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (idem, p. 403). Lack of personal knowledge about the importance and place of this topic in overall geometrical knowledge to be developed in school years (including secondary school) also implies a lack of vision on what exactly is important to teach about the topic. Without being clear about the core knowledge to be built, future teachers rely on the descriptions found in curricular materials, mainly in the Progression of Learning in Elementary School (MELS, 2016) and sources, such as websites and activity books. For example, in the Quebec Education Program (MELS, 2008), it is specified that students should be “observing and producing (grids, tracing paper) frieze patterns by means of reflections: reflection, line of reflection”. Without clear examples on *what for* and *how*, combined with lack of personal knowledge about the topic, future teachers are left to propose activities that do exactly, and nothing more than, what is suggested: "produce and observe", where *observe* often means *perceive*. In most cases, this translates into tasks at the visualization level. And that brings us to the competencies.

### 5.1.2 What reasoning?

With the purpose of producing lessons aligned with the curriculum, future teachers also want to promote the competencies specified in the provincial elementary curriculum. The Quebec mathematics curriculum is structured around three "competencies": solving situational problems, reasoning by using mathematical concepts and processes and communicate using mathematical language[18] (MELS, 2008, p. 141).

In the particular case of geometry, future teachers are aware of curricular specifications that foresee elementary students' progression from concrete to abstract. The following excerpt is from the Progression of Learning in Elementary School (MELS, 2016), chapter on geometry:

> *Throughout elementary school, by participating in activities and manipulating objects, students acquire the vocabulary of geometry and learn to get their bearings in space, identify plane figures and solids, describe categories of figures and observe their properties. Geometry in elementary school focuses on two-dimensional (plane) and three-dimensional figures and on key concepts, such as the ability to locate objects in space and observe their geometric and topological properties. Knowledge of vocabulary is not enough; the words must be closely tied to precise concepts such as shape, similarity, dissimilarity, congruency and symmetry. Thus, the use of varied activities and a wide range of objects is essential for students to develop spatial sense and geometric thought. This will allow students to progress from the concrete to the abstract, first by manipulating and observing objects, then by making various representations, and finally by creating mental images of figures and their properties.*
>
> *The ability to discern and recognize the properties of a geometric object or a category of objects must be developed before students can learn about the relationships among elements in a figure or among distinct figures. It is also required in order to develop the ability to identify new properties and use known or new properties in problem solving.*[19]

#### *5.1.2.1 Levels of the expected reasoning*

The last paragraph can be aligned with progressions along the van Hiele levels of geometric thinking, as it was already mentioned in Chapter 2. By this, I argue that future teachers are aware of the fact that it is expected from elementary students to advance from visualization level to the informal reasoning level of geometrical thinking by the end of primary school, which is grade 6 in Quebec. This is the main reason why it is important to look into the tasks they conceive for classroom use and analyze, on one hand, what level of thinking they entail and, on other, what purpose the future teachers say the activity would serve.

---

[18] http://www1.education.gouv.qc.ca/sections/programmeformation/primaire/index_en.asp?page=educprg2001h

[19] http://www1.education.gouv.qc.ca/progressionPrimaire/mathematique/index_en.asp?page=geometry

In chapter 4, in the detailed analysis of lessons on transformations, I already mentioned how most of the tasks remain at visualization level. Here, I will look, in general terms, at other activities proposed in geometry and identify the level of thinking those activities require. At the same time, I will take note of what is *said* the level to be in an attempt to establish the meanings future teachers associate with those expressions.

Establishing categories is a recurrent type of activity in the geometry section of the Problem Books. This is expected, considering that for grouping one must focus on characteristics of the figures and find commonalities and differences among them. This type of activity is quite common for progressing from visualization to analysis level. Along with categorization tasks, it is also common to find activities about a figure's membership in a category, or identifying the type of a certain geometrical object (for example, for polygons or angles). What level of geometrical thinking this requires depends on several aspects, among which I will mention: objects under analysis (for example, prototypical or not), complexity of criterion, tools allowed in performing verifications, etc.

It can be argued that if no tool is given and the students must decide the type of an object just from a drawing, the task remains at visualization level. For example, in case of the task in Figure 31, students must identify the type of angle, yet no tool is provided.

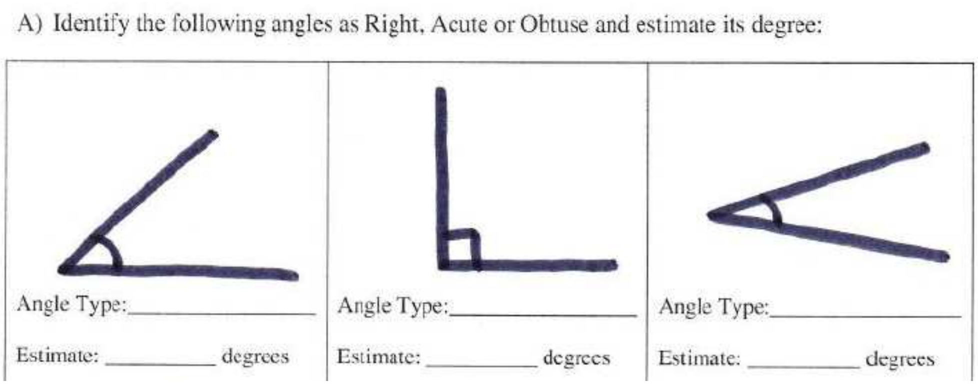



*Figure 31. A task about types of angles from a future teacher's problem book.*

The future teacher's stated instructional objectives include: "students will discover the properties of angles more in depth; student's knowledge of angles will expand."

There is a lack of articulation between what is intended and the level of activity – but, is this recognized by the future teacher? Or, as personally I think is the case, this is what "discovering properties" and "expanding knowledge" *mean* for the future teacher?

In a follow-up task, "Classifying various triangles" (Figure 32), the future teacher proposes another classification activity. The **Instructional objective** section contains the description "*Through this task, students will reason with the concept of angles at a higher cognitive level because they will learn to classify triangles not only by their angles, but by the length of their sides as well.*"

1. Classify the following triangles as Equilateral, Scalene or Isosceles and then classify them as Acute, Obtuse, or Right. Note that a triangle can be classified by both angle and length.

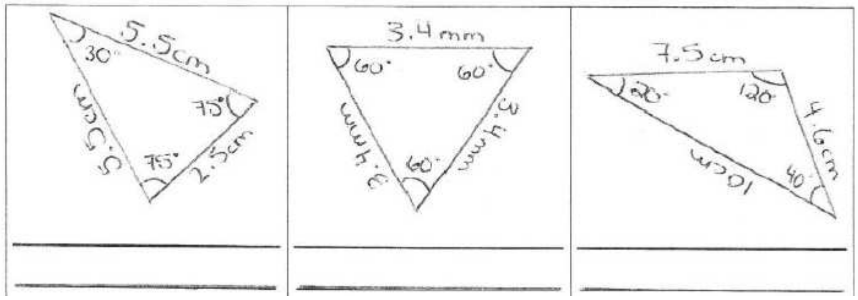



*Figure 32. A triangles classification tasks from a future teacher's problem book.*

It is expected from the student to do two, separate, categorizations on the same worksheet. But students were not expected to see possible relations between the two types of classifications. Therefore the task remained a simple naming. The expression "reasoning at higher conceptual level" is operationalized very differently from its usage in educational theory. At the same time, one can imagine different information (mainly, less) being provided which might have contributed to raise the task to the informal deduction level.

Another category of tasks refer to analysis of shapes to identify their properties. Once again, theoretically this is a type of activity supporting the transition from visualization to analysis level. The following specifications are part of the **Instructional objectives** section of a lesson plan: "*Through this activity, I expect students to discover the properties of different shapes; through this activity, I expect students to apply their knowledge of geometry to solve a problem.*"

The task requires replacing, in the configuration shown in Figure 33, the "Bouncy slide" games with some other "games". Students would be provided with cut-out shapes, so they can find different new configurations.


*Figure 33. A geometric game invented by a future teacher for her problem book.*

Most probably, students will enjoy trying to fit figures and combine them; however, will the activity help in *discovering properties*? They can *see* that some figures fit together and cover the same area, however it is questionable that students will (even) observe, let discover, some properties if the focus is not explicitly on this aspect.

From the point of view of level of thinking required by the task, it is still visualization: the solution is decided by perception. Yet, in the description of the lesson plan, the verb "*discover"* is used in a way that, I think, relates to *discovery learning* (Bruner, 1961). At the core of this learning theory is problem solving where the learner builds on their previous knowledge and current experience to discover new facts and relationship.

In other words, the future teachers' intention to follow the ideals of "child-centered education" in teaching children geometry is laudable, yet the way they design their tasks makes achieving that purpose unattainable.

Besides the peculiar way the future teachers thought they can help children *discover* geometric concepts and their properties, the tasks intended to help children *understand* something also raise question about the meaning they attach to this word.

Here is an example. The **Instructional objectives** of the activity were stated as: "*understand the concept of perimeter; understand the concept of area".* In the argumentation on how the activity helps achieving the instructional goal, the future teacher wrote:

> *The activity will provide students a basic graph with square grids and they will be asked to determine the perimeter of the shaded area by using their basic knowledge of the definition of what a perimeter is. The question will show the student a more visual representation of what a perimeter is.*

Then, the task was formulated as shown in Figure 34. Note the representation of the unit of length in the question about the perimeter.

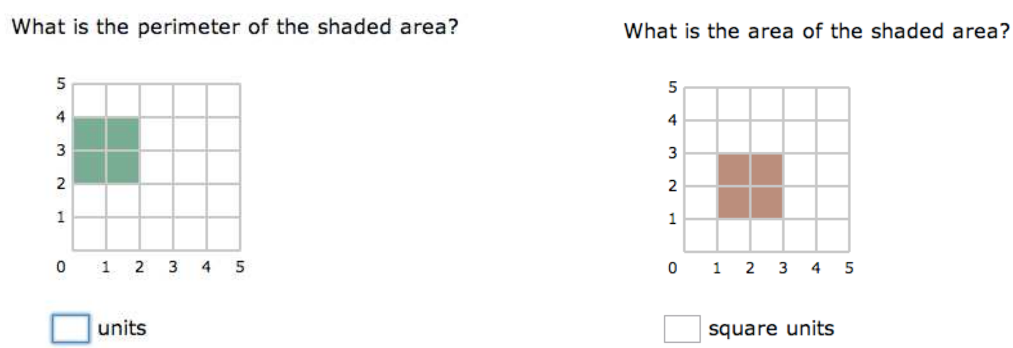



*Figure 34. A task intended to help students understand the concepts of area and perimeter.*

What does the future teacher mean by "*a more visual representation of what a perimeter is"?* And in what way the task helps to "*understand*" the concept of perimeter?

The treatment of the area task in the future teacher's **Expected solution** suggests some answers.

*The answer is four square units. But a student might be confused by the fact that the square ends on 3 on both sides. A student might add the two numbers or maybe even multiply them arriving at the answer that is incorrect. Although, it is important to see that if the student does multiply 3 and 3, there might be already an emerging understanding of area.*

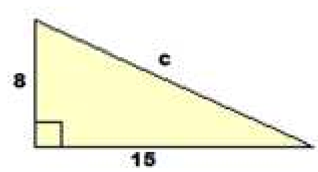
It is suggested in the last sentence that multiplying the two side lengths would be a sign of *emerging understanding* area. This leads to the idea that *understanding* the area / perimeter means to know a formula for calculating it.

It seems that there are several interpretations associated to the word *understanding* among future teachers. For example, in a different context the future teacher explains: "*In this activity, students will be given a list of geometric properties and figures that they must find in their immediate environment. This will allow them to understand geometric properties and see these properties in a real life context – their classroom.*" It is suggested that understanding geometrical properties is about "seeing" them (visually identify) in the environment; and it is enough to identify objects having certain properties (given on a list) to know what the property means. In another statement, "*Understanding that if a triangle has a 90 degree angle, this means that it is a right triangle*", *to understand* means just to know a "test" for deciding if a given object belongs to a certain category.[20] Or "*understanding the concept of calculating the length"* (mentioned in the first activity presented in this chapter) which means measuring on scale with a ruler.

*Reasoning* is yet another of those words teachers use when describing the activities, but its meaning is highly personalized. Here is an example.

***Instructional objectives of the activity:*** *The students will use mathematical reasoning while "deriving laws, rules and properties" (Q.E.P.) by becoming familiar with Pythagoras' Theorem.*

***Tasks for students:*** *Find the length of side C of the triangle.*




In this interpretation, *reasoning* means to correctly apply the theorem. Certainly, the task remains a direct application of a formula.

In conclusion, the level of geometric thinking required by the tasks from the problem books remain at visualization level. Understanding, reasoning often mean being able to visually recognize something, being able to count units, or to apply a formula or a "test".

At the same time, quite a few of the tasks could be modified to give an opportunity for moving on to a higher level of reasoning. Then, the question is: what do future teachers need in order to modify those

[20] Similarly as the "vertical line test" for functions in pre-calculus.

tasks accordingly? This is the very same question I posed in relation with transformation tasks, where I illustrated that it is possible to formulate rich tasks even in presence of constraints.

*I consider that there is a need to foster future teachers' content knowledge. Didactical expertise can be reached only after deep mathematical content knowledge; yet, in our case, it seems that future teachers' content knowledge is not sufficient to sustain their learning in a methods course.*

Moreover, there is a need to challenge future teachers' conceptions of educational terms such as *understanding*, *reasoning, discovery* or *expanding knowledge.* They are interpreted differently from how they are employed in educational theories. As a consequence, we are in a situation where at the level of discourse the meaning conveyed does not align with the original meanings of those words. In terms of the model of theoretical thinking - applied this time to theories of learning-, one aspect of analytical thinking, the use of specialized vocabulary, is lacking. Even though future teachers assert similar things, at the level of words, the meaning of what is said is at a personal level. We witness here a process of shifting meanings.

It must be said that the shift is part of a phenomenon at larger scale. We witness it also in other environments, like internet websites maintained by teachers to provide materials for other teachers[21], activity books published with the banner "In line with the Progression of Learning"[22], websites recommended by schoolboards[23] and some educational sites even integrated into teaching in schools[24]. In these, a concept is 'studied' through a set of exercises and problems; and, often they include an evaluation section consisting of identical problems. Then, passing the evaluation is considered that you mastered the concept and you can go on to something else. Consequently, understanding the concept is interpreted as the ability to solve a set of identical problems with those already seen. Meanings are very much shaped by what future teachers encounter in these resources. While such process is also normal, it needs to pass by an "inner filter" of the individual. Once more, I consider that content knowledge would be the primary element future teachers could use in the assessment of those materials. It would be also necessary, future teacher to have more examples on how concepts they know from educational theories would "look" when applied to certain domain.

#### *5.1.2.2 Looking at the tasks from the point of view of the Theoretical Thinking model*

In this section, I will revisit some ideas already advanced in Chapter 4, during the analysis of tasks. The structure for this discussion will follow the three features of theoretical thinking: reflective, systemic and analytic.

##### 5.1.2.2.1 Reflective thinking

As described in the conceptual framework section, reflective thinking refers to an individual's ability to reflect, investigate and extend ideas. I consider this thinking present when the future teacher thinks about the orchestration of different lessons, reflects on the continuity and the articulation of the ideas among them. In my analysis, only on two occasions have I came across a situation where the lesson plans were reported as being sequential in some sense, thus representing an intention to connect the lessons. Yet, what future teachers consider as being related is not necessarily related in the above sense.

[21] https://www.superteacherworksheets.com/
[22] https://www.cheneliere.ca/9953-livre-sommets-1er-cycle-2e-secondaire-.html
[23] http://csdm.ca/nouvelles/alloprof-videos-educatives/
[24] https://www.netmath.ca/fr-qc/

For example, one future teacher commented in her introduction to the geometry chapter of her problem book: "*The problems are all related in that they use regular polygons as a basis for exploration of their spatial qualities."* In this case, the connection is quite superficial since it is not about following up on previously introduced ideas about polygons; these are just the particular figures that appear in the task specification. In the other case, however, the succession of two lessons was thought in a way that the first lesson created a basis for the second one, by introducing the notions needed in the second.

The fact that there was generally no explicit focus on linking the activities in the problem books might be also explained by the fact that future teachers developed their activities along the semester and selected some among those for the problem book.

At the same time, reflective thinking could manifest itself also at a finer granularity: looking back at the solutions and proposing extension of the problems, in the frame of the same lesson. Future teachers were invited to extend their problems, yet those who proposed more problems mainly remained in the same context. It is not customary to think of extending ideas, connecting to other topics or domains. A potential reason might be the organization one encounters in textbooks and, in general, materials available to the future teachers. Usually, content is structured into units, sometimes changing from a topic of geometry to a topic of statistics for example (that is the case in grade 8 textbook where sectors of disks/circles are studied as preparation for constructing a pie-chart, (Boivin, 2006)). If there is a connection at all, this is not at the level of *mathematical* ideas, but at level of utility of certain concepts on treating other topics. In a certain sense, such approach also reduces the study of the concept to a section that is useful in the immediate future instead of a thorough treatment of it.

##### 5.1.2.2.2 Systemic thinking

The second feature of theoretical thinking is the systemic thinking which refers to the habit of thinking about system of concepts. In the frame of systemic thinking, three aspects are differentiated: a) meanings are established by definitions *(systemic-definitional thinking*); b) the validity of statements must be derived from knowledge already established within the system (*systemic-based on proofs*); and c) awareness of the hypothetical nature of knowledge. However, here I will refer only to the first two components for which there was enough data in the problem books.

If we are to describe in general terms the means future teachers used to introduce concepts in their lesson plans, it would be *metaphors*. Gestures and verbs from everyday language were associated with the concepts. As mentioned, in the process of understanding a concept, students often create a concept image, described by Vinner as "something non-verbal associated in our mind with the concept name" (Vinner, 1991, p. 68). Problems start when the concept image is used for solving tasks and becomes the concept in the mind of a student. In the case of transformations, the typical metaphors ("turn", "flip", and "slide") are used to introduce the concepts, and also to solve problems. Future teachers also rely on these images as we can see from the solutions given in the expected solutions sections. Yet, these metaphors do not cover the *mathematical* meaning of the concept and future teachers should work specifically to underline the differences between the two (concept image and concept). On the contrary, it seems, at least in the case of transformations, that the metaphors are institutionalized by the visuals future teachers create for their students, although also using the mathematical term. One of the detectable consequences of this is the fact that parameters of transformations are ignored, and, therefore, tasks remain at the visualization level. The metaphors create the impression that some new concepts are introduced while, in fact, the meaning is already captured by the everyday word.

Transformations are just one of the topics where the concepts are replaced by concept images. In some cases, the concept image is a prototypical representation of the concept where some essential feature of the concept is implicit.

The angle concept is another example of a concept that is introduced by an image: that of open arms. The passage from the physical image of the angle to the mathematical concept is not made – it is assumed a natural passage, with no issues to clarify. Yet, it is not at all clear how the two arms become two rays starting from the same vertex. In fact, what we observe from the lesson plans is that angles quickly become numbers through equating them with their measure. As a consequence, the meaning of the concept of angle depends on the context: sometimes it is a number of degrees (a measure) and sometimes it is a geometrical object. The following “definition” used by one of the future teachers is quite common: “right angles are 90 degrees” – a consequence of which is seen in the difficulty of creating a right angle without a square set or protractor, for example by folding a paper.

Beyond the examples invoked above, the question is: do future teacher *know* what definition in mathematics is? In the methods course, they were provided with formal definitions of all concepts that were used in the course. No explicit discussion has been carried out about the characteristics of mathematical definitions (for example, imperative features as mentioned in (Zaslavsky & Shir, 2005)); however, for all concepts they used in the problem books, they were given definitions. It is not necessary to have a class only on the characteristics of definitions; however future teachers should be aware of the specific role definitions play in the mathematical theory.

Yet, the lack of systemic definitional thinking has, as consequence, a variable meaning of a concept: one must be aware of the particular context in order to understand what it meant by the concept. As argued before, there is a consequence to this too: things are valid locally; they can’t be seen outside of the particular context.

The second aspect of systemic thinking refers to the fact that new truths are derived from ones already proven in the system; therefore, knowledge is organized into a hierarchy. For the purpose of discussion, I will consider here the case of “definitions” future teachers employed in the lesson plans on geometry. As illustrated by several examples for transformations, what is considered as definition, in fact, is a bundle of definition and properties, with no distinction between the two. In other words, the fact that some properties are *deduced* from definition and other, already proven, truths is not recognized.

By considering all elements with the same theoretical status, the knowledge becomes a flat structure, with elements having no or little connection among them.

#### 5.1.2.2.3 Analytic thinking

As for analytic thinking, I will look into two of its aspects: a) being sensitive to a specialized vocabulary and b) being sensitive to the structure and logic of mathematical language.

From the examples already presented in chapters 4 and 5, it follows that future teachers do not have a complete mathematical vocabulary. Instead, they rely on a mixture of everyday terms and mathematical ones. The language they use is telling us about their epistemology, about the nature of the objects they work with in mathematics lessons. We already mentioned the case of transformations and the replacement of mathematical terms with words from everyday language. We also mentioned the standard directions in the grid geometry, linked to a too-concrete understanding of the grid as a paper with innate directions (down, up, right, left). Or, on the contrary, they employ terms that belong to the mathematical vocabulary, yet the meaning associated to it remains linked to the everyday word.

Furthermore, I will focus on two cases here.

The first focuses on the mixture of words they use in defining the objects of mathematics or talking about these objects. Some examples, in this sense: (the shape) "what family it belongs to"; (topic of the lesson) "parts of shapes (vertices, edges)"; "quality of shapes"; "these *transformations are moving"; "*the slant and the height of trapezoids"; "A shape has symmetry when a line can be drawn that splits the shape exactly in two"; "a line of symmetry can go in any direction"…

All the everyday terms employed are coming with certain (everyday) connotations that will make it more difficult to create a mathematical understanding of the concept. The epistemic message they convey is: mathematics is just a fancy way of talking…

Second, I'll briefly present the case where specialized terms are used in wrong context. In one lesson plan, the future teacher proposed to introduce basic concepts prior to other activities. For this purpose, she introduced "definitions" for point, line etc. One of the "definitions" she used for *plane*, as taken from a website (mathisfun.com) is: "There are only two dimensions on a plane; e.g., length and height, x and y "[25] (this is not marked as definition on the website). The issue here is that she could not see the flaws of this description. Terms like "length" and "height" have a specific meaning in mathematics that links their usage to certain contexts. In the same lesson plan, she used the terms "vertex" and "point" as synonyms, which is not the case. In fact, this suggests that the mathematical terms are not perceived with all the specific meanings they stand for, but are used in some reduced way that fits the particular situation.

In the future teachers' use of language, we also encounter situations where they replace concepts with processes. They would say "the concept of translating, rotating and reflecting geometric figures". Once more, this suggests that transformation is a process, not yet abstracted into a concept.

One can imagine that the descriptions the future teachers include in their lesson plans are the ones they would use in the classroom. The overall structure of their talk is quite informal, like in the following example: *"In addition to learning the type of angles, the students will learn the parts of an angle (the vertex and the arms) in order to understand where to look (the angle between the two arms joined at the vertex) to know what type of angle it is."*

The frontier between everyday objects vs. mathematical objects is non-existent; references to objects, images from the physical environment are justified if they are to give an image that resembles the mathematical concept. In terms of epistemology, it suggests that it is possible to acquire "mathematical" knowledge through direct observation of everyday objects or through metaphors attached to verbs, gestures. No generalized meanings are to be created; in consequence, mathematics is just another form of common sense knowledge. Bernstein (1999) used the term *horizontal discourse* to designate a knowledge structure that is segmentally organized and differentiated, meaning, that is highly context dependent and specific.

[25] https://www.mathsisfun.com/geometry/plane.html

The way in which the future teachers engage with mathematical discourse is strongly influenced by their past experience as students in school, and their current experience as tutors. It is also constantly reinforced by what they encounter in their searches on Internet, in many books and in educational videos. From this point of view, future teachers are product and reproducers of a general mathematical culture present in the society.

## 5.2 HOW TO TEACH

In this section I will present and briefly discuss future teachers' conceptions of learning and teaching. Research on personal epistemology advanced the idea of a relation between teachers' views about the nature of knowledge and its acquisition and their conceptions on learning and teaching. The conceptions are defining a teacher's classroom behavior in the large sense: starting from selection of tasks, ways of conducting the class discussions and forms of evaluation (Gonzales Thompson (1984); Brown & Rose (1995); Waeytens, Lens, & Vandenberghe (2002)). Future teachers' geometry, as outlined in chapter 4, along with their beliefs about geometry, briefly characterized at the beginning of this chapter, constitute their views about knowledge ("epistemology" in the usual, philosophical sense). Elements of their conceptions about learning and teaching were mentioned in the task analysis (for example, in the discussion of didactical choices) and some were also highlighted above (for example, hands-on manipulations as way to gather observations about shapes). In the following, I would like to complete these by some new ones identified from the **Introduction** and **Do's and Don'ts** section of their problem books. The ideas enumerated below are not direct citations from their descriptions, but reformulations of recurrent comments. These were grouped by themes that emerged while compiling them.

On the teaching, some of the common ideas among future teachers were:

*Teachers should:*

a) With regard to the support of students during classroom interaction
   - appreciate students' effort (praise them, say encouraging words, never critique);
   - value students' ideas and strategies;
   - specify exactly what is expected from students in the solving of tasks.
b) Concerning the level of tasks selected for classroom use
   - give problems that students are able to solve and make sense of;
   - adapt the teaching style to the learning style of students;
   - adapt problems to the ability level of students.
c) Regarding the context of activities chosen for classroom
   - build on problems from everyday life for they promote authentic learning;
   - "make math alive" by using real life context and problems;
   - teach things that benefit students in life.
d) Regarding the overall organization of teaching
   - organize problems (in one lesson) and lessons (in a sequence of teaching) from simple to complex.

The above principles, when manifested in concrete teacher's actions, are often far from what the general formulation of the ideas suggests. I'll comment on two of them: supporting students during classroom interaction and the context of activities.

One form of support is praising and avoiding critique. In classroom practice, "praising students for effort" replaces an objective review of the student's solution. Here the word "objective" means evaluating *the solution inside* of mathematics. And, *inside* of mathematics is understood as an evaluation in the context of the mathematical theory where there are precise meanings to words, there is a systematic organization of concepts that gives rise to rules and theorems. Evaluation of the *solution* means that the focus of discussion is the solution and not the student. Teachers should discuss students' solutions; erroneous approaches because this is the way to bring awareness about certain *mathematical* aspects that were amiss. When teachers prefer to avoid this and simply "praise for effort", in fact, they take away from students a learning opportunity. In line with this idea, the question is how exactly future teachers *value students' ideas and strategies*.

In the lesson plans, especially in the **Expected solution**, future teachers suggest listening to students, acknowledging their efforts and then going on and present their own solution on the board. It seems that *valuing* ideas is understood as listening to students, letting them say what they did and not building on those ideas and orchestrating a discussion. This aspect is in line what research also reported on teachers' difficulties with conducting classroom discussion (for example, (Speer & Wagner, 2009).

A second comment concerns the context of activities. The reference to real-life problems is in line with the suggestions from the curricular documents where situational problems are to be set in real-life context. In analyzing the tasks proposed by future students, I found that "real-life" centered problems often require more reading skills than problem solving skills. Also, it often was the case that these are "real-life" by referring to objects from the environment, yet the question set up is very unrealistic and artificial. At the same time, there are many occasions where one can ask *mathematical* questions that make sense about a situation / object observed in real life. It is enough to consider the avalanche of information coming to us every single day, through advertising, news channels, and journals. It takes training the eye and the mind to assess them.

As a closing idea for these comments, I ask again what is needed for future teachers to build on students' solutions and see the opportunities to talk mathematics. I think the answer to the first part of the question is more straightforward: theoretical thinking. Seeing mathematics as a theory with its assumptions, and having an internal logic for deriving new results allows 'taking distance' from the student, classroom context and look at the solution only. As for the second part, having an inquisitive and critical view might be a starting point, yet this must be paired with a willingness to look for mathematical tools to treat the situations that were observed and questioned.

Future teachers also expressed their views of an ideal lesson.

*Lesson/activity should:*

- relate to students' interests
- be enjoyable, fun (by catchy titles to activities, by choosing contexts familiar to students);
- leave room to creativity and multiple solutions;

- involve the use of hands-on materials and concepts should be "act out", when possible (for example, transformations);

A comment about multiple solutions and creativity is due here. These seem to be highly appreciated, yet in the lesson plans the only activities where multiple solutions were accepted were on "shape hunting" or drawing examples of a certain type of triangle. As I tried to illustrate by the tasks I proposed in chapter 4, creativity is possible and tasks can leave space for multiple solutions. Yet, this also requires the ability to see the relations between features of the task and potential ways of dealing with it. It requires, once more, taking distance from the *actual* problem (in its concrete formulation), and see its structure. It requires "profound understanding of fundamental mathematics" in terms of Ma (1999).

Some further ideas, relevant to the organization of the teaching material or the teaching itself, are presented below grouped by recurrent themes:

a) Solving problems

- Give "tricks" to help students carry out the tasks;
- Before giving a task, the concepts needed for solving the task should be reviewed;
- The solution should be modeled by the teacher;

b) Source of students' difficulties and ways to react

- Students' difficulties are with labeling "*parts of shapes*" and, consequently, applying the formula since it is not clear which "part of the shape" correspond to what in the formula;
- Student can overcome difficulty by repeating the same kind of problem in different context;
- Teachers must tell the student on how to proceed if in difficulty.

c) Learning

- Learning a concept is knowing the formula;
- Students learn from each other during group activities;

d) Understanding and knowing

- Understanding (a transformation) is being able to perform it;
- Gestures help to remember concepts;
- Students understand a concept if they have a visual representation of it;
- Students understood a concept if they obtained the correct answer to a problem related to the concept;

e) Assessment

- Assessment should be almost identical to what was given in class.
- Students should not be assessed on what has not been taught explicitly in class.

An overall impression created by the above conceptions is that learning and knowing mathematics is about knowing procedures and knowing when and how to use them. All seems to revolve around it, it dictates all from what and how teacher should proceed in teaching problems solving till how to evaluate. One knows mathematics if one can carry out the procedures properly; therefore the teachers must ensure that the procedure is learnt. It suggest that mathematics is a toolbox where tools are specific items (formulas, recipes on dealing with problem types, etc.) to deal with specific issues.

Personally, I consider that it is in their conceptions about learning that their view of the nature of mathematics is best revealed. The contextual nature of the meanings of mathematical concepts leads to a set of locally valid ‘practical’ ‘theories’ which require that teaching and assessment be set exactly in the same context, in an almost identical situation.

# 6 CONCLUSIONS AND RECOMMENDATIONS

*Social access without epistemic access is merely to reproduce social inequality.*
LEESA WHEELAHAN, 2010

The thesis set out to construct a model future teachers' views of an ideal teaching of geometric transformations, and thereby gain an insight into what they consider important to teach in geometry, why and how, and what level of competency they aim at; in brief – an insight into the *future teachers' epistemology* . This goal was achieved through (a) an analysis of a group of future teachers' problem books, structured by means of the praxeology framework from the Anthropological Theory of the Didactic (Chevallard, 1999), and (b) an assessment of the level and nature of the knowledge planned to be taught in terms of the van Hiele levels (van Hiele P. , 1959) and the Theoretical Thinking model (Sierpinska, Bobos, & Pruncut, 2011).

The contribution of this research is in the detailed examples of future teachers' knowledge, in the specification of the limitations and potential obstacles this very knowledge creates and in the projection of its possible, long-term consequences. By clearly specifying the critical points where future teachers' knowledge may prove insufficient or inadequate, along with a detailed description of this knowledge, we could be able to see ways of improvement: concrete small elements that the preparation of teachers to teach mathematics could provide.

Rather than at the level of discourse, it is in the details that we can see the discrepancies between the future teachers' and a teacher educator's understanding of the tasks, their purpose, the didactics and pedagogy. Describing the future teachers' knowledge using general epistemological categories would probably miss out its complexity and leave us without tools to tackle the issues we sense the quality of this knowledge can raise.

But, yes, overall, we can invoke such general categories and state that future teachers' epistemology can be described as a horizontal discourse (Bernstein, 1996), characterized as 'local, segmented, context dependent, tacit, multi-layered, often contradictory across contexts but not within contexts' (idem, p. 170). This can be contrasted with the structure of academic disciplines that are vertical discourses. A vertical discourse 'takes the form of a coherent, explicit, systematically principled structure, hierarchically organized, or it takes the form of a series of specialized languages' (idem, p. 171). Then, the question becomes where and how future teachers' epistemology can prove limiting in the learning of their students.

Knowledge creates a world and a way of being in it through the problems it deals with, by the questions it raises, their nature, by the way it answers them, by what it considers as valid argument, etc. Ultimately, this is why identifying and understanding knowledge is essential: it has an impact on our relation with particular domains (such as mathematics), but also, by extension, on our relation with the social and physical environment. In the case of the studied group of future teachers' planned instructional activities, the mathematical knowledge allowed by them is rather not shared by teacher educators or by mathematicians and has little to do with the theoretical knowledge that mathematics is.

The aim of theoretical thinking is to allow us to detach from experience and reflect on it in order to understand not only the experience, but also the possible courses of action. In a succinct way, it allows understanding why we proceed as we do in a certain situation: it is interested in the fundamental assumptions underlying actions and the consequences of those assumptions. In order to create a distance from the practical situation at hand, we need generalized meanings, stated through definitions, and construct our understanding on them. We need to organize knowledge into a theory.

While there is a procedural aspect to mathematics, where we perform algorithms, we apply criteria, etc., in order to understand why and to what extent those apply, we must understand the underlying structure defined by concepts and relations among them.

Let us consider an example from elementary mathematics in order to clarify. The divisibility criterion by 3 states: "If the sum of digits of the number is divisible by 3 then the number is divisible by 3". In order to understand why this is true, one must understand the rules we adopted for writing numbers, the positional base-ten writing. If the link between the writing rules and the criterion is fully understood, one can ask, and answer: a) are there other ways to write numbers? b) will the same criterion work in this other way of writing?

Future teachers need theoretical thinking so that they can take ownership of their knowledge and, therefore, their teaching. Theoretical thinking is a basis for being critical and creative.

Future teachers must have the competency to foster the development of theoretical thinking in their students, so that their students have opportunities in the future to grow in the direction they wish. In the following, I'll bring some arguments on the crucial role teachers have in this process.

In the past decade, sociologists of education, especially those adhering to social realism, raised concerns about the place accorded to theoretical knowledge in the curriculum. In her book, *Why knowledge matters in curriculum,* Wheelahan (2010) argues that "access to abstract theoretical knowledge is an issue of distributional justice" and that "the principal goal of education should be to provide students with access to knowledge" (idem, p. 4); in other words – access to systems of meanings associated with disciplinary knowledge. Theoretical knowledge is a socially powerful knowledge; in her words, "unless students have access to theoretical knowledge they are denied the necessary means to participate in 'society's conversation'" (idem, p. 4).

Of course, ideally, future teachers should have been introduced to theoretical thinking and knowledge already as elementary school students. They have not. Yet, education at primary level has considerable importance: this is the moment where students should be made aware of different codes and 'languages' that exist in contrast with the 'language' they use in everyday conversations. In the following quote, Daniels (2001) draws attention to the consequences of failing to do so: "*Following Bernstein's (1999b) distinction between horizontal and vertical discourses should make us wary of providing learners with experiences which lead to their positioning within what he terms a segmented horizontal discourse, whereby participants are unlikely to access the analytical power or certainly the 'cultural capital' of scientific concepts."* p. 116)

If initiation into theoretical thinking does not happen even at the university level, and future teachers' knowledge remains a horizontal discourse, with its context-bound meanings and local coherence, the question is: how will it be possible for them to facilitate their students' acquisition of a vertical discourse, which is in terms of Bernstein theoretical knowledge? While it is not expected at primary level to acquire mathematics as theory, instruction in primary school must prepare students for doing so later on. A way to proceed is for future teachers to clearly separate between the world of mathematics and physical reality and they can do this by enhancing theoretical thinking. This can happen only if future teachers are aware of this distinction and they convey it through their classroom talk and interaction. Correct presentation of the concepts, and enforcing precise and meaningful mathematical language, on both semantic and syntactic levels, are conditions for facilitating students' recognition of a specialized language. Access to different contexts of schooling is "controlled" by the possession of these languages. This fact is one of possible explanations of why many students starting their tertiary education struggle with mathematics (Rylands & Coady, 2009; Varsavsky, 2010): they are not prepared for the change in 'language' and their theoretical thinking is not strong enough to sustain the change in code.

This explains, to some extent, why methods courses focusing on content knowledge or remedial mathematics courses for tertiary students in education have limited impact: they can only help those who are already prepared for this change. Then, how would be possible to break the circularity?

A content course in mathematics is needed, certainly, but how to design it, in view of the results of this research?

A first step is designing such course could be to look at the kinds of tasks that future teachers have difficulties to perform, and consequently teach, as a consequence of limited understanding of concepts. In case of geometry, isometric transformations and informal reasoning are hard topics for many. It is a rich area of study, a nexus of many elementary geometric concepts. Thus, it could be a desirable part of the content of the course. Making it seemingly easier for the future teachers by adopting an overly procedural approach set in a limited context (such as grid paper) will inevitably lead to a situation where most future teachers will not be able to perform isometric transformations beyond these standard contexts. Even if more challenging, a more flexible, theoretical approach is advisable, based on consideration of different approaches to a task, a systemic discussion of their mathematical validity, of the limitations of the task and the possibilities of its enrichment.

If a theoretical treatment is lacking, integration of new views of the same concept – as it is envisioned by a spiral curriculum – is not possible. A spiral curriculum relies on the idea that (school) mathematics has a 'vertical dimension' defined by integration and subsumption of existing knowledge (Maton, 2011). The integration happens at the level of meanings, thus requiring that meanings be freed from context through a process of abstraction and institutionalization. Concepts left at the procedural level, therefore, cannot be generalized for lack of supporting meaning. Such situation often leads to an accumulation of concepts into 'syncretic heaps' (Vygotsky, 1986) that cannot be integrated into a theory and, as a consequence, cannot properly reflect the nature of mathematical knowledge.

Discussions of concrete tasks would benefit from eventually reaching beyond the restricted topic at hand and addressing the more general questions of the nature of mathematics. Still, it is important to start from a task. Several factors contribute to the formation of the idea of a discipline, but it can be argued that the best vehicle for this purpose are tasks. On the one hand, the nature of a concept must be sustained by the nature of tasks given, and, on the other, evaluation should signal epistemic expectations (Shalem & Slonimsky, 2010). Through these elements, in time, the idea of discipline builds up: mathematics will be defined and clearly distinguished from other disciplines by the nature of mathematical knowledge, by its epistemology. Inside a discipline, at a finer granularity, tasks also delimitate the fundamental questions that structure situations under question.

How important is it, however, to change the future teachers' epistemology? Can one learn to teach mathematics so that the lessons convey knowledge in line with mathematical theory and epistemology if one's own ideas of what, why and how to teach are in conflict with the mathematical theory and epistemology? Different studies on teachers' knowledge argue that this cannot be the case; that, in fact student understanding is dependent on teacher understanding. In the following, I shall refer to the work of Liping Ma (1999) who conducted a comparative study of teachers from China and Unites States.

For this purpose, she introduced the notion of profound understanding of fundamental mathematics (PUFM). Fundamental mathematics serves as foundation for later learning and it contains advanced mathematical topics in rudimentary form. "Profound" comprises three related meanings: deep, broad and thorough. Understanding a topic with depth means "connecting it with more conceptually powerful ideas of the subject" (p. 121), while understanding with breadth is "to connect it with those of similar or less conceptual power" (ibid.). A thorough understanding is one that connects different parts and transforms knowledge of mathematics into a whole. For developing such understanding, content knowledge is a critical component. The development of specialized content knowledge relies on strong content knowledge (Shulman, 1986) where the links between concepts are emphasized horizontally and vertically.

Two directions for improvement can be identified here. One is specialization of teachers in mathematics, at least starting from grade 5 when concepts beyond the ubiquitous arithmetical operations are introduced. Specialization would give more time to develop content knowledge, prior to teaching and more time to reflect on teaching and ways of enhancing it further in the practicing of teaching. A second direction would target the epistemic aspect of the transmitted knowledge – and would consist in a concerted effort (among different levels of education) to teach mathematics closer to the nature of mathematical knowledge. Given that the first option would require considerable structural changes to the educational system, the second seems to be a more practical one. Teacher training programs could include content courses, yet this measure might not be efficient enough as most of cognitive structures are already in place when students arrive at university. An increase in exigencies, where the epistemic aspect of available textbooks and other teaching materials is concerned, properly sustained by end of cycle mandatory examinations could be a more practical first step. Restating the epistemic status of knowledge might be the way to cut a vicious circle of reproducing the different mathematics future teachers' knowledge defines.

A few words about the limitations of the research presented here are in order. First, one-to-one interviews with future teachers could be have been useful in clarifying certain aspects of their lesson plans. In preparing lesson plans in their weekly assignments future teachers benefited from detailed written feedback from the instructor and the teaching assistants. But face to face discussions would have given the researcher an opportunity to gain deeper insight into the beliefs the future teachers entertained about the purposes of teaching geometry and ways of accomplishing them.

A second limitation is that the future teachers did not have the opportunity to implement all their lesson plans included in the Problem Books. Although some choose to use these activities as source for their workshop presentations, and consequently, got feedback from their colleagues, most of activities were not taken to children. One can only suppose that, while trying to implement these lesson plans, the future teachers would have realized certain shortcomings of these and would have engaged in a reflection on the extent that the activity accomplished the claimed instructional objectives. We can only hope to have the opportunity to give these lesson plans to future generations of students and see their analyses and proposed improvements to them.

I must also underline that some additional data, from future teachers from a different university, but with very similar background, was available to me as a way to compare and contrast the findings. These were not included here, yet they served as confirmation of the fact that the identified praxeologies are shared among members of this generation of future teachers. In addition, data from the other institution also suggest that the limitations adopted by future teachers in the design of the transformation tasks (for example, the use of the grid support, vertical or horizontal lines of reflection or figures aligned to grid) are not only didactical choices, but choices defined by the future teachers' knowledge of transformations. Future teachers faced difficulties in performing transformations in situations that were not "standard" (for example, vector of translation defined by a vector oriented to the left, the line of reflection not passing through grid points or crossing the figure, etc.). These observations reinforce my earlier conclusion that future teachers need to be taught content and at a higher level than it is expected that they teach at.

What next?

I propose two directions of growth of my research.

First, it would be interesting to propose praxeological models for other topics in geometry and study the relations between the method and theory of different praxeologies of the same person. Such models could help to explain how future teachers deal with contradiction, how they integrate new information and maintain these geometries. The study of spontaneous epistemologies, the growth of knowledge and changes of epistemologies are in line with my interests.

Second, I would be interested in analyzing in-service teachers' ways of teaching geometry. The focus would be on identifying elements in their classroom talk that target the development of abstraction and generalized meanings. The concept of semantic density (Maton, 2014a) could be a useful framework for such investigation.

# References


Adam, R., & Chigeza, P. (2014). Beyond the Binary: Dexterous Teaching and Knowing in Mathematics Education. *Mathematics Teacher Education and Development*, *16(2)*, 108-125.

Artigue, M. (2002). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work. *International Journal of Computers for Mathematical Learning*, 245-274.

Ball, D., Thames, M., & Phelps, G. (2008). Content Knowledge for Teaching : What Makes It Special? *Journal of Teacher Education*, *59(5)*, 389-407.

Ball, L. D., & Bass, H. (2009). *With an eye on the mathematical horizon: Knowing matematics for teaching to learners' mathematical futures. Based on keynote address at the 43rd Jahrestagung fuer Didaktik der Mathematik, Oldenburg, Germany, March 1-4, 2009.* Retrieved from University of Dortmund - Mathematics: www.mathematik.tu-dortmund.de

Barbe, J., Bosch, M., Espinoza, N., & Gascon, J. (2005). Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools, *Educational Studies in Mathematics 59(1-3),* 235-268.

Bernstein, B. (1996). *Pedagogy, symbolic control and identity: Theory, research, critique.* London: Taylor & Francis.

Bernstein, B. (1999). Vertical and horizontal discourse: An essay. *British Journal of Sociology of Education, 20 (2)*, 157-173.

Berthelot, R., & Salin, M.-H. (1994). L'enseignement de la géometrie à l'école primaire. In G. N.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). Exemplification in Mathematics Education. In J. Novotna (Ed.) *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education.* V1, 126-154, Prague.

Bkouche, R. (2009). De l'enseignement de la géométrie. *Reperes-IREM 76*, 85-103.

Bloom, B. (1956). *Taxonomy of educational objectives. Handbook I: The Cognitive Domain.* New York: David McKay Co Inc.

Bobos, G. & Sierpinska, A. (2017). Measurement approach to teaching fractions: A design experiment in a pre-service course for elementary teachers. *International Journal for Mathematics Teaching and Learning 18*(2), 203-239.

Boivin, D. (2006). *Panoram@th, 1er cycle.* Les Éditions CEC inc.

Brousseau, G. (1986). La relation didactique: le milieu. *Actes de la 4e école d'été de didactique des mathématiques* (pp. 54-68). Paris: IREM de Paris 7.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dortrecht: Kluwer.

Brousseau, G. (1998). La théorie des situations didactiques. In M. Cooper, N. S. Balacheff, & V. Warfield (Eds.), *Recueil de textes de Didactique des mathematiques 1970-1990.* Grenoble: La Pensee Sauvage.

Brown, D. F., & Rose, T. J. (1995). Self-reported classroom impact of teachers' theories about learning and obstacles to implementation. *Action in Teacher Education, 17(1)*, 20–29.

Browning, C., Edson, A., Kimani, P., & Aslan-Tutak, F. (2014). Mathematical Content Knowledge for Teaching Elementary Mathematics: A Focus on Geometry and Measurement. *The Mathematics Enthusiast, 11(2)*, 333-384.

Bruner, J. (1961). The act of discovery. *Harvard Educational Review 31(1)*, 21-32.

Burger, W., & Shaughnessy, M. (1986). Characterizing the van Hiele Levels of Development in Geometry. *Journal for Research in Mathematics Education 17(1)*, 31-48.

Challita, Dalia (2013) Providing College Level Calculus Students with Opportunities to Engage in Theoretical Thinking. *Master's thesis*, Concordia University.

Chan, K., & Elliott, R. (2004). Relational analysis of personal epistemology and conceptions about teaching and learning. *Teaching and teacher education 20(8)*, 817-831.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques 19(2)*, 221-266.

Ching-shu Shen, C. (2015). Reflections on Learning How to Teach Mathematics: The Initial Training of Kindergarten Teachers. *Creative Education, 6*, 1328-1335 .

CIRTL Network (2012). *Planning a course: Defining Instructional Objectives.* Retrieved March 20, 2013, from Center for the Integration of Research, Teaching and Learning: http://www.cirtl.net/node/2503

Clements, D. H., Battista, M. T., & Sarama, J. (2001). Logo and Geometry. *Journal for Research in Mathematics Education Monograph Series 10*.

Clements, D., & Battista, M. (1992). Geometry and Spatial Reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: McMillan.

Colton, A., & Sparks-Langer, A. (1993). A Conceptual Framework to Guide the Development of Teacher Reflection and Decision Making. *Journal of Teacher Education, 44(1)*, 45-54.

Couto, A., & Vale, I. (2014). Pre-service teachers' knowledge on elementary geometry concepts. *Journal of the European Teacher Education Network , 9*, 57-73.

Crespo, S. (2003). Learning to pose mathematical problems: exploring changes in preservice teachers' practices. *Educational Studies in Mathematics 52(3)*, 243-270.

Crowley, M. L. (1987). The van Hiele Model of the Development of Geometric Thought. In M. Montgomery Lindquist (Ed.), *Learning and Teaching Geometry, K-12* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.

Daniels, H. (2001). *Vygotsky and Pedagogy.* London: Routledge.

Duval, R. (1995). *Sémiosis et pensée humaine registres sémiotique et apprentissages intellectuels.* Berne: Peter Lang.

Fennema, E., & Franke, L. M. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York: Macmillan.

Flick, U. (2014). *An introduction to qualitative research, Fifth edition.* SAGE Publications Ltd.

Fujita, T. (2001). The order of theorems in the teaching of Euclidean geometry: Learning from developments in textbooks in the early 20th Century. *ZDM 33(6),* 19-203.

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry between adolescents. *Journal for research in mathematics education Monograph 3*, 1-196.

Girnat, B. (2009). Ontological beliefs and their impact on teaching elementary geometry. *Proceedings of 33rd PME, Vol. 3* (pp. 89-96). Thessaloniki, Greece: PME.

Gomes, A., Ribeiro, M., Pinto, H., & Martins, F. (2013). Early Years' Trainee Teachers' Geometric Knowledge: the case of defining a rectangle. *Quaderni di Ricerca in Didattica (Mathematics) 23(1)*, 282-291.

Gonzales Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics 15*, 105-127.

Gutiérrez, A., Jaime, A., & Fortuny, J. (1991). An Alternative Paradigm to Evaluate the Acquisition of the van Hiele Levels. *Journal for Research in Mathematics Education 22(3)*, 237-251.

Hamersley, M., & Atkinson, P. (2007). *Ethnography: principles in practice.* London: Routledge.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level Calculus courses. *Educational Studies in Mathematics, 72*(3), 341-358.

Harper, S. (2003). Enhancing elementary pre-service teachers' knowledge of geometric transformations through the use of dynamic geometry computer software. *Proceedings of Society for Information Technology & Teacher Education International Conference* (pp. 2909–2916). Chesapeake: AACE.

Harris, D., & Sass, T. (2011). Teacher training, teacher quality and student achievement. *Journal of Public Economics 95(7-8)*, 798-812.

Hashweh, M. Z. (2005). Teacher pedagogical constructions: A reconfiguration of pedagogical content knowledge. *Teachers and Teaching: Theory and Practice*, 11(3), 273–292.

Herbst, P., Fujita, T., Halverscheid, S., Weiss, M. (2017). *The Learning and Teaching of Geometry in Secondary Schools: A Modeling Perspective.* London: Routledge.

Hewitt, D. (1999). Arbitrary and Necessary Part 1: A way of viewing the mathematics curriculum. *For the learning of Mathematics, 19*(3), 2-9.

Hewson, P.W., & Hewson, M.G. (1988). On Appropriate Conception of Teaching Science: a View from Studies of Science Learning. *Science Education 72 (5)*, 529-540.

Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction*, *26,* 430–511.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal 42(2)*, 371-406.

Hofer, B. K., & Pintrich, P. R. (1997). The development of epistemological theories: Beliefs about knowledge and knowing and their relation to learning. *Review of Educational Research, 67*, 88–140.

Houdement, C., & Kuzniak, A. (2003). Elementary geoemtry split into different geometrical paradigms. *Proceedings of CERME 3.* Bellaria.

Howe, R. (1999, September). Book review: Knowing and Teaching Elementary Mathematics. *AMS*, 881-887.

Jones, G. M., & Carter, G. (2007). Science teacher attitudes and beliefs . In S. Abell, & N. Lederman (Eds.), *Handbook of Research on Science Education* (pp. 1067-1104). New York: Routledge.

Jones, K. (2002). Issues in the Teaching and Learning of Geometry. In L. Haggarty (Ed.), *Aspects of Teaching Secondary Mathematics: perspectives on practice* (pp. 121-139). London: Routledge.

Kenney, R., Shoffner, M., & Norris, D. (2013). Reflecting to learn mathematics: supporting pre-service teachers' pedagogical content knowledge with reflection on writing prompts in mathematics education. *Reflective practice: International and Multidisciplinary Perspectives, 14(6)*, 787-800.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer, & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-168). Mahwah, NJ: Lawrence Erlbaum.

Liljedahl, P. (2005). Changing beliefs, changing intentions of practices: the re-education of preservice teachers of mathematics. *15th Study of the International Commission on Mathematics Instruction.* Aguas de Lindoia, Brazil.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* London: Laurence Erlbaum Associates.

Martinet, A., Raymond, D., & Gauthier, C. (2001). *La formation l'enseignement – Les orientations – Les compétences professionnelles.* Québec: Ministère de l'Éducation Direction de la formation et de la titularisation du personnel.

Maton, K. (2011). Theories and things: the semantics of disciplinarity. In F. Christie, & K. Maton (Ed.), *Disciplinarity: Functional Linguistic and Sociological Perspectives* (pp. 62-84). London: Continuum International Publishing Group.

Maton, K. (2014a). Building Powerful Knowledge: The Significance of Semantic Waves. In B. B., & R. E. (eds), *Knowledge and the Future of the Curriculum. Palgrave Studies in Excellence and Equity in Global Education.* (pp. 181-197). London: Palgrave Macmillan.

Maton, K. (2014b). *Knowledge and knowers: towards a realist sociology of education.* London: Routledge.

MELS. (2008). *Quebec Education Program: Preschool & Elementary Education. (QEP).* Retrieved from Ministère de l'Education.: http://www1.education.gouv.qc.ca/sections/programmeFormation/primaire/pdf/educprg2001bw/educprg2001bw.pdf

MELS. (2016). *Progression of Learning in Elementary School.* Retrieved 2016, from MELS: http://www1.education.gouv.qc.ca/progressionPrimaire/index_en.asp

Meredith, A. (1995). Terry's learning: Some limitations of Shulman's pedagogical content knowledge. *Cambridge Journal of Education*, 25(2), 175–187.

Ministere de l'Education, des Loisirs et du Sport. (2001). *Programme de formation de l'école Québecoise*. Retrieved from MELS Gouvernement du Quebec: http://www1.education.gouv.qc.ca/sections/programmeFormation/primaire/index.asp?page=prform2001h

Morris, A., Hiebert, M., & Spitzer, S. (2009). Mathematical Knowledge for Teaching in Planning and Evaluating Instruction:What Can Preservice Teachers Learn? *Journal for Research in Mathematics Education 40(5)*, 491-529.

Papademetri-Kachrimani, C. (2012). Revisiting van Hiele. *For the learning of mathematics, 32(3)*, 2-8.

Pegg, J. &. (1998). Interpreting student understanding in geometry: a synthesis of two models. In R. Lehrer, & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space* (pp. 109-135). Mahwah, NJ: Lawrence Erlbaum Associates.

Petrou, M., & Goulding, M. (2011). Conceptualising Teachers' Mathematical Knowledge in Teaching. In T. Rowland, & K. Ruthven (Eds.), *Mathematical Knowledge in Teaching* (pp. 9-25). Springer.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (pp. 102-119). New York: Macmillan.

Rogers, C. K. (2014). Journal Writing in a Mathematics Capstone Course for Prospective Secondary Teachers: Future Teachers Making Connections. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 24*(4), 465-479.

Roth, G., & Weinstock, M. (2013). Teachers' epistemological beliefs as an antecedent of autonomy-supportive teaching. *Motivation and Emotion 37*, 402–412.

Rowland, T. (2005). The Knowledge Quartet: A tool for developing mathematics teaching. *Proceedings of the 4th Mediterranean conference on mathematics education* (pp. 69–81). Nicosia: Cyprus Mathematical Society.

Rylands, L., & Coady, C. (2009). Performance of students with weak mathematics in first-year mathematics and science. *International Journal of Mathematical Education in Science and Technology, 40 (6)*, 741–753.

Rynhart, P. (2012). *The van Hiele model of geometric thinking.* Retrieved November 8, 2012, from Proactive Play: http://proactiveplay.com/the-van-hieles-model-of-geometric-thinking/

Samuel, D., & Ogunkola, B. (2015). Elementary School Teachers' Epistemological Beliefs as Predictors of Their Inquiry-Based Practices in Science Instruction. *International Journal of Elementary Education 4(6)*, 101-112.

Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational researcher, 15(2)*, 4-14.

Sierpinska, A., Bobos, G., & Pruncut, A. (2011). Teaching absolute value inequalities to mature students. *Educational Studies in Mathematics 78(3)*, 275-305.

Speer, N., & Wagner, J. (2009). Knowledge Needed by a Teacher to Provide Analytic Scaffolding during Undergraduate Mathematics Classroom Discussions. *Journal for Research in Mathematics Education, 40(5)*, 530-562.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity . *Educational Studies in Mathematics 12(2)*, 151-169.

Trigwell, K., Prosser, M., & Waterhouse, F. (1999). Relations between teachers' approaches to teaching and students' approaches to learning. *Higher Education 37(1)*, 57-70.

University of Cambridge. (2017). *The home of rich mathematics*. Retrieved from NRICH enriching mathematics: https://nrich.maths.org/

Usiskin, Z. (1982). *van Hiele levels and achievement in secondary school geometry.* University of Chicago.

van Hiele, P. (1959). *Levels of mental development in geometry.* Dortrecht.

van Hiele, P. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics 5(6)*, 310-316.

Varsavsky, C. (2010). Chances of success in and engagement with mathematics for students who enter university with a weak mathematics background. *International Journal of Mathematical Education in Science and Technology, 41(8)*, 1037-1049.

Vergnaud, G. (1983). Multiplicative structures. . In R. Lesh, & M. (. Landau, *Acquisition of mathematical concepts and processes* (pp. 127-174). New York: Academic press.

Vinner, S. (1991). The role of Definitions in the Teaching and Learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht: Kluwer Academic.

Vinner, S. (2011). The role of examples in the learning of mathematics and in everyday thought processes. *ZDM*, 43(2), 247–256.

Waeytens, K., Lens, W., & Vandenberghe, R. (2002). 'Learning to learn': teachers' conceptions of their supporting role. *Learning and Instruction 12* , 305–322.

Walle, v. d., & Lovin, L. H. (2006). *Teaching student-centered mathematics. Grades 5-8.* Toronto, ON: Pearson.

Wallrabenstein, H. (1973). Development and signification of a geometry test. *Educational Studies in Mathematics 5*, 81-89.

Wheelahan, L. (2010). *Why Knowledge Matters in Curriculum. A social realist argument.* London: Routledge.

Whiteley, W. (1999). The Decline and Rise of Geometry in 20th Century North America. *Proceedings of the 1999 CMESG Conference* (pp. 7-30). Brock: CMESG.

Wozniak, F., Bosch, M., & Artaud, M. (2012). *Yves Chevallard (English).* Retrieved November 12, 2016, from Association pour la recherche en didactique des mathematiques: http://www.ardm.eu/contenu/yves-chevallard-english

Yanik, H. (2011). Prospective middle school mathematics teachers' preconceptions of geometric translations. *Educational Studies in Mathematics 78(2)*, 231-260.

Yanik, H. B., & Flores, A. (2009). Understanding rigid geometric transformations: Jeff's learning path for translation . *The Journal of Mathematical Behavior 28 (1)*, 41-57.

Zachariades, T., Christou, C., & Pitta-Pantazi, D. (2013). Reflective, systemic and analytic thinking in real numbers. *Educational Studies in Mathematics 82*, 5–22.

Zaslavsky, O., & Shir, K. (2005). Students' Conceptions of a mathematical definition. *Journal for Research in Mathematics Education 36(4)*, 317–346.

Žilková, K., Gunčaga, J., & Kopáčová, J. (2015). (Mis)conceptions about geometric shapes in pre-service primary teachers . *Acta Didactica Napocensia 8(1)* , 27-36.