How do students know they are right and how does one research it?

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A Thesis

in

the Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

For the degree of Master in the Teaching of Mathematics at

Concordia University

Montreal, Quebec, Canada

January 2018

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

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Entitled: *How do students know they are right and how does one research it?*

and submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

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Abstract

How do students know they are right and how does one research it?

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Although standards of rigor in mathematics are subject to debate among philosophers, mathematicians and educators, proof remains fundamental to mathematics and distinguishes mathematics from other sciences. There is no doubt that the ability to appreciate, understand and construct proofs is necessary for students at all levels, in particular for students in advanced undergraduate and graduate mathematics courses. However, studies show that learning and teaching proof may be problematic and students experience difficulties in mathematical reasoning and proving.

This thesis is influenced by Lakatos' (1976) view of mathematics as a 'quasi-empirical' science and the role of experimentation in mathematicians' practice. The purpose of this thesis was to gain insight into undergraduate students' ways of validating the results of their mathematical thinking. How do they know that they are right? While working on my research, I also faced methodological difficulties. In the thesis, I included my earliest experiences as a novice researcher in mathematics education and described the process of choosing, testing and adapting a theoretical framework for analyzing a set of MAST 217 (Introduction to Mathematical Thinking) students' solutions of a problem involving investigation. The adjusted CPiMI (Cognitive Processes in Mathematical Investigation, Yeo, 2017) model allowed me to analyze students' solutions and draw conclusions about the ways they solve the problem and justify their results. Also I placed the result of this study in the context of previous research.

Acknowledgements

First and foremost I want to sincerely thank my supervisor Dr. Sierpinska, who has supported me throughout my studies with her patience and knowledge. Thank you for all of the help, guidance and insightful comments.

I would like to thank my committee members, Dr. Hardy and Dr. Stancu for agreeing to read my thesis.

I want to express my deep gratitude to my fellow students, Sarah, Hadas, Laura, and Mandana, who have always given me a hand. I thank the graduate students and professors who kindly agreed to participate in interviews for this research.

I would not be where I am today, without my family's support. I thank my parents and sister for their love and unwavering belief in me. I thank my lovely children, who have given me the extra strength and motivation to get things done. Finally, I thank my husband, who has supported and encouraged me to do my best and with whom I have shared moments of deep anxiety but also of big excitement while I was working on this thesis.

Table of Contents

List of Figures

List of Tables

Chapter 1. Introduction

If there were only one truth, you couldn't paint a hundred canvases on the same theme.

-Pablo Picasso, 1966

This thesis is the story of my journey as a novice researcher in mathematics education. I entered the MTM program with a strong intention to conduct research and make a contribution to mathematics education. Like many other novices, I was enthusiastic and ambitious. Conducting research in mathematics education has been, for me, both exciting and overwhelming.

In the research for my thesis, I was exploring, broadly speaking, how mathematicians and students know they are right in their mathematical results. This turned out to be a challenging task: How to select what is important in the data? What to focus on? How to analyze it? What exactly is my research question?

Finally, I settled on analyzing a set of MAST 217 (Introduction to Mathematical Thinking) students' solutions of a problem involving a mathematical investigation, from the point of view of the mathematical thinking and cognitive processes they engage in solving the problem and making sure they are right. Together with my supervisor, we also conducted introspective and "inter-spective" analyses of our own solutions to this problem. This explains the first part of the title of this thesis.

While struggling with the above-mentioned questions, I realized that it may be worthwhile sharing the story of my cyclic growth and describe my earliest experiences as a researcher. It could be useful for other novice researchers in mathematics education. This explains the second part of the title of the thesis. Thus, my thesis will take the reader behind the scenes, showing my personal feelings, struggles, doubts and successes.

1.1 Motivation for the research and development of the research questions

My personal experience in learning and teaching mathematics has directed me towards pursuing a career in mathematics education. I was absolutely new in the world of mathematics education research. I was fascinated and energized by a qualitative methods course, where I learned the basic principles of different qualitative research methodologies, data collection methods and approaches to analyzing the data. So I started looking for a “good” research problem. Researchers do not often share the reasons why (or circumstances in which) they decided to address the problem they write about in a paper. However, they agree that identifying a research problem is a challenging aspect of conducting research (Creswell, 2008). Mathematicians are particularly greedy in this respect, perhaps because finding a good problem to research (or a hypothesis to verify) is crucial part of their art. As Riemann once sighed: ”If only I had the theorems! Then I should find the proofs easily enough” (Riemann, quoted in Lakatos, 1976).

Schoenfeld argues that

> The hard part of being a mathematician is not solving problems; it’s finding one that you can solve, and whose solution the mathematical community will deem sufficiently important to consider an advance… In any real research (in particular, education research), the bottleneck issue is that of problem identification – being able to focus on problems that are difficult and meaningful but on which progress can be made. (as quoted by Selden and Selden, 2001, p.239)

The first ideas came from my teaching practice. I noticed that most of my students do not evaluate or analyze their solutions. They prefer to check if their answers are the same as those given at the back of the textbook or just ask the teacher if they are correct (or acceptable). My observations were not new and were pointed out in the literature (e.g. Sierpinska, 2007). The ability to justify, verify and analyze their own mathematical results becomes more critical for students who study advanced undergraduate and graduate mathematics courses. If you ask someone on the street: “How do you know that 2 plus 2 equals four?” you may get a variety of answers

- It is always true, just count…
- Because everyone knows it
- My mom told me…
- I don’t remember all the details, but my math teacher explained it to me, so I believe this is correct.
- I can prove it, look….

Indeed, in mathematics, we have proof. Proof is the cornerstone of mathematics and plays the central role in the practice of mathematicians. Schoenfeld (1994) stresses that “proof is not a thing separable from mathematics as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics” (p.76). At the same time, secondary and high school students’ experiences with proof are limited. Studies show that undergraduate (and even graduate) mathematics students experience difficulties understanding, constructing, and validating proofs (Martin & Harel, 1989; Moore, 1994; Selden & Selden, 2003; Alcock & Weber, 2005). One of the problems of mathematics courses is that they do not give students a feeling of how new results in mathematics can be discovered. Students have seen proofs in lectures and textbooks as a perfect chain of logical steps from conjecture to the theorem, since “deductivist style hides the struggles, hides the adventure” (Lakatos, 1976, p. 142). Presenting mathematical results in the form of definition-theorem-proof has become one of the hallmarks of mathematics.

> False starts, mistakes, revisions—these are all part of the creative process. But when the final result is published, we seldom see the enormous effort that was necessary for the creation; we see the polished product, the correct statement with a clean proof. This is more than a matter of simple etiquette; it's an important feature of mathematics. . . . We observe artistic etiquette because we have artistic goals. (John Ewing, as quoted by Csiszar, 2003, p. 244)

The initial goal of my study was to explore how students (‘novices’) and mathematicians (‘experts’) validate the results of their mathematical thinking. I had formulated the general research question:

How do students and mathematicians know that they are right?

I was working inductively, applying some elements of grounded theory methods (Strauss and Corbin, 1990). Based on my initial research question and literature search I began collecting data. Working closely with my supervisor, I decided to use, as data in my research, the homework assignments of students from the course MAST 217 – Introduction to mathematical thinking where I was a teaching assistant. Also, I conducted seven task-based interviews with ‘experts’: graduate students and mathematics professors.

After exploring the literature about proof and proving, teaching and learning proof, collecting students’ written responses and conducting a first round of semi-structured interviews with mathematicians and graduate students, I found myself with a huge amount of data which was very difficult to analyze. Thus, another problem surged:

How does one conduct research into how students and mathematicians know that they are right? How does one choose an appropriate framework for analyzing data?

So this thesis is mainly about my process of coming to terms with the second question; some answers to the first one will be obtained as a by-product of that process.

1.2 The structure of the thesis

This thesis consists of six chapters. In this Chapter 1, I introduce my study. This includes my motivation for the study and development of the research questions. Finally, I present the outline of my thesis.

In Chapter 2, I review the literature that inspired and informed me in my research. In particular, I provide a literature review on the epistemology and evolution of mathematical proof. I then discuss the functions of proof and the relationships between argumentation and proving. At the end of this chapter, I review studies that outline the difficulties that students experience in understanding and constructing proof.

In Chapter 3, I present the methodology and the setting of the study. I focus on the procedures and the description of a mathematical problem used in this study. I explain why, of all the homework assignments in the course MAST 217, I chose this particular problem to analyze students' solutions from the point of view of the question, "How do students know they are right?" The problem required finding a formula representing the outcome of a potentially infinite haggling process and justifying it (or, from the point of view of the student – making sure the formula is correct). Because, in the problem, the formula was not given but had to be found, solving it required engaging in a sort of small scale "mathematical investigation".

In Chapter 4, I describe the process of finding a theoretical framework for identifying the cognitive processes engaged in solving problems requiring some elements of mathematical investigation. One of the frameworks from the literature that I considered was "CPiMI" (Cognitive Processes in Mathematical Investigation, Yeo, 2017). I describe the difficulties I had in applying this framework to my concrete corpus of data, and I report on how, in an attempt to overcome these difficulties, we (myself and my supervisor), decided to first try to apply the framework to our own solutions, in a process we called the "introspective and inter-spective" analyses. I show how, in this process, we gained a better understanding of the CPiMI framework and found a way of adapting it to the analysis of students' solutions of a problem involving investigation as a process.

In Chapter 5, I interpret and analyze some students' solutions in detail, using the adapted CPiMI framework. Also, I present a summary of the results and accompanying discussion.

In Chapter 6, I discuss the results in the context of previous research, and I highlight the limitations of this study and its contribution to the field of mathematical education.

Chapter 2. Review of literature on mathematical proof

The purpose of this chapter is to review the literature on the role of proof in mathematics, its nature, the process of proving, different kinds of proof and issues related to the teaching and learning of proof. I start by looking at how the views on proof have changed during its history. Next, I consider modern perspectives on proof and proving. In the third section, I focus on the relationship between argumentation and proving, and then I provide an overview on the functions of proof. In the fifth section, I discuss the place of proof in mathematics education research; in particular, on students' conceptions of proof, as well as students' difficulties with proof, as outlined in the literature. Finally, I outline relationship between problem solving, proving and investigation as it has been discussed in the literature.

2.1 What is mathematical proof?

Until the 20th century the dominant view on proof, as a sequence of formal-deductive arguments that establish certain and infallible truths, had not been challenged. Thanks to the ancient Greeks, proof as deduction from a set of axioms became the cornerstone of (Western) mathematics. All three absolutist philosophies of mathematics that were developed in the first decades of 20th century - formalism, logicism and intuitionism - held this view on proof (Hanna, 1995; Tall, 1991; Ernest, 1991; Davis & Hersh, 1981). Formalists view mathematics as a formal system consisting of axioms, definitions, statements and proofs, therefore "the validity of any mathematical proposition rests upon the ability to demonstrate its truth through rigorous proof within an appropriate formal system" (Hanna, 1991, p.55). For proponents of logicism, mathematics is a branch of logic. This means that "all of mathematics can be expressed in purely logical terms and proved from logical principles alone" (Ernest, 1991, p. 9). Intuitionists reject some types of proofs - for instance, proof by contradiction - because they reject the law of excluded middle and claim that mathematical truths must be established by constructive methods. However, Gödel's incompleteness theorems demonstrate the limitations of formal systems; moreover, they show that proof is not capable of establishing all truths.

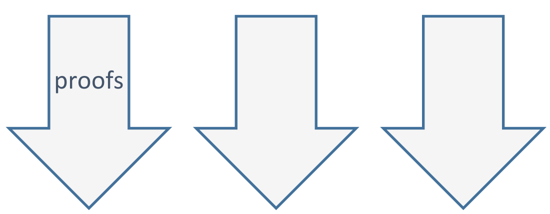
Raising questions that address mathematical practice and ways of thinking, as well as new discoveries and development of science and mathematics, led to accepting a quasi-empirical view on mathematics and re-evaluating the concept of proof.

“Quasi-empirical” nature of mathematics

Lakatos (1976) attacks formalism in mathematics and argues for the ‘quasi-empirical’ nature of mathematics. He states that attention should be on “growth and permanent revolution, not foundations and accumulation of eternal truths” (Lakatos, 1976, p. 207). Lakatos distinguishes two types of deductive systems: Euclidean and quasi-empirical. It is common for both that they take some statements as basic, then derive further statements in a deductive manner. The major difference between Euclidean and quasi-empirical systems is the direction of the flow of ‘truth’ and ‘falsity’.

In Euclidean systems, truth is injected at the ‘top’ (the level of axioms); therefore, truth flows ‘downward’ through the safe truth-preserving channels to the theorems (Figure 1). In contrast, in quasi-empirical systems, truth is injected at the ‘bottom’, at the level of theorems, which can be tested against experience (Figure 2). At the same time, truth cannot flow upwards; therefore falsity is inherited upwards from theorems at the ‘bottom’ to the set of axioms. In other words, the progress of quasi-empirical systems is pulled by refutations. Lakatos claims that mathematics is fallible: “we never know, we only guess”.

Axioms - primitive terms (at the top)



truth-value-injections

Figure 1. Euclidean theories

Science cannot be organized in this manner.

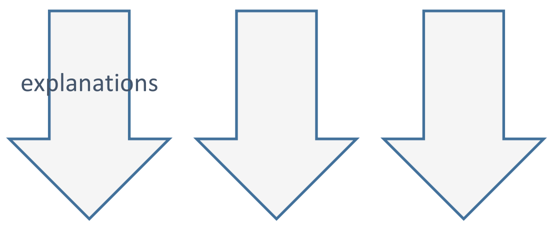


Figure 2. Scientific theories

The main statement of Lakatos' philosophy of mathematics is that proof plays a heuristic role in mathematics and can be used to improve mathematical conjectures. According to Lakatos (1976) mathematical development is driven by counterexamples. In *"Proofs and Refutations"*, he presents a new heuristic method for modifying mathematical ideas. At first he defines informal proof as a "thought experiment which suggests a decomposition of the original conjecture into subconjectures or lemmas, thus embedding it in a possibly quite distant body of knowledge" (Lakatos, p.9). Then, any of these subconjectures can be refuted by counterexamples. There are three kinds of counterexamples:

- 'Global but not local counterexamples' which refute the conjecture but do not refute the stated premises. They require the improvement of the proof as well as finding the 'hidden lemma'.
- 'Local but not global counterexamples' which refute some of the lemmas (subconjectures) but are not counterexamples to the conjecture. They require improvement of the proof by replacing the 'guilty lemma' with another one.
- 'Local and global counterexamples' which refute both the main conjecture and the premises. They require the improvement of the conjecture, by modifying the concepts and notions to find conditions the proof's validity.

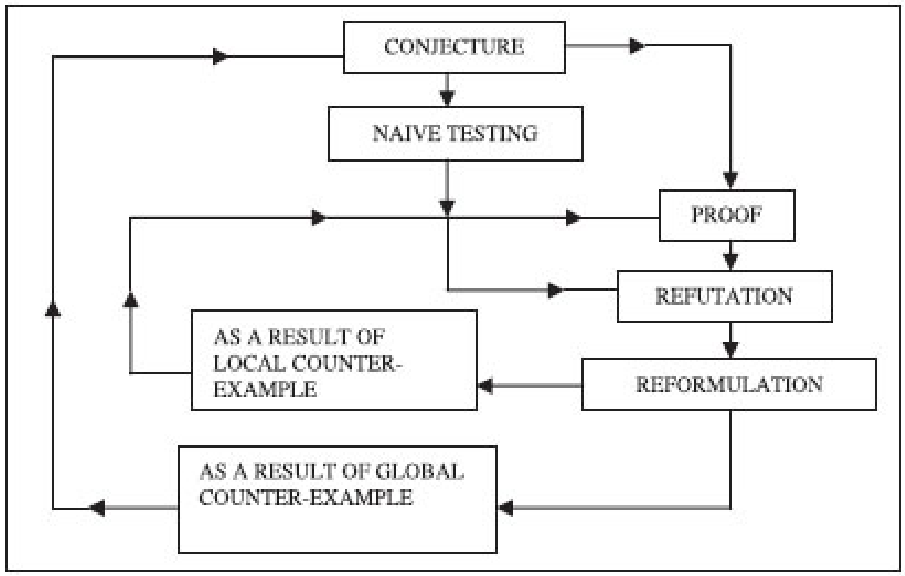


Figure 3 Lakatos' method summarized by Davis and Hersh (1980, p.292)

2.2 Modern perspectives on proof and proving

Accepting new forms of proof

The concept of proof was challenged not only by quasi-empirical views on mathematics but also by the development of computers and acceptance of new types of proofs (Hanna & Jahnke, 1996; Tymoczko, 1979; Thurston, 1994).

In *"Ongoing Value of Proof"* Gila Hanna (2007) points out that the use of computers in mathematical practice and new types of proof, such as a zero-knowledge proof and holographic proof, raise questions about the meaning of proof, and lead to predictions of the death of proof. Indeed, in his article *'The death of proof'*, Horgan (1993) states that traditional mathematical proofs will be replaced by experiments on computers. A zero-knowledge proof is an interactive protocol between two parties, called a prover and a verifier, and was first proposed by Goldwasser, Micali and Rackoff (1985). The prover convinces the verifier that some mathematical statement is true but does not reveal any details of the proof. As a result of the interaction, the verifier will be completely convinced that the statement is true; however, he will gain zero knowledge and will not be able to convince others (Hanna & Jahnke, 1996). A holographic proof "consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line" (Hanna & Jahnke, 1996, p. 882) Zero-knowledge proofs and holographic proofs are the opposite of the traditional view of mathematical proof, because it is impossible to verify every single line of the proof.

Another subject of controversy is computer-assisted proofs. One of the best known examples is the proof of the four-color theorem, introduced by Appel and Haken in 1976. A computer has been used to prove the reducibility lemma. Tymoczko (1979) argues that the use a computer in mathematical proving, such as the proof of the four-color theorem, has significant implications for the philosophy of mathematics. He considers three main characteristics of proofs:

- proofs are convincing to mathematicians;
- proofs are surveyable, in other words "a proof is a construction that can be looked over, reviewed, verified by a rational agent";
- proofs are formalizable or can be set into "…a finite sequence of formulas of a formal theory satisfying certain conditions".

Even though the majority of the mathematical community is satisfied with the first characteristic, most philosophers want a deeper explanation as to why mathematical proofs should be assumed to be convincing. Surveyability and formalizability explain why a proof is convincing to rational agents. According to Tymoczko, not all formalizable proofs are surveyable, and not all surveyable proofs are formalizable. For example, we can take a Gödel statement (surveyable) and show that it has no formal proof. In addition, there are many formal proofs that are too long to be checked by "a mathematician in a human lifetime", so they are not surveyable.

Therefore, to accept the proof of the four-color theorem, we need to modify our concept of proof by adding a new method (computer experiment) or to allow the inclusion of computer proofs into proofs. By discussing the four-color theorem, Tymoczko gives additional support to the idea that mathematics is quasi-empirical.

Accepting an argument as a proof

Mathematicians continue to discuss criteria of acceptable proof. The formalist perspective on proof was criticized by many philosophers, mathematicians and educators (e. g., Heinze, 2010;

Thurston, 1994; Tymoczko, 1986; Davis & Hersh, 1981; Hanna, 1995; Rav, 1999). According to Rav (1999), it is important to distinguish two types of proofs. The first type is a so-called 'derivation' or a formal proof, which includes the chain of statements according to rules of logical interference. It is possible to use a machine to verify such derivations. The second type of proof is a 'conceptual proof' or a kind of informal proof "of customary mathematical discourse, having an irreducible semantic content" (Rav, 1999, p.11). In other words, this type of proof includes rigorous arguments that can be accepted by the mathematical community. Even without the use of precise mathematical definitions, it is possible for mathematicians to verify the accuracy of each step. For instance, the majority of proofs published in mathematical journals are conceptual proofs (Hanna & Barbeau, 2010). Similarly, Thurston (1994) argues that "the humanly understandable and humanly checkable proofs" are different from formal proofs.

Furthermore, an argument is a proof if it is convincing to a mathematician (Weber, 2008; Tymoczko, 1979). For Davis & Hersh (1981), this mathematician is "a mathematician who knows the subject", while (Volminik, 1990) mentions "a reasonable skeptic". Moreover, Mason, Burton, and Stacey (1982) state that an argument is a proof if it would convince "an enemy". In addition, Hanna (1991) claims that some non-mathematical factors may affect acceptance of a proof. For example, the reputation of the prover may play a significant role.

Many mathematicians emphasize the social aspect of proof. According to Manin (1977), an argument becomes a proof after the social act of accepting it as a proof.

> Mathematical discovery rests on a validation called 'proof', the analogue of experiment in physical science. A proof is a conclusive argument that a proposed result follows from accepted theory. 'Follows' means the argument convinces qualified, skeptical mathematicians. Here I am giving an overtly social definition of 'proof'. (Hersh, 1997, p. 6)

> We call proof an explanation accepted by a given community at a given time. (Balacheff, 1987, translated from French)

I have, so far, briefly described some views on proof from the perspective of mathematicians and philosophers. Next, I will demonstrate that educators' views on an acceptable proof are based on goals of proving in mathematical practice, as well as in mathematics education.

2.3 Argumentation and proof

A significant body of research has investigated the relationship between argumentation and proof. Duval (1991) makes a distinction between argumentation and mathematical proof, and suggests major differences from a cognitive and logical point of view. While the role of argumentation is convincing somebody of the truth of a statement, by using rhetoric means, proof is considered as the derivation of a statement from a set of statements, according to logical rules.

> Deductive thinking does not work like argumentation. However, these two kinds of reasoning use very similar linguistic forms and proportional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proofs. (Duval, 1991, Abstract)

Balacheff (1987) also distinguishes mathematical proof ('démonstration' in French) from proof in everyday or legal sense ('preuve' in French), whose meaning is close to "evidence". Similarly, Hanna and De Villiers (2008) define justification as "reasoned discourse that is not necessarily deductive, but uses arguments of plausibility" while considering deductive proof as "a chain of well-organized deductive inferences that uses arguments of necessity" (p. 331).

On the other hand, some studies of the Italian school of research in mathematics education found a link between "argumentation as a process of producing a conjecture and constructing its proof", called 'cognitive unity' (Boero et al, 2010, p. 4). Commonly, argumentation is used to produce a conjecture. Therefore, sometimes it is possible to organize previously constructed arguments into a logical chain in order to produce a mathematical proof.

> During the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingling with the justification of the plausibility of his/ her choices.

> During the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain. (Boero et al. 1996, p. 113)

Moreover, Pedemonte (2002) develops the concept of 'cognitive unity' by distinguishing referential cognitive unity (using in argumentation and proof the same language, heuristics, drawings and theorems) and structural cognitive unity (using the same structure, such as deduction, abduction, and induction). While deductive reasoning moves from a general principle to individual instances, and inductive reasoning moves from several instances and observations to a general law, abductive reasoning moves from an incomplete set of observations to possible explanations. It is interesting that while continuity in referential system between argumentation and proof leads to the construction of proof, structural continuity may lead to errors and inconsistencies. The structure of argumentation is usually not deductive, therefore it is necessary to "overcome a structural distance" and change, for example, the abductive structure into a deductive one, in order to construct a correct proof (Mariotti, 2006).

Toulmin's model of argument

In *The Uses of Argument* (1958) Toulmin presents a model of informal reasoning, which is very different from traditional logical theory. There are three essential parts of an argument: the ***data*** (D), the ***claim*** (C) and the ***warrant*** (W). The claim (C) is the statement being argued. The data (D) are the facts or evidence used to support the claim (C). To justify the connection between the data (D) and the claim (C), the arguer uses the warrant (W). Moreover, the warrant might be supported by the ***backing*** (B) to present additional evidence. The ***qualifier*** (Q) is the statement that expresses the degree of confidence of the claim (C). In addition, the ***rebuttal*** (R) states the conditions (or provides a counter-argument) under which the claim (C) does not hold true. These six parts are often presented graphically, as shown in Figure 4.

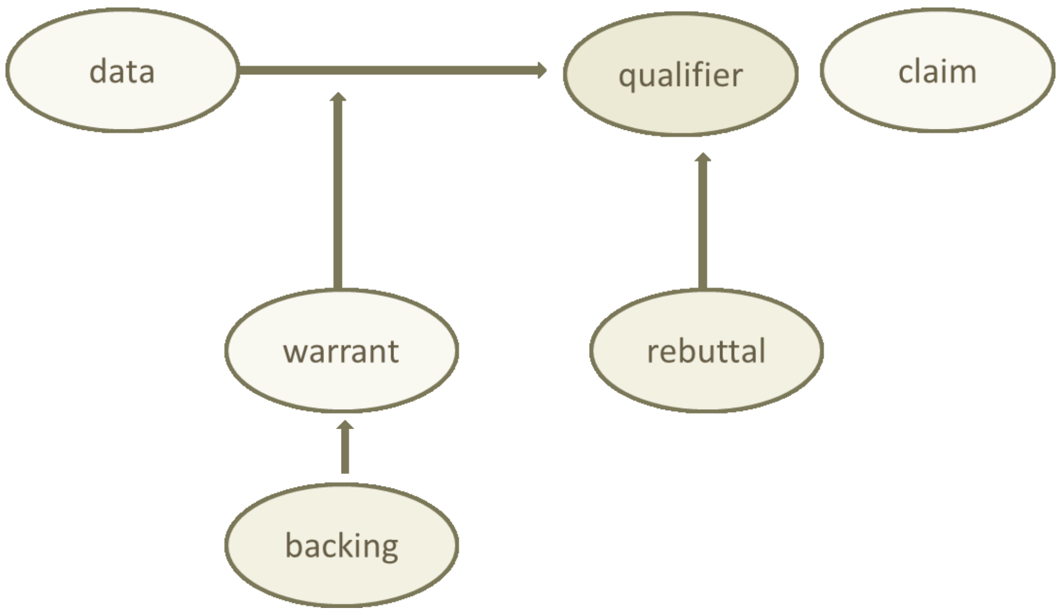


Figure 4. Toulmin's Model of Argument.

In recent years, Toulmin's scheme sparked much interest among mathematics educators (Alcock & Weber, 2005; Knipping, 2004; Yackel, 2001). Aberdein (2005) examined the applicability of Toulmin's model to mathematics and demonstrated that it can be applied to formal proofs. For example, Aberdein's decomposition of the proof that there exist irrational numbers α and β such α^{β} is rational is provided in Figure 5.

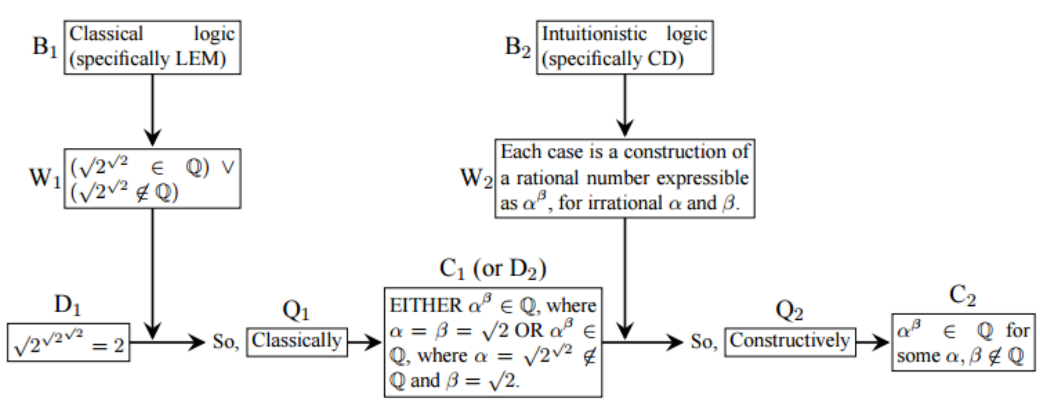


Figure 5. Aberdein's decomposition of a proof according to Toulmin's model.

Furthermore, some researchers have tried to extend Toulmin's Model of Argument. Weinstein (2006) admires Toulmin's work and highlights that "Toulmin is correct in rejecting mathematical logic as a theory of argument and logical empiricism as the philosophy of science" (Weinstein, 2006, p. 49). At the same time, he argues that there is an important place for formalism in the metatheory. His Model of Emerging Truth (MET) is an analogue of the metatheory of axiomatized mathematical theories, "which includes a function that maps from a deep explanatory base onto the theories upon which expectations are based" (ibid, p. 58).

2.4 The functions of proof

Apart from verification, which determines the truth of a statement, there are other roles of proof discussed in the literature (De Villiers, 1990, 2010; Hanna, 2000), such as explanation, systematization, discovery, communication and exploration of the meaning of a definition or the consequences of an assumption. Explanation is examining the proof in order to understand *why* a certain statement is true. Many researchers highlight the importance of this function of proof for mathematics education. Systematization is the organization of various results into a deductive system of axioms, major concepts, and theorems. Discovery function of proof appears when the process of proving leads unexpectedly to new results, models or theories. Communication is the transmission of mathematical knowledge in a clear manner. Exploration of the meaning of a definition or the consequences of an assumption means that proofs might show why it is adequate to use certain axioms and definitions. Different proofs of the same statement can play different roles. Hanna (2000) argues that some proofs are more explanatory than others, and a proof might not accomplish all functions. At the same time, educators (e. g., De Villiers, 1990; Hanna, 2000) agree that an acceptable proof must achieve at least one of the functions mentioned here.

2.5 Proof in mathematics education

Historically, in school, the concept of proof was introduced in geometry classes. In many cases classical proofs were simply memorized. This changed under the influence of constructivism, an educational theory based on the idea that people construct their own knowledge and understanding of the world through experiencing things and reflecting on those experiences. In his article, "Experimentation and Proof in Mathematics", Michael de Villiers (2010) points out that the educational system does not give students a feeling of how new results in mathematics are being discovered but just present the products of mathematical thought. As a result, students consider mathematics as developing in a systematic, deductive way from the beginning. Tall (1991) argues that undergraduate mathematics students should instead be engaged in developing processes of mathematical thinking. On the other hand, some mathematics educators support the view that learning deductive proof is not needed anymore, because informal justification, exploration and investigation play a more significant role in mathematics education today (Hanna, 2000). For example, MacKernan expresses the extreme viewpoint: "So, do we really need proof at all? Especially in schools? Why on earth can't we - the overwhelming majority – simply be allowed to accept that something is intuitive, or very probably true, or just simply obvious?" (Barnard et al., 1996, p. 16, quoting MacKernan).

Students' difficulties with proof

Studies have demonstrated that high school and university students have very little aptitude for proof and do not appreciate the importance of proof (Moore, 1994; Schoenfeld, 1995; Senk, 1985). Moreover, students cannot draw the line between informal argumentation and constructing a formal proof. "Often students do not see why a fact has to be proved, because in their view it is either obvious or sufficiently justified by actual measurements" (Hanna & Jahnke, 1996, p. 897).

Many empirical studies have focused on students` proof construction. These studies aim to characterize what students are doing as they construct arguments and proofs. Harel and Sowder (1998) studied college students' proof understanding, production and appreciation using interviews, tests and classroom observations. They define the process of proving as the process of removing or creating doubts about the truth of an observation. Authors divide the process of proving into two stages: ascertaining (convincing oneself) and persuading (convincing others). "A person's proof scheme consists of what constitutes ascertaining and persuading for that person […] As defined, ascertaining and persuading are entirely subjective and can vary from person to person, civilisation to civilisation, and generation to generation within the same civilisation" (Harel & Sowder 1998, p.242). Harel and Sowder (1998) described 17 different proof schemes and assigned them to three major classes: external conviction, empirical and analytical proof schemes. External conviction proof schemes are those in which students convince themselves and others by referring to external sources, such as the word of an instructor, a ritual or some symbolic manipulations. Empirical proof schemes involve using examples and specific cases, and can be either inductive or perceptual. Analytical proof schemes include the use of logical deduction and can be either transformational or axiomatic. Weber (2005) observed three categories of proof production: procedural, syntactic and semantic. He defines procedural proof as a proof that is created when a student uses "existing proof as a template for producing a new one" (p. 353). Syntactic proof production is characterized by manipulating mathematical statements and definitions without referring to intuitive representations. Semantic proof production is characterized by using informal representations to guide the creation of formal proof.

Other researchers have investigated students' views on proof. Healy and Hoyles (2000) pointed out that even though students preferred to use empirical arguments in their own proof constructions, they distinguished proofs that are convincing to themselves and proofs that would be accepted by a teacher and get the highest marks. Also 50% of students agreed that the main purpose of proof is establishing the truth and more than one third (35%) chose explanatory function.

Many undergraduates experience difficulties with constructing proofs. Moore (1994) analyzed difficulties university students face in learning formal mathematical proof and found that most of them had cognitive sources. He identified seven main sources of students' difficulties in writing proofs:

D1. The students did not know the definitions, that is, they were unable to state the definitions.

D2. The students had little intuitive understanding of the concepts.

D3. The students' concept images were inadequate for doing the proofs.

D4. The students were unable, or unwilling, to generate and use their own examples.

D5. The students did not know how to use definitions to obtain the overall structure of proofs.

D6. The students were unable to understand and use mathematical language and notation.

D7. The students did not know how to begin proofs. (Moore, 1994, p. 251)

Educators believe that the ability to validate proofs is related to the ability of construct them, since proof validation may include recalling theorems and definitions, asking, answering questions and constructing subproofs (Selden & Selden, 2003). At the same time, many of university students and even teachers of mathematics cannot determine whether mathematical arguments compose a valid proof (Martin & Harel, 1989; Selden & Selden, 2003; Alcock & Weber, 2005). The study conducted by Martin and Harel (1989) has shown that preservice elementary teachers accepted a proof mostly based on the form of the argument presented to them. For example, these preservice teachers rejected valid proofs written in paragraph form and accepted flawed proofs written in a traditional two column format. Alcock & Weber (2005) studied how undergraduate majors validate a flawed proof in Real Analysis. They reported that only 6 out of 13 students rejected the proof as invalid and only 2 of them supported their decision by legitimate mathematical reasons.

2.6 Problem solving, investigation and proving

Nowadays, educators stress the importance of proof and reasoning in mathematics education. For example, The National Council for Teachers of Mathematics (2000) states that "reasoning and proof should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12" (p. 56). However, as mentioned, learning and teaching proof can be problematic. In this section, I address studies which discuss incorporating investigative teaching methods in order to improve and develop students' mathematical thinking. Also I present studies that link problem solving, investigation and proving.

"Inquiry-based", "problem-based" learning

In recent years there has been a shift from lecture-based approach in teaching to "problem-based" or "inquiry-based" or "investigative" approaches (Friesen & Scott, 2013, Calleja, 2016, Mass & Artigue, 2013). Hattie (2009) broadly defines inquiry-based teaching as

> the art of developing challenging situations in which students are asked to observe and question phenomena; pose explanations of what they observe; devise and conduct experiments in which data are collected to support or contradict their theories; analyse data; drawn conclusions from experimental data; design and build models; or any combinations of these. (p.208)

Educators agree that inquiry-based approaches to learning and teaching may help in developing students' understanding of core concepts and procedures. However, the term "inquiry" has slightly different meanings across scientific disciplines. Calleja (2016) argues that "in science education, learning through inquiry is seen as the *process* of building understanding by collecting evidence and testing ideas" (p.2), while inquiry in mathematics includes many different forms of *activity* such as posing questions, modeling, exploring, conjecturing, reasoning, arguing and proving; defining and structuring; connecting, representing and communicating. Moreover, Schoenfeld and Kilpatrick (2013) point out that problem solving in mathematics and inquiry in science have similar meanings. They argue that inquiry in mathematics is seen as "finding connections between mathematical concepts and procedures by exploring how that mathematics might be used inside and outside school" (p. 908). On the other hand, mathematical problem solving involves conjecturing and reasoning that is similar to scientific inquiry, but an obtained solution must be presented "as a deduction from what was given in the problem to what was to be found or proved" (ibid.).

Mathematical problems and problem solving

Problem solving has been deeply discussed in mathematics education literature. This section clarifies the notion of problem solving in mathematics and provides a short overview of the work of key authors on mathematical problem solving. Not every mathematical task is a problem. According to Schoenfeld (1985),

> being a 'problem' is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. [...] If one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem. (p. 74)

In the context of this study I consider the term "problem" from Schoenfeld's point of view, as a task that is difficult for a person who does not know how to proceed directly to a solution. Therefore, solving a problem takes time and efforts. Pólya (1973) stated that "to understand mathematics means to be able to do mathematics" (p. 7). In 1945, he published a revolutionary book *"How to Solve It"* where he summed up general problem-solving heuristics and identified four major principles of problem solving:

1. Understanding the Problem
2. Devising a Plan
3. Carrying Out the Plan
4. Looking Back.

Schoenfeld (1985) developed Pólya's ideas about using heuristics and outlined four categories that determine the success in problem solving. Those categories are

- Resources (actual knowledge base),
- Heuristics,
- Control (metacognition),
- Beliefs.

Resources include procedural knowledge of mathematics and facts about mathematical ideas. Moreover, incorrect knowledge may also be a part of resources. Heuristics are strategies and technics for solving problems such as induction, drawing figures, specialization, analogy, variation, decomposition and recombining, working backwards. Schoenfeld (1985) argued that general heuristics do not help students to solve problems because they depend on both students' prior knowledge and on problems. Belief system is "one's mathematical world view" (Schoenfeld, 1985, p. 15). Control includes planning, monitoring and decision-making. Furthermore, Schoenfeld (1985) focused on decision-making behavior at the executive or control level. He analyzed students' and mathematicians' attempts to solve problems and identified six stages or episodes during problem solving: *read, analyze, explore, plan, implement and verify*. He then used a timeline to represent those episodes and analyze metacognitive control among novices and experts.

In "*Thinking Mathematically*" (1982), Mason, Burton and Stacey proposed a problem-solving model that includes three phases (Entry, Attack and Review). They also identified four fundamental processes (specializing, generalizing, conjecturing and justifying) involved in Attack phase and showed how those processes of mathematical thinking alternate between each other.

Proving as a part of problem solving

Researchers indicate an overlap between proving and problem solving. For many educators, proving is included in problem solving. Indeed, proof writing can be a problem for the person and requires applying different strategies and techniques. For example, Furinghetti and Morselli (2009) argue that "proof is considered as a special case of problem solving" (page 71). Weber (2005) considers "proof construction as a problem solving task" (p.351). Tall (1991) also linked proof and problem-solving by saying that "viewed as a problem-solving activity, we see that proof is actually the final stage of activity in which ideas are made precise" (p.16).

Mathematical investigation

In mathematics education the term 'investigation' is used in different situations and has sometimes different meanings. Ponte et al. (1992) pointed out that, in an investigation, "students are put in the role of mathematicians" (p.239).

> Mathematical investigations share common aspects with other kinds of problem solving activities. They involve complex thinking processes and require a high involvement and a creative stand from the student. However, they also involve some distinctive features. While mathematical problems tend to be characterized by well-defined givens and goals, investigations are much looser in that respect. The first task of the student is to make them more precise, a common feature that they share with the activity of problem posing. (ibid.)

Some educators insist that an investigation should be an open-ended problem without a clearly defined goal in its formulation. Moreover, in contrast to a closed mathematical problem, an open investigative task might have multiple correct answers (Bailey, 2007, Orton & Frobisher, 1996, as cited in Yeo & Yeap, 2010). While some researchers separate investigation and problem solving, others believe that investigation includes problem solving. Therefore an open investigative activity when students attempt an open investigative task involves both problem posing and problem solving. At the same time, some educators stress that investigation is primarily a process (Ernest, 1991). Indeed, processes similar to investigation, such as ‘heuristic reasoning’ (Pólya, 1973), ‘heuristic approach’ (Lakatos, 1976) and ‘exploration’ (Schoenfeld, 1992), are mentioned in the literature when researchers describe processes that occur during problem solving. Therefore, investigation as a process which is opposite to a deductive approach or rigorous proof can be considered as a part of problem solving. Separating investigation as an activity from investigation as a process and solving problem as an activity from problem solving as a process helps to resolve the conflict between statements: ‘investigation includes problem solving’ and ‘problem solving includes investigation’ (Yeo & Yeap, 2009).

Chapter 3. Methodology

In this chapter, I provide an overview of the research design and describe not only methods that I have applied in this thesis but also how I came to use certain research methods.

3.1 Research design

My research is exploratory and interpretive in nature. Thus the meaning of human actions is the focus of this study and my goal was to make interpretations in order to explain and understand. I aimed at conducting qualitative research and was open to data (Charmaz, 2006; Creswell, 2008). Data collected in researching my first question drove me to methodological questions. As a novice researcher I was overwhelmed by so much data. It was a challenge to decide where and how to start the analysis. Moreover, the data looked unrelated when I was trying to organize them. Therefore, I needed a tool to detect a structure in the data. The process of searching for a model for the analysis of students' solutions is described in the next chapter. I followed grounded theory methods (Strauss and Corbin, 1990), in the sense that the research process was interactive and cyclical. After I did a first round of analyses and formulated my second research question, a new set of data was collected and generated. The Model for Cognitive Processes in Mathematical Investigation (CPiMI) by Yeo (2013) and solutions for the haggling problem, produced by myself and my supervisor, were added for future analysis. In this phase of the study I was inside the research process and had a dual role. On the one hand, I was a researcher. On the other hand, I was a research subject: my solution became an object of analysis on a par with the MAST 217 students' solutions. Data collection, analysis and development of theory interacted very closely in my research. These interactions are illustrated in Figure 6.

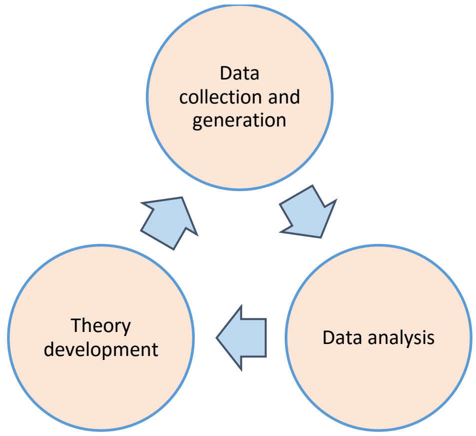


Figure 6: Interaction of data collection, analysis and development of theory

In contrast to grounded theory I did not generate theory from data. Instead of this, the CPiMI model was tested and adjusted for analysis and during analysis.

3.2 Collecting data

There are four methods of data collection used within the interpretive paradigm: participant observation, interviewing, a search for artifacts and researcher's introspection. Each of the above mentioned methods can give a different perspective on research. It is recommended to use multiple data sources for interpretive studies to increase the credibility (Eisenhart, 1988; Tobin, 2000). Interviews can take many forms that include informal conversation, long clinical interviews, semi-structured and highly structured interviews; they help to gain information about relevant historical events or participants' experiences in other settings. Artifacts from the field can be helpful in developing an extensive understanding of the context. Any information produced by participants or others may be considered as useful. Researcher's introspection involves collecting reflections on the research activities and context (Eisenhart, 1988).

The data for this study include transcripts from interviews with graduate mathematics students and mathematicians, written MAST 217 students' solutions to the haggling problem, and researchers' notes and solutions of the same problem. We also include, in the data, Yeo's (2013) Model for Cognitive Processes in Mathematical Investigation (CPiMI), and the results of the introspective and "inter-spective" interpretations of researchers' solutions and their coding in terms of CPiMI. First, each researcher (myself and my supervisor) wrote a description of their own processes of solving the haggling problem and coded it in terms of the CPiMI – this we called introspection. Then we gave each other our solutions without revealing the coding and we coded them independently – we call it “inter-spection”. Finally, we revealed to each other our personal coding. We discussed any points of disagreement. This resulted in re-coding some elements of a solution, or changing our interpretations of some CPiMI category, or in an addition of a category of cognitive processes to the model. Thus, in the process, the CPiMI model was also an object of analysis – this is why we consider it as part of the data. The modified version of the CPiMI model is considered to be one of the “results” of this research.

Collecting data from “Novices” – the MAST 217 students

MAST 217, *Introduction to Mathematical Thinking* is a transition-to-proof course for first year undergraduate students. This course is meant for students taking a Major in Mathematics and Statistics to prepare them for more advanced proof-oriented courses. The content of this course includes the language of mathematics, the logical structure of mathematical statements, different styles of proofs, and different techniques of problem solving. Assessment of the students is based on the weekly homework assignments, one midterm test and final examination. Each of the weekly assignments consists of two parts. The first part was graded electronically, the second part included a single problem and was marked manually. As a teaching assistant I was responsible for marking manually graded assignments and providing written feedback. In order to help students improve their future responses, I made minor corrections in structure and terminology, provided counter-examples to incorrect reasoning and suggested valid arguments. Thus, the first set of data came from students’ written responses to the eleven homework assignments. A list of tasks from homework assignments can be found in appendix A.

Next, I clarify the reasons I had when I chose, for analysis in my research, students’ solutions to the task #9 (the “haggling problem”).

I tried to find hidden patterns in students’ solutions while I collected the data. We met with my supervisor every week to discuss assignments, students’ progress and our observations. It became clear to me that there was a need to reduce the amount of data for use in my thesis. I analyzed the content of the homework assignments again, after going through students’ responses. I realized that some solutions should be omitted for two main reasons. First, some homework problems were only slightly different from examples from the class, so most of the students just repeated the procedure presented by the instructor. As a result, the final solutions did not provide any evidence of actual thinking processes; they just supported the procedural approach as it is outlined in literature (Weber, 2005). For example, a part of the task #6 (Appendix A) was

Prove that

(a) There is no rational number r *such that* $r^2 = 15$.

During the lecture preceding this assignment the instructor presented solutions for following problems:

Exercises 1. Prove that there is no rational number r such that

a) $r^2 = 3$
b) $r^2 = 5$
c) $r^2 = p$, where p is a prime number
d) $r^2 = 14$
e) $r^3 = p$, where p is a prime number

The analysis of the students' solutions have shown that 34 out of 35 students used proof by contradiction and assumed that there exists a rational number r such that $r^2 = 15$. In other words almost all students just repeated the steps from in class solutions and submitted very similar responses.

Another reason to eliminate a big chunk of the data was cheating. In the era of the Internet it is easy to find a solution to almost any standard problem for an undergraduate course, such as MATH 217. Also some students posted identical solutions as a result of collaboration. In addition we suspected that some students posted their tutors' solutions. I assumed that my study must be based on what students really do by themselves. Therefore I did not take those solutions into account and omitted identical or very similar solutions.

The haggling problem

Finally, I chose to discuss in my thesis in more detail students' solutions to assignment #9. Here is the text of the problem, as it was posted on the web page of the course.

> John is trying to sell Mark a bike for **a** dollars.
>
> Mark does not agree on the price and offers **b** dollars (0< b < a).
>
> John does not agree on this price but comes down to **(a + b)/2 = 1/2 a + 1/2 b**.
>
> Mark responds by offering **(b + (a + b)/2)/2 = 1/4 a + 3/4 b**.
>
> They continue haggling this way, each time taking the average of the previous two amounts.
>
> On what amount will they converge? Express the amount in terms of a and b.
>
> Explain your reasoning and justify your response.
>
> Have you tried to verify your answer? If yes, how?

Assignment #9 was given to students near the end of the course, in the 10th week of classes (the course lasts 13 weeks). By this time related topics such as geometric sequences and series, the notion of limit of a sequence, the theorem that increasing (decreasing) and bounded above (below) sequences are convergent in $\mathbb{R}$ and examples of convergent sequences related to computational algorithms were covered. Solving this non-routine problem required some mathematical investigation. It could be solved empirically by observing the numerical results and making a conjecture about the limit of the sequence. Students could try to verify the conjecture by drawing a diagram, by observing a link between the sequences involved in the amounts and geometric series or by using other means. Even though the haggling problem did not demand the formal construction of a proof, we expected that students will attempt to convince themselves and others. I believe that this question allows the researcher to view the means that participants use in order to be convinced that their answer is correct.

Collecting data from "Experts"

I also conducted seven "task-based interviews" with "experts" in proving. Four graduate students and three mathematicians volunteered to participate in this study. All graduate students (Masters and Doctoral) completed a number of advanced proof-oriented courses such as Analysis and Abstract Algebra. All mathematics professors were actively involved in mathematical research and had experience teaching advanced proof-oriented courses to undergraduate and graduate students. Participants were interviewed individually. Each interview was audio recorded and lasted between 45 and 60 minutes. The methodology of task-based interviews is outlined in Goldin (1997). Each interview with a graduate student or mathematician had two components: a task solving part where "experts" attempted two tasks and then reflected on those tasks, and a semi-structured interview part where questions about proving and validating were asked. The first task for the interviews was selected from the homework assignments for MAST 217. As a part of the pilot study I included the haggling problem in the task solving part of an interview with a graduate student. However, I realized that the haggling problem is not well suited for the interviews because working on this problem takes some time and may involve using a computer or calculator. The second task was selected from "Proofs and Refutations" by Lakatos (1976). The participants were asked to prove or disprove the flawed Cauchy's theorem that the limit of any sequence of continuous functions is continuous. A list of tasks and lists of questions for the interview can be found in appendices B and C. As this thesis focuses on addressing the second research question and only partially answers the first one, I do not present in this study the analysis of interviews. The interviews produced rich data which can be used for future research.

Chapter 4. Choosing a theoretical framework for analyzing the data

This chapter describes how a theoretical framework for this study was chosen and developed. I first explain methodological difficulties I experienced. Then I introduce Model for Cognitive Processes in Mathematical Investigation (CPiMI) (Yeo, 2013) and describe our attempt to understand it by means of introspective and inter-spective analyses. Finally, I present the final coding scheme for interpretation of students' solutions. We consider this chapter as presenting partial results of my research. It is not the classical "theoretical framework" chapter in a mathematics education paper. The theoretical framework I discuss here was a candidate for a tool to answer the first of my research questions but it became itself an object of study in dealing with the second. What we found about it thus became part of our results.

4.1 Searching for a model of mathematical thinking

As it was mentioned before in Chapter 2, a Lakatosian view of mathematics as a quasi-empirical science has influenced both the philosophy of mathematics and mathematics education. While I collected the data I was looking for a model of mathematical thinking that could help me to capture the "thought experiments" and "the logic of mathematical discovery" in students' responses in order to understand and explain the role of experimentation in their conjecturing and proving. I was not able to use Lakatos' model for analysis of collected data directly. At some point I was lost. I formulate the major problems I had at that time:

- huge amount of data,
- diversity of different types of data (written solutions to homework assignments, my field notes, interview transcripts),
- existing models of mathematical thinking seemed to be either very specific or too general.

I started to use open coding of students' solutions and interview transcripts using Lakatos' quasi-empirical view as a guideline. Also I continued to read relevant literature and search for a suitable model. I felt like I was looking for a needle in a haystack and discussed my concerns and doubts with my supervisor. I pointed out that I see a lot of similarities in struggles of novice mathematics education researchers and novice proof writers. Like many students who do not know how to begin proofs and how to apply known definitions, I could not decide how to use theory in my research and navigate the process. The result of our conversation was surprising, because at this point I arrived at my second research question:

> *How does one conduct a research into how do students and mathematicians know that they are right? How to choose an appropriate framework for analysis?*

It is a common situation in qualitative research that the existing frameworks are not applicable in new settings and must be refined and adjusted. My supervisor found that it might be interesting to describe the process of choosing, testing and adjusting a model for my study and include experiences of a novice researcher in mathematics education in my thesis.

4.2 The CPiMI model

There are a number of theoretical models developed to characterize thinking processes in problem solving, proving and investigation (Lakatos, 1976; Polya, 1973; Schoenfeld, 1985; Mason et al., 1982; Carlson & Bloom, 2005). A new framework called the Model for Cognitive Processes in Mathematical Investigation (CPiMI) was proposed recently by Yeo (2013).

I decided to apply this model for analysis of the students' solutions of the haggling problem (presented in section 3.2).

The model was proposed to analyze the interactions between cognitive processes when secondary school students attempted open investigative tasks such the following one:

> **Powers of 3 (Open Investigative Task)**
>
> Powers of 3 are $3^1, 3^2, 3^3, 3^4, 3^5, \ldots$ Investigate.

This task is an *open* investigative task because no question is posed for which there would be a clear-cut answer that could be evaluated as correct or incorrect; the goal is open and there are many correct answers (Yeo & Yeap, 2010, p. 2). Yeo & Yeap (2009) and Yeo (2013) further distinguish investigation as an activity from investigation as a process. They characterize mathematical investigation-as-a-process as “a process involving specialisation[1], conjecturing, justification and generalization” (Yeo & Yeap, 2009, p. Abstract) and distinguish it from investigation-as-an-activity by saying: “as a process, [mathematical investigation] can occur when solving problems with a closed goal and answer, while investigation as an activity involving open investigative tasks, includes both problem posing and problem solving.” (ibid.)

When students work on an open investigative task they perform investigation as an activity. Investigation as an activity involves several processes, such as understanding the task, problem posing, problem solving, checking solution and extension, whereas problem solving as a process might involve a process of investigation. Similarly, when students solve a mathematical problem they perform problem solving as an activity. Problem solving as an activity involves understanding the task, problem solving, checking solution and extension of the problem. At the same time the process of problem solving includes investigation as a process. In summary, the main difference between two models is the additional process of problem posing in the model for investigation as an activity.

It is worth noting that open investigative tasks are rare not only in school but also in the practice of mathematicians. Research in mathematics usually starts from existing or modified problems. “Novice mathematicians are immediately introduced into a problematique, a research program, with its central core of main unsolved problems and techniques that have been tried to attack them, and theories that have been built to support these techniques. So they are entering the field of mathematics via problems that someone else has already posed for them” (A. Sierpinska, personal communication, May 11, 2017).

Next, I describe and explain the CPiMI model. The diagram in Figure 7 is a reproduction of the figure 1 given by Yeo (2017, p. 339).

[1] Both in the sense of using special cases of a general statement and in the sense of specific examples.

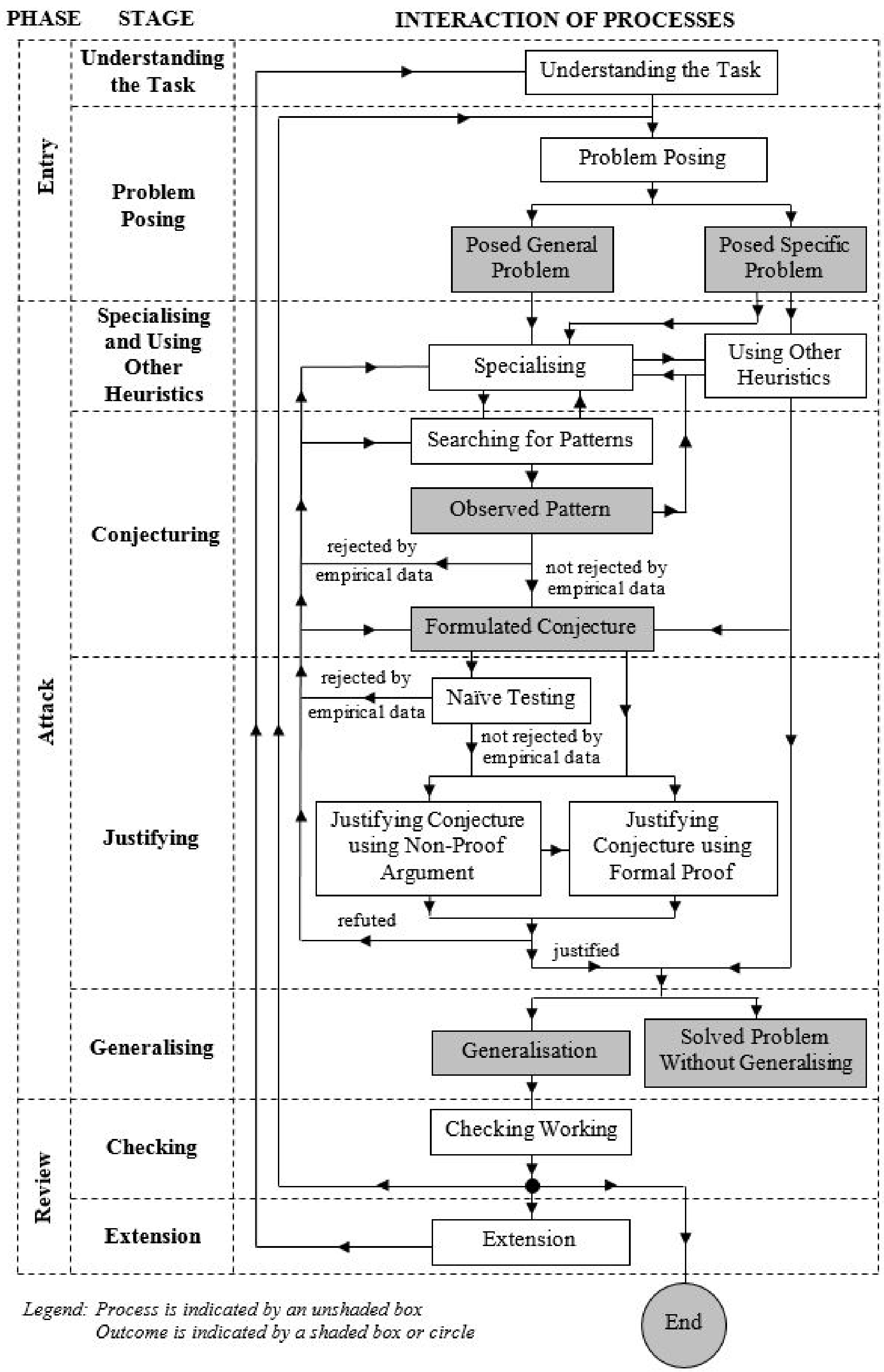


Figure 7. The CPiMI model (Yeo, 2017)

In the CPiMI model, investigation as an activity is divided into three phases: Entry, Attack and Review.

The Entry phase consists of two stages. During Stage 1, called Understanding the Task, students make sense of the task (Mason et al, 1985) by reading and analyzing the task, using examples, and visualizing the given information. The second stage is Problem Posing. This stage distinguishes investigation as an activity from problem solving as an activity. The model envisages two possible outcomes: a general problem such as "search for any pattern" is posed, or a specific problem to solve is posed.

The Attack Phase has four stages: Specializing and using other heuristics, Conjecturing, Justifying and Generalizing. During the Specializing and using other heuristics stage, students systematically try examples to search for or test patterns. However, some specific problems can be solved without specializing by using other heuristics only. Moreover, as indicated in the diagram (Figure 7) students may alternate between specializing and using other heuristics. In the next stage, Conjecturing, students may engage in the process of Searching for patterns, leading to observing a pattern, which, if not rejected by data, leads to the formulation of a conjecture. If the observed pattern is rejected by data, the student returns to searching for patterns. The Formulated Conjecture as an outcome, may, however, also be produced directly by the process of Using other heuristics in the previous stage.

According to the CPiMI model, the Justifying stage consists of three processes called: Naïve testing, Justifying conjecture using non-proof argument and Justifying conjecture using formal proof. Naïve testing was also a crucial part of Lakatos's model and its goal was to test the conjecture by looking for counterexamples to refute it. The CPiMI model assumes that if the conjecture is rejected, students may go back to reformulate the conjecture, search for new patterns or specialize. Another scenario is that students may construct a formal proof or justify the conjecture using non-proof arguments after naïve testing if they did not find a counterexample to the conjecture.

The stage called "Generalizing" in the CPiMI model is comprised of two outcomes: a generalization is obtained or the problem posed is solved but the result is not generalized. Yeo explains that Generalizing *as a process* takes place during the stages of Conjecturing and Justifying.

The Review phase contains two stages: Checking and Extension. During the Checking stage the students may check their work after solving a problem. They could have also been checking their work earlier, in other stages.

In the Extension stage, students may pose follow up questions or pose more problems to solve (Yeo, 2017).

I tried to use the categories of the CPiMI model to code the MAST 217 students' solutions of the haggling problem, in the aim of identifying the processes and the outcomes. Although I was able to label some data, many new questions arose. First, I could not code some parts of solutions in terms of the CPiMI model and felt that something is missing. Second, I was unsure about the meaning of the model's categories. For example, what is the difference between General Problem and Specific Problem? What does Generalization and Checking Working mean? Third, the order of the processes was also unclear. I found that Specializing may occur before Problem Posing and Justification may precede Formulated Conjecture. I surmised that my uncertainty and contradictions I noted were partly due to the type of data I was working with: written texts of students' solutions. Written texts do not reveal all cognitive processes occurred during solving problem and investigation. After discussing those concerns with my supervisor, we decided to add new artefacts to our research and try to analyze our own solutions of the haggling problem.

> Researchers build models of mathematical problem posing / solving / investigating based on observing students' mathematical behavior in very specific situations, those they have used in their clinical interviews or those the instructors happened to use while they were observing. Then they claim that this is what happens in ANY situation. So we are trying to use the CPiMI to people's behavior in a different situation and we immediately see that the model cannot be used as is. We have to adapt it. But is there anything in CPiMI that does apply to situations other than those Yeo has used? (A. Sierpinska, personal communication, May 11, 2017)

The next section presents the results of introspective, and what we called “inter-spective” analyses. The goal of these analyses was to test the CPiMI model and see how it can be adapted to describing a process of solving a problem that may involve investigation as a process.

4.3 Trying to understand the CPiMI model by means of introspective and inter-spective analyses

We (AS and NV) decided to examine the CPiMI model from analyzing our own solutions. Therefore in this part of research we had only two participants, an experienced professor and a novice researcher in mathematics education. The data available for analysis consisted of our personal notes of solutions to the haggling problem and the CPiMI model.

It was not easy for me to decide how to write my own solution for the haggling problem as I already went through all students solutions and discussed this task with my supervisor many times. I tried to reconstruct my initial solution and ideas behind it. As a teaching assistant in MAST 217 course I solved every problem from weekly homework assignments before I started to read students responses and grade them. I regret that I did not treat my own solutions as data for research and did not write detailed notes. By happy coincidence, I found my notes with a rough solution and used them to reconstruct my own investigative process.

General remarks about AS and NV’s solutions

AS’s solution is more detailed and contains comments about all the steps she was performing. So it is presented in a “thinking aloud” form. On the other hand, my solution looks more like a student’s work. Yes, I tried to make every step clear, but I skipped some thoughts and actions such as reading, writing and calculating. Some of those comments appeared in the second column where I explained cognitive processes in terms of the CPiMI model. Even though those comments are more about actions than interpretation, I was not sure whether or not they should be moved to the first column. My reason for leaving them in the second column was that the students’ responses are not very detailed, so reconstructing some non-written steps is a part of interpretation.

First, each of us completed an introspective analysis by using the CPiMI model. For this purpose a descriptive table was made which consisted of three columns: the solution of haggling problem, the explanation of processes in terms of CPiMI (NV) or classification of the action in terms of CPiMI categories for actions pertaining to investigation as an activity, and categories of Problem Solving as an activity (AS), and outcomes of the action in terms of CPiMI. Second, we interpreted each other's solution with our own interpretations and coding hidden. As a result, we had two solutions, two introspective analyses and two inter-spective analyses. Table 1 and Table 2 present the NV's and AS's solutions and their introspective and inter-spective interpretations.

Table 1: NV's solution of the haggling problem

		AS's interpretation		NV's interpretation	
Line	Action performed	Explanation of processes in terms of the CPiMI model	Outcome in terms of the CPiMI model	Explanation of processes in terms of the CPiMI model	Outcome in terms of the CPiMI model
	Understanding[2] the problem: making clear in one's mind the objects and relations that the problem is about				
1	The sequence of prices:	*Reading*[3] *the text of the problem and analyzing it, looking for relevant information: what is the problem about, what is given, what is to be found?*	**Posed a Specific Problem:** Re-posed the problem in mathematical terms: a sequence of numbers starts from two numbers and every next number is the arithmetic mean of the previous two. What is the limit of this sequence?	**Understanding the task**	
	Attacking the problem				
2	a b $\frac{a+b}{2}=\frac{1}{2}a+\frac{1}{2}b$ $\frac{b+\frac{a+b}{2}}{2}=\frac{1}{4}a+\frac{3}{4}b$	**Specializing:** Applying the rule given in the text of the problem: "each time taking the average of the previous two amounts" to re-calculate the third and fourth terms and calculate the fifth and sixth terms of the sequence. *Analyzing the problem*	*Computed* the first 6 terms of the sequence and represented them as combinations of a and b because that's what is suggested in the text.	**Specializing** Rewriting the initial statements.	**Observed pattern 1** Two sequences of coefficients; every time the sum of

[2] In bold, I (AS) highlight the categories of processes and outcomes that have been identified in the CPiMI model

[3] In italics, I (AS) mark actions and outcomes that characterize problem solving in general according to (Schoenfeld, 1992, p. 356): Read, Analyze, Explore, Plan, Implement and Verify. I consider the category of Explore as synonymous with Investigate, so Investigation is part of Problem Solving.

	$\frac{\frac{a+b}{2}+\frac{b+\frac{a+b}{2}}{2}}{2} = \frac{3}{8}a + \frac{5}{8}b$ $\frac{\frac{b+\frac{a+b}{2}}{2}+\frac{\frac{a+b}{2}+\frac{b+\frac{a}{2}}{2}}{2}}{2}$ $= \frac{5}{16}a + \frac{11}{16}b$		Realized that the main problem – what is the amount on which the haggling process will eventually converge – reduces to two subproblems: **Posed Specific Problems:** What is the limit of the sequence p_n of coefficients of a? What is the limit of the sequence q_n of coefficients of b?	Continue taking the average of the previous two amounts **Searching for Patterns**	coefficients of a and b is one I see that it is unclear from my solution, but I observed this pattern immediately and it may be formulated it as a conjecture. I did not use this fact later in my solution; however, after obtaining the result $\frac{1}{3}a + \frac{2}{3}b$ I made sure that $\frac{1}{3} + \frac{2}{3} = 1$
3	Coefficients of a $1, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \dots\dots$	**Problem posing**	**Posed Specific Problem**: Subproblem: What is the pattern in the sequence of coefficients of a?	**Understanding the Task** and **Problem posing** (as I turned from the initial task to analyzing the coefficients for a and b separately)	**Posed Specific Problems**
	1) 1 2) $0 = 1 - 1$ 3) $\frac{1}{2} = 0 + \frac{1}{2}$ 4) $\frac{1}{4} = \frac{1}{2} - \frac{1}{4}$ 5) $\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$	**Searching for patterns:** looking at ways that the next coefficient can be calculated from the previous one.	**Observed patterns**: the next coefficient is obtained from the previous one by subtracting or adding a power of one-half. For 2^{nd} coefficient, one subtracts; for the third – one adds. So	**Searching for Patterns** An alternating infinite series?	**Observed pattern 2**

			even index – minus, odd index – plus. Denominators are powers of 2, starting from the second coefficient: $\frac{0}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, etc.$		
5	Starting from $n = 2$ $p_n = p_{n-1} + (-1)^{n-1} \frac{1}{2^{n-2}}$	Generalizing and formalizing the observed pattern.	**Formulated conjecture**: a recursive formula for coefficients of a; the denominator of the n'th coefficient is 2^{n-2}.	Conjecturing	**Formulated conjecture** for coefficients of *a*
6	Coefficients of *b*: $0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \ldots$	**Problem posing**	**Posed Specific Problem**: What is the pattern in the sequence of coefficients of b?	**Problem posing**	**Posed Specific Problem**:
7	1) 0 2) $1 = 0 + 1$ 3) $\frac{1}{2} = 1 - \frac{1}{2}$ 4) $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ 5) $\frac{5}{8} = \frac{3}{4} - \frac{1}{8}$	**Searching for Patterns**	**Observed a pattern:** similarly as with the coefficients of a, the next coefficient of *b* is obtained from the previous one by adding or subtracting a power of one-half. But now even index corresponds to plus, and odd index – to minus.	**Searching for Patterns**	**Observed pattern 2**

			Denominators of coefficients of b are also powers of 2, with the same exponent as coefficients of a.		
8	$q_n = q_{n-1} + (-1)^n \frac{1}{2^{n-2}}$	Generalizing and formalizing the observed pattern.	**Formulated conjecture**: a recursive formula for coefficients of b; the denominator of the n'th coefficient is 2^{n-2}.	Conjecturing	**Formulated conjecture** for coefficients of *b*
	Stop, reflecting on the results obtained: <u>Planning</u>				
9	Rewrite the sequences of coefficients in terms of n only (without p_{n-1} , q_{n-1}).	*<u>Planning</u> Reflecting on the direction of the investigation; deciding that it may not be promising and changing direction* **Using other heuristics**: it may be hard to obtain the limit of a sequence from a recursive formula. A direct formula would be better.	**Posed specific problems**: to represent the n^{th} term of the sequence of coefficients of a (b) as a function of n.	**Using Other Heuristics**	
10	Look at numerators and denominators again of coefficients of a.	**Searching for patterns** in the relation between the numerators and the denominators of the coefficients of a.		**Searching for patterns**	
11	for $\frac{3}{8}$,	Observing a relationship when looking at the fifth and sixth terms.	**Observed pattern**: denominator = 3		**Observed pattern**

	$denominator = 3 \cdot numerator - 1$ for $\frac{5}{16}$, $denominator = 3 \cdot numerator + 1$		times the numerator plus or minus one		
12	for $1 = \frac{1}{1} (= p_1)$ $denominator = 3 \cdot 1 \pm 1 \neq 1$ So the pattern does not work for the first term for $0 = \frac{0}{1} (= p_2)$ $denominator = 1 = 3 \cdot 0 + 1$ for $\frac{1}{2} (= p_3)$ $denominator = 3 \cdot numerator - 1$ for $\frac{1}{4}$, $denominator = 3 \cdot numerator + 1$ So $denominator\ of\ p_n = 3 \cdot numerator + (-1)^n$	**Specializing**: checking the relationship for the first four terms **Searching for patterns**: observing that for the 3rd term there is minus 1 and for the 4th term there is plus 1.	**Observed pattern**: for coefficients with even index it is plus one; odd index corresponds to minus one	**Searching for patterns**	
13	Therefore $numerator\ of\ p_n = \frac{denominator + (-1)^{n-1}}{3}$	**Using other heuristics**: deduction by means of algebraic manipulation of an equation		**Using Other Heuristics**	
14	Denominators are powers of 2, starting from $n = 2$, as noticed before:	Generalizing and formalizing previously observed pattern	**Observed pattern**: relationship between the power of 2 in the		

	$p_n = \frac{numerator}{2^{n-2}}$		denominator and the index of the coefficient		
15	Therefore, for $n \geq 2$, $p_n = \frac{2^{n-2} + (-1)^{n-1}}{3 \cdot 2^{n-2}}$	**Using other heuristics**: algebraic substitution	**Formulated conjecture:** a formula for the n'th coefficient of a **Generalization**	**Using Other Heuristics**	**Re-Formulated conjecture** for coefficients of *a*
16	$p_6 = \frac{2^{6-2}+(-1)^{6-1}}{3 \cdot 2^{6-2}} = \frac{16-1}{3 \cdot 16} = \frac{5}{16}$ true	**Naïve testing**	**Not rejected by empirical data**	**Naïve Testing** (not rejected by empirical data)	
17	Coefficients of b will satisfy a similar relation with n $q_n = \frac{2^{n-1} + (-1)^n}{3 \cdot 2^{n-2}}$	**Using other heuristics:** analogy, taking account of the differences with the sequence of coefficients of *a*	**Formulated conjecture**: a formula for the n'th coefficient of b.	**Using Other Heuristics** Guessing the formula for coefficients of b in terms of n	
18	$n = 6$, $q_6 = \frac{5}{8} + (-1)^6 \frac{1}{2^{6-2}} = \frac{5}{8} + \frac{1}{16} = \frac{11}{16}$	**Naïve testing**	**Not rejected by empirical data**	**Naïve Testing** (not rejected by empirical data)	
19	Now we can calculate the limit of the sequence of amounts in the haggling process: $lim_{n \to \infty} p_n a + q_n b = ?$	*<u>Planning</u> what to do next: to compute a limit using properties of limits of sequences*			

20	$\frac{2^{n-2}+(-1)^{n-1}}{3\cdot 2^{n-2}}a + \frac{2^{n-1}+(-1)^{n}}{3\cdot 2^{n-2}}b = (\frac{1}{3}-\frac{(-1)^{n-2}}{3\cdot 2^{n-2}})a + (\frac{2}{3} + \frac{(-1)^{n-2}}{3\cdot 2^{n-2}})b$ Let $n \to \infty$, then $lim_{n\to\infty}[(\frac{1}{3}+\frac{(-1)^{n-1}}{3*2^{n-2}})a + (\frac{2}{3}+\frac{(-1)^{n}}{3*2^{n-2}})b] = \frac{1}{3}a + \frac{2}{3}b$	<u>*Implementing the plan*</u>	***Solved problem End***	**Justifying Conjecture using Formal Proof**	**Solved Problem**

Table 2: AS's solution of the haggling problem

		AS's interpretation		NV's interpretation	
Line	Action performed	Classification of the action in terms of CPiMI categories for actions pertaining to investigation as an activity, and categories of Problem Solving as an activity	Outcomes of the action in terms of CPiMI	Classification of the action in terms of CPiMI categories for actions pertaining to investigation as an activity	Outcomes of the action in terms of CPiMI
	Understanding[4] the problem: making clear in one's mind the objects and relations that the problem is about				
1	Reading the text of the problem, and stopping to reflect on the rule given in the problem text: "each time taking the average of the previous two amounts". Does this rule apply already to the third and fourth amounts?	*Questioning the claims made in the statement of the problem*[5]	**Posed specific question**: Does the rule "each time taking the average of the previous two amounts" apply to the 3rd and 4th amounts? (SP0)	**Understanding the Task** In particular, understanding "the rule of the game"	**Posed Specific Problem is "**On what amount will they converge? Express the amount in terms of **a** and **b**"
2	[Writing, calculating] a b $\frac{a+b}{2}=\frac{a}{2}+\frac{b}{2}$ $\frac{b+\frac{a}{2}+\frac{b}{2}}{2}=\frac{a}{4}+\frac{3b}{4}$	Solving SP0: *Re-calculating* the first four amounts	*Computed* the 3rd and 4th amounts in the haggling process according to the rule "each time taking the average of the previous two amounts" and representing them as combinations of a and b because that's	Rewriting in order to understand the problem and check that applying the rule leads to the same first amounts	

[4] Highlighted in bold are terms that belong to CPiMI.
[5] Highlighted in italics are categories of cognitive actions and processes involved in problem solving in general.

			how they appeared in the text.		
3	[Looking back at the values above and comparing them with the values given in the text of the problem]	**Checking** the claims of the author of the problem	*Verified* that the results obtained this way are the same as those given in the text of the problem. Solved SP0: Yes.		
4	Reading the question of the problem: "On what amount will they converge? Express the amount in terms of a and b."	**Understanding** the main question of the problem (saying it in different words)	*Re-formulated* the Main Question: **MQ** $lim_{n\to\infty} H_n =?$ where H_n are the successive amounts in the haggling process. $H_n = f(a,b)$	It is still **Understanding the Task**. At the same time, it is **Problem Posing**	
	Attacking the problem				
5	[Looking at the form of the first four terms calculated above] $1 \cdot a + 0 \cdot b$ $0 \cdot a + 1 \cdot b$ $\frac{1}{2}a + \frac{1}{2}b$ $\frac{1}{4}a + \frac{3}{4}b$	**Searching for patterns**	**Observed pattern**: the amounts are linear combinations of a and b. *Understood the intention behind the text of the problem*: Aha! So that's why the third and fourth amounts in the text of the problem were written in this weird way. The author of the problem was	**Searching for Patterns**	**Observed pattern**

			hinting at this pattern for us.		
6	[Looking at the coefficients by a and b in the lines] $1+0=1$ $0+1=1$ $\frac{1}{2}+\frac{1}{2}=1$ $\frac{1}{4}+\frac{3}{4}=1$	**Searching for patterns**	**Observed pattern**: the sum of the coefficients by a and b is equal to 1.	**Searching for Patterns**	**Observed pattern**
7	Fifth amount = $\frac{1}{2}((\frac{1}{2}a+\frac{1}{2}b)+(\frac{1}{4}a+\frac{3}{4}b))=$ $=\frac{1}{2}(\frac{1}{2}+\frac{1}{4})a+\frac{1}{2}(\frac{1}{2}+\frac{3}{4})b=$ $\frac{1}{2}(\frac{3}{4}a)+\frac{1}{2}(\frac{5}{4}b)=\frac{3}{8}a+\frac{5}{8}b$	**Specializing**: computing the fifth amount **Searching for patterns**: keeping record of the numbers added without writing the sum right away (so treating concrete numbers as variables)		**Specializing** Applying "the rule" to calculate fifth amount	
8	[Looking at the process of calculating the coefficient by a]	**Searching for patterns** in the coefficients by a.	**Observed pattern**: the next coefficient by a is half of the sum of the previous two.	**Searching for Patterns**	
9	$A_1=1$ $A_2=0$ $A_3=\frac{1}{2}$ $A_4=\frac{1}{4}$	**Specializing**: Checking the pattern on the first five terms.	Observed pattern **not rejected by data**: Confirmation of the pattern on the first five terms.	**Searching for Patterns** and observing a pattern from fifth line (coefficient A_5)	**Observed Pattern:** The coefficient by a is one half of previous two coefficients by a

	$A_5 = \frac{1}{2}(\frac{1}{2} + \frac{1}{4}) = \frac{3}{8}$				
10	$A_{n+2} = \frac{1}{2}(A_n + A_{n+1})$	*Formulating a conjecture*	**Formulated Conjecture** 1: starting from the third amount the next coefficient by a is the average of the previous two.	**Problem Posing**	**Formulated Conjecture 1 Posed Specific Problem (is Conjecture 1 a specific problem or sub-problem?)**
11	How can we prove analytically that Conjecture 1 is true?	**Problem posing**	**Posed a Specific Problem**: Sub -problem 1		
12	We will worry about this later. For now let's investigate the sequence assuming that the conjecture is true.	*Planning* what to do next		Making a decision about strategy Understanding a sub-problem	
13	This looks like a modified Fibonacci sequence. Is it convergent?	**Problem posing**	**Posed Specific Problem**: Sub-problem 2: Is the sequence of coefficients by a convergent?	Conjecturing **Searching for Patterns**	
14	Multiplying numbers by one half makes them smaller and smaller. Perhaps the sequence is strictly decreasing. Since it is bounded below by 0,	**Using other heuristics**: thinking about sufficient conditions for a sequence to convergent and asking if the sequence satisfies them.	**Formulated Conjecture:** Conjecture 2: The sequence of coefficients by a is strictly decreasing.		**Formulated in words Conjecture 2**

	then we would have proved that is convergent.				
15	But looking at the numbers $A_1, .., A_5$ we see that the sequence is not strictly decreasing; it is oscillating. Computing several more terms of the sequence with a computer algebra system (Maple), we see that the conjecture is false: $a1 := 1 :$ $a2 := 0 :$ **for** n **from** 1 **to** 10 **do** $a3 := \frac{1}{2} \cdot (a1 + a2) :$ $print(n + 2, a3, evalf(a3)) :$ $a1 := a2 :$ $a2 := a3 :$ **end do**: $3, \frac{1}{2}, 0.5000000000$ $4, \frac{1}{4}, 0.2500000000$ $5, \frac{3}{8}, 0.3750000000$ $6, \frac{5}{16}, 0.3125000000$	**Naïve testing** (two rounds)	Conjecture 2 **rejected by empirical data**	**Naïve Testing** Rejecting Conjecture 2 by empirical data **Specializing** Using numerical approach	

	$7, \frac{11}{32}, 0.3437500000$ $8, \frac{21}{64}, 0.3281250000$ $9, \frac{43}{128}, 0.3359375000$ $10, \frac{85}{256}, 0.3320312500$ $11, \frac{171}{512}, 0.3339843750$ $12, \frac{341}{1024}, 0.3330078125$				
16	[Looking at the decimal approximations] But it looks like the sequence is converging to one-third.	**Searching for patterns** *Observing a pattern* in the decimal digits of the approximations of the coefficients by a.	**Observed Pattern**: more and more three's in A_n as n grows larger	**Searching for Patterns**	**Observed Pattern**
17	$lim_{n \to \infty} A_n = \frac{1}{3}$	*Formulating a conjecture* about the convergence and limit of the sequence of coefficients [Conjecture 3]	**Formulated Conjecture** 3: the limit of the sequence of coefficients by a is $\frac{1}{3}$.		**Formulated conjecture 3**
18	How can we prove analytically that the sequence is convergent?	**Problem Posing**	**Re-Posed Specific Problem**: Sub-problem 2: To prove analytically that the sequence of coefficients by a is convergent.	**Problem Posing**	
19	We will try to solve the sub-problem 2 later.	*Planning the next step*		What is a sub-problem 2?	

20	If we know that the sequence is convergent, how can we prove analytically that the limit is $\frac{1}{3}$?	**Problem Posing**	**Posed Specific Problem**: Sub-problem 3: Assuming the sequence of coefficients by a is convergent, to prove that the limit is $\frac{1}{3}$. (based on Conjecture 3)	**Problem Posing**	
21	Let's try to prove that the limit is $\frac{1}{3}$. Using the Conjecture 1, and the technique of finding limits of sequences defined by recurrence relations, we could write: $L = \frac{1}{2}(L + L)$, where L is the limit. This gives an identity, from which nothing can be deduced about the limit.	**Justifying Conjecture 3/ solving Sub-problem 3 using formal proof**:	*Failed at proving* that if the sequence A_n is convergent then its limit is $\frac{1}{3}$. (inappropriate technique)	**Justifying** An attempt to justify Conjecture 3 using formal proof Refuting	
22	Perhaps there is a different recurrence relation between the terms. Let's look again at the first terms of the sequence of coefficients by a : $A_1 = 1$ $A_2 = 0$	**Searching for patterns**		**Searching for Patterns** by looking again at the results of previous specializing	

	$A_3 = \frac{1}{2}$ $A_4 = \frac{1}{4}$ $A_5 = \frac{1}{2}(\frac{1}{2} + \frac{1}{4}) = \frac{3}{8}$				
23	Observe: $A_5 = \frac{1}{2}(\frac{3}{4}) = \frac{1}{2}(1 - A_4)$		**Observed a pattern**		**Observed Pattern**
24	$\frac{1}{2}(1 - A_3) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4} = A_4$ $\frac{1}{2}(1 - A_2) = \frac{1}{2}(1 - 0) = \frac{1}{2} = A_3$ $\frac{1}{2}(1 - A_1) = \frac{1}{2}(1 - 1) = 0 = A_2$	**Specializing**		**Specializing**	
25	Generalize: $A_{n+1} = \frac{1}{2}(1 - A_n)$		**Formulated Conjecture** 4: The next term of the sequence of coefficients by a is one-half of the complement to 1 of the previous term.	Generalizing	**Formulated Conjecture 4**
26	This conjecture can be proved analytically, by induction, showing that the sequence obtained by the recurrence relation in Conjecture 4 is identical to the sequence obtained by the relation in	**Justifying Conjecture 4 using formal proof** (sketch only described here)		Making a decision about strategy Understanding a sub-problem	

	Conjecture 1. For the proof to be complete we would have to have proven Conjecture 1.				
27	The relation in Conjecture 4 gives an equation on the limit of the sequence A_n that has a single solution: $L = \frac{1}{2}(1 - L)$ is true for $L = \frac{1}{3}$.		**Solved problem:** Solution of Sub-problem 3. Formal proof of Conjecture 3.	**Justifying Conjecture 3 using formal proof**	**Solved problem**
28	But it is still not proved that the limit exists. It would help to have an expression of A_n as a function of n.	**Problem Posing**	**Posed a problem**: Sub-problem 4: To express the n^{th} coefficient by a as a function of n.	**Problem Posing** Filling the gap in justification for Conjecture 4	**Posed Specific Problem**
29	We will look again at the initial terms of the sequence and try of find a different pattern, depending on n and not on the previous term.	*Planning*		**Searching for Patterns**	
30	$A_4 = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{2}(\frac{2-1}{2}) = \frac{2-1}{2^2}$ $A_5 = \frac{1}{2}(1 - \frac{2-1}{2^2}) = \frac{2^2-2+1}{2^3}$	**Searching for patterns**: keeping record of the numbers added without writing the sum right away (so treating concrete numbers as variables)		**Specializing**	

	$A_6 = [calculations] = \frac{2^3-2^2+2-1}{2^4}$				
31	Probably: $A_7 = \frac{2^4-2^3+2^2-2+1}{2^5}$	**Observing a pattern**	**Formulated Conjecture 5:** the 7th coefficient is (as shown in the left column)	**Searching for Patterns** and Conjecturing	**Observed Pattern**
32	We compare the number on the right with the number obtained before with Maple program: both are equal to $\frac{11}{32}$.	**Naïve testing** of Conjecture 5	**Conjecture 5 justified**	**Searching for Patterns**	
33	Conjecture: $A_n = \frac{2^{n-3}-2^{n-4}+\cdots+(-1)^{n-3}}{2^{n-2}}$	*Generalizing*	**Generalization: Formulated Conjecture** 6	Generalizing	**Formulated Conjecture 5**
34	The numerator can be represented in the form of a closed expression (without dots), using the formula: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}))$	**Using other heuristics**: structure recognition		**Using Other Heuristics**	
35	$A_n = \frac{(2^{n-2}-(-1)^{n-2})}{3\cdot 2^{n-2}}$	**Using other heuristics:** representing an expression in a different way	Re-**formulated Conjecture 6** in the form of a closed	**Using Other Heuristics**	New form of **Formulated Conjecture 5**

			expression with variable n only.		
36	$A_n = \frac{1}{3}(1 - \frac{(-1)^{n-2}}{2^{n-2}})$	**Justifying Conjecture 6 using Formal proof**	First, re-**formulated Conjecture** 6 in a form convenient for showing that the sequence is convergent.	**Using Other Heuristics**	New form of **Formulated Conjecture 5**
37	Using properties of limits of sums and products of convergent sequences, we conclude that the sequence A_n is convergent and its limit is $\frac{1}{3}$.	*Proving*	Provided a **formal proof** of convergence and calculating the limit.	**Justifying Conjecture using Formal Proof**	
38	Writing the solution to the problem: 1. Proving that the sequence of coefficients of a as defined in the problem can be represented in the form obtained in Conjecture 6. 2. Proving that the sequence is convergent and that its limit is $\frac{1}{3}$. 3. Proving that the coefficients by b are equal to 1 minus the coefficients of a, and that, in the limit, the	*Writing up the solution, without describing the whole process of investigation.*	**Solved MQ**	Constructing formal proof to present to others	**Solved Problem**

	coefficient by b is equal to $1 - \frac{1}{3} = \frac{2}{3}$. 4. Concluding that the haggling process will converge to the amount of $\frac{1}{3}a + \frac{2}{3}b$.				

Discussion of NV's and AS's interpretations

After completing our analyses of NV's and AS's solutions, we compared them independently and wrote our own reflections. In this section, I discuss results of our introspective and inter-spective analyses and use fragments from our reflections to demonstrate what differences and similarities we pointed out. I analyzed our interpretations line by line and observed that we coded some parts almost identically; however, there were several significant differences in our interpretations. In the following I organize our agreements and disagreements about using the particular categories as a discussion of each stage described in the CPiMI model.

Stage 1: Understanding the Task

It is not easy to observe Understanding the Task in written solutions; however, both NV and AS agreed that Understanding the Task is an inevitable part of problem solving and coded the first lines of their solutions in similar way. According to Polya (1973), one may make sense of a problem by asking questions such as "What is the unknown? What are the data? What is the condition?" To be more specific, AS outlined categories that characterize problem solving in general according to Schoenfeld (1992) such as *Reading* and *Analyzing.*

Since Understanding the problem as a process looks very general in the CPiMI model, I decided to add a few sub-processes in the coding scheme, such as *Reading and rewriting in order to understand the problem, Analyzing, Reformulating the problem, Trying examples to understand the problem.* In his doctoral dissertation Yeo (2013) stated that there are three possible outcomes in the Understanding the task stage: 1) understood the task correctly, 2) misinterpreted the task and did not recover and 3) misinterpreted the task but recovered from the misinterpretation. As I analyze written responses I am not able to see the third possible outcome. At the same time it is important to indicate ***Errors or mistakes*** that occurred during in the Understanding the task stage. Therefore I added this category to the coding scheme.

Stage 2: Problem Posing

This is the most confusing part of coding. Even though both AS and NV, used the categories of Problem Posing for process and Posed Specific Problem for outcomes in a similar way in different parts of their solutions, I noticed that our understanding of Problem Posing is different from Yeo's view. Here is an excerpt from my notes:

> Table 1, Line 3. In terms of the CPiMI the subproblem "What is the pattern in the sequence of coefficients of a?" is **Posed General Problem**.
>
> … So they (students) may just set a general goal by searching for any pattern (Height, 1989). The latter can be called the posing of the general problem "Is there any pattern?" (Yeo and Yeap, 2010, p. 2)
>
> Table 2, Line 1. I suppose that Posed specific question is not an outcome in terms of CPiMI because it is not the same as **Posed Specific Problem**

I concluded that there is a need to clarify terms such as General Problem, Specific Problem, Subproblem in the context of my study.

The haggling problem is not an investigative task since the problem is already posed. It is a problem solving task which requires an investigation as a process. According to the CPiMI model, Stage 2 (Problem posing) occurs in the Entry Phase after Stage 1 (Understanding the Task) and before the process of investigation which involves specializing, conjecturing, justifying and generalizing. There are two possible outcomes: posed the general problem of searching for any pattern or posed a specific problem to solve. Formally Stage 2 should not be in the model for problem solving activity. However, solving the haggling problem may involve investigation as a process and requires posing sub-problems or/and reformulating the main problem. In our introspection and 'inter-spection' we (AS and NV) used the code Problem Posing many times in the same way. To resolve this problem I referred to the Schoenfeld's (1985) work on problem-solving. As it was mentioned in the literature review, he introduced four categories of knowledge necessary to be successful in problem-solving: *resources, heuristics, control, and belief systems.* Heuristics or problem solving strategies include induction, specialization, analogy, variation, decomposition and recombining, working backwards. Therefore, establishing sub-goals and solving sub-problems take place during Specializing and Using Other Heuristics Stage. It is necessary to point out that students may use both specializing and using other heuristics; or they can alternate between them. However, according to the CPiMI model, using other heuristics is not part of an investigative process.

Stage 3: Specializing and Using Other Heuristics

In my interpretation, I used the category of Specializing in more general sense that it is required the CPiMI model:

> Table 1, Line 2. I wrote "Specializing" and treated it as a part of Entry Phase, because I specialized in order to understand the problem. I do believe that Specializing can occur before Attack Phase. Moreover, in most cases Specializing helps not only to understand the problem but also pose or reformulate it. I agree that *Analyzing* the problem is the best explanation of the process behind this action.

Therefore I proposed to separate *Trying examples to understand the problem* from Specializing (*Trying examples to search for patterns).*

Even though we did not have difficulties in identifying the category Using Other Heuristics, I found that it would be helpful for analyzing students' solutions to specify what kind of heuristics is used.

> "We have both thought of this category here. So this category is rather clear" (A. Sierpinska, personal communication, June 8, 2017).

Stage 4: Conjecturing

Also introspection and 'inter-spection' have showed that it is not easy to distinguish the transition between Specializing and Searching for patterns.

> Table 2, Line 30. Since we calculate A_4, A_5, A_6 again and rewrite them in different form I would prefer to add Specializing to the interpretation of this process. Actually I think both Specializing and Searching for patterns take place here.

> Table 1, Line 12. "So Searching for pattern is not an easily observable process. We can only surmise that there has been a search for patterns if the student has written down a

formula or something to that effect: if a pattern has been observed*"* (A. Sierpinska, personal communication, June 8, 2017)

The CPiMI model does not include Conjecturing as a process. The outcome Formulated Conjecture is a result of Searching for Patterns or Using other Heuristics.

Table 1, Line 5. Conjecturing as a process is not a part of the CPiMI model. It can be explained as Generalizing, but according to the CPiMI model the Generalizing stage should come after the Justifying stage.

Table 2, Line 31. We interpreted this part of the solution differently. I identified only one Conjecture. Even with word "probably" I treated $\mathrm{A}_7 = \frac{2^4-2^3+2^2-2+1}{2^5}$ as an Observed pattern, not as a Conjecture.

Stage 5: Justifying

We used the code Naïve Testing identically then we analyzed our solutions, therefore we concurred that this category is pretty clear. However, the analysis of students' solutions made me think that the difference between Naïve Testing and Checking is not obvious. I will discuss this later.

Solving the haggling problem may involve posing subproblems and formulating more than one main conjecture. Analyzing our solutions we used the category of Justifying Conjecture using Formal Proof mostly to code proofs of small conjectures. The final result was obtained by implementing the plan.

Table 1, Line 20. Probably **Justifying Conjecture using Formal Proof** does not perfectly describe the action performed here. I would suggest it is **Using Other Heuristics** again. I think we can use *Planning and Implementing the plan* to interpret this part as it looks like problem solving, rather than mathematical investigation in the strict sense

Table 2, Line 36. I would like to be more precise, so from my point of view this step can be interpreted as **Justifying Conjecture 3:** the limit of the sequence of coefficients by a is $\frac{1}{3}$ **using Formal Proof**

Therefore, I adopted *Implementing the plan* as well as *Planning* from Schoenfeld's (1995) problem solving model.

Stage 6: Generalizing

As it was mentioned above, Conjecturing and Generalizing have similar meanings. For the haggling problem there are no outcomes: **Generalization** and **Solved problem Without Generalizing**. I proposed two codes for this stage: **Solved Sub-problem** and **Solved Problem.**

Stage 7: Checking

AS coded line 3 in Table 2 as "Checking the claim of the author of the problem." This is different from the meaning of Checking used in the CPiMI model. Yeo (2013) notes that "students can check all the working step by step, or they can just check the essential steps" (p. 78). So it is not clear what exactly students are doing and how we can see this process in written responses. In Polya's (1973) problem solving model the fourth phase calls "looking back" and includes examination of the solution by answering questions:

> Can you check the results? Can you check the argument?
>
> Can you derive the result differently?

Similarly, Schoenfeld (1985) stressed that Verification (checking) plays an important role in the problem solving process. Thus, "at a local level, you can catch silly mistakes. At a global level, by reviewing the solution process you can often find alternative solutions, discover connections to other subject matter …. and that can help you become a better problem solver " (p. 111). Solving the haggling problem encourages students do not ignore this stage by asking: Have you tried to verify your answer? If yes, how?

Therefore, I decided to apply the code Checking (Verifying) in the analysis of students' solutions in a very specific situation when students explain how they verified their answer.

Stage 8: Extension

We have agreed that students were not expected to extend the haggling problem, so we made no effort at the time to formulate possible extensions in our solutions. Surprisingly, one of the students[6] was very close to formulation an extension for the haggling problem, so I decided to keep this code.

Also we found that, for several lines of the solutions, we could not use the CPiMI model's categories. At the same time we coded those lines in similar manner. For instance, in Table 2, Line 12 AS states: "We will worry about this later. For now let's investigate the sequence assuming that the conjecture is true."

AS's interpretation was: *Planning* what to do next.

NV's interpretation was: Making a decision about strategy.

Thus another issue regarding analysis of haggling problem is that the CPiMI model cannot help to indicate metacognitive processes. Schoenfeld (1992) emphasized how important it is to control, monitor, and self-regulate our thinking. He found that students had spent much more time on exploring with calculation than on analyzing, planning, implementing or verifying a solution. On the contrary, mathematicians had spent a lot of time on planning and analyzing and had demonstrated the tendency to alternate between planning and analyzing. As the CPiMI model does not capture those important cognitive and metacognitive processes, I proposed to add more categories in the coding scheme such as *Planning* and *Monitoring*.

In conclusion, the introspective and inter-spective analyses have helped us to outline our difficulties in using the CPiMI model, identify gaps, and formulate the list of terms to clarify,

[6] See the case of student #009

and propose additional categories needed to adapt the model for coding problem-solving activities involving investigation as a process.

4.4 Arriving at a modified model of mathematical thinking

In this section, I present the final coding scheme I adopted for analyzing the MAST 217 students' solutions of the haggling problem. Based on the introspective and inter-spective analyses of our solutions the CPiMI model was modified to be used in the context of this study and additional codes were added. Once I compiled the list of cognitive and metacognitive processes, and outcomes I grouped them according to the stages of the CPiMI model. In Table 3, non-italicized codes represent processes outlined in the CPiMI model; codes in *italic* represent additional processes, sub-processes, and outcomes, and codes in **bold** represent outcomes.

Table 3. Final coding scheme

Phase	Stage	Code
Entry	**Understanding the Task**	*Reading and rewriting in order to understand the problem* *Analyzing* *Trying examples to understand the problem* *Reformulating the problem* ***Reformulated problem*** ***Error or mistake***
Attack	**Problem Posing**	*Planning* ***Decided on plan*** Posing sub-problem **Posed Sub-Problem**
	Specializing and Using Other Heuristics	Specializing (trying examples to search for patterns) ***Error or mistake*** Using Other Heuristics *Calculating* *Using Algebra* *Planning* *Implementing the plan*
	Conjecturing	Searching for patterns *Using Algebra* **Observed Patterns** *Planning* Using other heuristics

		Formulated Conjecture
	Justifying	*Planning* *Implementing the plan* *Calculating* Naïve testing Justifying Conjecture using formal proof Justifying Conjecture using reasoning *Monitoring*
	Generalizing	*Generalizing* ***Error or mistake*** **Solved Sub-Problem** **Solved Problem**
Review	**Checking**	*Checking* *Calculating* ***Error or mistake***
	Extension	Extension

As it was mentioned earlier there was constant comparison and interaction between data collection, analysis and development of theory. Thus the coding scheme presented in Table 3 is a result of several rounds of analyses of students' solutions during which the codes were revisited and adjusted.

Chapter 5. Application of the modified CPiMI model to analyzing students' solutions

In the next stage of my data analysis, I examined each student's response to the haggling problem. I chose four solutions of the haggling problem to present in this thesis in detail because they illustrate different investigative behaviors and approaches to solving the haggling problem. In addition, they demonstrate that the modified codes allow us to analyze both correct and incorrect solutions. A summary of the results is presented in the section 5.5 of this chapter.

5.1 The case of student #009

Student #009 was successful at solving the haggling problem. Table 4 shows an analysis of his[7] solution. He analyzed the problem, denoted the sequence of amounts and calculated several amounts applying the rule given in the text of the problem. He then searched for patterns and observed that the amounts are linear combinations of the initial amounts a and b and that the sum of coefficients of a and b is 1. The student used this conjecture later but did not seem to feel the need to prove or somehow justify it. He looked for the limit of the sequence of coefficients of a and clearly demonstrated that he planned and monitored his problem-solving process. He did not explain or justify the existence of the limit of the sequence. He made some mistakes in notations and was not precise in using his mathematical knowledge. For example, he recognized a geometric series with first term equal to $\frac{1}{2}$ and the ratio $-\frac{1}{2}$, but did not mention that the formula for the sum of the geometric series can be used since the ratio is less than 1. Despite those minor errors he arrived at the final (correct) answer. The student checked his solution by naïve testing and also by solving a special case of the problem in a different way. He observed that particular initial amounts are also partial sums of a convergent geometric series (in the case of first term 100 and ratio -1/2) and in fact formulated another conjecture. This can be considered as the beginning of an extension of the problem. If pushed a little bit, the student could be led to asking questions such as: Is it just a coincidence that the sequence of amounts, when a=100 and b=50, is identical with the sequence of partial sums of a geometric series with first term equal to a and an

[7] Here and throughout the analyses of students' solutions I use the pronoun "he" as a generic pronoun.

appropriate ratio q? Or is it more general? What conditions should a, b and q satisfy for the two sequences (one obtained by averages and the other – by partial sums of a geometric series) to be identical? In summary, student #009 did not justify every step of his solution; however, he definitely achieved the conviction that his answer is correct. His calculations together with the analysis of his solution removed all doubts.

Table 4: Analysis of the solution of student #009

	Action performed	Processes	Outcomes	Remarks
1	a(1)=a a(2)=b a(3)=(a+b)/2=(a(1)+a(2))/2=1/2a + 1/2b a(4)=1/4a+3/4b	*Reading and rewriting in order to understand the problem* *Analyzing* *Trying examples to understand the problem*		The student denotes the sequence of amounts obtained in the haggling process by "a(n)" and re-writes the values of the amounts given in the formulation of the problem using this notation ($n = 1, 2, 3, 4$). (Student used plain text to write his solution and submitted it online.)
2	a(5)=3/8a + 5/8b a(6)=5/16a +11/16b ...	Specializing		The student applies the rule "each time taking the average of the previous two amounts" to calculate a(5) and a(6).
3	xa+yb =1	Searching for patterns	**Observed Pattern 1** **Formulated conjecture 1** ***Error or mistake***	This statement is incorrect. Probably, the student observes that the amounts are linear combinations of the initial amounts a and b and that the sum of coefficients of a and b is 1. So he may mean that coefficient by a plus coefficient by b is equal to 1. He should has written x+y =1 where x and y are coefficients of a and b, but his notation is more like shorthand for a phrase than operational symbolism (Clement, 1981).

4	Let us look for x	Posing sub-problem *Planning*	**Posed sub-problem 1** ***Decided on plan***	At this point the student decides to focus on coefficients of *a* and finding the limit of the sequence of coefficients of *a*.
5	x=1,0,1/2,1/4,3/8,5/16...	Searching for patterns	***Error or mistake***	Student's notation is not correct but he probably tries to write a sequence of coefficients of a
6	x[n]=1-1+1/2-1/4+1/8-1/16... x(n+2)=1/2-1/4+1/8-1/16...		**Observed Pattern 2**	The student notices that starting from the third term, the coefficients of a seem to be partial sums of a geometric series with first term equal to $\frac{1}{2}$ and the ratio $-\frac{1}{2}$.
	Let us take the first term of x(n+2), and -1/2=q,	Using other heuristics	**Formulated conjecture 2**	The student formulates this conjecture almost explicitly.
7	for the limit of the geometric series $L = \frac{\frac{1}{2}}{1-(-\frac{1}{2})} = \frac{1}{3}$	Using other heuristics *Using algebra* *Calculating*	**Solved sub-problem 1**	The student implicitly assumes that in the final amount the coefficient of a will be the limit of the sequence of the coefficients of a in the haggling process, and calculates the sum of the geometric series using the formula: $a + aq + aq^2 + \cdots = \frac{a}{1-q}$ This is correct since the ratio of the series is less than 1 in absolute value, but we don't know how aware he is of this. He is probably aware of the legitimacy of ignoring the first

				two coefficients of a using the fact that their sum is 0.
8	x=1/3 at the limit =>y=2/3 The price will converge to 1/3a+2/3b	*Implementing the plan*	**Solved problem**	To formulate the conjecture, student uses the previous conjecture 1 Note that he did not justify the conjecture 1. He accepted the fact that the sum of coefficients of a and b is always 1.
9	a(1)=100 a(2)=50 a(3)=(100+50)/2=75 a(4)=75+50/2=62.5 ... This is a geometric series Sn=100-50+25...=100-50/2+25/4... L=100/1-(-1/2)=66+2/3 If we compute with 1/3a and 2/3b => 1/3(100) + 2/3(50)=66+2/3	Naïve testing Checking (by another solution) Extension Checking	 **Observed pattern** **Formulated conjecture**	He takes $a = 100$ and $b = 50$ and computes the first amounts in order to make sure that his conjecture is correct. In the process, he notices that the total amounts quoted in the haggling process with these particular initial amounts are also partial sums of a convergent geometric series (first term 100 and ratio -1/2) and formulates it as yet another conjecture. Then he calculates the sum of this geometric series using the formula S=a/(1-q) and tests it with calculations and with his previously formulated conjecture for the amount to which the haggling process converges, $\frac{1}{3}a + \frac{2}{3}b$, for the case of $a = 100$ and $b = 50$.

5.2 The case of student #010

Table 5 presents an analysis of student #10's solution. To solve the haggling problem this student decided to simplify it. He parametrized the segment between b and a on the number line and re-formulated the problem. It is likely that the student thought about convergence of the sequence of amounts when he proposed scaling of the number line and visualized the haggling process. He used different heuristics and demonstrated good problem-solving skills. At the same time the student appeared to have some difficulties in communicating his ideas. He was sloppy about mathematical notations. For example, he used the same letter 'n' to denote two different sequences. However, specializing led him to the conclusion that starting from the second term there is a geometric series with first term equal to -1 and ratio -1/2. Finally he calculated the sum of the geometric series and solved the problem. He planned his solution and had control over it. To validate his solution the student calculated sequences of amounts for several pairs of a and b in Excel. He was satisfied with the results of naïve testing and stated that "sure enough, for any values of a and b I experimented with, the final equation held". He did not consider numerical results as a proof (he used the word 'experimented'), but together with his solution they looked convincing for him.

Table 5: Analysis of the solution of student #010

	Action performed	Processes	Outcomes	Remarks
1	We aren't given the values of a and b, but we do know that the first value, a is the highest term in the sequence and the second term, b, is the lowest.	*Analyzing*		
2	We also know the amount they will converge on is somewhere between a and b. If we assume b to be 0 and a to be 1 we'll converge on a number between 0 and 1. If we call that number x and convert it to % then the final amount will be x% of the way from b (lowest term) to a (highest term).	*Analyzing* *Reformulating the problem (by representing the relations between the givens and the unknown in a different way)*		The student represents the given numbers a and b on the number line, putting its origin at b and 1 at a. This way, the distance between a and b $(a-b)$, since $a > b$) becomes the unit of distance on this number line. This allows him to represent the unknown price on which the haggling process will converge as a percent of the distance between a and b. This may look like "using examples" to understand the problem, but, in fact, the student uses the technique of convenient scaling of the number line. This can also be seen as parametrizing the segment between b and a on the number line.
3	This can be represented as b + x(a-b)		***Reformulated problem***	
4	n1 = 1 (our chosen value for a) n2 = 0 (our chosen value for b) n3 = ½ (the average of the last two terms) n4 = ¼ (the average of the last two terms)	Specializing		He seems to write the consecutive amounts quoted in the haggling process, in two ways: as values of the

	n5 = 3/8 (again the average of the last two terms)			coefficient x in $b + x(a - b)$, as if assuming $a = 0, b = 1$, and, in brackets, in terms of the variables $a,$ and b.
5	If for every n we look at the difference between n and n-1, we get a pattern with the first few terms as: n1 = 1 n2 = -1 n3 = +1/2 n4 = -1/4 n5 = 1/8	Searching for patterns	***Error or mistake***	Looking for the relationship between two consecutive terms of the sequence n_i, can we obtain the next one from the previous one? His notation is incorrect: he uses the same letters to mean different things. Only "n1" in line 5 means the same as "n1" in line 4. The expression "n2=-1" in line 5 probably means "n2 – n1 = 0 - 1" with n2 and n1 meaning the values in line 4. It may mean also that, to obtain n2, one has to subtract 1 from n1. In view of what the student writes in line 7 ("sum"), he may mean: n1 = 1 n2 = 1 – 1 n3 = 1 – 1 + ½ n4 = 1 – 1 + ½ - ¼ n5 = 1 – 1 + ½ - ¼ +1/8
6	In each case the next n is equal to the previous n multiplied by -1/2.		**Observed Pattern**	Here "the next n" appears to refer to the terms of the sum representing n_i: If $n_i = k_1 + k_2 + \cdots + k_i$ then, for $i \geq 3$,

				$k_1 = 1$ $k_2 = -1$ $k_{j+1} = k_j \cdot (-\frac{1}{2})$ for $j = 3, \ldots, i$
7	Taking the sum of all n from n=1 to n=infinity we can find what value of x should be used in the final answer b + x(a-b).	*Planning* Posing sub-problem	***Decided on plan*** **Posed sub-problem 1** ***Formulated conjecture***	There is an implicit conjecture here: $x = \sum_{j=1}^{\infty} k_j = \lim_{i\to\infty} n_i$ The sub-problem is to calculate the sum of the series.
8	Starting from n2, we have a geometric series with a = -1 and q = -1/2.		***Formulated conjecture***	Conclusion from Observed Pattern: $\sum_{j=2}^{\infty} k_j$ is a geometric series with first term equal to -1 and ratio -1/2.
9	For a geometric series of this type the sum of the term from n=2 to n=infinity can be represented as $a(\frac{1}{1-q})$. In this case we get -1(2/3) = -2/3. But this doesn't include n1, which has a value of 1. So the sum from n=1 to n=infinity = 1 – 2/3 = 1/3 (or 33.333% if expressed as a percent).	Using other heuristics *Implementing the plan* *Calculating*	***Solved sub-problem 1***	By saying "a geometric series of this type" the student probably means that since the ratio of the series is less than 1 in absolute value, we can use the formula for the sum of the geometric series : $a + aq + aq^2 + \cdots = \frac{a}{1-q}$
10	Thus the value for x in the equation b + x(a-b) is 1/3. So the final equation is: $b + \frac{a-b}{3}$.		**Solved problem**	
11	This can be verified using Microsoft Excel. I first entered arbitrary values for a and b in cells A1 and A2 respectively. In	Naïve testing *Calculating* *Checking*		

	cell A3 I used the formula =AVERAGE(A1,A2) which will give the average of a and b. I copied this formula down column a and when the same value kept repeating itself (technically each value is different but excel only displays a certain number of digits and they were all the same) it was clear that was the amount that we were converging on. And sure enough, for any values of a and b I experimented with, the final equation held.			

5.3 The case of student #003

Student #003 (Table 6) presented another type of solution. He correctly calculated several terms of the sequence of prices and wrote them as linear combinations of a and b. He searched for patterns and regularities in coefficients of a and b. The student summarized his observation and formulated three conjectures. He probably used references for the formula for the Jacobsthal numbers and uncritically trusted them. He assumed that the patterns he observed hold without looking for justification. It is possible that he tested the formula for Jacobsthal numbers to make sure that it works for observed patterns. So we can conclude that he obtained conviction by means of naïve testing. Then he decided to find the limit of the sequence. The student made some minor mistakes in using notations. For example, he calculated the limit of the sequence in infinity but he did not use the notation for the limit and continued to write the limit as n^{th} term of the sequence. Finally the student solved the haggling problem and formulated the correct answer. He chose $a = 100$ and $b = 50$ to verify his solution by naïve testing. Calculating sequences of amounts confirmed the result obtained analytically. It was enough for him to be convinced of the correctness of his solution.

Table 6: Analysis of the solution of student #003

	Action performed	Processes	Outcome	Remarks
1	First will calculate few terms of the price Pn P1 = 1/2 a + 1/2 b P2 = 1/4 a + 3/4 b	*Reading and rewriting in order to understand the problem* *Analyzing*		Student uses an indexed letter "Pn" to denote the price in the n^{th} step of the haggling process and rewrites the first terms of the sequence of prices with this symbol. He takes the first step of the haggling process to be the third number quoted (not a or b). This decision could have been made after he discovered the connection with Jacobsthal numbers (see line 4 of the solution below). The representation of the first two terms as linear combinations of a and b, is as given in the text of the haggling problem.
2	P3 = 3/8 a + 5/8 b P4 = 5/16 a + 11/16 b P5 = 11/32 a + 21/32 b P6 = 21/64 a + 43/64 b P7 = 43/128 a + 85/128 b P8 = 85/256 a + 171/256 b	Specializing Searching for patterns		In order to search for patterns in coefficients, student specializes in the sense of applying a general rule to specific values of a variable (here, the variable is the index n). He calculates several terms of the sequence of prices and writes them as linear combinations of a and b.
3	1. For both a and b coefficients in the denominator is 2^n		Observed Pattern Formulated conjecture 1	Student lists his observations. He finds the first pattern in denominators of the

				coefficients: the denominators are powers of 2
4	2. For a, coefficients in the nominator are 1, 1, 3, 5, 11, 21, 43, 85 ……. This series represent Jacobsthal number, which can be expressed as $[(-1)^{n-1} + 2^{n}]/3$[8]	Searching for patterns Using Other Heuristics (structure recognition) Generalizing	Observed Pattern Formulated conjecture 2	Searching for patterns in numerators of coefficients of a and b and probably, using references led him to Formulated conjecture 2. Does not mention the domain of the variable n: that $n = 1, 2, 3$ .... and does not justify his claims. Probably, trust in the authority of his references and naïve testing for a few initial values of n was enough for him.
5	3. For b coefficients in the nominator represent Jacobsthal number starting at n+1, which can be expressed as $[(-1)^{n} + 2^{n+1}]/3$	Searching for patterns Using Other Heuristics (analogy)	Observed Pattern Formulated conjecture 3	Formulated conjecture 3 using analogy.
6	substituting these coefficients Pn = $\frac{(-1)^{n-1}+2^{n}}{3\cdot 2^{n}} a + \frac{(-1)^{n}+2^{n+1}}{3\cdot 2^{n}}$	Using Other Heuristics (substitution)		The student used his formulated conjectures to write terms of the sequence in general form. No justification why the power of 2 in the denominator is the same as the power of 2 in the numerator in the coefficient of a.
7	Pn= $\left[\frac{(-1)^{n-1}}{3\cdot 2^{n}} + \frac{2^{n}}{3\cdot 2^{n}}\right] a + \left[\frac{(-1)^{n}}{3\cdot 2^{n}} + \frac{2^{n+1}}{3\cdot 2^{n}}\right] b$ Pn= $\left[\frac{(-1)^{n-1}}{3\cdot 2^{n}} + \frac{1}{3}\right] a + \left[\frac{(-1)^{n}}{3\cdot 2^{n}} + \frac{2}{3}\right] b$	Using Other Heuristics (algebraic manipulations) *Planning* Implementing the plan		Next he performed algebraic manipulations to present Pn as a function of n .

[8] For the sake of clarity, formulas in lines 4-7 were re-written by me in Equation Editor. They were written in plain text in the original.

8	When n -> infinity	*Planning*	*Decided on plan*	He decided to find the limit of the sequence.
9	Pn= {0+1/3} * a +{0 + 2/3} * b Pn= {1/3} * a + {2/3} * b	Implementing the plan	***Errors or Mistakes***	Notational inaccuracy: limit in infinity equated with n^{th} term of the sequence (possibly due to limitations of a software)
10	Pn= (a+2b)/3 this represent the amount that they will converge on.		Solved Problem	Finally, by calculating the limit he writes the answer to the haggling problem.
11	To verify let's assume a= $100 and b= $50	*Planning*		Student plans how to verify his solution.
12	so the price should be P= (100+2*50)/3 = 66.6666……	Naïve testing		He uses a particular pair of values for a and b and plugs them into his solution.
13	And if calculate few terms using excel and look what is the price that will converge we noticed that the correlation is correct a= 100 b=50 1. 75 2. 62.5 3. 68.75 4. 65.625 …………. 29. 66.6666667 30. 66.6666667	Checking		He performs calculations to find the limit of the sequence of amounts for a=100, b=50 numerically and compares the result with the value from previous step.

5.4 The case of student #014

An analysis of the unsuccessful solution of student #14 is presented in Table 7. This student made multiple mistakes and typos. It was not easy to guess what was his reasoning. He had difficulties in applying the rule given in the text of the problem: “each time taking the average of the previous two amounts”. He made mistakes in writing the $5^{th}$and 6^{th}amounts and probably tried to search for a pattern in the procedure. This approach did not give him a clue, so the student turned to computational approach and tried a particular pair of values for a and b to calculate a sequence of amounts. Unfortunately he made a mistake in calculations and noticed a contradiction with results of calculations in Excel. However, it did not help him in discovering his mistakes. As a result, he formulated an incorrect conjecture. He tried to verify his results by naïve testing, but he used the same pair of values for a and b and probably was satisfied because his wrong conjecture agreed with his incorrect numerical calculations. The student did not bother trying more examples or checking calculations.

Table 7: Analysis of the solution of student #014

	Action performed	Processes	Outcomes	Remarks
1	a b (a+b)/2 =1/2 a + 1/2 b (b+(a+b)/2)/2 =1/4 a + 3/4 b	*Reading and rewriting in order to understand the problem* *Analyzing*		Student writes down the first values of the sequence of prices
2	(a+(b+(a+b)/2)/2)/2 = 5/8 a + 3/8 b (b+(a+(b+(a+b)/2)/2)/2)/2 = 5/16 a + 11/8 b	*Trying examples to understand the problem* *Searching for patterns* (perhaps)	***Error or mistake***	Then he calculates next two terms of the sequence maybe intending to use the rule that every next term is the average of the previous two. The student does not use the results of his previous calculations to calculate the next. He may also be doing that to discover a <u>pattern in the procedure</u> (e.g., the number of divisions by 2 increases by one with each consecutive amount?). Nevertheless, he makes mistakes in writing $5^{th}$and 6^{th}amounts. The 5^{th} amount should be 3/8 a + 5/8 b. In the 6^{th} amount, there is probably a typo, so we should read 5/16 a + 11/18 b instead of 5/16 a + 11/8 b.
3	If we replace a by 200 and b by 100 we get * 200 * 100 * 150	Specializing *Calculating* Searching for patterns	***Error or mistake***	Student tries a particular pair of values for a and $b$and applies a numerical approach to search for patterns. He does not apply a wrongly

	* 175 * 162.5 * 168.75 * 165.625 * 167.1875 * 166.40625 * 166.796875 * 166.6015625 * 166.69921875 * 166.65039063 * ... * 166.666...			discovered pattern of calculation. The student makes a computational mistake because, starting from the 4th amount, the values are incorrect. The 4th amount should be 125.
4	numbers verified with Exel. If I made a mistake it would converge to 133.333…			Apparently he noticed a contradiction but this does not seem to make him to revise his solution.
5	Because they haggle for a cheaper or more expensive price the dollar amount will oscillate back and forth. With the price converging at 2/3 of the difference between the first suggested price.		**Observed Pattern** **Formulated Conjecture 1** ***Error or mistake***	From his specific numerical example student observes a pattern and formulates a conjecture in words. Here it looks as if he was claiming that the limit price is $b + \frac{2}{3}(a-b)$, which is incorrect.
6	John=200, Mark =100 John-Mark+2/3(John-Mark) 200-100+2/3(200-100) 100+66.666...166.666... a, b a-b+2/3(a-b) a-b+2a/3-2b/3 5a/3-5b/3	Conjecturing Justifying Conjecture using reasoning	**Formulated Conjecture 2** ***Error or mistake***	In fact, however, he thinks rather of the limit price being calculated using the formula $(a-b)+\frac{2}{3}(a-b)$, which is $\frac{5}{3}(a-b)$. If he doesn't notice the mistake, it is perhaps because for his particular a and b, $a-b=b$.

7	a=200, b=100 5(200)/3-5(100)/3 333.333...-166.666... 166.666... Converge at 5a/3 - 5b/3 with 0<b<a.	Naïve Testing		He uses the same pair of values for a and b, so for this particular case his wrong conjecture agrees with his incorrect numerical calculations. By using the same numbers he actually does not challenge the conjecture. It is obvious, that taking another example (a=10, b=1) could help him to reject his conjecture and go back to Specializing and Searching for Patterns stages.

5.5 Summary of the results

To address the first research question: how students do know that they are right and in particular, how they validate the results of their thinking while solving a problem that involves investigation as a process, I analyzed 32 solutions for the haggling problem. These results are summarized in Table 8. 'Successful' solutions are defined as solutions, where the code **Solved Problem** was used to indicate the answer to the haggling problem in the form $\frac{1}{3}\ a\ + \frac{2}{3}\ b$ (or an algebraically equivalent form). All other solutions are called 'unsuccessful', including completely wrong solutions, solutions of a misinterpreted problem, and incomplete solutions.

Table 8. Numbers of successful and unsuccessful solutions

Number of students (N =32)	Description
15 (47%)	'Successful' solutions
17 (53%)	'Unsuccessful' solutions

Throughout all students' solutions, I have noticed interesting tendencies in using examples. All students used examples while solving the haggling problem. In this section I describe *when* and *how* students used examples and *how* it helped them in solving the haggling problem and verification of their results.

In mathematics and mathematics education, the term "example" can be seen from different perspectives. By saying 'for example', we can present an algorithm for solving a problem, a type of problems, an object satisfying a given definition or a class of objects. In the context of my study, I consider the term "example" only as a mathematical object which illustrates a definition, concept or statement (Moore, 1994). Therefore, in solving the haggling problem students could use a particular pair of numbers for a and b, as well as particular terms of the sequence of prices in algebraic form which satisfy the given rule. Analysis of students' solutions revealed four distinct contexts in which examples were employed: to understand the problem, to specialize, to test a conjecture and to validate the results (Figure 88).

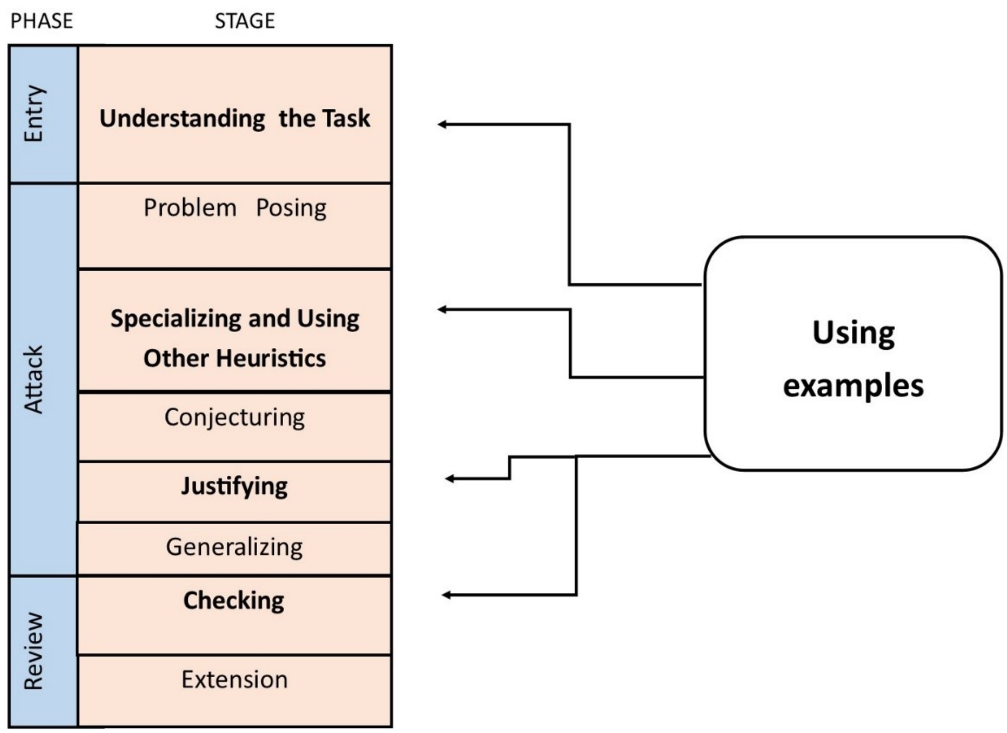


Figure 8. Using examples in solving the haggling problem

Understanding the Task stage: trying examples to make sense of the problem

Introspection and inter-spection analyses made us think that rewriting and recalculating the first four amounts and calculating the 5th amount occur in Understanding the Task stage, where one just applies the rule given in the text to analyze the problem (*Analyzing*) and plans the next steps (*Planning*). The transition between using examples to understand the problem and to search for patterns may be very smooth and cannot be easily observed from written solutions (Figure 9).

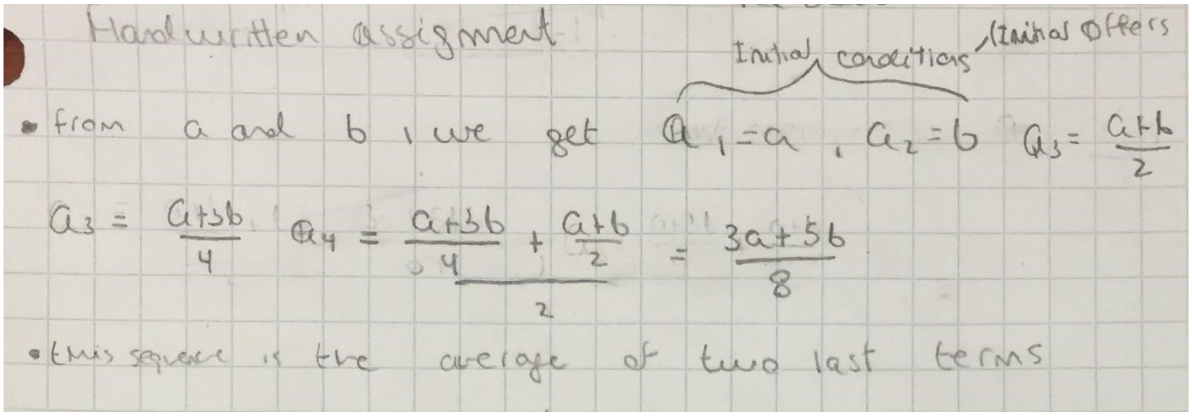


Figure 9. A part of Student's #005 solution (a)

Calculating the 5th amount correctly is an important indicator of understanding the haggling problem. The eight students who misinterpreted the problem either did not calculate the 5th amount or made a mistake in their calculations. To calculate the 5th amount, one has to take the average of the 3rd and 4th amounts (they are already given in the text as $\frac{a+b}{2}$ and $\frac{b+\frac{a+b}{2}}{2}$)

Some students calculated the average of the 4th and 2nd amounts and wrote a wrong expression for the 5th amount: $b+\frac{b+\frac{a+b}{2}}{2}$. As a result, they fixed the second amount b and instead of taking the average of the previous two amounts those students continued to calculate an average of the previous amount and the second amount b. In other words they tried to solve a different problem because they reformulated the haggling problem wrongly. Four students who made a mistake in writing the 5th amount solved the 'wrong' haggling problem and provided justification for their solutions. Here is an example of this type of solutions:

Student # 021

> They will eventually converge towards an amount _very close_ to b.
>
> We know from the information of 0<b<a that the sequence is bounded above and below, and it is decreasing.
>
> Each time they take the average of the previous two amounts, a gets smaller (closer to zero), and b increases (getting closer to its value).
>
> 1/2a + 1/2b

1/4a + 3/4b

1/8a + 7/8b [emphasis added]

1/16a + 15/16b

1/32a + 31/32b

and so on...

the limit is expressed as

lim a[n+1]= (a[n] +b[n])/2

n->inf

I've verified my answer by assuming that a=100$, and b= 55$; which respects that 0<b<a.

therefore getting to the repeating of:

((((((((100+55)/2)+55)/2)+55)/2)+55)/2)

=77.5; 66.25; 60.625; 57.8125; 56.40625...

So the limit of the sequence is the value of b.

Another way to make sense of the problem, simplify and reformulate it is to consider the situation where $b = 0$ and $a = 1$. For instance, student #010 (see section 5.2) assumed "b to be 0 and a to be 1" and concluded that "we'll converge on a number between 0 and 1". This kind of scaling helped him to plan the next steps and reformulate the problem.

Specializing Stage: trying examples systematically to search for patterns

As the analysis revealed, 26 students went thought Specializing stage and systematically explored examples to attack the haggling problem. It is possible that the remaining 6 students also used examples during this stage; however, we were unable to decide what strategies they used because they stated only an answer to the haggling problem without explanation how they arrived at the result. The analysis of a variety of uses of examples during Specializing and Using other Heuristics stage led me to distinguishing two main approaches in attacking the haggling problem: computational and analytical. Figure 10 shows how successful were students who appeared those approaches in their solutions.

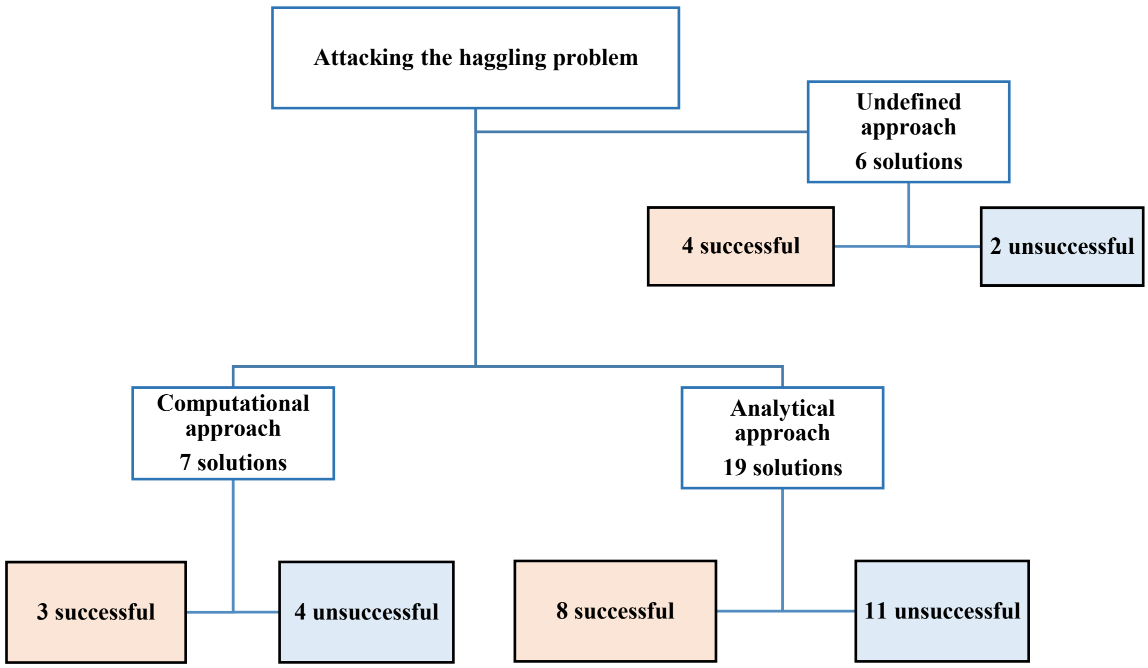


Figure 10. Attacking the haggling problem: computational and analytical approaches

Computational approach

By "computational approach" I mean arithmetical exploration of many concrete amounts or coefficients (usually using a computer software). Thus students carried out numerical experiments to show that a sequence of amounts converges. For example, student # 031 designed a code in Python and used several pairs of values for a and b in his calculations.

<u>Student # 031</u>

I have used Python language to program this question.

I attach the code in the following:

```
F={}
F[0]=0
```

```
F[1]=1
for i in range(2,30):
   F[i]=Fraction(F[i-1]+F[i-2])/2
   print i,F[i]
```

And after testing for several interval, for example, (a=0, b=1), (a=3, b=5) etc.

The result converge in the 2/3 of the interval length, for example, b=1, a =0

the result = 2/3, b=5, a=3, the result = 3+(5-3)*(2/3)

Note that this student reversed the role of a and b in his code for calculations; however, he reverted to the assumed roles of the letters in his final, correct answer:

On (2/3)b +(1/3)a will converge.

S0 = a

S1 = b

S2 = (b+a)/2

S3 = (b + (b+a)/2)/2

…..

Sn = (2/3)b + (1/3)a

Some students calculated coefficients directly. The example of Student #027's solution illustrates this:

Student # 027

ANALYSIS:

John will offer for a dollars

Mark will offer for b dollars, and we know 0< b < a

Then John will offer for (a+b)/2=0.5a+0.5b

Mark will offer for (a+3b)/4=0.25a+0.75b

John will offer for (3a+5b)/8=0.375a+0.625b

Mark will offer for (5a+11b)/16=0.3125a+0.6875b

John will offer for (11a+21b)/32=0.34375a+0.65625b

Mark will offer for (21a+43b)/64=0.3281a+0.6718b

John will offer for (43a+85b)/128=0.3359a+0.6640b

Mark will offer for (85a+171b)/256=0.3320a+0.6679b

John will offer for (171a+341b)/512=0.3339a+0.6660b

Mark will offer for (341a+683b)/1024=0.3330a+0.6669b

we can see easily in analysis above that the amount is converging to a/3+(2/3)b which is close to 0.333a+0.666b

Observation of numerical results did not always lead to the correct answer. For instance, student # 001 introduced two examples:

My answer has been verified with 2 examples.

One was with the a being $100.00 and b being $50.00 and the other pricing was done with $20.00 and $40.00

100	40
50	20
75	30
62.5	25
68.75	27.5
65.63	26.25
67.19	26.88
66.40	26.56
66.8	66.8
66.6	26.64
66.66=0.4444(a+b) =0.4444(150)	26.64=0.4444(a+b)=0.4444(60)

as the average is being continuously taken between John and Mark, the price will be settled at

$ 0.4444(a+b)

As can be seen from the above excerpt, both examples illustrate a special case when $b = \frac{1}{2}a$. In fact, the student's conjecture makes sense for all a and b such that $b = \frac{1}{2}a$. However, he did not try other examples or maybe he misinterpreted the problem. As a result he was not successful in solving the haggling problem.

Analytical approach

Analytical approach is assumed here to refer to an algebraical exploration of the sequence of coefficients, searching for patterns, and then using other heuristics, such as analogy, deduction, structure recognition, algebraic manipulations.

A majority of students (19) tried to solve the haggling problem by using this approach: they looked for a formula to represent a sequence of prices. Most of them thought about finding the limit of this sequence. Moreover some of them tried to show that the sequence is bounded and decreasing. Problem solving pathways of students who used analytical approach demonstrate that they alternated between Specializing, Conjecturing, Justifying and Using Other Heuristics.

Three students reformulated the haggling problem and considered an interval between b and a. The case of student #010 (see section 5.2) is an example of using this technique. The following example of student's solutionalso demonstrates this strategy:

Student # 029

Let x=(a-b)

first = a, second = b, third = b+(x/2), fourth = b+(x/2)-(x/4), fifth = b+(x/2)-(x/4)+(x/8), sixth = b+(x/2)-(x/4)+(x/8)-(x/16)...

As we can see from this excerpt, after reformulating the problem the student focused on writing first terms of the sequence by using new notation. Obviously there are some steps between the first and second lines. Presumably the student alternated between specializing and pattern searching and used other heuristics such as representing an expression in different way and structure recognition.

A number of students, after calculating several first terms of the sequence and representing coefficients of a and b as fractions, posed sub-problems to explore: to find relationships in numerators and denominators of the coefficients. Next, they summarised their observations to write a direct formula for n^{th} term of the sequence. The case of student #003 (see section 5.3) is an example of using this technique. Six students used Jacobsthal numbers in their solutions to write a formula for n^{th} term of the sequence and four of them successfully arrived to the correct answer. Jacobsthal numbers were not mentioned in class and they are not as famous as, for example, Fibonacci numbers. Therefore, I surmised that those students searched the Internet for references and accepted the closed form equation for the Jacobsthal number at a specific point in the sequence: $J_n = \frac{2^n-(-1)^n}{3}$ as a well-known fact. A solution illustrating this is provided below:

Student # 030

n = 1 => (a+b)/2
n = 2 => (a+**3**b)/4
n = 3 => (**3**a+**5**b)/8
n = 4 => (**5**a+**11**b)/16
n = 5 => (**11**a+**21**b)/32
n = 6 => (**21**a+**43**b)/64
n = 7 => (**43**a+**85**b)/128
Noticing the following:
- coefficient of a (in numerators) = [(2^n) - (-1)^n] / 3
- Similarly, coefficient of b (in numerators) = [(2^(n+1)) - (-1)^(n+1)] / 3
- denominator = 2^n
- coefficient of a + coefficient of b (in numerator) = denominator = 2^n
then the n'th term can be written as the following:

{[(2^n) - (-1)^n] / 3}*a + {[(2^(n+1)) - (-1)^(n+1)] / 3}*b
= ---

2^n

By taking the limit of the n^{th} term as n goes to infinity, student #030 was able to solve the problem. As many others, he said nothing about existence of the limit of this sequence.

It is worth noting that solving sub-problems (or finding a correct formula for n^{th} term of the sequence) does not guarantee that one ends up with correct answer. Two students were able to observe a pattern and recognize Jacobsthal numbers in coefficients, but they did not complete their solutions. Two solutions illustrating this category of answers are presented below:

Student # 013

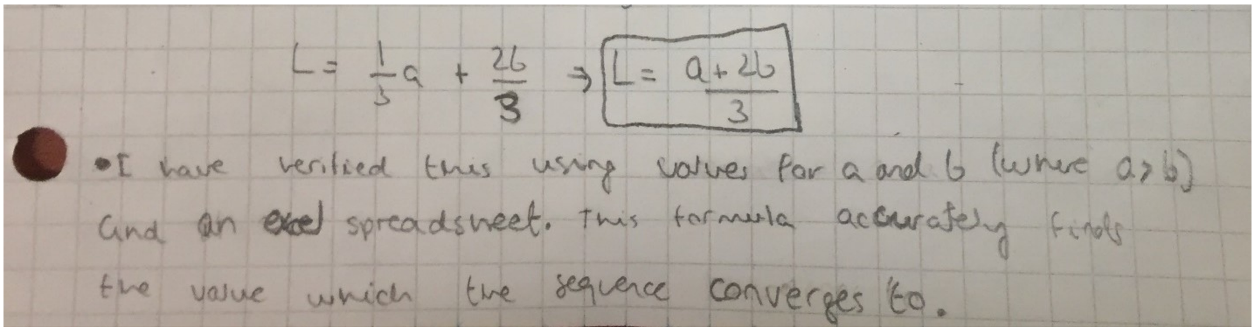
> Now I see a pattern that looks like the Jacobsthal sequence of 1,1,3,5,11,21,43,….
> (2^n –(-1)^n) / 3 Over the denominator of 2^n.
> Thus
> an = [((-1)^n-1 + 2^n)a /3 + ((-1)^n + 2^n+1) b /3] / 2^n   note that n-1 is used for the coefficient of a because it takes on the previous value. [end of the solution]

Student # 004

> We may notice that the coefficient of a and b correspond to the Jacobsthal sequence,
>
> which is J[No]=J(n-1)+2J(n-2).
>
> lim n-->+∞(J[No]a+J(n+1)b)/2^n.

Justifying Stage: trying examples to refute or validate a conjecture (naïve testing)

In most cases I merged Justifying and Checking (Validating) stages in my analysis. Students' written responses did not reveal other means of Checking than Validating results by naïve testing. Using particular pairs of numbers for ***a*** and ***b*** is a common strategy for this stage. Even successful students who used analytical approach and in fact justified the statement by deductive reasoning validated their results by examples. Overall, 18 students out of 32 checked their conjectures in this way. The following excerpt (Figure 11) illustrates this:



$L = \frac{1}{3}a + \frac{2b}{3} \rightarrow \boxed{L = \frac{a+2b}{3}}$

• I have verified this using values for a and b (where $a > b$) and an excel spreadsheet. This formula accurately finds the value which the sequence converges to.

Figure 11. A part of Student's #005 solution (b)

Discussion of students' solutions

The analysis revealed a variety of *resources* and *heuristics* that students applied to solve the haggling problem and justify their solution. The resources used by students were definitions, knowledge of properties of limits, sequences, series and algebraic manipulations. Examples of used heuristics (others than specializing) are reformulating a problem or formulating subproblems, scaling or parametrizing, analogy, structure recognition and substitution. I did not observe drawing diagrams or trying to visualize the problem in students' solutions. I surmise that some students (in particular those who scaled the interval between a and b) used this strategy but did not present it in submitted solutions. I think so because I was drawing a picture to understand what is going on when I was reading the haggling problem for the first time, also when I asked my 14 years son to solve the haggling problem he produced an empirical solution and visualized the problem. He considered a particular pair of values for a and b and then explored the problem by creating a code in Pascal (programming language). As a result he presented the diagram below (Figure 12) to justify his conjecture that the amounts converge to $\frac{1}{3}a + \frac{2}{3}b$.

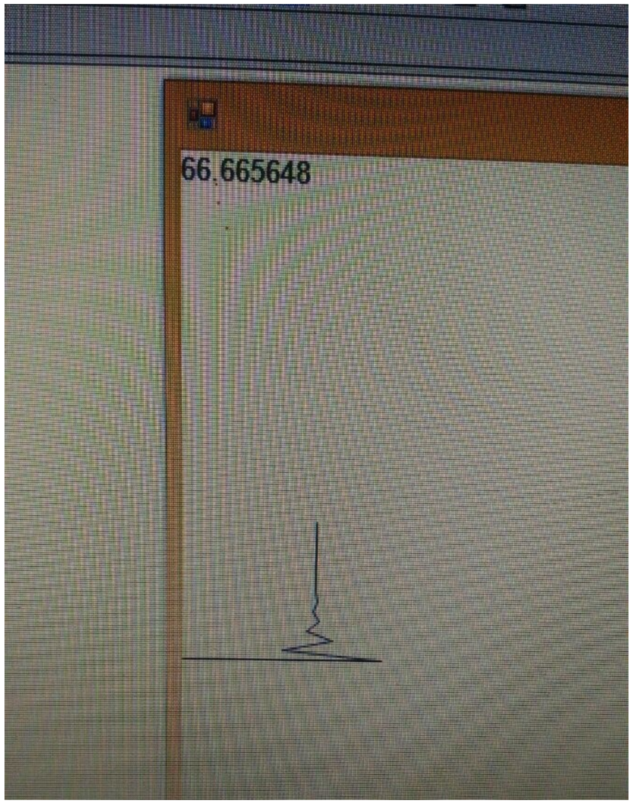


Figure 12. Visualization of convergence of the sequence of amounts

As mentioned in Chapter 4, the CPiMI model does not capture metacognitive processes. By adding additional codes from the Schoenfeld's (1985) model I was able to identify planning and monitoring processes in students' solutions and conclude that at least 12 students demonstrated *control* abilities (for instance, the cases of students #009, #010 and #003). However, observing errors and mistakes made me think that self-regulatory skills of most of the authors 'unsuccessful' solutions are weak. *Lack of sensitivity to contradictions* (Sierpinska, 2005) was an issue for many students (for example, the case of student #014).

Beliefs about proof and mathematics determine students' behavior in justification and verification. The data suggests that, by 'verify', the majority of students (at least 18 out of 32) mean the procedure of plugging in the concrete numbers for a and b into the obtained formula and comparing with numerical calculations of the limit of the sequence of amounts. Students' attempts to justify their conjectures shed light on their beliefs about proofs and justification. Three successful solutions (computational approach) did not contain deductive arguments but they can be accepted as a *generic example proofs* (Yopp at al., 2015). Several solutions could count as mathematical proofs (with gaps and minor mistakes). Students who observed patterns in coefficients of a and b generalized and made conjectures; however they did not try to prove them. Only one student tried to justify his conjecture that the sum of coefficients of a and b is 1 by using mathematical induction. Overall it seems that the majority of students did not feel the need to construct proof-like arguments because they were convinced of their results by other means (for example by naïve testing or confirming by another solution).

Chapter 6. Discussion and conclusions

In this chapter, I look back at my findings in the light of my research questions and relate them to previous research. In the following sections 6.1 and 6.2 I address the research questions in reverse order. I start from reflections on the process of finding a theoretical framework for this thesis and discuss the methodological contribution of this study to the field of mathematics education. Next, I point out how the results obtained during the analysis of students' solutions and presented in chapter 5 are related with findings outlined in literature. Finally, in section 6.3, I present some implications for teaching and my ideas for future research.

6.1 How does one conduct research into how students and mathematicians know that they are right?

Following the interpretive paradigm, the aim of my research was not only to describe things but interpret them to understand and see connections with other contexts. It is impossible to interpret and understand research findings without using some theoretical lens. In my thesis, I described the process of finding and adapting a theoretical framework for identifying the cognitive processes engaged in solving problems requiring some elements of mathematical investigation. I reported on difficulties in choosing and applying the CPiMI model to my concrete data and as a methodological contribution, this thesis presented the idea of using introspective and inter-spective analyses for testing and adjusting theoretical models. Introspection itself is a fruitful way of capturing researcher's cognitive processes (Eisenhart, 1988). Furthermore, inter-spection helped in establishing criteria for and ensured the objectivity of the data analysis, as it was a part of a triangulation method. Overall I feel that introspection and intra-spection allowed us to capture our mental processes and obtain a better understanding of the CPiMI model. But maybe more importantly, searching for an appropriate framework for the data analysis and in particular introspection and inter-spection triggered reflections on constructing theories and models in mathematics education. Going through mountains of readings about philosophical and social aspects of proof and proving; teaching and learning practices; empirical studies on problem solving, proving and investigation I realized that there is no consistency and unity in theories. I moved through the maze of terms and concepts where different terms were used for almost identical concepts and the same terms referring to different concepts. On the one hand, novice researchers face a diversity of perspectives, theories, frameworks and methods to conduct research. On the other hand, as Sierpinska (2002, p. 253) pointed out, “theories are not being sufficiently examined, tested, refined and expanded”. It seems that “novice” researchers in mathematics education prefer to create their own theory or elaborate their supervisor’s frameworks. I hope that by addressing my research question: “*How does one conduct research into how students and mathematicians know that they are right? How does one choose an appropriate framework for analyzing data?”* and presenting the results of testing and adjusting the CPiMI model in the context of my study I contributed in establishing the area of applicability of this model as well as in methodology of testing, examining, refining and expanding theoretical models in mathematics education.

6.2 How do students and mathematicians know that they are right?

Initially, I was interested to link students’ and mathematicians’ behaviors in justifying and validating the results of their mathematical thinking. The analysis of students’ solutions to the haggling problem, interviews with graduate students and professors, and review of related literature allowed me to draw some conclusions about differences in students’ and mathematicians’ attitudes to validating their mathematical results and a necessity of proof for them to being sure they are right. But before I start the discussion of the results of this study in the light of my first research question I would like to formulate a sub-question

Do students really want to know if they are right?

Looking back to the analysis of students’ solutions to the haggling problem we observed that ten out of 32 students (~31.2%) did not even try to verify their final answer or any step of their solutions[9]. Does it mean that those 10 students were not sure of the correctness of their results or were not willing to make their own decision about the correctness? Moreover, may we conclude that other 22 students who somehow tried to verify their solutions were interested in verification or justification? Unfortunately, this study does not answer these questions. However, my

[9] It is worth noting that only one solution among those 10 was successful and three solutions were incomplete but contained partial results.

impression was that most students looked for an answer to haggling problem and only a few of them were interested to know *why* their answer was correct.

Sierpinska (2005) stresses that “[t]heoretical thinking asks not only, *Is this statement true?* but also *What is the validity of our methods of verifying that it is true?* Thus theoretical thinking always takes a distance towards its own results” (p. 122). Some elements of theoretical thinking were present in students’ reasoning. Several students were reflective, they demonstrated self-regulation and monitored their work. It helped them to use heuristics and their knowledge effectively. On the other hand, many students were not able to solve the haggling problem and their reasoning was wrong; they could not be critical about their results. ***Lack of sensitivity to contradictions in mathematics*** stopped many students from arriving at a correct conclusion. For example, student # 014 noticed a contradiction in his calculations, but he did not revise the solution and did not discover mistakes. Eight students misinterpreted the problem and only locally checked their arguments without referring to the initial text of the problem. Sierpinska (2007) argues that “[t]he systemic character of theoretical thinking entails sensitivity to contradictions” (p. 122). Indeed, the results of this study confirm that without sensitivity to contradictions students cannot be successful in proving and justification and develop theoretical thinking.

Algebraic notations play a significant role in the development of mathematics (Sfard and Linchevsky, 1994). The results presented in this thesis also revealed multiple ***errors in using mathematical notation.*** For instance, student #009 used shorthand for a phrase to explain that the sum of coefficients of a and b is 1 (see the case of student #009). This is similar to the approach that Clement (1981) called as "word order match”. He analyzed calculus students’ responses to The Students-and-Professors Problem.

> Write an equation for the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors. (Clement, 1981, p. 288)

He found that direct mapping of the words into the symbols of algebra led students to the incorrect answer 6 S = P.

Researchers have confirmed that problems with mathematical notation are common among transition-to-proof courses students (Moore, 1994). Even students who were successful and produced valid arguments demonstrated some difficulties in using mathematical language and notation. They were not precise and consistent in using letters to name variables, defining sequences, terms of series and partial sums. But without this precision and consistency one cannot construct a mathematical proof.

Findings about ***using examples*** in solving the haggling problem resonate with previous studies (Alcock, 2004; Alcock and Weber, 2010). Alcock (2004) identified three instances in which mathematicians use example mathematical objects in reasoning. These are understanding a statement, generating an argument and checking an argument. The results of this thesis confirm that using examples can be useful for students in making sense of a problem, conjecturing and validating or checking a statement. The students appear to have better success with using examples to validate a conjecture (12 out of 15 'successful' students checked their conjecture by using concrete examples). On the other hand, this study revealed that students understand verification of results exclusively as checking specific examples or, in other words, as naïve testing. Even students who used analytical approach and whose solutions could be counted as mathematical proofs (with gaps and minor mistakes) used specific examples to validate their results. I found that concrete examples look very convincing for students even if they understand and accept that examples do not constitute proof. This agrees both with the literature (Healy and Hoyles, 2000) and my interviews with expert mathematicians. For instance, participant K. said:

> You know, I find that outside of mathematics people are very scientific. Right? So they are very convinced by many things that show you are right. That doesn't mean your [are right], but many things that showing truth are very convincing.

Then he added:

> …from looking at some examples you get [to] believe [that] something is true. Then you have to go back and actually prove it is true.

This is similar to other mathematicians' views on proof and validation.

> Proof is distinguished from other aspects of mathematical activity… by the fact that it belongs mainly to the verification stage of investigation. (Bell, 1979, p. 372)

Students in this study did not prove their conjectures about the general form of the sequences they were generalizing probably because they did not realize that there were only conjectures. The meaning of proof for them is restricted to proving exercises where they know *what* they need to do even if they do not know *how*. They do not consider justifications of the steps during problem solving as part of their conception of proof, as their conviction that the answer is correct comes from examples, guessing and scraps of analytical reasoning.

For one of my course projects, I conducted several interviews with mathematics students and one of the answers to the question "*What do you like the most about mathematics?*" was

> I like it that most of the time here there is an answer… either you get it right or you get it wrong. It's not from interpretation…. It is yes, you got it right because you know what you are doing and what steps to follow. So like there is a definite answer to the problems you are doing.

This might seem a common view that mathematics is a citadel of infallible certainty and the truth of a mathematical statement is objective. This view may lead to a belief that arriving at an answer means that thinking about the problem is finished. But for mathematicians it is not enough to find a correct answer; they are looking not only for certainty but also for understanding. Hanna (2000) pointed out that "...proof, valid as it may be in terms of formal derivation, actually becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding" (p.7). Can we say that proofs constructed by students in this study helped them in understanding why their final formula is correct? Maybe this is true for a few of them.

6.3 Teaching implications and suggestions for future research

Investigation (as a process) can be considered as a step in successful problem solving and proving. It is important for transition-to-proof courses to teach not only how to construct proofs but also to stimulate students to learn *via* proof by demonstrating other functions of proof besides conviction. Activities that invite students to make and test conjectures can be used as an opportunity to teach how to justify claims and why there is a need to do it. Inquiry-based teaching approaches offer many learning opportunities for students. One of them is that students can learn from each other. Encouraging students to present solutions in class with a goal not only to state and compare results but also to justify their reasoning may develop their abilities to communicate mathematics and see how proof can be helpful in exploration of mathematical properties and discovery of new results. For example, discussing different strategies of solving the haggling problem in class may be helpful in developing mathematical curiosity and skepticism in students.

One of obvious limitations of my study is the use of written solutions without additional follow-up interviews. At the very end of working on this thesis, my supervisor asked graduate students in a mathematics education course to solve the haggling problem. During the students' presentations of some of their paths to solutions, I was able to ask questions that helped me to understand deeper several confusing steps in written solutions of students from MAST217. Also, I observed how discussion of solutions may lead to raising more questions and open doors to new investigations.

I view this study as a first step toward a better understanding of students' viewpoint on the validity of their results. More research is needed to fully characterize undergraduate and graduate students' perceptions of proof. Another idea that surfaced during writing this thesis is how to assess and evaluate students' performance in investigation. Thinking about computational and analytical approaches that I described in this thesis led me to addressing new questions. Would I say that analytical approach is better or more successful than computational? I measured students' success very formally, but how to measure what students learned from the haggling problem?

References

Aberdein, A. (2005). The uses of argument in mathematics. *Argumentation*(19), 287–301.

Alcock, L. (2004). Uses of example objects in proving. *Proceedings of PME-28. 2*, pp. 17-24. Bergen, Norway: PME.

Alcock, L., & Weber, K. (2010). Undergraduates' example use in proof production: Purposes and effectiveness. *Investigations in Mathematical Learning, 3*, 1-22.

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 201-230). Dordrecht: Netherlands: Kluwer.

Balacheff, N. (1987). Processus de preuve et situations de validation. *Educational Studies in Mathematics*(18), 147–176.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Teachers and Children* (pp. 216–235). London: Hodder & Stoughton.

Barnard, T., Somson, A., Tahta, D., MacKernan, J., Pope, S., Reod, D., . . . Anderson, J. (1996). Teaching proof. *Mathematics Teaching, 155*, 6-39.

Bell, A. W. (1979). The learning of process aspects of mathematics. *Educational Studies in Mathematics*(10), 361-387.

Boero, P., Garuti, R., Lemut, E., & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig, & A. Gutierrez (Ed.), *Proceedings of 20th PME Conference. 2*, pp. 113–120. Valencia, Spain: PME.

Calleja, J. (2016). Teaching mathematics through inquiry. *Educational Designer*(3 (9)). Retrieved from http://educationaldesigner.org/ed/volume3/issue9/article30/index.htm

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An Emergent Multidimensional Problem-Solving Framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Charmaz, K. (2006). Grounded theory. In G. (. Ritzer, *Encyclopedia of sociology* . Cambridge, MA : Blackwell. .

Clement, J., Lochhead, J., & Monk, G. S. (1981). Translation Difficulties in Learning Mathematics. *The American Mathematical Monthly, 88*(4), 286-290. doi:10.2307/2320560

Creswell, J. W. (2008). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research.* Upper Saddle River, NJ: Pearson/Merrill Prentice Hall. .

Csiszar, A. (2003). Stylizing rigor: or, why mathematicians write so well. *Configurations*(11(2)), 239-268.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience.* Brighton, Sussex: Harvester Press.

De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*(24), 17-24.

De Villiers, M. (2010). Experimentation and proof in Mathematics. In G. Hanna, H. N. Janke, & H. Pulte (Eds.), *Explanation and Proof in Mathematics: Philisophical and Educational Perspectives* (pp. 205-221). New York: Springer.

Douek, N. (2007). Some remarks about argumentation and proof. In P. Boero (Ed.), *Theorems in School: From History, Epistemology and Cognition to Classroom Practice.* Rotterdam, Taipei: Sense Publishers.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics, 22*(3), 233-261.

Ernest, P. (1991). *The philosophy of mathematics education.* London; New York: Falmer Press.

Friesen, S., & Scott, D. (2013). *Inquiry based learning: a review of the research literature. Alberta Ministry of Education*. Retrieved from https://inspiring.education.alberta.ca/wp-content/uploads/2014/04/Inquiry basedLearning-A-Review-of-the-Research-Literature.pdf

Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and cognitive factors in proving. *Educational Studies of Mathematics, 70*, 71-90.

Goldin, G. (1997). Observing mathematical problem solving through task-based interviews. *Journal for Research in Mathematics Education, Monograph 9, Qualitative Research Methods in Mathematics Education*, 40-177.

Goldwasser, S., Micali, S., & Rackoff, C. (1985). The knowledge complexity of interactive proofs. *Proceedings of 17th Symposium on Theory of Computing*, (pp. 291-305). Providence.

Hanna, G. (1991). Mathematical Proof. In D. Tall (Ed.), *Advanced Mathematical thinking.* Dordrecht: Kluwer.

Hanna, G. (1995). Challenges to the Importance of Proof. *For the Learning of Mathematics, 15*(3), 42-49.

Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics, 44*(1-2), 5-23.

Hanna, G., & Barbeau, E. (2002). What is a proof? In B. Baigrie (Ed.), *History of modern science and mathematics* (pp. 36-48). New York: Charles Scribner's Sons.

Hanna, G., & Jahnke, H. (1996). Proof and proving. In A. Bishop et al. (Ed.), *International handbook of mathematics education* (pp. 877–908). Dordrecht: Kluwer.

Harel, G., & Sowder, L. (1998). Students proof schemes. *Research in Collegiate Mathematics Education, 3*, 234–282.

Harel, G., & Sowder, L. (2007). Towards a comprehensive perspective on proof. In *Second handbook of research on mathematical teaching and learning.* Washington, DC: NCTM.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analysis relating to achievement.* New York: Routledge.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Hersh, R. (1997). *What is mathematics, really?* Oxford, United Kingdom: Oxford University Press.

Horgan, J. (1993). The death of proof. *Scientific American, 269*(4), 93-103.

Knipping, C. (2004). Argumentations in proving discourses in mathematics classrooms. In G. Törner et al. (Ed.), *Developments in Mathematics Education in German-speaking Countries.* (pp. 73-84). Hildesheim: Verlag Franzbecker.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Lakatos, I. (1978). *Mathematics, science and epistemology.* Cambridge: Cambridge University Press.

Lakatos, I. (1986). A renaissance of empiricism in the recent philosophy of mathematics. In T. Tymoczko (Ed.), *Lakatos, I. (1986). A renaissance of empiricism in the recent philosophy New directions in the philosophy of mathematics: An anthology.* Boston: Birkhäuser.

Manin, Y. (1977). *A course in mathematical logic.* New York: Springer.

Mariotti, M. (2006). Proof and Proving in Mathematics Education. In A. Gutiérrez, & P. Boero (Eds.), *Mariotti M. (2006). Proof and Proving in Mathematics Handbook of Research on the Psychology of Mathematics Education. Past, present and future* (pp. 173-204). Rotterdam, Los Países Bajos: Sense Publishers.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*, 227–289.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically.* London; Reading, Mass.: Addison-Wesley Pub. Co. .

Mass, K., & Artigue, M. (2013). Mass, K. & Artigue, M. (2013) Implementation of inquiry-based learning in day-to-day teaching: A synthesis. *ZDM, 45*(6), 779-795.

Moore, R. M. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*, 249-266.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school.* Reston, VA: National Council of Teachers of Mathematics.

Pedemonte, B. (2002). Relation between argumentation and proof in mathematics: cognitive unity or break? *Proceedings of the 2nd Conference of the European Society for Research in Mathematics Education. 2*, pp. 70-80. Marienbad: ERME.

Polya, G. (1973). *How to solve it: A new aspect of mathematical method (2nd ed.).* Princeton, N.J.: Princeton University Press.

Ponte, J. P., & Matos, J. F. (1992). Cognitive processes and social interaction in mathematical investigations. In J. Ponte, F. Matos, J. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technologies: Research incontexts of practice* (pp. 239-254). Berlin: Springer.

Rav, Y. (1999). Why do we prove theorems. *Philosophia Mathematica, 7*(3), 5-41.

Schoenfeld, A. (1992). Learning to think mathematically: Problem-solving, metacognition and sense making in mathematics. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. (1994). What do we know about mathematics curricula? *Journal of Mathematical Behavior, 13*(1), 55-80.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, Fla.: Academic Press.

Schoenfeld, A. H., & Kilpatrick, J. (2013). A US perspective on the implementation of inquiry-based learning in mathematics. *ZDM - The International Journal on Mathematics Education, 45*, 901-909.

Selden, A., & Selden, J. (2001). Tertiary Mathematics Education Research and its Future. In D. Holton, M. Artigue, U. Kirchgräber, J. Hillel, M. Niss, & A. Schoenfeld (Eds.), *The Teaching and Learning of Mathematics at University Level.* Dordrecht: Springer.

Selden, A., & Selden, J. (2003). Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 36*, 4-36.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher, 78*(6), 448-456.

Sfard, A., & Linchevsky, L. (1994). The gains and pitfalls of reification – the case of algebra. *Educational Studies in Mathematics, 26*, 191–228.

Sierpinska, A. (2002). Editorial. *Educational Studies in Mathematics, 50*(1), V-Vi. Retrieved from http://www.jstor.org/stable/3483048

Sierpinska, A. (2005). On practical and theoretical thinking and other false dichotomies in mathematics education. In M. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign - Grounding Mathematics Education* (pp. 117-136). Dortrecht, The Netherlands: Kluwer Academic Publishers.

Sierpinska, A. (2007). I need the teacher to tell me if I am right or wrong. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education, July 8-13, 2007*, (pp. 45-64). Seoul, South Korea.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* London: SAGE.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 3-21). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics, 15*(1), 29-37.

Tobin, K. (2000). Interpretive research in science education. In A. E. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 487-512). Mahwah, NJ: Lawrence Erlbaum Associates.

Toulmin, S. E. (1969). *The uses of argument.* Cambridge: Cambridge University Press.

Tymoczko, T. (1979). The Four-Color Problem and its philosophical significance. *The Journal of Philosophy, 76*(2), 57-83.

Tymoczko, T. (Ed.). (1986). *New directions in the philosophy of mathematics.* Secaucus, NJ: Birkhauser.

Volminik, J. D. (1990). The nature and role of proof in mathematics education. *Pythagoras, 23*, 7-10.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24*, 351-360.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39*, 431–459.

Weinstein, M. (2006). A metamathematical extension of the Toulmin agenda. In D. Hitchcock, & J. Verheif (Eds.), *Arguing on the Toulmin Model: New Essays on Argument Analysis and Evaluation.* Dordrecht: Springer.

Wittmann, E. (1981). The complimentary roles of intuitive and reflective thinking in mathematics teaching. *Educational Studies in Mathematics*(12), 389-397.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. *Proceedings of the 25th international conference on the psychology of mathematics education* (pp. 9-23). Utrecht, Holland: GPME.

Yeo, B., & Yeap, B. H. (2010, October 5). Characterising cognitive processes in mathematical investigation. *International Journal for Mathematics Teaching and Learning, 1*, 1-10. Retrieved from http://www.cimt.org.uk/journal/jbwyeo.pdf

Yeo, J. B. (2013). *The nature and development of processes in mathematical investigation.* Singapore: National Institute of Education, Nanyang Technological University. doi:10.13140/RG.2.1.1280.0880

Yeo, J. B. (2017). Specialising and conjecturing in mathematical investigation. *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (pp. 337-344). Singapore: PME.

Yeo, J., & Yeap, B. H. (2009). *Mathematical investigation: Task, process and activity.* Singapore: National Institute of Education Singapore. doi:DOI: 10.13140/RG.2.1.3237.9760

Yopp, D., Ely, R., & Johnson-Leung, J. (2015). Generic example proving criteria for all. *For the Learning of Mathematics, 35*(3), 8-13.

Appendix A

Assignment 1.

The product of three consecutive positive integers is always a number that is divisible by 6.

Decide if the statement is true or false.

Justify your decision.

Assignment 2.

Prove:

If n is an integer greater than 2, then there is no integer m satisfying the equation $n + m = nm$ and divisible by n.

Assignment 3.

Choose ONE of the following 4 problems to solve and submit for grading.

PLEASE NOTE THAT:

1) The Triangle Inequality is to be assumed true. You can use it; it is not necessary to prove it in the assignment.

2) You are welcome (even encouraged) to try and solve all 4 problems but you are asked to submit for grading **ONLY ONE**. If you submit solutions of more than one problem, and do not state clearly which one you want to be graded, the marker will choose the shortest solution for grading (to have less work, and not because it will be the best!).

Assignment 4.

Let a and b be integers and let $d = \gcd(a, b)$.

Since d is a divisor of a then a/d is an integer.

Since d is also a divisor of b then b/d is an integer.

So it makes sense to speak of $\gcd(a/d, b/d)$.

Prove that $\gcd(a/d, b/d) = 1$.

Assignment 5.

Solve ONE of the two problems below - Problem 1 or Problem 2. Please state clearly which problem you are submitting for marking. Only one will be marked even if you upload both.

If you solve Problem 1 correctly, you obtain 5 marks.

If you solve part a of Problem 2, you obtain 4 marks. If you also solve part b of problem 2 correctly, you obtain 5 marks for the whole Problem 2.

Saying that you submit Problem 1 for grading, and then solving it partially correctly and also solving part a or part b of Problem 2 correctly will not increase your marks.

Problem 1.

Prove by Mathematical Induction:

For all integers n greater or equal 1, 4 divides (5^n – 1)

Problem 2.

(a) Prove by Mathematical Induction:

For all integers n greater or equal 1, 1 + 3 + 5 + … + (2n – 1) = n^2

In other words, the sum of the first n positive odd numbers is equal to the square of n.

(b) Prove the same statement directly, without using Mathematical Induction.

Assignment 6.

Prove that

(a) there is no rational number r such that r^2 = 15

(b) the number 1 - sqrt(15) is irrational.

Assignment 7.

A retailer purchased 38 gallons of canola oil and wants to put the oil in smaller cans (all of the same size) for sale. He knows his customers will NOT be interested in buying less than 3/5 of a gallon or more than 4/5 of a gallon of oil at a time.
He doesn't want to put the oil in 3/5 – gallon cans or 4/5 – gallon cans because this would not allow him to fill a whole number of cans to full capacity, and would leave him with some oil he would not be able to sell. Advise the retailer on the capacity of cans all of which he would be able to fill to full capacity, so that no oil is left. Explain how you arrived at an answer and how you made sure it was correct.

Assignment 8.

Let A be the set of all real numbers of the form 1/k +1/n, where k and n are natural numbers.

Justify all answers.

(a) Is A is bounded above? If yes, find Sup(A).

(b) Is A is bounded below? If yes, find Inf(A).

(c) True or false?

i. For every real number x there exists a number of the form 1/k + k/n that is larger than x.

ii. For every positive real number x there exists a number of the form 1/k + k/n that is less than x.

iii. The number sqrt(2) belongs to A.

iv. The number 5/6 belongs to A.

v. The number 6/5 belongs to A.

vi. The number 8/3 belongs to A.

Assignment 9.

John is trying to sell Mark a bike for **a** dollars.

Mark does not agree on the price and offers **b** dollars (0< b < a).

John does not agree on this price but comes down to **(a + b)/2 = 1/2 a + 1/2 b**.

Mark responds by offering **(b + (a + b)/2)/2 = 1/4 a + 3/4 b**.

They continue haggling this way, each time taking the average of the previous two amounts.

On what amount will they converge? Express the amount in terms of a and b.

Explain your reasoning and justify your response.

Have you tried to verify your answer? If yes, how?

Assignment 10.

Give three examples of a function f : R --> R with the following property:

for all x in R, f(x + 1) = f(x + 3)

For each example of function, justify why it has the required property.

Hint: the sine function has the property: sin(x + 0) = sin(x + 2π) for all x in R.

Assignment 11.

Choose between problems 1 and 2. The maximum marks for Problem 1 is 5 marks. For Problem 2 – the maximum is 4 marks. State clearly which Problem you have chosen and submit a solution to this problem only.

If you submit solutions to both problems and do not state clearly which one you want to be marked, only Problem 2 will be marked.

Problem 1 (max 5 marks)

Consider the function f: [-2, 2] $\longrightarrow$ [-1/2, 1/2]

defined by f(x)= $\frac{2x}{1+x^2}$

(a) Prove that f is surjective.

(b) Prove that f is injective.

(c) From the information obtained in (a) and (b) what conclusion can be drawn about the cardinalities of the intervals [-2,2] and [-1/2, 1/2] ?

Problem 2 (max 4 marks)

Prove that the intervals [-10, 100] and [0, 10] have the same cardinality by exhibiting a bijective function from one of these intervals to the other and proving that it is indeed bijective based on the definition of a bijective function (and not by reference to known properties of the kind of functions you will be using).

Appendix B

Tasks used for interviews with graduate students and professors.

Question 1.

Determine whether the following statement is true or false. Justify your decision.

The product of three consecutive positive integers is always a number that is divisible by 6.

Question 2.

Determine whether the following statement is true or false. Justify your decision.

A convergent series of continuous functions converges to a continuous function.

Appendix C

Follow-up questions:

At what point you made the decision to prove or disprove the statement?

How did you convince yourself that the statement is true or false? How would you convince an undergraduate student?

How would you present this statement and your decision about whether it is true or false in an undergraduate text? … In a graduate text?... In a lecture intended for the general public?

General questions about proofs:

What is a proof for you?

How is it different from the proving process?

What role proofs play in your research?

Questions related to validation:

Do you use examples to convince yourself if a general statement is true?

Do you ever use your intuition to help you verify results of your mathematical thinking?

Have you ever improved a conjecture by creating proofs and counterexamples?

After you have constructed a proof what means do you use to make sure it is correct?
Are you checking the proof line by line, the structure of the argument, testing it on examples or special cases, other?

Is this process different from the way you validate students' or other mathematicians' proofs? (as when you are reviewing a thesis or a manuscript submitted for publication)