Analysis on Infinite Trees and Their Boundaries

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ABSTRACT

Analysis on infinite trees and their boundaries

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The aim of this thesis is to understand the results of Björn, Björn, Gill and Shanmugalingam [BBGS], who give an analogue of the famous Trace Theorem for Sobolev spaces on the infinite K-ary tree and its boundary. In order to do so, we investigate the properties of a tree as a metric measure space, namely the doubling condition and Poincaré inequality, and study the boundary in terms of geodesic rays as well as random walks. We review the definitions of the appropriate Sobolev and Besov spaces and the proof of the Trace Theorem in [BBGS].

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Contents

Introduction

Analysis has evolved a lot throughout history. In 1902, Lebesgue presented a new theory of integration which was much more general than the one invented by Newton and Leibniz in the 17^{th} and later developed by Riemann in the 19^{th} century (we refer to [HW] for a nice exposition on the development of analysis throughout history). This theory, which is known as measure theory, is used in almost all fields of mathematics, particularly in probability theory and dynamical systems. Many of the classical results using measure theory are usually obtained on $\mathbb{R}^n$ or its subsets but it is interesting to further generalize these results to any metric measure space (X, d, μ) where (X, d) is a separable metric space equipped with a measure μ.

When dealing with these somewhat abstract spaces, it is nice to have some practical examples at hand for which generalizations of results on $\mathbb{R}^n$ to results on metric measure spaces can apply. Trees, which are widely used in mathematics (to name some examples : Galton-Watson trees in probability or decision trees in computer science) are a nice example of metric measure spaces for which we can obtain analogues to some classical theorems and inequalities on $\mathbb{R}^n$.

We are particularity interested in the theory of Sobolev spaces which are defined in $\mathbb{R}^n$ as functions in L^p, $1 \leq p < \infty$ whose derivatives belong to L^p. These derivatives are defined in the weak sense. Sobolev spaces play an important role in analysis, especially when working with partial differential equations and their boundary value problems, see [Bre]. This theory was originally developed by Sobolev for domains in $\mathbb{R}^n$ and can be extended to metric measure spaces.

When studying functions in Sobolev spaces, a natural question is to ask how they can be characterized in terms of behavior at the boundary. For this we need the notion of trace, which was first invented by Dedekind. For a nice historical exposition on this topic, we refer to [Pie].

When speaking about trace, we refer to boundary values, and the trace theorem, which gives a bounded linear operator from functions on a domain Ω to functions on the boundary $\partial\Omega$ gives a better description on boundary values of functions. On $\mathbb{R}^n$, if we have a domain Ω and a continuous function $u : \overline{\Omega} \to \mathbb{R}$, then we naturally have boundary values. But, when $u \in L^p(\Omega)$, there is no sufficient information to talk about u on $\partial\Omega$ since the Lebesgue measure of $\partial\Omega$ is 0. However, this is solved when $u \in W^{1,p}(\Omega)$.

The famous Trace Theorem on Sobolev spaces is a way to define boundary values even when functions are not continuous. This theorem was first developed by Gagliardo and Uspenskii (see [Mir], [MR]) and states that the trace of a function $u \in W^{1,p}(\Omega)$ belongs to $L^p(\partial\Omega)$. In addition, we can say that in fact, the trace of $u \in W^{1,p}(\Omega)$ is in a better space than $L^p(\partial\Omega)$ which is the Besov space $B^{1-(1/p)}_{p,p}(\partial\Omega)$, also called fractional Sobolev space which corresponds to some intermediate space between $L^p(\partial\Omega)$ and $W^{1,p}(\Omega)$.

We find a similar result to the one of continuous functions on $\mathbb{R}^n$ for the tree X. If $f : X \to \mathbb{R}$ is a Lipschitz function, we have $\mathrm{Trf} = \lim_{[0,\zeta)\ni \mathrm{x}\to\zeta} \mathrm{f(x)}$ and so we naturally have boundary values. For the more general case: in [BBGS], Björn, Björn, Gill and Shanmugalingam give an analogue to the above mentioned Trace Theorem (as well as the doubling condition and Poincaré inequality) on the infinite K-ary tree and its boundary.

The goal of this thesis is to understand the Trace result in [BBGS]. With regards to originality, we have corrected some inconsistencies and filled in many missing details throughout this thesis which were not included in the original work [BBGS]. We also provide a full background on all concepts that arise in the understanding of the aforementioned results and have added some general notions on metric measure spaces, trees, boundaries as well as some probabilistic notions and a study of the forward simple random walk on the tree. In addition, we have provided some useful examples and computations.

This thesis is organized as follows: In Chapter 1, we recall some basic notions on metric measure spaces and define Sobolev spaces on $\mathbb{R}^n$ and on metric measure spaces. We present different versions of the Poincaré inequality and define Besov spaces and the Trace Theorem. Chapter 2 is devoted to the study of the tree as a metric measure space equipped with the doubling condition and Poincaré inequality which are important notions in the theory of metric measure spaces. Chapter 3 consists in a general exposition on the notion of boundary of a tree, some boundary properties as well as a study of the different ways to get to the measure on the boundary. We include a section on random walks and the Martin boundary. Finally, we conclude with chapter 4, which brings a detailed proof of the Trace Theorem of [BBGS] on the infinite K-ary tree.

Chapter 1

Sobolev Spaces and Poincaré inequalities

1.1 Metric measure spaces

We recall briefly some metric measure space definitions and introduce some useful notations. The main references for this section are [AF], [Hei], [HKST].

A metric measure space is defined by a triplet (X, d, μ) (which we will denote simply by X later on) where (X, d) is a separable metric space and μ is a non trivial, locally finite Borel-regular measure on X. By locally finite, we mean that $\forall x \in X$, $\exists\, r > 0$ such that $\mu(B(x, r)) < \infty$, where $B(x, r)$ denotes a *ball* in X. We define

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

where $x \in X$ is called a *center* of the ball and r its radius, $0 < r < \infty$. When x, r are understood, we will use B instead of $B(x, r)$. We write λB to denote the *dilated ball* $B(x, \lambda r)$, for $\lambda > 0$.

Definition 1.1 (Doubling measure)**.** A measure μ on X is said to be a *doubling measure* if balls have finite and positive measure and if there exists a constant $C_\mu \geq 1$ such that

$$\mu(2B) \leq C_\mu\, \mu(B)$$

for all balls $B \in X$. This implies that $\forall \lambda \geq 1,\ \exists\ C_{\mu,\lambda}$ such that $\mu(\lambda B) \leq C_{\mu,\lambda}\, \mu(B)$. We call a metric measure space (X, d, μ) *doubling* if μ is a doubling measure.

A Borel measure μ on (X, d) is said to be *Q-regular* if there exists a constant $C > 0$ and a radius $r_0 > 0$ such that $C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q$ for every $B \subset X$ with $0 < r < r_0$. Furthermore, μ is said to be *Ahlfors Q-regular* if the above holds for every $B \subset X$ with $0 < r < 2\mathrm{diam}(X)$. A measure μ is *Ahlfors regular* if it is Ahlfors Q-regular for some $Q > 0$.

1.2 Sobolev spaces

In order to have a better understanding of the definition of Sobolev spaces on a metric measure spaces, we start by defining them on $\mathbb{R}^n$. In order to do that, we first need to define distributions and weak partial derivatives. We refer to [AF], [Ev], [HK], [HK1], [Ha], [Hal] [Hei], [HKST], [Sha] for the definitions and theorems in this section.

1.2.1 Distributions and weak derivatives

A test function space is a vector space $\mathcal{D}$ on $\mathbb{R}^n$ given by a set of functions with a notion of convergence of functions. For $\Omega \subset \mathbb{R}^n$, an open set and $f : \Omega \to \mathbb{C}$, we define the support of f by

$$\mathrm{supp}\ f := \overline{\{x \in \Omega, f(x) \neq 0\}},$$

and we denote by $\mathcal{D}(\Omega)$ the set of C^∞ functions with compact support in Ω, sometimes denoted $C_c^\infty(\Omega, \mathbb{C})$, with the following notion of convergence: $(\phi_n)_{n\geq 1} \in \mathcal{D} \to \phi \in \mathcal{D}$ if there exists a compact $K \in \Omega$ such that supp $\phi \subset K$, supp $\phi_n \subset K, \forall n \in \mathbb{N}$ and $D^\alpha \phi_n \to D^\alpha \phi$ uniformly on K for all $\alpha \in \mathbb{Z}_+^n$. Here D^α denotes the partial derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, where $\alpha = (\alpha_1, ..., \alpha_n), \alpha_i \in \mathbb{Z}_+$, is a multi-index and $|\alpha| = \alpha_1 + ... + \alpha_n$.

Definition 1.2 (Distribution)**.** We denote by $\mathcal{D}'$ the dual space of $\mathcal{D}$, that is, the continuous linear functionals on $\mathcal{D}$. A *distribution* is an element of $\mathcal{D}'$.

For $u \in L^1_{loc}(\Omega)$, the linear functional associated with u is defined by

$$< u, \phi >= \int_\Omega u\phi \quad \forall \phi \in \mathcal{D}(\Omega).$$

Definition 1.3 (Weak derivatives)**.** Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multi-index. We say that v is the α^{th}-weak partial derivative of u, written $D^\alpha u = v$, if

$$\int_\Omega u D^\alpha \phi dx = (-1)^\alpha \int_\Omega v\phi dx, \quad \phi \in \mathcal{D}(\Omega).$$

In other words, if we are given u and there is a function v which satisfies the above for all ϕ, we say that $D^\alpha u = v$ in the weak sense. If such a function does not exist, then u doesn't have a weak α^{th} partial derivative. This weak derivative if it exists is uniquely defined up to a set of measure zero.

1.2.2 Sobolev spaces on $\mathbb{R}^n$

We define function spaces whose members have weak derivatives of various orders lying in various L^p spaces, which are called Sobolev spaces.

Definition 1.4 (Sobolev space $W^{m,p}$)**.** Let $\Omega \subset \mathbb{R}^n$ be open , $1 \leq p \leq \infty$, and $m \in \mathbb{N}$. The Sobolev space $W^{m,p}(\Omega)$ consists of all locally integrable functions $u \in L^1_{loc}(\Omega)$ such that, for all multi-index α with $|\alpha| \leq m, D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$. That is

$$W^{m,p}(\Omega) = \{u \in L^1_{loc}(\Omega) \quad \text{such that} \quad D^\alpha u \in L^p \quad \text{for} \quad |\alpha| \leq m\}.$$

The *Sobolev norm* on $W^{m,p}(\Omega)$ is defined by

$$||u||_{m,p(\Omega)} = \begin{cases} \Big(\sum\limits_{|\alpha|\leq m} \int_\Omega |D^\alpha u(x)|^p dx\Big)^{1/p} & \text{if } 1 \leq p < \infty \\ \max\limits_{|\alpha|\leq m} \|D^\alpha u\|_\infty & p = \infty. \end{cases}$$

Equipped with this norm, the Sobolev space is a Banach space $\forall\ 1 \leq p \leq \infty,\ m \in \mathbb{N}$.

We now state some important inequalities known as the Sobolev embedding theorem which have a great importance in Sobolev space theory.

In the following, we will restrict ourselves to $m = 1$. We will use the notation ∇u to denote the vector $(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$ and $|\nabla u|$ for its Euclidean norm. An equivalent norm on $W^{1,p}$ is defined by

$$\|u\|_{1,p(\Omega)} = \|u\|_p + \|\nabla u\|_p.$$

Theorem 1.5 (Sobolev embedding theorem for $W^{1,p}$)**.** *For a function $u \in W^{1,p}(\mathbb{R}^n)$, there exist constants depending only on n and p such that we have:*

1. $$\|u\|_{p^*} \leq C_{n,p} \|\nabla u\|_p \tag{1.1}$$

   when $1 \leq p < n$ and $p^* = \frac{np}{n-p}$ is called the Sobolev conjugate of p. This shows that $W^{1,p}$ is continuously embedded into L^{p*}. This inequality is also called the Gagliardo-Nirenberg-Sobolev inequality.

2. *If $p > n$, then u has a continuous representative which satisfies*

   $$|u(x) - u(y)| \leq C_{n,p} |x - y|^{1-n/p} \|\nabla u\|_p \quad \forall x, y \in \mathbb{R}^n.$$

   This says that $W^{1,p}, p > n$ is continuously embedded into $C^{0,1-n/p}$, the space of Hölder continuous functions of order $1 - n/p$ in $\mathbb{R}^n$. This inequality is also called Morrey's inequality.

3. *If $p = n$, there are $\epsilon = \epsilon_n > 0$ and $C = C_n \geq 1$ such that*

   $$\int_\Omega \exp\left\{ \left(\frac{|u|}{\epsilon \|\nabla u\|_n} \right)^{n/n-1} \right\} \leq C|\Omega|,$$

   when u is compactly supported in an open set Ω, and $|\Omega|$ is the Lebesgue measure of Ω. This inequality is also called Trudinger's inequality.

If u is defined only on a bounded subset Ω, the Sobolev inequality (1.1) cannot hold. In fact, if for example u is a constant, then $\|\nabla u\|_p = 0$ but $\|u\|_{p^*} \neq 0$. A version of (1.1) can hold if we assume that u vanishes on the boundary. Alternatively, we can replace u by $|u - u_\Omega|$ where u_Ω is the mean value of u on Ω. For $\Omega = B$, a ball, we obtain the following Sobolev-Poincaré inequality:

Theorem 1.6 (Sobolev-Poincaré inequality for a ball)**.** *For every C^∞ function u in a ball $B \subset \mathbb{R}^n, 1 \leq p < n$, we have:*

$$\Big(\int_B |u - u_B|^{p^*} dx \Big)^{1/p^*} \leq C_{n,p} \Big(\int_B |\nabla u|^p dx \Big)^{1/p}.$$

This inequality extends to all $u \in W^{1,p}(B)$.

A corollary of this version of the Sobolev-Poincaré inequality is the Poincaré inequality for a ball.

Theorem 1.7 (Poincaré inequality for $B \subset \mathbb{R}^n$)**.** *Let $1 \leq p < \infty$ and let $B \subset \mathbb{R}^n$. Then there exists a constant C depending only on n and p such that for every function $u \in W^{1,p}(B)$, we have that*

$$\|u - u_B\|_{L^p(B)} \leq Cr\|\nabla u\|_{L^p(B)}$$

where $u_B = \frac{1}{|B|}\int_B u(y)dy$ is the average value of u over the ball B.

Finally, for a general domain, we have:

Theorem 1.8 (Poincaré inequality for $\Omega \subset \mathbb{R}^n$)**.** *Let $1 \leq p < \infty$ and let Ω be a bounded, connected, open subset of R^n with C^1 boundary $\partial\Omega$.*
Then there exists a constant C depending only on Ω, n and p such that for every function $u \in W^{1,p}(\Omega)$, we have that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$$

where $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u(y)dy$ is the average value of u over Ω.

To motivate the definition of Sobolev spaces on metric measure spaces, we will give a proof of the Poincaré inequality on a ball. We follow similar lines to [Hei], page 28.

We fix a unit vector $\omega \in \partial B(0,1) \subset \mathbb{R}^n$ and let u be a C^∞ function in the ball $B = B(x,r)$. If $y = x + s\omega$, we obtain by fundamental theorem of calculus,

$$\begin{aligned} |u(y) - u(x)| &= \Big|\int_0^s \frac{d}{dt}u(x+t\omega)dt\Big| \\ &= \Big|\int_0^s \nabla u(x+t\omega)\cdot\omega\, dt\Big| \\ &\leq \int_0^s |\nabla u(x+t\omega)|dt \quad \text{where } s = |x-y| \text{ and } \omega = \frac{y-x}{|y-x|}. \end{aligned}$$

Integrating over $y \in B$, we get:

$$
\begin{aligned}
\int_{y\in B} |u(y)-u(x)|dy &\leq \int_{y\in B}\int_0^s |\nabla u(x+t\omega)|\, dt\, dy \\
&= \int_0^{diam(B)} \int_{y\in B, |x-y|\geq t} |\nabla u(x+t\omega)| dy dt \quad \text{by Fubini's theorem} \\
&= \int_0^{diam(B)} \int_{y\in B, |x-y|\geq t} \int_{S^{n-1}} \chi_{\{x+s\omega\in B\}} |\nabla u(x+t\omega)|\, d\sigma\, s^{n-1} ds\, dt \\
&= \int_0^{diam(B)} \int_0^s \int_{S^{n-1}} \chi_{\{x+s\omega\in B\}} |\nabla u(x+t\omega)|\, d\sigma \frac{t^{n-1}}{t^{n-1}}\, dt\, s^{n-1}\, ds \\
&= \int_0^{diam(B)} \int_{B(x,s)\cap B\subset B} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\, dz\, s^{n-1}\, ds \qquad (1.2) \\
&\leq \left(\int_0^{diam(B)} s^{n-1} ds\right)\left(\int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz\right) \\
&= \frac{(diam(B))^n}{n} \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz.
\end{aligned}
$$

In (1.2), we used the change of variables $z = x + t\omega, |x - z| = t,\ dz = d\sigma t^{n-1}\, dt$.

Now we have that

$$
\begin{aligned}
|u(x) - u_B| = |u(x) - \fint_B u(y)dy| &\leq \frac{1}{|B|}\int_B |u(x)-u(y)|dy \\
&\leq \frac{1}{|B|}\frac{(diam B)^n}{n}\int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz \\
&= C_n \int_B \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz \qquad (1.3)
\end{aligned}
$$

where inequality (1.3) is called a potential estimate.

From (1.2) in the proof above, we can also see that

$$
|u(x) - u_B| \leq \sup_{s\leq diam B} \int_{B(x,s)\cap B} \frac{|\nabla u(z)|}{|x-z|^{n-1}} dz \qquad (1.4)
$$

which by an estimate of Hedberg, see [Hal] gives:

$$
|u(x) - u_B| \leq C(diam B) M(|\nabla u|\chi_B)(x). \qquad (1.5)
$$

We recall the Hardy-Littlewood maximal function defined by

$$
M_R(f)(x) := \sup_{r<R} \fint_{B(x,r)} |f(y)| dy
$$

and

$$M(f) = M_\infty(f)$$

which is bounded in L^p for $p > 1$

$$\int_{\mathbb{R}^n} |M(f)(x)|^p dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p dx. \tag{1.6}$$

Integrating (1.5) over the pth power on B gives

$$\begin{aligned}
\int_B |u(x) - u_B|^p &\leq C(diamB)^p \int_{\mathbb{R}^n} M(|\nabla u|\chi_B)^p(x)dx \\
&\leq C(diamB)^p \int_{\mathbb{R}^n} (|\nabla u|\chi_B)^p(x)dx \quad \text{where we used (1.6)} \\
&\leq C(diamB)^p \int_B (|\nabla u|(x))^p dx
\end{aligned}$$

which completes the proof of the Poincaré inequality.

Given $x, y \in \mathbb{R}^n$, let B be a ball containing x, y with $diamB \approx |x - y|$. Using (1.5), one can write (see [Hal]):

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u_B| + |u_B - u(y)| \\
&\leq C(diamB)(M|\nabla u|(x) + M|\nabla u|(y)) \\
&\leq C|x - y|(M|\nabla u|(x) + M|\nabla u|(y)).
\end{aligned}$$

Letting $g = CM|\nabla u|$, if $u \in W^{1,p}(\mathbb{R}^n), 1 < p < \infty$, then $g \in L^p(B)$ by the boundedness of the maximal function. Thus for some $g \in L^p$, we have

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)). \tag{1.7}$$

1.2.3 Sobolev spaces on metric measure spaces

Motivated by (1.7), in the context of $\mathbb{R}^n$, Hajłasz ([Ha], [Hal]) defines Sobolev spaces in the setting of an arbitrary metric space equipped with the Borel measure.

Definition 1.9 (Hajłasz-Sobolev spaces, $M^{1,p}$)**.** Let (X, d, μ) be a metric measure space. For $1 < p < \infty$, $M^{1,p}(X)$ is the set of functions $u \in L^p(X)$ such that there exists $g \in L^p(X)$ where (1.7) holds for a.e. $x, y \in X$.

$M^{1,p}(X)$ is a linear space and we equip it with the norm $\|u\|_{1,p} = \|u\|_p + \inf \|g\|_p$ where the infimum is taken over all g satisfying inequality (1.7).

Members of $M^{1,p}(X)$ are equivalence classes of L^p functions and hence defined only almost everywhere. One can show that $M^{1,p}(X) = W^{1,p}(X)$ if $1 < p < \infty$ and X is a smooth, bounded domain in $\mathbb{R}^n$ but in general, $M^{1,p}(X) \subset W^{1,p}(X)$ if $X \subset \mathbb{R}^n$.

$M^{1,p}(X)$ is a Banach space for all $1 \leq p < \infty$.

We define Sobolev spaces on metric measure spaces based on the notion of the upper gradient called the Newtonian space $N^{1,p}$, originally defined by Shanmugalingam in [Sha].

In order to define $N^{1,p}$ we will start by stating some useful definitions A *curve* γ in a metric space X is defined by a continuous mapping $\gamma : [a,b] \to X$. The *image of the curve* is denoted by $|\gamma| = \gamma([a,b])$. The *length of a curve* is defined by

$$l(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})), t_1 < t_2 < ... < t_{n-1} \text{with } t_1, t_2, ..., t_{n-1} \in [a,b], t_0 = a, t_n = b \right\}$$

A curve is said to be *rectifiable* if its length is finite. To each rectifiable curve γ, we associate a length mapping defined by $s_\gamma : [a,b] \to [0, l(\gamma)]$ given by $s_\gamma(t) = l(\gamma|_{[a,t]})$. This length function is nondecreasing and continuous and so is differentiable almost everywhere. We define $|\gamma'(t)| := s'_\gamma(t)$. If $\gamma : [a,b] \to X$ is a rectifiable curve, then there is a unique curve $\tilde{\gamma} : [0, l(\gamma)] \to X$ such that

$$\gamma = \tilde{\gamma} \circ s_\gamma.$$

Moreover, $l(\tilde{\gamma}|_{0,t}) = t$ for every $t \in [0, l(\gamma)]$ that is $|\tilde{\gamma}'| = 1$ with our previous notation. $\tilde{\gamma}$ is called the *arc-length parametrization* of γ. When γ is a rectifiable curve and $\rho : \gamma([a,b]) \to [0,\infty]$ is a Borel measurable function, we define

$$\int_\gamma \rho := \int_0^{l(\gamma)} \rho \circ \tilde{\gamma}(t) dt.$$

Definition 1.10. A Borel function $g : X \to [0,\infty]$ is an *upper gradient* of $u : X \to \mathbb{R}$, a Borel function if

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g \tag{1.8}$$

for every rectifiable curve $\gamma : [a,b] \to X$. An upper gradient g is said to be *minimal* if it is integrable and if $g \leq \sigma$ almost everywhere in X whenever σ is an upper gradient of u. We denote the minimal upper gradient of g by g_u.

Definition 1.11 (The Newtonian space $N^{1,p}$). Let $1 \le p < \infty$. The Newtonian space $N^{1,p}$ is defined as collection of all L^p integrable Borel functions on X that have an upper gradient in L^p. Alternatively, it is the collection of functions for which the following norm is finite.

$$\|u\|_{N^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p}$$

where the infimum is taken over all upper gradients of u. Note that the elements in $N^{1,p}$ are equivalence classes of functions that are identified if $\|u - v\|_{N^{1,p}} = 0$.

Following a result in [Hal], we have replaced weak upper gradient by upper gradient in the above definition.

1.2.4 Poincaré inequality on metric measure spaces

Let $X = (X, d, \mu)$ be a metric measure space. We define Poincaré inequalities for real valued and vector valued functions.

Definition 1.12. Let $1 \le p < \infty$. We say that a metric measure space X supports a p-Poincaré inequality if there exists constants C_n, depending only on n and $\lambda \ge 1$ such that for any Borel measurable function $u : X \to \mathbb{R}$ that is integrable on balls, with upper gradient $g : X \to [0, \infty]$, the following inequality

$$⨍_B |u - u_B| d\mu \le C_n (diam B) \Big(⨍_{\lambda B} g^p d\mu \Big)^{1/p} \tag{1.9}$$

holds for every open ball $B \in X$.

Recall $u_B = ⨍_B u d\mu$ stands for the mean value of u over the ball B. Following from Hölders inequality, we see that if a space supports a p-Poincaré inequality for some $p \ge 1$, then it supports a q-Poincaré inequality for all $1 \le p < q$. When u is a smooth function in $B \subset \mathbb{R}^n$, we have the following inequality

$$⨍_B |u - u_B| dx \le C_n (diam B) ⨍_B |\nabla u| dx \tag{1.10}$$

Note that (1.10) is Theorem 1.7 when $p = 1$. The Poincaré inequality (1.9) has 2 differences with the Poincaré inequality (1.10) which makes it a weaker inequality than (1.10): the right hand side is the averaged L^p integral instead of the averaged L^1 integral and also the integration on the right hand side is over a larger ball than the integration on the left hand side. When the metric measure space is doubling, we can characterize inequality (1.9) in terms of pointwise inequalities between functions and their upper gradients as we will see in the following theorem.

Theorem 1.13. *Let X be a doubling metric measure space and let $1 \leq p < \infty$. Let $u : X \to V$ be integrable on balls and let $g : X \to [0, \infty]$ be measurable. Then the following three conditions are equivalent.*

1. *There exist constants $C > 0$ and $\lambda \geq 1$ such that*

$$\fint_B |u - u_B| d\mu \leq C diam(B) \left(\fint_{\lambda B} g^p d\mu \right)^{1/p}$$

for every open ball B in X.

2. *There exist constants $C > 0$ and $\lambda \geq 1$ such that*

$$|u(x) - u_B| \leq C diam(B)(M_{\lambda diam(B)} g^p(x))^{1/p}$$

for every open ball B in X and for almost every $x \in B$.

3. *There exist constants $C > 0$ and $\lambda \geq 1$ and $A \subset X$ with $\mu(A) = 0$ such that*

$$|u(x) - u(y)| \leq C d(x, y)(M_{\lambda d(x,y)} g^p(x) + M_{\lambda d(x,y)} g^p(y))^{1/p}$$

for every $x, y \in X \backslash A$.

The constants C and λ are not necessarily the same in the 3 conditions above but they depend only on each other and on the doubling constant of μ.

Theorem 1.14. *Let X be a metric measure space with finite measure. Let $p \geq 1$. For all functions $u \in M^{1,p}(X)$ such that $g \geq 0$ satisfies inequality (1.7), we have that*

$$\int_X |u - u_X|^p d\mu \leq 2^p (diam X)^p \int_X g^p d\mu. \tag{1.11}$$

A proof of the above theorem is analogous to the proof of the Poincaré inequality given previously.

1.2.5 From Poincaré to Sobolev-Poincaré inequalities.

In [HK], Hajłasz and Koskela prove that a weak-Poincaré inequality (1.9) implies a Sobolev-Poincaré inequality. We will start by defining metric spaces satisfying a (λ, M)-chain condition and will state the Hajłasz-Koskela theorem.

Definition 1.15. Let $\lambda, M \geq 1$ and $a > 1$. We say that a bounded subset A of a metric space X satisfies a (λ, M, a)-chain condition with respect to a ball B_0 if $\forall x \in A$, there is a sequence of balls $\{B_i, i = 1, 2, ..\}$ such that

1. $\lambda B_i \subset A$ for $i = 0, 1, 2, ...$
2. B_i is centered at x for all i sufficiently large.
3. The radius r_i of B_i satisfies $M^{-1}a^{-i} diam A \leq r_i \leq M a^{-i} diam A$ for all $i \geq 0$.
4. $B_i \cap B_{i+1}$ contains a ball B_i' such that $B_i \cup B_{i+1} \subset M Bi'$ for all $i \geq 0$.

Theorem 1.16 (Hajłasz-Koskela theorem)**.** *Let X be a doubling space, B a ball in X and suppose $A \subset X$ satisfies a (λ, M, a)-chain condition. Suppose $\frac{\mu(B)}{\mu(A)} \geq 2^{-s}(\frac{r}{diam A})^s$ holds for some $s > 1$. If u and g are 2 locally integrable functions on A with g non negative satisfying (1.9) for some $1 \leq p < s$, $C \geq 1$ for all $B \subset X$ for which $\lambda B \subset A$, then for all $q < ps/(s-p)$, there exists a constant C' depending only on $q, p, s, \lambda, M, a, C$ and on the doubling measure μ such that*

$$\left(\fint_A |u - u_A|^q d\mu \right)^{1/q} \leq C' diam(A) \left(\fint_A g^p d\mu \right)^{1/p}. \tag{1.12}$$

1.3 Besov spaces and the Trace Theorem

The trace theorem attempts to assign values on the boundary of $\Omega \subset \mathbb{R}^n$ to a function u on Ω. When $u \in L^p(\Omega)$, there is no sufficient information to talk about u on $\partial\Omega$ since the Lebesgue measure of $\partial\Omega$ is 0. However, this is solved when $u \in W^{m,p}(\Omega)$. Thus, in the Sobolev space $W^{m,p}(\Omega)$, the notion of trace or restriction to the boundary can be defined on $\partial\Omega$ even for functions which are not continuous on $\overline{\Omega}$ as we will see in the following theorem. Our main references for this section are [AF], [Ev].

Theorem 1.17 (Classical Trace theorem, [Ev]). *Assume Ω is bounded and $\partial\Omega$ is smooth. Then there exists a bounded linear operator*

$$Tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that for all $u \in W^{1,p}$,

$$||Tr(u)||_{L^p(\partial\Omega)} \lesssim ||u||_{W^{1,p}(\Omega)}.$$

Furthermore, for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, we have that

$$Tr(u) = u|_{\partial\Omega}.$$

$Tr(u)$ is called the trace of u on $\partial\Omega$.

When looking at the classical trace theorem, we can say that in fact, the trace of a $u \in W^{m,p}$ is in a "better" space then L^p which is an intermediate space between $L^p(\Omega)$ and $W^{m,p}(\Omega)$. These are the Besov spaces $B^s_{p,q}$ which correspond to derivatives of fractional order. Besov spaces can be defined in many different ways, for example using Fourier transforms or via interpolation. For more information, we refer to [AF]. We will define Besov spaces on $\mathbb{R}^n$ using the L^p-modulus of continuity which measures the smoothness of functions.

Definition 1.18 (Modulus of continuity, [AF]). Let $\Omega \subset \mathbb{R}^n$ be open and u, a function in $L^p(\Omega), 0 < p \leq \infty$. Let h be a point in $\mathbb{R}^n$, let I denote the identity operator, τ_h the translation operator defined by

$$\tau_h(u, x) := u(x + h),$$

and $\Delta_h^m, m = 1, 2, ...,$ the difference operators defined by

$$\Delta_h^{(m)}(u) = (\tau_h - I)^m.$$

The *modulus of continuity* of order m of u is then

$$\omega_p^{(m)}(u; h) := \|\Delta_h^{(m)}\|_p.$$

When $u \in L^p(\mathbb{R}_+)$, we define

$$\omega_p^{(m)*}(u; h) := \sup_{|h| \leq t} \|\Delta_h^{(m)}\|_p.$$

A Besov space $B^s_{p,q}$ is a collection of functions u with common smoothness. The following theorem gives an intrinsic characterization of Besov spaces on $\mathbb{R}^n$.

Theorem 1.19. *Let $m > s > 0, 1 < p < \infty, 1 \leq q < \infty$. Let $u \in L^p(\mathbb{R}^n)$. The following conditions are equivalent.*

1. $u \in B^s_{p,q}(\mathbb{R}^n)$.
2. $\int_0^\infty [t^{-s}\omega_p^{m^*}(u;t)]^q \frac{dt}{t} < \infty$.
3. $\int_{\mathbb{R}^n} [|h|^{-s}\omega_p^m(u;h)]^q \frac{dh}{|h|^n} < \infty$.

When $q = \infty$, we replace the integrals in 1. *and* 2. *by the* supremum *of the quantities inside the brackets.*

Now, we can state a more refined version of the trace theorem using Besov spaces.

Theorem 1.20 (Trace theorem on $\mathbb{R}^n$)**.** *Let $1 < p < \infty$ and let u be a measurable function on $\mathbb{R}^n$. The following conditions are equivalent :*

1. *There is a function $U \in W^{m,p}(\mathbb{R}^{n+1})$ such that u is the trace of U.*
2. $u \in B^{m-(1/p)}_{p,p}(\mathbb{R}^n)$.

For a proof of this theorem, we refer to [AF].

Chapter 2

Trees

In this chapter, we study the tree as a metric measure space equipped with the doubling measure and Poincaré inequality. We will begin by reviewing some basics of graph theory and the particular case of the tree. Our main references for this chapter are: [BBGS], [BH] [BHK], [Ev1], [Pa]. Figures in Chapters 2 and 3 we produced using Latex, Mathematica and Geogebra. The last case in figure 5 was taken from: https://www.philipvanegmond.nl/wiskunde/dim4-e.htm

2.1 Graph theory

A *graph* G is a pair of sets (V, E) where V is the set of vertices and $E \subseteq V \times V$ is the set of edges of the graph G. A graph is called *directed* if the edges between vertices have an implied direction and is *undirected* otherwise. For G, an undirected graph and $x, y \in V$, we consider (x, y) and (y, x) to be the same edge. Two vertices $x, y \in V$ are *adjacent* or *neighbors* if there is an edge joining them, i.e. $(x, y) \in E$. We will also use the notation $x \sim y$ The degree of a vertex $d_G(x)$ is the number of its neighbors. In a directed graph, if $(x, y) \in E$, we call the vertex x, the origin or initial vertex and the vertex y, the terminal vertex. A *loop* is an edge with only one end i.e. $(x, x) \in E$ for some $x \in V$. A graph G is said to be *simple* if it has no loops and no parallel edges (incident to the same vertices). A graph G is said to be *complete* if it is simple and all vertices of G have strictly positive degree. If all the vertices of a graph G are of same degree, say K, then G is said to be K-*regular* or just *regular*.

A *path* connecting two vertices $x, y \in V$ in a graph G is a non-empty subgraph $P = (V', E')$ where $V' = \{x_0, ..., x_k\} \subset V, E' = \{(x_0, x_1), ..., (x_{k-1}, x_k)\} \subset E$ with $x_i \neq x_j$ when $i \neq j$ for all $0 < i, j \leq k$. If we denote the path P from x to y by concatenation of vertices, we can write $P = x_0 x_1 ... x_k$ where $x = x_0$ and $y = x_k$. The *length of a path* is the number of edges it contains. The distance $d(x, y)$ between two vertices $x, y \in V$ in a graph G is the length of the shortest path from x to y if one exists; otherwise, $d(x, y) = \infty$. The *diameter of a graph* is the maximal distance between two vertices, $diam(G) = \max_{x,y \in G} d(x, y)$ (could be ∞). A graph G is said to be *connected* if for any two distinct vertices $x, y \in V$, there is a path from x to y. A *cycle* is a non trivial closed path. In the following, we will restrict ourselves to simple undirected, infinite graphs with all vertices of finite degree.

2.2 Trees

We are now able to define the setting of the tree. A *tree* T is an *acyclic connected graph*, i.e. for any pair of vertices $x, y \in T$, there is a unique path of distinct edges connecting x to y. Since a path between two vertices $x, y \in T$ is unique, it is therefore the shortest path between these vertices, which is called a *geodesic* and is denoted by $[x, y]$.

It is often convenient to designate one vertex as the root of the tree. Such a tree is called a rooted tree and each edge is implicitly directed away from the root, but we will still consider it to be an undirected graph. We will denote the root of a tree T by 0_T. The choice of a root defines a natural hierarchy on the vertices of a rooted tree according to their distance from the root.

Let T be a rooted tree. For $x \in T$, let $|x|$ be the distance from the root 0 to x, i.e. the number of edges in the geodesic from 0 to x. It is also called the *depth or level* of x. The *height of a rooted tree* is the length of the longest path from the root.

The *parent* z of x is the unique neighbor of x on the geodesic from the root 0_T to x. All other neighbors are called *children*. Each vertex has exactly one parent, except for the root. Vertices having the same parent are called *siblings*. A vertex y is called a *descendant* of a vertex x (and x is called an *ancestor* of y) if x is on the geodesic connecting the root to y. We denote this by $x \leq y$. We write $x < y$ if $x \leq y$ and $x \neq y$.

The length of the geodesic $[x, y]$ is denoted by $|x - y|$. If x and y are descendants of two different children of the root, then $|x - y| = |x| + |y|$. When y is a descendant of x, $|x - y| = |y| - |x|$.

We will consider rooted trees such that each vertex other than the root is of degree 3 or greater; the root will be of degree at least 2.

A *K-ary tree*, $K \geq 2$ is a rooted tree in which every vertex has at most K children. From here and on, we will follow the exposition in [BBGS] and restrict ourselves to *regular K-ary trees*, for a fixed K, i.e. rooted trees for which each vertex has exactly K children.

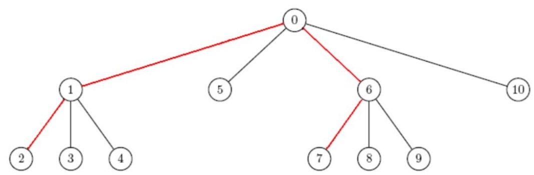

figure 1: tree and geodesic rays

2.3 The tree as a metric measure space

In order to define a metric and a measure on the tree, we want to view it no longer as a discrete tree. We consider each edge to be an isometric copy of the open unit interval and the vertices to be the endpoints of this interval. In addition to the vertices, we allow points to be taken along the edges of the tree. We will denote the tree, considered now as the set of all these points, by X, also consistent with [BBGS]. More formally, this is consistent with the notion of $\mathbb{R}$-tree.

Definition 2.1. A metric measure space X is called an $\mathbb{R}$-tree if it verifies the following properties: For all $x, y \in X$, there exists a unique topological segment that joins them and this topological segment is the image of a geodesic path $r : [a, b] \to X$. By topological segment, we mean a subset of X that is the homeomorphic image of a closed interval of $\mathbb{R}$.

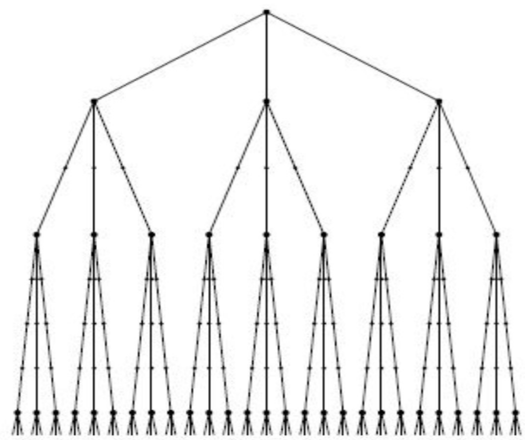

figure 2: tree as a metric measure space

We can now define $|z|$ when we are considering z to be some point along one of the edges of the tree, whereas until now, we have defined the norm $|x|$ to be the length of the geodesic between the root and the vertex x, which is necessarily an integer. This gives us the canonical metric of X , $|x-y|$ with which $diam(X) = \infty$. In order to give the tree a finite diameter, we fix $\epsilon > 0$ and define a uniformizing metric on X (in the sense of Bonk-Heinonen-Koskela [BHK]) as follows:

$$d(x,y) = \int_{[x,y]} e^{-\epsilon|z|} d|z|$$

where $[x,y]$ is the geodesic connecting x to y and $d|z|$ is the measure which gives each edge Lebesgue measure 1. The metric $d(x,y)$ is the conformal metric derived from the continuous density $e^{-\epsilon|z|}$ by conformal deformation of the canonical metric of X, $|x-y|$. With this metric, we have that $diam(X) = 2\int_0^\infty e^{-\epsilon t} dt = \frac{2}{\epsilon}$, which is finite. We can extend the definition of $x \leq y$ from vertices to any points along the edges. Let

$$B(x,r) = \{y \in X : d(x,y) < r\}$$

be an open ball in X with respect to d, and let

$$F(x,r) = \{y \in X : y \geq x, d(x,y) < r\}$$

be the downward directed half ball.

Lemma 2.2. *For every $x \in X$ and $r > 0$, we have*

$$F(x,r) \subset B(x,r) \subset F(z,2r),$$

where $z \leq x$ and

$$|z| = max\{|x| - \frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|}), 0\}.$$

Proof. The first inclusion $F(x,r) \subset B(x,r)$ is clear and true for all r by the above definitions of $F(x,r)$ and $B(x,r)$. For the second inclusion, we differentiate 2 cases: If $r \leq \frac{1-e^{-\epsilon|x|}}{\epsilon}$, we have that

$$\begin{aligned}
\epsilon r \leq (1-e^{-\epsilon|x|}) &\Leftrightarrow \epsilon r e^{\epsilon|x|} \leq e^{\epsilon|x|} - 1 \\
&\Leftrightarrow 1 + \epsilon r e^{\epsilon|x|} \leq e^{\epsilon|x|} \\
&\Leftrightarrow \log(1+\epsilon r e^{\epsilon|x|}) \leq \epsilon|x| \\
&\Leftrightarrow \frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|}) \leq |x| \\
&\Leftrightarrow |x| - \frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|}) \geq 0,
\end{aligned}$$

which gives us $|z| = |x| - \frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|})$. Also,

$$\begin{aligned}
d(x,z) &= \int_{|x|}^{|z|} e^{-\epsilon t} dt \\
&= -\frac{1}{\epsilon}e^{-\epsilon|x|} + \frac{1}{\epsilon}e^{-\epsilon|z|} \\
&= \frac{1}{\epsilon}e^{-\epsilon|x|}(e^{-\epsilon(|z|-|x|)} - 1) \\
&= \frac{1}{\epsilon}e^{-\epsilon|x|}(\epsilon r e^{\epsilon|x|}) \\
&= r.
\end{aligned}$$

At the same time, if $r \geq \frac{1-e^{-\epsilon|x|}}{\epsilon}$, then $|z| = 0$ and so

$$
\begin{aligned}
d(x,z) &= d(x,0) \\
&= \int_0^{|x|} e^{-\epsilon t} dt \\
&= -\frac{1}{\epsilon} e^{-\epsilon|x|} + \frac{1}{\epsilon} \\
&= \frac{1}{\epsilon}(1 - e^{-\epsilon|x|}) \\
&\leq r.
\end{aligned}
$$

Then for all $r > 0$ and for all $y \in B(x,r)$, the following inequalities are satisfied

$$d(y,z) \leq d(x,y) + d(x,z) \leq r + r < 2r.$$

If $z = 0$, all y satisfy $z \leq y$. Otherwise, we have $z \leq x$ and $d(x,z) = r$, so we must have $z \leq y$ whenever $d(x,y) < r$. $\square$

2.4 The doubling condition on the tree

We define a weighted measure on the tree by

$$d\mu(x) = e^{-\beta|x|} d|x| \tag{2.1}$$

where $\beta > \log K$ is fixed. Recall that $d|x|$ is the one-dimensional Lebesgue measure on X which gives each edge measure 1. One of the nice properties of the measure (2.1) is its doubling condition, which we will show. Note that if $\beta \leq \log K$, then $\mu(X) = \infty$ for the regular K-ary tree and since X is bounded, μ wouldn't be a doubling measure. This will become evident in the proof of Lemma 2.3. We will give the sequence of Lemmas leading to Corollary 2.9 showing that the measure on the tree is a doubling measure. Our starting point will be an estimation of the measure of balls in X. For the sake of completeness, we will reproduce some of the proofs in [BBGS], for which we have included some additional explanations and points of interest when necessary.

Lemma 2.3. *For $z \in X$ and $0 < r \leq \frac{e^{-\epsilon|z|}}{\epsilon}$, $\mu(F(z,r)) \simeq e^{(\epsilon-\beta)|z|} r$.*

Proof. Let $\rho > 0$ be such that

$$\int_{|z|}^{|z|+\rho} e^{-\epsilon t} dt = \frac{1}{\epsilon} e^{-\epsilon |z|}(1 - e^{-\epsilon \rho}) = r$$

We look at each t such that

$$|z| \leq t \leq |z| + \rho.$$

When representing a point on the tree as a center of a ball of radius r, we include all points which lie on the geodesic from the root to this point. Recall that the tree is K-ary, and so the number of points $y \in F(z, r)$ satisfying $|y| = t$ is approximately $K^{t-|z|}$. Thus, when estimating the measure of the downward directed half ball $F(z, r)$, we will sum over all those points. Since this sum isn't discreet, we will approximate it by an integral.

We get the following approximation:

$$\begin{aligned}
\mu(F(z,r)) &\simeq \int_{|z|}^{|z|+\rho} K^{t-|z|} e^{-\beta t} dt \\
&= K^{-|z|} \int_{|z|}^{|z|+\rho} K^{t} e^{-\beta t} dt \\
&= K^{-|z|} \int_{|z|}^{|z|+\rho} e^{t \log K} e^{-\beta t} dt \\
&= K^{-|z|} \int_{|z|}^{|z|+\rho} e^{t(\log K - \beta)} dt \qquad (2.2) \\
&= K^{-|z|} \frac{1}{\beta - \log K} e^{(\log K - \beta)|z|} \left(1 - e^{(\log K - \beta)\rho}\right) \\
&= \frac{e^{-\beta |z|}}{\beta - \log K} (1 - e^{(\log K - \beta)\rho}) \\
&= \frac{e^{-\beta |z|}}{\beta - \log K} \left((1 - (1 - \epsilon r e^{\epsilon |z|})^{\frac{\beta - \log K}{\epsilon}}\right)
\end{aligned}$$

by the choice of ρ. The right hand side of equality (2.2) shows that if z were the root, we would get the measure of the downward ball that is $e^{t(\log K - \beta)} dt$ and as $\rho \to \infty$, we would get the measure of whole tree. If $\beta \leq \log K$, this integral will blow up to infinity so a necessary condition is that $\beta > \log K$. We will now use the fact that when $\sigma > 0$ and $t \in [0, 1]$, we have that

$$\min\{1, \sigma\} t \leq 1 - (1 - t)^{\sigma} \leq \max\{1, \sigma\} t \qquad (2.3)$$

where we will take $t = \epsilon r e^{\epsilon |z|}$ and $\sigma = \frac{\beta - \log K}{\epsilon}$. This gives us

$$\mu(F(z,r)) \simeq \frac{e^{-\beta |z|}}{\beta - \log K} \frac{\beta - \log K}{\epsilon} \epsilon r e^{\epsilon |z|} \simeq e^{(\epsilon - \beta)|z|} r$$

From Lemma 2.2 and Lemma 2.3, we get the following:

Corollary 2.4 ([BBGS]). *If $0 < r \leq e^{-\epsilon|x|}/\epsilon$ then $\mu(B(x,r)) \simeq e^{(\epsilon-\beta)|x|}r$.*

Lemma 2.5. *Let $x \in X$ and $\frac{e^{-\epsilon|x|}}{\epsilon} \leq r \leq \frac{1}{\epsilon}(1-e^{-\epsilon|x|})$. Then $\mu(B(x,r)) \simeq r^{\frac{\beta}{\epsilon}}$.*

Proof. We use Lemma 2.2 and let $r \leq \frac{1}{\epsilon}(1-e^{-\epsilon|x|})$. This gives us that $B(x,r) \subset F(z,r) \subset F(z,\infty) = F(z, \frac{e^{-\epsilon|z|}}{\epsilon})$ We now use Lemma 2.3 which gives

$$\mu(F(z,\infty)) \simeq e^{(\epsilon-\beta)|z|}\frac{e^{-\epsilon|z|}}{\epsilon} \lesssim e^{(\epsilon-\beta)|z|}e^{-\epsilon|z|} \simeq e^{-\beta|z|}. \tag{2.4}$$

Now, $\frac{e^{-\epsilon|x|}}{\epsilon} \leq r$ implies that $1 \leq \epsilon r e^{\epsilon|x|}$ and so $1 + \epsilon r e^{\epsilon|x|} \leq 2\epsilon r e^{\epsilon|x|}$. Now by using the definition of $|z|$, we obtain that

$$\begin{aligned} e^{-\beta|z|} &= e^{-\beta|x|}(1+\epsilon r e^{\epsilon|x|})^{\frac{\beta}{\epsilon}} \\ &\leq e^{-\beta|x|}(2\epsilon r e^{\epsilon|x|})^{\frac{\beta}{\epsilon}} \\ &= (2\epsilon r)^{\frac{\beta}{\epsilon}}. \end{aligned}$$

We now insert the above into (2.4) which gives us the proof for the upper bound. For the lower bound, we use once again, the definition of $|z|$ and so

$$\begin{aligned} \mu(B(x,r)) &\geq \int_{|z|}^{|x|} e^{-\beta t}dt \\ &= -\frac{1}{\beta}e^{-\beta|x|} + \frac{1}{\beta}e^{-\beta\left(|x|-\frac{1}{\epsilon}\log(1+\epsilon r e^{\epsilon|x|})\right)} \\ &= \frac{e^{-\beta|x|}}{\beta}((1+\epsilon r e^{\epsilon|x|})^{\frac{\beta}{\epsilon}} - 1) \end{aligned}$$

We now use the function $f(t) = ((1+t)^{\frac{\beta}{\epsilon}} - 1)/t^{\frac{\beta}{\epsilon}}$ for $t = \epsilon r e^{\epsilon|x|}$ which is monotone with $\lim_{t\to\infty} f(t) = 1$. Since $\epsilon r e^{\epsilon|x|} \geq 1$, we obtain that $f(\epsilon r e^{\epsilon|x|}) \geq \min\{1, f(t)\} \simeq 1$ which finally gives us the desired approximation for the lower bound :

$$\mu(B(x,r)) \gtrsim e^{-\beta|x|}(\epsilon r e^{\epsilon|x|})^{\frac{\beta}{\epsilon}} \simeq r^{\frac{\beta}{\epsilon}}$$

□

Lemma 2.6 ([BBGS]). *Let $x \in X$ and $d(x,0) = (1-e^{-\epsilon|x|})/\epsilon \leq r \leq 2diamX$ then, $\mu(B(x,r)) \simeq r$. In particular, if $x = 0$, then this estimate holds for every $r \geq 0$.*

Following from Corollary 2.4, Lemmas 2.5 and 2.6, we have the following proposition:

Proposition 2.7. *Let $x \in X$ and $0 < r \leq 2diam(X)$ and let $R_0 = e^{-\epsilon|x|}/\epsilon$. We differentiate 2 cases based on the value of $|x|$.*

1. *If $|x| \leq (\log 2)/\epsilon$, then $\mu(B(x,r)) \simeq r$.*

2. *If $|x| \geq (\log 2)/\epsilon$, then* $\mu(B(x,r)) \simeq \begin{cases} e^{(\epsilon-\beta)|x|}r & \text{if } r \leq R_0 \\ r^{\frac{\beta}{\epsilon}} & \text{if } r \geq R_0 \end{cases}$

Corollary 2.8. *The following dimension condition holds for all balls $B(x,r)$ and $B(x',r')$ with $x' \in B(x,r)$ and $0 < r' \leq r$.*

$$\frac{\mu(B(x',r'))}{\mu(B(x,r))} \gtrsim \left(\frac{r'}{r}\right)^s$$

where $s = \max\{1, \frac{\beta}{\epsilon}\}$ is the best possible.

Proof. We first study the case of $x' = x$ and $r \leq 2$ diam$X$ and will show the general case later. We look at the possibles values of $|x|$.

1. $|x| \leq (\log 2)/\epsilon$

2. $|x| \geq (\log 2)/\epsilon$ and r and r' belong to the same interval.

   In the above cases, we have an immediate result by Proposition 2.7.

3. $|x| \geq (\log 2)/\epsilon$ and $r' \leq R_0 \leq r$.

For this last case, we have that

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \simeq \frac{e^{(\epsilon-\beta)|x|}r'}{r^{\frac{\beta}{\epsilon}}} \simeq \begin{cases} \left(\frac{R_0}{r'}\right)^{\frac{\beta}{\epsilon}-1}\left(\frac{r'}{r}\right)^{\frac{\beta}{\epsilon}} & \text{if } \beta \geq \epsilon \\ \left(\frac{R_0}{r}\right)^{\frac{\beta}{\epsilon}-1}\left(\frac{r'}{r}\right) & \text{if } \beta \leq \epsilon \end{cases}$$

Since $R_0/r' \geq 1 \geq R_0/r$, we have our result. This also shows that (2.8) cannot hold for any $s < \max\{1, \beta/\epsilon\}$.

We now study the general case of $x' \in B(x,r)$ and $0 < r' \leq r \leq$ diam$X$. This gives us that $\overline{B(x,r)} \subset B(x',2r)$ and so by the above, we get

$$\frac{\mu(B(x',r'))}{\mu(B(x,r))} \geq \frac{\mu(B(x',r'))}{\mu(B(x',2r))} \gtrsim \left(\frac{r'}{2r}\right)^s.$$

Finally, if $r \geq$ diam X, Then $B(x,r) = X = B(x', diamX)$ and we are done:

$$\frac{\mu(B(x',r'))}{\mu(B(x,r))} = \frac{\mu(B(x',r'))}{\mu(B(x',\mathrm{diam}X))} \gtrsim \left(\frac{r'}{\mathrm{diam}X}\right)^s \geq \left(\frac{r'}{r}\right)^s.$$

□

As a direct consequence, we have the following Corollary:

Corollary 2.9 ([BBGS]). *The measure μ is doubling i.e.*

$$\mu(B(x,2r)) \lesssim \mu(B(x,r)).$$

2.5 Poincaré inequality on the tree

We have shown that the tree is a metric measure space and that the measure on the tree is doubling. Recall that a Borel function $g : X \to [0,\infty]$ is an upper gradient of $u \in L^1_{loc}(X)$ if

$$|u(z) - u(x)| \leq \int_\gamma g d_X s$$

when $z, x \in X$ and γ is the geodesic from z to y and $d_X s$ denotes the arc length measure with respect to the metric d_X. In fact, in the setting of a tree, any geodesic connecting z to y is a rectifiable curve with endpoints z and y. This gives us the following Lemma which will lead to the Poincaré inequality on the tree.

Lemma 2.10 ([BBGS]). *Let $B = B(x,r)$ be a ball in X, and let z be defined as in Lemma 2.2.Then for every $u : B \to \mathbb{R}$ and every upper gradient g of u in B, the following inequality is satisfied*

$$\int_B |u(y) - u(z)| d\mu(y) \leq \int_B g(\omega) e^{(\beta-\epsilon)|\omega|} \mu(\{y \in B : y \geq \omega\}) d\mu(\omega) \tag{2.5}$$

Theorem 2.11 ([BBGS]). *The space X supports a 1-Poincaré inequality*

Proof. To get to the Poincaré inequality, we use (2.5) and the following estimate in [BBGS] for $\omega \in B = B(x,r)$ when $|\omega| \geq |x|$:

$$|w| < |x| - \frac{1}{\epsilon}\log(1 - \epsilon r e^{\epsilon|x|}) \tag{2.6}$$

as well as Lemma 2.3. We estimate the measure of the downward half ball when $y \in B$ such that $y \geq \omega$, depending on the values of r.

When $r \leq e^{-\epsilon|x|}/3\epsilon$, we have that $2r \leq \frac{2e^{-\epsilon|x|}}{3\epsilon} < \frac{e^{-\epsilon|\omega|}}{\epsilon}$ by using (2.6). From Lemma 2.3, we get that $\mu(\{y \in B : y \geq \omega) \lesssim e^{(\epsilon-\beta)|\omega|}r$ which together with (2.5) gives

$$\int_B |u(y) - u(z)| d\mu(y) \lesssim r \int_B g(\omega) d\mu(\omega). \tag{2.7}$$

Now, for $r \geq e^{-\epsilon|x|}/3\epsilon$, we obtain through the same process that

$$\int_B |u(y) - u(z)| d\mu(y) \lesssim e^{-\epsilon|z|} \int_B g(\omega) d\mu(\omega)$$

which by the choice of r and of $|z|$ as in Lemma 2.2 gives us again (2.7). We now use the fact that the mean oscillation of a function on the ball is bounded by its oscillation with respect to any constant to get

$$⨍_B |u - u_B| \leq 2 ⨍_B |u(y) - u(z)| d\mu(y) \lesssim 2r ⨍_B g(\omega) d\mu(\omega)$$

where $u(z)$ is constant on B. This gives us the 1-Poincaré inequality for the tree and concludes the proof. □

A corollary of the above is the p-Poincaré inequality.

Corollary 2.12 ([BBGS]). *The space X supports a p-Poincaré inequality*

$$⨍_B |u - u_B|^p d\mu \leq Cr ⨍_B g^p d\mu.$$

Chapter 3

The boundary of the tree

In this chapter, we will describe and study the properties of ∂X, the boundary of the tree X. We refer to [BBGS] [BH], [Edg], [Pa], [Woe1] for the definitions and examples in this chapter. We define ∂X by completing X with respect to the metric d_X. Equivalently, we can construct the boundary as follows: a point $\zeta \in \partial X$ is identified with an infinite geodesic in X starting at the root 0. This geodesic is denoted by concatenation of vertices i.e. $\zeta = 0x_1x_2x_3...$ where x_i is a vertex in X at distance i from the root and x_{i+1} is a child of x_i.

We define the distance between points 2 points $\zeta, \xi \in \partial X$ as the length of the infinite geodesic $[\zeta, \xi]$ between them, with respect to the metric d_X. If this infinite geodesic lies at distance k from the root (i.e. in order to get from ζ to ξ, we start from ζ and "climb up" the tree till we reach the common parent to ζ and ξ which lies as distance k from the root, and then we "go down" to ξ), then $d_X(\zeta, \xi) = 2\int_k^\infty e^{-\epsilon t} dt = \frac{2}{\epsilon} e^{-\epsilon k}$. The restriction of the metric d_X to the boundary ∂X is called the visual metric on ∂X.

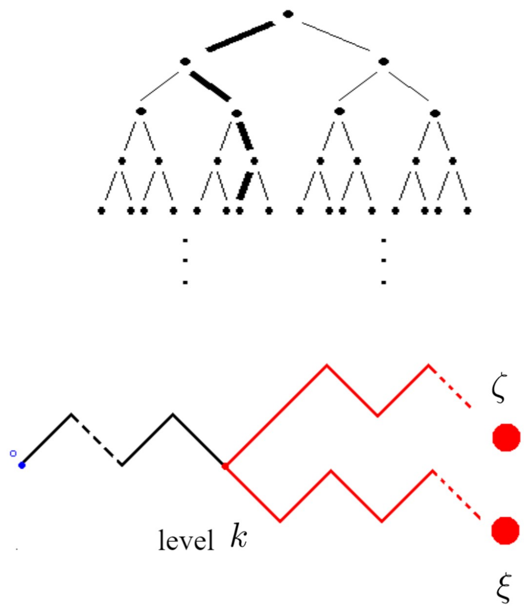

figure 3: the boundary

The goal of this section is to identify the boundary with the infinite geodesics. We will start by defining more rigorously a general notion of boundary.

3.1 The visual boundary

We define the boundary of a metric space X as the collection of equivalence classes of geodesic rays in X where rays are equivalent if they are asymptotic. We will start by defining asymptotic geodesic rays and then explain how the boundary is constructed.

Definition 3.1. A *geodesic ray* in a metric space X is an isometry

$$r : [0, \infty) \to X.$$

We say that $r(0)$ is the *origin* of r or that r starts at $r(0)$.

Two geodesics rays $r_1 : [0,\infty) \to X$ and $r_2 : [0,\infty) \to X$ are *asymptotic* if there exists a constant $C \in \mathbb{R}$ such that for all $t \geq 0$, we have

$$d_X(r_1(t), r_2(t)) \leq C.$$

Note that in the case of a regular K-ary tree, we can say that a geodesic ray r is a path that starts at any vertex and goes to infinity without backtracking. Using the notations in the previous chapter, $r = x_0x_1...$ such that $x_{k-1} \neq x_{k+1}$ $\forall k \in \mathbb{N}$ and $x_k \sim x_{k+1}$. The asymptoticity of geodesic rays is a very useful generalization to arbitrary metric spaces of the notion of parallel lines in the Euclidean space. In order to better understand the notion of geodesic rays, we give some examples:

1. In $\mathbb{R}$, each point is the origin of exactly 2 geodesic rays which are not asymptotic.

2. Let $X \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating around the z-axis the subset Γ of the xz-plane defined as the union of the half line $\{x = -1, z \geq 0\}$ with the affine segment joining $(0,-1)$ to $(-1,0)$. We equip X with the metric induced from $\mathbb{R}^3$. In this space, Γ is the image of a geodesic ray r starting at $(0,-1)$ and the rays asymptotic to r are those obtained by composing r with the rotation around the z-axis.

3. In an $\mathbb{R}$-tree, 2 geodesic rays are asymptotic if and only if their images coincide up to a compact segment, i.e. $\exists\, t_1, t_2 \geq 0$ such that $r_1([t_1,\infty)) = r_2([t_2,\infty))$. In particular, two geodesic rays emanating from the same point are asymptotic if and only if they are the same ray.

We are now ready to construct the boundary.

Definition 3.2. Let X be a geodesic metric space. For each point $p \in X$, we consider the set $\mathcal{R}_pX$ of geodesic rays in X starting at p, equipped with the topology of uniform convergence on compact sets in $[0,\infty)$. We then associate to p the space ∂_pX which is defined as the quotient space of $\mathcal{R}_pX$ by the equivalence relation that identifies 2 geodesic rays if and only if they are asymptotic. The space ∂_pX, equipped with the quotient topology induced from that of $\mathcal{R}_pX$, is called the *visual boundary* of X at p.

We illustrate with some examples of visual boundaries:

1. If $X = \mathbb{R}^n$, then for every $p \in X$, the visual boundary $\partial_p X$ coincides with the space of geodesic rays $\mathcal{R}_p X$ and is homeomorphic to the sphere $\mathbb{S}^{n-1}$.

2. When X is a regular K-ary tree, $K \geq 2$, we have the canonical identification: $\mathcal{R}_p X \simeq \partial_p X$ for every $p \in X$ and this space is totally disconnected. This is because the tree is a *Busemann space* where each equivalence class of geodesic rays originating at a given point p is reduced to one element (see 3. in previous set of examples). We have seen that the restriction of d_X to ∂X is called the visual metric on ∂X. Combined with the metric on the tree, we can define the metric on $\overline{X} = X \cup \partial X$. In fact, we consider the boundary where $p = 0$ is fixed and look at the rays emanating from the root: Let y be the common ancestor to x and ζ at distance k from the root. The distance from x to ζ is given as follows:

   (a) When $x \neq y$, then $d(x, \zeta) = \int_k^\infty e^{-\epsilon t} dt + \int_k^{|x|} e^{-\epsilon t} dt = \frac{2}{\epsilon} e^{-\epsilon k} - \frac{1}{\epsilon} e^{-\epsilon |x|}$

   (b) When $x = y$, i.e. x is on the geodesic from the root to ζ, then $k = |x|$ and we have, $d(x, \zeta) = \frac{1}{\epsilon} e^{-\epsilon |x|}$ which goes to 0 as $x \to \zeta$.

3.2 Ultrametric spaces

We will now show that the boundary equipped with the uniformizing metric on the boundary is an ultrametric space.

Definition 3.3. A metric space (Z, d_Z) is an *ultrametic space* if for each triple of points $x, y, z \in Z$, we have

$$d_Z(x, z) \leq \max\{d_Z(x, y), d_Z(y, z)\}.$$

The ultrametric property is called “the strong triangle inequality”.

Lemma 3.4. *The metric space $(\partial X, d_X)$ is an ultrametric space and consequently, for every $\zeta \in \partial X, r > 0$ and $\xi \in (B(\zeta, r))$, we have that $B(\zeta, r) = B(\xi, r)$. In other words, every point in a given ball on the boundary is actually a center to this ball.*

Proof. We choose 3 points $\zeta, \eta, \xi \in \partial X$ and define k, k_1, k_2 respectively as the number of edges in the shortest path connecting the root 0 to the infinite geodesic $[\zeta, \xi], [\zeta, \eta], [\xi, \eta]$. Then, we have that $k \geq \{k_1, k_2\}$ and we obtain that $e^{-\epsilon k} \leq \max\{e^{-\epsilon k_1}, e^{-\epsilon k_2}\}$. Now, by using the metric on the boundary,

$$d_X(\zeta, \xi) = 2\int_k^\infty e^{-\epsilon t} dt = \frac{2}{\epsilon} e^{-\epsilon k}$$

and replacing it in the above, we get its ultrametric property:

$$d_X(\zeta, \xi) \leq \max\{d_X(\zeta, \eta), d_X(\xi, \eta)\}.$$

The last part of the lemma is a direct consequence of the ultrametric property. In fact, given a point $\zeta \in \partial X$ and a distance r, the ball $B(\zeta, r)$ is represented in an ultrametric tree by the set of all leaves in the subtree descending from a certain vertex. Suppose ξ is an arbitrary point in $B(\zeta, r)$, then $B(\xi, r)$ is represented by the set of leaves in the subtree descending from the unique vertex above ξ at level r. But, this vertex is the same as the one above ζ at level r, giving us the same ball. □

We follow Holly's exposition in [Ho] for a nice example of an ultrametric space visualized as the tree of $\mathbb{Q}_p$, the field of p-adic numbers. We will give some useful definitions to the understanding of this example.

Definition 3.5. Let $b \in \mathbb{Q}$. We write $b = \frac{r}{s} p^n$ where $r, s, n \in \mathbb{Z}$ and p is a *fixed prime* which doesn't divide r or s. The *p-adic norm* on $\mathbb{Q}$ is defined by $|b|_p = \frac{1}{p^{ord_p(b)}}$ where $ord_p(b) = n$. Note that when p is a prime, its p-adic norm decreases as its positive powers increase since $|p^n|_p = 1/p^n$. The *p-adic metric* on $\mathbb{Q}$ is the metric induced by the p-adic norm. We denote by $|x - y|_p$, the *p-adic distance* between $x, y \in \mathbb{Q}$. Under the p-adic metric, every $x \in \mathbb{Q}$ can be written as

$$x = \sum_{k=n}^{\infty} b_k \, p^k$$

for some $n \in \mathbb{Z}, b_k \in \{0, 1, ..., p-1\} \forall k \geq n$. We say that this series represents the *p-adic expansion* of the number which is finite for positive integers and infinite for negative integers and many non-integers and converges under the p-adic metric.

For the sake of simplicity, we will study the tree for $\mathbb{Z}_p$ which is the completion of Zin $\mathbb{Q}_p$ with respect to the p-adic metric. The structure of the tree for $\mathbb{Z}_p$ is as follows:

1. The levels or heights on the tree correspond to the different values of $ord_p(b)$.

2. The edges of the tree correspond to the choices of coefficients b_k in the p-adic expansion of an integer.

3. The vertices are given by a truncated form of the p-adic expansion of an integer. For a vertex x at level j in the tree, we have that $x = \sum_{k=n}^{j} b_k p^k$.

4. The distance between any two integers corresponds to the distance from the root to their common ancestor and is given by the p-adic distance.

5. Finally, the boundary corresponds to the infinite sequences which are given by the p-adic expansion of the integers as described before.

For an example and illustration of this tree under the 3-adic metric, see [Ho]. Since $\mathbb{Z}_p$ corresponds to the integers or the unit balls in $\mathbb{Q}_p$, another way to visualize the boundary of the tree for $\mathbb{Z}_p$ is with balls as we will explain in the next section. We refer to figure 4.

3.3 The measure on the boundary

We will show that there are 3 ways to get to the measure of the boundary. We will start by defining the notions of Hausdorff measure and Hausdorff dimension.

Definition 3.6. Let F be a subset of S, a metric space, and $Q \in \mathbb{R}_+$, then for any $\delta > 0$

$$\mathcal{H}_\delta^Q(F) = \inf \Big\{ \sum_{i=1}^{\infty} (\text{diam B}_i)^Q : \{B_i\} \text{ is a } \delta\text{-cover of F} \Big\}.$$

By δ-cover, we mean a cover by ball with diameter less than δ. Let

$$\mathcal{H}^Q(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^Q(F).$$

This limit exists for all subsets of S, a metric space and is called the Q-dimensional Hausdorff measure of F.

Definition 3.7. The value

$$Q = \inf\{Q : \mathcal{H}^Q(F) = 0\} = \sup\{Q : \mathcal{H}^Q(F) = \infty\}$$

such that

$$\mathcal{H}^Q(F) = \begin{cases} \infty & \text{if } Q < dim_H F \\ 0 & \text{if } Q > dim_H F \end{cases}$$

is called the Hausdorff dimension of the set F and is denoted $dim_H F$. If $Q = dim_H F$, $\mathcal{H}^Q(F)$ may be zero or infinite or may satisfy

$$0 < \mathcal{H}^Q(F) < \infty.$$

The uniform measure ν on the boundary of the tree ∂X is obtained by putting mass 1 uniformly on the vertices of level n in the tree and then taking weak limits as $n \to \infty$. The relation between Q-dimensional Hausdorff measure and uniform measure on the boundary will be explained in proof of following lemma.

Lemma 3.8. *The boundary of the K-ary tree X, ∂X, equipped with the uniform measure ν is an Ahlfors Q-regular space with*

$$Q = \frac{\log K}{\epsilon}.$$

This means that there exists a constant $C > 0$ such that

$$C^{-1} r^Q \leq \nu(B(x,r)) \leq C r^Q$$

for every $B \subset X$ with $0 < r < \operatorname{diam}(X)$. Therefore, $\nu \approx \mathcal{H}^Q$, the Q-dimensional Hausdorff measure.

Proof. Fix $n \in \mathbb{N}$. Since X is a K-ary tree, we have K^n descendants of the root at level n. For each one of these K^n vertices, if we consider the geodesic rays from the root going through it, these corresponds to a ball on the boundary. Any 2 points in these balls are connected by an infinite geodesic lying at distance at least n from the root and therefore the ball has radius $r_n = \frac{2}{\epsilon} e^{-\epsilon n}$. Moreover, the balls corresponding to the different vertices are all disjoint and lie at distance r_{n-1} from each other. This defines the boundary ∂X as the union of these K^k disjoint open balls. Using the ultrametric property of the boundary ∂X

which tells us that any point of a ball on the boundary is a center of this ball, we obtain that $\nu(B(\zeta, r_k)) = K^{-k}$ for every $\zeta \in \partial X$. From the definitions of r_k and Q, we obtain that

$$\nu(B(\zeta, r_k)) \simeq r_k^Q. \tag{3.1}$$

In fact, a simple computation of the above gives

$$r_k^Q = \left(\frac{2}{\epsilon} e^{-\epsilon k}\right)^Q = K^{-k} \left(\frac{2}{\epsilon}\right)^{\frac{\log K}{\epsilon}}. \tag{3.2}$$

Any $0 < r < diam(\partial X) = \frac{2}{\epsilon}$ must satisfy that $r_{k+1} < r \leq r_k$ for some $k \in \mathbb{N}$. Because of the discrete nature of the distances between points on boundary, the ball $B(\zeta, r)$ is equal to the ball $B(\zeta, r_k)$. We get the Hausdorff condition $\nu(B(\zeta, r_k)) = \nu(B(\zeta, r)) \simeq r_k^Q \simeq r^Q$. By extension, for all measurable sets $A \subset \partial X$, we have that $\nu(A) \approx \mathcal{H}^Q(A)$ where $\mathcal{H}^Q$ is the normalized Q-dimensional Hausdorff measure on the boundary. $\square$

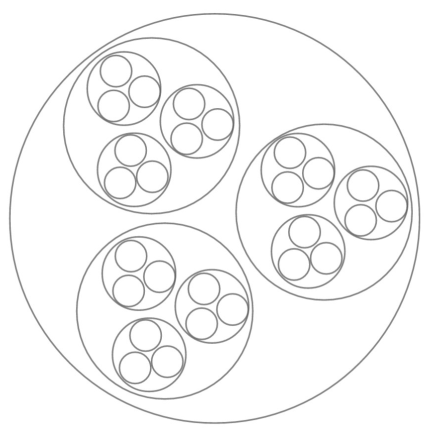

figure 4: the boundary of a 3-ary tree

We can view the computation of the uniform measure on the boundary, following [Ba], [BBGS][example 5.3], using the notion of similarity.

Definition 3.9. Let A be closed set in Y, a metric space. The transformations

$$S_i : A \to A,\ i = 1, ..., m,\ m \geq 2$$

are called *similarities*, or *contracting similarities* if there exist constants $0 < c_i < 1$ such that

$$|S_i(x) - S_i(y)| = c_i |x - y|$$

for $x, y \in Y$. The constant c_i is called the *ratio* of S_i.

We identify K, the number of children of each vertex with K similarities going from the tree X to a sub-tree, rooted at a child of the original root. When going from the tree to the subtree, the length of each edge is multiplied by a similarity ratio of $e^{-\epsilon}$. As such, the boundary ∂X can be identified with a totally disconnected fractal regular set F where each of the subtrees obtained are the "parts" of the fractal and all the distances are shrunk by the similarity ratio. The projection of each subtree on the boundary corresponds to a ball. As discussed above, we have geodesics passing through the root of each subtree which gives every ball $(B(\zeta, r))$ on boundary diameter $e^{-\epsilon k}\mathrm{diam}(X)$ where k is the largest level of the root of the subtree identified with this given ball. Note that the ball corresponding to the original root is the whole boundary and has diameter $2/\epsilon$.

Since we can re-write $K = e^{\left(-\epsilon \frac{\log K}{\log e^{-\epsilon}}\right)} = e^{\left(-\epsilon \frac{\log K}{-\epsilon}\right)}$, we get that

$$\nu(B(\zeta, r)) \simeq (\mathrm{diam}\, B(\zeta, r))^Q$$

with $Q = \log K/\log \epsilon$, the Hausdorff dimension of F, which is consistent with the proof of the lemma above.

1. Now with $K = 2$ and $\epsilon = \log 3$, we have a binary tree where at level n, each of the two similarities has contraction ratio 3^{-n} and we obtain the ternary Cantor set.

2. For $K = 3$ and $\epsilon = \log 3$, we have a 3-ary tree where at level n, each on the three similarities has contraction ratio 3^{-n} and we obtain the Sierpinski dust: this is achieved by splitting an equilateral triangle into 9 smaller congruent triangles and picking the 3 triangles containing the vertices of the original triangle and repeating these steps recursively.

3. For $K = 3$ and $\epsilon = \log 2$, we have a 3-ary tree where at level n, each on the three similarities has contraction ratio 2^{-n} and we obtain a snow-flaked version of the Sierpinski dust.

4. For $K = 4$ and $\epsilon = \log 4$, we have a 4-ary tree where at level n, each on the four similarities has contraction ratio 4^{-n} and we obtain the one-dimensional Garnett-Ivanov set also called the four corner set. This is achieved by splitting a square into 16

smaller congruent squares and picking the 4 corner squares and repeating the process recursively for each of the 4 corner squares obtained.

Each of these fractal sets can be mapped out as a K-ary tree with K vertices and length of edges corresponding to the contraction ratio at that given level.

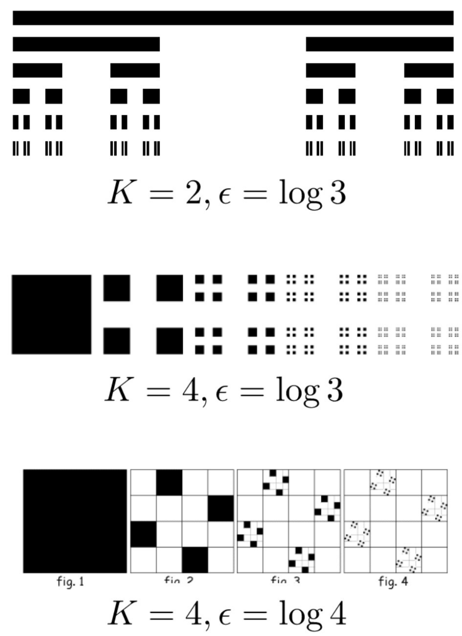

figure 5: the boundary as a fractal regular set

3.4 Random walks on the tree

Finally, we can get to the measure on the boundary through probabilistic notions.

A *random process* $(X_i)_{i\in\mathbb{N}}$ is a family of random variables where X_i denotes the position of a particle at time i. It is a *Markov chain* if it satisfies the Markov condition:

$$\mathbb{P}(X_n = s|X_0 = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = s|X_{n-1} = x_{n-1})$$

One of the most encountered random processes is the random walk.

Definition 3.10. A *random walk* on a graph G is a Markov chain on its set of vertices V which means that $\forall n_o \in \mathbb{N}$, conditionally on its present value X_{n0}, the future of the path $(X_{n_0+1}, X_{n_0+2}, ...)$ is independent of the past $(X_0, X_1, ..., X_{N_0})$. We denote the law of the random walk starting at x_0 (i.e. such that $X_0 = x_0$) by $\mathcal{P}_{x_0}$. *The transition probabilities* of the random walk which correspond to the probabilities of transitioning from x to y in one step will be denoted by

$$p_1(x,y) := \mathcal{P}_{x_0}(X_{n+1} = y | X_n = x).$$

A random walk is said to be a *simple* if at time i, a particle moves from its current position at vertex x to any of its neighbors y with equal probability. The transition probabilities are then defined by

$$p(x,y) = \begin{cases} \dfrac{1}{deg(x)} & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases}$$

For any vertex $x \in V$, we have that

$$\sum_{x,y} p(x,y) = 1.$$

The transition probabilities give the *transition operator*

$$Pf(x) = \sum_{y \in X} p(x,y) f(y),$$

where f is a function on the graph G. The *Laplacian operator* is then defined by

$$\Delta = P - I,$$

where I is the identity operator.

We say that a function f on a G is *harmonic* if $\Delta f = 0$ or equivalently, with the above definition, if $Pf = f$ for all $x \in G$.

Let $p_n(x,y) := \mathcal{P}_x(X_n = y)$, the probability to transition from x to y in exactly n steps where $n \geq 1$. We define the Green function on $V \times V$ by

$$G(x,y) := \sum_{n=0}^{\infty} p_n(x,y) \quad \forall\, x, y \in G.$$

Starting at x, $G(x,y)$ is the expected number of visits to the vertex y during the lifetime of the random walk. Note that we can write $G(x,x)$ in terms of p_x, the probability that the random walk starting at x returns to x, at least once:

$$\mathcal{P}_x(\text{returning exactly n times to x}) = p_x^n\,(1-p_x), \quad \forall\, n \in \mathbb{N}.$$

When $p_x = 1$, we have $G(x,x) = \infty$, and x is said to be recurrent. If $p_x < 1$, x is said to be transient and we have

$$G(x,x) = \sum_{n=0}^{\infty}(n+1)\,p_x^n(1-p_x) = \frac{1}{1-p_x}.$$

The points of a graph will either all be recurrent or they will all be transient. We can define de notions of recurrence and transience for a random walk as well.

Definition 3.11. A random walk (X_n) on a graph G is said to be *transient* if

$$\mathcal{P}_x(X_\infty = \lim_{n\to\infty} X_n \in \partial X) = 1 \quad \forall x \in G.$$

This means that eventually, (X_n) will permanently leave any finite subset of G. When this is not the case, i.e. when

$$\mathcal{P}_x(X_n = y \quad \text{for infinitely many} \quad n \geq 0) = 1 \quad \forall x, y\ \in G,$$

the random walk is said to be *recurrent*.

We are interested in the *simple forward random walk* on the infinite K-ary tree X which is a particular case of the simple random walk for which the next position of a particle is chosen uniformly among the children of the current vertex. In this case, the transition probabilities are given by

$$p(x,y) = \frac{1}{K} \quad \text{and} \quad p_n(x,y) = K^{-n}$$

when y is a descendant of x and 0 otherwise. By letting $n \to \infty$, we get the uniform measure on the boundary of the tree ∂X.

The forward random walk will always be transient. We would like to identify the uniform measure on the boundary of the tree X with the harmonic measure on the boundary. In order to do that, we will need the following definition:

Definition 3.12. A function f on the set of vertices of X is called a *flow* if $f \geq 0$ and for all $x \in X$,

$$f(x) = \sum_{x \leq y} f(y). \tag{3.3}$$

These flows can be identified with positive Borel measures ν_x on the boundary ∂X via

$$f(x) = \nu_x(\zeta \in \partial X : \ x \text{ is an ancestor of } \zeta).(3.4)$$

A result in [LP], [LPP] states that we can identify equally splitting flows on X which in our case corresponds to the uniform measure on the vertices with the harmonic measure ν_0 on the boundary, where ν_x is the hitting measure on the boundary from the converging random trajectories starting at x. For more information on this topic, we refer to [Woe], [Woe1], [LP], [LPP].

The Laplacian and corresponding harmonic functions are basic and important objects associated to a graph and by extension to the space of the tree. They are closely connected with an associated random walk, and the Markov chain defined by the transition probabilities as we discussed above. This connection is given by boundary theory: In fact, suppose that X is compactified with a boundary ∂X and that almost every trajectory of the random walk converges to some point in ∂X. Then the Poisson formula gives us harmonic functions h on X from any boundary value. It also gives a connection with the harmonic measure on the boundary.

Theorem 3.13 (Poisson formula)**.** *Suppose that (X_n) defines a transient simple forward random walk on the tree X with transition matrix P as defined above. If $h : X \to \mathbb{R}$ is of the form*

$$h(x) = \int_{\partial X} f(\zeta) d\nu_x(\zeta)$$

where ν_x is the harmonic measure, then h is harmonic with respect to P.

This is consistent with the result stated above where ν_x represent the flow functions.

The converse of the above is the Fatou Lemma, see [Mou]. Note that studying harmonic functions on graphs are also relevant to electrical network theory. In fact, if we let each edge correspond to 1 Ohm resistance, then by Kirchoff's laws, we find that the passive currents are exactly the differentials of harmonic functions f which belong to the space of Dirichlet finite functions. We refer to [LP] for a nice exposition on this topic.

We now discuss the connection between the visual boundary defined previously which can also be called the geometric boundary and the Martin boundary which is defined using normalized Green kernels:

$$K_y := \frac{G(., y)}{G(o, y)} \quad \text{for } y \in V.$$

These kernels are harmonic everywhere but at y and we want to send this singularity to infinity. In order to do this, we consider the sequence $(y_n), n \in \mathbb{N}$ such that the sequence of functions $(K_{y_n}), n \in \mathbb{N}$ converges pointwise to a harmonic function on V. The Martin boundary is defined as the quotient space of these sequences by the equivalence relation $(y_n) \sim (y'_n)$ when $\lim_{n\to\infty} K_{y_n} - K_{y'_n} = 0$.

When comparing the geometric boundary to the Martin boundary, one is able to see that the geodesic rays define a point of the Martin boundary. We identify the set of vertices with the set of normalized Green kernels and the Martin boundary with the set of limit harmonic functions and equip these sets with the topology of pointwise convergence. This gives us a new compactification of V which is called the Martin compactification. With this topology, the sequences $(K_{y_n}), n \in \mathbb{N}$ converge to the point of the Martin boundary which they represent.

A theorem by Cartier (see [Mou]) gives the following result:

Theorem 3.14. *For a transient tree, the Martin compactifification coincides with the Geometric compactification. Furthermore, all the points belonging to the Martin boundary are extremal.*

As a corollary, we have the Poisson formula stated above.

Chapter 4

Besov space and Trace / Extension on trees

The goal of this chapter is to show that if a function u on X - a regular K-ary tree - is well behaved, then it has a trace on ∂X and that a function on ∂X has an extension on X, by using Besov spaces. Natural functions on the tree X that have a trace on ∂X are the Lipschitz functions. They are analogue to continuous functions and have a trace on the boundary because of the property of Cauchy sequences. In fact, on the tree X, sequences assigned to vertices will be Cauchy so Lipschitz functions will take sequences of vertices to Cauchy sequences of values meaning the limit along the rays going to the boundary exists. We will start by reviewing some useful definitions for this chapter. Most of the material for this chapter follows from [BBGS]. Recall the definitions in section 1.3:

Theorem 4.1 (Trace/Extension theorem on $\mathbb{R}^n$)**.** *Let $1 < p < \infty$ and let f be a measurable function on $\mathbb{R}^n$. The following conditions are equivalent :*

1. *There is a function $u \in W^{1,p}(\mathbb{R}^{n+1})$ such that f is the trace of u.*

2. *$f \in B^{1-(1/p)}_{p,p}(\mathbb{R}^n)$.*

3. *$\int_{\mathbb{R}^n}[|h|^{-(1-(1/p))}\|f(x+h)-f(x)\|_p]^p \frac{dh}{|h|^n} < \infty$.*

4.1 Besov spaces

Definition 4.2. Let $f : \partial X \to \mathbb{R}$. Let ν denote the normalized Q-dimensional Hausdorff measure on ∂X. For $t > 0$ and $p \geq 1$, we set

$$E_p(f,t) := \Big(\int_{\partial X} \fint_{B(\zeta,t)} |f(\zeta) - f(\xi)|^p d\nu(\xi) d\nu(\zeta) \Big)^{1/p}$$

and for $\theta > 0$ and $q \geq 1$,

$$\|f\|_{B^\theta_{p,q}(\partial X)} := \left(\int_0^\infty \left\{ \frac{E_p(f,t)}{t^\theta} \right\}^q \frac{dt}{t} \right)^{1/q}.$$

The Besov space $B^\theta_{p,q}(\partial X)$ consists of all $f \in L^p(\partial X)$ for which this semi-norm is finite.

Note the analogy of $E_p(f,t)$ in the above definition to the modulus of continuity in the case of $\mathbb{R}^n$. In this section, we only deal with Besov spaces for which p=q. The expression

$$\|f\|_{\tilde{B}^\theta_{p,p}(\partial X)} := \|f\|_{L^p(\partial X)} + \|f\|_{B^\theta_{p,p}(\partial X)}$$

defines a norm on $B^\theta_{p,p}(\partial X)$.

4.2 Trace Theorem on trees and proof

In this section, we assume that X is a regular K-ary tree. Recall the definitions of the metric

$$d_X := \int_{[x,y]} e^{-\epsilon|z|} d|z|$$

and measure on X:

$$d\mu(x) = e^{-\beta|x|} d|x|$$

We have seen that the Newtonian space $N^{1,p}(X)$ is defined as the collection of functions for which the following norm is finite:

$$\|u\|_{N^{1,p}(X)} := \left(\int_X u^p d\mu \right)^{1/p} + \inf_g \left(\int_X g^p d\mu \right)^{1/p}$$

where $g = g_f$ is the minimal p-weak upper gradient of $f \in N^{1,p}(X)$.

Theorem 4.3 (Trace Theorem [BBGS]). *Assume $\beta > \log K$ and $p \geq 1$. Then for every θ satisfying $0 < \theta \leq 1 - \frac{\beta - logK}{p\varepsilon}$, there is a bounded linear trace operator,*

$$\mathrm{Tr} : N^{1,p}(X) \to B^{\theta}_{p,p}(\partial X)$$

such that for $f \in N^{1,p}(X)$,

$$\|\mathrm{Tr} f\|_{L^p(\partial X)} \leq |f(0)| + C\|g_f\|_{L^p(X)}$$

and

$$\|\mathrm{Tr} f\|_{B^{\theta}_{p,p}(\partial X)} \lesssim \|g_f\|_{L^p(X)}$$

In particular,

$$\|\mathrm{Trf}\|_{\tilde{\mathrm{B}}^{\theta}_{\mathrm{p,p}}(\partial \mathrm{X})} \lesssim \|\mathrm{f}\|_{\mathrm{N}^{1,\mathrm{p}}(\mathrm{X})}$$

Furthermore, for Lipschitz functions $f : X \to \mathbb{R}$, we have that $\mathrm{Tr} f = f|_{\partial X}$.

Proof. Let $f \in N^{1,p}(X)$, We begin by showing that the trace

$$\mathrm{Tr} f := \tilde{f}$$

is defined by the limit

$$\tilde{f}(\zeta) = \lim_{[0,\zeta) \ni x \to \zeta} f(x) \tag{4.1}$$

taken along the geodesic ray $[0, \zeta]$ and that this limit exists for ν-a.e. $\zeta \in \partial X$.

We choose an arbitrary $\zeta \in \partial X$, denote by $x_j = x_j(\zeta)$ its ancestor with $|x_j| = j$, and let $f_n(\zeta) = f(x_n(\zeta))$ and show that f_n is Cauchy in $L^p(\partial X)$ and therefore has a limit $\tilde{f} \in L^p(\partial X)$ and a subsequence which converges to $\tilde{f}$ for ν-a.e. $\zeta \in \partial X$.

Let $r_j = 2/\epsilon e^{-\epsilon j}$ as in the previous chapter and recall that $ds = e^{-\epsilon|x|}dx = e^{-(\beta-\epsilon)|x|}d\mu(x)$ which gives us the following approximation on the edge $[x_j, x_{j-1}]$,

$$ds \simeq e^{(\beta - \varepsilon)j} d\mu \simeq r_j^{1-\frac{\beta}{\varepsilon}} d\mu.$$

We fix $n \in \mathbb{N}$ and let $m \geq n$ be arbitrary. We first get an estimate for $|f(x_m) - f(x_n)|$. In the second inequality, we use the definition of the upper gradient and then the above approximation for ds. We get the following:

$$\begin{aligned}|f(x_m)-f(x_n)| &\leq \sum_{j=n}^{m-1}|f(x_{j+1})-f(x_j)| \\ &\leq \sum_{j=n}^{\infty}\int_{[x_j,x_{j+1}]} g_f\, ds \\ &\simeq \sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}}\int_{[x_j,x_{j+1}]} g_f\, d\mu.\end{aligned}$$

We now choose $0<\kappa<\theta p$ and insert $r_j^{\frac{\kappa}{p}} r_j^{-\frac{\kappa}{p}}$ into the above sum. We get

$$\sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}}\int_{[x_j,x_{j+1}]} g_f\, d\mu = \sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}} r_j^{\frac{\kappa}{p}} r_j^{-\frac{\kappa}{p}}\int_{[x_j,x_{j+1}]} g_f\, d\mu$$

For $p>1$, we now apply Hölder's inequality to the above integral i.e. for q s.t. $\frac{1}{p}+\frac{1}{q}=1$

$$\int_{[x_j,x_{j+1}]} g_f\, d\mu \leq \left(\int_{[x_j,x_{j+1}]} g_f^p\, d\mu\right)^{\frac{1}{p}} (\mu([x_j,x_{j+1}])^{\frac{1}{q}}$$

We have that $\mu([x_j,x_{j+1}])=\int_j^{j+1} e^{-\beta|x|}d|x| \approx e^{-\beta j}\approx r^{\frac{\beta}{\epsilon}}$ and so we can use the approximation $\mu([x_j,x_{j+1}])^{\frac{1}{q}} \simeq r_j^{\frac{\beta}{\varepsilon q}} = r_j^{\frac{\beta}{\varepsilon}(1-\frac{1}{p})}$ so that $r_j^{1-\frac{\beta}{\varepsilon}} r_j^{\frac{\beta}{\varepsilon}(1-\frac{1}{p})} = r_j^{1-\frac{\beta}{\varepsilon p}}$.

We then get

$$\sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}} r_j^{\frac{\beta}{\varepsilon q}} r_j^{\frac{\kappa}{p}} r_j^{-\frac{\kappa}{p}} (\int_{[x_j,x_{j+1}]} g_f^p\, d\mu)^{\frac{1}{p}} = \sum_{j=n}^{\infty} r_j^{\frac{\kappa}{p}} r_j^{1-\frac{\beta}{\varepsilon p}-\frac{\kappa}{p}} (\int_{[x_j,x_{j+1}]} g_f^p\, d\mu)^{\frac{1}{p}}$$

We now apply Hölder's inequality to the above sum as follows:

$$\begin{aligned}\sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}} r_j^{\frac{\beta}{\varepsilon q}} r_j^{\frac{\kappa}{p}} r_j^{-\frac{\kappa}{p}} &\leq \left(\sum_{j=n}^{\infty}(r_j^{\frac{\kappa}{p}})^q\right)^{\frac{1}{q}}\left(\sum_{j=n}^{\infty}(r_j^{1-\frac{\beta}{\varepsilon p}-\frac{\kappa}{p}})^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{j=n}^{\infty} r_j^{\frac{\kappa}{p-1}}\right)^{\frac{1}{q}}\left(\sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa}\right)^{\frac{1}{p}}\end{aligned}$$

Note that in the above inequality, we used the equality $q=\frac{p}{p-1}$ since $\frac{1}{p}+\frac{1}{q}=1$.

Inserting into the above, we finally get that

$$\sum_{j=n}^{\infty} r_j^{1-\frac{\beta}{\varepsilon}}\int_{[x_j,x_{j+1}]} g_f\, d\mu \lesssim r_n^{\frac{\kappa}{p}}\left(\sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa}\int_{[x_j,x_{j+1}]} g_f^p\, d\mu\right)^{\frac{1}{p}}$$

where we used the properties of the geometric sequence to obtain $r_j = r_n e^{(n-j)\varepsilon}$. From here, it follows that

$$|f(x_m) - f(x_n)|^p \lesssim r_n^{\kappa} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \int_{[x_j,x_{j+1}]} g_f^p \, d\mu \tag{4.2}$$

We now integrate the above over all $\zeta \in \partial X$ and use Fubini's theorem to get

$$\begin{aligned}
\int_{\partial X} |f(x_m(\zeta)) - f(x_n(\zeta))|^p \, d\nu(\zeta) &\lesssim r_n^{\kappa} \int_{\partial T} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \int_{[x_j,x_{j+1}]} g_f^p \, d\mu \, d\nu(\zeta) \\
&= r_n^{\kappa} \int_X g_f(x)^p \int_{\partial X} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \mathbb{1}_{[x_j(\zeta),x_{j+1}(\zeta)]} \, d\nu(\zeta) \, d\mu(x) \\
&\lesssim r_n^{\kappa} \int_X g_f(x)^p \int_{\partial X} r_{j(x)}^{p-\frac{\beta}{\varepsilon}-\kappa} \chi_{\{x<\zeta\}} \, d\nu(\zeta) \, d\mu(x) \\
&\lesssim r_n^{\kappa} \int_X g_f(x)^p \, r_{j(x)}^{p-\frac{\beta}{\varepsilon}-\kappa} \nu(E(x)) \, d\mu(x) \\
&\lesssim r_n^{\kappa} \int_X g_f(x)^p \, r_{j(x)}^{p-\frac{\beta}{\varepsilon}-\kappa+Q} d\mu(x) \\
&\lesssim r_n^{\kappa} \int_X g_f(x)^p \, d\mu.
\end{aligned}$$

We used the notation $E(x) = \{\zeta \in \partial X : \zeta > x\}$ and $j(x)$ is the largest integer such that $j(x) \leqslant |x|$. For every x, the only term that appears in the sum is the one with $j = j(x)$ and $\mathbb{1}_{[x_j(\zeta),x_{j+1}(\zeta)]}(x)$ is non zero only if $j \le |x| \le j+1$ and $x < \zeta$. We also used Lemma 3.8 which gives us that $\nu(E(x)) \lesssim r_{j(x)}^{Q}$ and $p - \frac{\beta}{\varepsilon} - \kappa + Q > 0$ by the choice of $\kappa < \theta p \le p - \frac{\beta}{\epsilon} + \frac{\log K}{\epsilon}$. Since $r_n \lesssim e^{-\epsilon n}$, the right hand side of the last inequality goes to zero as $n \to \infty$.

Hence, we showed that the sequence of functions $f_n(\zeta) = f(x_n(\zeta))$ is Cauchy in the L^p norm on ∂X. By the completeness of $L^p(\partial X)$, it converges to a function $\tilde{f} \in L^p(\partial X)$. As a consequence, there is a subsequence which converges to $\tilde{f}$ almost everywhere on ∂X.

By letting $n = 0$, we also get that

$$\int_{\partial X} |\tilde{f}(\zeta) - f(0)|^p \, d\nu(\zeta) = \lim_{m\to\infty} \int_{\partial X} |f_m(\zeta) - f_0(\zeta)|^p \, d\nu(\zeta) \lesssim r_0^{\kappa} \int g_f^p \, d\mu \lesssim \int g_f^p \, d\mu$$

and thus

$$\|\tilde{f}(\zeta)\|_{L^p(\partial X)} \le |f(0)| + C\|g_f\|_{L^p(X)}.$$

We now estimate $\|\tilde{f}(\zeta)\|_{B^{\theta}_{p,p}(\partial X)}$. Fixing n and taking m_k such that $f(m_k) \to \tilde{f}$ a.e., we get for $\nu - a.e.\zeta \in \partial X$,

$$|\tilde{f}(\zeta) - f(x_n(\zeta))|^p \lesssim r_n^{\kappa} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \int_{[x_j(\zeta),x_{j+1}(\zeta)]} g_f^p \, d\mu.$$

By taking $\xi \in \partial X$ such that $d_X(\zeta,\xi) = r_n$ and replacing $x_j = x_j(\zeta)$ by the ancestor y_j of ξ, where $x_n = y_n$ is the common ancestor at level n, we get a similar estimate for ξ. Combining the estimates for ζ and for ξ, we obtain

$$|\tilde{f}(\xi - \tilde{f}(\zeta)|^p \lesssim r_n^\kappa \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \left(\int_{[x_j,x_{j+1}]} g_f^p \, d\mu + \int_{[y_j,y_{j+1}]} g_f^p \, d\mu \right)$$

where $n = n(\zeta,\xi) \approx -\log(\varepsilon d_X(\zeta,\xi)/2)$ is the level of the largest common ancestor of ζ and ξ. We will now insert the above inequality in the following approximation given by Lemma 5.4 in [BBGS]:

$$\|f\|^p_{B^\theta_{p,p}(\partial X)} \simeq \int_{\partial X} \int_{\partial X} \frac{|f(\zeta) - f(\xi)|^p}{d_X(\zeta,\xi)^{\theta p}\, \nu(B(\zeta, d_X(\zeta,\xi)))} \, d\nu(\xi)\, d\nu(\zeta).$$

This gives us

$$\|\tilde{f}\|^p_{B^\theta_{p,p}}(\partial X) \simeq \int_{\partial X} \int_{\partial X} \frac{r_n^\kappa}{d_X(\zeta,\xi)^{\theta p + Q}} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \left(\int_{[x_j,x_{j+1}]} g_f^p \, d\mu + \int_{[y_j,y_{j+1}]} g_f^p \, d\mu \right) d\nu(\xi) d\nu(\zeta).$$

We again use the fact that $n = n(\zeta,\xi)$ which depends on ζ and ξ . The roles of ζ and ξ are symmetric in the above formula, therefore it suffices to estimate the expression with the integral over $[x_j, x_{j+1}]$. We write $\partial X = \bigcup_{n=0}^\infty A_n$ where $A_n = \{\xi \in \partial X : d_X(\zeta,\xi) = r_n\}$ and use Lemma 3.8 which gives $\nu(A_n) \lesssim r_n^Q$. We recall that $\mathbb{1}_{[x_j(\zeta),x_{j+1}(\zeta)](x)}$ is non zero only if $n \leqslant j \leq |x| \leq j+1$ and $x < \zeta$ and that the edge $[x_j(\zeta), x_{j+1}(\zeta)]$ belongs to the geodesic ray connecting the root 0 to ζ. We obtain,

$$\begin{aligned}
\|\tilde{f}\|^p_{B^\theta_{p,p}}(\partial X) &\simeq \int_{\partial X} \sum_{n=0}^{\infty} r_n^{-\theta p - Q + \kappa} \int_{A_n} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \int_{[x_j(\zeta),x_{j+1}(\zeta)]} g_f^p \, d\mu \, d\nu(\xi)\, d\nu(\zeta) \\
&\simeq \int_{\partial X} \sum_{n=0}^{\infty} r_n^{-\theta p - Q + \kappa} \sum_{j=n}^{\infty} r_j^{p-\frac{\beta}{\varepsilon}-\kappa} \int_X g_f(x)^p \mathbb{1}_{[x_j(\zeta),x_{j+1}(\zeta)]} \, d\mu \, d\nu(\xi)\, d\nu(\zeta) \\
&\simeq \int_{\partial X} \sum_{n=0}^{\infty} r_n^{-\theta p + \kappa} \int_X g_f(x)^p {r_{j(x)}}^{p-\frac{\beta}{\varepsilon}-\kappa} \mathbb{1}_{\{z\in X: \zeta > z\}}(x) \mathbb{1}_{\{z \in X: |z| \geq n\}}(x) \, d\mu \, d\nu(\xi)\, d\nu(\zeta) \\
&\simeq \int_{\partial X} \int_X g_f(x)^p {r_{j(x)}}^{p-\frac{\beta}{\varepsilon}-\kappa} \mathbb{1}_{\{z\in X: \zeta > z\}}(x) \sum_{n=0}^{j(x)} r_n^{-\theta p + \kappa} \, d\mu(x)\, d\nu(\zeta)
\end{aligned}$$

since $|x| \geq n$ which gives $n \leq j(x)$. Finally, by Fubini's theorem, we get

$$\|\tilde{f}\|^p_{B^\theta_{p,p}(\partial X)} \simeq \int_X g_f(x)^p \, r_{j(x)}^{p-\frac{\beta}{\varepsilon}-\kappa} \nu(E(x)) \sum_{n=0}^{j(x)} r_n^{-\theta p + \kappa} d\mu(x)$$

where we used the definition of the set $E(x)$ given above. Choosing $\kappa < \theta p$ and using the properties of geometric series, we have that $\sum_{n=0}^{j(x)} r_n^{-\theta p+\kappa} \approx r_{j(x)}^{-\theta p+\kappa}$. Now again, since by Lemma 3.8, we have that $\nu(E(x)) \lesssim r_{j(x)}^{Q}$ with $p - \frac{\beta}{\varepsilon} - \theta p + Q \geq 0$, we get the desired result:

$$
\begin{aligned}
\|\tilde{f}\|^p_{B^\theta_{p,p}}(\partial X) &\simeq \int_X g_f^p(x) r_{j(x)}^{p-\frac{\beta}{\varepsilon}-\theta p+Q} d\mu(x) \\
&\lesssim \int_X g_f^p d\mu.
\end{aligned}
$$

□

4.3 Extension Theorem on trees and proof

Theorem 4.4 (Extension Theorem [BBGS]). *Let X be a regular K-ary tree with the metric d_X and the measure μ. Let $p \geq 1$. Suppose that*

$$
\theta \geq 1 - \frac{\beta - \log K}{p\epsilon} \quad \textit{and} \quad \theta > 0. \tag{4.3}
$$

Then there is a bounded linear extension operator

$$
\mathrm{Ext} : B^\theta_{p,p}(\partial X) \to N^{1,p}(X)
$$

such that for $u \in B^\theta_{p,p}$, we have $\mathrm{Tr}(\mathrm{Ext}(u)) = u \quad \nu$ a.e. where Tr is the trace operator constructed in the Trace theorem. Furthermore, for ν a.e. $\zeta \in \partial X$ and a geodesic γ in X terminating at ζ, we have $lim_{t\to\infty}\mathrm{Ext}(u)(\gamma(t)) = u(\zeta)$. Moreover, with $\tilde{u} = \mathrm{Ext}(u)$, we have

$$
\|g_{\tilde{u}}\|_{L^p(X)} \lesssim \|u\| B^\theta_{p,p(\partial X)}
$$

$$
\|\tilde{u}\|_{N^{1,p}(X)} \lesssim \|u\|_{L^p(\partial X)} + \|u\|_{B^\theta_{p,p(\partial X)} = \|u\|_{\tilde{B}^\theta_{p,p}(\partial X)}}
$$

Note also that if , $u \in B^\theta_{p,p}(\partial X)$ is continuous, then we have, $\mathrm{Tr}(\mathrm{Ext}(u)) = u$ everywhere.

Proof. Let $u \in B^\theta_{p,p}(\partial X)$. For $x \in X$, with $|x| = n \in \mathbb{N}$, let

$$
\tilde{u}(x) = \fint_{B(\zeta, r_n)} u \, d\nu \tag{4.4}
$$

where $r_n = \frac{2e^{(-n)\epsilon}}{\epsilon}$ and $\zeta \in \partial X$ is any descendant of x. Recall that the ball $B(\zeta, r_n)$ consists of all points in ∂X that have x as an ancestor. In other words, the geodesics connecting the root 0 to these points pass through x. The function $\tilde{u}$ was only defined on vertices until now. We now define $\tilde{u}$ on any point along the edges as well.

Given $x \in X$, and letting $y \in X$ be a child of x, we extend $\tilde{u}$ to the edge $[x, y]$ by the following steps:

1. By the ultrametric property of ∂X , every point in the ball $B(\zeta, r_n)$ is a center of this ball. We can then choose $\zeta \in \partial X$ such that ζ is a descendant of x and of y as well.

2. For each $t \in [x, y]$, we define

$$g_{\tilde{u}}(t) = \frac{|\tilde{u}(x) - \tilde{u}(y)|}{d_X(x, y)} = \frac{\epsilon |u_n(\zeta) - u_{n+1}(\zeta)|}{(1 - e^{-\epsilon})e^{-\epsilon n}} \tag{4.5}$$

   and

$$\tilde{u}(t) = \tilde{u}(x) + g_{\tilde{u}}(t) d_X(x, t)$$

   i.e $g_{\tilde{u}}$ is constant and $\tilde{u}$ is linear with respect to the metric d_X on the edge $[x, y]$. Note that by definition of the upper gradient, it follows that $g_{\tilde{u}}$ is the minimal upper gradient of $\tilde{u}$ on the edge $[x, y]$.

With this definition, $\tilde{u}$ is continuous and can be approximated by piecewise constant and continuous functions $u_n(\zeta)$. Now, by (4.1) and (4.4), we have:

$$\mathrm{Tr}(\tilde{u}(\zeta)) = \lim_{[0,\zeta)\ni x \to \zeta} \tilde{u}(x) = \lim_{[0,\zeta)\ni x \to \zeta} \fint_{B(\zeta, r_n)} u d\nu = \lim_{r_n \to 0} \fint_{B(\zeta, r_n)} u d\nu = u(\zeta)$$

whenever $\zeta \in \partial X$ is a Lebesgue point.

Using (4.5), we now obtain that

$$\begin{aligned} \int_{[x,y]} g_{\tilde{u}}^p \, d\mu &\simeq \int_n^{n+1} \left(\frac{|u_n(\zeta) - u_{n+1}(\zeta)|}{e^{-\epsilon n}} \right)^p e^{-\beta \tau} d\tau \\ &\simeq e^{(\epsilon p - \beta)n} |u_n(\zeta) - u_{n+1}(\zeta)|^p. \end{aligned}$$

Note that ζ can be replaced by any choice of $\xi \in B(\zeta, r_n)$, by the ultrametric property of ∂X.

By integrating the above over this smaller ball, we get

$$\nu(B(\zeta, r_{n+1})) \int_{[x,y]} g_{\tilde{u}}^p \, d\mu \simeq e^{(\epsilon p-\beta)n} \int_{B(\zeta, r_{n+1})} |u_n(\zeta) - u_{n+1}(\zeta)|^p \, d\nu(\zeta).$$

The next step is to sum over all edges in X connecting vertices at level n to vertices at level $n+1$, the above then becomes comparable to

$$\sum_{|x|=n, |y|=n+1} \int_{[x,y]} g_{\tilde{u}}^p \, d\mu = e^{(\epsilon p-\beta)n} \int_{\partial X} \frac{|u_n(\zeta) - u_{n+1}(\zeta)|^p}{\nu(B(\zeta, r_{n+1}))} \, d\nu(\zeta).$$

We now sum over all $n \in \mathbb{N}$ and write $|u_n(\zeta) - u_{n+1}(\zeta)| \leq |v_n(\zeta)| + |v_{n+1}(\zeta)|$ where $v_n = u_n - u$, and we obtain that

$$\begin{aligned}
\int_X g_{\tilde{u}}^p \, d\mu &\lesssim \sum_{n=0}^{\infty} e^{(\epsilon p-\beta)n} \int_{\partial X} \frac{(|v_n(\zeta)| + |v_{n+1}(\zeta)|)^p}{\nu(B(\zeta, r_{n+1}))} \, d\nu(\zeta) \\
&\lesssim \sum_{n=0}^{\infty} \frac{e^{(\epsilon p-\beta)n}}{r_n^Q} \int_{\partial X} \fint_{B(\zeta, r_n)} |u(\chi) - u(\zeta)|^p \, d\nu(\chi) \, d\nu(\zeta) \\
&\simeq \sum_{n=0}^{\infty} \frac{e^{(\epsilon p-\beta)n} r_n^{\theta p}}{r_n^Q} \left(\frac{E_p(u, r_n)}{r_n^\theta} \right)^p.
\end{aligned}$$

Since $r_n \simeq e^{-n\epsilon}$, we can use Lemma 5.4 in [BBGS] which gives us

$$\|u\|_{B_{p,p(\partial X)}^\theta}^p \simeq \sum_{n=0}^{\infty} \frac{e^{(\epsilon p-\beta)n} r_n^{\theta p}}{r_n^Q} \left(\frac{E_p(u, r_n)}{r_n^\theta} \right)^p \tag{4.6}$$

when $\frac{e^{(\epsilon p-\beta)n} r_n^{\theta p}}{r_n^Q} \simeq e^{(\epsilon p - \beta - \epsilon(\theta p - Q))n} \leq C \quad \forall n \in \mathbb{N}$. This is satisfied when $\epsilon p - \beta - \epsilon(\theta - Q) \leq 0$ which is the same as (4.3). From here it follows that

$$\|g_{\tilde{u}}\|_{L^p(X)} \lesssim \|u\|_{B_{p,p(\partial X)}^\theta}.$$

Now using the above and the p-Poincaré inequality in Corollary 2.8 which holds for for $\tilde{u} \in L^1(X)$ and incorporating it in the definition of $\|\tilde{u}\|_{N^{1,p}(X)} = \|\tilde{u}\|_{L^p(\partial X)} + \inf_{g_{\tilde{u}}} \|g_{\tilde{u}}\|_{L^p}$ we obtain the desired result:

$$\|\tilde{u}\|_{N^{1,p}(X)} \lesssim \|u\|_{L^p(\partial X)} + \|u\|_{B_{p,p(\partial X)}^\theta}.$$

$\square$

Bibliography

[An] Anandam, V., Harmonic functions and Potentials on Finite or Infinite Networks, Lecture Notes of the Unione Matematica Italiana, 12, Springer, Berlin, 2011.

[AF] Adams, R.A., Fournier, J.J.F, Sobolev spaces, second edition, Academic Press, Oxford, 2003.

[Ba] Bandt, C., Geometry of Self-similar Sets, in "Fractals, Wavelets, and their Applications", pp. 21–36, Springer Proceedings in Mathematics & Statistics, Vol **92**, Springer, Cham, 2014.

[BBGS] Björn, A., Björn, J., Gill, J. T., Shanmugalingam, N., *Geometric analysis on Cantor sets and trees*, J. Reine Angew. Math. **725** (2017), 63-114.

[BH] Bridson, M., Haefliger, A., Metric spaces of Non-positive curvature, Grundlehren, der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. **319** Springer, Berlin (1999).

[BHK] Bonk, M., Heinonen, J., Koskela, P., Uniformizing Gromov hyperbolic spaces, Asterisque, 270, Soc. Math. de France, Paris, 2001.

[Bre] Brezis, H., Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.

[DMcS] Dafni, G., McCann, R.J., Stancu, A., Analysis and Geometry of Metric Measure Spaces, Lecture Notes of the 50th Séminaire de Mathématiques Supérieures (SMS), Montréal, 2011, CRM Proceedings & Lecture Notes, Vol. **56**, American Math. Soc. 220 pp. 2013

[Edg] Edgar, G., Measure, Topology, and Fractal Geometry, Second Edition, Springer, 2008.

[Ev] Evans, L. C., Partial Differential Equations, Graduate Series in Mathematics, Vol. 19, American Mathematical Society, 2010.

[Ev1] Evans, L. C., Probability and Real Trees, Lecture notes in mathematics, Springer, 2008.

[Fal] Falconer, K., Fractal Geometry, Third Edition, Wiley, 2014.

[Ga] Garnett, J., Positive length but zero analytic capacity, Proc. Amer. Math. Soc. 24 (1970), 696–699; errata, ibid. 26 (1970), 701.

[Ha] Hajłasz, P., Sobolev spaces on an arbitrary metric space, Potential Anal. **5**, pp. 403–415, 1996.

[Hal] Hajłasz, P., Sobolev spaces on metric-measure spaces, in "Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)", Contemp. Math. 338, Amer. Math. Soc. Providence, RI 2003, pp. 173–218.

[Hei] Heinonen, J., Lectures on Analysis on Metric Spaces, Universtext. Springer, New York, 2001.

[HK] Hajłasz, P., Koskela, P., Sobolev met Poincaré, Memoirs Amer. Math. Soc. **145** (2000), no. 688, x+101 pp.

[HK1] Hajłasz, P., Koskela, P., Sobolev meets Poincaré, C. R. Acad. Sci. Paris Ser. I Math. **320**, pp. 1211–1215, 1995.

[HKST] Heinonen, J., Koskela, P., Shanmugalingam, N., Tyson, J.T,. Sobolev Spaces on Metric Measure spaces, An Approach Based on Upper Gradients, Cambridge University Press, Cambridge, 2015.

[Ho] Holly, J.E., Pictures of ultrametric spaces, the p-adic numbers, and valued fields, in "Ultrametric spaces, p-adic numbers, valued fields", pp. 721–728, The mathematical association of America, Monthly **108**, October 2001.

[HW] Hairer, E., Wanner, G., L'analyse au fil de l'histoire, Scopos, Springer, Berlin, 2001

[LP] Lyons, R., Peres, Y., Probability on Trees and Networks, Cambridge University Press, 1997-2016.

[LPP] Lyons, R., Pemantle, R., Peres, Y., Ergodic Theory on Galton-Watson Trees: Speed of Random Walk and Dimension of Harmonic Measure, Ergodic Theory Dynamical Systems, **15**, 1995, pp.93–619

[Mir] Mironescu, P., Note on Gagliardo's theorem "$\mathrm{TrW}^{1,1} = \mathrm{L}^1$". Annals of the University of Bucharest. Mathematical series, Bucureti: Editura Universităii din Bucureti, 2015, 6 (LXIV) (1), pp.9–103.

[Mor] Moran, M. A., Metrics on visual boundaries of CAT(0) spaces, Geometriae Dedicta, Volume **183**, Issue 1, pp. 123–142, Springer, Berlin, 2016.

[Mou] Mouton, F., Comportement asymptotique des fonctions harmoniques sur les arbres, Séminaire de Probabilités (Strasbourg), tome 34 (2000), pp. 353-373.

[MR] Mironescu, P., Russ, E., Traces of weighted Sobolev spaces. Old and new. Nonlinear Analysis: Theory, Methods and Applications, Elsevier, 2015, 119, pp.354–381.

[Pa] Papadopoulos, A., Metric spaces, Convexity and Nonpositive Curvature (Second Edition), European Mathematical Society, 6, 320 pp. IRMA Lectures in Mathematics and Theoretical Physics Vol. 6.

[Pie] Pietsch, A., Acta et Commentationes Universitatis Tartuensis de Mathematica, Volume 18, Number 1, June 2014.

[Saw] Sawyer, S. A., Martin Boundaries and Random Walks, Koranyi (Ed.), Harmonic functions on trees and buildings, Contemporary Mathematics **206**, American Mathematical Society, Providence, 1997, pp. 17–44

[Sha] Shanmugalingam, N., Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoam. 16 (2000), 243–279.

[Woe] Woess, W., Random walks on infinite graphs and groups, Cambridge tracts in mathematics **138**, Cambridge university Press, Cambridge, 2000.

[Woe1] Woess, W., Denumerable Markov chains, Generating functions, boundary theory, random walks on trees, EMS textbooks in Mathematics, Zürich, 2009.