Critical L-values of Primitive Forms Twisted by Dirichlet Characters

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Abstract

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Let f be a primitive form of weight $k > 1$ with nebentypus ε and χ be a primitive Dirichlet character. Then, we consider the twist of f by χ and its Dirichlet L-series denoted by $L(f, s, \chi)$. Those central L-values (or even vanishings and nonvanishings of them) are believed to encode important arithmetic invariants of algebraic objects over various fields.

In this thesis, we mainly study vanishings and nonvanishings of the central L-values of primitive form f of weight $k \geq 2$ twisted by χ of a prime order l. More precisely, firstly, assume that $k > 2$, $2 \mid k$ and $l > 2$. Then, as a generalisation of the nonvanishing theorem for $k = 2$ of Fearnley, Kisilevsky and Kuwata, for the case that $L(f, k/2) \neq 0$ we prove that for all but finitely many primes l there exist infinitely many of twists of order l such that $L(f, k/2, \chi) \neq 0$. We also present numerical results on vanishings of twists of $l = 3, 5,$ and 7 for some primitive forms of $k > 2$, and based on the random matrix theory, make a conjecture that for a primitive form of even weight $k > 2$, there exist only a finite number of vanishings of twists of order $l > 2$.

Secondly, assume that $k = 2$ and $l = 3$. Then, inspired by the work of Fiorilli for quadratic twists, we estimate the average analytic rank of twists for the group family under some hypotheses including the generalised Riemann hypothesis, which implies that there exist infinitely many cubic twists such that $L(f, 1, \chi) \neq 0$.

Lastly, consider quadratic twists of an elliptic curve E over $\mathbb{Q}$ of conductor N associated with quadratic characters χ_d of fundamental discriminants d with a prime $|d|$. Then, by controlling 2-Selmer groups of E and its quadratic twist by χ_d using the method of Mazur and Rubin, we show that for some E satisfying some conditions there exist a set of residue classes $|d| \bmod N$ such that $L(E, 1, \chi_d) \neq 0$ under the Birch and Swinnerton-Dyer conjecture.

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Symbols and Notations

$\overline{K}$	a choice of algebraic closure of a field K
$\dot{\cup}$	disjoint union
$\langle \cdot, \cdot \rangle$	the Petersson inner product or the integration pairing of modular symbols and modular forms depanding on the context
$\mathcal{C}_l$	Family of all primitive Dirichlet characters of order an odd prime l
$\mathcal{C}_l(X)$	$\{\chi \in \mathcal{C}_l \mid \chi^l = \chi_0, \chi^n \neq \chi_0 \text{ for } 0 < n < l \text{ and } \mathfrak{f}_\chi \leq X\}$
$\mathcal{C}_{l,f}(X)$	$\{\chi \in \mathcal{C}_l(X) \mid (N, \mathfrak{f}_\chi) = 1 \text{ and } \mathfrak{f}_\chi \leq X\}$ for level N of f
$\mathcal{C}_{l,f}$	$\{\chi \in \mathcal{C}_l \mid (N, \mathfrak{f}_\chi) = 1\} \cup \{\chi_0\}$ for level N of f
Δ	the discriminant of an elliptic curve E
$\delta_\nu(E, F)$	$\dim_{\mathbb{F}_2}(E(\mathbb{Q}_\nu)/\mathrm{Norm}(E(F_\omega))$ for local extension $\mathbb{F}_\omega/Q_\nu$
$\mathfrak{f}_\chi$	the conductor of a Dirichlet character χ
$d(n)$	the divisor function: $d(n) = \sum_{d\mid n} 1$
V^*	the dual space of a vector space V over a field F, i.e. $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$
$\mathrm{End}_F(V)$	the endomorphism ring of V over F
$\mathcal{F}_l(k, X)$	$\{L(f, k/2, \chi) \mid \chi \in \mathcal{C}_{l,f}(X)\}$
$\mathcal{F}_l^0(k, X)$	$\{L(f, k/2, \chi) \in \mathcal{F}_l(k, X) \mid L(f, k/2, \chi) = 0\}$
$(\cdot, \cdot)$	the greatest common divisor
$\mathrm{Gal}(K/F)$	the Galois group of the Galois extension K/F
$\mathcal{G}_f(X)$	$\{\chi \in \mathcal{C}_f \mid \chi = \prod_{0 \leq j \leq t} {\chi_{p_j}}^{\alpha_j} \text{ for } p_j \in \mathcal{P}(X) \text{ and } \alpha_j \in \{0, 1, 2\}\}$
$H^n(G, M)$	n-th Galois cohomology of Galois group G with coefficients in M
$H^1_f(K_\nu, E[n])$	the image of κ_ν of $E(K_\nu)/2E(K_\nu)$ into $H^1(K_\nu, E[n])$
$(\cdot, \cdot)_\nu$	the quadratic Hilbert norm residue symbol for a place ν
κ_ν	the local Kummer map

$\left(\frac{\cdot}{\cdot}\right)$	the Legendre symbol
loc	the localisation map $H^1(\mathbb{Q}, E[2]) \to \bigoplus_{\nu} H^1(\mathbb{Q}_\nu, E[2])$
$\mathrm{loc}_{\mathcal{W}}$	the $\mathcal{W}$ components of loc for given set of places $\mathcal{W}$
$(\mathcal{M}g)(s)$	the Mellin transform of an $\mathbb{R}$-valued integrable function g
$M_k(N, \varepsilon)$	the space of modular forms of weight k and level N with nebentypus ε, also denoted by $M_k(\Gamma_0(N), \varepsilon)$
$\mathbb{M}_k(N)$	the space of modular symbols of weight k and level N, also denoted by $\mathbb{M}_k(\Gamma_0(N))$
$\mu(n)$	the Möbius function
$\nu(n)$	the number of distinct prime factors of n or the valuation of n for a place ν
$\mathcal{O}_K$	the ring of integers of a number field K
$\mathcal{P}$	the set of all rational primes
$\mathcal{P}(X)$	$\{p \in \mathcal{P} \mid p \leq X \text{ and } p \equiv 1 \bmod 3\} \cup \{9\}$
$\phi(n)$	the Euler totient function
q	$\exp(2\pi i z)$
$\mathbb{Q}(\chi)$	the number field adjoining the values of character χ
E_d/E^F	elliptic curve of E over $\mathbb{Q}$ twisted by a quadratic character χ_d of fundamental discriminant d associated to $F = \mathbb{Q}(\sqrt{d})$
$r_{\mathrm{an}}(f)$	the analytic rank of $L(f, s)$: the order of vanishing at the central point
$r_{\mathrm{al}}(E)$	the algebraic rank of E: the $\mathbb{Z}$-rank of the Mordell-Weil group of E
$Ш(E/K)$	the Tate-Shafarevich group of an elliptic curve E defined over a field K
$\mathrm{Sel}_m(E/K)$	the m-Selmer group of an elliptic curve E defined over a field K
$S_k(N, \varepsilon)$	the space of cusp forms of weight k and level N with nebentypus ε, also denoted by $S_k(\Gamma_0(N), \varepsilon)$
$\mathbb{S}_k(N)$	the space of cuspidal modular symbols of weight k and level N, also denoted by $\mathbb{S}_k(\Gamma_0(N))$
$L(\mathrm{Sym}^k f_\chi, s)$	L-function of the symmetric k-th power of f_χ
$L(f_\chi \otimes f_\chi, s)$	L-function of the Rankin-Selberg convolution of f_χ
Θ_f	the rational period mapping of $\mathbb{M}_k(N, \varepsilon)$ with respect to f

T_p, $\mathbb{T}$	the Hecke operator at prime p, the Hecke algebra acting on $M_k(\Gamma_1(N))$ respectively
$\mathcal{H}$ and $\mathcal{H}^*$	the upper half plane of $\mathbb{C}$ and $\mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$, respectively
G_{tor}	the torsion subgroup of a group G
$f = O(g)$	$\limsup_{n\to\infty} \lvert f(n)\rvert / g(n) < \infty$
$f = o(g)$	$\lim_{n\to\infty} f(n)/g(n) = 0$
$f \ll g$	$f \in O(g)$
$f \sim g$	$f - g \in o(g)$
$U(n)$	the topological group of $n \times n$ unitary matrices over $\mathbb{C}$
$\mathbb{Z}[\varepsilon]$	$\mathbb{Z}$-module generated by the values of a character ε

Chapter 1

Introduction

In this chapter, we summarise basics of primitive forms which are a special type of modular forms and their L-functions. Main references of theories of modular forms and L-functions we refer are [Miy89], [Ste07], [Was82] and [Iwa97]. In the last section of this chapter, we discuss some results on vanishings and nonvanishings of the central L-values of primitive forms twisted by primitive Dirichlet characters of prime order.

1.1 Modular forms

Before defining modular forms, we need to study group actions of $GL_2(\mathbb{R})$ on the upper half-plane of $\mathbb{C}$, denoted by $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Definition. For $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ and $z \in \mathcal{H}$,

$$\gamma z := \frac{az + b}{j(\gamma, z)}$$

Note that $j(\gamma, z) = cz + d$, called the automorphy factor, enjoys the cocycle relation, i.e. $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)j(\gamma_2, z)$ for $\gamma_1, \gamma_2 \in GL_2(\mathbb{R})$ and $z \in \mathcal{H}$. Consider the following sequence of subgroups of $GL_2(\mathbb{R})$:

$$SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) \subset GL_2^+(\mathbb{R}) \subset GL_2(\mathbb{R})$$

Note that firstly, $GL_2^+(\mathbb{R}) = SL_2(\mathbb{R}) \times \mathbb{R}$ and $\mathbb{R}$ trivially acts on $\mathcal{H}$ as elements in $GL_2^+(\mathbb{R})$ and secondly, $SL_2(\mathbb{Z})$ can be generated by two elements

$$\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note also that $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$.

Definition. A holomorphic function f on $\mathcal{H}$ is called a weakly-modular of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ if for every $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathcal{H}$,

$$f(z) = j(\gamma, z)^{-k} f(\gamma z).$$

It is convenient to use the slash operator on a holomorphic function f on $\mathcal{H}$ defined as for $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$,

$$(f|\gamma)(z) := \rho(\gamma, z)^k f(\gamma z), \tag{1.1}$$

where $\rho(\gamma, z) := \det(\gamma)^{1/2} j(\gamma, z)^{-1}$. Then, the automorphy condition of a holomorphic functions f of weight k for $SL_2(\mathbb{Z})$ is $(f|\gamma)(z) = \rho(\gamma, z)^k f(\gamma z) = f(z)$ for every $\gamma \in SL_2(\mathbb{Z})$. Since $\tau \in SL_2(\mathbb{Z})$, we have the periodicity $f(z+1) = f(z)$. It implies that f has a Laurent expansion which can be represented as

$$f(z) = \sum_{n=\infty}^{\infty} a_n q^n$$

where $q := \exp(2\pi i z)$. We say a weakly-modular function f is meromorphic at ∞ if there exists an integer m such that

$$f(z) = \sum_{n=m}^{\infty} a_n q^n$$

and holomorphic at ∞ if such $m = 0$.

Definition. A weakly-modular function f of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ is called a modular form if f is holomorphic on $\mathcal{H} \cup \{\infty\}$.

Definition. A modular form f of weight $k \in \mathbb{Z}$ for $SL_2(\mathbb{Z})$ is called a cusp form if it vanishes at ∞.

Thus, a cusp form f has the q-expansion of form $f(z) = \sum_{n \geq 1} a_n q^n$.

Let N be a positive integer. Consider the natural group homomorphism $\psi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$. A congruence subgroup Γ is defined as a subgroup of $SL_2(\mathbb{Z})$ containing $\Gamma(N) := \ker \psi$. More precisely, we have

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid b, c \equiv 0 \bmod N \text{ and } a, d \equiv 1 \bmod N \right\}.$$

For a congruence subgroup Γ, there exists the smallest such N, called the level of Γ. The congruence subgroups we are interested in are

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \bmod N \right\}$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c \equiv 0 \bmod N \text{ and } a, d \equiv 1 \bmod N \right\}.$$

We can extend the definition of the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ to $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ where $\mathbb{P}^1(\mathbb{Q})$ is the rational projective line as following: for $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{P}^1(\mathbb{Q})$,

$$\gamma z = j(\gamma, z)^{-1}(az + b) \text{ if } z \neq \infty \text{ and } \gamma\infty = \frac{a}{c}.$$

For a congruence subgroup Γ, the set of cusps is defined as the set of Γ-orbits of $\mathbb{P}^1(\mathbb{Q})$. For examples, the only cusp for $SL_2(\mathbb{Z})$ is just the class of ∞ and the cusps for $\Gamma_0(p)$ with a prime p are the classes of ∞ and 0. We can naturally extend the definition of a holomorphic weakly-modular function on $\mathcal{H}$ for a congruence subgroup.

Definition. Let Γ be a congruence subgroup. A holomorphic function f on $\mathcal{H}$ is called a weakly-modular of weight $k \in \mathbb{Z}$ for Γ if for every $\gamma \in \Gamma$ and $z \in \mathcal{H}$,

$$f(z) = j(\gamma, z)^{-k} f(\gamma z).$$

Let $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Then, we also extend the definition of a modular form for a congruence subgroup.

Definition. Let Γ be a congruence subgroup. A weakly-modular function f of weight $k \in \mathbb{Z}$ for Γ is called a modular form if f is holomorphic on $\mathcal{H}^*$.

Definition. A modular form f of weight $k \in \mathbb{Z}$ for Γ is called a cusp form if it vanishes at every cusps of Γ.

Let f and g be modular forms (cusp forms) of weight k for a congruence subgroup Γ and $s \in \mathbb{C}$. Then, it is clear that $f + g$ and sf are also modular forms of weight k (cusp forms, respectively); note that 0 is trivially a cusp form. Thus, we can consider the space of modular forms (cusp forms) of weight k for Γ as a vector space over $\mathbb{C}$ and denote it by $M_k(\Gamma)$ ($S_k(\Gamma)$, respectively). It is also clear that for two congruence subgroups $\Gamma \subset \Gamma'$, $M_k(\Gamma') \subset M_k(\Gamma)$ and $S_k(\Gamma') \subset S_k(\Gamma)$. It is well-known that $M_k(\Gamma)$ is a finite dimensional using the complex analysis and the Valence Formula and the dimension formulae have been found for $S_k(\Gamma)$ for $\Gamma = \Gamma_0(N)$ and $\Gamma_1(N)$ and $k > 1$ (see §2.5 in [Miy89]).

It turns out that there exists an inner product due to Petersson (see §2.1 in [Miy89]) for $M_k(\Gamma)$ with a congruence subgroup Γ as a postive defnite Hermitian form

$$\langle \cdot, \cdot \rangle : M_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$$

defined by

$$\langle f, g \rangle := \int_{\Gamma \backslash \mathcal{H}} f(z)\overline{g(z)}\tau^{k-2} d\sigma d\tau$$

where $\overline{g(z)}$ is the complex conjugate of $g(z) \in \mathbb{C}$ and $\Gamma \backslash \mathcal{H}$ is a fundamental domain of Γ. In this definition, we miss the normalizing factor in the integral above. Note that it is well-defined since $f(z)\overline{g(z)}\tau^k$ is bounded in $\mathcal{H}$ and $f(z)\overline{g(z)}\tau^{k-2}d\sigma d\tau$ is invariant under the action of $SL_2(\mathbb{Z})$ (moreover, the measure $\tau^{-2}d\sigma d\tau$ on $\mathcal{H}$ is invariant under the action of $GL_2^+(\mathbb{R})$).

1.2 Dirichlet characters and Gauss sums

Let G be a finite abelian group. Then, we call a group homomorphism $\chi : G \to \mathbb{C}^\times$ a character of G. Furthermore, let $\widehat{G}$ be the set of all characters of G and define the multiplication of two characters χ and χ' as $(\chi\chi')(g) = \chi(g)\chi'(g)$ for all $g \in G$. Then, $\widehat{G}$ is also a finite abelian group with the identity element which is the trivial group homomorphism called the trivial character. For a finite abelian group G, we have the non-canonical isomorphism $\widehat{G} \simeq G$ (see Lemma 3.1 in [Was82] for the proof) and the canonical isomorphism $\widehat{\widehat{G}} \simeq G$ (see Corollary 3.2 in [Was82] for the proof).

Consider a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for a positive integer N. We extend it to a function on $\mathbb{Z}$ called a Dirichlet character χ mod N as

Definition.

$$\chi(n) = \begin{cases} \chi(n \bmod N) & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that a Dirichlet character χ is completely multiplicative on $\mathbb{Z}$. There is the unique Dirichlet character mod 1 which sends all n to 1, called the trivial Dirichlet character and denoted by χ_1. Also, for each positive integer N there exists the Dirichlet character χ mod N such that $\chi(n) = 1$ if $(n, N) = 1$ and $\chi(n) = 0$ otherwise, called the principal character modulo N and denoted by χ_0. Furthermore, for each positive integer N and a Dirichlet character χ mod N, there is a Dirichlet character χ' mod d that induces χ for some positive integer $d \mid N$. More precisely, for every integer n, we have

$$\chi(n) = \chi'(n \bmod d)\chi_0(n).$$

In other words, let π be a natural map $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times$. Then, the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{Z}/N\mathbb{Z})^\times \arrow[r, "\pi"] \arrow[d, "\chi"'] & (\mathbb{Z}/d\mathbb{Z})^\times \arrow[dl, "\chi'"] \\
\mathbb{C}^\times &
\end{tikzcd}$$

Thus, we can take the minimum of those d's for each χ mod N and call it the conductor of χ mod N and denote it by $\mathfrak{f}_\chi$. Furthermore, we call χ a primitive Dirichlet character mod N if $\mathfrak{f}_\chi = N$. We can identify a Dirichlet character $\chi \pmod N$ with a unique primitive Dirichlet character χ' of conductor $\mathfrak{f}_\chi$ by $\chi = \chi'\chi_0$. In particular, χ_1 induces χ_0 mod N for all N.

As a special case of proposition 3.8 (a) in [Was82], we can view a Dirichlet character mod N as a homomorphism : $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to \mathbb{C}^\times$ by the following proposition:

Proposition 1.1. Let N be a positive integer and $\mathbb{Q}(\zeta_N)$ be the cyclotomic field where ζ_N is a primitive N-th root of unity. Then,

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times .$$

Proof. See theorem 2.5 in [Was82]. □

Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Then, proposition 1.1 implies that $|\widehat{G}| = \phi(N)$ where ϕ is the Euler totient function. In summary, we have the following isomorphisms:

$$\widehat{G} \simeq G \simeq (\mathbb{Z}/N\mathbb{Z})^\times \simeq (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times.$$

Definition. For a positive integer N and a Dirichlet character χ mod N, the Gauss sum $\tau(\chi)$ of χ is defined as

$$\tau(\chi) := \sum_{n=0}^{N-1} \chi(n)\exp(2\pi i n/N).$$

We need some properties of the Gauss sum of a Dirichlet character χ mod N. Let $\overline{\chi}$ be the complex conjugate of χ.

Lemma 1.2. For a positive integer N and a primitive Dirichlet character χ mod N,

$$|\tau(\chi)|^2 = \chi(-1)\tau(\chi)\tau(\overline{\chi}) = N.$$

Proof. The first equality follows from

$$\begin{aligned}
\overline{\tau(\chi)} &= \sum_{n=0}^{N-1} \overline{\chi}(n)\exp(-2\pi i n/N) \\
&= \sum_{m=0}^{N-1} \chi(-1)\overline{\chi}(m)\exp(2\pi i m/N), \text{ by rearranging the summation} \\
&= \chi(-1)\tau(\overline{\chi}).
\end{aligned}$$

For the second equality,

$$\begin{aligned}
|\tau(\chi)|^2 = \tau(\chi)\overline{\tau(\chi)} &= \sum_{n=0}^{N-1}\overline{\chi}(n)\sum_{m=0}^{N-1}\chi(m)\exp(2\pi i m/N)\exp(-2\pi i n/N) \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\chi(m/n)\exp(2\pi i(m/n)n/N)\exp(-2\pi i n/N) \\
&= \sum_{m=0}^{N-1}\chi(m)\sum_{n=0}^{N-1}\exp(2\pi i(m-1)n/N), \text{ by rearranging the summation} \\
&= N \text{ since } \sum_{n=0}^{N-1}\exp(2\pi i(m-1)n/N) = 0 \text{ for } m \neq 1.
\end{aligned}$$

$\square$

Note that for a primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$ and of an odd prime order, $\chi(-1) = 1$, and lemma 1.2 implies that

$$\overline{\tau(\chi)} = \chi(-1)\tau(\overline{\chi}) = \tau(\overline{\chi}). \tag{1.2}$$

Let l be an odd prime and X be a positive real number. Consider the number of distinct primitive Dirichlet characters χ of order l and conductor $\mathfrak{f}_\chi$. David, Fearnley, and Kisilevsky [DFK04] showed that there are $2^{\nu(\mathfrak{f}_\chi)}$ distinct cubic primitive Dirichlet characters of conductor $\mathfrak{f}_\chi$ where $\nu(n)$ is the number of distinct prime factors of n. With the same arguments, we can generalize this result for primitive Dirichlet characters of an odd prime order as follows.

Lemma 1.3. Let l be an odd prime and χ be a primitive Dirichlet character of order l and conductor $\mathfrak{f}_\chi$. Then, there are $(l-1)^{\nu(\mathfrak{f}_\chi)}$ distinct characters of order l and conductor $\mathfrak{f}_\chi$.

Proof. For a primitive Dirichlet character χ of order l and conductor

$$\mathfrak{f}_\chi = \prod_{1 \le n \le \nu(\mathfrak{f}_\chi)} p_n^{a_n}$$

where p_n's are pairwise distinct primes and $a_n \in \mathbb{Z}_{\ge 1}$, χ can be factorized by

$$\chi = \prod_{1 \le n \le \nu(\mathfrak{f}_\chi)} \chi_n$$

where χ_n is a character of order l and conductor $p_n^{a_n}$ for $1 \le n \le \nu(\mathfrak{f}_\chi)$. However, a_n should be 2 if $p_n = l$ and 1 if $p_n \ne l$ since, otherwise, there is a character of order l and conductor p_n inducing that χ_n. Furthermore, each χ_n is a character of order l and conductor $p_n \ne l$ if and only if $p_n \equiv 1 \bmod l$. It is due to the Lagrange theorem of group theory and the fact that the group of Dirichlet characters of conductor $p_n^{a_n}$ has order $\phi(p_n^{a_n}) = p_n^{a_n-1}(p_n - 1)$ for an odd prime l and $a_n > 1$. Therefore, since for each χ_n, there are $l-1$ distinct characters of order l, the proof is completed. $\square$

1.3 Hecke operators and primitive forms

Let Γ be a congruence subgroup. In this section, we study Hecke operators on $M_k(\Gamma)$ as linear transforms on a vector space. Hecke operators can be defined as double coset operators as below.

Definition. Let Γ_1 and Γ_2 be congruence subgroups and $\alpha \in GL_2^+(\mathbb{Q})$. Then, the double coset is

$$\Gamma_1 \alpha \Gamma_2 := \{\gamma_1 \alpha \gamma_2 \mid \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 \in \Gamma_2\}.$$

Note that the multiplications of Γ_1 (Γ_2) on $\Gamma_1\alpha\Gamma_2$ are left (right, respectively) action. It turns out that $\Gamma_1\alpha\Gamma_2 = \dot\bigcup_j \Gamma_1\alpha\beta_j$ where β_j's some choices of coset representatives of a congruence subgroup $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ in Γ_2: it is easy to show that $\alpha^{-1}\Gamma_1\alpha \cap \Gamma_2$ is a congruence subgroup and the index of this in Γ_2 is finite (See Lemma 2.7.1 in [Miy89]). Now, we define the action of the double coset on $f \in \mathcal{M}_k(\Gamma_1)$ as

$$f|(\Gamma_1\alpha\Gamma_2) = \sum_j f|(\alpha\beta_j).$$

Note that this definition in well-defined, i.e. it is independent of choice of β_j since $\Gamma_1\alpha\beta_j = \Gamma_1\alpha\beta_j'$ if and only if $\beta_j'\beta_j \in \alpha^{-1}\Gamma_1\alpha$. An important fact of this action is that $\Gamma_1\alpha\Gamma_2$ linearly maps $M_k(\Gamma_1)$ to $M_k(\Gamma_2)$, in particular it maps $S_k(\Gamma_1)$ to $S_k(\Gamma_2)$, (See Lemma 2.7.2 in [Miy89] for the proof). Now, we define Hecke operators using the double coset $\Gamma\alpha\Gamma$ for $\Gamma = \Gamma_1(N)$.

Definition. Let $N \geq 1$ be an integer and p be a prime. Denote $\Gamma := \Gamma_1(N)$. Then, the Hecke operator at p, denoted by T_p, on $M_k(\Gamma)$ is defined as for $f \in M_k(\Gamma)$,

$$T_p f := f|(\Gamma\alpha\Gamma)$$

where $\alpha := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.

There is also a linear operator, called the diamond operator and denoted by $\langle d\rangle$ for a positive integer d with $(d, N) = 1$, acting on $M_k(\Gamma_1(N))$ defined as

Definition. Let N be a positive integer and d be a positive integer with $(d, N) = 1$. Then, the diamond operator for d on $M_k(\Gamma_1(N))$ is defined as for $f \in M_k(\Gamma_1(N))$,

$$\langle d\rangle f := f|(\begin{pmatrix} a & b \\ c & d' \end{pmatrix} \in \Gamma_0(N))$$

where $d' \equiv d \bmod N$.

Note that it is well-defined since $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$. We now use this diamond operator to establish the decomposition of $M_k(\Gamma_1(N))$ as $\mathbb{C}$-vector space.

Definition. Let N be a positive integer and d be a positive integer with $(d, N) = 1$. Then, the space of modular forms of weight k and for $\Gamma_0(N)$ with a Dirichlet character, called nebentypus, ε modulo N is defined as

$$M_k(\Gamma_0(N), \varepsilon) := \{f \in M_k(\Gamma_1(N)) \mid \langle d\rangle f = \varepsilon(d)f \text{ for } d \in (\mathbb{Z}/N\mathbb{Z})^\times\}.$$

Similarly, $S_k(\Gamma_0(N), \varepsilon)$ denotes the subspace of cusp forms in $M_k(\Gamma_0(N), \varepsilon)$.

Proposition 1.4. As $\mathbb{C}$-vector space, we have

$$M_k(\Gamma_1(N)) = \bigoplus_{\varepsilon \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} M_k(\Gamma_0(N), \varepsilon) \text{ and } S_k(\Gamma_1(N)) = \bigoplus_{\varepsilon \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} S_k(\Gamma_0(N), \varepsilon).$$

Proof. See Lemma 4.3.1 in [Miy89]. □

Note that since $-I \in \Gamma_0(N)$, for $f \in S_k(\Gamma(N), \varepsilon)$, $(-1)^k f = f|(-I) = \varepsilon(-1)f$. Thus, if $\varepsilon(-1) \neq (-1)^k$, then $f = 0$. We define the modular form $f(z) = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \varepsilon)$ twisted by a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ as

$$f_\chi(z) := \sum_{n\geq 0} \chi(n) a_n q^n.$$

Then, the direct verification shows that $f_\chi \in M_k(\Gamma_0(M), \varepsilon\chi^2)$ where M is the least common multiple of N, $\mathfrak{f}_\varepsilon \mathfrak{f}_\chi$ and $\mathfrak{f}_\chi$. Moreover, if $f \in S_k(\Gamma_0(N), \varepsilon)$, then $f_\chi \in S_k(\Gamma_0(M), \varepsilon\chi^2)$ (See the proof of Theorem 7.4 in [Iwa97]). Denote $\varepsilon(\gamma) := \varepsilon(d)$ for $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Going back to the Hecke operator T_p at a prime p acting on $f \in M_k(\Gamma_1(N))$ we can explicitly write the image $T_p f$, where f has the following Fourier q-expansion:

$$f(z) = \sum_{n\geq 0} a_n q^n.$$

Proposition 1.5. For a prime p and $f(z) = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \varepsilon)$ we have

$$T_p f = U_p f + \varepsilon(p) p^{k-1} V_p f,$$

where $(U_p f)(z) = \sum_{n\geq 0} a_{np} q^n$ and $(V_p f)(z) = \sum_{n\geq 0} a_{n/p} q^n$ with $a_{n/p} = 0$ if $p \nmid n$.

Proof. See §4.5 in [Miy89]. □

The straightforward computation shows the following relations:

$$T_{nm} = T_n T_m \text{ if } (n, m) = 1$$

$$T_{p^r} = \begin{cases} T_p^r & \text{if } p|N \\ T_{p^{r-1}} T_p - \varepsilon(p) p^{k-1} T_{p^{k-2}} & \text{if } p \nmid N \end{cases}$$

Let $\mathbb{T}$ be the Hecke algebra acting on $M_k(\Gamma_1(N))$ generated by all T_p at prime p and $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then, $\mathbb{T}$ is a $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(M_k(\Gamma_1(N)))$ and it turns out that $\mathbb{T}$ is commutative. If $f \in S_k(\Gamma_1(N))$ is an eigenvector for every T_p such that $p \nmid N$, then f is called an eigenform. Next, we define the space of anti-holomorphic cusp forms of weight k and level N. Shimura yields the isomorphism theorem of the space of cuspidal modular symbols, which is defined in the next chapter, with respect to the product of this space and $M_k(\Gamma_1(N))$, refer to [Shi59].

Definition. The space of anti-holomorphic cusp forms with respect to $S_k(\Gamma_1(N))$ is defined as

$$\overline{S}_k(\Gamma_1(N)) := \{\overline{f} \mid f \in S_k(\Gamma_1(N))\}. \tag{1.3}$$

Moreover, that with respect to $S_k(N, \varepsilon)$ is defined as

$$\overline{S}_k(N, \varepsilon) := \{\overline{f} \mid f \in S_k(N, \overline{\varepsilon})\}.$$

Note that $\overline{f}(z) = \overline{f(z)}$ and the action of $\gamma \in GL_2^+(\mathbb{R})$ on $g \in S_k(\Gamma_1(N))$ is defined as

$$(g|\gamma)(z) := \rho^k(\gamma, \overline{z})g(\gamma z).$$

Moreover, the actions of $\langle d \rangle$ and T_p on $\overline{S}_k(\Gamma_1(N))$ are defined by

$$\langle d \rangle \overline{f} = \overline{\langle d \rangle f} \text{ and } T_p \overline{f} = \overline{T_p f}.$$

Consider an eigenspace for $\langle d \rangle$ and T_p for all primes p in $S_k(\Gamma_1(N))$. Then, it turns out that there exists the eigenspace of the complex conjugate eigenvalues of the same dimension of $S_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$ can be decomposed into eigenspaces with the same eigenvalues in the exactly same way for $S_k(\Gamma_1(N))$. Note that such eigenspace is twice the dimension of its restriction to $S_k(\Gamma_1(N))$.

Fix a positive integer N. For each divisor M of N, consider a positive divisor m of N/M. Let $\psi_m : S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$ defined by $f(z) \mapsto f(mz)$. Then, combine ψ_m for M so that we define the map

$$\Psi_M = \bigoplus_{m|(N/M)} \psi_m.$$

Then, we define the subspace of $S_k(\Gamma_1(N))$ generated by the images of Ψ_M for every $M|N$ and $M \neq N$ and denote it by $S_k(\Gamma_1(N))^{\text{old}}$. Also, we define the complementary subspace to $S_k(\Gamma_1(N))^{\text{old}}$ in $S_k(\Gamma_1(N))$ with respect to the Petersson inner product $\langle \cdot, \cdot \rangle$ and denote it by $S_k(\Gamma_1(N))^{\text{new}}$. Note that if N is a prime, then

$$S_k(\Gamma_1(N)) = S_k(\Gamma_1(N))^{\text{new}}.$$

We call a form that is an eigenform in $S_k(\Gamma_1(N))^{\text{new}}$ for every Hecke operator by a new-form and if its Fourier coefficient $a_1 = 1$, it is called a normalized form. Now, we define a primitive form of $S_k(\Gamma_0(N), \varepsilon)$.

Definition. A cusp form of $S_k(\Gamma_0(N), \varepsilon)$ is called a primitive form if $f \in S_k(\Gamma_0(N), \varepsilon)^{\text{new}}$, f is the Hecke eigenform for every T_p for every prime $p \nmid N$ and normalized.

Martin in [Mar05] derived the closed formulae for the dimensions of $S_k(\Gamma_0(N))$ and $S_k(\Gamma_1(N))^{\text{new}}$ for weight $k \geq 2$ by using the linear combinations of multiplicative functions of level N. He also derived the asymptotic averages of those dimensions. Stein showed the effective algorithms to compute $M_k(\Gamma)$, $S_k(\Gamma)$ and their old and new subspaces in [Ste07] using modular symbols. We use those algorithms to study vanishings and nonvanishings of the special L-values of primitive forms. For more detailed information on the algorithms implemented, refer Sage [S$^+$17].

1.4 L-functions of primitive forms

Let N and k be positive integers and l be an odd prime. In this section we define L-functions of primitive forms of weight k for $\Gamma_0(N)$ with a nebentypus ε modulo N, study some important properties and discuss vanishings and nonvanishings of the central L-values.

Definition. Let $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_0(N), \varepsilon)$ be a primitive form. Then, the L-function of f is defined as for $\mathrm{Re}(s) > c$ with a positive real number c,

$$L(f, s) := \sum_{n\geq 1} \frac{a_n}{n^s}.$$

Moreover, for a primitive l-th Dirichlet character χ of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$, the L-function of f twisted by χ is defined as for $\mathrm{Re}(s) > c$ with a positive real number c,

$$L(f, s, \chi) := L(f_\chi, s) = \sum_{n\geq 1} \frac{\chi(n) a_n}{n^s}.$$

The constant c in the definition above is depending on an upper bound of the Fourier coefficients a_n of f and from the upper bound, we can rewrite the series of $L(f, s)$ into a product, called the Euler product. More precisely, suppose that $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_0(N), \varepsilon)$ be a primitive form and $|a_n| \ll n^\alpha$ for some α. Then, $L(f, s)$ absolutely converges for $\mathrm{Re}(s) > c$ for some constant c. It implies that we have the Euler product for $\mathrm{Re}(s) > c$ as

$$L(f, s) = \prod_{p \text{ prime}} \left(1 - \frac{a_p}{p^s} + \frac{\varepsilon(p) p^{k-1}}{p^{2s}}\right)^{-1}. \tag{1.4}$$

In fact, Deligne in [Del74] showed that for a primitive form $f \in S_k(\Gamma_0(N), \varepsilon)$,

$$|a_n| \leq d(n)n^{(k-1)/2}, \text{(particulary } |a_p| \leq 2p^{(k-1)/2} \text{ for a prime } p)$$

where $d(n)$ is the divisor function. Thus, for $\mathrm{Re}(s) > (k+1)/2$, we have (1.4). Note that $L(f,s)$ may be considered as the Mellin transform $(\mathcal{M}f)(s)$ of f. Indeed,

$$L(f,s) = \frac{(2\pi)^s}{\Gamma(s)}(\mathcal{M}f)(s) = \frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty f(it)t^s\frac{dt}{t}. \tag{1.5}$$

Hecke showed that every $f \in S_k(\Gamma_0(N), \varepsilon)$ admits analytic continuation to the whole complex plane and satisfies a certain functional equation. For his theorem, we need two linear operators acting on $S_k(\Gamma_0(N), \varepsilon)$ called the Fricke involution and the complex conjugation on the Fourier coefficients also denoted by W and K, respectively.

Definition. For each $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_0(N), \varepsilon)$, the operators are defined as

$$(Wf)(z) := (f|\gamma)(z) = (\sqrt{N}z)^{-k} f(\frac{-1}{Nz}) \text{ where } \gamma := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix},$$

$$(Kf)(z) := \overline{f}(-\overline{z}) = \sum_{n\geq 1} \overline{a_n} q^n,$$

$$(\overline{W}f)(z) := (KWf)(z) = (K(Wf))(z).$$

Notice that $W^2 = (-1)^k$ and $K^2 = 1$ and that since $(Wf)(z) \in S_k(\Gamma_0(N), \overline{\varepsilon})$ and $K : S_k(\Gamma_0(N), \varepsilon) \to S_k(\Gamma_0(N), \overline{\varepsilon})$, $\overline{W}$ is well-defined and its images are in $S_k(\Gamma_0(N), \varepsilon)$. It also turns out that $\overline{W}^2 = 1$ and for a Hecke eigenform $f \in S_k(\Gamma_0(N), \varepsilon)$, $\overline{W}f = \eta f$ for some $\eta \in \mathbb{C}$ with $|\eta| = 1$ (See Section 6.7 in [Iwa97] for the properties of those operators). In particular, if $f \in S_k(\Gamma_0(N), \varepsilon)$ is a primitive form, then

$$\eta = \tau(\overline{\varepsilon})a_N/N^{k/2} \tag{1.6}$$

(this is Theorem 6.29 in [Iwa97]). Now, we present the Hecke's theorem mentioned above.

Theorem (Hecke)**.** Let $f \in S_k(\Gamma_0(N), \varepsilon)$. Then, $L(f,s)$ has analytic continuation to $\mathbb{C}$, and it is entire. Moreover, we have the following functional equation as

$$\Lambda(f,s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(f,s) = i^k\overline{\eta}\Lambda(\overline{f}, k-s),$$

where η is the eigenvalue of f under $\overline{W}$.

Proof. See the proofs of Theorem 7.1 and 7.2 in [Iwa97] or Theorem 14.17 in [IK04]. □

Hecke actually proved the similar theorem as above for a general modular form satifying a polynomial boundness of the Fourier coefficients and there the corresponding L-function has meromorphic continuation instead of analytic continuation (possible simple poles at $s = 0$ and k; see Theorem 7.3 in [Iwa97]). The version of theorem for $L(f, s, \chi)$ for a primitive Dirichlet character χ is following.

Proposition 1.6. Let $f \in S_k(\Gamma_0(N), \varepsilon)$ and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. Then, $L(f, s, \chi)$ has analytic continuation to $\mathbb{C}$, and it is entire. Moreover, we have the following functional equation as

$$\Lambda(f, s, \chi) := \left(\frac{\mathfrak{f}_\chi \sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f, s, \chi) = i^k \psi(\chi) \overline{\eta} \Lambda(\overline{f}, k - s, \overline{\chi}), \tag{1.7}$$

where $\psi(\chi) = \varepsilon(\mathfrak{f}_\chi)\chi(N)\tau(\chi)^2/\mathfrak{f}_\chi$ and η is the eigenvalue of f under $\overline{W}$.

Proof. See the proofs of Theorem 7.6 in [Iwa97] or Proposition 14.20 in [IK04]. □

For a general level N and character ε, the formula for $\eta(f)$ is not known yet, however, under some conditions on N and ε, we have a formula for $\eta(f)$ as shown in Proposition 14.15 and Proposition 14.16 in [IK04].

Lemma 1.7. Let N be a positive integer and ε be a Dirichlet character modulo N. Then, for a primitive form f in $S_k(N, \varepsilon)$, we have

$$\eta = \begin{cases} \tau(\overline{\varepsilon}) a_N N^{-k/2} & \text{if } \varepsilon \text{ is primitive} \\ \mu(N) a_N N^{1-k/2} & \text{if } N \text{ is square free and } \varepsilon \text{ is trivial} \end{cases}$$

where μ is the Möbius function and a_N is the N-th Fourier coefficient of f.

Proof. See the proof of Theorem 6.29 in [Iwa97]. □

Why is $L(f, s, \chi)$ important in number theory? There are two important algebro geometric and arithmetic aspects on it. For the first aspect, let L-function of an elliptic curve E defined over $\mathbb{Q}$ with conductor N, i.e. for $\mathrm{Re}(s) > 3/2$,

$$L(E, s) := \prod_{p \mid N} \left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p \nmid N} \left(1 - \frac{a_p}{p^s} + \frac{p}{p^{2s}}\right)^{-1} = \sum_{n \geq 1} \frac{a_n}{n^s}, \tag{1.8}$$

where a_p is the trace of Frobenius endomorphism (more precisely, $a_p = 1+p-|E(\mathbb{F}_p)|$). Then, by the works of Wiles, Breuil, Conrad, Diamond, and Taylor in [Wil95], [TW95] and [BCDT01], there exists a primitive form $f_E \in S_2(\Gamma_0(N), \varepsilon_0)$ where ε_0 is trivial modulo N such that $L(E,s) = L(f_E, s)$ up to the isogeny class of E and $L(E,s)$ admits analytic continuation and the following functional equation:

$$\Lambda(f_E, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f_E, s) = \omega(E)\Lambda(f_E, 2-s), \tag{1.9}$$

where $\omega(E) = \pm 1$ is the eigenvalue of f_E under the Fricke involution W. In other words, roughly speaking, this modularity theorem relates the Galois representations associated with elliptic curves to modular forms. More generally, the Langlands reciprocity conjecture deals with the relation between Artin L-functions associated with finite dimensional Galois representations of number fields and those from automorphic cuspidal representations.

For the other aspect, from the studies of classical degree 1 L-functions such as the Riemann zeta function and Dirichlet L-function to the Birch and Swinnerton-Dyer conjecture on L-functions of elliptic curves, we naturally expect that special L-values of those L-function may encode some important arithmetic invariants on number fields and elliptic curves through the class number formula for number fields and the formula in the second part of the Birch and Swinnerton-Dyer conjecture shown below, respectively. Let E, $r_{\mathrm{al}}(E)$ and $r_{\mathrm{an}}(f_E)$ be an elliptic curve over $\mathbb{Q}$, the algebraic rank of E and the analytic rank of $L(E,s)$, respectively.

Conjecture 1.4.1 (Birch and Swinnerton-Dyer)**.** Let E be an elliptic curve over $\mathbb{Q}$ with conductor N. Then, $r_{\mathrm{al}}(E) = r_{\mathrm{an}}(E)$. Moreover, for the r-th derivative of $L(E,s)$, we have

$$\frac{L^{(r_{\mathrm{al}})}(E,1)}{(r_{\mathrm{al}})!} = \frac{\Omega_E R_E |\text{Ш}(E)|}{|E_{\mathrm{Tor}}|^2} \prod_{p|N} c_p,$$

where Ω_E is the canonical real period, R_E is the regulator, $\text{Ш}(E)$ is the Tate-Shafarevich group, E_{Tor} is the tortion subgroup of the Modell-Weil group and the c_p's are the Tamagawa numbers of E.

Gross and Zagier [GZ86], Kolyvagin [Kol89] and Breuil et al. [BCDT01] showed that the first assertion of the Birch and Swinnerton-Dyer conjecture for $r_{\mathrm{al}}(E) \leq 1$. More recently, Bhargava and Shankar [BS15] showed that the average rank of the

Modell-Weil group of $E/\mathbb{Q}$ has bounded above by 7/6. We will study $Ш(E)$ for a certain subfamily of quadratic twists of E defined over $\mathbb{Q}$ in Chapter 4 to investigate nonvanishings of $L(E,1,\chi)$.

Let E/K be an elliptic curve defined over a number field K. Denote the ring of integers of K by $\mathcal{O}_K$. Then, the L-function of E/K is defined as

$$L(E/K,s) := \prod_{\mathfrak{p}|\Delta}\left(1-\frac{a_{\mathfrak{p}}}{(N\mathfrak{p})^s}\right)^{-1}\prod_{\mathfrak{p}\nmid\Delta}\left(1-\frac{a_{\mathfrak{p}}}{(N\mathfrak{p})^s}+\frac{N\mathfrak{p}}{(N\mathfrak{p})^{2s}}\right)^{-1},$$

where Δ is the discriminant of E/K and N is the field norm for $K/\mathbb{Q}$. Here, $a_{\mathfrak{p}} \in \{-1,0,1\}$ depanding on the reduction type at $\mathfrak{p} \mid \Delta$ and $a_{\mathfrak{p}} = 1 + N\mathfrak{p} - |\tilde{E}(\mathcal{O}_K/\mathfrak{p})|$ for $\mathfrak{p} \nmid \Delta$ where $\tilde{E}$ is the reduction of E/K at $\mathfrak{p}$.

Now, let $K/\mathbb{Q}$ be a finite abelian extension with Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ and the conductor $\mathfrak{f}$ of $K/\mathbb{Q}$. Then, since $K/\mathbb{Q}$ is finite abelian, via the Kronecker-Weber theorem, G is isomorphic to a quotient of $(\mathbb{Z}/\mathfrak{f}\mathbb{Z})^\times$. Thus, we can associate the character group $\widehat{G}$ of G to a group of primitive Dirichlet characters of $(\mathbb{Z}/\mathfrak{f}\mathbb{Z})^\times$ with the trivial character χ_0 of conductor 1 as the identity. Consider a primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$ as an element of $\widehat{G}$ and the cyclic subgroup $\langle\chi\rangle$ generated by $\chi \in \widehat{G}$. Let $\mathbb{Q}(\chi)$ be a cyclotomic field adjoining the values of χ. Then, the action of $\sigma \in G_\chi := \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ on $\langle\chi\rangle$ is given by $\chi^\sigma(n) = \sigma(\chi(n))$ for $n \in (\mathbb{Z}/\mathfrak{f}_\chi\mathbb{Z})^\times$. Therefore, $\langle\chi\rangle$ can be generated by the images of G_χ, and we can rewrite

$$L(E/K,s) = \prod_{\chi\in\widehat{G}} L(E,s,\chi) = \prod_{\langle\chi\rangle\subset\widehat{G}}\left(\prod_{\sigma\in G_\chi} L(E,s,\chi^\sigma)\right). \tag{1.10}$$

For example, given an elliptic curve over $\mathbb{Q}$ and a quadratic number field $K := \mathbb{Q}(\sqrt{D})$ with the fundamental discriminant D, its Galois group G is of order 2 and there exists a unique quadratic Dirichlet character χ_D of conductor $|D|$. Then,

$$L(E/K,s) = L(E,s)L(E,s,\chi_D),$$

Thus, assuming the Birch and Swinnerton-Dyer conjecture, we can predict the algebraic rank of E/K from the vanishings of $L(E,s)$ and $L(E,s,\chi_D)$. Goldfeld [Gol79] made a conjecture that the average analytic rank of $L(E,s,\chi_D)$ in the family of quadratic twists of $E/\mathbb{Q}$ is 1/2. There are many results on quadratic twists of elliptic curves, however comparing to quadratic twists we poorly know about vanishings and nonvanishings of the higher order twists of elliptic curves (more generally primitive forms). One of main obstacles to obtain a powerful vanishing and nonvanishing theorem for higher order twists is due to the non-integral root number, i.e. the sign in the functional equation. Despite of poor understanding of higher order twists, David, Fearnley and Kisilevsky [DFK04] and Fearnley, Kisilevsky and Kuwata [FKK12] were able to obtain some results on vanishings and nonvanishings of twists of odd prime orders of elliptic curves using two distinct techniques. In the next section, we will summarise the known motivating results on vanishings and nonvanishings of $L(f, s, \chi)$ for some special settings of f, s and χ and our new results.

1.5 Summaries of known results and corresponding new results

In the following Chapter 2, 3 and 4, we show vanishings and nonvanishings theorems on critical (or central) L-values of primitive forms f twisted by Dirichlet characters χ with some special settings of them. Those results are generalisations by known results and tools of some authors. In this section, we summarise the known results and tools of those authors in order to give the reader a big picture on this thesis.

For Chapter 2, Fearnely, Kisilevsky and Kuwata in [FKK12] showed that the number of nonvanishings of $L(f, 1, \chi)$ for a primitive form of weight $k = 2$ and the families of primitive Dirichlet characters of order an odd primes l in the following theorem.

Theorem (Fearnley, Kisilevsky and Kuwata)**.** Let $E/\mathbb{Q}$ be an elliptic curve and let l be a prime. Suppose that $L^{\mathrm{alg}}(E, 1) \not\equiv 0 \bmod l$. Then, there exists a set of primes S of positive density such that $L(E, 1, \chi) \neq 0$ for any characters χ of order l with conductor $\mathfrak{f}_\chi$ supported on S.

They use the algebraic parts of $L(f, 1, \chi)$ which is defined in (2.4) and modular symbols to obtain congruence relations modulo a prime $\mathfrak{l}$ in $\mathbb{Q}(\chi)$ above rational prime l. We apply the same methods to obtain similar relations for primitive forms of even weight $k > 2$ and whose Fourier coefficients are essentially in $\mathbb{Q}$ or $\mathbb{Z}[\chi]$. Then, we extend their nonvanishing result for $k > 2$ as presented in Theorem 2.12. Note that for $k > 2$, homogeneous polynomials of degree $k-2$ are involved in modular symbols, hence one needs more care in computing the images of Hecke actions on modular symbols. Furthermore, we present the computational statistical results on the number of vanishings of twists of weight $k > 2$ and characters of order $l = 3, 5, 7$. The data suggests that vanishings of twists occur less often as l and k increase. Therefore, we make Conjecture 2.7.2 in Section 2.7 on the number of vanishings depending on the order of twists and the weight of primitive forms using the random matrix theory as David, Fearnley and Kisilevsky [DFK04] and [DFK06] for twists of weight $k = 2$.

For Chapter 3, we consider cubic twists of primitive forms of weight 2 and obtain the asymptotic of the number of analytic rank of f twisted characters in a special family (called group family) of primitive cubic Dirichlet characters under all Riemann hypothesis (GRH) needed and a hypothesis on the number non-trivial and non-real zeroes of cubic twists for group family. It gives us asymptotics of vanishings 3.12 and nonvanishings 3.13. The methods in this work are based on those of Fiorilli [Fio16] in estimating the analytic ranks of quadratic twists of an elliptic curve E defined over $\mathbb{Q}$. He obtained the following average number of analytic ranks of quadratic twists of E for the usual family of primitive quadratic Dirichlet characters of fundamental discriminants d ordered by $|d|$ under the hypothesis below.

Hypothesis. There exsists $0 < \delta < 1$ such that for $D^{2-\delta} \leq x \leq 2D^{2-\delta}$ we have

$$\sum_{(d,N)=1}{}' \omega(\frac{d}{D}) \sum_{\rho_d \notin \mathbb{R}} \frac{x^{\rho_d}}{\rho_d(\rho_d+1)} = o(D\sqrt{x})$$

Theorem (Fiorilli)**.** Assume all GRH's necessary and the above hypothesis for some non-negative Schwartz weight function ω with $\omega(0) > 0$. Then,

$$\lim_{D\to\infty} \frac{1}{N(D)} \sum_{\substack{0<|d|\leq D \\ (d,N)=1}}{}' r_{an}(E, \chi_d) = \frac{1}{2},$$

For Chapter 4, we consider quadratic prime twists E^{χ_d} parameterised by their fundamental discriminants d with a prime $|d|$ and the even root number of an elliptic curve E defined over $\mathbb{Q}$ with square-free conductor N. Tables 2, 3, 4 and 5 shows that for E with the following elliptic invariants:

$$d_2(E(\mathbb{Q})[2]) > 0, \text{Ш}(E/\mathbb{Q})[2] = 1, \text{ and } r_{\mathrm{al}}(E(\mathbb{Q})) = 0$$

non-existence of vanishings of quadratic twists occurs depending on the residue class $|d| \bmod N$. In order to explain these phenomena, we use the methods of Kramer [Kra81] and Mazur and Rubin [MR10] to control 2-Selmer groups of E and E^{χ_d}. More precisely, considering the images of $E(\mathbb{Q}_p)$ and $E^{\chi_d}(\mathbb{Q}_p)$ under the local Kummer map κ_p denoted by $H^1_f(\mathbb{Q}_p, E[2])$ and $H^1_f(\mathbb{Q}_p, E^{\chi_d}[2])$, respectively, as subgroups of $H^1(\mathbb{Q}, E[2])$, it turns out that they differ only at a finite number of places. Therefore, we can control the 2-Selmer groups by choosing a proper finite set of places at which $H^1_f(\mathbb{Q}_p, E[2])$ and $H^1_f(\mathbb{Q}_p, E^{\chi_d}[2])$ only differ and using the Tate local and Poitou-Tate global dualities on Galois cohomologies. Then, we eventually obtain the rank of E^{χ_d} is 0. Then, the Birch and Swinnerton-Dyer conjecture asserts $L(E, 1, \chi_d) \neq 0$ for those E^{χ_d} as shown in Theorem 4.9. However, it should be noted that this methods do not working for every elliptic curve above but only for those satisfying the extra conditions in Theorem 4.9, for examples $E := 17a1, 42a1, 70a1$ in Table 2 and 5.

Chapter 2

Modular symbols for higher order twists

In this chapter, we study modular symbols for $M_k(N,\varepsilon) := M_k(\Gamma_0(N),\varepsilon)$ and $S_k(N,\varepsilon) := S_k(\Gamma_0(N),\varepsilon)$ which are used in computing the critical L-values for a primitive form f and the random matrix theory and present some vanishing and nonvanishing theorems using those two techniques. The main references for this chapter are [MTT86], [Ste07], [FKK12] and [DFK04].

2.1 Algebraic parts of the critical L-values

For each positive integer $k \geq 2$, let $f \in S_k(\Gamma_1(N))$ be a primitive form of weight k and level N. Also let $P_k(\mathbb{C})$ be the space of polynomials of variable z and degree less than or equal to $k-2$ over $\mathbb{C}$. Recall the automorphy factor $\rho(\gamma,z) := \det(\gamma)^{1/2} j(\gamma,z)^{-1}$. Define the action of $GL_2^+(\mathbb{R})$ on $P_k(\mathbb{C})$ as $\gamma \in GL_2^+(\mathbb{R})$ and $h(z) \in P_k(\mathbb{C})$

$$(h|\gamma)(z) := \rho(r,z)^{2-k} h(\gamma z).$$

Note that the restricted action of $SL_2(\mathbb{Z})$ preserves $P_k(\mathbb{Z}) \in P_k(\mathbb{C})$.

The algebraic parts of $L(f,n,\chi)$ for integers $1 \leq n \leq k-1$, called the critical L-values associated with the critical integers n, were introduced by Mazur, Tate and Teitelbaum in [MTT86]. It plays a crucial role in study of critical values of $L(f,n,\chi)$. In order to define the algebraic parts we first define the modular integral.

Definition. For $h \in P_k(\mathbb{C})$ and $r \in \mathbb{P}^1(\mathbb{Q})$, define

$$\Phi(f,h,r) := 2\pi i \int_{i\infty}^{r} f(z)h(z)dz = \begin{cases} 2\pi \int_0^\infty f(r+it)h(r+it)dt & \text{if } r \neq \infty \\ 0 & \text{if } r = \infty \end{cases}.$$

Notice that for every $r \in \mathbb{P}^1(\mathbb{Q})$, $\Phi(f,h,r)$ is bilinear in f and h over $\mathbb{C}$. Moreover, observe that for each $\gamma \in GL_2^+(\mathbb{R})$,

$$(f|\gamma)(z)(h|\gamma)(z)dz = f(\gamma z)h(\gamma z)d(\gamma z). \tag{2.1}$$

Using this observation with (1.1) and Cauchy's theorem, we have

$$\Phi(f|\gamma, h|\gamma, r) = \Phi(f,h,\gamma r) - \Phi(f,h,\gamma\infty). \tag{2.2}$$

Let $R(N) := \{\gamma_1, \dots, \gamma_m\}$ be a set of coset representatives of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$ with $m := [SL_2(\mathbb{Z}) : \Gamma_0(N)]$.

Proposition 2.1. Choose $f \in S_k(N,\varepsilon)$ and denote the the ring generated by the values of ε by $\mathbb{Z}[\varepsilon]$. Then, for $h \in P_k(\mathbb{C})$ and $r \in \mathbb{P}^1(\mathbb{Q})$,

$$\Phi(f,h,r) = \sum_{1 \le j \le m} \sum_{0 \le i \le k-2} a_{i,j}(\Phi(f,z^i,r_j\infty) - \Phi(f,z^i,r_j 0))$$

where $a_{i,j} \in \mathbb{Z}[\varepsilon]$ for $1 \le j \le m$ and $0 \le i \le k-2$.

Proof. Only brief proof is present here. Refer to Proposition in §2 of Chapter I in [MTT86] for detailed proof. For $\gamma \in \Gamma_0(N)$, $(f|\gamma)(z) = \varepsilon(\gamma)f(z)$. Then, $\mathbb{C}$-bilinearity of Φ and (2.2) implies that the images of $P_k(\mathbb{C}) \times \mathbb{P}^1(\mathbb{Q})$ is a $\mathbb{Z}[\varepsilon]$-module. Now, write a given $r \in \mathbb{P}^1(\mathbb{Q})$ as $r = a/b$ with $a, b \in \mathbb{Z}$ in the lowest terms if $b \neq 0$. Then, the rest of proof follows by straightforward computations with $R(N)$ and induction on b. □

In particular, if ε is trivial, i.e. $\varepsilon = \varepsilon_0$, then $\Phi(f,h,r)$ is in a $\mathbb{Z}$-module generated by the summands appearing in the above proposition. Recall (1.5) and connect it with the modular integral. Then, we have the following relation for $1 \le n \le k-1$ as

$$L(f,n) = \frac{(-2\pi i)^{n-1}}{(n-1)!}\Phi(f,z^{n-1},0). \tag{2.3}$$

Let $f(z) = \sum_{n\ge1} a_n q^n \in S_k(N,\varepsilon)$ and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Then, recall the twist of f by χ is given by $f_\chi(z) = \sum_{n\ge1}\chi(n)a_n q^n$. We show the following relation between $\Phi(f,h,r)$ and $\Phi(f_{\overline{\chi}},h,r)$.

Lemma 2.2. Let $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(N,\varepsilon)$ and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Then, for $h \in P_k(\mathbb{C})$ and $r \in \mathbb{P}^1(\mathbb{Q})$,

$$\Phi(f_{\overline{\chi}}, h, r) = \frac{1}{\tau(\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a)\Phi(f, h|\begin{pmatrix} 1 & -a/\mathfrak{f}_\chi \\ 0 & 1 \end{pmatrix}, r + \frac{a}{\mathfrak{f}_\chi}).$$

Proof. Using the Gauss sum, simple computations show that

$$f_{\overline{\chi}}(z) = \frac{1}{\tau(\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) f(z + \frac{a}{\mathfrak{f}_\chi}).$$

Then, by $\mathbb{C}$-linearity in $S_k(N,\varepsilon)$, we have

$$\begin{aligned}
\Phi(f_{\overline{\chi}}, h, r) &= 2\pi i \int_{i\infty}^{r} f_{\overline{\chi}}(z) h(z) dz \\
&= \frac{1}{\tau(\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) 2\pi i \int_{i\infty}^{r} f(z + \frac{a}{\mathfrak{f}_\chi}) h(z) dz \\
&= \frac{1}{\tau(\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) \Phi(f|\begin{pmatrix} 1 & a/\mathfrak{f}_\chi \\ 0 & 1 \end{pmatrix}, h, r) \\
&= \frac{1}{\tau(\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) \Phi(f, h|\begin{pmatrix} 1 & -a/\mathfrak{f}_\chi \\ 0 & 1 \end{pmatrix}, r + \frac{a}{\mathfrak{f}_\chi}) \text{ by (2.2)},
\end{aligned}$$

which completes the proof. □

As equation (2.3), from Lemma 2.2, we can deduce the relation of $L(f,n,\chi)$ and modular integrals.

Proposition 2.3. Let $f(z) = \sum_{n\geq 1} a_n q^n \in S_k(N,\varepsilon)$ and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Then, for $1 \leq n \leq k-1$,

$$L(f,n,\chi) = \frac{(-2\pi i)^{n-1}}{(n-1)!} \frac{\tau(\chi)}{\mathfrak{f}_\chi^n} \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a) \Phi(f, (\mathfrak{f}_\chi z + a)^{n-1}, -\frac{a}{\mathfrak{f}_\chi}).$$

Proof. As equation (1.5), we have

$$L(f,n,\chi) = \frac{(-2\pi i)^{n-1}}{(n-1)!} (2\pi i \int_{i\infty}^{0} f_\chi(z) z^{n-1} dz) = \frac{(-2\pi i)^{n-1}}{(n-1)!} \Phi(f_\chi, z^{n-1}, 0).$$

By observing that

$$z^{n-1}|\begin{pmatrix} 1 & -a/\mathfrak{f}_\chi \\ 0 & 1 \end{pmatrix} = \frac{1}{\mathfrak{f}_\chi^{n-1}} (\mathfrak{f}_\chi z - a)^{n-1}$$

and using Lemma 2.2 for $r = 0$ and $\mathbb{C}$-linearity in $P_k(\mathbb{C})$ we have

$$\begin{aligned} L(f,n,\chi) &= \frac{(-2\pi i)^{n-1}}{(n-1)!} \frac{1}{\tau(\overline{\chi})\mathfrak{f}_\chi^{n-1}} \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a)\Phi(f, (\mathfrak{f}_\chi z - a)^{n-1}, \frac{a}{\mathfrak{f}_\chi}) \\ &= \frac{(-2\pi i)^{n-1}}{(n-1)!} \frac{\tau(\chi)}{\mathfrak{f}_\chi^{n}} \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a)\Phi(f, (\mathfrak{f}_\chi z + a)^{n-1}, -\frac{a}{\mathfrak{f}_\chi}) \end{aligned}$$

In the last line, we used $\overline{\chi}(-1)\tau(\overline{\chi})\tau(\chi) = \mathfrak{f}_\chi$ and arranged the sum. $\square$

The following theorem by Shimura is the fundamental theorem in the discretisation of critical L-values. Denote $\mathrm{sign}(\chi) = \chi(-1) \in \{-1, 1\}$.

Theorem (Shimura)**.** Let $f \in S_k(N, \varepsilon)$ be a primitive form and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. Then, there exist non-zero complex numbers Ω^+ and Ω^-, called the periods, depending only on f such that for $1 \leq n \leq k-1$,

$$\frac{(n-1)!}{(-2\pi i)^{n-1}} \frac{\mathfrak{f}_\chi^n}{\tau(\chi)} \frac{L(f,n,\chi)}{\Omega^\pm} \in \mathbb{Q}(f)\mathbb{Q}(\chi)$$

where $\mathbb{Q}(f)$ and $\mathbb{Q}(\chi)$ is the number fields adjoining the values of Fourier coefficients of f and χ, respectively and $\mathbb{Q}(f)\mathbb{Q}(\chi)$ is the compositum of them. Moreover, it takes Ω^+ if $\mathrm{sign}(\chi) = (-1)^{n-1}$ and Ω^- if $\mathrm{sign}(\chi) = (-1)^n$.

Proof. See the proof of Theorem 1 in [Shi77]. $\square$

Note that the choice of sign of Ω depends on both the sign of χ and the parity of the critical integers $1 \leq n \leq k$.

Definition. Let $k \geq 2$ be an integer. Given $f \in S_k(N, \varepsilon)$ and a primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$, the algebraic parts of $L(f, n, \chi)$ for $1 \leq n \leq k-1$ is defined by

$$\begin{aligned} L^{\mathrm{alg}}(f,n,\chi) &:= \frac{1}{\Omega^\pm} \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a)(\Phi(f, (\mathfrak{f}_\chi z + a)^{n-1}, -\frac{a}{\mathfrak{f}_\chi}) \pm \Phi(f, (\mathfrak{f}_\chi z - a)^{n-1}, \frac{a}{\mathfrak{f}_\chi})) \\ &= \frac{(n-1)!}{(-2\pi i)^{n-1}} \frac{\mathfrak{f}_\chi^n}{\tau(\chi)} \frac{L(f,n,\chi)}{\Omega^\pm}. \end{aligned} \tag{2.4}$$

Note that the definition of the algebraic parts and the Shimura theorem implies that for $1 \leq n \leq k-1$,

$$L^{\mathrm{alg}}(f,n,\chi) = 0 \text{ if and only if } L(f,n,\chi) = 0. \tag{2.5}$$

The Shimura theorem also implies that there exist $\Omega^{\pm}$ so that $L^{\mathrm{alg}}(f,n,\chi)$ are algebraic integers by taking some integral multiple of the periods. We take those $\Omega^{\pm}$ in the definition above. For example, if the Fourier coefficients of f are all in $\mathbb{Q}$, then $L^{\mathrm{alg}}(f,n,\chi) \in \mathbb{Z}[\chi]$ for $1 \le n \le k-1$.

We present the functional equation of the algebraic parts in the following proposition.

Proposition 2.4. Let $f \in S_k(N,\varepsilon)$ be a primitive form and χ be a primitive Dirichlet charactor of conductor $\mathfrak{f}_\chi$ with $(N,\mathfrak{f}_\chi) = 1$. Then, for the integers n such that $1 \le n \le k-1$ we have the following functional equation of $L^{\mathrm{alg}}(f,n,\chi)$.

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^n \psi(\chi)\overline{\eta} N^{k/2-n} \frac{\tau(\overline{\chi})}{\tau(\chi)} L^{\mathrm{alg}}(Kf, k-n, \overline{\chi}). \tag{2.6}$$

If all Fourier coefficients a_m of f are rational and ε is a primitive Dirichlet character modulo N, then

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^n \varepsilon(-\mathfrak{f}_\chi)\tau(\varepsilon)\frac{a_N}{N^n}\chi(-N) L^{\mathrm{alg}}(f, k-n, \overline{\chi}), \tag{2.7}$$

and if all Fourier coefficients a_m of f are rational, N is square-free and ε is trivial, then

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^n \mu(N)\frac{a_N}{N^{n-1}}\chi(-N) L^{\mathrm{alg}}(f, k-n, \overline{\chi}), \tag{2.8}$$

where μ is the Möbius function.

Proof. The first functional equation (2.6) can be easily derived from the functional equation (1.7) and the definition of $L^{\mathrm{alg}}(f,n,\chi)$.
From now on, assume that the Fourier coefficients a_m of f are rational, hence $Kf = f$. By Lemma 1.7, if ε is a primitive Dirichlet character modulo N, then we have $\eta = \tau(\overline{\varepsilon}) a_N N^{-k/2}$ so that

$$\psi(\chi)\overline{\eta} = \varepsilon(\mathfrak{f}_\chi)\chi(N)\frac{\tau(\chi)^2}{\mathfrak{f}_\chi}\overline{\tau(\overline{\varepsilon})}\frac{a_N}{N^{k/2}} = \varepsilon(-\mathfrak{f}_\chi)\tau(\varepsilon)\frac{a_N}{N^{k/2}}\chi(-N).$$

Here we use the assumptions that ε and χ are primitive so that $\varepsilon(-1)\tau(\varepsilon)\tau(\overline{\varepsilon}) = N = \tau(\varepsilon)\overline{\tau(\varepsilon)}$ and $\chi(-1)\tau(\chi)\tau(\overline{\chi}) = \mathfrak{f}_\chi$. Put this $\psi(\chi)\overline{\eta}$ into (2.6) and the functional equation (2.7) follows. Lastly, if N is square-free and ε is a trivial charactor modulo N, then, again by Lemma 1.7, we have $\eta = \mu(N) a_N N^{1-k/2}$ so that

$$\psi(\chi)\overline{\eta} = \mu(N) a_N N^{1-k/2}\chi(-N).$$

Again put this $\psi(\chi)\overline{\eta}$ into (2.6), we can easily obtain the functional equation (2.8) as desired. This completes the proof. $\square$

Theorem 4.6.17 in [Miy89] asserts that if N is square-free and ε is a trivial character modulo N, then the value of the N-th Fourier coefficient a_N of f satisfies $a_N^2 = N^{k-2}$.

2.2 Modular symbols

In this section, we study the theory of modular symbols of weight $k \geq 2$ and for congruence subgroup Γ and present algorithms to compute the algebraic parts introduced in the previous section.

Denote the torsion subgroup of a group G by G_{tor}. Define the free abelian group $\mathbb{M}_2 := \mathbb{M}/\mathbb{M}_{\text{tor}}$, where

$$\mathbb{M} := \langle \{\alpha, \beta\} \mid \alpha, \beta \in \mathbb{P}^1(\mathbb{Q})\rangle / \langle \{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} \mid \alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})\rangle.$$

The relation above is called the 3-term relation. Fix an integer $k \geq 2$. Let $\mathbb{Z}[X, Y]_{k-2}$ be the abelian group of homogeneous polynomials of variables X and Y and degree $k-2$. Define

$$\mathbb{M}_k := \mathbb{Z}[X, Y]_{k-2} \otimes_{\mathbb{Z}} \mathbb{M}_2.$$

Note that each element in $\mathbb{M}_k$ is an equivalence class, however it is denoted by $h(X, Y) \otimes \{\alpha, \beta\} \in \mathbb{M}_k$ for an abuse of notation. Also note that the relation above implies that for $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, $\{\alpha, \beta\} = -\{\beta, \alpha\}$ and $\{\alpha, \alpha\} = 0$. Let Γ be a subgroup of $SL_2(\mathbb{Z})$ with a finite index. For a given $h(X, Y) \in \mathbb{Z}[X, Y]_{k-2}$, consider z as the column vector of (X, Y) so that $h(z) := h(X, Y)$. Then, define the action of $GL_2^+(\mathbb{Q})$ on $\mathbb{M}_k \otimes \mathbb{Q}$ as follows: given $h((X, Y)) = h(X, Y) \in \mathbb{Z}[X, Y]_{k-2}$, $\{\alpha, \beta\} \in \mathbb{M}_2$, for $A \in GL_2^+(\mathbb{Q})$

$$A(h(X, Y) \otimes \{\alpha, \beta\}) := h(\det(A)A^{-1}(X, Y)^{\mathrm{T}}) \otimes \{A\alpha, A\beta\},$$

where $(X, Y)^{\mathrm{T}}$ is the transpose of the row vector (X, Y).

For an example, let $A := \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$. Then, we have

$$A(XY \otimes \{0, \infty\}) = (X - Y)(3X - 2Y) \otimes \{1, 2/3\} = (3X^2 - 5XY + 2Y^2) \otimes \{1, 2/3\}.$$

We can relate the actions of $GL_2^+(\mathbb{Q})$ on $P_k(\mathbb{C})$ and $\mathbb{Z}[X,Y]_{k-2}$ as follows. Let $h(X,Y) \in \mathbb{Z}[X,Y]_{k-2}$ and $A \in GL_2^+(\mathbb{Q})$. Write

$$h(X,Y) := \sum_{j=0}^{k-2} a_j X^j Y^{k-2-j} \text{ and } A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, by replacing $X = z$ and $Y = 1$ we have

$$\begin{aligned} A(h(z,1)) &= \sum_{0 \le j \le k-2} a_j (dz - b)^j (-cz + a)^{k-2-j} \\ &= (-cz+a)^{k-2} \sum_{0 \le j \le k-2} a_j (\det(A) A^{-1} z)^j \\ &= (-cz+a)^{k-2} h(\det(A) A^{-1} z, 1) \\ &= \rho^{2-k}(\det(A) A^{-1}, z) h(\det(A) A^{-1} z, 1) \\ &= (h|(\det(A)A^{-1}))(z,1) \\ &= (h|A^{-1})(z,1). \end{aligned} \tag{2.9}$$

Note that in the last line, $h(z,1)$ is considered an element in $P_k(\mathbb{C})$.

Definition. Let $k \ge 2$ be an integer and Γ be a congruence subgroup. Denfine

$$\mathbb{M}_k^\Gamma := \mathbb{M}_k / \langle Ax - x \mid A \in \Gamma, x \in \mathbb{M}_k \rangle.$$

Then, define the space of modular symbols of weight k and for Γ as

$$\mathbb{M}_k(\Gamma) := \mathbb{M}_k^\Gamma / (\mathbb{M}_k^\Gamma)_{\text{tor}}.$$

Again, each element in $\mathbb{M}_k(\Gamma)$ is an equivalence class, however it is denoted by $h(X,Y) \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma)$ for an abuse of notation. Note that if $\Gamma_1(N) \in \Gamma$, then $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \Gamma$ for every $r \in \mathbb{Z}$. Therefore, for $h(z) \otimes \{0, a/b\}, h(z) \otimes \{0, a'/b\} \in \mathbb{M}_k(\Gamma)$ with $b \neq 0$,

$$h(z) \otimes \{0, a/b\} = h(z) \otimes \{0, a'/b\} \text{ if } a \equiv a' \bmod b.$$

We extend the definition of $\mathbb{M}_k(\Gamma)$ over $\mathbb{Z}$ to a ring R by taking tensor product as

$$\mathbb{M}_k(\Gamma; R) := \mathbb{M}_k(\Gamma) \otimes_{\mathbb{Z}} R.$$

Recall the diamond operator $\langle d \rangle$ on $M_k(\Gamma_1(N))$. Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character of order l and $\mathbb{Z}[\varepsilon])$ is the ring generated by the values of ε. We define the diamond operator $\langle d \rangle$ on $M_k(\Gamma_1(N); \mathbb{Z}[\varepsilon])$ with $(d, N) = 1$ in the same manner as for $h(z) \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma_1(N); \mathbb{Z}[\varepsilon])$

$$\langle d \rangle (h(z) \otimes \{\alpha, \beta\}) := h(z) \otimes \{d\alpha, d\beta\}.$$

Then, define

$$\mathbb{M}_k^{\Gamma_0(N)}(N, \varepsilon) := \mathbb{M}_k(\Gamma_1(N); \mathbb{Z}[\varepsilon]) / \langle \langle d \rangle x - \varepsilon(d)x \mid d \in (\mathbb{Z}/N\mathbb{Z})^\times, x \in \mathbb{M}_k(\Gamma_1(N); \mathbb{Z}[\varepsilon]) \rangle$$

define the space of modular symbols $\mathbb{M}_k(N, \varepsilon)$ of weight $k \geq 2$ for $\Gamma_1(N)$ with nebentypus ε, also a torsion-free $\mathbb{Z}[\varepsilon]$-module as

$$\mathbb{M}_k(N, \varepsilon) := \mathbb{M}_k^{\Gamma_0(N)}(N, \varepsilon) / (\mathbb{M}_k^{\Gamma_0(N)}(N, \varepsilon))_{\mathrm{tor}}.$$

Now, we define the subspace of $\mathbb{M}_k(\Gamma)$ called the cuspidal subspace associated with $S_k(\Gamma)$ via so called the boundary map. Let $k \geq 2$ be an integer and Γ be a congruence subgroup. Similary to $\mathbb{M}_k$ we define the free abelian group

$$\mathbb{B} := \langle \{\alpha\} \mid \alpha \in \mathbb{P}^1(\mathbb{Q}) \rangle \text{ and } \mathbb{B}_k := \mathbb{Z}[X, Y]_{k-2} \otimes_{\mathbb{Z}} \mathbb{B}.$$

Again, define the action of $GL_2^+(\mathbb{Q})$ on $\mathbb{B}_k \otimes \mathbb{Q}$ by for $A \in GL_2^+(\mathbb{Q})$ and $h(X, Y) \otimes \{\alpha\} \in \mathbb{B}_k$,

$$A(h(X, Y) \otimes \{\alpha\}) := h(\det(A)A^{-1}(X, Y)^{\mathrm{T}}) \otimes \{A\alpha\}.$$

Define $\mathbb{B}_k^\Gamma := \mathbb{B}_k / \langle Ax - x \mid A \in \Gamma, x \in \mathbb{B}_k \rangle$ and $\mathbb{B}_k(\Gamma) := \mathbb{B}_k^\Gamma / (\mathbb{B}_k^\Gamma)_{\mathrm{tor}}$. Then, the boundary map $\partial : \mathbb{M}_k(\Gamma) \to \mathbb{B}_k(\Gamma)$ is defined and linearly extended by

$$\partial(h(z) \otimes \{\alpha, \beta\}) := h(z) \otimes (\{\beta\} - \{\alpha\}) = h(z) \otimes \{\beta\} - h(z) \otimes \{\alpha\}.$$

Definition. Let $k \geq 2$ be an integer and Γ be a congruence subgroup. Then, the space of cuspidal modular symbols $\mathbb{S}_k(\Gamma)$ is defined as a subspace of $\mathbb{M}_k(\Gamma)$ by

$$\mathbb{S}_k(\Gamma) := \ker \partial.$$

$\mathbb{M}_k(N) := \mathbb{M}_k(\Gamma_0(N))$ and $\mathbb{S}_k(N) := \mathbb{S}_k(\Gamma_0(N))$ also can be interpreted as holomogy groups on the modular curve $X_0(N)$. Refer to Chapter 3 and 8 in [Ste07] and [Man72] for the holomogical aspects of the space of modular symbols. Roughly speaking, they can be considered as the spaces of the linear combinations of geodesics with end points in $\mathbb{P}^1(\mathbb{Q})$ modulo the relations mentioned above.

2.3 Hecke operators and pairing on $\mathbb{M}_k(\Gamma)$

In this section, we define a pairing on $\mathbb{M}_k(\Gamma_1(N))$ of weight $k \geq 2$ for a congruence subgroup $\Gamma_1(N)$ and show the duality of the signed subspaces of $\mathbb{S}_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ (or $\overline{S}_k(\Gamma_1(N))$ defined in (1.3)). As on $M_k(\Gamma_1(N))$, there exist the Hecke operators acting on $\mathbb{M}_k(\Gamma_1(N))$ in the exactly same manner as Section 1.3.

Definition. Let $k \geq 2$ and $N \geq 1$ be integers and ε be a Dirichlet character modulo N. Let p be a prime. Then, the Hecke operator T_p at p is defined as for $h \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma_1(N))$,

$$T_p h \otimes \{\alpha, \beta\} := \Gamma_1(N) A_p \Gamma_1(N)(h \otimes \{\alpha, \beta\})$$

where $A_p := \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Moreover, a Hecke operator T_p on $\mathbb{M}_k(N, \varepsilon)$ is defined as a linear map for $h \otimes \{\alpha, \beta\} \in \mathbb{M}_k(N, \varepsilon)$,

$$T_p(h(z) \otimes \{\alpha, \beta\}) := \sum_{0 \leq r \leq p-1} \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix} (h(z) \otimes \{\alpha, \beta\}) + \varepsilon(p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} (h(z) \otimes \{\alpha, \beta\}). \tag{2.10}$$

Moreover, T_1 is defined as just the trivial map.

This is well-defined, i.e. it is equivalent for a choice of right coset representatives for the double coset in the same way for $M_k(\Gamma_1(N))$. Thus, there exists the commutative Hecke algebra $\mathbb{T} := \mathbb{Z}[T_1, T_2, \ldots] \subset \mathrm{End}(\mathbb{M}_k(\Gamma_1(N)))$. For a composite n, Merel defined T_n on $\mathbb{M}_k(\Gamma_1(N))$ by computing a concrete finite right coset R_n representative set of $\Gamma_1(N)$ in Δ_n for the double coset $\Gamma_1(N)\Delta_n\Gamma_1(N)$ where

$$\Delta_n := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & n \end{pmatrix} \bmod N\}. \tag{2.11}$$

Refer to Lemma 1 of §2.3 in [Mer94] for the detailed arguments on this.

Definition. Let $k \geq 2$ and $N \geq 1$ be integers. Then, the integration pairing is a bilinear map

$$\langle \cdot, \cdot \rangle : (S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))) \times \mathbb{M}_k(\Gamma_1(N)) \to \mathbb{C}$$

defined by

$$\langle f + g, h(X, Y) \otimes \{\alpha, \beta\} \rangle := 2\pi i (\int_\alpha^\beta f(z) h(z, 1) dz + \int_\alpha^\beta g(z) h(\overline{z}, 1) d\overline{z}). \tag{2.12}$$

Let $f \in S_k(\Gamma_1(N))$ and $g \in \overline{S}_k(\Gamma_1(N))$. Let $A \in \Gamma_1(N)$ and $h \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma_1(N))$. Then,

$$\begin{aligned}\langle f+g, A(h \otimes \{\alpha,\beta\})\rangle &= 2\pi i(\int_{A\alpha}^{A\beta} f(z)h(A^{-1}z,1)dz + \int_{A\alpha}^{A\beta} g(z)h(A^{-1}\overline{z},1)d\overline{z}) \\ &= 2\pi i(\int_{\alpha}^{\beta} f(z)h(z,1)dz + \int_{\alpha}^{\beta} g(z)h(\overline{z},1)d\overline{z}) \\ &= \langle f+g, h \otimes \{\alpha,\beta\}\rangle.\end{aligned}$$

The equality in the middle line above follows from the assumption that f and g satisfies the automorphy condition for $\Gamma_1(N)$. Therefore, the integration pairing is well-defined. Moreover, this integration pairing can be extended to any finite index subgroup of $SL_2(\mathbb{Z})$, refer to Chapter 8 in [Ste07]. However, since our interest can be restrict to $M_k(\Gamma_1(N))$, the above definition is enough to be considered.

Consider $S_k(\Gamma_1(N))$ and $\overline{S}_k(\Gamma_1(N))$ as finite dimensional $\mathbb{C}$-vector spaces. In general, the integraion pairing $\langle\cdot,\cdot\rangle$ for $\mathbb{M}_k(\Gamma_1(N),\mathbb{C})$ is degenerate pairing over $\mathbb{C}$; there exists the subspace of $\mathbb{M}_k(\Gamma_1(N),\mathbb{C})$ called the space of Eisenstein modular symbols such that the pairings vanish at the elements in that subspace. However, Shokurov [Sho81] showed that the integration pairing restricted to $\mathbb{S}_k(\Gamma_1(N),\mathbb{C})$ is non-degenerate.

Theorem (Shokurov)**.** The integration pairing $\langle,\rangle$ restricted to

$$\langle\cdot,\cdot\rangle : (S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))) \times \mathbb{S}_k(\Gamma_1(N),\mathbb{C}) \to \mathbb{C}$$

is a non-degenerate pairing over $\mathbb{C}$.

Proof. This theorem is a restatement of the Shokurov's theorem 0.2 in [Sho81] consisting of the homological groups on the modular curve for $\Gamma_1(N)$ in terms of $\mathbb{S}_k(\Gamma_1(N),\mathbb{C})$. Also, see §1.5 in [Mer94] for the remarks on his proof. □

It implies the dimensional relation between $\mathbb{S}_k(\Gamma_1(N),\mathbb{C})$ and $S_k(\Gamma_1(N))$.

Corollary 2.5. Let $k \geq 2$ and $N \geq 1$ be integers. Then, we have

$$\dim_{\mathbb{C}} \mathbb{S}_k(\Gamma_1(N),\mathbb{C}) = 2\dim_{\mathbb{C}} S_k(\Gamma_1(N)) = 2\dim_{\mathbb{C}} \overline{S}_k(\Gamma_1(N)).$$

Proof. Since $\dim_{\mathbb{C}} S_k(\Gamma_1(N)) = \dim_{\mathbb{C}} \overline{S}_k(\Gamma_1(N))$, the proof immediately follows from the theorem above. □

The integration pairing possesses a nice property that it is compatible with the Hecke operators T_n for each integer $n \geq 1$. Note that T_n acts on $f_1+f_2 \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$ by

$$T_n(f_1 + f_2) = (T_n f_1 + T_n f_2) \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N)).$$

Proposition 2.6. Let $k \geq 2$ and $N \geq 1$ be integers. Let T_n be a Hecke operator for an integer $n \geq 1$. Given $f := f_1 + f_2 \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$ and $h \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma_1(N))$,

$$\langle T_n f, h \otimes \{\alpha, \beta\}\rangle = \langle f, T_n(h \otimes \{\alpha, \beta\})\rangle$$

for every integer $n \geq 1$.

Proof. Choose $f := f_1 + f_2 \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$ and $h \otimes \{\alpha, \beta\} \in \mathbb{M}_k(\Gamma_1(N))$. When $n = 1$, T_1 is trivial, hence it is clear. Suppose that $n \geq 2$. Consider the finite set R_n of right coset representatives of $\Gamma_1(N)$ in Δ_n shown in Lemma 1 of §2.3 in [Mer94]. Then, we have

$$\begin{aligned}
\langle T_n f, h \otimes \{\alpha, \beta\}\rangle &= 2\pi i\Big(\int_\alpha^\beta (T_n f_1(z))h(z,1)dz + \int_\alpha^\beta (T_n f_2(z))h(\overline{z},1)d\overline{z}\Big) \\
&= 2\pi i\Big(\int_\alpha^\beta \big(\sum_{r\in R_n}(f_1|r)(z)\big)h(z,1)dz + \int_\alpha^\beta \big(\sum_{r\in R_n}(f_2|r)(z)\big)h(\overline{z},1)d\overline{z}\Big) \\
&= 2\pi i \sum_{r\in R_n}\Big(\int_\alpha^\beta (f_1|r)(z)h(z,1)dz + \int_\alpha^\beta (f_2|r)(z)h(\overline{z},1)d\overline{z}\Big).
\end{aligned}$$

Consider $h(z, 1)$ as a polynomial $\in P_k(\mathbb{C})$ and use the equality (2.1). Then, for each $r \in R_n$,

$$\begin{aligned}
\int_\alpha^\beta (f_1|r)(z)h(z,1)dz &= \int_\alpha^\beta (f_1|r)(z)((h|r^{-1})|r)(z,1)dz \\
&= \int_\alpha^\beta f_1(rz)(h|r^{-1})(rz,1)d(rz) \text{ by (2.1)} \\
&= \int_{r\alpha}^{r\beta} f_1(z)(h|r^{-1})(z,1)dz \text{ by changing the variable} \\
&= \int_{r\alpha}^{r\beta} f_1(z)(r(h(z,1)))dz \text{ by (2.9)} \\
&= \langle f_1, r(h\otimes\{\alpha,\beta\})\rangle.
\end{aligned}$$

The same arguments hold for the integral against f_2 and it completes the proof. $\square$

Define the linear map ι on $S_k(\Gamma_1(N))\oplus\overline{S}_k(\Gamma_1(N))$ by $\iota(f_1(z)+f_2(z)) := (f_1(-\overline{z})+f_2(-\overline{z}))$. Clearly, ι is an involution and an isomorphism. Moreover, define $\iota^* : \mathbb{S}_k(\Gamma_1(N),\mathbb{C}) \to \mathbb{S}_k(\Gamma_1(N),\mathbb{C})$ by

$$\iota^*(h(X,Y)\otimes\{\alpha,\beta\}) = -h(-X,Y)\otimes\{-\alpha,-\beta\}.$$

Then, ι^* is an involution. We can define the eigenspaces of $\mathbb{S}_k(\Gamma_1(N),\mathbb{C})$ with the eigenvalues +1 and -1 under ι^* by $\mathbb{S}_k(\Gamma_1(N))^+$ and $\mathbb{S}_k(\Gamma_1(N))^-$, respectively. Observe that for $f := (f_1+f_2) \in S_k(\Gamma_1(N))\oplus\overline{S}_k(\Gamma_1(N))$ and $h\otimes\{\alpha,\beta\}\in\mathbb{S}_k(\Gamma_1(N))$

$$\begin{aligned}\langle\iota(f), h\otimes\{\alpha,\beta\}\rangle &= 2\pi i\Big(\int_\alpha^\beta f_1(-\overline{z})h(z,1)dz + \int_\alpha^\beta f_2(-\overline{z})h(\overline{z},1)d\overline{z}\Big)\\ &= 2\pi i\Big(-\int_{-\alpha}^{-\beta} f_1(z)h(-\overline{z},1)d\overline{z} - \int_{-\alpha}^{-\beta} f_2(z)h(-z,1)dz\Big)\\ &= \langle f, \iota^*(h\otimes\{\alpha,\beta\})\rangle.\end{aligned}$$

Therefore, ι^* is the adjoint operator of ι with respect to $\langle\cdot,\cdot\rangle$.

The following corollary directly implied from the Shokurov's theorem above, Proposition 2.6 and the adjointness of ι and ι^* shows that $S_k(\Gamma_1(N))$ and $\overline{S}_k(\Gamma_1(N))$ are dual to $\mathbb{S}_k(\Gamma_1(N))^+$ and $\mathbb{S}_k(\Gamma_1(N))^-$, respectively, over $\mathbb{C}$.

Corollary 2.7. Let $k\geq 2$ and $N\geq 1$ be integers. Then, the integration pairing $\langle\cdot,\cdot\rangle$ induces non-degenerate Hecke compatible pairings over $\mathbb{C}$

$$S_k(\Gamma_1(N))\times\mathbb{S}_k(\Gamma_1(N))^+\to\mathbb{C} \text{ and } \overline{S}_k(\Gamma_1(N))\times\mathbb{S}_k(\Gamma_1(N))^-\to\mathbb{C}.$$

Let $f\in S_k(\Gamma_1(N))$, $h(z)\in P_k(\mathbb{C})$ and $\alpha,\beta\in\mathbb{P}^1(\mathbb{Q})$ such that $h\otimes\{\alpha,\beta\}\in\mathbb{M}_k(\Gamma_1(N))$. Consider $h(z)$ as $h(X,Y)\in\mathbb{Z}[X,Y]_{k-2}$ with $X=z$ and $Y=1$. Then, we can obtain the relationship between $\langle\cdot,\cdot\rangle$ and Φ as

$$\begin{aligned}\langle f, h\otimes\{\alpha,\beta\}\rangle &= 2\pi i\int_\alpha^\beta f(z)h(z,1)dz\\ &= 2\pi i\Big(\int_{i\infty}^\beta f(z)h(z,1)dz - \int_{i\infty}^\alpha f(z)h(z,1)dz\Big)\\ &= \Phi(f,h,\beta)-\Phi(f,h,\alpha).\end{aligned}\tag{2.13}$$

Here, $\langle\cdot,\cdot\rangle$ is restricted to $S_k(\Gamma_1(N))$ and the Cauchy theorem is used.

2.4 Nonvanishing theorems of $L(f, n, \chi)$

Throughout this section, let $f \in S_k(N, \varepsilon)$ be a primitive form and let l be an odd prime and χ be a primitive dirichlet character of order l and conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. In this section, we show some nonvanishing thoerems for $L(f, n, \chi)$ for $1 \le n \le k-1$ where $k > 2$ with $2|k$. For $k = 2$, Fearnley, Kisilevsky and Kuwata showed the nonvanishing theorem 1.5 on $L(f, 1, \chi) = L(E, 1, \chi)$ for a modular elliptic curve $E := E/\mathbb{Q}$ in [FKK12].

For non-central critical integers when $k > 2$, i.e. integers $1 \le n \le k-1$ with $n \ne k/2$ if $2|k$ and $n \ne (k \pm 1)/2$ if $2 \nmid k$, we can show that $L(f, n, \chi) \ne 0$ by using the Euler product and the Deligne bound on the Fourier coefficients a_n of f. Note that it is a well-known result, see [IK04] for an example, however, we reproduce the proof for these specific settings.

Lemma 2.8. Let $P := \prod_{n \ge 1}(1 + s_n)$ where $s_n \in \mathbb{C}$ for every $n \ge 1$. Suppose that $S := \sum_{n \ge 1} |s_n|$ converges. Then, P converges as well. Moreover, $P = 0$ if and only if there exists some $n \ge 1$ such that $s_n = -1$.

Proof. Consider the partial product $P_N := \prod_{1 \le n \le N}(1 + s_n)$. Note that $1 + x \le e^x$ for $x \ge 0$. Thus, for every $N \ge 1$, $\prod_{1 \le n \le N}(1 + |s_n|) \le e^S$. Choose a real number $\varepsilon > 0$. Then, there exists an integer $N_\varepsilon \ge 1$ such that $\sum_{n \ge N_\varepsilon} |s_n| \le \varepsilon/(2e^S) < 1/2$. For $M \ge N \ge N_\varepsilon$, $|P_M - P_N| = |P_N||\prod_{n=N}^{M}(1 + s_n) - 1|$. Observe that

$$\begin{aligned}
|\prod_{N<n\le M}(1+s_n) - 1| &= |\sum_{\substack{I \subset \{N,\dots,M\} \\ I \ne \emptyset}} \prod_{j \in I} s_j| \\
&\le \sum_{\substack{I \subset \{N,\dots,M\} \\ I \ne \emptyset}} \prod_{j \in I} |s_j| \text{ by the triangle inequality} \\
&= \prod_{N<n\le M}(1+|s_n|) - 1 \le \exp(\sum_{N<n\le M}|s_n|) - 1 \\
&\le \exp(\sum_{n \ge N_\varepsilon}|s_n|) - 1 \le \exp(\frac{\varepsilon}{2e^S}) - 1 < \frac{\varepsilon}{e^S}
\end{aligned}$$

where I is a non-empty subset of $\{N, N+1, \dots, M\}$. The last inequality follows from the fact that if $0 < x < 1/2$, then $e^x - 1 < 2x$. Therefore, we have

$$|P_M - P_N| = |P_N||\prod_{n=N}^{M}(1+s_n) - 1| < \varepsilon.$$

Therefore, $\{P_n\}_{n\geq 1}$ is a Cauchy sequence, hence P converges. Moreover, take $N = N_\varepsilon$ in the above equality. Then, for every $M \geq N_\varepsilon$,

$$|P_{N_\varepsilon}| - |P_M| \leq |P_M - P_{N_\varepsilon}| = |P_{N_\varepsilon}|\left|\prod_{n\geq N_\varepsilon}^{M}(1+s_n) - 1\right| \leq |P_{N_\varepsilon}|(\exp(\frac{\varepsilon}{2e^S}) - 1).$$

Therefore, $|P_M| \geq (2-\exp(\varepsilon/(2e^S)))|P_{N_\varepsilon}|$, hence we have $P = 0$ if and only if $P_{N_\varepsilon} = 0$ if and only if there exsits $n \geq 1$ such that $s_n = -1$. It completes the proof. $\square$

Proposition 2.9. Let $f \in S_k(N, \varepsilon)$ be a primitive form of weight $k > 2$ and l be an odd prime. Also, let χ be a primitive dirichlet character of order l and conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. Then, $L(f, n, \chi) \neq 0$ for integers $1 \leq n \leq k-1$ with $n \neq k/2$ if $2|k$ and $n \neq (k \pm 1)/2$ if $2 \nmid k$.

Proof. Let a_n be the n-th Fourier coefficient of f. Then, recall the Deligne bound $|a_n| \leq d(n)n^{(k-1)/2}$ for every $n \geq 1$. In particular, we have $|a_p| \leq 2p^{(k-1)/2}$ for each prime p. Since $d(n) \ll n^\varepsilon$ for every $\varepsilon > 0$. Then, $L(f, s, \chi) = \sum_{n\geq 1}\chi(n)a_n/n^s$ absolutely converges for $\mathrm{Re}(s) > (k+1)/2 + \varepsilon$ since

$$|L(f, s, \chi)| \leq \sum_{n\geq 1}\frac{|\chi(n)||a_n|}{n^{\mathrm{Re}(s)}} \ll n^{\frac{k-1}{2}-\mathrm{Re}(s)+\varepsilon}.$$

Then, the Euler product of $L(f, s, \chi)$ is definable for $\mathrm{Re}(s) > (k+1)/2 + \varepsilon$ as

$$\begin{aligned} L(f, s, \chi) &= \prod_{p|N}\left(1 - \frac{a_p}{p^s}\right)^{-1}\prod_{p\nmid N}\left(1 - \frac{\chi(p)a_p}{p^s} + \frac{\varepsilon(p)\chi^2(p)p^{k-1}}{p^{2s}}\right)^{-1} \\ &= \prod_{p|N}\left(1 - \frac{a_p}{p^s}\right)^{-1}\prod_{p\nmid N}\left(\left(1 - \frac{\chi(p)\alpha_p}{p^s}\right)\left(1 - \frac{\chi(p)\beta_p}{p^s}\right)\right)^{-1} \end{aligned}$$

where $\alpha_p, \beta_p \in \mathbb{C}$ such that $\alpha_p + \beta_p = a_p$, $\alpha_p\beta_p = \varepsilon(p)p^{k-1}$ and $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$. The existences of such α_p and β_p follows the work of Deligne in [Del74]. Suppose that $2|k$. Then, for integers n with $k/2 + 1 \leq n \leq k-1$, $L(f, n, \chi)$ admits the Euler product above at $s = n$. Then, by using the Deligne bound we obtain

$$\left|\frac{a_p}{p^n}\right| \leq 2p^{\frac{k-1}{2}-n} \leq 2p^{-\frac{3}{2}} < 1 \text{ and } \left|\frac{\alpha_p}{p^n}\right| = \left|\frac{\beta_p}{p^n}\right| < p^{-\frac{3}{2}} < 1.$$

By Lemma 2.8, for $k/2 + 1 \leq n \leq k-1$, $L(f, n, \chi) \neq 0$. For the remaining ciritical integers $1 \leq n \leq k/2 - 1$, using the functional equation of $L(f, s, \chi)$, the desired results immediately follows for $2|k$. If $2 \nmid k$, then the central point in the critical strip is a rational number $k/2$. Then, by the same arguments for $2|k$, the proof is immediate. $\square$

From now on, fix a primitive form $f \in S_k(N, \varepsilon)$ of weight k and let l be an odd prime and χ be a primitive dirichlet character of order l and conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. Also, let n be an integer such that $1 \leq n \leq k-1$. Notice that by the relation (2.13) and the Cauchy theorem, $L^{\mathrm{alg}}(f, n, \chi)$ can be written in two ways, using Φ and $\langle \cdot, \cdot \rangle$, as

$$\begin{aligned} L^{\mathrm{alg}}(f, n, \chi) &:= \sum_{a \bmod \mathfrak{f}_\chi} \frac{\overline{\chi}(a)}{\Omega^{\pm}} (\Phi(f, (\mathfrak{f}_\chi z + a)^{n-1}, -\frac{a}{\mathfrak{f}_\chi}) \pm \Phi(f, (\mathfrak{f}_\chi z - a)^{n-1}, \frac{a}{\mathfrak{f}_\chi})) \\ &= \sum_{a \bmod \mathfrak{f}_\chi} \frac{\overline{\chi}(a)}{\Omega^{\pm}} (\langle f, (\mathfrak{f}_\chi z + a)^{n-1} \otimes \{\infty, -\frac{a}{\mathfrak{f}_\chi}\} \rangle \pm \langle f, (\mathfrak{f}_\chi z - a)^{n-1} \otimes \{\infty, \frac{a}{\mathfrak{f}_\chi}\} \rangle) \end{aligned}$$

For simplicity, for an integer $m > 0$, $a \in \mathbb{Z}/m\mathbb{Z}$ and $1 \leq n \leq k-1$, denote

$$c^{\pm}(a, m, n; f) := \frac{1}{\Omega^{\pm}} (\langle f, (\mathfrak{f}_\chi z + a)^{n-1} \otimes \{\infty, -\frac{a}{\mathfrak{f}_\chi}\} \rangle \pm \langle f, (\mathfrak{f}_\chi z - a)^{n-1} \otimes \{\infty, \frac{a}{\mathfrak{f}_\chi}\} \rangle)$$

so that we have

$$L^{\mathrm{alg}}(f, n, \chi) = \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a) c^{\pm}(a, \mathfrak{f}_\chi, n; f). \tag{2.14}$$

Since the order of χ is odd, $\mathrm{sign}(\chi) = \chi(-1) = 1$. Thus, from the Shimura theorem in Section 2.1, we choose Ω^+ if n is odd and Ω^- otherwise. Let T_p be the Hecke operator at a prime p. Then, observe that

$$\begin{aligned} &T_p((\mathfrak{f}_\chi z \pm a)^{n-1} \otimes \{\infty, \mp\frac{a}{\mathfrak{f}_\chi}\}) \\ &= (\sum_{r=0}^{p-1} \begin{pmatrix} 1 & r \\ 0 & p \end{pmatrix} + \varepsilon(p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})((\mathfrak{f}_\chi z \pm a)^{n-1} \otimes \{\infty, \mp\frac{a}{\mathfrak{f}_\chi}\}) \\ &= \sum_{r=0}^{p-1} (\mathfrak{f}_\chi(pz - r) \pm a)^{n-1} \otimes \{\infty, \mp\frac{a \mp r\mathfrak{f}_\chi}{p\mathfrak{f}_\chi}\} + \varepsilon(p) p^{k-2} (\mathfrak{f}_\chi(\frac{z}{p}) \pm a)^{n-1} \otimes \{\infty, \mp\frac{pa}{\mathfrak{f}_\chi}\} \\ &= \sum_{r=0}^{p-1} (\mathfrak{f}_\chi pz \pm (a \mp r\mathfrak{f}_\chi))^{n-1} \otimes \{\infty, \mp\frac{a \mp r\mathfrak{f}_\chi}{p\mathfrak{f}_\chi}\} + \varepsilon(p) p^{k-n-1} (\mathfrak{f}_\chi z \pm pa)^{n-1} \otimes \{\infty, \mp\frac{pa}{\mathfrak{f}_\chi}\}. \end{aligned}$$

Note that in the last term of middle equality we take p^{k-2} because we need the image of the homogeneous polynomial under the action to be in $\mathbb{Z}[X, Y]_{k-2}$. Then, acting the Hecke oeprator T_p for a prime p on f, we have

$$\begin{aligned} a_p c^{\pm}(a, \mathfrak{f}_\chi, n; f) &= c^{\pm}(a, \mathfrak{f}_\chi, n; T_p f) \\ &= \sum_{r=0}^{p-1} c^{\pm}(a \mp r\mathfrak{f}_\chi, p\mathfrak{f}_\chi, n; f) + \varepsilon(p) p^{k-n-1} c^{\pm}(pa, \mathfrak{f}_\chi, n; f) \end{aligned} \tag{2.15}$$

where a_p is the p-th Fourier coefficient of f.

For positive integers m and t and a Hecke operator T_p, denote

$$S_m^{\pm}(t,n) := \sum_{\substack{a \bmod t \\ (a,m)=1}} c^{\pm}(a,t,n;f)$$
$$S_m^{\pm}(t,n)|T_p := \sum_{\substack{a \bmod t \\ (a,m)=1}} c^{\pm}(a,t,n;T_p f).$$

Let $\mathfrak{l}$ be a prime dividing l in $\mathbb{Q}(\zeta)$, the number field adjoining a primitive l-th root of unity ζ. Then, for an integer a, $\chi(a) \equiv 1 \bmod \mathfrak{l}$ if $(a, \mathfrak{f}_\chi) = 1$ and $\chi(a) = 0$ otherwise. Therefore, we have

$$L^{\mathrm{alg}}(f,n,\chi) = \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a) c^{\pm}(a,\mathfrak{f}_\chi,n;f) \equiv \sum_{\substack{a \bmod \mathfrak{f}_\chi \\ (a,\mathfrak{f}_\chi)=1}} c^{\pm}(a,\mathfrak{f}_\chi,n;f) \equiv S_{\mathfrak{f}_\chi}^{\pm}(\mathfrak{f}_\chi,n) \bmod \mathfrak{l}.$$

Now, we obtain the congruence relation between an algebraic part and its twist at the central point for a primitive form of even weight.

Proposition 2.10. Let $N \geq 1$ and $k \geq 2$ be integers with $2|k$ and l be an odd prime. Also, let $f \in S_k(N,\varepsilon)$ be a primitive form where the nebentypus ε is the trivial, a quadratic with $\varepsilon(-1) = 1$ or a character of order l. Let χ be a primitive Dirichlet character of conductor $m := \mathfrak{f}_\chi$ and order dividing l. If χ_p is a primitive Dirichlet character of order l and a prime conductor $\mathfrak{f}_{\chi_p} = p$ with $(p,m) = 1$, then

$$L^{\mathrm{alg}}(f,k/2,\chi\chi_p) \equiv (a_p - 1 - \varepsilon(p)) L^{\mathrm{alg}}(f,k/2,\chi) \bmod \mathfrak{l}.$$

where a_p is the p-th Fourier coefficient of f. If χ_{l^2} is a primitive Dirichlet character of order l and conductor $\mathfrak{f}_{\chi_{l^2}} = l^2$ with $(l,m) = 1$, then we have

$$L^{\mathrm{alg}}(f,k/2,\chi\chi_{l^2}) \equiv \begin{cases} (a_l - 1)(a_l - \varepsilon(l)) L^{\mathrm{alg}}(f,k/2,\chi) & \text{if } k = 2 \\ a_l^2 L^{\mathrm{alg}}(f,k/2,\chi) & \text{if } k \geq 4 \end{cases} \bmod \mathfrak{l}.$$

where a_l is the l-th Fourier coefficient of f.

Proof. For abuse of notation, denote $c^{\pm}(a,m) := c^{\pm}(a,m,k/2;f)$ and $S_m^{\pm}(t) := S_m^{\pm}(t,k/2)$. Note that $L^{\mathrm{alg}}(f,k/2,\chi) \equiv S_m^{\pm}(m) \bmod \mathfrak{l}$ and $L^{\mathrm{alg}}(f,k/2,\chi\chi_p) \equiv S_{pm}^{\pm}(pm) \bmod \mathfrak{l}$. By equation (2.15), we have

$$\begin{aligned} a_p S_m^{\pm}(m) &= S_m^{\pm}(m)|T_p \\ &= \sum_{\substack{a \bmod m \\ (a,m)=1}} (\sum_{r=0}^{p-1} c^{\pm}(a \mp rm, pm) + \varepsilon(p) p^{\frac{k}{2}-1} c^{\pm}(pa, m)) \\ &= \sum_{\substack{b \bmod pm \\ (b,m)=1}} c^{\pm}(b, pm) + \varepsilon(p) p^{\frac{k}{2}-1} \sum_{\substack{a \bmod m \\ (a,m)=1}} c^{\pm}(a, m) \\ &= S_m^{\pm}(pm) + \varepsilon(p) p^{\frac{k}{2}-1} S_m^{\pm}(m). \end{aligned} \tag{2.16}$$

Note here that the critical integer $n = k/2$. Moreover, we have

$$\begin{aligned} S_m^{\pm}(pm) &= \sum_{\substack{a \bmod pm \\ (a,m)=1}} c^{\pm}(a, pm) \\ &= \sum_{\substack{a \bmod pm \\ (a,pm)=1}} c^{\pm}(a, pm) + \sum_{\substack{a \bmod pm \\ (a,pm)=p}} c^{\pm}(a, pm) \\ &= S_{pm}^{\pm}(pm, n) + p^{\frac{k}{2}-1} \sum_{\substack{a \bmod m \\ (a,m)=1}} c^{\pm}(a, m) \\ &= S_{pm}^{\pm}(pm, n) + p^{\frac{k}{2}-1} S_m^{\pm}(m, n). \end{aligned} \tag{2.17}$$

Combining equations (2.16) and (2.17), we have

$$S_{pm}^{\pm}(pm) = (a_p - (1 + \varepsilon(p)) p^{\frac{k}{2}-1}) S_m^{\pm}(m), \tag{2.18}$$

which proves the first statement by noting $p \equiv 1 \bmod l$. Now, suppose that $\mathfrak{f}_\chi = l^2$ with $(l, m) = 1$. Notice that $L^{\mathrm{alg}}(f, k/2, \chi\chi_{l^2}) \equiv S_{l^2m}^{\pm}(l^2 m) \bmod \mathfrak{l}$. Observe that

$$\begin{aligned} S_m^{\pm}(lm, n)|T_l &= \sum_{\substack{a \bmod lm \\ (a,m)=1}} c^{\pm}(a, lm, k/2; T_l f) \\ &= \sum_{\substack{a \bmod lm \\ (a,m)=1}} (\sum_{r=0}^{l-1} c^{\pm}(a \mp rlm, l^2 m) + \varepsilon(l) l^{k-2} \sum_{\substack{a \bmod lm \\ (a,m)=1}} c^{\pm}(a, m)) \\ &= \sum_{\substack{a \bmod l^2 m \\ (a,m)=1}} c^{\pm}(a, l^2 m) + \varepsilon(l) l^{k-1} \sum_{\substack{a \bmod m \\ (a,m)=1}} c^{\pm}(a, m) \\ &= S_m^{\pm}(l^2 m) + \varepsilon(l) l^{k-1} S_m^{\pm}(m). \end{aligned} \tag{2.19}$$

Acting T_l twice on $S_m^{\pm}(m)$ and using the above equation and using (2.19), we have

$$\begin{aligned} a_l^2 S_m^{\pm}(m) &= (S_m^{\pm}(m)|T_l)|T_l = S_m^{\pm}(lm)|T_l + \varepsilon(l) l^{k/2-1} S_m^{\pm}(m)|T_l \\ &= S_m^{\pm}(l^2 m) + \varepsilon(l) l^{k-1} S_m^{\pm}(m) + \varepsilon(l) l^{k/2-1} S_m^{\pm}(lm) + \varepsilon^2(l) l^{k-2} S_m^{\pm}(m) \\ &= S_m^{\pm}(l^2 m) + \varepsilon(l) l^{k/2-1} S_m^{\pm}(lm) + \varepsilon(l) l^{k-2}(l + \varepsilon(l)) S_m^{\pm}(m). \end{aligned}$$

Again, by equation (2.17) for l, we have

$$S_m^{\pm}(lm) = S_{lm}^{\pm}(lm) + l^{\frac{k}{2}-1} S_m^{\pm}(m)$$

and

$$\begin{aligned} S_m^{\pm}(l^2 m) &= \sum_{\substack{a \bmod l^2 m \\ (a,m)=1}} c^{\pm}(a, l^2 m) \\ &= \sum_{\substack{a \bmod l^2 m \\ (a,l^2 m)=1}} c^{\pm}(a, l^2 m) + \sum_{\substack{a \bmod l^2 m \\ (a,l^2 m)=l}} c^{\pm}(a, l^2 m) + \sum_{\substack{a \bmod l^2 m \\ (a,l^2 m)=l^2}} c^{\pm}(a, l^2 m) \\ &= S_{l^2 m}^{\pm}(l^2 m) + l^{\frac{k}{2}-1} S_{lm}^{\pm}(lm) + l^{k-2} S_m^{\pm}(m). \end{aligned}$$

Combining the above three equations and using equation (2.18) for l, we have

$$S_{l^2 m}^{\pm}(l^2 m) = (a_l^2 - (1 + \varepsilon(l)) l^{\frac{k}{2}-1} + \varepsilon(l) l^{k-2}(1 - l)) S_m^{\pm}(m),$$

which completes the proof. $\square$

Proposition 2.11. Let $N \geq 1$ and $k \geq 2$ be integers with $2|k$ and l be an odd prime. Also, let $f \in S_k(N, \varepsilon)$ be a primitive form where the nebentypus ε is the trivial, a quadratic or a character of order l with $\varepsilon(-1) = 1$. Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ and order l. Let a_n be the n-th Fourier coefficient of f. If $l \nmid \mathfrak{f}_\chi$, then

$$L^{\mathrm{alg}}(f, k/2, \chi) \equiv L^{\mathrm{alg}}(f, k/2) \prod_{p | \mathfrak{f}_\chi} (a_p - 1 - \varepsilon(p)) \bmod \mathfrak{l}.$$

If $l | \mathfrak{f}_\chi$ and $k \geq 4$, then

$$L^{\mathrm{alg}}(f, k/2, \chi) \equiv L^{\mathrm{alg}}(f, k/2) a_l^2 \prod_{l \neq p | \mathfrak{f}_\chi} (a_p - 1 - \varepsilon(p)) \bmod \mathfrak{l}.$$

If $l | \mathfrak{f}_\chi$ and $k = 2$, then

$$L^{\mathrm{alg}}(f, k/2, \chi) \equiv L^{\mathrm{alg}}(f, k/2)(a_l - 1)(a_l - \varepsilon(l)) \prod_{l \neq p | \mathfrak{f}_\chi} (a_p - 1 - \varepsilon(p)) \bmod \mathfrak{l}.$$

Proof. As shown in the proof of Lemma 1.3, any primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$ and order l can be written as a product of a primitive character of order l and conductor l^2 or a prime $p|\mathfrak{f}_\chi$ and $p \neq l$. Note that $p \equiv 1 \bmod l$. Taking the trivial character of conductor 1 for χ in the proof of Proposition 2.10 and iterating the results for those primitive characters of conductor dividing $\mathfrak{f}_\chi$, the proof completes. □

The above congruence relations modulo $\mathfrak{l}$ depending on the values of Fourier coefficients of f. Fortunately, we can use the work of Serre, cf. §2 and §6 in [Ser76] using the l-adic Galois represention. Suppose that $f = \sum_{n\geq 1} a_n q^n$ is a simultaneous Hecke eigenform in $S_k(N,\varepsilon)$ where all a_n belong to the ring of integers $\mathcal{O}_K$ of a number field K. Let $\mathfrak{m}$ be a prime ideal of $\mathcal{O}_K$ for which norm is m. Then, there exists a set of rational primes $p \equiv 1 \bmod m$ with a positive arithmetic density in the set of all rational primes such that $a_p \equiv a \bmod \mathfrak{m}$ for a fixed $a \in \mathcal{O}_K/\mathfrak{m}$. Using this argument and Proposition 2.11, we obtain the following theorem and corollary similar to Theorem 3.9 and Theorem A, respectively, of Fearnley, Kisilevsky and Kuwata [FKK12].

Let $\mathcal{P}$ be the set of all rational primes, l be an odd prime and $\mathcal{C}_l$ be the family of all primitive Dirichlet characters of order an odd prime l. Also, let

$$\mathcal{C}_l(X) = \{\chi \in \mathcal{C}_l \mid \chi^l = \chi_0, \chi^n \neq \chi_0 \text{ for } 0 < n < l \text{ and } \mathfrak{f}_\chi \leq X\}$$
$$\mathcal{C}_{l,f}(X) = \{\chi \in \mathcal{C}_l(X) \mid (N, \mathfrak{f}_\chi) = 1 \text{ and } \mathfrak{f}_\chi \leq X\}$$

Theorem 2.12. Let $N \geq 1$ and $k \geq 2$ be integers with $2|k$. Also, let $f \in S_k(N,\varepsilon)$ be a primitive form where the nebentypus ε is the trivial, a quadratic with $\varepsilon(-1) = 1$, or a character of order l. Suppose that $L(f, k/2) \neq 0$. Then, for all but finite number of primes l, there exists a set of primes $\mathcal{P}_l \subset \mathcal{P}$ depending on l of positive density such that $L(f, k/2, \chi) \neq 0$ where χ is a primitive Dirichlet character of order l and conductor $\mathfrak{f}_\chi$ supported on $\mathcal{P}_l$.

By the prime number theorem, the existence of positive density of $\mathcal{P}_l$ with respect to $\mathcal{P}$ immediately implies the following corollary.

Corollary 2.13. With the same assumptions above, we have

$$|\{\chi \in \mathcal{C}_{l,f}(X) \mid \mathfrak{f}_\chi \in \mathcal{P}, L(f, k/2, \chi) \neq 0\}| \gg X/\log X.$$

2.5 Computing the algebraic parts via period mapping

Manin in [Man72] introduced the Manin symbols which are basically same as the modular symbols however much simplier to deal with in computations.

Definition. Let $k \geq 2$ be an integer and Γ be a congruence subgroup. Then, a Manin symbol of weight k for Γ is an element in $\mathbb{M}_k(\Gamma)$ such that

$$[h, \gamma] := \gamma(h(z) \otimes \{0, \infty\}), \text{ where } \gamma \in SL_2(\mathbb{Z}).$$

This is well-defined up to Γ since $[h, \gamma] \in \mathbb{M}_k(\Gamma)$ by the definition i.e. it is invariant under the action of Γ. In other words, if $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$ such that $\Gamma\gamma_1 = \Gamma\gamma_2$, then

$$[h, \gamma_1] = [h, \gamma_2].$$

Let $\mathrm{R}(\Gamma)$ be a set of right coset representatives of a congruence subgroup Γ which is finite. Then, the space of Manin symbols can be generated by the following finite set over $\mathbb{Z}$ with the identity 0, so that it has a finite rank,

$$\{[X^{k-2-i}Y^i, r] \mid 0 \leq i \leq k-2, r \in \mathrm{R}(\Gamma)\}.$$

Proposition 2.14. Let $k \geq 2$ be an integer and Γ be a congruence subgroup. Then, the space of Manin symbols generates $\mathbb{M}_k(\Gamma)$ over $\mathbb{Z}$. Hence, the space of Manin symbols is the same as $\mathbb{M}_k(\Gamma)$.

Proof. By definition, each Manin symbol belongs to $\mathbb{M}_k(\Gamma)$. Thus, if we prove that the space of Manin symbols generates $\mathbb{M}_k(\Gamma)$ over $\mathbb{Z}$, then the last statement follows. Moreover, by the 3-term relation it suffices to prove that each $h(z) \otimes \{0, \alpha\}$ for $\alpha \in \mathbb{P}^1(\mathbb{Q})$ can be written as a linear combination of Manin symbols over $\mathbb{Z}$. Write $\alpha = a/b$ in the lowest terms (if $\alpha = 0($ or $\infty)$ then let $\alpha = 0/1($ or $1/0$, respectively)). Then, consider the continued fraction convergents of a/b:

$$\frac{a_0}{b_0} = \frac{0}{1}, \frac{a_1}{b_1} = \frac{1}{0}, \frac{a_1}{b_1}, \dots, \frac{a_n}{b_n} = \frac{a}{b}$$

Then, $a_j b_{j-1} - a_{j-1} bj = (-1)^{j+1}$ for $1 \leq j \leq n$. Let $\gamma_j = \begin{pmatrix} (-1)^{j+1}a_j & a_{j-1} \\ (-1)^{j+1}b_j & b_{j-1} \end{pmatrix}$.

Then, each $\gamma_j \in SL_2(\mathbb{Z})$ and we have

$$h(z) \otimes \{0, \alpha\} = \sum_{1 \leq j \leq n} \gamma_j((\gamma_j^{-1}h)(z) \otimes \{0, \infty\}) = \sum_{1 \leq j \leq n} [\gamma_j^{-1}h, \gamma_j].$$

For each γ_j, $\gamma_j^{-1}h$ is a finite $\mathbb{Z}$-linear combination of finite generators of $\mathbb{Z}[X,Y]_{k-2}$. Therefore, by the linearity in $\mathbb{Z}[X,Y]_{k-2}$, the proof follows. $\square$

The finiteness of the space of Manin symbols is essentially enable us to compute the space of modular symbols for a given weight $k \geq 2$ and a finite index (or congruence) subgroup of $SL_2(\mathbb{Z})$ within a finite computational time. There are also the finite set of relations in the space of Manin symbols which correspond to the relations for the space of modular symbols so that both spaces are isomorphic as a free abelian grop. For the theorems and algorithms regarding the isomorphism and computations of those spaces, refer to Chapter 3 and 8 in [Ste07] and §1.2 and 1.3 in [Mer94].

Fix integers $N \geq 1$ and $k \geq 2$. Assume that $S_k(N,\varepsilon)$ has a character ε with $\varepsilon^2 = 1$ so that the values of ε are in $\mathbb{Q}$. Also, fix a primitive form $f = \sum_{n\geq 1} a_n q^n \in S_k(N,\varepsilon)$. For the rest of this section, we develop the algorithm to compute $L^{\mathrm{alg}}(f,n,\chi)$ for a primitive Dirichlet character χ of conductor $\mathfrak{f}_\chi$ and a critical integer $1 \leq n \leq k$. In order to compute $L^{\mathrm{alg}}(f,n,\chi)$ we need so called the rational (or integral) period mapping for the $\mathbb{Z}$-module of $\mathbb{M}_k(N,\varepsilon)$. Stein defined the rational period mapping and developed an algorithm to compute it for the general case when f admits its non-trivial $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ conjugates, refer to Chapter 10 in [Ste07]. The algorithm has been implemented in Sage [S$^+$17]. He also developed an algorithm to compute the canonical period $\Omega^{\pm}$ for the critical L-values for general weight k in the same chapter. However, it is implemented in Sage only for $k = 2$ by now.

Let $\mathbb{T}$ be the Hecke algebra associated to $M_k(N,\varepsilon,\mathbb{C})$ and V_f be the subspace of $S_k(N,\varepsilon,\mathbb{C})$ generated by f over $\mathbb{C}$. Denote the dual space of V_f by V_f^*. Note that in this case

$$\dim_{\mathbb{C}} V_f = 1 = \dim_{\mathbb{C}} V_f^*.$$

The integration pairing $\langle \cdot, \cdot \rangle$ induces a $\mathbb{T}$-equivariant homomorphism

$$\Phi_f : \mathbb{M}_k(N,\varepsilon,\mathbb{C}) \to V_f^*,$$

where $t \in \mathbb{T}$ acts on $\varphi \in V_f^*$ as $(t\varphi)(x) = \varphi(tx)$ for every $x \in \mathbb{M}_k(N,\varepsilon,\mathbb{C})$. Note that Proposition 2.6 implies that Φ_f is $\mathbb{T}$ stable. Let I_f be the annihilator of f in $\mathbb{T}$, i.e.

$$I_f := \{t \in \mathbb{T} \mid tf = 0\}.$$

Also, let

$$\mathbb{M}_k^*(N,\varepsilon)[I_f] := \{\varphi \in \mathbb{M}_k^*(N,\varepsilon) \mid t\varphi = 0 \text{ for all } t \in I_f\}.$$

Note that since f is a primitive form, $\mathbb{M}_k^*(N,\varepsilon)[I_f]$ is a $\mathbb{Q}$-vector space over of dimension 1. Let $\mathbb{M}_K^0$ be the subspace of $\mathbb{M}_k(N,\varepsilon)$ such that

$$\mathbb{M}_k^0 := \{x \in \mathbb{M}_k(N,\varepsilon) \mid \varphi(x) = 0 \text{ for all } \varphi \in \mathbb{M}_k^*(N,\varepsilon)[I_f]\}.$$

Definition. The rational period mapping Θ_f of $\mathbb{M}_k(N,\varepsilon)$ with respect to f is a natural quotient map defined as

$$\Theta_f : \mathbb{M}_k(N,\varepsilon) \to \mathbb{M}_k(N,\varepsilon)/\mathbb{M}_k^0.$$

On restricting the domain of Φ_f to $\mathbb{M}_k(N,\varepsilon)$ observe that

$$\begin{aligned} x \in \ker(\Theta_f) &\iff \varphi(x) = 0 \text{ for every } \varphi \in \mathbb{M}_k^*(N,\varepsilon)[I_f] \\ &\iff (t\varphi)(x) = \varphi(tx) = 0 \text{ for every } \varphi \in \mathbb{M}_k^*(N,\varepsilon) \text{ and } t \in I_f \\ &\iff \langle g, tx\rangle = \langle tg, x\rangle = 0 \text{ for every } g \in S_k(N,\varepsilon) \oplus \overline{S}_k(N,\varepsilon) \text{ and } t \in I_f \\ &\iff x \in \ker(\Phi_f) \cap \mathbb{M}_k(N,\varepsilon). \end{aligned}$$

Algorithm 2.15. This algorithm computes Θ_f of $\mathbb{M}_k(N,\varepsilon)$ with respect to f.

a) Compute $\mathbb{M}_k(N,\varepsilon)$ of dimension d as a space of column vectors.

b) Choose a prime p (small enough) and compute T_p as a $d \times d$ matrix over $\mathbb{Q}$.

c) Compute a basis B of $\mathbb{M}_k^*(N,\varepsilon)[I_f] = \ker((T_p)^{\mathrm{T}} - a_p)$.

d) Write the elements of B as row vectors and return it for Θ_f.

Example Consider the following primitive form in $S_6(\Gamma_0(3))$:

$$f(z) = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6)$$

Note that $\mathbb{M}_6(\Gamma_0(3))$ is of dimension 4 and T_2 with respect to a basis of $\mathbb{M}_6(\Gamma_0(3))$ is given by

$$T_2 = \begin{pmatrix} 33 & 3 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 6 & \frac{27}{2} & \frac{27}{2} \\ 0 & 12 & \frac{15}{2} & \frac{27}{2} \\ 0 & 12 & \frac{27}{2} & \frac{15}{2} \end{pmatrix}.$$

Its characteristic polynomial is $(X+6)^2(X-33)^2$ and $\ker((T_2)^{\mathrm{T}} + 6)$ is spanned by $(1,-9,0,8)$ and $(0,0,1,-1)$. Therefore,

$$\Theta_f = \begin{pmatrix} 1 & -9 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

2.6 Numerical results on the algebraic parts

In this section, we present the numerical results on vanishings and nonvanishings of the algebraic parts of the critical L-values for some primitive forms $f \in S_k(N, \varepsilon)$ with $\varepsilon^2 = 1$ and primitive Dirichlet characters χ of order $l \in \{3, 5, 7\}$ with $(N, \mathfrak{f}_\chi) = 1$ using the period mapping of modular symbols in Sage [S$^+$17]. Those L-values (or even values of their derivatives) also can be computed by the Dokchitser's L-function calculator wrapped in Sage [S$^+$17] and twist it by a given χ. It can compute the critical L-values of motivic L-functions and moreover, their derivatives with arbitrary precisions. It uses the functional equation of L-functions and convergence of exponential decaying series. For the Dokchitser's algorithm to obtain the special values of the motivic L-functions, refer to [Dok04].

Since each modular symbols in that sum have $\mathbb{Z}$-values of the perfect pairing with a primitive form, equation (2.4) implies that the twisted algebraic parts for the Galois orbits of a l-th primitive Dirichlet character are Galois-equivariant. More specifically, let $\mathrm{Gal}(\chi)$ be the Galois group associated with a l-th primitive Dirichlet character χ generated by σ. Then, $L(f, n, \chi^{\sigma^j}) = \sigma^j L(f, n, \chi)$ for $j = 0, \ldots, l-1$, where χ^{σ^j} and $\sigma^j L(f, n, \chi)$ are the actions of σ^j.

For $k = 2$, David, Fearnley and Kisilevsky in [DFK04] suggested asymptotes of the number of vanishings of twists for $\mathcal{C}_{l,f}(X)$ using the random matrix theory. The numerical results suggest that the number of vanishings of the twists depends not only the order of twists but also the weight of primitive forms. For examples, there exists no vanishings found so far for $k > 2$ if $l \geq 5$ and for $k > 6$ if $l = 3$. Hence, it is interested in the statistics of vanishings for $k = 4$ and $l = 3$, which is presented in the following table. Suppose that k is even. Let

$$\mathcal{F}_l(k, X) := \{L(f, k/2, \chi) \mid \chi \in \mathcal{C}_{l,f}(X)\},$$
$$\mathcal{F}_l^0(k, X) := \{L(f, k/2, \chi) \in \mathcal{F}_l(k, X) \mid L(f, k/2, \chi) = 0\}.$$

Note that $\mathcal{F}_l^0(k, X)$ and $\mathcal{F}_l(k, X)$ includes all Galois conjugates $\mathrm{Gal}(\chi)$ of a given $L^{\mathrm{alg}}(f, k/2, \chi)$.

N	$\lvert\mathcal{F}_l^0(4,X)\rvert$	$\lvert\mathcal{F}_l(4,X)\rvert$	X
5	4	10800	34083
6	0	9200	35473
7	0	9000	36513
8	4	10800	34083
9	0	9000	34681
10	24	10400	32787

Table 1: The number of vanishings of cubic twists for primitive form f of $k = 4$

Let $\eta(z) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ with $q = e^{2\pi i z}$. Also, let ε_0 is the trivial character modulo N. In the following examples, we verified the functional equations of the algebraic parts for 6 primitive forms of $S_k(N,\varepsilon)$ with $k > 2$ and present some statistical results on vanishings.

Example Let $\Delta(z) = \eta^{24}(z) = q\prod_{n\geq 1}(1-q^n)^{24} \in S_{12}(1,\varepsilon_0)$ be the modular discriminant. It is the primitive form and has $N = 1, k = 12, \varepsilon = \varepsilon_0$ and $a_N = 1$. Thus, From (2.8) for each integer n such that $1 \leq n \leq 11$, we have the following functional equation:

$$L^{\text{alg}}(f,n,\chi) = (-1)^n L^{\text{alg}}(f,12-n,\overline{\chi}).$$

For $\mathcal{C}_{3,f}(1129)$, $\mathcal{C}_{5,f}(1891)$ and $\mathcal{C}_{7,f}(2059)$, there exists no vanishing found among 134, 81 and 51, respectively, twists up to Galois conjugates.

Example Let $f(z) := \eta^8(z)\eta^8(2z) \in S_8(2,\varepsilon_0)$. It is the primitive form and has $N = 2, k = 8$ and $a_N = -8$. From (2.8), for each integer $1 \leq n \leq 7$, we have the following functional equation:

$$L^{\text{alg}}(f,n,\chi) = (-1)^n 8^{2-n}\chi(2)L^{\text{alg}}(f,8-n,\overline{\chi}).$$

For $\mathcal{C}_{3,f}(1009)$, $\mathcal{C}_{5,f}(311)$ and $\mathcal{C}_{7,f}(2381)$, there exists no vanishing found at $n = 4$.

Example Let $f(z) := \eta^6(z)\eta^6(3z) \in S_6(3,\varepsilon_0)$. It is the primitive form and has

$$N = 3, k = 6, \varepsilon = \varepsilon_0 \text{ and } a_N = 9.$$

From equation (2.8), for each integer n such that $1 \leq n \leq 5$, we have the following functional equation

$$L^{\text{alg}}(f,n,\chi) = (-1)^{n-1}3^{3-n}\chi(3)L^{\text{alg}}(f,6-n,\overline{\chi}).$$

The canonical periods $\Omega^{\pm}$ for f can be computed in the precision 53 bits using a choosen algebraic parts in §1.9 in [MTT86] and its critical values of $L(f,n,\chi)$ for a primitive quadratic character χ computed by the Dokchitser's L-function calculator as $\Omega^{+} = 0.00218245319471487$ and $\Omega^{-} = 0.0177792140769595i$.

For $\mathcal{C}_{3,f}(1027)$, there exists only one vanishing at $n = 3$ for $\mathfrak{f}_{\chi} = 127$ among 107 cubic twists up to Galois conjugates. For $\mathcal{C}_{5,f}(1741)$ and $\mathcal{C}_{7,f}(1597)$, there exists no vanishing found at $n = 3$.

Example Let $f(z) := \eta^4(z)\eta^2(2z)\eta^4(4z) \in S_5(4,\varepsilon)$ be the primitive form with the primitive quadratic Dirichlet character ε of conductor 4. It is the primitive form and has $N = 4, k = 5, a_N = 16, \varepsilon(-1) = -1$ and $\tau(\varepsilon) = 2i$. Note that it has the complex multiplication by $\mathbb{Q}(i)$ and there is no central integer since k is odd. From (2.7), we have the following functional equation:

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^{n-1}(2i)\varepsilon(\mathfrak{f}_{\chi})\chi(4)L^{\mathrm{alg}}(f,5-n,\overline{\chi}).$$

For $\mathcal{C}_{3,f}(1009)$, $\mathcal{C}_{5,f}(2201)$ and $\mathcal{C}_{7,f}(1471)$, there exists no vanishing found at $n = 2, 3$.

Example Let $f(z) := \eta^4(z)\eta^4(5z) \in S_4(5,\varepsilon_0)$. It is the primitive form and has $N = 5, k = 4$ and $a_N = -5$. From (2.8), we have the following functional equation:

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^n 5^{2-n}\chi(5)L^{\mathrm{alg}}(f,4-n,\overline{\chi}).$$

For $\mathcal{C}_{3,f}(34083)$, there are only two vanishings at $n = 2$ for $\mathfrak{f}_{\chi} = 223, 259$ among 5400 cubic twists up to Galois conjugates. For $\mathcal{C}_{5,f}(2161)$ and $\mathcal{C}_{7,f}(2969)$, there exists no vanishing found at $n = 2$.

Example Let $f(z) := \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z) \in S_4(6,\varepsilon_0)$. It is the primitive form and has $N = 6, k = 4$ and $a_N = 6$. From (2.8), we have the following functional equation:

$$L^{\mathrm{alg}}(f,n,\chi) = (-1)^n 6^{2-n}\chi(6)L^{\mathrm{alg}}(f,4-n,\overline{\chi}).$$

For $\mathcal{C}_{3,f}(35473)$, $\mathcal{C}_{5,f}(1301)$ and $\mathcal{C}_{7,f}(2633)$, there exists no vanishing found at $n = 2$.

Example Let $f(z) := \eta^3(z)\eta^3(7z) \in S_3(7,\varepsilon)$ with ε is the primitive quadratic Dirichlet character of conductor 7. It is the primitive form and has $N = 7, k = 3, \varepsilon(-1) = -1, \tau(\varepsilon) = i\sqrt{7}$ and $a_N = -7$. Note that it has the complex multiplication by

$\mathbb{Q}(\sqrt{-7})$ and there is no central integer since k is odd. From (2.7), we have the following functional equation:

$$L^{\mathrm{alg}}(f, n, \chi) = (-1)^n (i\sqrt{7}) 7^{1-n} \varepsilon(\mathfrak{f}_\chi) \chi(7) L^{\mathrm{alg}}(f, 3-n, \overline{\chi}).$$

For $\mathcal{C}_{3,f}(1009)$, $\mathcal{C}_{5,f}(2141)$ and $\mathcal{C}_{7,f}(2521)$, there exists no vanishing found at $n = 3, 4$.

2.7 Conjecture on vanishings of twists for higher weight via random matrix theory

The computational data presented in the previous section suggests that the number of vanishings for the central L-values of primitive forms of even weights $k > 2$ twisted by characters of order an odd prime l is finite. For the rest of this section, we let k be an even integer with $k > 2$ and assume a primitive form $f \in S_k(N, \varepsilon_0)$ as in Theorem 2.12, i.e. $f \in S_k(N, \varepsilon)$ is a primitive form where the nebentypus ε is the trivial, a quadratic with $\varepsilon(-1) = 1$, or character of order l. In this section, we will also support our expections on the number of vanishings for such primivie forms by using the random matrix theory for families of such twists by following the arguments of David, Fearnley and Kisilevsky [DFK04] and [DFK06] for twists of primitive forms of weight 2.

Let N be a positive integer and l be an odd prime. Let k be an even integer with $k > 2$, and fix a positive real number X and a primitive form f as above. From Proposition 2.4, we can write the functional equation of $L^{\mathrm{alg}}(f, k/2, \chi)$ for a character $\chi \in \mathcal{C}_{l,f}(X)$ as

$$L^{\mathrm{alg}}(f, k/2, \chi) = \omega(f, \chi) L^{\mathrm{alg}}(f, k/2, \overline{\chi}) = \omega(f, \chi) \overline{L^{\mathrm{alg}}(f, k/2, \chi)}, \tag{2.20}$$

where $\omega(f, \chi)$ is a primitive n-th root of unity where $n \mid 2l$. In [DFK04] and [DFK06], it is shown that the values of $|L^{\mathrm{alg}}(f, 1, \chi)|$ for $\chi \in \mathcal{C}_{l,f}(X)$ for a weight 2 primitive form f are discretely distributed on the positive real line. Then, we have the following discretisation of the value of $|L^{\mathrm{alg}}(f, k/2, \chi)|$ for higher weight. Let $K := \mathbb{Q}(\zeta_l)$, $K^+ := \mathbb{Q}(\zeta_l + \zeta_l^{-1})$, and $\mathcal{O}_{K^+}$ be the ring of integers of K^+.

Proposition 2.16. Let $\zeta := \omega(f,\chi)$ and ζ_l be a primitive l-th root of unity. Then,

$$|L^{\mathrm{alg}}(f,k/2,\chi)| = \begin{cases} |n_\chi| & \text{if } \zeta \neq -1 \\ |\zeta_l - \zeta_l^{-1}||n_\chi| & \text{if } \zeta = -1 \end{cases},$$

where $n_\chi \in \mathcal{O}_{K^+}$.

Proof. The proof is exactly same as that of Theorem 2.1 in [DFK06]. □

Now, for n_χ in in the above proposition, we can obtain a bound of $|L(f,k/2,\chi)|$ to ensure its vanishing by using the geometry of numbers. Consider an embedding $\pi : \mathcal{O}_{K^+} \to \mathbb{R}^{(l-1)/2}$ defined for each $\alpha \in \mathcal{O}_{K^+}$

$$\alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_{(l-1)/2}(\alpha)) \in \mathbb{R}^{(l-1)/2},$$

where $\sigma_i \in \mathrm{Gal}(K^+/\mathbb{Q})$ for $1 \leq i \leq (l-1)/2$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{(l-1)/2}\}$ be an integral basis of $\mathcal{O}_{K^+}$. Then, the images of $\mathcal{O}_{K^+}$ is a lattice in $\mathbb{R}^{(l-1)/2}$ generated by

$$\beta_i := \pi(\alpha_i) \text{ for } 1 \leq i \leq (l-1)/2.$$

Therefore, by proposition 2.16, we have

$$L(f,k/2,\chi) = 0 \iff L^{\mathrm{alg}}(f,k/2,\chi) = 0 \iff n_\chi = 0 \iff \pi(n_\chi) \in R, \tag{2.21}$$

where $R := \{r_1\beta_1 + \ldots + r_{(l-1)/2}\beta_{(l-1)/2} \mid -1 < r_i < 1 \text{ for } 1 \leq i \leq (l-1)/2\}$.

Proposition 2.17. Assume f, χ and K as above. Then, for every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$,

$$|L(f,k/2,\chi^\sigma)| = \frac{A(f,l)}{\mathfrak{f}_\chi^{(k-1)/2}}|\sigma(n_\chi)|, \tag{2.22}$$

where χ^σ is the character obtain by the action of σ on χ and $A(f,l)$ is a constant depending on f and l.

Proof. Using (2.14) for the Galois equivariance of $L^{\mathrm{alg}}(f,k/2,\chi^\sigma)$, we have

$$\begin{aligned} \sigma(L^{\mathrm{alg}}(f,k/2,\chi)) &= \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}^\sigma(a)c^\pm(a,\mathfrak{f}_\chi,k/2;f) \\ &= \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi^\sigma}(a)c^\pm(a,\mathfrak{f}_\chi,k/2;f) = L^{\mathrm{alg}}(f,k/2,\chi^\sigma). \end{aligned}$$

Then, use the definition of the algebraic parts (2.4) and Proposition 2.16 to complete the proof. □

For $l = 3$, since $n_\chi \in \mathbb{Q}$ and $\mathrm{Gal}(K^+/\mathbb{Q})$ is trivial, Proposition 2.17 and equalities (2.21) imply that

$$|n_\chi| = 0 \iff |n_\chi| = |\pi(n_\chi)| < 1 \iff |L(f, k/2, \chi)| < \frac{A(f,3)}{\mathfrak{f}_\chi^{(k-1)/2}}. \tag{2.23}$$

For $l = 5$, since $n_\chi \in \mathcal{O}_{K^+} = \mathbb{Z}[(1+\sqrt{5})/2]$ and $\mathrm{Gal}(K^+/\mathbb{Q}) = \langle \sigma \rangle$ where σ is the automorphism sending $\sqrt{5}$ to $-\sqrt{5}$. Thus, $\pi(\mathcal{O}_{K^+})$ in $\mathbb{R}^2$ is the lattice generated by

$$\beta_1 := ((1+\sqrt{5})/2, (1-\sqrt{5})/2) \text{ and } \beta_2 := ((1-\sqrt{5})/2, (1+\sqrt{5})/2).$$

Let $R_1 = \{(r_1, r_2) \in \mathbb{R}^2 \mid -1 < r_1, r_2 < 1\}$ and $R_2 = \{(r_1, r_2) \in \mathbb{R}^2 \mid -\sqrt{5} < r_1, r_2 < \sqrt{5}\}$. Then, $R_1 \subset R \subset R_2$ and Proposition 2.17 and equalities (2.21) imply that

$$\begin{aligned}
(n_\chi, \sigma(n_\chi)) \in R_1 &\iff |n_\chi|, |\sigma(n_\chi)| < 1 \iff |L(f,k/2,\chi)|, |L(f,k/2,\chi^\sigma)| < \frac{A(f,5)}{\mathfrak{f}_\chi^{(k-1)/2}}, \\
(n_\chi, \sigma(n_\chi)) \in R_2 &\iff |n_\chi|, |\sigma(n_\chi)| < \sqrt{5} \iff |L(f,k/2,\chi)|, |L(f,k/2,\chi^\sigma)| < \frac{A(f,5)\sqrt{5}}{\mathfrak{f}_\chi^{(k-1)/2}}.
\end{aligned} \tag{2.24}$$

For $k \geq 7$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_{(l-1)/2}\}$ be an integral basis of $\mathcal{O}_{K^+}$ and $\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})$ be a resticted automorphism to $\mathrm{Gal}(K^+/\mathbb{Q})$ for $1 \leq i \leq (l-1)/2$. Consider the region $\tilde{R}$ in $\mathbb{R}^{(l-1)/2}$

$$\tilde{R} := \{(r_1, \ldots, r_{(l-1)/2} \in \mathbb{R}^{(l-1)/2} \mid -M < r_i < M \text{ for } 1 \leq i \leq (l-1)/2\},$$

where $M = \max_{1 \leq i \leq (l-1)/2} \sum_{j=1}^{(l-1)/2} |\sigma_i(\alpha_j)|$. Then, $R \subset \tilde{R}$ and

$$n_\chi = 0 \implies \pi(n_\chi) \in \tilde{R} \iff |L(f, k/2, \chi^{\sigma_i})| < \frac{A'(f,l)}{\mathfrak{f}_\chi^{(k-1)/2}} \tag{2.25}$$

for every $1 \leq i \leq (l-1)/2$ and a constant $A'(f,l)$ depending only on f and l. We use the bounds (2.23), (2.24) and (2.25) to obtain the probability that $L(f, k/2, \chi) = 0$ via the random matrix theory.

Let $U(n)$ be the group $n \times n$ unitary matrices whose entries in $\mathbb{C}$ and the group operation is the matrix multiplicaton with the identity I_n. Then, it is well-known that it is not an abelian group for $n > 1$ and as linear transformations, $A \in U(n)$ preserves the inner product in $\mathbb{C}^n$ and the elements of the sequence of eigenvalues of

A lie on the unit circle in $\mathbb{C}$. Moreover, $U(n)$ is a topological group and there exists a unique Haar measure μ for $U(n)$ up to constants, see Theorem 6.8 and Proposition 6.15b in [Kna05] for the proof. We define the s-th moments of $|\det(A - I_n)|$ for $A \in U(n)$ by

$$M_U(s,n) := \int_{U(n)} |\det(A - I_n)|^s d\mu.$$

Keating and Snaith in [KS00] showed that

$$M_U(s,N) = \prod_{j=1}^{n} \frac{\Gamma(j)\Gamma(j+s)}{\Gamma^2(j+s/2)}$$

, hence it is analytic for $\mathrm{Re}(s) > -1$ and has meromorphic continuation to $\mathbb{C}$, and its probability density function $P_U(x,n)$ is the Mellin transform of $M_U(s,N)$ and determined by the first pole of $M_U(s,N)$ at $s = -1$. Hence, it is given by

$$\begin{aligned} P_U(x,n) &\sim \frac{1}{\Gamma(n)} \prod_{j=1}^{n} \frac{\Gamma^2(j)}{\Gamma^2(j-1/2)} \quad \text{as } x \to 0 \\ &\sim n^{\frac{1}{4}} G^2(1/2) \qquad\qquad \text{as } n \to \infty, \end{aligned} \tag{2.26}$$

where G is the Barnes G-function:

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right)^j \exp\left(\frac{z^2}{2j} - z\right),$$

where γ is the Euler-Mascheroni constant.

Also, define the t-th moments of $L(f, k/2, \chi)$ over $\chi \in \mathcal{C}_{l,f}(X)$ as

$$M_f(s,X) := \frac{1}{|\mathcal{C}_{l,f}(X)|} \sum_{\chi \in \mathcal{C}_{l,f}(X)} |L(f, k/2, \chi)|^s.$$

Then, the random matrix theory relates $M_U(t,n)$ and $M_f(t,X)$ by the following Keating, Sanith, David, Fearnley and Kisilevsky conjecture for twists of higher even weights by characters of order l. Note that the detailed arguments on their conjecture for the case $k = 2$ and the justification of $n \sim 2\log X$ by equating the mean densities of eigenvalues of elements in $U(n)$ and non-trivial zeroes of $L(f, k/2, \chi)$ for $\chi \in \mathcal{C}_{l,f}(X)$ at a fixed height in the critical strip can be found in [DFK04] and [DFK06].

Conjecture 2.7.1. Let f be a primitive form of weight $k \geq 4$. Then, we have

$$M_f(s,X) \sim c_f(s) M_U(s,n), \text{ as } n \sim 2\log X \to \infty,$$

where $c_f(s)$ is an arithmetic factor depending only on f.

Now, the probability density function $P_f(x, X)$ is also the Mellin transform of $M_f(s, X)$ and by Conjecture 2.7.1 and equation (2.26) it is given by

$$\begin{aligned}
P_f(x, X) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_f(s, X)x^{-s}\frac{ds}{s} \\
&\sim \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} c_f(s)M_U(s, n)x^{-s}\frac{ds}{s} \\
&\sim c_f(-1)n^{\frac{1}{4}}G^2(1/2) \qquad \text{for } x < \frac{1}{2\log X} \text{ as } n \to \infty \\
&\sim d_f \log^{\frac{1}{4}} X \qquad \text{for some constant } d_f.
\end{aligned}$$

Note that in the last two lines above, we use $n \sim 2\log X$. For large enough $\mathfrak{f}_\chi$ such that $x < c/\mathfrak{f}_\chi^{(k-1)/2} < (2\log \mathfrak{f}_\chi)^{-1}$ with a constant c, we have $P_f(x, X) \sim d_f \log^{\frac{1}{4}} \mathfrak{f}_\chi$. Therefore, the probability that $|L(f, k/2, \chi)| < c/\mathfrak{f}_\chi^{(k-1)/2}$ can be given by

$$\begin{aligned}
\text{Prob}\left(|L(f, k/2, \chi)| < c/\mathfrak{f}_\chi^{(k-1)/2}\right) &\sim \int_0^{c/\mathfrak{f}_\chi^{(k-1)/2}} d_f \log^{\frac{1}{4}} \mathfrak{f}_\chi dx \\
&= cd_f\frac{\log^{\frac{1}{4}} \mathfrak{f}_\chi}{\mathfrak{f}_\chi^{(k-1)/2}}.
\end{aligned} \tag{2.27}$$

Lemma 2.18. Let l be an odd prime. Then, for a primitive form f of level N and positive real number X,

$$|\mathcal{C}_{l,f}(X)| \sim b_{l,f}X$$

where $b_{l,f}$ is a constant depending only on l and N.

Proof. Refer to Section 5 in [DFK04] for the proof when $l = 3$. Moreover, for the proof when $l > 3$, refer to Cohen, Diaz and Olivier [CyDO02]. □

Recall that we assumed that $k \geq 4$ and $2 \mid 4$. Now, we estimate the number of vanishings of twists of order l using the probabilistic model given by (2.27) and the discretisations of $|L(f, k/2, \chi)|$ with the bounds (2.23), (2.24) and (2.25). Firstly, let $l = 3$. Then, by (2.23) and (2.27) we have

$$\text{Prob}\left(|L(f, k/2, \chi)| = 0\right) = \text{Prob}\left(|L(f, k/2, \chi)| < A(f, 3)/\mathfrak{f}_\chi^{(k-1)/2}\right) \sim d_{3,f}\frac{\log^{\frac{1}{4}} \mathfrak{f}_\chi}{\mathfrak{f}_\chi^{(k-1)/2}},$$

where $d_{3,f} = A(f, 3)d_f$. Summing the probabilities of twists in family $\mathcal{F}_3(k, X)$ and using the partial summation and Lemma 2.18, we have

$$|\mathcal{F}_3^0(k, X)| \sim d_{3,f}\sum_{\mathfrak{f}_\chi \leq X}\frac{\log^{\frac{1}{4}} \mathfrak{f}_\chi}{\mathfrak{f}_\chi^{(k-1)/2}} = O(1).$$

Let $l=5$. Then, since $R_1 \subset R \subset R_2$, by using (2.24), we have

$$\begin{aligned}
&\mathrm{Prob}\left(|L(f,k/2,\chi)| < A(f,5)/\mathfrak{f}_\chi^{(k-1)/2}\right) \mathrm{Prob}\left(|L(f,k/2,\chi^\sigma)| < A(f,5)/\mathfrak{f}_\chi^{(k-1)/2}\right)\\
&\le \mathrm{Prob}\left(|L(f,k/2,\chi)| = 0\right)\\
&\le \mathrm{Prob}\left(|L(f,k/2,\chi)| < A(f,5)\sqrt{5}/\mathfrak{f}_\chi^{(k-1)/2}\right) \mathrm{Prob}\left(|L(f,k/2,\chi^\sigma)| < A(f,5)\sqrt{5}/\mathfrak{f}_\chi^{(k-1)/2}\right).
\end{aligned}$$

Assume that $|L(f,k/2,\chi)|$ and $|L(f,k/2,\chi^\sigma)|$ are independent identical random variable. Then, by (2.27), we have

$$\mathrm{Prob}\left(|L(f,k/2,\chi)| = 0\right) \sim d_{5,f}\frac{\log^{\frac{1}{2}}\mathfrak{f}_\chi}{\mathfrak{f}_\chi^{k-1}}$$

for a constant $d_{5,f}$.Then, again using the partial summation and Lemma 2.18, we have

$$|\mathcal{F}_5^0(k,X)| \sim d_{5,f}\sum_{\mathfrak{f}_\chi \le X}\frac{\log^{\frac{1}{2}}\mathfrak{f}_\chi}{\mathfrak{f}_\chi^{k-1}} = O(1).$$

Let $l \ge 7$. Then, since $R \subset \tilde{R}$, by using (2.25), we have

$$\mathrm{Prob}\left(|L(f,k/2,\chi)| = 0\right) \le \prod_{1\le i\le (l-1)/2}\mathrm{Prob}\left(|L(f,k/2,\chi^{\sigma_i})| < \frac{A'(f,l)}{\mathfrak{f}_\chi^{(k-1)/2}}\right).$$

Assume that $|L(f,k/2,\chi^{\sigma_1})|, |L(f,k/2,\chi^{\sigma_2})|, \ldots, |L(f,k/2,\chi^{\sigma_{(l-1)/2}})|$ are independent identical random variable. Then, by (2.27), we have

$$\mathrm{Prob}\left(|L(f,k/2,\chi)| = 0\right) \sim d_{l,f}\frac{\log^{\frac{l-1}{8}}\mathfrak{f}_\chi}{\mathfrak{f}_\chi^{(l-1)(k-1)/4}}$$

for a constant $d_{l,f}$.Then, again using the partial summation and Lemma 2.18, we have

$$|\mathcal{F}_l^0(k,X)| \sim d_{l,f}\sum_{\mathfrak{f}_\chi \le X}\frac{\log^{\frac{l-1}{8}}\mathfrak{f}_\chi}{\mathfrak{f}_\chi^{(l-1)(k-1)/4}} = O(1).$$

Therefore, we make the following conjecture for the number of vanishings of $L(f,k/2,\chi)$ in family $\mathcal{F}_l(k,X)$ for $k \ge 4$ and $2 \mid k$ and an odd prime l.

Conjecture 2.7.2. Let f be a primitive form of weight $k \ge 4$ and $2 \mid k$ and l be an odd prime. Then, we have

$$|\mathcal{F}_l^0(k,X)| = O(1),$$

where the implicit constant depending on k, l and f.

Chapter 3

Analytic result on cubic twists of a weight 2 primitive form

3.1 Motivation and introduction

In this chapter, we estimate an upper bound of vanishings of a special family of cubic twists of L-function of a weight 2 primitive form with integral Fourier coefficients, which is defined in the next section, under some hypotheses including the generalized and grand Riemann Hypotheses (GRH). Our work is inspired by Fiorilli's work [Fio16]. He showed that under some hypotheses, stronger than the Riemann hypothesis for E, the average of analytic (algebraic as well) rank of the usual family of quadratic twists is 1/2. Here the usual family of quadratic twists is the family of quadratic twists ordered by the conductors of primitive quadratic Dirichlet characters.

Throughout this chapter, denote $\sigma := \mathrm{Re}(s)$ and $\tau := \mathrm{Im}(s)$ for $s \in \mathbb{C}$. Choose an elliptic curve E defined over $\mathbb{Q}$ of conductor N. Recall that by the modularity theorem by Wiles, Breuil, Conrad, Diamond, and Taylor in [Wil95], [TW95] and [BCDT01], there exists a primitive form f for $S_2(N) := S_2(N, \varepsilon_0)$,where ε_0 is trivial, such that $L(E, s + 1/2) = L(f, s)$. Here, we normalise the Fourier coefficients, i.e. for $\sigma > 1$,

$$L(f,s) = \prod_{p\nmid N}\left(1 - \frac{a_p}{p^{s+1/2}} + \frac{1}{p^2 s}\right)^{-1}\prod_{p|N}\left(1 - \frac{a_p}{p^{s+1/2}}\right)^{-1}$$
$$= \prod_{p}\left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

where $a_p/p^{1/2} = \alpha_p + \beta_p$ such that if $p \nmid N$, $\alpha_p = \overline{\beta_p} \in \mathbb{C}$ and $|\alpha_p| = |\beta_p| = 1$, and if $p \mid N$, $\alpha_p \in \mathbb{R}$ and $\beta_p = 0$. Thus, the critical strip of $L(f,s)$ is $\{\sigma + i\tau \mid 0 < \sigma < 1 \text{ and } \tau \in \mathbb{R}\}$ and its critical line is at $\sigma = 1/2$. Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Then, we also have

$$L(E, s+1/2, \chi) = L(f_\chi, s) = \prod_p \left(1 - \frac{\chi(p)\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)\beta_p}{p^s}\right)^{-1}.$$

Moreover, by 1.6 for $k = 2$, $L(f,s,\chi)$ can be analytically continued to $\mathbb{C}$ and has the functional equation:

$$\Lambda(f,s,\chi) := \left(\frac{\mathfrak{f}_\chi \sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{1}{2}\right) L(f,s,\chi) = \omega(f)\chi(N)\frac{\tau(\chi)^2}{\mathfrak{f}_\chi}\Lambda(f, 1-s, \overline{\chi}), \quad (3.1)$$

where $\omega(f) = \pm 1$, called the root number of f. Note that by the factor $\Gamma(s+1/2)$ in (3.1), there exist the simple trivial zeroes of $L(f,s,\chi)$ at $s = -j + 1/2$ for every positive integers j.

In order to investigate the central L-value, we take the product of the logarithmic derivative of $L(f,s,\chi)$ and some suitable smooth weight function $h(s)$, i.e. for an $\mathbb{R}$-valued integrable function g on the positive real numbers,

$$h(s) := (\mathcal{M}g)(s) := \int_0^\infty x^s g(x) \frac{dx}{x} \text{ and } g(x) = (\mathcal{M}^{-1}h)(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}h(s)ds$$

In addition, we should apply some appropriate conditions on g:

Property 3.1. a) $(\mathcal{M}g)(s)$ converges for $\Re(s) \geq 0$.

b) For any $n \geq 1$ and $x \geq 1$, $g(x) \ll_n 1/x^n$.

c) $(\mathcal{M}g)(s)$ can be analytically continued to a meromorphic function with possible poles of order at most one at the points $s = j$ for non-positive integer j.

d) Uniformly for $|\sigma| \leq 1$ and $|\tau| \geq 1$, $(\mathcal{M}g)(s) \ll 1/\tau^2$.

We can take $g_k(x) := \max\{1 - x^k, 0\}$ for positive integer k for examples. Note that

$$(\mathcal{M}g_k)(s) = \int_0^\infty x^s g(x^k)\frac{dx}{x} = \frac{1}{k}\int_0^\infty t^{s/k} g(t)\frac{dt}{t} = \frac{1}{k}(\mathcal{M}g)\left(\frac{s}{k}\right).$$

Choose g satisfying Property 3.1 and take the inverse Mellin transform of the product of the logarithmic derivative of $L(f,s,\chi)$ and the Mellin transform of g to get the following equalities:

$$\begin{aligned}
\mathcal{M}^{-1}\left(-\frac{L^{'}}{L}(f,s,\chi)(\mathcal{M}g)(s)\right)\left(\frac{1}{x}\right) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L^{'}}{L}(f,s,\chi)x^{s}(\mathcal{M}g)(s)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\sum_{k\geq 1}\sum_{p}\frac{\chi^{k}(p)(\alpha_p^k+\beta_p^k)\log p}{p^{ks}}x^{s}(\mathcal{M}g)(s)ds \\
&= \sum_{k\geq 1}\sum_{p}\chi^{k}(p)(\alpha_p^k+\beta_p^k)\log p\left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{p^k}{x}\right)^{-s}(\mathcal{M}g)(s)ds\right) \\
&= \sum_{k\geq 1}\sum_{p}\chi^{k}(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/x),
\end{aligned} \tag{3.2}$$

for some $c>1$. By taking the contour integral along the vertical line at $\sigma = c$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L^{'}}{L}(f,s,\chi)x^{s}(\mathcal{M}g)(s) &= -\sum_{\rho_\chi}x^{\rho_\chi}(\mathcal{M}g)(\rho_\chi) \\
-\frac{L^{'}}{L}(f,s,\chi)\underset{s=0}{Res}(\mathcal{M}g)(s) &+ O(x^{-\frac{1}{3}}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\left|\frac{L^{'}}{L}(f,s,\chi)\right|\frac{|ds|}{|s|^2}),
\end{aligned} \tag{3.3}$$

where ρ_χ's are that the non-trivial zeroes of $L(f,s,\chi)$ with multiplicities, i.e. r_{an}. From the functional equation of $L(f,s,\chi)$ in (3.1), we have the following functional equation of the logarithmic derivative of $L(f,s,\chi)$:

$$\frac{L^{'}}{L}(f,s,\chi) = \frac{\Gamma^{'}}{\Gamma}(\frac{3}{2}-s)-\frac{\Gamma^{'}}{\Gamma}(\frac{1}{2}+s)-2\log\left(\frac{\mathfrak{f}_\chi\sqrt{N}}{2\pi}\right)+\frac{L^{'}}{L}(f,1-s,\overline{\chi}).$$

Then, on the vertical line $\sigma=\frac{4}{3}$, $\frac{L^{'}}{L}(f,\frac{4}{3}+i\tau,\chi)$ is bounded by an absolute constant. Furthermore, by the asymptotic equation of $\frac{\Gamma^{'}}{\Gamma}(s)\sim\log(s)$ for $\{s\in\mathbb{C} \mid |\arg(s-1)| < \pi-\delta\}$ given a fixed $\delta>0$, we have

$$\frac{L^{'}}{L}(f_\chi,-\frac{1}{3}+i\tau)\ll_g \log(\mathfrak{f}_\chi^2 N|\tau|)$$

so that the error term in (3.3) has a following bound:

$$x^{-\frac{1}{3}}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\left|\frac{L^{'}}{L}(f_\chi,s)\right|\frac{|ds|}{|s|^2}\ll_g x^{-\frac{1}{3}}\log(\mathfrak{f}_\chi^2 N). \tag{3.4}$$

Moreover, from the Hadamard product of $L(f,s,\chi)$ and the functional equation of $\frac{L'}{L}(f,s,\chi)$, refer to (5.28) on p. 103 in [IK04] and note that the order of zero or pole $r=0$ in the formula of (5.28), we have

$$\frac{L'}{L}(f,1,\chi) \ll \sum_{|1-\rho_\chi|<1} \frac{1}{|1-\rho_\chi|} + \log(\mathfrak{f}_\chi^2 N).$$

In particular, assuming GRH for $L(f,s,\chi)$ at the above bound,

$$\frac{L'}{L}(f_\chi,1) \ll \sum_{|1-\rho_\chi|<1} 2 + \log(\mathfrak{f}_\chi^2 N_E),$$

hence, using the functional equation we obtain

$$\frac{L'}{L}(f,0,\chi) \ll \log(\mathfrak{f}_\chi^2 N). \tag{3.5}$$

Combining (3.5) and (3.4) into (3.2) and (3.3) and equating them, we have the following lemma.

Lemma 3.2. Under GRH for $L(f,s,\chi)$, we have

$$\sum_{k\geq 1}\sum_{p} \chi^k(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/x) = -\sum_{\rho_\chi} x^{\rho_\chi}(\mathcal{M}g)(\rho_\chi) + O(\log(\mathfrak{f}_\chi^2 N)). \tag{3.6}$$

Lemma 3.3. Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ and order an odd prime l. Then, under GRH of $L(s,\chi)$, for $c>1$ we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi)x^s(\mathcal{M}g)(s)\frac{ds}{s} \ll x^{\frac{1}{2}}\log \mathfrak{f}_\chi,$$

Proof. Since the order of χ is odd, $\chi(-1)=1$. Thus, $L(s,\chi)$ has the functional equation:

$$\Lambda(s,\chi) := \left(\frac{\mathfrak{f}_\chi}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi) = \frac{\tau(\chi^2)}{\sqrt{\mathfrak{f}_\chi}}\Lambda(1-s,\overline{\chi}).$$

Observe that $L(s,\chi)$ has a simple zero at $s=0$ since the right hand side is finite, hence its logarithmic derivative has a simple pole at $s=1$. By taking the contour integral at c and moving the vertical line to the left until $\sigma=-1/3$, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi)x^s(\mathcal{M}g)(s)\frac{ds}{s} = -\sum_{\rho_\chi} x^{\rho_\chi}(\mathcal{M}g)(\rho_\chi)$$
$$-\operatorname*{Res}_{s=0}\left(\frac{L'}{L}(s,\chi)\mathcal{M}g(s)\right) + O\left(x^{-\frac{1}{3}}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\left|\frac{L'}{L}(s,\chi)\right|\frac{|ds|}{|s|^2}\right),$$

where ρ_χ in the first sum are the non-trivial zeroes of $L(s,\chi)$. By Property 3.1 of g, $\frac{L'}{L}(s,\chi)\mathcal{M}g(s)$ has at most double pole at $s=0$. Then, applying the Riemann-von Mangoldt formula for $L(s,\chi)$, refer to Theorem 5.24 in [IK04]) to the sum over ρ_χ above with GRH for $L(s,\chi)$, we have

$$\sum_{\rho_\chi} x^{\frac{1}{2}+i\tau_{\rho_\chi}}(\mathcal{M}g)(\rho_\chi) \ll x^{\frac{1}{2}}\log \mathfrak{f}_\chi$$

where τ_{ρ_χ} is the imaginary part of ρ_χ. Bounds for other two terms can be easily obtained by the same arguments, only differ by the analytic conductor, shown for Lemma 3.2 and it completes the proof. □

We also believe to obtain a bound for the line integral without assuming GRH of $L(s,\chi)$ above by using the zero-free region of it, refer to Theorem 5.26 in [IK04]) even though it is not a better bound than one above.

Lemma 3.4. Let $f \in S_2(N)$ be a primitive form. Then, under GRH of $L(f,s)$, for $c>1$ we have

$$\sum_p (\alpha_p+\beta_p)(\log p)g(p/x) \ll x^{\frac{1}{2}}\log N + \log N$$

Proof. Take the logarithmic derivative formula of $L(f,s)$. Then, use the exactly same arguments for Lemma 3.2 and 3.3 except taking the trivial bounds of α_p and β_p and considering the trivial zeroes at $s=-(1+n)/2$ for integers $n\geq 0$. □

3.2 Symmetric square and cube of primitive form

We follow all the notations and definitions of the previous section. Fix a primitive form $f \in S_2(N)$ and a primitive cubic Dirichlet character χ of conductor $\mathfrak{f}_\chi$ with $(N,\mathfrak{f}_\chi)=1$. In the this section, we will obtain a bound of the left hand side of Lemma 3.2 for $k=2,3$ using the theory of L-function of the symmetric square and cube of f_χ which is a primitive form of f twisted by χ.

Definition. For a primitive form f_χ, the L-function of the symmetric k-th power of f_χ is defined as

$$L(\mathrm{Sym}^k f_\chi, s) := \prod_p \prod_{0\leq j\leq k}\left(1-\frac{\chi^k(p)\alpha_p^j\beta_p^{k-j}}{p^s}\right)^{-1}.$$

Note that we normalised the Fourier coefficients so that the critical strip is

$$\{s \in \mathbb{C} \mid 0 < \sigma < 1\}.$$

In the case of $k = 2$, by Rankin and Selberg, f_χ admits the Rankin-Selberg convolution:

$$L(f_\chi \otimes f_\chi, s) := \prod_{p \nmid N} \left(1 - \frac{\chi^2(p)\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{\chi^2(p)\beta_p^2}{p^s}\right)^{-1} \left(1 - \frac{\chi^2(p)}{p^s}\right)^{-2} \times \prod_{p \mid N} \left(1 - \frac{\chi^2(p)\alpha_p^2}{p^s}\right)^{-1}.$$

Refer to Section 5.12 in [IK04] for backgrounds for the Rankin-Selberg convolution and the symmetric square of a primitive form. The straight-forward computations show that

$$L(\mathrm{Sym}^k f_\chi, s) = L(f_\chi \otimes f_\chi, s) L(s, \chi_0\chi^2)^{-1} \prod_{p \mid N} \left(1 - \frac{\chi^2(p)}{p^s}\right)^{-1},$$

where χ_0 is the trivial character modulo N. From Remark 2.1 in [Fio16], $L(\mathrm{Sym}^2 f_\chi, s)$ is holomorphic and non-zero at $s = 1$. Moreover, Shimura [Shi75] showed that $L(\mathrm{Sym}^2 f_\chi, s)$ can be analytically continued to $\mathbb{C}$ and has the following functional equation:

$$\begin{aligned} \Lambda(\mathrm{Sym}^2 f_\chi, s) &:= q(\mathrm{Sym}^2 f_\chi)\pi^{-3s/2}\Gamma^2\left(\frac{s}{2} + \frac{1}{2}\right)\Gamma\left(\frac{s}{2} + 1\right) L(\mathrm{Sym}^2 f_\chi, s) \\ &= \omega(\mathrm{Sym}^2 f_\chi)\Lambda(\mathrm{Sym}^2 f_\chi, 1 - s), \end{aligned} \tag{3.7}$$

where $q(\mathrm{Sym}^2 f_\chi, s)$ is the analytic conductor and $\omega(\mathrm{Sym}^2 f_\chi)$ is the root number of $L(\mathrm{Sym}^2 f_\chi, s)$. Note that the trivial zeroes of $L(\mathrm{Sym}^2 f_\chi, s)$ are simple zeroes at $s = -2n$ and double zeroes at $s = 1 - 2n$ for every integer $n \geq 1$. Similarly, Kim and Shahidi [KS02a] and [KS02b] showed the similar results as above on $L(\mathrm{Sym}^3 f_\chi, s)$. It has the following functional equation:

$$\begin{aligned} \Lambda(\mathrm{Sym}^3 f_\chi, s) &:= 4q(\mathrm{Sym}^3 f_\chi)^{-\frac{s}{2}}(2\pi)^{-2s-2}\Gamma\left(s + \frac{1}{2}\right)\Gamma\left(s + \frac{3}{2}\right) L(\mathrm{Sym}^3 f_\chi, s) \\ &= \omega(\mathrm{Sym}^3 f_\chi)\Lambda(\mathrm{Sym}^3 f_\chi, 1 - s), \end{aligned} \tag{3.8}$$

where $q(\mathrm{Sym}^3 f_\chi, s)$ is the analytic conductor and $\omega(\mathrm{Sym}^3 f_\chi)$ is the root number of $L(\mathrm{Sym}^3 f_\chi, s)$. Note that both of symmetric square and cubes has non-zero at $s = 1$.

Lemma 3.5. Let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ and order an odd prime l. Then, under GRH of $L(\mathrm{Sym}^2 f_\chi, s)$ and $L(s,\chi^2)$, we have

$$\sum_p \chi^2(p)(\alpha_p^2+\beta_p^2)(\log p)g(p/x) \ll_g x^{\frac{1}{2}}\log(\mathfrak{f}_\chi^2 N).$$

Proof. Using the similar arguments for Lemma 3.2, we have

$$\sum_{k\geq 1}\sum_p \chi^{2k}(p)(\alpha_p^{2k}+1+\beta_p^{2k})(\log p)g(p^k/x) = -\sum_\rho x^\rho(\mathcal{M}g)(\rho) - \frac{L'}{L}(\mathrm{Sym}^2 f_\chi, 0)\operatorname*{Res}_{s=0}(\mathcal{M}g)(s) + O\left(x^{-\frac{1}{3}}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\left|\frac{L'}{L}(\mathrm{Sym}^2 f_\chi, s)\right|\frac{|ds|}{|s|^2}\right), \tag{3.9}$$

where the sum over ρ is that over the non-trivial zeroes of $L(\mathrm{Sym}^2 f_\chi, s)$ with multiplicities. Applying the Riemann-von Mangoldt formula, refer to Theorem 5.8 in [IK04]) to the first term in the right hand side in (3.9) and assuming GRH for $L(\mathrm{Sym}^2 f_\chi, s)$, we have

$$\sum_\rho x^\rho(\mathcal{M}g)(\xi) \ll x^{\frac{1}{2}}\log q(\mathrm{Sym}^2 f_\chi, \frac{1}{2}) \ll x^{\frac{1}{2}}\log \mathfrak{f}_\chi^2 N. \tag{3.10}$$

For the second term of the right hand side in (3.9), use the same argument in Lemma 3.2 with GRH for $L(\mathrm{Sym}^2 f_\chi, s)$ to obtain

$$\frac{L'}{L}(\mathrm{Sym}^2 f_\chi, 0)\operatorname*{Res}_{s=0}(\mathcal{M}g)(s) \ll \log(\mathfrak{f}_\chi^2 N). \tag{3.11}$$

For the last term of the right hand side in (3.9), use the functional equation of (3.7) to obtain

$$\frac{L'}{L}(\mathrm{Sym}^2 f_\chi, -1/3+i\tau) \ll_g \log(\mathfrak{f}_\chi^2 N|\tau|).$$

Hence,

$$x^{-\frac{1}{3}}\int_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\left|\frac{L'}{L}(\mathrm{Sym}^2 f_\chi, s)\right|\frac{|ds|}{|s|^2} \ll x^{-\frac{1}{3}}\log(\mathfrak{f}_\chi^2 N|\tau|). \tag{3.12}$$

Combining all three bounds (3.10), (3.11) and (3.12) into (3.9), we have

$$\sum_{k\geq 1}\sum_p \chi^{2k}(p)(\alpha_p^{2k}+1+\beta_p^{2k})(\log p)g(p^k/x) \ll x^{\frac{1}{2}}\log \mathfrak{f}_\chi^2 N. \tag{3.13}$$

Now, we use $|\alpha_p| = |\beta_p| = |\chi(p)| = 1$ to compute the trivial bound of the left hand side of the above asymptotic equation for $k > 2$. Then, we have

$$\begin{aligned}
&\sum_{k\geq 1}\sum_{p} \chi^{2k}(p)(\alpha_p^{2k} + 1 + \beta_p^{2k})(\log p) g(p^k/x) \\
&\quad = \sum_{p} \chi^2(p)(\alpha_p^2 + \beta_p^2)(\log p) g(p/x) + \sum_{k\geq 1}\sum_{p} \chi^{2k}(p)(\log p) g(p^k/x) \\
&\qquad + \sum_{k\geq 2}\sum_{p} \chi^{2k}(p)(\alpha_p^{2k} + \beta_p^{2k})(\log p) g(p^k/x) \\
&\quad = \sum_{p} \chi^2(p)(\alpha_p^2 + \beta_p^2)(\log p) g(p/x) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi^2) x^s (\mathcal{M}g)(s)\frac{ds}{s} \qquad (3.14)\\
&\qquad + O(\sum_{k\geq 2}\sum_{p^k\leq x} \log p + \sum_{p|\mathfrak{f}_\chi N} \log p) \\
&\quad = \sum_{p} \chi^2(p)(\alpha_p^2 + \beta_p^2)(\log p) g(p/x) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{L'}{L}(s,\chi^2) x^s (\mathcal{M}g)(s)\frac{ds}{s} \\
&\qquad + O(x^{\frac{1}{2}} + \log \mathfrak{f}_\chi^2 N).
\end{aligned}$$

In the second equality above, we use the logarithmic derivative of $L(s, \chi^2)$ as Lemma 3.2. In the last equality above, we use the bound of the von-Mangolt function Λ and partial summation to obtain

$$\sum_{k\geq 2}\sum_{p^k\leq x} \log p = \sum_{k\geq 2}\sum_{n\leq x^{1/k}} \Lambda(n) \ll x^{\frac{1}{2}} \text{ and } \sum_{p|\mathfrak{f}_\chi N} \log p \ll \log \mathfrak{f}_\chi^2 N. \qquad (3.15)$$

Using Lemma 3.3 for the second term of right hand side of (3.14) and equating (3.13) and (3.14), the proof follows. □

Notice that the integral against the logarithmic derivative of $L(s, \chi^2)$ in the above formula differs to that against $\zeta(s)$ for quadratic twists, cf. equation (15) in [Fio16].

Lemma 3.6. Let $f \in S_2(N)$ be a primitive form and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ and order an odd prime l. Then, under GRH of $L(\mathrm{Sym}^3 f_\chi, s)$, we have

$$\sum_{p\nmid(\mathfrak{f}_\chi)} (\alpha_p^3 + \beta_p^3)(\log p) g(p/x) \ll_g x^{\frac{1}{2}} \log(\mathfrak{f}_\chi^2 N).$$

Proof. The proof is almost same as that of Lemma 3.5. Take the logarithmic derivative of $\Lambda(\mathrm{Sym}^3 f_\chi)$. Then, using the same arguments for Lemma 3.5 under GRH of

$L(\mathrm{Sym}^3 f_\chi, s)$, we have

$$\sum_{k\geq 1}\sum_{p} \chi^3(p)(\alpha_p^{3k}+\alpha_p^{2k}\beta_p^k+\alpha_p^k\beta_p^{2k}+\beta_p^{3k})(\log p)g(p^k/x) \ll x^{\frac{1}{2}}\log(\mathfrak{f}_\chi^2 N).$$

By taking the trivial bound for the partial sum for $k\geq 2$ of the above sum using (3.15) and $\alpha_p\beta_p = 1$, we have

$$\begin{aligned}&\sum_{k\geq 1}\sum_{p\nmid \mathfrak{f}_\chi}(\alpha_p^{3k}+\alpha_p^{2k}\beta_p^k+\alpha_p^k\beta_p^{2k}+\beta_p^{3k})(\log p)g(p^k/x)\\ &\quad \ll \sum_{p\nmid \mathfrak{f}_\chi}(\alpha_p^3+\beta_p^3)(\log p)g(p/x)+\sum_{p\nmid \mathfrak{f}_\chi}(\alpha_p+\beta_p)(\log p)g(p/x)+x^{\frac{1}{2}}\log(\mathfrak{f}_\chi N).\end{aligned}$$

Finally, use Lemma 3.4 for the second sum of the right hand side of the above equation and the proof completes. □

3.3 Group family of cubic twists

Now choose g satisfying Property 3.1. In this section, we introduce a special family of cubic twists of a fixed prmitive form and define a prime sum over this family. Fix a positive real number X, a positive integer N and a primitive form $f=\sum_{n\geq 1} a_n q^n \in S_2(N)$ where $a_n\in\mathbb{Z}$ for every n. Recall that $\mathcal{C}_3$ is the set of all primitive cubic Dirichlet character χ of conductor f and $\mathcal{P}$ is the set of all rational primes. Let χ_0 be the trivial character of conductor 1 and

$$\begin{aligned}\mathcal{C}_{l,f} &:= \{\chi\in\mathcal{C}_l \mid (N,\mathfrak{f}_\chi)=1\}\cup\{\chi_0\},\\ \mathcal{P}(X) &:= \{p\in\mathcal{P}\mid p\leq X \text{ and } p\equiv 1 \bmod 3\}\cup\{9\}.\end{aligned}$$

We can write $\mathcal{P}(X)=\{p_0=9,p_1,p_2,\ldots,p_t\leq X\}$ in the increasing order for $p_1,p_2,\ldots,p_t$. Denote a primitive cubic Dirichlet character of prime conductor $p\in P(X)$ by χ_p. Note that for each $p\in\mathcal{P}(X)$ there are exactly two primitive cubic characters χ_p and χ_p^2 of conductor p such that $\chi_p^2=\overline{\chi_p}$. Define a subset of $\mathcal{C}_{3,f}$ by

$$\mathcal{G}_f(X) := \{\chi\in\mathcal{C}_{3,f}\mid \chi=\prod_{0\leq j\leq t}{\chi_{p_j}}^{\alpha_j} \text{ for } p_j\in\mathcal{P}(X) \text{ and } \alpha_j\in\{0,1,2\}\}.$$

Note that $\mathcal{G}(X)$ is a multiplicative group with identity χ_0 for every X and by the prime number theorem in arithmetic progression we have

$$t\sim\frac{X}{2\log X} \text{ and } |\mathcal{G}_f(X)| = 3^{t+1} = 3^{\frac{X}{2\log X}(1+o(1))}.$$

The right hand side in the above relations imiplies that

$$X \ll \log |\mathcal{G}_f(X)| \log\log |\mathcal{G}_f(X)|. \tag{3.16}$$

Notice that if $X \le Y$, then $\mathcal{G}_f(X)$ is a subgroup of $\mathcal{G}_f(Y)$. In particular, $\mathcal{G}_f(X) \to \mathcal{C}_{3,f}$ as $X \to \infty$. This family has an advantage of orthogonality and will be used to estimate a bound of the character sum in this family over primes. It is well-known for the Chebyshev function, refer to Corollary 11.20 in [MV07], that for some constant c

$$\vartheta(X;1,3) := \sum_{\substack{p\in\mathcal{P}(X)\\ p\neq 9}} \log p = \frac{X}{2} + O(Xe^{-c\sqrt{\log X}}) = \frac{X}{2}(1+O(e^{-c\sqrt{\log X}})). \tag{3.17}$$

This implies that the maximum conductor $\mathfrak{f}_\chi$ in $\mathcal{G}_f(X)$ is as $X \to \infty$

$$\mathfrak{f}_\chi = \prod_{j=0}^{t} p_j = 9e^{\vartheta(X;1,3)} = 9\exp\left(\frac{X}{2}(1+O(e^{-c\sqrt{\log X}}))\right). \tag{3.18}$$

Observe that for each $0 \le j \le t$, the number of $\chi \in \mathcal{G}_f(X)$ whose conductor $\mathfrak{f}_\chi$ is divisible by p_j is $3^{t+1} - 3^t = 2(3^t)$. Hence, applying (3.17), we have

$$\begin{aligned}\sum_{\chi\in\mathcal{G}_f(X)} \log \mathfrak{f}_\chi &= 2(3^t)\sum_{j=0}^{t}\log p_j \\ &= 2(3^t)\left(\log 9 + \vartheta(X;1,3)\right) \\ &= 2(3^t)(\log 9 + \frac{X}{2}(1+O(e^{-c\sqrt{|\log X|}}))) \\ &\ll X|\mathcal{G}_f(X)|.\end{aligned} \tag{3.19}$$

Lemma 3.7. Suppose that $|\mathcal{G}_f(X)|^{2-\delta} \le P \le 2|\mathcal{G}_f(X)|^{2-\delta}$ for some $0<\delta<1$. Then,

$$P^{-\frac{1}{4}} \sum_{\chi\in\mathcal{G}_f(X)} \log(\mathfrak{f}_\chi^2 N) \ll_f |\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}+\varepsilon}.$$

Proof. Note that the assumption implies that

$$2^{-\frac{1}{4}}|\mathcal{G}_f(X)|^{-\frac{1}{2}+\frac{\delta}{4}} \le P^{-\frac{1}{4}} \le |\mathcal{G}_f(X)|^{-\frac{1}{2}+\frac{\delta}{4}}. \tag{3.20}$$

By (3.19) and (3.16), $\sum_{\chi\in\mathcal{G}_f(X)} \log(\mathfrak{f}_\chi^2 N) \ll |\mathcal{G}_f(X)| \log|\mathcal{G}_f(X)| \log\log|\mathcal{G}_f(X)|$. Therefore, by using (3.20) and noting that $0<\delta<1$, we have for any $\varepsilon > 0$,

$$P^{-\frac{1}{4}} \sum_{\chi\in\mathcal{G}_f(X)} \log(\mathfrak{f}_\chi^2 N) \ll_f |\mathcal{G}_f(X)|^{(\frac{1}{2}+\frac{\delta}{4})} \log|\mathcal{G}_f(X)| \log\log|\mathcal{G}_f(X)|, \tag{3.21}$$

which completes the proof. $\square$

Consider $\mathcal{G}_f(X)$ as the group of characters on $\mathcal{R}(X)/\mathcal{R}^3(X)$ where

$$\begin{aligned}\mathcal{R}(X) &:= (\mathbb{Z}/p_0\mathbb{Z})^\times \times (\mathbb{Z}/p_1\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_t\mathbb{Z})^\times, \\ \mathcal{R}^3(X) &:= \left((\mathbb{Z}/p_0\mathbb{Z})^\times\right)^3 \times \left((\mathbb{Z}/p_1\mathbb{Z})^\times\right)^3 \times \cdots \times \left((\mathbb{Z}/p_t\mathbb{Z})^\times\right)^3, \\ \mathcal{R}^3(X,j) &:= \mathcal{R}^3(X)/\left((\mathbb{Z}/p_j\mathbb{Z})^\times\right)^3 \text{ for } 1 \le j \le t,\end{aligned}$$

where $p_j \in \mathcal{P}(X)$ for $0 \le j \le t$. Then, we have the following orthogonality of characters of finite abelian group: for each $p \in \mathcal{P}$,

$$\sum_{\chi \in \mathcal{G}_f(X)} \chi(p) = \begin{cases} |\mathcal{G}_f(X)|/3 = 3^t & \text{if } p \in \mathcal{P}(X), p \in \mathcal{R}^3(X,j) \text{ for some } 1 \le j \le t \\ |\mathcal{G}_f(X)| = 3^{t+1} & \text{if } p \notin \mathcal{P}(X), p \in \mathcal{R}^3(X) \\ 0 & \text{otherwise} \end{cases}. \tag{3.22}$$

Lemma 3.8. Suppose that $|\mathcal{G}_f(X)|^{2-\delta} \le P \le 2|\mathcal{G}_f(X)|^{2-\delta}$ for some $0 < \delta < 1$. Then,

$$\sum_{\substack{p \in \mathcal{P}, p \le P \\ p \notin \mathcal{P}(X) \\ p \in \mathcal{R}^3(X)}} 1 \sim \frac{P}{2|\mathcal{G}_f(X)| \log P} \quad \text{and} \quad \sum_{\substack{1 \le j \le t \\ p_j \in \mathcal{R}^3(X,j)}} 1 \ll |\mathcal{G}_f(X)|^{\varepsilon - 1} = o(1)$$

for every $\varepsilon > 0$.

Proof. Note that p is a prime, $p = \pm 1 \bmod 3$ and $p \in \left((\mathbb{Z}/9\mathbb{Z})^\times\right)^3$ if and only if $p \equiv \pm 1 \bmod 9$, respectively. Since for each prime $p \equiv 1 \bmod 3$ there are $(p-1)/3$ cubic residue classes modulo p. Moreover, for $p = 9$ there are two cubic residue classes modulo 9. Thus, by the Chinese remainder theorem, we have

$$\frac{|\{p_j \in \mathcal{R}^3(X,j) \mid 1 \le j \le t\}|}{|\{p \in \mathcal{P}(X) \mid p \ne 9\}|} \sim \frac{1}{6}\left(\frac{1}{3}\right)^{t-1}.$$

Thus, since $t \asymp \log |\mathcal{G}_f(X)|$, as $X \to \infty$ for every $\varepsilon > 0$,

$$\sum_{\substack{1 \le j \le t \\ p_j \in \mathcal{R}^3(X,j)}} 1 \sim \frac{t}{6}\left(\frac{1}{3}\right)^{t-1} \ll \frac{\log |\mathcal{G}_f(X)|}{|\mathcal{G}_f(X)|} \ll |\mathcal{G}_f(X)|^{\varepsilon - 1}.$$

Moreover, since $|\mathcal{G}_f(X)| = 3^{t+1}$, $t = \log |\mathcal{G}_f(X)| / \log 3 - 1$, hence $t \ll \log P$. Then, by the prime number theorem,

$$|\{p \in \mathcal{P} \mid p \le P\}| - t = \frac{P}{\log P}(1 + o(1)) - t \sim \frac{P}{\log P}.$$

Then, we have

$$\frac{|\{p \in \mathcal{P} \mid p \leq P, p \notin \mathcal{P}(X), p \in \mathcal{R}^3(X)\}|}{|\{p \in \mathcal{P} \mid p \leq P, p \notin \mathcal{P}(X)\}|} = \frac{|\{p \in \mathcal{P} \mid p \leq P, p \notin \mathcal{P}(X), p \in \mathcal{R}^3(X)\}|}{|\{p \in \mathcal{P} \mid p \leq P\}| - t}$$
$$\sim \frac{1}{6}\left(\frac{1}{3}\right)^t.$$

Therefore, we have following asymptotic equation:

$$|\{p \in \mathcal{P} \mid p \leq P, p \notin \mathcal{P}(X), p \in \mathcal{R}^3(X)\}| \sim \frac{P}{6 \log P}\left(\frac{1}{3}\right)^t \sim \frac{P}{2|\mathcal{G}_f(X)| \log P}$$

By the assumption, the proof completes. □

3.4 Nonvanishing theorem on cubic twists

Define the following prime sum over the special family $\mathcal{G}_f(X)$ of cubic twists of a primitive form $f \in S_2(N)$.

$$S(X, P) := -\sum_{\chi \in \mathcal{G}_f(X)} \sum_{p} \frac{\chi(p) a_p \log p}{\sqrt{p}} g\left(\frac{p}{P}\right). \tag{3.23}$$

In this section, we will establish an explicit formula involving the sum of the analytic ranks for $S(X, P)$ and show a nonvanishing theorem of $\mathcal{G}_f(X)$ under GRH and the following extra hypothesis.

Choose g satisfying Property 3.1 and a primitive form $f \in S_2(N)$. Consider

$$\sum_{\chi \in \mathcal{G}_f(X)} \sum_{\rho_\chi \notin \mathbb{R}} P^{\rho_\chi} \mathcal{M}g(\rho_\chi),$$

where ρ_χ are non-trivial zeroes of $L(f, s, \chi)$ and a real number $P > 1$. We need to estimate this sum in order to find a bound of the number of vanishings of the cubic twists for $\mathcal{G}_f(X)$. The non-trivial zeroes in the critical strip have poorly been studies and their distributions seems very difficult problem in present. Thus, we make the following hypothesis on the above sum, cf. Hypothesis M in [Fio16] for quadratic twists. He supported Hypothesis M with some statistical arguments in Appendix A.

Hypothesis 3.9. There exist $0 < \delta < 1$ and $0 < \eta < 1/2$ such that for $|\mathcal{G}_f(X)|^{2-\delta} \leq P \leq 2|\mathcal{G}_f(X)|^{2-\delta}$,

$$\sum_{\chi \in \mathcal{G}_f(X)} \sum_{\rho_\chi \notin \mathbb{R}} P^{\rho_\chi} \mathcal{M}g(\rho_\chi) = O_f\left(|\mathcal{G}_f(X)|^{1-\eta} P^{\frac{1}{2}}\right),$$

where ρ_χ are non-trivial zeroes of $L(f,s,\chi)$.

Denote a non-trivial zero $\rho_\chi = \sigma_\chi + i\tau_\chi$ of $L(f,s,\chi)$. Choose $g(x) = \max_{x\in(0,\infty)}\{1-x,0\}$ and put $P = e^y$. Then, under GRH, for a fixed X, we have

$$h_X(y) := \sum_{\chi\in\mathcal{G}_f(X)} \sum_{\rho_\chi\notin\mathbb{R}} P^{\rho_\chi - \frac{1}{2}} \mathcal{M}g(\rho_\chi) = \sum_{\chi\in\mathcal{G}_f(X)} \sum_{\rho_\chi\notin\mathbb{R}} \frac{e^{iy\tau_\chi}}{\rho_\chi(\rho_\chi+1)},$$

which can be considered as a $\mathbb{R}$-valued function of y since

$$L(f,\rho_\chi,\chi) = 0 \text{ if and only if } L(f,\overline{\rho_\chi},\overline{\chi}) = 0.$$

As Fiorilli support his similar hypothesis, cf. Hypothesis M in [Fio16], we can show that $h_X(y)$ possess a limiting distribution, and can compute the first and second moments of $h_X(y)$ assuming the boundedness of the multiset of all the nontrivial zeroes of $L(f,s,\chi)$ for $\chi\in\mathcal{C}_3$ and GRH. Then, considering $h_X(y)$ as a random variable, by the central limit theorem, we may obtain Hypothesis 3.9, refer to Appendix A in [Fio16], for these arguments for quadratic twists.

Proposition 3.10. Suppose that g is an $\mathbb{R}$-valued function on $[0,\infty)$ satisfying Property 3.1. Assume Hypothesis 3.9. Then, we have for some $0<\delta<1$,

$$P^{-\frac{1}{2}} S(X,P) \ll |\mathcal{G}_f(X)|^{1-\frac{\delta}{2}}$$

Proof. By taking the Hasse bound, i.e. $|a_p| \le 2\sqrt{p}$ and the trivial bound on $\log p$ and using the orthogonality (3.22) and Lemma 3.8, we have

$$\begin{aligned} S(X,P) &:= -\sum_{\chi\in\mathcal{G}_f(X)} \sum_{p} \frac{\chi(p)a_p\log p}{\sqrt{p}} g\left(\frac{p}{P}\right) = -\sum_{p} \frac{a_p\log p}{\sqrt{p}} g\left(\frac{p}{P}\right) \sum_{\chi\in\mathcal{G}_f(X)} \chi(p) \\ &\ll |\mathcal{G}_f(X)| \sum_{\substack{p\in\mathcal{P}, p\le P \\ p\notin\mathcal{P}(X) \\ p\in\mathcal{R}^3(X)}} \frac{a_p\log p}{\sqrt{p}} + \frac{|\mathcal{G}_f(X)|}{3} \sum_{\substack{1\le j\le t \\ p_j\in\mathcal{R}^3(X,j)}} \frac{a_p\log p}{\sqrt{p}} \\ &\ll |\mathcal{G}_f(X)| \log P \Big(\sum_{\substack{p\in\mathcal{P}, p\le P \\ p\notin\mathcal{P}(X) \\ p\in\mathcal{R}^3(X)}} 1 + \frac{1}{3} \sum_{\substack{1\le j\le t \\ p_j\in\mathcal{R}^3(X,j)}} 1 \Big) \ll P + |\mathcal{G}_f(X)|^{\varepsilon} \log P \end{aligned} \tag{3.24}$$

for any $\varepsilon > 0$. Note that $\log P \ll \log|\mathcal{G}_f(X)|$ and by the assumption, for some $0<\delta<1$,

$$|\mathcal{G}_f(X)|^{1-\frac{\delta}{2}} \le P^{\frac{1}{2}} \le \sqrt{2}|\mathcal{G}_f(X)|^{1-\frac{\delta}{2}}.$$

Therefore, dividing the both sides for $S(X,P)$ by $P^{1/2}$, the proof completes by following

$$P^{-\frac{1}{2}}S(X,P) \ll P^{\frac{1}{2}} + P^{-\frac{1}{2}}|\mathcal{G}_f(X)|^{\varepsilon}\log P \ll P^{\frac{1}{2}} \ll |\mathcal{G}_f(X)|^{1-\frac{\delta}{2}}.$$

□

We are now ready to establish an explicit formula for $S(X,P)$.

Proposition 3.11. Suppose that g is an $\mathbb{R}$-valued function on $[0,\infty)$ satisfying Property 3.1. Assume Hypothesis 3.9 and GRH for $L(f,s)$, $L(\mathrm{Sym}^m f_\chi, s)$, $L(s,\chi)$ and $L(f_\chi, s)$ for $m = 2,3$ and every $\chi \in \mathcal{G}_f(X)$. Then, for any $\varepsilon > 0$ and some η with $0 < \eta < 1/2$

$$P^{-\frac{1}{2}}S(X,P) = \mathcal{M}g\left(\frac{1}{2}\right)\sum_{\chi\in\mathcal{G}_f(X)} r_{\mathrm{an}}(f_\chi) + O_{f,g}\left(|\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}+\varepsilon} + |\mathcal{G}_f(X)|^{1-\eta}\right).$$

Proof. For each $\chi \in \mathcal{G}_f(X)$, we can write

$$\begin{aligned}\sum_{k\geq 1}\sum_p \chi^k(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/P) &= \sum_p \chi(p)(\alpha_p+\beta_p)(\log p)g(p/P)\\ &\quad+\sum_p \chi^2(p)(\alpha_p^2+\beta_p^2)(\log p)g(p^2/P) + \sum_{p\nmid\mathfrak{f}_\chi}(\alpha_p^3+\beta_p^3)(\log p)g(p^3/P)\\ &\quad+\sum_{k\geq 4}\sum_p \chi^k(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/P)\end{aligned}$$

Note that for the term for $k=2$ of the right hand side, we use $g(p^2/P) = g_2(p/\sqrt{P})$ and Lemma 3.5 to obtain

$$\sum_p \chi^2(p)(\alpha_p^2+\beta_p^2)(\log p)g(p^2/P) = \sum_p \chi^2(p)(\alpha_p^2+\beta_p^2)(\log p)g_2(p/\sqrt{P}) \ll_g P^{\frac{1}{4}}\log(\mathfrak{f}_\chi^2 N).$$

Moreover, for the term for $k=3$ of the right hand side, we use $g(p^3/P) = g_3(p/\sqrt[3]{P})$ and Lemma 3.6 to obtain

$$\sum_{p\nmid\mathfrak{f}_\chi}(\alpha_p^3+\beta_p^3)(\log p)g(p^3/P) = \sum_{p\nmid\mathfrak{f}_\chi}(\alpha_p^3+\beta_p^3)(\log p)g_3(p/\sqrt[3]{P}) \ll_g P^{\frac{1}{6}}\log(\mathfrak{f}_\chi^2 N),$$

and for the last term of the right hand side, we use (3.15) for $k \geq 3$ to obtain

$$\sum_{k\geq 4}\sum_p \chi^k(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/P) \ll P^{\frac{1}{4}} + \log(\mathfrak{f}_\chi^2 N).$$

On the other hand, we can rewrite Lemma 3.2 after replacing $x = P$ as

$$\sum_{k\geq 1}\sum_{p}\chi^k(p)(\alpha_p^k+\beta_p^k)(\log p)g(p^k/P)$$
$$= -r_{\mathrm{an}}(f_\chi)P^{\frac{1}{2}}(\mathcal{M}g)\left(\frac{1}{2}\right) - \sum_{\rho_\chi\notin\mathbb{R}} P^{\rho_\chi}(\mathcal{M}g)(\rho_\chi) + O(\log(\mathfrak{f}_\chi^2 N)).$$

Note that for each $\chi \in \mathcal{G}_f(X)$, the possible zero at $s = 1/2$ of $L(f, s, \chi)$ contributes exactly $P^{1/2}(\mathcal{M}g)\,(1/2)$ with multiplicity $r_{\mathrm{an}}(f_\chi)$ in the above sum. Combining all the bounds obtained above and putting these into the left hand side of the above equation, we have

$$\sum_{p}\chi(p)(\alpha_p+\beta_p)(\log p)g(p/P) = -\,r_{\mathrm{an}}(f_\chi)P^{\frac{1}{2}}(\mathcal{M}g)\left(\frac{1}{2}\right) - \sum_{\rho_\chi\notin\mathbb{R}} P^{\rho_\chi}(\mathcal{M}g)(\rho_\chi)$$
$$+\, O_g(P^{\frac{1}{4}}\log(\mathfrak{f}_\chi^2 N)) + O_g(P^{\frac{1}{4}}).$$

Summing this quantity over $\mathcal{G}_f(X)$ and dividing both sides by $P^{1/2}$, we have

$$P^{-\frac{1}{2}}S(X,P) = (\mathcal{M}g)\left(\frac{1}{2}\right)\sum_{\chi\in\mathcal{G}_f(X)} r_{\mathrm{an}}(f_\chi) + \sum_{\chi\in\mathcal{G}_f(X)}\sum_{\rho_\chi\notin\mathbb{R}} P^{(\rho_\chi-\frac{1}{2})}(\mathcal{M}g)(\rho_\chi)$$
$$+\, O_g(P^{-\frac{1}{4}}\sum_{\chi\in\mathcal{G}_f(X)}\log(\mathfrak{f}_\chi^2 N)) + O_{f,g}(P^{-\frac{1}{4}}\sum_{\chi\in\mathcal{G}_f(X)} 1).$$

Applying Hypothesis 3.9 and Lemma 3.7 to the second and third terms, respectively, we have for any $\varepsilon > 0$,

$$P^{-\frac{1}{2}}S(X,P) = (\mathcal{M}g)\left(\frac{1}{2}\right)\sum_{\chi\in\mathcal{G}_f(X)} r_{\mathrm{an}}(f_\chi)$$
$$+\, O_{f,g}(|\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}+\varepsilon} + P^{-\frac{1}{4}}|\mathcal{G}_f(X)| + |\mathcal{G}_f(X)|^{1-\eta}).$$

Finally, the assumption in Hypothesis 3.9 implies that

$$P^{-\frac{1}{4}}|\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}} \leq P^{-\frac{1}{4}}|\mathcal{G}_f(X)| \leq |\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}},$$

so that $|\mathcal{G}_f(X)|^{\frac{1}{2}+\frac{\delta}{4}} \ll |\mathcal{G}_f(X)|^{\frac{3}{4}}$ and the proof completes. $\square$

Now, we present the nonvanishing theorem on cubic twists of a primitive form $f \in S_2(N)$. Proposition 3.10 and 3.11 immediately implies the following theorem and corollary (also by the modularity theorem on elliptic curve).

Theorem 3.12. Let f be a primitive form in $S_2(N)$. Suppose that g is an $\mathbb{R}$-valued function on $[0,\infty)$ satisfying Property 3.1. Assume Hypothesis 3.9 and GRH for $L(f,s)$, $L(\mathrm{Sym}^m f_\chi, s)$, $L(s,\chi)$ and $L(f_\chi, s)$ for $m=2,3$ and every $\chi \in \mathcal{C}_{3,f}$. Then, as $X \to \infty$,

$$\sum_{\chi \in \mathcal{G}_f(X)} r_{\mathrm{an}}(f_\chi) \ll_{f,g} |\mathcal{G}_f(X)|^M, \text{ where } M = \max\left\{1-\frac{\delta}{2}, \frac{1}{2}+\frac{\delta}{4}+\varepsilon, 1-\eta\right\}.$$

Corollary 3.13. Let f be a primitive form in $S_2(N)$. Suppose that g is an $\mathbb{R}$-valued function on $[0,\infty)$ satisfying Property 3.1. Assume Hypothesis 3.9 and GRH for $L(f,s)$, $L(\mathrm{Sym}^m f_\chi, s)$, $L(s,\chi)$ and $L(f_\chi, s)$ for $m=2,3$ and every $\chi \in \mathcal{C}_{3,f}$. Then, there exist infinitely many $L(f,s,\chi)$ where $\chi \in \mathcal{C}_3$ such that $L(f,1/2,\chi) \neq 0$. In particular, for a given elliptic curve E defined over $\mathbb{Q}$, there exist infinitely many $L(E,s,\chi)$ where $\chi \in \mathcal{C}_{3,f}$ such that $L(f,1/2,\chi) \neq 0$.

3.5 Remarks on the result and further research

In addition to the result on nonvanishings of cubic twists of a fixed primitive form $f \in S_2(N)$ under Hypothesis 3.9 on the large set of non-trivial zeros and GRH, there are some remarks on vanishings from Theorem 3.12 for our special family $\mathcal{G}_f(X)$.

$\mathcal{G}_f(X)$ is ordered by the maximum prime $p \le X$, equivalently t-th prime p_t, such that $p \equiv 1 \bmod 3$. Let $Y(t) := \prod_{p \in \mathcal{P}(X)} p$, which is the maximum conductor of characters in $\mathcal{G}_f(X)$, and $G(t) := |\mathcal{G}_f(X)| = 3^{t+1}$. Then, by Theorem 3.12, we have the following average number of vanishings for $\mathcal{G}_f(X)$ on $G(t)$

$$\frac{1}{G(t)} \sum_{\substack{\chi \in \mathcal{G}_f(X) \\ L(f,\frac{1}{2},\chi)=0}} 1 \le \frac{1}{G(t)} \sum_{\chi \in \mathcal{G}_f(X)} r_{\mathrm{an}}(f_\chi) \ll_{f,g} G(t)^{M'}$$

where $M' = \max\{-\delta/2, -1/2+\delta/4+\varepsilon, -\eta\}$ for any $\varepsilon > 1$, some $0 < \delta < 1$ and $0 < \eta < 1/2$. The sharpest bound can be obtained when $\delta = 2/3 - \varepsilon$ and $\eta = 1/3 - \varepsilon$ so that

$$\frac{1}{G(t)} \sum_{\substack{\chi \in \mathcal{G}_f(X) \\ L(f,\frac{1}{2},\chi)=0}} 1 \ll_{f,g} G(t)^{-\frac{1}{3}+\varepsilon}. \tag{3.25}$$

Note that if we take the trivial estimate of

$$\sum_{k \ge 3} \sum_p \chi^k(p)(\alpha_p^k + \beta_p^k)(\log p) g(p^k/P), \tag{3.26}$$

we will get $-1/3+\delta/6$ as the exponent of $G(t)$ instead of $-1/3+\delta/4$ in the above average equation. Then, we can not obtain this sharper bound, $-1/3+\varepsilon$ on average.

While the usual family $\mathcal{C}_{3,f}(Y)$ is ordered by the maximum conductor $\mathfrak{f}_\chi \leq Y$. David, Fearnley and Kisilevsky in [DFK04] and [DFK06] expect that $N(Y) := \sum_{\chi\in\mathcal{C}_{3,f}(Y)} 1 \sim c_3 Y$, where $c_3 = 0.3170564\ldots$ Moreover, they showed using the geometry of numbers of the algebraic part of the central L-values and the random matrix theory that

$$\sum_{\substack{\chi\in\mathcal{C}_{3,f}(Y)\\ L(f,\frac{1}{2},\chi)=0}} 1 \sim b_f Y^{\frac{1}{2}} \log^\alpha Y \tag{3.27}$$

for some constants b_f depending on f and α. Then, the average number of vanishings on $N(Y)$ is given by

$$\frac{1}{N(Y)} \sum_{\substack{\chi\in\mathcal{C}_{3,f}(Y)\\ L(f,\frac{1}{2},\chi)=0}} 1 \sim b_f' Y^{-\frac{1}{2}} \log^\alpha Y$$

for some constant b_f'. Comparing this to (3.25), their bound $-1/2+\varepsilon$ is sharper than ours $-1/3+\varepsilon$ on average. We suspect that taking the trivial bound on the Fourier coefficients and the logarithms in (3.24) mainly contributes this higher bound. However, our bound is actually the bound of average analytic ranks.

$L(f,s,\chi)$ is a degree 2 L-function. We can generalise the special family of cubic twists for higher odd prime order l twists. Thus, we may be able to show the similar nonvanishing results for twists of order l under the similar hypotheses. However, for a sharper bound of the average number of vanishings using the special family for twists of order l as David, Fearnley and Kisilevsky [DFK04] and [DFK06], we may need to use $L(\mathrm{Sym}^k f_\chi, s)$ for $k=2,3,\ldots,l$ to compute non-trivial bounds of (3.26) for $k\leq 2$.

Chapter 4

Galois cohomology for special family of quadratic prime twists

4.1 Motivation and introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ of a square-free conductor N and discriminant Δ and χ_d be a quadratic Dirichlet character of fundamental discriminant d and χ_0 be the trivial Dirichlet character. Fix a positive real number X. Consider the family of primitive quadratic Dirichlet characters χ_d of fundamental discriminant d such that $|d| < X$:

$$\mathcal{X}(E, X) := \{\chi_d \mid \chi_d^2 = \chi_0 \text{ and } \chi_d \neq \chi_0 \text{ with } (N, d) = 1 \text{ and } |d| < X\} \tag{4.1}$$

Recall the L-function of E twisted by χ_d

$$L(E, s, \chi_d) := \sum_{n=1}^{\infty} \frac{\chi_d(n) a_n}{n^s}$$

Then, as seen in Chapter 1, it has the analytic continuation on $\mathbb{C}$ and the following functional equation:

$$\Lambda(E, s, \chi_d) := \left(\frac{|d|\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(E, s, \chi_d) = \omega(E)\chi_d(-N)\Lambda(E, 2-s, \chi_d),$$

where $\omega(E) \in \{-1, 1\}$ is the root number of E. This functional equation implies that if $\omega(E)\chi_d(-N) = -1$, then $L(E, 1, \chi_d) = 0$. So, we are interested in the following family of $L(E,s,\chi)$ with $\omega(E)\chi_d(-N) = +1$, which is called $L(E,s,\chi_d)$ with even functional equation:

$$\mathcal{S}(E,X) = \{L(E,s,\chi_d) \mid \chi_d \in \mathcal{X}(E,X) \text{ and } \omega(E)\chi_d(-N) = +1\} \tag{4.2}$$

Also, define the following families of quadratic prime twists of E in $\mathcal{S}(E,X)$: for each $a \in (\mathbb{Z}/N\mathbb{Z})^\times$,

$$\begin{aligned}
P_i(a,E) &:= \{L(E,s,\chi_d) \in \mathcal{S}(E,X) \mid |d| \text{ is a prime }, d < 0, -d \equiv a \bmod N\}\\
P_r(a,E) &:= \{L(E,s,\chi_d) \in \mathcal{S}(E,X) \mid |d| \text{ is a prime }, d > 0, d \equiv a \bmod N\}\\
P_i^0(a,E) &:= \{L(E,s,\chi_d) \in P_i(a,E) \mid L(E,1,\chi_d) = 0\}\\
P_r^0(a,E) &:= \{L(E,s,\chi_d) \in P_r(a,E) \mid L(E,1,\chi_d) = 0\}.
\end{aligned}$$

While investigating the distributions of nonvanishings for $\mathcal{S}(E,X)$ for various E with square-free conductor N and $X = 10^5$ using Rubinstein's L-function calculator built in Sage [S$^+$17], we found an interesting phenomenon as shown in Table 2 and 3 for examples. Note that the elliptic curves in the tables follow the Cremona's notation in [LMF13]. The data show that for E with some elliptic invariants, $L(E,1,\chi_d) \neq 0$ for some residue classes $|d|$ modulo N, more precisely depending on either the quadratic residue or non-residue modulo N. Note that, for the those elliptic curves E, half of residue classes of $(\mathbb{Z}/N\mathbb{Z})^\times$ have even functional equation for each real and imaginary quadratic prime twists.

Due to the large number of residue classes modulo N for $\phi(N)$ large, define the set of the residue classes with non-vanishings for imaginary and real quadratic prime twists of each E denoted by

$$N_i := \{a \in (\mathbb{Z}/N\mathbb{Z})^\times \mid P_i^0(a,E) = \emptyset\} \text{ and } N_r := \{a \in (\mathbb{Z}/N\mathbb{Z})^\times \mid P_r^0(a,E) = \emptyset\}$$

Table 5 shows the two disjoint lists of $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $a \in N_i$ and $a \notin N_i$ ($a \in N_r$ and $a \notin N_r$) for imaginary (resp. real) quadratic prime twists of each of the following 14 elliptic curves:

$$38b1, 39a1, 42a1, 43a1, 46a1, 55a1, 62a1, 65a1, 66a1, 66b1, 66c1, 69a1, 70a1, 77c1$$

We have the algebraic version of the quadratic twist of an elliptic curve E defined over a number field K. Since K has characteristic 0, one may write E as a Weierstrass equation of form

$$E : y^2 = x^3 + ax + b, \text{ where } a, b \in K.$$

Then, the quadratic twist of E by χ_d can be defined as

$$E^{\chi_d} : y^2 = x^3 + ad^2x + bd^3.$$

Note that each K-rational point (x, y) of E can be mapped to $(dx, d^{3/2}y)$ of E^{χ_d}, hence E^{χ_d} can be considered as an elliptic curve defined over $F := \sqrt{d}$. This implies that there exists a canonical isomorphism $\psi : E \to E^{\chi_d}$ over $\overline{K}$. For a number field K (or the completion K_ν of K at a place ν), denote the 2-torsion subgroup over $\overline{K}$ (or $\overline{K_\nu}$) by $E[2]$. Then, we can consider χ_d as a character

$$\chi_d : G := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(F/\mathbb{Q}) \to \{-1, 1\}$$

satisfying $\psi(P)^\sigma = \chi_d(\sigma)\psi(P^\sigma)$, where $\sigma \in G$ acts a point $P \in E(\overline{K})$ and $\psi(P) \in E^{\chi_d}(\overline{K})$ component-wisely. Thus, ψ induces an isomorphism (as G-modules) between $E[2]$ and $E^{\chi_d}[2]$.

To explain the phenomenon mentioned in the above paragraph, we can use the arguements of Kramer [Kra81] and Mazur and Rubin [MR10]. The idea of them is that we can control the 2-Selmer group for the image of Galois coholomogy of the abstract Galois group $G := \mathrm{Gal}(\overline{K}/K)$ with coefficients in the 2-torsion G-module under the localisation map, defined in the next section, and consider the Tate-Shafarevich group $Ш(E/K)$ and all of their dimensions as $\mathbb{F}_2$-vector spaces to restrict the ranks for E and E^{χ_d}. However, unfortunately, it turns out that their method seems very limited to explain this phenomenon and it works only for very special set of elliptic curves, see Theorem 4.9 for these special cases.

4.2 2-Selmer group and Tate-Shafarevich group

In this section, we first summarise definitions and properties of the zeroth and first group cohomology and then study the 2-Selmer group and Tate-Shafarevich group of an elliptic curve.

Let G be a group and M be a G-module which are both equipped with the discrete topology. Then, the zeroth group cohomology of G with coefficients in M is defined as

$$H^0(G, M) := M^G := \{m \in M \mid \sigma m = m \text{ for every } \sigma \in G\}.$$

A crossed homomorphism (or called 1-cocycle) f is defined as a map: $G \to M$ satisfying

$$f(\sigma\tau) = \sigma f(\tau) + f(\sigma) \text{ for every } \sigma, \tau \in G.$$

Let $Z^1(G, M)$ be the abelian group of those crossed homomorphisms. A principal crossed homomorphism (or called 1-coboundary) is defined as a crossed homomorphism such that

$$f(\sigma) = \sigma m - m \text{ for some } m \in M.$$

Let $B^1(G, M)$ be the set of all the principal crossed homomorphisms. Then, it is easy to see that $B^1(G, M)$ is a subgroup of $Z^1(G, M)$. Define the first group cohomology of G with coefficients in M is defined as

$$H^1(G, M) := Z^1(G, M)/B^1(G, M).$$

For an example, if G trivially acts on M, then $H^1(G, M) = \mathrm{Hom}(G, X)$, continuous homomorphisms of groups. For the general definitions of n-th cohomology groups for $n \in \mathbb{Z}$, refer to §2 in Chapter I in [Ser97]. For a number field K (or a completion K_ν of K by a place ν) and a $G_K := \mathrm{Gal}(\overline{K}/K)$ (or $G_{K_\nu} := \mathrm{Gal}(\overline{K_\nu}/K_\nu)$)-module M, denote $H^n(K, M) := H^n(G_K, M)$ (or $H^n(K_\nu, M) := H^n(G_{K_\nu}, M)$) for $n = 0, 1$.

Fix a number field K, an elliptic curve E defined over K and E^{χ_d} twisted by a quadratic character χ_d associated to F. Let ν be a place for K and K_ν be the completion of K by ν. Then, the canonical inclusions $G_{K_\nu} \subset G_K$ and $E(\overline{K}) \subset E(\overline{K_\nu})$ give us a restriction map loc on their cohomology groups and we define the Tate-Shafarevich group $Ш(E/K)$ as its kernel, i.e.

$$Ш(E/K) := \bigcap_\nu \ker\{\mathrm{loc}_\nu : H^1(G_K, E(\overline{K})) \to H^1(G_{K_\nu}, E(\overline{K_\nu}))\},$$

where $\mathrm{loc} = \oplus_\nu \mathrm{loc}_\nu$. Moreover, for each integer $m \geq 1$, the multiplication by m maps on $E(\overline{K})$ and $E(\overline{K_\nu})$ yield the following diagram of cohomologies:

$$\begin{tikzcd}
0 \arrow[r] & E(K)/mE(K) \arrow[r] \arrow[d] & H^1(K, E[m]) \arrow[r] \arrow[d, "\mathrm{loc}_\nu"] & H^1(K, E)[m] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E(K_\nu)/mE(K_\nu) \arrow[r, "\kappa_\nu"] & H^1(K_\nu, E[m]) \arrow[r] & H^1(K_\nu, E)[m] \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

Note that the second map of the bottom exact sequence is the local m-Kummer map $\kappa_\nu : E(K_\nu) \to H^1(K_\nu, E[m])$ defined as follows: for each $P \in E(K_\nu)$, since

$mE(\overline{K_\nu}) = E(\overline{K_\nu})$, there exists $\mathbb{Q} \in E(\overline{K_\nu})$ such that $P = mQ$. Then, for every $\sigma \in G_\nu := \mathrm{Gal}(\overline{K_\nu}/K_\nu)$, define

$$\kappa_\nu(P) = Q^\sigma - Q \in H^1(K_\nu, E[m]).$$

By checking the 1-cocycle and 1-coboundary conditions on $H^1(K_\nu, E[m])$, it is well-defined. Note that the kernel of κ_ν is $mE(K_\nu)$. Denote $H^1_f(K_\nu, E[2])$ the image of κ_ν. Note that $H^1_f(K_\nu, E[2])$ depends not only $E[2]$ but also E. Moreover, denote $Ш(E/K)[m]$ the kernel of multiplcation by m in $Ш(E/K)$, which is the intersection over all places ν of the kernels of the right-most vertical maps in (4.3). Define the m-Selmer group denoted by $\mathrm{Sel}_m(E/K)$ by the following exact sequence:

$$0 \longrightarrow \mathrm{Sel}_m(E/K) \longrightarrow H^1(K, E[m]) \xrightarrow{\mathrm{loc}} \bigoplus_\nu H^1(K_\nu, E[m])/H^1_f(K_\nu, E[m]) \quad (4.4)$$

Consult VIII.2 and X.4 in [Sil09] for the more details of the Kummer map for a number field K and Tate-Shafarevich and Selmer groups. From the above definitions, we have the following exact sequence:

$$0 \longrightarrow E(K)/mE(K) \longrightarrow \mathrm{Sel}_m(E/K) \longrightarrow Ш(E/K)[m] \longrightarrow 0$$

When m is a prime, each term in the above sequence is $\mathbb{F}_m$-vector space and we have the dimensional equality as

$$\dim_{\mathbb{F}_m}(\mathrm{Sel}_m(E/K)) = r_{\mathrm{al}}(E(K)) + \dim_{\mathbb{F}_m}(E(K)[m]) + \dim_{\mathbb{F}_m}(Ш(E/K)[m]). \quad (4.5)$$

Therefore, $r_{\mathrm{al}}(E(K)) \leq \dim_{\mathbb{F}_m}(\mathrm{Sel}_m(E/K)) - \dim_{\mathbb{F}_m}(E(K)[m])$. In particular, if $\dim_{\mathbb{F}_2}(Ш(E/K)[2]) < \infty$, then

$$r_{\mathrm{al}}(E(K)) \equiv \dim_{\mathbb{F}_m}(\mathrm{Sel}_m(E/K)) + \dim_{\mathbb{F}_m}(E(K)[m]) \bmod 2$$

by a consequence of the Cassels-Tate perfect pairing on $Ш(E/K)[2]$. Since E and E^{χ_d} are isomorphic over $\overline{K}$, $E(\overline{K})[2]$ and $E^{\chi_d}(\overline{K})[2]$ are isomorphic as G_K-modules. Hence, we can consider $S_2(E^{\chi_d}/K)$ and $H^1_f(K_\nu, E^{\chi_d}[2])$ as subgroups of $H^1(K, E[2])$ and $H^1(K_\nu, E[2])$, respectively, for every place ν. It implies that the difference between $\mathrm{Sel}_2(E/K)$ and $\mathrm{Sel}_2(E^{\chi_d}/K)$ depends only on the collection of differences between $H^1_f(K_\nu, E[2])$ and $H^1_f(K_\nu, E^{\chi_d}[2])$ over ν.

Denote $d_m V := \dim_{\mathbb{F}_m} V$ for the dimension of a $\mathbb{F}_m$-vector space V. Now, fix an elliptic curve E defined over $\mathbb{Q}$ of a square-free conductor N and discriminant Δ and the twist E^{χ_d} of E by a primitive quadratic Dirichlet character χ_d associated with $F := \mathbb{Q}(d)$. For every place ν of $\mathbb{Q}$, choose a place ω of F above ν. Then, by choosing $\sigma \in G_\nu := \mathrm{Gal}(F_\omega/\mathbb{Q}_\nu)$ such that $\sigma \neq 1$, the norm map $\mathrm{Norm} : E(F_\omega) \to E(\mathbb{Q}_\nu)$ is defined as for every $P \in E(F_\omega)$

$$\mathrm{Norm}(P) = P + P^\sigma.$$

Denote $\delta_\nu(E, F) := d_2(E(\mathbb{Q}_\nu)/\mathrm{Norm}(E(F_\omega)))$. Then, from the bottom exact sequence of (4.3), $E(\mathbb{Q}_\nu)/2E(\mathbb{Q}_\nu)$ is embedded to $H^1(\mathbb{Q}_\nu, E[2])$, more precisely

$$H^1_f(\mathbb{Q}_\nu, E[2]) \cong E(\mathbb{Q}_\nu)/2E(\mathbb{Q}_\nu). \tag{4.6}$$

We relate $\mathbb{F}_2$-dimensions of $H^1_f(\mathbb{Q}_\nu, E[2])$ and $E(\mathbb{Q}_\nu)[2]$ for every ν except for 2 and ∞.

Lemma 4.1. For $\nu \in \mathcal{V}$ such that $\nu \nmid 2\infty$, we have

$$d_2 H^1_f(\mathbb{Q}_\nu, E[2]) = d_2 E(\mathbb{Q}_\nu)[2].$$

Proof. For a finite prime $l > 2$ which is the characteristic of the residue field of $\mathbb{Q}_\nu$, it is a consequence of the topological structure of $E(\mathbb{Q}_\nu)$. More precisely, we have

$$H^1_f(\mathbb{Q}_\nu, E[2]) = E(\mathbb{Q}_\nu)/2E(\mathbb{Q}_n u).$$

Then, the proof immediately follows. Refer to Lemma 2.2 [MR07] for the proof. $\square$

Moreover, the following lemma allows us to identify an element in the intersection of $H^1_f(\mathbb{Q}_\nu, E[2])$ and $H^1_f(\mathbb{Q}_\nu, E^{\chi_d}[2])$ with that in $\mathrm{Norm}(E(F_\omega))/2E(\mathbb{Q}_\nu)$, which are proved by Mazur and Rubin [MR07] or Kramer [Kra81].

Lemma 4.2. Identifying $H^1_f(\mathbb{Q}_\nu, E[2]) = E(\mathbb{Q}_\nu)/2E(\mathbb{Q}_\nu)$, we have

$$H^1_f(\mathbb{Q}_\nu, E[2]) \cap H^1_f(\mathbb{Q}_\nu, E^{\chi_d}[2]) = \mathrm{Norm}(E(F_\omega))/2E(\mathbb{Q}_\nu).$$

Proof. Refer to Proposition 7 in [MR07] and Proposition 5.2 in [Kra81] for proof. $\square$

4.3 Local conditions

Follow the notations and assumptions in the previous section. In this section, we use the work of Mazur and Rubin [MR10] and Kramer [Kra81] to determine $\delta_\nu(E,F)$ relating the quadratic Hilbert norm residue symbols for $\mathbb{Q}$ with values of χ_d.

Let $\mathcal{V}$ be the set of all places ν of $\mathbb{Q}$, i.e.

$$\mathcal{V} := \{p \in \mathbb{Q} \mid p \text{ is a prime } \} \cup \{\infty\}.$$

Define the following subsets of $\mathcal{V}$:

$$\begin{aligned}
\mathcal{V}_\infty &:= \{\infty\} \\
\mathcal{V}_d &:= \{|d|\} \\
\mathcal{V}_1 &:= \{p \in \mathcal{V} \mid p \neq \infty, \text{ and } \chi_d(p) = 1\} \\
\mathcal{V}_{-1,g} &:= \{p \in \mathcal{V} \mid p \neq \infty, \chi_d(p) = -1 \text{ and } p \nmid N\} \\
\mathcal{V}_{-1,m} &:= \{p \in \mathcal{V} \mid p \neq \infty, \chi_d(p) = -1 \text{ and } p \mid N\}
\end{aligned}$$

Note that $|d|$ is a rational prime and $\mathcal{V}$ is the disjoint union of all subsets above. Note that for every $\nu \in \mathcal{V}\backslash(\mathcal{V}_\infty \cup \mathcal{V}_d)$, ν is never ramified in $F_\omega/\mathbb{Q}_\nu$. To see the effects of the group of quadratic residue modulo N for a square-free positive integer, we review the relations between χ_d and the quadratic Hilbert norm residue symbol. One can find detailed properties of the Hilbert norm residue symbols in Chapter III in [Ser73]. Recall that for each $\nu \in \mathcal{V}$, the quadratic Hilbert norm residue symbol is as a map

$$(\cdot,\cdot)_\nu : \mathbb{Z}\backslash\{0\} \times \mathbb{Z}\backslash\{0\} \to \{-1,1\}$$

defined by, for $a, b \in \mathbb{Z}\backslash\{0\}$,

$$(a,b)_\nu = \begin{cases} 1 & \text{if } ax^2 + by^2 = z^2 \text{ has a non-trivial solution in } \mathbb{Q}_\nu \\ -1 & \text{otherwise} \end{cases}.$$

Note that $(a,b)_\infty = 1$ if $ax^2+by^2 = z^2$ has a non-trivial solution in $\mathbb{R}$ and -1 otherwise. It is clear that

$$\begin{aligned}
(\Delta, d)_\nu = 1 &\iff \Delta x^2 + dy^2 = z^2 \text{ has a non-trivial solution in } \mathbb{Q}_\nu \\
&\iff \Delta \in \mathrm{Norm}_{F_\omega/\mathbb{Q}_\nu}(F_\omega),
\end{aligned}$$

where $\mathrm{Norm}_{F_\omega/\mathbb{Q}_\nu}$ is the field norm map for $F_\omega/\mathbb{Q}_\nu$. Also, by the local-global principle for ternary quadratic forms on $\mathbb{Q}$, $ax^2 + by^2 = z^2$ has a non-trivial solution in $\mathbb{Q}$ if and only if $(a,b)_\nu = 1$ for every $\nu \in \mathcal{V}$. Moreover, local class field theory implies that $(\cdot,\cdot)_\nu$ is a bi-multiplicative map, i.e. for $a,b,c \in \mathbb{Z}\backslash\{0\}$,

$$(ab,c)_\nu = (a,c)_\nu(b,c)_\nu \text{ and } (a,bc)_\nu = (a,b)_\nu(a,c)_\nu,$$

refer to Theorem 2 of Chapter III in [Ser73] for the proof. Indeed, it is a nondegenerate pairing on the $\mathbb{F}_2$-vector space $K^\times/(K^\times)^2$. Let $\nu \in \mathcal{V}$ be a finite place associated with a rational prime p. Write nonzero integers $a = p^{\nu(a)a'}$ and $b = p^{\nu(b)}b'$ with $p \nmid (a'b')$. Then, the following explicit formula gives us a relation between Hilbert norm residue symbol and primitive quadratic Dirichlet character.

$$(a,b)_\nu = \begin{cases} (-1)^{\nu(a)\nu(b)\varepsilon(p)}\left(\frac{a'}{p}\right)^{\nu(b)}\left(\frac{b'}{p}\right)^{\nu(a)} & \text{if } p \neq 2 \\ (-1)^{\varepsilon(a')\varepsilon(b')+\nu(a)\rho(b')+\nu(b)\rho(a')} & \text{if } p = 2 \end{cases}, \tag{4.7}$$

where $\varepsilon(n) \equiv (n-1)/2 \bmod n$ and $\rho(n) \equiv (n^2-1)/8 \bmod 2$ for $2 \nmid n$. Note that $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol, however, a p-adic unit on top of it is the image of the natural reduction modulo p, for example,

$$\left(\frac{a'}{p}\right) := \left(\frac{a' \bmod p}{p}\right).$$

Refer to Theorem 1 Chapter III in [Ser73] for the proof.

For the rest of this section, let E be an elliptic curve defined over $\mathbb{Q}$ of discriminant Δ and χ_d be a primitive quadratic Dirichlet character associated with $|d|$ is an odd prime and $F := \mathbb{Q}(\sqrt{d})$ such that $(\Delta, d) = 1$. Let $\widetilde{\Delta}$ be the product of $\operatorname{sign}(\Delta)$ and the square-free part of $|\Delta|$, i.e. for each of the largest prime factor p^m of Δ, if m is even, $p \nmid \widetilde{\Delta}$, and if m is odd, $p \mid \widetilde{\Delta}$ but $p^2 \nmid \widetilde{\Delta}$. Thus, we have $\operatorname{sign}(\Delta) = \operatorname{sign}(\widetilde{\Delta})$ and $(\Delta,d)_\nu = (\widetilde{\Delta},d)_\nu$.

Lemma 4.3. Let $\nu \in \mathcal{V}$ associated with a prime p. If $p \neq 2$, then we have

$$(\Delta,d)_\nu = \begin{cases} 1 & \text{if } p \nmid (\widetilde{\Delta}d) \\ \left(\frac{\widetilde{\Delta}}{p}\right) = \left(\frac{\Delta}{p}\right) & \text{if } p \mid d \\ \left(\frac{d}{p}\right) & \text{if } p \mid \widetilde{\Delta} \end{cases}.$$

If $p = 2$, then we have

$$(\Delta,d)_\nu = \begin{cases} -1 & \text{if } p \mid \widetilde{\Delta} \text{ and } d \equiv 5 \bmod 8 \\ 1 & \text{otherwise} \end{cases}.$$

Proof. Recall that $(\Delta, d)_\nu = (\widetilde{\Delta}, d)_\nu$ for every place ν and $(\Delta, d) = 1 = (\widetilde{\Delta}, d)$. Suppose that $p \neq 2$. Then, if $p \nmid (\widetilde{\Delta} d)$, then $\nu(\widetilde{\Delta}) = 0 = \nu(d)$. If $p \mid d$, then $\nu(\widetilde{\Delta}) = 0$ and $\nu(d) = 1$. If $p \mid \widetilde{\Delta}$, then $\nu(\widetilde{\Delta}) = 1$ and $\nu(d) = 0$. Hence, applying (4.7), we prove the first part for $p \neq 2$. Suppose that $p = 2$. Note that since d is a fundamental discriminant and $|d|$ is a prime, $d \equiv 1 \bmod 4$ so that $\varepsilon(d) \equiv 0 \bmod 2$. It also implies that $d \equiv 1$ or $5 \bmod 8$. Therefore, $(\Delta, d)_\nu = -1$ only if $p \mid \widetilde{\Delta}$ and $d \equiv 5 \bmod 8$ and it completes the proof. □

Kramer [Kra81] and Mazur and Rubin [MR10] computed $\delta_\nu(E, F)$ for each $\nu \in \mathcal{V}$ according to ramification of ν in $F/\mathbb{Q}$ and reduction type of E using Tate curve, local class field theory and etc. Their results are depending on the parity of Δ, the values of $(\Delta, d)_\nu$, and we translate their results into Legenre symbols using Lemma 4.3 in the following proposition.

Proposition 4.4. Let E be an elliptic curve defined over $\mathbb{Q}$ of a square-free conductor N and a discriminant Δ. Let χ_d be the primitive quadratic Dirichlet characters of fundamental discriminant d with $(N, d) = 1$. Then, we have

a) For $\nu \in \mathcal{V}_\infty$,

$$\delta_\nu(E, F) = \begin{cases} 1 & \text{if } \Delta > 0 \text{ and } \chi_d(-1) = -1 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

b) For $\nu \in \mathcal{V}_1 \cup \mathcal{V}_{-1,g}$,

$$\delta_\nu(E, F) = 0$$

c) For $\nu \in \mathcal{V}_{-1,m}$,

$$\delta_\nu(E, F) = \begin{cases} 1 & \text{if } E \text{ has non-split or split multiplicative reduction at } \nu \\ & \quad \text{and } \nu(\Delta) \equiv 0 \bmod 2 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

d) For $\nu \in \mathcal{V}_d$,

$$\delta_\nu(E, F) = \begin{cases} 0 \text{ or } 2 & \text{if } \left(\frac{\Delta}{|d|}\right) = 1 \\ 1 & \text{if } \left(\frac{\Delta}{|d|}\right) = -1 \end{cases}$$

Proof. Let p be a finite prime associated with a finite place $\nu \in \mathcal{V}$ and ω be a choice of valuation above ν.

a) For $\nu \in \mathcal{V}_\infty$, if $\chi_d(-1) = 1$, i.e. F is a totally real quadratic field, then $E(F_\omega) = E(\mathbb{R})$. So, it is clear that $\delta_\nu(E, F) = 0$. Suppose that $\chi_d(-1) = -1$, i.e. F is a totally complex quadratic field. Then, since $\mathrm{Norm}(E(\mathbb{C})) = 2E(\mathbb{R})$ under the complex conjugate and $\mathrm{sign}(\Delta)$ determines the connectedness of $E(\mathbb{R})$, we obtain the desired result.

b) For $\nu \in \mathcal{V}_1$, p is split in $F/\mathbb{Q}$. Thus, $\mathrm{Norm}(E(F_\omega)) = E(\mathbb{Q}_\nu)$, which immediately implies the desired result. For $\nu \in \mathcal{V}_{-1,g}$, the desired result is shown in Corollary 4.4 of [Maz72].

c) Suppose that $\nu \in \mathcal{V}_{-1,m}$. If E has non-split multiplication at ν, then by Proposition 1 in [Kra81],

$$\delta_\nu(E, F) = \begin{cases} 1 & \text{if } \nu(\Delta) \equiv 0 \bmod 2 \\ 0 & \text{if } \nu(\Delta) \equiv 1 \bmod 2 \end{cases}$$

Moreover, if E has split multiplication at ν, then by Proposition 2 in [Kra81],

$$\delta_\nu(E, F) = \begin{cases} 1 & \text{if } (\Delta, d)_\nu = 1 \\ 0 & \text{if } (\Delta, d)_\nu = -1 \end{cases}$$

Then, we apply Lemma 4.3 to obtain the desired results. Note that for these finite prime p we have $\chi_d(p) = -1$ so that we discard the cases that $\left(\frac{d}{p}\right) = 1$ for odd primes p and $d \not\equiv 5 \bmod 8$ for $p = 2$, which imply $(\Delta, d)_\nu = 1$.

d) For $\nu \in \mathcal{V}_d$, by Proposition 3 in [Kra81],

$$\delta_\nu(E, F) = d_2(\widetilde{E}(\mathbb{F}_{|d|})[2]) \equiv \begin{cases} 0 \bmod 2 & \text{if } (\Delta, d)_\nu = 1 \\ 1 \bmod 2 & \text{if } (\Delta, d)_\nu = -1 \end{cases}$$

Note that by the arguments of Serre in §5.3 of [Ser72],

$$\begin{aligned} d_2(\widetilde{E}(\mathbb{F}_{|d|})[2]) \text{ is even. } &\iff \Delta \text{ is square in } \mathbb{F}_{|d|} \text{ on reduction modulo } |d|. \\ &\iff (\Delta, d)_\nu = 1. \end{aligned}$$

Finally, by considering the structure of 2-torsion subgroup of $\widetilde{E}(\mathbb{F}_{|d|})$ and Lemma 4.3 for the case when $p \mid d$, the desired result also follows.

□

4.4 Tate local and Poitou-Tate global dualities

In this section, we study the Tate local duality and Poitou-Tate global duality and use it with the results in the previous section to deduce the nonvanishing theorem, which will be presented in the next section, adopting the arguments to control 2-Selmer groups in the work of Mazur and Rubin [MR10].

Let $\mathcal{W}$ be a finite set of $\mathcal{V}$ for $\mathbb{Q}$. Define the localisation map restricted on $\mathcal{W}$ as

$$\mathrm{loc}_{\mathcal{W}} : H^1(\mathbb{Q}, E[2]) \to \bigoplus_{\nu \in \mathcal{W}} H^1(\mathbb{Q}_\nu, E[2]).$$

Also, define the subgroups $S^{\mathcal{W}}$ and $S_{\mathcal{W}}$ of $H^1(\mathbb{Q}, E[2])$ by the exact sequences below:

$$0 \longrightarrow S^{\mathcal{W}} \longrightarrow H^1(\mathbb{Q}, E[2]) \longrightarrow \bigoplus_{\nu \notin \mathcal{W}} \left(H^1(\mathbb{Q}_\nu, E[2])/H^1_f(\mathbb{Q}_\nu, E[2])\right),$$

$$0 \longrightarrow S_{\mathcal{W}} \longrightarrow S^{\mathcal{W}} \xrightarrow{\mathrm{loc}_{\mathcal{W}}} \bigoplus_{\nu \in \mathcal{W}} H^1(\mathbb{Q}_\nu, E[2]).$$

It is clear from the definition of 2-Selmer group, see (4.4), that

$$S_{\mathcal{W}} \subset \mathrm{Sel}_2(E/\mathbb{Q}) \subset S^{\mathcal{W}} \subset H^1(\mathbb{Q}, E[2]).$$

Let K be a number field and ν be a place for K. Let E be an elliptic curve defined over K. Then, we have the following theorems known as the Tate local duality and the Poitou-Tate global duality for the Galois cohomology of K and E. The proofs of them can be found in Rubin's book on Euler systems [Rub00]. In Tate local duality, we need the cup product of two Galois cohomology groups, which is defined as follows: For a given finite or pro-finite group G, let M_1, M_2 and M_3 be G-modules and $\Psi : M_1 \otimes M_2 \to M_3$ be a G-module homomorphism. Then, the cup product is a map $H^i(G, M_1) \times H^j(G, M_2) \to H^{i+j}(G, M_3)$. Since we will be interested in the case where $i + j = 2$, we illustrate the definition of the cup product for the cases where $i = 0$ and $j = 2$ and $i = j = 1$ here. For $i = 0$ and $j = 2$, choose $\phi \in H^2(G, M_2)$ and $m \in H^0(G, M_1) = M_1^G$. Then, $(m, \phi) : G \times G \to M_3$ is defined as for $g, h \in G$,

$$(m, \phi)(g, h) := \Psi(m \otimes \phi(g, h)).$$

When $i = 2$ and $j = 0$, the definition of the cup product is similar. For $i = j = 1$, the cup product of $\phi_1 \in H^1(G, M_1)$ and $\phi_2 \in H^1(G, M_2)$ is defined as for $g, h \in G$,

$$(\phi_1, \phi_2)(g, h) := \Psi(\phi_1(g) \otimes g(\phi_2(h))).$$

It is clear that their images are in $H^2(G, M_3)$.

Theorem 4.5. (Tate local duality)

a) For every $n \geq 0$, $H^n(K_\nu, E[2])$ is finite. Moreover, $H^n(K_\nu, E[2]) = 0$ if $n \geq 3$.

b) For $n = 1, 2$, the following cup product is a non-degenerate pairing:

$$H^n(K_\nu, E[2]) \times H^{2-n}(K_\nu, E[2]) \to H^2(K_\nu, \overline{K_\nu}^\times) \cong \mathbb{Q}/\mathbb{Z}$$

Note that $E[2]$ is self-dual and the $\mathrm{Gal}(\overline{K_\nu}/K_\nu)$-module homomorphism on $E[2] \otimes E[2]$ is given by the Weil pairing. For $n = 1$ and $K = \mathbb{Q}$, the Tate local duality theorem implies that the cup product is a perfect pairing

$$\langle \cdot, \cdot \rangle_\nu : H^1(\mathbb{Q}_\nu, E[2]) \times H^1(\mathbb{Q}_\nu, E[2]) \to (H^2(\mathbb{Q}_\nu, \mu_2) \cong \mathbb{Z}/2\mathbb{Z})$$

where μ_2 is the primitive quadratic root of unity. Moreover, $H^1_f(\mathbb{Q}_\nu, E[2])$ is orthogonal complement of itself in $H^1(\mathbb{Q}_\nu, E[2])$ with respect to this cup product, refer to Proposition 4.2 in [Rub00] for the proof. Hence, we have

$$d_2 H^1(\mathbb{Q}_\nu, E[2]) = 2 d_2 H^1_f(\mathbb{Q}_\nu, E[2]). \tag{4.10}$$

Keeping the notations in the above theorem, we have the Poitou-Tate global duality as below.

Theorem 4.6. (Poitou-Tate global duality) Let $\mathcal{W}$ be a finite set of places of $\mathcal{V}$ for a number field K and $\mathcal{W}_0 \subset \mathcal{W}$. Then, we have

a) Following sequences are exact:

$$0 \longrightarrow S^{\mathcal{W}_0} \longrightarrow S^{\mathcal{W}} \xrightarrow{\mathrm{loc}_{\mathcal{W}}} \bigoplus_{\nu \in \mathcal{W} \setminus \mathcal{W}_0} (H^1(K_\nu, E[2])/H^1_f(K_\nu, E[2]))$$

$$0 \longrightarrow S_{\mathcal{W}} \longrightarrow S_{\mathcal{W}_0} \xrightarrow{\mathrm{loc}_{\mathcal{W}}} \bigoplus_{\nu \in \mathcal{W} \setminus \mathcal{W}_0} H^1_f(K_\nu, E[2])$$

b) The images of $S^{\mathcal{W}}$ and $S_{\mathcal{W}_0}$ of the rightmost maps in the above sequences are mutually orthogonal complements with respect to $\sum_{\nu \in \mathcal{W} \setminus \mathcal{W}_0} \langle \cdot, \cdot \rangle_\nu$.

Proof. Refer to Theorem 7.3 in [Rub00] for the proof. $\square$

Let $\mathcal{W}_0 = \emptyset$. Then, the definitions of $S^{\mathcal{W}_0}$ and $S_{\mathcal{W}_0}$ implies that $S^{\mathcal{W}_0} = S_{\mathcal{W}_0} = \mathrm{Sel}_2(E/K)$. Furthermore, the images of $S^{\mathcal{W}}$ and $\mathrm{Sel}_2(E/K)$ of the rightmost maps in the above sequences are isomorphic to $S^{\mathcal{W}}/\mathrm{Sel}_2(E/K)$ and $\mathrm{Sel}_2(E/K)/S_{\mathcal{W}}$, respectively. Then, b) in Theorem 4.6 asserts that

$$d_2(S^{\mathcal{W}}/\mathrm{Sel}_2(E/K)) + d_2(\mathrm{Sel}_2(E/K)/S_{\mathcal{W}}) = \sum_{\nu \in \mathcal{W}} d_2(H^1(K_\nu, E[2])/H^1_f(K_\nu, E[2]))$$

Therefore, by the Tate local duality (4.10), we have

$$d_2 S^{\mathcal{W}} - d_2 S_{\mathcal{W}} = \sum_{\nu \in \mathcal{W}} d_2 H^1_f(K_\nu, E[2]). \tag{4.11}$$

Keeping the notations introduced before in this section for $\mathbb{Q}$, we need the Kramer's congruence relation on 2-Selmer groups for $E/\mathbb{Q}$ and $E^{\chi_d}/\mathbb{Q}$ with $\delta_\nu(E, F)$ for all $\nu \in \mathcal{V}$ as shown below in controlling 2-Selmer groups.

Theorem 4.7. (Kramer [Kra81]) we have

$$d_2\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q}) \equiv d_2\mathrm{Sel}_2(E/\mathbb{Q}) + \sum_{\nu \in \mathcal{V}} \delta_\nu(E, F) \bmod 2.$$

Proof. Refer to Theorem 2.7 in [MR10] as a consequence of Theorem 1 and 2 of [Kra81] for the proof. □

4.5 Controlling 2-Selmer groups and nonvanishing theorem

Following the arguments in Proposition 3.3 of of Mazur and Rubin [MR10], we can restrict $\mathrm{Sel}_2(E^{\chi_d}\mathbb{Q})$ and obtain $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q}))$ for special cases of E as we saw in Table 2, 3, 4 and 5. Note that in their arguments they only consider the ramified places ν dividing possibly composite conductor of quadratic characters.

However, in our case, we only take prime twists and consider the (possibly infinite and finite) places ν dividing $(N\infty)$ such that $d_2 H^1_f(\mathbb{Q}_\nu, E[2]) = \delta_\nu(E, F) = 1$. Note that under this condition we can explain only partial residue classes modulo the conductor of those elliptic curves with $r_{\mathrm{al}}(E^{\chi_d}/\mathbb{Q})$.

For the rest of this section, we assume that E is an elliptic curve defined over $\mathbb{Q}$ of discriminant Δ and square-free conductor N, and fix χ_d, the primitive quadratic

Dirichlet character of fundamental discriminant d with an odd prime $|d|$, $(N, d) = 1$ and $\omega(E)\chi_d(-N) = 1$. Lastly, denote $F := \mathbb{Q}(\sqrt{d})$. Let p be a (finite or infinite) prime for $\mathbb{Q}$ with $p \neq 2$ and ω be a choice of valuation above p for completion of F. The following lemma shows the intersection of two local conditions at each $p \neq 2$ depending on local invariants δ_p and global invariants Δ and $d_2(E(\mathbb{Q}_p)/2E(\mathbb{Q}_p))$. Denote $I_f := H^1_f(\mathbb{Q}_p, E[2]) \cap H^1_f(\mathbb{Q}_p, E^{\chi_d}[2])$.

Lemma 4.8. Assume the settings of E and χ_d as above. Then, we have

a) For every prime p,

$$H^1_f(\mathbb{Q}_p, E[2]) = H^1_f(\mathbb{Q}_p, E^{\chi_d}[2]) \text{ if and only if } \delta_p(E, F) = 0. \tag{4.12}$$

b) For every $p \nmid (2\infty)$,

$$d_2 I_f = 0 \text{ if and only if } \delta_p(E, F) = d_2(E(\mathbb{Q}_p)[2]). \tag{4.13}$$

In particular, if $p = |d|$, then we have

$$d_2 I_f = 0 \text{ and } d_2 H^1_f(\mathbb{Q}_p, E[2]) = \delta_p(E, F). \tag{4.14}$$

c) If $p = \infty$, then we have

$$d_2 H^1_f(\mathbb{Q}_p, E[2]) = 1 \text{ if } \Delta > 0 \text{ and } 0 \text{ otherwise.} \tag{4.15}$$

Moreover, if either $\Delta < 0$ or $\Delta > 0$ and $d < 0$, then we have $d_2 I_f = 0$.

d) If $p \neq 2$ with place ν such $p \mid N$, $\nu(\Delta) \equiv 0 \bmod 2$ and $\chi_d(p) = -1$, then we have

$$d_2 I_f = \begin{cases} 0 & \text{if } d_2(E(\mathbb{Q}_p)/2E(\mathbb{Q}_p)) = 1 \\ 1 & \text{if } d_2(E(\mathbb{Q}_p)/2E(\mathbb{Q}_p)) = 2 \end{cases}. \tag{4.16}$$

Proof. a) From Lemma 4.2, the proof follows by observing

$$H^1_f(\mathbb{Q}_p, E[2]) = H^1_f(\mathbb{Q}_p, E^{\chi_d}[2]) \iff \mathrm{Norm}(E(F_\omega)) = E(\mathbb{Q}_p) \iff \delta_p(E, F) = 0.$$

b) Observe that

$$d_2 I_f = 0 \iff \mathrm{Norm}(E(F_\omega)) = 2E(\mathbb{Q}_p) \iff \delta_p(E, F) = d_2(E(\mathbb{Q}_p)[2]).$$

Therefore, the result of the first part follows by Lemma 4.1. If $p = |d|$, then by following Proposition 3 of [Kra81], we have $E(\mathbb{Q}_{|d|})/\mathrm{Norm}(E(F_\omega)) \cong \tilde{E}(\mathbb{F}_p)/2\tilde{E}(\mathbb{F}_p)$. Moreover, observe that

$$\begin{aligned} P \in \mathrm{Norm}(E(F_\omega)) &\iff \tilde{P} \in 2\tilde{E}(\mathbb{F}_p) \\ &\iff \tilde{P} = 2\tilde{Q} \text{ for some } Q \in E(\mathbb{Q}_p) \\ &\iff P \in 2E(\mathbb{Q}_p). \end{aligned}$$

Therefore, $\mathrm{Norm}(E(F_\omega)) = 2E(\mathbb{Q}_p)$, the second part follows directly from the first part.

c) If $p = \infty$, it is well-known, refer to V.2 of Silverman [Sil94], that

$$d_2(E(\mathbb{R})/2E(\mathbb{R})) = 1 \text{ if } \Delta > 0 \text{ and } 0 \text{ if } \Delta < 0.$$

Therefore, if $\Delta < 0$, then $d_2 I_f = 0$ by (4.8). Moreover, the hypothesis ensures that $\delta_p(E, F) = 1$. Therefore, the proof immediately follows by (4.13).

d) The hypothesis implies from (4.9) that $\delta_p(E, F) = 1$. Thus, the desired result follows from (4.13).

□

Theorem 4.9. Let E and χ_d be as Lemma 4.8 such that

$$d_2(E(\mathbb{Q})[2]) > 0, Ш(E/\mathbb{Q})[2] = 1, \text{ and } r_{\mathrm{al}}(E(\mathbb{Q})) = 0. \tag{4.17}$$

Suppose that E and χ_d satisfy the following conditions:

a) 2 splits in $F/\mathbb{Q}$ or E has good reduction at 2.

b) $\left(\frac{\Delta}{|d|}\right) = -1$ and 1 if $d_2(E(\mathbb{Q})[2]) = 1$ and 2, respectively.

c) if $d_2(E(\mathbb{Q})[2]) = 2$, $d < 0$ and $\Delta > 0$, then there exists an odd prime $p \mid N$ such that $d_2(E(\mathbb{Q})[2]) = 2$.

d) if $d_2(E(\mathbb{Q})[2]) = 1$ and either $\Delta < 0$ or $\Delta > 0$ and $d > 0$, then there exists an odd prime $p \mid N$.

Then, for those E and χ_d, $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q})) = 0$. Moreover, there exists a residue classes $|d| \bmod N$ such that $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q})) = 0$ for every such χ_d, and in particular, there are infinitely many χ_d such that $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q})) = 0$. Under the Birch and Swinnerton-Dyer conjecture 1.4.1, for those E^{χ_d}, $L(E, 1, \chi_d) \neq 0$.

Proof. We use the arguments in Proposition 3.3 of Mazur and Rubin [MR10]. Let $\mathcal{W}$ be a finite set of (finite or infinite) primes for $\mathbb{Q}$ such that $\delta_p(E,F)=0$ for all other primes $p\notin\mathcal{W}$. Denote

$$\begin{aligned}
\mathrm{S}_2 &:= \mathrm{Sel}_2(E/\mathbb{Q})\cap\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})\\
V_{\mathcal{W}} &:= \mathrm{loc}_{\mathcal{W}}(\mathrm{Sel}_2(E/\mathbb{Q}))\subset\bigoplus_{p\in\mathcal{W}}H^1_f(\mathbb{Q}_p,E[2])\\
V_{\mathcal{W}}^{\chi_d} &:= \mathrm{loc}_{\mathcal{W}}(\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q}))\subset\bigoplus_{p\in\mathcal{W}}H^1_f(\mathbb{Q}_p,E^{\chi_d}[2]).
\end{aligned}$$

Then, by (4.12), $H^1_f(\mathbb{Q}_p,E[2])=H^1_f(\mathbb{Q}_p,E^{\chi_d}[2])$ for $p\notin\mathcal{W}$, in particular, for $p=2$ by hypothesis a), hence we have

$$S_{\mathcal{W}}\subset\mathrm{S}_2\subset\mathrm{Sel}_2(E/\mathbb{Q}),\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})\subset S^{\mathcal{W}}.$$

Moreover, we have the following two exact sequences:

$$0\longrightarrow S_{\mathcal{W}}\longrightarrow\mathrm{Sel}_2(E/\mathbb{Q})\xrightarrow{\mathrm{loc}_{\mathcal{W}}}V_{\mathcal{W}}\longrightarrow 0$$

$$0\longrightarrow S_{\mathcal{W}}\longrightarrow\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})\xrightarrow{\mathrm{loc}_{\mathcal{W}}}V_{\mathcal{W}}^{\chi_d}\longrightarrow 0$$

Therefore, we have

$$d_2\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})-d_2\mathrm{Sel}_2(E/\mathbb{Q})=d_2V_{\mathcal{W}}^{\chi_d}-d_2V_{\mathcal{W}}. \tag{4.18}$$

It implies that $d_2V_{\mathcal{W}}^{\chi_d}=d_2V_{\mathcal{W}}$ if and only if $d_2\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})=d_2\mathrm{Sel}_2(E/\mathbb{Q})$, and the equality implies that $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q}))=0$ as well. Then, in the other hand, we can write

$$\begin{aligned}
d_2V_{\mathcal{W}}+d_2V_{\mathcal{W}}^{\chi_d} &= d_2(\mathrm{Sel}_2(E/\mathbb{Q})/S_{\mathcal{W}})+d_2(\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q}/S_{\mathcal{W}})\\
&= d_2(\mathrm{Sel}_2(E/\mathbb{Q})/\mathrm{S}_2)+d_2(\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q})/\mathrm{S}_2)+2(d_2\mathrm{S}_2-d_2S_{\mathcal{W}})\\
&= d_2((\mathrm{Sel}_2(E/\mathbb{Q})+\mathrm{Sel}_2(E^{\chi_d}/\mathbb{Q}))/\mathrm{S}_2)+2(d_2\mathrm{S}_2-d_2S_{\mathcal{W}})\\
&\le d_2(S^{\mathcal{W}}/S_{\mathcal{W}})+d_2\mathrm{S}_2-d_2S_{\mathcal{W}}\\
&= d_2(S^{\mathcal{W}})-d_2(S_{\mathcal{W}})+d_2\mathrm{S}_2.
\end{aligned} \tag{4.19}$$

Now, notice that since $d_2(E(\mathbb{Q})[2])>0$, $d_2(E(\mathbb{Q}_{|d|}[2])=\delta_{|d|}(E,F)>0$. Therefore, $|d|$ should be always in $\mathcal{W}$ and by (4.13), $H^1_f(\mathbb{Q}_p,E[2])\cap H^1_f(\mathbb{Q}_p,E^{\chi_d}[2])=0$. Then, by hypothesis b) we have $d_2\mathrm{Sel}_2(E/\mathbb{Q})=\delta_{|d|}(E,F)=d_2H^1_f(\mathbb{Q}_{|d|},E[2])$. Again, since $d_2(E(\mathbb{Q})[2])>0$, we have $d_2V_{\mathcal{W}}>0$, hence $d_2\mathrm{S}_2=S_{\mathcal{W}}=0$. Therefore, from (4.19) and (4.11), we have

$$d_2V_{\mathcal{W}}^{\chi_d}-d_2V_{\mathcal{W}}\le\sum_{p\in\mathcal{W}}H^1_f(\mathbb{Q}_p,E[2])-2d_s\mathrm{Sel}_2(E/\mathbb{Q}). \tag{4.20}$$

Therefore, if the right hand side of the above inequality is 0 or 1 and satisfies the Kramer's congruence relation (4.7), then Under the hypothesis, we will choose a suitable set of (finite or infinite) primes of $\mathbb{Q}$ for which $d_2V_{\mathcal{W}}^{\chi_d} = d_2V_{\mathcal{W}}$. In order to do that we need consider the two cases for the signs of Δ and d.

For the case when $d < 0$ and $\Delta > 0$, it is always that $\delta_\infty(E, F) = d_2H^1_f(\mathbb{R}, E[2]) = 1$ by (4.8) and (4.15). By hypothesis b) and c), we can choose $\mathcal{W} := \{\infty, |d|\}$ if $d_2(E(\mathbb{Q})[2]) = 1$ and $\mathcal{W} := \{\infty, p, |d|\}$ such that p is an odd prime with $p \mid N$ and $d_2(E(\mathbb{Q}_p)[2]) = 2$ if $d_2(E(\mathbb{Q})[2]) = 2$. Since in this case, the right hand side of (4.20) is 0 and 1 if $d_2(E(\mathbb{Q}_p)[2]) = 1$ and 2, respectively. It is also easy to check that they respect (4.7).

For the other case, i.e. when $d_2(E(\mathbb{Q})[2]) = 1$ and either $d > 0$ and $\Delta > 0$ or $\Delta < 0$, it is always that $\delta_\infty(E, F) = 0$. By hypothesis d), choose $\mathcal{W} := \{p, |d|\}$ with $p \mid N$. Then, the right hand side of (4.20) is 0 and 1 if $d_2(E(\mathbb{Q}_p)[2]) = 1$ and 2, respectively. Again, (4.7) is fully respected in any case above. Note that those choices of $\mathcal{W}$ are the only possible ones to obtain $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q})) = 0$ using the above arguments. Now the existence of such χ_d implies that $r_{\mathrm{al}}(E^{\chi_d}(\mathbb{Q})) = 0$, hence $L(E, 1, \chi_d) \neq 0$ under the Birch and Swinnerton-Dyer conjecture 1.4.1. The last statement follows since the hypotheses guarantee that there exists a residue class $|d|$ modulo N. $\square$

4.6 Examples and tables

We present some examples for Theorem 4.9 and tables of nonvanishings of quadratic twists per residue classes in this section. Unfortunately, there are only a few of E and residue classes modulo N which satisfy the hypotheses as in Theorem 4.9. For example, we only can find the examples applicable by Theorem 4.9 for E with $d_2(E(\mathbb{Q})[2]) = 1$. For abuse of notations, we use ∞ or a rational prime p for each place $\nu \in \mathcal{V}$ and let $F := \mathbb{Q}(\sqrt{d})$. Note that the labels of elliptic curves are due to the Cremona's.

Example ($E := 17a1$) We have the following invariants for E:

$$N = 17, \omega(E) = 1, \Delta = -17^4, E(\mathbb{Q})[2] \cong \mathbb{Z}/4\mathbb{Z}, r_{\mathrm{al}}(E(\mathbb{Q})) = 0, Ш(E/\mathbb{Q})[2] = 0.$$

It follows that $d_2\mathrm{Sel}_2(E/\mathbb{Q}) = d_2(E(\mathbb{Q})[2]) = 1$. Hypothesis a) is satisfied since $2 \nmid N$. Since $\left(\frac{\Delta}{|d|}\right) = \left(\frac{-1}{|d|}\right) = -1$ for $d < 0$, $\delta_{|d|}(E, F) = 1 = d_2(E(\mathbb{Q})[2])$ satisfying hypothesis b). Choose $\mathcal{W} = \{17, |d|\}$ such that $d < 0$ satisfying hypothesis d). Using Sage [S+17], we obtain $d_2(E(\mathbb{Q}_{17})[2]) = 2$. Then, the right hand side of equation (4.20) is

$$d_2 H^1_f(\mathbb{Q}_{17}, E[2]) + \delta_{|d|}(E, F) - 2d_2\mathrm{Sel}_2(E/\mathbb{Q}) = 1.$$

By the Kramer congruence relation 4.7, we have

$$\delta_{17}(E, F) + \delta_{|d|}(E, F) + 1 \equiv 0 \bmod 2.$$

We can find the residue classes $|d|$ modulo 17 such that $\left(\frac{-1}{|d|}\right) = -1$ and $\chi_d(-17) = 1$. By using the Chinese remainder theorem, we obtain the following residue classes as shown in Table 2:

$$|d| \equiv 3, 5, 6, 7, 10, 11, 12, 14 \bmod 17$$

Example ($E := 42a1$) We have the following invariants for E:

$$N = 42, \omega(E) = 1, \Delta = -2^8 3^2 7, E(\mathbb{Q})[2] \cong \mathbb{Z}/8\mathbb{Z}, r_{\mathrm{al}}(E(\mathbb{Q})) = 0, Ш(E/\mathbb{Q})[2] = 0.$$

It follows that $d_2\mathrm{Sel}_2(E/\mathbb{Q}) = d_2(E(\mathbb{Q})[2]) = 1$. Hypothesis a) is satisfied if we take χ_d such that $\chi_d(2) = 1$. Observe that $\left(\frac{\Delta}{|d|}\right) = \left(\frac{-7}{|d|}\right) = \mathrm{sign}(d)\chi_d(7)$. Choose $\mathcal{W} = \{3, |d|\}$. Using Sage [S+17], we obtain $d_2(E(\mathbb{Q}_3)[2]) = 1$. So far, we chose χ_d such that $\chi_d(2) = 1$ and $\chi_d(3) = -1$. So, following the functional equation, if $d < 0$, we need to take χ_d with $\chi_d(7) = 1$, and if $d > 0$, take χ_d with $\chi_d(7) = -1$. By choosing those χ_d, we have $\delta_{|d|}(E, F) = 1 = d_2(E(\mathbb{Q})[2])$ and satisfy hypothesis b). Then, the right hand side of equation (4.20) is

$$d_2 H^1_f(\mathbb{Q}_3, E[2]) + \delta_{|d|}(E, F) - 2d_2\mathrm{Sel}_2(E/\mathbb{Q}) = 0.$$

By the Kramer congruence relation 4.7, we have

$$\delta_3(E, F) + \delta_{|d|}(E, F) + 1 \equiv 0 \bmod 2.$$

We can find the residue classes $|d|$ modulo 42 such that either $\chi_d(-1) = -1$ and $\chi_d(7) = 1$ or $\chi_d(-1) = 1$ and $\chi_d(7) = -1$. By using the Chinese remainder theorem, we obtain the following residue classes as shown in Table 5:

$$|d| \equiv 13, 19, 31 \bmod 42 \text{ for } d < 0$$

$$|d| \equiv 5, 17, 41 \bmod 42 \text{ for } d > 0$$

Example $(E := 70a1)$ We have the following invariants for E:

$$N = 70, \omega(E) = 1, \Delta = -2^4 5^2 7, E(\mathbb{Q})[2] \cong \mathbb{Z}/4\mathbb{Z}, r_{\text{al}}(E(\mathbb{Q})) = 0, Ш(E/\mathbb{Q})[2] = 0.$$

It follows that $d_2\text{Sel}_2(E/\mathbb{Q}) = d_2(E(\mathbb{Q})[2]) = 1$. Hypothesis a) is satisfied if we take χ_d such that $\chi_d(2) = 1$. Observe that $\left(\frac{\Delta}{|d|}\right) = \left(\frac{-7}{|d|}\right) = \text{sign}(d)\chi_d(7)$. Choose $\mathcal{W} = \{5, |d|\}$. Using Sage [S$^+$17], we obtain $d_2(E(\mathbb{Q}_5)[2]) = 1$. So far, we chose χ_d such that $\chi_d(2) = 1$ and $\chi_d(5) = -1$. So, following the functional equation, if $d < 0$, we need to take χ_d with $\chi_d(7) = 1$, and if $d > 0$, take χ_d with $\chi_d(7) = -1$. By choosing those χ_d, we have $\delta_{|d|}(E, F) = 1 = d_2(E(\mathbb{Q})[2])$ and satisfy hypothesis b). Then, the right hand side of equation (4.20) is

$$d_2 H^1_f(\mathbb{Q}_5, E[2]) + \delta_{|d|}(E, F) - 2d_2\text{Sel}_2(E/\mathbb{Q}) = 0.$$

By the Kramer congruence relation 4.7, we have

$$\delta_5(E, F) + \delta_{|d|}(E, F) + 1 \equiv 0 \bmod 2.$$

We can find the residue classes $|d|$ modulo 70 such that either $\chi_d(-1) = -1$ and $\chi_d(7) = 1$ or $\chi_d(-1) = 1$ and $\chi_d(7) = -1$. By using the Chinese remainder theorem, we obtain the following residue classes as shown in Table 5:

$$|d| \equiv 3, 13, 17, 27, 33, 47 \bmod 70 \text{ for both } d < 0 \text{ and } d > 0$$

E	a	$\lvert P_i^0(a,E)\rvert$	$\lvert P_i(a,E)\rvert$	a	$\lvert P_r^0(a,E)\rvert$	$\lvert P_r(a,E)\rvert$
$11a1$	1	23	479	1	18	466
	3	25	493	3	24	470
	4	27	479	4	36	484
	5	22	478	5	28	485
	9	27	478	9	26	475
$14a1$	1	39	404	1	45	385
	3	0	399	3	0	408
	5	0	406	5	0	398
	9	40	397	9	37	396
	11	35	401	11	49	390
	13	0	395	13	0	398
$15a1$	1	0	604	1	79	585
	2	0	607	2	0	604
	4	0	609	4	82	584
	8	0	597	8	0	603
$17a1$	3	0	300	1	24	298
	5	0	301	2	25	307
	6	0	304	4	22	302
	7	0	289	8	29	291
	10	0	296	9	27	294
	11	0	314	13	23	293
	12	0	299	15	18	301
	14	0	308	16	19	288
$19a1$	1	19	262	1	13	263
	4	19	267	4	14	266
	5	19	272	5	16	270
	6	15	255	6	19	272
	7	14	276	7	15	259
	9	20	260	9	8	269
	11	13	273	11	16	262
	16	10	266	16	14	257
	17	11	273	17	16	263
$21a1$	2	0	399	1	57	383
	8	0	403	4	48	388
	10	0	401	5	0	406
	11	0	406	16	49	392
	13	0	395	17	0	404
	19	0	409	20	0	396
$30a1$	1	46	300	1	61	286
	7	0	295	7	0	299
	11	0	296	11	0	300
	13	0	304	13	0	303
	17	0	304	17	0	311
	19	56	306	19	63	292
	23	0	295	23	0	297
	29	0	303	29	0	307

Table 2: The number of vanishings of quadratic prime twists of E with even functional equation for the residue classes modulo N

E	a	$\lvert P_i^0(a,E)\rvert$	$\lvert P_i(a,E)\rvert$	a	$\lvert P_r^0(a,E)\rvert$	$\lvert P_r(a,E)\rvert$
33a1	5	0	238	1	24	224
	7	0	226	2	0	240
	10	0	252	4	29	238
	13	0	235	8	0	237
	14	0	254	16	30	245
	19	0	252	17	0	245
	20	0	238	25	23	238
	23	0	237	29	0	242
	26	0	236	31	30	229
	28	0	243	32	0	240
34a1	1	16	152	1	17	148
	3	0	156	3	0	150
	5	0	156	5	0	152
	7	0	141	7	0	147
	9	13	153	9	10	145
	11	0	151	11	0	144
	13	14	144	13	12	151
	15	13	152	15	18	149
	19	16	152	19	10	159
	21	17	148	21	11	142
	23	0	157	23	0	145
	25	12	154	25	12	143
	27	0	146	27	0	148
	29	0	150	29	0	166
	31	0	151	31	0	152
	33	25	151	33	16	141
37a1	1	7	129	2	14	127
	3	6	135	5	20	139
	4	8	150	6	14	139
	7	8	133	8	10	127
	9	6	133	13	14	129
	10	6	130	14	18	140
	11	6	135	15	16	125
	12	7	128	17	17	135
	16	8	138	18	16	134
	21	4	128	19	14	132
	25	8	132	20	14	135
	26	7	138	22	18	129
	27	7	129	23	24	136
	28	8	134	24	15	136
	30	6	137	29	12	138
	33	8	132	31	12	135
	34	3	141	32	13	135
	36	6	140	35	20	138

Table 3: The number of vanishings of quadratic prime twists of E with even functional equation for the residue classes modulo N

E	a	$\lvert P_i^0(a,E)\rvert$	$\lvert P_i(a,E)\rvert$	a	$\lvert P_r^0(a,E)\rvert$	$\lvert P_r(a,E)\rvert$
42a1	5	0	238	1	24	224
	7	0	226	2	0	240
	10	0	252	4	29	238
	13	0	235	8	0	237
	14	0	254	16	30	245
	19	0	252	17	0	245
	20	0	238	25	23	238
	23	0	237	29	0	242
	26	0	236	31	30	229
	28	0	243	32	0	240
34a1	1	16	152	1	17	148
	3	0	156	3	0	150
	5	0	156	5	0	152
	7	0	141	7	0	147
	9	13	153	9	10	145
	11	0	151	11	0	144
	13	14	144	13	12	151
	15	13	152	15	18	149
	19	16	152	19	10	159
	21	17	148	21	11	142
	23	0	157	23	0	145
	25	12	154	25	12	143
	27	0	146	27	0	148
	29	0	150	29	0	166
	31	0	151	31	0	152
	33	25	151	33	16	141
37a1	1	7	129	2	14	127
	3	6	135	5	20	139
	4	8	150	6	14	139
	7	8	133	8	10	127
	9	6	133	13	14	129
	10	6	130	14	18	140
	11	6	135	15	16	125
	12	7	128	17	17	135
	16	8	138	18	16	134
	21	4	128	19	14	132
	25	8	132	20	14	135
	26	7	138	22	18	129
	27	7	129	23	24	136
	28	8	134	24	15	136
	30	6	137	29	12	138
	33	8	132	31	12	135
	34	3	141	32	13	135
	36	6	140	35	20	138

Table 4: The number of vanishings of quadratic prime twists of E with even functional equation for the residue classes modulo N

E	$[a \in N_i]$	$[a \notin N_i]$	$[a \in N_r]$	$[a \notin N_r]$
$38b1$		$[1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29, 31, 33, 35, 37]$		$[1, 3, 5, 7, 9, 11, 13, 15, 17, 21, 23, 25, 27, 29, 31, 33, 35, 37]$
$39a1$	$[2, 5, 8, 11, 20, 32]$	$[1, 4, 10, 16, 22, 25]$	$[2, 5, 8, 11, 20, 32]$	$[1, 4, 10, 16, 22, 25]$
$42a1$	$[1, 5, 13, 17, 19, 25, 31, 37, 41]$	$[11, 23, 29]$	$[5, 11, 13, 17, 19, 23, 29, 31, 41]$	$[1, 25, 37]$
$43a1$		$[2, 3, 5, 7, 8, 12, 18, 19, 20, 22, 26, 27, 28, 29, 30, 32, 33, 34, 37, 39, 42]$		$[2, 3, 5, 7, 8, 12, 18, 19, 20, 22, 26, 27, 28, 29, 30, 32, 33, 34, 37, 39, 42]$
$46a1$	$[5, 7, 11, 15, 17, 19, 21, 33, 37, 43, 45]$	$[1, 3, 9, 13, 25, 27, 29, 31, 35, 39, 41]$	$[5, 7, 11, 15, 17, 19, 21, 33, 37, 43, 45]$	$[1, 3, 9, 13, 25, 27, 29, 31, 35, 39, 41]$
$55a1$	$[2, 7, 8, 13, 17, 18, 28, 32, 43, 52]$	$[1, 4, 9, 14, 16, 26, 31, 34, 36, 49]$	$[2, 7, 8, 13, 17, 18, 28, 32, 43, 52]$	$[1, 4, 9, 14, 16, 26, 31, 34, 36, 49]$
$62a1$	$[3, 11, 13, 15, 17, 21, 23, 27, 29, 37, 43, 53, 55, 57, 61]$	$[1, 5, 7, 9, 19, 25, 33, 35, 39, 41, 45, 47, 49, 51, 59]$	$[3, 11, 13, 15, 17, 21, 23, 27, 29, 37, 43, 53, 55, 57, 61]$	$[1, 5, 7, 9, 19, 25, 33, 35, 39, 41, 45, 47, 49, 51, 59]$
$65a1$	$[2, 7, 8, 18, 28, 32, 33, 37, 47, 57, 58, 63]$	$[1, 4, 9, 14, 16, 29, 36, 49, 51, 56, 61, 64]$		$[3, 6, 11, 12, 17, 19, 21, 22, 23, 24, 27, 31, 34, 38, 41, 42, 43, 44, 46, 48, 53, 54, 59, 62]$
$66a1$	$[5, 7, 13, 19, 23, 43, 47, 53, 59, 61]$	$[1, 17, 25, 29, 31, 35, 37, 41, 49, 65]$	$[5, 7, 13, 19, 23, 43, 47, 53, 59, 61]$	$[1, 17, 25, 29, 31, 35, 37, 41, 49, 65]$
$66b1$	$[5, 7, 13, 17, 19, 23, 29, 35, 41, 43, 47, 53, 59, 61, 65]$	$[1, 25, 31, 37, 49]$	$[5, 7, 13, 19, 23, 43, 47, 53, 59, 61]$	$[1, 17, 25, 29, 31, 35, 37, 41, 49, 65]$
$66c1$	$[5, 7, 13, 19, 23, 43, 47, 53, 59, 61]$	$[1, 17, 25, 29, 31, 35, 37, 41, 49, 65]$	$[5, 7, 13, 19, 23, 43, 47, 53, 59, 61]$	$[1, 17, 25, 29, 31, 35, 37, 41, 49, 65]$
$69a1$	$[7, 10, 19, 22, 28, 34, 37, 40, 43, 61, 67]$	$[2, 8, 26, 29, 32, 35, 41, 47, 50, 59, 62]$	$[5, 11, 14, 17, 20, 38, 44, 53, 56, 65, 68]$	$[1, 4, 13, 16, 25, 31, 49, 52, 55, 58, 64]$
$70a1$	$[3, 13, 17, 19, 23, 27, 31, 33, 37, 41, 43, 47, 53, 57, 59, 61, 67, 69]$	$[1, 9, 11, 29, 39, 51]$	$[3, 13, 17, 19, 23, 27, 31, 33, 37, 41, 43, 47, 53, 57, 59, 61, 67, 69]$	$[1, 9, 11, 29, 39, 51]$
$77c1$	$[3, 5, 12, 20, 26, 27, 31, 34, 38, 45, 47, 48, 59, 69, 75]$	$[2, 8, 18, 29, 30, 32, 39, 43, 46, 50, 51, 57, 65, 72, 74]$	$[6, 10, 13, 17, 19, 24, 40, 41, 52, 54, 61, 62, 68, 73, 76]$	$[1, 4, 9, 15, 16, 23, 25, 36, 37, 53, 58, 60, 64, 67, 71]$

Table 5: Lists of $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $a \in N_i$, $a \notin N_i$, $a \in N_r$ and $a \notin N_r$ for quadratic prime twists of E

Chapter 5

Future researches for twists of higher weight forms

5.1 Vanishing and nonvanishing conditions

Let l be an odd prime and f be a primitive form in $S_k(N, \varepsilon)$ with $k \geq 2$ and $2|k$. Also, let χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$. If we might be able to find conditions for vanishing and nonvanishing of $L(f, k/2, \chi)$ in stead of the congruence relations in Proposition (2.10) and (2.11), then not only we could prove nonvanishing theorem 2.12 but also might be able to make a progress toward the number of vanishings (or nonvanishings) and prove conjectures (3.27) and (2.7.2).

Let G_χ be the kernel of χ in $(\mathbb{Z}/\mathfrak{f}_\chi Z)^\times$. Then, we can choose a prime $q \notin G_\chi$ so that

$$(\mathbb{Z}/\mathfrak{f}_\chi Z)^\times = \dot{\bigcup_{0 \leq j \leq l-1}} q^j G_\chi$$

where $\dot{\bigcup}$ is the disjoint union. For finding such an exact condition, we can rewrite equation 2.14 for the algebraic parts and an odd prime l as

$$L^{\mathrm{alg}}(f, k/2, \chi) = \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a) c^+(a, \mathfrak{f}_\chi, k/2; f) = \sum_{0 \leq j \leq l-1} \zeta_l^j S_j,$$

where ζ_l is a primitive l-th root of unity and $S_j = \sum_{a \in G_\chi} c^+(q^j a, \mathfrak{f}_\chi, k/2; f) \in \mathbb{Z}$. Then, one can easily show that

$$L(f, k/2, \chi) = 0 \iff L^{\mathrm{alg}}(f, k/2, \chi) = 0 \iff S_i = S_j \text{ for every } 0 \leq i, j \leq l-1.$$

Moreover, by the functional equation (2.20) and Proposition 2.16, it is immediate that there exist at least $(l-1)/2$ pairs of (S_i, S_j) for $0 \leq i, j \leq l-1$ and $i \neq j$ such that $S_i = S_j$. As the root number determines those such pairs of (S_i, S_j), the next goal is to find another condition(s) to identify the equalities among the rest of S_j's.

5.2 Motives and Bloch-Kato conjecture

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N. Fearnley and Kisilevsky [FK10] and [FK12] examined and supported by numerical data the central values of r-th derivative of $L(E, 1, \chi$ where r_χ is its analytic rank and χ is a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$ with $(N, \mathfrak{f}_\chi) = 1$ and of an odd prime order $l \geq 3$ based on Birch and Swinnerton-Dyer conjecture 1.4.1. More precisely, let $K/\mathbb{Q}$ be a cyclic extension of degree l with Galois group G. Then, considering E also as an elliptic curve defined over K, $L(E/K, s)$ can be written by (1.10) as

$$L(E/K, s) = L(E, s) \prod_{1 \leq j \leq l-1} L(E, s, \chi^j).$$

Then, they suggested that for $\chi \neq \chi_0$,

$$\frac{L^{(r_\chi)}(E, 1, \chi)}{r_\chi!} = \frac{\tau(\chi)}{\mathfrak{f}_\chi} \Omega^+ \lambda_\chi(P_1, \dots, P_r) \alpha_\chi^+(P_1, \dots, P_r) z_\chi,$$

where r is the algebraic rank of $E(K)$, $\{P_1, \dots, P_r\} \subset E(K)$ are independent points of infinite order with trace 0 to $\mathbb{Q}$, Ω^+ is the real period of E, $\lambda_\chi(P_1, \dots, P_r)$ is a positive real number obtained from the Néron-Tate canonical height pairing on $E(\overline{\mathbb{Q}})$, $\alpha_\chi^+(P_1, \dots, P_r)$ and z_χ are algebraic integers in $\mathbb{Q}(\chi)$.

We may want to extend their work for cubic twists of higher weight primitive forms, in particular weight 4 forms since the existences of vanishings from the data for vanishings presented in Section 2.6 in Chapter 2. For example, the following unique primitive form $f \in S_4(10, \varepsilon_0)$ has 24 vanishings of cubic twists for $\mathfrak{f}_\chi \leq 32787$:

$$f(z) := q + 2q^2 - 8q^3 + 4q^4 + 5q^5 - 16q^6 + O(q^7).$$

In order to examine the derivatives of central L-values for weight 4 case, we need an algebraic model as elliptic curves for weight 2 case. Moreover, we also need a generalised Birch and Swinnerton-Dyer conjecture. Unfortunately, at this time, we do not have those concrete models. However, there exist an abstract theoretical model, named motive, and a generalised Birch and Swinnerton-Dyer conjecture, named the Block-Kato conjecture. There also some examples for concrete geometric model as algebraic varieties over $\mathbb{C}$, named Calabi-Yau threefolds M such that their L-function $L(M, s)$ can be defined and

$$L(M, s) = L(f, s)$$

for a weight 4 primitive forms f of some specific level. Refer to Schütt in [Sch04] for those examples. Note that those examples are given only for weight 4 primitive forms of a small number of distinct levels including 10. For the primitive form f of weight 4 and level 10 above, Schütt in [Sch04] found the corresponding (rigid) Calabi-Yau threefolds defined by the projective small resolution $\hat{W}$ of a fibre product $W := (ES_1(6), \mathrm{proj}) \times_{\mathbb{P}^1} (ES_1(6), \pi \circ \mathrm{proj})$, where $ES_1(6)$ is the modular elliptic surface over compact modular curve $X(\Gamma_1(6))$, proj is the natural projection and the automorphism (twist) $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$t \mapsto \frac{t}{t-1}.$$

It is also known that $ES_1(6)$ can be represented as a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ by

$$s(x+y)(y+z)(z+x) = txyz,$$

where $[x : y : z] \in \mathbb{P}^2$ and $[s : t] \in \mathbb{P}^1$. Since the Bloch-Kato conjecture is written by the languages of distinct categorical cohomology groups, our next task would be to interpret this conjecture into concrete algebraic terms so that we can examine the central values of derivatives of twists of those primitive forms of weight 4.

Bibliography

[BCDT01] C. Breuil, B. Conrad, F. Diamond, and R. Taylor. On the modularity of elliptic curves over $\mathbb{Q}$: Wild 3-adic exercises. *J. Amer. Math. Soc.*, 14(4):843–939, 2001.

[BS15] M. Bhargava and A. Shankar. Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0. *Ann. of Math.*, 181(2):587–621, 2015.

[CyDO02] Henri Cohen, Francisco Diaz y Diaz, and Michel Olivier. A survey of discriminant counting. In *In Proceedings of the 5th International Symposium on Algorithmic Number Theory*, pages 80–94, London, UK, 2002. ANTS-V, Springer-Verlag.

[Del74] Pierre Deligne. La conjecture de Weil. I. *Inst. Hautes Études Sci. Publ. Math.*, (43):273–307, 1974.

[DFK04] Chantal David, Jack Fearnley, and Hershy Kisilevsky. On the vanishing of twisted L-functions of elliptic curves. *Experiment. Math.*, 13(2):185 – 198, 2004.

[DFK06] Chantal David, Jack Fearnley, and Hershy Kisilevsky. Vanishing of l-functions of elliptic curves over number fields. In *Clay Mathematical Institute Special Week on Ranks of Elliptic Curves and Random Matrix Theory*, volume Special Volume, pages 33–46. Cambridge University Press, 2006.

[Dok04] T. Dokchitser. Computing special values of motivic L-functions. *Experiment. Math.*, 13(2):137–149, 2004.

[Fio16] D. Fiorilli. A conditional determination of the average rank of elliptic curves. *Journal of the London Mathematical Society*, 94(3):767–792, 2016.

[FK10] Jack Fearnley and Hershy Kisilevsky. Critical Values of Derivatives of Twisted Elliptic L-Functions. *Experiment. Math.*, 19(2):149–160, 2010.

[FK12] Jack Fearnley and Hershy Kisilevsky. Critical Values of Higher Derivatives of Twisted Elliptic L-Functions. *Experiment. Math.*, 21(3):213–222, 2012.

[FKK12] Jack Fearnley, Hersy Kisilevsky, and Masato Kuwata. Vanishing and non-vanishing Dirichlet twists of L-functions of elliptic curves. *Journal of the London Mathematical Society*, 86:539–557, 2012.

[Gol79] D. Goldfeld. Conjectures on elliptic curves over quadratic fields. In *Number theory, Carbondale 1979*, volume 751 of *Lecture Notes in Math.*, pages 108–118. Proc. Southern Illinois Conf., Southern Illinois Univ., Carbondale, III, 1979, Springer, Berlin, 1979.

[GZ86] B. Gross and D. Zagier. Heegner points and derivatives of L-series. *Inv. Math.*, 84(2):225–320, 1986.

[IK04] H. Iwaniec and E. Kowalski. *Analytic Number Theory*. Number v. 53 in American Mathematical Society colloquium publications. American Mathematical Society, 2004.

[Iwa97] H. Iwaniec. *Topics in Classical Automorphic Forms*, volume 17 of *Graduate Studies in Mathmatics*. American Mathematical Society, 1997.

[Kna05] Anthony W. Knapp. *Advanced Real Analysis Along with a companion volume Basic Real Analysis*. Birkhäuser, 2005.

[Kol89] V. Kolyvagin. Finiteness of $E(\mathbb{Q})$ and $X(E, \mathbb{Q})$ for a class of Weil curves. *Math. USSR Izv.*, 32(3):523–541, 1989.

[Kra81] Kenneth Kramer. Arithmetic of elliptic curves upon quadratic extension. *Trans. Amer. Math. Soc.*, 264:121–135, 1981.

[KS00] J. P. Keating and N. C. Snaith. Random matrix theory and $\zeta(1/2 + it)$. *Comm. Math. Phys.*, 214:57–89, 2000.

[KS02a] H.H. Kim and F. Shahidi. Cuspidality of symmetric powers with applications. *Duke Math. J.*, 112(1):177–197, 2002.

[KS02b] H.H. Kim and F. Shahidi. Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2. With an appendix by C. J. Bushnell and G. Henniart. *Ann. of Math.*, 155(3):837–893, 2002.

[LMF13] The LMFDB Collaboration. The L-functions and modular forms database. `http://www.lmfdb.org`, 2013. [Online; accessed 16 September 2013].

[Man72] J. I. Manin. Parabolic points and zeta functions of modular curves. *Math. USSR-Izv*, 6(1):19 – 64, 1972.

[Mar05] G. Martin. Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$. *J. Number Theory*, 112(2):298–331, 2005.

[Maz72] B. Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18:183–266, 1972.

[Mer94] L. Merel. Universal Fourier expansions of modular forms. In *On Artin's conjecture for odd 2-dimensional representations*, pages 59–94. Springer, 1994.

[Miy89] T. Miyake. *Modular Forms*. Springer-Verlag, 1989.

[MR07] B. Mazur and K. Rubin. Finding large Selmer rank via an arithmetic theory of local constants. *Ann. Math.*, 166:581–614, 2007.

[MR10] B. Mazur and K. Rubin. Ranks of twists of elliptic curves and hilbert's tenth problem. *Inventiones mathematicae*, 181(3):541–575, Sep. 2010.

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. Math.*, 84(1):1–48, 1986.

[MV07] H.L. Montgomery and R.C. Vaughan. *Multiplicative Number Theory I: Classical Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2007.

[Rub00] K. Rubin. *Euler Systems*. Annals of Mathematics Studies. Princeton University Press, 2000.

[S$^+$17] W. A. Stein et al. *SageMath, the Sage Mathematics Software System (Version 8.0)*. The Sage Development Team, http://www.sagemath.org, 2017.

[Sch04] Matthias Schütt. New examples of modular rigid calabi-yau threefolds. *Collectanea Mathematica*, 55(2):219–228, 2004.

[Ser72] J.-P. Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiquies. *Inv. mathematicae*, 15:259–331, 1972.

[Ser73] J.-P. Serre. *A Course In Arithmetic*, volume 7 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1973.

[Ser76] J.-P. Serre. Divisibilité de certaines fonctions arithmétiques. *L'Ensein. Math.*, 22:227–260, 1976.

[Ser97] J.-P. Serre. *Galois Chomology translated from the French by Patrick Ion.* Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 1997.

[Shi59] G. Shimura. Sur les intégrales attachées aux formes automorphes. *Journal of the Mathematical Society of Japan*, 11:291–311, 1959.

[Shi75] G. Shimura. On the holomorphy of certain dirichlet series. *Proc. London Math. Soc.*, 31(1):79–98, 1975.

[Shi77] G. Shimura. On the periods of modular forms. *Mathematische Annalen*, 229:211–221, 1977.

[Sho81] V. V. Shokurov. A study of the homology of Kuga varieties. *Math. USSR Izv.*, 16(2):399–418, 1981.

[Sil94] Joseph H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag New York, 1994.

[Sil09] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*, volume 106 of *Graduate Texts in Mathematics*. Springer New York, 2nd edition, 2009.

[Ste07] W. Stein. *Modular Forms, a Computational Approach*, volume 79 of *Graduate Studies in Mathmatics*. American Mathematical Society, 2007.

[TW95] R. Taylor and A. Wiles. Ring theoretic properties of certain Hecke algebras. *Ann. of Math*, 141(3):553–572, 1995.

[Was82] L. Washington. *Introduction to Cyclotomic Fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, 1982.

[Wil95] A. Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math*, 3(141):443–551, 1995.