Absolutely Continuous Invariant Measures for Piecewise Convex Maps of Interval with Infinite Number of Branches

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ABSTRACT

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The main result of this Master's thesis is the generalization of the existence of absolutely continuous invariant measure for piecewise convex maps of an interval from a case with the finite number of branches to one with infinitely many branches. We give a similar result for piecewise concave maps as well. We also provide examples of piecewise convex maps with a finite and infinite number of branches without ACIM.

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Background

1.1 Review of Necessary Facts from Measure Theory and Functional Analysis

In this section, we presented a review of some key concepts in measure theory and functional analysis. For more details, we refer the readers to Rudin's introductory text [10].

Definition 1.1.1. *Assume X is a topologcal space. Let $\mathscr{B}$ be a collection of subsets of X with the following properties:*

- $\emptyset \in \mathscr{B}$.
- $B \in \mathscr{B} \Longrightarrow (X \setminus B) \in \mathscr{B}$.
- $\{B\}_{n=1}^{\infty} \subset \mathscr{B} \Longrightarrow \cup_{n=1}^{\infty} B_n \in \mathscr{B}$.

Then we say $\mathscr{B}$ is a ***σ-algebra****.*

Definition 1.1.2. *The smallest σ-algebra containing all open subsets (or equivalently by the closed sets) of X is called the* ***Borel σ-algebra*** *of X. Elements of this σ-algebra are called* ***Borel sets****.*

Let's now define a measure which is a fundamental tool used in the study of dynamical systems. A measure can be thought of as a function assigning a notion of size to the sets in a σ-algebra.

Definition 1.1.3. *Let μ be a set-function defined on $\mathscr{B}$ such that*

- $\mu(\emptyset) = 0$,

- *$\mu(B) \geq 0$ for all $B \in \mathscr{B}$, and*
- *for every sequence of pairwise disjoint sets $\{B_n\}_{n=1}^{\infty} \subset \mathscr{B}$,*

$$\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

Then we say μ is a measure on $\mathscr{B}$.

The last criterion is called **countable additivity**. For a given measure μ, we call the collection $(X, \mathscr{B}, \mu)$ is a measure space. Also, we call the collection $(X, \mathscr{B}, \mu)$ is a **compact measure space** when X is compact. This definition is important as the spaces we consider in this text are exclusively compact. When $\mu(X) = 1$, we say that the measure μ has been *normalized* or that μ is a **probability measure**. All probabilities on a space of events must sum to 1 and so $\mathscr{B}$ can be thought of a set of events with μ providing the probability of each event occurring. A **probability space** is a measure space with total measure one.

We now wish to define a space of measures and establish a few results for spaces of measures. First we start with a foundation of normed linear spaces [4].

Definition 1.1.4. *Let $(X, \mathscr{B}, \mu)$ be a measurable space and p a real number, $1 \leq p < \infty$. The family of all possible real-valued measurable functions $f : X \to \mathbb{R}$ satisfying*

$$\int_X |f(x)|^p \mu dx < \infty$$

*is the $\mathscr{L}^p(X, \mathscr{B}, \mu)$ **space**. Note that if $p = 1$ then $\mathscr{L}^1$ **space** consists of all possible integrable functions.*
The integral appearing above is very important for an element $f \in \mathscr{L}^p$. Thus it is assigned the special notation

$$||f||_{\mathscr{L}^p} = \left[\int_X |f(x)|^p \mu dx \right]^{\frac{1}{p}}$$

*and is called the $\mathscr{L}^p$ **norm** of f.*

Definition 1.1.5. *Let $\mathscr{L}$ be a linear space over $\mathbb{R}$. Then we define a norm on $\mathscr{L}$ to be a function $\| \cdot \|$ satisfying*

- $\|f\| = 0 \iff f \equiv 0$,

- $\|\alpha f\| = |\alpha|\|f\|$,
- $\|f+g\| \leq \|f\| + \|g\|$,

for all f, $g \in \mathscr{L}$ and any scalar $\alpha \in \mathbb{R}$. We say the pair $(\mathscr{L}, \|\cdot\|)$ is a ***normed linear space****.*

Definition 1.1.6. *Let $\mathscr{L}$ be a normed linear space. Then we call the space of all continuous linear functionals on $\mathscr{L}$ the* ***adjoint or dual space*** *of $\mathscr{L}$ and denote it as $\mathscr{L}^*$.*

A normed linear space and its adjoint are connected in the sense of convergence. The following definition establishes how this connection works.

Definition 1.1.7. *We say a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathscr{L}$ converges weakly to $x \in \mathscr{L}$ if and only if for any $F \in \mathscr{L}^*$ we have that*

$$\lim_{n\to\infty} F(x_n) = F(x).$$

In the other direction, we say that a sequence of functionals $\{F_n\}_{n=1}^{\infty}$ converges in the weak topology to a functional $F \in \mathscr{L}^*$ if and only if for all $x \in \mathscr{L}$*

$$\lim_{n\to\infty} F_n(x) = F(x).$$

The important theorem we will use in the proof of the next proposition is Banach-Alaoglu theorem (see V.4.2. in [4]).

Theorem 1.1.1. *If $\mathscr{L}$ is a Banach space, then any bounded subset of $\mathscr{L}^*$ is precompact in weak* topology.*

Proof. For the proof we refer to [4]. ■

Theorem 1.1.2. *(Th. IV.8.1 and Th. IV.8.8 of [4]) If $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there is an isometric isomorphism between $(\mathscr{L}^p)^*$ and $\mathscr{L}^q$ in which the corresponding elements $F \in (\mathscr{L}^p)^*$ and $g \in \mathscr{L}^q$ are related by the identity*

$$F(f) = \int_X fg d\mu, \qquad f \in \mathscr{L}^p.$$

Proof. For the proof we refer to [4]. ■

Proposition 1.1.1. *Let $(X, \mathscr{B}, \mu)$ be a finite measure space. Let A be a subset of $\mathscr{L}^1$ which is bounded in $\mathscr{L}^\infty$. Then A is weakly precompact in $\mathscr{L}^1$.*

Proof. Since A is bounded in $\mathscr{L}^\infty$, it is also bounded subset of $\mathscr{L}^2$. As $(\mathscr{L}^2)^* = \mathscr{L}^2$ by Banach-Alaoglu theorem, A is precompact in weak* topology of $\mathscr{L}^2$. It means that there exists a sequence $\{f_n\} \subset A$ and an $f_0 \in \mathscr{L}^2 \subset \mathscr{L}^1$, such that for any function $g \in \mathscr{L}^2$ we have,

$$\int_X f_n g d\mu \to \int_X f_0 g d\mu.$$

Since $\mathscr{L}^\infty \subset \mathscr{L}^2$ this shows that A is weakly precompact in $\mathscr{L}^1$. ■

Theorem 1.1.3 (**Monotone Convergence Theorem**). *[10] Suppose $E \in \mathbb{R}$. Let $\{f_n\}$ be a sequence of measurable function such that for every $x \in E$*

$$0 \leq f_1(x) \leq f_2(x) \leq ...$$

Let f be defined by

$$f_n(x) \to f(x)$$

as $n \to \infty$. Then

$$\int_E f_n d\mu \to \int_E f d\mu$$

as $n \to \infty$.

Theorem 1.1.4 (***Hahn Decomposition Theorem***). *[4] Let μ be a signed measure on $(X, \mathscr{B})$. Then there is a positive set $V \in \mathscr{B}$ and a negative set $W \in \mathscr{B}$, so that $X = V \cup W$ and $V \cap W = \phi$.*

Theorem 1.1.5 (**Yosida-Kakutani Theorem**). *[14] Let T be a bounded linear operation which maps a Banach space $\boldsymbol{B}$ into itself. Let us further assume that*

1. *there exists a constant C such that $\|T^n\| \leqq C$ for $n = 1, 2, \dots,$ and that*

2. *for any $x \in \boldsymbol{B}$, the sequence $\{x_n\}$, $n = 1, 2, \dots,$ where $x_n = \frac{1}{n}(T + T^2 + \cdots + T^n)x$, contains a subsequence which converges weakly to a point $\bar{x} \in \boldsymbol{B}$.*

   Under these assumptions,

3. *the sequence $\{x_n\}$, $n = 1, 2, \dots$ converges strongly to a point $\bar{x}$, and if we denote by T_1 the operation $x \to \bar{x}$, then T_1 is a bounded linear operation which maps $\boldsymbol{B}$ into itself and*

4. *$TT_1 = T_1T = T_1^2 = T_1, \quad \|T_1\| \leqq C.$*

Proof. For the proof we refer to the original paper [13]. ■

We now formally define a space of measures and equip it with a norm, hence creating a normed linear space of great interest for this text. For this definition we consider a more general definition of measure.

Definition 1.1.8. *Let $(X, \mathscr{B}, \mu)$ be a measurable space. Then μ is called a* ***σ-finite measure*** *if there existes a sequence $\{B_n \in \mathscr{B} : n \in \mathbb{N}\}$ such that $\cup_{n\in\mathbb{N}} B_n = X$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$.*

Definition 1.1.9. *Let $(X, \mathscr{B})$ be a measurable space. A* ***signed measure*** *on $(X, \mathscr{B})$ is a function $\mu : \mathscr{B} \to \mathbb{R} \cup \{-\infty, \infty\}$ such that*
1) μ takes on at most one of the values $-\infty$ or ∞
2) $\mu(\phi) = 0$
3) If $\{B_n\}_{n=1}^{\infty} \subset \mathscr{B}$ is a sequence of pairwise disjoint sets then $\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$

Definition 1.1.10. *Let μ be a signed measure on $(X, \mathscr{B})$. Let $V, W, M \in \mathscr{B}$. Then we say that:*
1) V is positive if $\mu(B \cap V) \geq 0$ for all $B \in \mathscr{B}$.
2) W is negative if $\mu(B \cap W) \leq 0$ for all $B \in \mathscr{B}$.
3) M is null if $\mu(B \cap M) = 0$ for all $B \in \mathscr{B}$.

Definition 1.1.11. *If $P_1, P_2, \ldots$ are pairwise disjoint and $\bigcup_{i=1}^{\infty} P_i = \mathscr{P}$, then the collection $P_1, P_2, \ldots$ forms a* ***partition*** *of $\mathscr{P}$.*

Definition 1.1.12. *Let $\mathscr{M}$ denote the space of all measures on $(X, \mathscr{B})$. Then we define the norm of $\mu \in \mathscr{M}$, as*

$$\|\mu\| = \sup_{P\in\mathscr{P}} \{|\mu(P_1)| + \cdots + |\mu(P_N)|\},$$

where $\mathscr{P}$ is the set of all finite partitions of X and $P = \{P_1, P_2, ..., P_N\}$ is a partition of X. This norm is known as the total variation norm [4].

Definition 1.1.13. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Let ν be a measure defined on all sets in $\mathscr{B}$. Then we say ν is* ***absolutely continuous*** *with respect to μ if and only if for every set $B \in \mathscr{B}$*

$$\mu(B) = 0 \Longrightarrow \nu(B) = 0.$$

We write $\nu \ll \mu$.

Indeed, this is not the form of an absolutely continuous measure that we will typically be working with. This definition forms a rudimentary basis for a topic which we will expand upon in greater detail in section 1.4. In particular, we note that the Radon-Nikodym theorem, Theorem 1.4.1, which builds further upon the idea of an absolutely continuous measure, will play a key role in the development of the Frobenius-Perron operator, an essential tool in dynamical systems.

1.2 Measure-Preserving Transformations

In the study of dynamical systems, we are generally working with functions which under iteration, transform the space X. We aptly call these functions transformations. To expand upon this idea we now define what it means for a transformation to be measurable and for a measure to be invariant under iterations of a transformation.

Definition 1.2.1. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ be a transformation of X. Then τ is said to be a **measurable transformation** if and only if*

$$B \in \mathscr{B} \implies \tau^{-1}(B) \in \mathscr{B}.$$

Definition 1.2.2. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ be a measurable transformation. Then μ is said to be τ**-invariant** if and only if for every $B \in \mathscr{B}$,*

$$\mu(\tau^{-1}(B)) = \mu(B). \tag{1.1}$$

It is often difficult or sometimes impossible to verify equation (1.1) for every set in the collection $\mathscr{B}$. This is why in the following definition we establish the notion of a π-system. Having established the π-system we will be able to prove invariance of μ for a collection of simpler sets and then through Theorem 1.2.1 which appears in [1], conclude that μ is invariant for all sets in $\mathscr{B}$.

Definition 1.2.3. *We say that Ξ is a π**-system** generating the σ-algebra $\mathscr{B}$ if and only if for every E_1 and E_2 in Ξ, $E_1 \cap E_2$ is in Ξ and $\mathscr{B}$ is the smallest σ algebra containing all sets of Ξ.*

Theorem 1.2.1. *[1] Let $(X, \mathscr{B}, \mu)$ be a compact measure space and τ be a measurable transformation. Let Ξ be a π-system generating $\mathscr{B}$. If*

$$\mu(\tau^{-1}(E)) = \mu(E)$$

for every $E \in \Xi$. Then τ preserves μ on $\mathscr{B}$.

Now define dynamical system, the central topic in this text.

Definition 1.2.4. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Let τ be a transformation of X with μ being a τ-invariant measure. Then we call the collection $(X, \mathscr{B}, \mu, \tau)$ a **dynamical system**.*

Definition 1.2.5. *We say a measurable transformation $\tau : X \to X$ is **non-singular** on a measure space $(X, \mathscr{B}, \mu)$ if and only if for all $B \in \mathscr{B}$*

$$\mu(B) = 0 \Longrightarrow \mu(\tau^{-1}(B)) = 0.$$

Finally, we conclude this section by defining an *absolutely continuous invariant measure* or ACIM. Since many invariant measures are uninteresting (point measures, zero measure, etc.), we often look for ACIMs of a dynamical system.

Definition 1.2.6. *Let $(X, \mathscr{B}, \mu, \tau)$ be a dynamical system. Let ν be a measure defined on all sets in $\mathscr{B}$ which is invariant under τ and absolutely continuous with respect to μ. Then we say ν is an **absolutely continuous invariant measure (ACIM)** for τ.*

1.3 Markov Operator

In this section we are introducing Markov operator which we shall use to establish the result on Piecewise Concave Maps in Chapter 4. For more details, we refer the reader to [8].

Definition 1.3.1. *Let $(X, \mathscr{B}, \mu)$ be a measure space. Any linear operator $P : \mathscr{L}^1 \to \mathscr{L}^1$ satisfying:*

$$\begin{aligned} &\text{(a)} \quad Pf \geq 0 \qquad \text{for } f \geq 0, f \in \mathscr{L}^1; \text{ and} \\ &\text{(b)} \quad \|Pf\| = \|f\| \qquad \text{for } f \geq 0, f \in \mathscr{L}^1, \end{aligned} \tag{1.2}$$

*is called a **Markov Operator.***

Definition 1.3.2. *Let $(X, \mathscr{B}, \mu)$ be a measure space and the set $D(X, \mathscr{B}, \mu)$ be defined by $D(X, \mathscr{B}, \mu) = \{f \in \mathscr{L}^1 : f \geq 0$ and $||f|| = 1\}$. Any function $f \in D(X, \mathscr{B}, \mu)$ is called a **density**.*

Definition 1.3.3. *Let $(X, \mathscr{B}, \mu)$ be a measure space and $P : \mathscr{L}^1 \to \mathscr{L}^1$ a Markov Operator. Then $\{P^n\}$ is said to be **asymptotically stable** if there exists a unique $f^* \in D$ such that $Pf^* = f^*$ and*

$$\lim_{n\to\infty} \|P^n f - f^*\| = 0 \quad \text{for every } f \in D. \tag{1.3}$$

Definition 1.3.4. *Let $(X, \mathscr{B}, \mu)$ be a finite measure space. A Markov Operator P is called **constrictive** if there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which*

$$\int_E P^n f(x)\mu(dx) \leq \kappa \quad \text{for} \quad n \geq n_0(f) \quad \text{and} \quad \mu(E) \leq \delta. \tag{1.4}$$

1.4 The Frobenius-Perron Operator

The Frobenius-Perron Operator is a linear operator which describes the probabilistic behaviour of successive iterations of a dynamical system. The Operator was first studied by Kuzmin in [6]. Developing the Frobenius-Perron operator will provide us with an essential tool for uncovering the absolutely continuous invariant measures of dynamical systems.

1.4.1 The Radon-Nikodym Theorem

The existence and uniqueness of the Frobenius-Perron operator follows as a result of the Radon-Nikodym theorem, a theorem in measure theory which establishes the existence of a function called the Radon-Nikodym derivative. The Radon-Nikodym derivative can be interpreted as a density function.

Theorem 1.4.1 (***The Radon-Nikodym Theorem***)**.** *Let $(X, \mathscr{B}, \mu)$ be a measure space with μ being a σ-finite measure and ν be a finite measure absolutely continuous with respect to μ. Then there exists a unique non-negative measurable function $f : X \to [0, \infty)$ such that for every $A \in \mathscr{B}$*

$$\nu(A) = \int_A f d\mu.$$

*We call the function f the **Radon-Nikodym derivative**.*

Proof. We assume that $(X, \mathscr{B}, \mu)$ is a finite measure space and show the existence and uniqueness of the Radon-Nikodym derivative. Suppose μ and ν are both finite-valued non-negative measures. The general case is proved considering the partition of X into subsets of finite measure.

Let $\mathscr{F} = \{f \text{ is measurable } | \nu(E) \geq \int_E f d\mu, \text{ for all } E \in \mathscr{B}\}$. $\mathscr{F}$ is nonempty, since it contains at least the zero function. Let $s = \sup_{f \in \mathscr{F}} \int_X f d\mu$. Then there is a sequence $\langle h_n \rangle$ in $\mathscr{F}$ such that $\lim_{n\to\infty} \int_X f_n d\mu = s < \infty$.

Let $f_1, f_2 \in \mathscr{F}$, then for any $E \in \mathscr{B}$,

$$\begin{aligned}
\int_E \max\{f_1, f_2\}\, d\mu &= \int_{\{x\in E | f_1(x) \geq f_2(x)\}} \max\{f_1, f_2\}\, d\mu + \int_{\{x\in E | f_1(x) < f_2(x)\}} \max\{f_1, f_2\}\, d\mu \\
&= \int_{\{x\in E | f_1(x) \geq f_2(x)\}} f_1\, d\mu + \int_{\{x\in E | f_1(x) < f_2(x)\}} f_2\, d\mu \\
&\leq \nu(\{x \in E | f_1(x) \geq f_2(x)\}) + \nu(\{x \in E | f_1(x) < f_2(x)\}) \\
&= \nu(E).
\end{aligned}$$

Therefore, $\max\{f_1, f_2\} \in \mathscr{F}$.

Let f_n be a sequence of functions in $\mathscr{F}$. By replacing f_n with the maximum of the first n functions h_n, we can say $\langle f_n \rangle$ is a nonnegative increasing sequence in $\mathscr{F}$ and $\lim_{n\to\infty} \int_X f_n d\mu = s$. Define g by $g(x) = \lim_{n\to\infty} f_n$ for $x \in X$. Then by the monotone convergence theorem, for any $E \in \mathscr{B}$,

$$\int_E g d\mu = \lim_{n\to\infty} \int_E f_n d\mu \leq \nu(E).$$

This shows $g \in \mathscr{F}$ and $\int_X g d\mu = \lim_{n\to\infty} \int_X f_n d\mu = s$.

Therefore, the function ν_0 defined on $\mathscr{B}$ by $\nu_0(E) = \nu(E) - \int_E g d\mu$ is a measure. We want to show that $\nu_0 = 0$ and then g is the desired function. Suppose ν_0 is not zero. Since $\nu_0(X) > 0$ and $\mu(X) < \infty$, there is $\epsilon > 0$ such that $\nu_0(X) - \epsilon\mu(X) > 0$. Let $\{A, B\}$ be a Hahn decomposition for the signed measure $\nu_0 - \epsilon\mu$. Then for every $E \in \mathscr{B}$, $\nu_0(A \cap E) - \epsilon\mu(A \cap E) \geq 0$. So,

$$\begin{aligned}
\nu(E) = \nu_0(E) + \int_E g d\mu &\geq \nu_0(E \cap A) + \int_E g d\mu \\
&\geq \epsilon\mu(E \cap A) + \int_E g d\mu = \int_E (g + \epsilon\chi_A) d\mu.
\end{aligned}$$

Therefore, $g + \epsilon\chi_A$ is also in $\mathscr{F}$. However, if $\mu(A) > 0$, then

$$\int_X (g + \epsilon\chi_A) d\mu = \int_X g d\mu + \epsilon\mu(A) > \int_X g d\mu = s,$$

which is a contradiction. In fact, if $\mu(A) = 0$, since $\nu \ll \mu$, $\nu(A) = 0$. So $\nu_0(A) = \nu(A) - \int_A g d\mu \leq \nu(A) = 0$. Hence $\nu_0(A) = 0$. Consequently, $\nu_0(X) - \epsilon\mu(X) = \nu_0(B) - \epsilon\mu(B) \leq 0$, contradicting that $\nu_0(X) - \epsilon\mu(X) > 0$.

Therefore, $\nu_0 = 0$, which means $\nu(E) = \int_E g d\mu$ for every $E \in \mathscr{B}$.

To show uniqueness, let $\nu(E) = \int_E f d\mu = \int_E g d\mu$. Then $\int_E (f-g) d\mu = 0$. Since E is arbitrary, $\int_{\{f-g\geq 0\}} (f-g) d\mu = 0$. This shows $f = g$ with respect to μ, hence also ν on $\{x \in X | f(x) \geq g(x)\}$. Similarly $f = g$ a.e. on $\{x \in X | f(x) < g(x)\}$. Hence $f = g$ a.e. on X.

■

1.4.2 Motivation

We use the same motivation as given in [2].

Definition 1.4.1. *A* ***random variable*** Y *is a measurable function from a probability space to* $\mathbb{R}$*, such that for every borel set* B*,*

$$Y^{-1}(B) = \{Y \in B\} \in \mathscr{B}.$$

Suppose we are working with some random variable Y, on an interval $X = [a, b]$. Suppose that the random variable has probability density function f, i.e., for all measurable sets $A \in X$,

$$\text{Prob}\{Y \in A\} = \int_A f d\lambda,$$

where λ is the normalized Lebesque measure on X. Now suppose that we have a measurable transformation $\tau : I \to I$, then $\tau(Y)$ is also a random variable, and thus we may inquire about its probability density function. To obtain the density function for $\tau(Y)$, we must be able to write

$$\text{Prob}\{\tau(Y) \in A\} = \int_A \phi d\lambda,$$

for some function ϕ. If such a ϕ exists, it would depend on τ and f.

We begin our derivation by assuming Y is a random variable having probability density function $f \in \mathscr{L}^1$, τ is non-singular and we define for any measurable set A,

$$\mu(A) = \text{Prob}\{\tau(Y) \in A\} = \text{Prob}\{Y \in \tau^{-1}(A)\} = \int_{\tau^{-1}(A)} f d\lambda.$$

Since τ is non-singular, $\lambda(A) = 0 \implies \lambda(\tau^{-1}(A)) = 0$. Thus, $\mu(A) = 0$ implies

$$\mu(A) = \int_{\tau^{-1}(A)} f d\lambda = 0.$$

Therefore, μ is absolutely continuous with respect to λ. Now it follows by Theorem 1.4.1 that there exists a $\phi \in \mathscr{L}^1$ such that,

$$\mu(A) = \int_A \phi d\lambda$$

for any measurable set A. Furthermore, as a result of Theorem 1.4.1, ϕ is unique up to sets of zero measure. Thus we set,

$$P_\tau f = \phi$$

and call this function the Frobenius-Perron operator associated with the transformation τ. As our motivating derivation implies, the Frobenius-Perron operator transforms the probability density function of a random variable Y into the density function for $\tau(Y)$. As such, it can be used as a tool in our analysis of dynamical systems to find absolutely continuous invariant measures.

1.4.3 The Frobenius-Perron Operator

We now formally define the Frobenius-Perron operator.

Definition 1.4.2. *Let $(X, \mathscr{B}, \mu)$ be a measure space and let $\tau : X \to X$ be a non-singular transformation. Then the Frobenius-Perron operator $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$, is defined to be the almost everywhere unique $\mathscr{L}^1$ function satisfying:*

$$\int_A P_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu,$$

for any $A \in \mathscr{B}$.

We continue by establishing several properties of the Frobenius-Perron operator.

Proposition 1.4.1. ***Linearity**: $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ is a linear operator.*

Proof. Let $f, g \in \mathscr{L}^1$. Suppose α is a scalar and $A \subset X$ is a measurable set.

Then,

$$\begin{aligned}\int_A (P_\tau(\alpha f + g))d\mu &= \int_{\tau^{-1}(A)} (\alpha f + g)d\mu \\ &= \alpha \int_{\tau^{-1}(A)} f d\mu + \int_{\tau^{-1}(A)} g d\mu \\ &= \int_A (\alpha P_\tau f + P_\tau g)d\mu.\end{aligned}$$

Thus it follows that

$$P_\tau(\alpha f + g) = \alpha P_\tau f + P_\tau g,$$

μ almost everywhere. ■

Proposition 1.4.2. ***Positivity***: *Let $f \in \mathscr{L}^1$ with $f \geq 0$. Then, $P_\tau f \geq 0$.*

Proof. Let $A \in \mathscr{B}$ be arbitrary. Then,

$$\int_A P_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu \geq 0,$$

since $f \geq 0$. Therefore since A was arbitrary, $P_\tau f \geq 0$. ■

Proposition 1.4.3. ***Preservation of Integrals***. *Let $f \in \mathscr{L}^1(X)$. Then,*

$$\int_X P_\tau f d\mu = \int_X f d\mu.$$

Proof. Let $f \in \mathscr{L}^1(X)$. Then,

$$\int_X P_\tau f d\mu = \int_{\tau^{-1}(X)} f d\mu = \int_X f d\mu.$$

Since f was arbitrary, the proposition holds. ■

Proposition 1.4.4. ***Contraction***: *$P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ is contraction, i.e. the inequality*

$$\|P_\tau f\|_1 \leq \|f\|_1$$

holds for all $f \in \mathscr{L}^1(X)$.

Proof. Let $f \in \mathscr{L}^1(X)$ be arbitrary. Then it is possible to decompose f into two non-negative functions,

$$\begin{aligned}f_- &= \max(0, -f), \\ f_+ &= \max(0, f),\end{aligned}$$

both of are also be in $\mathscr{L}^1(X)$. Then for $f^+, f^- \in \mathscr{L}^1$,

$$f = f_+ - f_-,$$

and,

$$|f| = f_+ + f_-.$$

So applying Proposition 1.4.1, the property of linearity, we have that,

$$\begin{aligned} P_\tau f &= P_\tau(f_+ - f_-) \\ &= P_\tau f_+ - P_\tau f_-. \end{aligned}$$

Therefore,

$$|P_\tau f| \leq |P_\tau f_+| + |P_\tau f_-|,$$

which by the result of Proposition 1.4.2, positivity, is

$$\begin{aligned} &= P_\tau f_+ + P_\tau f_- \\ &= P_\tau(f_+ + f_-) \\ &= P_\tau |f|. \end{aligned}$$

Thus, taking the norm of the operator, we get that,

$$\begin{aligned} \|P_\tau f\|_1 &= \int_X |P_\tau f| d\mu \\ &\leq \int_X P_\tau |f| d\mu. \end{aligned}$$

Finally, applying Proposition 1.4.3, the preservation of integrals, we get,

$$\begin{aligned} \|P_\tau f\|_1 &\leq \int_X |f| d\mu \\ &= \|f\|_1, \end{aligned} \tag{1.5}$$

which proves the result. ∎

A direct consequence of Proposition 1.4.4 is that the Frobenius-Perron operator is continuous.

Corollary 1.4.1.1. ***Continuity in the norm topology****:* $P_\tau f : \mathscr{L}^1 \to \mathscr{L}^1$ *is a continuous operator with respect to the norm topology.*

Proof. Let $f, g \in \mathscr{L}^1$. Then,

$$\|P_\tau f - P_\tau g\|_1 \leq \|f - g\|_1,$$

which implies $\|P_\tau f - P_\tau g\|_1 \to 0$ as $\|f - g\|_1 \to 0$. ∎

We now present the property of composition for the Frobenius-Perron operator:

Proposition 1.4.5. ***Composition***: *Let $(X, \mathscr{B}, \mu)$ be a measure space and $\tau : X \to X$ and $\sigma : X \to X$ be a non-singular transformations. Then $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$ (μ almost everywhere), and in particular, $P_{\tau^n} f = P_\tau^n f$.*

Proof. Let τ and σ be non-singular transformations. Then, $\tau \circ \sigma$ is non-singular as, for every $A \in \mathscr{B}$ such that $\mu(A) = 0$, $\mu(\tau^{-1}(A)) = 0$ since τ is non-singular. Then

$$\begin{aligned}\mu((\tau \circ \sigma)^{-1}(A)) &= \mu(\sigma^{-1}(\tau^{-1}(A))) \\ &= 0,\end{aligned}$$

as σ is also non-singular. Now let $f \in \mathscr{L}^1$ and $A \in \mathscr{B}$, then

$$\begin{aligned}\int_A P_{\tau\circ\sigma} f d\mu &= \int_{(\tau\circ\sigma)^{-1}A} f d\mu = \int_{\sigma^{-1}(\tau^{-1}A)} f d\mu \\ &= \int_{\tau^{-1}A} P_\sigma f d\mu = \int_A P_\tau(P_\sigma f) d\mu.\end{aligned}$$

Therefore, $P_{\tau\circ\sigma} f = P_\tau P_\sigma f$, μ almost everywhere. It then follows by induction that $P_{\tau^n} f = P_\tau^n f$, μ almost everywhere. ∎

Proposition 1.4.6. ***Adjoint***: *If $f \in \mathscr{L}^1$, $g \in \mathscr{L}^\infty$, then $\langle P_\tau f, g\rangle = \langle f, U_\tau g\rangle$, i.e.*

$$\int_X (P_\tau f) g d\mu = \int_X f U_\tau(g) d\mu, \tag{1.6}$$

where $U_\tau(g)$ is the Koopman Operator, defined as $U_\tau(g) = g \circ \tau$.

Proof. Let $A \in \mathscr{B}$ and set $g = \chi_A$. Then,

$$\begin{aligned}\int_X (P_\tau f) g d\mu &= \int_A P_\tau f d\mu \\ &= \int_{\tau^{-1}(A)} f d\mu = \int_X f \chi_{\tau^{-1}(A)} d\mu = \int_X f \cdot (\chi_A \circ \tau) d\mu.\end{aligned}$$

This verifies equation (1.6) for characteristic functions. Since linear combinations are dense in $\mathscr{L}^\infty$, we can conclude that the result holds for any $g \in \mathscr{L}^\infty$. ∎

The following proposition is particularly useful in our research on absolutely continuous invariant measures. It states that a measure $\nu = f^*\mu$, absolutely continuous with respect to μ is τ-invariant if and only if it is a fixed point of the Frobenius-Perron operator, i.e. $P_\tau f^* = f^*$. In other words, it provides an equivalent definition for an ACIM in terms of the Frobenius-Perron operator.

Proposition 1.4.7. *Let $\tau : X \to X$ be non-singular. Let the measure ν be defined by,*

$$\nu(A) = \int_A f^* d\mu,$$

where $f^* \in \mathscr{L}^1$, $f^* \geq 0$, and $\|f\|_1 = 1$. Then ν is τ-invariant (Definition 1.2.2) if and only if,

$$P_\tau f^* = f^*.$$

Note: the definition of ν makes it absolutely continuous with respect to μ.

Proof. $\Longrightarrow$
Assume ν is τ-invariant. Then,

$$\nu(A) = \nu(\tau^{-1}(A)),$$

for every measurable set A. Therefore, on an arbitrary measurable set A, we have,

$$\begin{aligned}\nu(\tau^{-1}(A)) &= \int_{\tau^{-1}(A)} f^* d\mu \\ &= \int_A P_\tau f^* d\mu,\end{aligned}$$

and

$$\nu(A) = \int_A f^* d\mu.$$

Thus, by assumption

$$\int_A P_\tau f^* d\mu = \int_A f^* d\mu.$$

Since A was arbitrary, $P_\tau f^* = f^*$, μ almost everywhere.

$\Longleftarrow$
Assume $P_\tau f^* = f^*$, μ almost everywhere. Then

$$\begin{aligned}\int_A P_\tau f^* d\mu &= \int_{\tau^{-1}(A)} f^* d\mu \\ &= \nu(\tau^{-1}(A)),\end{aligned}$$

which by assumption,

$$= \int_A f^* d\mu$$
$$= \nu(A).$$

Thus,

$$\nu(A) = \nu(\tau^{-1}(A)).$$

And since A was arbitrary, we conclude ν is τ-invariant. ■

Proposition 1.4.8. Continuity : *Let $(X, \mathscr{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ be non-singular. Then $P_\tau : \mathscr{L}^1 \to \mathscr{L}^1$ is continuous in the weak topology on $\mathscr{L}^1$.*

Proof. In order for P_τ to be continuous in the weak topology of $\mathscr{L}^1$, we must have the following condition:

$$f_n \to f \text{ weakly} \implies P_\tau f_n \to P_\tau f \text{ weakly},$$

where we say $f_n \to f$ weakly in $\mathscr{L}^1$ if and only if

$$\int_X f_n g d\mu \to \int_X f g d\mu$$

for all $g \in \mathscr{L}^\infty$. Thus, we assume that $f_n \to f$ weakly and use Proposition 1.4.6, giving

$$\int_X (P_\tau f_n) g d\mu = \int_X f_n (g \circ \tau) d\mu.$$

Now the composition $g \circ \tau \in \mathscr{L}^\infty$ and by assumption, $f_n \to f$ weakly. Thus,

$$\int_X f_n (g \circ \tau) d\mu \to \int_X f (g \circ \tau) d\mu$$
$$= \int_X (P_\tau f) g d\mu.$$

Therefore,

$$\int_X (P_\tau f_n) g d\mu \to \int_X (P_\tau f) g d\mu,$$

as $n \to \infty$. Hence, $P_\tau f_n \to P_\tau f$ weakly in $\mathscr{L}^1$. ■

1.5 Piecewise Monotonic Maps and Representation of Frobenius-Perron Operator

For a special class of piecewise monotonic transformations, the Frobenius-Perron operator has a convenient representation, which will be of great use in the sequel.

Definition 1.5.1. *The function τ is said to be of class C^r if the derivatives $\tau', \tau'', ..., \tau^r$ exist and are continuous.*

Let $I = [a, b]$ and let λ denotes the normalized Lebesgue measure on I. The transformation $\tau : I \to I$ is called piecewise monotonic if there exists a partition of I, $a = a_0 < a_1 \cdots < a_n = b$ and a number $r \geq 1$ such that,

(1) τ is a C^r function on (a_{i-1}, a_i); $i = 1, \ldots, n$, which can be extended to a C^r function on $[a_{i-1}, a_i]$; $i = 1, \ldots, n$ and

(2) $|\tau'(x)| > 0$ on (a_{i-1}, a_i), $i = 1, \ldots, n$.

If the condition (2) is replaced by $|\tau'(x)| \geq \alpha > 1$, then τ is called piecewise monotonic and expanding. Let the transformation τ be piecewise monotonic on the partition $\mathscr{P} = \{a_0, a_1, \ldots, a_n\}$. Denote $\tau_{|[a_{i-1},a_i)}$ by τ_i and $B_i = \tau([a_{i-1}, a_i]), i = 1, \ldots, n$. Then for any measurable set $A \subset I$,

$$\tau^{-1}(A) = \bigcup_{i=1}^{n} \tau_i^{-1}(A \cap B_i).$$

It is obvious that the sets $\left\{\tau_i^{-1}(A \cap B_i)\right\}_{i=1}^{n}$ are mutually disjoint and depending on A, may even be empty. We can separate the integral and change the variable:

$$\begin{aligned}
\int_A P_\tau f(x) d\lambda &= \int_{\tau^{-1}(A)} f(x) d\lambda \\
&= \sum_{i=1}^{n} \int_{\tau_i^{-1}(A\cap B_i)} f(x) d\lambda \\
&= \sum_{i=1}^{n} \int_{A\cap B_i} f(\tau_i^{-1}(x)) \left|\left(\tau_i^{-1}(x)\right)'\right| d\lambda \\
&= \sum_{i=1}^{n} \int_{A} f(\tau_i^{-1}(x)) \left|\left(\tau_i^{-1}(x)\right)'\right| \chi_{B_i}(x) d\lambda \\
&= \int_A \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{\left|\tau'\left(\tau_i^{-1}(x)\right)\right|} \chi_{B_i}(x) d\lambda.
\end{aligned}$$

Since A is arbitrary, we can write

$$P_\tau f(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{\left|\tau'\left(\tau_i^{-1}(x)\right)\right|} \chi_{\tau([a_{i-1},a_i])}(x). \tag{1.7}$$

Example. Let τ be the tent map as shown in Figure 1.1. τ is a piecewise function on $[0,1]$:

$$\tau(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2}, \\ -2x+2, & \frac{1}{2} \le x < 1. \end{cases}$$

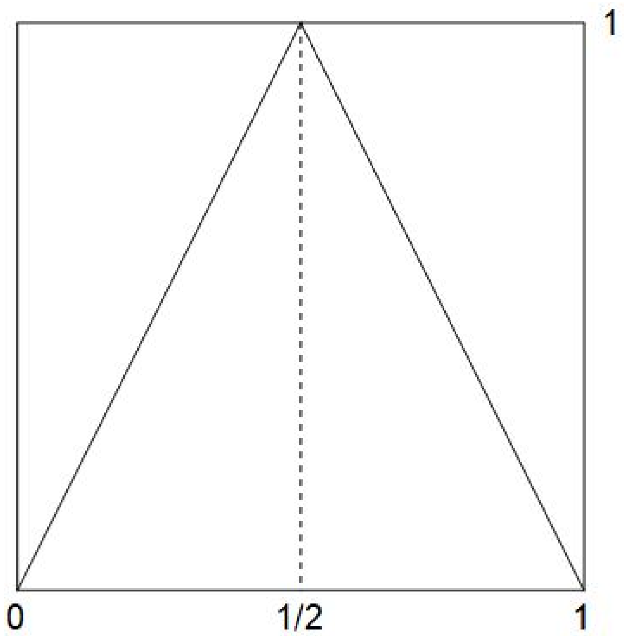

Figure 1.1: The tent map

We denote the left branch as τ_1 and τ_2 for the right one. Then,

$$\begin{aligned} &\tau_1'(x) = 2, \qquad \tau_2'(x) = -2, \\ &\tau_1^{-1}(x) = \frac{1}{2}x, \qquad \tau_2^{-1}(x) = -\frac{1}{2}(x-2), \\ &P_\tau f = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(1 - \frac{x}{2}\right). \end{aligned}$$

We can see that $\rho(x) = 1$ is the invariant density for the tent map since $P_\tau \rho = \rho$.

Chapter 2

Piecewise Convex Maps

In this section we consider transformations that are not necessarily expanding, i.e., their derivatives may be smaller than 1, but they possess another property which makes them very special, namely piecewise convexity. The proof that such transformations possess absolutely continuous invariant measures follows from the ideas of [7] and [2].

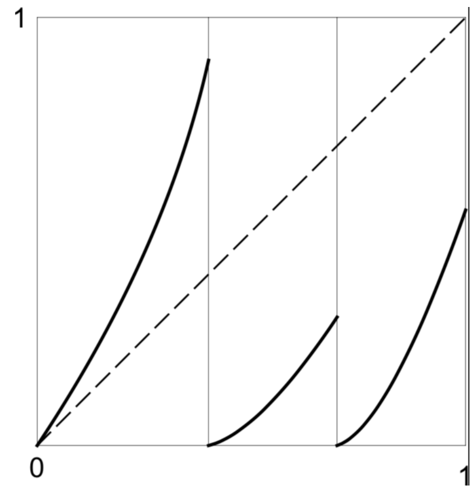

Figure 2.1: Piecewise Convex Map

Let $I = [0, 1]$. We say that $\tau \in \mathcal{T}_{pc}(I)$ if it satisfies the following conditions:

(i) There exists a partition $0 = a_0 < \cdots < a_q = 1$ such that $\tau_{|[a_{i-1},a_i)}$ is continuous and convex, $i = 1, \ldots, q$;

(ii) $\tau(a_{i-1}) = 0$, $\tau'(a_{i-1}) > 0$, $i = 1, \ldots, q$;

(iii) $\tau'(0) = \alpha > 1$.

Let us recall that $\tau : J \to \mathbb{R}$ is ***convex*** if and only if for any points $x, y \in J$ and for any $0 \leq \eta \leq 1$,

$$\tau(\eta x + (1-\eta)y) \leq \eta\tau(x) + (1-\eta)\tau(y).$$

It can be proved that a convex function is differentiable except at a countable set of points and that its derivative τ' is nondecreasing. In particular, this means that (ii) implies

$$\tau'(x) > \tau'(a_{i-1}) > 0\,,\ x \in [a_{i-1}, a_0),$$

and $\tau_{|[a_{i-1},a_i)}$ is increasing for $i = 1, \dots, q$. An example of $\tau \in \mathcal{T}_{pc}(I)$ is shown in Figure 2.1.

First we prove the following useful property of transformations in $\mathcal{T}_{pc}(I)$:

Proposition 2.0.1. *Let $\tau \in \mathcal{T}_{pc}(I)$ and let f be a nonincreasing function. Then $P_\tau(f)$ is also nonincreasing.*

Proof. We have

$$P_\tau(f)(x) = \sum_{i=1}^{q} f(\tau_i^{-1}(x)) \frac{1}{\tau'(\tau_i^{-1}(x))} \chi_{\tau([a_{i-1},a_i))}(x).$$

Let $0 \leq x < y \leq 1$. We will show that, for any $i = 1, \dots, q$,

$$\begin{aligned} f(\tau_i^{-1}(x)) &\frac{1}{\tau'(\tau_i^{-1}(x))} \chi_{\tau([a_{i-1},a_i))}(x) \\ &\geq f(\tau_i^{-1}(y)) \frac{1}{\tau'(\tau_i^{-1}(y))} \chi_{\tau([a_{i-1},a_i))}(y). \end{aligned} \tag{2.1}$$

Let us fix $1 \leq i \leq q$. Since $\tau_{|[a_{i-1},a_i)}$ is increasing and $\tau(a_{i-1}) = 0$, if $\chi_{\tau([a_{i-1},a_i))}(x) = 0$ then $\chi_{\tau([a_{i-1},a_i))}(y) = 0$. Thus,

$$\chi_{\tau([a_{i-1},a_i))}(x) \geq \chi_{\tau([a_{i-1},a_i))}(y).$$

If they are both nonzero, we have

$$f(\tau_i^{-1}(x)) \geq f(\tau_i^{-1}(y)),$$

since f is nonincreasing and $\tau_i^{-1}(x) < \tau_i^{-1}(y)$. Also

$$\frac{1}{\tau'(\tau_i^{-1}(x))} \geq \frac{1}{\tau'(\tau_i^{-1}(y))},$$

since τ' is nondecreasing and $\tau_i^{-1}(x) < \tau_i^{-1}(y)$. Hence (2.1) is proved. Summing up (2.1) completes the proof. ■

We now make the following observation:

Lemma 2.0.1. *If $f \geq 0$ and f is nonincreasing, then $f(x) \leq \frac{1}{x}\lambda(f)$, for $x \in [0,1]$, where*

$$\lambda(f) = \int_I f d\lambda.$$

Proof. For any $0 < x \leq 1$ and assume $f(0) = 1$. We have,

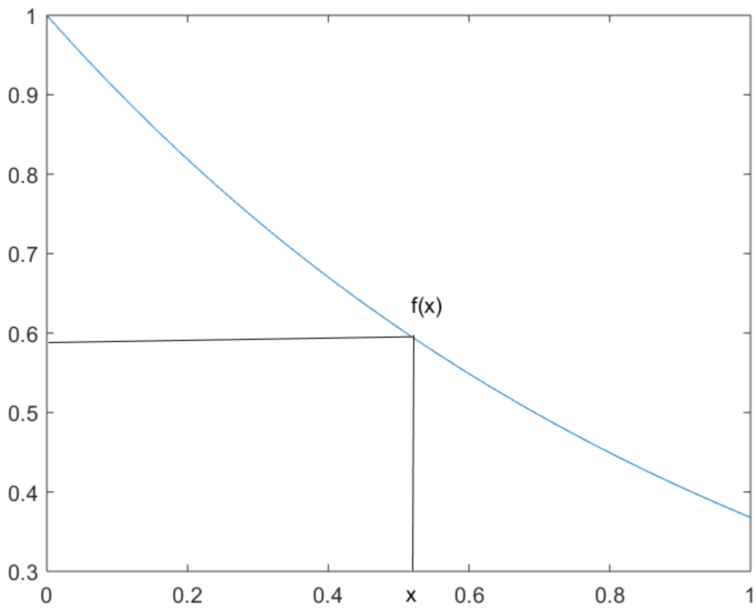

Figure 2.2: Inequality of Lemma 3.0.1

$$\lambda(f) \geq \int_0^x f(y)d\lambda(y) \geq x \cdot f(x).$$

■

Now, we will prove an inequality that closely resembles the Lasota-Yorke inequality [9]

Proposition 2.0.2. *Let $\tau \in \mathcal{T}_{pc}(I)$. If $f : [0,1] \to \mathbb{R}^+$ is nonincreasing, then*

$$||P_\tau f||_\infty \leq \frac{1}{\alpha}||f||_\infty + C||f||_1, \tag{2.2}$$

where $C = \sum_{i=2}^{q}(a_{i-1} \cdot \tau'(a_{i-1}))^{-1}$.

Proof. Since f is nonincreasing, we have $f(0) \geq ||f||_\infty$, and by Proposition 2.0.1, $P_\tau f(0) \geq ||P_\tau f||_\infty$. Recall that $\alpha = \tau'(0) \geq 1$. Hence, by condition (ii) and Lemma 2.0.1, we have

$$P_\tau f(0) = \frac{1}{\tau'(0)} f(0) + \sum_{i=2}^{q} \frac{f(\tau_i^{-1}(0))}{\tau'(\tau_i^{-1}(0))} = \frac{1}{\alpha} f(0) + \sum_{i=2}^{q} \frac{f(a_{i-1})}{\tau'(a_{i-1})}$$
$$\leq \frac{1}{\alpha} f(0) + \sum_{i=1}^{q} \frac{\lambda(f)}{a_{i-1}} \frac{1}{\tau'(a_{i-1})} \leq \frac{1}{\alpha} ||f||_\infty + C||f||_1.$$

∎

We are now ready to prove the main result of this section

Theorem 2.0.2. *Let* $\tau \in \mathcal{T}_{pc}(I)$. *Then* τ *admits an absolutely continuous invariant measure,* $\mu = f^*\lambda$, *and the density* f^* *is nonincreasing.*

Proof. Let $f \equiv 1$. f is nonincreasing. Then by Proposition 2.0.2, we can apply inequality (2.2) iteratively. We obtain

$$||P_\tau^n f||_\infty \leq \frac{1}{\alpha^n} ||f||_\infty + C(1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{n-1}})||f||_1 \leq 1 + C\frac{1}{1 - \frac{1}{\alpha}}.$$

Thus, the sequence $\{P_\tau^n f\}_{n=1}^\infty$ is uniformly bounded and thus weakly compact in $\mathcal{L}^1$ (Proposition 1.1.1) . By the Yosida-Kakutani Theorem (Theorem 1.1.5), the sequence $\frac{1}{n}\sum_{i=1}^{n-1} P_\tau^i f$ converges in $\mathcal{L}^1$ to a P_τ-invariant function f^*. It is nonincreasing since it is the limit of nonincreasing functions. ∎

There are cases where it is possible to allow zero-derivatives at certain endpoints a_{i-1}. In that case the invariant density will not be bounded but still absolutely continuous with respect to Lebesgue measure.

Example: In this example we will discuss more about the cases mentioned above.

Let $\tau : [0, 1] \to [0, 1]$ be a piecewise function as follows:

$$\tau_1(x) = 2x,$$
$$\tau_2(x) = 4\left(x - \frac{1}{2}\right)^2.$$

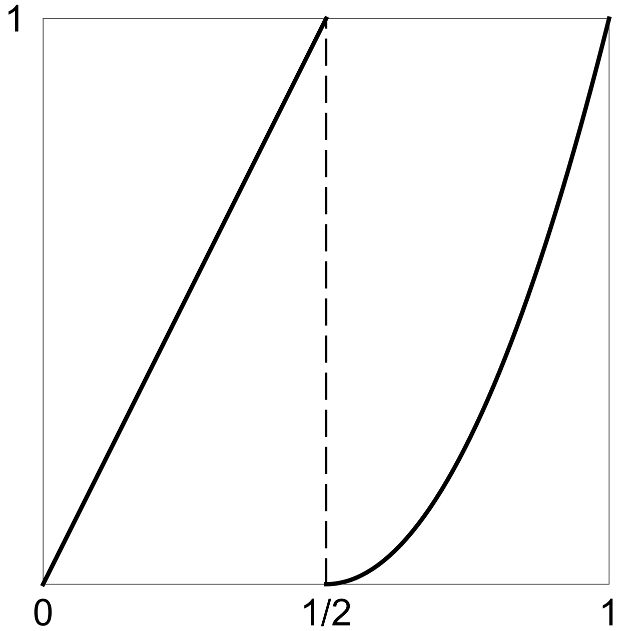

Figure 2.3: Graph of τ

We obtain that:

$$(P_\tau f)(x) = \frac{f\left(\frac{x}{2}\right)}{2} + \frac{f\left(\frac{1}{2} + \frac{\sqrt{x}}{4}\right)}{4\sqrt{x}}.$$

We will show that τ has a finite acim, using conjugation (see the definition and the proposition below) by the diffeomorphism

$$\sigma(x) = x^2.$$

Let $\tilde{\tau} = \sigma^{-1} \circ \tau \circ \sigma$. So,

$$\tilde{\tau}_1(x) = \sqrt{2x^2} = \sqrt{2}x.$$

$$\tilde{\tau}_2(x) = \sqrt{4\left(x^2 - \frac{1}{2}\right)^2} = 2\left(x^2 - \frac{1}{2}\right)$$

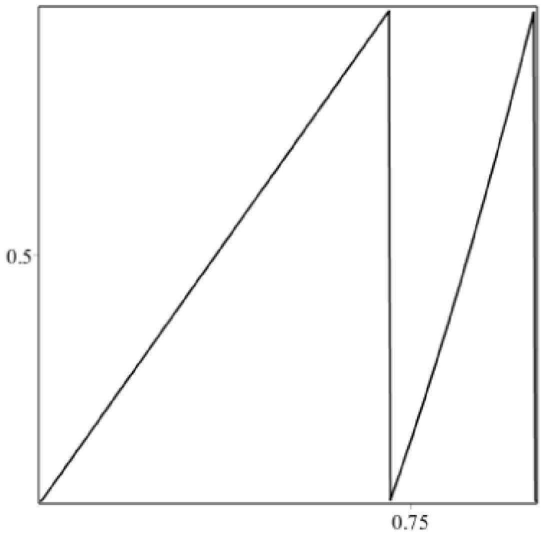

Figure 2.4: Graph of $\tilde{\tau}$

with $\tilde{\tau}_1$ defined on $[0, \sqrt{\frac{1}{2}}]$ and $\tilde{\tau}_2$ defined on $[\sqrt{\frac{1}{2}}, 1]$. We note that $\tilde{\tau}_1$, $\tilde{\tau}_2$ are convex and satisfy the hypotheses of Theorem 2.0.2. So for $\tilde{\tau}$ we can find an acim of the form $\tilde{f}(x)$, $\tilde{f} \geq 0$, bounded and decreasing.

Then $f(x) = \frac{\tilde{f}(\sqrt{x})}{2\sqrt{x}}$ is an invariant density for τ and it is integrable so the measure fdx is finite.

In the above example we have used the following proposition.

Definition 2.0.1. *Two transformations $\tau : I \to I$ and $\tilde{\tau} : J \to J$ on intervals I and J are called* ***conjugate*** *if there exists a homeomorphism σ, such that*

$$\tau(x) = \sigma^{-1}(\tilde{\tau}[\sigma(x)]).$$

Then the map σ is called the ***conjugation.***

Proposition 2.0.3. *Let, $\tau : I \to I$ be non singular and $\sigma : I \to I$ be a diffeomorphism. Then $P_\tau f = f$ implies $P_{\tilde{\tau}} g = g$, where $\tilde{\tau} = \sigma \circ \tau \circ \sigma^{-1}$ and $g = (f \circ \sigma^{-1}) \cdot |(\sigma^{-1})'|$.*

Proof. The fact that σ is a diffeomorphic, implies σ is monotonic. By equation 1.7,

$$\begin{aligned} P_\sigma f &= \sum_{i=1}^{n} (f \circ \sigma_i^{-1}) |(\sigma_i^{-1})'| \chi_{[a_{i-1}, a_i]} \\ &= (f \circ \sigma^{-1}) |(\sigma^{-1})'| \\ &= g. \end{aligned}$$

Now, using the composite relation we obtain,

$$\begin{aligned} P_{\tilde{\tau}} g &= P_{\tilde{\tau}}(P_\sigma f) \\ &= P_{\sigma \circ \tau \circ \sigma^{-1}}(P_\sigma f) \\ &= P_\sigma \circ P_\tau \circ P_{\sigma^{-1}}(P_\sigma f) \\ &= P_\sigma \circ P_\tau (f) \\ &= P_\sigma (f) \\ &= g. \end{aligned}$$

■

Chapter 3

Piecewise Concave Maps

In this section, we prove the existence of ACIM theorem for a class of mapping that are piecewise concave. We also prove that they are exact. For more details, we refer the readers to [3] and [8]

Definition 3.0.1. *Let $(X, \mathscr{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ be measure preserving such that $\tau(A) \in \mathscr{B}$ for each $A \in \mathscr{B}$. If*

$$\lim_{n\to\infty} \mu(\tau^n A) = 1$$

*for every $A \in \mathscr{B}$, $\mu(A) > 0$, then τ is **exact**.*

Theorem 3.0.1. *[2] Let $(X, \mathscr{B}, \mu)$ be a normalized measure space and let $\tau : (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be measure preserving. Then τ is exact if and only if*

$$\mathscr{B}^T = \bigcap_{n=0}^{\infty} \tau^{-n}(\mathscr{B})$$

is trivial, that is , the tail σ-algebra consists of the sets of μ-measure 0 or 1.

Proof. Let us assume that $A \in \mathscr{B}^T$, $0 < \mu(A) < 1$ and let $B_n \in \mathscr{B}$ be such that $A = \tau^{-n} B_n$, $n = 1, 2, \ldots$. ($\tau^{-n}(\mathscr{B})$ be the σ-algebra consisting for the set of the form $\tau^{-n} B_n$). Since τ preserves μ, we have $\mu(B_n) = \mu(A)$, $n = 1, 2, \ldots$. We also have $\tau^n(A) = \tau^n(\tau^{-n} B_n) \subset B_n$. Hence, $\mu(\tau^n(A)) \leq \mu(A) < 1$ for $n = 1, 2, \ldots$, which contradicts the exactness of τ. Let $A \in \mathscr{B}$ and $\mu(A) > 0$. If $\lim_{n\to+\infty} \mu(\tau^n A) < 1$, we may assume that for some $a < 1$, $\mu(\tau^n(A)) \leq a < 1$, $n = 1, 2, \ldots$. For any $n \geq 0$ we have $\tau^{-(n+1)}(\tau^{n+1} A) \supset \tau^{-n}(\tau^n A)$. Thus, the set $B = \bigcup_{n=0}^{\infty} \tau^{-n}(\tau^n A)$ belongs to $\mathscr{B}^T$. Since $B \supset A$ and $\mu(B) \geq \mu(A) > 0$, $\mu(B) = 1$. On the other hand,

$$\mu(B) = \lim_{n\to+\infty} \mu(\tau^{-n}(\tau^n A)) = \lim_{n\to+\infty} \mu(\tau^n(A)) \leq a < 1.$$

■

Theorem 3.0.2. *[8]*

Let P be a constrictive Markov operator. Assume there is a set $A \subset X$ of nonzero measure, $\mu(A) > 0$, with the property that for every $f \in D$ there is an integer $n_0(f)$ such that

$$P^n f(x) > 0 \tag{3.1}$$

for almost all $x \in A$ and all $n > n_0(f)$. Then $\{P^n\}$ is asymptotically stable.

Theorem 3.0.3. *Let $\tau : [0,1] \to [0,1]$ satisfy the following:*

(I) *there is a partition $0 = a_0 < a_1 < \cdots < a_m = 1$ of $[0,1]$ such that $\tau|_{(a_{i-1},a_i]}$ is of $\mathbb{C}^2$ for each $i = 1, \ldots m$*

(II) *$\tau'(x) > 0$ and $\tau''(x) \leq 0$ for all $x \in [0,1]$, where $\tau'(a_i)$ and $\tau''(a_i)$ are left derivatives.*

(III) *$\tau(a_i) = 1$ for each integer, $i = 1, \ldots, m$.*

(IV) *$\tau'(1) = \alpha > 1$.*

Then the corresponding Frobenius-Perron operator P maps increasing functions to increasing ones and τ has a normalized ACIM. Its density f^* is increasing.

Proof. From the definition of the Frobenius-Perron operator P,

$$Pf(x) = -\frac{d}{dx}\int_{\tau^{-1}([x,1])} f d\mu \tag{3.2}$$

Let τ_i be the restriction of τ to the interval $(a_{i-1}, a_i]$ and let

$$g_i(x) = \begin{cases} a_{i-1}, & x \in [0, \tau(a_{i-1}+0)] \\ \tau_i^{-1}(x), & x \in [\tau(a_{i-1}+0), 1] \end{cases}$$

for $i = 1, \ldots, m$, where $\tau(a_{i-1}+0)$ is the right limit. Then for any $x \in [0,1]$,

$$\tau^{-1}([0,x]) = \bigcup_{i=1}^{m} [g_i(x), a_i],$$

from which the F-P operatior is thus,

$$Pf(x) = -\frac{d}{dx}\sum_{i=1}^{m}\int_{g_i(x)}^{a_i} f d\mu = \sum_{i=1}^{m} g_i'(x) f(g_i(x)).$$

Since τ_i is increasing, so is g_i. And g_i' is increasing since $g_i'' = -\tau''/(\tau')^2 \geq 0$. Thus, Pf is a non-negative increasing function if f is non-negative and increasing, and

$$\begin{aligned} Pf(x) = \sum_{i=1}^{m} g_i'(x) f(g_i(x)) &\leq \sum_{i=1}^{m} g_i'(1) f(g_i(1)) \\ &= \sum_{i=1}^{m-1} \left[g_i'(1) f(a_i) + g_m'(1) f(1) \right]. \end{aligned}$$

Now let $f \in \mathscr{L}^1(0,1)$ be an increasing density. Then

$$1 \geq \int_x^1 f(t)dt \geq \int_x^1 f(x)dt = f(x)(1-x) \tag{3.3}$$

which implies $f(x) \leq \frac{1}{1-x}$.

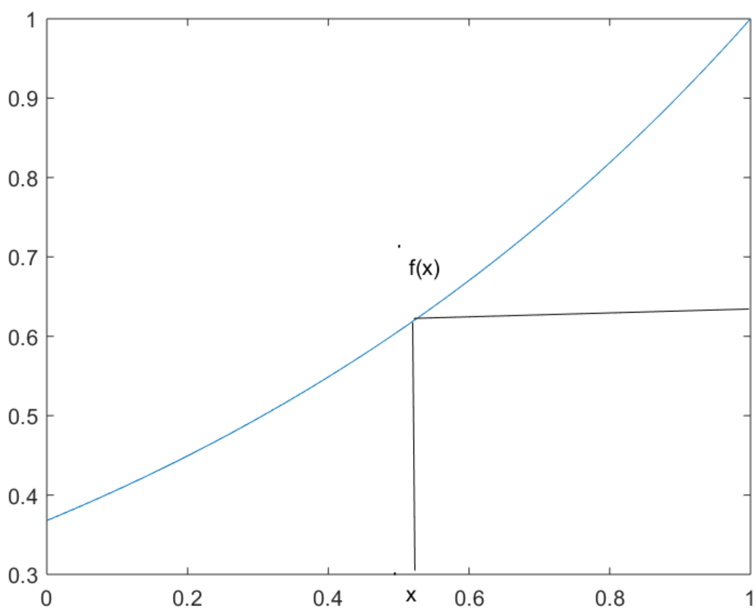

Figure 3.1: Inequality 3.3

Hence for $1 \leq i \leq m-1$,

$$g_i'(1) f(a_i) \leq \frac{g_i'(1)}{1-a_i}.$$

Noting that $g'_m(1) = 1/\tau'(1) = 1/\alpha < 1$, we have

$$Pf(x) \leq \sum_{i=1}^{m-1} \frac{g'_i(1)}{1-a_i} + \frac{1}{\alpha}f(1) - \frac{1}{\alpha}f(1) + M, \tag{3.4}$$

where

$$M = \sum_{i=1}^{m-1} \frac{g'_i(1)}{1-a_i}.$$

It follow that

$$P^n f(x) \leq \frac{1}{\alpha^n}f(1) + \frac{\alpha M}{\alpha - 1} \leq f(1) + K, \tag{3.5}$$

where $K = \alpha M/(\alpha - 1)$ is independent of f. Since $\|P\| = 1$ and the set

$$\{h \geq 0 | h(x) \leq f(1) + K, \, x \in [0,1]\}$$

is weakly compact in $\mathscr{L}^1(0,1)$, by a standard compactness argument for $\mathscr{L}^1$-spaces and by Yosida-Kakutani Theorem (Theorem 1.1.5), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k-0}^{n-1} P^k f = f^*$$

in $\mathscr{L}^1$ norm, where f^* is invariant density of P. It is obvious that f^* is increasing since f is increasing. ■

Theorem 3.0.4. *[3] Under the conditions of theorem 3.0.3, the invariant density f^* is unique. Furthermore, the iterative sequence $\{P^n\}$ is asymptotically stable, that is $\lim_{n\to\infty} P^n f = f^*$ for every density $f \in \mathscr{L}^1(0,1)$.*

Proof. Integrating both sides of (3.5) over measurable set $E \subset [0,1]$ for any increasing density f, we have

$$\int_E P^n f d\mu \leq \frac{1}{\alpha^n} \int_E f(1) d\mu + \int_E k d\mu = \left[\frac{1}{\alpha^n} f(1) + k\right] \mu(E).$$

Now let f be a density bounded variation. Then $f = f_1 - f_2$, where f_1 and f_2 are nonnegative and increasing. Hence

$$\begin{aligned}\int_E P^n f d\mu &= \int_E P^n f_1 d\mu - \int_E P^n f_2 d\mu \leq \int_E P^n f_1 d\mu \\ &= \|f_1\| \int_E P^n \frac{f_1}{\|f_1\|} d\mu \leq \|f_1\| \left[\frac{1}{\alpha^n} \frac{f_1(1)}{\|f_1\|} + k\right] \mu(E).\end{aligned}$$

Therefore, there exists $0 < k < 1$ and $\delta > 0$ such that, for any density f of bounded variation, there is an integer $N(f)$ for which

$$\int_E P^n f d\mu \leq k$$

for $n \geq N(f)$ and $\mu(E) < \delta$. Since any density function is a limit of a sequence of densities of bounded variation, P is constrictive. Since each g_i' in (3.3) is positive, if f is a density such that supp$f \supset [a_{m-1}, 1]$, then

$$Pf(x) = \sum_{i=1}^{m-1} g_i'(x) f(g_i(x)) + g_m'(x) f(g_m(x)) \geq g_m'(x) f(g_m(x)) > 0$$

on $[a_{m-1}, 1]$ since $\tau'(1) > 1$ and τ' is decreasing on $[a_{m-1}, 1]$ which means that supp$Pf \supset$ supp$f$.

Given any density f, there is an integer k such that supp$P^* f \supset [a_{m-1}, 1]$ since $\tau'(1) > 1$ and $\tau' > 0$ is decreasing on each $[a_{i-1}, a_i]$.

Hence for every density f, there is an integer $N(f)$ such that

$$P^n f(x) > 0 \quad \forall x \in [a_{m-1}, 1],\ n \geq N(f). \tag{3.6}$$

By the theorem 3.0.2, $\lim_{n\to\infty} \|P^n f - f^*\| = 0$ for all densities f. ■

3.1 Exactness of a Piecewise Concave Dynamical Systems

In this section we prove exactness of Piecewise Concave Map τ. We shall prove that the strong convergence of $P_\tau^n f$ to f^* implies exactness of τ.

Let, $||P_\tau^n f - f^*||_1 \to 0$. Assume $\mu(A) > 0$. Define the sequence α_n by $\alpha_n = ||P_\tau^n f - f^*||_1$. Then

$$\begin{aligned}
\mu(\tau^n(A)) &= \int_{\tau^n(A)} f^* d\mu = \int_{\tau^n(A)} [P_\tau^n f_A - (P_\tau^n f_A - f^*)] d\mu \\
&\geq \int_{\tau^n(A)} P_\tau^n f_A d\mu - \int_{\tau^n(A)} |P_\tau^n f_A - f^*| d\mu \\
&\geq \int_{\tau^n(A)} P_\tau^n f_A d\mu - \alpha_n = \int_{\tau^{-n}(\tau^n(A))} f_A d\mu - \alpha_n \\
&\geq \int_A f_A d\mu - \alpha_n = 1 - \alpha_n.
\end{aligned}$$

Since $\alpha_n \to 0$, we obtain

$$\lim_{n\to\infty} \mu(\tau^n(A)) = 1.$$

Which implies exactness of τ.

Chapter 4

ACIM for Piecewise Maps With Infinite Number of Branches

In this section we will generalize the main result of Section 3 to the case where τ has infinite number of branches.

We say $\tau \in \mathcal{T}_{pc}^{\infty}(I)$, $I = [0,1]$ if there exists a countable partition of I:

$$0 = a_0 < a_1 < a_2 < \dots$$
$$\lim_{j\to\infty} a_j = 1.$$

So that:
1) for $i = 1, 2, \dots$ $\tau_i = \tau_{|[a_{i-1}, a_i)}$ is continuous and convex.
2) $\tau(0) = 0$, $\tau'(0) = \alpha > 1$,
3) If $i > 1$: $\tau(a_{i-1}) = 0$, $\tau'(a_{i-1}) > 0$,
4) $\sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})} < \infty$.

In this version we allow infinitely many branches, but require that the derivatives at the endpoints increases rather rapidly.

For $\tau \in \mathcal{T}_{pc}^{\infty}(I)$, $f \in \mathcal{L}^1(I)$, $f \geq 0$, define:

$$P_\tau(f)(x) = \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))} \chi_{\tau[a_{i-1}, a_i)}(x). \tag{4.1}$$

Under the assumption that $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ none of the denominators are equal to 0. We have the following proposition analogous to Proposition 2.0.1.

Proposition 4.0.1. *Let $\tau \in \mathcal{T}_{pc}^{\infty}(I)$, $f \in \mathcal{L}^1(I)$, $f \geq 0$, f non-increasing. Then:*

1) $P_\tau(f) \in \mathcal{L}^1(I)$,
2) $P_\tau(f) \geq 0$,
3) $P_\tau(f)$ is non-increasing,
4) $||P_\tau(f)||_\infty \leq C||f||_\infty$.

Proof. As in the proof of proposition 2.0.1 note that each of the branches

$$\frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_{i-1},a_i)}(x)$$

is non-negative and non-increasing, establishing (2) and (3).

To establish (1)

$$\int_I P_\tau(f)dx = \int_{\tau^{-1}(I)} f dx = \int_I f dx.$$

Therefore, $P_\tau(f) \in \mathcal{L}^1(I)$.

(4). We have

$$\begin{aligned} P_\tau f(x) &\leq \sum_{i=1}^{\infty} \frac{||f||_\infty}{\tau'(\tau_i^{-1}(x))} \leq \sum_{i=1}^{\infty} \frac{||f||_\infty}{\tau'(a_{i-1})} \\ &= \left(\frac{1}{\alpha} + \sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})}\right) ||f||_\infty. \end{aligned}$$

Therefore with

$$C = \frac{1}{\alpha} + \sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})};$$

$$||P_\tau f||_\infty \leq C||f||_\infty.$$

∎

Lemma 4.0.1. *If $f \geq 0$ and f is non-increasing, f finite for all x. Then:*

$$f(x) \leq \frac{1}{x}\lambda(f) = \frac{1}{x}\int_I f dx.$$

The following is analogue to Lemma 2.0.1

Proposition 4.0.2. *If* $f : [0,1] \to \mathbb{R}^+$ *is non-increasing and* $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ *then:*

$$||P_\tau f||_\infty \leq \frac{1}{\alpha}||f||_\infty + C||f||_1,$$

where:

$$C = \sum_{i=2}^{\infty} \frac{1}{a_{i-1}\tau'(a_{i-1})}.$$

Note:

$$C < \frac{1}{a_1}\sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})} < \infty.$$

Proof. The proof of proposition 2.0.2 verbatim. ∎

Theorem 4.0.2. *Let* $\tau \in \mathcal{T}_{pc}(I)$. *Then* τ *admits an absolutely continuous invariant measure:*

$$\mu = f^*\lambda$$

with f^* *is non-increasing.*

Proof. Let $f \equiv 1$ and consider the sequence $P_\tau^n f$. As in Theorem 2.0.2 :

$$||P_\tau^n f||_\infty \leq 1 + \frac{C}{1 - \frac{1}{\alpha}}.$$

So again the sequence $\{P_\tau^n f\}$ is uniformly bounded and weakly compact (Proposition 1.1.1). By Yosida-Kakutani Theorem (Theorem 1.1.5), $\frac{1}{n}\sum_{i=1}^{n} P_\tau^i f$ converges in $\mathcal{L}^1$ to a P_τ invariant function f^*. It is non-increasing since it is the limit of non-increasing functions.

∎

4.0.1 Examples

Example-1: Let,

$$a_n = \frac{1}{2^n}, \qquad n = 0, 1, 2, \ldots.$$

Let $\tau : [a_n, a_{n+1}) \to [0,1]$ be linear. It is easy to see that Lebesgue measure is τ-invariant.

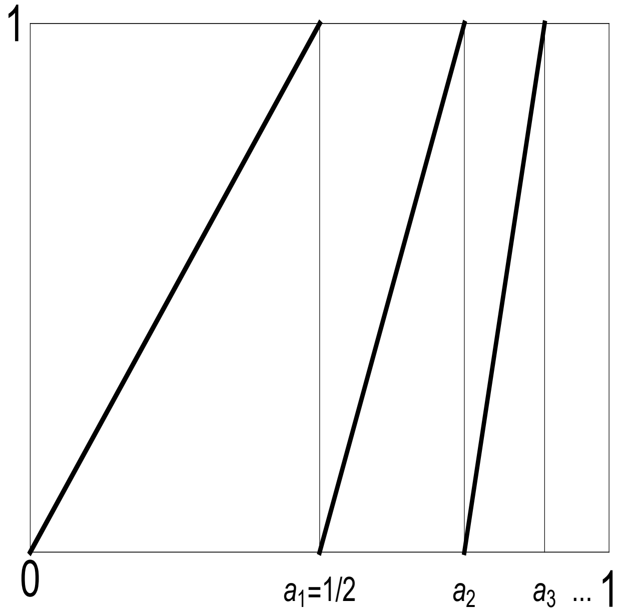

Figure 4.1: Piecewise linear map with infinite number of branches.

Proof. The slope of τ on the segment $[a_n, a_{n+1}]$ is equal to $\frac{1}{(a_{n+1}-a_n)}\cdot$ So the inverse of τ has slope $(a_{n+1} - a_n)$. Then with density $f = 1$,

$$P_\tau(f) = \sum_{n=1}^{\infty}\{a_{n+1} - a_n\} \cdot 1 = 1,$$

using telescopic sum so,

$$P_\tau(f) = f.$$

∎

Example-2: Let us consider the transformation

$$\tau(x) = \frac{x}{1-x} \pmod 1$$

with countably many branches. We will show that it preserves the density $f = \frac{1}{x}$.

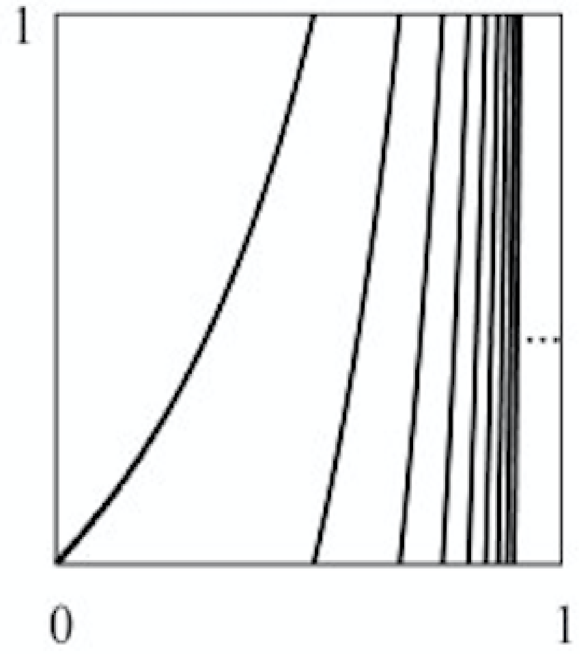

Figure 4.2: Example 2

Proof. Let $n \in \mathbb{N}$, then

$$\frac{x}{1-x} = n \Leftrightarrow x = n - nx \Leftrightarrow x = \frac{n}{1+n}.$$

Now, let

$$a_k = \frac{k}{1+k},$$

and

$$\tau_k = \tau|_{[a_k, a_{k+1})}.$$

If

$$y \in [0,1], \text{ and } y = \tau_k(x).$$

Then,

$$\frac{x}{1-x} = y + k.$$

So,

$$x = y + k - x(y+k).$$

Therefore,

$$x = \frac{y+k}{1+y+k} = \tau_k^{-1}(y).$$

Furthermore,

$$(\tau_k^{-1})'(y) = \frac{(1+y+k)\cdot 1 - (y+k)}{(1+y+k)^2} = \frac{1}{(1+y+k)^2}.$$

Then,

$$P_\tau\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty} \frac{1+x+k}{x+k} \cdot \frac{1}{(1+x+k)^2} = \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{1+x+k}\right) = \frac{1}{x}.$$

■

The series $\sum_{k=0}^{\infty} \frac{1}{\tau'(a_k)} = \sum_{k=0}^{\infty} \frac{1}{(1+k)^2} < \sum_{k=0}^{\infty} \frac{1}{k^2} < \infty$ is convergent using integral test.The invariant measure is infinite since $\tau'(0) = 1$ and condition 2 fails.

Example-3:

In this example we will show the preservation of measure with density $\frac{1}{x}$ by Lasota-Yorke example

$$\tau_1(x) = \frac{x}{1-x},$$
$$\tau_2(x) = 2x - 1.$$

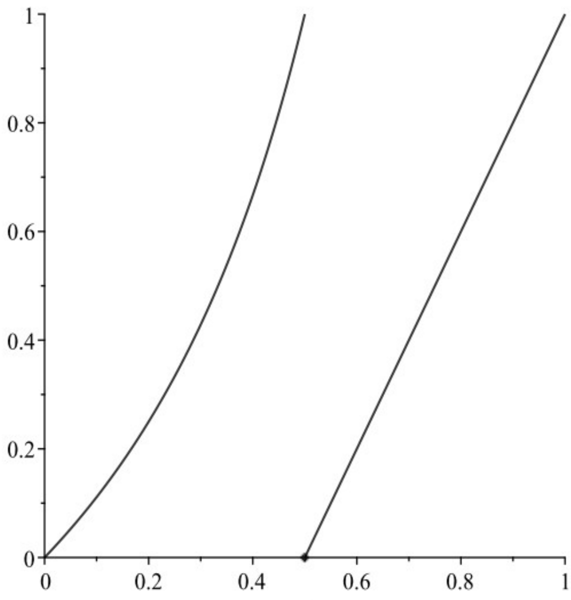

Figure 4.3: Example 3

We will also present the construction of the induced map of τ along with the definition of induced transformation and prove that the induced map is piecewise convex.

Proof. We have the derivatives of the piecewise functions,

$$\tau^{'}(x) = \begin{cases} \frac{1}{(1-x)^2}, \\ 2. \end{cases}$$

Also, the inverses of the piecewise funtions,

$$\tau^{-1}(x) = \begin{cases} \frac{x}{(x+1)}, \\ \frac{(x+1)}{2}. \end{cases}$$

$$P_\tau f(x) = P_\tau \left(\frac{1}{x}\right) = \frac{\frac{(x+1)}{x}}{|(x+1)^2|} + \frac{\frac{2}{(x+1)}}{|2|} = \frac{1}{x}.$$

So $\tau(x)$ preserves measure with density $\frac{1}{x}$. $\blacksquare$

Definition 4.0.1. *Let $\tau : X \to X$ be a measurable transformation preserving a normalized measure μ. Let $A \in \mathscr{B}$ and $\mu(A) > 0$. According to Kac's Lemma [2], the first return-time function $n = N_A$ is integrable and we define a transformation*

$$\tau_A(x) = \tau^{n(x)}(x), x \in A.$$

The transformation $\tau_A : A \to A$ is called an ***induced transformation*** *or the first return transformation.*

Now we will construct the induced map of $\tau(x)$ on interval $[0, \frac{1}{2}]$, also known as first return map, i.e., $\tau_{\text{induced}}(x) = \tau^{n(x)}(x)$, where $n(x)$ is smallest integers such that, $\tau^{n(x)}(x) \in [0, \frac{1}{2}]$. We show the induced map in the Figure 4.4.

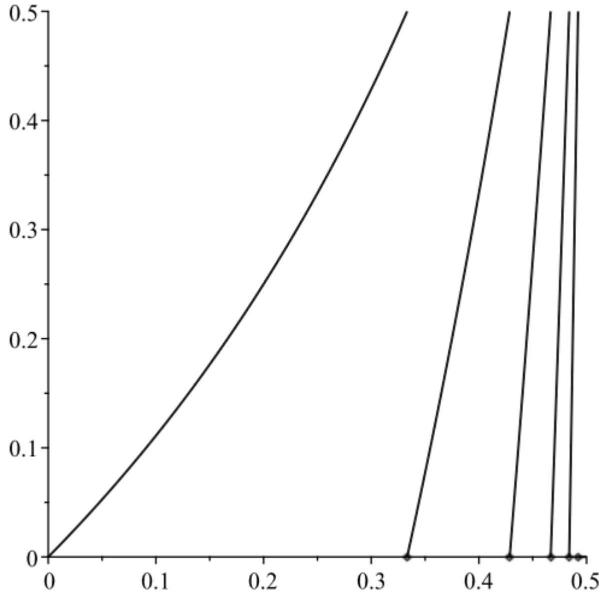

Figure 4.4: Induced Map

We have $\tau_1^{-1}(\frac{1}{2}) = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$. So, on $[0, \frac{1}{3})$, $\tau_{\text{induced}}(x) = \tau_1(x)$, while $[\frac{1}{3}, \frac{1}{2}]$ is mapped to $[\frac{1}{2}, 1]$.

Also, τ_2 maps $[\frac{1}{2}, 1)$ to $[0, 1)$, i.e., $\tau_2^{-1}(\frac{1}{2}) = \frac{\frac{1}{2}+1}{2} = \frac{3}{4}$. So, if $x \in [\frac{1}{3}, \frac{1}{2})$ is so that, $\tau_1(x) \in [\frac{1}{2}, \frac{3}{4})$. Then $n(x) = 2$ and $\tau_{\text{induced}}(x) = \tau_2 \circ \tau_1(x)$.

Now $\tau_1^{-1}(\frac{3}{4}) = \frac{\frac{3}{4}}{1+\frac{3}{4}} = \frac{3}{7}$. Therefore, on $[\frac{1}{3}, \frac{3}{7})$, $\tau_{\text{induced}}(x) = \tau_2 \circ \tau_1(x)$.

Next, $\tau_2^{-1}(\frac{3}{4}) = \frac{\frac{3}{4}+1}{2} = \frac{7}{8}$ and $\tau_1^{-1}(\frac{7}{8}) = \frac{\frac{7}{8}}{1+\frac{7}{8}} = \frac{7}{15}$.

Therefore, on $[\frac{3}{7},\frac{3}{15})$, $\tau_{\text{induced}}(x) = \tau_3 \circ \tau_2 \circ \tau_1(x)$. We see that the points generated $\frac{1}{2},\frac{3}{4},\frac{7}{8},\ldots$ are of the form $(1-\frac{1}{2^n})$ and $\tau_1^{-1}(1-\frac{1}{2^n}) = \frac{1-\frac{1}{2^n}}{1+1-\frac{1}{2^n}}$. Hence, on interval of the form $[\frac{2^n-1}{2.2^n-1},\frac{2^{n+1}-1}{2.2^{n+1}-1})$, $\tau_{\text{induced}}(x) = \tau_2^n \circ \tau_1(x)$.

Since,

$$\begin{aligned}
\tau_2 &\text{ is linear,} \\
\tau_2^{'} > 0, \tau_2^{''} &= 0, \\
\tau_1^{'} &> 0, \\
\tau_1^{''} &> 0.
\end{aligned}$$

We see $\frac{d^2}{dx^2}\tau_{\text{induced}}(x) > 0$. Hence, $\tau_{\text{induced}}(x)$ is piecewise convex.

Also, using Proposition 3.6.1 from [2] we conclude that measure with density $\frac{1}{x}$ is τ_{induced}-invariant on $[0,\frac{1}{2}]$.

Chapter 5

Examples without ACIM

In this section, we have shown some examples where omission of some conditions in Theorem 2.0.2 results in non-existence of absolutely continuous invariant measure conditions fails.

5.1 No ACIM examples with finite number of branches

Example 1.

In this example Condition-3 fails: (3) If $i > 1$: $\tau(a_{i-1}) = 0$. This allows the right-most branch to be a uniform contraction with an attracting fixed point at 1 and there is no ACIM.

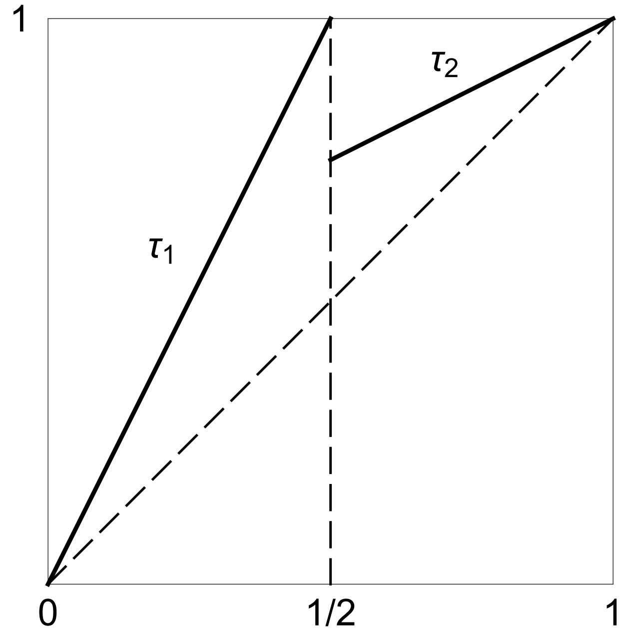

Figure 5.1: Example 1

Let's define $\tau : [0,1] \to [0,1]$ as follows:

$$\tau_1(x) = 2x, \quad 0 \le x < \frac{1}{2},$$
$$\tau_2(x) = \frac{1}{2}(x-1) + 1, \quad \frac{1}{2} \le x \le 1.$$

For any point $x \in [\frac{1}{2}, 1]$, $\tau^n(x) \to 1$, since $x = 1$ is the attracting fixed point in $[\frac{1}{2}, 1]$.

$$\tau_2(1) = 1, \tau_2'(1) = \frac{1}{2}\cdot$$

So $[\frac{1}{2}, 1]$ is a trapping region.
Let $0 < x < \frac{1}{2}\cdot$
Claim: For n large enough

$$\tau^n(x) \in [\frac{1}{2}, 1].$$

Since $0 < x < \frac{1}{2}$, we can find n so that:

$$\frac{1}{2^n} \le x < \frac{1}{2^{n-1}}.$$

Then, $\frac{1}{2} = \tau^{n-1}\left(\frac{1}{2^n}\right) \le \tau^{n-1}(x) \le 1\cdot$ We conclude that $\tau^{n+m}(x) \to 1$ as $m \to \infty$. Hence for all $x > 0$, $\tau^n(x) \to 1$ and as a result there can be no ACIM.

Example 2.

In this example one of the branches fails to be convex which allows the fixed point 1 to be attractive, and there is again no ACIM. Here condition-1 fails: (1) for $i = 1, 2, \ldots$ $\tau_i = \tau_{|[a_{i-1}, a_i)}$ is continuous and convex.

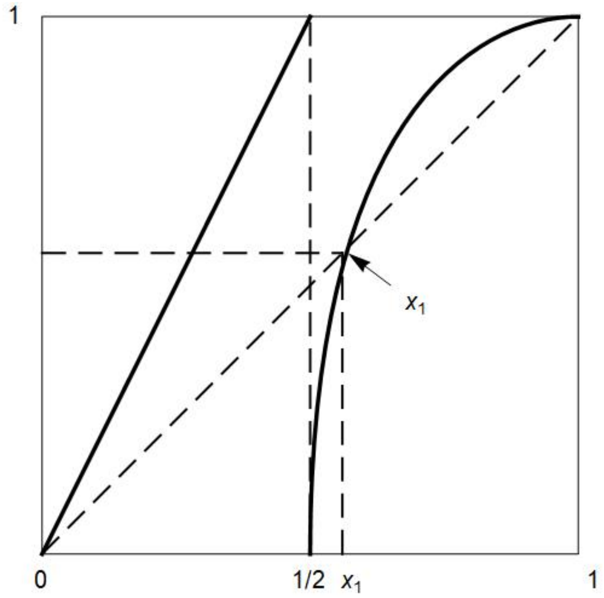

Figure 5.2: Example 2(1)

We define $\tau : [0,1] \to [0,1]$ as follows:

$$\begin{aligned} &\tau_1(x) = 2x, \\ &\tau_2(x) \text{ concave as shown}, \\ &\tau_2'(\frac{1}{2}) > 1. \end{aligned}$$

Then τ has an attracting fixed point at 1, and a repelling fixed point at x_1, see Figure 5.2. For any $x \in (x_1, 1]$, $\tau^n(x) \to 1$.

Now consider τ on the interval $[0, x_1]$.

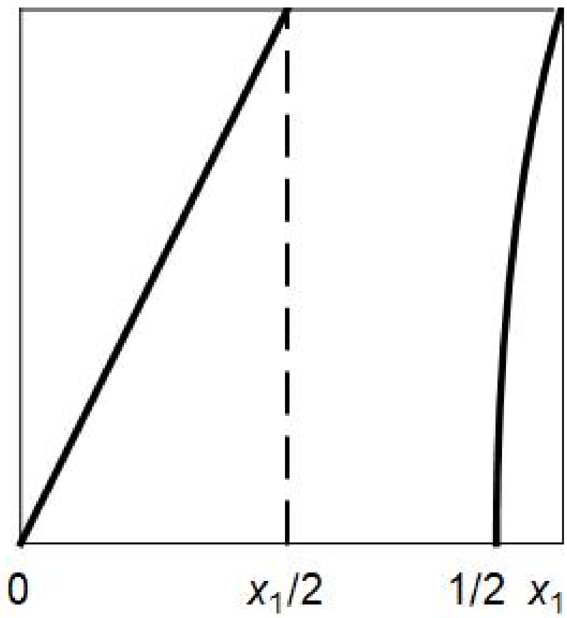

Figure 5.3: Example 2(2)

Any point in $[\frac{x_1}{2}, \frac{1}{2})$ is mapped into $(x_1, 1)$ and so for such x, $\tau^n(x) \to 1$. Now the set of points $x \in [0, x_1]$, which never land in $(\frac{x_1}{2}, \frac{1}{2})$ is a cantor set of measure 0, because the set is non-empty, compact and consist of 2^n intervals.

Therefore, for almost all $x \in [0, x_1)$, $\tau^n(x) \in (\frac{x_1}{2}, \frac{1}{2})$ for some n. So, $\tau^{n+m}(x) \to 1$ as $m \to \infty$ for almost all x. Hence, the non-wandering set has measure 0 and there is then no ACIM.

Example 3.
In this example we have a failure of condition-4: (4) $\sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})} < \infty$. This allows almost every point to tend to the fixed point at 0, and there is again no ACIM.

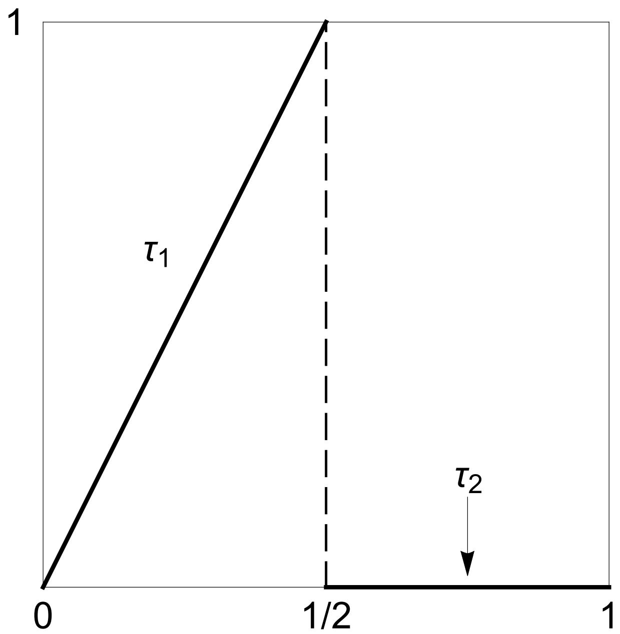

Figure 5.4: Example 3

We obtain $\tau : [0,1] \to [0,1]$ as follows:

$$\tau_1(x) = 2x, \quad 0 \le x < \frac{1}{2}$$
$$\tau_2(x) = 0, \quad \frac{1}{2} < x \le \frac{1}{2}.$$

For any $x \in [\frac{1}{2}, 1]$, $\tau(x) = 0$. Since 0 is a fixed point of τ, for all $n \ge 1$, $\tau^n(x) = 0$. Now if $0 < x \le \frac{1}{2}$ then there exists n so that

$$\left(\frac{1}{2}\right)^n < x \le \left(\frac{1}{2}\right)^{n-1}.$$

So, $\tau^{n-1}(x) \in [\frac{1}{2}, 1]$. Then, $\tau^n(x) = 0$ for all n large enough. This implies then that there is no ACIM.

5.2 No ACIM example with infinite number of branches

Example: In this example we have a failure of condition-4: (4) $\sum_{i=2}^{\infty} \frac{1}{\tau'(a_{i-1})} < \infty$ with infinite number of branches. This allows almost every point to tend to the fixed point at 0 and there is again no ACIM.

Let,

$$a_n = 1 - \frac{1}{2^n}, \quad a_0 = 0, \quad a_1 = \frac{1}{2}, \quad \tau|_{[a_0,a_1]} = 4x^2,$$

and τ linear on $[a_{n-1}, a_n] \to [0, 1]$.

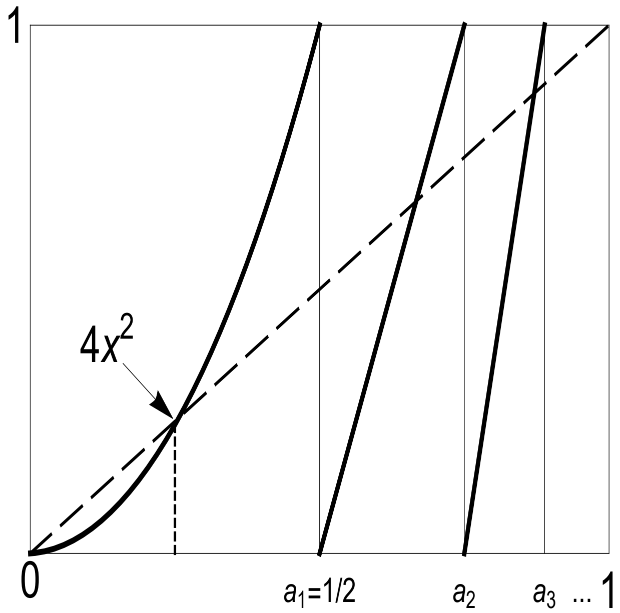

Figure 5.5: Map with infinite number of branches

$$\tau\left(\frac{1}{4}\right) = \frac{1}{4}.$$

Let $J_0 = \left[0, \frac{1}{4}\right)$. If $x \in J_0$: $\tau^n(x) \to 0$.

Now let (as before) $C_n = \{x : x \notin J_0, \tau(x) \notin J_0, \dots, \tau^n(x) \notin J_0\}$. $C_\infty = \bigcap_n C_n$. C_∞ cantor set. If $x \notin C_\infty$ then $\tau^n(x) \to 0$ as $n \to \infty$ and C_∞ is cantor set of measure 0. So τ has no ACIM.

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