On Properties of Ruled Surfaces and Their Asymptotic Curves

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ABSTRACT

On Properties of Ruled Surfaces and Their Asymptotic Curves

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Ruled surfaces are widely used in mechanical industries, robotic designs, and architecture in functional and fascinating constructions. Thus, ruled surfaces have not only drawn interest from mathematicians, but also from many scientists such as mechanical engineers, computer scientists, as well as architects. In this paper, we study ruled surfaces and their properties from the point of view of differential geometry, and we derive specific relations between certain ruled surfaces and particular curves lying on these surfaces. We investigate the main features of differential geometric properties of ruled surfaces such as their metrics, striction curves, Gauss curvature, mean curvature, and lastly geodesics. We then narrow our focus to two special ruled surfaces: the rectifying developable ruled surface and the principal normal ruled surface of a curve. Working on the properties of these two ruled surfaces, we have seen that certain space curves like cylindrical helix and Bertrand curves, as well as Darboux vector fields on these specific ruled surfaces are important elements in certain characterizations of these two ruled surfaces. This latter part of the thesis centers around a paper by Izmuiya and Takeuchi, [4], for which we have considered our own proofs. Along the way, we also touch on the question of uniqueness of striction curves of doubly ruled surfaces.

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First of all, I would like to express my deep and sincere gratitude to my research supervisor, Dr. Alina Stancu, for her expertise assistance, invaluable guidance, motivation, and patience throughout the process of writing this thesis. She has not only guided me academically, but also reached out to me in dealing with my personal problems as well. It was my great privilege and honor to study my most favourite subject under her supervision. Her persistent help has touched me at different levels and I could not have imagined having a better supervisor for my graduate studies at Concordia University.

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Chapter 1

Introduction and Prerequisites

1.1 Introduction

Mathematics and architecture are related, if only, in the fundamental way that architecture has given a wonderful interpretation of the beauty of geometry used to define all forms of construction structures. One of the most popular forms used by architects around the world to enhance the structural elegance of architecture is a ruled surface. Ruled surfaces have been seen widely in architecture for centuries, probably as early as the construction of the first hyperboloid structure, one of Shukhov Water Towers, built by Vladimir Shukhov for the 1896 All-Russian Exhibition in Russia. The advantages of geometric properties of ruled surfaces have facilitated the architects to link ruled surfaces to contemporary free-form architectural forms by breaking down a complex shape into small regions of many single patches or strips of ruled surfaces.

Personally, it was an eye-opening experience for me to do research on ruled surfaces as I am able to relate this topic with visual stimulus of my environment. While doing the research on ruled surface, I re-encountered a very intriguing building structure, Walt Disney Concert Hall, Los Angeles, California, USA. When I lived in Los Angeles in 2007, I happened to pass by this beautiful building everyday without even realizing that it actually has a name as a ruled surface until I have done this research.

Figure 1.1: Walt Disney Concert Hall, Los Angeles, California, USA. Released to CC0 Public Domain by Jean Beaufort.

In this thesis, our goal was to understand geometric properties of ruled surfaces, as well as the characterizations of ruled surface that is rectifying developable of a curve and the principal normal ruled surface of a curve corresponding to properties of curves on them. Knowing that any point p on a ruled surface is a point on a straight line that is completely lying on the ruled surface led us to conclude that the surface does not curve when moving from one point to another in the direction of these straight lines (rulings). As expected, we have identified simple geometric forms that we have seen in almost all of our classical mathematics textbooks such as cylinders, cones, hyperbolic paraboloids, hyperboloids of one sheet, as well as more complex forms as helicoids, Möbius strips and the Plücker's conoid as the simplest examples of ruled surfaces. Along the way, we analysed the intrinsic properties of ruled surfaces by computing their Gauss and mean curvature, and investigating their geodesics as well. We have shown that the Gauss curvature of ruled surfaces is non-positive, which has linked ruled surfaces to developable surfaces when the Gauss curvature is zero, and to their asymptotic directions at the point p when the Gauss curvature is negative (where we have locally the saddle shape). This asymptotic direction suggests that we have another tangential direction of zero normal curvature besides the ruling direction at point p of ruled surfaces. We also brought into focus that the Gauss curvature of ruled surface that is rectifying developable of a curve is associated to the cylindrical helix curve on it, the mean curvature of the principal normal ruled surface corresponds to the Bertrand curve on it, and that its geodesics can become its striction curves under certain conditions. Next, we have studied and have drawn the conclusion that a ruled surface is rectifying developable of a curve α if and only if α is a geodesic of the ruled surface which is traversal to rulings and Gauss curvature vanishes along α. We also show that a regular ruled surface is a developable surface provided the existence of its cylindrical helix geodesic curve with non-zero curvature. We reproved that a ruled surface is a principal normal surface of a space curve α if and only if α is an asymptotic curve of the ruled surface which is transversal to rulings and mean curvature vanishes along α. We finally provided a characterization of Bertrand curve as curves on ruled surfaces by the following concept. If there exists two curves on a regular ruled surface such that they are disjoint asymptotic curves on this ruled surface, both are transversal to rulings and mean curvature of the ruled surface vanishes along these curves, then these two curves are Bertrand mate of each other. The latter part of the thesis centers around a paper by Izmuiya and Takeuchi, [4], for which we have considered however our own proofs. Finally in the Appendix, we have directed our attention to the uniqueness of striction curves of the two specific doubly ruled surfaces: the hyperbolic paraboloid and hyperboloid of one sheet. Lastly, we could show that the striction curve of non-cylindrical ruled surface is unique to each parametrization for a doubly ruled surface, but as a ruled surface can have two distinct ruled parametrizations (which are not reparametrizations of each other), we show the example of a doubly ruled surface with two distinct striction curves.

The thesis is structured as follows. We continue this chapter by introducing the definition, some examples, and properties of ruled surfaces in general. Next, we proceed to define the striction curves for non-cylindrical ruled surfaces along with their properties. Lastly, in the same first chapter, we define doubly ruled surfaces and study some examples. In Chapter 2, we examine certain curves on two specific ruled surfaces, the ruled surface that is rectifying developable of a curve and a principal normal surface of a curve, from the view point of geometry of curves on ruled surfaces. We also derive the characterizations of these ruled surfaces via the properties of the curves on them.

In this chapter, we have used several sources such as [1], Chapter 14, and [2], [5], but whenever possible we have presented our own proofs.

1.2 Definition and Examples of Ruled Surfaces

Definition 1.2.1. *[2], [5] A surface $\mathcal{M}$ in $\mathbb{R}^3$ is called a ruled surface if it admits a parametrization $X : I \times J \to \mathcal{M}$ which consists of a collection of a one-parameter family of straight lines indexed by v, of the form of $X(u, v) = \alpha(u) + v\beta(u)$, where $u \in I$, an open interval in $\mathbb{R}$ and $v \in J$, a possibly different open interval in $\mathbb{R}$. The curve $I \mapsto \alpha(u) \in \mathbb{R}^3$ is called the directrix or the base curve, and $I \mapsto \beta(u) \in \mathbb{R}^3$ is called the director curve. The straight lines are called the rulings or the generators of the ruled surface, while X is called a ruled patch.*

In our work, we assume that α and β are regular, smooth curves.

Remark 1.2.1. *In general, we distinguish two types of ruled surfaces: cylindrical or non-cylindrical. We say that a ruled surface is cylindrical if the rulings are parallel to each other. In other words, $\beta(u) \times \beta'(u)$ is identically the zero vector. Similarly, a non-cylindrical ruled surface is any ruled surface for which the rulings are always* *changing direction. Equivalently, a non-cylindrical ruled surface is a ruled surface for which the vector $\beta(u) \times \beta'(u)$ is never the zero vector.*

We will discuss these concepts in detail in a latter section.

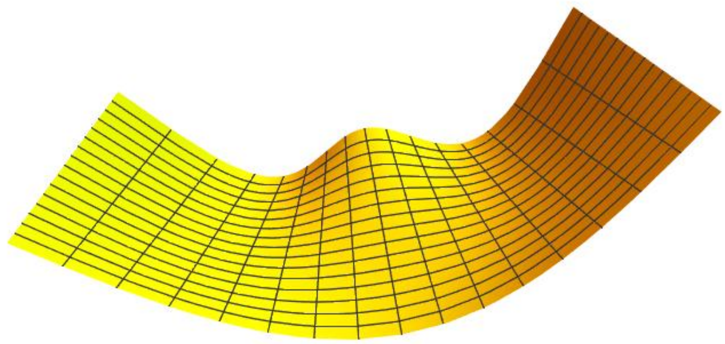

Figure 1.2: Example of Ruled Surface, where $\alpha(u) = (u \cos u, \sin u, 0)$, $\beta(u) = (\cos u/2, (\cos 2u \sin u)/2, 1)$, $u \in (-2\pi, 2\pi)$, and $v \in (0, 1)$.

Examples: Planes, cylinders, cones, helicoids, hyperbolic paraboloids, hyperboloids of one sheet, Möbius strips, as well as Plücker's conoids are common examples of ruled surfaces.

We choose to start by presenting the Möbius strips and the Plücker's conoid as examples of ruled surfaces.

The parametric equation of a **Möbius strip** with radius r and height h is

$$X(u,v) = \left(r \cos u + v \cos \frac{u}{2} \cos u,\ r \sin u + v \cos \frac{u}{2} \sin u,\ v \sin \frac{u}{2}\right), \tag{1.1}$$

where $u \in (-2\pi, 2\pi)$ and $v \in (-h, h)$. We have

$$\begin{aligned} X(u,v) &= \left(r \cos u + v \cos \frac{u}{2} \cos u,\ r \sin u + v \cos \frac{u}{2} \sin u,\ v \sin \frac{u}{2}\right) \\ &= \left(r \cos u,\ r \sin u, 0\right) + v\ (\cos \frac{u}{2} \cos u,\ \cos \frac{u}{2} \sin u,\ \sin \frac{u}{2}\Big). \end{aligned} \tag{1.2}$$

Therefore, a Möbius strip can be written as a ruled surface via $X(u,v) = \alpha(u) + v\beta(u)$, where the circle with radius r, $\alpha(u) = (r \cos u,\ r \sin u,\ 0)$, is the directrix curve and the generators of the Möbius strip are the straight segments of length $2h$ that move along the director curve $\beta(u) = (\cos\frac{u}{2}\cos u,\ \cos\frac{u}{2}\sin u,\ \sin\frac{u}{2})$.

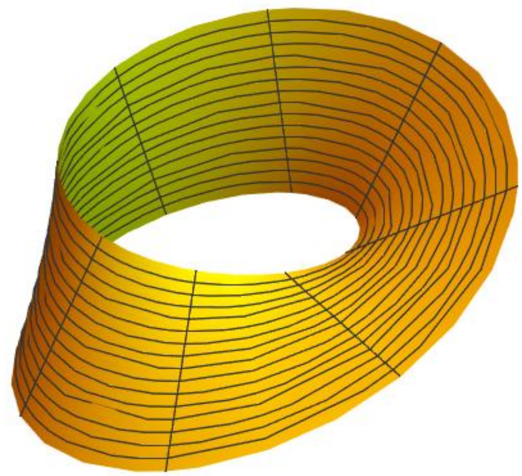

Figure 1.3: Möbius strip with $u \in (-2\pi, 2\pi)$ and $v \in (-1, 1)$.

Next, we study **Plücker's conoid**. In the same manner, where x, y are in the xy-plane without the origin, Plücker's conoid can be defined parametrically in the Euclidean space $\mathbb{R}^3$ by

$$X(x, y) = \left(x,\ y,\ \frac{2xy}{x^2 + y^2}\right). \tag{1.3}$$

It is not straightforward to see that Plücker's conoid is a ruled surface unless we convert this parametrization into a polar parametrization with $x = r\cos\theta$ and $y = r\sin\theta$, $r \in (0, +\infty)$ and $\theta \in (-2\pi,\ 2\pi)$. Therefore, in its new form, the Plücker's conoid is

$$\begin{aligned} X(r\cos\theta,\ r\sin\theta) &= (r\cos\theta,\ r\sin\theta,\ 2\cos\theta\sin\theta) \\ &= (0,\ 0,\ \sin 2\theta) + r\ (\cos\theta,\ \sin\theta,\ 0), \end{aligned}$$

and we recognize that Plücker's conoid is a ruled surface, $\tilde{X}(\theta, r) = \alpha(\theta) + r\beta(\theta)$, whose base curve $\alpha(\theta) = (0,\ 0,\ \sin 2\theta)$ is in the direction of z-axis, and director curve $\beta(\theta) = (\cos\theta,\ \sin\theta,\ 0)$ is a circle in the xy-plane. The one-parameter family of straight lines for this ruled surface is indexed by r.

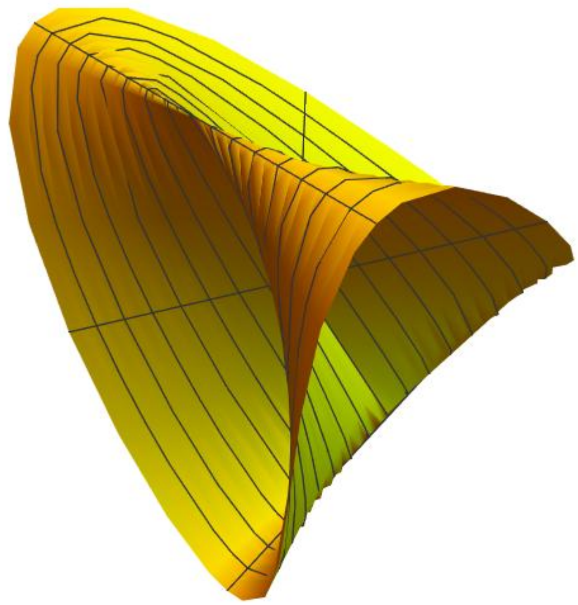

Figure 1.4: Plücker's conoid with $\theta \in (-2\pi, 2\pi)$ and $r \in (0, 1)$.

$\square$

For the rest of this section, we will discuss in detail the properties of three simplest ruled surfaces: the generalized cylinder, the generalized cone and, respectively, the ruled surface that is tangent developable of a curve. Lastly, we will explain how the hyperbolic paraboloid and the hyperboloid of one sheet are examples of special ruled surfaces which are called doubly ruled surfaces.

1.2.1 The Generalized Cylinder

Definition 1.2.2. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a ruled surface. We say that $\mathcal{M}$ is a generalized cylinder over a curve α if and only if $\mathcal{M}$ can be parametrized as*

$$X : I \times J \to \mathbb{R}^3,\ X(u, v) = \alpha(u) + v\boldsymbol{a}, \tag{1.4}$$

where, as before, the directrix curve $\alpha(u)$ is any curve in $\mathbb{R}^3$ and the director curve $\boldsymbol{a}$ is a constant curve whose image is a fixed vector in $\mathbb{R}^3$, $\beta(u) = \boldsymbol{a}$ for all $u \in I$.

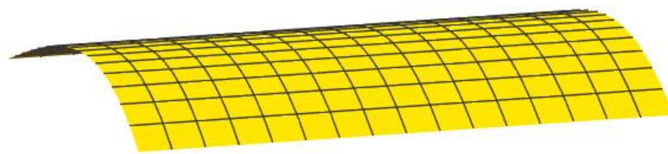

Figure 1.5: Generalized Cylinder where $\alpha(u) = (2^u, \sin u \cos u, u^4 + 1)$, $\mathbf{a}$ = (2,1,0), $u \in (0, 1)$ and $v \in (-1, 1)$.

Remark 1.2.2. *Referring to the definition of the generalized cylinder, we can also assert that the generalized cylinder is a cylindrical ruled surface as in Remark 1.2.1. As seen above, the director vector is a constant vector, $\boldsymbol{a}$. Therefore, $\boldsymbol{a} \times \boldsymbol{a}' = \vec{0}$ for all $u \in I$.*

The next proposition provides a relationship between the director, respectively the directrix, curve of a generalized cylinder.

Proposition 1.2.1. *For any generalized cylinder $\mathcal{M}$ with parametrization $X(u, v) = \alpha(u) + v\boldsymbol{a}$ as in equation (1.4), where $\boldsymbol{a}$ is taken to be a unit vector parallel to the rulings, we may assume that the curve $\alpha(u)$ lies in $\mathcal{P}$, where $\mathcal{P}$ is a plane orthogonal to $\boldsymbol{a}$.*

Proof. Let $\tilde{\alpha}(u) = \alpha(u) - (\alpha(u) \cdot \mathbf{a})\mathbf{a}$ be the perpendicular projection of $\alpha(u)$ in a plane $\mathcal{P}$ orthogonal to $\mathbf{a}$, which we may choose to pass through a fixed point of the surface. We will now show that $X(u, v) = \alpha(u) + v\mathbf{a}$ can be reparametrized by $\tilde{X}(u, \tilde{v}) = \tilde{\alpha}(u) + \tilde{v}\mathbf{a}$, where $\tilde{v} = v + \alpha(u) \cdot \mathbf{a}$ and that $X(u, v) = X(u, \tilde{v})$. Therefore, we can conclude that we may assume that the directrix curve $\alpha(u)$ of a generalized cylinder is in the plane $\mathcal{P}$ as well.

First, we will show that $\tilde{X}(u,\tilde{v}) = \tilde{\alpha}(u) + \tilde{v}\,\mathbf{a}$ where $\tilde{v} = v + \alpha(u)\cdot\mathbf{a}$ is a reparametrization of $X(u,v) = \alpha(u) + v\,\mathbf{a}$ by showing that the map $\Phi : (u,\tilde{v}) \to (u,v)$ is a smooth homeomorphism and that the Jacobian matrix of this map is invertible.

To prove that the reparametrization map $\Phi : (u,\tilde{v}) \to (u,v)$ is a smooth homeomorphism, we have that $\tilde{v} = v + \alpha(u)\cdot\mathbf{a}$ is a smooth function of $(u,\tilde{v})$ since $\tilde{v}_u(u,v) = \alpha'(u)\cdot\mathbf{a}$ and $\tilde{v}_v(u,v) = 1$. Therefore, $\tilde{v}$ has continuous partial derivatives of all orders with respect to u and v provided that $\alpha(u)$ is a smooth curve. Clearly when u is fixed, $\Phi(u,\tilde{v})$ is a bijective map since it is a linear function in v.

Next, we check $J(\Phi)$, the Jacobian of Φ, is an invertible matrix. We have:

$$J(\Phi)(u,v) = \begin{bmatrix} 1 & 0 \\ \alpha' \times \mathbf{a} & 1 \end{bmatrix}.$$

We conclude that $J(\Phi)$ is an invertible matrix since the determinant of $J(\Phi) = 1 \neq 0$. Thus, Φ^{-1} exists and, similarly, is smooth as well. As the result, Φ is a smooth homeomorphism.

Finally, we need to show that $X(u,v) = \tilde{X}(u,\tilde{v}) \circ \Phi^{-1}$.

Consider

$$\begin{aligned}
\tilde{X}(u,\tilde{v}) \circ \Phi^{-1} &= \tilde{\alpha}(u) + \tilde{v}\,\mathbf{a} \\
&= \alpha(u) - (\alpha(u)\cdot\mathbf{a})\,\mathbf{a} + (v + \alpha(u)\cdot\mathbf{a})\,\mathbf{a} \\
&= \alpha(u) - (\alpha(u)\cdot\mathbf{a})\,\mathbf{a} + v\,\mathbf{a} + (\alpha(u)\cdot\mathbf{a})\,\mathbf{a} \\
&= \alpha(u) + v\,\mathbf{a} \\
&= X(u,v).
\end{aligned}$$

Thus, we have proved that $\tilde{X}(u,\tilde{v})$ is a reparametrization of $X(u,v)$.

Finally, for completeness, we want to show that $\tilde{\alpha}(u) = \alpha(u) - (\alpha(u)\cdot\mathbf{a})\mathbf{a}$ is in the plane $\mathcal{P}$ perpendicular to $\mathbf{a}$, thus we must show that $< \tilde{\alpha}(u), \mathbf{a} > = 0$.

Consider

$$\begin{aligned} < \tilde{\alpha}(u), \mathbf{a} > &= < (\alpha(u) - (\alpha(u) \cdot \mathbf{a})\mathbf{a}),\ \mathbf{a} > \\ &= < \alpha(u),\ \mathbf{a} > - < (\alpha(u) \cdot \mathbf{a})\mathbf{a}, \mathbf{a} > \\ &= < \alpha(u),\ \mathbf{a} > - < \alpha(u),\ \mathbf{a} >< \mathbf{a}, \mathbf{a} > \\ &= < \alpha(u),\ \mathbf{a} > - < \alpha(u),\ \mathbf{a} > \|\mathbf{a}\|^2. \end{aligned}$$

We have that $\mathbf{a}$ is a fixed vector and, by hypothesis, $\|\mathbf{a}\|^2 = 1$. We now can conclude that

$$< \tilde{\alpha}(u), \mathbf{a} > = 0.$$

Thus, $\tilde{\alpha}(u) = \alpha(u) - (\alpha(u) \cdot \mathbf{a})\,\mathbf{a}$ is in the plane $\mathcal{P}$ perpendicular to $\mathbf{a}$. This completes the proof of the proposition. $\square$

Note that if we start with a base curve $\alpha(u) = (r\cos u,\ r\sin u,\ 0)$, a circle in the xy-plane with $-2\pi < u < 2\pi$, and take the director curve to be the fixed unit vector $\mathbf{a} = (0,\ 0,\ 1)$ parallel to the z-axis, and $v \in \mathbb{R}$, we obtain an infinitely long circular cylinder with radius r in our standard Euclidean space $\mathbb{R}^3$. Moreover, the directrix curve is in the xy-plane which is perpendicular the director curve, the line parallel to the z-axis.

The regularity condition for the generalized cylinder will be discussed in the following lemma.

Lemma 1.2.1. *A generalized cylinder $\mathcal{M}$ of parametrization $X(u,v) = \alpha(u) + v\boldsymbol{a}$ as in the equation* (1.4) *is regular if and only if, for any $u \in I$, we have that $\alpha'(u) \times \boldsymbol{a} \neq \vec{0}$.*

Proof. Note that a surface $\mathcal{M}$ is regular, if and only if the vector $X_u(u,v) \times X_v(u,v) \neq \vec{0}$, where $X_u(u,v)$ is the partial derivative of $X(u,v)$ with respect to u and $X_v(u,v)$ is the partial derivative of $X(u,v)$ with respect to v. These two derivatives, form a basis of the tangent plane at each point of the surface. We thus consider $X_u(u,v) \times X_v(u,v)$

for the surface patch of $\mathcal{M}$, $X(u,v) = \alpha(u) + v\mathbf{a}$. Then $X_u(u,v) = \alpha'(u)$ and $X_v(u,v) = \mathbf{a}$ and, therefore, $X_u(u,v) \times X_v(u,v) = \alpha'(u) \times \mathbf{a}$. We conclude that having a regular generalized cylinder is equivalent to having $\alpha'(u) \times \mathbf{a} \neq \vec{0}$, $\forall u \in I$. $\square$

1.2.2 The Generalized Cone

Definition 1.2.3. *We say that a ruled surface $\mathcal{M} \subset \mathbb{R}^3$ is a generalized cone if and only if $\mathcal{M}$ can be parametrized as*

$$X : I \times J \to \mathbb{R}^3,\ X(u,v) = p + v\beta(u), \tag{1.5}$$

where, as before, $u \mapsto \beta(u) \in \mathbb{R}^3$ is a director curve, and p is a fixed point in $\mathbb{R}^3$ called the vertex of the generalized cone.

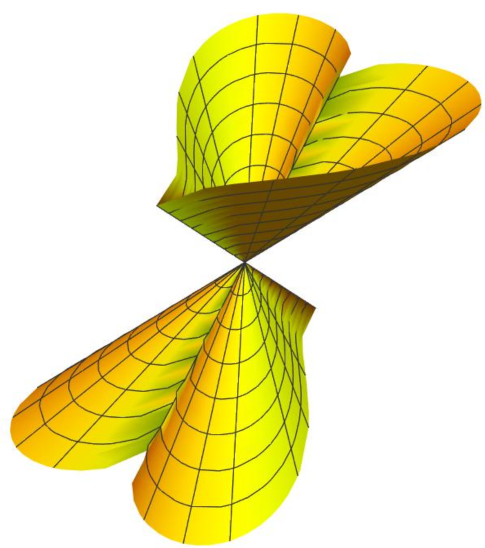

Figure 1.6: Generalized Cone, where $\beta(u) = (\cos u, \sin u + \sqrt{|\cos u|}, 1)$, $p = (1,1,1)$, $u \in (-2\pi, 2\pi)$ and $v \in (-1,1)$.

Note that if we let the director curve $\beta(u) = (r\cos u,\ r\sin u,\ 0)$ be a circle in the xy-plane with radius $r \in (0,\infty)$ and $u \in (-2\pi,\ 2\pi)$, the fixed point $p = (0,\ 0,\ h)$

be the vertex of this cone and, respectively, $v \in J$ is the parameter along the z-axis. We can see that this ruled surface is the lateral part (without its base) of a right cone of height h and radius r in our standard Euclidean space $\mathbb{R}^3$. Under the same assumptions, if we now let the director curve $\beta(u) = (a \cos u, b \sin u, 0)$ be an ellipse, where $a, b \in (0, \infty)$, the resulting ruled surface is called an elliptic cone.

In the following lemma, we will remark on the regularity of generalized cones.

Lemma 1.2.2. *For any generalized cone $\mathcal{M}$ with parametrization $X(u,v) : I \times J \to \mathbb{R}^3$, $X(u,v) = p + v\beta(u)$ as in the equation* (1.5)*, we have that $\mathcal{M}$ is regular at all points where $v\beta(u) \times \beta'(u) \neq \vec{0}$, thus $\mathcal{M}$ is never regular at its vertex p (if it belongs to the surface).*

Proof. As mentioned earlier, we only need to check whether the vector $X_u(u,v) \times X_v(u,v) \neq \vec{0}$ to infer on the regularity of $\mathcal{M}$. Thus, we will compute $X_u(u,v) \times X_v(u,v)$ for this ruled surface. Since $X(u,v) = p + v\beta(u)$, we have that $X_u(u,v) = v\beta'(u)$, and $X_v(u,v) = \beta(u)$, thus $X_u(u,v) \times X_v(u,v) = v\beta'(u) \times \beta(u)$. Therefore, $\mathcal{M}$ is regular if $X_u(u,v) \times X_v(u,v) = v\beta'(u) \times \beta(u) \neq \vec{0}$ whenever $v \neq 0$. If $v = 0$, then $X_u(u,v) \times X_v(u,v) = \vec{0}$, so $\mathcal{M}$ is not regular at this point. We can then conclude that the generalized cone is regular whenever $v\beta'(u) \times \beta(u) \neq \vec{0}$ and it is not regular at its vertex if $0 \in J$. $\square$

1.2.3 Ruled Surfaces that are Tangent Developable of a Curve

Definition 1.2.4. *A ruled surface $\mathcal{M} \subset \mathbb{R}^3$ is said to be tangent developable of a curve $u \mapsto \alpha(u) \in \mathbb{R}^3$ if and only if $\mathcal{M}$ can be parametrized by*

$$X : I \times J \to \mathbb{R}^3, \ X(u,v) = \alpha(u) + v\alpha'(u), \tag{1.6}$$

where $\alpha'(u)$ is the tangent vector to the curve α at point $\alpha(u)$.

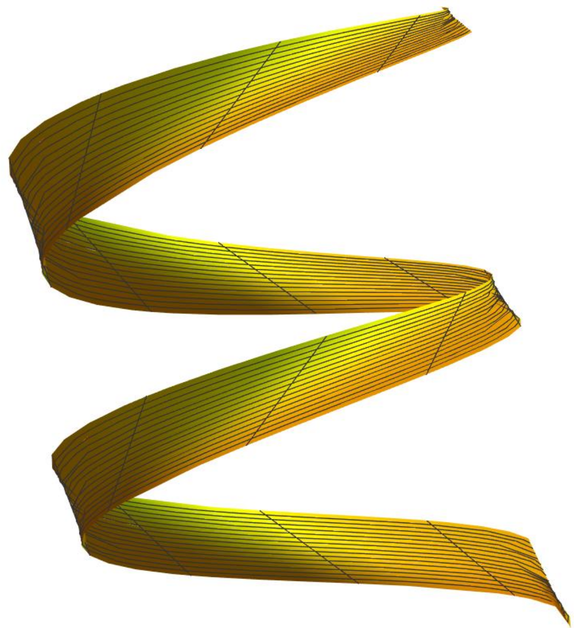

Figure 1.7: Surface developable to α, where $\alpha(u) = (5\cos u, 3\sin u, u)$, $u \in (-2\pi, 2\pi)$, $\alpha'(u) = (-5\sin u, 3\cos u, 1)$, and $v \in (-1, 1)$.

The following lemma gives a criterion for the regularity of surfaces that are tangent developable to a curve.

Lemma 1.2.3. *Given any ruled surface $\mathcal{M}$ that is tangent developable to a curve $\alpha(u)$, with parametrization $X(u,v) = \alpha(u) + v\alpha'(u)$ as in the equation* (1.6)*, we have that $\mathcal{M}$ is regular everywhere except along the curve α provided that the curvature of $\alpha(u)$ never vanishes, i.e. $k(u) \neq 0$, for all $u \in I$.*

Proof. As in previous proofs, we need to check if the vector field $X_u \times X_v$ vanishes at any points $X(u,v)$. We have $X(u,v) = \alpha(u) + v\alpha'(u)$, thus $X_u(u,v) = \alpha'(u) + v\alpha''(u)$ and $X_v(u,v) = \alpha'(u)$. Therefore,

$$X_u \times X_v = (\alpha'(u) + v\alpha''(u)) \times \alpha'(u) = (\alpha'(u) \times \alpha'(u)) + v(\alpha''(u) \times \alpha'(u)).$$

Since $\alpha'(u) \times \alpha'(u) = \vec{0}$, $\forall \alpha'(u) \Rightarrow X_u(u,v) \times X_v(u,v) = v(\alpha''(u) \times \alpha'(u))$.

Furthermore, we know from the formula of the curvature of a curve that

$$k(\alpha(u)) = \frac{\|\alpha''(u) \times \alpha'(u)\|}{\|\alpha'(u)\|^3} \neq \vec{0} \qquad \Leftrightarrow \qquad \|\alpha''(u) \times \alpha'(u)\| \neq \vec{0}. \tag{1.7}$$

If $v \neq 0$, we have $X_u(u,v) \times X_v(u,v) = v(\alpha''(u) \times \alpha'(u)) \neq \vec{0}$, which is equivalent to having that the surface $\mathcal{M}$ is regular.
If $v = 0$, then $X(u,v) = \alpha(u)$ and $X_u(u,v) \times X_v(u,v) = v(\alpha''(u) \times \alpha'(u)) = \vec{0}$, which implies that $\mathcal{M}$ is not regular along $\alpha(u)$. Therefore, the ruled surface that is tangent developable to a curve $\alpha(u)$ of non-zero curvature is regular everywhere except along its base curve $\alpha(u)$. □

In the next proposition, we will calculate the first fundamental form of the surface that is tangent developable to a curve. For simplicity of calculations, we may consider that the surface's base curve is parametrized by arc-length. Recall that the first fundamental form of a surface provides a distance on the surface and this distance, also called metric, leads to an area form which allows us to calculate areas of domains on the surface.

Proposition 1.2.2. *Suppose that $\alpha : (a,b) \to \mathbb{R}^3$ is a unit speed curve. Then the first fundamental form of a regular ruled surface that is tangent developable to α, of parametrization $X(u,v) = \alpha(u) + v\alpha'(u)$ as in (1.6), is*

$$E\,du^2 + 2F\,du\,dv + G\,dv^2, \tag{1.8}$$

with $E(u,v) = 1 + v^2k^2(u)$, $F(u,v) = 1$, $G(u,v) = 1$.

Proof. Consider $X_u(u,v) = \alpha'(u) + v\alpha''(v)$ and $X_v(u,v) = \alpha'(u)$. Since $\alpha(u)$ is a unit-speed curve, we have $\|\alpha'(u)\| = 1$, $< \alpha'(u), \alpha''(u) > = 0$, and $\|\alpha''(u)\| = k(u)$, where $k(u)$ is the curvature of the curve at the point $\alpha(u)$. We then have

$$\begin{aligned}
E &= \|X_u(u,v)\|^2 \\
&= \|\alpha'(u) + v\alpha''(u)\|^2 \\
&= < \alpha'(u) + v\alpha''(u), \alpha'(u) + v\alpha''(u) > \\
&= \|\alpha'(u)\|^2 + \|v\alpha''(u)\|^2 + 2v < \alpha'(u), \alpha''(u) > \\
&= 1 + v^2k^2(u) + 0.
\end{aligned}$$

Therefore,

$$E = 1 + v^2k^2(u), \tag{1.9}$$

and, similarly,

$$G = \|X_v(u,v)\|^2 = \|\alpha'(u)\|^2 = 1. \tag{1.10}$$

Furthermore,

$$\begin{aligned}
F &= < X_v(u,v), X_u(u,v) > \\
&= < \alpha'(u), (\alpha'(u) + v\alpha''(u)) > \\
&= < \alpha'(u), \alpha'(u) > \ + v \ < \alpha'(u), \alpha''(u) > \\
&= \|\alpha'(u)\|^2 + 0 = 1,
\end{aligned}$$

since $< \alpha'(u), \alpha''(u) > = 0$ and $\|\alpha'(u)\|^2 = 1$.

Hence

$$F = 1, \tag{1.11}$$

concluding the form of the metric on $\mathcal{M}$ known as the first fundamental form. $\square$

1.2.4 Hyperbolic Paraboloid and Hyperboloid of One Sheet

Hyperbolic paraboloid and hyperboloid of one sheet are special ruled surfaces because they are doubly ruled.

Definition 1.2.5. *A surface $\mathcal{M}$ is called doubly ruled if and only if $\mathcal{M}$ has two distinct ruled patches that is through every one of its points there are two distinct lines that lie on the surface.*

We now present the hyperbolic paraboloid in Euclidean space $\mathbb{R}^3$ as an example of a doubly ruled surface. The Cartesian equation of hyperbolic paraboloid is

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}, \tag{1.12}$$

where a, b are constants and x, y are in the xy-plane of our standard Euclidean space. For any $u, v \in (-\infty, \infty)$, we can then parametrize the hyperbolic paraboloid with two different surface patches.

The first surface patch is

$$X_1(u, v) = (a(u+v),\ bv,\ u^2 + 2uv) = (au,\ 0,\ u^2) + v\ (a,\ b,\ 2u), \tag{1.13}$$

with its base curve $\alpha(u) = (au,\ 0,\ u^2)$, and director curve $\beta(u) = (a,\ b,\ 2u)$.

Another surface patch is

$$X_2(u, v) = (a(u+v),\ -bv,\ u^2 + 2uv) = (au,\ 0,\ u^2) + v\ (a,\ -b,\ 2u), \tag{1.14}$$

where the base curve is $\alpha(u) = (au,\ 0,\ u^2)$, and the director curve is $\beta(u) = (a,\ -b,\ 2u)$. In both cases, $u, v \in \mathbb{R}$. Therefore, the hyperbolic paraboloid is a doubly ruled surface with two distinct ruled patches $X_1(u, v) = (au,\ 0,\ u^2) + v(a,\ b,\ 2u)$ and $X_2(u, v) = (au,\ 0,\ u^2) + v(a,\ -b,\ 2u)$.

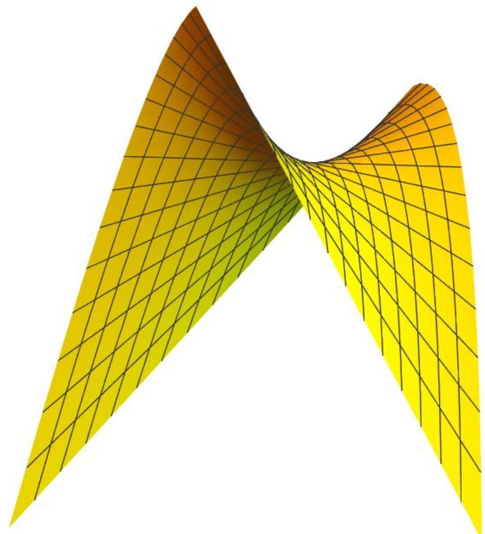

Figure 1.8: Hyperbolic Paraboloid, where $\alpha(u) = (u, 0, u^2)$, $u \in (-1, 1)$, $\beta(u) = (1, -1, 2u)$, and $v \in (-1, 1)$.

Similarly, we can draw the same conclusion that the hyperboloid of one sheet in $\mathbb{R}^3$ is doubly ruled as well. Hyperboloid of one sheet is defined non-parametrically by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{1.15}$$

where constants $a, b, c > 0$ and the standard reparametrizations can be represented by two different surface patches.

For any $u \in (-2\pi, 2\pi)$ and $v \in (-\infty, \infty)$, the first surface patch of hyperboloid of one sheet is $X_1(u, v) = (a(\cos u - v \sin u),\ b(\sin u + v \cos u),\ cv)$. Thus,

$$X_1(u, v) = (a \cos u,\ b \sin u,\ 0) + v\ (-a \sin u,\ b \cos u,\ c), \tag{1.16}$$

where $u \mapsto (a \cos u,\ b \sin u,\ 0)$ is its base curve, and $u \mapsto (-a \sin u,\ b \cos u,\ c)$ is its director curve.

The second ruled patch is $X_2(u,v) = (a(\cos u + v\sin u),\ b(\sin u - v\cos u),\ cv)$. Thus,

$$X_2(u,v) = (a\cos u,\ b\sin u,\ 0) + v\ (a\sin u, -b\cos u,\ c), \tag{1.17}$$

as before, $u \mapsto (a\cos u,\ b\sin u,\ 0)$ is its base curve, and $u \mapsto (a\sin u,\ -b\cos u,\ c)$ is its director curve. Hence, the hyperboloid of one sheet is a doubly ruled surface with two distinct ruled patches $X_1(u,v)$ and $X_2(u,v)$ as in the above equations.

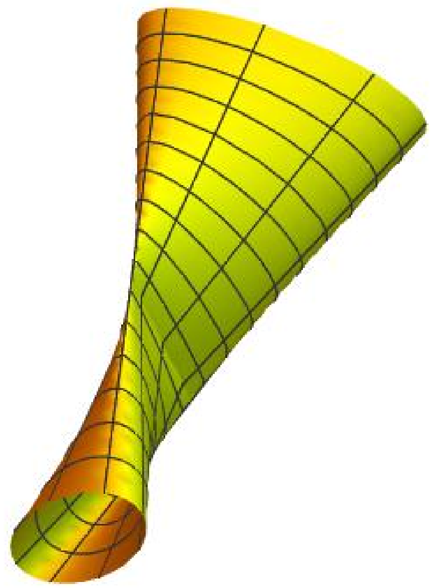

Figure 1.9: Hyperboloid of One Sheet, where $\alpha(u) = (\cos u, 3\sin u, 0)$, $u \in (-2\pi, 2\pi)$, $\beta(u) = (\sin u, -3\cos u, 2)$, and $v \in (-5, 5)$.

1.3 Properties of Ruled Surfaces

In this section, we first look at one of the common properties shared by all ruled surfaces. This pertains to the sign of the Gaussian curvature of ruled surface in general. We will then note the characterization of flat ruled surfaces and point out some special cases. Next, we define the striction curve which plays an important role for non-cylindrical ruled surfaces. Lastly, we proceed to calculate the mean curvature, and as well as the geodesics on non-cylindrical ruled surfaces.

1.3.1 Gaussian Curvature of Ruled Surfaces

The Gaussian curvature of a surface in the Euclidean space $\mathbb{R}^3$ is interesting because it represents an intrinsic geometric property of the surface and it does not change according to the way we immerse the surface in the Euclidean space, thus how the surface bends. Roughly speaking, as long as the surface is not distorted, stretched nor compressed, the Gaussian curvature of the surface remains the same. On the other hand, the curvature of a curve in $\mathbb{R}^3$ is an extrinsic geometric property since its curvature is determined by the way the curve itself bends in the Euclidean space.

There are different approaches to calculate the curvature of a surface. In our work, we consider the Gauss curvature of a surface computed directly from the first and second fundamental forms of a surface patch. We recall a classical result that states that the Gaussian curvature of an oriented surface can be determined by the first and second fundamental form of its surface patch

Suppose that $\mathcal{M}$ is a surface in $\mathbb{R}^3$ and a diffeomorphism map $X(u,v)$ is the surface patch of $\mathcal{M}$. We say that the expression $Edu^2 + 2Fdudv + Gdv^2$ is the first fundamental form of the surface patch $X(u,v)$, where the coefficients E, F, G and the linear maps du, dv depend on the choice of each surface patch of $\mathcal{M}$. Similarly, the expression $Ldu^2 + 2Mdudv + Ndv^2$ is called the second fundamental form of the surface $\mathcal{M}$ where, as before, the coefficients L, M, N and the linear maps du, dv depend on the choice of each surface patch of $\mathcal{M}$ as well, and they are defined as follows. Let $\mathbf{N} = \mathbf{N}(u,v)$ be a unit normal vector at point $X(u,v)$ of the surface $\mathcal{M}$,

$$\mathbf{N}(u,v) = \frac{X_u(u,v) \times X_v(u,v)}{\|X_u(u,v) \times X_v(u,v)\|},$$

then $L = < X_{uu}(u,v),\ \mathbf{N} >$, $N = < X_{vv}(u,v),\ \mathbf{N} >$, $M = < X_{uv}(u,v),\ \mathbf{N} >$.

The Gaussian curvature K of the surface $\mathcal{M}$ at each point $X(u,v) \in \mathcal{M}$ is

$$K(u,v) = \frac{LN - M^2}{EG - F^2},$$

where, recall from earlier computations that $E = \|X_u(u,v)\|^2$, $F = < X_u(u,v), X_v(u,v) >$, $G = \|X_v(u,v)\|^2$. For simplicity, we have dropped the arguments u and v from $\mathbf{N}$, and the coefficients of the first, and second, fundamental form, though these coefficients generally depend on the point as we mentioned earlier.

We are now ready to state the result on the Gaussian curvature of a ruled surface.

Proposition 1.3.1. *The Gaussian curvature of a regular ruled surface is everywhere non-positive.*

Proof. Let $X(u,v) = \alpha(u) + v\beta(u)$ be the surface patch of a ruled surface $\mathcal{M}$. Then $X_u(u,v) = \alpha'(u) + v\beta'(u)$, $X_v(u,v) = \beta(u)$, $X_{vv}(u,v) = \vec{0}$, $X_{uu} = \alpha''(u) + v\beta''(u)$, and $X_{uv} = \beta'(u)$.

Consequently, we have

$$\mathbf{N} = \frac{X_u(u,v) \times X_v(u,v)}{\|X_u(u,v) \times X_v(u,v)\|} = \frac{(\alpha'(u) + v\beta'(u)) \times \beta(u)}{\|(\alpha'(u) + v\beta'(u) \times \beta(u)\|}, \tag{1.18}$$

$$N = < X_{vv}(u,v),\ \mathbf{N} > = < \vec{0},\ \frac{X_u(u,v) \times X_v(u,v)}{\|X_u(u,v) \times X_v(u,v)\|} > = 0, \tag{1.19}$$

and

$$\begin{aligned} M &= < X_{uv}(u,v),\ \mathbf{N} > \\ &= < \beta'(u),\ \frac{X_u(u,v) \times X_v(u,v)}{\|X_u(u,v) \times X_v(u,v)\|} > . \end{aligned}$$

Thus,

$$M = < \beta'(u), \ \frac{(\alpha'(u) + v\beta'(u)) \times \beta(u)}{\|(\alpha'(u) + v\beta'(u) \times \beta(u)\|} > . \tag{1.20}$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 - M^2}{EG - F^2} = -\frac{M^2}{EG - F^2}. \tag{1.21}$$

For all (u, v), we know that $-M^2 \leq 0$ and also that

$$\begin{aligned} EG - F^2 &= < (X_u(u,v), X_u(u,v) > < (X_v(u,v), X_v(u,v) > \ -(< X_u(u,v), X_v(u,v) >)^2 \\ &= < (X_u(u,v) \times X_v(u,v)), (X_u(u,v) \times X_v(u,v)) > . \end{aligned}$$

Therefore,

$$EG - F^2 = \|X_u(u,v) \times X_v(u,v)\|^2. \tag{1.22}$$

As the surface is regular, for all (u, v), we have $EG - F^2 = \|X_u(u,v) \times X_v(u,v)\|^2 > 0$, concluding that the Gaussian curvature of the ruled surface is negative or zero for all values of (u, v). □

We will discusss soon in greater detail the ruled surfaces whose Gauss curvature is zero everywhere. To do so, we need first to introduce the notion of striction curve of ruled surfaces.

1.3.2 Striction Curve of Non-Cylindrical Ruled Surfaces

As mentioned earlier, there are two types of ruled surfaces: cylindrical and non-cylindrical ruled surfaces. In this section, we are interested in studying in detail certain properties of non-cylindrical ruled surfaces because even though the directions of the rulings on a non-cylindrical ruled surface are different at each point of the surface, we can find a useful reference curve called striction curve which is unique to non-cylindrical ruled surfaces. However, the cylindrical ruled surfaces could have different striction curves possibly infinitely many.

Let us start by defining formally a cylindrical, respectively a non-cylindrical, ruled surface, then introducing the notion of striction curve of a non-cylindrical ruled surface.

Definition 1.3.1. *A ruled surface $\mathcal{M} \subset \mathbb{R}^3$ parametrized by $X(u,v) = \alpha(u) + v\beta(u)$ is called a cylindrical ruled surface if and only if the vector $\beta(u) \times \beta'(u)$ vanishes everywhere on the surface.*

Definition 1.3.2. *A ruled surface $\mathcal{M} \subset \mathbb{R}^3$ parametrized by $X(u,v) = \alpha(u) + v\beta(u)$ is called a non-cylindrical ruled surface if and only if the vector $\beta(u) \times \beta'(u)$ never vanishes for all $u \in J$.*

Definition 1.3.3. *Any base curve $\alpha(u)$ of a non-cylindrical ruled surface $\mathcal{M} \subset \mathbb{R}^3$ of the form of $X(u,v) = \alpha(u) + v\beta(u)$ that satisfies the properties that*

$$< \alpha'(u), \beta'(u) > = 0 \quad \textit{and} \quad ||\beta(u)|| = 1 \tag{1.23}$$

is called the striction curve of $X(u,v)$.

Example: We will examine here the striction curve of a circular helicoid. Let $\mathcal{M}$ be a circular helicoid with parametrization $X(u,v) = (v\cos u,\ v\sin u,\ au)$ where $a \in \mathbb{R}$ is a fixed constant, and $v \in \mathbb{R}$, while $-2\pi < u < 2\pi$.

This parametrization can be written as

$$X(u,v) = (0,\ 0,\ au) + v(\cos u,\ \sin u,\ 0).$$

As seen, this is a ruled surface with $\alpha(u) = (0,0,au)$ as its base curve, and $\beta(u) = (\cos u, \sin u, 0)$ as its director curve, parametrized by arc-length. Furthermore, we can easily verify that $||\beta(u)|| = 1$ and that $< \alpha'(u), \beta'(u) >= 0$. By definition, we can now say that $\alpha(u)$ is a striction curve of this circular helicoid.

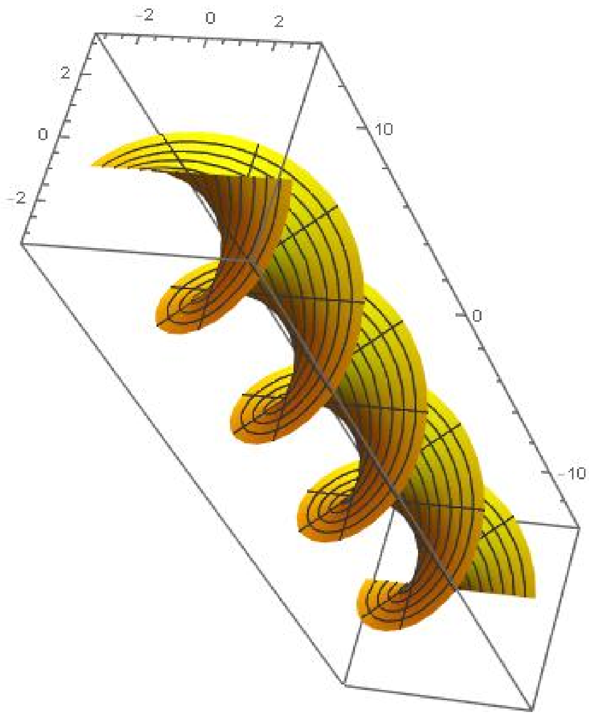

Figure 1.10: Circular Helicoid, where $\alpha(u) = (0, 0, 3u)$, $\beta(u) = (\cos u, \sin u, 0)$, $a = 1$, $u \in (-2\pi, 2\pi)$, and $v \in (-3, 3)$.

This striction curve is a very useful reference curve for non-cylindrical ruled surface since it is uniquely determined by the parametrization. Only if a non-cylindrical ruled surface admits two different parametrizations, such as doubly ruled surfaces, then the uniqueness is, in general, no longer true. We will discuss this situation in the Appendix and see two different possible cases.

Further, we present a lemma which shows that any surface patch of a non-cylindrical ruled surface can be reparametrized by its striction curve.

Lemma 1.3.1. *For any non-cylindrical ruled surface $\mathcal{M}$ with parametrization*

$$X(u, v) = \alpha(u) + v\beta(u),$$

then $\mathcal{M}$ can be reparametrized in the form of

$$\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u), \tag{1.24}$$

where $\sigma(u)$ is the striction curve of $X(u,v)$, and

$$\tilde{v} = v + \frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}. \tag{1.25}$$

Proof. We will prove this lemma in two parts. First, we will find the striction curve $\sigma(u)$ of this patch $X(u,v)$ and, then, we will show that $\tilde{X}(u,\tilde{v})$ is the reparametrization of $X(u,v)$.

Let us start by finding the striction curve of $X(u,v)$. Since we are looking for a curve $\sigma(u)$ on the surface $X(u,v)$, then

$$\sigma(u) = \alpha(u) + (v(u)||\beta(u)||)\,\frac{\beta(u)}{||\beta(u)||} =: \alpha(u) + \bar{v}(u)\gamma(u), \tag{1.26}$$

where $\gamma(u) = \beta(u)/||\beta(u)||$ and $\bar{v}(u)$ will be determined in order for σ to satisfy the first property from the definition of the striction curve. Note that, with the given notations, $\sigma'(u) = \alpha'(u) + \bar{v}'(u)\gamma(u) + \bar{v}(u)\gamma'(u)$, then $< \sigma'(u), \gamma'(u) > = < \alpha'(u), \gamma'(u) > + \bar{v}'(u) < \gamma(u), \gamma'(u) > + \bar{v}(u) < \gamma'(u), \gamma'(u) >$.

Since $\|\gamma(u)\| = 1$, or $< \gamma(u), \gamma(u) > = 1$, we infer that $< \gamma'(u), \gamma(u) >= 0$ and $< \gamma'(u), \gamma'(u) >= \|\gamma'(u)\|^2$.

Therefore,

$$\begin{aligned} < \sigma'(u), \gamma'(u) > &= < \alpha'(u), \gamma'(u) > + 0 + \bar{v}(u)\|\gamma'(u)\|^2 \\ &= < \alpha'(u), \gamma'(u) > + \bar{v}(u)\|\gamma'(u)\|^2, \end{aligned}$$

and, for $\sigma(u)$ to be a striction curve of $X(u,v)$, we must have $< \sigma'(u), \gamma'(u) >= 0$. This implies the choice of $\bar{v}$ as $< \alpha'(u), \gamma'(u) > + \bar{v}(u)\|\gamma'(u)\|^2 = 0$.

$$\Rightarrow \bar{v}(u) = -\frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}.$$

We replace

$$\bar{v}(u) = -\frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}$$

in the equation (1.26) and obtain

$$\sigma(u) = \alpha(u) - \frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}\gamma(u). \tag{1.27}$$

To prove $\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u)$ is the reparametrization of $X(u, v) = \alpha(u) + v\beta(u)$, first we prove that $\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u)$ is the reparametrization of $\tilde{\tilde{X}}(u, v) = \alpha(u) + v\gamma(u)$, and next we show that $\tilde{\tilde{X}}(u, v) = \alpha(u) + v\gamma(u)$ is the reparametrization of $X(u, v) = \alpha(u) + v\beta(u)$.

Now, prove that $\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u)$ is the reparametrization of $\tilde{\tilde{X}}(u, v) = \alpha(u) + v\gamma(u)$. Let us start by first verifying that $\tilde{X}(u, \tilde{v}) = \tilde{\tilde{X}}(u, v)$.
Consider

$$\begin{aligned}
\tilde{X}(u,\ \tilde{v}) &= \sigma(u) + \tilde{v}\gamma(u) \\
&= \alpha(u) - \frac{\alpha'(u).\gamma'(u)}{\|\gamma'(u)\|^2}\gamma(u) + v\gamma(u) + \frac{\alpha'(u).\gamma'(u)}{\|\gamma'(u)\|^2}\gamma(u) \\
&= \alpha(u) + v\gamma(u) = \tilde{\tilde{X}}(u, v).
\end{aligned}$$

Next we will show that $\tilde{v}$ is a smooth function of (u, v). We have as in equation (1.25),

$$\tilde{v}(u, v) = v + \frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}.$$

$$\text{Then, } \tilde{v}_v = 1 \text{ and } \tilde{v}_u = \left[\frac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}\right]'.$$

From these calculations, and the assumption that all curves considered are smooth and regular, we can conclude that $\tilde{v}(u, v)$ has derivatives of all orders with respect to u and v, which also implies that $\tilde{v}(u, v)$ is a smooth function of u and v.

Lastly, $J(\Phi)$ is an invertible matrix with Φ: $(u, v) \mapsto (u, \tilde{v})$. We have that

$$J(\Phi) = \begin{bmatrix} 1 & 0 \\ \tilde{v_u} & 1 \end{bmatrix}$$

is an invertible matrix since its determinant is not zero. Therefore, $\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u)$ is the reparametrization of $\tilde{\tilde{X}}(u, v) = \alpha(u) + v\gamma(u)$.

However, $X(u, v) = \alpha(u) + v\beta(u)$ is also a reparametrization of $\tilde{\tilde{X}}(u, v)$ as

$$\tilde{\tilde{X}}(u, v) = \alpha(u) + v\gamma(u) = \alpha(u) + \frac{v}{\|\beta(u)\|}\beta(u) = X\left(u, \frac{v}{\|\beta(u)\|}\right).$$

We can easily check that $v_1 = \dfrac{v}{\|\beta(u)\|}$ is the smooth map in u and v and the Jacobian of the map $\Phi: \ (u, v) \mapsto (u, v_1)$is an invertible matrix as well since we have

$$J(\Phi) = \begin{bmatrix} u_u & u_v \\ v_{1u} & v_{1v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v_{1u} & \dfrac{1}{\|\beta(u)\|}. \end{bmatrix}$$

Thus, $\tilde{X}(u, \tilde{v}) = \sigma(u) + \tilde{v}\gamma(u)$ is the corresponding parametrization of $X(u, v) = \alpha(u) + v\beta(u)$. $\square$

Remark 1.3.1. *As we have seen in the proof of Lemma 1.3.1, assuming the director curve of the ruled surface satisfies $||\gamma(u)|| = 1, \ \forall u \in I$, we can derive the formula for the striction curve $\sigma(u)$ as in the equation (1.27), which, for convenience, we repeat below*

$$\sigma(u) = \alpha(u) - \frac{< \alpha'(u), \gamma'(u) >}{||\gamma'(u)||^2}\gamma(u).$$

From this result, we infer that the striction curve of a non-cylindrical ruled surface is uniquely determined by its parametrization. However, for the case of a non-cylindrical surface that is doubly ruled, though the striction curve is unique for each parametriza-

tion of its surface patch, it is not necessarily unique as curve on the surface, see the examples of Appendix A1 and A2.

The next lemma demonstrates that the striction curves do not depend on the base curves of the non-cylindrical ruled surfaces.

Lemma 1.3.2. *The striction curve of a non-cylindrical ruled surface $\mathcal{M}$ does not depend on the choice of the base curve.*

Proof. Suppose that $\mathcal{M}$ has two different base curves $\alpha(u)$ and $\tilde{\alpha}(u)$, respectively. Thus, $\mathcal{M}$ can be reparametrized by $\alpha(u) + v\beta(u)$ and $\tilde{\alpha}(u) + \tilde{v}(v)\beta(u)$, where $\tilde{v}(v)$ is some function v. Thus, we have

$$\alpha(u) + v\beta(u) = \tilde{\alpha}(u) + \tilde{v}(v)\beta(u), \tag{1.28}$$

$$\alpha - \tilde{\alpha} = (v - \tilde{v}(v))\beta, \tag{1.29}$$

$$\alpha' - \tilde{\alpha}' = (v - \tilde{v}(v))\beta'. \tag{1.30}$$

Let us denote by $\sigma(u)$ and $\tilde{\sigma}(u)$ the two corresponding striction curves of $\mathcal{M}$. By equation (1.27), we can express $\sigma(u)$ and $\tilde{\sigma}(u)$ as bellow:

$$\sigma(u) = \alpha(u) - \frac{< \alpha'(u), \beta'(u) >}{\|\beta'(u)\|^2}\beta(u), \text{ and } \tilde{\sigma}(u) = \tilde{\alpha}(u) - \frac{< \tilde{\alpha}'(u), \beta'(u) >}{\|\beta'(u)\|^2}\beta(u).$$

Thus, dropping the argument u for simplicity of writing, we note that

$$\sigma - \tilde{\sigma} = (\alpha - \tilde{\alpha}) - \frac{< (\alpha' - \tilde{\alpha}'), \beta' >}{\|\beta'\|^2}\beta.$$

Furthermore, replacing $\alpha - \tilde{\alpha}$ and $\alpha' - \tilde{\alpha}'$ by the calculations above, we have

$$\sigma - \tilde{\sigma} = [(v - \tilde{v}(v))\beta] - \frac{[(v - \tilde{v}(v))\beta'] \cdot \beta'}{\|\beta'\|^2}\beta, \tag{1.31}$$

and, consequently, $$\sigma = \tilde{\sigma} + [(v - \tilde{v}(v))\beta] - \frac{[(v - \tilde{v}(v))\beta'] \cdot \beta'}{\|\beta'\|^2}\beta, \tag{1.32}$$

which only depends on v and β, concluding the proof. $\square$

1.3.3 Flat Ruled Surfaces

Definition 1.3.4. *A flat surface, sometimes also called a developable surface, is a smooth surface whose Gaussian curvature is zero everywhere.*

Roughly speaking, it is a surface that can be flattened into a planar domain without distortion. Planes, and, as we will show next, generalized cylinders, generalized cones, and surfaces that are tangent developable of a curve are some of the classical examples of flat ruled surfaces. We will later show, in a characterization result, that these are precisely the only flat ruled surfaces.

Proposition 1.3.2. *Generalized cylinders, generalized cones, and surfaces that are tangent developable of a curve are flat ruled surfaces.*

Proof. These surfaces' Gaussian curvatures can be computed using the equation (1.21) and equation (1.20):

$$\begin{aligned} K &= -\frac{M^2}{EG-F^2} \\ &= -\frac{1}{EG-F^2} < \beta'(u),\ \frac{(\alpha'(u)+v\beta'(u))\times\beta(u)}{\|(\alpha'(u)+v\beta'(u)\times\beta(u)\|} > \\ &= \frac{<\beta'(u),\ (\alpha'(u)+v\beta'(u))\times\beta(u)>^2}{(EG-F^2)(\|(\alpha'(u)+v\beta'(u)\times\beta(u)\|)^2}. \end{aligned} \tag{1.33}$$

We will show that for each of the surfaces mentioned in the statement of the proposition, the scalar product in the numerator of the formula for K is zero.

For generalized cylinders, $\beta'(u)=\mathbf{a}'=\vec{0}$ implies that $<\beta'(u),\ (\alpha'(u)+v\beta'(u))\times\beta(u)>=0$, and then the curvature of the generalized cylinder is 0.

For generalized cones, $\alpha'(u) = 0$ implies that

$$\begin{aligned} < \beta'(u),\ (\alpha'(u) + v\beta'(u)) \times \beta(u) > &=< \beta'(u),\ v\beta'(u) \times \beta(u) > \\ &=< \beta(u),\ v\beta'(u) \times \beta'(u) > \\ &= 0, \end{aligned}$$

which is the same as saying that the curvature of generalized cones is zero.

For any ruled surface that is tangent developable to a curve, the director curve $\beta(u)$ is $\alpha'(u)$, thus $\beta'(u) = \alpha''(u)$ implies that

$$\begin{aligned} < \beta'(u),\ (\alpha'(u) + v\beta'(u)) \times \beta(u) > &=< \alpha''(u),\ (\alpha'(u) + v\alpha'(u)) \times \alpha'(u) > \\ &=< \alpha''(u),\ \vec{0} > \\ &= 0, \end{aligned}$$

which is equivalent to the fact that the Gauss curvature of any ruled surface that is tangent developable to a curve is zero as well. $\square$

The following theorem allows us to conclude whether a flat ruled surface is a cone, a cylinder, or a surface that is tangent developable of its striction curve based on the characteristics of their directrix and director curves.

Theorem 1.3.1. *Suppose that* $X : I \times \mathbb{R} \to \mathbb{R}^3,\ X(u, v) = \alpha(u) + v\beta(u)$ *with* $\|\beta(u)\| = 1,\ \forall u \in I$, *is the parametrization of a flat ruled surface* $\mathcal{M}$. *Then*

i. *If* $\alpha'(u) \equiv \vec{0}$, *then* $\mathcal{M}$ *is a cone.*

ii. *If* $\beta'(u) \equiv \vec{0}$, *then* $\mathcal{M}$ *is a cylinder.*

iii. *If both* $\alpha'(u)$ *and* $\beta'(u)$ *never vanish, then* $\mathcal{M}$ *is the tangent developable surface of one of its striction curves.*

Proof. The proofs of these statements of Theorem 1.3.1 are as follows:

Proving part *i.* and *ii.* are immediate from the formula (1.33), and from the definition of the cone, respectively cylinder.

Recall that $\mathcal{M}$ is a cone if and only if $\alpha(u)$ is a fixed point in $\mathbb{R}^3$. For Theorem 1.3.1.*i.*, we have $\alpha'(u) \equiv \vec{0}$ which implies that $\alpha(u)$ is a fixed point in $\mathbb{R}^3$. Then $\mathcal{M}$ is a cone by definition.

Similarly for a cylinder, we only need to show that the director curve is a fixed vector in $\mathbb{R}^3$. For Theorem 1.3.1.*ii.*, we have $\beta'(u) \equiv \vec{0}$, then $\beta(u)$ is a constant vector in $\mathbb{R}^3$. By definition, $\mathcal{M}$ is a cylinder.

Proving part *iii.* is a little bit more involved.

Let $\mathcal{M}$ be a flat ruled surface with $X(u,v) = \alpha(u) + v\beta(u)$ and $\|\beta(u)\| = 1$, for the simplicity of calculation, we may assume that $\alpha(u)$ is a unit-speed striction curve of this surface such that $\alpha' \cdot \beta' = 0$. We are able to assert this assumption without loss of generality as we can always reparametrize the ruled surfaces $X(u,v) = \alpha(u) + v\beta(u)$ by the arc-length of its striction curve while holding the conditions that $\alpha' \cdot \beta' = 0$ and $\|\beta(u)\| = 1$ true. First, let us reparametrize the surface $\mathcal{M}$ by the arc-length of its striction curve as $\bar{X}(u,v) = \sigma(u) + v\beta(u)$, where $\sigma(u)$ is its striction curve and s is the arc-length parametrization of the curve σ. We then can reparametrize the curve $\sigma(u)$ by it arc-length s. Let $\sigma(u(s) = \tilde{\sigma}(s)$ be the arc-length parametrization of $\sigma(u)$, and $\tilde{\beta}(s)$ be the reparametrization of $\beta(u(s))$. We then can reparametrize the surface $\mathcal{M}$ again by $\tilde{\sigma}(s)$ while holding the other conditions satisfied by $\tilde{X}(s,\tilde{v}) = \tilde{\sigma}(s) + \tilde{v}\tilde{\beta}(s)$, where $\tilde{v} = \dfrac{v}{\|\tilde{\beta}(s)\|}$ and thus $\|\tilde{\beta}(s)\| = 1$. Since $s, \tilde{v}$ are arbitrary, we can always assume that $X(u,v) = \alpha(u) + v\beta(u)$ with α is a unit-speed curve such that $\alpha' \cdot \beta' = 0$ and $\|\beta\| = 1$.

Since $\mathcal{M}$ is flat, we have that $K(u,v) = 0$ at all points $X(u,v)$ of the surface. By Proposition 1.3.1, see also later Proposition 1.3.3 for the explicit calculation, we have that $< \beta'(u), (\alpha'(u) \times \beta(u)) > = 0$. Thus, provided that $< \alpha'(u), \beta'(u) >= 0$

and $< \beta'(u), (\alpha'(u) \times \beta(u)) >= 0$, we have $\beta(u)$ is parallel to $\alpha'(u)$, which means that $\beta(u) = \lambda(u)\alpha'(u)$. Moreover, $\|\beta(u)\| = 1$ and $\|\alpha'(u)\| = 1$ both, thus we have $|\lambda(u)| = 1$. We may conclude that $\alpha'(u)$ and $\beta(u)$ are, at each point, collinear vectors, and, up to orientation, there is a unique choice so that $X(u, v)$ can be expressed as $X(u, v) = \alpha(u) + v\alpha'(u)$. By definition, $\mathcal{M}$ is a surface tangent developable of its striction curve α.

□

1.3.4 Gauss and Mean Curvature of Non-Cylindrical Ruled Surfaces

Though we have calculated the Gauss curvature of an arbitrary ruled surface earlier, in this section, we will calculate the Gaussian curvature $K(u, v)$ of a non-cylindrical ruled surface in terms of its striction curve and director curve. We will simplify the formula by expressing it in terms of a new parameter λ and we will analyse the characteristics of this function $K(u, v)$ with respect to the change of the parameter $\lambda(u)$. Lastly, we calculate the mean curvature of a non-cylindrical surface as well.

Proposition 1.3.3. *Let $\mathcal{M} \subset \mathbb{R}^3$ be a non-cylindrical ruled surface with the paramatrization $X(u, v) = \sigma(u) + v\gamma(u)$, where $\sigma(u)$ is its striction curve and $\|\gamma(u)\| = 1$. Then the Gaussian curvature of $\mathcal{M}$ is*

$$K(u, v) = -\frac{< \gamma'(u), (\sigma'(u) \times \gamma(u)) >^2}{\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2} \cdot \frac{1}{\|\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - < \sigma'(u), \gamma(u) >^2}. \tag{1.34}$$

Proof. From the equation (1.21), the Gaussian curvature of a non-cylindrical is

$$K = -\frac{M^2}{EG - F^2}.$$

As seen in this formula, we need to compute the first and second fundamental forms of the surface $\mathcal{M}$ in order to evaluate the Gaussian curvature K.

We have $X(u,v) = \sigma(u) + v\gamma(u)$ we then can calculate $X_u = \sigma'(u) + v\gamma'(u)$, $X_v = \sigma(u)$, $X_{uv} = \sigma'(u)$, $X_{uu} = \sigma''(u) + v\gamma''(u)$, and $X_{vv} = 0$.

First, we proceed with the calculation of the first fundamental form of the surface $\mathcal{M}$. We have

$$\begin{aligned} E = \|X_u\|^2 &= \|\sigma' + v\gamma'\|^2 \\ &= < \sigma' + v\gamma', \sigma' + v\gamma' > \\ &= \|\sigma'\|^2 + v^2\|\gamma'\|^2 + 2v < \sigma', \gamma' > . \end{aligned}$$

By hypothesis, the fact that $\sigma(u)$ is a striction curve of $X(u,v)$ implies $< \sigma', \gamma' > = 0$. Therefore,

$$E = \|\sigma'\|^2 + v^2\|\gamma'\|^2. \tag{1.35}$$

We also have $G = \|X_v\|^2 = \|\gamma\|^2$, but recall that $\|\gamma\|^2 = 1$.

Therefore,

$$G = 1. \tag{1.36}$$

Similarly, we have

$$F = < X_u, X_v > = < \sigma' + v\gamma', \gamma > = < \sigma', \gamma > + v < \gamma', \gamma > .$$

By hypothesis, $\|\gamma\| = 1$, thus $< \gamma, \gamma > = \|\gamma\|^2 = 1^2$ and this implies

$$\begin{aligned} < \gamma', \gamma > + < \gamma, \gamma' > &= 0 \\ < \gamma', \gamma > &= 0. \end{aligned}$$

Therefore,

$$F = < \alpha', \gamma > . \tag{1.37}$$

We are now ready to calculate $EG - F^2$,

$EG - F^2 = (\|\sigma'\|^2 + v^2\|\gamma'\|^2)1 - (< \sigma', \gamma >)^2 = (\|\sigma'\|^2 + v^2\|\gamma'\|^2) - (< \sigma', \gamma >)^2.$

$$\Rightarrow EG - F^2 = (\|\sigma'\|^2 + v^2\|\gamma'\|^2) - (< \sigma', \gamma >)^2. \tag{1.38}$$

Next, we compute the standard unit normal vector $\mathbf{N}$ to this ruled surface $\mathcal{M}$ at point (u, v), where

$$\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

We have $X_u = \sigma'(u) + v\gamma'(u)$ and $X_v = \gamma(u)$ so

$$X_u \times X_v = (\sigma' + v\gamma') \times \gamma = (\sigma' \times \gamma) + v(\gamma' \times \gamma),$$

and

$$\begin{aligned}\|X_u \times X_v\|^2 &= \|(\sigma' \times \gamma) + v(\gamma' \times \gamma)\|^2 \\ &= < (\sigma' \times \gamma) + v(\gamma' \times \gamma), (\sigma' \times \gamma) + v(\gamma' \times \gamma) > \\ &= \|\sigma' \times \gamma\|^2 + v^2\|\gamma' \times \gamma\|^2 + 2v < (\sigma' \times \gamma), (\gamma' \times \gamma) > .\end{aligned}$$

Now we need to calculate $< (\sigma' \times \gamma), (\gamma' \times \gamma) >$ and $\|\gamma' \times \gamma\|$ separately. First consider $< (\sigma' \times \gamma), (\gamma' \times \gamma) >$. We have $< \sigma', \gamma' > = 0$, which implies that $\sigma' \perp \gamma'$ as well as $(\sigma' \times \gamma) \perp (\gamma' \times \gamma)$. It is the same as $< (\sigma' \times \gamma), (\gamma' \times \gamma) > = 0$. We compute $\|\gamma' \times \gamma\|$ as follows:

$\|\gamma' \times \gamma\| = \|\gamma'\|\|\gamma\| \sin\theta = \|\gamma'\|$ because $\|\gamma\| = 1$ and $\sin\theta = 1$ as $\gamma' \perp \gamma$.

Therefore,

$$\|X_u \times X_v\|^2 = \|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2. \tag{1.39}$$

To that end, we have

$$\mathbf{N} = \frac{(\sigma' \times \gamma) + v(\gamma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}}. \tag{1.40}$$

The coefficients of the second fundamental form of the surface $\mathcal{M}$ are as follows. First,

$$\begin{aligned} M &= < X_{uv}, \mathbf{N} > \\ &= < \gamma', \frac{(\sigma' \times \gamma) + v(\gamma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}} > \\ &= \frac{< \gamma', (\sigma' \times \gamma) > + v < \gamma', (\gamma' \times \gamma) >}{[\|\sigma' \times \gamma\|^2 + v^2\|\beta'\|^2]^{1/2}}. \end{aligned}$$

We also have that $< v\gamma', (\gamma' \times \gamma) > = \; v < \gamma, (\gamma' \times \gamma') > = 0$ since $\gamma' \times \gamma' = 0$.

Therefore,

$$M = \frac{\gamma'.(\sigma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}}. \tag{1.41}$$

$$\begin{aligned} L &= < X_{uu}, \; \mathbf{N} > \\ &= < \sigma'' + v\gamma'', \; \frac{(\sigma' \times \gamma) + v(\gamma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}} > . \end{aligned}$$

Thus, we have

$$L =< \sigma'' + v\gamma'', \; (\sigma' \times \gamma) + v(\gamma' \times \gamma) > \cdot \frac{1}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}}. \tag{1.42}$$

Also,

$$N = < X_{vv}, \; \mathbf{N} > = < 0, \; \frac{(\sigma' \times \gamma) + v(\gamma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}} > = 0. \tag{1.43}$$

Hence, $LN = 0$.

Now we are ready to compute the Gauss curvature.

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{0 - M^2}{EG - F^2} \\ &= \frac{-M^2}{EG - F^2} \\ &= -\frac{[\gamma'.(\sigma' \times \gamma)]^2}{\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2}.\frac{1}{\|\sigma'\|^2 + v^2\|\gamma'\|^2 - <\sigma', \gamma>^2}. \end{aligned}$$

We have proved the formula claimed as

$$K(u, v) = -\frac{<\gamma'(u), (\sigma'(u) \times \gamma(u))>^2}{\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2}.\frac{1}{\|\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - <\sigma'(u), \gamma(u)>^2}.$$

$\square$

To simplify the above formula of $K(u, v)$, we introduce a new parameter λ, which is sometimes also called distribution parameter, and we will express $K(u, v)$ in terms λ.

Lemma 1.3.3. *Suppose $\mathcal{M}$ be a non-cylindrical ruled surface as mentioned in Proposition 1.3.3, then we have*

$$K(u, v) = -\frac{\lambda^2(u)}{[\lambda^2(u) + v^2]^2},$$

where

$$\lambda(u) = \frac{<\gamma'(u), (\sigma'(u) \times \gamma(u))>}{\|\gamma'(u)\|^2}.$$

Moreover, we have the following properties of the Gauss curvature that we can express via the distribution parameter:

a. Along a ruling (where u is fixed), $\lim_{v\to\infty} K(u, v) = 0$*;*

b. $K(u,v) = 0 \Leftrightarrow \lambda(u) = 0$*;*

c. If $\lambda(u) \neq 0$*, and for any fixed* u *the Gauss curvature* $K(u,v)$ *is continuous, then* $|K(u,v)|$ *attains its maximum at* $v = 0$ *and the max value is* $|K(u,0)| = \dfrac{1}{\lambda^2(u)}$

d. If $\lambda(u)$ *has an absolute minimum at* u_0 *then* $|K(u,v)|$ *has a maximum at* $(u_0, 0)$ *and the value is* $|K(u,v)| = \dfrac{1}{\lambda(u_0)}$*.*

Proof. For the first part of the proof, we will show that $K(u,v) = -\dfrac{\lambda^2(u)}{[\lambda^2(u) + v^2]^2}$. By equation (1.21), we have that the Gaussian curvature of the noncylindrical $\mathcal{M}$ is $K(u,v) = -\dfrac{M^2}{EG - F^2}$, where $M = \dfrac{< \gamma'(u), (\sigma'(u) \times \gamma(u)) >}{[\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2]^{1/2}}$ by equation (1.41), and $EG - F^2 = (\|\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2) - (< \sigma'(u), \gamma(u) >)^2$ by equation (1.38).

Next, we want to express M and $EG - F^2$ in terms of $\lambda(u)$.
We have $\lambda(u) = \dfrac{< \gamma'(u), (\sigma'(u) \times \gamma(u))}{\|\gamma'(u)\|^2}$.

Then,

$$\begin{aligned}
\lambda(u)\|\gamma'(u)\|^2 &=< (\sigma'(u) \times \gamma(u)), \gamma'(u) > \\
&= \|\sigma'(u) \times \gamma(u)\|\|\gamma'(u)\| \cos\theta,
\end{aligned}$$

where θ is the angle between the vector $[\sigma'(u) \times \gamma(u)]$ and the vector $\gamma'(u)$.
Since $(\sigma'(u) \times \gamma(u)) \perp \gamma(u)$ and $\gamma'(u) \perp \gamma(u)$, we can infer that $(\sigma'(u) \times \gamma(u))$ is parallel to vector $\gamma'(u)$. It is the same as saying that $\cos\theta = \pm 1$.
Consequently,

$\lambda(u)\|\gamma'(u)\|^2 = \pm\|\sigma'(u) \times \gamma(u)\|\|\gamma'(u)\|$

$\Rightarrow \lambda(u)\|\gamma'(u)\| = \pm\|\sigma'(u) \times \gamma(u)\|$

$\Rightarrow \pm\lambda(u)\|\gamma'\| = \|\sigma'(u) \times \gamma(u)\| = \|\sigma'(u)\|\|\gamma(u)\| \sin\phi$.

Therefore,

$\sin\phi = \pm\dfrac{\lambda(u)\|\gamma'(u)\|}{\|\sigma'(u)\|}$, where ϕ is the angle between $\sigma'(u)$ and $\gamma(u)$.

Now we are ready to compute $EG - F^2$ in terms of $\lambda(u)$.

We have:

$$\begin{aligned} EG - F^2 &= \|(\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - \|(\sigma'(u)\|^2 \cos^2\phi \\ &= \|\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - \|(\sigma'(u)\|^2(1 - sin^2\phi) \\ &= \|\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - \|(\sigma'(u)\|^2[1 - \frac{\lambda^2(u)\|\gamma'(u)\|^2}{\|\sigma'(u)\|^2]}] \\ &= \|(\sigma'(u)\|^2 + v^2\|\gamma'(u)\|^2 - \|(\sigma'(u)\|^2 + \lambda^2(u)\|\gamma'(u)\|^2 \\ &= v^2\|\gamma'(u)\|^2 + \lambda^2(u)\|\gamma'(u)\|^2 \\ &= [\lambda^2(u) + v^2]\|\gamma'(u)\|^2. \end{aligned}$$

Therefore,

$$EG - F^2 = [\lambda^2(u) + v^2]\|\gamma'(u)\|^2. \tag{1.44}$$

Computing M in terms of $\lambda(u)$.

Following the calculation results of the proof of Proposition 1.3.3, we have

$$M = \frac{< \gamma'(u), (\sigma'(u) \times \gamma(u)) >}{[\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2]^{1/2}} = \frac{< \gamma'(u), (\sigma'(u) \times \gamma(u)) >}{\|X_u \times X_v\|},$$

where $(\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2)^{1/2} = \|X_u \times X_v\|$.

We also have: $\pm\lambda(u)\|\gamma'\| = \|\sigma'(u) \times \gamma(u)\|$ from the above calculation

$\Rightarrow \|\sigma'(u) \times \gamma(u)\|^2 = \lambda^2(u)\|\gamma'\|^2$.

Therefore,

$$\begin{aligned} \|X_u \times X_v\| &= (\|\sigma'(u) \times \gamma(u)\|^2 + v^2\|\gamma'(u)\|^2)^{1/2} \\ &= (\lambda^2(u)\|\gamma'\|^2 + v^2\|\gamma'(u)\|^2)^{1/2} \\ &= (\lambda^2(u) + v^2)^{1/2}\|\gamma'(u)\|, \end{aligned}$$

and

$$\|X_u \times X_v\| = (\lambda^2(u) + v^2)^{1/2}\|\gamma'(u)\|. \tag{1.45}$$

Lastly, we can calculate $K(u,v)$.

$$\begin{aligned} K(u,v) = -\frac{M^2}{EG-F^2} &= -\frac{<\gamma'(u), (\sigma'(u) \times \gamma(u))>^2}{\|X_u \times X_v\|^2}\frac{1}{EG-F^2} \\ &= -\frac{\lambda^2(u)\|\gamma'(u)\|^4}{[\lambda^2(u)+v^2]\|\gamma'(u)\|^2}\frac{1}{[\lambda^2(u)+v^2]\|\gamma'(u)\|^2} \\ &= -\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}. \end{aligned}$$

We have thus proved that

$$K(u,v) = -\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}. \tag{1.46}$$

In the following steps, we will prove part $a.$, $b.$, $c.$, $d.$ of Lemma 1.3.3.

1. Proof of Lemma 1.3.3 a.

   We have

   $$K(u,v) = -\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}.$$

   The Gauss curvature $K(u,v)$ is the function of v when u is fixed along a ruling. Assume first that $\lambda(u) \neq 0$, thus, for any fixed u,

   $$\lim_{v\to\infty} K(u,v) = -\lim_{v\to\infty}\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2} = 0.$$

   If $\lambda(u) = 0$, then $K \equiv 0$ everywhere and the limit is trivially true.

2. Proof of Lemma 1.3.3 b.

As seen in the previous proof, we have

$$K(u,v) = -\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}.$$

Thus $K(,v) = 0 \Leftrightarrow \lambda(u) = 0$ as well.

3. Proof of Lemma 1.3.3 c.

   Firstly, we note that $K(u,v)$ is a continuous function of v for each fixed u. Given $\lambda(u) \neq 0$, we can conclude that

$$K(u,v) = -\frac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}$$

   is well-defined for all value of (u,v), and all order partial derivatives with respect to v of $K(u,v)$ exist and are continuous.

   We thus conclude that $K(u,v)$ is a smooth function on the domain where $\lambda(u) \neq 0$.

   Secondly, we check that $|K(u,v)|$ has the minimum at $v = 0$ and this minimum value is $|K(u,0)| = \dfrac{1}{\lambda^2(u)}$.

   Again we have $|K(u,v)| = \dfrac{\lambda^2(u)}{[\lambda^2(u)+v^2]^2}$, then

$$\begin{aligned}
K_u &= \frac{[2\lambda(u)\lambda'(u)][\lambda^2(u)+v^2]^2 - \lambda^2(u)2[\lambda^2(u)+v^2]2\lambda(u)\lambda'(u)}{[\lambda^2(u)+v^2]^4} \\
&= \frac{2\lambda(u)\lambda'(u)[\lambda^2(u)+v^2][\lambda^2(u)+v^2-2\lambda^2(u)]}{[\lambda^2(u)+v^2]^4} \\
&= \frac{2\lambda(u)\lambda'(u)[v^2-\lambda^2(u)]}{[\lambda^2(u)+v^2]^3}.
\end{aligned}$$

   Similarly, $K_v = -\dfrac{4v\lambda^2(u)}{[\lambda^2(u)+v^2]^3}$.

   We know that any critical points of $K(u,v)$ must satisfy both equations $K_u = 0$, and $K_v = 0$. Thus, we have

$$\frac{2\lambda(u)\lambda'(u)[v^2 - \lambda^2(u)]}{[\lambda^2(u) + v^2]^3} = 0, \text{ and } -\frac{4v\lambda^2(u)}{[\lambda^2(u) + v^2]^3} = 0.$$

Since $\lambda(u) \neq 0$, then we have $\lambda'(u) = 0$ and $v = 0$. Therefore, the critical point of $|K(u,v)|$ is at $v = 0$ and $\lambda'(u) = 0$. The critical value of $K(u,0)$ is $|K(u,0)| = \frac{\lambda^2(u)}{[\lambda^2(u) + 0]^2} = \frac{1}{\lambda^2(u)}$.

Thirdly, we need to check whether this critical value is a maximum or minimum value.

We have $\lambda^2(u) \leq \lambda^2(u) + v^2$, for all (u,v), so

$$\frac{\lambda^2(u)}{[\lambda^2(u) + v^2]^2} \leq \frac{\lambda^2(u) + v^2}{[\lambda^2(u) + v^2]^2} = \frac{1}{\lambda^2(u) + v^2} \leq \frac{1}{\lambda^2(u)}.$$

Therefore, we have that $\frac{\lambda^2(u)}{[\lambda^2(u) + v^2]^2} \leq \frac{1}{\lambda^2(u)}$ for all (u,v).

$\Rightarrow \frac{1}{\lambda^2(u)}$ is the max point of $|K(u,v)|$ whenever u is fixed and $\lambda(u) \neq 0$.

4. Proof of Lemma 1.3.3 d.

   We have $|K(u,0)| = \frac{1}{\lambda^2(u_0)}$. Suppose $\lambda(u)$ has an absolute minimum at u_0, then $|K(u_0,0)|$ reaches its maximum at $(u_0,0)$ and its value is $|K(u_0,0)| = \frac{1}{\lambda^2(u_0)}$ as calculated in the above proof.

$\square$

We wrap-up this section by investigating the mean curvature of non-cylindrical ruled surface. Besides the intrinsic Gauss curvature, a surface has other types of curvature. Among them is the mean curvature, denoted by H, which is, in fact, very relevant to the way the surface is immersed in the Euclidean space which is why, we will see later, mean curvature is related to the notion of minimal surfaces. Both of the Gaussian curvature and the mean curvature at a point can be calculated from the second fundamental form (normalized by the first fundamental form) and they are in fact related through the so called principal curvatures. We add below a few more details, while for a comprehensive coverage we refer the reader to [5].

Let $\mathbf{p}$ be a point on a surface S in $\mathbb{R}^3$ and let $\mathcal{W}_{\mathbf{p}} = -d\mathbf{N}_{\mathbf{p}} : T_{\mathbf{p}}S \to T_{\mathbf{p}}S$ be the Weingarten map of S at the point $\mathbf{p}$. Note that the Weingarten map is a linear map. The principal curvatures k_1 and k_2 at the point $\mathbf{p}$ are the eigenvalues of $\mathcal{W}$, $\mathcal{W}_{\mathbf{p}}(\mathbf{t}_1) = k_1\mathbf{t}_1$ and $\mathcal{W}_{\mathbf{p}}(\mathbf{t}_1) = k_1\mathbf{t}_1$, where the eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are called the principal directions at the point and they form a basis of the tangent plane of $\mathcal{M}$ at $\mathbf{p}$. Then the mean curvature is, by definition, the trace of the Weingarten map and the Gauss curvature is its determinant, hence if k_1, k_2 be the principal curvatures of a surface at a point $\mathbf{p}$, then the mean and Gaussian curvatures at that point are given by $H(\mathbf{p}) = \dfrac{(k_1 + k_2)}{2}$ and $K(\mathbf{p}) = k_1 k_2$.

As we mentioned earlier, the mean curvature can be calculated from the first and second fundamental forms of a given surface patch $X(u, v)$ and we provide below the formula in terms of their coefficients, similarly with the formula we have stated for the Gauss curvature:

$$H(\mathbf{p}) = \frac{LG - 2MF + NE}{2(EG - F^2)}, \tag{1.47}$$

where $\mathbf{p} = X(u_0, v_0)$ for some (u_0, v_0) in the domain of the patch.

Definition 1.3.5. *[5] A surface is called a minimal surface if its mean curvature is zero everywhere.*

Roughly speaking, the mean curvature zero means that the surface is immersed in $\mathbb{R}^3$ most efficiently, minimizing the surface area of any domain of $\mathbb{R}^3$ with boundary on the surface. We state below an important result of differential geometry pertinent to ruled surfaces. Its proof is very involved and is outside the techniques of this thesis.

Theorem 1.3.2 (Catalan 1842)**.** *[2] The only minimal ruled minimal surfaces are the plane or the helicoid (or, of course, parts of them).*

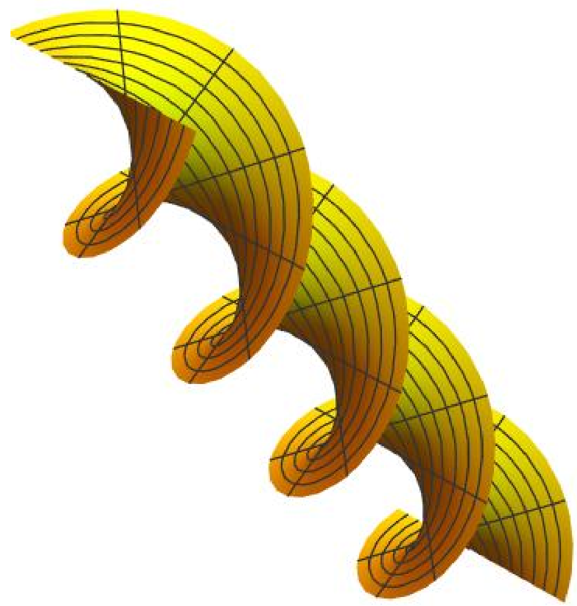

Figure 1.11: Circular Helicoid, where $\alpha(u) = (0, 0, 3u)$, $\beta(u) = (\cos u, \sin u, 0)$, $a = 1$, $u \in (-2\pi, 2\pi)$, and $v \in (-3, 3)$.

Heuristically, a part of this theorem is not hard to imagine as the Gauss curvature of ruled surfaces is non-positive. Thus, if the product of principal curvatures $k_1 k_2 \leq 0$ and $H = k_1 + k_2 = 0$ everywhere, we must have $k_1 = -k_2$. If one of the k_i's is zero, so is the other, and we obtain the plane. The helicoid is a ruled surface with strictly negative Gauss curvature such that $k_1 = -k_2 \neq 0$.

We return now to the mean curvature of a non-cylindrical ruled surface via the following proposition.

Proposition 1.3.4. *The mean curvature of a non-cylindrical ruled surface $\mathcal{M}$ with $X(u, v) = \sigma(u) + v\gamma(u)$, where $\sigma(u)$ is its striction curve and $\|\gamma(u)\| = 1$, is*

$$H = \frac{(< \sigma'' + v\gamma'', (\sigma' \times \gamma + v(\gamma' \times \gamma) > -2(< \sigma', \gamma >)(< \gamma', (\sigma' \times \gamma >)}{2(EG - F^2)^{\frac{3}{2}}}. \tag{1.48}$$

Proof. We know that the mean curvature of a surface in $\mathbb{R}^3$ can be calculated using the equation 1.47 where $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)}$ where, as before, E, F, G and L, M, N are the coefficients of the first and, respectively, second fundamental forms of $\mathcal{M}$. By Proposition 1.3.3 and Lemma 1.3.3, for non-cylindrical ruled surfaces, we have the following formulas: $E = \|\sigma'\|^2 + v^2\|\gamma'\|^2$ as in the equation (1.35), $G = 1$ as in the equation (1.36), $F =< \sigma(u)', \ \gamma(u) >$ as in the equation (1.37), $M = \dfrac{\gamma'.(\sigma' \times \gamma)}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}}$ as in the equation (1.41), $L =< \sigma'' + v\gamma'', \ (\sigma' \times \gamma) + v(\gamma' \times \gamma) > \cdot \dfrac{1}{[\|\sigma' \times \gamma\|^2 + v^2\|\gamma'\|^2]^{1/2}}$ as in the equation (1.43), $N = 0$ as in the equation (1.43), and $EG - F^2 = \|X_u \times X_v\|^2$ as in the equation (1.22).

Therefore,

$$\begin{aligned}
H &= \frac{LG - 2MF + NE}{2(EG - F)^2} \\
&= \frac{(< \sigma'' + v\gamma'', (\sigma' \times \gamma + v(\gamma' \times \gamma) > -2(< \sigma', \gamma >)(< \gamma', (\sigma' \times \gamma >) + 0}{2(EG - F^2)^2(EG - F^2)} \\
&= \frac{(< \sigma'' + v\gamma'', (\sigma' \times \gamma + v(\gamma' \times \gamma) > -2(< \sigma', \gamma >)(< \gamma', (\sigma' \times \gamma >)}{2(EG - F^2)^{\frac{3}{2}}}.
\end{aligned}$$

$\square$

1.3.5 Geodesics on Non-Cylindrical Ruled Surfaces

Definition 1.3.6. *[5] A curve $\sigma(t)$ on a surface $\mathcal{M}$ in $\mathbb{R}^3$, with t in some open interval, is called a geodesic if and only if $\sigma''(t) = \vec{0}$, or, for all values of the parameter t, the vector $\sigma''(t)$ is perpendicular to the tangent plane of the surface at the point $\sigma(t)$.*

Particularly for a ruled surface $\mathcal{M}$ with $X(u, v) = \alpha(u) + v\beta(u)$, a unit-speed curve $\sigma(u)$ on $\mathcal{M}$ defined thus by $\sigma(u) = \alpha(u) + v(u)\beta(u)$ where u is the arc-length of $\sigma(u)$, is a geodesic of $\mathcal{M}$ if and only if $\sigma''(u)$ is perpendicular to both X_u and X_v.

Thus, by definition, σ is a geodesic on $\mathcal{M}$ if and only if

$$\begin{cases} \sigma''(u) \perp [\alpha'(u) + v(u)\beta'(u)] \\ \sigma''(u) \perp \beta(u) \end{cases},$$

or, equivalently,

$$\begin{cases} \sigma''(u) \cdot [\alpha'(u) + v(u)\beta'(u)] = 0 \\ \sigma''(u) \cdot \beta(u) \qquad\qquad\qquad = 0. \end{cases}$$

Since $\sigma'(u) = \alpha'(u) + v(u)\beta'(u) + v'(u)\beta(u)$, note that we have,

$$\begin{aligned} \sigma''(u) \cdot \sigma'(u) &= \sigma''(u) \cdot [\alpha'(u) + v(u)\beta'(u) + v'(u)\beta(u)] \\ &= \sigma''(u) \cdot [\alpha'(u) + v(u)\beta'(u)] + \sigma''(u) \cdot v'(u)\beta(u). \end{aligned}$$

Thus,

$$\sigma''(u) \cdot \sigma'(u) = \sigma''(u) \cdot [\alpha'(u) + v(u)\beta'(u)] + v'(u)\sigma''(u) \cdot \beta(u). \tag{1.49}$$

From above, for any unit-speed geodesic curve, $\sigma(u)$, we have that $\sigma'' \cdot \sigma' = 0$. Vice versa, if $\sigma(u)$ is a unit speed curve of the ruled surface $\mathcal{M}$ with $\sigma''(u) \cdot \beta(u) = 0$, then $\sigma''(u) \cdot [\alpha'(u) + v\beta'(u)] = 0$ also holds for all u, since $\sigma'' \cdot \sigma' = 0$ due to the unit speed condition. Thus, we conclude the lemma below:

Lemma 1.3.4. *A curve $\sigma(u)$ parametrized by arc-length is a geodesic of a ruled surface $\mathcal{M}$ with parametrization $X(u, v) = \alpha(u)+v\beta(u)$ if and only if $\sigma''(u)\cdot\beta(u) = 0$.*

In stating the next result, we will follow Izumiya-Takeuchi and call the statement Bonnet's theorem for non-cylindrical surfaces. We also point out that its assertion is true for general ruled surfaces as well.

Theorem 1.3.3. *[4] Suppose $\mathcal{M}$ is a ruled surface with parametrization of the form of $X(u, v) = \alpha(u) + v\gamma(u)$ with $\|\gamma(u)\| = 1$ and that $\sigma(u) = \alpha(u) + v(u)\gamma(u)$ is a curve on $X(u, v)$, where u is the arc-length of $\sigma(u)$. We consider the following three* *conditions on $\sigma(u)$:*

1. *$\sigma(u)$ is a line of striction of $X(u)$, i.e. $\sigma'(u) \cdot \gamma'(u) = 0$.*

2. *$\sigma(u)$ is a geodesic of $X(u, v)$, i.e. $\sigma''(u) \cdot \gamma(u) = 0$.*

3. *The angles between $\sigma'(u)$ and $\gamma(u)$ are constant, i.e. $\sigma'(u) \cdot \gamma(u) =$ constant.*

Suppose any two of the above three conditions hold, then the other condition holds.

Proof. Assume that conditions 1. and 2. of this proposition hold, i.e. if $\sigma(u)$ is a line of striction and, simultaneously, a geodesic of $X(u, v)$, then we must show that the angle between $\sigma'(u)$ and $\gamma(u)$ is constant as a function of u. We have

$$(\sigma' \cdot \gamma)'(u) = \sigma''(u) \cdot \gamma(u) + \sigma'(u) \cdot \gamma'(u) = 0, \tag{1.50}$$

because $(\sigma' \cdot \gamma')(u) = 0$, for all u, as σ is a line of striction and $(\sigma'' \cdot \gamma)(u) = 0$, for all u, as σ is a geodesic of $X(u, v)$. Thus, $\sigma'(u) \cdot \gamma(u) = ||\sigma'(u)|| \, ||\gamma(u)|| \cos\theta(u) = \cos\theta(u) =$ constant, concluding that the angle, $\theta(u)$, between these two vectors is constant.

Now, suppose that conditions 2. and 3. are satisfied, thus we have $(\sigma'' \cdot \gamma)(u) = 0$ and $(\sigma' \cdot \gamma)(u) =$ constant, for all u. Hence

$$(\sigma' \cdot \gamma)'(u) = (\sigma'' \cdot \gamma + \sigma' \cdot \gamma')(u) = 0.$$

By hypothesis, $(\sigma'' \cdot \gamma)(u) = 0$ implies that $(\sigma' \cdot \gamma)'(u) = (\sigma' \cdot \gamma')(u) = 0$. We have that $\|\gamma(u)\| = 1$ and $(\sigma' \cdot \gamma')(u) = 0$. Thus σ is a striction curve of $X(u, v)$ by definition.

Lastly, suppose that $(\sigma' \cdot \gamma) =$ constant, and $(\sigma' \cdot \gamma') = 0$, where $||\gamma|| = 1$, in other words, condition 1. and 3. hold. Similarly to the previous proof, by equation (1.50), we obtain $\sigma'' \cdot \gamma = 0$. By Lemma 1.3.4, we infer that σ is a geodesic of $X(u, v)$.

□

Chapter 2

Relationship between certain Ruled Surfaces and their Curves

In this chapter, we will investigate the characterizations of certain ruled surfaces that are related to the characteristics of certain curves on them. We structure Chapter 2 as follows:

We start the chapter by introducing a ruled surface called *rectifying developable of a curve*. In the same first section, we will study the special curves on such ruled surfaces and we will detail the characteristics of the ruled surfaces that are rectifying developable of a curve.

Next, we will investigate the geodesics on cylindrical ruled surfaces. As seen in Chapter 1, there are two categories of ruled surfaces, cylindrical and non-cylindrical. We introduce the geodesics of cylindrical ruled surfaces in this section and we will show that any ruled surface that is rectifying developable of a curve is a cylindrical ruled surface. From the study of properties of ruled surfaces that are rectifying developable of curve, we are able to draw certain conclusions on the geodesics on cylindrical ruled surfaces as well.

Third, we will introduce another special type of ruled surface called *principal* *normal surfaces of a curve.* Analogous with the previous case, we will introduce certain special curves on these ruled surfaces and their properties.

Lastly, we will examine the asymptotic curves on ruled surfaces. After introducing the asymptotic curves as well as some of their properties on general ruled surface, we will focus on the study of asymptotic curves on the ruled surfaces that are principal normal surfaces of a curve. We will wrap-up the section, as well as the chapter, by giving the characterizations of minimal asymptotic curves on ruled surfaces.

This chapter follows closely the paper by Izumiya and Takeuchi, [4]. However, whenever possible, we have provided our own proofs filling in all steps left to the reader, even when following the main idea of the paper.

2.1 Ruled Surfaces that are Rectifying Developable of a Curve

In this section, we focus on a specific ruled surface called rectifying developable of its base curve. We start this section by introducing the cylindrical and circular helix, as well as Darboux and modified Darboux vector fields on this specific ruled surface. We will prove that a rectifying developable is regular if and only if its base curve is a cylindrical helix and, in this case, the surface is a cylindrical ruled surface as well. We will also show that a ruled surface rectifying developable of a curve σ is regular if and only if the curve σ is a geodesic of the ruled surface, transversal to rulings, and the Gaussian curvature of the ruled surface vanishes along the curve σ.

Let us start this section by introducing cylindrical helix, circular helix, Darboux vector field and modified Darboux vector field.

2.1.1 Special Curves on Ruled Surfaces that are Rectifying Developable of a Curve

The Darboux vector field is a very practical tool to geometrically illustrate the curvature and torsion of a unit-speed curve $\alpha(u)$. In this respect, the curvature of $\alpha(u)$, $k(u)$, at each point on the curve is the measure of the rotation of the Frenet frame about the bi-normal unit vector $\mathbf{b}(u)$, while the torsion, $\tau(u)$, is the measure of the rotation of the Frenet frame about the tangent unit vector, $\mathbf{t}(u)$, with the precise definition to follow shortly.

Recall that for any regular unit-speed curve $\alpha : I \mapsto \mathbb{R}^3$, we denote $\mathbf{t}(u) = \alpha'(u)$ as its unit tangent vector, $k(u) = \|\alpha''(u)\|$ as its curvature, $\mathbf{n}$(u)= $\dfrac{1}{k(u)}\alpha''(u)$ as its unit principal normal vector, and $\mathbf{b}(u) = \mathbf{t}(u) \times \mathbf{n}(u)$ as its unit bi-normal vector of α at point $\alpha(u)$, respectively. Note that we also have the Frenet-Serret equations of α with nowhere vanishing curvature as below:

$$\mathbf{t}'(u) = k(u)\mathbf{n}(u) \tag{2.1}$$

$$\mathbf{n}'(u) = -k(u)\mathbf{t}(u) + \tau(u)\mathbf{b}(u) \tag{2.2}$$

$$\mathbf{b}'(u) = -\tau(u)\mathbf{n}(u), \tag{2.3}$$

where $\tau = \dfrac{(\alpha' \times \alpha'') \cdot \alpha'''}{\|\alpha' \times \alpha''\|^2}$ is the torsion of α at point $\alpha(u)$.

Definition 2.1.1. *For any regular unit-speed curve $\alpha : I \mapsto \mathbb{R}^3$, we define the Darboux vector field along α as the vector field defined by*

$$D(u) = \tau(u)\boldsymbol{t}(u) + k(u)\boldsymbol{b}(u). \tag{2.4}$$

Additionally, assuming that the curve α has non-zero curvature, we also define the *vector field*

$$\tilde{D}(u) = \left(\frac{\tau}{k}\right)(u)\boldsymbol{t}(u) + \boldsymbol{b}(u) \tag{2.5}$$

which is called the modified Darboux vector field of α.

Definition 2.1.2. *A space curve $\alpha : I \mapsto \mathbb{R}^3$ with curvature $k(u) \neq 0$ is called a cylindrical helix if and only if its tangent lines make a constant angle with a fixed line in space and the ratio $\left(\frac{\tau}{k}\right)(u) =$ constant, where $\tau(u)$ is the torsion of the curve α. Equivalently, it can be shown that a curve α as above is a circular helix if and only if both its torsion $\tau(u)$ and curvature $k(u)$ are constant along the curve.*

In other words, since we are assuming that the curve is non-trivially a space curve, hence $\tau \neq 0$, the cylindrical helix in $\mathbb{R}^3$ resembles a spring and is a curve on a vertical cylinder of revolution where the curve forms a constant angle with respect to the axis of the cylinder, taken, for example, to be the z-axis. The Cartesian parametrization of this cylindrical helix is $\alpha(u) = (a\cos(u),\ \pm a\sin(u),\ bu)$, where the parameter $u \in \mathbb{R}$, $a > 0$ is the radius of the helix, $b \neq 0$ is the slope or incline of the helix, respectively. The constant angle formed by its tangent with xy-plane is $2\pi b$, in other words as u increase by 2π, the point $(a\cos(u),\ a\sin(u),\ bu)$ rotates once round the z-axis and moves parallel to the z-axis by $2\pi b$. The positive number $2\pi|b|$ is called the pitch of the helix.

The circular helix is simply a right-handed cylindrical helix with the parametrization $\alpha(u) = (a\cos(u),\ a\sin(u),\ bu)$, where as the left-handed cylindrical helix is $\alpha(u) = (a\cos(u),\ -a\sin(u),\ bu)$. Recall that the torsion of this particular circular helix is

$$\tau(u) = \frac{b}{a^2 + b^2} \tag{2.6}$$

and its curvature is

$$k(u) = \frac{a}{a^2 + b^2}, \tag{2.7}$$

and note that they are both constant functions along the curve.

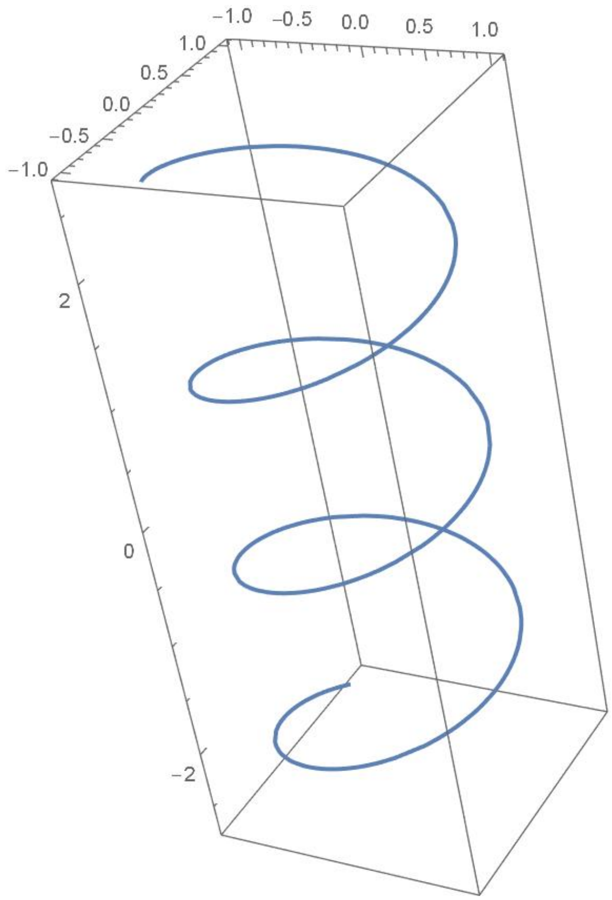

Figure 2.1: Circular Helix, where $\alpha(u) = (\cos u, \sin u, \frac{u}{4})$, $a = 1$, $u \in (-10, 10)$, $a = 1$, and $b = \frac{1}{4}$.

2.1.2 Properties of Ruled Surfaces that are Rectifying Developable of a Curve

We are now ready to define a ruled surface that is rectifying developable of a curve and prove certain properties of this ruled surface as mentioned before.

Definition 2.1.3. *A ruled surface $\mathcal{M}$ is called rectifying developable of a unit-speed curve α if it can be expressed in the form*

$$X(u, v) = \alpha(u) + v\tilde{D}(u), \tag{2.8}$$

where $\tilde{D}(u)$ is the modified Darboux vector field along its directrix curve α.

The lemma below characterizes the points on a ruled surface that is rectifying developable of a curve α where the surface is singular.

Lemma 2.1.1. *For any ruled surface that is rectifying developable of its directrix curve $\alpha(u)$ as defined by equation (2.8), the point (u_0, v_0) is its singular point if and only if*

$$\left(\frac{\tau}{k}\right)'(u_0) \neq 0, \tag{2.9}$$

and

$$v_0 = -\frac{1}{\left(\frac{\tau}{k}\right)'(u_0)}. \tag{2.10}$$

Proof. Recall that a point is a singular point $X(u_0, v_0)$ of a surface $X(u, v)$ (in particular for $X(u, v) = \alpha(u) + v\tilde{D}(u)$ as in equation (2.8)) if and only if $\frac{\partial X}{\partial u}(u_0, v_0) \times \frac{\partial X}{\partial v}(u_0, v_0) = \vec{0}$.

The definition of $X(u_0, v_0)$ to be a singular point of $\mathcal{M}$ implies that

$$\begin{aligned}\frac{\partial X}{\partial u}(u_0, v_0) \times \frac{\partial X}{\partial v}(u_0, v_0) &= \left[\alpha'(u_0) + v_0\tilde{D}'(u_0)\right] \times \tilde{D}(u_0) \\ &= \left[\mathbf{t}(u_0) \times \tilde{D}(u_0)\right] + v_0\left[\tilde{D}'(u_0) \times \tilde{D}(u_0)\right] = \vec{0}.\end{aligned}$$

Thus, the fact that $X(u_0, v_0)$ is a singular point of $X(u, v)$ as in equation (2.8) is equivalent to the condition

$$\left[\mathbf{t}(u_0) \times \tilde{D}(u_0)\right] + v_0\left[\tilde{D}'(u_0) \times \tilde{D}(u_0)\right] = \vec{0}. \tag{2.11}$$

Furthermore, recalling the definition of the modified Darboux vector of α, and by

Frenet-Serret equations, we have the following equations:

$$\begin{aligned}
\tilde{D} &= \left(\frac{\tau}{k}\right)\mathbf{t} + \mathbf{b}, \\
\tilde{D}' &= \left(\frac{\tau}{k}\right)'\mathbf{t} + \left(\frac{\tau}{k}\right)\mathbf{t}' + \mathbf{b}', \\
\tilde{D}' &= \left(\frac{\tau}{k}\right)'\mathbf{t} + \frac{\tau}{k}k\mathbf{n} - \tau\mathbf{n}, \\
\tilde{D}' &= \left(\frac{\tau}{k}\right)'\mathbf{t}.
\end{aligned}$$

Thus,

$$\tilde{D}'(u_0) = \left(\frac{\tau}{k}\right)'(u_0)\mathbf{t}(u_0). \tag{2.12}$$

Substituting equation (2.8) and equation (2.12) into equation (2.11), and evaluating it at (u_0, v_0), we have

$$\begin{aligned}
\left[\mathbf{t} \times \tilde{D}\right] + v_0\left[\tilde{D}' \times \tilde{D}\right] &= \vec{0}, \\
\mathbf{t} \times \left[\frac{\tau}{k}\mathbf{t} + \mathbf{b}\right] + v_0\left[\frac{\tau}{k}\mathbf{t} + \mathbf{b}\right] \times \left[\left(\frac{\tau}{k}\right)'\mathbf{t}\right] &= \vec{0}, \\
\left(\frac{\tau}{k}\right)(\mathbf{t} \times \mathbf{t}) + (\mathbf{t} \times \mathbf{b}) + v_0\left[\left(\frac{\tau}{k}\right)\left(\frac{\tau}{k}\right)'(\mathbf{t} \times \mathbf{t}) + \left(\frac{\tau}{k}\right)'(\mathbf{b} \times \mathbf{t})\right] &= \vec{0}.
\end{aligned}$$

Applying Frenet-Serret formulas,

$$\begin{aligned}
\vec{0} - \mathbf{n}(u_0) + v_0\left(\frac{\tau}{k}\right)(u_o)\mathbf{n}(u_0) &= \vec{0}, \\
\mathbf{n}(u_0)\left[-1 + v_0\left(\frac{\tau}{k}\right)\right] &= \vec{0}.
\end{aligned}$$

Consequently, since for all u_0, $\mathbf{n}(u_0) \neq \vec{0}$ we conclude that (u_0, v_0) is a singular point of $X(u,v)$, a ruled surface that is rectifying developable of α, if $\left(\frac{\tau}{k}\right)'(u_0) \neq 0$ and $v_0 = -\dfrac{1}{\left(\frac{\tau}{k}\right)'(u_0)}$.

For the sufficiency condition, it is very simple to check that $\dfrac{\partial X}{\partial u}(u_0, v_0) \times \dfrac{\partial X}{\partial v}(u_0, v_0) =$

$\vec{0}$ since

$$\begin{aligned}
\frac{\partial X}{\partial u}(u_0,v_0) \times \frac{\partial X}{\partial v}(u_0,v_0) &= [\alpha'(u_0) + v_0\tilde{D}(u_0)] \times \tilde{D}(u_0) \\
&= \left[\mathbf{t}(u_0) - \frac{1}{\left(\frac{\tau}{k}\right)'(u_0)}\left(\frac{\tau}{k}\right)'(u_0)\mathbf{t}(u_0)\right] \times \tilde{D}(u_0) \\
&= \vec{0} \times \tilde{D}(u_0) \\
&= \vec{0}.
\end{aligned}$$

Therefore, (u_0, v_0) is a singular point of a ruled surface that is rectifying developable of α if and only if $\left(\frac{\tau}{k}\right)'(u_0) \neq 0$ and $v_0 = -\frac{1}{\left(\frac{\tau}{k}\right)'(u_0)}$. □

The proposition below will characterize ruled surfaces that are rectifying developable of a curve α.

Proposition 2.1.1. *Let $u \in I \mapsto \alpha(u)$ be a unit-speed curve with non-zero curvature everywhere, $k(u) \neq 0$. Then the following conditions are equivalent:*

1. *The ruled surface that is rectifying developable of the curve $\alpha(u)$ as in the equation (2.8), where $X(u,v) = \alpha(u) + v\tilde{D}(u)$, is a regular surface.*

2. *The curve $\alpha(u)$ is a cylindrical helix.*

3. *The ruled surface that is rectifying developable of $\alpha(u)$ is a cylindrical surface.*

Proof. We will first prove that statement 1. implies statement 2. that is the same as saying that, for any regular ruled surface that is rectifying developable of its base curve $\alpha(u)$, we have that $\alpha(u)$ is a cylindrical helix.

By the characterization of singular points of a rectifying developable of a curve from Lemma 2.1.1, we have that $\mathcal{M}$ is singular if and only if $v_0 = -\frac{1}{\left(\frac{\tau}{k}\right)'(u_0)}$ and $\left(\frac{\tau}{k}\right)'(u_0) \neq 0$. On the contrary, $\mathcal{M}$ is regular if and only if there are no points (u_0, v_0) that satisfy the conditions above i.e., given that $v \in \mathbb{R}$, for all $u \in I$, $\left(\frac{\tau}{k}\right)'(u) = 0$, so that $v \neq -\dfrac{1}{\left(\frac{\tau}{k}\right)'(u)}$. Then for all (u, v), we have that $\left(\dfrac{\tau}{k}\right)(u)$ is constant on I. Hence, by Definition 2.1.2, the curve α is a cylindrical helix.

We will prove next that for any ruled surface that is rectifying developable of a cylindrical helix is a cylindrical surface, or that statement 2. implies statement 3. Consider $\tilde{D}'$ as in equation (2.12), where $\tilde{D}'(u_0) = \left(\dfrac{\tau}{k}\right)'(u_0)\,\mathbf{t}(u_0)$. By hypothesis, $\alpha(u)$ is a cylindrical helix, thus we have that the ratio $\left(\dfrac{\tau}{k}\right)(u)$ is constant. Thus, $\left(\dfrac{\tau}{k}\right)'(u) = 0$, which implies that $\tilde{D}'(u_0) = \left(\dfrac{\tau}{k}\right)'(u_0)\,\mathbf{t}(u_0) = \vec{0}$. Therefore, for any ruled surface that is rectifying developable to its base curve α with parametrization $X(u, v) = \alpha(u) + v\tilde{D}(u)$ as in equation (2.8), we have that $\tilde{D}'(u) \times \tilde{D}(u) = \vec{0} \times \tilde{D}(u) = \vec{0}$. Hence, this ruled surface is a cylindrical ruled surface by definition.

Lastly, we show that any cylindrical ruled surface that is rectifying developable to its base α is a regular ruled surface. Since $X(u, v) = \alpha(u) + v\tilde{D}(u)$ is a cylindrical surface, then $\tilde{D}'(u) \times \tilde{D}(u) = \vec{0}$. By the definition of the modified Darboux vector field in equation (2.5) of Definition 2.1.1, we have $\tilde{D}(u) = \left(\dfrac{\tau}{k}\right)(u)\,\mathbf{t}(u) + \mathbf{b}(u)$. We have seen that $\tilde{D}(u) \neq \vec{0}$, $\forall u \in I$ since $\mathbf{t}(u) \neq 0$ and $\mathbf{b}(u) \neq 0$, and they are non-collinear vectors fields. Hence, $\tilde{D}'(u) \times \tilde{D}(u) = \vec{0}$ if and only if $\tilde{D}'(u)$ is either the zero vector or is parallel to $\tilde{D}(u)$. Since, for all $u \in I$, we have $\tilde{D}'(u) = \left(\dfrac{\tau}{k}\right)'(u)\,\mathbf{t}(u)$, it follows that $\left(\dfrac{\tau}{k}\right)'(u) = 0,\ \forall u \in I$.

Therefore, no point $(u, v) \in I \times J$ is a singular point of this ruled surface $X(u, v)$, which implies that $X(u, v)$, a rectifying developable of its base curve α, is a regular ruled surface. $\square$

We finish this section by stating the main theorem that characterizes rectifying developable surfaces via the geodesics on them in the sense that any regular ruled surface is rectifying developable of a curve σ if and only if the curve σ is a geodesic of this ruled surface that is transversal to rulings and the Gaussian curvature of the ruled surface vanishes along the curve σ.

Theorem 2.1.1. *Let $\mathcal{M}$ with $X(u,v) = \alpha(u) + v\beta(u)$ be a regular ruled surface with $\|\beta(u)\| = 1$ and let $\sigma(u) = \alpha(u) + v(u)\beta(u)$ be a curve on $X(u,v)$ with its curvature $k(u) \neq 0$, where u is the arc-length of $\sigma(u)$. Then, the following conditions are equivalent:*

a. The surface $X(u,v)$ is rectifying developable of $\sigma(u)$.

b. The curve $\sigma(u)$ is a geodesic of $X(u,v)$ which is transversal to rulings and $X(u,v)$ is a developable surface.

c. The curve $\sigma(u)$ is a geodesic of $X(u,v)$ which is transversal to rulings and the Gaussian curvature of $X(u,v)$ vanishes along $\sigma(u)$.

Proof. To prove that statement $a.$ is equivalent to $b.$ we start with a regular ruled surface $\mathcal{M}$ which is a rectifying developable of $\sigma(u)$, as in $a.$ and we show that $\sigma(u)$ is a geodesic of $\mathcal{M}$ which is transversal to rulings, thus it also implies that $\mathcal{M}$ is a developable surface.

Let $X(u,v) = \sigma(u) + v\tilde{D}(u)$ be the rectifying developable surface of $\sigma(u)$, so $X_u(u,v) = \sigma'(u) + v\tilde{D}'(u)$ and $X_v(u,v) = \tilde{D}(u)$. Consider $\sigma''(u) \cdot X_v(u,v)$:

$$\begin{aligned}
\sigma''(u) \cdot X_v(u,v) &= \sigma''(u) \cdot \tilde{D}(u) \\
&= [k(u)\mathbf{n}(u)] \cdot \left[\left(\frac{\tau}{k}\right)(u)\mathbf{t}(u) + \mathbf{b}(u)\right] \\
&= \left(\frac{\tau}{k}\right)(u)k(u)[\mathbf{n}(u) \cdot \mathbf{t}(u)] + k(u)[\mathbf{n}(u) \cdot \mathbf{b}(u)] \\
&= \tau(u)[\mathbf{n}(u) \cdot \mathbf{t}(u)] + k(u)[\mathbf{n}(u) \cdot \mathbf{b}(u)] \\
&= 0 + 0 = 0.
\end{aligned}$$

Hence, $\sigma''(u)$ is perpendicular to $X_v(u,v)$. Similarly, we will show $\sigma''(u) \cdot X_u(u,v) = 0$ as well using the Frenet-Serret formulas and the equation (2.12) of the proof of Lemma

2.1.1. We have

$$
\begin{aligned}
\sigma''(u) \cdot X_u(u,v) &= \sigma''(u) \cdot [\sigma'(u) + v\tilde{D}'(u)] \\
&= 0 + v[\sigma''(u) \cdot \tilde{D}'(u)] \\
&= \; v[k(u)\mathbf{n}(u)] \cdot \left[\frac{\tau}{k(u)})'(u)\mathbf{t}(u)\right] \\
&= \; vk(u)\left(\frac{\tau}{k}\right)'(u)[\mathbf{n}(u) \cdot \mathbf{t}(u)] \\
&= 0.
\end{aligned}
$$

Thus, $\sigma''(u)$ is perpendicular to $X_u(u,v)$ as well. We can conclude that the acceleration $\sigma''(u)$ of the unit-speed curve σ is perpendicular to the tangent plane of $\mathcal{M}$ at each point $\sigma(u)$. Therefore, $\sigma(u)$ is a geodesic of $X(u,v)$ as in the claim of Theorem 2.1.1.

Moreover, the Darboux vector field of the curve σ is $\tilde{D}(u) = \left(\frac{\tau}{k}\right)(u)\,\mathbf{t}(u) + \mathbf{b}(u)$ which represents also the rulings of the ruled surface $\mathcal{M}$ is always transverse to $\sigma(u)$ because, as seen in the formula above, the modified Darboux vector field contains the tangent and bi-normal vector of the curve $\sigma(u)$. Thus, $\tilde{D}(u)$ is in the rectifying plane of σ, and we conclude that the curve σ is transversal to the rulings of $\mathcal{M}$ and $X(u,v)$ is the parametrization of a developable surface.

Recalling now equation (1.21), the Gauss curvature is $K = -\frac{M^2}{EG - F^2}$, where $M = \tilde{D}'(u) \cdot \left[\frac{(\sigma'(u) + v\tilde{D}'(u)) \times \tilde{D}(u)}{\|(\sigma'(u) + v\tilde{D}'(u)) \times \tilde{D}(u)\|}\right]$, as derived in equation (1.20). Thus, we only need to verify that the numerator of M is zero to conclude that $K = 0$. We have $\tilde{D}' \cdot [(\sigma' + v\tilde{D}') \times \tilde{D}] = \tilde{D}' \cdot [(\sigma' \times \tilde{D}) + v(\tilde{D}' \times \tilde{D})] = \tilde{D}' \cdot [\sigma' \times \tilde{D}]$ because $\tilde{D}' \times \tilde{D}$ is, by the nature of the vectorial product, a vector perpendicular to $\tilde{D}'$. Since σ is a geodesic, then $\sigma'' \perp X_v = \tilde{D}$ and also we have that $\sigma'' \perp \sigma'$, thus we conclude that σ' is parallel to $\tilde{D}$ which implies that $\sigma' \times \tilde{D} = \vec{0}$. Therefore, $\tilde{D}' \cdot [\sigma' \times \tilde{D}] = 0$, which concludes that $K(u,v) = 0$ along the curve σ and $\mathcal{M}$ is a flat ruled surface.

The statement c. follows directly from b. since for any developable ruled surface, we have its Gaussian curvature vanishes everywhere on $\mathcal{M}$. Hence the Gaussian curvature of $X(u,v)$ vanishes along the base curve $\sigma(u)$ as well.

For the last part of the proof, we need to show that $X(u,v) = \alpha(u) + v\beta(u)$ is a rectifying developable of $\sigma(u)$ provided that $\sigma(u)$ is a geodesic of $X(u,v)$ which is transversal to the rulings with vanishing Gaussian curvature along the curve $\sigma(u)$. Since $\sigma(u)$ is transverse to the rulings, and the Gaussian curvature $K(u,v) = 0$ along $\sigma(u)$, we may assume that $\sigma(u) = \alpha(u)$. Therefore, the ruled surface $\mathcal{M}$ can be expressed as $X(u,v) = \sigma(u) + v\beta(u)$. Recall that a curve β is a geodesic of a surface $X(u,v)$ then the direction of β at a point on $X(u,v)$ is the same as the direction of the normal vector at that point as well. As the results, $\beta(u)$ is in a rectifying plane of σ at the point $\sigma(u)$. Note that a rectifying plane is the plane that spanned by the bi-normal $\mathbf{b}(u)$ and tangent vector $\mathbf{t}(u)$, where $\mathbf{n}(u)$ is a normal vector to this plane. Then, there exists $\mu_1(u)$ and $\mu_2(u)$ such that

$$\beta(u) = \mu_1(u)\mathbf{t}(u) + \mu_2(u)\mathbf{b}(u), \tag{2.13}$$

where $\mathbf{t}(u)$ and $\mathbf{b}(u)$ are the tangent and bi-normal vector of $\sigma(u)$, respectively. Then we have

$$\begin{aligned}
\beta' &= \mu_1'\mathbf{t} + \mu_1\mathbf{t}' + \mu_2'\mathbf{b} + \mu_2\mathbf{b}' \\
&= \mu_1'\mathbf{t} + \mu_1 k\mathbf{n} + \mu_2'\mathbf{b} - \mu_2\tau\mathbf{n} \\
&= \mu_1'\mathbf{t} + \mu_2'\mathbf{b} + [\mu_1 k - \mu_2\tau]\mathbf{n}.
\end{aligned}$$

Consider

$$
\begin{aligned}
\beta' \cdot (\sigma' \times \beta) &= (\beta' \times \mathbf{t}) \cdot \beta \\
&= < [\mu_1'\mathbf{t} + \mu_2'\mathbf{b} + (\mu_1 k - \mu_2 \tau)\mathbf{n}] \times \mathbf{t},\ \beta > \\
&= < \vec{0} + \mu_2'(\mathbf{b} \times \mathbf{t}) + [(\mu_1 k - \mu_2 \tau)(\mathbf{n} \times \mathbf{t})],\ \beta > \\
&= < -\mu_2'\mathbf{n} + (\mu_1 k - \mu_2 \tau)\mathbf{b},\ \beta > \\
&= < -\mu_2'\mathbf{n} + (\mu_1 k - \mu_2 \tau)\mathbf{b},\ \mu_1\mathbf{t} + \mu_2\mathbf{b} > \\
&= \ 0 + 0 + 0 - \mu_2(\mu_1 k - \mu_2 \tau) < \mathbf{b},\ \mathbf{b} > \\
&= -\mu_2(\mu_1 k - \mu_2 \tau).
\end{aligned}
$$

Recall that, by hypothesis, the Gaussian curvature $K(u, v) = 0$, that is

$$
K(u, v) = 0 \Leftrightarrow \beta' \cdot (\sigma' \times \beta) = 0,
$$

$$
\beta' \cdot (\sigma' \times \beta) = 0 \Leftrightarrow -\mu_2(\mu_1 k - \mu_2 \tau) = 0.
$$

Therefore, either $\mu_2 = 0$ or $\mu_1 k = \mu_2 \tau$.

On the other hand, from equation (2.13), we have that $\beta(u) = \mu_1(u)\mathbf{t}(u) + \mu_2(u)\mathbf{b}(u)$. If $\mu_2 = 0$, then there exists a point $\beta(u)$ such that $\beta(u) = \mu_1(u)\mathbf{t}(u)$, which contradicts the fact the β is transversal to σ. Hence, $\mu_1 k = \mu_2 \tau$ or $\mu_1 = \dfrac{\mu_2 \tau}{k}$.

From equation (2.13), let us now note that

$$
\begin{aligned}
\beta &= \mu_1\mathbf{t} + \mu_2\mathbf{b} \\
&= \frac{\mu_2 \tau}{k}\mathbf{t} + \mu_2\mathbf{b} \\
&= \mu_2(\frac{\tau}{k}\mathbf{t} + \mathbf{b}).
\end{aligned}
$$

Thus,

$$\beta(u) = \mu_2(u)\tilde{D}(u). \tag{2.14}$$

Therefore, $X(u,v) = \sigma(u) + v\beta(u)$ can be reparametrized by $X(u,\tilde{v}) = \sigma(u) + \tilde{v}\tilde{D}(u)$, where $\tilde{v} = \mu_2 v$, concluding that it is a ruled surface that is rectifying developable of the curve σ. □

With the result from Proposition 2.1.1 and Theorem 2.1.1, we can see that the rectifying developable of a cylindrical helix is a cylindrical surface and also a surface rectifying developable surface of a curve if and only if this curve is a geodesic of the ruled surface. We will use these facts in the next section to draw a connection between cylindrical helices and geodesics on cylindrical ruled surface.

Since the definition of the surface that is rectifying developable of a curve is related to the modified Darboux vector field, from now on, note that we will say only Darboux vector field when referring to the surface parametrization, even if we actually are referring to the modified Darboux vector field. Moreover, for convenience, we can always assume the condition that the norm of the director curve of the ruled surface is equal to 1, by a possible reparametrization of v, and this assumption is applied for the rest of this chapter.

2.2 Geodesics on Cylindrical Ruled Surfaces

In Chapter 1, we have introduced Bonnet's theorem (Theorem 1.3.3) which relates the line of striction of a non-cylindrical ruled surface to a geodesic of this ruled surface. However, for cylindrical ruled surfaces, their geodesics provide an even stronger characterization as the corollary of Theorem 1.3.3 shows:

Corollary 2.2.1. *Consider a ruled surface $X(u,v) = \alpha(u) + v\beta(u)$ with two distinct geodesics $\sigma_i(u)$, $i = 1,2$, such that the corresponding angles between $\sigma_i'(u)$ and $\beta(u)$*

are constant. Then, the ruled surface $X(u,v)$ is a cylindrical ruled surface and both of $\sigma_i(u)$ are cylindrical helices. Moreover, the direction of $\beta(u)$ is the direction of the Darboux vector of σ_i.

Proof. Suppose σ_1 and σ_2 are two distinct geodesics on the surface $X(u,v)$ such that angle between σ_1 and β, and also, the angle between σ_2 and β is constant. By Theorem 1.3.3, then we have that both σ_1 and σ_2 are two distinct striction curves of this ruled surface $X(u,v) = \alpha(u) + v\beta(u)$. By Remark 1.3.1, if a ruled surface is non-cylindrical, then the surface admits a unique striction curve. However, we have just inferred that the given ruled surface has two distinct striction curves σ_1 and σ_2. We thus conclude that $X(u,v)$ is a cylindrical ruled surface.

We will now prove that the geodesics of the cylindrical ruled surface that satisfy the above conditions are cylindrical helices. Let $i \in \{1,2\}$ be arbitrary, but fixed. Since σ_i is a geodesic of the cylindrical ruled surface, then for any point on σ_i, the direction of the normal vector to σ_i is the same as the direction of the normal vector to the surface at that point i.e. the rectifying plane of σ_i is the tangent plane of $X(u,v)$, at the corresponding point, as well. Since β is the director curve of $X(u,v)$, and σ_i is geodesic, we have by Theorem 2.1.1 that $X(u,v)$ is the surface rectifying developable of the curve σ_i: $X(u,\tilde{v}) = \sigma_i(u) + \tilde{v}\tilde{D}_i(u)$. Therefore, by Proposition 2.1.1, σ_i is a cylindrical helix and the direction of $\beta(u)$ is the direction of the Darboux vector of σ_i. Since i was chosen arbitrarily, the conclusion follows. $\square$

Similarly, we have another characterization of cylindrical surfaces as in the following proposition.

Proposition 2.2.1. *Suppose that $X(u,v) = \alpha(u) + v\beta(u)$ is a regular developable surface. If there exists a cylindrical helix with non-zero curvature on $X(u,v)$ which is a geodesic of $X(u,v)$, then $X(u,v)$ is a cylindrical ruled surface.*

Proof. By hypothesis, $X(u,v)$ is a regular developable ruled surface with a geodesic that is a cylindrical helix where $k \neq 0$, then this cylindrical helix σ must be transverse to the rulings of $X(u, v)$. Thus, by Theorem 2.1.1, $X(u, v)$ is a ruled surface that is rectifying developable of the curve σ. Then, Proposition 2.1.1 implies that $X(u, v)$ is a cylindrical ruled surface. $\square$

Before introducing another characterization of cylindrical ruled surface, we introduce the notion of line of curvature of a surface. Recall our earlier discussion on principal curvatures, the eigenvalues of the Weingarten map. To each principal curvature, we have associate a principal direction, the eigenvector of the Weingarten map. Thus, we have the following definition, [2], [5].

Definition 2.2.1. *A curve γ is a line of curvature of a surface if and only if its tangent at every point is aligned along a principal curvature direction.*

Remark 2.2.1. *A curve α on a surface $\mathcal{M}$ is a line of curvature if and only if $(\boldsymbol{N}(\alpha(u)))' = -k\alpha'(u)$. where $\boldsymbol{N}$ is the standard unit normal vector to surface at a point and k is the corresponding principal curvature. This is obvious since $\mathcal{W}_{\boldsymbol{p}}(\alpha') = k\alpha' = -\boldsymbol{N}' =: -d\boldsymbol{N}$. This formula is called Rodrigue's formula, [5].*

Corollary 2.2.2. *Let $X(u, v) = \alpha(u) + v\beta(u)$ be a regular ruled surface. If there exists $\sigma(u)$, a unit-speed planar geodesic of $X(u, v)$ with non-zero curvature which is perpendicular to the rulings at every point, then $X(u, v)$ is a cylindrical surface.*

Proof. Let $\mathbf{N}$ be a standard unit normal vector to a surface at a point p. Let $\sigma(u)$ be a planar geodesic of $X(u, v)$ with curvature $k \neq 0$. Thus, its torsion $\tau(u)$ is zero. Consider one of the Frenet-Serret formulas of the curve σ, namely

$$\mathbf{n}' = -k\mathbf{t} + \tau\mathbf{b} = -k\mathbf{t}.$$

Since σ is a geodesic of $X(u, v)$, σ'' is parallel to $\mathbf{N}$. We also have that $\sigma'' = k\mathbf{n}$, i.e. σ'' is parallel to $\mathbf{n}$. Then we can conclude that $\mathbf{N}$ is parallel to $\mathbf{n}$, and specifically,

$\mathbf{n} = \pm\mathbf{N}$. Recall that $\mathcal{W}_{\mathbf{p}}(\alpha') = k_1\alpha' = k_1\mathbf{t}$, where k_1 is the principal curvature as mentioned above. But we also have that $\mathcal{W}_{\mathbf{p}}(\alpha') =: -d\mathbf{N} = -\mathbf{N}'$. Therefore, $\mathcal{W}_{\mathbf{p}} = -d\mathbf{N} = -\mathbf{N}' = \pm\mathbf{n}' = \pm k\mathbf{t} = k_1\mathbf{t}$. Thus, we have $k = \pm k_1$ and also $-\mathbf{N}' = -k_1\mathbf{t}$. By Remark 2.2.1, σ is the line of curvature of $X(u, v)$, i.e. the direction of σ is the same as the principal direction. Given that σ is perpendicular to the rulings β at every points, then the direction of the rulings is also the principal direction. But the rulings are lines, so one of the principal curvatures is zero and the Gauss curvature, as the product of principal curvatures, is also zero. Thus, $X(u, v)$ is a developable ruled surface.

Given that σ is a curve in plane, then we can assume that it is a helix since for all plane curves, we have tnat $\frac{\tau}{k}(u)$ is constant for all value of u. By Proposition 2.2.1, $X(u, v)$ is a cylindrical ruled surface.

□

In the next section, we introduce another type of ruled surface, the principal normal ruled surface of a curve.

2.3 Principal Normal Ruled Surface of a Curve

The principal normal ruled surface of a curve α is defined by α as the base curve and the normal vector of α as the director curve of this ruled surface, with the rigorous definition to follow shortly.

We open this section by studying Bertrand curves with the goal of presenting the main result of Izumiya and Takeuchi, [4], in which Bertrand curves are special curves on ruled surfaces. By the construction of the principal normal ruled surface of a curve α, we will show that this specific ruled surface has a connection between a Bertrand curve and its mean curvature, similar with the ruled surface that is rectifying developable of a curve, as seen in previous section, that has a connection to a circular helix and its Gaussian curvature. We finish this section by examining the properties of the principal normal surface of a curve such as the singularity and regularity conditions, its mean curvature and lastly the minimal locus of a principal normal surface of a Bertrand curve.

2.3.1 Bertrand Curves and Their Properties

Bertrand curves, or Bertrand mates, are space curves associated to each other via their principal normal vector. A Bertrand curve is a curve that admits a Bertrand mate and can be viewed as generalization of the circular helix. Its ratio of torsion to curvature may not be constant, but these two quantities that identify the curve completely satisfy an affine relation as it will be explained later.

Definition 2.3.1. *A space curve $\alpha : I \to \mathbb{R}^3$ with the curvature $k(u) \neq 0$ is a Bertrand curve if and only if there exists a curve $\tilde{\alpha} : I \to \mathbb{R}^3$ such that $\alpha(u)$ and $\tilde{\alpha}(u)$ have the same principal normal line at corresponding points $u \in I$, where I is an open interval in $\mathbb{R}$. In this case, $\tilde{\alpha}(u)$ is called a Bertrand mate or Bertrand conjugate of $\alpha(u)$. Both curves $\alpha(u)$ and $\tilde{\alpha}(u)$ are called a pair of Bertrand curves.*

Next, we explore some fundamental properties of Bertrand curves.

Proposition 2.3.1. *Let $\alpha : I \to \mathbb{R}^3$ be a Bertrand curve with curvature $k \neq 0$, and $\tilde{\alpha} : I \to \mathbb{R}^3$ be its Bertrand mate. Then, the following hold:*

a. The Bertrand mate $\tilde{\alpha}(u)$ can be parametrized by

$$\tilde{\alpha}(u) = \alpha(u) + A\boldsymbol{n}(u), \tag{2.15}$$

where $A \in \mathbb{R}$ is a non-zero constant representing the distance between corresponding points of $\alpha(u)$ and $\tilde{\alpha}(u)$ in the direction of the unit normal vector $\boldsymbol{n}(u)$ of the curve α at u.

b. The angle between the tangent vectors of a Bertrand pair at the corresponding points $\alpha(u)$ and $\tilde{\alpha}(u)$ is constant.

Proof. To prove part a., note that, by definition, since a Bertrand mate of α lies on its normal line, $\tilde{\alpha}$ must be of form

$$\tilde{\alpha}(u) = \alpha(u) + A(u)\mathbf{n}(u),$$

where $A(u)$ represents the distance from corresponding points $\alpha(u)$ and $\tilde{\alpha}(u)$, a priori depending on the parameter $u \in I$. We will show that this distance function $A(u)$ is constant A that does not change depending on the choice of parameter u.
Indeed, let $l(a)$ be the normal line at point $\alpha(u)$ with the vector equation

$$l(a) = \alpha(u) + a\mathbf{n}(u).$$

By hypothesis, $\alpha(u)$ and $\tilde{\alpha}(u)$ are Bertrand mates, which implies that the point $\tilde{\alpha}(u) \in l(a)$ as well. The point $\tilde{\alpha}(u) \in l(a)$ means that there exists $A(u) > 0$, the value of the distance from point $\alpha(u)$ to point $\tilde{\alpha}(u)$, such that

$$\tilde{\alpha}(u) = \alpha(u) + A(u)\mathbf{n}(u). \tag{2.16}$$

We will now show that the distance function $A(u)$ is a constant. We start by taking the derivative with respect to u of the equation (2.16) above to obtain

$$\tilde{\alpha}'(u) = \alpha'(u) + A'(u)\mathbf{n}(u) + A(u)\mathbf{n}'(u).$$

Furthermore, this implies

$$
\begin{aligned}
<\tilde{\alpha}'(u), \mathbf{n}(u)> &= <\alpha'(u), \mathbf{n}(u)> \ + < A'(u)\mathbf{n}(u), \mathbf{n}(u)> \ + < A(u)\mathbf{n}'(u), \mathbf{n}(u)> \\
&= <\alpha'(u), \mathbf{n}(u)> \ + A'(u) < \mathbf{n}(u), \mathbf{n}(u)> \ + A(u) < \mathbf{n}'(u), \mathbf{n}(u)>,
\end{aligned}
$$

but $<\alpha'(u), \mathbf{n}(u)> = 0$ because the unit normal vector $\mathbf{n}(u)$ is perpendicular to the tangent vector $\alpha'(u)$ of α at u. Similarly, $<\tilde{\alpha}'(u), \mathbf{n}(u)> = 0$ since the unit normal vector $\mathbf{n}(u)$ is, by the definition of the Bertrand mate, perpendicular to the tangent vector $\tilde{\alpha}'(u)$ of $\tilde{\alpha}(u)$ at u. As $\mathbf{n}(u)$ is a unit vector field of the curve α, we have $<\mathbf{n}(u), \mathbf{n}(u)> = \|\mathbf{n}(u)\|^2 = 1$, thus $<\mathbf{n}'(u), \mathbf{n}(u)>= 0$ as well. Therefore, $A'(u) = 0$ and $A(u)$ is a constant A, completing the proof of part a. of Proposition 2.3.1.

Before we begin the proof of part b., recall that, by hypothesis, $k(u) \neq 0$. For simplicity of the calculation, we assume that $\alpha(u)$ and $\tilde{\alpha}(\tilde{u})$ are the parametrizations by arc-length u and $\tilde{u}(u)$ respectively. Let $\mathbf{t}(u) = \alpha'(u)$ be the unit tangent vector of α at u and $\tilde{\mathbf{t}}(\tilde{u}) = \tilde{\alpha}'(\tilde{u})$ be the unit tangent vector of $\tilde{\alpha}(\tilde{u})$ at $\tilde{u}$.

Consider the derivative of the dot product below:

$$
\frac{\partial < \mathbf{t}(u), \tilde{\mathbf{t}}(\tilde{u}) >}{\partial u} = < \frac{\partial \mathbf{t}(u)}{\partial u}, \ \tilde{\mathbf{t}}(\tilde{u}) > + < \mathbf{t}(u), \ \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial u} > .
$$

Since α and $\tilde{\alpha}$ are Bertrand mates then $\mathbf{n}(u) = \pm\tilde{\mathbf{n}}(u)$. We have

$$
\begin{aligned}
< \frac{\partial \mathbf{t}(u)}{\partial u}, \ \tilde{\mathbf{t}}(\tilde{u}) > &= < \mathbf{n}(u), \tilde{\mathbf{t}}(\tilde{u}) > \\
&= < \pm\tilde{\mathbf{n}}(u), \ \tilde{\mathbf{t}}(\tilde{u}) > \\
&= < \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial u}, \ \tilde{\mathbf{t}}(\tilde{u}) > \\
&= \frac{\partial \tilde{u}}{\partial u} < \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial \tilde{u}}, \ \tilde{\mathbf{t}}(\tilde{u}) > \\
&= 0.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
< \mathbf{t}(u),\ \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial u} > &= < \mathbf{t}(u),\ \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial u} > \\
&= \frac{\partial \tilde{u}}{\partial u} < \mathbf{t}(u),\ \frac{\partial \tilde{\mathbf{t}}(\tilde{u})}{\partial \tilde{u}} > \\
&= \frac{\partial \tilde{u}}{\partial u} < \mathbf{t}(u),\ \tilde{\mathbf{n}}(u) > \\
&= \frac{\partial \tilde{u}}{\partial u} < \mathbf{t}(u),\ \pm \mathbf{n}(u), > \\
&= 0.
\end{aligned}$$

Therefore,

$$\frac{\partial < \mathbf{t}(u), \tilde{\mathbf{t}}(\tilde{u}(u)) >}{\partial u} = 0$$

implies that

$$< \mathbf{t}(u),\ \tilde{\mathbf{t}}(\tilde{u}) >= c,$$

where c is a constant. In other words, since $\|\mathbf{t}(u)\| = \|\tilde{\mathbf{t}}(\tilde{u})\| = 1$,

$$< \mathbf{t}(u),\ \tilde{\mathbf{t}}(\tilde{u}) > = \ \|\mathbf{t}(u)\| \|\tilde{\mathbf{t}}(\tilde{u})\| \cos\theta = \cos\theta = c,$$

where θ is the angle between the tangent vectors of α and $\tilde{\alpha}$ at u.

Hence, we have proved that the angle between the two tangent vectors of α and $\tilde{\alpha}$ at corresponding points is constant. □

The next proposition proposes the affine characterization of a Bertrand curve mentioned in the introduction of this section.

Proposition 2.3.2. *Suppose that I is an open interval in $\mathbb{R}$ and $\alpha : I \to \mathbb{R}^3$ is a space curve with curvature $k(u) \neq 0$, and torsion $\tau(u) \neq 0$. Then, the following statements are true:*

i. $\alpha(u)$ is a Bertrand curve if and only if there exist nonzero real numbers A, B such *that* $Ak(u) + B\tau(u) = 1$*, for all* $u \in I$*.*

ii. $\alpha(u)$ *is a Bertrand curve if and only if there exists nonzero real number* A *such that* $A[(\tau'(u)k(u) - k'(\alpha(u))\tau(u)] - \tau'(u)) = 0$*, for all* $u \in I$*.*

iii. If $\alpha(u)$ *is a Bertrand curve, and* $\tilde{\alpha}(u)$ *is its Bertrand mate, then* $\tau(u)\tilde{\tau}(u)$ *is a non-negative constant, where* $\tilde{\tau}(u)$ *is the torsion of* $\tilde{\alpha}$ *at* $\tilde{\alpha}(u)$*.*

Proof. As before, for the simplicity of calculations, we may assume that $\alpha(u)$ and $\tilde{\alpha}(\tilde{u})$ are parametrized by arc-length u and $\tilde{u}$, respectively.

To prove the necessary condition for part *i.*, consider the dot product of tangent vectors of the Bertrand mates α and $\tilde{\alpha}$ at point u that was calculated earlier, in the form

$$< \mathbf{t},\ \tilde{\mathbf{t}} >= \left\langle \mathbf{t},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} \right\rangle = \cos\theta. \tag{2.17}$$

By hypothesis, α and $\tilde{\alpha}$ are Bertrand mates, then let $\tilde{\alpha}(u) = \alpha(u) + A\mathbf{n}(u)$, where $A > 0$ is a constant. Then

$$\begin{aligned} \frac{\partial \tilde{\alpha}}{\partial u} &= \frac{\partial \alpha}{\partial u} + A\frac{\partial \mathbf{n}}{\partial u} \\ &= \mathbf{t} + A(-k(\alpha)\mathbf{t} + \tau\mathbf{b}) \\ &= \mathbf{t} - Ak\mathbf{t} + A\tau\mathbf{b}. \end{aligned}$$

Thus,

$$\frac{\partial \tilde{\alpha}}{\partial u} = (1 - Ak)\mathbf{t} + A\tau\mathbf{b}. \tag{2.18}$$

Consider $\left\langle \mathbf{t},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} \right\rangle$.

We have $\left\langle \mathbf{t},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} \right\rangle = \left\langle \mathbf{t},\ \frac{\partial \tilde{\alpha}}{\partial u} \right\rangle \frac{\partial u}{\partial \tilde{u}}$.

By equation (2.18), we have that

$$
\begin{aligned}
\left\langle \mathbf{t},\ \frac{\partial\tilde{\alpha}}{\partial u}\frac{\partial u}{\partial\tilde{u}}\right\rangle &= <\mathbf{t},\ (1-Ak)\mathbf{t}+A\tau\mathbf{b}>\frac{\partial u}{\partial\tilde{u}}\\
&=<\mathbf{t},\ (1-Ak)\mathbf{t}>\frac{\partial u}{\partial\tilde{u}}+<\mathbf{t},A\tau\mathbf{b}>\frac{\partial u}{\partial\tilde{u}}\\
&=<\mathbf{t},\ \mathbf{t}>(1-Ak)\frac{\partial u}{\partial\tilde{u}}+0\\
&=(1-Ak)\frac{\partial u}{\partial\tilde{u}},
\end{aligned}
$$

concluding, from equation (2.17), that

$$\cos\theta = (1-Ak)\frac{\partial u}{\partial\tilde{u}}. \tag{2.19}$$

Consider now

$$
\begin{aligned}
\mathbf{t}\times\tilde{\mathbf{t}} = \ &\mathbf{t}\times\left(\frac{\partial\tilde{\alpha}}{\partial u}\frac{\partial u}{\partial\tilde{u}}\right)\\
&=\left[\mathbf{t}\times((1-Ak)\mathbf{t}+A\tau\mathbf{b})\right]\frac{\partial u}{\partial\tilde{u}}\\
&=\left[\mathbf{t}\times(1-Ak)\mathbf{t})\right]\frac{\partial u}{\partial\tilde{u}}+\left[\mathbf{t}\times(A\tau\mathbf{b})\right]\frac{\partial u}{\partial\tilde{u}}\\
&=0+A\tau(\mathbf{t}\times\mathbf{b})\frac{\partial u}{\partial\tilde{u}},
\end{aligned}
$$

thus

$$\|\mathbf{t}\times\tilde{\mathbf{t}}\| = A\tau \mid \frac{\partial u}{\partial\tilde{u}} \mid \|\mathbf{t}\|\|\mathbf{b}\|\sin\frac{\pi}{2} = A\tau\frac{\partial u}{\partial\tilde{u}},$$

where we have assumed, without any loss of generality, that $\frac{\partial u}{\partial\tilde{u}} > 0$. On the other hand,

$$\|\mathbf{t}\times\tilde{\mathbf{t}}\| = \|\mathbf{t}\|\|\tilde{\mathbf{t}}\|\sin\theta = \sin\theta,$$

thus

$$\sin\theta = A\tau \frac{\partial u}{\partial \tilde{u}}. \tag{2.20}$$

Dividing equation (2.20) and (2.19), we have that

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1 - Ak}{A\tau}. \tag{2.21}$$

By Proposition 2.3.1.b., the angle between $\mathbf{t}$ and $\tilde{\mathbf{t}}$ at a corresponding point is constant which implies that $\cot\theta$ is constant, so

$$\begin{aligned} \frac{1 - Ak}{A\tau} &= \text{Constant}, \\ \frac{1 - Ak}{\tau} &=: B, \\ 1 - Ak &= B\tau \\ Ak + B\tau &= 1, \end{aligned}$$

where B is, by defintion, a constant.

Next, we prove the sufficient condition of 2.3.2.i. Let $\tilde{\alpha}(u) = \alpha(u) + A\mathbf{n}(u)$ be a space curve in $\mathbb{R}^3$ where A is the constant from hypothesis such that for A, and a constant B, we have that the curve α satisfies the condition $Ak(u) + B\tau(u) = 1$. We will show that $\tilde{\alpha}$ is the Bertrand mate of α. To assist with the calculations, we assume that α is parametrized by the arc-length u, while, as before, $\tilde{\alpha}(u)$ is not necessary an arc-length parametrization. From equation (2.18), we have

$$\begin{aligned} \tilde{\alpha}_u = \frac{\partial\tilde{\alpha}}{\partial u} &= (1 - Ak(u))\mathbf{t}(u) + A\tau(u)\mathbf{b}(u) \\ &= B\tau(u)\mathbf{t}(u) + A\tau(u)\mathbf{b}(u), \end{aligned}$$

where $k(u), \tau(u), \mathbf{b}(u)$, and $\mathbf{t}(u)$ are the curvature, torsion, unit bi-normal vector, and unit tangent vector of α, respectively. Let $\tilde{\mathbf{t}}(u)$ be the unit tangent vector of the curve

$\tilde{\alpha}_u$ (with one of the two choices of orientation):

$$\tilde{\mathbf{t}}(u) = \frac{\tau(u)[B\mathbf{t}(u) + A\mathbf{b}(u)]}{\sqrt{(B\tau(u))^2 + (A\tau(u))^2}} = \frac{\tau(u)[B\mathbf{t}(u) + A\mathbf{b}(u)]}{\tau(u)\sqrt{A^2 + B^2}}$$

Hence,

$$\tilde{\mathbf{t}}(u) = \frac{A\mathbf{b}(u) + B\mathbf{t}(u)}{\sqrt{A^2 + B^2}}, \tag{2.22}$$

and, consequently,

$$\frac{\partial\tilde{\mathbf{t}}(u)}{\partial u} = \frac{1}{\sqrt{A^2 + B^2}}\left[B\frac{\partial\mathbf{t}(u)}{\partial u} + A\frac{\partial\mathbf{b}(u)}{\partial u)}\right],$$

and, by Frenet-Serret formulas,

$$\frac{\partial\tilde{\mathbf{t}}(u)}{\partial u} = \frac{1}{\sqrt{A^2 + B^2}}[Bk(u)\mathbf{n}(u) - A\tau(u)\mathbf{n}(u)]. \tag{2.23}$$

Denoting by $\tilde{u}$ the arc-length of $\tilde{\alpha}$, we have

$$\frac{\partial\tilde{\mathbf{t}}(u)}{\partial u} = \frac{\partial\tilde{\mathbf{t}}(u)}{\partial\tilde{u}}\frac{\partial\tilde{u}}{\partial u},$$

and, since $\dfrac{\partial\tilde{\mathbf{t}}(u)}{\partial\tilde{u}} = \tilde{k}(u)\tilde{\mathbf{n}}(u)$, once more by Frenet-Serret formula applied now to the curve $\tilde{\alpha}$, we conclude that

$$\frac{\partial\tilde{\mathbf{t}}(u)}{\partial u} = \tilde{k}(u)\tilde{\mathbf{n}}(u)\frac{\partial\tilde{u}}{\partial u}. \tag{2.24}$$

We want to evaluate now the expression of the change of variables $\dfrac{\partial\tilde{u}}{\partial u}$, thus recall that

$$\frac{\partial\tilde{\alpha}}{\partial u} = \frac{\partial\tilde{\alpha}}{\partial\tilde{u}}\frac{\partial\tilde{u}}{\partial u} = B\tau(u)\mathbf{t}(u) + A\tau(u)\mathbf{b}(u).$$

Consider

$$\begin{aligned}
< \frac{\partial \tilde{\alpha}}{\partial \tilde{u}},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} > &= < \frac{\partial \tilde{\alpha}}{\partial \tilde{u}},\ B\tau(u)\mathbf{t}(u) + A\tau(u)\mathbf{b}(u) > \frac{\partial u}{\partial \tilde{u}} \\
&= < \tilde{\mathbf{t}},\ B\tau\mathbf{t} > + < \tilde{\mathbf{t}},\ A\tau\mathbf{b} > \frac{\partial u}{\partial \tilde{u}} \\
&= [B\tau < \tilde{\mathbf{t}},\ \mathbf{t} > + A\tau < \tilde{\mathbf{t}},\ \mathbf{b} >]\frac{\partial u}{\partial \tilde{u}}.
\end{aligned}$$

However, we also have that $< \frac{\partial \tilde{\alpha}}{\partial \tilde{u}},\ \frac{\partial \tilde{\alpha}}{\partial \tilde{u}} > = \|\tilde{\mathbf{t}}(\tilde{u})\| = 1 = < \frac{\partial \tilde{\alpha}}{\partial \tilde{u}},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} >$, and $< \tilde{\mathbf{t}},\ \mathbf{t} > = (1 - Ak)\frac{\partial u}{\partial \tilde{u}}$, by equation (2.17) and equation (2.19), and furthermore

$$\begin{aligned}
< \mathbf{b},\ \tilde{\mathbf{t}} > &= < \mathbf{b},\ \frac{\partial \tilde{\alpha}}{\partial u}\frac{\partial u}{\partial \tilde{u}} > \\
&= < \mathbf{b},\ (B\tau\mathbf{t} + A\tau\mathbf{b}) > \frac{\partial u}{\partial \tilde{u}} \\
&= [< \mathbf{b},\ B\tau\mathbf{t} > + < \mathbf{b},\ A\tau\mathbf{b} >]\frac{\partial u}{\partial \tilde{u}} \\
&= A\tau\frac{\partial u}{\partial \tilde{u}}\mathbf{b}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
1 &= \left[B\tau(1 - Ak)\frac{\partial u}{\partial \tilde{u}} + A\tau A\tau\frac{\partial u}{\partial \tilde{u}}\mathbf{b}\right]\frac{\partial u}{\partial \tilde{u}} \\
&= [B^2\tau^2 + A^2\tau^2]\left(\frac{\partial u}{\partial \tilde{u}}\right)^2,
\end{aligned}$$

implies that

$$\left(\frac{\partial \tilde{u}}{\partial u}\right)^2 = (A^2 + B^2)\tau^2.$$

Thus,

$$\frac{\partial \tilde{u}}{\partial u} = \pm\tau\sqrt{A^2 + B^2}. \tag{2.25}$$

Using equation (2.25) into equation (2.24), we obtain the following equality

$$\frac{\partial \tilde{\mathbf{t}}(u)}{\partial u} = \tilde{k}(u)\tilde{\mathbf{n}}(u)\frac{\partial \tilde{u}}{\partial u}$$

$$\frac{\partial \tilde{\mathbf{t}}(u)}{\partial u} = \pm\tau\sqrt{A^2+B^2}\,\tilde{k}(u)\tilde{\mathbf{n}}(u). \tag{2.26}$$

Equate equation (2.23) and equation (2.26) to conclude

$$\pm\tau\sqrt{A^2+B^2}\tilde{k}(u)\tilde{\mathbf{n}}(u) = \frac{1}{\sqrt{A^2+B^2}}[Bk(u) - A\tau]\mathbf{n}(u).$$

We conclude that, up to scaling, $\tilde{\mathbf{n}}(u) = \pm\mathbf{n}(u)$, thus α and $\tilde{\alpha}$ are Bertrand mates, completing the proof of Proposition 2.3.2.i.

Next, we prove part ii. From Proposition 2.3.2.i., a space curve α is a Bertrand curve if and only if there exist constants $A, B \neq 0$ such that, for all $u \in I,\ Ak+B\tau = 1$, thus, since $\tau \neq 0$,

$$B = \frac{1-Ak}{\tau}$$

and so, differentiating with respect to u,

$$0 = \frac{-Ak'\tau - (1-Ak)\tau'}{\tau^2}$$
$$-A\tau k' - \tau' + Ak\tau' = 0,$$

with k, τ as functions of u. Hence, α is a Bertrand curve if and only if there exists a non-zero constant A such that $A[\tau'(u)k(u) - k'(u)\tau(u)] - \tau'(u) = 0$, for all $u \in I$.

Lastly, we prove part iii. From equation (2.18),

$$\tilde{\mathbf{t}} = \frac{\partial\tilde{\alpha}}{\partial\tilde{u}} = \frac{\partial\tilde{\alpha}}{\partial u}\frac{\partial u}{\partial\tilde{u}} = ((1-Ak)\mathbf{t} + A\tau\mathbf{b})\frac{\partial u}{\partial\tilde{u}}, \tag{2.27}$$

which, further combined with equations (2.19) and (2.20), implies

$$\tilde{\mathbf{t}} = \cos\theta\mathbf{t} + \sin\theta\mathbf{b}. \tag{2.28}$$

Consider

$$\begin{aligned}\tilde{\mathbf{b}} &= \tilde{\mathbf{t}} \times \tilde{\mathbf{n}} \\ &= (\cos\theta\mathbf{t} + \sin\theta\mathbf{b}) \times \tilde{\mathbf{n}} \\ &= (\cos\theta\mathbf{t} + \sin\theta\mathbf{b}) \times \mathbf{n} \\ &= \cos\theta\mathbf{b} - \sin\theta\mathbf{t}.\end{aligned}$$

Differentiating with respect to u both sides of $\tilde{\mathbf{b}} = \cos\theta\mathbf{b} - \sin\theta\mathbf{t}$, we obtain

$$\frac{\partial\tilde{\mathbf{b}}}{\partial u} = \cos\theta\mathbf{b}' - \sin\theta\mathbf{t}' = -\tau\mathbf{n}\cos\theta - k\mathbf{n}\sin\theta = (-\tau\cos\theta - k\sin\theta)\,\mathbf{n}.$$

However, from equation (2.21),

$$\begin{aligned}\cot\theta &= \frac{1-Ak}{A\tau}, \\ A\tau\cot\theta + Ak &= 1, \\ \tau\cos\theta + k\sin\theta &= \frac{\sin\theta}{A}.\end{aligned}$$

Thus,

$$\frac{\partial\tilde{\mathbf{b}}}{\partial u} = (-\tau\cos\theta - k\sin\theta)\,\mathbf{n} = -\frac{\sin\theta}{A}\,\mathbf{n}, \tag{2.29}$$

but, on the other hand,

$$\frac{\partial\tilde{\mathbf{b}}}{\partial u} = \frac{\partial\tilde{\mathbf{b}}}{\partial\tilde{u}}\frac{\partial\tilde{u}}{\partial u} = -\tilde{\tau}\tilde{\mathbf{n}}\frac{\partial\tilde{u}}{\partial u} = -\tilde{\tau}\frac{A\tau}{\sin\theta}\,\mathbf{n}, \tag{2.30}$$

using again that $\sin\theta = A\tau\dfrac{\partial u}{\partial\tilde{u}}$, by equation (2.20). Equating equations (2.29) and

(2.30), and using the parts $i., ii.$ previously proved, there exists a constant $A \neq 0$ and a constant angle $\theta \in (0, \pi)$ such that $\dfrac{\sin\theta}{A} = \tilde{\tau}\dfrac{A\tau}{\sin\theta}$. Thus, for all θ such that $\theta \in (0, \pi)$,

$$\tilde{\tau}\tau = \frac{(\sin\theta)^2}{A^2} > 0, \tag{2.31}$$

concluding the proof of part $iii.$ that the product of the corresponding torsions of two Bertrand mates is a non-negative constant. $\square$

The last property of Bertrand curves suggests a deeper connection between Bertrand curves and circular helices illustrated by the following proposition.

Lemma 2.3.1. *Given a space Bertrand curve $\alpha : I \to \mathbb{R}^3$ with curvature $k(u) \neq 0$ and $\tau(u) \neq 0$, for all u, the following statements hold:*

1. *If $\alpha(u)$ admits more than one Bertrand mates, then it admits infinitely many Bertrand mates.*

2. *The curve $\alpha(u)$ admits infinitely many Bertrand mates if and only if $\alpha(u)$ is a circular helix.*

Proof. Assume the hypothesis of part 1., namely that a Bertrand curve α admits more than one Bertrand mate. In particular, let us assume that $\alpha(u)$ has two distinct Bertrand mates $\alpha_1(u)$ and $\alpha_2(u)$, respectively,

$$\alpha_1(u) = \alpha(u) + A_1\mathbf{n}(u)$$
$$\alpha_2(u) = \alpha(u) + A_2\mathbf{n}(u),$$

where $A_1 \neq A_2$ are non-zero constants. Since the torsion of the curve is non-zero, we have, by equation (2.21), that

$$\frac{1 - Ak}{A\tau} = \cot\theta = \text{Constant},$$

furthermore, by the lemma's assumption, we then have that two constants $C_1 \neq C_2$ such that

$$\begin{cases} \dfrac{1 - A_1 k}{A_1 \tau} = C_1 \\ \dfrac{1 - A_2 k}{A_2 \tau} = C_2 \end{cases} \Rightarrow \begin{cases} -A_1 k + 1 = A_1 C_1 \tau \\ \\ -A_2 k + 1 = A_2 C_2 \tau \end{cases}$$

and, by differentiating,

$$\begin{cases} -A_1 k'(u) = A_1 C_1 \tau'(u) \\ \\ -A_2 k'(u) = A_2 C_2 \tau'(u) \end{cases} \Rightarrow \begin{cases} k'(u) = -C_1 \tau'(u) \\ \\ k'(u) = -C_2 \tau'(u). \end{cases}$$

Given that $C_1 \neq C_2$, we can infer that $k'(u) = \tau'(u) = 0$, for all u. Thus, k and τ are constant.

Since α has k and τ constant, we can conclude that there are infinitely many pairs of real numbers $(A, B) \neq (0, 0)$ such that $Ak + B\tau = 1$, which is equivalent to the fact that α admits infinitely many Bertrand mates as well (each at distance A from α along its normal vector field).

Proving the necessary condition for Lemma 2.3.1.2. is straight forward from the proof of Lemma 2.3.1.1. Having constant curvature and torsion, the Bertrand curve α is a circular helix.

For the sufficient condition, let α be a circular helix. Since α is a circular helix, then its curvature and torsion k, τ are constant. There is a *line* of points (A, B) such that $Ak + b\tau = 0$, thus α admits infinitely many Bertrand mates. $\square$

In the following section, we will define and investigate the properties of a principal normal surface of a curve α, which will be used later to determine the characterization of the asymptotic curve of ruled surfaces in general.

2.3.2 Properties of Ruled Surfaces that are Principal Normal Surface of a Curve

Definition 2.3.2. *A ruled surface $\mathcal{M}$ with a parametrization of the form of*

$$X(u,v) = \alpha(u) + v\boldsymbol{n}(u) \tag{2.32}$$

is defined to be the principal normal surface along a curve α in $\mathbb{R}^3$ if the rulings are given by the principal normal vector $\boldsymbol{n}(u)$ of the curve at $\alpha(u)$.

Remark 2.3.1. *Note that the principal normal surface of a curve α is a cylindrical ruled surface since $\boldsymbol{n}(u) \times \boldsymbol{n}'(u) = \vec{0}$ at all points.*

In the following lemma, we will study the singularities of the principal normal surface of a curve.

Lemma 2.3.2. *Let $\mathcal{M}$ be a principal normal ruled surface with a parametrization of the form of $X(u,v) = \alpha(u) + v\boldsymbol{n}(u)$ as in the equation (2.32), where $\alpha(u)$ is a unit speed curve with curvature $k(u) \neq 0$. Then $(u_0,\ v_0)$ is a singular point of $\mathcal{M}$ if and only if $\tau(u_0) = 0$ and $v_0 = \dfrac{1}{k(u_0)}$.*

Proof. As mentioned several times earlier, we only need to check that

$$\frac{\partial X}{\partial u}(u_0,v_0) \times \frac{\partial X}{\partial v}(u_0,v_0) = \vec{0},$$

where $X(u,v) = \alpha(u) + v\mathbf{n}(u)$ as in equation (2.32).

If (u_0,v_0) is a singular point of $\mathcal{M}$, then

$$\begin{aligned}\frac{\partial X}{\partial u}(u_0,v_0) \times \frac{\partial X}{\partial v}(u_0,v_0) &= [\alpha'(u_0) + v_0\mathbf{n}'(u_0)] \times \mathbf{n}(u_0) \\ &= [\mathbf{t}(u_0) \times \mathbf{n}(u_0)] + v_0\,[\mathbf{n}'(u_0) \times \mathbf{n}(u_0)] = \vec{0}.\end{aligned}$$

Using Frenet's equations, we have further that

$$[\mathbf{t}(u_0) \times \mathbf{n}(u_0)] + v_0 [-k(u_0)\mathbf{t}(u_0) + \tau(u_0)\mathbf{b}(u_0)] \times \mathbf{n}(u_0) = \vec{0},$$

$$[1 - v_0 k(u_0)][\mathbf{t}(u_0) \times \mathbf{n}(u_0)] + v_0 \tau(u_0)[\mathbf{b}(u_0) \times \mathbf{n}(u_0)] = \vec{0},$$

$$[1 - v_0 k(u_0)]\mathbf{b}(u_0) - v_0 \tau(u_0)\mathbf{t}(u_0) = \vec{0}.$$

By successively, taking the dot product of the above equality with $\mathbf{t}(u_0)$, then $\mathbf{b}(u_0)$, we conclude that (u_0, v_0) is a singular point of the surface if $\tau(u_0) = 0$ and $v_0 = \dfrac{1}{k(u_0)}$.

Note from above that, for arbitrary (u, v), we have

$$\frac{\partial X}{\partial u}(u,v) \times \frac{\partial X}{\partial v}(u,v) = [1 - vk(u)]\mathbf{b}(u) - v\tau(u)\mathbf{t}(u). \tag{2.33}$$

Thus, for (u_0, v_0) with $\tau(u_0) = 0$ and $v_0 = \dfrac{1}{k(u_0)}$, we have $\dfrac{\partial X}{\partial u} \times \dfrac{\partial X}{\partial v}(u_0, v_0) = \vec{0}$, concluding the sufficiency condition. This completes the proof of the lemma. $\square$

Remark 2.3.2. *From this lemma we can conclude that a principal normal surface of a curve $X(u,v)$ is regular if and only if the directrix curve has nonzero torsion everywhere.*

Furthermore, the above lemma gives an immediate characterization of a singular principal normal surface of a Bertand curve in our Euclidean space.

Corollary 2.3.1. *Let $\alpha(u) : I \mapsto \mathbb{R}^3$ be a Bertrand curve. The principal normal surface of $\alpha(u), X(u,v) = \alpha(u) + v\boldsymbol{n}(u)$, has a singular point if and only if $\alpha(u)$ is a planar curve.*

Proof. Suppose (u_0, v_0) is a singular point of $X(u,v) = \alpha(u) + v\mathbf{n}(u)$, then $\tau(u_0) = 0$. Since α is a Bertrand curve, let $\tilde{\alpha}$ be its Bertrand mate with $\tilde{\tau}$ its torsion. Then $\tau(u_0)\tilde{\tau}(u_0) = 0$ at the point u_0. Suppose that $\alpha(u)$ is a space curve in $\mathbb{R}^3$. Since α is a Bertrand curve, then by the equation (2.31) of Proposition 2.3.2.iii, we have that for all u, $\tau(u)\tilde{\tau}(u) > 0$. However, their exists a point u_0 such that $\tau(u_0)\tilde{\tau}(u_0) = 0$, which contradicts Proposition 2.3.2.iii. Thus α must be a plane curve.

Conversely, we need to show that if $\alpha(u)$ is a planar curve, then $X(u,v) = \alpha(u)+v\mathbf{n}(u)$ is a singular ruled surface. Recall that a singular point of a principal normal ruled surface corresponds to the point $u_0 \in I$ such that $\tau(u_0) = 0$ and $v_0 = \dfrac{1}{k(u_0)}$ by Lemma 2.3.2. Given that $\alpha(u)$ is a planar curve, then $\tau(u) = 0$ for all u. By equation (2.33) where, $\dfrac{\partial X}{\partial u}(u,v) \times \dfrac{\partial X}{\partial v}(u,v) = [1 - vk(u)]\mathbf{b}(u) - v\tau(u)\mathbf{t}(u)$, we are able to find at least a point (u_0, v_0) where $\tau(u_0) = 0$ and $v_0 = \dfrac{1}{k(u_0)}$ that satisfies the equation $\dfrac{\partial X}{\partial u}(u,v) \times \dfrac{\partial X}{\partial v}(u,v) = 0$. Thus, $X(u,v) = \alpha(u) + v\mathbf{n}(u)$ is a singular ruled surface. $\square$

The next lemma gives us the formula of the mean curvature of the principal normal surface of a curve.

Lemma 2.3.3. *Given a principal normal ruled surface $\mathcal{M}$ with parametrization $X(u,v) = \alpha(u) + v\boldsymbol{n}(u)$ as in equation (2.32), where u is the arc-length of $\alpha(u)$, the mean curvature of the surface at $X(u,v)$ is equal to*

$$H(u,v) = \frac{v[\tau'(u) + v(k'(u)\tau(u) - \tau'(u)k(u))]}{2(EG - F^2)^{\frac{3}{2}}(u,v)}, \tag{2.34}$$

where the curvature and torsion of α, respectively, k, τ, are non-zero, and E, F, G are the coefficients of the first fundamental form of the surface with the metric induced by $X(u,v)$.

Proof. Given $X(u,v) = \alpha(u)+v\mathbf{n}(u)$, then we have $X_u(u,v) = \alpha'(u)+v\mathbf{n}'(u) = \mathbf{t}(u)+ v\mathbf{n}'(u)$, $X_v(u,v) = \mathbf{n}(u)$, $X_{uu}(u,v) = k\mathbf{n} + v\mathbf{n}''(u)$, $X_{vv}(u,v) = 0$, $X_{uv}(u,v) = \mathbf{n}'(u)$. For simplicity of exposition, we will drop the argument u whenever we consider that there is no risk of confusion. Thus, we have

$$E = \|X_u\|^2 = \|\alpha' + v\mathbf{n}'\|^2$$
$$G = \|X_v\|^2 = \|\mathbf{n}\|^2 = 1$$
$$F = \ X_u \cdot X_v \ = \ (\alpha' + v\mathbf{n}') \cdot \mathbf{n} = 0.$$

Next, we compute the standard unit normal vector $\mathbf{N}$ to $\mathcal{M}$ at point $X(u, v)$, where $EG - F^2 = \|X_u \times X_v\|^2$ by equation (1.22). We have

$$\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \ \frac{[1 - vk]\mathbf{b} - v\tau\mathbf{t}}{\sqrt{EG - F^2}},$$

since $X_u \times X_v = [1 - vk(u)]\mathbf{b}(u) - v\tau(u)\mathbf{t}(u)$ by equation (2.33).

The coefficients of the second fundamental form of $\mathcal{M}$ are:

$$L = X_{uu} \cdot \mathbf{N} = \frac{(\alpha'' + v\mathbf{n}'') \cdot [(1 - vk)\mathbf{b} - v\tau\mathbf{t}]}{\sqrt{EG - F^2}},$$
$$M = X_{uv} \cdot \mathbf{N} = \frac{\mathbf{n}' \times (\alpha' \cdot \mathbf{n})}{\sqrt{EG - F^2}} = 0,$$
$$N = X_{vv} \cdot \mathbf{N} = \vec{0} \cdot \mathbf{N} = 0.$$

From equation (1.47), we have $H = \dfrac{LG - 2MF + NE}{2(EG - F^2)} = \dfrac{L}{2(EG - F^2)}$. Using Frenet's equations, we see that the numerator of L is:

$$(\alpha'' + v\mathbf{n}'') \cdot [(1 - vk)\mathbf{b} - v\tau\mathbf{t}] = [k\mathbf{n} + v(-k\mathbf{t} + \tau\mathbf{b})'] \cdot [(1 - vk)\mathbf{b} - v\tau\mathbf{t}]$$

$$= v(-k'\mathbf{t} - k\mathbf{t}' + \tau'\mathbf{b} + \tau\mathbf{b}') \cdot [(1 - vk)\mathbf{b} - v\tau\mathbf{t}] = v^2k'\tau + vk(1 - vk)\mathbf{t}' \cdot \mathbf{b} + v\tau'(1 - vk) - v^2\tau^2\mathbf{b}' \cdot \mathbf{t}.$$

Moreover, since $\mathbf{t} \cdot \mathbf{b} = 0$, we have $\mathbf{t}' \cdot \mathbf{b} = -\mathbf{t} \cdot \mathbf{b}' = 0$.

Thus,

$$
\begin{aligned}
H &= \frac{v^2k'\tau + (1-vk)v\tau'}{2(EG-F^2)^{\frac{3}{2}}} \\
&= \frac{v^2k'\tau + \tau'v - v^2\tau'k}{2(EG-F^2)^{\frac{3}{2}}} \\
&= \frac{v(\tau' + vk'\tau - v\tau'k)}{2(EG-F^2)^{\frac{3}{2}}} \\
&= \frac{v[\tau' + v(k'\tau - \tau'k)]}{2(EG-F^2)^{\frac{3}{2}}}.
\end{aligned}
$$

Hence, we have obtained the formula for the mean curvature of a principal normal ruled surface. $\square$

For the rest of this section, we define a minimal locus of a ruled surface and proceed to examine the minimal locus of the principal normal surface.

Definition 2.3.3. *A curve $\alpha : I \to \mathbb{R}^3$ of a ruled surface $\mathcal{M}$ with parametrization $X(u,v) = \alpha(u) + v\beta(u)$ is called a minimal locus of $X(u,v)$ if and only if the mean curvature of $X(u,v)$ vanishes on $\alpha(u)$.*

Proposition 2.3.3. *Let α and $\tilde{\alpha}$ be non-planar Bertrand mates. Then $\tilde{\alpha}$ is the minimal locus of $\mathcal{M}$, the principal normal surface of α with parametrization $X(u,v) = \alpha(u) + v\boldsymbol{n}(u)$.*

Proof. Suppose that $\tilde{k}, \tilde{\tau}$ are the curvature and torsion of the Bertrand curve $\tilde{\alpha}$. Given that α and $\tilde{\alpha}$ are Bertrand mates, they both have the same normal direction, and by Proposition 2.3.1, we have $\tilde{\alpha} = \alpha + A\mathbf{n}$, where A is a non-zero constant. We then can reparametrize $\mathcal{M}$, the principal normal surface of α, of the form of $X(u,v) = \alpha(u) + v\mathbf{n}(u)$, as $X(u,A) = \tilde{\alpha}(u) + A\mathbf{n}(u)$. Thus, by equation (2.34), the mean curvature of $\mathcal{M}$ is

$$
H(\tilde{\alpha}) = H(u,A) = \frac{A[\tilde{\tau}' + A(\tilde{k}'\tilde{\tau} - \tilde{\tau}'\tilde{k})]}{2(EG-F^2)^{\frac{3}{2}}}.
$$

However, by Proposition 2.3.2.ii, one of the properties of Bertrand curve, there exists a non-zero constant A such that $A(\tilde{\tau}'\tilde{k}-\tilde{k}'\tilde{\tau})-\tilde{\tau}'=0$, for all $u \in I$, which is the same as having $\tilde{\tau}' + A(\tilde{k}'\tilde{\tau} - \tilde{\tau}'\tilde{k}) = 0$. Thus $H(\tilde{\alpha}) = H(u, A) = 0$ for all $u \in I$, concluding that $\tilde{\alpha}$ is the minumal locus of $\mathcal{M}$. □

The next section will focus on asymptotic curves on the ruled surfaces in general, which in turn will enable us to investigate asymptotic curves on ruled surfaces that are the principal normal surfaces of a curve.

2.4 Asymptotic Curves on Ruled Surfaces

2.4.1 Asymptotic Curves

Recall that for any unit-speed curve $\alpha : I \to \mathbb{R}^3$ with curvature at point $\alpha(u)$ being $k(u) = \|\alpha''(u)\|$, the unit principal normal vector of α at a point $\alpha(u)$ is $\mathbf{n}(u)$ such that $\alpha''(u) = k(u)\mathbf{n}(u)$. Suppose that the curve α is on a surface $\mathcal{M}$ and denote the unit normal vector of the surface $\mathcal{M}$ by $\mathbf{N}$. We also define α'' as a linear combination of the vectors $\mathbf{N}$ and $\mathbf{N} \times \alpha'$ as follows

$$\alpha'' = k_n\mathbf{N} + k_g(\mathbf{N} \times \alpha'), \tag{2.35}$$

where $k_n = \alpha'' \cdot \mathbf{N} = k\cos\phi$ and $k_g = \alpha'' \cdot (\mathbf{N} \times \alpha') = k\sin\phi$ are called the normal curvature and, respectively, the geodesic curvature of the curve α. Here, ϕ is the angle between $\mathbf{N}$ and $\mathbf{n}$ and, from the definition, We note that

$$k^2 = k_n^2 + k_g^2. \tag{2.36}$$

Definition 2.4.1. *A curve $\alpha : I \to \mathbb{R}^3$ on a surface $\mathcal{M}$ is called an asymptotic curve if and only if its normal curvature of $\alpha(u)$ is everywhere zero, in other words,*

$k_n(u) = \alpha''(u) \cdot \boldsymbol{N}(u, \alpha(u)) = 0$ *for all* u *in* I.

Proposition 2.4.1. *Let curve* $\alpha : I \to \mathbb{R}^3 \in \mathcal{M}$ *a regular surface be a unit-speed curve and* $\boldsymbol{N}(p)$ *be the unit normal vector to* $\mathcal{M}$ *at a point* p *of* $\mathcal{M}$. *Then, we have* $\alpha(u)$ *is an asymptotic curve of* $\mathcal{M}$ *if and only if*

$$\alpha'(u) \cdot \boldsymbol{N}'(u, \alpha(u)) = 0, \quad \forall u \in I. \tag{2.37}$$

Proof. Since $\alpha(u) \in \mathcal{M}$ for all $u \in I$, then α' is always tangent to $\mathcal{M}$, so $\alpha' \cdot \mathbf{N} = 0$. We differentiate both side of the equation above to obtain

$$(\alpha' \cdot \mathbf{N})' = 0$$
$$\alpha'' \cdot \mathbf{N} + \alpha' \cdot \mathbf{N}' = 0.$$

By Definition 2.4.1, α is an asymptotic curve if and only if $\alpha'' \cdot \mathbf{N} = 0$. Therefore $\alpha' \cdot \mathbf{N}' = 0$. The converse is immediate from the same argument. □

2.4.2 Characteristics of Asymptotic Curves on Ruled Surfaces

As seen earlier, an asymptotic curve α of a surface has zero principal normal curvature. The lemma below is preparing the proof of the next proposition which will characterize further asymptotic curves on a ruled surface.

Lemma 2.4.1. *[4] Let* $\boldsymbol{e}_1 = (0, 1)$, $\boldsymbol{e}_2 = (1, 0)$ *be the canonical basis of the Euclidean vector space* $\mathbb{R}^2$ *and let* $\boldsymbol{v}_1, \boldsymbol{v}_2$ *be also unit vectors in* $\mathbb{R}^2$ *such that we describe* $\boldsymbol{v}_1 = \lambda(\boldsymbol{e}_1 + \mu\boldsymbol{e}_2)$ *for some constants* $\lambda > 0$ *and* $\mu \in \mathbb{R}$. *Then,* $\boldsymbol{v}_2 = \lambda(\boldsymbol{e}_1 - \mu\boldsymbol{e}_2)$ *if and only if*

$$\boldsymbol{v}_2 \cdot \boldsymbol{e}_1 = \boldsymbol{v}_1 \cdot \boldsymbol{e}_1 \text{ and } \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \frac{1 - \mu^2}{1 + \mu^2}.$$

Proof. We have $\mathbf{v}_1 \cdot \mathbf{e}_1 = \lambda(\mathbf{e}_1 + \mu\mathbf{e}_2) \cdot \mathbf{e}_1 = \lambda\mathbf{e}_1 \cdot \mathbf{e}_1 + \lambda\mu\mathbf{e}_1 \cdot \mathbf{e}_2 = \lambda$ and, simultaneously, $\mathbf{v}_2 \cdot \mathbf{e}_1 = \lambda(\mathbf{e}_1 - \mu\mathbf{e}_2) \cdot \mathbf{e}_1 = \lambda\mathbf{e}_1 \cdot \mathbf{e}_1 - \lambda\mu\mathbf{e}_1 \cdot \mathbf{e}_2 = \lambda$. Thus, $\mathbf{v}_2 \cdot \mathbf{e}_1 = \mathbf{v}_1 \cdot \mathbf{e}_1$.

Next, we consider

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 &= \lambda^2(\mathbf{e}_1 + \mu\mathbf{e}_2) \cdot (\mathbf{e}_1 - \mu\mathbf{e}_2) \\
&= \lambda^2(\|\mathbf{e}_1\|^2 - \mu\mathbf{e}_1 \cdot \mathbf{e}_2 + \mu\mathbf{e}_1 \cdot \mathbf{e}_2 - \mu^2\|\mathbf{e}_2\|^2) \\
&= \lambda^2(1 - \mu^2).
\end{aligned}$$

We also have

$$\begin{aligned}
1 = \|\mathbf{v}_1\|^2 &= \mathbf{v}_1 \cdot \mathbf{v}_1 \\
&= \lambda^2(\mathbf{e}_1 + \mu\mathbf{e}_2) \cdot (\mathbf{e}_1 + \mu\mathbf{e}_2) \\
&= \lambda^2(\|\mathbf{e}_1\|^2 + \mu\mathbf{e}_1 \cdot \mathbf{e}_2 + \mu\mathbf{e}_1 \cdot \mathbf{e}_2 + \mu^2\|\mathbf{e}_2\|^2) \\
&= \lambda^2(1 + \mu^2).
\end{aligned}$$

Then $\lambda^2 = \dfrac{1}{1 + \mu^2}$.

Thus,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{1 - \mu^2}{1 + \mu^2}.$$

For sufficient condition, we have

$$\begin{aligned}
\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{1 - \mu^2}{1 + \mu^2} &= \lambda^2(1 - \mu^2) \\
&= \lambda^2(\|\mathbf{e}_1\|^2 - \mu^2\|\mathbf{e}_2\|^2) \\
&= \lambda(\mathbf{e}_1 + \mu\mathbf{e}_2) \cdot \lambda(\mathbf{e}_1 - \mu\mathbf{e}_2).
\end{aligned}$$

Since $\mathbf{v}_1 = \lambda(\mathbf{e}_1 + \mu\mathbf{e}_2)$, then $\mathbf{v}_2$ must be $\lambda(\mathbf{e}_1 - \mu\mathbf{e}_2)$.

$\square$

Proposition 2.4.2. *[4] Suppose that $\mathcal{M}$ with parametrization $X(u,v) = \alpha(u) + v\beta(u)$ is a regular ruled surface on a unit-speed curve $\alpha(u)$ that is transversal to the rulings. Then, $\alpha(u)$ is an asymptotic curve of $X(u,v)$ if and only if*

$$\alpha'(u) \cdot \boldsymbol{e}_1(u) = \beta(u) \cdot \boldsymbol{e}_1(u), \tag{2.38}$$

and

$$\alpha'(u) \cdot \beta(u) = \frac{k_1(u) + k_2(u)}{k_1(u) - k_2(u)}, \tag{2.39}$$

where $\mathbf{e}_1, \mathbf{e}_2$ are the principal directions along $\alpha(u)$ with $\|e_1\| = \|e_2\| = 1$, and $k_1(u), k_2(u)$ are the principal curvatures of the surface along $\alpha(u)$, in the direction of e_1, e_2, respectively.

Proof. Since $\mathbf{e}_1, \mathbf{e}_2$ are principal directions along $\alpha(u)$, then $\mathbf{e}_1(u), \mathbf{e}_2(u)$ form a basis of the tangent plane of the surface $\mathcal{M}$ along the curve α at each point $\alpha(u)$. Let us denote by $\mathbf{v}_1, \mathbf{v}_2$ two tangent vectors at point $\alpha(u)$ as in Lemma 2.4.1, for $\lambda = 1$ and arbitrary constant μ, thus $\mathbf{v}_1 = \mathbf{e}_1(u) + \mu\mathbf{e}_2(u)$ and $\mathbf{v}_2 = \mathbf{e}_1(u) - \mu\mathbf{e}_2(u)$. By Proposition 1.3.1, we know that the Gaussian curvature of a ruled surface is non-positive. Assuming $K \neq 0$, the two principal curvatures $k_1(u)$ and $k_2(u)$ along the principal direction $\mathbf{e}_1, \mathbf{e}_2$ of the curve α must have opposite sign. We can then conclude that $-\dfrac{k_1(u)}{k_2(u)} > 0$ and chose the constant μ to be $\mu := \sqrt{-\dfrac{k_1(u)}{k_2(u)}}$.
We thus have,

$$\mathbf{v}_1 = \mathbf{e}_1(u) + \sqrt{-\frac{k_1(u)}{k_2(u)}}\,\mathbf{e}_2(u), \text{ and } \mathbf{v}_2 = \mathbf{e}_1(u) - \sqrt{-\frac{k_1(u)}{k_2(u)}}\,\mathbf{e}_2(u).$$

Recall that if $\mathcal{W} = -d\mathbf{N}$ is the Weingarten map of a surface $\mathcal{M}$ at a point $X(u,v)$ of $\mathcal{M}$, then $k_1(u), k_2(u)$, the principal curvatures of $\mathcal{M}$, with principal direction vectors $\mathbf{e}_1, \mathbf{e}_2$, are the eigenvalues of the Weingarten map with eigenvectors $\mathbf{e}_1, \mathbf{e}_2$. The eigenvectors are orthogonal to each other and form a basis of the tangent plane of $\mathcal{M}$ at the point $X(u,v)$: $\mathcal{W}(\mathbf{e}_1) = k_1\mathbf{e}_1$ and $\mathcal{W}(\mathbf{e}_2) = k_2\mathbf{e}_2$.

We will now apply the Weingarten map to the tangent vector $\mathbf{v}_1$, $\mathcal{W}(\mathbf{v}_1) = \mathcal{W}\left(\mathbf{e}_1 + \sqrt{-\frac{k_1}{k_2}}\mathbf{e}_2\right) = \mathcal{W}(\mathbf{e}_1) + \sqrt{-\frac{k_1}{k_2}}\,\mathcal{W}(\mathbf{e}_2) = k_1\mathbf{e}_1 + \sqrt{-\frac{k_1}{k_2}}k_2\mathbf{e}_2$. Thus, $\mathcal{W}(\mathbf{v}_1) \cdot \mathbf{v}_1 = \left(k_1\mathbf{e}_1 + \sqrt{-\frac{k_1}{k_2}}k_2\mathbf{e}_2\right) \cdot \left(\mathbf{e}_1 + \sqrt{-\frac{k_1}{k_2}}\mathbf{e}_2\right) = k_1 + 0 - \frac{k_1}{k_2}k_2 = 0$. Similarly, we can derive that $\mathcal{W}(\mathbf{v}_2) \cdot \mathbf{v}_2 = 0$. This means that $\mathbf{v}_1$ and $\mathbf{v}_2$ gives asymptotic directions of $\mathcal{M}$ at the point $\alpha(u)$, i.e. there exists a tangential direction of zero normal curvature at α.

Having finished the preparation, let us recall that, by hypothesis, α is an asymptotic curve of the surface $X(u,v) = \alpha(u) + v\beta(u)$ and that α is transversal to rulings. Thus α', the direction of the directrix curve α, is an asymptotic direction as well. Furthermore, we have that the rulings are asymptotic lines of the surface $\mathcal{M}$ as well. Suppose $a(v) = \alpha + v\beta$ is an arbitrary ruling of the surface $\mathcal{M}$. Since rulings are straight lines on the surface, then $a''(v) = 0$, which also implies that $k_n = \alpha'' \cdot \mathbf{N} = 0$. Hence, the ruling is asymptotic and β gives an asymptotic direction of $\mathcal{M}$ at a point $\alpha(u)$. Earlier, we have shown that v_1 and v_2 of the form above give asymptotic directions of $\mathcal{M}$. Then β and α' correspond to tangent vectors v_1 and v_2 as above that give the asymptotic directions of the surface $\mathcal{M}$. (Up to sign, there are only two asymptotic directions at any point.) Since α' and β are unit vectors, we may assume that $\beta = \lambda v_1 = \lambda\left(e_1 + \sqrt{-\frac{k_1}{k_2}}e_2\right)$, and $\alpha' = \lambda v_2 = \lambda(u)\left(e_1 - \sqrt{-\frac{k_1}{k_2}}\right)$, where $\lambda = \frac{1}{\sqrt{1 + \frac{k_1}{k_2}}}$ is the norm of vector v_1 and v_2. Now we can apply Lemma 2.4.1 to the unit vectors β and α', concluding that $\alpha'(u) \cdot \mathbf{e}_1(u) = \beta(u) \cdot \mathbf{e}_1(u)$, and $\beta \cdot \alpha' = \frac{1-\mu^2}{1+\mu^2} = \frac{1 - (-\frac{k_1}{k_2})}{1 + (-\frac{k_1}{k_2})} = \frac{k_2 + k_1}{k_2 - k_1}$, where $\mu := \sqrt{-\frac{k_1}{k_2}}$.

$\square$

The corollary below follows directly from the Proposition 2.4.2. and, according to its authors, is analogous to Bonnet's theorem on geodesics of ruled surfaces.

Corollary 2.4.1. *[4] Let $\mathcal{M}$ with parametrization $X(u,v) = \alpha(u)+v\beta(u)$ be a regular ruled surface on $\alpha(u)$, where $\alpha(u)$ is an asymptotic curve of the surface. We denote $k_1(u), k_2(u)$ are principal curvatures of the surface along $\alpha(u)$, respectively. Then, the following conditions are equivalent:*

a. The angle between $\alpha'(u)$ and $\beta(u)$ is constant along α.

b. The ratio $\dfrac{k_1}{k_2}(u)$ is constant along the curve α.

Proof. By Proposition 2.4.2., $\alpha(u)$ is an asymptotic curve of $X(u,v)$ if and only if

$$\alpha'(u) \cdot \beta(u) = \frac{k_1(u) + k_2(u)}{k_2(u) - k_1(u)}.$$

But, the angle between $\alpha'(u)$ and $\beta(u)$ is constant means that $\alpha'(u) \cdot \beta(u)$ is constant because the length of each of the vectors is equal to one. Thus, $\dfrac{k_1(u) + k_2(u)}{k_2(u) - k_1(u)}$ is constant. Let C be a constant in $\mathbb{R}$.

Then

$$\begin{aligned}
\frac{k_1(u) + k_2(u)}{k_2(u) - k_1(u)} &= C \\
k_1(u) + k_2(u) &= Ck_2(u) - Ck_1(u) \\
\frac{k_1(u)}{k_2(u)} + 1 &= C - \frac{k_1(u)}{k_2(u)}C \\
\frac{k_1(u)}{k_2(u)} = \frac{C-1}{C+1} &= \text{ Constant.}
\end{aligned}$$

$\square$

2.4.3 Asymptotic Curves on Principal Normal Surfaces of a Curve

In this section, we will present the main characterization of the principal normal ruled surfaces of a curve α. It will be shown that, for any principal normal surface of a curve α, the curve α is not only the asymptotic curve on this ruled surface, but also a minimal asymptotic curve which is transversal to rulings and the mean curvature of the ruled surface vanishes along α.

Definition 2.4.2. *A curve α is called a minimal asymptotic curve of a surface $\mathcal{M}$ in $\mathbb{R}^3$ if and only if α is an asymptotic curve and the mean curvature vanishes along the curve α.*

Theorem 2.4.1. *[4] Consider a ruled surface $\mathcal{M}$ with parametrization $X(u,v) = \alpha(u) + v\beta(u)$ and suppose that $\sigma(u)$ is a unit speed curve on $\mathcal{M}$. Then, the following conditions are equivalent:*

a. $X(u,v)$ is the principal normal surface of $\sigma(u)$.

b. The curve $\sigma(u)$ is a minimal asymptotic curve of $X(u,v)$ which is transversal to rulings.

Proof. First we will prove that for any principal normal surface of $\sigma(u)$, then $\sigma(u)$ is a minimal asymptotic curve of $X(u,v)$ which is transversal to rulings. Given that $X(u,v) = \alpha(u) + v\beta(u)$ is a principal normal surface of $\sigma(u)$ and $\sigma(u)$ is a curve on $\mathcal{M}$ then we can reparametrized $\mathcal{M}$ as $\tilde{X}(u,v) = \sigma(u) + v\mathbf{n}(u)$ where $\mathbf{n}(u)$ is the unit normal of $\sigma(u)$. Let us denote $\mathbf{N}$ the unit normal to $\mathcal{M}$ with the parametrization $\tilde{X}(u,v) = \sigma(u) + v\mathbf{n}(u)$. Then $\mathbf{N}$ is perpendicular to the tangent plane of $\mathcal{M}$. Thus $\mathbf{N}(u,v) \perp X_v(u,v)$ and the latter is in fact the normal to the curve $X_v(u,v) = \mathbf{n}(u)$, more precisely, $\mathbf{n}$ is the unit normal vector of σ. Thus

$$\begin{aligned} \mathbf{N} \cdot \mathbf{n} &= 0 \\ \mathbf{N} \cdot \left(\frac{1}{k} \sigma'' \right) &= 0 \\ \mathbf{N} \cdot \sigma'' &= 0, \end{aligned}$$

concluding that σ is, by definition, an asymptotic curve of the ruled surface $\mathcal{M}$. To show that the curve is minimal, we need to verify that the mean curvature of $\mathcal{M}$ vanishes along the curve σ. As before, by equation (2.34), we have the mean curvature of a principal normal surface of a curve σ as follows:

$$H(u, v) = \frac{v[\tau' + v(k'\tau - \tau' k)]}{2(EG - F^2)^{\frac{3}{2}}}.$$

Along the curve σ, we have that $v = 0$. Thus, $H(u, 0) = 0$ for all u. $H(u, 0) = 0$ corresponds to the mean along σ vanishes along its asymptotic curve σ for all u. Thus σ is a minimal asymptotic curve which is transversal to rulings. Note that a curve σ is transversal to rulings means that at each point of the surface $\mathcal{M}$, the curves σ intersects orthogonally the rulings since $\sigma' \cdot \mathbf{n} = 0$. Hence, we have proved that for any base curve σ of a principal normal of the σ is an asymptotic curve which is transversal to rulings.

Conversely, we will show that if $\sigma(u)$ is a minimal asymptotic curve of $X(u, v) = \alpha(u) + v\beta(u)$ which is transversal to rulings, then $X(u, v)$ is a principal normal surface of $\sigma(u)$. Since $\sigma(u)$ is a minimal asymptotic curve of $\mathcal{M}$, then $H(u, v) = 0$ along $\sigma(u)$. Thus, at each point $X(u, v)$ along σ, though we drop the parameters, the surface has zero mean curvature

$$H = \frac{k_1 + k_2}{2} = 0$$
$$k_1 + k_2 = 0,$$

where k_1 and k_2 are the principal curvature along σ. Since σ is an asymptotic curve that is transversal to rulings, then we can reparametrize $X(u, v) = \alpha(u) + v\beta(u)$ as $\tilde{X}(u, v) = \sigma(u) + v\tilde{\beta}(u)$, where we may assume that $||\tilde{\beta}|| = 1$. By Proposition 2.4.2, we have that $\sigma'(u) \cdot \tilde{\beta}(u) = \dfrac{k_1(u) + k_2(u)}{k_1(u) - k_2(u)} = 0$ for all u. Hence, $\sigma' = \mathbf{t}$ is perpendicular to $\tilde{\beta}$. Thus, at each point along σ, the ruling direction is of the form $\tilde{\beta} = a\mathbf{n} + b\mathbf{b}$, for some real numbers a, b. However, σ is asymptotic, thus $\sigma'' \cdot \mathbf{N} = 0$, where, from the parametrization, along the curve σ, we have $\mathbf{N} = \mathbf{t} \times \tilde{\beta} = a\mathbf{b} - b\mathbf{n}$. On the other hand, from Frenet equations, $\sigma'' = \mathbf{t}' = k\mathbf{n}$, so $\sigma'' \cdot \mathbf{N} = 0$ implies $b = 0$. Thus $\tilde{\beta}$ is parallel to $\mathbf{n}$, and by choosing earlier the norm of $\tilde{\beta}$ to be one, we have $a = 1$ and $\tilde{X}(u, \tilde{v}) = \sigma(u) + v\mathbf{n}(u)$, concluding that the surface is the principal normal ruled surface of $\sigma(u)$. □

2.4.4 Minimal Asymptotic Curves on Ruled Surfaces

Having studied the characteristics of minimal asymptotic curves on principal normal surfaces of a curve, we then can turn our attention to the characteristics of minimal asymptotic curves on ruled surface in general. The following proposition gives us the characterization of a ruled surface which is a helicoid with circular helices as their minimal asymptotic curves.

Proposition 2.4.3. *[4] Any regular ruled surface $\mathcal{M} : X(u, v) = \alpha(u) + v\beta(u)$ with three disjoint minimal asymptotic curves which are transversal to rulings is a helicoid, and these minimal asymptotic curves are circular helices.*

Proof. Let σ_i of $\mathcal{M}$ be the three minimal asymptotic curves which are transversal

to rulings, where $i = 1, 2, 3$. By equation (1.48) of Proposition 1.3.4, we have the mean curvature for a regular ruled surface with $X(u, v) = \alpha(u) + v\beta(u)$ is quadratic function of variable v. Thus, if there exists three disjoint minimal asymptotic curves on $X(u, v) = \alpha(u) + v\beta(u)$ which are transversal to rulings, then the mean curvature $H(u, v)$ vanishes in any direction. Therefore, $\mathcal{M}$ is a ruled minimal surface. Recall that a minimal surface is a surface whose mean curvature is zero everywhere.
Recall that for any ruled minimal surface, it is either a plane or helicoid. By hypothesis, $\mathcal{M}$ is a ruled surface, thus it must be helicoid which is can be expressed as $X(u, v) = (v\cos(u),\ v\sin(u),\ bu)$, where $\alpha(u) = (0,\ 0,\ bu)$ and $\beta(u) = \cos(u),\ \sin(u),\ 0)$. For each cases of minimal asymptotic curves on the surface $\mathcal{M}$, then $\sigma(u)$ can be expressed as $\sigma(u) = (v\cos(u),\ v\sin(u),\ bu)$, which is the parametrization of circular helix as well. $\square$

We close this section by presenting the characterization of Bertrand curves as the minimal asymptotic curves which are transversal to rulings in any ruled surfaces.

Proposition 2.4.4. *[4] Any two disjoint minimal asymptotic curves which are transversal to rulings of a regular ruled surface $\mathcal{M} : X(u, v) = \alpha(u) + v\beta(u)$ are a pair of Bertrand curves.*

Proof. Let $\sigma_1(u)$, and $\sigma_2(u)$ be two disjoint minimal asymptotic curves which are transversal to rulings of $\mathcal{M}$, then we can express them as $\sigma_1(u) = \alpha(u) + v_1(u)\beta(u)$ and $\sigma_2(u) = \alpha(u)+v_2(u)\beta(u)$. We have $\sigma_1(u) = \alpha(u)+v_1(u)\beta(u)$, then $\sigma' = \alpha'+v_1'\beta+ v_1\beta' = X_u + v'X_v$, where $X_u = \alpha' + v\beta', X_v = \beta, X_{uu} = \alpha'' + v\beta'', X_{uv} = \beta', X_{vv} = 0$. For the fact the $\sigma_1(u)$ is an asymptotic curve of $X(u, v)$, then we have the normal curvature of $\sigma_1(u)$, $k_n = << \sigma_1'(u),\ \sigma_1'(u) >> = Ldu^2 + 2Mdudv + Ndv^2 = 0$, where L, M, N are the coefficients of the second fundamental form of $X(u, v)$ along $\sigma_1'(u)$. We compute these below:

$$\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{(\alpha' + v_1\beta') \times \beta}{\|X_u \times X_v\|},$$

$$M = X_{uv} \cdot \mathbf{N} = \frac{\beta' \cdot [(\alpha' + v_1\beta') \times \beta]}{\|X_u \times X_v\|} = \frac{\beta \cdot (\alpha' \times \beta)}{\|X_u \times X_v\|} = \frac{\beta' \cdot (\alpha' \times \beta')}{\|X_u \times X_v\|} = \frac{\det(\beta', \alpha', \beta)}{\|X_u \times X_v\|},$$

$$L = X_{uu} \cdot \mathbf{N} = \frac{(\alpha'' + v_1\beta'') \cdot [(\alpha' + v_1\beta') \times \beta]}{\|X_u \times X_v\|} = \frac{\det((\alpha'' + v_1\beta'', \alpha' + v_1\beta', \beta)}{\|X_u \times X_v\|},$$

$$N = X_{vv} \cdot \mathbf{N} = 0.$$

Then $Ldu^2 + 2Mdudv + Ndv^2 = Ldu^2 + 2Mdudv = 0$. We have $\sigma' = u'\sigma_u + v'\sigma_v$. Then $<< \sigma', \sigma' >> = Lu'^2 + 2Mu'v' = 0$, for an asymptotic curve, $v' = \dfrac{dv_i}{du} := v_i'$, $i = 1, 2$. Thus, for $i = 1$, we have

$$2 \det\,(\beta', \alpha', \beta)\; dv_1 + \det\,(\alpha'' + v_1\beta'', \alpha' + v_1\beta', \beta) = 0. \tag{2.40}$$

On the other hand, by hypothesis, $\sigma_1(u)$ is a minimal asymptotic curve of $\mathcal{M}$, then the mean curvature along $\sigma_1(u)$ vanishes. By equation (1.48) of the mean curvature of ruled surface, then we have

$$-2(\alpha' \cdot \beta) \det\,(\beta', \alpha', \beta) + \det\,(\alpha'' + v_1\beta'', \alpha' + v_1\beta', \beta)\; = 0. \tag{2.41}$$

Equating the equation (2.40) and equation (2.41), we have $v_1' = -2(\alpha' \cdot \beta)$. Similarly, we can apply the second fundamental form on $\sigma_2(u)$. We obtain $v_2' = -2(\alpha' \cdot \beta)$. Hence, $v_1' - v_2' = 0$, which implies that $v_1(u) - v_2(u)$ is constant. Now, suppose that there exists a non-zero constant A such that $v_1(u) = v_2(u) + A$. Replace $v_1(u)$ into the equation $\sigma_1(u) = \alpha(u) + v_1(u)\beta(u) = \alpha(u) + v_2(u)\beta(u) + A\beta(u) = \sigma_2(u) + A\beta(u)$.

Since σ_i are the minimal asymptotic curve of $\mathcal{M}$ which are transversal to rulings, by Theorem 2.4.1, $X(u, v) = \alpha(u) + v\beta(u)$ is the principal normal surface of $\sigma_1(u)$ and $\sigma_2(u)$. Consider the principal normal surface along σ_2, $X(u, v) = \sigma_2(u) + v\beta(u)$, and take u the arclength parameter of σ_2 so $\beta(u)$ is the unit normal vector field of $\sigma_2(u)$.

Moreover, we know that the mean curvature of principal normal surface of σ_2 along $\sigma_1(u)$ must be zero, as they are both minimal asymptotic curves. By equation (2.34), the mean curvature formula is

$$H(u,v) = \frac{v[\tau'(u) + v(k'(u)\tau(u) - \tau'(u)k(u))]}{2(EG-F^2)^{\frac{3}{2}}} = 0,$$

thus, we have $\sigma_1(u) = \sigma_2(u) + A\beta(u)$ and $A[\tau'(u) + v(k'(u)\tau(u) - \tau'(u)k(u))] = 0$. By Proposition 2.3.2.ii., we can then conclude that $\sigma_1(u)$ and $\sigma_2(u)$ are Bertrand mates.

$\square$

Appendix A

Striction Curves of Doubly Ruled Surfaces

We investigate the uniqueness of striction curves of doubly ruled surfaces by finding the striction curve for the hyperbolic paraboloid and, respectively, the hyperboloid of one sheet using their two distinct surface patches. We will show that the striction curve for doubly ruled surfaces is different for each parametrization, and thus depends on the parametrization, though it was shown earlier that, for each parametrization, the striction curve is unique.

A1 Striction Curves of Hyperbolic Paraboloid

As mentioned in Chapter 1, the hyperbolic paraboloid with equation $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ is a doubly ruled surface where the two distinct surface patches are $X_1(u, v) = (au,\ 0,\ u^2) + v\ (a,\ b,\ 2u)$, and $X_2(u, v) = (au,\ 0,\ u^2) + v\ (a,\ -b,\ 2u)$, where $u, v \in \mathbb{R}$, as in the equation (1.13), and (1.14), respectively.

To simplify the calculations, we will consider the hyperbolic paraboloid with $a = b = 1$.

For the first surface patch of the hyperbolic paraboloid, we have

$$X_1(u,v) = (u,\ 0,\ u^2) + v\ (1,\ 1,\ 2u),$$

where $\alpha(u) = (u, 0, u^2)$ and $\beta(u) = (1, 1, 2u)$. Since $\beta \times \beta' = (1, 1, 2u) \times (0, 0, 2) \neq \vec{0}, \forall u \neq 0$, this hyperbolic paraboloid is a non-cylindrical ruled surface. By Remark 1.3.1, the striction curve of the non-cylindrical can be computed by $\sigma(u) = \alpha(u) - \dfrac{< \alpha'(u), \gamma'(u) >}{\|\gamma'(u)\|^2}\gamma(u)$, where $\|\gamma(u)\| = 1$. Therefore, since $\|\beta(u)\| = \sqrt{2+4u^2}$, we calculate $\gamma = \left(\dfrac{1}{\sqrt{2+4u^2}}, \dfrac{1}{\sqrt{2+4u^2}}, \dfrac{2u}{\sqrt{2+4u^2}}\right)$,
$\gamma' = \left(\dfrac{-4u}{\sqrt{(2+4u^2)^3}}, \dfrac{-4u}{\sqrt{(2+4u^2)^3}}, \dfrac{4}{\sqrt{(2+4u^2)^3}}\right)$, $\|\gamma'\|^2 = \dfrac{8}{(2+4u^2)^2}$, and

$$\begin{aligned}
\alpha' \cdot \gamma' &= (1, 0, 2u) \cdot (\frac{-4u}{\sqrt{(2+4u^2)^3}}, \frac{-4u}{\sqrt{(2+4u^2)^3}}, \frac{4}{\sqrt{(2+4u^2)^3}}) \\
&= \frac{-4u}{\sqrt{(2+4u^2)^3}} + \frac{8u}{\sqrt{(2+4u^2)^3}} \\
&= \frac{4u}{\sqrt{(2+4u^2)^3}}.
\end{aligned}$$

Thus, the striction curve of the first patch of hyperbolic paraboloid is

$$\begin{aligned}
\sigma_1(u) &= \alpha - \frac{\alpha' \cdot \gamma'}{\|\gamma'\|^2}\gamma \\
&= (u, 0, u^2) - \frac{4u}{\sqrt{(2+4u^2)^3}} \frac{(2+4u^2)^2}{8} \left(\frac{1}{\sqrt{2+4u^2}}, \frac{1}{\sqrt{2+4u^2}}, \frac{2u}{\sqrt{2+4u^2}}\right) \\
&= (u, 0, u^2) - \frac{u\sqrt{2+4u^2}}{2} \left(\frac{1}{\sqrt{2+4u^2}}, \frac{1}{\sqrt{2+4u^2}}, \frac{2u}{\sqrt{2+4u^2}}\right) \\
&= (u, 0, u^2) - \left(\frac{u}{2}, \frac{u}{2}, u^2\right) \\
&= \left(\frac{u}{2}, -\frac{u}{2}, 0\right), \quad u \in \mathbb{R}.
\end{aligned}$$

Therefore, the striction curve is the planar curve $y = -x$.

For the second surface patch of hyperbolic paraboloid, we have

$$X_2(u,v) = (u,\ 0,\ u^2) + v\ (1,\ -1,\ 2u),$$

where $\alpha(u) = (u, 0, u^2)$ and $\beta(u) = (1, -1, 2u)$. We have the following: $\|\beta(u)\| = \sqrt{2+4u^2}$, $\gamma = \left(\frac{1}{\sqrt{2+4u^2}},\ -\frac{1}{\sqrt{2+4u^2}},\ \frac{2u}{\sqrt{2+4u^2}}\right)$,
$\gamma' = \left(\frac{-4u}{\sqrt{(2+4u^2)^3}},\ \frac{4u}{\sqrt{(2+4u^2)^3}},\ \frac{4}{\sqrt{(2+4u^2)^3}}\right)$, $\|\gamma'\|^2 = \frac{8}{(2+4u^2)^2}$ as before, and

$$\begin{aligned}
\alpha' \cdot \gamma' &= (1, 0, 2u) \cdot (\frac{-4u}{\sqrt{(2+4u^2)^3}},\ \frac{4u}{\sqrt{(2+4u^2)^3}},\ \frac{4}{\sqrt{(2+4u^2)^3}}) \\
&= \frac{-4u}{\sqrt{(2+4u^2)^3}} + \frac{8u}{\sqrt{(2+4u^2)^3}} \\
&= \frac{4u}{\sqrt{(2+4u^2)^3}}.
\end{aligned}$$

Thus, the striction curve of the second patch of hyperbolic paraboloid is

$$\begin{aligned}
\sigma_2(u) &= \alpha - \frac{\alpha' \cdot \gamma'}{\|\gamma'\|^2}\gamma \\
&= (u, 0, u^2) - \frac{4u}{\sqrt{(2+4u^2)^3}} \frac{(2+4u^2)^2}{8} \left(\frac{1}{\sqrt{2+4u^2}},\ -\frac{1}{\sqrt{2+4u^2}},\ \frac{2u}{\sqrt{2+4u^2}}\right) \\
&= (u, 0, u^2) - \frac{u\sqrt{2+4u^2}}{2} \left(\frac{1}{\sqrt{2+4u^2}},\ -\frac{1}{\sqrt{2+4u^2}},\ \frac{2u}{\sqrt{2+4u^2}}\right) \\
&= (u, 0, u^2) - \left(\frac{u}{2}, -\frac{u}{2}, u^2\right) \\
&= \left(\frac{u}{2}, \frac{u}{2}, 0\right), \quad u \in \mathbb{R}.
\end{aligned}$$

Thus, the second striction curve is the planar curve $y = x$, hence the two curves are the two (different) bisectors of the quandrant II, respectively quadrant I, of the xy-plane, as seen also from the Mathematica plot below.

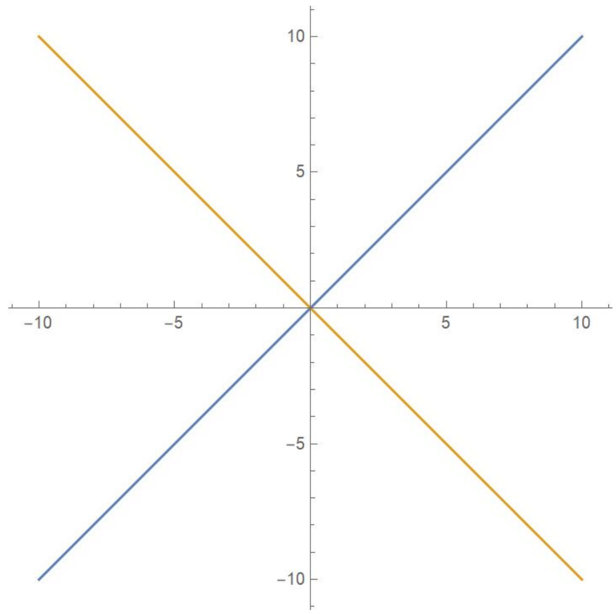

Figure A.1: Striction curves of Hyperbolic Paraboloid, where the blue graph represents the striction curve σ_1 and the yellow one represents the striction curve σ_2, (here $u \in (-20, 20)$)

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A2 Striction Curves of Hyperboloid of One Sheet

Similarly, hyperbolic paraboloid with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ is a doubly ruled surface with the two surface patches $X_1(u, v) = (a\cos u,\ b\sin u,\ 0) + v\ (-a\sin u,\ b\cos u,\ c)$, and $X_2(u, v) = (a\cos u,\ b\sin u,\ 0) + v\ (a\sin u, -b\cos u,\ c)$ as in the equation (1.16) and (1.17), respectively.

Again, for a simpler calculation, we assume that $a = b = c = 1$ so this specific hyperbolic paraboloid is defined by $x^2 + y^2 - z^2 = 1$.

For the first surface patch of hyperboloid of one sheet, we have $X_1(u, v) = (\cos u,\ \sin u,\ 0) + v\ (-\sin u,\ \cos u,\ 1)$, where $\alpha(u) = (\cos u,\ \sin u,\ 0)$ and $\beta(u) = (-\sin u,\ \cos u,\ 1)$. As before, since $\beta \times \beta' = (-\sin u,\ \cos u,\ 1) \times (-\cos u,\ -\sin u,\ 0) \neq \vec{0}, \forall u \neq 0$, then this hyperbolic paraboloid is a non-cylindrical ruled surface. We have the following: $\|\beta\| = \sqrt{2},\ \gamma = \left(-\dfrac{\sin u}{\sqrt{2}},\ \dfrac{\cos u}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}}\right), \gamma' = \left(-\dfrac{\cos u}{\sqrt{2}},\ -\dfrac{\sin u}{\sqrt{2}},\ 0\right)$,

$\|\gamma'\|^2 = \dfrac{1}{2}$, and $\alpha' \cdot \gamma' = (-\sin u,\ \cos u,\ 0) \cdot \left(-\dfrac{\cos u}{\sqrt{2}},\ -\dfrac{\sin u}{\sqrt{2}},\ 0\right) = 0$. Thus the striction curve of the first patch of hyperbolic paraboloid is $\sigma_1(u) = \alpha(u) = (\cos u,\ \sin u,\ 0)$.

For the second surface patch of hyperboloid of one sheet where $a = b = c = 1$, we have $X_2(u, v) = (\cos u,\ \sin u,\ 0) + v\,(\sin u,\ -\cos u,\ 1)$, where $\alpha(u) = (\cos u,\ \sin u,\ 0)$ and $\beta(u) = (\sin u,\ -\cos u,\ 1)$. We have the following:
$\|\beta\| = \sqrt{2}$, $\gamma = \left(\dfrac{\sin u}{\sqrt{2}},\ -\dfrac{\cos u}{\sqrt{2}},\ \dfrac{1}{\sqrt{2}}\right)$, $\gamma' = \left(\dfrac{\cos u}{\sqrt{2}},\ \dfrac{\sin u}{\sqrt{2}},\ 0\right)$, $\|\gamma'\|^2 = \dfrac{1}{2}$, and $\alpha' \cdot \gamma' = (-\sin u,\ \cos u,\ 0) \cdot \left(\dfrac{\cos u}{\sqrt{2}},\ \dfrac{\sin u}{\sqrt{2}},\ 0\right) = 0$. Thus the striction curve of the second patch of hyperbolic paraboloid is $\sigma_2(u) = \alpha(u) = (\cos u,\ \sin u,\ 0)$.

We have that $\sigma_1(u) = \sigma_2(u) = (\cos u,\ \sin u,\ 0)$ which is precisely the base curve and it represents a unit circle centered at the origin in the xy-plane. In the case of the hyperboloid of one sheet, we can conclude that the striction curve is unique independently of the parametrizations of the surface patch.

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