The Development of (Non-)Mathematical Practices through Paths of Activities and Students' Positioning: The Case of Real Analysis

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Abstract

The development of (non-)mathematical practices through paths of activities and students' positioning: The case of Real Analysis

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Previous research has found that within elementary university courses in single variable and multivariable Calculus, the activities proposed to students may enable and encourage the development of non-mathematical practices. More specifically, the research has shown that students can obtain good passing grades by learning highly routinized techniques for a restricted set of task types, with little to no understanding of the mathematical theories that justify the choice and validity of the techniques. We were interested in knowing what happens as students progress to more advanced courses in Analysis. The study presented in this thesis focussed on a first Real Analysis course at a large urban North American university. To frame our study, we turned to the Anthropological Theory of the Didactic, which offers theoretical tools for modelling practices as they exist within and across institutions. We analyzed various course materials to develop models of practices students may have been expected to learn in the course. These were then used to inform our construction of a task-based interview that would allow us to elicit and model practices students had actually learned. Interviews were conducted with fifteen students shortly after they passed the course. In our qualitative analyses of the resulting data, we found that students' practices were (non-)mathematical in different ways and to varying degrees. Moreover, this seemed to be linked not only to the kinds of activities students had been offered in the course, but also to the characteristically different ways in which students may have interacted with those activities. As theoretical tools for thinking about these links, we introduce the notion of a path to a practice and a framework of five positions that students may adopt in a university mathematics course institution: the Student, the Skeptic, the Mathematician in Training, the Enthusiast, and the Learner. We discuss the possibility of designing paths of activities that might perturb students' positioning and encourage the development of practices that are more mathematical in nature.

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Introduction

This thesis originated from a general interest in the process of becoming a mathematician, given the considerable gap that seems to exist between what students learn in university mathematics courses and what mathematicians do to solve new mathematics problems. Personal experience and research led us to question if students really learn to "behave mathematically" (i.e., like a mathematician) as they progress through the coursework of an undergraduate (or even a graduate) degree in mathematics. If students do learn to behave mathematically, we wondered when and how this occurs throughout the progression of coursework. We were also curious about the kinds of circumstances that would most effectively cultivate desired mathematical behaviour in most students, and whether we would find these in existing courses.

Since "behaving mathematically" is such a broad term, the first challenge we faced was to determine what we wanted it to mean in the context of this thesis. Over the years, scholars have proposed many different elements of a potential definition (e.g., Burton, 2004; Cuoco et al., 1996; Hardy et al., 2013; Lockhart, 2009; Papert, 1993; Schoenfeld, 1987, 1992; Sierpinska et al., 2002). A recent master's thesis (Murray, 2016) combines several of such theories into one possible model of mathematical behaviour, which includes actions and dispositions such as: being curious, seeking challenges, thinking in terms of systems of concepts, being concerned with consistency and validity, conjecturing, justifying, regulating one's own actions, collaborating, persisting, having confidence in one's abilities, and being emotionally connected to the subject matter. We need only evoke two contradictory images to see the significance of such a definition: on the one hand, university students passively receiving, memorizing, and regurgitating lecture presentations of the polished results of mathematics past, and on the other, mathematicians actively engaging in the messy processes of inquiry aimed at developing new mathematics. This said, we have come to question the necessity and suitability of the adjective "mathematical" in describing the general behaviours mentioned above: for example, are not curiosity, collaboration, and confidence important in most inquiry-oriented endeavours (not only those pursued by mathematicians)? In our work, we wanted to find a way of focussing on "mathematical behaviour" that is intrinsically "mathematical." This is why we turned to the *Anthropological Theory of the Didactic* (ATD) developed by Yves Chevallard (1985, 1992, 1999, 2002).

The ATD enabled us to think about "mathematical behaviour" that is fundamentally "mathematical" through its notion of *praxeology*: a model of *mathematical knowledge*, which, in the context of the theory, is equated to *mathematical practice* (Chevallard, 1999). According to the model, the ever-evolving collection of mathematical knowledge (or practices) can be characterized by the *types of tasks* mathematicians know how to solve, the *techniques* they have for solving them, and the *theoretical discourses* they use to describe and justify the techniques. Think, for example, of the various techniques mathematicians have developed for solving the type of task "find $\lim_{x \to a} f(x)$, where $f: \mathbb{R} \to \mathbb{R}$ is a specified function and $a \in \mathbb{R}$," as well as the numerous definitions, theorems, and proofs they call upon to explain how and why those techniques work. Notice that these components are indeed "mathematical": that is, they represent parts of mathematicians' behaviours that are fundamentally theirs and could not be used to characterize any other field of inquiry.

In this thesis, we say that an individual *has mathematical knowledge* if they can *enact a mathematical practice*: i.e., if they can identify a given mathematical task as an example of a certain type of task, choose and implement a technique for solving the task, and explain how and why the technique works, all in a *mathematical* manner (i.e., in a way that would be judged as acceptable and appropriate by mathematicians faced with the same task).

In posing the above definition, our general interest in the process of becoming a mathematician was narrowed to a focus on the development of mathematical practices. We wondered:

> Do students develop mathematical practices as they progress through university mathematics coursework? If so, when? And under which circumstances?

Note that we are not suggesting that the only goal of university mathematics coursework is the development of mathematical practices. We do, however, see it as one important aim of any training in mathematics.

Since the birth of the field of university mathematics education, researchers from around the world have built up literature suggesting that in the context of single variable Calculus courses, students are enabled and encouraged to develop *non-mathematical practices* (Bergqvist, 2007; Cox 1994; Hardy, 2009; Lithner, 2000, 2003, 2004; Orton, 1983; Selden et al., 1999; Tallman et al., 2016). These studies have shown that students can obtain good passing grades in such courses by learning to apply highly routinized techniques (often in the form of an arbitrary list of steps) to solve a limited set of task types (often characterized through superficial means), with little to no understanding of underlying mathematical discourses (often replaced by explanations of the form: "I do this because that's what my professor told me to do for this kind of question"). A recent master's thesis (Brandes, 2017) suggests that the same situation may arise in the context of the multivariable Calculus courses that typically follow single variable Calculus in North American universities.

It seems reasonable to predict that as students progress further in the *Analysis Stream* of university coursework (including, for example, courses in single variable Calculus, multivariable Calculus, Real Analysis, Metric and Functional Spaces, and so on), their practices should become more mathematical in nature. Researchers have made various predictions about what may happen in Real Analysis courses. Theoretically speaking, it has been proposed that such courses invite students to (a) learn the mathematical discourses that may have been missing from the types of tasks and techniques learned in Calculus; and (b) develop new proof-based practices characterised by increased levels of rigour and the necessity of supporting mathematical discourses (Winsløw, 2006). On a practical level, however, scholars express doubt that students ever rework the non-mathematical practices they learned in Calculus (e.g., Kondratieva & Winsløw, 2018; Winsløw et al., 2014), pinpoint several obstacles to students' development of formal proof practices (e.g., Bergé, 2008; Maciejewski & Merchant, 2016; Raman, 2002, 2004; Sfard, 1991; Tall, 1992; Timmermann, 2005), and suggest that it may be possible to succeed in Real Analysis courses by memorizing a particular subset of definitions, theorems, and proofs (e.g., Darlington, 2014), or by learning new kinds of superficial and routinized practices (e.g., Weber, 2005a, 2005b). It seemed there was a need to investigate such predictions through a holistic empirical study. This is what we hoped to contribute in this thesis.

Based on the interests and results outlined above, we devised a doctoral research project that aimed to gain an understanding of

1. the nature (mathematical or otherwise) of the practices developed by students in a first Real Analysis course; and
2. the factors that may be shaping the development of such practices.

With our first research objective, we hoped to contribute to investigating how students' practices evolve as they advance in their training of mathematics; in particular, as they transition from Calculus to Analysis. More specifically, we wished to examine if and in what sense students' practices become more mathematical (i.e., more like mathematicians'), or not. With our second research objective, we aimed to provide some possible explanations for our findings by looking into some of the factors that may be shaping the development of students' practices. This reflected our more general interest in thinking about the circumstances under which the development of mathematical practices may (or may not) take place.

To be able to address the above objectives, we needed to specify not only what we meant by "practices" and their "nature" (i.e., "mathematical" or not), but also how we would think about "factors" that might shape the development of students' practices in a first course on Real Analysis. As explained above, we turned to the ATD – in particular, its notion of praxeology (Chevallard, 1999) – to propose a definition of a "mathematical practice." Then we combined the ATD with some other theories to elaborate a framework of three interconnected layers of influence on the development of students' practices in any university mathematics course:

- the broad institutional context of the course;
- the activities offered to students by the professor of the course; and
- the positions adopted by students in the course, which dictate how they interact with the activities offered to them.

The outermost layer – the broad institutional context – reflects our choice to see the teaching and learning of university mathematics from an institutional perspective inspired by the ATD (Chevallard, 1985, 1992) and the *Institutional Analysis and Development* (IAD) *Framework* (Ostrom, 2005). According to this perspective, any university mathematics course takes place in a relatively stable social structure called a *didactic institution*, whose defining features (e.g., intended outcomes, participants, and rules) differ significantly from the *professional institutions* on which it is based. More specifically, the institution of *University Mathematics* – where professors and students come together in courses to study a certain transformed subset of existing mathematical practices – operates according to a different set of conditions and constraints than the institution of *Scholarly Mathematics* – where mathematicians use and produce mathematical practices to solve new problematic tasks. For example, the interactions between professors and students are often heavily influenced by the curricular organization of knowledge, time constraints, and examination procedures. A key assumption underlying the entirety of our work is that such contextual features shape the practices to be (and actually) taught by professors, and the practices to be (and actually) learned by students.

The two other layers listed above represent hypotheses we made based on our own experiences with university mathematics courses and the research we read (including the many studies on Calculus cited above). As students progress through any university mathematics course, they encounter numerous *activities*: in lectures, tutorials, textbooks, assignments, (past) midterms, (past) final exams, and so on. We hypothesized that the nature of those activities (and what is made explicit about them, in professors' solutions, course textbooks, and so on) can play a crucial role in shaping the nature of the practices actually learned by students. More specifically, we expected that the activities posed in assessment situations (like assignments, past midterms, and past final examinations) would communicate to students a minimal core of the practices to be learned in order to receive a good passing grade in the course. Moreover, these "minimal" practices may not be the ones considered desirable by the professor, or, more generally, by mathematicians (e.g., Hardy, 2009). This said, we also recognized that students might interact with the activities offered to them in significantly different ways (theorized in terms of the different *positions* they occupy in the course institution), and we hypothesized that this may also influence the nature of the practices students learn. For instance, some students may interact with the activities they are offered from a position of *Student*: i.e., they may see mathematics as a course to be passed and work hard to identify and master the minimal core of practices to be learned. Other students, in contrast, may take on a position of *Learner*: i.e., they may see mathematics primarily as a mental endeavour to be shared with their professor and interact with any activity offered to them with the principal aim, not of passing the course, but of enriching their understanding (e.g., Sierpinska et al., 2008).

Under these hypotheses, we specified our research objectives as follows:

We aimed to gain an understanding of

1. the nature (mathematical or otherwise) of the practices actually learned by students who are deemed successful in a first Real Analysis course in the institution of University Mathematics; and

2. how these practices are shaped by the positions that students adopt and the activities they are offered.

Our doctoral project would systematically investigate the nature of students' practices and the influence of two specific factors: namely, the positions students adopt and the activities they are offered. We would reflect on the influence of the broad institutional context more generally when it came time to discuss our results. That is, once we identified potential relationships between the nature of students' practices, the activities students are offered, and the positions students adopt, we planned to take a step back and consider how these relationships are established in the context of a didactic institution. With this, we hoped to make a contribution to anthropological research on university mathematics education, all the while producing results that might inform the future instruction of courses or have implications for curriculum development.

The study we designed to address the above objectives focussed on a core mathematics course – Real Analysis I (RA I) – at one large urban North American university. To address our first research objective, we aimed to model and characterize the practices actually learned by successful RA I students. To reveal these practices, and their nature, we designed a *task-based interview* (Goldin, 1997, 2000). Crucial to the design of the interview was a choice of tasks that would be both *recognizable* and *deceptive* to successful RA I students. On the one hand, the students needed to be able to recognize the tasks as being ones that could be solved using a practice they had developed during the RA I course. On the other hand, the task needed to have some deceptive quality so that when students tried to enact a practice they had developed, any superficiality or routinization of the practice would be revealed. To assist us in creating these kinds of tasks, we engaged in an analysis of the assessment activities (i.e., activities in assignments, past midterms, and past final exams) that were given to RA I students during two recent academic terms.[1]

We gathered over 200 assessment activities from the professor who was teaching the course at this time. Then we modelled the practices that students may have been expected to develop by engaging in these activities based on the resources made available to them: i.e., the activities, written solutions to the activities, and the course textbook. During our analysis, we noticed that

[1] We do not reveal the exact terms in order to protect the confidentiality of the participants of our study.

some activities were *isolated*, while others formed what we called a *path to a practice*: in other words, some activities seemed disconnected from all other activities, while others formed a collection of related activities that might gradually lead students towards the development of a certain kind of practice (Broley & Hardy, 2018). We characterized the practices that students may have learned by following the paths, including any superficiality or routinization that may have occurred. Then we chose a subset of paths on which to base our interview tasks. Our selection was influenced by several factors, including time constraints (e.g., we aimed for a two-hour interview), the high volume of material covered in RA I, and our interest in covering a wide range of topics (so that we might be able to account for variation in the nature of students' practices from one topic to the next). Given our interest in the evolution of students' practices from Calculus to Real Analysis, as sketched above, we were also drawn to topics (and tasks) that students might link with practices they had developed previously, in Calculus courses. In doing this, we thought we might get a glimpse of how students coordinate the practices they learn in their progression through the Analysis Stream; in particular, we might see if succeeding in a first course in Real Analysis would have led students to work on some of the potentially non-mathematical practices they had learned in Calculus.

Our final selection of six interview tasks (listed in Appendix A) relate to the following topics:

- proving (by contradiction) that a real number is irrational;
- proving that a function $f: \mathbb{R} \to \mathbb{R}$ has a certain number of zeros;
- proving that the supremum (or infimum) of a set of real numbers is M (or m);
- finding and/or proving limits of sequences of real numbers; and
- determining and proving the (dis-)continuity and (non-)differentiability of piecewise real-valued functions on $\mathbb{R}$.

To address our second research objective, we also needed to ensure that our task-based interview would enable inferences about a student's positioning during RA I (e.g., whether they had acted as a Student or as a Learner) and how this, along with the assessment activities that had been offered in the course, may have shaped the practices that the student had developed. We could reflect on the influence of assessment activities by comparing our models of (a) the practices students may have been expected to develop by engaging with the activities and (b) the practices students had actually developed, as exhibited in the task-based interviews. The nature of students' practices and the degree to which the assessment activities seemed to play a role in shaping them would also assist us in characterizing the positions students had adopted in the course. For instance, while a Student's practices would closely reflect the assessment activities they were offered, including any superficiality or routinization that may have been enabled or encouraged by the activities, a Learner's practices would be much less predictable. In addition to this, we decided to elicit a student's positioning by supplementing our observations of their spontaneous task solving with soliciting their responses to reflective questions about their perceptions of mathematics, the RA I course, their own actions in relation to these, and so on.

We conducted our interview with fifteen students, shortly after they successfully completed RA I (i.e., obtained a final grade of 50% or higher). In our qualitative analyses of the resulting data, we found that some of the mathematical tasks we posed elicited mainly practices that were expected based on our analysis of assessment activities from RA I, and that these were non-mathematical or mathematical in different ways and to varying degrees. In contrast, there were some tasks where participants exhibited a wider variety of practices than we had expected, some of which had a procedural and intuitive flavour reflecting practices learned in previous Calculus courses. In the end, we found that we could make some sense of our findings – of the varying nature of participants' practices and how this seemed to be linked to the assessment activities that were offered in RA I – through the identification and characterization of five theoretical positions: not only the Student and the Learner (as already identified in the literature), but also the Skeptic, the Enthusiast, and the Mathematician in Training. While most participants exhibited signs of positioning themselves as a Student in RA I and developing practices that were, in some way, non-mathematical, there were participants who seemed to strongly occupy a position other than Student, adopt a hybrid positioning including the Student, or have an ability to shift out of the Student position at moments where they struggled in solving a task or were prompted by the interviewer. Moreover, these different kinds of positioning seemed to be productive, in different ways, towards the development of practices that are more mathematical in nature. This leads to the question of whether it could be possible to construct activities that productively perturb students' positioning, and whether this possibility could be realized in courses operating under the conditions and constraints of University Mathematics. We return to these ideas at the end of the thesis.

We begin this thesis, in Chapter 1, with a literature review aimed at showing how our research fits within the context of existing work and may contribute to moving the field forward. In Chapter 2, we explain our choice of the ATD as our foundational theory, introduce each component of the theoretical framework we built to specify our research objectives, and clarify some of the theoretical contributions we have made. We dedicate Chapter 3 to describing the relevant particularities of the institutional context where we conducted our study, including the features of the specific course, RA I, that we considered. Then, in Chapter 4, we outline the three stages of our methodology: the collection and analysis of assessment activities from RA I, the creation of a task-based interview and its implementation with successful RA I students, and the qualitative analyses of the resulting data. We present our results in two chapters. The goal of Chapter 5 is to present one of the main theoretical contributions of our study: a positioning framework that could be used to try to understand how the positions students adopt in any mathematics course institution may shape the practices that they learn. We use this framework in our analyses throughout Chapter 6, where we present our results for each task from our task-based interview. This includes constructing models of practices to be learned and practices actually learned for each task (as synthesized in Appendices B – G). It also includes some initial reflections aimed at addressing our research objectives. Such reflections serve as the springboard to our final discussion and conclusions in Chapter 7.

Chapter 1: Literature Review

As laid out in the Introduction, our doctoral work aimed to gain an understanding of the nature of the practices (types of tasks, techniques, and theoretical discourses) developed by students in a first Real Analysis course, as well as the factors that may be shaping the development of those practices. The goal of this chapter is to place these general research objectives in the context of existing literature, and to set the foundation for highlighting how our study contributes to pushing the field forward.

We consider our work as falling within the broad research domain of university mathematics education; and, more specifically, within the large subset of research concerning student learning in what we have termed the "Analysis Stream" of university coursework (including, for example, courses in single variable Calculus, multivariable Calculus, Real Analysis, Metric and Functional Spaces, and so on). Student learning of Calculus and Analysis has and continues to be a dominant area of research, and it would be unfeasible for us to review all the related work here (for some more comprehensive reviews of research on university mathematics education, see Artigue, 2016; Artigue et al., 2007; Nardi, 2017). In what follows, we focus on two themes that are pertinent to understanding how our study fits within the landscape of preceding theories and results.

First, we provide a review of work that has documented how students develop non-mathematical practices in the context of Calculus courses (Section 1.1). We discuss how researchers from across time and around the world have shown that the kinds of activities proposed to Calculus students can lead them to learn superficial types of tasks and highly routinized techniques, with little to no understanding of underlying mathematical theories – in particular, with little to no understanding of why a certain technique should be chosen over others or why such a technique accomplishes a given task. We consider two lines of inquiry that arise naturally from the reviewed literature. On the one hand, research might inquire into the redesign of Calculus courses to encourage the development of practices that are more mathematical in nature. On the other hand, one might wonder if students' practices are eventually required to become more mathematical in the progression of existing university mathematics coursework. The latter is the line of inquiry behind our work.

The second theme we consider is how students' practices might evolve as they progress from Calculus to Analysis. We start by presenting a theoretical model of this evolution, proposed by Winsløw (2006), which depicts how students might develop practices that are more mathematical in nature as they progress further in the Analysis Stream (Section 1.2). Then we review some key observations concerning the evolution from Calculus to Analysis, which enable various predictions about whether students' practices (are required to) become more mathematical or not in the context of a first Real Analysis course (Section 1.3).

In the end, we highlight the scientific pertinence of conducting a holistic empirical study that investigates both the nature of the practices students actually learn in a first Real Analysis course and the factors that may be shaping them. We finish the chapter by restating the general research objectives we presented in the Introduction (Section 1.4).

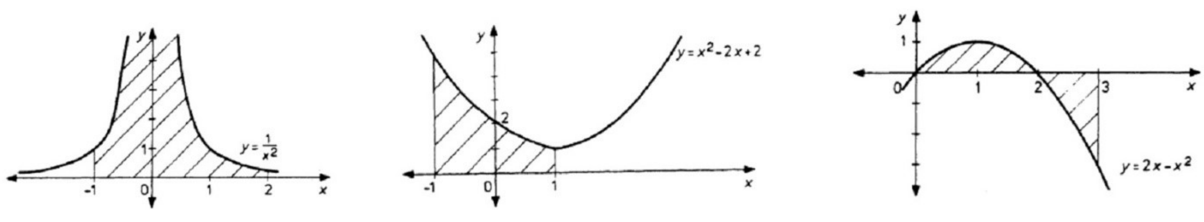
1.1 Calculus Students' Development of Non-Mathematical Practices

The issue of students learning non-mathematical practices in Calculus courses has been documented by mathematics education research since at least the 1980s. The earliest paper we read on the subject was inspired by complaints from British teachers, who noticed that their students were skilled in differentiating and integrating specific functions, but they lacked an understanding of the underlying processes. Orton's (1983) task-based interviews[2] with 110 students confirmed the teachers' observations: While the tasks on integral calculations were accomplished, on average, rather successfully by the participants, those that directly probed at students' awareness of how the integral is defined (as a limit of sums), were challenging for even the best students. Moreover, a qualitative analysis of students' difficulties on certain tasks allowed Orton to exemplify how students seemed to be operating according to "rules without reasons" (or "routinized techniques without mathematical theoretical discourses"). For instance, to find the area of the shaded region in Graph A in Figure 1.1, some students calculated

$$\int_{-1}^{2}\left(\frac{1}{x^2}\right) = -\frac{1}{x}\bigg|_{-1}^{2} = -\frac{1}{2} - \left(-\frac{1}{-1}\right) = -\frac{3}{2}.$$

[2] A description of this research tool, as conceptualized by Goldin (1997, 2000), will be given in Chapter 4 (see Section 4.2), where we introduce the methodology adopted in our study.

Other students made mistakes in their calculation for Graph B, which led to negative answers; but the students did not question their results. And in reference to Graph C, Orton (1983) claimed: "The most striking feature of responses was that many students appeared to know what to do [i.e., calculate the area in two parts], but, when questioned about their method, didn't really know why they were doing it" (p. 8). In place of a mathematical explanation, some students gave a non-mathematical reason for their rule: e.g., "That's the way I was taught how to do it."



Graph A Graph B Graph C

Figure 1.1 Areas students were asked to calculate in task-based interviews by Orton (1983).

Shortly after Orton's (1983) study, similar results concerning a variety of Calculus topics were published by other researchers from different parts of the world (e.g., Artigue et al., 1990, in France; Cox, 1994, in Britain; Selden et al., 1994, in the United States; White & Mitchelmore, 1996, in Australia), some of whom began to look more carefully at the factors that may have been leading students to learn "rules without reasons."

Cox's (1994) report brought to light the potential influence of students' approaches to learning and how these are shaped by the context within which learning takes place. After administering multiple choice tests to small groups of students who had successfully completed a Calculus course, Cox engaged in discussions with the students, who disclosed their reliance on what he called *strategic learning*: i.e., trying to get the best grades possible by tailoring what they learn to the kinds of questions that are typically posed on final examinations. Cox also had conversations with teachers, which further suggested that instructional practices might be encouraging this kind of learning. Teachers reported using final exams to determine which topics to drill and were ultimately not surprised that Cox found students' practices to be specified to such topics. In reflecting on his findings, Cox noted that the situation was made even more complex by the massive amount of material that appeared in the curriculum; it was perhaps impossible for the teachers to cover any topic in any considerable depth. From experience, he suggested that the same kinds of observations could be made in any university course: i.e., broadening curricula, overloading syllabi, and tailoring teaching to exams to avoid embarrassing failure rates – "in short, we encourage strategic learning in our own students" (Cox, 1994, p. 20).

Around the same time as Cox's (1994) study, a series of studies conducted by the Selden et al. (1994) group began to highlight how the nature of the tasks students are invited to solve may also be shaping what they learn. The researchers introduced a distinction between two kinds of tasks: namely,

- a given task is *routine* for a given solver if they can solve the task by carrying out a method they know; and
- a given task is *nonroutine* for a given solver if solving the task requires them to develop a method they do not already know.

Based on these definitions, the researchers constructed two tests: one with ten tasks they assumed would be routine for students who passed a traditionally taught Calculus course (e.g., "find the maximum value of $f(x) = -2 + 2x - x^2$"), and the other with five tasks assumed to be nonroutine for such students (e.g., "find at least one solution to the equation $4x^3 - x^4 = 30$ or explain why no such solution exists"). The researchers administered the tests to small groups of students who had received good passing grades in Calculus courses. The results for one A-level student were illustrative: While his score of 90% on the routine test reflected his A grade, he obtained only 16% on the nonroutine test and got only one out of the five nonroutine tasks substantially correct. Such results suggested that an emphasis on routine tasks in the instruction and evaluation of Calculus courses may have led students to learn a very particular set of practices that enabled them to obtain high grades, even if they lacked a good conceptual understanding of the course material (an understanding that would have been required for solving the nonroutine tasks; Selden et al., 1999).

As the field progressed, researchers, practitioners, and policy makers spoke about the "superficial," "procedural," or "rote" nature of what students were learning in Calculus courses. But there was a lack of theories that served to clearly define these concepts and to characterize the quality of students' practices. This gap sparked an extensive research program in Sweden, initiated by Johan Lithner (2000, 2003, 2004, 2008). The foundation of the conceptual framework built up by Lithner (2008) is a model of task solving that includes four basic steps:

1. Meeting a (sub)task;

2. Choosing a strategy;
3. Implementing the strategy; and
4. Obtaining a conclusion.

Lithner (2008) specifies that the steps are not necessarily linear and could occur several times depending on how many subtasks or strategy attempts are required for a given solver to solve a given task. Moreover, "choosing a strategy" is to be understood in a wide sense: "choosing" could mean recalling, discovering, constructing, guessing, etc., and "strategy," like the ATD's "technique," could include anything from local procedures to general approaches. Essential to characterizing students' practices is also the specification that the steps may be accompanied by some kind of "argumentation" (or "theoretical discourse"): in particular, answers to the questions "Why will the strategy solve the task?" (at step 2) or "Why did the strategy solve the task?" (at step 3). Such argumentation is comprised in what Lithner (2008) more generally defines as *reasoning*: i.e., "the line of thought adopted to produce assertions and reach conclusions in task solving," which "is not necessarily based on formal logic, thus not restricted to proof, and may even be incorrect as long as there are some kinds of sensible (to the reasoner) reasons backing it" (p. 257).

Lithner's early work characterized Calculus students' practices as more or less mathematical through the distinction between well-founded and superficial reasoning. Lithner (2000) began by observing students solving tasks in exam-like situations, where they had no access to external aids. This allowed him to identify two types of reasoning:

- *Plausible Reasoning* (PR), where the argumentation is founded on intrinsic mathematical properties of the components involved in the reasoning; and
- *Reasoning based on Established Experiences* (EE), where the argumentation is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment (ibid., p. 167).

In the interviews he conducted, Lithner (2000) found that students' strategy choices were based primarily on EE, and that when their chosen strategies did not work in the ways they had expected, their abilities in PR were too weak to assist them in coherently solving the given tasks. Through the lens of the ATD, the students had developed highly routinized techniques void of supporting mathematical theoretical discourses (i.e., non-mathematical practices).

As an example, consider the task of maximizing $p(x) = x^2 - 1300x + 420000$ on $[400, 600]$. When faced with this task, one student immediately differentiated p, found that $p'(x) = 0$ when $x = 650$, and considered the sign of $p''(650)$; but upon finding $p''(650) > 0$, the student became stumped. He was looking for a maximum, and his memory of the second derivative test told him (correctly) that $p''(650) > 0$ implies that p has a minimum at $x = 650$. Nevertheless, the student's reliance on memory and lack of understanding made him doubt this implication and inhibited his ability to proceed in a mathematical manner. He did not consider any intrinsic mathematical properties of the components of the task. For example, the task requests that p be maximized on a closed interval, $[400, 600]$, which means that the student's chosen technique (i.e., "find x where $p'(x) = 0$ and show that $p''(x) < 0$") would not necessarily work (since p could obtain its maximum at the boundaries of the interval). Alternatively, the student may have noticed that p is a quadratic function with a positive second-degree term, which means that its graph is an upward facing parabola (so it makes sense that it has a minimum at $x = 650$, and a maximum at one of the boundaries of the given interval). Instead, the student's way forward on solving the task was to search through his established experiences for other methods of checking critical points (e.g., the first derivative test); and he eventually became convinced that $x = 650$ is not the location of a maximum only once he calculated $p(650)$ and saw that it is negative (a negative maximum must have also been outside of his established experiences).

Although a key result of Lithner's (2000) case studies was the dominance of superficial reasoning in Calculus students' task solving, the researcher also found that some students were able to proceed skillfully based on sound mathematical reasoning when they were guided to do so. He echoed the researchers that came before him (e.g., Cox, 1994 and Selden et al., 1999) when he explained his findings as follows:

> If one supposes that PR is not perceived by students as one of the main tools when solving tasks, perhaps because PR is not emphasised in the teaching practice they experience, and that the students hence obtain very limited experience in constructing PR, then their behaviour is perfectly natural. (Lithner, 2000, p. 188)

In other words, the kinds of practices students learn depend on the task-solving experiences they have, which in turn depend on the context within which those experiences take place. Since the time of Lithner's (2000) initial paper, more and more studies appeared that systematically analyzed pertinent contextual factors and how they may condition the way in which Calculus students (learn to) solve tasks.

Lithner's (2003, 2004) subsequent studies focussed on investigating the nature of the exercises posed in Calculus textbooks, which are often a key component of the learning environment in Calculus courses. This led him to introduce a third reasoning type: i.e.,

> *Reasoning based on Identification of Similarities* (IS), where the strategy choice is founded on identifying similar surface properties in an example, theorem, rule, or some other situation described earlier in the text; and the strategy implementation is carried through by mimicking the procedure from the identified situation (Lithner, 2003, p. 35).

Based on an analysis of 598 exercises from single variable sections of three popular Calculus textbooks, Lithner (2004) determined that the majority of the exercises (in some cases, up to 70%) could be solved by IS reasoning alone, and that PR was often required only to determine if an identified procedure could be copied as is, or if a few local steps needed to be modified. Moreover, the exercises that required a significant amount of PR either seemed way beyond the level of the students or appeared among the last exercises of a section (and were therefore likely to be skipped by teachers). In accordance with these findings, Lithner (2003) also found that in task-based interviews where the use of textbooks was permitted, students spent almost all their time trying to solve the given tasks through IS reasoning. They searched the pages of their textbook for surface similar examples whose solutions could be "blindly" copied. In fact, many of them complained about the shortage of surface similar examples in their textbooks. Based on this, Lithner (2003) predicted that students were likely to depend primarily on IS reasoning during homework completion, whereby they would not develop sufficient PR and would therefore be forced to rely almost exclusively on EE reasoning in exams. In the words of the ATD, Lithner's (2003, 2004) studies suggested that when Calculus students interact with the activities in their textbooks, they are encouraged to develop non-mathematical practices formed of superficial task types (i.e., task types defined by surface rather than intrinsic properties) and algorithmic techniques (i.e., techniques formed by a step-by-step solution pattern); and that they are not required to develop supporting mathematical theoretical discourses (since the steps are endorsed by the authority of textbooks, and students' established experiences in solving the exercises therein).

We note that over time, Lithner's framework has evolved and in its current state (as described in Lithner, 2008), the opposition between superficial and well-founded reasoning seems to have become secondary to the opposition between imitative and creative reasoning. *Imitative reasoning* (IR), as indicated in the name, is based on recall (of facts, proofs, algorithms, etc.) or guidance from outside sources (e.g., the teacher or the textbook). *Creative reasoning* (CR) is characterized by novelty: that is, the reasoning produced by the reasoner is in some way new to them (ibid., p. 266). In essence, the distinction between IR and CR reflects the notions of routine and nonroutine tasks described by Selden et al. (1994): In the presence of a routine task, students solve the task by applying IR, while a nonroutine task requires some degree of CR. Note that in either case, the students may be able to provide a coherent mathematical explanation of what they are doing and why. For instance, a memorized proof is not necessarily a misunderstood proof; a recalled algorithm is not necessarily performed mindlessly, or without understanding of how and why it works. What is stressed by Lithner (2008) is that, in the case of IR, it is possible that all analytical and conceptual thinking processes are missing, whereas in the case of CR, such processes are required. Tasks that require CR (i.e., nonroutine tasks) are therefore more likely to push students towards operating according to mathematically well-founded reasoning; and if IR is sufficient to pass exams, then it may be possible to pass a mathematics course through the development of superficial and routinized practices.

The importance of final examinations in determining the sufficient knowledge of passing students has led several researchers to focus their studies on the nature of exam tasks. As an extension of Lithner's research program, one of his students (Bergqvist, 2007) studied over 200 tasks taken from 16 Calculus exams given at four Swedish universities in one academic year (2003-2004). At these universities, students' final results were determined solely by their exam grade. Bergqvist (2007) systematically classified each of the tasks as requiring IR or CR by analysing the tasks that students had been practicing in their courses, based on the textbooks they had used and the materials they had received from their teachers. For instance, if in a Calculus course students did not practice tasks asking them to construct an example with certain continuity properties (e.g., "give an example of a function that is right continuous, but not left continuous, at $x = 3$"), then such a task would be classified as requiring CR from those students; the same task, in a course where the students had practiced similar tasks, would be classified as requiring less CR or possibly completely IR. In the end, Bergqvist (2007) found that it was possible to solve about 70% of the exam tasks by IR reasoning alone and that 15 of the 16 exams could be passed with distinction (i.e., with a grade of 70% or higher) without using any substantial amount of CR. She concluded that it was possible for students to pass the Calculus courses she studied through the recall of algorithms based on tasks' surface properties (p. 367).

Studies have confirmed and extended the results outlined above in the North American context. Hardy (2009) contributed a smaller-scale study of one North American Calculus course and focussed on limit finding tasks, which enabled her to provide detailed qualitative analyses of the nature of both the examination tasks typically posed in the course and the practices developed by successful students. Based on task-based interviews with 28 students who recently passed the course, she demonstrated how the emphasis on routine tasks in the exams had led the students to develop non-mathematical practices (we provide more details about Hardy's findings when we construct our theoretical framework in Chapter 2). The larger-scale study of examination tasks performed by Tallman et al. (2016) suggested that the same situation may be arising for most Calculus topics and in most Calculus courses across North America. These researchers studied a random sample of 150 final exams given in Calculus courses across the United States, which implied an analysis of 3735 tasks. Based on their combined experiences of what typically occurs in American Calculus courses, the researchers classified each task according to the most cognitively-demanding intellectual behaviour that would be required in solving it among seven possible behaviours (adapted from Bloom's taxonomy): remember, recall and apply procedure, understand, apply understanding, analyze, evaluate, and create. The results resembled Bergqvist's (2007): about 85% of all exam tasks were found to be solvable by retrieving rote knowledge from memory or recalling and applying a procedure, and of the 150 exams, 90% had 70% or more tasks coded as "remember" or "recall and apply procedure." Tallman et al. (2016) concluded that most of the Calculus exams "promote memorization of procedures for answering specific problem types and do not encourage students to understand or apply concepts" (p. 131). In other words, to successfully complete most of the Calculus courses, it was sufficient for students to develop non-mathematical practices.

In our reading of the literature on students' learning in introductory Calculus courses, we have found a consensus amongst researchers from across time and around the world. We expect many researchers today would agree with Cox (1994), who, over 25 years ago, noted how Calculus students ended their courses, not with the essence of mathematics, but with "a rag-bag of soon-to-be-forgotten techniques, and the misapprehension that mathematics is an uninspiring cookbook without rhyme or reason" (Cox, 1994, p. 17). Much of the research described above showed systematically how the activities proposed to students in such courses might lead to the development of non-mathematical practices. It seems that it is often possible to receive good passing grades in single variable Calculus courses by learning to identify a restricted set of tasks (often through superficial means) and to apply highly routinized techniques, with little to no understanding of the supporting mathematical theories – in particular, with little to no understanding of why a given technique is chosen over others, nor why such a technique accomplishes the proposed task.

A recent study has shown that this may also occur in the context of multivariable Calculus courses, which typically follow single variable Calculus in North American university curricula. In her master's research, Brandes (2017) studied the course outlines and textbook used in the multivariable Calculus courses at one institution to develop extensive models of the practices that were expected to be taught in those courses. She then studied twelve exams given in the courses over a span of three years to create a model of the minimal core of the practices that students must typically learn in order to succeed in the courses (the exams tend to be very similar from year to year, and the students' final exam grades can almost entirely determine their success in the courses). Ultimately, Brandes found that a superficial grasp of procedures sufficed (see Brandes & Hardy, 2018 for a concise description of the research and its results). In the exams, there was no need for students to justify or explain the techniques they used; in some cases, they were even told which technique to use. The students needed to know enough terms, formulas, and concepts to be able to recognize a routine task and implement the corresponding technique. In short, students' practices were not required to become any more mathematical than demanded by their previous courses in single variable Calculus.

There are at least two lines of questioning that arise naturally from the above selection of literature. The first concerns the possibility of modifying the way Calculus is currently taught so that students are required to develop practices that are more mathematical in nature. Maciejewski and Star (2016) provide an interesting recent example of a study that has pursued this kind of questioning. The researchers point out how Calculus reform movements, since the 1980s, have aimed to shift students away from the rote reproduction of mathematical procedures principally by encouraging "concepts-first" instruction. In line with this is the view that procedural knowledge is, by definition, superficial, and hence of lesser value than conceptual knowledge. What makes Maciejewski and Star's (2016) paper stand out is their aim to argue, contrary to the dominant reform models, for a renewed focus on procedures in Calculus. After all, it is not the focus on procedures in and of itself that makes Calculus students' practices non-mathematical: procedural knowledge can be either superficial (as in the rote execution of a technique void of a mathematical theoretical discourse) or deep (in that students know how and why techniques work, and can apply them flexibly and innovatively in nonroutine tasks). Maciejewski and Star (2016) are interested in designing instructional experiences that lead to the latter kind of knowledge. They are driven by the assumptions that (a) teaching procedures is a necessary and important part of introductory Calculus; and (b) the students who take such courses – for the large part non-mathematics majors – typically arrive ill-equipped to engage with instruction that has a predominant conceptual focus. We would add that one of the reasons students in service disciplines take Calculus is so that they can become proficient in some of the techniques on which their later work will rely. One might even question the degree to which such students, who stop their mathematical studies at Calculus, need to be taught how to mathematically explain and validate the techniques that they learn. In any case, is this not the goal of subsequent courses in Analysis?

This leads to the second line of questioning, the one that inspired this doctoral thesis: Are students' practices eventually required to become more mathematical as they progress further in university mathematics coursework? In the next two sections, we review what past theoretical and empirical research tells us about the development of students' practices in the context of a first Real Analysis course. We start, in Section 1.2, by presenting a model of how students' practices might evolve as they progress further in the Analysis Stream of university coursework. Then, in Section 1.3, we review some key observations concerning the evolution from Calculus to Analysis, which lead to various predictions about how students' practices may actually evolve.

1.2 The Evolution of Students' Practices from Calculus to Analysis: A Model

As students progress in university mathematics coursework, they are expected to work with more abstract objects and to solve more theoretical tasks. To model the way in which their practices may be expected to evolve, Carl Winsløw (2006) turned to the ATD's notion of *praxeology*. As indicated previously, this is a key notion underlying our work. As such, it is part of our theoretical framework and is described in Chapter 2 (see Section 2.2). For now, we recall that a student's practices can be thought of as comprising the *types of tasks* they know how to solve, the *techniques* they have for solving them, and the *theoretical discourses* they use to describe, explain, and justify the techniques. Together, the types of tasks and techniques known to the student form the practical part of their knowledge (also called their *praxis*) – we will denote it by Π – and the theoretical discourses form the theoretical part of their knowledge (also called their *logos*) – we will denote it by Λ. Based on these ideas, Winsløw (2006) suggested that, in principle, as students progress in university mathematics coursework, their practices undergo at least two transitions, as depicted in Figure 1.2.

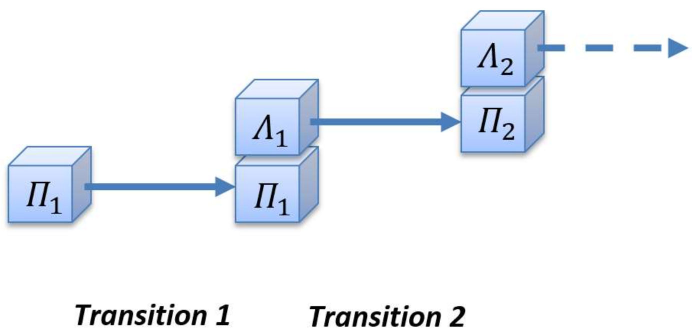


Figure 1.2 Winsløw's (2006) model of transitions in students' practices as they progress in university mathematics coursework.

The block labelled by Π_1 represents a stage where students' practices are expected to be based primarily on recognizing types of tasks and applying appropriate techniques. The practices students are expected to learn in elementary university courses in single variable and multivariable Calculus are typically considered to be of this form. For instance, students are expected to learn the techniques for solving tasks about the calculation of limits and derivatives; but they are not expected to learn the underlying logos at this point. Even if teachers introduce the theory in class, the activities independently completed by students focus their attention on the development of techniques for solving types of tasks. This said, as students progress further in the Analysis Stream, they are expected to work with and on the theory that supports these techniques, which may bring them through the two transitions in Figure 1.2.

Accomplishing the first transition in the model (Transition 1) means learning the logos, Λ_1, that corresponds to the previously developed praxis, Π_1. Hence, the blocks labelled by (Π_1, Λ_1)

represent a stage where students are expected to be able to explain the techniques they learned before. For example, they would not only know how to calculate $\lim_{x\to 1}\frac{x^2-1}{x-1}$ and $\frac{d}{dx}(\sin(x)+x^2)$; they would also be able to describe the techniques they used and to justify why those techniques solve the given tasks, including an awareness of pertinent definitions, theorems, and proofs.

Accomplishing the second transition in the model (Transition 2) means that students go beyond knowing the theoretical block, Λ_1, to being able to work with the more abstract objects therein: i.e., to be able to solve types of tasks containing those more abstract objects. The blocks labelled by (Π_2, Λ_2) represent a stage where students know how to construct proofs in a more general sense. For example, they not only know how to prove the limit and differentiability laws; they also know how to solve a task like

Let $|f(x)| \leq g(x)$ and $\lim_{x\to 0}\frac{g(x)}{x} = 0$. Show that f is differentiable at 0.

Unlike in the second stage (i.e., (Π_1, Λ_1)), Winsløw (2006) suggests that the practical block at the third stage cannot exist without the corresponding theoretical block. Consider, for instance, how solving the above task requires a direct application of the definition of differentiability. In comparison, it is possible to isolate the calculation of $\frac{d}{dx}(\sin(x)+x^2)$ using differentiation rules from the definition of differentiability used to validate these rules. In other words, the techniques at the second stage are weakly tied to the corresponding theoretical discourse.

Overall, the transition model proposed by Winsløw (2006) suggests that students' practices are expected to become more mathematical as they progress throughout university mathematics coursework. First, students should become capable of basing their practical work on a more solid mathematical foundation; then they shift into more abstract practical work, where knowing the related mathematical theory is necessary. In light of this model, one might speculate that a successful completion of a first course in Real Analysis – typically the theoretical counterpart of some previously taken Calculus course – should correspond to making the first transition, and possibly also the second, with respect to some subset of practices.

We now turn to reviewing what other theories and empirical results say about how students' practices may evolve as they progress through a first course on Real Analysis.

1.3 The Evolution of Students' Practices from Calculus to Analysis: Observations and Predictions

1.3.1 An Observation of Curricula

In a later paper, Kondratieva and Winsløw (2018) critique Winsløw's (2006) original model (presented in Section 1.2) and propose a new way of thinking about the evolution of students' practices from Calculus to Analysis based on a key observation of how university curricula typically organize the practices students are expected to learn. The researchers explain that, in principle, courses in Calculus and Analysis can be seen as inviting students to learn practices, (Π, Λ), that are close in standards to the practices held by present-day mathematicians. In Calculus, students are expected to learn techniques for solving tasks (i.e., the praxis, Π) about the calculation of limits, derivatives, and integrals, the continuity of functions, the convergence of series, and so on. In Analysis, students are expected to learn the theoretical underpinnings (i.e., the logos, Λ) involving the definitions, theorems, and proofs that explain and validate these techniques. This said, the main tenet of the paper by Kondratieva and Winsløw (2018) is that the reality of university teaching of Calculus and Analysis makes it unlikely that students will develop complete mathematical practices of the form (Π, Λ). The researchers claim that the *desynchretisation of practices in university curricula* – that is, the separation of Calculus and Analysis into distinct courses, themselves divided into rather isolated compartments of topics – leads to the teaching (and hence learning) of isolated practices that students would not link naturally. In terms of the transition model in Figure 1.2, the researchers essentially suggest that students may never be explicitly invited to develop the practices depicted at the second stage (i.e., (Π_1, Λ_1)). In other words, students are not invited to "update" the practices learned in Calculus with sound and complete mathematical descriptions, explanations, and justifications.

The new model proposed by Kondratieva and Winsløw (2018) suggests that:

- the practices students are expected to learn in Calculus courses are of the form (Π, L): the techniques resemble those used (for tasks of the same type) by professional mathematicians, while the logos is either absent or limited to informal explanations;

- the practices students are expected to learn in Analysis courses are of the form (P, Λ): the logos blocks are consistent with that of present-day mathematicians, while the praxis blocks are didactic "supplements" constructed simply so that students will acquire Λ; and
- students are left to their own devices to make the links and modifications necessary for developing practices of the form (Π, Λ).

A previous paper by Winsløw (2016) exemplifies the model for a specific topic: angles and trigonometry. Winsløw (2016) notes that in pre-Calculus and Calculus courses, the cosine and sine functions are introduced in three distinct contexts (as tools in solving triangle problems, in terms of the unit circle, and as functions through tables and graphs), none of which offer students a rigorous mathematical definition of the functions. Students' learning of techniques for solving tasks related to angles and trigonometry is therefore supported by an informal understanding of the related formulae and functions. In other words, students are expected to learn practices of the form (Π_{AT}, L_{AT}), where the praxis Π_{AT} is akin to that held by professional mathematicians (for the same types of tasks), and the logos L_{AT} is more informal in nature. This informal theoretical block is supported by students' natural conception of "angle measure" as the space that simply exists between two crossing line segments. To construct a more formal understanding of this concept (as the "arc length" of the unit circle) would require the theory of curve length and natural parametrizations of curves, which is typically taught later, in Analysis courses. Winsløw (2016) describes how this theory can be used to define rigorously the notions of "angle," "width opened between rays," and the functions "cosine" and "sine," thereby supporting the development of a theoretical block, Λ_{AT}, closer in nature to that held by the community of mathematicians. He also shows how the textbook he used when teaching a first Real Analysis course in Denmark could, in theory, support this development. Nevertheless, the course he taught did not include any activities that specifically aimed for students to develop Λ_{AT}. While the focus in Calculus was the practical work in "angles and trigonometry," Π_{AT}, the focus in Real Analysis was a seemingly distant theoretical discourse on "curve length," Λ_{CL}. Winsløw (2016) concluded that students were on their own to make the connection between Λ_{CL} and Π_{AT} and modify Λ_{CL} into Λ_{AT}.

1.3.2 Observations of the Activities Proposed to Students

In Section 1.1., we highlight how researchers of university mathematics education eventually began to make sense of Calculus students' non-mathematical practices by analysing the students' learning environments: in particular, the kinds of textbooks they use and the kinds of tasks that are proposed in homework exercises and final examinations. There have been several studies that extend such analyses to later courses in the Analysis Stream (e.g., Darlington, 2014; Bergé, 2008; Raman, 2002, 2004). At first, the general trends reported seem to validate Winsløw's (2006) model, as presented in Section 1.2: students' practices are expected to become more mathematical as they progress from elementary courses in Calculus to more advanced courses in Analysis. This said, the studies also suggest some nuances in the evolution of expected practices and bring to light some didactic and institutional obstacles to students' development of mathematical practices in a first Real Analysis course.

One particularly relevant study, by Bergé (2008), investigated the practices students were expected to learn in a sequence of four courses (Course I, II, III, and IV) representing the Analysis Stream at an Argentinian university. Bergé (2008) focussed on one core topic: the completeness property of the real numbers, which manifested itself in tasks involving the notions of supremum and infimum. To model the practices students were expected to learn, Bergé analyzed course syllabi, recordings of lectures, students' notes, and the sets of homework tasks given to students. A birds-eye view of the results led to the conclusion that

> tasks evolve from computational and other technical exercises ("find," "decide," "determine") in Course I, to tasks requiring more and more sophisticated forms of justification: "justify" in Course II; "prove," "give a formal proof" in Courses III and IV. This forces the evolution of the techniques in the same direction, from applications to more creative proofs. (p. 232)

It seemed that the practices students were expected to develop, in relation to the topic of completeness, followed the transition model proposed by Winsløw (2006). From Course I to Course II, the activities completed by students required them to change their justification standards for the same types of tasks (first transition); and upon entering Course III, the concepts of supremum and infimum went from "numbers to be computed" to "abstract objects to be manipulated theoretically and used conceptually in proofs" (second transition). However, Bergé's

(2008) detailed analyses of homework tasks brought to light a crucial difference between the evolution of expectations she observed and the evolution as modelled by Winsløw (2006).

In Winsløw's (2006) model, the first transition corresponds to the development of a mathematical theoretical block for previously developed techniques (the techniques remain unchanged). In contrast, Bergé's (2008) results showed that for some types of tasks, the first transition may actually correspond to the development of new techniques. As an example, consider one homework task that might be posed in Courses I, II, and III at the university studied by Bergé (2008, p. 230):

Find the supremum and infimum of the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

In Course I, an acceptable response would include a representation of the set A on a number line and an inference, based on the picture, that $\sup A = 1$ and $\inf A = 0$. In contrast, the same conclusion would require more explicit and formal reasoning in Courses II and III, along the lines of:

A is bounded: $0 < \frac{1}{n}$ since $\frac{1}{n}$ is a ratio of two positive numbers.

$\frac{1}{n} \leq 1$ since every natural number satisfies $n \geq 1$.

Since $1 \in A$ the number 1 is a maximum, hence a supremum.

To show 0 is the infimum: Suppose that $\epsilon > 0$.

There exists an n such that $n > \frac{1}{\epsilon} \Leftrightarrow \epsilon > \frac{1}{n}$.

Thus, no other positive real number can be a lower bound. ■

Before constructing the above solution, it may be necessary to think about A informally, for example, by picturing its elements on a number line. But this kind of thinking is no longer sufficient to solve the task. In fact, as depicted above, it may no longer be considered part of the solution. We could imagine the same kind of situation arising for other types of tasks typically covered in Calculus and Analysis courses: not only "find the supremum and infimum of a specified set," but also "find the limit of a specified sequence or function," "determine if a specified function is continuous at a specified point," and so on.

Bergé (2008) explains that although this transition – the development of new formal techniques for old types of tasks – may make sense from the institution's point of view, it may not make sense to students. Ideally, by keeping the types of tasks constant, students would be able to work with familiar concrete objects, thereby allowing them to ease into the new element of using theory to construct proofs. This said, developing the new formal techniques not only requires students to adapt to a new methodic formal style of reasoning; it also asks them to replace what might seem to them as a perfectly good, simple, intuitive technique with a whole bunch of complicated formality, for no apparent reason. Indeed, when students are asked to prove "obvious" results, the validation standards are changing, not as a necessary solution to unreliable intuition, but because that is just what they are expected to do in this particular course. This kind of change, imposed by didactic contract alone, may lead students to see the activities as arbitrary and meaningless, rather than as an important introduction to mathematicians' practices. Bergé (2008) predicts that students may be reluctant to engage seriously with such activities. As a result, they might not develop the related mathematical practices.

A study by Raman (2002, 2004) of commonly used textbooks in North American pre-Calculus, Calculus, and Analysis courses corroborates Bergé's (2008) results for another core topic – continuity – and brings up another crucial issue in the evolution of students' (expected) practices: the (lack of) coordination of informal and formal approaches. Raman (2004) found that in the pre-Calculus text by Demana, Waits, and Clemens (1993), students are expected to determine if a function is continuous or not through the informal technique of looking at its graph and deciding if it is possible to draw it without lifting a pencil. Then, in Stewart's (1998) Calculus textbook, students are expected to adopt a more formal technique based on checking if the function satisfies the limit definition of continuity. Like Bergé (2008), Raman (2004) notes that the informal technique becomes replaced by the formal one for no apparent reason: that is, students are instructed to use a strict formal technique in the textbook by Stewart (1998), even if the continuity of most functions can still be determined through graphing. Raman (2002) points out that after studying with such a textbook, students may not spontaneously think to use graphs to understand a problem, even if it could help tremendously; and such ignorance of useful informal approaches may be further perpetuated in the textbooks used in Analysis courses. For instance, in the Analysis textbook by Rudin (1976), informal explanations and interpretations are almost entirely absent, and students are expected to use formal techniques based on the ϵ-δ definition of continuity in the context of general metric spaces. Raman (2004) notes that the formal techniques are now essential to solving many of the tasks in the textbook; in particular, the ones that invite students to deduce properties that have continuity in the hypothesis or conclusion. Raman (2004) predicts, nevertheless, that developing the formal proof techniques may be more challenging if students do not realize that informal thinking can still be useful (e.g., to interpret tasks, choose and implement appropriate techniques, and even explain how the techniques work).

In the UK, a study of examination tasks confirmed that there are drastic changes in the kinds of techniques students are expected to develop as they progress to courses in Real Analysis. At the same time, the study showed that it may not be necessary to adapt to such changes in order to obtain passing grades. Darlington (2014) compared the final examination tasks given in two pre-Analysis courses between 2006 and 2010 with the final examination tasks given in a university Real Analysis course between 2006 and 2012. She characterized the tasks in three different groups, according to what they seemed to require from students: i.e., applying techniques for solving routine tasks (A), using existing knowledge in new ways (B), or applying conceptual knowledge to construct arguments (C). She found that in the pre-Analysis courses, the majority of the marks were awarded for Group A questions, while in the Real Analysis course, the majority of the marks were awarded for Group C questions. It seemed that students' practices were expected to shift quite suddenly from mainly the recall and application of well-practiced and predictable procedures (in pre-Analysis courses), to including the ability to draw implications and justify conclusions based on mathematical theory. This said, Darlington (2014) also found that the Real Analysis course still had a significant percentage of the grade attributed to tasks that require "lower level routine skills" classified in Group A: namely, tasks that ask for the statement of facts or the reproduction of proofs from lecture notes. In fact, students could earn a passing grade by successfully solving these tasks alone. Hence, it may be possible for students to succeed in the Real Analysis course by learning specific definitions and theorems, without being able to connect them with previously learnt practices or to engage in general proof construction. In other words, it may be possible to pass a first course in Real Analysis without having developed mathematical practices.

1.3.3 Observations of Students' Approaches to Learning

A Canadian study by Maciejewski and Merchant (2016) contributes some key observations of how students' approaches to learning may come into play in the transitions depicted in Winsløw's (2006) model. Although the study does not focus on the Analysis Stream, we expect the results would apply to the evolution from Calculus to Analysis. One reason for this is that part of the study involved an analysis of examination tasks, which led to findings consistent with the UK study by Darlington (2014), mentioned above. More specifically, Maciejewski and Merchant (2016) used Bloom's taxonomy to classify the level of cognitive demand required in the final examination tasks given in four first-year courses, five second-year courses, and six third- or fourth-year courses (content unspecified) in the same semester at various major Canadian universities. Like Darlington (2014), the researchers found that the percentage of examination tasks asking students to recall and apply well-practiced and predictable procedures decreased drastically from first- and second-year courses (e.g., in Calculus) to third- and fourth-year courses (e.g., in Analysis), where there was increased emphasis on tasks that require students to justify conclusions, judge the truth of statements, and arrange ideas in novel ways to construct arguments. The researchers even made the same observation as Darlington (2014) that in the more advanced courses, there was an unexpectedly large number of tasks requiring simple recall of definitions or theorems (Maciejewski & Merchant, 2016, p. 383).

What was added by Maciejewski and Merchant (2016) were observations of how the evolution of the nature of final examination tasks might relate to the evolution of students' approaches to learning. The researchers consider students' approaches to learning through two types of study approaches (deep vs. surficial), which they expect to be linked with two types of conceptions of mathematics (cohesive vs. fragmented). As the names suggest, a *deep study approach* aims at long-term retention and an understanding of the connections between practice and theory, while a *surficial study approach* aims at retaining facts and procedures just long enough to perform well on assessment tasks (much like the notion of *strategic learning* introduced by Cox, 1994). In close connection with this, a person is said to possess *cohesive conceptions of mathematics* if they see the subject as a cohesive logical system, while someone with *fragmented conceptions of mathematics* perceives the subject as a collection of isolated facts and procedures.

In addition to analyzing the tasks on final exams, Maciejewski and Merchant (2016) collected information about 322 students who had taken those exams: 169 students from the first-year courses, 100 from the second-year courses, and 53 from the third- or fourth-year courses. The information collected included the students' final grades and their responses to two questionnaires that could be used to classify their study approaches and their conceptions of mathematics. As expected, the students' questionnaire responses showed positive correlations between fragmented conceptions and surficial approaches; and, similarly, between cohesive conceptions and deep approaches. More interesting were the correlations between students' grades and their self-reported study approaches. The researchers found that in the first-year courses, "deep approaches are rewarded slightly, but surficial approaches are not discouraged" (p. 383); that is, students could do very well in the courses using surficial approaches alone. In contrast, in the more advanced courses, "the more surficial the approach, the poorer the outcome" (384); i.e., the more students seemed to rely on memorizing facts and procedures for the sake of passing exams, the lower their grade. When combined with the observed evolution in the nature of examination tasks, such results indirectly suggest that in more advanced university coursework (e.g., in Analysis), the practices of more successful students may indeed be required to become more mathematical.

The issue highlighted by Maciejewski and Merchant (2016) is that achieving such success (and the related mathematical practices) may be very difficult for most students. Indeed, the emphasis on rote procedural work in first-year courses (e.g., in Calculus) can reinforce and encourage fragmented conceptions of mathematics and surficial study approaches, which are further validated when students use such approaches to obtain good grades on exams. This need not mean that their approaches to learning cannot change:

> A likely situation is that a student experiences success with a surficial approach in the first year and starts their second year off with this same approach. The higher level assessments present in the second year are an impetus for the student to change their approach to study, but there is inertia to overcome. These students end up under-performing because the approach to study that worked for them in the previous year proves no longer adequate. (ibid., p. 385)

Other studies confirm that students' approaches to learning are particularly difficult to change, even through extensive, custom-made remediation (e.g., Cox, 1994). It would seem reasonable to predict, then, that in a first course in Real Analysis, some students may continue using surficial approaches that lead to the development of non-mathematical practices.

1.3.4 Observations of Students' Conceptual Development

The studies outlined in the previous three sections all offer observations of how the expectations of students may change in the progression from introductory Calculus courses to more advanced university courses in Analysis. In our search for studies that have actually observed students – how they adapt to changes in expectations and the nature of the practices they develop – we ended up reading many papers reporting on so-called *cognitively oriented research* (Tall, 1992). Although such research focusses on students' "understanding of concepts," we have found that its theoretical characterization of the transition from Calculus to Analysis aligns well with Winsløw's (2006) model and that its theories and results provide some insight, however partial or indirect, into the practices developed by students and the potential influence of epistemological or cognitive factors.

Foundational to the "cognitive way" of thinking about the transition from Calculus to Analysis are the notions of concept definition and concept image, which were put forward by Shlomo Vinner in the 1970s (e.g., in 1976). Given any concept (e.g., function, limit, derivative, continuity), the related *concept definition* comprises the words used to define (i.e., to specify) the concept. It can be *formal*, as formulated by the mathematical community at large, or *personal*, as stated by an individual. In comparison, a *concept image* is typically considered on an individual basis and consists of everything the individual has in their mind in relation to the concept. On top of their personal concept definition, this could include prototypical examples, pictures, properties, procedures, links between them, and so on. Cognitively oriented research has shown that the personal concept definitions and concept images developed by students in elementary mathematics courses (e.g., courses in Calculus) can be far from the corresponding formal concept definitions (Tall, 1992). This said, as students progress further in university mathematics coursework (e.g., to courses in Analysis), they are typically expected to develop what early researchers in university mathematics education coined *advanced mathematical thinking*: i.e., the kind of thinking where concepts are specified through their formal definitions and their properties are reconstructed through logical deduction (ibid.). The development of this kind of thinking could be seen as part of making the transitions in Winsløw's (2006) model: that is, to develop a mathematical logos for the praxis typically learned in Calculus, and then develop more formal proof practices in Analysis, students would also need to develop advanced mathematical thinking.

To give a brief common example, consider the well-researched teaching and learning of the limit concept. Several studies have shown that after successfully completing an introductory Calculus course, students tend to define and view limits, not in terms of the formal ϵ-δ or ϵ-N concept definitions, but in terms of a variety of intuitive, dynamic, procedural, and erroneous concept images (e.g., Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991). For instance, students may think of a limit of a function or sequence as a list of values or a graph that gets closer and closer to a particular number. For such students, the following two statements may lack meaning or may simply seem obvious (Tall & Vinner, 1981):

$$a_n \to a \neq 0 \Rightarrow \frac{1}{a_n} \to \frac{1}{a}$$

$$\lim_{x \to a} f(x) = b \wedge \lim_{y \to b} g(y) = c \Rightarrow \lim_{x \to a} g\big(f(x)\big) = c$$

Understanding the statements, producing a formal proof of the first, and questioning the truth of the second, as might be expected in a first course in Real Analysis, would require a cognitive shift towards understanding limits through their formal definitions (ibid.).

Several empirical studies have shown that the transition to advanced mathematical thinking – and, by extension, the transition to practices that are more mathematical in nature – may not occur for students who succeed in Real Analysis courses (e.g., Edwards, 1997; Edwards & Ward, 2004; Moore, 1994; Przenioslo, 2004). Some students may still struggle to state and understand formal concept definitions. Others may be able to state such definitions and even provide coherent mathematical explanations of them, without appreciating their importance in specifying concepts or being able to operationalize them in proofs. It seems that even after taking university courses in Analysis, students may still favour the informal or erroneous images they had developed previously. They may remain convinced that such images determine the meaning of concepts and continue to rely on them most when solving tasks (even if they are inefficient). Essentially, these studies show that students may not develop a "mathematical understanding of concepts" in the context of a first course on Real Analysis, which indirectly suggests that students may also not be developing mathematical practices.

Cognitively oriented literature has explained students' difficulties in developing advanced mathematical thinking – and, by extension, mathematical practices – by identifying epistemological and cognitive obstacles that make the development of such thinking inherently challenging. For instance, the development of advanced mathematical thinking is generally seen as requiring a demanding shift from operational to structural conceptions (Tall, 1992). *APOS Theory*, which emerged in the field of university mathematics education, models this shift in four stages (as described in Cottrill et al., 1996):

1. **A**ctions;

2. **P**rocesses (actions that can be described and reflected upon without performing them);
3. **O**bjects (processes that are considered in their totality, whose properties can be investigated, and on which new actions can be performed); and
4. **S**chema (organized collections of actions, processes, objects, and links between them, which can be called upon in problem situations).

Sfard (1991), who proposed a similar model involving three stages (interiorization, condensation, and reification), argued that although well-trained minds (e.g., those of mathematicians) might be able to jump to a structural way of thinking about a concept, the most natural course of learning a concept is to understand it operationally before structurally (i.e., in terms of actions and processes before objects and schema). Sfard (1991) also stressed that moving from an operational to a structural conception is particularly challenging because it requires the student to see something familiar in a completely new way. Moreover, if students are to be able to solve more complex mathematical problems, they will need to gain a certain cognitive flexibility to move between different types of conceptions. Based on her theory, Sfard (1991) concluded that it is perfectly natural for students to linger in stages of insufficient understanding and mechanical drill. It would make sense, then, that the development of mathematical practices might be a lengthy process for some students, perhaps unattainable in the duration of a first course in Real Analysis.

1.3.5 Observations of Students' Practices

In our search through existing literature, we found very few empirical studies that directly explore (the nature of) students' actual practices after they succeed in a first course in Real Analysis.

We found two small-scale studies of students' practices that confirmed a basic prediction mentioned several times above: namely, the two transitions modelled by Winsløw (2006) might not be accomplished in a first Real Analysis course. To complement his study of the curricular organization of practices related to angles and trigonometry (as described in Section 1.3.1), Winsløw (2016) performed a semi-structured interview with one A-level master's student who had also served as a teaching assistant in his Analysis course. Recall that this Analysis course did not include any activities that explicitly intended to bring students through the first transition in Winsløw's (2006) model; in particular, there were no activities that invited students to use the theory studied in the course to construct the theoretical blocks for previously learned practical knowledge about angles and trigonometry. The results of the interview led Winsløw (2016) to conclude that most students, like the student he interviewed, would be unable to make this kind of transition on their own. Hence, a first course in Real Analysis may not bring students through the first transition for any topic, unless there are specific activities dedicated to this. Another small-scale study, by Timmermann (2005), confirmed that the development of more abstract Analysis techniques involved in making the second transition in Winsløw's (2006) model may take more time than is offered in a first Real Analysis course. He observed two pairs of students solving the following task:

Find all the functions $f: \mathbb{R} \to \mathbb{R}$ which satisfy $f(x) - f(y) \leq (x-y)^2$ for all $x, y \in \mathbb{R}$.

The pair that had just completed their first course in Analysis struggled with the task, approaching it with algebraic manipulation strategies that do not apply. The pair of students at the end of their second Analysis course, on the other hand, were able to approach and solve the task successfully. Timmermann (2005) suggests that the difference can be explained by the simple fact that the second pair had experienced more tasks and practiced more abstract techniques, whereby they were better equipped to understand and use mathematical definitions.

We located only one study that offered some characterization of the nature of the practices developed by students in a first course on Real Analysis. Although Weber's (2005a, 2005b) study stemmed from the cognitively oriented work described in Section 1.3.4, his results are more directly related to students' practices due to his choice to consider proof through the lens of task solving. Weber (2005b) characterized students' proof practices through the introduction of three *types of proof production*:

- *Procedural Proof Production*, where a student locates a proof of a statement that is similar in form and uses this as a template for producing a new one;
- *Syntactic Proof Production,* where a student logically manipulates mathematical statements and draws logical inferences without examining the representations of the objects to which the proof pertains; and
- *Semantic Proof Production*, where a student considers informal or intuitive representations of relevant concepts to guide their formal work and see why the statement to be proven is true or false.

Note that these types of proof production could be seen as belonging to the larger collections of reasoning types identified by Lithner (2008). Procedural and syntactic proof production seem like examples of imitative reasoning (in particular, reasoning based on identification of similarities and delimiting algorithmic reasoning, respectively); and semantic proof production seems like an example of plausible reasoning. Echoing Lithner (2000, 2003), Weber (2005b) pointed out that an imbalance in favour of procedural and syntactic proof production can deny students the opportunity to develop mathematical practices. Indeed, students who produce proofs in a primarily procedural manner "avoid difficult processes of deductive reasoning and understanding formal mathematical concepts by reducing proofs to following templates, changing variables, and filling in algebraic manipulations" (ibid., 354). While students taking such an approach may gain some practice in applying a technique, it is unlikely that they will develop the ability to generalize it to solve similar proof tasks or understand why the technique works to solve those tasks. Similarly, in a syntactic proof production, students may increase their facility in applying rules of inference, but they may still have no idea what they are doing and why.

In one study, Weber (2005a) investigated the evolution of students' actual proof productions as they progressed through their first Real Analysis course. With six students, he conducted individual semi-structured task-based interviews every two weeks, for a total of eight interviews per student. In his analysis of the interviews, he noticed how some students could eventually achieve a sophisticated understanding of a proof technique, even if they started by approaching it procedurally. One student, for example, started by learning the proof by induction technique by rote because it was so alien to him. At first, his only way of knowing if his own proof was correct or not was by thinking about how superficially similar it was to the ones he had previously observed in lectures. For instance, he found a question on his midterm to be unfair because he had only practiced induction with sums, but not with products. However, about twelve weeks into the course, this same student had acquired more general techniques for dealing with proof by induction tasks that enabled him to solve tasks that he had not encountered before; and two weeks later, he could give a convincing mathematical explanation of how and why the proof technique works. In contrast with this exemplary case study, Weber briefly notes that several students who first attempted to understand a certain proof technique procedurally did not know how it worked by the end of the course. The success rate of these students is not described, although Weber does mention a student who was able to get perfect marks on a midterm question (a proof of a certain limit) by applying a process for which she could not explain the mathematical validity. In sum, Weber's study points to the possibility that in a first course on Real Analysis, some students may indeed develop expected mathematical practices, while others may still be able to do well on exams using routinized techniques supported by limited and possibly non-mathematical theoretical discourses. This said, the study does not holistically characterize students' practices or provide explanations of why differences in students' practices may have occurred – for example, by analysing the activities that were provided to students, or the various approaches the students may have taken towards those activities.

1.3.6 Synthesis of Predictions

The literature concerning the evolution from Calculus to Analysis can be pieced together to paint a complex picture. Researchers seem to agree on one thing: There is a big difference between what students are expected to learn in introductory Calculus courses and more advanced courses in Analysis. One prediction is that, as a result of the compartmentalization of curricula, students experience a first course in Real Analysis, not as an invitation to update and extend their Calculus practices, but as an introduction to an entirely new set of practices, which is characterised by increased levels of rigour and formalism, and the necessity of mathematical theoretical discourses. When students take their first course in Real Analysis, almost everything about their learning environment can change: the nature of the textbooks used, the types of tasks proposed (on homework and exams), and the kinds of techniques permitted. And yet, the students may sometimes feel a strong familiarity with what they learned before; perhaps even experiencing learning Real Analysis as a process of relearning Calculus in a different, more challenging, way. Numerous obstacles may arise. On the one hand, students may hesitate to adopt the new practices, especially if they are not convinced of the need to do so: for example, if they feel they are simply being asked to replace previously developed informal techniques with more complicated formal techniques for solving the same type of task. On the other hand, students may interpret the formal representation of the practices in textbooks and lectures as falsely indicating a need to abandon informal reasoning altogether. To thrive, students will not only need to develop new ways of thinking about mathematical concepts (e.g., as objects and processes), but they may also need to change particularly resilient study habits (e.g., shifting from surficial to deep approaches). Such shifts may take time, leading students to underperform or fail the first time around. Nevertheless, there are some studies that suggest that it could be possible to pass a first Real Analysis course having memorized a particular subset of definitions, theorems, and proofs, or having developed new kinds of superficial and routinized practices.

All this said, and as far as our reading took us, it seemed that there was a lack of studies that holistically investigate both the nature of the practices developed by students in a first Real Analysis course and the factors that may be shaping the development of such practices. This is what we hoped to contribute in this thesis.

1.4 General Research Objectives

As stated in the Introduction, the present research project aimed to gain an understanding of

1. the nature (mathematical or otherwise) of the practices developed by students in a first Real Analysis course; and
2. the factors that may be shaping the development of such practices.

With our first research objective, we hoped to contribute to investigating the many predictions made above concerning how students' practices are expected to evolve as they advance in their training of mathematics; in particular, as they transition from Calculus to Analysis. More specifically, we wanted to examine if and in what sense students' practices become more mathematical (or not). We also wanted to look into some of the factors that may be shaping the development of those practices (second objective), with the aim of offering some possible explanations for our findings about the nature of students' practices. This would enable us to contribute to the field of university mathematics education, while obtaining results that might inform future instruction of courses or have implications for curriculum development.

The next chapter is dedicated to outlining the theoretical framework that explains and specifies the above objectives.

Chapter 2: Theoretical Framework

In this chapter, we present our theoretical framework, which is largely based on the *Anthropological Theory of the Didactic* (ATD) established by Yves Chevallard (1985, 1992, 1999, 2002). We start the chapter by portraying the essence of the ATD and why we chose it to frame our study (Section 2.1). Then we describe in detail the theoretical tools that we have used to make sense of our two general research objectives:[3] that is, we specify how we think about *the nature of the practices developed by students in a first Real Analysis course* (first research objective, Section 2.2) and *the factors that may be shaping the development of such practices* (second research objective, Section 2.3). We conclude by reformulating our objectives in light of the language and concepts afforded by the theoretical framework (Section 2.4).

Note that we have supplemented tools provided by the ATD with additional notions:

- we pose a definition of *having developed a mathematical practice* based on Chevallard's (1999) *praxeological model of knowledge as practice* (Section 2.2);
- we complement Chevallard's (1991, 1992) theories about the *institutionalization of practices* with the precise characterization of an institution offered by Ostrom's (2005) *Institutional Analysis and Development* (IAD) *Framework* (Section 2.3.1);
- we specify what we have come to see as an important distinction between *activity* and *practice*; in particular, we distinguish between didactic activities that remain *isolated* and those that form a *path to a practice* (Section 2.3.2); and
- we adopt a more nuanced interpretation of a student's *positioning* in a course institution than is provided by the ATD and the IAD framework, following the example of researchers (e.g., Sierpinska et al., 2008) who discuss the distinction between *studenting* and *learning* (Section 2.3.3).

These additional notions are further explained in the relevant subsections below.

[3] The ATD offers many more theoretical tools than we describe in this chapter. We chose a subset of tools that we thought would be helpful in achieving our research objectives, based on the previous work that inspired our study.

2.1 The ATD and Why We Chose it to Frame Our Study

When first learning about the ATD, it can be a challenge to acquire an understanding of what it is all about. One approach that we have found useful is to think about the significance of the title in parts. The ATD is "anthropological" because it seeks to understand human behaviour by analysing the contextual conditions that create and maintain said behaviour. The ATD is a "theory" in that it offers a collection of tools (i.e., models, concepts, and ideas) that can produce certain kinds of understandings of the human behaviour in question. Finally, the ATD is "of the didactic" since it was built to assist in understanding a particular subset of human behaviour: that which takes place when there is an intention to teach some subset of people a particular subset of some institutionalized knowledge (like mathematics).

There are several reasons why we turned to the ATD as the main framework for our study. One reason is that the ATD has been used by several researchers in university mathematics education whose work is foundational to ours (e.g., Bergé, 2008; Hardy, 2009; Winsløw, 2006). We outline some of this work throughout Chapters 1 and 2. Another reason is that the ATD provides theoretical constructs that enabled us to elaborate and specify both of our general research objectives. Recall that our first objective was to gain an understanding of the nature (mathematical or otherwise) of the practices students develop in a first course on Real Analysis. To do this, we needed to clarify what we meant by "practices" and how we would think about their "nature"; in particular, how we would characterize them as "mathematical" (or not). The ATD is helpful in this respect because it not only offers a general definition of a "practice"; through its notion of *praxeology* (Chevallard, 1999), it also provides a concrete way of (a) modeling "practices" that are fundamentally mathematical (i.e., that are enacted principally by mathematicians, as opposed to other inquirers), and (b) pinpointing whether and in what sense students' practices are mathematical or not. To assist us with our second research objective – trying to understand the factors that may be shaping students' practices – the ATD offers an *institutional perspective* (Chevallard, 1991, 1992): that is, it helps us to see the teaching and learning of mathematics as occurring within educational institutions whose aims and functioning impose conditions and constraints on the practices that can be taught and the practices that can be learned. Such ideas served as a foundation to our thinking about how a course institution (like a first course in Real

Analysis) enables and encourages students to develop certain kinds of (potentially non-mathematical) practices.

In the next sections, we elaborate on the theoretical constructs mentioned above and how we adapted them to specify our research objectives.

2.2 Mathematical Practices

As presented above, we wanted to think about *the practices students develop in a first Real Analysis course* in the sense of whether they are of *mathematical nature* or not. For this, we needed to specify what we mean by "practices" and what it means for them to be "mathematical" or not. This is the focus of this section.

Generally speaking, by "practices" we mean regularized and purposeful human actions. Practices can be thought of as "personal" (as in the practices developed by an individual) or "institutional" (as in the practices created, encouraged, and enforced in the context of *social institutions*). We provide a detailed description of an institution in Section 2.3.1. For now, it is sufficient to think of an institution as a relatively stable structural element of a society that has been established to organize human (inter)actions and orient them towards the achievement of certain outcomes. For instance, any profession (pure mathematics research, actuarial science, engineering, etc.) or form of education (primary school mathematics, secondary school mathematics, university mathematics, etc.) can be thought of as an institution. An individual is said to "know" something in relation to a certain institution if they can enact an institutional practice; that is, if they have developed a personal practice that is judged to be acceptable and worthwhile within that institution. It is in this sense that we equate "knowledge" and "practice" in the context of this thesis.

Theoretically speaking, Chevallard (1999) offers the notion of *praxeology* to model knowledge as practice. According to the model, any piece of knowledge (or practice) can be represented by a quadruplet $[T, \tau, \theta, \Theta]$ – called a "praxeology" – involving four interconnected, essential components:

- a *type of task,* T, which generates the need for the practice;
- the corresponding collection of *techniques,* τ, created to accomplish T;

- the discourses used to describe, justify, explain, and produce the techniques (i.e., their *technologies*[4], θ); and
- the *theories*[5], Θ, that serve as a foundation of the technological discourse.

Crucial to this representation of practice is that it comprises both a practical block (the know-how), $[T, \tau]$, called the *praxis*, and a theoretical block (the rational discourse), $[\theta, \Theta]$, called the *logos*.

To illustrate the praxeological model of a practice, consider the problem of finding the limit of a rational function at a point. In response, mathematicians (as members of a professional institution we refer to as Scholarly Mathematics) have produced a practice, including both the techniques that should be called upon to solve tasks of this type, and the discourses that enable them to communicate to and convince themselves and other members of the institution how and why the techniques work. A model of this practice as a praxeology is shown in Table 2.1.

T_M	Find $\lim\limits_{x \to c} \frac{P(x)}{Q(x)}$ where c is a fixed constant and $P(x)$ and $Q(x)$ are polynomials.
τ_M	Substitute c for x. If $\frac{P(c)}{Q(c)}$ exists, then that is the value of the limit. Otherwise, there are two possibilities: 1. $Q(c) = 0$ and $P(c) \neq 0$, whereby the limit does not exist; or 2. $Q(c) = 0$ and $P(c) = 0$, in which case factor $P(x)$ and $Q(x)$ and cancel common factors of the form $(x - c)$ to get a new rational function $\frac{R(x)}{S(x)}$. Repeat the process with this function.
θ_M	If $f(x)$ is a rational function and c is a real number such that $f(c)$ exists, then $\lim\limits_{x \to c} f(x) = f(c)$. If $f(x) = \frac{P(x)}{Q(x)}$ is such that $Q(c) = 0$ and $P(c) \neq 0$, then there is an infinite limit at $x = c$. If two functions f and g agree in all but the value c, then $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$.
Θ_M	The ϵ-δ definition of limits can be used to construct the relevant theorems and proofs, which rely more generally on the theories of real numbers, sets, and functions.

Table 2.1 A praxeological representation of a practice in the institution of Scholarly Mathematics.

[4] Chevallard (1999) uses the word "technology" to mean "rational discourse (or logos) of the technique."

[5] Our use of the word "theory" is abundant throughout this thesis, and it does not always reflect this particular meaning. Chevallard (1999) uses the word "theory" to refer specifically to the component of a practice that serves to justify the technology. In this thesis, we often use the word "theory" in a more general sense. For instance, when we say that a student is or is not operating according to the rules of "mathematical theory," we mean that the student is or is not proceeding based on "the definitions, concepts, theorems, principles, etc., that would be considered appropriate by mathematicians."

Now that we have introduced and exemplified the concept of a practice, we look at its nature in the sense of being (or not) mathematical. Based on the model of praxeology, we say that an individual *has developed a mathematical practice* if they can:

> Identify a given mathematical task, t, as belonging (or not) to a general mathematical type of task, T;
>
> Choose (recall, discover, construct, guess, etc.) and implement an appropriate mathematical technique, τ, to accomplish the task;
>
> Describe, in a mathematical discourse (called a technology), θ, how and why the technique works; and
>
> Acknowledge the existence and importance of mathematical theory, Θ, that supports the discourse.

The abundant use of the word "mathematical" throughout this definition is meant to emphasize that every component of the practice is put in relation with the institution of Scholarly Mathematics. Essentially, we are saying that an individual has developed a mathematical practice if for a task belonging to some area of mathematics, they can enact a practice in which each component would be considered sensible and suitable according to the rules and standards of mathematicians working in that area.

Note that enacting a mathematical practice, by our definition, only requires an *acknowledgement* of foundational mathematical theory, as opposed to, for example, a *description* of that theory. We made this choice because we see the theory component of a mathematical practice as including absolutely everything (i.e., *all* mathematical theory, up to the axioms) that has been created to support mathematicians' technologies. We decided that expecting any individual (including a mathematician) to describe all such theory is both impractical and nonsensical. In fact, we have found that an individual's acknowledgement of theory is often included in, intimately linked with, or difficult to distinguish from their technology (this is illustrated in the examples provided in the next paragraphs). Because of this, we tend to group the technology and theory components of a practice together, referring to the pair as the theoretical, rational, or explanatory discourse that supports the praxis. This is evident, for example, in certain parts of our methodology (as described in Chapter 4; see, e.g., Section 4.1.3 and Section 4.3).

To exemplify the above definition, suppose that a person is faced with the mathematical task of finding the limit: $\lim_{x \to 1} \frac{x-1}{x^2+x}$. If, in response, they enact $[T_M, \tau_M, \theta_M, \Theta_M]$ shown in Table 2.1, then we would say that they have developed a mathematical practice. We can imagine, for instance, that if a mathematician were asked to find $\lim_{x \to 1} \frac{x-1}{x^2+x}$, they would identify the task as one about finding the limit of a rational function at a particular point, which, in this case, can be solved using the technique of direct substitution. Moreover, if prompted to explain how and why their chosen technique works, they might say something like:

> The given function is rational and well-defined at the point of interest. So we can just substitute the point of interest into the function to find the value of the limit. There's a theorem that allows us to do this. It says that if $f(x)$ is a rational function and c is a real number such that $f(c)$ exists, then $\lim_{x \to c} f(x) = f(c)$. This can be proved using the limit laws, which are built up using the ϵ-δ definition of limits.

This entire discourse is an example of a mathematical technology: i.e., a mathematical answer to the question of how and why the chosen technique works to accomplish the given task. Within the discourse we also find examples of acknowledging mathematical theory, such as "a theorem," which can be proved using "limit laws," which are based on the "ϵ-δ definition of limits." As mentioned in the previous paragraph, notice that the acknowledgement of mathematical theory is intimately linked with the mathematical technology in this case. We might therefore simply refer to the above fictional citation as a mathematical theoretical (or rational or explanatory) discourse.

Previous research, some of which is outlined throughout Chapter 1, has identified different ways in which students' practices can be *non-mathematical*. Hardy (2009), Lithner (2000), and Orton (1983), for example, documented instances of Calculus students having learned to:

- identify types of tasks and choose techniques through superficial means (rather than the intrinsic properties of the objects involved);
- implement techniques based on a recall of an arbitrary set of steps to be followed (rather than the use of plausible reasoning);
- justify chosen techniques through perceived classroom norms (rather than a mathematical discourse); and/or

- trust in the authority of teachers, textbooks, and solutions to past exams (rather than the foundations set by mathematical theory).

As an example, we contrast the actions of an imagined mathematician described above with the actions of the students in Hardy's (2009) study. When asked to find $\lim_{x \to 1} \frac{x-1}{x^2+x}$, some of the students identified the task with a type characterized by an easily factorable rational expression, which necessitated some sort of algebraic technique. 20 of the 28 students tried factoring, seven of which did direct substitution first. Hardy (2009) specifies: "It seems that students were doing substitution not to find the limit or to characterize an indetermination, but because that is 'what you do first'" (p. 351). In general, the students' explanatory discourses were of the sort: "We do this because that's what our teacher showed us, and that's what we normally do for this kind of problem." Hardy (2009) concludes that such students had learned to behave "normally" rather than "mathematically." By our definition above, we would say that they had developed non-mathematical practices.

We expect that students develop practices that are more mathematical in nature as they progress further in their training in mathematics. We were interested in seeing if, to what extent, and in what sense this occurs in a first course on Real Analysis.

Of course, learning does not exist in a vacuum: there are a complex collection of factors that influence the kinds of practices that are developed by students in a given course. We now turn to describing how we think about such factors in the context of this thesis.

2.3 Factors Shaping the Development of Students' Practices

Our second general research objective was to better understand the factors that may be shaping the development of students' practices – in particular, their mathematical or non-mathematical nature, as defined in the previous section – in a first Real Analysis course. This section is dedicated to presenting the factors we chose to consider in this doctoral work.[6]

[6] We acknowledge the complexity of the phenomenon under study and the existence of other factors. Our choice of factors is based on taking the ATD as our foundational framework and being inspired to build on work that came before ours.

To assist us in characterizing the factors shaping the development of students' practices in any university mathematics course, we have found it useful to think of three interconnected layers of influence. The outermost layer is the broad institutional context responsible for the establishment of the course as an institution in and of itself (Section 2.3.1). The subsequent layer comprises the didactic activities implemented by professors to orchestrate the learning of students in the course (Section 2.3.2). And, finally, there are different ways in which a given student may position themselves in the course institution, which governs how they interact with the didactic activities offered to them (Section 2.3.3). We say that these layers are "interconnected" to highlight that we see them as mutually influential. For instance, the conditions and constraints imposed by the institutional context of a course may enable and encourage certain kinds of didactic activities (offered by professors) and positioning (adopted by students).

In what follows, we discuss each of the three layers in turn and exemplify how they might be thought of as shaping the nature of the practices (that can be) developed by students in a university mathematics course.

2.3.1 The Institutionalization of Practices

According to the ATD, all human practices are shaped by (and concurrently shape) the *institutions* where they live (Chevallard 1991, 1992). The teaching and learning of mathematics, for example, is seen as taking place in so-called *didactic institutions* (e.g., University Mathematics), whose aims, structure, and functioning serve to permit, promote, and perpetuate practices of certain types; in particular, practices that are based on, but may also differ from those produced and used in the related *professional institutions* (e.g., Scholarly Mathematics). After clarifying and exemplifying what we mean by an "institution," this section is dedicated to specifying how we see the specific didactic institution of University Mathematics as shaping the nature of the practices that are developed by students in a university mathematics course.

In a broad sense, we define an *institution* as a relatively stable structure that has been established by a society to direct human behaviour towards achieving certain outcomes. Our thinking about what constitutes an institution is based on a combination of the ATD (Chevallard, 1991, 1992), the *Institutional Analysis and Development* (IAD) *Framework* described by policy analyst Elinor Ostrom (2005), and empirical work that has used the ATD and the IAD framework to characterize their institutional theoretical perspective (e.g., Sierpinska et al., 2008). We were drawn to Ostrom's IAD framework because it offers a detailed, compact, and simple description of the universal building blocks of any institution. According to the framework, an institution is essentially made up of the *(inter)actions* between members who seek to collectively achieve certain *outcomes*. There are *guidelines* (put) in place, either implicitly or explicitly, which seek to direct the members, their actions and interactions, towards the outcomes. These guidelines may be of different natures: they may be regulatory (i.e., laid down by some authority as being required by some members), instructional (i.e., offering possible plans of action), or cultural (i.e., relating to the ideas, customs, and social behaviour of a community). To distinguish between these, Ostrom (2005) uses the terms *rules* (regulatory sense), *strategies* (instructional sense), and *norms* (community precept sense). Together, rules, strategies, and norms specify which classes of members (*positions*) are required, permitted, or forbidden to do certain things in relation to certain outcomes. Thus, the functioning of an institution – its actions and outcomes – have some order and predictability to them. There is an expectation that members will abide by the rules, or else face some sort of punishment for not doing so. Indeed, an institution will have some kind of *evaluation* mechanism put in place, which judges its success and that of its members.

In this thesis, an important example of an institution is what the ATD refers to as *Scholarly Mathematics* (see, e.g., Chevallard, 1991). Members of this institution, whom we simply refer to as mathematicians[7], (inter)act with the intended outcome of producing and using mathematical practices to solve novel problematic tasks. They are guided (and ultimately evaluated by their peers) on the basis of the strict rules of mathematical theory, varied strategies of mathematical problem solving, and norms such as insisting on validating the consistency of an argument or searching for a more elegant proof. In principle, members are positioned as peers, according to the kind of valuable expertise they contribute to the field.

When compared to Scholarly Mathematics, the defining features of the institution of *University Mathematics* are quite different. First, the intended outcome of University Mathematics is an instruction in a selection of *already-existing practices* that have been produced by mathematicians, often long ago. Second, the functioning of University Mathematics depends on

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[7] We recognize the diversity of those who produce and use mathematical practices. For instance, we acknowledge the complexity of the boundaries between mathematicians and some other professionals, or between mathematics and some other disciplines. For the purposes of explaining our theoretical perspective, we think this complexity can be ignored.

two complementary, albeit asymmetrical positions: the Professor, who is expected to teach (because they already know), and the Student, who is expected to learn (because they do not know yet). Third, the rules, strategies, and norms governing the (inter)actions of professors and students are *considerably coloured by institutional constraints* such as the curricular organization of knowledge, time limitations, and evaluation procedures. Think, for example, of how the need to evaluate students from year to year may lead the creators of a course to aim for consistency in final examinations, and how, as a result, professors may perpetuate norms about the kinds of commonly tested tasks, students may adopt strategies for tailoring what they learn to perceived norms, and it may become possible to succeed in the course (in the sense of receiving a passing grade) by following the rules of mimicking memorized solutions (rather than the rules of mathematical theory). This is not to say that the behaviour of mathematicians is not also shaped by institutional constraints (e.g., deadlines and publishing demands); it is simply to recognize what we perceive to be an imbalance in the power of such constraints in influencing the nature of the practices that are developed.

With his *Theory of Didactic Transposition*, Chevallard (1985) brought to light the transformation of practices as they migrate from a professional institution like Scholarly Mathematics into a didactic institution like University Mathematics. To model this transformation, he and other researchers (e.g., Bosch et al., 2005; Hardy, 2009) have since discussed five categories of practices, as outlined in Table 2.2.

Scholarly Mathematical Practices:	The depersonalized practices produced and used by mathematicians (i.e., members of Scholarly Mathematics).
Practices to be Taught:	The practices professors are expected to teach, as prescribed by curricula, course outlines, and textbooks.
Practices Actually Taught:	The practices professors actually teach, which can be gleaned by the interactions they have with their students, e.g., in lectures.
Practices to be Learned:	The practices students are expected to develop, the minimal core of which is indicated by assessments, e.g., assignments, midterms, and final exams.
Practices Actually Learned:	The practices students actually develop, which can be approximated by observation and analysis of student work.

Table 2.2 Different categories of practices that arise from the transposition of practices from Scholarly Mathematics into University Mathematics.

To exemplify this model, consider the praxeology in Table 2.1, which represents the *scholarly mathematical practice* generated by the problem of finding the limit of a rational function at a point (as discussed in Section 2.2). When transposed into University Mathematics, this practice is typically compartmentalized so that the praxis (i.e., $[T_M, \tau_M]$) is the main focus in Calculus courses, and the logos (i.e., $[\theta_M, \Theta_M]$) is addressed more deeply in subsequent Real Analysis courses. In particular, the ϵ-δ definition of limits and related formal proofs are usually not part of the *practices to be taught* in Calculus courses. Moreover, depending on the actual progression of a given course (i.e., what has been taught, the time remaining, and what is left to teach), a Calculus professor will often have the choice of which mathematical technologies they will include or skip in their lectures. In some cases, they might also support the technologies with informal explanations (e.g., they may convince students of certain theorems using graphical evidence or a worked-out example). While these (more or less mathematical) discourses are part of the *practices actually taught*, they are normally not part of the *practices to be learned* by Calculus students. Generally speaking, the students are expected to focus on mastering the techniques for finding limits of various types, including T_M (i.e., limits of rational functions at a point). The way in which student success is evaluated in a course may nevertheless indicate a more specified subset of minimal expectations. For instance, Hardy (2009) found that to do well on final examinations and ultimately pass the Calculus course she studied, students were only required to master the more particularized praxis represented in Table 2.3. Given this possibility of achieving success through a routinized praxis, void of a mathematical logos, Hardy was able to understand why the *practices actually learned* by the students she interviewed tended to be non-mathematical in nature (i.e., with superficial types of tasks, highly algorithmic techniques, and explanatory discourses of the sort "I do it this way because that's how I was taught to do it," as described in Section 2.2).

T_D	Find $\lim_{x\to c} \frac{P(x)}{Q(x)}$ where c is a fixed constant and $P(x)$ and $Q(x)$ are polynomials such that the factor $(x - c)$ occurs in both.
τ_D	Substitute c for x and recognize the indetermination $0/0$. Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

Table 2.3 A praxeological representation by Hardy (2009) of a minimal practice to be learned, resulting from a transformation of the scholarly mathematical practice in Table 2.1.

In this thesis, we aim to better understand the factors shaping the nature of the practices developed (i.e., the practices actually learned) by students in a first Real Analysis course. This section has introduced one layer of influence we consider: namely, the institution of University Mathematics, whose ecology has the power to shape the nature of the practices to be (and actually) taught by professors and the practices to be (and actually) learned by students. In particular, the mechanism for evaluating the "success" of a student in University Mathematics (i.e., their passing of courses, continued survival in the institution, and eventual entry into related professional institutions) can create a set of minimal expectations that permit and promote the development of certain, potentially non-mathematical, practices. In the following two sections, we zoom in on further layers of influence. First, we describe our theory on how minimal expectations may be set by professors and met by students in the progression of activities offered throughout a course. Then we theorize how, within the same institutional context and faced with the same collection of didactic activities, students may still develop practices of different natures depending on the positioning they adopt.

2.3.2 Paths to Practices formed by Activities

When scholars use the ATD, they typically employ the words "activity" and "practice" as synonyms. This may be because Chevallard (1999) describes praxeology in this way: i.e., as a model of any regularly accomplished human activity or practice. In our work, we have found that the semantic distinction between activity and practice is crucial. One of our key assumptions is that as students progress throughout any course in University Mathematics, they encounter numerous *activities* that progressively determine the kind of *practices* they develop. These activities may occur in students' independently driven work, in lectures and tutorials, in recommended exercises, or in past or present assessments. This said, experience, empirical research (e.g., Cox, 1994; Hardy, 2009), and theory in mathematics education (e.g., Chevallard, 2002) tell us that assessment activities can play a very special role: namely, in anticipation of evaluating students, professors typically offer a collection of activities that are flavoured to indicate the minimal core of what needs to be known, which in turn moulds the kinds of practices that students develop. Such practices may or may not be the ones considered desirable by the professor and/or professional institutions. Regardless, we hypothesize that the practices students develop are indeed shaped by the quantitative and qualitative nature of the activities they are offered. In what follows, we specify this idea by introducing a distinction between activities that are *isolated* and those that form a *path to a practice* (as we have done before, in Broley & Hardy, 2018). Then we discuss and illustrate how paths of activities might shape the nature of students' practices.

To define the notions of an *isolated* activity and activities forming a *path to a practice*, we consider a hypothetical example. Suppose that in an introductory Calculus course, students are asked to engage in the following three activities (taken from Stewart, 2008, p. 107):

> a_1: Evaluate the limit $\lim_{x \to 2} \frac{2x^2+1}{x^2+6x-4}$ and justify each step by indicating the appropriate Limit Law(s).
>
> a_2: Evaluate the limit $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$, if it exists.
>
> a_3: Evaluate the limit $\lim_{x \to 2} \frac{x^2-x+6}{x-2}$, if it exists.

If the students are completing such activities independently for the first time (e.g., as recommended exercises or as part of an assignment), we would expect their actions to be relatively localized and particular, driven by the singular goal of creating a solution to the problem at hand. As the students participate in more activities, an exposure to more problems of the same type may urge them to progress in the development of a related practice. Imagine, for instance, that the Calculus students are offered (in lectures, tutorials, textbook exercises, assignments, etc.) more activities like a_2: e.g., (also taken from Stewart, 2008, p. 107),

> Evaluate the limit, if it exists:
>
> $\lim_{x \to 4} \frac{x^2+5x+4}{x^2+3x-4}$; $\lim_{x \to 4} \frac{x^2-4x}{x^2-3x-4}$; $\lim_{t \to -3} \frac{t^2-9}{2t^2+7t+3}$; $\lim_{x \to 1} \frac{x^3-1}{x^2-1}$; $\lim_{x \to -1} \frac{x^2+2x+1}{x^4-1}$; …

As students complete more and more activities like this, we would expect them to recognize common characteristics that enable them to formulate a type of task and to identify future activities as tasks belonging (or not) to that type. In observing solutions (e.g., proposed by professors or in textbooks) and carrying out their own, the students might also extrapolate a general technique and begin to master its implementation. All the while, the students would also develop rational discourses that guide how they identify the type of task, and choose, implement, and validate the corresponding technique. Seeing activities like a_2 show up in review materials (e.g., past exams or review classes) might further encourage the students to study and routinize their developed practices in preparation for their evaluations. We say that such activities – that exist in relatively high quantity and in situations that are pertinent to a student's success (e.g., not only practice exercises, but also assessments) – form a *path to a practice*: they indicate to students that some kind of practice should be developed. In contrast, certain activities may be encountered by students only in rare or seemingly non-relevant (e.g., non-tested) situations. The action of accomplishing those activities may hence remain isolated and particular, never contributing to the development of a practice. Imagine, for example, if activities like a_1 were only encountered in a few recommended exercises at isolated, unique moments in a course. We would say that such activities are *isolated* (as opposed to forming a path to a practice).

The nature of the practice suggested by a path depends on many characteristics. It depends on the nature of the proposed activities, as well as the context within which the various activities take place. Activities appearing in previous exams or in a review class, for example, may have a stronger influence on the suggested practice than activities appearing in a textbook or on an assignment. The nature of the suggested practice also depends on what is made explicit about the activities and what is not: for example, the steps included (or skipped) and the discourses present (or absent) in professors' solutions. To illustrate what we mean, consider the problem of finding the limit of a rational function at a point. Suppose that, in relation to this problem, the past final exams of an introductory Calculus course only ever included activities like a_2. This was the case for the Calculus course in Hardy's (2009) study. Even if students had initially been exposed to some other kinds of activities about finding the limit of a rational function at a point (e.g., like a_1, a_3 or others), Hardy (2009) found that the emphasis on activities like a_2 in final exams shaped the practices developed by students. Students came to expect the rational functions in limit finding tasks to be factorable, and they expected to be able to solve the tasks by factoring, canceling common factors, and plugging in. Hardy (2009) also observed that the solutions provided to students did not always include an initial step of checking for an indeterminant form or a limiting value; and, in general, she found that none of the final exam activities required students to provide mathematical justifications for the steps they took or the techniques they used. Hence, she could make some sense of why students did not understand the usefulness of plugging in before factoring, nor could they formulate a mathematical explanation for their technique. Hypothetically speaking, we could imagine a student from Hardy's (2009) study explaining their solution to a_2 (i.e., evaluate

$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$) by saying something like: “First, I plug in two: because that’s what you do first. Then, I factor: if you can factor, you should factor. Finally, I cancel the common factors and plug in two again. That’s always how we do this kind of problem.” It is not surprising, then, that a moderately nonroutine activity such as “evaluate $\lim_{x \to 1} \frac{x - 1}{x^2 + x}$” elicited strange behaviour from Hardy’s (2009) participants, including plugging in one and then trying to factor. As discussed in Section 2.2., this behaviour was indicative of a superficial and routinized praxis void of a mathematical logos: i.e., a non-mathematical practice. Using the terminology introduced above, we would suggest that the practice was shaped by a path of activities that encouraged students to identify a type of task based on the algebraic form of the rational function being factorable, and to master a technique composed of an arbitrary list of steps that just seemed to work, at least well enough to succeed in the course.

Recall that our second research objective was to understand the factors that might shape the practices students develop in a first Real Analysis course. In this section, we focussed on one possible factor: namely, the types of activities offered to students throughout the course. Although students may seem to be learning mathematical practices, they may in fact be engaging in isolated activities or following paths to practices of non-mathematical nature. We conjecture that by changing the characteristics of the paths available in a course, it is possible to encourage practices that are more mathematical in nature. More generally, we believe that a certain collection of activities encourages the development of certain kinds of practices (and the non-development of others). Such a belief is reminiscent of Brousseau’s (2002) *Theory of Didactical Situations*, a first reading of which could be interpreted as implying that professors can create classroom scenarios that *necessarily* lead students to develop an aimed understanding (or, in our case, practice). We agree with Brousseau that creators and professors of courses have the important responsibility of reflecting on the objectives they would like to attain and carefully crafting a collection of activities that can guide students to reach those objectives. This said, we also recognize that students might take different approaches towards the activities offered to them, which may also influence the nature of the practices they develop. We elaborate on this idea further in the following section.

2.3.3 Students' Positioning

In the characterization of an "institution" proposed by the ATD (Chevallard, 1991, 1992) and the IAD (Ostrom, 2005), the notion of *position* is key. A position can be thought of as a title or a status, which defines a particular way of participating in the institution for any member who occupies the position. More specifically, when a member of an institution adopts a certain position, they are expected to engage in certain (inter)actions, abide by certain rules, strategies, and norms, and have certain aims. Depending on the intended outcomes of an institution, some positions will be explicitly recognized and offered to its members. As discussed in Section 2.3.1, the functioning of the institution of University Mathematics relies on the existence of two basic positions: the Professor (who is expected to teach) and the Student (who is expected "to learn"). This said, literature in mathematics education (e.g., Sierpinska et al., 2008) has suggested that there may be other positions, other than Student, available to students[8] taking a university mathematics course; moreover, what is learned by students may vary according to which of these positions they adopt. In the following, we turn to the literature to exemplify the different kinds of positioning a student may occupy in a course and how this may shape the way in which the student interacts with the activities offered to them, which can in turn shape the nature of the practices they develop. The literature we read leads us to focus on two contrasting positions: the *Student* (who *students*) and the *Learner* (who *learns*). While conducting our study, we nevertheless remained open to the existence of other possible positions. We provide definitions of the five positions we identified in our data analysis when we present our positioning framework in Chapter 5.[9]

In their work on student frustration in pre-requisite university mathematics courses, Sierpinska et al. (2008) introduced four positions that a student may occupy when speaking about their experiences in a course institution: they may identify as

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[8] We use "Student" with a capital "S" when we are speaking about the position. We use "student" with a lower case "s" when we are speaking about any person enrolled in a course, regardless of the position they adopt.

[9] As alluded to in this paragraph, our use of the word "position" does not reflect the use of the word in the ATD. As such, it may have been appropriate to introduce a different term. We were inspired to use the word "position" based on its use in the work by Sierpinska et al. (2008) and its conceptualization in the work of Ostrom (2005). While the ATD refers to the positions formally or explicitly defined by the didactic institution, teacher and student, our use and conceptualization of the term refers to the availability of more (unexpected and essentially different) positions, other than the institutionally recognized position of student. We plan to reflect on these ideas further in future work.

- a *Person*: a member of society at large, for whom mathematics is part of the world;
- a *Learner*: a cognitive subject, for whom mathematics is a mental activity to be shared with the teacher;
- a *Student*: a subject of a school institution, who must abide by its rules and norms, and for whom mathematics is a course; or
- a *Client*: a customer of the services of the classroom and the product of mathematics, who has the right to evaluate their quality.

The researchers suggest that a student could adopt several positions to varying degrees and in different situations. They emphasize, nonetheless, how adopting a particular position in relation to a set of activities can shape the nature of the practices developed. They note, for instance, how the participants in their study seemed to prefer not to validate their solutions to tasks (about absolute value inequalities) or to make explicit the theoretical underpinnings of their calculations. They argue that "from the position of Student, it is not mathematical truth that matters but 'a sufficient number of sufficiently good answers' on a test, obtained not by [mathematical] reasoning but by knowing which procedure to apply in a given problem" (p. 315). The procedure, as institutionalized in the didactic activities that had been offered to students, did not *need* (*need* in the sense of to succeed on the test) to be accompanied by a mathematical discourse. Having approached such activities from a position of Student (with the aim of passing the course), the participants of Sierpinska et al.'s (2008) study had developed non-mathematical practices. We predict that an orientation of Learner (with the aim of gaining understanding) towards the same set of activities, would have led to the development of practices of a different nature.

The potential disparity between the behaviours of a Student and a Learner has been explored by other researchers and theorists. Liljedahl and Allan (2013a, 2013b), for example, identified behaviours that mathematics students may adopt in their independent work (either in-class or during homework) that they classified as *studenting,* but do not actually lead to mathematical *learning*. These researchers based their studies on Gary Fenstermacher's (1994) distinction between what it means "to student" and what it means "to learn." Although Fenstermacher's theories seem to have been developed independently of the ATD and the subsequent refinements made by researchers like Sierpinska et al. (2008), they are consistent with the positions outlined above. In particular, Fenstermacher (1994) defines *studenting* as "a series of performances whose character is strongly influenced by the institutional and organizational properties of the school setting" (p. 1). In comparison, *learning* is "a series of performances whose character is determined by the properties of a discipline or subject matter, and the methods of inquiry appropriate to that discipline or subject matter" (ibid., p. 1). Combining these with the positions defined by Sierpinska et al. (2008), we could say that a Student *students* and a Learner *learns*. As a result, the Student develops practices that are strongly shaped by features of the course institution; for example, the paths of activities indicating the minimal core of practices to be learned, which may be more or less mathematical (as discussed in Section 2.3.2). The Learner, in contrast, develops practices that are comparable to mathematicians (i.e., the related scholarly practices), irrespective of the paths to practices offered to them.

To contrast the way in which a Student and a Learner may approach a given activity, Liljedahl and Allan (2013a, 2013b) introduce a distinction between *mimicking* and *reasoning.* Common classroom norms, such as taking notes during professors' demonstrations and then practicing similar examples in preparation for exams, can lead the Student to engage in *mimicking*. *Mimicking* means solving an activity, not by relying on understanding, but by following a solution pattern of some given example: "Students determine what goes in line N of their solution more from what is in line N of the worked example than from what is in line N-1 of their own work" (Liljedahl & Allan, 2013a, p. 6). As Sierpinska et al. (2008) would say, the rules of such a Student's game are not those of mathematical theory, but those of a solution pattern to be copied. This is not to say that a person who is *reasoning* may not also depend on prior examples. Inherent to the praxeological definition of practice (introduced in Section 2.2) is the belief that learning a mathematical practice requires some sort of recognition of activities as being similar (i.e., as belonging to the same type of task), and a certain degree of routinization of corresponding techniques. The difference proposed by Liljedahl and Allan (2013b) is that a person who is reasoning considers prior examples in a more holistic sense, rather than as a line-by-line set of instructions to mimic for the next problem. Moreover, when prompted to do so, they demonstrate "a good understanding of the mathematical relationships and skills at play" (Liljedahl & Allan, 2013b, p. 6). In the words of Sierpinska et al. (2008), the Learner plays by the rules of the mathematicians' game. In doing so, they may develop practices that are mathematical in nature.

Lithner's (2000, 2003, 2004, 2008) task solving framework, which is described throughout Chapter 1 (see Section 1.1), offers some useful extensions and nuances to Liljedahl and Alan's

(2013a, 2013b) concepts. What Liljedajl and Allan (2013a) call *mimicking* is comprised in Lithner's (2003) definition of *reasoning based on identification of similarities* (IS). In IS reasoning, it is through mimicking that the student implements a strategy for solving a given task. What Lithner's (2003) definition adds is a conceptualization of how a student may identify the example to be copied: i.e., through searching their notes, textbooks, or other resources (e.g., the internet) for examples that are similar in some superficial (rather than mathematically significant) way. In addition to this, Lithner (2000) introduces how we may think of mimicking when a student is not in the presence of a worked example: they may *reason based on their established experiences* (EE), i.e., their memory of the notions and procedures used in the learning environment. In fact, Lithner (2003) predicts that when a student does their independent work through IS reasoning alone, they will have no choice but to rely on EE reasoning in exams (or in task-based interviews). In Lithner's (2000, 2003) work, this situation is contrasted with students who regularly employ *plausible reasoning* (PR) when independently solving tasks: namely, students who regularly choose and implement strategies by carefully considering the intrinsic mathematical properties of the components involved in the solving of the task. This seems to align with what Liljedahl and Allan (2013b) simply refer to as *reasoning*. A subtle but significant note is that the specific word "reasoning" is used in a slightly different way by the two groups. While Liljedahl and Allan (2013b) use "reasoning" to mean drawing inferences and conclusions in a way that is logical and sensible *from the perspective of mathematicians*, Lithner (2008) uses "reasoning" to mean drawing inferences and conclusions in a way that is logical and sensible to *the reasoner*. We prefer the latter use of the word because it recognizes that the practices developed through both studenting and learning are supported by rational discourses; that is, both the Student and the Learner have (good) reasons for drawing the inferences and conclusions that they do (even if these reasons may be of a different nature).

Note that when we say that a person has adopted a particular position, we recognize that this may or may not have resulted entirely from a conscious choice. A crucial idea shared by all the researchers mentioned above is that the way in which a university mathematics course functions may encourage students to act like a Student rather than a Learner. Sierpinska et al. (2008) suggest that the fast pace of courses may leave little to no time for students to learn anything beyond the procedures emphasized in the activities offered to them. They also note, like Liljedahl and Allan (2013a, 2013b), that students may be drawn to mimicking over mathematical reasoning if professors urge them to take notes of the problems solved in class and to practice similar looking problems in preparation for examinations. Hardy's (2009) research adds that if students' success depends heavily on the grades obtained in final exams, and final exams tend to remain the same from year to year, then students may be incentivized to look for patterns and engage in strategic studying that does not lead to mathematical learning. All of this said, we also recognize that studenting (or learning) could require certain tendencies and skills that not all students possess. For instance, not all students may be as able to recognize patterns in past exams or be as willing to devote the time and effort required for doing so. In a similar vein, not all students may have the foundation in mathematics that may be needed to interact with activities as a Learner. Ultimately, we see the position a person adopts as possibly originating from a combination of sources, including conscious choice, institutional context, and the person's background, skills, and interests.

In this doctoral work, we seek to understand the factors shaping (the nature of) the practices developed by students in a first Real Analysis course. In the previous section (Section 2.3.2), we explain how we see the activities offered throughout the course as progressively encouraging the development of certain practices. In this section, we specified how students may actually develop different kinds of practices depending on the positioning they adopt in relation to the activities offered to them. Based on previous work in mathematics education, we introduced four possible positions: Person, Learner, Student, and Client. Then we discussed the existing theories on how the behaviours of a Student (i.e., studenting) might differ from the behaviours of a Learner (i.e., learning), and how this may result in practices of different natures. In general, we expect that in a given university mathematics course institution, there may be several positions available to a student. Moreover, the different positions may or may not coincide with the development of mathematical practices. In the next section, we synthesize the three factors we decided to consider in our research in terms of three principle underlying assumptions.

2.3.4 Synthesis of the Principle Assumptions Underlying our Work

The tools described in the previous three sections assist us in forming a theoretical perspective about the factors that may be shaping the development of students' practices in a first course in Real Analysis. This perspective can be summarized by the following three key assumptions:

The practices developed by students in a university mathematics course are shaped by

1. the conditions and constraints of the didactic institution of University Mathematics, which does not replicate the professional institutions (e.g., Scholarly Mathematics) on which it is based;
2. the activities offered by the professor of the course; in particular, the activities that form paths to practices and indicate the minimal core of what needs to be learned to be deemed successful in the course (i.e., to do well on assessments); and
3. the positioning students adopt (by conscious choice or not), which defines how they interact with the activities offered to them.

These assumptions are supported by empirical research in mathematics education (e.g., Cox, 1994; Hardy, 2009; Liljedahl & Allan, 2013a; Sierpinska et al. 2008), some of which is outlined throughout Chapters 1 and 2.

2.4 Reframing our General Research Objectives

At the end of the previous chapter (in Section 1.4), we outline our general research objectives. Based on the language and notions introduced in this chapter, we can reframe these objectives as follows:

In this doctoral work, we aim to gain an understanding of

1. the nature (mathematical or otherwise) of the practices actually learned by students who are deemed successful in a first Real Analysis course in the institution of University Mathematics; and
2. how these practices are shaped by the positions that students adopt and the activities they are offered.

Our theoretical framework was built through our understanding of the teaching and learning of Calculus, which has been studied extensively by researchers in mathematics education. The results presented in this thesis may contribute an understanding of the discussed concepts in the context of subsequent Real Analysis courses, allowing us to partake in the ongoing discussion of how and why students may develop practices of various natures as they progress in the institution of University Mathematics. Theoretically speaking, we expect our results to expand what the field knows about the ways in which students' practices may be (non-)mathematical, the kinds of positioning that might be available to students in a mathematics course, and how the design of paths of activities may lead students in different positions to develop different kinds of (un)desirable practices. When we discuss our results in Chapter 7, we also reflect on how the conditions and constraints of University Mathematics may be working to support or encourage students to adopt certain positions and professors to design certain kinds of activities. All the while, we keep in mind how the theoretical choices we made may have coloured the way in which we collected, analyzed, and interpreted our research data.

The study we conducted to address the above research objectives focussed on a particular Real Analysis course at a particular university. The next chapter is dedicated to presenting the particularities of this institutional context and describing how the particular course can be seen as an institution in and of itself.

Chapter 3: Institutional Context

Our study focussed on a core course (Real Analysis I, or RA I for short) offered at one large urban North American research university (hereafter called the University). Our choice to locate our study here was opportunistic and out of convenience. This chapter is dedicated to making explicit the relevant characteristics of this institutional context. We do this with several aims in mind. First, we hope to provide any reader with the information they need to be able to make informed critical analyses of our work. Second, such a detailed description of the context is necessary for those who wish to consider our results within other kinds of contexts or make comparisons with the results of other research. Lastly, we would like to be able to take into consideration the institutional context when we discuss our results in Chapter 7; in particular, and as mentioned at the end of Chapter 2 (see Section 2.4), we would like to be able to reflect on how the context may have played a role in influencing the positions adopted by students, the activities offered to them, and the resulting practices.

We begin this chapter by briefly describing the mathematics study options available at the University, which portrays the potential diversity of students who may take the RA I course (Section 3.1). Then we place RA I in the context of the curriculum, where it can be seen as one of eight mandatory courses for obtaining a specialization degree in mathematics (Section 3.2), and one course in the Analysis Stream (Section 3.3). This allows us to highlight how the University has deemed RA I to be a fundamental component of a student's training for work with advanced mathematics and mathematical research, no matter which specialization they have chosen, or how far they will progress in the Analysis Stream. Finally, we detail the characteristics of RA I that make it an institution: i.e., the intended outcomes, positions, and guidelines that govern the functioning of the course each time it is given (Section 3.4). We also address the fact that we studied particular instances of when the RA I course was given, in two recent, consecutive fall and winter terms.

3.1 Mathematics Study Options at the University

The University provides several options for studying mathematics through the Department of Mathematics and Statistics (hereafter the Department). Any student in the University can take any course offered by the Department, including RA I, if they are given permission to do so (such permission is normally granted by an academic advisor in the Department or the undergraduate/graduate program director). Alternatively, students can choose to study mathematics through a program: i.e., a collection of courses that enable one to attain either a minor, a (joint) major, or a specialization in mathematics. The major and specialization programs can lead to a Bachelor of Arts or a Bachelor of Science and are meant to be completed in three years by full-time students. In terms of specializations, the Department essentially offers four: Actuarial Mathematics (ACTU), Mathematical and Computational Finance (COMPFIN), Statistics (STAT), and Pure and Applied Mathematics (P&A). Only a specialization program can be completed as a so-called "honours program," i.e., involving a research project supervised by a professor in the Department, typically in the third and final year.

Evidently, the different mathematics study options are aimed at different populations of students. The courses required for the minor are meant simply to provide some foundational skills in mathematics and complement a student's main degree (e.g., Computer Science). The major program is principally geared towards students who know they will enter the workforce after graduating (i.e., not pursuing graduate studies). As a result, it has an applied flavor (e.g., emphasizing topics in modeling and symbolic computation) and includes several courses that involve the use of computational software. It also has less stringent admission requirements than the specialization programs. The ACTU and COMPFIN programs are particularly competitive, sometimes requiring students to have an average[10] of 85 - 90% (A- or A) to be admitted. In comparison, the minimum admission grade for the STAT and P&A programs is around 75% (B-). All specialization programs are designed to prepare students for graduate work (i.e., working with advanced mathematics and mathematical research) in the area of specialization.

3.2 The Core Courses of a Specialization in Mathematics

As mentioned above, any student in any program at the University can be permitted to take the RA I course we studied (provided they are deemed to have the pre-requisites). From a curricular point

[10] In general, the average is calculated based on the grades the student received in a subset of university-oriented courses taken within the last 1 – 2 years of study. The number of courses considered in the average can vary depending on where the student received their pre-university education. A variety of courses may also be considered (e.g., grades in English courses are typically considered), although the Department indicates that they pay special attention to grades in both mathematics and science.

of view, however, the course is part of the core courses in the four specialization programs offered by the Department. Table 3.1 lists these core courses, according to the year and term in which students in specialization programs are ideally expected to take them (of course, the actual progression of a student may not fit this model). Note that for the North American university student, an academic year is typically composed of two terms: the fall term (September – December) and the winter term (January – April). During each term, full-time students typically attend five different courses. Hence, Table 3.1 illustrates that students begin three-year mathematics specialization programs by attending the same kinds of courses; beyond this, the courses taken may vary greatly from specialization to specialization (this is of course what allows students to "specialize" in different areas).

Year 1		Year 2	
Fall Term	Winter Term	Fall Term	Winter Term
Multivariable Calculus I Linear Algebra I Statistics I	Multivariable Calculus II Linear Algebra II Statistics II	Real Analysis I	Real Analysis II

Table 3.1 The mathematics courses common to every specialization program offered by the Department of Mathematics and Statistics at the University.

The first year is dominated by common courses, which aim to introduce students to the areas of Multivariable Calculus, Linear Algebra, and Statistics. Statistics I and II are entry-level courses. Indeed, the only requirement to enrol in Statistics I is that students have taken or are taking a course equivalent to Linear Algebra I. The focus of Statistics I is Probability, including topics like combinatorial probability, discrete and continuous distributions, (conditional) expectation, random sampling, and sampling distributions. It is a prerequisite for Statistics II, which focuses on point and interval estimation, hypothesis testing, likelihood ratio tests, correlation, and regression. In comparison, the Linear Algebra sequence builds on knowledge related to systems of linear equations, matrices, and vectors in $\mathbb{R}^n$ (there are related courses listed as the prerequisites). Linear Algebra I is dedicated to studying general vector spaces over the reals and related topics: e.g., subspaces, linear dependence and independence, basis and dimension, linear transformations and isomorphisms, matrices (including diagonalizability), and systems of equations. Linear Algebra II expands on some of these topics in the context of vector spaces over the complex numbers. It also continues with the study of topics in general vector spaces: e.g., invariant subspaces, inner products and norms, operators (normal, adjoint, unitary, orthogonal), and Jordan canonical forms. The

Multivariable Calculus courses also build on prerequisite knowledge: in this case, the topics typically covered in single variable differential and integral prerequisite Calculus courses are generalized to $\mathbb{R}^n$. Multivariable Calculus I targets topics such as parametric equations, conic sections, curves and surfaces, limits and continuity in $\mathbb{R}^n$, the matrix representation of derivatives, partial derivatives, tangent planes, linear approximation, gradients, extrema (including Lagrange multipliers), and the classification of critical points. Multivariable Calculus II introduces techniques for calculating double and triple integrals, including switching between different coordinate systems (polar, cylindrical, spherical), as well as vector fields, line integrals, curl and divergence, parametric surfaces, and surface integrals.

As of the second year, only one course per term is required for all students in a specialization program. Together, Real Analysis (RA) I and II form an introduction to Analysis of single variable real-valued functions. Most of the topics listed in the course outlines are identical to those covered in typical single variable Calculus courses: e.g., sequences of real numbers, limits of functions, continuity, and derivatives (in RA I), and Riemann integration, series, sequences and series of functions, and power and Taylor series (in RA II). The difference is the expectation (made explicit in curricular documents) that the courses will introduce students to mathematical rigour and proofs. The beginning of RA I is dedicated to elements of proofs, set theory, and an axiomatic construction of the real numbers, and the course ends by introducing certain elements of topology. Note that in three of the four specializations (i.e., ACTU, STAT, and P&A), there is one other common second-year course: Numerical Analysis. This course serves as an introduction to topics such as: numerical algorithms, solutions of non-linear equations, fixed-point iterations, rate of convergence, interpolations and approximations, Legendre polynomials, and numerical integration and quadrature.

We do not speculate here why the Department has designated the eight courses in Table 3.1 as mandatory for all students in a specialization program. Whatever the reasons may be, we believe this sends the message that these courses – including RA I – form an important (in fact, necessary) component of a student's preparation for work with advanced mathematics and mathematical research, no matter which specialization they have chosen.

3.3 The Analysis Stream in the Department

In the Introduction and Chapter 1, we use the term "Analysis Stream" to vaguely refer to a collection of courses that occur in succession and relate to the general area of Analysis: e.g., courses in single variable Calculus, multivariable Calculus, Real Analysis, Metric and Functional Spaces, and so on. This made sense given that our work was founded on (a) research that has shown that students can do well in single variable and multivariable Calculus courses by developing non-mathematical practices (as discussed in Section 1.1); (b) the different observations and predictions of what may happen in subsequent Real Analysis courses (as discussed in Sections 1.2 and 1.3); and (c) our general interest in how students' practices evolve as they progress further in university mathematics coursework. This said, the Analysis Stream does not necessarily need to be restricted to a single sequence of courses and could naturally include courses that students take concurrently: e.g., courses in Complex Analysis, Numerical Analysis, Convex Analysis, or Measure Theory.

With either definition, the students who take RA I may have vastly different journeys through the Analysis Stream. As examples, consider the students who are enrolled in specialization programs. Before entering RA I, all such students will have presumably passed through the same checkpoints within the Analysis Stream: each will have been deemed successful in single variable Calculus and multivariable Calculus courses. In a similar way, due to prerequisites set out by the Department, they will all be expected to pass RA I before entering RA II. But beyond this, the students' journeys within the Analysis Stream can differ depending on their specialization. For instance, for students specializing in ACTU, COMPFIN, or STAT, RA I and II may mark the end of their journey in the Analysis Stream. For ACTU students, none of the courses for which RA II is a prerequisite can be counted for credit towards their degree. The COMPFIN students have the option to take some of the subsequent courses for credit: namely, courses in Metric and Functional Spaces, Measure Theory, or Convex Analysis. The STAT students can also choose to obtain a certain number of credits for such courses. The difference is that they can make their selection from a larger bank of courses for which RA II is a prerequisite: e.g., courses in the History of Mathematics, Calculus of Variations, Dynamical Systems, or Topology. The only students who are required by their program to continue in the Analysis Stream (beyond RA I and II) are those with a specialization in P&A, who must take the course in Metric and Functional Spaces. They also have a much larger number of credits that could be dedicated to continuing their study of Analysis, if they so choose.

3.4 Real Analysis I: The Institution and the Instances Studied

As stated at the end of Chapter 2 (see Section 2.4), we aimed to study the practices actually learned – their nature and how they are shaped by the positions students adopt and the activities they are offered – in a *first Real Analysis course* in University Mathematics. Hence why we focussed on RA I.

RA I can be thought of as an institution in the sense outlined in Chapter 2 (see Section 2.3.1). The course normally runs with two sections in the fall term and one section in the winter term, with about 50 students per section. The structure and functioning of the course remain relatively stable across sections and across terms. As indicated in the course outline, the intended outcome is to provide instruction in the following topics: proofs, set theory, real number axioms, sequences of real numbers, limits, continuity, and differentiability of single variable real-valued functions, and elements of topology. Members of the institution fill one of four basic positions: Professor, Student[11], Course Examiner, and Marker. Each section is led by a Professor, who is typically a full-time mathematics professor engaging in both teaching and research activities in the Department. Throughout a term (a period of 13 weeks), the Professor is required to provide 1.5-hour lectures twice a week or 2.5-hour lectures once a week at predetermined times. The Professor must also designate 1.5 hours each week towards being available in their office to answer their students' questions. As mentioned above, the position of Student can be filled by any student at the University who is given permission to do so, although it is often filled by a student in a specialization program. The Student is not required to attend lectures or "office hours." They are evaluated through weekly assignments, a midterm occurring about halfway through the term, and a final exam occurring sometime after the term has ended. The Student is provided with an overall grade by taking the best result from two possible distributions:

1. 10% assignments, 30% midterm, 60% final exam; or

[11] These are the basic positions available from the institution's point of view. As discussed in Chapter 2 (Section 2.3.3), literature in mathematics education (e.g., Sierpinska et al., 2008) has shown that there may be positions other than Student available to those who take a mathematics course. Part of our study aims to identify these different positions in RA I and how they may influence the nature of the practices developed.

2. 10% assignments, 90% final exam.

The Student is considered successful (i.e., to have passed the course), if they receive an overall grade of 50% or higher. The position of Course Examiner is also usually filled by a full-time mathematics professor, who has previously been in the position of Professor. The Course Examiner is responsible for ensuring that there is some consistency across sections and across terms to the manner in which the Student is assessed. Finally, the Marker assists the Professor with the grading and is typically a mathematics graduate student in the Department.

The guidelines outlined above ensure that there is a certain order and predictability to what occurs in a particular term and a particular section, when a particular professor implements the course with a particular group of students. At the time when our study took place, the two sections given in the fall term were taught by two different professors, who decided to give the same assignments, midterm, and final exam. In addition to this, they provided students with past midterms and past final exams that could be used to guide their studying. One of these two professors also taught the section offered in the winter term and provided students with similar assignments, (past) midterms, and (past) final exams. Despite the regularity described above, we recognize that our results may be coloured by the particular professors who were teaching RA I at the time of our study. Therefore, one has to be careful if generalizing our results to any other instance of RA I.

Nevertheless, our data and qualitative analyses, which are outlined in the following chapter, allowed us to systematically examine not only the nature of the practices developed by successful RA I students, but also how these practices may relate to the assessment activities (in assignments, past midterms, and past final exams) that were offered to the students, and the way in which the students positioned themselves towards those activities. We therefore expect our results to be informative and useful to professors of any course in Real Analysis (or, more generally, any course in University Mathematics): informative about the different kinds of positioning students may be taking in the course, and useful for the design of activities that may lead students to develop desired practices.

Chapter 4: Methodology

As outlined at the end of Chapter 2, our research aimed to gain an understanding of:

1. the nature (mathematical or otherwise) of the practices actually learned by students who are deemed successful in a first Real Analysis course in the institution of University Mathematics; and
2. how these practices are shaped by the positions that students adopt and the activities they are offered.

Our approach to addressing our first objective was to model practices developed by successful students of the Real Analysis I (RA I) course institution described in Chapter 3. To reveal such practices, we implemented a task-based interview (Goldin, 1997, 2000) with students who recently passed RA I. The interviews were also designed with our second objective in mind, allowing the interviewer to ask questions that would elicit the positions the students may have adopted in RA I and providing the researchers with data to reflect on how those positions, the activities offered to students, and the broad institutional context may have contributed to shaping the practices students had developed.

In what follows, we describe the steps we took to carry out the above approach. We started by analysing activities in assessment documents provided to students of RA I (Section 4.1). The purpose of this analysis was two-fold. On the one hand, we needed to understand the types of tasks that students are expected to solve in RA I to develop appropriate tasks for our task-based interview. On the other hand, we needed to understand the kinds of practices students might be expected to develop while engaging in RA I activities to reflect on how these may contribute to shaping the practices students actually develop. We then proceeded to creating our task-based interview, recruiting participants, and conducting interviews (Section 4.2). We conducted interviews, each approximately two- to three-hours in length, with fifteen students who passed RA I in recent, consecutive fall (ten students) and winter (five students) terms. Finally, we analysed the data we collected (Section 4.3).

4.1 Analysis of Assessment Activities

As explained above, our goal in carrying out an analysis of activities was twofold. Our main goal in analysing activities students are exposed to in RA I was to reflect on how these activities may contribute to shaping students' practices. However, this analysis was also key to our development of task-based interviews that would elicit practices actually learned by successful RA I students.

The sections below describe how we went about gathering and analyzing activities. We communicated with a professor of RA I to gather assessment activities (i.e., activities in assignments, past midterms, and past final exams) that were provided to students (Section 4.1.1). At first, we analyzed all the activities (Section 4.1.2), modelling the practices that students might be expected to develop based on what is available to them in the course (i.e., the activities, their solutions, and the course textbook). During this process, we noticed that there were two main types of activities: those that are *isolated* and those that contribute to a *path to a practice* (see Section 2.3.2 for our theoretical description of these notions). Since we are interested in the practices developed by students, we discarded the isolated activities and focussed on identifying and characterizing the paths (Section 4.1.3). Due to the high volume of paths and the time constraints we faced (completing a doctoral thesis and aiming to construct a two-hour long interview), we ultimately made a choice of paths, and activities within the paths, on which to base our task-based interview. Considering our research objectives, we selected paths and activities that we felt could have a significant influence on the nature of the practices developed by students. We also aimed to cover a range of topics so that we could see if the nature of students' practices might change from one topic to the next. Given our interest in how students' practices evolve as they progress further in university mathematics coursework – in particular, as they transition from Calculus to Analysis – several of the tasks we selected were also related to the practices students may have learned in previous Calculus courses.

4.1.1 Gathering Assessment Activities

As explained in Chapter 2 (see Sections 2.3.2 and 2.3.4), we are operating under the assumption that the activities offered to students by their professors – e.g., in lectures, assignments, (past) midterms, and (past) final exams – progressively shape the practices they develop. Given the constraints of completing a doctoral thesis, we decided that it would be unfeasible to study *all* such activities. For instance, we decided against conducting observations of lectures, although we thought such observations would have been relevant and interesting. Ultimately, we chose to base our analysis on the activities students received on weekly assignments and in the past midterms and past final exams they were given to use as study aids. We felt this was sufficient because it would allow us to model a significant subset of the practices communicated to students as being potentially relevant for succeeding in the course. In other words, we would be able to identify (some of) the practices that successful students were expected to develop.

In the summer term prior to the fall and winter terms when we conducted our study, we began communicating with one of the professors who would be teaching RA I in those fall and winter terms. From this professor, we gathered the assignments, (past) midterms, and (past) final exams that they had used when they previously taught the course (a few years back); and later on, the ones they used in the fall and winter terms corresponding to our study. The design of our task-based interview was based on the activities from a few years back and the fall term (recall that we ran our interviews at the end of the term, which is why we could use the activities offered *during* the term in the design of the task-based interview). The assessment activities from the winter term were sufficiently similar to the ones from the fall term for us to proceed without changing our analysis and the resulting task-based interview (for the purposes of interviewing students who succeeded in RA I in the winter term).

To offer a better idea of the materials gathered and activities analyzed, here is a list of the main data sources collected from the RA I course taking place in the fall term:

- the course outline and textbook;
- the eleven assignments that students were expected to complete on a weekly basis, each with 7 – 14 activities for a total of 103 activities;
- the seven midterms and six final exams that were given to students as study aids, each with 6 – 14 activities for a total of 110 activities (58 for the midterms and 52 for the finals); and
- documents containing solutions to assignments or study aids, whenever they were available.

Note that the numbers of activities indicated above are meant solely to provide a rough idea of the extent of our analyses. They will not be used to perform quantitative analyses. In fact, many of the activities contained several parts, each of which could have been counted as an activity on its own.

In the next two sections, we explain and illustrate how we went about analysing the above materials to model practices students might be expected to learn in RA I.

4.1.2 Modeling Practices to be Learned in Assessment Activities

Our first approach to analyzing the assessment activities we gathered was to model all the practices that a successful RA I student might be expected to develop based on what is available to them in the course: in particular, the activities we gathered, their solutions, and the course textbook.[12]

We examined the activities in the order that students would have presumably experienced them, as indicated in the course outline, alternating between reading the relevant chapters of the textbook and studying the activities and their solutions. For a given activity, we would read the statement and start thinking about the type of task it might represent. Then we would read through the proposed solution to try to uncover the intended technique. The solutions would sometimes include elements of justification, which explicitly indicated portions of expected technologies. However, the solutions often lacked steps and justifications. To better grasp the technologies students might be expected to develop, as well as the theories they might be expected to use and acknowledge in formulating their technologies, we filled in gaps in steps and justifications based on the material provided in the course textbook and the practices modelled in previous activities.[13]

To exemplify our process, consider the following assignment activity and the solution provided to students, with our additions in double brackets ($\mathrm{card}(A)$ denotes the cardinality of a set A):

Activity: Let I be a set of disjoint open intervals in $\mathbb{R}$. Prove that I is countable.

[12] Since we did not speak with the professors of RA I, we cannot claim that our model reflects what they would expect the students to learn.

[13] This process was not straightforward or linear, although it may seem to be described that way. It was not always clear to us how to phrase the type of task or technique represented in an activity and its solution, or what to include in the corresponding theoretical blocks. As with any modelling process, we were forced to make choices. The initial model we created served mainly to help us explore potential practices to be learned. In the next stage of analysis, as described in the next section, we refined the model to better serve our research objectives.

Solution: If $I_\alpha \in I$, ((since I_α is an open interval and $\mathbb{Q}$ is dense in $\mathbb{R}$,)) then I_α contains a rational number q_α.

Since the intervals are disjoint, if $I_\alpha \neq I_\beta$ then $q_\alpha \neq q_\beta$. (*) ((Suppose, by contradiction, that $q_\alpha = q_\beta$. Then $q_\alpha \in I_\alpha$ and $q_\alpha = q_\beta \in I_\beta$. So $q_\alpha \in I_\alpha \cap I_\beta \neq \emptyset$, which contradicts the assumption that I_α and I_β are disjoint.))

Thus $f: I \to \mathbb{Q}$ defined by $f(I_\alpha) = q_\alpha$ is an injection of I into $\mathbb{Q}$.

((Let $f(I_\alpha) = f(I_\beta) \Leftrightarrow q_a = q_b$. Then the contraposition of (*) implies that $I_\alpha = I_\beta$. Hence f is an injective function by definition.))

((So $\text{card}(I) \leq \text{card}(\mathbb{Q}) = \text{card}(\mathbb{N})$.)) Thus I is countable. ■

In the context of RA I, given the resources made available to students around the topic of cardinality, we proposed that the *type of task* represented by the above activity is "prove that a concrete set S is countable" and the *technique* illustrated in the solution is "construct an injection from S into some countable set C." Our additions to the solution show that in carrying out this technique, students might be expected to justify their steps by using or acknowledging various elements of the theory of proofs, sets, and functions: e.g., what it means for a set to be dense in $\mathbb{R}$, the density of $\mathbb{Q}$ in $\mathbb{R}$, the principles of contradiction and contraposition, the definition of an injective function, and the cardinality of $\mathbb{Q}$. This said, when trying to establish a *technology*, we thought specifically about why the modelled technique accomplishes the modelled type of task. Notice that in the above solution provided to students (i.e., without our additions in double brackets), there is no trace of these reasons. This is when we would turn to the course textbook to look for relevant theoretical elements. For the above technique, we identified the following two definitions as useful:

Definition 1: Let A and B be two sets. If there is an injective function $f: A \to B$, we say $\text{card}(A) \leq \text{card}(B)$.

Definition 2: When $\text{card}(A) \leq \text{card}(\mathbb{N})$, then A is said to be a countable set.

To construct a complete justification for the technique, as we had written it, we decided that we also needed a theorem, which was not stated in the textbook, but could be proved based on the theory therein (i.e., Definition 1 and the definition of an injective function, which can be used to prove that the composition of two injective functions is injective):

Theorem: Let A, B, and C be sets. If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

Using these theoretical elements, we could formulate a technology, which includes an acknowledgement of underlying *theory*: e.g., "If there is an injection from S into C, then we have $\text{card}(S) \leq \text{card}(C)$ (by Definition 1). Since C is countable, $\text{card}(C) \leq \text{card}(\mathbb{N})$ (by Definition 2). Thus, $\text{card}(S) \leq \text{card}(\mathbb{N})$ (by the Theorem). So S is also countable (by Definition 2)."

The result of carrying out the above process for every activity we gathered was a massive model containing 94 types of tasks, each with 1 – 7 corresponding techniques. We kept track of our model using a mind mapping software called iMindQ. For illustration purposes, a segment of the map is depicted in Figures 4.1 and 4.2 (the type of task related to the activity discussed above is in Figure 4.1 in the blue box labelled T2.6). As we analyzed the activities, we identified eight overarching themes: Proofs, Set Theory, Functions, Numbers, Bounds of Sets, Sequences, Limits of Functions and Continuity, and Differentiability. Attached to each theme, we listed all the types of tasks students may have encountered in the related activities (in blue); attached to each type of task, we listed all the techniques demonstrated in the solutions (in green); and attached to these, we indicated theoretical elements based on the content of the textbook that might be used or acknowledged in constructing the corresponding technologies (in orange and white).

Making this initial model of practices to be learned was helpful for a few reasons. First off, it gave us some training in constructing models of practices in the area of Real Analysis. Secondly, it allowed us to gain an overview of what was happening in RA I: i.e., the types of tasks that students were being asked to solve, the techniques they were being instructed to use, and the related theory that they could either access directly in their textbook or construct based on the content therein. Of course, it was very satisfying to see all the potential practices to be learned from the activities we gathered mapped in one massive document. This said, as we progressed further in constructing our model, we started to question its usefulness. More specifically, we wondered if every activity was really meant to introduce students to a type of task and a related practice to be mastered for exams (i.e., for succeeding in the course). We noticed, for example, that the number of activities related to a given practice could vary greatly, as could the context (i.e., assignment or past midterm/final exam) within which the activities were posed. Moreover, our mind map did not keep track of all the characteristics of the activities and their solutions that might influence the development of students' practices (e.g., patterns in activity statements and solution steps that might shape the types of tasks, techniques, and discourses actually developed by students). These realizations led us to a new stage of analysis, which we describe in the next section.

Figure 4.1 Our initial model of practices to be learned in RA I in relation to the theme of Set Theory. Types of tasks are in blue, techniques are in green, and references to underlying theoretical elements (shown in Figure 4.2) are in orange.

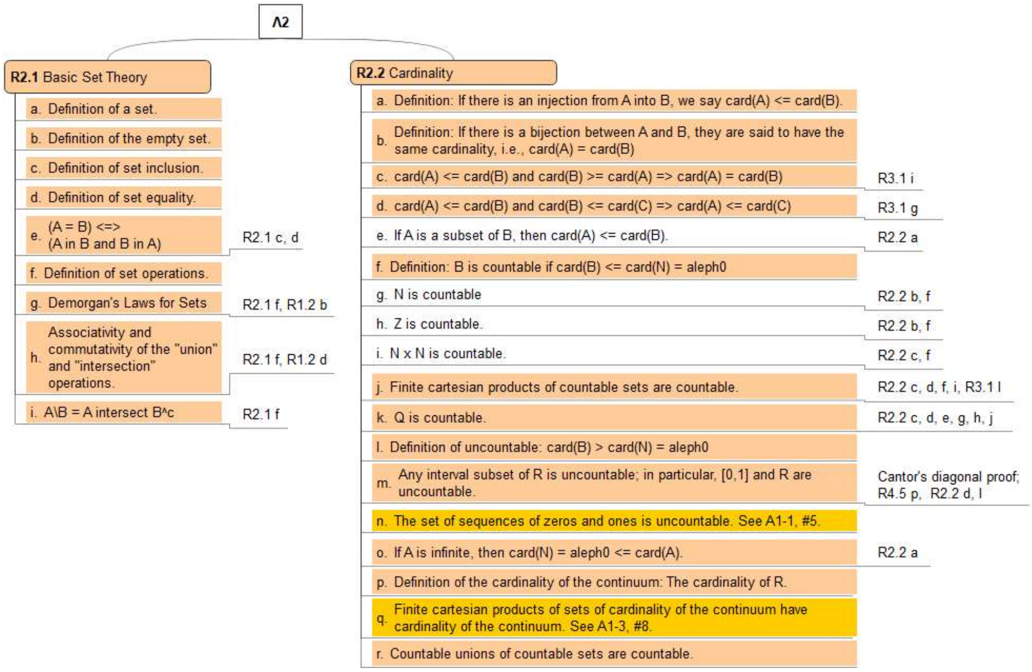

Figure 4.2 Theoretical elements in our initial model of practices to be learned in RA I in relation to the theme of Set Theory.

4.1.3 Identifying and Characterizing Paths to Practices

During our analysis of all the activities we gathered, we realized that there were two key types: those that are *isolated* and those that form a *path to a practice*. We define and exemplify these two notions in Chapter 2 (see Section 2.3.2), where we construct our theoretical framework. Since we are interested in the practices actually developed by successful RA I students, we decided to discard isolated activities and focus on the paths to practices. This section is dedicated to outlining how we went about identifying and characterizing the paths. We discuss how we determined if an activity was isolated or contributed to a path to a practice. We also illustrate how we characterized the practice suggested by a path. Near the end of the section, we list the general types of tasks representative of the numerous paths we considered before we ultimately selected a subset of the paths, and activities therein, to inspire the creation of our task-based interviews. Recall that our choices were influenced by a combination of factors, including our research goal of seeing how activities could influence the nature of practices developed by students, our interest in covering a range of topics (some of which are connected to topics covered in previously-taken Calculus courses), and the time constraints we faced (not only completing a PhD, but also conducting two-hour interviews with students).

In general, if a practice was represented in only one or two activities, which occur only on assignments (and not in past midterms or past final exams), then we considered those activities to be *isolated*, and unlikely to contribute to the development of the practice. As an example, consider the following two activities, which were posed on an assignment ("the axioms given in class" refer to those used to construct the real numbers as an ordered complete field):

Activity 1: Prove $[ax = a \text{ and } a \neq 0] \Rightarrow x = 1$ using the axioms given in class.

Activity 2: Prove using axioms given in class, $a, b, c \in \mathbb{R}$:

a) $a < b \Leftrightarrow a + c < b + c$;
b) $(a < b) \wedge (b < c) \Rightarrow a < c$;
c) $(a < b) \wedge (c > 0) \Rightarrow ac < bc$; and
d) $(a < b) \wedge (c < 0) \Rightarrow ac > bc$.

During our initial modelling process outlined in the previous section (Section 4.1.2), these two activities led to the creation of T4.5 (and the related links) depicted in the mind map portion shown in Figure 4.3 (below). Within our new approach to analysis, we labelled the above activities as isolated and concluded that successful RA I students might not have been expected to develop the modelled practice.[14] For the same reason, we made the same conclusion about the practices generated by T4.3 and T4.4, also depicted in Figure 4.3.

In contrast with the activities we deemed isolated, we identified collections of activities that seemed to be forming paths to practices. To illustrate what we mean, consider the collection of activities in Table 4.1 (below). Based on the theoretical perspective outlined in Chapter 2 (see Section 2.3.2), we expected students' actions in solving the assignment activities (first row of Table 4.1) to be relatively localized, driven principally by the aim of constructing a particular solution for the problem at hand. However, as students observe similar activities show up in the past midterms and past final exams used as study aids (such as those in rows two to four of Table 4.1), we think they are encouraged to develop a practice.

[14] Activities that are isolated in our sense could serve a purpose other than the development of a practice. They could, for example, expose students to new mathematical objects, invite students to engage in building specific theoretical blocks, illustrate to students how a particular theorem can be used to solve different types of tasks, or contribute to students' development of more general skills in constructing proofs.

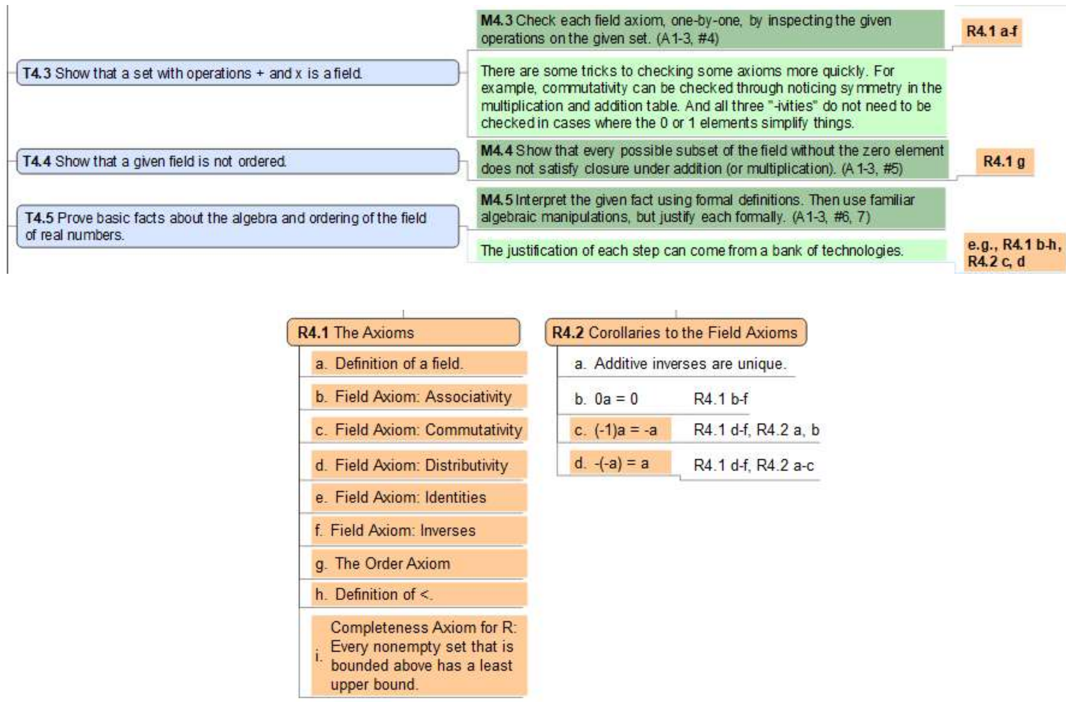

Figure 4.3 Some models of practices related to isolated activities and deemed unlikely to be mastered by successful RA I students.

Activities in Assignments	Prove that the set $A = \{a + b\sqrt{2} + c\sqrt{3}: a, b, c \in \mathbb{Q}\}$ is countable.	Let S be the set of all squares in $\mathbb{R}^2$ such that the sides are parallel to the coordinate axes, the center has rational coordinates, and the length of the side is a rational number. Prove that S is countable.
Activities in Past Midterms or Past Final Exams	Prove that the set $B = \{a + b\sqrt{2} + c\sqrt{3}: a, b, c \in \mathbb{Z}\}$ is countable.	Consider a family B of rectanges in $\mathbb{R}^2$ with sides parallel to the axes, rational centers, and side lengths also rational. Prove that the cardinality of B is at most $\aleph_0$.
	Prove that the set $S = \{x = a + b\sqrt{2}: a \in \mathbb{Q}, b \in \mathbb{Z}\}$ is countable.	Let P be the set of all parabolas in the plane which satisfy the equation $y = ax^2 + bx + c$ for some $a, b, c \in \mathbb{Q}$. By parabola we mean its graph: $\{(x, y) \in \mathbb{R}^2: y = ax^2 + bx + c, x \in \mathbb{R}\}$. Show that $\mathrm{card}(P) = \aleph_0$. Does the union of all parabolas in P cover the whole plane $\mathbb{R}^2$?
	Is the set $S = \{n\sqrt{2} : n \in \mathbb{N}\}$ countable or uncountable? Explain.	Let T be the set of all triangles in the plane which have all vertices at rational points. Show that $\mathrm{card}(T) = \aleph_0$. What would be the cardinality if we allow one non-rational vertex?

Table 4.1 Some of the activities in the path related to constructing proofs about the cardinality of sets (examples of two activities in assignments and six activities in past midterms or past final exams).

Once we identified a path of activities, we aimed to characterize the suggested practice. To do this, we would record the specific characteristics of the activity statements and solutions that we felt might shape the practices developed by students. For instance, when faced with the collection of activities shown in Table 4.1, we recorded the following observations:

- Type of Task: Students are expected to be able to prove that a concrete set S is countable or has cardinality (at most) $\aleph_0$. The elements of S are often defined by a finite number of parameters in $\mathbb{Q}$ and/or $\mathbb{Z}$ and/or $\mathbb{N}$.
- Technique: Students are instructed to construct an appropriate function between S and some countable set C (an injection from S into C, a surjection from C onto S, or a bijection between S and C). There is often an obvious choice of C: i.e., the Cartesian product of the sets corresponding to the parameters that define S. From this, students can construct a surjection from C onto S and conclude that $\mathrm{card}(S) \leq \aleph_0$ (i.e., S is countable). If needed, they can also state that S is infinite, whereby $\mathrm{card}(S) = \aleph_0$. (**)
- Theoretical Discourses:[15] Solutions are inconsistent in which assertions and justifications should be made explicit. While the countability of sets like $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ is typically taken for granted, solutions often contain notes such as "C is countable because the product of countable sets is countable," which might be justified as "a theorem proved in class." Students are expected to state the nature of the function they construct and the implication that follows. Some justification for the nature of the function is often, although not always, provided (e.g., the natural surjection from C onto S is claimed to be a surjection "since S is defined by the parameters"). Students also do not need to explain why the constructed function or S being infinite implies the conclusions about the cardinality of S (e.g., by referring to definitions and theorems like the ones in Section 4.1.2).

[15] As discussed in Chapter 2 (see Section 2.2), we often prefer to group the technology and theory components of a practice together. We did this at different moments in our methodology, not only because it was difficult to pull apart the two closely related components, but also because we did not see the use of trying to do so at that moment. In fact, when we tried to separate technology from theory, our choice of boundaries often seemed arbitrary. In any case, we expected that when it came time to present and discuss our results, we would be able to separate the components if it was helpful for addressing our research objectives.

We are not suggesting that these are the only interesting observations that can be made about the activities shown in Table 4.1. They are examples of the kinds of observations we made in attempt to characterize the practices that students were being led to develop through a path.

The activities shown in Table 4.1 are representative of a rather narrow path: that is, a path that might lead to the development of a superficial type of task and a routinized technique lacking a mathematical explanatory discourse. Evidently, identifying and characterizing such narrow paths was useful for our goal of constructing task-based interviews that would enable us to inquire into whether the practices developed by students were mathematical or not and how these practices may have been shaped by the activities students were offered. This said, the paths were not always as narrow as Table 4.1 suggests. There were often several additional activities and solutions that could serve to expand the type of task, expose students to other possible techniques and technologies, and even invite them to develop relevant theoretical discourses. We made note of these activities and the kinds of expansion or exposure they may have offered. For instance, in relation to the example discussed above, there was one seemingly isolated activity – prove that if a set S is infinite, then it contains a countably infinite subset – which can be used to support a theoretical discourse related to (**) on the previous page (since it implies that if a set S is infinite, then $\aleph_0 \leq \mathrm{card}(S)$). There were also a couple solutions where a set was shown to be countable, not by the technique of constructing an appropriate function, but by the technique of rewriting it as a countable union of countable sets. In addition to this, there were activities that invited students to prove either the countability of other kinds of sets (an example was given in Section 4.1.2), or other kinds of statements about the cardinality of sets: i.e., that a set is uncountable or, more generally, that two sets have the same cardinality. In one such activity, students were asked to work with sets described through abstract properties alone, which needed to be unpacked and used in carrying out the technique (i.e., prove that if $\mathrm{card}(A\backslash B) = \mathrm{card}(B\backslash A)$, then $\mathrm{card}(A) = \mathrm{card}(B)$).

We engaged in the process described above for many other collections of activities. To illustrate the extent of our analyses, each type of task listed below is representative of a wide path we identified and characterized:

- Negate a formal mathematical statement.
- Determine the truth of a formal mathematical statement (using truth tables).

- Prove a statement about the cardinality of sets.
- Determine and/or prove the (ir)rationality of a specified or abstract number. (*)
- Prove a statement (by induction).
- Find (and prove) the supremum and infimum of specified sets. (*)
- Prove abstract relationships between the supremum and/or infimum of abstract sets.
- Find and/or prove limits of specified or abstract sequences. (*)
- Show that a sequence with a particular property exists. (*)
- Find the limit superior and limit inferior of a specified sequence.
- Prove abstract relationships between the limsup and/or liminf of abstract sequences.
- Find and/or prove limits of specified or abstract functions.
- Prove a function is continuous at a point (by the Cauchy or Heine definition). (*)
- Prove a function is differentiable (or not) at a point, or on $\mathbb{R}$. (*)
- Prove a functional inequality on a certain domain (using the Mean Value Theorem).
- Prove a function is (or is not) uniformly continuous on a certain domain.
- Show that a function has a certain number of fixed points, or an equation has a certain number of solutions, or two functions have a certain number of intersections, or a function has a certain number of zeros on a given domain. (*)

As mentioned before, due to the high volume of paths we identified and the time constraint of conducting a two-hour task-based interview with students, we eventually chose a varied subset of paths on which to base our study. These are marked by a (*) in the above list.

Recall that we had two aims when we performed our analysis of activities in assessment documents. First, we wanted to be able to design task-based interviews that would elicit practices developed by successful RA I students. Second, we wanted to be able to reflect on how the activities may have contributed to shaping the nature of those practices. Throughout Chapter 6 (in Sections $6.i.1$ for $i = 1, 2, 3, 4, 5, 6$), we present our analysis of the paths to practices related to each interview task, with the goal of reflecting on how they may lead students to develop certain kinds of practices. In the next section, we discuss the creation and implementation of our task-based interview.

4.2 Task-Based Interview

As the name implies, the distinguishing feature of a *task-based interview* is that the interactions between the interviewer and the interviewee are centered around one or more tasks. The tasks are introduced by the interviewer to the interviewee in some preplanned way, and the latter is expected to think aloud (i.e., to express verbally their thoughts) as they engage with the tasks. To maximize the observation of the interviewee's spontaneous behaviour in solving tasks, the interviewer refrains from intervening. This said, interventions of various kinds are often carefully planned and implemented to gain more information than the interviewee offers naturally. To better understand the task-based interview approach and its relevance in mathematics education research, we turned to the pioneering work of Gerald Goldin (1997, 2000). Goldin (2000) argues that

> in comparison with conventional, paper-and-pencil test-based methods, task-based interviews make it possible to focus research attention more directly on the subjects' processes of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce. (p. 520)

It is exactly this possibility – of focussing our attention on the interviewee's task solving processes – that made the task-based interview an obvious methodological tool for our research.

Indeed, task-based interviews would enable us to systematically observe students in real-time, as they identify a given activity as a task belonging (or not) to a particular type of task, choose and implement a technique (perhaps among several considered) for accomplishing such a task, and determine whether or not they have truly accomplished the task (through some theoretical discourse). Ideally, the "think aloud" norm would help us bear witness to all sorts of phenomena that would be impossible to observe by looking only at students' written solutions: for instance, how students come to decide on a technique (e.g., by relying on their established experiences or by considering intrinsic properties of the objects involved), their way of carrying out the technique (e.g., by proceeding mainly through memory or by using mathematical reasoning), and the degree to which they mathematically validate their techniques (if at all). For cases where students forget to reveal their thoughts or certain elements of their descriptive or explanatory discourses do not naturally come to mind during the solving process, we could plan interventions aimed at soliciting them. The interview context would also afford us the opportunity to ask students more general reflective questions about how they perceive mathematics, the RA I course, their relationship to these, and so on. In the end, we would have a rich data source for inferring not only some of the practices the students had developed, but also the positions they had occupied in RA I. Based on these inferences, we could then characterize the nature of the practices (our first research objective); and, combined with our analysis of assessment activities (described in Section 4.1), we could reflect on how students' positioning and the activities they were offered may have played a role in shaping those practices (our second research objective).

When introducing the concept of a task-based interview, Goldin (1997, 2000) suggests that it can be considered as "scientific" an approach as large-scale quantitative studies, capable of producing generalized understandings of mathematics learning, insofar as it is replicable. He urges researchers not only to carefully design, conduct, and analyze the interviews, but also to provide detailed descriptions of the entire process so that it can be critiqued, improved, and applied in other studies. This is what we intend to do in the remainder of this chapter. We start, in the next two sections, by describing how we created our task-based interview (Section 4.2.1) and implemented it with a particular group of students (Section 4.2.2).

4.2.1 Creation

We aimed to create a two-hour[16] task-based interview that, when implemented with students who recently passed RA I, would elicit (a) (the nature of) the practices they had developed; and (b) the position(s) they had adopted. Crucial to (a) was a choice of interview tasks that would be recognized by successful RA I students as requiring practices they had developed, while at the same time allowing us to detect the nature of those practices; in particular, whether they are mathematical or not. In what follows, we discuss how we used the analysis of assessment activities outlined in Section 4.1 to choose such tasks, which are *recognizable*, but also *deceptive* (Section 4.2.1.1). Then we describe in detail the interview protocol we created to guide the interactions between the interviewer (i.e., the doctoral student), an interviewee (i.e., a successful RA I student), and the chosen tasks (Section 4.2.1.2). In light of (b) – our interest in eliciting students' positioning in RA I – the interview would not only center around the interviewee solving the chosen tasks; the

[16] We thought that two hours was enough time for us to collect a sufficient amount of data for addressing our research objectives, but not too much time to expect students to maintain their engagement in solving mathematical tasks and responding to reflective questions.

interviewer would also ask reflective questions with the goal of soliciting the interviewee's perceptions, of mathematics, of RA I, of their own actions in relation to these, and so on.

4.2.1.1 The Tasks

The Tasks we Chose for our Task-Based Interview	Related Paths to Practices (as listed in Section 4.1.3)
1. Is $\sqrt{8}$ rational or irrational?	Determine and/or prove the (ir)rationality of a specified or abstract number.
2. Show that the function $f(x) = e^x - 100(x-1)(2-x)$ has 2 zeros.	Show that a function has a certain number of fixed points, or an equation has a certain number of solutions, or two functions have a certain number of intersections, or a function has a certain number of zeros on a given domain.
3. Let $A = \{x_n \in \mathbb{R} : n = 1, 2, \ldots\}$. a) Under what conditions is $\lim_{n\to\infty} x_n = \sup A$? b) Give an example of A where the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$.	Find and/or prove limits of specified or abstract sequences. Show that a sequence with a particular property exists. Find (and prove) the supremum and infimum of specified sets.
4. Let $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \ldots\right\}$. a) Prove that for any $p > 1$, A is unbounded above. b) Prove that for $p = 1$, $\sup A = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$.	
5. Find the limit $$\lim_{n\to+\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n}$$ where $a \in \mathbb{R}$ and $(x_n)_{n\in\mathrm{N}}$ is any sequence of real numbers.	Find and/or prove limits of specified or abstract sequences.
6. a) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ For what values of p is g continuous on $\mathbb{R}$? b) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ For what values of p is g differentiable on $\mathbb{R}$? c) What if $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$? For what values of p is g continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?	Prove a function is continuous at a point (by the Cauchy or Heine definition). Prove a function is differentiable (or not) at a point, or on $\mathbb{R}$.

Table 4.2 The tasks we chose for our task-based interview and the related paths we analyzed.

Of all the controlled variables in the design of a task-based interview, the tasks are perhaps the most fundamental in determining the kinds of behaviours that can be observed and the degree to which such behaviours are meaningful in a specific research study. In the words of Goldin (1997),

> we simply have the choice of proceeding unscientifically, choosing tasks that seem interesting and just "seeing what happens," or trying to proceed systematically with tasks explicitly designed to elicit behaviors that are to some degree anticipated. (p. 58)

In our study, we wanted a task-based interview that would enable us to examine the nature (mathematical or not) of the practices developed by successful RA I students. Following the example of previous research (e.g., Hardy, 2009), this meant that our interview tasks needed to be:

- *recognizable* to successful RA I students (so that the students would be encouraged to consider using practices that they had developed in RA I); but also
- *deceptive* to successful RA I students (so that when the students would try to use practices they had developed in RA I, any superficiality or routinization of the practices would be revealed).

Generally speaking, we chose tasks that mirrored activities within the paths to practices that we analyzed (to achieve recognizability), but that also differed from those activities in some significant way (to achieve deception). Thus, the design of our interview tasks would also support our reflections on how the activities offered to students may have played a role in shaping the nature of their practices.

To illustrate how we designed tasks for our task-based interview, consider the first task (Task 1) listed in Table 4.2 above: Is $\sqrt{8}$ rational or irrational? We expected Task 1 to be *recognizable* to successful RA I students because it resembles activities in a path we identified and represented in Section 4.1.3 by the general type of task: Determine and/or prove the (ir)rationality of a specified or abstract number. More specifically, our analysis of this path found that successful RA I students would have developed a practice for solving the specific type of task "prove that a given real number c involving a root is not a rational number" (see Section 6.1.1) and we expected the students to call upon this practice when faced with Task 1. To help us examine the nature of this practice (and how it may have been influenced by the nature of activities offered to students), there are two key characteristics of Task 1 that make it *deceptive*:

1. the phrasing of the task. We could have phrased Task 1 as: Prove that $\sqrt{8}$ is not a rational number. All the assessment activities that resembled Task 1 were either phrased in this way or could have been (because students were always asked to consider an irrational number). We chose to phrase Task 1 as a question so that we could see if successful RA I students would think about the (ir)rationality of $\sqrt{8}$ to inform their choice to prove that $\sqrt{8} \notin \mathbb{Q}$. Upon seeing Task 1, we expected students to be deceived into ignoring the question and immediately starting to implement a proof technique they had developed (without thinking about if $\sqrt{8}$ is rational or not). This would provide a hint that (a) students may have developed superficial means for identifying the given task with a praxis they had learned (e.g., seeing the square root of a number and the word "rational"); and (b) students may lack the understanding required to base their choice of technique on the intrinsic properties of the given number (e.g., an understanding of basic types of irrational numbers and how algebraic operations can be used to form more). This could be confirmed through follow-up questions about (a) whether students thought about the (ir)rationality of $\sqrt{8}$ before jumping into their proof; and (b) whether they had other ways of solving Task 1 (i.e., of determining if $\sqrt{8}$ is rational or irrational).
2. the form of the number considered in the task. In our analysis of assessment activities, we observed a narrow path that may have led successful RA I students to develop a routinized praxis for proving the irrationality of numbers of two particular forms: $\sqrt{p}$ (or $\sqrt{p \cdot q}$) and $\sqrt{p} + \sqrt{q}$ (or $\sqrt{\sqrt{p} + \sqrt{q}}$), where p and q are specified distinct prime numbers. Table 4.3 (below) illustrates the kinds of steps students would have seen in solutions they were provided. Mathematically speaking, we could model the corresponding technique as "assume the number is rational and derive a contradiction," where the manner of deriving a contradiction differs: either
    - the number is assumed to be an irreducible fraction, m/n, but it is shown that m and n share a factor greater than 1 (this is the approach for $\sqrt{p}$); or

- the number is assumed to be a fraction, m/n, but algebraic manipulations lead to the conclusion that a known irrational number is also rational (this is the approach for $\sqrt{p} + \sqrt{q}$).

We expected the form of $\sqrt{8}$ to deceive students into trying to use the approach for $\sqrt{p}$, even if the approach for $\sqrt{p} + \sqrt{q}$ can lead to a relatively simple solution (if one takes for granted that $\sqrt{2} \notin \mathbb{Q}$). In any case, it would not be possible to successfully prove that $\sqrt{8} \notin \mathbb{Q}$ by simply repeating the steps for numbers of the form $\sqrt{p}$ or $\sqrt{p} + \sqrt{q}$ (Table 4.3 shows the adaptations needed for $\sqrt{8}$). Hence, we would gain insight into the degree to which students had developed a technique that was a rigid set of memorized steps lacking mathematical explanatory discourse.

Sections $6.i.1$ of Chapter 6 ($i = 1, 2, 3, 4, 5, 6$) are dedicated to providing a detailed discussion of each task in Table 4.2, how it relates to the assessment activities and paths to practices we analyzed, and what we expected to observe as successful RA I students tried to solve the task. In doing this, we develop models of practices to be learned and reflect on how this may have contributed to shaping the practices actually learned by successful RA I students.

As mentioned above, our ultimate choice of interview tasks was guided by several constraints (e.g., the volume of material covered in RA I and our choice to conduct interviews of approximately two hours in length), as well as our interest in covering a range of topics so that we could see if the nature of students' practices might change from one topic to the next. We guessed that different students may have different strengths; thus, covering as many topics as possible would also increase the opportunities students would have to demonstrate their knowledge. We also wanted the tasks to progressively increase in complexity: the first task needed to serve as a "warm-up" (i.e., relatively easy for every participant), while the last task could be challenging for any participant. Note that the collection of tasks we chose is not intended to be representative of *all* the practices to be learned in RA I, nor the *most important* ones (whatever that could mean according to different perspectives). In fact, our overarching interest in how students' practices evolve throughout university mathematics coursework (as sketched in the Introduction and throughout Chapter 1) led us to include several tasks that students might link with practices studied in Calculus courses. In this way, we might observe how students coordinate the practices to be learned in RA I with the practices they had developed previously. While this is not the focus of our study, we expected this to further support our reflections on how students' practices are shaped by their positioning, the activities they are offered, and the broad institutional context. We discuss this at different instances of our analysis in Chapter 6 and our discussion in Chapter 7.

<table>
<tr><th colspan="2">Steps for Solving Specific Types of Tasks in RA I Assessment Activities</th></tr>
<tr><td>Prove that $\sqrt{p}$ is not a rational number, where p is a specified prime number.</td><td>Prove that $\sqrt{p}+\sqrt{q}$ is not a rational number, where p and q are specified distinct prime numbers.</td></tr>
<tr><td>Assume $\sqrt{p}=\frac{m}{n}$ where m and n are integers without common factors.
Then $p=m^2/n^2$ or $pn^2=m^2$.
So m^2 is divisible by p and m is also divisible by p. $m=pk$.
We obtain $pn^2=p^2k^2$ or $n^2=pk^2$.
So n^2 is divisible by p and n is also divisible by p.
We showed that m and n have a common factor p. Contradiction.
Thus, $\sqrt{p}$ is not a rational number. ■</td><td>Assume $\sqrt{p}+\sqrt{q}=\frac{m}{n}$ where $m,n\in\mathbb{N}$.
Then $p+2\sqrt{pq}+q=\frac{m^2}{n^2}$ or $\sqrt{pq}=\frac{m^2/n^2-p-q}{2}$.
Thus, we showed that $\sqrt{pq}$ is a rational number.
Let $\sqrt{pq}=r/s$ where r,s are natural numbers without common divisors.
Then $pq=r^2/s^2$ or $pqs^2=r^2$ and r^2 is divisible by p. Thus, r is divisible by p and $r=pk$ where k is a natural number.
We have $pqs^2=p^2k^2$ or $qs^2=pk^2$.
Thus, s^2 is divisible by p and then s is also divisible by p, which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{p}+\sqrt{q}$ is not a rational number. ■</td></tr>
<tr><th colspan="2">Adaptations of the Above Steps for Solving Task 1 of our Task-Based Interview</th></tr>
<tr><td colspan="2">Prove that $\sqrt{8}$ is not a rational number.</td></tr>
<tr><td>Assume $\sqrt{8}=\frac{m}{n}$, where m and n are integers without common factors.
Then $8=m^2/n^2$ or $2\cdot 2\cdot 2n^2=m^2$.
So m^2 is even and m is also even. $m=2k$.
We obtain $2\cdot 2\cdot 2n^2=4k^2$ or $2n^2=k^2$.
So k^2 is even and k is also even. $k=2l$.
We obtain $2n^2=4l^2$ or $n^2=2l^2$.
So n^2 is even and n is also even.
We showed that m and n have a common factor 2. Contradiction.
Thus, $\sqrt{8}$ is not a rational number. ■</td><td>Assume that $\sqrt{8}=m/n$ where $m,n\in\mathbb{N}$.
Then $2\sqrt{2}=m/n$ or $\sqrt{2}=m/(2n)$.
Thus, we showed that $\sqrt{2}$ is a rational number.
Let $\sqrt{2}=r/s$ where r,s are natural numbers without common divisors.
Then $2=r^2/s^2$ or $2s^2=r^2$ and r^2 is even.
Thus, r is even and $r=2k$ where k is a natural number. We have $2s^2=4k^2$ or $s^2=2k^2$.
Thus, s^2 is even and then s is also even, which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{8}$ is not a rational number. ■</td></tr>
</table>

Table 4.3 Steps that may have been learned by successful RA I students, and how these steps would need to be adapted for solving Task 1 of our task-based interview.[17]

[17] The differences in language used in the proofs (e.g., integers vs. natural numbers, factors vs. divisors) reflect differences we observed in solutions provided to students, which we are mimicking in this table.

4.2.1.2 The Protocol

Once we had designed our interview tasks, we had to think carefully about how they would be implemented in an actual interview: that is, we needed to determine a set of rules that would govern the interactions between the interviewer (i.e., the doctoral student), an interviewee (i.e., a successful RA I student), and the interview tasks. We hence created a fourteen-page interview protocol (included in Appendix A), which summarized procedures for introducing the interview, dealing with general scenarios that might arise throughout the interview, and intervening during potential task-specific scenarios. As we describe these procedures below, it is important that readers keep in mind that the main purpose of creating the protocol was to spend a significant amount of time predicting what could happen during an interview and determining the kinds of interventions we wanted to allow based on our research goals. The protocol was not used in a strict manner. Although the interviewer studied the protocol before each interview, they did not refer to it while conducting the interviews. Given the exploratory nature of our study, we decided that the interviewer should maintain some freedom to ask questions that arose naturally from the (possibly unexpected) behaviour of an interviewee. In this way, we hoped to achieve the twin goals of reproducibility and flexibility, as suggested by Goldin (1997, 2000): namely, "good" research is not just that which can be replicated in other contexts, but that which remains open to new or unforeseen possibilities.

The introduction was the most consistent part of the interview. Each interview began by offering the interviewee time to read a consent form (included in Appendix A)[18] and ask questions. After the consent form was signed, the interviewer gave a short explanation of how the interview would proceed. The interviewee was told that they would be given some "problems" to solve. They were asked to "think aloud," i.e., to say what they were thinking, as they solved the problems. In hopes of making the interviewee more at ease, the interviewer also emphasized that the interview was not an evaluation, but a way of seeing how they solve problems. Once an interviewee had no more questions, the interviewer would start the audio recording, hand over a standard-sized sheet of paper with the first problem printed on the top, and say: "Here is the first problem." We hoped that it would take no more than ten minutes to get to this point.

[18] A certificate of ethics approval was obtained.

After being exposed to a problem, an interviewee would be given time to engage in what Goldin (2000) calls "free problem solving": i.e., spontaneous actions towards solving the problem, without any guidance from the interviewer. As Goldin (2000) explains,

> this technique permits explorations of the subjects' freely chosen strategies, representations, and so forth, maximizing the information gained; whereas providing premature guidance results in the loss of information. (p. 542)

There were a few minimally directional interventions that we permitted during free problem solving. We anticipated that some interviewees would be better at "thinking aloud" than others; hence, we allowed the interviewer to gently remind an interviewee to say what they were thinking at any point of the interview, especially if there were long silences. The interviewer also found that they needed to adjust their "active listening presence" depending on the interviewee: some interviewees seemed more able, willing, and even interested in treating the interviewer like a fly on the wall, while others insisted on acting like they were explaining their solution directly to the interviewer. In the latter case, the interviewer could offer some physical or verbal cue that they were listening (e.g., nods, or short phrases such as: "Ok."). Otherwise, the interviewer would try to resist intervening in any significant way while spontaneous action took place.

We expected the initial spontaneous actions of an interviewee to correspond with one or more of the six scenarios described in Table 4.4 below. As is also depicted in the table, we decided that the interviewer's response should vary depending on the scenario they judged to be occurring for a given interviewee during their attempt at a given problem.

In sum, we anticipated that an interviewee might "get stuck" and we planned to respond to this situation by providing guidance. For example, if an interviewee seemed stuck in starting to solve a given problem, the interviewer might ask the interviewee if they recalled solving a similar kind of problem, urge them to think about the meaning of the objects involved, or provide a hint towards a possible solution approach. If, on the other hand, an interviewee seemed stuck while implementing an approach, the interviewer might ask the interviewee what they are hoping to happen, encourage them to think of other ways of solving the problem, or offer a hint for progressing to the next step. When an interviewee identified a useful definition, theorem, or property, but forgot their statements, the interviewer would offer to provide these. We also anticipated that an interviewee might struggle with the generalized nature of some problems, in which case the interviewer would ask if it would help to specify a parameter. For the last, most challenging problem, we permitted the interviewer to be particularly generous in offering guidance. As mentioned above, such interventions can result in a loss of information; but when they occur at a blockage, they can actually lead an interviewee to "demonstrate competencies that otherwise he or she would never 'get to' during the problem solving, which adds to the information gained" (Goldin, 1997). In our study, interviewer intervention could enable us to gain more information about the kinds of practices developed by students, as well as the way in which the students may have positioned themselves towards similar problems during RA I. While analyzing our data, we nevertheless remained attentive to the fact that our interventions may have led to actions and explanations that would not have been observed otherwise.

In addition to the six scenarios listed in Table 4.4, we also expected some interviewees to explicitly solicit guidance from the interviewer. This expectation was quickly confirmed within the first interviews. If an interviewee asked which approaches they were allowed to use, the interviewer would tell them that they could use any approach they wanted to solve the problem. If an interviewee asked if it was a good idea to use a particular approach, the interviewer would provide honest answers: e.g., "Yes, that is a way I would go about solving this problem," "I didn't think about solving the problem in that way. So I'm not sure if that approach will work or not," or "That approach will not be helpful in this case. Can you think of another way to solve the problem?" This said, we also wanted to avoid an interviewee becoming dependant on the interviewer or adopting the interviewer's understandings as their own. Where necessary (e.g., when an interviewee would ask about the correctness of their solution), the interviewer would remind them of our goal: we were interested in seeing how students think about a problem, not if they can get a complete correct answer. The interviewer could be honest with an interviewee about not wanting to influence the way they think about the problems during the interview. Moreover, if an interviewee was ever interested, the interviewer would be willing to more openly share their opinions and understandings once the interview had been completed.

As shown in Table 4.4, no matter which scenario arose (with the exception of a complete blockage), the interviewer would ask some reflective follow-up questions. The two main types of questions that were consistently asked either sought to clarify something an interviewee said or did while trying to solve a problem, or to solicit any unexplained reasoning behind their actions (either in carrying out specific steps, or in choosing the whole approach). In line with Goldin's

(2000) principles, we wanted to provide as much opportunity as possible for an interviewee to correct themselves. Thus, the interviewer would often ask for clarification about seemingly "incorrect" or "incomplete" statements to see if the interviewee could recognize their error or complete their thinking. All the while, the interviewer attempted to avoid "professor-like" questions that might make an interviewee feel like they were being tested (e.g., rather than asking "What is the definition of a rational number?" the interviewer would ask "What comes to your mind when you think of a rational number?").

Scenario	If the interviewee…	The interviewer will…
Immediately Stuck	Is unable to spontaneously start solving the problem.	Provide a series of hints that get progressively more directive, to see if the interviewee can get unstuck. If they cannot, the interviewer will suggest to move on to the next problem and inform the interviewee that if there is time later, they may come back to this problem if they want.
Off Track (and Stuck)	Seems to be approaching the problem in a way that will not lead to a solution, and persists in using that approach for some time, but does not make substantial progress.	Ask the interviewee to clarify the approach they are taking and why. If they are using an approach that will not be successful, the interviewer may explain this and ask if they know of another approach.
Stuck During	Starts solving the problem in a promising way, but stops making progress (either they express that they do not know how to proceed, or they spend a significant amount of time figuring out the next step).	Act as in the first row of this table, with the additional action, before moving on to the next problem, of considering asking some follow-up questions about any partially unexpected or meaningful production the interviewee has made (see the last two rows of this table).
Unexplained or Unclear Thinking	Seems to have skipped an explanation of part of their thinking or gives an explanation that is incomprehensible.	Either interrupt with a question aimed at getting the interviewee to reveal or clarify their thinking immediately or, if the interviewee is in a good flow, take a note and choose to ask a follow-up question about it once their thinking has stopped.
Unexpected Production	Seems to accept as a solution something that is very different from what would have been expected in RA I (based on our analysis of assessment activities).	Ask questions regarding the acceptability of the solution in three contexts: the interviewee's personal work, RA I, and a community of mathematicians. E.g.: Would their solution receive full marks by their RA I professor? Would it be accepted by mathematicians?
Meaningful Production	Produces a meaningful solution to the problem (i.e., one where they have exhibited an appropriate RA I approach, at least in part).	Pose follow-up questions to clarify unexplained or unclear thinking, solicit reasons behind certain steps or the whole approach, ask about the acceptability of the solution in different contexts (see the above row), and/or inquire further into the interviewee's perceptions of mathematics, RA I, themselves in relation to these, etc.

Table 4.4 Six anticipated interview scenarios and how the interviewer would respond.

Depending on how an interviewee responded to initial follow-up questions, the interviewer was permitted to ask more questions, keeping in mind that we were interested in eliciting not only the interviewee's practices, but also their positioning in RA I. To be able to characterize an interviewee's positioning, we wanted them to share their perceptions of mathematics, of the RA I course, of how they approached the activities in the course, and so on. One specific line of questioning we planned as a springboard for gaining access to these perceptions concerned the acceptability of a solution in different contexts. For any problem, the interviewer might ask an interviewee how their solution would differ depending on whether they wanted to convince a community of mathematicians, convince their RA I professor to give them full marks, or simply convince themselves of the response. In some cases, the interviewer offered alternative solutions to see what the interviewee thought about them. Posed with an open and non-judgemental tone, we hoped that these kinds of follow-up questions would encourage an interviewee to reveal how they might act differently in different contexts, thus helping us reflect on the position(s) they would have taken in RA I. If an interviewee mentioned anything else that might give insight into their positioning (their general approach to study in RA I, what they found easy/challenging, interesting/boring, relieving/stressful, etc.), the interviewer either followed up on this right away, or took note of it so that it could be discussed after all the problems had been considered.

Before moving from one problem to the next, the interviewer aimed to observe an interviewee formulate at least one solution approach, and a reason for approaching the problem in that way. As much time as possible was given for an interviewee to actually solve a problem, including overcoming any blockages. This said, the interviewer always had their eye on the time and prioritized getting through all the problems, rather than an interviewee obtaining complete, correct solutions. If the interviewer thought it was time to move on to the next problem, they would attempt to bring it up naturally in the flow of discussion. Once an interviewee agreed to move on, the interviewer would move the paper they had used for the previous problem out of the way and hand them a new sheet of paper with the next problem printed on the top. Only the sixth interview task was split into three separate pieces of paper, which were intended to be revealed one-by-one as though they each corresponded to a different problem. That is, only once an interviewee had attempted Task 6(a) would the interviewer reveal Task 6(b), and only once the interviewee had attempted Task 6(b) would the interviewer reveal Task 6(c).

If an interviewee got through all the problems, we planned to spend any remaining time on asking exploratory questions that might assist us in gaining more insight into the interviewee's positioning in RA I. If it had not been done already, the interviewer would ask the interviewee to talk about how they felt about the problems in RA I: e.g., which ones they liked and which ones they did not like. In hopes of connecting these feelings to the interviewee's actions throughout the interview, the interviewer would encourage the interviewee to reflect on the specific problems they had solved during the interview. The interviewer would also typically ask an interviewee to compare Calculus and Analysis and to describe their transition from one into the other. Finally, depending on the specific interactions that arose in each interview and the time remaining, the interviewer would follow up on various other elements. For example, if an interviewee got stuck on a problem, the interviewer would ask them if they wanted to try again and/or how they would go about getting unstuck (e.g., if it had been a problem posed on an assignment in RA I). If an interviewee said they would use the internet in some way, the interviewer might ask them to use a computer to illustrate what they meant. If an interviewee seemed quite comfortable (or, in contrast, quite anxious) throughout the interview, the interviewer would mention this and ask the interviewee if they thought this resulted from their experience in their RA I course. Once all questions had been asked or two hours had passed, the interviewer would bring the interview to a close by thanking the interviewee and turning off the audio recording.

Now that we have explained how we created our task-based interview, we turn to outlining how we went about implementing it with a particular group of students.

4.2.2 Implementation: Recruiting Participants and Conducting Interviews

As stated previously, we considered the instances of RA I taught in recent consecutive fall and winter terms. Hence, there were two rounds of recruitment: one at the beginning of the winter term with students enrolled in RA II (for the previous RA I fall cohort), and one at the end of the winter term with students enrolled in RA I (for that RA I winter cohort). The recruiting process was facilitated by the professors of RA I and RA II, who agreed to send students emails about the research and allowed us to attend a class to give a short advertisement of the project in person. In all advertisements, students were given a brief summary of the research, including what their participation would entail: a two-hour interview, which could be scheduled at their convenience. Students were told that no preparation (e.g., studying) was required. They were also assured that their involvement in the study would remain strictly confidential: After completing an interview, all data pertaining to them would be labelled with a code (S1, S2, S3, …), and only the doctoral student would be aware of their identity. We provided an email address and asked students to send an email if they were interested in participating.

From the fall cohort, we received ten emails from interested students, all of whom eventually participated in the research. From the winter cohort, seven students sent initial emails showing interest, and five responded to follow-up emails (the other two simply stopped responding without explanation). It is not surprising that we recruited less participants from the winter cohort: Not only are there significantly less students taking RA I during the winter term, but we were also conducting the corresponding interviews at the beginning of what is typically a summer vacation for undergraduate students in North America. In the end, we had a total of fifteen participants.

Participant	Final Grade in RA I	Program of Study
S1	B	Actuarial Mathematics
S2	B	Actuarial Mathematics
S3	B-	Pure and Applied Mathematics
S4	D-	Actuarial Mathematics
S5	A-	Not Reported
S6	C	Actuarial Mathematics
S7	A	Actuarial Mathematics
S8	B+	Actuarial Mathematics
S9	A-	Actuarial Mathematics
S10	A+	Joint Major: Mathematics and Statistics, and Philosophy
S11	A	Actuarial Mathematics
S12	A+	Engineering
S13	A+	Statistics
S14	A+	Pure and Applied Mathematics
S15	B	Pure and Applied Mathematics

Table 4.5 The final grades and programs of study reported by the fifteen participants of our study.

At the time of correspondence by email, the interested students were informed of our only condition for continuing to participate in the research: they must have received a passing grade in RA I (i.e., at least a 50%, or, equivalently, a D-). This stipulation came from our interest in characterizing the practices developed by students who had been deemed successful in the RA I course institution (i.e., students who, in the eyes of the institution, had learned enough to be permitted to continue their journey in University Mathematics). We decided to rely on the students to confirm whether they met the condition or not, rather than verifying the students' grades by some other means (e.g., by requesting the grades from professors). Such a verification would have been difficult to obtain without breaching the confidentiality of the students involved. All interested students confirmed that they had passed RA I; thus, none were turned away from the research due to this condition. During the interview, the students were asked to report their final grade in the course, as well as their program of study. Our main goal in collecting this data was to offer a rough sketch of the group of students we interviewed. We did not intend to use the final grades in any significant way during our analysis. We guessed that the programs of study could be helpful in our reflections on how the broader institutional context may be shaping students' practices; and we did not see any reason for a student to lie about this. We therefore deemed self-reporting as an appropriate means to obtain this information. A summary is provided in Table 4.5 above.

The date, time, and location of each interview was determined on a one-on-one basis via email. Depending on the level of engagement and flexibility of each participant, the duration of an interview ranged from about two to three hours. The interviews took place in a small room at a large table. The interviewer sat close enough to a participant, on the opposite side of the hand with which they wrote, so that they could follow what was being written. On the table, near where the participant worked, was a set of resources that could be used if needed: a stack of blank white paper, a calculator, a pencil, an eraser, and a ruler. A computer was also on the table. It was plugged in and powered on in case the interviewer decided to introduce its use into the interview. When not in use, it was on the other side of the interviewer and directed away from the participant, so that it would not be distracting. The eight pieces of paper with problems on them were also on the other side of the interviewer, face-down and ordered so that the next one could be easily grabbed and presented to a participant. At the beginning of each interview, the interviewer made sure to put a "do not disturb" sign on the door to the room, and to have in front of them a blank consent form, a pen, and a pad of paper with a basic checklist for introducing the interview. The pad of paper was also used by the interviewer to keep notes throughout the interview.

To facilitate our data analysis, we decided that we would audio record the interviews. The Voice Memos app on the interviewer's iPhone turned out to be a convenient and effective way of accomplishing this. This iPhone was the only other object on the table during the interview. After ensuring that the iPhone was in airplane mode (so that the recording would not be interrupted by calls, texts, etc.), making a test recording with the participant, and officially beginning the interview, the iPhone was turned over so that the screen was not visible. We decided that video recording was unnecessary for our research goals, which we felt could be addressed through capturing what the participants said and wrote; and this would be accomplished through the audio recording and the collection of participants' written work. One challenge that arose was that the participants would often refer to parts of their written work or the problem statement as "this" or "that" (i.e., they would not always provide specific objects in their verbal explanations of their thinking). At times, they would also write things without saying what they were writing. This posed an issue when we were eventually analysing the interviews: It was not always clear what a participant was referring to or where they were in their written work. Wherever possible, we would rely on the context (e.g., what was said before and after) to infer the missing links. And once we became aware of the issue, we tried our best to make the necessary specifications occur in the interview, either by stating them ourselves in follow-up questions, or by asking the participant to clarify what they were doing or what they meant. Nevertheless, we feel that this is an important issue to remember for the next time we engage in such research methods.

In the next section, we describe how we analyzed the data we collected through our task-based interview – i.e., the audio recordings and the written work of the fifteen participants we interviewed – and we outline how we used this, along with the analysis of assessment activities discussed in Section 4.1, to address our research objectives.

4.3 Data Analysis

When analysing data collected through task-based interviews, Goldin (1997) says that "it is crucial to maintain carefully the scientific distinction between that which is observed and inferences that are drawn from observations" (p. 52). In relation to his own research, he explains that

> we (at best) are able to *observe* children's verbal and nonverbal behavior, as captured on videotape during the sessions. From these observations, we (and others who use similar methods) seek to *infer* something about the children's internal representations, thought processes, problem-solving methods, or mathematical understandings. We cannot "observe" any of the latter constructs. (p. 52; our emphasis)

In our study, we wanted to characterize the practices developed and the positions adopted by participants. Such practices and positions are model-dependent constructs, which are descriptive of phenomena internal to participants. Hence, they can only be inferred based on observed behaviours. In what follows, we aim to provide as many details as possible about the kinds of observations we made and how we drew inferences from them to address our research objectives.

To identify the kinds of observations made possible by the interviews we conducted, we began with a sort of pre-analysis of the data. We used the audio recordings to transcribe each of the fifteen interviews verbatim. We included any significant elements of participants' written work within the transcriptions so that the material to be analyzed was virtually reduced to the transcriptions alone. The final documents were, on average, 40 pages each. Creating such documents facilitated our analysis, while ensuring participants' confidentiality. In addition to eliminating the possibility of identifying a participant by their voice, we blinded any references they made throughout the interview that may have served to identify them. Each participant was referred to only through their code (i.e., S1, S2, S3, …) from this point onwards. This was important since the documents created during the analysis were accessible to both the doctoral student and the supervisors, who were not supposed to know the participants' identities. We started by reading and discussing the collection of transcripts pertaining to each interview task, one by one. Throughout our discussions of the first few tasks, we devised an initial plan for analyzing the data, which was revised several times as we progressed in our analysis.

To address our first research objective – gaining an understanding of the nature of the practices actually learned by students who succeed in a first Real Analysis course – we wanted to be able to infer the practices actually learned in RA I by the participants of our study. One way we supported these inferences was by creating what we called a "praxeology table" for each participant and each interview task (Table 4.6 below is an example). Each praxeology table consisted of rows in which we recorded any observations that could assist us in inferring the type of task (T), technique(s) (τ), and theoretical discourses (θ/Θ)[19] that had been developed by a participant with respect to a given task. More specifically, in row T we inferred the type of task that the participant would try to solve using their chosen approach and any supporting observations. Row τ comprised a step-by-step summary of what the participant did (and sometimes what they said) in attempt to solve the given task, including all approaches considered, where they got stuck, and what they did (if anything) to get unstuck. We re-emphasize that this is not a

[19] See footnote 15 on page 77.

technique in the sense of the ATD; it is a set of observations that would enable us to infer the technique(s) a participant would spontaneously choose for solving the given task (and others of the type T), as well as the degree to which the participant was capable of implementing that technique. Similarly, in row θ/Θ, we recorded mostly direct quotations from the participant as they described or justified their approach, which we used to infer the discourses they had to support their inferred technique(s).

T	Prove that $\sqrt{anything}$ (or, e.g., $\sqrt{2}+\sqrt{5}$) is not rational. "Root two, root eight, anything, would pretty much be the same. I know in an assignment we had something like this. [He writes $\sqrt{2}+\sqrt{5}$.] Which basically once you multiply them together boils down to something that's similar."
τ	"I'll use the method that we learned in class." Assume $\sqrt{8}=\frac{m}{n}, m, n \in \mathbb{Z}$ and have no common factors. $\Rightarrow 8=\frac{m^2}{n^2} \Rightarrow 8\,n^2=m^2 \Rightarrow 2\cdot 4n^2=m^2$ $\Rightarrow m^2$ even $\Rightarrow m$ even STUCK: "And now I want to show that n is even. How did I used to do that?" "trying things": $2\cdot 4n^2=m^2$, $4n^2=\frac{m^2}{2}$, $n^2=\frac{m^2}{8}$ STUCK: Forgets how to show n is even. But knows that if n is even, then m and n share a common factor (2) and the assumption does not hold. Then, by contradiction, $\sqrt{8}$ is irrational.
θ $/\Theta$	"Honestly a problem like this I'd do it, um… I'd remember how we did square root of two and just do it exactly the same way. [-] I feel that we're grinded to do so many questions really quickly. So we need to associate problems to a solution [-] really fast. [-] As soon as I see something like this, I'll associate it right away – 'Oh, we did it like this.' [-] Cause I don't really have the time to analyze the problem and try different things during an exam. So I grind problems at home. And when I get in an exam, I see the problem and I say 'Ok, that's exactly the kind of problem…it goes down to this.' So that's why I do it that way." "I know that I could do two square root of two [he writes $2\sqrt{2}$], which equals to this [he underlines $\sqrt{8}$]. [-] And I know that that one's not [rational] [he is pointing to $\sqrt{2}$] because we did it in class. And… an irrational times a rational stays irrational. So that's why I'd say that it's not [i.e., $\sqrt{8}$ is not rational]. So I'll go down that road. And if we… If ever it is, well I'll just restart." S1 says that he knows that an irrational times a rational stays irrational, but he does not know how to prove it; and he would need to be able to prove it if he wanted to use this explanation to solve Task 1 on an exam in RA I. "if m is even and n is even, then they share a common factor which is two, and the assumption doesn't hold. So by contradiction, it would be irrational. We're assuming that it's rational. [-] It has to be irrational if it's not rational."

Table 4.6 The praxeology table for S1 and Task 1: Is $\sqrt{8}$ rational or irrational?

Note that the kinds of elements in row θ/Θ were not always consistent across participants or across tasks, either because we asked some follow-up questions to a subset of the participants, or because the participants understood our follow-up questions in different ways. For instance, for Task 1, we asked most participants how they knew the truth of the implication "m^2 even $\Rightarrow m$ even," but we did not ask this of S1 (as evidenced in Table 4.6). One concern that was raised during our analysis was that when we asked participants "Why did you choose that approach?" we may have (mis)led some of them to give non-mathematical reasons like "Because that's the approach I learned," or "Because I like that approach." A question like "How do you know that this approach solves the problem?" might have been better suited for our interests. Nevertheless, we think that we observed enough of the participants' descriptions and justifications to be able to classify the kinds of theoretical discourses they had developed.

Once we had all fifteen praxeology tables for a task, we engaged in a classification of participants according to criteria that we felt were relevant to achieving our research objectives. More specifically, we used criteria that we expected to help us in inferring not only the different kinds of practices participants had developed, but also the different positions that, in combination with the assessment activities offered in the course, may have led to the development of those practices. The criteria varied from one task to the next and were not limited to "type of task solved," "technique(s) considered," or "technologies/theories used." For instance, for Task 2 – show that the function $f(x) = e^x - 100(x-1)(2-x)$ has 2 zeros – we used the following eight criteria:

- the first thing a participant spoke about or did upon receiving the task;
- the task they chose to solve;
- how they chose which x values to plug in (to locate sign changes in the function);
- components of the Intermediate Value Theorem they exhibited;
- technologies/theories they used to support a technique for showing f has at most two zeros;
- how much they depended on a calculator;
- how they used graphing; and
- how they perceived proof / convincing.

To exemplify how we classified participants according to such criteria, Table 4.7 below exhibits our classification for the third criterion in the above list: i.e., how participants chose which x values to plug in (to locate sign changes in the function). This criterion, which was specific to Task 2, came about because most participants decided to solve the task by looking for sign changes in f and, in doing so, exhibited various approaches for choosing x values. Classifying participants according to these approaches helped us describe the different kinds of techniques they had developed, as well as the theoretical discourses some of them had for producing a more productive approach (these are discussed in Section 6.2.2.2). Looking at Table 4.7 also assisted us in pinpointing collections of participants (including singletons) that may have been representative of different positions. For each task, we constructed several tables of this sort, which enabled an efficient visualization of our inferred groupings and facilitated our reporting of results. These tables were our principal source of guidance for writing Sections 6.i.2 ($i = 1, 2, 3, 4, 5, 6$, about the practices actually learned for each task), although we constantly used praxeology tables and original transcripts to verify our inferences and offer examples of the observations that led to them.

Calculates limits at $\pm\infty$ (or considers $f(x)$ at large $\pm$ x values).	Goes "randomly" (or by "trial and error," maybe considering ease of calculation or variance).	Tries to solve $f'(x) = 0$ or use f' in some way.	Picks "small" x (because e^x grows much faster than polynomials or e^x will be small there).	Pays attention to how the size of $f(x)$ is changing as x changes.	Studies an analytic expression for f and how different parts affect the sign.	Uses knowledge of graphs for parts of the function.
S1	S1					
	S2*					S2*
S3						S3
	S4			S4	S4	
	S5			S5	S5	
	S6			S6		
S7	S7	S7***				S7
	S8	S8***	S8			
S9**	S9					S9
S10**	S10	S10**				
	S11	S11**	S11**			
						S12
S13	S13				S13***	
	S14		S14**			
	S15*					

* Was looking for sign changes in $f'(x) = e^x - 100(3 - 2x)$, rather than f.
** Hinted at the possibility of using the approach but did not use it.
*** It is unclear what they were doing.

Table 4.7 Our classification of participants according to the criterion: How they chose which x values to plug in (to locate sign changes in $f(x) = e^x - 100(x-1)(2-x)$).

The final stage of analysis for addressing our first research objective was to characterize the nature of participants' inferred practices. This required us to synthesize the many results of the previously described analyses into models (presented in Appendices B – G) and compare these with our definition of *having developed a mathematical practice*, as introduced in Section 2.2 of our theoretical framework. In other words, we reflected on the degree to which participants had identified tasks as belonging (or not) to general mathematical types of tasks, chosen (recalled, discovered, constructed, guessed, etc.) and implemented appropriate mathematical techniques to accomplish the tasks, and used mathematical discourses to explain how and why the techniques work. We used our own knowledge of mathematics for doing so.

Recall that in our research, we also aimed to provide some explanation for the nature of the practices developed by students who succeed in a first Real Analysis course. Throughout Section 2.3 of our theoretical framework, we introduce our focus on three interacting factors that may shape the development of students' practices in any university mathematics course: the broad institutional context of the course, the didactic activities offered therein, and the positions adopted by students, which dictate how they interact with the activities. Our inclusion of the broad institutional context intends to recognize the various layers of conditions and constraints that have historically been imposed on a university course institution and which, as researchers taking an anthropological approach, must be considered. We do this in a general sense when we discuss our results in Chapter 7. In contrast, our analysis of the influence of the other two factors – activities offered by professors and positions adopted by students – was more systematic. Hence why our second research objective was specified (in Section 2.4) as an aim of gaining an understanding of how the practices actually learned by students who succeed in a first Real Analysis course are shaped by the activities offered to students and the positions students adopt.

To address this second research objective, we wanted to be able to reflect on how the practices actually learned by the participants of our study may have been shaped by the activities they were offered in RA I. As discussed in Section 4.1 earlier in this chapter, we engaged in an analysis of assessment activities that had been offered to participants when they took and were deemed successful in RA I. The results of this analysis – i.e., our inferences about the kinds of practices to be learned from those assessment activities – are provided throughout Chapter 6 (in Sections 6.i.1 for $i = 1, 2, 3, 4, 5, 6$). A comparison of these with the results presented in Sections 6.i.2 ($i = 1, 2, 3, 4, 5, 6$) – i.e., our inferences about the kinds of practices actually learned by participants – allowed us to make conjectures about how the practices actually learned may have been shaped by the assessment activities.

To completely address our second research objective, we also needed to be able to infer the positions that participants had adopted in RA I. This required us to identify and define different possible positions, and then classify participants according to these. We started with the positions defined in existing literature (as presented in Section 2.3.3); in particular, the Student[20] and the Learner[21]. As we progressed in the analyses described above, it seemed that there were three other positions that were also available to participants in the RA I course institution: i.e., the Skeptic, the Mathematician in Training, and the Enthusiast. Because we consider the five positions we identified and their defining features to be a significant theoretical contribution of this thesis, we introduce them as a positioning framework in their own chapter (Chapter 5). There are, however, several details that are pertinent to describing our methodology at this stage: i.e., how we identified and defined the five positions, and then used them to classify participants.

Theoretically speaking, we have come to see the position one adopts in a course as an idealized orientation towards (or relationship with) what one perceives to be the practices to be learned in the course, which is intrinsically linked with how one interacts with the activities offered to them (more on this in Chapter 5). For example, a student in the position of Student aims to identify the minimal subset of the practices to be learned that is required for getting a good grade in the course; hence, they will listen for hints during activities presented in lectures and exert a great deal of effort looking for patterns in assessment activities. During a task-based interview, a given participant can show signs of having "occupied (or adopted) a position" (i.e., of having oriented themselves towards the practices to be learned in the way defined by the position). Throughout our analyses, there were five participants who began to stand out as exhibiting strong signs of five different positions. To confirm this, and to better characterize the positions and their corresponding signs, we wrote a first set of definitions for the positions and performed a case study of each of the five standout participants. More specifically, we reread each of their transcripts and

[20] In Section 2.3.3, a Student is defined as a subject of a school institution, who must abide by its rules and norms, and for whom mathematics is a course.

[21] In Section 2.3.3, a Learner is defined as a cognitive subject, for whom mathematics is a mental activity to be shared with the teacher.

created descriptions of how they exhibited the position they seemed to embody, and how this stood in opposition with the other four positions. We then engaged in another round of synthesis and analysis of our transcripts: for each participant, we reread their entire transcript, reconsidered some of the related analyses, and, for each task, gathered evidence that could be used to classify them in one or more of the positions we had identified. The evidence was of three main forms:

1. The nature of a participant's inferred practices and how these compared with the assessment activities we analyzed (e.g., a Student's practices would be characterized by the paths to practices we had identified, including any superficiality, routinization, or lack of mathematical discourse that may have been permitted in following the paths; a Learner's practices would not be so predictable and would reflect more their personal understanding);
2. What a participant said as they propelled themselves to the next step of their solution – in particular, what they said and did when they got stuck – which allowed us to infer how much they depended on memory of established experiences versus on mathematical reasoning (e.g., when stuck, a Student would blame lack of recall and seek assistance from outside sources such as their course notes; a Learner might try different things or specify a lack of intuition, understanding, and/or time); and
3. A participant's reflections on their experiences in RA I and, more generally, in learning mathematics, from which we inferred their perceptions of the practices to be learned in RA I and various characteristics of how they interacted with the activities that were offered to them (e.g., a Student would perceive RA I as a course to be passed and would adopt strategies like reading the teacher or looking for patterns in past exams; a Learner would perceive RA I as an invitation to think and would base their interactions with activities on an interest in gaining personal understanding).

As mentioned above, Chapter 5 is dedicated to providing more information about all five positions we identified: their definitions and how these fit in the context of existing work. Throughout Chapter 6 (in Sections $6.i.3$ for $i = 1, 2, 3, 4, 5, 6$), we then use these definitions and the analyses outlined above to present our reflections on how different positionings seemed to manifest themselves in the behaviour we observed. This serves as a springboard to our more general discussion, in Chapter 7, of how the positioning of students, in combination with the activities offered to them, and the broader institutional context, may work together to shape the practices students develop.

In sum, when faced with studying something as complex as human behaviour, we have tried to not shy away from the complexity. We have sought to understand the behaviour we observed in our interviews through several iterations of qualitative analysis: i.e., reading the transcripts; synthesizing in a new way; reflecting, writing, and discussing. We addressed our first research objective by extracting so-called "praxeology tables" for each interview task and each participant (i.e., 6×15 tables), sorting participants according to various criteria (including, but not limited to, type of task solved, techniques considered, and technologies/theories used), inferring the practices developed by participants, and characterizing their nature based on our definition of a mathematical practice. To address our second research objective, we then reflected on how these practices may have been shaped by the assessment activities we analyzed, the positions we identified, and the larger institutional context. Throughout this entire process, we nevertheless remained attentive to the possibility that other variables may have been responsible for the behaviours we observed. Of particular importance when conducting the task-based interviews is that we did not ignore the fact that what we observed was participants' behaviour in the context of a structured intervention, including the presence of and interactions with an interviewer (Goldin, 2000, p. 521). The "unnatural" interview environment may be considered a limitation of our research. In fact, a few participants said that they were acting differently in the interview than if they had been working on their own. However, we share with Goldin (2000) the belief that the interview context is only a limitation insofar as it remains an unrecognized variable in the analysis. We would even argue that the inclusion of this variable in the analysis can deepen our reflections on the nature of the practices being taught and being learned, and the way in which those practices are taught and assessed. For example, the interview environment may allow us to infer that a student's written solution on an exam can receive full marks, even if the student is applying a non-mathematical practice to construct the solution. In our own reflections, we have also considered issues such as the mental, emotional, or physical state of the participant, the relationship established between the participant and the interviewer (e.g., the former's interest in pleasing the latter, or the influence of the latter's interventions on the former), and the restrictions posed by the interview on the time or resources available to solve a task. In the remaining chapters, we make notes about these when we think it is important and/or interesting.

Chapter 5: Positioning Framework

The purpose of this chapter is to present a key theoretical contribution of our doctoral work: a framework of students' positioning, which could serve as a tool for thinking about the development of students' practices in any mathematics course institution. In the context of this thesis, we use the framework throughout Chapter 6 when we analyse the practices actually learned by successful RA I students. Then, in Chapter 7, we rely on the framework to address our second research objective and reflect more generally on how the nature of the practices actually learned by students in a first Real Analysis course may be shaped by the positions students adopt, the activities they are offered, and the broad institutional context.

In Chapter 2, we introduce the notion of students' positioning in a course institution, and how it can shape the practices a student develops by dictating how they interact with the activities offered to them (see Section 2.3.3). In our study, this meant that we could infer the positioning of the fifteen students we interviewed based on the practices they used in our task-based interview, and how these compared with the assessment activities that we analyzed (as discussed in Section 4.3). Recall that we had also designed our interview to elicit participants' positioning through reflective questions aimed at soliciting their perceptions of mathematics, the RA I course, and of themselves in relation to these (see Section 4.2.1.2 for more details). Based on these two things – practices and perceptions – we identified five significantly different positions that seemed to have been available to students in the RA I course institution.

In addition to the positions of Student and Learner that have been discussed in previous work (e.g., Fenstermacher, 1994; Sierpinska et al., 2008), we discovered three other positions: the Skeptic, the Mathematician in Training, and the Enthusiast. We have found it most useful to distinguish between the positions in terms of idealized relationships with the practices to be learned in a given course, which we see as intrinsically linked with the nature of practices actually learned by a student who occupies the position (as presented in Table 5.1). This said, we have also been inspired by definitions in the literature and the behaviour we observed in our task-based interviews to create more elaborated definitions for each position.

In the following sections (Section 5.1 – Section 5.5), we proceed position by position. For each position, we offer an elaborated definition that aims to characterize the position in comparison with the other four. Then we provide a concrete illustration of the position in RA I by describing the perceptions of a participant who seemed to have strongly embodied the position. Finally (Section 5.6), we provide some comments on how our framework fits within the context of existing work, and why it constitutes a theoretical contribution.

Position	**The nature of the practices actually learned is shaped primarily by…**
Student	doing what is needed to get a good grade in the course; i.e., identifying the minimal subset of the practices to be learned that is required to succeed in assessments.
Skeptic	questioning what they perceive to be the practices to be learned in the course in the sense of critically reflecting on whether they should be learning them or not.
Mathematician in Training	incorporating the practices to be learned in the course into a larger collection of practices they perceive as important for participating in a community of mathematicians.
Enthusiast	devoting themselves to what they perceive to be the practices to be learned in the course.
Learner	seeking their own understanding of the practices to be learned in the course.

Table 5.1 The five positions in our framework, distinguished according to an idealized relationship with the practices to be learned in a given course.

We have three important notes for the reader:

1. Human beings are too complex to be completely and perfectly represented by a position. The positioning framework is a model: i.e., a tool, not for accurately representing reality, but for assisting us in better understanding it. Our main goal is to use this tool to help us better understand how different students may develop practices of different natures when offered the same collection of didactic activities. The examples we give in the following are intended to provide some concrete illustration, not to perfectly represent a participant. Most participants showed signs related to more than one position.
2. When a student adopts a position in a given course institution, this can be the result of a conscious choice; but it may also be the result of other factors. Some positions may not be accessible to all students (depending, for example, on their varied backgrounds, abilities, and interests). Some positions may be encouraged more than others depending on the institutional context. We return these ideas in Chapter 7.
3. As already mentioned in Chapter 2 (see Section 2.3.3), we acknowledge that our use of the word "position" does not reflect the original meaning of the word within

the ATD. We were inspired by previous work (e.g., Ostrom, 2005; Sierpinska et al., 2008) to maintain this term, and we see it as a way of highlighting how the conditions of didactic institutions like University Mathematics can lead to the availability of unexpected and essentially different positions, other than the institutionally recognized position of student. In future work, we plan to reflect further on the use of the term "position" as we conceive it versus how it is conceived within the ATD.

5.1 The Student

5.1.1 Definition

The Student is best understood as a subject of an educational institution – in this case, the University – which deems it necessary for the Student to successfully complete the course in question (e.g., because it is part of the requirements of the program of study in which the Student is enrolled). Unlike the Skeptic and the Enthusiast, the Student does not exhibit particularly strong negative or positive feelings towards the practices to be learned in the course. The Student's energy is directed towards figuring out what should be done in order to achieve a good grade. This means that, in complete opposition with the Leaner, what the Student learns depends on how this grade will be determined. If, for example, the examinations of the course tend to remain the same from year to year, then the Student will dedicate serious time and effort to analyzing past exams in search of patterns (mathematical or otherwise) and then doing similar problems to develop corresponding practices. It is not necessarily important to the Student if such practices are mastered for future use. In contrast with the Mathematician in Training, the practices are seen, not necessarily as a means to participating in a related mathematical community, but as a way of achieving success in the course.

5.1.2 Example

S11 seemed to be the epitome of Student when he described his interactions with the activities provided in RA I. All such interactions seemed to be with the same goal in mind: namely, enabling what S11 called "selective studying" at the time of examinations. S11 referred to himself several times as "the resourceful guy in the program." He claimed, for example, that he did not even touch the assignments throughout the semester – "Like no effort. Like not even reading the questions."

– because he had found a way to copy the solutions from other students. When it came time to study, he said that he would take a closer look at the assignment questions: “And then try to find every question that I can in that type of question. And then do all of them.” In addition to this, S11 explained that an essential feature of his approach to study was analyzing the teacher: “I want to know what type of teacher you are. I want to know what type of person you are. I want to know what type of questions you like. [-] That’s why I go to class.” During regular lectures and review classes, S11 said his principal aim was to take “crazy notes”:

> [Professors] write stuff. They say stuff. But they don’t write the stuff that they say, right? And those are sometimes important. [-] I’m not trying to understand what you’re saying. I’m really not. I’m just writing everything down.

S11 suggested that teachers typically give hints and that he would make sure to star these portions of his notes for later reference. Finally, S11 spoke of having obtained the final exams for RA I for the past seven years, which he seemed to use as his ultimate guide to his “selective studying” process. “I went over the professor’s old finals and saw what’s reoccurring and what’s not,” he explained; if something was judged to be non-reoccurring, then he would skip over it.

5.2 The Skeptic

5.2.1 Definition

Like the Student, the Skeptic is best understood to be a subject of an educational institution – the University – which has stipulated the course in question as a requirement for the degree sought. The difference is that the Skeptic has doubts about this requirement. Unlike the Mathematician in Training, the Skeptic is not convinced that the practices to be learned contribute to their training in a significant way. Unlike the Enthusiast, the Skeptic also does not simply accept the practices to be learned due to some intrinsic interest in the course material. If we think of the Student-Learner spectrum as representing how much one’s learning is shaped by the assessment activities given in the course, with the Student’s learning entirely shaped by such activities and Learner’s almost not at all shaped by them, the Skeptic might lie somewhere in the middle. The Skeptic is pushed into behaving like a Student because, on the one hand, they do not think the practices to be learned are of value to them; and, on the other, they need to pass the course in order to obtain their chosen degree of study. This said, what the Skeptic learns is likely more than minimal since they are able to rationally question and argue about the worth of the practices to be learned in relation to the practices they learned previously, as well as their personal interests and goals.

5.2.2 Example

The participant who most strongly embodied the Skeptic role in relation to RA I was S9, for whom the interview context served as a forum for complaint: “I don’t want to dismiss the course. Cause I can see its utility. But I find that it maybe shouldn’t be [-] necessary for like specific bachelor degrees.” S9 specified that he thought RA I would be useful for those who intended to pursue a career in research mathematics, but not for those, like himself, who were studying in more applied areas, such as actuarial sciences. For actuarial students, S9 felt that taking RA I was “a step backwards”:

> We already knew Hospital’s Rule. We already knew how to use the derivatives and stuff. Instead of building on this, we were instead just like reinforcing these foundations. But to the cost of building on top of it. [-] We didn’t learn any new rules. We just learned why this rule that we learned already works.

S9 felt frustrated by his perception that, after successfully completing RA I, he still had the same set of “tools” as his peers who had taken previous courses in Calculus and Linear Algebra with him, but who had not taken any courses in Analysis. He understood mathematical proofs as being important and necessary for showing that a rule works in general; what he questioned was the need for actuarial graduates to have seen the proofs of the basic rules they only needed to be able to use, not to mention learning to engage in producing similar types of proofs. When it came to additional proof tasks suggested by the professor, S9 said that he would not engage with them, not because they were outside of the practices to be studied to do well on exams (a potential Student justification), but because “they’re not easy to do. They’re not easy to find. And even once I make it, it’s not going to change the way I use that formula.” S9 did not want to spend energy on activities that he felt had no pay off for him personally. His bitterness about having no choice when it came to RA I can be summed up in his reference to the course as a “bullet I have to bite.”

5.3 The Mathematician in Training

5.3.1 Definition

The Mathematician in Training is a peripheral subject of some community of mathematicians (or some subset of the institution of Scholarly Mathematics). In comparison with the Skeptic and the Enthusiast, the Mathematician in Training is not particularly skeptical or devoted to the practices to be learned in the course in question; rather, they seek to integrate its elements into the practices they have. Like the Student, they diligently study the practices that are taught to them. The difference is that their goal lies in the more distant future: while the Student aims to learn what is needed to succeed in the course, the Mathematician in Training aims to learn what will prepare them for eventual success as a full participant of the community of mathematicians. This means that the Mathematician in Training puts much more effort into learning; they are not, however, a Learner. Unlike the Learner, the Mathematician in Training is attentive to the value placed on the course by their community of interest and, by extension, takes the practices to be learned very seriously. In other words, the practices they learn are shaped by the activities they receive in the course; but not only the patterns present in assessments.

5.3.2 Example

The participant who inspired the Mathematician in Training position, S13, saw RA I as a turning point in her education of "mathematics":

> Calculus looks so boring to me. [-] Cause it's the kind of things that a technician, an engineer, they can do it. [-] The mathematician, they do the proofs, they do the understanding. And then they give you the theory.

The comparison made by S13 between Calculus (or engineering) and Analysis (or mathematics) seemed to be rooted in personal experience. She told us that she had originally been trained in both micro-electronics and financial engineering and was working in a related field while pursuing another degree. She expressed strong regrets for not choosing mathematics in the first place: "I love mathematics. I should have studied mathematics. [-] If I want to forgive myself, I have to go back and do what I like." When asked what she loves about mathematics, S13 said:

> Oh you have to use your brain. And it's beautiful. And it's fair. [-] You can use your emotion to understand it. But then, once you have to prove it, there's no emotion, and there's no approximation, and it's truths and only truths.

The way S13 described how she interacted with the activities given in RA I reflected a dedication to learning a new craft. For instance, in addition to doing assignments, going through the textbook, and researching related questions on the internet, S13 said that she approached her professor for more problems: "The professor gave me a book. Three volumes. On Analysis. And they said that once you finish this, then you will be very good." S13 said she would finish going through the entire book in the summer, when she would have more time. For her, the time she spent outside of class, independently doing as many problems and proofs as possible, was crucial to improving her own vision of mathematics. Class time, then, was for seeking inspiration from her professor. S13 told stories of disappointment when previous lecturers would simply reproduce the textbook she already studied on her own. She was appreciative when, in RA I, the professor did not do this: "They solve hard questions. [-] They keep the course interesting. [-] They have a good understanding, a big picture of what's happening. And they were trying to teach us, just one shot at a time of that."

5.4 The Enthusiast

5.4.1 Definition

The Enthusiast is the only position of the five that we feel is best understood as being a subject of the course in question. While interacting with the activities of the course, the Enthusiast is driven, not by a desire to get a certain grade (like the Student) or to gain personal understanding (like the Learner), but by a keen interest in experiencing what they perceive to be the practices to be learned. In opposition to the Skeptic, the Enthusiast exhibits a positive disposition towards the possibility of taking the course, whether the University has deemed it a requirement of their degree or not. This does not mean that the Enthusiast is without critique: they have certain expectations of what the practices to be learned will be like and are disappointed if they feel these expectations are not being met. The Mathematician in Training also expresses a certain degree of enthusiasm for the course; but they are not devoted to it in the same way as the Enthusiast. For instance, while the Mathematician in Training would turn to any practice to solve a given task, the Enthusiast would rather spend time trying to solve the task using the practices to be learned from the course. For them, this constitutes a fascinating challenge.

5.4.3 Example

The participant who led us to introduce the Enthusiast position was S10, whose relationship to the RA I course was encapsulated in the statement that: “It’s a fun class, but it’s also frustrating.” The fun part came from RA I offering the opportunity for S10 to explore something he personally found very interesting. Of the fifteen students we interviewed, S10 was our unique double major: Mathematics and Statistics, on the one hand, and Philosophy, on the other. He spoke to us about his interest in philosophy – the birth of which he attributed to reading the philosopher Kant – and how this was linked with a personal curiosity of trying to reduce things to basic assumptions that are clear by virtue of intuition of space and time. What he looked forward to in RA I was gaining insight into the reduction of things he already more or less knew “to some really really basic ideas that nobody feels like they can dispute. Things like $x + y$ is equal to $y + x$.” Hence, he was excited by his interpretation of the didactic contract for the course:

> The first day, they said ‘Ok, everything needs to be proven in this course. Everything that we do in the lecture, you can take for granted. And anything, let’s say the propositions in the textbook, anything in the sections that we’ve covered, you can use that. But then anything else, you have to prove that it’s the case.’

S10’s frustration came from what he felt were breaches to this contract.

When describing his interactions with the activities given in RA I, S10 emphasized these breaches. He spoke, for instance, of how the solutions provided for assignments had gaps that could not be filled simply by looking at what was proved in the class or provided in the textbook. S10 considered the possibility that his professors intended for students to fill in the gaps on their own:

> But then, I think also the implication is that if you gave that as your own homework, then that would also count as full marks, which is confusing. Because then, all of a sudden, it’s not clear what counts as a satisfactory assignment.

To follow the rules that were stipulated at the beginning of the course, S10 felt that the given solutions were not satisfactory. He also considered the possibility that creating complete solutions would be too time consuming: “But then, if we start skipping that, then I don’t know what the point of doing Real Analysis is in the first place.” It seems that for S10, the stipulation to develop complete logical arguments was not an arbitrary part of the didactic contract in RA I; it was a necessity of the related scholarly domain. He referred more generally to “mathematics” as being “one of these special domains where we get to be like really super specific as what counts as a successful attempt at something or demonstration. And what’s unsuccessful.” When S10 judged his professors’ actions as being contradictory in this regard, he decided that the contract would need to be fulfilled through his own independent work. He spoke of searching in various resources to try to fill the gaps and said it could take him days to “spell out all the logic” for some problems.

It is important to note that S10’s interpretation of the didactic contract seemed to clash with the assessment structure of RA I; and this created an excessive amount of stress for him. Consider, for example, how he compared the process of studying in RA I with how he had been able to study in previous courses:

> For Calculus, I’d sit down with a practice exam, and I’d give myself three hours, and I’d see, oh, you know, I did like eighty percent of it. And so, that’s pretty good. I’ve got a couple more days to study. But I started doing that with Real Analysis, and it stressed me out so much I had to stop. Like the idea that I could possibly do an exam in three hours?

For S10, RA I was about developing complete logical statements, which sometimes took him days to figure out; but the final examination did not last for days. After feeling horribly at the midterm (even though he received a perfect score), S10 said he had to change his tactics: he practiced doing final exams where he would not complete the problems. This, he said, was particularly frustrating, given that it did not reflect what he was hoping to learn in the course.

5.5 The Learner

5.5.1 Definition

The Learner is first and foremost a cognitive subject, whose learning is governed, not by their current or future membership in an educational or mathematical institution, but by their own conscious thoughts and judgements. In comparison with the Skeptic and Enthusiast, the Learner does not exhibit a strong emotional stance towards the practices to be learned in the course in question. If they want to learn, they will learn; if they do not want to learn, they will not. Indeed, the Learner is typically not a very good Student. For instance, they are not willing to dedicate a ton of time and energy to studying patterns in past examinations for the sake of mastering exactly what is needed to do well in the course. What they learn is shaped by what they seek to understand, rather than what is assessed. The Learner sees the course as an invitation to think and build their own understanding. Unlike the Mathematician in Training, the Learner also does not care too much about being able to communicate what they understood to others (e.g., within a community of mathematicians). In fact, when the Learner tries to explain what they know, it can sometimes seem as though they lack understanding, even if, in their reality, they have completely understood.

5.5.2 Example

The closest we had to a pure Learner among the participants of our study was S3, who said of the interview: "This is fun actually. [-] There's interest in what I think, so that makes me feel important." For S3, it seemed that doing mathematics was an individual mental activity and the interview context was meant for the interviewer to try to gain access to it. This view of mathematics was also present in the way S3 described his interactions with the activities in RA I, which stood in complete opposition with the stories shared by S11 (the Student). For instance, while S11 claimed to go to class to study the teacher and take note of everything they say and do, S3 said his main goal during lectures was to understand the mathematics the teacher was saying and doing; and, if he succeeded, there was no need to take notes. In cases where he was unable to follow a lecture, S3 explained:

> I get very frustrated and I start writing down everything. And that's when my notes come out really messy and completely useless. So either way, I don't need my notes. Because if I need my notes, my notes are not helpful. If I don't need my notes, I don't have them.

This explanation was given by S3 to support his claim that if he ever got stuck on a problem, he would never refer to his notes. He spoke, instead, about taking breaks, referring to textbooks, and maybe doing a couple searches on the internet. If the problem was part of an assignment with an imminent due date, however: "Then I probably wouldn't do it," he said, "I'm not the person who just stays up until four in the morning trying to solve these things. [-] Fighting doesn't work. It just makes me more frustrated, more angry, and more likely not to get it." While S3 referred to getting stuck as "unfortunate," he also recognized it as "part of the process. [-] That's also what makes it fun when we do get it right."

5.6 Our Framework in the Context of Existing Theories

When we outlined our theoretical framework in Chapter 2, we referred to the literature that inspired our interest in students' positioning and how it may shape the practices students develop in a mathematics course institution (see Section 2.3.3). Sierpinska et al. (2008) introduced four positions – the Person, the Learner, the Student, and the Client – and defined each of them as a particular type of subject with a particular view of mathematics. Liljedahl and Allan (2013a, 2013b) recognized two positions – the Student and the Learner – and defined them according to the kinds of actions they perform, based on Fenstermacher's (1994) distinction between studenting and learning. We summarize these contributions in Table 5.2.

Position	Type of Subject	View of Mathematics	Actions are Shaped By
Person	Member of Society at Large	It's part of the world.	–
Learner	Cognitive Subject	It's a mental activity to be shared with the teacher.	The properties, methods of inquiry, etc., of the discipline or subject matter.
Student	Subject of a School Institution	It's a course to be passed.	The institutional and organizational properties of the school setting.
Client	Customer of the Classroom	It's a product, and I have the right to evaluate its quality.	–

Table 5.2 The defining features of positions identified in past work.

The elaborated definitions we present throughout the previous sections were inspired by these contributions. For instance, we also characterize each of the five positions we identified as a particular type of subject and we discuss the kinds of views and actions one would expect from a student in that position. Our definition of the Student was particularly influenced by the work that we read and aligns well with the definitions presented above. Our definition of the Learner only partially reflects these definitions due to our identification of another, somewhat similar, but significantly different, position: the Mathematician in Training. When the Learner learns and the Mathematician in Training trains, they may both act according to the properties and methods of the discipline or subject matter. However, we see the Learner's actions as most strongly shaped by their own interests, thoughts, judgements, and understandings; and we see the Mathematician in Training's actions as most strongly shaped by the practices they feel are pertinent to the community of mathematicians to which they seek membership.

We did not identify the Person or the Client in our data analysis. The Person seems far from any of the positions that we identified. We note that the Client may seem to resemble the

Skeptic since both are defined by taking a judgemental stance towards the subject matter. This said, the Skeptic does not evaluate the quality of mathematics (the product) or teaching (how the product is delivered to them) in general; they evaluate whether the practices to be learned in a course are worth learning for them. Ultimately, we did not find that the general views of mathematics as part of the world or as a product to be evaluated played a crucial role in shaping the practices actually learned by the participants of our study.

There are a few important features of our framework that we feel add to the existing theories. First, we have identified three new positions – the Skeptic, the Mathematician in Training, and the Enthusiast – which recognize that students' positioning may be influenced by their broader goals and interests. Second, our definitions are stated in relation to a given course (rather than in relation to mathematics in general), since we predict that a student's positioning (and the nature of their practices) could change drastically from one course to another (depending, for example, on the subject matter and how it relates to the students' goals and interests). These two features seem crucial especially when we are thinking about university mathematics education, where students have chosen a particular program of study with their broader goals and interests in mind. Lastly, our framework clarifies a new way of distinguishing between the positions: in terms of idealized relationships with the practices to be learned in a given course (see Table 5.1 at the beginning of this chapter). We have come to see these different relationships as most helpful in understanding how students in different positions learn practices of different natures when given the same set of didactic activities.

In the next chapter, we present our task-by-task results about the practices to be learned by a successful RA I student and the practices actually learned by the fifteen participants of our study. We use the positioning framework outlined above to assist us in formulating some initial reflections on how the nature of the practices actually learned may have been shaped by the activities participants had been offered in RA I and the positions they may have adopted when interacting with those activities.

Chapter 6: Task-by-Task Results and Analysis

In this chapter, we present and analyze the empirical results of our study. Our presentation is organized according to the six tasks we posed in our task-based interview. For each task, there are three sections.

In Section 6.i.1 ($i = 1, 2, 3, 4, 5, 6$), we construct a model of practices to be learned in RA I most relevant for Task i. In Chapter 2, we introduce the notion of the "practices to be learned" in a university mathematics course (see Section 2.3.1), and we theorize that the minimal core of these practices are communicated to students through collections of didactic activities that form paths to practices (as defined in Section 2.3.2). The models of practices to be learned are based on our analysis of the assessment activities that were offered to the participants of our study when they were enrolled in the RA I course; in particular, on what we think an RA I student would be expected to learn by engaging in those activities, as gleaned from the activities themselves, the solutions offered to students, and the theories appearing in the recommended textbook for the course (see Section 4.1 for more details). We use our models to make conjectures about the kinds of practices that a successful RA I student would try to use to solve each of our six interview tasks. Recall that we designed these tasks to be both recognizable and deceptive to a successful RA I student: that is, we wanted such a student to recognize the task as requiring a practice they had developed in RA I, but we also wanted the student to be potentially deceived by some characteristic of the task, so that any superficiality or routinization of their practice would be revealed (we discuss this further in Section 4.2.1.1). Hence, in 6.i.1, we also highlight the deceptive nature of Task i in terms of the difficulties a successful RA I student might face in solving the task, and the corresponding ways in which the practices they actually learned might be non-mathematical.

In Section 6.i.2, we construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task i. The theoretical construct of the "practices actually learned" in a university mathematics course is also introduced in Section 2.3.1 of our theoretical framework. Essentially, Section 6.i.2 presents the practices fifteen successful RA I students seemed to be using when they solved Task i in the task-based interview we constructed (as described in Section 4.2). We begin Section 6.i.2 by recalling what we had expected a successful RA I student to do when faced with Task i (i.e., we provide a summary of

$6.i.1$), and we briefly compare this with what participants actually did. Then we describe and exemplify different components of the practices that participants used to solve the task. The organization and nature of results varies from task to task, depending on the criteria that emerged during our data analysis as fruitful (towards our research objectives) for grouping participants (as discussed in Section 4.3). We have attempted to provide some harmony in the overall structure by using headings that reflect our definition of having developed a practice (introduced in Section 2.2): e.g., identifying the task (as belonging or not to a particular type), choosing (or recalling, discovering, constructing, guessing) a technique, implementing a technique, describing and justifying a technique, and acknowledging underlying theory. But even these are not entirely uniform from one task to the next, since our analysis led us to focus on different parts of practices for different tasks.

Finally, in Section $6.i.3$, we offer some reflections on the results presented in Sections $6.i.1$ and $6.i.2$, which aim to address our research objectives. Based on the models of practices actually learned (constructed in $6.i.2$), we propose some ways of characterizing the nature of the practices (first research objective). In particular, we specify and discuss different ways in which participants' practices may be thought of as being mathematical or not. Then we think about the practices actually learned in relation to our model of practices to be learned (constructed in $6.i.1$) and our positioning framework (outlined in Chapter 5). More specifically, we reflect on how the practices may have been shaped by the activities participants had been offered in RA I and the positions they may have adopted in the course (second research objective). These reflections serve as the foundation to our more general discussion and conclusions in Chapter 7.

We have three important notes for the reader:

1. Our presentation of results is interpretative and selective. Although our discourse may at times seem direct, we acknowledge that our descriptions, characterizations, and reflections are shaped by our own understandings and interests. Moreover, we do not provide an account of all possible results, but only those that we feel are most pertinent to the research objectives we sought to address. While reading Chapter 6, the reader may identify several threads of thought on students' practices that we have not followed in depth or at all (e.g., fluency on pre-Calculus concepts, students' perceptions of what counts as a mathematical proof, the validity of

software to derive conclusions about the behaviour of functions, etc.). We have not followed these threads in depth or at all either because we did not see them or because we made conscious choices to narrow the scope of our reflections to what we deemed necessary to address our research goals.

2. We encourage the reader to have the appendices readily available as they read this chapter. The chapter is quite substantial due to the richness of our data and our interest in providing ample examples in support of the inferences we have made. To help readers manage the amount of material presented, we include some syntheses throughout. In addition to this, we have created some useful appendices, including a list of the tasks from our task-based interview (included in Appendix A) and, for each of these tasks, our models of practices to be learned and practices actually learned in table form (Appendix B – Appendix G).
3. We provide numerous direct quotations from the interview transcripts to exemplify and support our inferences. For the sake of brevity and readability, we sometimes omit or slightly alter certain parts of these quotations. We often remove filler words such as "um," "uh," or "like," unless we think they are important to the point we are trying to make. We enclose words with square brackets if we have added them or used them to replace an unspecified pronoun with the object to which it relates. We specify when a significant section of speech has been eliminated by replacing it with [-]. In contrast, we use […] to indicate when a participant made a significant pause, and … to designate when a sentence was not completed by the speaker. We use the code "L" to refer to the interviewer.

6.1 Task 1

Is $\sqrt{8}$ *rational or irrational?*

6.1.1 Practices to be Learned[22]

If Task 1 were posed on a final exam in RA I, we would expect a student to attempt to provide a solution equivalent to one of the two presented in Figure 6.1 below. Even though it is phrased as

[22] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 1. This model is synthesized in table form in Appendix B.

a question, the student would identify the given task with t_1: Prove that $\sqrt{8}$ is not a rational number, and they would not necessarily indicate (or perhaps even think about) whether $\sqrt{8}$ is rational or irrational before starting their proof. They would begin their solution with the assumption that $\sqrt{8}$ is a rational number or that $\sqrt{8} = m/n$, possibly with m and n having no common factors (or divisors). Then they would try to reach a contradiction in some way, making explicit their deductions and the moment when the contradiction is reached, before stating the conclusion: by contradiction, $\sqrt{8}$ is not a rational number. The difference between the two solution types is the way in which the contradiction is obtained.

Solution 1:

Assume that $\sqrt{8}$ is a rational number.
Then $\sqrt{8} = m/n$, where m and n are integers without common factors.
Then $8 = m^2/n^2$ or $m^2 = 2 \cdot 2 \cdot 2n^2$.
So m^2 is even and m is also even.
$m = 2k$.
We obtain $4k^2 = 2 \cdot 2 \cdot 2n^2$ or $k^2 = 2n^2$.
So k^2 is even and k is also even.
$k = 2l$.
We obtain $4l^2 = 2n^2$ or $2l^2 = n^2$.
So n^2 is even and n is also even.
We showed that m and n have a common factor 2. Contradiction.
Thus, $\sqrt{8}$ is not a rational number. ■

Solution 2:

Assume that $\sqrt{8} = m/n$ where m and n are natural numbers.
Then $2\sqrt{2} = m/n$ or $\sqrt{2} = m/(2n)$.
Thus, we showed that $\sqrt{2}$ is a rational number.
Let $\sqrt{2} = r/s$ where r, s are natural numbers without common divisors.
Then $2 = r^2/s^2$ or $2s^2 = r^2$ and r^2 is even.
Thus, r is even and $r = 2k$ where k is a natural number.
We have $2s^2 = 4k^2$ or $s^2 = 2k^2$.
Thus, s^2 is even and then s is also even, which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{8}$ is not a rational number. ■

Figure 6.1 Examples of the two kinds of solutions anticipated for Task 1.[23]

The contradiction in Solution 1 is found by assuming that $\sqrt{8} = m/n$ with m and n having no common factors and showing, on the contrary, that m and n have a common factor. The student does not need to make explicit why they can assume that m and n have no common factor (i.e., that this is a way of uniquely defining rational numbers). To reach the contradiction, they must implement basic algebraic manipulations and certain properties about integers (in particular, properties about evenness). They can take for granted (i.e., without justification) the property: If

[23] The differences in language used in the proofs (e.g., integers versus natural numbers, factors versus divisors) reflect differences that we observed in solutions provided to students, which we are mimicking in this figure.

m is an integer and m^2 is even, then m is also even. Note that all the solutions we analyzed used this property at most twice (like in the proof that $\sqrt{2}$ is irrational, as shown in Solution 2), while Solution 1 of Figure 6.1 contains three iterations of using the property. Because of this, we expected a successful RA I student may encounter some difficulty in completing a solution of this type. In the least, they would be required to engage in some creative thinking in implementing the technique: a line-by-line copying of the solutions we analyzed would not be enough.

The contradiction in Solution 2 is reached by assuming that $\sqrt{8} = m/n$ and using basic algebraic manipulations to find that a known irrational number ($\sqrt{2}$) is equal to a rational number ($m/(2n)$). We expected this approach to be less common among participants since it was, in the activities we analyzed, typically applied to numbers of a different algebraic form (e.g., $\sqrt{p} + \sqrt{q}$ where p and q are distinct prime numbers). It is, nonetheless, a somewhat simpler approach, especially if the irrationality of $\sqrt{2}$ were to be taken for granted (in such a case, it requires only basic algebra and an awareness of what rational and irrational numbers look like). In at least one solution provided to students, the fact that $\sqrt{2}$ is irrational was taken for granted because it was "proved in class." This said, the solutions we analyzed were not consistent in terms of whether the irrationality of certain numbers could be taken for granted or not. In an exam situation, we would expect many students to reproduce the proof that $\sqrt{2} \notin \mathbb{Q}$, as it is done in Solution 2, to ensure that they get all the points.

Based on our analysis of assessment activities (i.e., activities in assignments and past midterms and final examinations, as well as their solutions), we could model the practice to be learned in RA I that is most relevant to Task 1 as shown in Table 6.1. In the RA I course, students were invited to prove the irrationality of a subset of numbers, which we describe in T_1 rather vaguely as "involving a root." Paths of activities indicate that a successful RA I student would be able to prove that $\sqrt{p}$ and $\sqrt{p} + \sqrt{q}$ (and maybe $\sqrt{p \cdot q}$ and $\sqrt{\sqrt{p} + \sqrt{q}}$) are not rational numbers for any specified distinct prime numbers p and q using τ_{11} and τ_{12}, respectively, as well as the more general property: If m is an integer and m^2 is divisible by a prime number p, then m is also divisible by p. For these specific types of numbers, the techniques can be specified to a collection of specific steps (given in Table 6.1), which can be copied line by line. As mentioned previously, Task 1 cannot be solved through such an approach. To show that $\sqrt{8}$ is not a rational number, one would need to adapt the specific steps shown in Table 6.1. The essential characteristics of the techniques are, however, the same. We chose $\sqrt{8}$ purposefully to see if the specifications in commonly tested practices had any effect on the practices developed by the participants of our study. We wondered, for instance, if they had learned a general by-contradiction technique, or if they had learned a set of steps. Moreover, in the latter case, we wanted to know if students would be able to adapt these steps or not.

T_1: Prove that a given real number c involving a root is not a rational number.	
τ_{11}: Assume $c = m/n$ where m and n are integers without common factors. Show that m and n have a common factor. Commonly tested was $c = \sqrt{p}$ for a specified prime number p: Assume $\sqrt{p} = \frac{m}{n}$ where m and n are integers without common factors. Then $p = m^2/n^2$ or $pn^2 = m^2$. So m^2 is divisible by p and m is also divisible by p. $m = pk$. We obtain $pn^2 = p^2k^2$ or $n^2 = pk^2$. So n^2 is divisible by p and n is also divisible by p. We showed that m and n have a common factor p. Contradiction. Thus, $\sqrt{p}$ is not a rational number. ■	τ_{12}: Assume $c = m/n$ where m and n are natural numbers. Use algebra to find that a known irrational number is equal to a rational number. Then apply τ_{11} to the irrational number. Commonly tested was $c = \sqrt{p} + \sqrt{q}$ for specified distinct prime numbers p and q: Assume that $\sqrt{p} + \sqrt{q} = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Then $p + 2\sqrt{pq} + q = \frac{m^2}{n^2}$, or $\sqrt{pq} = \frac{m^2/n^2 - p - q}{2}$. Thus, we showed that $\sqrt{pq}$ is a rational number. Let $\sqrt{pq} = r/s$ where r, s are natural numbers without common divisors. Then $pq = r^2/s^2$ or $pqs^2 = r^2$ and r^2 is divisible by p. Thus, r is divisible by p and $r = pk$ where k is a natural number. We have $pqs^2 = p^2k^2$ or $qs^2 = pk^2$. Thus, s^2 is divisible by p and then s is also divisible by p, which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{p} + \sqrt{q}$ is not a rational number. ■
θ_1: "By contradiction."	

Table 6.1 The practice to be learned in RA I most relevant to Task 1.

Among the assessment activities we analyzed, there were some that offered exposure to adaptations of the steps in Table 6.1. There were single instances of several other number types: $\sqrt{p} + \sqrt{n^2}$ where p is a specified prime and n is a specified integer, $a\sqrt{p} + b\sqrt{q}$ where p and q are specified primes and a and b are general rational numbers with $(a, b) \neq (0,0)$, $n\sqrt{2}$ where n is a general natural number, and $\sqrt[n]{p}$ where p is a specified prime number and n is a specified natural number other than 2. If a student considered such activities seriously in their studying, then they might have learned a more general by-contradiction technique that would enable them to prove the irrationality of a larger range of numbers, including $\sqrt{8}$.

A closely related collection of activities concerned more general statements about the products and sums of rational and irrational numbers. For instance, one activity asked students if it is true that if $a \cdot b$ is rational, then both a and b must be rational. Another invited them to judge the truth of the statement: The sum of a rational number and an irrational number is irrational. In the case of a false statement, the student was expected to provide simple counterexamples; and for a true statement, they were expected to produce a proof. In the activities we analyzed, no explicit connection was made between the more specific "prove that" tasks mentioned in the previous paragraph and the more general "is it true that?" tasks described in this paragraph. For example, the statement mentioned above – "the sum of a rational number and an irrational number is irrational" – is true and would be expected to be proved through an implementation of τ_{12}. In a solution to a separate activity, $\sqrt{3}+\sqrt{4}$ was also proved to be irrational using an implementation of τ_{12}. An alternative would have been to argue, more generally, that $\sqrt{3}+\sqrt{4}=\sqrt{3}+2 \notin \mathbb{Q}$ since it is a sum of a rational number and an irrational number. We did not observe these kinds of arguments in the solutions we studied. Hence, in the case of $\sqrt{8}$, while the mathematician might rewrite $\sqrt{8}=2\sqrt{2}$ and argue (with ability to prove) that "the product of a non-zero rational number and an irrational number is irrational," we would not necessarily expect a successful RA I student to be drawn to such an approach. Our choice to phrase Task 1 as a question would nevertheless afford us the opportunity to inquire into whether participants had other ways (mathematical or not) of solving Task 1, i.e., of convincing themselves that $\sqrt{8}$ is irrational.

Another general manner of arguing that we might expect from the mathematician, but not from a successful RA I student, would be to simply note that $\sqrt{8}$ is the square root of a non-square number. More specifically, the mathematician would know that $\sqrt{N} \notin \mathbb{Q}$ whenever N is a natural number and $\nexists a \in \mathbb{N}$ such that $N=a^2$. They would also be able to sketch a proof. For example (gcd stands for greatest common divisor):

Suppose $\sqrt{N} \in \mathbb{Q}$. Then $\sqrt{N}=\frac{m}{n}$, with $m, n \in \mathbb{N} \wedge \gcd(m,n)=1$.

$$\Rightarrow N=\frac{m^2}{n^2} \wedge \gcd(m^2,n^2)=1$$

$\Rightarrow Nn^2 = m^2$

The Fundamental Theorem of Arithmetic (FTA) tells us that the prime factorization of N, $P(N)$, is contained in $P(m^2)$.

Since $\nexists\ a \in \mathbb{N}$ such that $N = a^2$, $\exists t \in P(N)$ of odd multiplicity.

In $P(m^2)$, t has even multiplicity.

So $t \in P(n^2)$.

Contradiction. Hence $\sqrt{N} \notin \mathbb{Q}$. ■

Our analysis of assessment activities identified a proof like the one shown above as outside the practices to be learned in RA I in at least one way: a successful RA I student does not seem to be expected to know about the Fundamental Theorem of Arithmetic. More generally, we are unsure of the extent to which the successful RA I student would be able to characterize irrational numbers. Although they would have engaged in proving that $\sqrt{N} \notin \mathbb{Q}$ for several specified $N \in \mathbb{N}$, they were not required to reflect more generally on the nature of N for which $\sqrt{N} \notin \mathbb{Q}$. Hence why we chose to describe T_1, the type of task studied, in a rather vague manner.

We would expect a successful RA I student to have some awareness of the mathematical theory underlying $[T_1, \tau_{11,2}, \theta_1]$. In one assignment activity, students were asked to engage in proving the principle that justifies the by-contradiction technology. The proof provided in the solution is shown in Table 6.2. This solution uses the so-called "table method," an algorithm aimed at constructing a table that considers all possible truth combinations of the parts of a given formal sentence and checks if, in every case, they lead to the truth of the entire sentence. Essentially, if the column related to the sentence of interest contains only 1s, then the statement is true; otherwise, it is false. We identified a path of activities that indicated this to be a commonly tested proof technique in RA I: that is, we would expect a successful RA I student to be able to determine the truth of any given mathematical statement expressed in the language of formal mathematical logic. We would, for example, expect them to be able to easily reconstruct the proof of the sentence $[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$. This said, a successful RA I student might not be able to construct this sentence on their own, or spontaneously think of the sentence if asked to discuss how they know θ_1. By the time students were learning about techniques for proving the irrationality of numbers, it seemed their theoretical discourse was not required to be so extensive and formal. There were, however, a couple early assignment activities where the principle of contradiction was explicitly acknowledged and formally used.

Θ_1: Proof of the principle of contradiction, i.e., $[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$.

α	β	$\gamma \wedge \neg\gamma$	$\alpha \wedge \neg\beta$	$(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)$	$\alpha \Rightarrow \beta$	$[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$
0	1	0	0	1	1	1
0	0	0	0	1	1	1
1	1	0	0	1	1	1
1	0	0	1	0	0	1

Thus, the statement is true. ■

Λ_1: (m is an integer and m^2 is even) ⇒ (m is even).

Proof, using the principle of contradiction:
i.e., $[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$

Let $\alpha =$ "m^2 is even" and $\beta =$ "m is even."
We assume α and $\neg\beta$.
i.e., m^2 is even and m is odd.
If m is odd, then $m = 2k + 1$ and
$m^2 = 4k^2 + 4k + 1 = 2(2k + 2)k + 1$
So m^2 is odd.
If we set $\gamma =$ "m^2 is even," we have γ and $\neg\gamma$.
We have a contradiction. ■

Proof, using the principle of contraposition:
i.e., $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$

Let $\alpha =$ "m^2 is even" and $\beta =$ "m is even."
Then $\neg\beta =$ "m is odd" and $\neg\alpha =$ "m^2 is odd."
If m is odd, then $m = 2k + 1$ and
$m^2 = 4k^2 + 4k + 1 = 2(2k + 2)k + 1$
So m^2 is odd.
We proved $\neg\alpha$.
The theorem has been proved by contraposition. ■

Table 6.2 Some mathematical theory studied in RA I that underlies $[\boldsymbol{T_1}, \boldsymbol{\tau_{11,2}}, \boldsymbol{\theta_1}]$.

If we zoom out on all of the practices to be learned that we modelled, we would find an overarching "prove that…" genre of task, which was enriched throughout RA I with an abundance of task types and five general techniques: the table method, proof by contradiction, proof by contraposition, direct proof, and proof by induction. During our analysis of assessment activities, there were some activities in early assignments that were difficult to typify beyond "prove a statement of the form $P \Rightarrow Q$ (using one of the five general techniques)". For example, in two assignment activities, students were asked to prove, first by contradiction and then by contraposition, that if m is an integer and m^2 is odd, then m is also odd. In relation to Task 1, we could see such activities as inviting students to gain the skills that would enable them to prove a very similar element of mathematical theory: If m is an integer and m^2 is even, then m is also even. If we mimic the solutions that had been made available to students, such a proof might look like the ones presented in Table 6.2. Notice, as alluded to earlier, that these proofs rely directly on the principles of contradiction and contraposition represented in the language of formal mathematical logic. If asked, in retrospect of their solution to Task 1, how they knew that m^2 being even meant m was also even, we would expect the successful RA I student to attempt to provide such a proof. That is, they would not only acknowledge the existence of such mathematical theory, they would try to demonstrate their ability to build it.

6.1.2 Practices Actually Learned[24]

Although Task 1 is phrased as a question (Is $\sqrt{8}$ rational or irrational?), we expected a successful RA I student to interpret it as t_1: Prove that $\sqrt{8}$ is not a rational number, and to choose one of two techniques to construct a proof. In both techniques, the student would assume that $\sqrt{8} \in \mathbb{Q}$, i.e., $\sqrt{8} = m/n$, and then they would try to derive a contradiction: either by assuming that the fraction is irreducible and finding that, on the contrary, m and n have a common factor (we denoted this as τ_{11}), or by using some intelligent algebra to manipulate the equality $\sqrt{8} = m/n$ and find that some irrational number is equal to a rational number (we denoted this as τ_{12}).

As we expected, all fifteen of the students we interviewed seemed to spontaneously identify Task 1 with t_1, with ten having the instinct to choose τ_{11}. Interestingly, two of these students decided against their initial choice, either because they came to identify the task as one that they did not know how to solve (because 8 is not prime), or because they decided that they had a "better" technique (i.e., τ_{12}). The remaining five participants constructed solutions centred on proving that $\sqrt{2}$ is not a rational number (using τ_{11}), but not all of these participants were implementing τ_{12}; three participants spontaneously called upon the property "a rational times an irrational is irrational." We discuss the details of how participants seemed to identify Task 1 and choose a proof technique in 6.1.2.1. To our surprise, participants also seemed to have different techniques for convincing someone (but not proving) that $\sqrt{8}$ is irrational; we present some of these at the end of 6.1.2.1. Another surprising feature of our interviews was that many participants got stuck while implementing τ_{11}, even if they seemed to know their goal in carrying out the technique We discuss this further in 6.1.2.2, where we present an analysis of the means by which participants seemed to

[24] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 1. These models are synthesized in table form in Appendix B.

be propelling themselves through each step of implementing their chosen proof technique. One such step, which was not recalled by all students, relies on the statement: If $m \in \mathbb{Z}$ and $p = 2$ (or, more generally, p is prime), then $p|m^2 \Rightarrow p|m$. Recall that we expected a successful RA I student to try to provide a mathematical proof of this statement if asked how they knew it was true: that is, they would not only acknowledge the theory (which we called Λ_1), they would engage in trying to build it. Of the ten students who offered justifications, two built the expected theoretical block. We summarize the behavior of all students with regard to this statement in 6.1.2.3.

6.1.2.1 Identifying the Task and Choosing a Technique

When it came to solving Task 1, participants acted as expected in at least one way: they identified the given task – Is $\sqrt{8}$ rational or irrational? – with

$$t_1\text{: Prove that } \sqrt{8} \text{ is not a rational number.}$$

In 6.1.2.1.1, we outline the different ways that participants chose to solve (or not solve) t_1. Then, in 6.1.2.1.2, we describe the techniques participants might have used to solve a different task:

$$\widehat{t_1}\text{: Convince someone that } \sqrt{8} \text{ is not a rational number.}$$

One student, S4, offered an insightful description of the difference between a "prove that" and a "convince that" task:

> When you show something, it's a proof. Like there's nothing that can be said against it, assuming that the math written down or shown is correct. But to convince someone is just to give enough, not necessarily evidence, but enough reason to say that ok, this can't be expressed as a fraction of two integers. [-] If someone doesn't know a lot about math, [-] that person can be more easily convinced than someone who knows a lot about math and has actually taken a course about Analysis.

We could predict, as alluded to in the above quote, that the more someone knows about mathematics, the more $\widehat{t_1}$ and t_1 are one in the same. For instance, to convince a mathematician that $\sqrt{8}$ is not a rational number, one would need to prove that $\sqrt{8}$ is not a rational number (in the least, the mathematician would need to know that some mathematical proof exists). S4 suggests that to convince a student who has passed RA I would be more difficult than convincing someone who has had less exposure to mathematics. It is true that all our participants attempted to construct a proof in response to Task 1; however, not all required it (or its existence) to convince themselves that $\sqrt{8} \notin \mathbb{Q}$.

6.1.2.1.1 It's a "prove that" task.

6.1.2.1.1.1 Prove $\sqrt{8} \notin \mathbb{Q}$ by treating it like $\sqrt{2}$.

To solve Task 1, ten participants (S1, S4, S5, S6, S8, S10, S11, S12, S14 and S15) seemed to have an instinct to recall τ_{11}: i.e.,

> Assume $\sqrt{8} = \frac{m}{n} \Leftrightarrow 8n^2 = m^2$, where m and n are coprime (i.e., $\sqrt{8} \in \mathbb{Q}$).
>
> Reach a contradiction by showing that m and n have a common factor.

The reasons they gave for choosing this technique were very similar.

It seemed these participants had quickly identified Task 1 with a type of task and technique they had studied in RA I. Upon seeing the task, S1's first words were "Ok. So I'll use the method that we learned in class"; S6 specified that they "actually had a lot of problems like this"; and S11 added that "even in our final, we had [to] prove if [a number is] rational or irrational." S12, who was particularly good at thinking aloud, gave the following description of what was going through his mind when he first saw Task 1:

> So, my thought process for this question is largely based on what I've seen before for other questions. There's a methodology I've seen for this kind of question, that I just approach blindly for any, any irrational, or any, any root of any number.

This "blind approach" seemed to be common among the participants in this group, most of whom did not explicitly consider the nature of $\sqrt{8}$ before assuming $\sqrt{8} \in \mathbb{Q}$ and writing $\sqrt{8} = m/n$. When S10 realized that this is what he had done, he stopped:

> Uh, wait a second. That's uh, that's pretty silly, isn't it? [-] I actually didn't ask myself in my head before I started if [-] there was a square root of eight. [-] Like you could have given me square root of four and I would have started this way.

It seems that seeing the square root of a number and the words "rational or irrational" is what initially triggered these students to choose the technique that they did.

Identifying the task and choosing a technique in this way seemed to result from participants' experiences of the learning environment in RA I. S15, who had written $\sqrt{8} = 2\sqrt{2}$ and contemplated making an argument using the fact that $2 \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$, said:

> I know that maybe it can be simpler to use this argument. But at that point, I don't remember necessarily all the details of this result. And usually the teachers emphasize more [the other] approach.

S15 explained his ultimate choice of τ_{11} by saying "That's the way we were taught"; S11 called it "a habit" learned through problems seen in class and on exams; and S12 said that "This is a way of proving [-] that the teacher has used that [-] I've used to regurgitate back to him." These students recognized τ_{11} as being something they needed to master if they were to be deemed a successful student of RA I. S1 further explained why the technique needed to be mastered to the point of being automatic:

> I feel that we're grinded to do so many questions really quickly. So we need to associate problems to a solution [-] really fast. [-] Cause I don't really have the time to analyze the problem and try different things during an exam. So I grind problems at home. And when I get in an exam, I see the problem and I say, "Ok, that's exactly the kind of problem…it goes down to this."

Note that such references to the learning environment to justify a choice of technique do not imply a lack of mathematical discourse for describing how the technique works. Among the participants we identified as recalling τ_{11}, some seemed to possess such a discourse; and others did not. We expand on this further in 6.1.2.2.

Two students (S11 and S14), whose first steps of solving Task 1 seemed to reflect a recall of τ_{11}, eventually abandoned the approach. They had extremely different reasons for doing so. We describe these and the alternate techniques they demonstrated in the following two subsections.

6.1.2.1.1.2 I know how to prove the irrationality of a different kind of number.

Like the participants mentioned above, S11 started his solution to Task 1 by assuming that $\sqrt{8} = m/n$ and squaring both sides to get $8n^2 = m^2$. At this step, several participants did as we had expected, deducing that $2|m^2$ and invoking some version of the statement: If m is an integer and $2|m^2$, then $2|m$. In contrast, when S11 reached $8n^2 = m^2$, he was compelled to write as a next step: $8|m^2 \Rightarrow 8|m$. "But eight is not a prime number," he noted. "You need a prime number for you to be able to say that m is also divisible by 8." When asked what he would do if 8 were prime, S11 described the steps for showing that $\sqrt{p} \notin \mathbb{Q}$ for p prime, which rely on the more general statement: If m is an integer and m^2 is divisible by a prime number p, then m is also divisible by p. It seems, therefore, that S11 had learned the specified version of τ_{11} that would solve the more restricted, most commonly tested, type of task: Prove that $\sqrt{p} \notin \mathbb{Q}$ for p prime. Moreover, he clearly identified Task 1 as not belonging to that type (since 8 is not prime). In fact, when he recognized that he had learned a technique specified to prime numbers, S11 doubted if $\sqrt{8}$ is irrational and used a different approach to check (more on this in 6.1.2.1.2 below). In contrast, we would expect the students mentioned in the previous section to try to apply τ_{11} to solve any task of the form "prove that $\sqrt{c} \notin \mathbb{Q}$."

When compared to the other students we interviewed, S11's behaviour was interesting in at least two ways. First, he did not succeed in solving Task 1, not because he claimed to have forgotten the steps, as was the case for many of his peers (see 6.1.2.2 below), but because he identified the task as not belonging to the collection of practices he had learned in RA I (recall that S11 was our epitome of a student in the position of Student; see Section 5.1.2). Second, S11 was uninterested, unwilling, and/or unable to see how the pieces of the practice he had developed might be used to solve Task 1. For instance, he did not contemplate a manipulation such as $8n^2 = m^2 \Leftrightarrow 2(2n)^2 = m^2$ or the specification of the statement he used to a more useful case (i.e., $p = 2$).

6.1.2.1.1.3 Prove $\sqrt{8} \notin \mathbb{Q}$ by proving $\sqrt{2} \notin \mathbb{Q}$.

Upon recognizing that one can rewrite $\sqrt{8}$ as $2\sqrt{2}$, six students (S2, S3, S7, S9, S13, and S14) chose to solve Task 1 by showing that $\sqrt{2}$ is irrational. Like the students mentioned in 6.1.2.1.1.1, their solutions reflected a choice of τ_{11}; but they applied it to $\sqrt{2}$ rather than $\sqrt{8}$. Mathematically speaking, such an approach needs to be accompanied by an explanation as to why $\sqrt{2} \notin \mathbb{Q} \Rightarrow 2\sqrt{2} \notin \mathbb{Q}$. The students in this group had different ways of considering this implication.

Two students, S2 and S9, struggled to justify the implication $\sqrt{2} \notin \mathbb{Q} \Rightarrow 2\sqrt{2} \notin \mathbb{Q}$. While they were solving the task, they did not explicitly address the implication; and when the interviewer asked about it afterwards, both students became unsure of their chosen technique. These students seemed to recall learning certain rules regarding the multiplication of two irrational numbers or two rational numbers. They lacked confidence, however, in the statement that the product of a rational number and an irrational number is irrational. Consider, for instance, what S9 said, based on the example $\sqrt{2} \cdot \sqrt{2} = 2$:

> If you multiply an irrational number with another irrational number, it doesn't necessarily mean it's going to be irrational. And if you multiply an irrational with a rational, I guess the same thing applies.

After looking to the interviewer for guidance as to whether her chosen technique was right or wrong, S2 noted that

> any rational number times a rational number, then I think the result must be a rational number. [-] But I forgot some knowledge I had learned last semester, so I'm not sure [if a rational number times an irrational number is an irrational number].

Both S2 and S9 ended up suggesting alternative techniques. S2, for example, said that if the interviewer told her that her chosen technique was wrong, then she would attempt to solve the task using τ_{11}. S9 eventually concluded: "I know it's irrational. I just don't know how to… get to it. I didn't really like RA I. Maybe that could be good for your notes."

This begs the question of why these students thought they could solve Task 1 by showing that $\sqrt{2}$ is irrational. It is possible that their choice of approach came primarily from the recognition that $\sqrt{8}$ includes a number, $\sqrt{2}$, whose irrationality they had studied before. When explaining why he did what he did, S9 said that "when we simplify it to two square root two, I remember like proving that square root two was irrational. So, I guess I tried to recreate that." It is also possible that these students were recalling a partial version of τ_{12} (the other technique to be learned in RA I).

Like S2 and S9, S7 did not explain how he went from concluding that $\sqrt{2}$ is irrational to concluding that $\sqrt{8}$ is irrational during his solution. The major difference is that, when the interviewer solicited this unexplained reasoning, S7 confidently said: "A rational multiplied by an irrational is irrational. That can be proved by contradiction as well." In what we assume was a search for a proof of this general property, S7 constructed an argument of the specified implication, $\sqrt{2} \notin \mathbb{Q} \Rightarrow 2\sqrt{2} \notin \mathbb{Q}$, which can be summarized as follows:

Assume $2\sqrt{2} \in \mathbb{Q}$.

If we divide a rational by a rational, then we should get a rational.

But then, $(2\sqrt{2} \in \mathbb{Q} \wedge 2 \in \mathbb{Q}) \Rightarrow \frac{2\sqrt{2}}{2} = \sqrt{2} \in \mathbb{Q}$.

But we just proved that $\sqrt{2}$ is an irrational number.

Contradiction.

So $2\sqrt{2}$ is irrational. ■

Two other students either exhibited an awareness of this kind of solution (S10) or chose to produce a similar proof to solve the task (S14); but S7 was the only one to explicitly discuss how his proof came from a recall of the activities in RA I. He explained his choice of technique as follows:

> I'm a visual person. So I try to remember visually what we learned in class. And I remember there were a lot of examples like this in the sample midterms and finals. [-] They were mostly you had to add two numbers to prove that they were irrational. So then I tried to apply it with what I knew about multiplication, and this is how I got to the result.

S7 seemed to recall two main types of numbers showing up in the activities in RA I, each of which required its own technique: presumably, he recalled that there were not only numbers like $\sqrt{2}$ (requiring τ_{11}), but there were also numbers like $\sqrt{2} + \sqrt{3}$ (requiring τ_{12}). Moreover, S7 was able to recall the solutions for the latter number type and adapt them to Task 1 using what he knew about the multiplication (or division) of rational numbers. In other words, he seemed to understand the main idea in τ_{12}: namely, use rules of algebra to find that an irrational number is rational.

S3 and S13 differed from the students mentioned above in at least two important ways. First, they spontaneously included in their solution the mathematical technology on which it was based. In fact, the first words uttered by S3 after seeing Task 1 expressed that he would be taking the following statement "as a given":

$$\text{If } a \in \mathbb{Q} \text{ and } b \in \mathbb{R}\backslash\mathbb{Q}, \text{ then } ab \in \mathbb{R}\backslash\mathbb{Q}.$$

This statement is not entirely correct: it is missing the assumption $a \neq 0$. Such precision was attained by S13, who followed her proof that $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$ by explaining: "We know that any rational number multiplied by an irrational number, unless it's zero, it's an irrational." The second unique characteristic of these students' solving of Task 1 was that they were confident in their ability to construct a proof for their technology; and they demonstrated this ability in the interview. For S3, it seemed to be the first time he was constructing such a proof, and he seemed excited to try it: "How do I know that a rational times an irrational is irrational? [-] I don't know. Ok, let's see!" In comparison, when she was asked how she knew that a non-zero rational multiplied by an irrational is irrational, S13 said rather calmly: "Well this is very easy to prove." She then demonstrated what she meant by quickly producing the following argument:

> Let's say $q \cdot r$ is rational, where r is irrational and q is rational.
> So $q \cdot r = \frac{a}{b} \Rightarrow r = \frac{a}{b \cdot q}$ (we assume $q \neq 0$ so we can write this).
> But we cannot write irrationals like this.
> So the first assumption is wrong: $q \cdot r \notin \mathbb{Q}$. ■

Note that S3 eventually sketched a similar argument. Notice also how this is essentially a generalized version of S7's proof that $2\sqrt{2} \notin \mathbb{Q}$ (just let $q = 2$ and $r = \sqrt{2}$).

Another unique characteristic of S3's task solving behaviour was that after he had successfully solved Task 1 using the approach described above, he interrupted the interviewer to announce: "Actually! [-] I could probably do this without proving the square root of two [is irrational]." It seemed to be out of sheer interest that S3 then embarked on trying to prove $\sqrt{8} \notin \mathbb{Q}$ by treating it like $\sqrt{2}$ (i.e., by trying to recall and implement τ_{11}). Not surprisingly, S3's actions in implementing τ_{11} were also noticeably different from all students mentioned in 6.1.2.1.1.1 (we return to this in 6.1.2.2).

S14 also stood out from all other participants, but in a completely opposite manner when compared to S3. S14 seemed to begin solving Task 1 by recalling τ_{11}: i.e., he started by assuming that $\sqrt{8} \in \mathbb{Q}$, whereby $\sqrt{8} = \frac{m}{n} \Leftrightarrow 8n^2 = m^2$. Then he stopped: "Ah, you know what? Actually, I don't like this proof. I got an easier proof." S14 insisted on abandoning his first choice of technique for the one demonstrated above, by S7. When asked why he did this, he said the following:

> As someone who kind of leans towards the pure math side of things, I actually always prefer a more elegant proof. So if I can find something simpler, more elegant, "prettier," as it were, I will always go with that. [-] It's more like being a bit more creative, a bit more quick on your feet. Rather than just going like: "Oh, I saw these, how to do this problem in class, or in a book or whatever. And I'm just going to like robot proof it to like get through."

S14 claimed that his choice of technique was not based solely on what he remembered or what he knew: it was about what he thought would be "prettier," more "elegant," or more "creative." Moreover, he attributed his preference to his personality, as someone who "leans towards the pure math side of things." By extension, he seemed to be suggesting that criteria like "beauty," "elegance," or "creativity" might serve to judge a practice as more mathematical than another, at least in pure mathematics. For S14, τ_{11} (applied to $\sqrt{8}$) was a less mathematical technique in this sense; and those who applied it were behaving like "robots."

We would agree with S14 that some students' behaviour when solving Task 1 was "robotic." Moreover, we hypothesize that as students developed such "automatic practices," it may have inhibited a certain level of fluency in the by-contradiction proving technique; e.g., the level required to mathematically transform the task of proving that $\sqrt{8}$ is irrational into the task of showing that $\sqrt{2}$ is irrational. This could explain why some students were limited to recalling τ_{11} in steps or could not provide a mathematical explanation for their actions (i.e., why proving $\sqrt{2} \notin \mathbb{Q}$ proves that $\sqrt{8} \notin \mathbb{Q}$).

6.1.2.1.2 It's a "convince that" task.

Recall that in our analysis of assessment activities from RA I, we noticed that for tasks like Task 1, the corresponding solutions typically began with the first line of the proof: i.e., Assume $\sqrt{8} \in \mathbb{Q}$ $\Leftrightarrow \sqrt{8} = m/n$. In light of this, it is perhaps not surprising that many participants did not seem to think about if $\sqrt{8}$ is rational or irrational before jumping into their proof. Consider, for example, the following exchange between the interviewer and S5, which occurred in retrospect of S5's solution attempt:

> L: "Before you jumped into here [pointing at the line $\sqrt{8} = m/n$], did you think that root eight was irrational? Or rational? Did you think about that?"
>
> S5: "Oh, no, I didn't think about it. No."
>
> L: "Ok."
>
> S5: "But, when you look at it, you feel, I mean, you know it's irrational. I don't know. Like I would feel that it's irrational."

In her last comment, S5 provides a hint that she has a different (albeit tacit) technique, other than the by-contradiction proof, that she can use to convince herself that $\sqrt{8} \notin \mathbb{Q}$. This seemed to also be the case for most of her peers. Most participants, in retrospect of their proof attempts, claimed that they did not need a proof to convince themselves that $\sqrt{8}$ is irrational. This section summarizes the other ways that participants may have answered Task 1 had they identified it with $\widehat{t_1}$: Convince someone that $\sqrt{8}$ is not a rational number, rather than t_1: Prove that $\sqrt{8}$ is not a rational number.

It seemed that some students (S2, S5, S9, and S15) would solve $\widehat{t_1}$ by simply saying that it is obvious that $\sqrt{8}$ is irrational, with little to no additional explanation. "I just know it," claimed S2, who saw the proof as a means of exercising her logical thinking; S15 said, in a similar vein, "It's kind of evident. But we must be formal"; and S9 added that doing the proof "is necessary to get full marks." As mentioned previously (in the case of S2 and S9), these three students were not sure if the product of a rational and an irrational number is irrational; but they said that they just knew that $\sqrt{8} = 2\sqrt{2} \notin \mathbb{Q}$. As alluded to in the excerpt shown above, S5 also held this belief. Looking at $\sqrt{8}$, she explained that "it's a square root. So usually it's irrational." Then, having simplified it to $2\sqrt{2}$, she added that "it still has a square root of two. And I don't think you can write it as a rational number." We conjecture that these students based their belief on a combination of previous experience (e.g., square roots indicate irrationality) and how the number looks (e.g., algebra cannot be used to eliminate the square root symbol).

Many other students (S1, S3, S7, S8, S10, and S13) said that they knew $\sqrt{8} \notin \mathbb{Q}$ as soon as they saw that $\sqrt{8} = 2\sqrt{2}$ and recalled that $\sqrt{2} \notin \mathbb{Q}$. The difference was that they had more confidence that the product of a rational and an irrational is irrational. Recall that three of these students (S3, S7, and S13) had demonstrated an ability to sketch mathematical proofs of either general or specified versions of this statement (see 6.1.2.1.1.3); they did not indicate that they had any other manner of convincing themselves of the irrationality of $\sqrt{8}$. In comparison, the remaining three students (S1, S8, and S10) had chosen to implement τ_{11}: i.e., to let $\sqrt{8} = m/n \Leftrightarrow 8n^2 = m^2$ and show that m and n have a common divisor (see 6.1.2.1.1.1). All three of these students exhibited a different way of convincing themselves that $\sqrt{8} \notin \mathbb{Q}$ during a pause from implementing τ_{11}. Just like S3, S7, and S13, S10 exhibited an awareness of τ_{12}: in his brief pause from τ_{11}, he noticed:

> This all would have been a lot easier just letting $\sqrt{8}$ be equal to $2\sqrt{2}$. [-] And then, if we assume that this is a rational number, 2 is a rational number, so we can divide by 2, and $\sqrt{2}$ is a rational; and then show that $\sqrt{2}$ is irrational.

S1 and S8 seemed to have less fluency in the by-contradiction proof technique. S1, for instance, said that he was convinced of his statement that "an irrational times a rational stays irrational," but he did not know how to prove it. Moreover, he felt he would need to prove it to solve t_1. Hence, after mentioning the statement, he returned to his implementation of τ_{11}, explaining "I'll go down that road. And if [-] ever [$\sqrt{8}$] is [rational], well I'll just restart."

Four participants (S4, S6, S12, and S14) pinpointed the non-square nature of 8 to explain how they knew that $\sqrt{8} \notin \mathbb{Q}$. Their conviction seemed to be of different strengths and to come from different sources. S14 confidently told the interviewer that he immediately knew that $\sqrt{8} \notin \mathbb{Q}$: "It's a theorem that [-] the square root of a non-perfect square is irrational." S12 made a similar statement, though he claimed to accept its truth, not because it is a proved theorem, but because he had built up a certain intuition through prior experience. This also seemed to be the case for S4 and S6, who said they would use this more intuitive approach if they were trying to convince someone who does not know a lot about math. S6 explained the approach, with some hesitation, as follows:

> If I don't see a square number, then I would assume it's irrational, like just instinct I guess. I wouldn't know if it's right or wrong. [-] Let's say someone who doesn't know anything about Analysis, I don't think [the proof] would be necessary [for them]. Cause, I mean, basic common knowledge, like you know, maybe fourth grade, like multiplication, [-] if you don't see a square number, then most probably it is irrational and you would just go based on that.

We could write the technique proposed explicitly by S6, and implicitly by the others, as: When given $\sqrt{c}$, check if c is a square number; if not, conclude $\sqrt{c}$ is irrational. S6 suggests that such a technique could be taught in earlier mathematics courses. All four of the students mentioned here also seemed to see the technique as useful for convincing themselves that $\sqrt{8} \notin \mathbb{Q}$; but it was not appropriate for proving that $\sqrt{8} \notin \mathbb{Q}$.

We identified three other techniques used by participants to convince themselves that $\sqrt{8}$ is irrational. All three were unexpected and each were exhibited by only one participant. Recall that S11 explicitly indicated that he had learned a practice for showing that $\sqrt{p} \notin \mathbb{Q}$ for p prime, which led him to conclude that he did not know how to solve Task 1 (see 6.1.2.1.1.2). In fact, upon recognizing that his practice was specified to prime numbers, S11 expressed doubt that $\sqrt{8}$ is irrational. He was the only one to clarify his doubt by turning to his calculator. After using it, he confirmed: "No, it would be irrational." Although we did not inquire directly into what S11 did with his calculator, we assume that he typed in $\sqrt{8}$, pressed the fraction button, and found that the calculator did not provide a response. When asked later if he needed a proof to convince himself that $\sqrt{8}$ is irrational, S11 said:

> Well, doesn't that kind of come back to the calculator thing? [-] If you know that a rational number is something that you can write in m/n form, right? Then if you can figure out that you can't, then that's pretty much [it]… Like you don't need to see the whole proof. But if you tell me to prove it, then yeah.

While S11 seemed convinced that $\sqrt{8}$ is irrational by pressing a button on his calculator, he did not think that this technique could "prove" that $\sqrt{8} \notin \mathbb{Q}$.

Another surprising technique was exhibited by S12 when he was stuck at the step $8n^2 = m^2$ in his implementation of τ_{11}. Like S11, S12 noted that 8 is not prime: "Often if we end up with a prime number here, we can just end up saying [-] by the uniqueness of prime factorization, this isn't possible." S12 exemplified what he meant by reproducing an argument that $\sqrt{3} \notin \mathbb{Q}$. Then he used this to produce a similar argument about the $\sqrt{8}$ case. More specifically, S12 leveraged the property that any number can be broken into factors of primes, in a unique way, to explain why an equality like $8n^2 = m^2$ would not be able to hold:

> I could factor out eight as being three twos. [-] If we factor out n into primes, we would end up with pairs of primes, and pairs on this side [i.e., m^2]. And there would be no way to end up with another triplet of twos on this side [i.e., m^2].

Based on his argument for $\sqrt{3}$, we think that S12 understood that there would be an odd number of twos on the left and that there could only be an even number of twos on the right of the equality

$8n^2 = m^2$, whereby the equality could not be true. He concluded, nonetheless: “To prove that is a little different.” After deeming his argument as unworthy of “proof” status, S12 returned to the classroom methodology.

Finally, there was S4, who had also chosen τ_{11} and, after getting stuck, exhibited another kind of argument: namely, $\sqrt{8} = \left(\sqrt{2}\right)^3$ is irrational because $\sqrt{2}$ is irrational. When asked about how he knew this, S4 said that “any power of an irrational number is also irrational.” Shortly thereafter, the interviewer presented him with a possible counterexample, $(\sqrt{2})^2$, to which S4 quickly responded: “Okay, that is not irrational. [-] Yeah, I don’t know.” S4 had seemed to be on the verge of an interesting mathematical technology (capable of producing a mathematical technique): i.e., the odd power of an irrational square root is irrational. But S4 seemed to also lack some of the tools required to get there: not only in terms of awareness (e.g., of the nature of different types of numbers), or practices (e.g., techniques that could assist in proving the technology), but perhaps also in terms of confidence and problem solving skills. As exhibited throughout the rest of this section, S4 was not the only student who seemed to be in this situation.

6.1.2.2 Implementing and Describing a Technique: Proving that $\sqrt{c} \notin \mathbb{Q}$ using τ_{11}.

In 6.1.2.1, we outline the different techniques participants seemed to have developed for arguing that a given number is not rational. In spite of the differences we observed, there was one commonality: every proof attempt included the aim of proving $\sqrt{c} \notin \mathbb{Q}$ for a specified c using the technique we denoted as τ_{11}: i.e.,

Assume $\sqrt{c} = \frac{m}{n} \Leftrightarrow cn^2 = m^2$, where $m, n \in \mathbb{N}$ are coprime (i.e., $\sqrt{c} \in \mathbb{Q}$).

Reach a contradiction by showing that m and n have a common factor.

One group of students (S2, S3, S7, S9, S13, and S14) chose to implement τ_{11} for $c = 2$; the rest (S1, S4, S5, S6, S8, S10, S11, S12, and S15) chose to implement τ_{11} for $c = 8$; and one student (S3), after implementing τ_{11} for $c = 2$, tried to see if he could also implement τ_{11} for $c = 8$. In this section, we focus on participants’ implementation of τ_{11}, and we juxtapose this with the descriptions some of them offered of the technique.

Four of the six students who applied τ_{11} for $c = 2$ were able to succeed without any significant struggle. This is perhaps not surprising for at least two reasons. First, we suspect that these students had studied the proof that $\sqrt{2} \notin \mathbb{Q}$ enough to commit it to memory; in this sense, perhaps some of them were reproducing a proof, rather than implementing a technique. For instance, when S3 first saw Task 1, he said that "it's an easy one"; and he later explained why: "I've heard about the proof that the square root of two is irrational. I've seen it many times, so…" Second, the implementation of τ_{11} for $c = 2$ is more straightforward than for $c = 8$. This said, there were two students (S7 and S9) whose implementation was incomplete in some way. The students began as expected:

> Suppose $\sqrt{2} = \frac{m}{n}$, where the fraction is irreducible (i.e., $\sqrt{2}$ is rational).
>
> Then $2n^2 = m^2$.

At this point, S9 got stuck. S7 went one step further:

> $2n^2 = m^2 \Rightarrow m^2$ is even $\Rightarrow m$ is even.

But S7 was also unable to complete the proof. Both students offered a similar explanation of what they were trying to accomplish. S7, for example, explained:

> I'm actually trying to prove that n is also even, which I know it is, but uh… [-] If n is even, then we know that [-] m over n [-] has a common multiplier. [-] So by contradiction, [-] it cannot be a rational.

S7 and S9 recalled the key component of their proof that $\sqrt{2} \notin \mathbb{Q}$: that is, if $\sqrt{2} = m/n$, then one can show that both m and n are even. They could not remember how to show it.

Most students who attempted to implement τ_{11} for $c = 8$ seemed to struggle due to a similar lack of recall of certain steps. When he got stuck, S1 explained that "this problem, I do it by memory, so… I just forgot the… Well forgot-ish the way how to do it." S12 offered a more detailed description of what "doing it by memory" meant for him:

> Often I'm not thinking as I'm going. [-] I'm almost recalling a solution that I've seen related to this, that I'm trying to use to prove it. [-] And because I'm struggling to conjure that image, I'm having trouble proving this.

Some of these students were affected emotionally by their struggle. S5, for instance, said that she felt stupid because "it was such an easy common question. [-] I would just proceed on following the steps mostly. And then, I understood how to do it. But now I don't know why I'm stuck." In response to the interviewer's question of how she might get unstuck, S5 said: "I would just go back into my notes and look at it. Like how the teacher proceeded. Because they did it many times." In a similar vein, when S9 got stuck in proving $\sqrt{2} \notin \mathbb{Q}$, he claimed: "If I was at home, I'd check online, how to do the proof."

	S8's proof that $\sqrt{2} \notin \mathbb{Q}$.	S8's steps to show that $\sqrt{8} \notin \mathbb{Q}$.
1	Let $\sqrt{2} = \frac{m}{n}, m, n \in \mathbb{N}, \gcd(m, n) = 1$.	Let $\sqrt{8} = \frac{m}{n}, m, n \in \mathbb{N}, \gcd(m, n) = 1$.
2	$2n^2 = m^2$	$8n^2 = m^2 \Leftrightarrow 2(2n)^2 = m^2$
3	$\Rightarrow m^2$ is even $\Rightarrow m$ is even; $m = 2k, k \in \mathbb{N}$	$\Rightarrow m^2$ is even $\Rightarrow m$ is even; $m = 2k, k \in \mathbb{N}$
4	$2n^2 = (2k)^2$ $\Leftrightarrow 2n^2 = 4k^2$ $\Leftrightarrow n^2 = 2k^2$	$2(2n)^2 = (2k)^2$ $\Leftrightarrow 2(2n)^2 = 4k^2$ $\Leftrightarrow (2n)^2 = 2k^2$
5	$\Rightarrow n^2$ is even $\Rightarrow n$ is even; $n = 2q, q \in \mathbb{N}$	$\Rightarrow (2n)^2$ is even $\Rightarrow 2n$ is even; $2n = 2q, q \in \mathbb{N}$ $\Rightarrow n = q$ "that's not really… useful."
6	$\Rightarrow n$ and m are multiples of 2 Contradiction. "So therefore, you can't write root two as a rational. So it must be irrational." ■	Go back and try to expand 4: $4n^2 = 2k^2$ $\Leftrightarrow 2n^2 = k^2$
7		k^2 is even $\Rightarrow k = 2q, q \in \mathbb{N}$ "But then I'm just sort of going around in circles a bit."
8		$\Rightarrow 2n^2 = 4q^2$ $\Leftrightarrow n^2 = 2q^2$ "Oh, which would mean…"
9		$n = 2v, v \in \mathbb{N}$
10		$\Rightarrow n$ is a multiple of 2, m is a multiple of 4 But 2 and 4 have common factors. Contradiction. "I think that would be it. [-] I'm convinced. I'm not sure where else I would go from that, but I think that's right. To some extent."

Table 6.3 How S8 adapted the proof that $\sqrt{2} \notin \mathbb{Q}$ to prove that $\sqrt{8} \notin \mathbb{Q}$.

One interviewee demonstrated what using "an outside source" to solve the given task might look like. When S8 got stuck, the interviewer asked if the task would have been easier if the number was different. "Yeah, I think root two I could do," responded S8. His memory of the proof that $\sqrt{2} \notin \mathbb{Q}$ was strong enough that he could reproduce it completely (see the left column of Table 6.3 above). Note also that this further supports our conjecture that some of the students who proved $\sqrt{2} \notin \mathbb{Q}$ were reproducing a proof, rather than implementing a technique). With the $\sqrt{2}$ proof completed, S8 then returned to Task 1: "I'm going to try to use the same logic, like in my head that seems to work, so…" We can represent the subsequent steps taken by S8 as shown in the right column of Table 6.3 (we have also included some direct quotes from S8 throughout). It was particularly interesting to observe what happened at the moment where the two proofs differed. At line 5, it seemed that S8 was hoping to be able to conclude that n is even, so to mimic the steps of the $\sqrt{2}$ proof; but, of course, he was unable to do so. He did not give up: Even if he felt that he was "going around in circles" and seemed to lack confidence in his conclusion, S8 persisted to the point of correctly adapting the proof to the $\sqrt{8}$ case.

We wonder how other students would have acted if given the $\sqrt{2}$ proof. S6, for example, made it all the way to the equivalent of line 6 in S8's proof that $\sqrt{8} \notin \mathbb{Q}$; and then she stopped, claiming that she "forgot" the rest. Perhaps seeing the $\sqrt{2}$ proof would have supported her in realizing that a certain adaptation was needed. In other words, it is not possible to proceed simply by remembering the $\sqrt{2}$ proof.

There were two students (S10 and S12) who, like S8, were eventually able to complete their implementation of τ_{11} on $\sqrt{8}$, not only because they remembered the steps, but also because they figured out how to adapt them to solve Task 1. S10 and S12 constructed solutions that were very similar to S8's, as depicted in Table 6.3. Notice that this solution was the adaptation we had expected from successful RA I students. S3, in comparison, constructed an unexpected adaptation of τ_{11}, which did not rely on showing that m and n have a common factor. At first, his work resembled that of his peers: i.e.,

$$\sqrt{8} = \frac{m}{n} \Leftrightarrow 8n^2 = m^2 \Rightarrow 2|m^2 \Rightarrow 2|m \Rightarrow m = 2k \Rightarrow 8n^2 = 4k^2 \Rightarrow 2n^2 = k^2$$

Then, after some thought, S3 proclaimed: “Oh… This is interesting! [-] Because that implies that the square root two is rational. [-] And I know that that’s not true!” Notice that the last step shown above can be manipulated to $\sqrt{2} = \frac{k}{n} \in \mathbb{Q}$. S3 concluded that, although there might be a way to do the proof without using the fact that $\sqrt{2} \notin \mathbb{Q}$ (e.g., by showing that m and n have a common factor), he simply did not know it yet.

In contrast with the students mentioned above, several students were unable to get unstuck in the context of the interview (like S7 and S9). As already mentioned in 6.1.2.1.1.2, one student (S11) seemed unable to get unstuck because he could not see how to adapt the particular steps he recalled for proving that $\sqrt{p} \notin \mathbb{Q}$ for p prime. Most other students (S1, S4, S5, S6, and S15) seemed unable to get unstuck due to their lack of memory of the steps.

Among these latter students, we observed vastly different kinds of descriptions of τ_{11}. S4, for instance, seemed unaware of the goal of τ_{11}. After writing $8m^2 = n^2$, he somehow concluded that m and n are even; but he did not stop to explicitly recognize that this completed his solution to the task (he continued trying to argue why $\sqrt{8}$ is irrational). Accordingly, when S4 described the technique in the interview, his discourse seemed limited to the following sentence: “Well, to verify whether it is a rational or an irrational, it is about a fraction of two integers.” Compare this with S1, who did not know how to show that both m and n are even, but explained more clearly how τ_{11} works:

> If m is even and n is even, then they share a common factor which is two, and the assumption doesn’t hold. So by contradiction, it would be irrational. We’re assuming that it’s rational. [-] It has to be irrational if it’s not rational.

S6 and S15 exhibited very similar descriptions of what they were trying to accomplish, even if they did not know how to get there.

At least one question arose for us when we noticed the specificity of the conclusion these students were trying to reach: How did they know that showing $\sqrt{8} \notin \mathbb{Q}$ relied on showing that m and n are both *even*? In other words, what compelled students, when they saw $8n^2 = m^2$ to note that this implies $2|m^2$? It is possible that, like S8, the students were recalling the specific steps of the proof that $\sqrt{2}$ is irrational. S1, for instance, said: “Honestly, a problem like this [-] I’d remember how we did square root of two and just do it exactly the same way.” In connection with this, perhaps some students' techniques were particularized to using the technology "m^2 even $\Rightarrow m$ even $\forall\, m \in \mathbb{Z}$," as opposed the more general version: "$p|m^2 \Rightarrow p|m$ for any prime number p and $m \in \mathbb{Z}$." Whatever the case, all students who chose $2|m^2$ did so quickly, which makes us wonder if they based their choice solely upon 8 being an even number. S6 was the only student who was asked to clarify her reasoning: "We have an eight here. So if it was like any even number: two, four, six, … Um, I don't know, I learnt it somewhere. I don't know which class exactly. But then you would have an even number." We wonder how such students would react to a task such as showing that $\sqrt{12} \notin \mathbb{Q}$. At the same step, $12n^2 = m^2$, we think some students might make a similar choice, $2|m^2$, which would not be productive in this case. Recall that only one student (S12) demonstrated an awareness of and inclination towards studying an equality like $12n^2 = m^2$ using the notion of prime factorization, which could lead to a more productive choice (i.e., $3|m^2$). Because of this, we suspect that several participants may have developed a more specified technique than the one required for showing that $\sqrt{c} \notin \mathbb{Q}$ for any non-square number c.

6.1.2.3 Acknowledging or Building Theory: $p|m^2 \Rightarrow p|m$ for $p = 2$ or p prime.

One of the technologies required in the implementation of participants' chosen proof techniques is that if m is an integer and m^2 is even, then m is also even; or, more generally, if m is an integer and p is a prime number such that $p|m^2$, then $p|m$. Some participants showed no signs of recalling any version of this theorem at the time of the interview (S1, S4, S5, and S9), or at least did not state it clearly during their solution (S6); not surprisingly, these students were unable to fully implement their chosen techniques (as discussed in 6.1.2.2 above). Eight of the ten remaining students relied on the property that if m^2 is even, then m is also even; and two students cited the more general version: if $p|m^2$, then $p|m$. All ten of these students offered justifications for how they knew the property they used. We summarize these below.

Three students (S8, S10, and S11) referenced a past experience to justify their belief in the property they used. S11 said he recalled his professor mentioning something: "Isn't there like a theorem that says like… Like let's say m^2 is divisible by 8… Oh, but by 7, let's say. Because 7 is a prime number, then you can say that [-] m is also divisible by 7." S8 said he remembered seeing a proof in RA I that if you have something like $m^2 = 2n^2$, then m has to be even: "I can't quite remember the proof, but yeah… I do remember seeing that, yeah. So I know I'm allowed to [use it]." S10 added that his belief in the theorem came from the memory of having done it. [-] It's like enough in the recent past, it's probably in the past year that I remember showing this to myself. [-] I guess in my head I've sort of checked it off as something that I saw at one point, and I now have confidence in.

It is important to note that although S10's past experience gave him confidence that the theorem is true, he insisted that his solution was incomplete without a proof. In contrast, most of his peers seemed to be okay with taking the theorem for granted (the exception was S15, who included his proof of the theorem in his solution); and when the interviewer inquired how they knew the property they used, most seemed compelled to try to give a mathematical explanation.

Two students (S2 and S7) offered a logically unsound explanation. During her proof that $\sqrt{2} \notin \mathbb{Q}$, S2 noted that m^2 is even implies that m is even "because even times even is equal to even." In a similar vein, S7 explained: "If you multiply two even numbers, you have to get an even number. Actually… Uh, yeah, anything multiplied by an even number is even." The mathematical statements made by these students are correct and can easily be proved using the definition of an even number and the closure of the integers under multiplication. For example, if m is even, then, by definition, it is equal to $2k$ for some integer k, whereby $m^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is also even because $2k^2$ is an integer. Notice, however, that the students' statements are linked, not to the theorem they invoked, but to its converse. More specifically, the students justified "m^2 even $\Rightarrow$ m even" by saying, in essence, that "m even $\Rightarrow$ m^2 even."

This confusion between the statement used and its converse also seemed to be present in another students' discourse. Before S8 concluded that he did not recall the proof that "m^2 even $\Rightarrow$ m even," he suggested that it might go by induction: "You could test two, like that's your n equals one case. You could say: Well, if you square two, you get four, which is a multiple of two." What S8 seemed to be envisioning is a process that considers each even number, squares it, and checks if the result is also even; that is, a process that verifies "m even $\Rightarrow$ m^2 even." Compare this with S12, who explained: "If you consider the set of all square numbers, you have one, four, nine, sixteen, twenty-five, … All the ones that are even are products of even numbers." Indeed, to check if "m^2 even $\Rightarrow$ m even," a more appropriate process would be to list all the square numbers, consider all the even ones, and check if their square roots are even. Notice that S12 did not link this process to a proof technique, like S8's "induction." He seemed to be operating on a more intuitive level.

After providing the above explanation, S12 chose to argue that "m^2 even $\Rightarrow m$ even" in a different manner. In contrast with S2, S7, and S8, both S12 and S15 seemed to understand that it was not enough to state that if m is even, then m^2 is also even. S12 reasoned as follows: "Either m is odd or m is even. If m is odd, then m^2 is odd. If m is even, m^2 is even. So, going backwards, we have m^2 is even. We know that m is even." When asked how he knew that if m is odd (or even), then m^2 is odd (or even), S12 said: "I think that's just a pattern that I've just seen enough to recognize, and to identify." Once again, he seemed to be proceeding rather intuitively. An alternative would have been to provide a written proof, as S15 did: i.e.,

> If m is odd, then $m = 2k + 1$ for some $k \in \mathbb{Z}$ and $m^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd.
> If m is even, then $m = 2k$ for some $k \in \mathbb{Z}$ and $m^2 = 4k^2 = 2(2k^2)$, which is even.
> Hence, if m^2 is even, then m is also even. ■

S15 also gave a nice summary of the key idea behind both his and S12's explanations: "We cannot have an odd integer, and when we [-] put it to the square, it cannot give an even integer." In other words, it is not possible to find an odd integer m such that m^2 is even. Thus, if m^2 is even, m has to be even.

S13's Proof by Contradiction that m^2 even $\Rightarrow m$ even.	S14's Proof by Contradiction that m^2 even $\Rightarrow m$ even.
If $m^2 = 2k$ but $m = 2k' + 1$, then $m^2 = 4k'^2 + 4k' + 1 = 2k + 1$. So the first assumption was wrong. ■	Say (m^2 even) $\wedge$ (m odd). This is $\neg(\alpha \Rightarrow \beta) = \alpha \wedge \neg\beta$. Then $2k = (2l + 1)^2 = 4l^2 + 4l + 1 = 2(2l^2 + 2l) + 1$ They're not of the same form. It's a contradiction. ■

Table 6.4 Examples of participants building theory underlying their practice for Task 1.

Only two students (S13 and S14) acted closely to our expectations for a successful RA I student and produced proofs by contradiction, as shown in Table 6.4 above. Although both students' proofs are missing some steps, we get the sense that they understood the proof technique they were applying. Note that S14's reaction to the question of how he knew "m^2 even $\Rightarrow m$ even" was very different from S13's. While S13 reacted by providing the proof in Table 6.4, S14 reacted, like some of the students mentioned previously, by referencing a past experience: "That was from a computer science course I did. It was uh… [Sigh.] Ah, you told me not to prepare for this." The difference with S14, when compared to the previously mentioned students (like S8, S10, and S11), was that he felt compelled to figure out the proof, and eventually succeeded in doing so. He was the only student to introduce any hint of the language of formal mathematical logic at this point in the interview. He also deliberately chose to show an "if and only if"; i.e., in addition to the proof shown in Table 6.4, he gave the short proof that "m even $\Rightarrow m^2$ even."

The one student we have not mentioned up until this point is S3, who stood out from the rest in his way of justifying the property he used: i.e., $(2 \text{ prime} \wedge 2|m^2) \Rightarrow 2|m$. He explained that if we have a prime number p and $p|a \times b \times c$, then

> p must divide either a or b or c. [-] If a prime number divides the product, then it must divide one of the numbers individually. Because otherwise that would contradict the Fundamental Theorem of, of Numbers, which is that each of them have a unique whatever, combination of primes. Like every uh composite number is the unique, has, is the product of unique primes.

What distinguishes S3 from the rest of the students we interviewed is that he demonstrated an awareness of the "Fundamental Theorem of Arithmetic," which he called the "Fundamental Theorem of Numbers" (FTN). He struggled to put together a precise formulation of the theorem: i.e., that any integer greater than 1 is either a prime or can be expressed as a product of prime numbers in a unique way, up to the order of the factors. Nonetheless, he clearly expressed his argument: If p is prime and $p|m^2$ or $p|m \times m$, then we must have $p|m$ because, if this were not the case, we would contradict the FTN. It remains unclear what S3 had in mind when he claimed that there would be a "contradiction" with the FTN. We suspect that his conclusion was based more on an intuition that the result follows from the FTN, rather than a precise formal proof by contradiction (i.e., assuming $p|m^2 \wedge p \nmid m$, and reaching a contradiction based on the FTN). This said, given S3's enthusiasm throughout solving Task 1, it is possible that he would have tried to construct such a proof, had the interviewer prompted him to do so.

6.1.3 Reflections on the Nature of Practices, Positioning, and Activities[25]

Non-Mathematical: Identify task with technique in an automatic fashion. VS. *Non-Mathematical*: Reformulate the task, without performing the assessment: "Will this work? Is this better?" VS. *Mathematical*: Study the task, think of equivalent formulations or different possible techniques, and perform the assessment: "Will this work? Is this better?"

It seemed many participants had learned non-mathematical practices in the sense of automatically identifying the given task with a technique. The participants saw the square root of some number and the words "rational or irrational" and were triggered to write "let $\sqrt{8} = m/n$." Such behaviour seems natural for the Student, who may rely on the strategy of automatically identifying tasks with techniques for doing well on exams. This said, we have also observed this behaviour in participants who seem to have occupied other positions (e.g., the Enthusiast or the Mathematician in Training). We have found that different positioning becomes more evident when analysing participants' abilities (rather than instincts) to go in anything but the automatic direction.

Most participants who exhibited the automatic behaviour described above studied Task 1 more closely when they were convincing themselves that $\sqrt{8} \notin \mathbb{Q}$. For instance, they were convinced that $\sqrt{8} \notin \mathbb{Q}$ because $\sqrt{8} = 2\sqrt{2}$ (e.g., S1, S5, S8, and S15), 8 is a non-square number (e.g., S6, S12, and S14), or $\sqrt{8} = \left(\sqrt{2}\right)^3$ (S4). These different ways of thinking about the irrationality of $\sqrt{8}$ suggests a potential for transitioning towards mathematical practices in the sense depicted in the above box; a potential that is supported by the nature of Task 1, since it can be solved in several different ways. This said, most participants did not leverage their ways of convincing themselves to produce techniques for proving that $\sqrt{8} \notin \mathbb{Q}$. This may be linked to the participants' interpretation of the didactic contract in RA I. For instance, seeing a particular

[25] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.1.1 and 6.1.2 that begin to address our research objectives: characterizing the nature (mathematical or not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

technique emphasized in professors' solutions may have led students to think that they were always expected to use that particular technique. Many participants, however, seemed to be held back in other ways. Some participants lacked a fluency in characterizing irrational numbers: for example, they did not know for sure if a rational multiplied by an irrational is irrational (e.g., S5 and S15), or if any irrational to a power is irrational (S4), or if the square root of every non-square number is irrational (e.g., S6). Other participants had more fluency in such characterizations but did not know how to prove them (e.g., S1). It seemed these participants did not have the tools to do anything but go in the automatic direction. We think this could be linked to a Student positioning towards activities that encouraged the practicing of particular techniques (in opposition, we might imagine posing tasks that can be solved in more than one way and inviting students to try to come up with multiple solutions). In contrast, there were two students (S10 and S14) who were able to shift out of their automatic choice of technique, notice that $\sqrt{8} = 2\sqrt{2}$, and exhibit what they called an "easier" and "better" technique based on the fact that $\sqrt{2} \notin \mathbb{Q}$. We think this could be linked to S10 and S14 having adopted positions other than Student in RA I (e.g., the position of Enthusiast or the position of Mathematician in Training, respectively).

There were two students (S2 and S9) who exhibited a non-mathematical practice in the sense that they reformulated the given task based on noticing that $\sqrt{8} = 2\sqrt{2}$, but they did not perform an assessment of whether the reformulation would work. More specifically, they decided to show that $2\sqrt{2} \notin \mathbb{Q}$ by showing $\sqrt{2} \notin \mathbb{Q}$; but they could not explain why showing $\sqrt{2} \notin \mathbb{Q}$ means that $2\sqrt{2} = \sqrt{8} \notin \mathbb{Q}$. Like some of the participants mentioned above, S2 and S9 seemed to lack fluency in characterizing irrational numbers (e.g., they were not sure if a rational multiplied by an irrational is irrational). When confronted with their inability to explain their technique, S2 and S9 showed signs of having occupied different positions in RA I. S2 revealed her positioning as a Student when she relied on the interviewer to tell her if her technique was right or wrong, and then decided that she could just use the technique of letting $\sqrt{8} = m/n$ instead (presumably, the latter technique was validated through established experiences from the RA I course). S9, in contrast, showed signs of being a Skeptic when he said he just *knew* that $\sqrt{8} = 2\sqrt{2} \notin \mathbb{Q}$ and started to complain about not liking the RA I course. It seemed S9 did not see the point of learning a technique for proving "obvious" results; hence, his Skeptic positioning was not challenged by tasks like Task 1.

The three participants (S3, S7, and S13) who decided that $\sqrt{8} \notin \mathbb{Q} \Leftrightarrow 2\sqrt{2} \notin \mathbb{Q} \Leftrightarrow \sqrt{2} \notin \mathbb{Q}$ exhibited a mathematical practice in that they not only took the time to study the given task and to think of equivalent formulations, but they were also able to perform an assessment of these new formulations. All three students knew that a rational multiplied by an irrational is irrational and thought they would be able to construct a proof. We think this can be linked to their positioning as a Learner, Student-Learner, and Mathematician in Training, respectively.

Non-mathematical: Construct a proof by recalling each step of a proof one has seen or done before. Vs. *Mathematical:* Implement a general proof technique.

Many participants seemed to exhibit a non-mathematical practice in the sense that they constructed their proof that $\sqrt{8} \notin \mathbb{Q}$ based on recalling each step of a solution they had seen or done before. Some differences in the participants' positioning could be observed at the moments where they struggled to make progress based on memory alone.

A collection of participants showed strong signs of occupying the Student position. S11, for instance, got stuck because he could not apply one step of the solution he was recalling: i.e., $8|m^2 \Rightarrow 8|m$ because 8 is prime. This revealed that he had learned a very particular and rigid technique for proving that $\sqrt{p} \notin \mathbb{Q}$ for p prime. Other participants got stuck because they could not remember the steps of the proof that $\sqrt{2} \notin \mathbb{Q}$ (e.g., S1, S4, S5, S6, S9, and S15), a situation that could be fixed by looking back at their class notes. It seemed these participants were limited to relying on their memory because they lacked some fluency in: (a) the general proof technique for showing that $\sqrt{c} \notin \mathbb{Q}$ for $c \neq a^2\ \forall a \in \mathbb{N}$; and/or (b) strategies for constructing proofs. For instance, some participants did not seem to really understand why proving $\sqrt{8} \notin \mathbb{Q}$ depended on showing that if $\sqrt{8} = m/n$, then m and n are both even. Participants also seemed to lack the habit of trying to deduce a next possible step based on a combination of local and global mathematical reasoning: e.g., after determining that m is even, it seemed participants were trying to remember the next step, rather than think about what m being even means, and how that meaning might be used to make progress in reaching the overall goal.

Four participants (S7, S8, S10, and S12) showed signs of being able to shift out of the Student position, or of having occupied other positions in the RA I course, when they persevered in their adaptation of a recalled solution and successfully showed that $\sqrt{8} \notin \mathbb{Q}$. Recall that S7 seemed to be remembering (and adapting) solutions for numbers like $\sqrt{2} + \sqrt{3}$ and S8, S10, and S12 seemed to be remembering (and adapting) solutions for numbers like $\sqrt{2}$. We think that S8's hesitancy in trying out steps that differed from the $\sqrt{2}$ proof showed that the adaptation could be a real challenge for those who are used to succeeding through a more direct copy of solution steps. This is, nevertheless, a productive challenge in that it can show students the insufficiency of memorizing steps; and this, in turn, might encourage a shift from studenting to learning, and to practices that are more mathematical in nature. Such shifts are supported by the nature of Task 1, which necessarily requires an adaptation of the solutions given in RA I. We predict that some of the participants mentioned in the previous paragraph may have experienced this productive challenge in the interview had they been given the proof that $\sqrt{2} \notin \mathbb{Q}$.

There were some participants (e.g., S3, S13, and S14) who seemed to have learned practices that were more mathematical in nature in that they showed an ability to implement more general proof techniques. S13 seemed to distinguish herself as a Mathematician in Training, characterized by a love of learning how to build theory and a great facility in constructing proofs, when she easily and clearly implemented the by-contradiction technique for proving three different statements:

- $\sqrt{2} \notin \mathbb{Q}$;
- $q \cdot r \notin \mathbb{Q}$ if $q \in \mathbb{Q}\backslash\{0\}$ and $r \notin \mathbb{Q}$; and
- if $m \in \mathbb{Z}$ and m^2 is even, then m is even.

We note that only one other participant (S14) provided clear formal by-contradiction proofs of these three statements, with $q = 2$ and $r = \sqrt{2}$; two participants (S10, S12) showed some facility in proving two out of the three types of statements represented above; and most other participants seemed to present difficulty in constructing well-understood proofs for just one statement. The exception was S3, whose actions and attitudes in relation to Task 1 seemed to distinguish him as a Learner.

We think S3's positioning as a Learner was evident in the way he seemed to thoroughly understand and enjoy playing with by-contradiction proofs about the particular topic in question: i.e., the irrationality of numbers. Recall that after successfully arguing that $\sqrt{8} = 2\sqrt{2} \notin \mathbb{Q}$ based on proofs of (1) and (2) above, S3 told the interviewer that he thought he could prove that $\sqrt{8} \notin \mathbb{Q}$ without relying on the fact that $\sqrt{2} \notin \mathbb{Q}$. His initial written work resembled that of a Student: i.e.,

$$\sqrt{8} = \frac{m}{n} \Leftrightarrow 8n^2 = m^2 \Rightarrow 2|m^2 \Rightarrow 2|m \Rightarrow m = 2k \Rightarrow 8n^2 = 4k^2 \Rightarrow 2n^2 = k^2$$

S3 then distinguished himself as a Learner when he showed his ability to think on the spot and complete the proof in an unexpected way: i.e., by noticing that the last implication means that $\sqrt{2} \in \mathbb{Q}$ (but we know that $\sqrt{2} \notin \mathbb{Q}$). S3 also distinguished himself as a Learner when he did not blame a lack of memory for his inability to show that $\sqrt{8} \notin \mathbb{Q}$ without using the fact that $\sqrt{2} \notin \mathbb{Q}$; he simply noted that there could be a way to do it, and he did not know it *yet*. Such openness to thinking on the spot and not knowing *yet* may be discouraged among those who take a Student position towards activities that can be solved by copying professors' solutions. For such students, it may seem like the tasks they are given can (or should) always be solved in this manner.

6.2 Task 2

Show that the function $f(x) = e^x - 100(x-1)(2-x)$ *has* 2 *zeros.*

6.2.1 Practices to be Learned[26]

We expected a successful RA I student to attempt to construct the top half of the solution in Figure 6.2 below when presented with Task 2: that is, they would identify the task as requiring a search for two locations where f changes sign and an explicit reference to the Intermediate Value Theorem (IVT). Notice that the solution contains no indication of how the sign changes were found. This was also the case in all the solutions we analyzed. The difference between Task 2 and the type of task we saw in RA I is that in the latter, the task statement almost always included a hint of where the zeros could be found. More specifically, students were almost always told to show the function has zeros in a specified interval of the form $[a, b]$, where a and b are integers;

[26] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 2. This model is synthesized in table form in Appendix C.

furthermore, the sign changes would result from calculating $f(a)$, $f(b)$, and, if needed, $f(c)$, where c is either the midpoint of (a, b) or another integer therein. Task 2, in comparison, does not specify an interval where the zeros can be found; moreover, the sign changes cannot be found by simply calculating $f(x)$ for integer values of x. We therefore expected a successful RA I student might struggle to find the sign changes in Task 2. Since it is not required in the solution, it is also unclear if a successful RA I student would be able to provide mathematical justifications for their use of the IVT (e.g., "because the zeros of f cannot be found analytically") or explain, in a mathematical discourse, how the IVT works (e.g., "if f is a continuous function on an interval $[a, b]$ and $D \in (f(a), f(b))$, then there is some number $c \in (a, b)$ such that $f(c) = D$"). A successful RA I student might include in their solution a note that f is continuous and possibly provide some sort of reason for this (e.g., "f is the sum of an exponential function and a polynomial function, both of which are known to be continuous"); the solutions made available in RA I did not consistently include this note, which verifies the main condition of the IVT.

$f(1) = e > 0$
$f(1.5) = e^{1.5} - 25 < 0$
$f(2) = e^2 > 0$
By the Intermediate Value Theorem, f has at least 2 zeros – one in $(1, 1.5)$ and one in $(1.5, 2)$.

$f'(x) = e^x - 100(1)(2 - x) - 100(-1)(x - 1) = e^x - 100(3 - 2x)$
$f''(x) = e^x + 200 > 0$ So f'' has no zeros.
Assume f has 3 zeros. Then by Rolle's Theorem, f' has 2 zeros and f'' has 1 zero. Contradiction.
Thus, f has at most 2 zeros. ■

Figure 6.2 An example of the kind of solution anticipated for Task 2.

When asked how their solution would change if Task 2 had been rephrased to "show that f has *exactly* two zeros," a successful RA I student might bring several possible techniques to mind. The one we anticipated them to choose is represented by the steps shown in the remainder of the solution in Figure 6.2. Calculating the first and second derivatives of f and finding that f'' has no zeros would justify the choice to use Rolle's Theorem (RT) and the related by-contradiction proof technique. The solutions we analyzed did not consistently have this calculation and study of derivatives before the assumption (e.g., that f has three zeros) was made. Once again, we would expect a successful RA I student to make an explicit reference to the theorem used (i.e., RT). They would also be required to clearly indicate the steps of their proof, including the assumption, contradiction, and conclusion. We do not know the degree to which a successful RA I student would be able to provide a more detailed mathematical explanation of how and why RT works. As depicted in Figure 6.2, this was not a necessary part of completing a solution to tasks like Task 2.

<table>
<tr><td colspan="2" align="center">T_2: Prove that a function $f(x)$ has exactly n zeros on a domain D.
Typically: $n \in \mathbb{N}$ is small (e.g., $1, 2, 3$, or 4) and D is a specified interval $[a, b]$ with $a, b \in \mathbb{Z}$.
$T_2 = T_{2_a} \wedge T_{2_b}$</td></tr>
<tr><td>T_{2_a}: Prove that $f(x)$ has at least n zeros on D.</td><td>T_{2_b}: Prove that f has at most n zeros on D.</td></tr>
<tr><td>τ_{2_a}: Find n sign changes of $f(x)$ in D.
Typically: calculate $f(a)$ and $f(b)$, and maybe $f(c)$ for c equal to integers in (a, b) or midpoints between integers.</td><td>$\tau_{2_{b_2}}$: Assume that f has $n + 1$ zeros and derive a contradiction. More specifically, argue that f' has n zeros, f'' has $n - 1$ zeros, …, and f^n has 1 zero; and show f^n has no zeros.</td></tr>
<tr><td>θ_{2_a}: “By the Intermediate Value Theorem.”</td><td>$\theta_{2_{b_2}}$: “By Rolle’s Theorem and by contradiction.”</td></tr>
</table>

Table 6.5 The practice to be learned in RA I most relevant to Task 2.

We could model the practice studied in RA I and most relevant to Task 2 as shown in Table 6.5. This said, there was much more that the successful RA I student might have learned (or at least been exposed to) while following the path of activities related to proving that a function f has exactly n zeros on a domain D (a more comprehensive model is given in Appendix C). As specified in our description of T_2, the activities we analyzed all had n equal to $1, 2, 3$ or 4. In the cases when $n = 2$, $\tau_{2_{b_2}}$ was the technique illustrated in solutions for proving that f has at most n zeros; but when $n = 1, 3$, or 4, we identified other techniques in the proposed solutions. In the two activities where the solutions involved showing that a given f has 3 or 4 zeros, f was a polynomial and students were exposed to the technique of considering the degree (or order) of f, along with explanatory discourse such as: “If f is a polynomial of order 4, then f has at most 4 zeros” (we model this as $[\tau_{2_{b_3}}, \theta_{2_{b_3}}]$ in Appendix C). In the two activities where students were asked to show that a specified function f has at most one zero on a domain D, they were exposed to the technique of showing that f' is strictly positive (or strictly negative) on D and a corresponding justification of the sort: “If $f' > 0$ (or < 0) on D, then f is strictly increasing (or strictly decreasing) on D and can cross the line $y = 0$ at at most one point” (we model this as $[\tau_{2_{b_1}}, \theta_{2_{b_1}}]$ in Appendix C). In sum, the activities we analyzed offered some alternatives to $\tau_{2_{b_2}}$. We expected a successful RA I student to be aware of these variations and to choose $\tau_{2_{b_2}}$ because it was the illustrated technique when $n = 2$, like in Task 2. We nevertheless wondered if the diversity of techniques would have posed problems for the participants of our study; in particular, if they would have struggled to routinize them all and, at the time of the interview, be able to choose one that is appropriate for the given task.

In contrast, we anticipated that $[T_{2_a}, \tau_{2_a}, \theta_{2_a}]$ would be much more routinized among participants due to the large number of activities in which it was present. There were some interesting variations in T_{2_a}, which may have expanded students' experiences. In a couple of assignment activities, the task was presented in a motivating form (e.g., a tourist walking back and fourth along the same path), which required students to engage in some mathematization before being able to solve the task. In addition to this, there were several activities (on assignments and past exams) where the functions were more general in nature, thereby exposing students to the idea of using certain abstract properties to solve the task (rather than concrete function values): e.g.,

> Let f and g be continuous functions on $[a, b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in (a, b)$. (You may use the Intermediate Value Theorem.)[27]

There were also three activities that could be interpreted as suggesting an alternative to τ_{2_a}. Consider, for example, the following activity, whose solution used Rolle's Theorem (rather than the IVT) to show that a function has at least a certain number of zeros in a specific interval (the other two activities were similar):

> Let $f(x) = 100(x - 1)(x - 3)(x - 5)(x - 7)(x - 100)$. Prove that all the zeros of the second derivative $f''(x)$ are in the interval $(1,100)$.

Of course, there is a significant mathematical difference between Task 2 and the above activity: i.e., the latter gives a nice analytic expression for f, which is used to make a conclusion about the zeros of f'' (not f – the zeros of f are immediately known). We wondered if the varied use of Rolle's Theorem could add to a students' difficulty in routinizing the practices depicted in Table

[27] In solving this activity, the student would be expected to introduce a new function $h = f - g$ and prove that h has at least one zero in (a, b). Hence why the activity can be seen as representative of T_{2_a}. We discuss equivalent task types in more detail later in this subsection.

6.5 (both τ_{2_a} and τ_{2_b}). Nonetheless, the most commonly illustrated and tested technique for showing that a function has *at least* n zeros was to locate n sign changes in the function and cite the IVT. For this reason, we expected a successful RA I student to call upon this technique when presented with Task 2.

As mentioned above, we purposefully constructed Task 2 so that, even if the choice of τ_{2_a} was easy, its implementation might be challenging for a successful RA I student. One key difference between Task 2 and the activities we analyzed from RA I is the domain, D, on which the given function is indicated to have a certain number of zeros. In Task 2, a domain is not specified; but we expected participants to assume $D = \mathbb{R}$.[28] In the activities we analyzed, D was almost always an interval (as noted in Table 6.5). Compare, for example, the following assignment activity with the statement of Task 2:

> Assignment activity: Prove that the function $f(x) = e^x - 100x(1-x)$ has exactly two zeros in the interval $[0,1]$.
>
> Task 2: Show that the function $f(x) = e^x - 100(x-1)(2-x)$ has 2 zeros.

As exemplified by the assignment activity, it is possible that the successful RA I student had learned a specified version of τ_{2_a}, whereby sign changes are found by plugging in the endpoints of the given interval (typically integers) and some distinct points in between (typically also integers and/or the midpoint of the interval). We were interested in seeing if this characteristic of the activities offered to students had led participants to develop specified practices; and, if so, we wanted to know if the participants would be able to develop other, more mathematical, strategies for locating sign changes in f. This is why we designed Task 2 so that it does not indicate an interval on which the zeros can be found. We also purposefully constructed the function, $f(x) = e^x - 100(x-1)(2-x)$, so that plugging in integer values of x would always lead to positive values of $f(x)$ and therefore would not be enough to locate sign changes. A much more efficient strategy for finding sign changes would be to consider Task 2 in a different, albeit equivalent form.

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[28] Task 2 could have been used to challenge this assumption and inquire, for example, into students' understanding of continuity or the notion of the domain of a function. These are, after all, important components of the mathematical theory underlying the practice in Table 6.5. Our study did not address this.

Within the path of activities we identified as related to Task 2, students were actually introduced to a collection of four equivalent task types, including T_2: i.e.,

a) Prove that a function f has (exactly) n zeros on a domain D;
b) Prove that a function g has (exactly) n fixed points on a domain D;
c) Prove that the equation $g(x) = h(x)$ has (exactly) n solutions on a domain D; and
d) Prove that two functions, g and h, intersect at (exactly) n points on a domain D.

The equivalence of these task types follows directly from the definitions of zeros of functions, fixed points, solutions to equations, and intersection points. In RA I, students were given some activities similar to the following (an example of task type (c)):

Prove that the equation $e^x = 100(x-1)(2-x)$ has exactly 2 solutions on ℝ.

In the solutions we analyzed, students were instructed to introduce a function $f(x) = e^x - 100(x-1)(2-x)$ and to argue that f has exactly 2 zeros by implementing the practice in Table 6.5. We did not observe any activities where the equivalence was leveraged in the other direction. For instance, to prove that f has exactly 2 zeros, it can be productive to think of the equivalent task concerning the solutions of $e^x = 100(x-1)(2-x)$, or the task of proving that $g(x) = e^x$ and $h(x) = 100(x-1)(2-x)$ intersect exactly twice (an example of task type (d)). Indeed, this last task triggers the use of basic knowledge of graphs, which could lead to a simple solution like the one depicted in Figure 6.3. While we would not expect a successful RA I student to spontaneously construct such a solution, we think a mathematician would argue in this way. In the least, they would be able to call upon this kind of knowledge of graphs to efficiently try to locate the x values (e.g., $1, 1.5,$ and 2) that would demonstrate the sought-after sign changes in f.

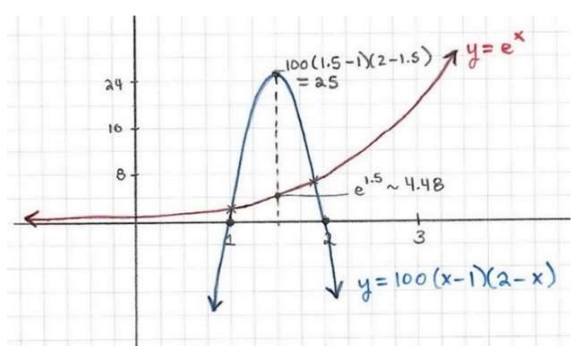

Figure 6.3 An unanticipated graphical solution to Task 2.

6.2.2 Practices Actually Learned[29]

As discussed in the previous section, 6.2.1, we expected a successful RA I student to identify Task 2 – show that the function $f(x) = e^x - 100(x-1)(2-x)$ has two zeros – with two subtasks: t_{2_a}, prove that the given function has *at least* two zeros (by showing that it has two sign changes); and, subsequently, t_{2_b}, prove that the given function has *at most* two zeros (by assuming that it has three and reaching a contradiction).

In 6.2.2.1, we present how the participants of our study came to identify Task 2 with one or both of these subtasks. As expected, most participants were triggered to cite the Intermediate Value Theorem and to search for sign changes in the given function (i.e., to implement $[\tau_{2_a}, \theta_{2_a}]$). In 6.2.2.2, we outline the different ways – unproductive and productive – in which the participants conducted their search for sign changes. When it came to solving t_{2_b}, choosing a technique seemed to be much less automatic for participants. Surprisingly, none of them solved t_{2_b} in the expected manner (i.e., using $\tau_{2_{b_2}}$). In 6.2.2.3, we outline the different techniques they seemed to be considering, and the theoretical discourses they had (or did not have) for justifying said techniques.

6.2.2.1 Identifying the Task and Choosing a Technique

For most of the students we interviewed (all but S2, S3, and S9), the identification of Task 2 with the need to use a special "theorem," "property," or "method" seemed almost immediate. For instance, the first words spoken by S5 upon seeing the task were:

> Ok, for this one here… […] […] I know I would use a method. I don't remember which one. [-] I'm not sure if I would use the Bisection Method, or the like Fixed Point Method. Oh no, Intermediate Value Theorem I think.

We identified ten other students (S1, S4, S6, S7, S8, S10, S11, S12, S13, and S14) as speaking about the use of the Intermediate Value Theorem (IVT) within the initial moments after receiving Task 2. Some recalled more than the IVT. S6, for instance, said: "This one would be IVT, Intermediate Value Theorem. I think. [-] I think Rolle's Theorem as well." S14 explained: "Ok

[29] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 2. These models are synthesized in table form in Appendix C.

right off the bat I would use the derivative and the Intermediate Value Theorem." Other students did not recall the name of the IVT exactly or seemed to refer to it as the Mean Value Theorem; but it was clear that they were intending on choosing an IVT-inspired technique based on their subsequent actions and discourse (i.e., they looked for sign changes in f). The only student not already mentioned in this paragraph is S15, for whom Task 2 also triggered the need to "remember the theorems." The difference with S15 is that he was unsure which theorems to use and took time to try to recall complete statements of each. He was unable to decide which would be helpful for Task 2 in the context of the interview.

These students seemed to be acting based on a routine they had developed for solving similar looking tasks. For instance, when asked how he knew that the IVT was applicable in the given situation, S1 explained: "I know that it's applicable in this situation. [-] Why do I know? Well, I'm cheating. Cause I know that that's how we used to solve it. [-] Cause we did it in class." S4, S5, and S6 also referred to their previous experiences as students to justify their choice of using the IVT. S5 said: "Honestly, that's the only way I could think of." S4 noted: "It's just having repeated it so often, whether it be assignments, class, practice, …" In addition to saying that she had "done that like pretty often," S6 specifically identified the part of the task that triggered the use of the IVT for her:

> "Show that it has zero" is IVT for sure. Like the fact that, ok, so like if it's a continuous function, and, you know, so you plug in some values, so you get a negative, then positive, then negative. Ok so, it must cross the line, like the x-axis at some point. So it does have a zero.

Many students were able to provide explanations of the IVT comparable to S6's; that is, they exhibited a certain procedural understanding of how the IVT works, including, in many cases (but not all), a note about the continuity condition. More specifically, six students (S1, S3, S4, S5, S7, and S11) did not mention continuity throughout their entire solving of Task 2; all others mentioned continuity in relation to their use of the IVT. This said, some of the students' discourse made it seem as though knowing how the IVT worked was sufficient reason to use it. For example, when asked "What about this problem made you think IVT?", S8 responded: "I understand that the Intermediate Value Theorem works like that. [-] If I find one that's positive and one that's negative, I knew that I could find a zero… [-] That's why I felt like using that."

Only one student provided some additional explanation for his choice to use the IVT *before* starting to use it: S9 explained that

> we could find when does zero equal [$f(x)$]. And it would be hard because we have the e^x. [-] It's not as simple as just isolating the x. [-] So in this case, we have to use one of those theorems we saw in Analysis.

In saying this, S9 seemed to be imagining setting $f(x) = 0$ and solving for x, which he decided would be "hard." Some other students made it clear that they had similar feelings. S1, for example, paused his search for sign changes momentarily to look back at the given expression for f. Like S9, he identified e^x as the problem, noting that if it "wasn't here, it'd be pretty easy to find the two zeros. But since there's this now, we have to do the… the non-high school way." S3 noted similarly that f is

> a classic example where you cannot use a lot of easy things to find the root. [-] I can't really factor it. [-] This is a regular quadratic function, here I have the exponential, and I don't really know… Maybe there is a way to find it nicely. But I can't think of one right now.

Although these students would have been open to learning a technique for solving $f(x) = 0$, they did not know of one at the time of the interview; hence, they needed to use the IVT. It seemed other students had not developed this kind of explanatory discourse: i.e., the kind where choosing the IVT is justified as a necessity, due to the nature of the given function. As will be discussed later in this section, S3 distinguished himself from the other students we interviewed when his thinking about $f(x) = 0$ led him to productively reinterpret the task.

Given that most students identified Task 2 with using the IVT, it is not surprising that most of them chose to solve t_{2_a}: Show that $f(x) = e^x - 100(x-1)(2-x)$ has *at least* two zeros. Only two students (S9 and S10) spontaneously restricted their solution to solving this task. S10 explicitly noted that "the question here is not that it has exactly two zeros. It's show that it has two zeros." This is actually what we expected from a successful RA I student; in fact, we had planned a follow-up question about how their technique might change if the task had asked them to show that f has *exactly* two zeros. To our surprise, nine students (S1, S2, S3, S4, S6, S7, S11, S12, and S13) spontaneously identified the task in this way, without an intervention from the interviewer. For instance, in complete opposition to S10, S4 interpreted the wording of the task as "saying that it has two zeros, not at least." Most of the students who made progress in showing that f has *exactly* two zeros chose to solve t_{2_a} and t_{2_b}: Show that f has *at most* two zeros, i.e., *only* the two zeros that were identified in t_{2_a}, and no more. S1, who often reflected our expectations for the successful RA I student, clearly outlined his plan right from the beginning: "I need to show that there's exactly two [zeros]. I'll show that [-] there's at least two and at most two and then we'll have two."

The four students who were not mentioned in the previous paragraph (S5, S8, S14, and S15) sought the interviewer's input as to how they should identify the given task. Shortly after being presented with Task 2, both S8 and S15 asked if the question was about showing the function has exactly two zeros. S15 said: "I need a clarification. Do we want to show that [-] it has only two zeros? Exactly two zeros?" S8 asked similarly: "Is it exactly two zeros? Is that the question?" For S5 and S14, the question came later, but was essentially the same. S14 seemed frustrated by the wording of the task: "It says 'has two zeros.' To me, that's worded a little weird in the sense that I don't know if it has at least two zeros or has exactly two zeros." Other students had different ways of interpreting the wording of the task and dealing with their interpretation. S13, for example, explained that "if it asks you, you know, how many zeros it has, then it's harder. [-] But once it says that it has two zeros, then your job is easier." Like most of her peers, S13 interpreted the wording of the task as indicating that the given function has exactly two zeros. S12, in contrast, explained: "It says that the function has two zeros. Not that it has exactly two zeros. That's a different question. Which it said on our final exam." S12 nevertheless proceeded to try to solve both $t_{2_a}$and t_{2_b} because he convinced himself that f did indeed have exactly two zeros.

One possible reason that students asked which task they should solve is that they did not immediately recall a technique for showing that a function has exactly two zeros and they wanted to know how much work was required of them. After S15 asked the interviewer for clarification on the task, the following interaction ensued:

S15: "Do we want to show that [-] it has only two zeros? Exactly two zeros?"

L: "Uh, exactly two zeros? Would it change your solution?"

S15: "Uh… Let me think about it. […] I don't remember specifically. [-] Maybe. [-] I'm just trying to remember. [-]"

L: "Ok. No problem. [-] Do you think that the function only has two zeros?"

S15: "At this point, I would say uh… Well, yes."

L: "Yes?"

S15: "Because we have a polynomial of degree two."

As will be discussed in 6.2.2.3, S15, S14, and S8 eventually devised somewhat unexpected techniques for arguing that f has exactly two zeros. In comparison, as she embarked on trying to solve t_{2_b}, S5 stopped herself: "Or do we need to show that there can't be more than two zeros? Or the question is only asking to show two zeros? [-] I honestly don't know how to show that there's only two zeros."

At the beginning of this section, we noted that there were three students who did not immediately talk about the IVT upon seeing Task 2. We already discussed S9, who considered other possibilities (e.g., solving $f(x) = 0$) before deciding that he would need to use the IVT. The remaining two students (S2 and S3) significantly reinterpreted the task. S2's reflex upon receiving Task 2 was to find the derivative of f. After doing so, she engaged with the interviewer as follows:

S2: "I think there is a theorem. [-] It says that if you have *n* zeros for $f(x)$, then that means you have $n - 1$ zeros for the derivative. That's what I thought."

L: "[-] If you show that the derivative has $n - 1$ zeros, does that mean that the original function had n zeros?"

S2: "Yeah. That's what I thought. Mm… But I don't know if I'm right. I'll just try this way."

Throughout her solution, S2 seemed to be trying to solve the task:

t_2: Show that $f(x) = e^x - 100(x-1)(2-x)$ has *exactly* two zeros.

Based on the interaction shown above, it seemed S2 reinterpreted this task (incorrectly) as:

${t_2}'$: Show that $f'(x) = e^x - 100(3-2x)$ has *exactly* one zero.

It is possible that S2 did not clearly understand the implication on which she based her reinterpretation of the task; in the least, she did not take the time to carefully contemplate it. First of all, the statement "f has n zeros ⇒ f' has $n - 1$ zeros" is ambiguous: while it is true that "f has n zeros ⇒ f' has at least $n - 1$ zeros," it is not true that "f has exactly n zeros ⇒ f' has exactly $n - 1$ zeros." Secondly, the converse of a true statement does not necessarily hold: although it is true that “f has n zeros $\Rightarrow f'$ has at least $n-1$ zeros,” it is not true that “f' has $n-1$ zeros $\Rightarrow f$ has n zeros.” Such statements can be checked by considering simple examples in graphical form. It is possible that S2 did not feel the need to engage in such checking since reinterpreting the task in this way is what she recalled learning in RA I. Note that if we disregard S2’s choice to solve t_2', then her solution is comparable to the ones constructed by the participants who seemed to be using Rolle’s Theorem to solve t_{2_b} (see 6.2.2.3.2), and whose practices can be seen as specified or routinized versions of the one we had expected from a successful RA I student. Like her peers, S2 seemed to recall the importance of the derivative and Rolle’s Theorem. But she did not recall exactly how the derivative was used. Moreover, her vague understanding of Rolle’s Theorem led her to use the derivative in an erroneous manner.

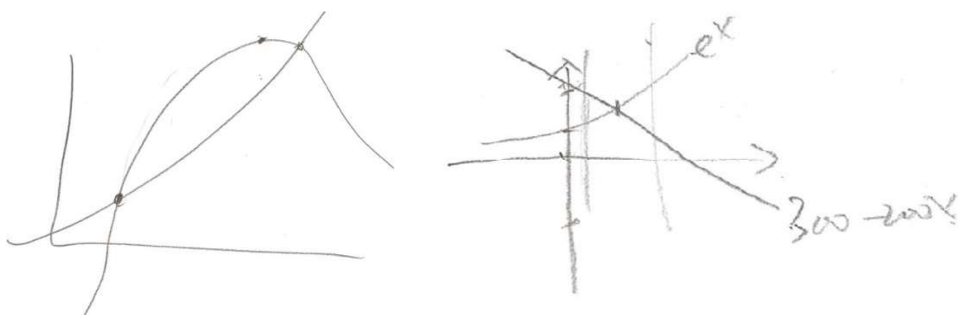

Figure 6.4 Sketches by S3 (left) and S2 (right), illustrating how they reinterpreted tasks about zeros as tasks about intersections between graphs.

S3, in contrast, seemed to be acting out of careful analytical thought, rather than a vague memory of a past solution. He read and reread the statement of Task 2 before concluding: “The question basically says that this [i.e., $e^x = 100(x-1)(2-x)$] has two solutions.” In other words, S3 almost immediately reinterpreted the given task as:

$\widetilde{t_2}$: Show that $e^x = 100(x-1)(2-x)$ has exactly two solutions.

We infer that seeing Task 2, a task about finding “zeros of functions,” made S3 think about solving $f(x) = 0$. After rearranging the equation to get a task about finding “solutions of equations,” S3 was quick to give another interpretation. He drew the sketch shown in Figure 6.4 and explained: “What I’m trying to prove is that when I graph it [-] it’s going to cross twice.” S3 had therefore identified Task 2 as a task about “intersections between graphs:”

$\widehat{t_2}$: Show that the graphs of e^x and $100(x-1)(2-x)$ intersect twice.

A few other students exhibited an awareness of these different kinds of tasks. S1 and S2, for example, seemed to reinterpret the task of showing $f'(x) = e^x - 300 + 200x$ has exactly one zero as a task about showing that e^x and $300 - 200x$ intersect exactly once (see S2's sketch in Figure 6.4). S7 also seemed to be thinking about a graph like the one S3 drew when he was eventually trying to get unstuck in finding sign changes for f (more on this in the next section: 6.2.2.2). This said, S3 was the only student to dedicate time at the beginning of his solution to reinterpret the task in this productive way.

6.2.2.2 Implementing a Technique: Showing that a function f has at least n zeros by finding n sign changes.

In every interview we conducted, the student ended up plugging values of x into $f(x)$ or $f'(x)$ during their solution to Task 2. For most students (eleven out of fifteen), the goal was to locate two sign changes in f, so to show that the function has at least two zeros. In other words, the students were implementing the IVT-based technique they had chosen. The remaining four students were not solving the exact same task: S3 was looking for two places where e^x and $100(x-1)(2-x)$ cross, S4 seemed to be trying to find values of x for which $f(x) = 0$, and both S2 and S15 were looking for sign changes in f'. This said, we have decided to assume that the methods these students exhibited for choosing x values to plug in are somewhat representative of the ways in which they would have spontaneously approached finding sign changes for f. Under this assumption, we describe below the different methods for finding sign changes that we identified and how they could contribute to the students' success in solving the task, t_{2_a}: Show that $f(x) = e^x - 100(x-1)(2-x)$ has at least two zeros.

Four students (S1, S3, S7, and S13) started their search for sign changes by considering the "general" behaviour of f. S1, S7, and S13 all attempted to calculate the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, while S3 simply thought about what would happen for large positive and negative x values (like negative and position a billion). Although such an approach could be a fruitful way of using the tools of RA I, it does not lead to information about the zeros of the given function. More specifically, since $\lim_{x\to\pm\infty} f(x) = \infty$ for $f(x) = e^x - 100(x-1)(2-x)$, no sign changes can be located using this approach.

Almost every student we interviewed seemed to, at some point, choose x values at random (the exceptions were S3 and S12, who made careful decisions about the x values they chose; we discuss their behaviour later in this section). After calculating the two limits of f at infinity, S1 explicitly marked his change of approach: "I'd like to see if there's a negative. [-] So I'd just try random numbers." He used his calculator to calculate $f(0), f(100)$, and $f(-5)$, finding positive values each time. "So I'll… I'll keep trying," he said. Recall that Task 2 was designed purposefully so that plugging in random integers would lead students to find positive values (see Section 6.2.1, page 154, for more details). We were interested in seeing how students would adapt to this. Would they have or be able to find other, more productive, ways of locating the required x values? S1 justified why he was "just going randomly" at the time of the interview: "There's probably a better way. I know there's a better way. But I forgot the better way. So right now I'm just trying out numbers." Not surprisingly, "trying out numbers" (or, more specifically, "trying out integers") did not lead S1 to solve the task. In fact, any student who made progress in solving the task (i.e., locating an x-value for which $f(x)$ is negative) exhibited at least one other approach (other than at random) for locating sign changes.

In addition to S1, there were others who did not seem to implement an approach beyond randomly selecting integers. S14 echoed S1 when he was asked how he was choosing values for x: "Not really that methodically right now. I haven't done something like this in a little while, so… I don't know." S11 seemed particularly held back by the expectation that plugging in integer values would be sufficient for solving a task like t_{2_a}. He explained, for instance, that if the task had been rephrased to include the interval [0,5] as the location of the two zeros, then he would just plug in "zero, one, two, three, four, five. In this function. And you'll see which one alternates between negative and positive. And you'll figure out how many zeros you have." It seemed that S11 had developed a very particular practice, whereby the number of zeros for a given function f on a given interval $[a, b], a, b \in \mathbb{Z}$, can be found by counting the number of sign changes in the list $f(a), f(a+1), f(a+2), \dots, f(b-2), f(b-1), f(b)$. In discussion with the interviewer, S11 was nevertheless able to imagine some other possible ways of looking for sign changes.

When the interviewer asked "Would there be a different way of studying the sign of this function?", S11 responded:

> Well, if you take the derivative, [-] at different points, then you can see if the slope of the tangent line is negative. Then you know it's decreasing at that point. And then you might want to check the intervals around that.

Recall that most students were finding only positive values of f. Hence, looking for where the function is decreasing could help. As S11 started to move forward with this idea, he came up with another: "Could you like study the behaviour of this [e^x] and then the behaviour of this [$-100(x-1)(2-x)$] on their own?" S11 eventually recalled that

> exponentials grow a lot faster than polynomials do. [-] So e^x is going to be going huge. And then you have your polynomial that's not going to be going as big. [-] So that if it's hitting zero, it's going to be in the early phases.

In saying this, S11 seemed to realize that for large values of x, e^x would dominate and make the function, $f(x) = e^x - 100(x-1)(2-x)$ positive. Hence, he had the feeling that $f(x)$ could only be negative for small values of x. For a general function of the form $e^x + h(x)$, where h is a polynomial, this is not true: it is possible to construct $h(x)$ so that the function becomes negative at any fixed value of x that we choose. For the function in Task 2, however, it turns out that S11's idea can lead to a solution. It was not S11 who demonstrated this to us. Immediately after reasoning as shown above, S11 deemed himself stuck: "It just comes down to I don't know where you check where exactly it is zero."

In complete opposition to S11 was S8, who did not hesitate to test out several non-integer values for x. He chose $x = 0.5, -0.5, 2.5$, and 1.5; and, despite making several calculation errors along the way, this enabled him to solve the task. In retrospect, he told the interviewer that he was picking his x values through "smart guessing," which seemed to be based on the same idea that S11 had. S8 explained:

> What was going through my head was I know that e^x grows quite quickly. Much quicker than like a polynomial will. So I figured it kind of had to be small values of x so that I can get negatives in this function in the first place.

As mentioned above, noticing that e^x grows faster than a polynomial is not reason enough to conclude that a function of the form $e^x + h(x)$, where h is a polynomial, is only negative for "small" values of x. For example, if $g(x) = e^x - 10^7(x-10)(20-x)$, then $g(x)$ is negative for $x = 15$; and we suspect that S8 would not call this value "small." To find the interval (or section of the real numbers) where this function is negative, one has to reflect on where the polynomial $10^7(x-10)(20-x)$ is positive and "beats" the exponential function so that g becomes negative.

A more systematic, though perhaps inefficient, alternative would be to pay attention to how the size of the value of $f(x)$ changes as one varies the value of x. This seemed to be sufficient for S6 to locate the zeros of the given function, $f(x) = e^x - 100(x-1)(2-x)$. Like her peers, S6 started by trying out several integer values for x. Her conclusion that she was "doing it wrong" because she was "just getting positive values" apparently came too soon, for within the next breath she announced: "Ok, no, I got it." When asked by the interviewer to reveal what she was thinking at that time, S6 explained:

> I saw that the value at [zero] was two hundred and something, then at one it got, it started, um… like it was decreasing. [-] So I was like, "Ok, there's something going on." Like it has to decrease. Cause it's a continuous function, right? [-] So it's getting smaller and smaller this way. [-] So I tried one point five, and it turns out it went negative there.

S6 had quickly used her calculator to find $f(x)$ for several values of x (e.g., $x = 0, 5, 1, 2$) before having the inkling to try $x = 1.5$. Based on her reflection in the above quote, it seems that S6 was keeping a mental record of the results produced by her calculator and eventually noticed the decrease from $f(0) = 201$ to $f(1) = e = 2.718$ .... As a result, S6 seemed to predict that as x increases from 1, f would continue decreasing towards zero, and possibly into the negatives. For the given function, the prediction is correct (of course, it is possible to construct functions where such a technique might be extremely inefficient). Perhaps if other students had more methodically kept track of the values of $f(x)$ for different values of x, they would have also been able to successfully use this approach. On the contrary, and as expected, many students chose to ignore the value of f and keep track only of its sign. S1, for instance, explicitly noted that he did not know the value of $f(10)$; "But it's positive, that's all I really care about."

Two other students (S4 and S5) chose to keep track of the value of f and seemed to use it to guide their subsequent choices of x. They nevertheless exhibited an additional way of thinking about where f might change sign: namely, by studying the signs of different parts of an analytic expression for the function. S4 decided to expand the given expression for f to get

$$f(x) = 100x^2 - 300x + 200 + e^x.$$

After his use of trial and error to choose x values led to finding only positive values for $f(x)$, S4 seemed to more carefully study the above expression. He noticed that $f(x)$ could only be negative if the term $-300x$ remained negative; hence, x needed to be positive. He then proceeded somewhat like S6, noticing that $f(10)$ is "way too big," $f(5)$ is "getting closer, but not close enough to zero," and ultimately, upon calculating $f(1) = e$ and $f(2) = e^2$, that he "went fishing way too far." Although he came very close, S4 did not succeed in finding sign changes in the context of the interview. We suspect that this is because he had already spent a significant amount of time using trial and error and got a bit lost in all the calculations that he was making.

In comparison, S5 almost immediately located a negative value for f once she took a moment to study the given expression: $f(x) = e^x - 100(x-1)(2-x)$. She too engaged in the common approach of choosing x values at random. After calculating $f(1), f(0), f(2), f(-1)$, and $f(-0.1)$, she concluded: "I'm nowhere near my goal, because I'm only getting positives." She also realized that the values of f were "going down" at some point, which led her to try $f(5)$; yet again, a positive. Finally, she looked back at the given expression for f. She justified her finding of $f(1.5) < 0$ by explaining that $x = 1.5$ is

> more than one and less than two. So basically, the negative signs [in $-100(x-1)(2-x)$] won't cancel out and it won't become a positive sign. Or else I'd always end up with getting a positive number.

It seems that S5 was thinking along the following lines: since e^x is always positive, she needed to figure out how to get the term $-100(x-1)(2-x)$ to be negative. She also seemed to realize that the negative sign in front of the 100 could be maintained if the two factors, $(x-1)$ and $(2-x)$, were both positive; i.e., if there was no other negative to cause a "cancellation." Consequently, S5 chose an x value "more than one and less than two."

Of course, there are other, non-algebraic, ways to locate where the function $h(x) = -100(x-1)(2-x)$ is negative. S3, S7, S9, and S12 relied on what they knew about parabolas. S7 and S9 were similar to the students mentioned above in that they both tried plugging in numbers at random before realizing that they needed to change their approach. S3 and S12 distinguished themselves from all other students that we interviewed in that they never plugged in

x values at random. In other words, their instinct was to base their choice of x on a careful a priori analysis of f. They took slightly different approaches to this analysis. S12, for instance, considered f as it was given – $f(x) = e^x + h(x)$ – and noted that since h is an upward facing parabola with roots at $x = 1$ and $x = 2$, its minimum (i.e., its most negative value) would occur at $x = 1.5$. Based on this, he predicted that "$f(1.5)$ is likely going to be negative. And then I'd show that [-] $f(1)$ and $f(2)$ are both positive." As mentioned above, one key difference with S3 was that he reinterpreted the task to be about intersections, between e^x and $-h(x) = -100x^2 + 300x - 200$. He used the negative sign in front of x^2 to justify that $-h$ would be negative (and hence below e^x) for large negative and large positive x values. He then located the maximum value of $-h$ (i.e., its most positive value, where it would likely be above e^x) by finding where its derivative is zero. In either case, the student efficiently identified the x-value, $x = 1.5$, that is key to finding the sign changes they sought.

x	0	1	1.5	2	3	3.5	5
Sign of f	+	+	-	+	+	-	+

Table 6.6 The sign changes found by S6.

x	-0.5	0.5	1	1.5	2.5
Sign of f	-	+	+	-	+

Table 6.7 The sign changes found by S8.

We were surprised at how challenging it was for most students to complete the task of locating sign changes in f. As outlined above, only five students eventually demonstrated the ability to study the given expression for f and use it to efficiently pinpoint the useful values of x. The remaining students relied solely on inefficient approaches, all of which included a certain degree of arbitrariness in choosing x values. Even those who decided to pay attention to the value of f or focus on small values of x (because that's when e^x is small), ended up finding appropriate x values somewhat by chance. In fact, many of these students made several errors when calculating $f(x)$. Consider, for instance, Tables 6.6 and 6.7, which represent the data sought and found by S6 and S8, respectively. Both students did not realize that their errors ($f(3.5) < 0$ for S6 and $f(-0.5) < 0$ for S8) implied that f actually has more than two zeros.

One student, S10, was unique in speaking out about avoiding the complexity of plugging in values of x by hand. After calculating $f(1)$, $f(2)$, and $f(0)$, he explained: "One thing I could do is just grab the calculator and test for some values. [-] But that's not very satisfying. [-] Because I know that there's a way to do it without using a calculator." Although he suspected to find zeros between $x = 0$ and $x = 2$, S10 seemed to recall not needing to use a calculator in RA I. He pointed out, furthermore, that

> we've got a lot of different very powerful methods of computation that would be really easy to use. You know, I could have set up Mathematica and listed values between zero and two. I could have done a hundred of them and [-] get it to tell me how many times in those it found values less than zero, [-] and how many times it switched.

It is true that if S10 were to use software like Mathematica, he could easily write a program to systematically and efficiently check the sign of $f(x)$ for a large number of x values. Such an approach is comparable to choosing x-values at random; the difference is that it takes advantage of the power of computer tools to perform the grunt work calculation at a much higher speed and with much greater precision. There is no doubt that S10 would have been able to locate two zeros for the given f in this way. Moreover, had the function been more complex, S10 may have been able to deal with the complexity by simply increasing the number of $f(x)$ values found; something that would be very difficult to do by hand. All this said, S10 explained that since he had already spent a lot of time "fumbling around with Mathematica," he wanted to try to think of a different, more theoretical approach. Since S10 was ultimately unable to solve Task 2, it is unclear what a more theoretical approach would look like in this case. We wonder if S10 would have been satisfied with any of the arguments made by other students since they all relied heavily on calculation and/or visualization. If he would have been satisfied with such arguments, then it is possible that S10 lacked a certain fluency in analysing a function and its algebraic or graphical properties; we predict that this was the case for many of his peers. If, on the other hand, S10 would not have been satisfied with such arguments, then perhaps his interests as a student and his experience in RA I led him to favour the theoretical over the concrete to an impractical degree.

6.2.2.3 Choosing a Technique: Showing that a function f has at most n zeros.

To show that $f(x) = e^x - 100(x-1)(2-x)$ has at most two zeros, recall that we expected a successful RA I student to apply a technique, denoted by $\tau_{2_{b_2}}$, based on two key theories: the

Principle of Contradiction and Rolle's Theorem. More precisely, we expected them to provide an argument like the following:

> Assume that f has three zeros.
> By two applications of Rolle's Theorem, f' has at least two, and f'' has at least one.
> However, $f''(x) = e^x + 200 > 0$; and so f'' cannot have any zeros.
> Contradiction.
> f cannot have three zeros. It can have at most two. ■

To our surprise, none of the fifteen participants in our study argued in this way. We identified four students (S4, S5, S6, and S10) as not clearly exhibiting a technique for arguing that the given f has at most two zeros, either because they claimed to not know how (S4, S5, and S6) or because they had chosen to show only that f has at least two zeros (S10), and they were not prompted to go further by the interviewer.[30] Among the remaining eleven students, there seemed to be three ways of attempting to produce a technique: using the degree of the polynomial in f (6.2.2.3.1), Rolle's Theorem (6.2.2.3.2), or what f' and/or f'' tell us about the shape of f's graph (6.2.2.3.3). Note that some of these students' approaches seemed to be based on recalling the techniques to be learned in RA I ($\tau_{2_{b_3}}$, $\tau_{2_{b_2}}$, and $\tau_{2_{b_1}}$, respectively), while others seemed to be based on recalling the techniques learned in previous Calculus courses. In the last part of this section (6.2.2.3.4), we remind readers of the very straightforward graphical approach to showing the given function f has exactly two zeros, and we reflect on why students may not have taken such an approach.

6.2.2.3.1 By the degree of the polynomial.

Three students (S7, S11, and S15) thought that the function $f(x) = e^x - 100(x-1)(2-x)$ has at most two zeros because it contains a polynomial of degree two. S11 said almost exactly that: "It has a polynomial of degree two. So I know it has at most two zeros. [-] Isn't that a theorem?" "I think it's the Fundamental Theorem of Algebra," noted S15. S6 provided a more general description of the technique she knew, saying that it is "based off the highest degree of the

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[30] As specified in Section 4.2.1.2 of our Methodology, the interviewer prioritized getting through all six of the interview tasks during the planned two-hour duration of the interview. This is an example of a moment where the interviewer prioritized moving to the next interview task over prompting students to think more about how they might show that the given f has at most two zeros. Some of these students (S4, S6, and S10) nevertheless briefly hinted at possible approaches. Descriptions of their behaviour is included in the relevant subsections below.

polynomial you get. [-] Like that's the max number of zeros you can get." The difference was that S6 did not identify f as part of her established experiences. She explained that usually "we would only get some easy polynomial. [-] But the fact that there's an e^x is kind of like something I'm not used to seeing." Such a memory reflects the fact that the Fundamental Theorem of Algebra is a theorem about polynomials, which f is not. Hence, the theorem cannot be used on its own to produce a technique for arguing about the maximum number of zeros in this case. Recall that some of the activities we analyzed used the degree of a polynomial to deduce the maximum number of zeros it could have. We suspect that some of the students in this group were recalling this; but they had not learned (or could not recall) that the corresponding technique applied only to polynomials (not functions containing polynomials).

Interestingly, both S15 and S7 tried to think more deeply about why the degree of $h(x) = -100(x-1)(2-x)$ would be relevant to concluding the maximum number of zeros f can have. They noted, as many of their peers did, that h has two zeros (at $x = 1$ and $x = 2$); then they sought to justify why adding e^x would have no effect on the maximum number of zeros. S15 highlighted, for example, that "the exponential is strictly positive on the reals. So it doesn't change sign." S7 had the feeling that e^x would essentially "shift" the graph of h, though he was unsure and unable to graph f to verify his conclusion.

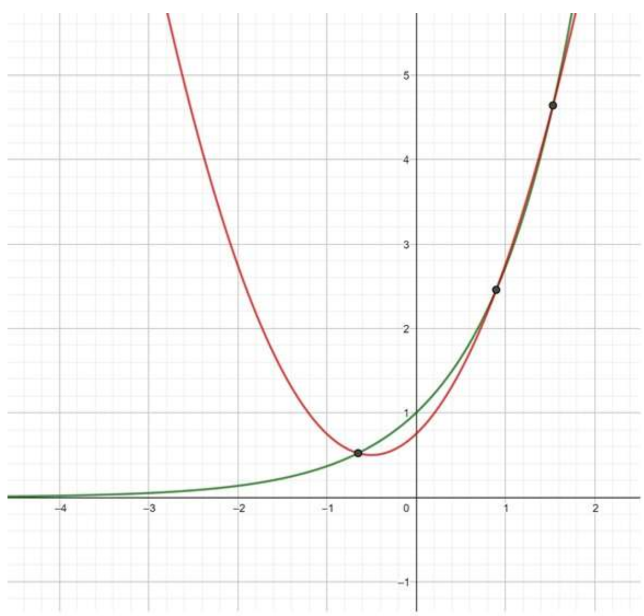

Figure 6.5 An example of $h(x) = (x + 0.5)^2 + 0.5$ (in red) that intersects e^x (in green) three times; i.e., the function $f(x) = e^x - h(x)$ has exactly three zeros.

It is not true in general that if $f(x) = e^x - h(x)$, where $h(x)$ is a polynomial of degree two, then f has at most two zeros. It is even possible that $h(x)$ has no zeros and $f(x)$ has three. Consider, for example, if $h(x) = (x + 0.5)^2 + 0.5$ (the graphs of e^x and h are shown in Figure 6.5). Had the students in this group reinterpreted Task 2 as one about showing that e^x and $h(x) = -100(x - 1)(2 - x)$ have two intersections, and they were able and willing to explore what that would mean visually, perhaps they would have realized this possibility. Furthermore, this may have led them to build on their theoretical discourse, which could actually be seen as incomplete rather than incorrect. Indeed, it is true that $f(x) = e^x - h(x)$ has at most two zeros if $h(x)$ is a polynomial of degree two *and* $h(x)$ has a negative x^2 term.

6.2.2.3.2 By "Rolle's Theorem."

Although S15 seemed convinced that $f(x) = e^x - 100(x - 1)(2 - x)$ has exactly two zeros because it contains a polynomial of degree two, he was not satisfied with this argument. He proceeded to recall some of the theorems he learned in RA I, explaining: "I want to plug these theorems because [-] it seems more reliable a result than just saying this function has two zeros because we have a polynomial." S15 was able to write out rather complete formal statements of several theorems, along with visual representations of them (Figure 6.6 depicts what he wrote for Rolle's Theorem). The issue was that he could not determine, in the context of the interview, how he could use the theorems to solve the given task. For instance, after writing out the statement shown in Figure 6.6, S15 briefly tried to interpret it and concluded: "I hesitate on the method because it's a bit uh, it's far in my head at this point."

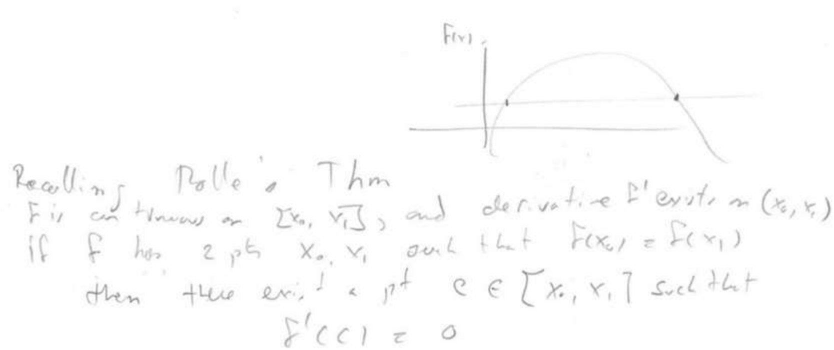

Figure 6.6 What S15 remembered about Rolle's Theorem, which he was unable to use to solve Task 2 in the context of the interview.

Four students (S1, S2, S7, and S13) seemed to have a better memory of the method to which S15 was referring. They all chose to show that f' has exactly one zero. S1 explained his choice by simply saying: “If the derivative has one, [-] the original has at most two.” It can require some complex logical thinking to go from Rolle’s Theorem to this statement: e.g.,

> To show: f' has exactly one zero $\Rightarrow f$ has at most two zeros.
> Consider the contrapositive: f has more than two zeros $\Rightarrow f'$ does not have exactly one zero. Or, more specifically: f has more than two zeros $\Rightarrow f'$ has more than one.
> Indeed, if f has more than two zeros, then it has at least three.
> I.e., there are three points, $x_0 < x_1 < x_2$, such that $f(x_0) = f(x_1) = f(x_2) = 0$.
> Then, by Rolle’s Theorem (assuming f satisfies the continuity and differentiability conditions), there are points $c \in (x_0, x_1), d \in (x_1, x_2)$ so that $f'(c) = f'(d) = 0$.
> I.e., f' has at least two zeros, I.e., f' has more than one zero. ■

None of the students explicitly expressed any reasoning remotely comparable to this, or a simpler version: e.g., If we show f' has exactly one zero, then f must have at most two, otherwise f would have three and, by Rolle’s Theorem, f' would have two (contradiction). On the one hand, S15 recalled a complete mathematical technology (Rolle’s Theorem), but not its relation to a practical block capable of proving that a given function has at most two zeros. On the other, S1, S2, S7, and S13 seemed to be acting based on a memory of a routinized practical block, for which they lacked a complete mathematical technology.

In place of such a technology, the students may have developed others, which could mislead them to think their technique is more powerful than it is. Consider, for instance, the behaviour of S7, who made a mistake in calculating $f'(x)$: he found $f'(x) = e^x - 200x - 300$ (it should be $f'(x) = e^x + 200x - 300$). The derivative calculated by S7 can be shown to have exactly two zeros by simply representing e^x and $200x + 300$ on the same set of axes. And yet, after locating one zero, S7 stopped: “Oh, there can only be one by Rolle’s. So yeah. This is the only one there is. Since here [for f] there’s at most two, that’s what you have to prove.” S7’s conclusion brings up at least two interconnected issues. First, the students may not have learned the nuances of Rolle’s Theorem: it is not true, as S7 may have thought, that if f has at most two zeros, then f' has at most one (this relates back to our interpretation of S2’s actions as using her version of Rolle’s Theorem to incorrectly reinterpret the task; see 6.2.2.1). Second, the students may have come to expect that a function f can be shown to have at most two zeros by always showing that f' has exactly one. Of course, it is not difficult to imagine a function where this technique would not work (e.g., try drawing a function with exactly two zeros, whose derivative has infinitely many). In fact, there exist functions of the same form as the function in Task 2, $f(x) = e^x - 100(x-1)(2-x)$, where the technique would not work. For instance, if $g(x) = e^x - ((x+0.5)^2 + 0.75)$, then g has exactly two zeros, and so does g'. This can be verified through the computer-generated image in Figure 6.7, which plots e^x, $(x+0.5)^2 + 0.75$, and $2(x+0.5)$ on the same set of axes (notice that e^x and $(x+0.5)^2 + 0.75$ intersect exactly twice, as do e^x and $2(x+0.5)$). Ultimately, S7's miscalculation of f' should have led him to find that f' has exactly two zeros, which should have in turn led him to question his chosen technique; it seemed S7 was too caught up in recalling the routine from RA I to be able to follow this line of reasoning.

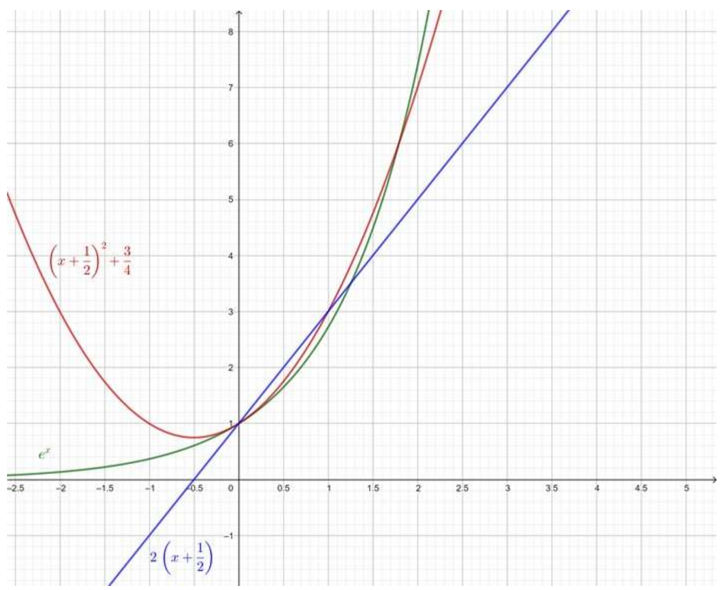

Figure 6.7 Verification that $g(x) = e^x - ((x+0.5)^2 + 0.75)$ and $g'(x) = e^x - 2(x+0.5)$ both have exactly two zeros, whereby Rolle's Theorem cannot be used to conclude that g has at most two zeros.

6.2.2.3.3 By the relationships between a function's derivatives and the shape of its graph.

We identified five students (S3, S8, S9, S12, and S14) as building a technique for showing that f has at most two zeros based on the curve sketching theorems they had likely first encountered in Calculus courses, and had seen again in RA I.

Two students (S12 and S14) tried to produce a technique based principally on the relationship between the sign of $f'(x) = e^x + 200x - 300$ and the monotonicity of f. As depicted in their sketches shown in Figure 6.8, these students expected f to have a global minimum. The plan they devised could be described as follows: locate the minimum by (a) finding x_m such that $f'(x_m) = 0$; and (b) arguing that $f'(x) < 0$ (i.e., f is strictly decreasing) on $(-\infty, x_m)$ and $f'(x) > 0$ (i.e., f is strictly increasing) on (x_m, ∞). This would prove that f could have at most 2 zeros because on each of the intervals where f is monotone, it could have at most one zero. It is possible that S12 and S14 were acting based on some of the activities that we analyzed from RA I. Recall that some activities had used a similar argument to show that a given function f has exactly one zero on a domain. Such activities relied on the result: if f is strictly increasing (or strictly decreasing) on a domain, then it can cross the x-axis at most once on that domain. Although this result is rather intuitive, when asked how he knew it, S12 provided a more formal explanation:

> If a function is [-] increasing strictly, it means that [-] if I have two points, a and b, where $a < b$, then that entails that $f(a) < f(b)$. So if I have some point that is a zero, on this function, [-] we'll say b is that point. [-] Any point that is greater than b is going to have to be greater than zero. So that shows that no value c greater than b is actually going to give something that's a zero in our function. Similarly, no value less than b will give us a value of zero. [-] The same is true for decreasing functions.

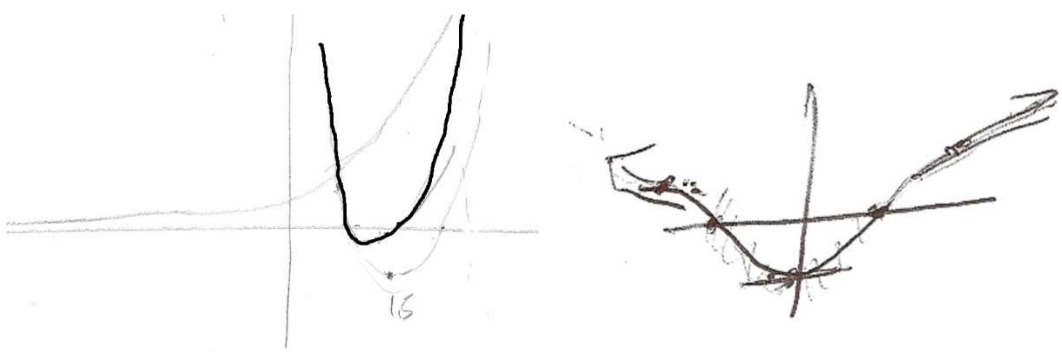

Figure 6.8 Sketches made by S12 (left) and S14 (right) as they explained how they would show that f has at most two zeros. On S12's sketch, we have bolded the curve representing f.

Both S12 and S14 were unable to carry out their technique because it unnecessarily relied on the need to find a value for where the minimum is located: i.e., x_m such that $f'(x_m) = 0$. When confronted with trying solve $e^{x_m} + 200x_m - 300 = 0$, both students got stuck. "I honestly don't know how I would solve that," said S14. S12 made a stronger claim: "This is not a function that you can solve analytically." In the context of the interview, the students did not seem to realize that it is possible and sufficient to argue for the existence of x_m. We suspect that they had the necessary ingredients for doing so. For instance, S14 pointed out that $e^x + 200x$ is increasing, and 300 is just a constant. Hence, by a result equivalent to the one that S12 justified above, this means that the graph of $y = e^x + 200x$ can cross the graph $y = 300$ at most once. In fact, by the IVT, which both students seemed to understand, it is possible to show that the graphs do indeed cross: e.g.,

If $x = 0$, then $e^0 + 200(0) = 1 < 300$.
If $x = 2$, then $e^2 + 200(2) = e^2 + 400 > 300$.
Hence, by the IVT, $\exists x_m$ such that $e^{x_m} + 200x_m = 300$

$$\Leftrightarrow e^{x_m} + 200x_m - 300 = 0$$

$$\Leftrightarrow f'(x_m) = 0. \blacksquare$$

Another way that S12 and S14 may have gotten unstuck was to turn to the second derivative, as the remaining students in this group did.

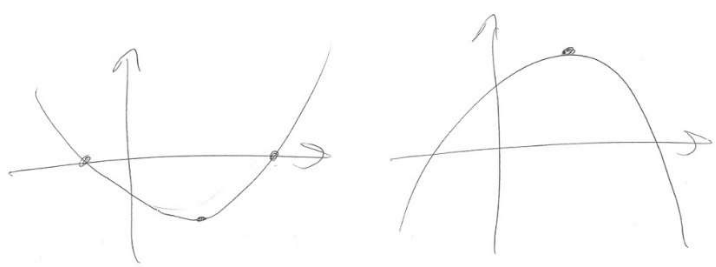

Figure 6.9 S8's sketches of the possible graphical forms for f based on his observation that (a) f has at least two zeros and (b) f has only one min or max.

S3, S8, and S9 all used the sign of f'' to argue that f can have at most two zeros. Upon calculating $f''(x) = e^x + 200 > 0$, S8 concluded that f' is always increasing, whereby it could only cross the x-axis once. In other words, f could have only one min or max. Having already shown that f has at least two zeros, S8 concluded that the graph essentially had to look like one of the two images shown in Figure 6.9. When he realized that the second derivative is positive, S9 made a conclusion equivalent to S8's – the slope of the tangent is always increasing – but he used it to argue in a slightly different manner. S9 had noted that since $f'(1) < 0$ and $f'(2) > 0$, the graph of f is "going downward, then it goes up," just like the graph on the left in Figure 6.9. S9 further noted that f could go from the positives into the negatives; however, "if it goes back into the positives at some point, it can't go back into the negatives, because that would mean that the slope would have to start decreasing." Hence f could have at most two zeros.

We see at least two reasons that S8 and S9 were able to produce the technique that they did. First, they had developed a strong fluency in using the derivative to picture the shape of a graph. Second, they identified that this fluency might be helpful in solving Task 2, even if it was not gained in RA I. This was not the case for those students who considered producing a technique in the same way and failed to do so. For instance, after calculating $f'(x)$, S4 reasoned as follows:

> If I were to set that at zero, it would give me the maximum or minimum; so that's not what I want to do. [-] If I take the second derivative, it just gives me the direction of the tangent; that's not what I want.

Although S4 seemed to have a sense of how the derivative could help him visualize certain elements of the graph of f, he did not see the connection between this and the task at hand: i.e., showing that f has two zeros. It is possible that he was intent on recalling the techniques learned in RA I, like the students in the previous group, since he still had the feeling that "the derivative has something to do with it." In contrast, after receiving Task 2, S10 declared with confidence that "there's clearly [-] a very straightforward way to do this [-] if I give myself rules of Calculus and kind of a visual representation of this function." This said, S10 was unable to use the rules of Calculus with the same facility as S8 and S9. After realizing that he could not easily solve $f'(x) = 0$, S10 decided to reassess his approach and, echoing S4, said: "I'm trying to think why… […] why the derivative mattered here."

6.2.2.3.4 By the look of graphs.

Recall that S3 was the only student to reinterpret Task 2 as:

$\widehat{t_2}$: Show that the graphs of e^x and $100(x-1)(2-x)$ intersect exactly twice.

Suppose we take for granted certain technologies about how the graphs of these two types of functions look. For instance, $100(x-1)(2-x)$ is a downward-facing parabola, with roots at 1 and 2; and e^x is a strictly increasing function that approaches 0 as $x \to -\infty$, has a y-intercept at

1, and approaches ∞ as $x \to \infty$. Based on these kinds of technologies, we can sketch the parabola $100(x-1)(2-x)$ and e^x, rather accurately, on the same set of axes (see the sketch on the left in Figure 6.10). It is then quite easy to solve Task 2: So long as we check that the graph of the parabola lies above the graph of e^x at some point (e.g., at $x = 1.5$, as shown in the sketch), then the task has been solved. No additional action is required. As alluded to in the previous section (6.2.2.3.3), this is not how S3 solved Task 2.

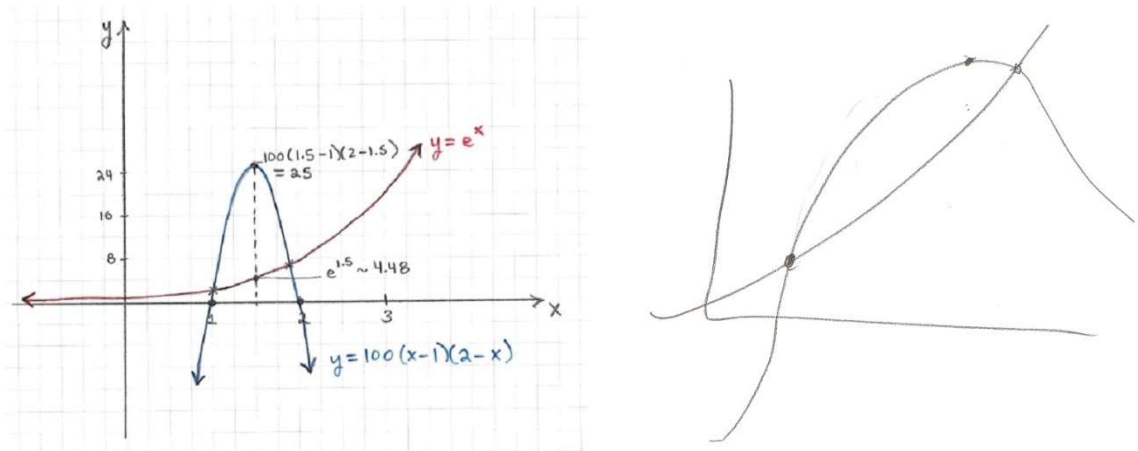

Figure 6.10 A graphical solution to Task 2 (left, reproduced from Figure 6.3) and S3's graphical representation of $\widehat{t_2}$ (right, reproduced from Figure 6.4).

While S3 seemed to take the graph of e^x for granted, when it came to the graph of the parabola, he claimed: "I have no idea yet how this works, how this one looks like." His entire solution was based on using Calculus to study the expression $h(x) = 100(x-1)(2-x) = -100x^2 + 300x - 200$ and make deductions about the graph of h. More specifically, S3 noted that the coefficient of x^2 is negative, whereby h would be negative for negative and positive a billion; he found that $h'(x) = 0$ at $x = 1.5$, whereby h has a "maximum" there; and he calculated $h''(x) < 0$ to conclude that the graph of h "always goes down," and therefore has "only one max." If S3 really required such a detailed analysis to determine what the graph of h looks like, then we would say that he was lacking the level of fluency in quadratic functions that we would expect of students taking a course like RA I. On the contrary, we suspect that S3 almost immediately knew what the graph of h looks like; in fact, he seemed to be using it to guide his collection of data. Recall, for instance, that while S3 was reinterpreting Task 2, he drew the sketch on the right in Figure 6.10. Moreover, after producing this image, S3 used it to clearly articulate his goal:

I need to show that there is a point where the derivative would be zero, and that point would be above the function [e^x]. So that will show that it has at least two roots. But I need to show that it has only two roots. That would mean that I need to show that it has only one maximum.

As S3 said this, it seemed that he already knew that h is a downward facing parabola. Why then did he feel the need to provide additional explanation?

One possible reason can be found in analysing students' responses to a series of reflective questions concerning whether a computer-generated graph of $f(x) = e^x - 100(x - 1)(2 - x)$ would show that the function has exactly two zeros. In general, the students found the graph (shown in Figure 6.11) to be rather convincing and believed that it would suffice to convince some people that f has two zeros; but they did not think it would be acceptable as a solution to Task 2 in the context of a mathematics course. S3, for instance, explained as follows:

Maybe this is just my brain being programmed: When I see the word "show," my last instinct is to go find a computer and graph it, because we do it in exams, and [-] we're being trained to do it rigorously. [-] If I meet someone who doesn't know the derivatives and stuff like that, then of course I would just show them on a computer.

In his explanation, S3 points out that in an exam, mathematics students are not expected to use computer tools to solve tasks; they are expected to solve the tasks "rigorously." Other students expanded on what S3 may have meant by this.

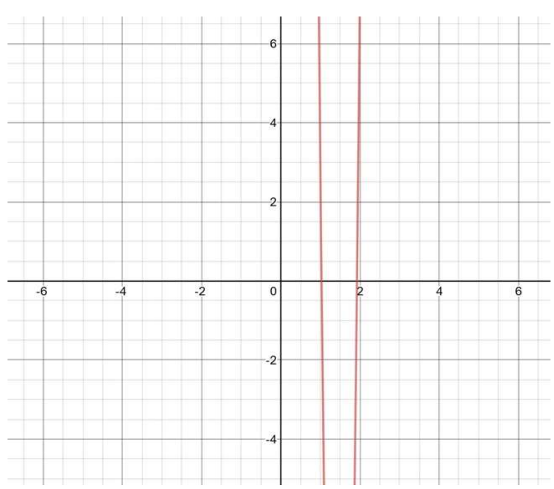

Figure 6.11 A computer generated graph of $f(x) = e^x - 100(x - 1)(2 - x)$, which most participants found convincing, but not acceptable as a solution to Task 2.

When S1 was trying to show that $f'(x) = e^x - 300 + 200x$ has exactly one zero, he eventually realized the usefulness of the equation $f'(x) = 0 \Leftrightarrow 300 - 200x = e^x$. He concluded that since $e^x > 0$ and $300 - 200x$ is linearly decreasing, the functions would cross at exactly one point; and he provided the image shown in Figure 6.12 to illustrate his argument. It is not true in general that a strictly positive function and a linearly decreasing function cross at exactly one point. Hence, S1's argument was either missing some conditions on the functions or depended on his sketch. Later in the interview, S1 revealed that he did not intend the latter. When he was shown the computer-generated graph of f in Figure 6.11, he claimed that it was insufficient because "we need to show how the graph was drawn." Referring to his sketch in Figure 6.12, he continued: "If I just did this, it's not really a proof. Cause I drew this with my hand and it doesn't mean anything." S7 and S8 also spoke about how a graph could only be a proof if it is accompanied by additional explanation. To solve Task 2, S7 said that "the answer, it's not an image itself, but explaining how you get the image." S8 specified further: "I know the RA I prof would want me to do more to answer the question." He spoke, for example, of needing to discuss the continuity of f, which ensures that the lines in Figure 6.11 are an accurate representation of the function's behaviour (e.g., the function really passes through the axis at the points suggested by the image). Other students also mentioned the fact that the graph only depicts a small window of the function's behaviour and does not indicate what its overall behaviour is like. Based on these kinds of comments, it seems that some students had developed a standard of rigour that would not accept as a solution to Task 2 a by-hand sketch of the graphs of e^x and $h(x) = 100(x-1)(2-x)$ (as in Figure 6.10); further written justification, like that provided by S3, would likely be required.

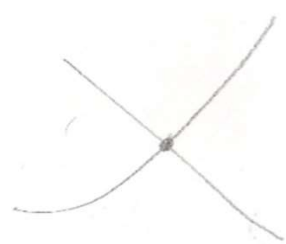

Figure 6.12 The sketch used by S1 to illustrate (but not prove) that e^x and $300 - 200x$ intersect at exactly one point, whereby $f'(x) = e^x - 300 + 200x$ has exactly one zero.

An interconnected reason for S3's choice to analyze the behaviour of a quadratic function using Calculus was that this is what he had learned to do when solving similar types of tasks in previous courses. When asked why he took the approach he did to solve Task 2, S3 simply said:

> Because I did Numerical Analysis. And Numerical Analysis involves a lot finding roots. Proving that there are roots. [-] A lot of times the question started: "Show that there is a root and then find it." [-] When I did this [referring to his solution to Task 2], like when I tried to do the derivative, I was already trying to think of things that I learned in Numerical Analysis.

Note that there were other students who saw Task 2 as more closely connected with courses other than RA I (e.g., Numerical Analysis or Calculus). What is more important for the point we are making here is that S3's experience in previous mathematics courses was clearly shaping his decision to analyze h using its first and second derivative. Moreover, he seemed to see this analysis as part of solving Task 2 "rigorously." He did not consider, for instance, that such an analysis might be excessive for the given function, whose sketch could be efficiently produced if one takes for granted certain technologies concerning the graphs of parabolas (technologies that were expected to be mastered in previous courses). The situation is completely different, for example, with a function like $f(x) = e^x - 100(x-1)(2-x)$, which is not easy to graph, even with the rules of Calculus. Several of the students we interviewed hinted that they had not yet contemplated such subtleties. They had so far learned that graphing cannot prove, no matter what kind of function is being considered:

> L: "In RA I, would it be an acceptable approach to use graphing in this case?"
> S9: "No. [-] They don't like graphs. [-] Cause a graph isn't a proof. That's what they say. And it's all about proofs, that course."

6.2.3 Reflections on the Nature of Practices, Positioning, and Activities[31]

Non-Mathematical: Identify task with technique in an automatic fashion. VS. *Mathematical*: Study the task, think of equivalent formulations or different possible techniques, and perform the assessment: "Will this work? Is this better?"

It seemed the practices developed by most participants were non-mathematical in at least one

[31] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.2.1 and 6.2.2 that begin to address our research objectives: characterizing the nature (mathematical or

sense: namely, they began with an identification of the given task with a technique in an automatic fashion based on surface properties of the task. The students saw a task about "showing zeros" of some function and immediately thought: "I'm going to use the IVT." This behaviour is most obviously linked with the Student position since being able to identify a task with a technique in an automatic fashion is very useful for doing well on exams. This said, we also think this behaviour can be linked to the Enthusiast and the Mathematician in Training, depending on the nature of the given task and the activities offered in the related course institution. Recall that these positions are characterized by a dedication to learning what they perceive to be the practices to be learned in the course. In RA I, the activities emphasized solving tasks like Task 2 using a particular practice (based on the IVT). Hence, it would not be surprising to find that Enthusiasts (e.g., S10) and/or Mathematicians in Training (e.g., S13) would automatically think to use this practice. The difference is that they may be more interested in or able to provide a mathematical justification for the practice: e.g., noting the continuity condition and providing both informal and formal explanations of the IVT.

There were some participants (e.g., S1 and S9) who seemed to be in between the two behaviours in the box above and might be thought of as being in transition (or with the potential to be in transition) to practices that are more mathematical in nature. These students either stopped to question their automatic choice of technique and/or considered one or two alternative "easier" techniques briefly before deciding to use the IVT. They showed an inclination to accompany their identification of task with technique with a response to the questions: "Do I really need to use this technique? Might there be a simpler approach?" We think this behaviour can be linked with the Student-Learner (who might feel tension between wanting to student and wanting to learn) and the Skeptic (who has a tendency to think about how their previously learned practices might suffice in solving the task). When S9 decided that he actually needed to use a practice he had learned in RA I, his Skeptic positioning was challenged.

Only one student (S3) exhibited a mathematical practice in the sense that he took time to study the given task, to think of equivalent formulations, and perform an assessment of the form: "Will this work? Is this better?" We think this is linked with S3's positioning as a Learner and the

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not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

nature of the task. On the one hand, one of the features of the Learner (by our definition) is that they are not a particularly good Student. On the other hand, Task 2 is almost identical to some of the activities offered in RA I. It is possible that S3's position as a Learner enabled him to look at Task 2 in an entirely different way than his peers.

Non-Mathematical: Implement a technique based on patterns in solutions that just seemed to work. VS. *Mathematical*: Implement a technique based on a qualitative study of the objects involved; in particular, using the essential characteristics of those objects.

When it came to implementing the technique based on the IVT (i.e., solving the task of finding two sign changes in $f(x) = e^x - 100(x-1)(2-x)$), it seemed many students were acting based on recalling a pattern in the solutions they had seen in the RA I course: namely, the task can be solved by plugging in some x values "at random," and possibly also checking limits at infinity. Plugging in some x values and checking limits at infinity are mathematical techniques that can be useful in making progress in finding sign changes of a function. What made the students' practices non-mathematical was their automatic inclination towards and dependence on those techniques. They chose the techniques without thinking about the objects involved in the task, and they struggled when the techniques did not work. Once again, the automatic identification of task with technique did not distinguish between students in different positions. We found that distinctions between positions were more evident in what students did to cope with their struggle.

S10 seemed to distinguish himself as *the* Enthusiast when he accepted being stuck in the interview because (a) he felt there should be a way to think about Task 2 by reducing it to basic assumptions (i.e., by using what he perceived to be the practices to be learned in RA I); and (b) he would need more time to figure out how to do this. He refused to pull out his calculator and continue trying numbers, as many of his peers did, because he had already developed a more mathematical practice for doing that (using the power and precision of computer technology).

Some participants showed strong signs of being a Student. S11, for example, blamed his struggle in finding sign changes on knowing how to solve a more particular type of task to which Task 2 did not belong (because it did not specify the interval where sign changes can be found).

S1 simply noted that he could not recall the "better way" and decided he would continue in plugging in numbers. Note that the practices developed by these students – including S11's highly non-mathematical practice – would be sufficient for solving the assessment activities from RA I. We therefore suspect that the behaviour exhibited by S1 and S11 can be linked to both the students' dedication to learning solution patterns and the existence of such patterns in the activities they were offered.

In comparison, there were students who, in response to getting stuck, made a change to their technique and developed a practice that was more mathematical nature. The alternative practices were mathematical in the sense that students supported them with mathematical discourses: they weren't just going "at random," they were making systematic choices for mathematical reasons. There were still important differences in the nature of the techniques. Consider, for example, the two techniques: "pay attention to how the value of $f(x)$ is changing as x changes" and "study the sign of $f'(x)$ for particular x." Both rely on local studies of the function's monotonicity (either from one point to the next or based on the sign of the derivative at a point) to make predictions about whether or not the function will change sign somewhere nearby. When considered in isolation from a task, these techniques are mathematical. When we consider the techniques applied to Task 2, they might be seen as less mathematical since they involve a quantitative (as opposed to qualitative) study of the function and they do not take advantage of the essential features of the function. This is not the case for the other three, more mathematical, techniques: "compare the growth of different parts of f," "compare the signs of different parts f," and "graph the different parts of f," all of which are based on studying an expression for f and trying to figure out some of its essential characteristics that can be used to understand the behaviour of the function more globally.

We think this is another example where students showed that they were (or had potential to be) in transition to mathematical practices, while showing they had (or could have) shifted away from a strict Student position. We think the nature of Task 2 could support the shift from studenting to learning since it purposefully led students to get stuck. It is also possible that Task 2 could support this shift for more students since it was accessible: students could rely principally on arithmetic and algebra or could call upon knowledge of graphs and functions. We suspect that we could construct similar tasks that would have encouraged students to also realize a need to shift from primarily quantitative local strategies, to more qualitative global approaches that consider essential features of the function. The activities offered to students in RA I may not have encouraged this transition for some students because they could be solved in a routine manner.

Only one student (S12) had developed the mathematical routine of finding sign changes by performing a qualitative study of the given function. Generally speaking, this student seemed to lie somewhere in the Student-Learner-Mathematician-in-Training realm.

Non-Mathematical: Explain a technique with a distant, disconnected acknowledgement of theory. VS. *Mathematical*: Produce a technique using a combination of informal understandings and formal theory.

To show that $f(x) = e^x - 100(x-1)(2-x)$ has at most two zeros, several students applied practices that were non-mathematical in the sense that they comprised routinized techniques that were based on some distant, disconnected acknowledgement of theory. S11, for example, chose to implement a technique where you note the degree of the polynomial based on some theorem (lacking the essential condition that f should be a polynomial). Others (e.g., S1, S2, and S13) chose to implement a technique where you show the derivative has exactly one less than the desired number of zeros; also based on some theorem. The explanatory discourses of these students were not entirely absent or non-mathematical: they relied on the acknowledgement of some theorem. But the students either did not seem to really know the theorem; or, if they knew it, they did not seem to be able to use it to provide an explanation of how and why the techniques work. This is another behaviour that can be naturally linked with the Student position and activities that do not require students to justify the routines they are being led to develop. This said, we also think this behaviour could be linked with other positions, such as the Mathematician in Training. The key difference with the latter is that they might be more open and able to make the connection between the theory and the technique, independently of the activities provided to them.

S7 was a student who seemed to be in-between the two behaviours in the box above and might have been showing signs of being (or having the potential to be) in transition towards mathematical practices (in general, we tended to classify him as a Student-Learner). S7 showed an inclination towards the development of mathematical practices when his instinct and intuition told him to use the degree of the polynomial, but he did not want to use it without finding a more mathematical explanation of why this would work. Interestingly, when he recalled a routinized practice based on established experiences and the acknowledgement of Rolle's Theorem, the work of thinking about his initial choice of technique stopped. This is one example that shows how activities that can be solved using very routine practices might impede the transition towards practices that are more mathematical in nature.

The students who used curve sketching theorems to solve the task (i.e., S3, S8, S9, S12, and S14) exhibited practices that were mathematical in the sense that they relied on a combination of sound mathematical theoretical discourses about the properties of functions, and informal understandings of the looks of graphs. This behaviour seemed possible for students who had occupied the Learner, the Skeptic, or the Mathematician in Training positions. This includes students who not only had a good understanding of practices learned in previous Calculus courses, but also an openness to use those practices to solve the given task, even if they may not have been emphasized in RA I. This would not be the case for those lying primarily in the Student (e.g., S4) or Enthusiast (e.g., S10) positions, both of whom would be inclined to try to connect the given task with what they perceived to be the practices learned in RA I.

6.3 Task 3

Let $A = \{x_n \in \mathbb{R} : n = 1, 2, \dots\}$.

a) *Under what conditions is* $\lim_{n\to\infty} x_n = sup\, A$*?*
b) *Give an example of A where the limit* $\lim_{n\to\infty} x_n$ *exists but does not equal* $sup\, A$.

6.3.1 Practices to be Learned[32]

When faced with Task 3(a), we expected a successful RA I student to provide two conditions: $(x_n)_{n\in\mathbb{N}}$ increasing and bounded above. We also thought that they might also be compelled to provide a proof that if these conditions are true, then $\lim_{n\to\infty} x_n = \sup A$, as shown in Figure 6.13. The successful RA I student would at least be aware that the statement can be proved and be able

[32] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 3. This model is synthesized in table form in Appendix D.

to make some progress if asked to prove it. It is possible that they would face challenges in constructing the proof since it implements several powerful technologies. It uses the bounded condition and the Completeness Axiom to conclude that $\sup A = M$ is a real number. It calls upon the definition of sequence convergence to construct the main body of the proof. Implicit in the solution is also a formal definition of $\sup A$, which we can represent as:

$$\sup A = M \Leftrightarrow ((\forall \epsilon > 0, \exists a \in A, a > M - \epsilon) \wedge (\forall a \in A, a \leq M)).$$

Finally, the proof uses the condition that $(x_n)_{n\in\mathbb{N}}$ is increasing and, implicitly, the corresponding definition: i.e., $x_n \leq x_{n+1}\ \forall n \in \mathbb{N}$.

In contrast with our expectations for part (a), we anticipated the successful RA I student's solution to part (b) to be short and sweet, including a description of the exemplar A, the value of $\lim_{n\to\infty} x_n$, the value of $\sup A$, and an indication that they are not equal. In other words, we did not expect the student to spontaneously offer a proof of their statements when asked to "give an example." This said, a successful RA I student would be able to construct relevant proofs if asked to do so (we outline the types of proofs the student would be expected to construct for a specified limit and a specified supremum when we discuss Task 4 in Section 6.4.1 below).

a) If $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup A$.
Proof: Let $(x_n)_{n\in\mathbb{N}}$ be increasing and bounded above.
Then $A = \{x_1, x_2, x_3, \ldots\}$ is non-empty and bounded above.
So by the Completeness Axiom, $\sup A = M$ is a finite real number.
We will show that $x_n \to M$.
We have to show $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, |x_n - M| < \epsilon$.
Let us take an $\epsilon > 0$.
Then we can find an element of A, say x_N, such that $x_N > M - \epsilon$.
Since $(x_n)_{n\in\mathbb{N}}$ is increasing, $\forall n \geq N, x_n \geq x_N > M - \epsilon$.
On the other hand, all x_n's satisfy $x_n \leq M$.
Thus, for $n \geq N$, we have $M - \epsilon < x_n < M + \epsilon$.
We proved that $x_n \to M$. ∎

b) $A = \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$
$\lim_{n\to\infty} \frac{1}{n} = 0 \neq 1 = \sup A$.

Figure 6.13 An example of the kind of solution anticipated for Task 3.

At first glance, part (a) of Task 3 – under what conditions is $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ – does not represent a type of task that generates a practice studied in RA I. In the activities we analyzed, students were not invited to engage in devising conditions that make a certain statement hold. Rather, the expected conditions and corresponding statement in this case – an equivalent version of the so-called Monotone Sequence Theorem (MST) – were used in solving other types of tasks for which the successful RA I student would have developed practices. In other words, Task 3(a) relates to a piece of the practices to be learned – a theoretical discourse – which can be modelled as a theoretical block:

> Λ_3: If $(x_n)_{n\in\mathbb{N}}$ is an increasing (or decreasing) bounded above (or below) sequence and $A = \{x_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} x_n = \sup A$ (or $\inf A$). A proof is shown in Figure 6.13.

There are two types of tasks in RA I where students might be expected to call upon Λ_3. One type is: Show that that a specified recursively defined sequence is convergent. Examples of such sequences are:

$$x_1 = 4 \text{ and } x_{n+1} = \sqrt{10x_n}, n = 1, 2, \ldots \text{ or } x_1 = 3 \text{ and } x_{n+1} = 2 - \left(\frac{1}{x_n}\right), n = 1, 2, \ldots .$$

The corresponding activity statements typically included a request to prove that the given sequence is increasing (or decreasing) and bounded above (or below), and to refer to a theorem (i.e., the MST) to conclude its convergence. The other type of task where students might be expected to use the MST is: Prove that the supremum (or infimum) of a specified set, often of the form $A = \{x_n : n = 1, 2, \ldots\}$, is M (or m). As mentioned earlier, we provide more details about this second practice in Section 6.4.1. Here, we simply want to demonstrate why we expected a successful RA I student to think about Λ_3 upon receiving Task 3: the paths to practices we characterized indicated that it was a key theorem to be recalled, used, and stated in solutions to commonly tested tasks.

All this said, it is also possible that a successful RA I student would be expected to be able to provide the proof referred to in Λ_3; and this is not just because RA I is a proof-heavy course. One assignment activity asked students to prove part of the MST: namely, that a decreasing bounded below sequence is convergent. Within the solution provided, a decreasing bounded below sequence $(x_n)_{n\in\mathbb{N}}$ is shown to be convergent by proving that the sequence converges to the infimum of $A = \{x_n : n \in \mathbb{N}\}$. In other words, the task of proving that a sequence is convergent is transformed into the task of proving the limit of a sequence. When we classified this assignment activity, we saw it not only as engaging students in producing an important theoretical block they would use in solving other types of tasks; we also saw it as representative of a type of task that a successful RA I student should be able to recognize and attempt to solve: i.e., Prove the limit of a specified or more generally defined sequence using the formal definition. Indeed, there were several other activities that invited students to prove both specified and general limits using the formal definition: e.g.,

$$\lim_{n\to\infty} \frac{3n}{n+2} = 3; \text{ and}$$

$$\text{If } \lim_{n\to\infty} a_n = A \text{ and} \lim_{n\to\infty} b_n = B, \text{ then } \lim_{n\to\infty} (a_n + b_n) = A + B.$$

We describe the related practice to be learned in Section 6.4.1.

Of course, one could argue that the technique used to prove the second limit above is so specific that it should be linked to its own type of task: e.g., Prove a limit law for sequences. There is, nevertheless, a significant similarity between such a proof and the proof that $\lim_{n\to\infty} x_n = \sup A$ if $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded above, and $A = \{x_1, x_2, x_3, \dots\}$: in both activities, one must figure out how to satisfy the formal definition of limit convergence using the general properties of a sequence and related technologies. When compared to the proof of a limit law, the proof shown in Figure 6.13 is perhaps more complex and more challenging to independently construct, even for a very successful RA I student. This might be why the limit law activity shown above was given on an exam, while the task of proving one part of the MST was given on an assignment. We wondered if the participants in our study would insist on overcoming this challenge and developing a proof, or if they would be satisfied with other, perhaps simpler ways of justifying the conditions in Λ_3. Moreover, if participants had other ways of convincing themselves, we were interested in knowing if the corresponding theoretical discourses could be classified as mathematical or not.

Part (b) of Task 3 – give an example of $A = \{x_n \in \mathbb{R}: n \in \mathbb{N}\}$ where the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$ – is also related to a more general type of task, as well as a theoretical discourse (i.e., Λ_3). When we were modelling practices, we noticed a path of activities that invited students to come up with examples of sequences that satisfy certain properties. No technique was exhibited for solving such a task; solutions would typically comprise the examples and some indication that they satisfied the required properties. We were interested in verifying our assumption that students would use trial and error and we wanted to see how easy they found such an approach. Usually, the activities that requested examples also related to some important theoretical discourse. For instance, one activity asked students to give examples of two sequences such that each of them does not converge, but their sum converges. Such an activity relates to the theorem mentioned above: i.e.,

$$\text{If } \lim_{n\to\infty} a_n = A \text{ and} \lim_{n\to\infty} b_n = B, \text{ then } \lim_{n\to\infty}(a_n + b_n) = A + B.$$

One possible aim or at least by-product of the "give an example" activity is an illustration of how the conditions of a forward implication do not necessarily need to be met to satisfy the conclusion. In the particular example given, the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ do not need to converge to say that their sum converges. However, the convergence of the sequences is a simple and useful sufficient condition, which assists in constructing a limit arithmetic. We wondered if participants would make these kinds of reflections, even if they were not directly invited to do so in the assessment activities we analyzed from the RA I course. Note that Task 3(b) relates to (a) in a slightly different, perhaps simpler manner: that is, it invites students to think about sequences that do not satisfy a conclusion of a forward implication. We actually expected this to be the easiest task we posed (a sort of "confidence booster" in the middle of the interview): once a successful RA I student stated "increasing" as a condition for (a), we expected them to choose a decreasing sequence for (b). Due to its simplicity and popularity among the activities analyzed, $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ seemed like a likely response.

6.3.2 Practices Actually Learned[33]

Task 3 intended to engage the participants of our study in the building and exploration of a theoretical discourse, which might serve them in producing a technique for solving Task 4. In part (a) – under what conditions is $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ – we expected a successful RA I student to exhibit, and possibly try to prove, the following technology: "If $(x_n)_{n\in\mathbb{N}}$ is an increasing bounded above sequence and $A = \{x_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} x_n = \sup A$." In part (b) – give an

[33] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 3. These models are synthesized in table form in Appendix D.

example of $A = \{x_n : n \in \mathbb{N}\}$ where $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$ – we expected a successful RA I student to easily provide an example of a decreasing sequence (e.g., $x_n = 1/n$).

Students	Conditions
S1*, S2, S3, S4, S7, S8*, S9, S12*, S13*, S14, S15	$(x_n)_{n\in\mathbb{N}}$ increasing.
S13	$(x_n)_{n\in\mathbb{N}}$ eventually increasing.
S8	$(x_n)_{n\in\mathbb{N}}$ not exactly increasing.
S1, S12	No x_n can be greater than $\alpha = \sup A$ / the asymptote
S6, S12	$(x_n)_{n\in\mathbb{N}}$ has an asymptote above that it approaches from below.
S2, S11, S14	$\lim_{n\to\infty} x_n$ exists. / $(x_n)_{n\in\mathbb{N}}$ convergent.
S1*, S3, S7*, S8*, S9, S11, S13*, S15	A has an upper bound. / $(x_n)_{n\in\mathbb{N}}$ is bounded above. / A is bounded. / $\sup A$ is finite
S11**, S10*	$\forall \epsilon > 0, \exists\, n \in N, \sup A - \epsilon < x_n \leq \sup A$
S10	$\forall n \in N, x_n \leq \sup A$
S11**, S12**, S10	$\forall \epsilon > 0, \exists N, \forall n \geq N \sup A - \epsilon < x_n \leq \sup A$
S4*	All limits of the subsequences of $(x_n)_{n\in\mathbb{N}}$ are equal to $\sup A$.

* indicates that the student rejected the condition.
** indicates that the student was unable to formulate the condition.

Table 6.8 The conditions considered by participants to have $\boldsymbol{lim_{n\to\infty}(x_n) = sup\{x_n\}}$.

We provide a summary of the conditions considered for part (a) in Table 6.8. Not surprisingly, the most commonly considered conditions were “increasing” and “bounded above.” What was interesting were the different ways in which participants thought about, justified, and/or rejected these conditions. Only a couple of the participants spontaneously considered constructing a proof. Surprisingly, some participants decided against the expected conditions in favour of others or offered conditions of an entirely different kind. In these latter cases, participants seemed to be aiming, not to find simple sufficient conditions that imply $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, but to characterize all sequences $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, or to provide a characterization of the equality $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ in formal terms. We were also surprised to find some participants struggling to solve the given task, not only part (a), but also part (b). Throughout the first six sections below, we present the different conditions that seemed to be considered by participants while solving part (a), and the kinds of theoretical discourses they used to argue for or against them. In 6.3.2.7, we mention the behaviour of students who seemed to struggle to engage with Task 3 because they did not (immediately) identify it with a type of task they knew how to solve. We finish, in 6.3.2.8, by presenting some of our observations for Task 3(b).

6.3.2.1 The sequence should be increasing...

Eleven of the fifteen participants (S1, S2, S3, S4, S7, S8, S9, S12, S13, S14, and S15) explicitly considered "increasing" as a condition for part (a).

For S7, this condition seemed automatic and obvious. After reading the statement of Task 3(a), he said: "So, x_n would have to increase," and, without any further justification, he went on to part (b). When the interviewer later inquired about how "increasing" came to his mind, S7 provided the following explanation:

> So if it doesn't increase and it decreases, then, if… Well, yeah, let's start with decreasing. If it's decreasing, then the first one would be the highest one. Because from then on it would just go down and down and down. [-] And if it's constant, well they're all equal, so that's… I guess that would work. And if it's… If it's cyclical, or if the x_n, if the sequence behaves in like, there's no pattern, it could just go up and one point in the sequence could be the sup. Or if it's cyclical like sine of x_n, then [-] there's no limit.

Three other students (S2, S3, and S15) presented similar explanations: namely, they considered paradigmatic examples of sequences that *do not* increase and argued that they *might not* converge to the supremum. Like S7, S15 pointed out that the sequence could be constant, in which case it would go to both the infimum and the supremum; but it could not be decreasing, because then it would not converge to the supremum. To justify her choice of "increasing," S2 simply said that if $(x_n)_{n\in\mathbb{N}}$ is decreasing, then the limit might equal the infimum. In a similar vein, S3 said that $(x_n)_{n\in\mathbb{N}}$ should be increasing "cause otherwise, [-] like if it goes down, I have no reason to assume that getting further, going further along in the sequence will bring me closer to my sup." Notice that none of these arguments completely justify "increasing" as either a sufficient or a necessary condition since

a) to show "increasing" is a sufficient condition means showing the statement: $(x_n)_{n\in\mathbb{N}}$ increasing $\Rightarrow \lim_{n\to\infty} x_n = \sup A$. But the arguments provided by the students support the *inverse* of this statement: i.e., $(x_n)_{n\in\mathbb{N}}$ not increasing $\Rightarrow \lim_{n\to\infty} x_n \neq \sup A$; and
b) showing the latter statement is equivalent to showing its *contrapositive*: i.e., $\lim_{n\to\infty} x_n = \sup A \Rightarrow (x_n)_{n\in\mathbb{N}}$ increasing. This would show that "increasing" is a necessary condition. However, the types of sequences considered by students – decreasing, constant, cyclical – do not cover all possible sequences that are "not increasing." In other words, the students did not show that $(x_n)_{n\in\mathbb{N}}$ not increasing $\Rightarrow \lim_{n\to\infty} x_n \neq \sup A$, whereby they did not show its contrapositive.

Some participants offered more informal arguments that could support "increasing" as a sufficient condition. Two students (S3 and S9) used gestures and/or described dynamic images of an increasing sequence as "crawling towards the supremum" or "continuously growing" so that the "final n" is the biggest. In a similar manner, S4 said that "by definition" $(x_n)_{n\in\mathbb{N}}$ increasing means that "as n approaches [-] its maximal value, in this case being infinity, then that limit would have to be the maximum value that would be in A, so the supremum of A." S14 was the only one to support the "increasing" condition through a single exemplary image, as shown in Figure 6.14. As he described the image, he made it clear that it was meant to be a model: "I'm just talking about the intuition behind it," he said.

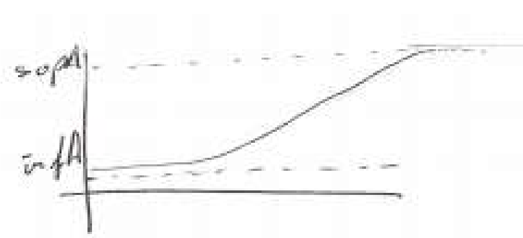

Figure 6.14 The sketch used by S14 to explain the intuition behind "increasing" as a condition for $\boldsymbol{lim_{n\to\infty} x_n = sup\{x_n: n \in \mathbb{N}\}}$.

For Task 3(a), only two students (S1 and S13) spoke spontaneously about formal proof. Both students took very little time to state the condition of "increasing" and both claimed to have reached it through picturing sequences in their heads, like some of the students mentioned above. Then S1, without any prompting from the interviewer, tried to construct a proof. We summarize below the thoughts S1 seemed to be expressing to us, which are not entirely coherent and do not clearly come together to form a proof:

> Let $\alpha = \sup A \Rightarrow \alpha + \epsilon \notin A, \alpha - \epsilon \in A$.
>
> $x_n < \alpha \wedge \rightarrow \alpha$
>
> Proof: Let α be a number C.
>
> Then $x_{n-1} < \alpha$ (otherwise $x_{n-1} > \alpha \ \Rightarrow \ x_{n-1} \notin A$).
>
> Then $x_{n-1} < x_n \ldots x_n$ has to be the largest.
>
> $x_0 < x_1 < x_2 < \cdots < \ x_{n-1} < x_n$

It is possible that S1 was not entirely sure what he was proving. After the second line shown above, he stopped and said: "Actually, I can't prove it cause I don't have anything. [-] Actually, no I do, cause my statement is that it is increasing." Shortly thereafter, S1 confirmed to the interviewer that he was trying to prove that if $(x_n)_{n\in\mathbb{N}}$ is increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. Hence, it is also possible that S1 had not quite mastered the general technique of direct proof: i.e., using what is given in a statement P (e.g., $(x_n)_{n\in N}$ increasing and $\sup A = M$) to prove the implication of another statement Q (e.g., $\lim_{n\to\infty} x_n = M$). When the interviewer asked S1 if he recalled using the definition of convergence to prove a limit, S1 swiftly wrote out the definition ($\forall\epsilon > 0$, $\exists N \geq 1, \forall n \geq N\ |x_n - L| < \epsilon$); then he concluded: "No I don't see how I could do it that way."

In contrast with S1, S13 did not spontaneously construct a proof, but claimed: "I can prove it." Then, when the interviewer asked if her solution would be different for a final exam in RA I, S13 provided the following argument:

> $\sup\{A\} = M \Rightarrow \forall\epsilon > 0\ \exists n_0 \ni M - \epsilon < x_{n_0}$.
>
> $\{x_n\}$ is increasing $\Rightarrow\ \forall n > n_0\ \ M - \epsilon < x_n \Leftrightarrow M - x_n < \epsilon$.
>
> "It means that if these situations exist, then this [$(x_n)_{n\in\mathbb{N}}$] goes to supremum A."

Due to a lack of details, we might call this argument a "sketch" of a proof, rather than a "proof." Nevertheless, the argument relies on relevant technologies to unpack a statement P and make deductions to get to a statement Q. It seemed that S13 had mastered the direct proof technique more than S1. It is also possible that S13 had studied this exact proof before and had a better memory of it than S1.

There could be several reasons why the other thirteen participants did not spontaneously speak about formal proof when faced with Task 3(a). Some students may have been acting in response to the statement of the task, which did not explicitly request a proof. Consider, for example, the following exchange between S2 and the interviewer, which took place after S2 stated her conditions:

> L: "Do you think you'd have to prove it?"
>
> S2: "Nope."
>
> L: "No? Ok. Why not?"
>
> S2: "Because it doesn't say I should prove it."

It is also possible that students were acting based on knowing a theorem that was proved in class. Towards the end of his solution, S15 explicitly noted that he ultimately knew "increasing" was a condition because he knew the "Theorem of Convergence." It seemed many students simply did not want to engage in constructing a proof. For instance, after S3 stated his conditions, he sought the interviewer's input on whether they were correct or not, and the interviewer responded by asking S3 how he might go about convincing himself. S3 decided to come up with an example – $x_n = \frac{n}{n+1}$, the sequence from Task 4(b) – and concluded:

> I feel a little bit more convinced. But I know that showing an example is not sufficient. So let's say if I have to prove that if a sequence [-] is always increasing that the limit brings me to the sup. […] Yeah, like… Do you want… I don't want to prove it.

Shortly after saying this, S3 clarified that he did want to prove the statement, just not in the context of the interview. He claimed that it would "take too much energy" and that it would require him to "go open a book to just see the very rigorous definition of a sup." Hence, students may have been avoiding proof, not only because they did not want to do it, but also because they lacked the required level of fluency in this type of proof and relevant theories.

To our surprise, both S1 and S13 did not stop at proving that if $(x_n)_{n \in \mathbb{N}}$ is increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$: they also tried to prove that if $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, then $(x_n)_{n \in \mathbb{N}}$ is increasing. In other words, they both spontaneously chose to prove an equivalence. Almost immediately, S1 noticed that the second implication is not true: "Let's say that [the sequence] drops and comes back up. [-] So it doesn't have to be increasing." As a result, S1 decided to change his response to Task 3(a) completely: he replaced his condition of "increasing" with the condition that "no x_n can be greater than $\alpha = \sup A$" (i.e., $\forall n \in \mathbb{N}, x_n \leq \sup A$). S1 did not realize that, in doing this, he had exchanged a condition that is only sufficient (but not necessary) for one that is necessary (but not sufficient). S13, in contrast, clarified her condition to "$(x_n)_{n\in\mathbb{N}}$ is *eventually* increasing" and started to think of a possible proof approach for the statement: If $\lim_{n\to\infty} x_n = \sup\{x_n: n \in \mathbb{N}\}$, then $(x_n)_{n\in\mathbb{N}}$ is eventually increasing. She decided to prove the contrapositive statement, i.e., if $(x_n)_{n\in\mathbb{N}}$ is not eventually increasing, then $\lim_{n\to\infty} x_n \neq \sup\{x_n: n \in \mathbb{N}\}$:

> The idea is to find a subsequence of the, you know, of the agents, that it doesn't uh, that it's smaller than the neighbours, and show that if [-] that subsequence exists, then it cannot converge to your supremum. [-] It converges to a number smaller than the supremum, right? So then that doesn't exist. Yeah. I think I'm convincing myself, yeah.

While it was relatively easy for S13 to sketch a proof of the other implication (perhaps because she had proved it before, possibly several times), this implication required a bit more thought. At first, S13 was not sure it is true. But as she started to visualize sequences and devise a plan – i.e., find a subsequence that converges to something that is less than the supremum – she started to become more convinced. "I want to go home, write it down, and make sure," she said as she finished up with Task 3: "I'm sure that I can prove it."

Two other students (S8 and S12) also seemed to be in search of both necessary and sufficient conditions for $\lim_{n\to\infty} x_n = \sup\{x_n: n \in \mathbb{N}\}$. At first, S8 chose "increasing" and gave justifications that resembled some of the ones we outlined at the beginning of this section. He noted, for instance, that "if [$(x_n)_{n\in\mathbb{N}}$] decreases, then if you go along in the sequence, you wouldn't reach the supremum, that doesn't make sense." This said, S8 also expressed that he was bothered by his response because it did not take into consideration the cases where there are points in the sequence that break the increasing pattern. To illustrate what he meant, he sketched the points depicted in Figure 6.15. Then he struggled to think of a name for a condition that would take such cases into account: "There must be like a way of characterizing it, or labelling it. In the effort to be more precise, I guess. Yeah. Monotone, but not exactly. That's what I'll call it. For now." It is not clear if S8's "monotone, but not exactly" is the same as S13's "eventually increasing," or if it is closer to S12's characterization, which we outline in a later section (6.3.2.3).

Figure 6.15 Sketches by S8 (left) and S12 (right), which supported their choice to modify or reject the condition of "increasing" for Task 3(a).

In relation to the topic we are discussing here, it is important to note that S12 saw something that S13 did not immediately see, which led him to reject the "increasing" condition, and even the "eventually increasing" variation. At one point, S12 asked himself: "Does [the sequence] have to be increasing? I don't know if that's the case." Then he drew the sketch shown in Figure 6.15. "The magnitude of oscillation could decrease gradually and still reach [the supremum]," he explained, "it's not necessarily always increasing or decreasing." Based on this example, S12 concluded that "increasing" is not a necessary condition for all sequences $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. "Saying it's increasing just seems like an arbitrary choice," he said. Note that it is not too difficult to come up with an analytic expression for an example like the one S12 drew. In fact, the interviewer and S13 worked together at the end of her interview to generate and validate such an example, which showed that it would not be possible to prove that if $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, then $(x_n)_{n\in\mathbb{N}}$ is eventually increasing. The sequence was constructed by taking elements in alternation from $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ and $\left(\frac{2n}{2n+1}\right)_{n\in\mathbb{N}}$. The resulting sequence could be represented as:

$$x_n = \begin{cases} \dfrac{\left(\frac{n+1}{2}\right)}{\left(\frac{n+1}{2}\right)+1} & \text{if } n \text{ is odd} \\[2ex] \dfrac{2\left(\frac{n}{2}\right)}{2\left(\frac{n}{2}\right)+1} & \text{if } n \text{ is even} \end{cases}$$

S13 easily convinced herself that this sequence "goes up, then it goes down, then it goes up, then it goes down"; she concluded that she would need to think more about her solution to Task 3(a).

In sum, seven students (S2, S3, S4, S7, S9, S14, and S15) chose the condition “increasing,” but did not always make it clear if they saw it as a sufficient and/or necessary condition. Four students (S1, S8, S12, and S13) considered the “increasing” condition and either modified it (S8 and S13) or rejected it (S1 and S12) because they seemed to be aiming to characterize all sequences $(x_n)_{n\in\mathbb{N}}$ with the property $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. S8 chose the condition “increasing, but not exactly,” and S13 chose the condition “eventually increasing,” which she was led to question with the help of the interviewer. S1 chose “no x_n can be greater than $\alpha = \sup A$,” thereby replacing a condition that is only sufficient with one that is only necessary. S12 described his conditions in a different way, which will be discussed in 6.3.2.3 below. Before that, we outline how the participants in our study felt about accompanying the “increasing” condition with the condition of “bounded above.”

6.3.2.2 …and bounded above.

Participants did not agree on whether the conditions should be “increasing and bounded above,” or only “increasing.”

Three students (S3, S9, and S15) explicitly decided to accompany their “increasing” condition with the condition that $(x_n)_{n\in\mathbb{N}}$ (or A) should be bounded above. “If not, there’s no sup,” claimed S9: “That’s what I remember.” S3 and S15 also seemed to believe that the existence of $\sup A$ depends on A having an upper bound. They also argued that if A is not bounded above, then “it” will go to infinity, and the supremum of a set cannot be infinite. S15 explained more specifically that “if the set is increasing, [-] but not bounded, we don’t have a sup. Well, sup is infinite. But, it’s not in the real line. [-] I am assuming that the supremum is in the real line.” Both S3 and S15 insisted that the supremum is a finite real number. Moreover, their conviction seemed to be an integral part of their personal concept definition of $\sup A$ as a least upper bound. For instance, when the interviewer asked if $\sup A$ could be infinite, S3 said: “By the definition of a supremum, no. A supremum is a least upper bound. So, if it’s infinity, it’s not an upper bound. Infinity implies that there is no bound.”

Two students (S2 and S14) contemplated a seemingly different condition: i.e., that $\lim_{n\to\infty} x_n$ exists. Since convergent sequences are bounded, but not all bounded sequences are convergent, this condition is technically stronger than the one given by the students mentioned above. This said, when combined with the condition that $(x_n)_{n\in\mathbb{N}}$ should be increasing, the conditions

"$(x_n)_{n\in\mathbb{N}}$ bounded above" and "$\lim_{n\to\infty} x_n$ exists" are interchangeable. It is not surprising, then, that S2 was led to the "limit exists" condition in a very similar manner as the students in the above paragraph. She too made it clear that she wanted to avoid the cases where $\lim_{n\to\infty} x_n = \infty$. For instance, the first thing she did when presented with Task 3 was sketch a sequence that diverges to infinity and claim that, in such a case, $\lim_{n\to\infty} x_n \neq \sup A$. In reference to this sketch, she later specified that "if the limit does not exist, just like this one, then the supremum does not exist." In conversation with the interviewer, S2 also confirmed that she did not think that the supremum of a set could be infinite. Although S14 did not explicitly address the $\sup A = \infty$ case, his first thoughts upon seeing Task 3(a) were similar to S2's: i.e., the sequence should be convergent. Note also that the "limit exists" condition is implicit in the model sequence S14 drew to represent the "increasing" condition (see Figure 6.14).

Three students (S7, S8, and S13) chose "increasing and bounded above/limit exists," and then questioned the "bounded above/limit exists" condition. For S7 and S8, the rejection of the condition was almost immediate. S7's first words after reading Task 3(a) were: "So x_n would have to increase and would have to [-] converge to a certain limit. Or… [-] No, it could actually converge to infinity and it could be the sup." In a similar vein, S8 said: "So I think it has to be bounded for sure, as well. Um… Oh, does it though? No it doesn't have to be. Because you could have the supremum as infinity." Both students exemplified what they meant through the same example: $x_n = n$. It is possible, however, that they were not entirely convinced of their choice to reject the "bounded above" condition. "I would have to go back to my notes," S7 explained: "[-]. I would have to review, I'm not sure a hundred percent."

S13 was not as quick to abandon the "bounded above" condition; but she seemed to start moving towards that direction once she made explicit her personal concept definition of the supremum. After S13 provided a sketch of the proof that if $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ (see 6.3.2.1 for the details), the interviewer asked why the "bounded above" condition was important. The following exchange ensued:

S13: "Why it matters that it's bounded?

L: "Yeah."

S13: "Then it wouldn't be, uh, converging."

L: “Ok. Could the limit be infinite?”

S13: “Well that is what I, uh… When you have sup A, it means that it is not… To me, it means that it’s not infinite.” [-]

L: “Ok. What is your definition of supremum of A?”

S13: “Yeah. The biggest number… All the, all the, mm… All the, uh… What do you call it? All the, the agents, are less than the supremum of A. I understand that it can be infinite. But then I just separate the cases. I decide them separately.”

S13 seemed to know that if she did not have the “bounded above” condition on $(x_n)_{n\in\mathbb{N}}$, then the sequence would not converge. This would not align with her proof (described in 6.3.2.1), in which she showed that an increasing bounded above sequence $(x_n)_{n\in\mathbb{N}}$ converges to a real number $M = \sup\{x_n : n \in \mathbb{N}\}$. Notice that S13 did not directly point to the expected use of the “bounded above” condition in her proof, which allows one to invoke the Completeness Axiom of $\mathbb{R}$ to conclude that $\sup A = M \in \mathbb{R}$ exists. Notice also that S13’s personal concept definition of the supremum – i.e., all the elements of A are less than $\sup A$ – does not exclude the possibility that $\sup A = \infty$. This may be why she decided, in the end, that the supremum could be infinite, and that she needed to deal with this case separately. She did not demonstrate how she would do this during the interview.

S1 was the only student to explicitly consider and allow the case $\sup A = \infty$ from the beginning. He agreed with S13 that this case should be treated separately:

> If this [i.e., $(x_n)_{n\in\mathbb{N}}$] goes to infinity, then obviously it’s the greatest… I don’t know how to say it… [-] I feel like it’s only worth proving if [the sequence] converges, cause if it diverges it’s natural that [the limit] will be the supremum of A. [-] If it diverges, then of course it’s going to be the greatest number.

Like S13, S1 seemed to struggle to verbalize his concept definition of the supremum, which he decided to call “the greatest number.” It is possible that S1’s struggle was linked to the contradiction between wanting to allow $\sup A = \infty$, but not wanting to call infinity a “number.” Whatever the case, the possibility of an infinite supremum was part of S1’s concept definition, unlike those students who saw the supremum as a “bound,” and a bound as a “finite real number.” S1 claimed that when $\lim_{n\to\infty} x_n = \infty$, then it is obvious that $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$; no proof is required.

There was one student, S4, who discussed the "increasing" condition, but did not give any trace, during his solution to Task 3, of the "bounded" condition. S4, along with the other five students who were not mentioned in this section (i.e., S5, S6, S10, S11, and S12) stood out to us as acting in particularly unexpected ways when presented with Task 3. In comparison with most students discussed above, we could characterize these six students by saying that they did not have the reflex to choose the "increasing" condition and/or they offered conditions of a completely different kind. We describe these conditions in the following sections. In terms of the "bounded above" condition, these students acted with the same diversity as the students discussed in this section: i.e., they either did not consider it at all (S4 and S5), tacitly included it (S6, S10, and S12), or explicitly included it (S11).

6.3.2.3 The sequence should approach an asymptote from below.

Two of the students (S6 and S12) who did not immediately think of the "increasing" condition chose to characterize the sequences $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ by thinking of the supremum as an asymptote that the sequence approaches from below. Both students represented their characterization graphically, as shown in Figure 6.16. Notice that these images look a lot like the one produced by S14 (see Figure 6.14). In particular, they tacitly include the condition that the sequence should be bounded above. The difference is that S6 did not mention the "increasing" condition; and, as discussed in 6.3.2.1, S12 chose against it. While making her drawing, S6 simply said: "Any function that has [-] [an] asymptote. That's all I can think of. So any function that, you know, like goes like this." As will be further discussed in 6.3.2.7, S6 required a lot of prompting from the interviewer to get to this point. With more time, she may have formulated a condition like "increasing," which her image seems to imply. Alternatively, she may have provided a more nuanced interpretation of her image, like S12. As he drew his image, S12 explained that

> for it to be an asymptote, it means that as this function continues along infinity, it will be getting closer and closer to this thing, to the asymptote. But it also means that, for it to be the supremum, it can never go beyond it, it can never be above it. So it can have any kind of behaviour here [at the beginning]. But eventually it needs to flatten out around this [the asymptote] um... without going above it. It can't overshoot.

Recall that S12 eventually considered the condition of "increasing" and rejected it because he could imagine an oscillating sequence that approaches the asymptote (see Figure 6.15). In other words, S12 knew that the sequence need not be eventually increasing, as is the case for the sequence depicted in his initial drawing (in Figure 6.16). By the end of the time S12 spent working on Task 3(a) in the interview, the way he formulated his conditions had not changed much from the explanation given above. He decided that to characterize $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, he would say that "it approaches [an] asymptote from below. It's rising up until it reaches this asymptote. [-] And at no point in the function is it ever greater than that value."

Figure 6.16 Sketches by S6 (left) and S12 (right) to explain their response to Task 3(a).

Note that in between his initial explanation of his drawing and his last formulation of conditions, S12 attempted to write out the conditions more formally, but was unable to do so. In particular, and as evidenced by the additional markings on his drawing in Figure 6.16, S12 attempted to recall the formal definition of sequence convergence. He first recalled the importance of a certain x_N such that all the ones that come after are bounded in a certain zone. As shown in Figure 6.17, he also recalled the beginning of the formal definition and, with the additional sketch, described it as

> a game such that no matter what epsilon I'm given, I can find some value N along this series in which, um, all… [-] I can find points, a set of points, that are all greater than N that are bound within the limits of ϵ.

Despite all the understanding S12 seemed to demonstrate, he was unable to write out the rest of the formal definition in the context of the interview. He claimed that he wished his approach to the task was "more related to like the conventions of set theory," but that he "never really picked up the vocabulary or the terminology well for it." One other student, whose actions are described in the next section, had developed a similar interest in breaking things down into the formal language of set theory. As will be shown, he had also developed a greater fluency in the language.

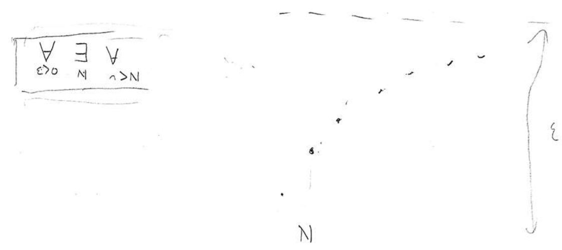

Figure 6.17 What S12 remembered about the formal definition of sequence convergence.

6.3.2.4 $(\forall \boldsymbol{\epsilon} > \mathbf{0}, \exists \mathbf{N} \in \mathbb{N}, \forall \mathbf{n} \geq \mathbf{N}, \sup \mathbf{A} - \boldsymbol{\epsilon} < \mathbf{x_n}) \wedge (\forall \mathbf{n} \in \mathbb{N}, \mathbf{x_n} \leq \sup \mathbf{A})$

When faced with Task 3(a) – under what conditions is $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ – S10 was the only participant to provide solely formal conditions; in particular, he did not have the reflex to consider the conditions of "increasing" and "bounded above," as expected. He started by trying to recall two formal conditions that would ensure that "$\sup A$" is the supremum of the set $A = \{x_n : n \in \mathbb{N}\}$. With the support of the small sketch in Figure 6.18, he described the first condition, both verbally and in written symbols: "So for any epsilon greater than zero, there's going to have to be an element of A that's greater than supremum minus epsilon, but it's also less than or equal to the supremum," i.e., (1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \sup A - \epsilon < x_N \leq \sup A$. S10 continued: "And I think the second condition is just that the supremum is an upper bound of the set," i.e., (2) $\forall n \in \mathbb{N}, x_n \leq \sup A$. Assuming that $\sup A \neq \infty$, notice that conditions (1) and (2) are necessary, but not sufficient to have $\lim_{n\to\infty} x_n = \sup A$ where $A = \{x_n : n \in \mathbb{N}\}$. S10 did not realize this immediately; he decided to move on to part (b).

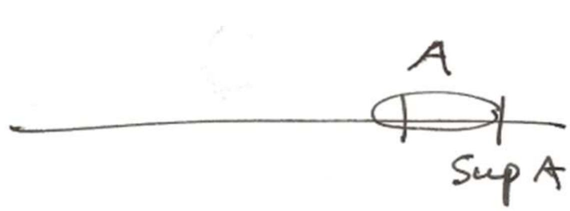

Figure 6.18 S10's memory aid for recalling the formal definition of the supremum.

The interviewer seemed to (unintentionally) alert S10 to the insufficiency of his chosen conditions when she asked him to explain more about how he "verified that if these conditions hold, then we for sure have that the limit is equal to the supremum." "So now, now that you ask that, I'm questioning whether these are sufficient," he responded: "[-] I'm just realizing that this was a stronger claim than I thought initially." S10 eventually started to wonder if he was missing something:

> What I'm wondering now is if we have to add this condition actually: [-] for any epsilon, hold this n constant and then say that this [i.e., $\sup A - \epsilon < x_n \leq \sup A$] is actually true for all natural numbers larger and equal to it, for this to be the case. Um… […] But, I actually don't think that I need that condition maybe.

S10 was correct in thinking that to characterize $\lim_{n\to\infty} x_n = \sup A$, where $A = \{x_n : n \in \mathbb{N}\}$, he could simply modify his first condition to be: (1) $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \sup A - \epsilon < x_n \leq \sup A$. At the time of solving Task 3(a), S10 seemed unsure if this modification was needed.

It is not entirely clear why S10 doubted the need to modify his initial conditions. One possible reason is that he may have been operating under the additional assumption that the limit, $\lim_{n\to\infty} x_n$, exists. He claimed, for instance, that he would need to add the additional condition described above if the sequence "doesn't have a limit. But it does have a limit." Mathematically speaking, if one assumes that $\lim_{n\to\infty} x_n$ exists, then the first version of condition (1) offered by S10 is sufficient to show that there is a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to $\sup A$, whereby the sequence also converges to $\sup A$. S10 did not provide any additional reasoning of this sort. Nevertheless, by the time he had progressed to Task 4(b), his actions seemed to reflect an acceptance of the modified version of (1). Indeed, when faced with the particular set $A = \left\{x_n = \frac{n}{n+1} : n \in \mathbb{N}\right\}$, and the task of showing that $\sup A = 1 = \lim_{n\to\infty} x_n$, S10 was able to reduce his work to showing:

1. $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - 1| < \epsilon$; and
2. $\forall n \in \mathbb{N}, x_n \leq 1$.

We return to the details of S10's solution to Task 4 in 6.4.2 (see 6.4.2.3.3). In the context of the interview, S10 did not clearly verbalize if his actions in Task 4 changed his mind about his conditions in Task 3. While solving Task 3, he said that he might try to prove the conditions he chose by going back to the definition of a supremum as a least upper bound. He suspected, however, that this would take him several pages of work. Hence, he did not try in the interview.

6.3.2.5 The sequence should be bounded above and convergent.

S11 was the only student to choose the two conditions: bounded above and convergent. His choice of the "bounded above" condition was almost immediate. When the interviewer later asked about how he knew this condition, S11's discourse resembled the explanations given by the students mentioned at the beginning of 6.3.2.2. He said that

> the supremum of A is actually a number, right? For it to reach a certain point, you need to have a bound. That's why I'm saying it has to be bounded above. Cause if it goes to infinity, then it doesn't really work.

Like his peers, S11 seemed to be suggesting that if the set $A = \{x_n : n \in \mathbb{N}\}$ is not bounded above, then "it" will go to infinity; and the supremum of A cannot be infinity since it has to be a "number." In other words, S11 chose the "bounded above" condition to avoid the $\sup A = \infty$ case.

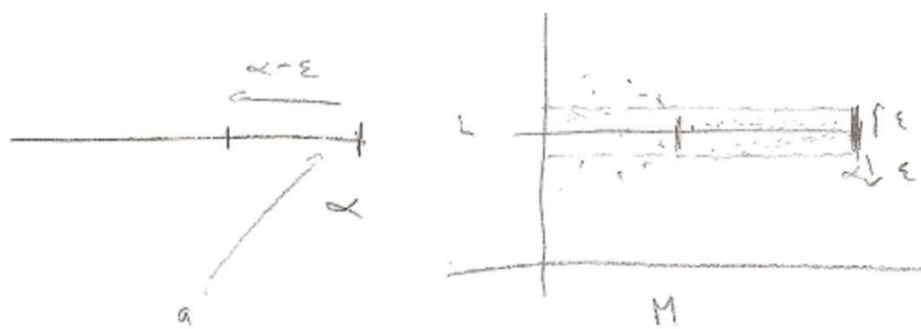

Figure 6.19 Two sketches made by S11 in search of conditions for Task 3(a).

S11's choice of the "convergent" condition was much less automatic. In search of other conditions, he provided sketches (shown in Figure 6.19) and exhibited discourses that were very similar to the students mentioned in the previous two sections. Like S10 (see Figure 6.18), S11 drew the sketch on the left in Figure 6.19 and said that if you can find an element a in the interval he depicted, then the supremum of the set A would be equal to α. Like S12 (see Figures 6.16 and 6.17), S11 also drew the sketch on the right in Figure 6.19 and tried to recall the formal definition of limit convergence. The difference was that S11 seemed to be thinking about the meaning of each side of the equality $\lim_{n\to\infty} x_n = \sup A$ on its own; he was not (yet) thinking about the meaning of the equality itself. Consider, for instance, how the sequence depicted in the sketch on the right in Figure 6.19 has a limit that exists, but does not satisfy the equality $\lim_{n\to\infty} x_n = \sup A$.

When the interviewer asked if he was thinking of another condition through these images, S11 said:

> Well there has to be the M that I'm talking about. [-] Like let's say your sequence minus the limit, it has to be like between the bounds of that epsilon. [-] Like it will also get closer to the limit basically. [-] So I'm guessing it also has to be convergent?

S11 decided that the sequence "has to converge. And it has to be bounded above. But like one means the other kind of." In saying this, S11 seemed to have an inkling that the two conditions he had chosen – the sequence should be convergent and the sequence should be bounded above – are not entirely independent of one another. It is not clear if S11 knew that convergent sequences are bounded. He explained, rather, that if the sequence is not bounded above, then "you're diverging to infinity, I think." What S11 did not seem to realize was that the conditions he gave are not sufficient for the equality $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ to hold. Indeed, it is easy to show examples of convergent (and bounded above) sequences that do not converge to their supremum.

6.3.2.6 All subsequences should converge to the supremum.

Before S4 started explicitly mentioning the monotonicity of the sequence $(x_n)_{n\in\mathbb{N}}$, he recalled a theorem that he learned in class: namely, that a sequence converges to a number M if and only if all of its subsequences also converge to M. S4 described his condition for $\lim_{n\to\infty} x_n = \sup A$, where $A = \{x_n : n \in \mathbb{N}\}$, as follows:

> You could technically make subsequences out of the sequence x_n and find the limit of each, [-] and if they would all be equal to the supremum of A, then that would mean that the limit of the sequence x_n would be the supremum of A.

Notice that S4's description reflects more a process (i.e., checking that all subsequences converge to $\sup A$) than an abstract theory (i.e., if all subsequences converge to $\sup A$, then…). We suspect that this enabled him to recognize his condition as "really complicated," and ultimately "not practical." He also recalled his professor telling him that "because there are unlimited amounts of subsequences for most sequences," the theorem he recalled is used in solving a different type of task: rather than proving the limit of a sequence, it is used to show that a sequence does not converge. "So that's why I said that it could be done, but it's just not practical," S4 concluded. In comparison with some of the other behaviour that we observed, we found S4's behaviour particularly interesting. It is perhaps because he did not immediately recognize Task 3 as a type he knew how to solve that he brought to mind a totally unexpected condition, and was able to judge the practicality of it.

6.3.2.7 I don't know how to solve tasks of this type.

The only student whose solution has not been mentioned up until this point is S5. S5's behaviour seemed to reflect not only a weak memory, but also the recognition of Task 3 as not being one of the types of tasks that she studied in RA I. Upon seeing Task 3, S5 immediately claimed: "I don't remember anything about sup. [-] Yeah, all of this stuff: I don't remember at all." Then, after reading the statement of the task more carefully, she asked: "Are you asking to find a sequence? I'm not sure what the question is asking for." Four other students (S2, S4, S6, and S11) also expressed uncertainty about what they were supposed to do to solve Task 3(a). S11 said: "It just seems too simple. Like I don't exactly know what to answer." "I get it, you know, kind of," claimed S6: "[-] But then what do they want specifically?" In a similar vein, S4 asked the interviewer for clarification: "What do you mean by 'under what conditions'?" And S2 sought the interviewer's input on the specific kind of conditions she should provide. She inquired, for instance, if she needed to give conditions that included things like "$\delta > 0$" and "$\epsilon > 0$."

Through interactions with the interviewer, S2, S4, S6, and S11 eventually discussed different types of conditions, which we have outlined in the previous sections. This said, they were not always confident in these conditions, or even aware that they had solved the task. For instance, once S6 seemed to be nearing the conclusion that any sequence $(x_n)_{n\in\mathbb{N}}$ that approaches an asymptote from below would satisfy $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$, she started to express doubts: "Under what conditions? I wouldn't know how to like name them specifically. This is just like, you know, like I'm desperate and I'm trying to get points." S4, in comparison, exhibited a clear awareness that a monotonic increasing sequence converges to its supremum, but he did not see this awareness as useful in solving the given task: "Nowhere does it say that it's monotonic increasing. [-] Personally, when I do exercises, I don't like to assume information that isn't given." S4's comments remind us that the students may have been most habituated with using given information to make deductions, rather than providing the information needed to make deductions. More generally, these students seemed inexperienced and uncomfortable with trying to characterize mathematical objects having certain properties.

S5 was the only student to not discuss any conditions. After the interviewer tried to reformulate Task 3(a), S5 said: "I'm just confused. [-] I'm still not sure what to do." Then, during her attempt at solving Task 3(b), S5 began to reveal that her struggles may have been coming from the way in which she was hoping to solve the given task: that is, by recalling a common task type from RA I, and trying to apply the corresponding technique. In response to Task 3(b) – give an example of $A = \{x_n : n \in \mathbb{N}\}$ such that the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$ – S5 said:

> I know there's one example that we used a lot [she wrote $\frac{1}{n} + \sup A$]. But that's like not what we're asking for here. We want something that does not equal $\sup A$. [-] I honestly don't know how to do it.

The interviewer attempted to remind S5 of assignment activities that had invited her to "give an example," and inquired if she remembered how she went about solving such tasks. In her response, S5 revealed that, in relation to Task 3, she had a stronger memory of a different type of task:

> S5: "I know usually we would use the Archimedean Property to solve these problems. [-] But like, usually for these types of problems, we would have a sequence, like written in form, like in this form."
>
> L: "Ok. An actual sequence."
>
> S5: "Yeah, exactly. And then we would have to plug in values and see what the sup and inf is. And then we would have to go about proving the inf or the sup."

From this exchange, it seemed that Task 3 led S5 to think about tasks of the type: Find the infimum and supremum of a set $A = \{x_n : n \in \mathbb{N}\}$, where $(x_n)_{n\in\mathbb{N}}$ is a specified sequence. This is not surprising, since Task 3 is indeed related to this task type. Nevertheless, it seemed that S5 was drawn more to engaging in this task type (i.e., employing a practical block she had learned in RA I), rather than engaging with Task 3 (i.e., building and exploring a corresponding theoretical block).

Other students also deemed Task 3(b) as a type that they did not know how to solve. When S11 got stuck in finding an appropriate example, he claimed: "I know the theory around this. I just don't.... [-] I'm used to like numerical problems." After being presented with Task 3, S6 said: "I've seen this. You know, where we learned about like sequences, and you know, like finding their sup and inf"; then, when she started to attempt part (b), she continued her line of thought:

I understand the question. But then I don't know how to approach it. Like, I don't know where to start from. [-] It's either like I don't study properly, so I miss out on the details that I would need. Or I just don't have it. Like, you know, I've never seen it before.

S6 claimed that she did not know how to solve Task 3(b), either because she missed something in her studies, or because she had never seen it before. Recall that Task 3(b) simply asks for an example of $A = \{x_n : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$. With support from the interviewer, S6 eventually displayed an ability to analyze $x_n = \frac{1}{n}$ as such an example. It seemed that not knowing how to immediately approach the task put S6 and S5 in a position of discomfort that inhibited them from being able to make progress in solving the task. They may not have been alone. S14, for example, seemed overwhelmed and frustrated after receiving Task 3. He sighed heavily and said: "So I should be killing this. I got like a ninety-six on the final."

6.3.2.8 Observations for Task 3(b): Giving an example.

Four students (S2, S4, S5, and S11), all mentioned in the previous section, did not succeed in solving Task 3(b). They each offered one candidate for a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n$ exists and does not equal $\sup\{x_n : n \in \mathbb{N}\}$:

S2: $x_n = \sin(n)$;
S4: $x_n = -n$;
S5: $x_n = \frac{1}{n} + \sup A$; and
S11: $0, \frac{1}{2}, 2, 0, \frac{1}{4}, 2, 0, \ldots.$

S2, S5, and S11 each realized that their candidates did not satisfy the required conditions; but they did not offer any other options. All three expressed a lack of confidence in their ability to come up with the required example. S2, for instance, asked the interviewer: "Do I need to find one? Because I don't think I can." As alluded to in the previous section, these students may have been stuck partially because they (thought that they) had not developed techniques for solving the given task type. Another, potentially related, possibility is that they lacked fluency in sequences. Notice that the sequences proposed by S5 and S11 each contain the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$, which would solve Task 3(b); and S2 did not know if the limit of $(\sin(n))_{n\in\mathbb{N}}$ existed or not without the help of Google. In the case of S4, the issue seemed more a lack of attention to detail. He purposefully chose $x_n = -n$

because it is a "decreasing function," and was able to coherently explain that $\lim_{n\to\infty} x_n = -\infty \neq -1 = \sup\{x_n : n \in \mathbb{N}\}$. He even noted that the limit does not exist. It seems he simply did not realize that this meant his candidate would not solve the task.

The remaining eleven participants correctly solved Task 3(b). Eight of them chose the expected example, $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$, and three of them chose other examples: $x_n = \frac{1}{n-3}, n = 4,5,6,...$ (S13), $x_n = \sin\left(\frac{1}{n}\right)$ (S8), and $x_n = \frac{\sin(n)}{n}$ (S3). Most of these students seemed to find the task easy to solve, and none of them provided formal proofs for all of their statements. Perhaps the most formal arguments were provided by S10 and S12, both of whom explained that 1 is the maximum of the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (i.e., $\frac{1}{n} \leq 1, \forall n \in \mathbb{N}$ and $1 \in A$), whereby it is the supremum. S10 mentioned that, in RA I, he had proved and used the property: the maximum of a set is the supremum of the set. He also described the main idea of the proof he had constructed: i.e., "There can't be any upper bound that's lower than that maximum. Because that would mean that the maximum was not part of the set." Nonetheless, when it came to the limit of $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$, S10 simply stated it to be zero, and S12 wrote $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$, while citing a memory from Calculus.

Just like for Task 3(a), there could be several reasons why students did not choose to provide formal proofs of their statements for Task 3(b). One possible reason is that the statements were obvious. This seemed to be the case for S13. After choosing $x_n = \frac{1}{n-3}$, she sketched the image shown in Figure 6.20 and explained:

> You see, this function is like this. [-] So the supremum of this is one. [-] I can see that it's one. But then the limit goes to zero. [-] I can mathematically prove it also. But, yeah. I feel it, I see it, I understand it.

Given S13's actions throughout the rest of the interview, we do not doubt her claim that she could provide mathematical proofs of her statements. She did not need to reproduce these to convince herself of something like $\lim_{n\to\infty} \frac{1}{n-3} = 0$, for which she also possessed simple graphical and intuitive explanations: e.g., "You have one apple, you want to divide it between billions of people, they almost get nothing."

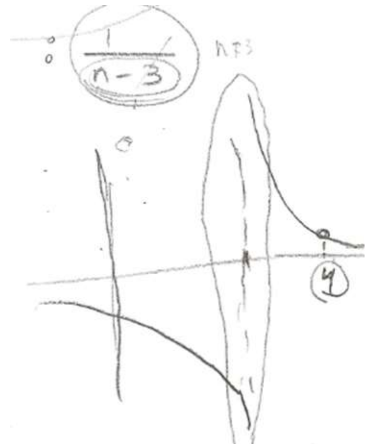

Figure 6.20 The sketch used by S13 to explain her example for Task 3(b).

It seemed other students would agree with S13 that calculating the limits and suprema of simple sequences could be done intuitively. But there were other reasons why students seemed to be avoiding formal proof. As mentioned for Task 3(a), we suspect that some students' actions were guided by the specific statement of the task, which did not explicitly ask them to "prove." "You just have to come up with an example," said S7, as he reflected on why he did not provide proofs. S7 validated the candidate $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ for Task 3(b) by saying: "For n is equal to one, you get one. And then it just decreases. And the limit of x_n as n tends to infinity is just zero. So your sup would be one. But the limit of x_n would be zero." Several other students provided similar types of arguments. Had the statement of Task 3(b) been phrased differently, perhaps more students would have attempted to provide formal proofs. S1, for example, provided formal proofs in Task 4, the statement of which explicitly requests students to "prove." He did not, however, prove his statements in Task 3(b). In reflecting on this, he explained that "it's just by reflex: When we're asked to 'Give an example…' we don't prove it. So I didn't prove it. But if I was asked to prove, I would have used the same thing [as in Task 4]." In 6.4.2, we describe how participants behaved for "prove that…" tasks related to limits and suprema of sequences. To our surprise, most participants actually did not react to the phrasing "prove that…" by providing formal proofs.

A last potential reason why students did not construct more formal arguments was that they did not want to and/or did not think they could, even if they felt they should. S6, for example, eventually solved Task 3(b) by writing out the first few terms of $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ – $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ – and noticing that "it gets smaller and smaller," whereby the supremum is one. This, she claimed, was not “the right way” to approach the task: “You’d have to like prove using the definition. Which like, I wasn’t really comfortable with.” S9 solved the task in a similar manner to S6, and then he addressed the interviewer with a similar concern: “Maybe you’re going to think I went fast on this. [-] I’m not going to do the definition.” It seems that S9 was feeling some pressure to provide some sort of formal proof. “That’s the whole RA I course,” he explained: “[-] Every statement, you had to back it up.” When it came to tasks about finding/proving limits, like Task 3(b), S9 was particularly frustrated by this:

> At first I was like: “Oh yeah, I remember how to do this.” [-] And I remember enjoying that in Calculus. And then it was no longer enjoyable in Analysis when we had to use like epsilon and stuff. [-] That’s what I didn’t like about the Analysis course: It’s like relearning stuff.

6.3.3 Reflections on the Nature of Practices, Positioning, and Activities[34]

Non-Mathematical: Seek to identify task with technique in an automatic fashion.

VS.

Mathematical: Be open to a nonroutine task about mathematical theory,
to studying the task and the properties of the objects involved in the task,
to coming up with (possibly wrong) new ideas on-the-spot.

It seemed some participants had developed non-mathematical practices in the sense that they relied on being able to automatically identify a given task with a well-practiced technique. These participants struggled to engage with both parts of Task 3 because they were unsolvable through a routinized praxis. We only observed this behaviour among participants who seemed to strongly occupy a Student position. This makes sense since (a) the Student is characterized by an aim of mastering the minimal core of the practices to be learned in a course; and (b) Task 3 was related to building (and possibly questioning and verifying) a theoretical discourse underlying commonly

[34] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.3.1 and 6.3.2 that begin to address our research objectives: characterizing the nature (mathematical or not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

tested techniques. We think that for these participants, the minimal core included the techniques, but not the discourses.

Some of these participants (S2, S6, S4, and S11) demonstrated a potential to be in transition towards mathematical practices in the sense depicted in the above box. With significant prompting from the interviewer, they tried to study the task and come up with ideas on-the-spot. S11, for instance, tried to recall the definitions of sequence convergence and suprema to see if they would help him propose conditions for $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. S4 and S6 thought of unexpected conditions: i.e., "all subsequences of $(x_n)_{n\in\mathbb{N}}$ must converge to the supremum" and "the sequence $(x_n)_{n\in\mathbb{N}}$ must have an asymptote," respectively. In general, however, these students seemed to be uncomfortable with the given task and lacked the confidence, tools, and/or strategies needed to make progress in solving it. We predict that posing more "devise conditions" tasks could support students in shifting to practices that are more mathematical in nature. Such tasks can show students that it is not always possible to solve a task through highly routinized or algorithmic techniques. They can also expose students to a type of task that is, by nature, "very mathematical": i.e., that focusses on the building, questioning, and verifying of mathematical theory.

Most participants seemed to have developed mathematical practices in the sense that they were open to engaging with a nonroutine task about mathematical theory. In terms of positioning, we observed more interesting distinctions in the kinds of explanations students gave for their chosen conditions.

Non-Mathematical: Theoretical discourses
based on established experiences, inert knowledge,
and/or taking one's own understanding for granted.

VS.

Mathematical: Theoretical discourses
based on clarifying, questioning, and verifying one's own understanding.

Most participants seemed to have developed non-mathematical practices in the sense that their theoretical discourses were based primarily on inert knowledge, tacit assumptions, and/or taking their own understanding for granted.

Consider, for example, the participants (e.g., S2, S7, and S15) who justified "increasing" as a condition for $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ by considering paradigmatic examples of "non-increasing" sequences. Coming up with examples and non-examples is an important part of mathematics. What made the students' practices non-mathematical was their lack of attention to and assessment of how they were using the examples: they did not try to clarify the statement they were seeking to justify and/or they did not question if the justifications they were giving really served to "justify" that statement. We suspect that these participants were already convinced of their conditions based on citing or knowing the statement of the Monotone Sequence Theorem; and the examples they gave served more to confirm this established experience or inert knowledge than to clarify, question, or verify it. We think this behaviour is linked to the Student position for at least two reasons. First, the Student aims to get as many points as possible on exams, which often corresponds to writing down an answer as quickly as possible and moving on to the next question (without fully clarifying, questioning, or verifying one's own understanding). Second, in the RA I course, the Student may have perceived the minimal core of practices to be learned as including routinized techniques for solving particular tasks, with some acknowledgement of underlying theory; but not necessarily the building, questioning, and verifying of that theory.

Next, consider the participants (e.g., S3, S4, S9, and S14) who justified "increasing" as a condition for $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ based on intuitive or informal explanations: for instance, drawing a graph (like the one in Figure 6.14) or saying something like "an increasing sequence means that it will grow continuously until it reaches its maximal value." Once again, it is important to note that thinking intuitively and informally is an essential part of mathematics and that some of the participants (S3 in particular) exhibited a very deep intuitive understanding of many concepts. What made the students' practices non-mathematical was that their justifications seemed to be based on tacit assumptions and/or taking their own understanding for granted. They offered justifications that confirmed their own thinking, but they did not question their thinking: e.g., Am I making some additional assumptions here? Are there other ways that a sequence "increases" or "grows"? What do I mean when I say "growing continuously" or "reaching its maximal value"? What do these things mean in mathematical terms? Could I prove this? What statement am I trying to prove? Does it solve the proposed task? For similar reasons mentioned above, we think this behaviour might be linked to the Student position. We also think it could be linked to the positions of Learner and Skeptic, depending on the nature of the task. Recall that the Learner is characterized by seeking personal understandings based on their own thoughts and whims. Hence, a participant who occupied a position of Learner in a course (e.g., S3 in RA I) might have developed a very deep personal understanding of a topic, but may struggle to go beyond that understanding in cases where they lack the motivation or tools for doing so. The Skeptic, in comparison, is characterized by questioning what they perceive to be the practices to be learned. Hence, a participant who occupied a position of Skeptic in a course (e.g., S9 in RA I) might not be convinced that they should go beyond their own personal understandings in cases where such understandings seem to be sufficient for solving the task.

It seems that being able to transition to practices that are more mathematical in nature in the sense depicted in the above box requires not only the tools to be able to clarify, question, and verify, but also the sensitivity to which situations might require more (or less) clarification, questioning, and verification. Mathematicians would probably consider it reasonable to make a quick intuitive or informal argument in the situation of giving a specific simple example. Several of the students mentioned above did this when solving Task 3(b); and they did it well. It seemed some of them did not see why a more intuitive or informal approach might be sufficient for Task 3(b), but not necessarily for a "devise conditions" task like Task 3(a). This behaviour seems naturally linked to the Student (whose decision to be more precise and formal is based on whether they think their professor has asked them to do it that way or not) and the Skeptic (who fights against precision and formality unless they are convinced that it is necessary). To support such students in the transition, we think it is necessary to not only pose activities that demonstrate the limitations of imprecise and informal thinking, but also to invite students to think more explicitly about situations where it is necessary and situations where it might not be necessary to argue in a more precise and formal manner.

There were some participants who seemed to have learned practices that were more mathematical in nature in that their theoretical discourses were based on clarifying, questioning, and/or verifying their own understanding. There were, for instance, several students (S1, S8, S12, and S13) who (a) questioned the "increasing" condition by thinking up examples of sequences $(x_n)_{n\in\mathbb{N}}$ that break the increasing pattern but still satisfy $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$; and (b) engaged in trying to more precisely clarify another condition. Some of these students also sought to verify their understanding in a more formal manner, either by constructing a proof or thinking about their conditions in terms of formal definitions. As alluded to above, we think the desire to prove or to think in formal terms could be linked with the Student (in cases where they think proof and/or formality would be necessary to get full marks) or the Learner (in cases where they want to further their personal understanding by thinking in more precise and formal terms). In the case of Task 3, developing and supporting personal understanding confidently, clearly, and easily with formal proof or formal language seemed strongly linked to the positions of Mathematician in Training (i.e., S13) or Enthusiast (i.e., S10); in particular, going above and beyond what was expected in the assessment activities offered in RA I.

6.4 Task 4

Let $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \dots\right\}$.

a) Prove that for any $p > 1$*,* A *is unbounded above.*

b) Prove that for $p = 1$*,* $\sup A = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$.

6.4.1 Practices to be Learned[35]

$A = \left\{\frac{n^p}{n+1} : n = 1, 2, \dots\right\}$.

a) Prove that for any $p > 1$, A is unbounded above:

For any $M > 0$, we can find an n such that $\frac{n^p}{n+1} > M$.

Since $\frac{n^p}{n+1} > \frac{n^p}{2n}$ for $n > 1$, it is enough to find n such that $\frac{n^p}{2n} > M$ or $\frac{n^{p-1}}{2} > M$ or $n > (2M)^{\frac{1}{p-1}}$.

Its existence follows by the unboundedness of $\mathbb{N}$. ■

Figure 6.21 An example of the kind of solution anticipated for Task 4(a).

When solving Task 4, we would expect a successful RA I student to attempt to provide formal proofs. Our analysis of assessment activities suggests that the student would be able to prove that a given set A has an infinite supremum for a specified p (e.g., $p = 2$). Mimicking a corresponding solution, we obtain the work shown in Figure 6.21, which is also a solution to Task 4(a). To get to

[35] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 4. This model is synthesized in table form in Appendix E.

this solution, a student would need to connect the given task (about the unboundedness of a set), to a type they know well, but that is phrased differently: i.e., Find the supremum and infimum of a specified set. We thought this connection might be facilitated by the exploration encouraged by Task 3, as well as the similarity between the given set A and the majority of sets in the assessment activities we analyzed (mostly rational sequences comprised of polynomials of low degree). It is also possible that a successful RA I student would make the connection by recalling the definition of infinite supremum from their textbook: namely, a set is said to have an infinite supremum if and only if it is not bounded above, i.e., if it is unbounded above. If needed, we expected the student to be able to obtain the formal definition of unbounded above by negating the definition of bounded above: i.e.,

A is bounded above $\Leftrightarrow \exists B \in \mathbb{R}, \forall a \in A, a \leq B$;

thus, A is unbounded above (i.e., not bounded above) $\Leftrightarrow \forall B \in \mathbb{R}, \exists a \in A, a > B$.

Indeed, one path of activities we characterized required the replication of formal statements like the one shown above and/or writing their negation. A version of this definition of unbounded above, which limits B to positive real numbers, remains implicit in the solution in Figure 6.21. To receive full marks, a student would be expected to clearly indicate what they are looking for (i.e., an n such that $\frac{n^p}{n+1} > M$ for fixed $M > 0$), and then they would manipulate inequalities until being able to claim the existence of n based on a taken for granted fact: i.e., $\mathbb{N}$ is unbounded.

We can model the practice to be learned in RA I and most relevant to Task 4(a) as shown in Table 6.9 below. Notice that Task 4 would not only appear to a successful RA I student as being of a slightly different form than T_{4_a}; it would also seem more general due to the inclusion of the variable p. In all the assessment activities we analyzed, the rational sequence, $\frac{r(n)}{s(n)}$, was composed of specified polynomials. Moreover, whenever the sequence had an infinite supremum, the degrees of the polynomials satisfied the relation: $\deg(r(n)) = \deg(s(n)) + 1$. Consider, for instance, the following sets that showed up within assignment and past midterm activities that asked students to find the supremum and infimum of the set:

$$\left\{\frac{n^2+1}{n} : n = 1, 2, \ldots\right\} \text{ and } \left\{\frac{n^5+25}{n^4+4} : n = 1, 2, \ldots\right\}.$$

Note that once a student identified the supremum to be ∞, they would then engage in a proof (i.e., in solving a task of type T_{4_a}). Having worked with such specified sequences in RA I, we thought a successful RA I student might struggle with the implementation of τ_{4_a} in Task 4. In particular, we were interested in seeing how participants would deal with the generality of p.

T_{4_a}: Prove that $\sup A = \infty$, where $A = \left\{\frac{r(n)}{s(n)} : n = 1, 2, ...\right\}$ and $r(n)$ and $s(n)$ are specified polynomials with $\deg(r(n)) > \deg(s(n))$.
τ_{4_a}: For $M > 0$, show that there exists an n such that $\frac{r(n)}{s(n)} > M$. Trick: find a sequence smaller than $\frac{r(n)}{s(n)}$ that will make the inequality easier to manipulate. Manipulate the inequality until finding something of the form $n > f(M)$. Conclude the existence of n from the unboundedness of $\mathbb{N}$. Commonly tested was $\frac{r(n)}{s(n)} = \frac{n^{p+1}+c}{n^p+d}$, where $c, d \in \mathbb{R}$ and $p \in \mathbb{N}$: For any $M > 0$, we can find an n such that $\frac{n^{p+1}+c}{n^p+d} > M$. Since $\frac{n^{p+1}+c}{n^p+d} > \frac{n^{p+1}}{2n^p}$ for large enough n, it is enough to find n such that $\frac{n}{2} > M$ or $n > 2M$. Its existence follows from the unboundedness of $\mathbb{N}$. $\blacksquare$
θ_{4_a}: Implicit: $\sup A = \infty \Leftrightarrow A$ is not bounded above $\Leftrightarrow \neg(\exists M, \forall a \in A, a \leq M)$ $\Leftrightarrow \forall M, \exists a \in A, a > M$ $\Leftrightarrow \forall M > 0, \exists a \in A, a > M$

Table 6.9 The practice to be learned in RA I most relevant to Task 4(a).

While we identified only one technique for proving that a specified set has an infinite supremum, the assessment activities we analyzed exposed students to several techniques for proving that $\sup A$ is equal to a finite number M. As alluded to above, the activities typically invited students to find the supremum and infimum of a specified set, which, according to the corresponding solutions, came with the expectation to also prove that the specified set has that supremum and infimum. In Task 4(b) – prove that for $p = 1$, $\sup\left\{\frac{n^p}{n+1} : n = 1, 2, ...\right\} = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$ – we anticipated a successful RA I student to extract a task of this type: namely, prove that $\left\{\frac{n}{n+1} : n = 1, 2, ...\right\}$ has supremum 1. They would then aim for a solution equivalent to any one of the three shown in Figure 6.22 below. Each of these solutions relies on a different key technology, as summarized in the subsequent table. Notice that the technologies are made explicit to varying degrees, as are the justifications for various steps, which reflects an inconsistency we observed in solutions. For instance, the principle underlying "theorem" is mentioned and described in Solution 1, while the definitions of the supremum used in Solution 2 and Solution 3 remain implicit. Notice also that in Solution 1, the increasing nature of the sequence is proved by definition, while the limiting value is simply stated as known. Within the progression of the assessment activities given in the RA I course, we noticed that statements that once required a proof (like a limit), were often assumed in later solutions. We were interested in seeing how much and which kind of justification participants would spontaneously give, and why.

$A = \left\{\frac{n}{n+1} : n = 1, 2, \ldots\right\}$.

b) Prove that $\sup A = 1$:

Solution 1:

$\frac{1}{2} = 0.5, \frac{2}{3} = 0.666\ldots, \frac{3}{4} = 0.75, \ldots$

It seems the sequence is increasing.

We prove this:

$$\frac{n+1}{n+2} > \frac{n}{n+1}$$
$$(n+1)^2 > n(n+2)$$
$$n^2 + 2n + 1 > n^2 + 2n$$
$$1 > 0$$

which is always true.

We know that $\frac{n}{n+1} \to 1$ as $n \to \infty$.

We also proved a theorem saying that an increasing bounded above sequence converges to its supremum.

Thus, $\sup A = 1$. ■

Solution 2:

$\frac{n}{n+1} \leq 1 \; \forall n = 1, 2, \ldots$

So 1 is an upper bound for A.

We will show that for an arbitrary $\epsilon > 0$ we can find an element $\frac{n}{n+1} \in A$ such that

$$\frac{n}{n+1} > 1 - \epsilon$$
$$n > n + 1 - n\epsilon - \epsilon$$
$$n\epsilon > 1 - \epsilon$$
$$n > \frac{1-\epsilon}{\epsilon}$$

Since natural numbers are unbounded, for arbitrary $\epsilon > 0$ we can find an n satisfying this inequality.

This proves $\sup A = 1$. ■

Solution 3:

$\frac{n}{n+1} \leq 1 \; \forall n = 1, 2, \ldots$

So 1 is an upper bound for A.

Assume a lower number $1 - \epsilon$ ($\epsilon > 0$) is also an upper bound.

Then

$$\frac{n}{n+1} \leq 1 - \epsilon \; \forall n \in \mathbb{N}$$
$$n \leq n - n\epsilon + 1 - \epsilon$$
$$n\epsilon \leq 1 - \epsilon$$
$$n \leq \frac{1-\epsilon}{\epsilon}$$

which contradicts that $\mathbb{N}$ is unbounded. ■

Figure 6.22 Examples of the three kinds of solutions anticipated for showing the sup in Task 4(b).

Table 6.10 below depicts a model of the practice to be learned in RA I and most relevant to showing the sup in Task 4(b). The assessment activities we analyzed presented students with a slightly more diverse set of tasks for a non-infinite supremum (hence why we have generalized the sequence to $(x_n)_{n\in\mathbb{N}}$ in Table 6.10, as opposed to $\frac{r(n)}{s(n)}$ used in Table 6.9). The majority were still of a rational form; e.g.,

$$\left\{\frac{n+1}{n} : n = 1,2, \ldots\right\} \text{ or } \left\{\frac{n^3+3}{n^4+16} : n = 1,2, \ldots\right\}.$$

But students were also exposed to finding the infimum and supremum of sequences like $\frac{n+1}{(-1)^n n}$ or $\sqrt[n]{n}$. As evidenced by the table, there was also a certain diversity in the techniques shown to students. In fact, a successful RA I student would also have in their arsenal an additional $[\tau, \theta]$ pair: namely,

$\tau_{4_{b_{14}}}$: Argue that $M \in A$ and $x_n \leq M \; \forall n \in \mathbb{N}$ (i.e., M is an upper bound for A).

$\theta_{4_{b_{14}}}$: "If A has a greatest element, then this greatest element is the supremum."

As exemplified by the sequences mentioned above, $\tau_{4_{b_{14}}}$ was actually the most common technique demonstrated in the solutions we analyzed for proving the supremum of a specified set. There was also an assignment activity that invited students to build the related mathematical theoretical block: i.e., to prove $\theta_{4_{b_{14}}}$. The techniques shown in Table 6.10 typically appeared in their complementary forms for proving that the infimum of A is some specified number m. We were not sure if a successful RA I student would experience a challenge in choosing an appropriate technique for Task 4 or adapting the technique to proving a supremum. This said, we also expected that the exploration in Task 3 might lead the student to favour $\tau_{b_{11}}$. Indeed, one of our reasons for posing both Task 3 and Task 4 was to see if participants would make connections between the two.

Notice that in the solution corresponding to $\tau_{4_{b_{11}}}$ (see Figure 6.22 above), the sequence $(x_n)_{n \in \mathbb{N}} = \left(\frac{n}{n+1}\right)_{n \in \mathbb{N}}$ is shown to be increasing by simply considering the inequality $x_{n+1} > x_n$ and then manipulating it into equivalent, simpler versions, until a known inequality (in this case $1 > 0$) enables the conclusion. This is not the only technique in the practices to be learned for showing a sequence is increasing. As mentioned in Section 6.3.1, one type of task that a successful RA I student would be able to solve involved showing that a specified recursively-defined sequence is convergent using $\theta_{4_{b_{11}}}$ (or what we called Λ_3 in 6.3.1). The related solutions almost always implemented a proof by induction to show that the sequence is increasing (or decreasing) and bounded above (or below). According to our analysis, these subtasks were part of a larger practice to be learned, which was generated by the type of task: Prove (by induction) a mathematical statement that depends on the natural numbers. In addition to monotonicity and boundedness, a successful RA I student would be able to prove "by induction" other inequalities, division statements, or formulas (e.g., for the sum of the first n integers). In the latter cases, the activity statement typically included an explicit direction to use induction. In comparison, when students had to prove that a sequence is increasing, using induction was a choice. Of course, for a recursively-defined sequence, induction may have been chosen out of routine (or the goal of developing one). But for a sequence like the one in Task 4(b) – i.e., $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ – we wondered what the participants of our study would do. A mathematician, we thought, would see induction as unnecessary for proving a simple inequality like $\frac{n+1}{n+2} > \frac{n}{n+1}$.

$T_{4_{b_1}}$: Prove that $\sup A = M$, where $A = \{x_n : n = 1, 2, \ldots\}$ and x_n is a specified sequence.		
$\tau_{4_{b_{11}}}$: List elements of A to check if the sequence seems to be increasing. If so, show $x_{n+1} > x_n \; \forall n \in \mathbb{N}$. Argue that $x_n \to M$ as $n \to \infty$.	$\tau_{4_{b_{12}}}$: Argue that $x_n \leq M \; \forall n \in \mathbb{N}$ so that M is an upper bound for A. Show that for an arbitrary $\epsilon > 0$ one can find an $x_n \in A$ such that $x_n > M - \epsilon$. Manipulate the inequality to find $n > f(\epsilon)$. Conclude the existence of n since $\mathbb{N}$ is unbounded.	$\tau_{4_{b_{13}}}$: Argue that $x_n \leq M \; \forall n \in \mathbb{N}$ so that M is an upper bound for A. Assume there is a lower number $M - \epsilon$ ($\epsilon > 0$) that is also an upper bound. Then $x_n \leq M - \epsilon \; \forall n \in \mathbb{N}$. Manipulate the inequality to find $n \leq f(\epsilon) \; \forall n \in \mathbb{N}$. Conclude that this contradicts the unboundedness of $\mathbb{N}$.
$\theta_{4_{b_{11}}}$: "We proved a theorem that an increasing bounded above sequence converges to its supremum."	$\theta_{4_{b_{12}}}$: Implicit: $\sup A = M \Leftrightarrow (\forall a \in A, a \leq M) \wedge (\forall \epsilon > 0, \exists a \in A, a > M - \epsilon)$	$\theta_{4_{b_{13}}}$: Implicit: $\sup A = M \Leftrightarrow M$ is a least upper bound for $A \Leftrightarrow (\forall a \in A, a \leq M) \wedge [(a \leq B \; \forall a \in A) \Rightarrow M \leq B)]$ $\therefore$ if $\forall a \in A, a \leq M$, $\neg(\sup A = M) \Rightarrow \exists B < M, (a \leq B, \forall a \in A)$

Table 6.10 The practice to be learned in RA I most relevant to showing the sup in Task 4(b).

We purposefully phrased Task 4(b) – prove that for $p = 1$, $\sup\left\{\frac{n^p}{n+1} : n = 1, 2, \ldots\right\} = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$ – so that a successful RA I student would also extract the task: Prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Faced with such a limit, we expected the student to turn to the formal definition of sequence convergence and attempt to construct a solution like the one shown in Figure 6.23 (E is the greatest integer function). There is nothing particularly challenging about this limit when compared to the ones students were asked to prove in the assessment activities we analyzed: e.g., $\lim_{n\to\infty} \frac{n^3+2}{n^3-2} = 1$, $\lim_{n\to\infty} \frac{1}{n^2} = 0$, or $\lim_{n\to\infty} \frac{3n}{n+2} = 3$. As usual, not all solutions had the same amount of explicit justification, and we wondered how well participants would be able to explain what they were doing and why. We provide a model of the corresponding practice to be learned in Table 6.11.

> $A = \left\{\frac{n}{n+1} : n = 1, 2, \ldots\right\}$.
>
> b) Prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$:
>
> We need to show $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, |x_n - 1| < \epsilon$.
>
> We fix an $\epsilon > 0$.
>
> We need $\left|\frac{n}{n+1} - 1\right| < \epsilon$ or $\left|\frac{n-n-1}{n+1}\right| < \epsilon$ or $\left|\frac{-1}{n+1}\right| < \epsilon$ or $\frac{1}{\epsilon} < n + 1$.
>
> Since for all $n \in \mathbb{N}$, we have $n < n + 1$, it is enough to have $\frac{1}{\epsilon} < n$.
>
> We define $N = E\left(\frac{1}{\epsilon}\right) + 1$.
>
> If $n \geq N$, then $n > \frac{1}{\epsilon}$ and $n + 1 > \frac{1}{\epsilon}$ which is equivalent to $\left|\frac{n}{n+1} - 1\right| < \epsilon$. ■

Figure 6.23 An example of the kind of solution anticipated for showing the limit in Task 4(b).

$T_{4_{b_2}}$: Prove by the definition: $\lim_{n\to\infty} x_n = L$, where $(x_n)_{n\in\mathbb{N}}$ is a simple specified sequence (typically of rational or polynomial form), and L is its specified limit.
$\tau_{4_{b_2}}$: Need to show $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, \|x_n - L\| < \epsilon$. Fix $\epsilon > 0$ and write the specified inequality $\|x_n - L\| < \epsilon$. Manipulate the inequality to find $n > f(\epsilon)$. Sometimes a smaller sequence can be used to make the inequalities easier to manipulate. Choose $N = E\big(f(\epsilon)\big) + 1$ (E is the greatest integer function) or assert the existence of $N > f(\epsilon)$ by citing the Archimedean Property, or the unboundedness of $\mathbb{N}$. Conclude that if $n \geq N$, then $n > f(\epsilon)$ and the inequality holds.
$\theta_{4_{b_2}}$: Implicit: By definition, $\lim_{n\to\infty} x_n = L \Leftrightarrow \forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, \|x_n - L\| < \epsilon$.

Table 6.11 The practice to be learned in RA I most relevant to showing the limit in Task 4(b).

In sum, we expected “prove that” tasks to trigger the successful RA I student to construct formal proofs based on the definitions and/or theorems that were emphasized in the RA I course. Nevertheless, a key reason that we chose Task 4 was the abundance of possible solution approaches; not only the ones described above, but also ones that might be less formal in nature. We were interested in seeing which kinds of techniques participants possessed and which they would favour for “proving” the suprema of sets and the limits of sequences, after having successfully completed RA I.

6.4.2 Practices Actually Learned[36]

To solve the collection of tasks that constitute Task 4, we expected a successful RA I student to be drawn to developing formal proofs based on the definitions and/or theorems emphasized in the RA I course. More specifically, we expected such a student to solve Task 4(a) – prove that for any $p > 1$, $A = \left\{\frac{n^p}{n+1} : n \in \mathbb{N}\right\}$ is unbounded above – by recalling that a set of real numbers A is "unbounded above" if and only if its supremum is infinite, i.e., $\forall M > 0, \exists a \in A, a > M$. Then we expected them to recognize in Task 4(b) – prove that for $p = 1$, $\sup\left\{\frac{n^p}{n+1} : n \in \mathbb{N}\right\} = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$ – the need to develop two proofs: (1) a proof that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ by the formal definition of sequence convergence; and (2) a proof that $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$, either using a formal definition of the supremum, or Λ_3 (i.e., an increasing bounded above sequence converges to its supremum). We expected a successful RA I student to show that the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing by simply transforming the corresponding inequality $\frac{n}{n+1} < \frac{n+1}{n+2}$ into an equivalent inequality that is clearly true (e.g., $0 < 1$). We recognized, however, that the student might also be triggered to recall the "by induction" technique that they were supposed to master in RA I.

To our surprise, only two students attempted to solve Task 4(a) and prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$ in the expected "proof-based" manners. All other participants chose different techniques, which we present in 6.4.2.1 and 6.4.2.2. Although there was a dominant overall technique for proving that the set A in 4(a) is unbounded above (i.e., arguing that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ when $p > 1$), both sections highlight the diverse collection of techniques participants had developed for justifying the limits of rational sequences like $\left(\frac{n^p}{n+1}\right)_{n\in\mathbb{N}}$. Indeed, when it came to "proving a limit," it seems participants were drawn more to the calculation techniques they had presumably learned in Calculus courses, rather than the formal proof techniques they were supposed to learn in RA I. In a similar vein, only a couple students chose an approach based on a formal definition to prove the

[36] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 4. These models are synthesized in table form in Appendix E.

supremum of A in 4(b). This said, the most common technique was the one we had expected (i.e., $\tau_{4_{b_{11}}}$ based on Λ_3) and was part of the practices to be learned in RA I. We present the chosen techniques for proving the supremum in 6.4.2.3. As will be exemplified, no participant chose to show that the sequence is increasing exactly as we had anticipated: While several techniques were demonstrated, the most common approach turned out to be induction.

6.4.2.1 Choosing a Technique: Proving that a set A is unbounded above.

Among the students we interviewed, we identified four techniques for solving the type of task:

Prove that $A = \{x_n : n \in \mathbb{N}\}$ is unbounded above.

The techniques, as well as the students who seemed to be employing them, are listed in Table 6.12. Note that all students seemed to employ some technique while solving Task 4(a), and one student (S13) demonstrated two different techniques: one ($\tau_{4_{a_2}}$), which she used to convince herself, and the other ($\tau_{4_{a_4}}$), which she explicitly deemed to be more mathematical (we provide examples of her discourse in the relevant subsections below). As mentioned above, only two students (S1 and S10) chose the expected "formal definition" approach (here we call it $\tau_{4_{a_3}}$), while the other techniques were, in some way or another, unexpected. Notice that $\tau_{4_{a_1}}$ is not appropriate for solving the above type of task; and $\tau_{4_{a_2}}$, the most commonly chosen technique, works on a particular subset of the type of task to which Task 4(a) belongs (in comparison, $\tau_{4_{a_3}}$ could be seen as working on any task of the above type). In what follows, we describe each of the techniques and any interesting characteristics of how students seemed to implement and/or explain them.

Students	Techniques
S4	$\tau_{4_{a_1}}$: Show that $(x_n)_{n\in\mathbb{N}}$ increases by induction.
S2, S3, S5, S6, S7, S8, S9, S11, S12, S13, S14, S15	$\tau_{4_{a_2}}$: Argue that $\lim_{n\to\infty} x_n = \infty$.
S1, S10	$\tau_{4_{a_3}}$: Show that $\forall M \in \mathbb{R}, \exists n \in \mathbb{N}, x_n > M$. (the expected technique, denoted τ_{4_a} in 5.1.4)
S13	$\tau_{4_{a_4}}$: Assume that $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, x_n \le M$ and derive a contradiction.

Table 6.12 Participants' techniques for proving that $A = \{x_n : n \in \mathbb{N}\}$ is unbounded above.

6.4.2.1.1 Show the elements of A are increasing by induction.

Only one student (S4) chose to show that $A = \left\{\frac{n^p}{n+1}: n \in \mathbb{N}, p > 1\right\}$ is unbounded above by showing that $\left(\frac{n^p}{n+1}\right)_{n\in\mathbb{N}, p>1}$ increases by induction. Other students may have considered this technique. For instance, while reflecting on his solution, S8 said that he had thought about showing that $\left(\frac{n^p}{n+1}\right)_{n\in\mathbb{N}, p>1}$ is increasing by induction: "Cause usually when you have something with integers and you want to prove it for all, then you have to do induction. But the induction approach just seemed longer to me." For S8, using techniques for evaluating limits, which he had learned in previous Calculus courses, seemed "quicker" and "easier," even if the induction approach seemed more "rigorous" and "precise" (i.e., "more like a proof"). Neither S8 nor S4 realized the technique was also inadequate for solving the task. When asked why he used induction, S4 explained:

> Because p has to be greater than one and can't be equal to one, that means that whatever is in the numerator will consistently increase. And the bigger that p is, the bigger it will increase by. [-] So in my mind, it's telling me that it's going to be increasing, for all n. So if I can show that by induction, starting at $n = 1$, and then assuming that for n it's true, and then showing it applies for $n + 1$, then it would show that A is unbounded above.

At the beginning of this statement, S4 seemed to be comparing the growth of the numerator and the denominator in the sequence $\left(\frac{n^p}{n+1}\right)_{n\in\mathbb{N}, p>1}$. It is possible that he was noticing that the numerator will increase faster than the denominator. One way of using this observation would be to conclude that the limit of the sequence is infinite, whereby the corresponding set is unbounded above (like the students in 6.4.2.1.2.2). Instead, S4 decided to show that the sequence is "increasing," which led him to the by-induction approach. We suspect that when S4 used the term "increasing" here, he was thinking solely about a sequence that "increases without bound"; that is, he was overlooking the possibility of bounded increasing sequences, which would explain why showing $(x_n)_{n\in\mathbb{N}}$ is increasing does not necessarily show that $A = \{x_n: n \in \mathbb{N}\}$ is unbounded above.

When S4 attempted to carry out the "by induction" proof technique he had chosen, it became clear that he had mastered the initial steps of such a proof, but that he did not know how to reach the conclusion. In fact, he successfully completed two cycles of the initial steps, as summarized below:

Prove $\left(\frac{n^p}{n+1}\right)_{n\in\mathbb{N},p>1}$ increases by induction:

First step: $n = 1: \frac{1}{2}, n = 2: \frac{2^p}{3} > \frac{1}{2}$ because $p > 1$

The assumption: $\frac{(n-1)^p}{n} < \frac{n^p}{n+1}$

To prove: $\frac{n^p}{n+1} < \frac{(n+1)^p}{n+2} \Leftrightarrow \frac{n+2}{n+1} < \frac{(n+1)^p}{n^p}$

Prove $\frac{n+2}{n+1} < \frac{(n+1)^p}{n^p}$ by induction:

First step: $n = 1: \frac{3}{2} < 2^p$ because $p > 1$

The assumption: $\frac{n+2}{n+1} < \frac{(n+1)^p}{n^p}$ (*)

To prove: $\frac{n+3}{n+2} < \frac{(n+2)^p}{(n+1)^p}$

At this last inequality, S4 was stuck: "I am blanking on how I would show that." Interestingly, he claimed that he did not need to complete the proof to convince himself of the conclusion. Referring to the inequality at (*), S4 said: "For me, this is obvious that it's true for any n. But I know that that's not sufficient."

6.4.2.1.2 Show the elements of A form a sequence that diverges to infinity.

The most common technique used to solve Task 4(a) – prove that for any $p > 1$, $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \ldots\right\}$ is unbounded above – was to argue that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$. As shown in Table 6.12, we identified twelve of fifteen participants as using this technique. The implication

$$\lim_{n\to\infty} x_n = \infty \Rightarrow A = \{x_n : n \in \mathbb{N}\} \text{ is unbounded above}$$

on which the technique is based seemed to be obvious to most; in some cases, so obvious that it need not be stated. S3, for example, ended his solution after arguing that the sequence goes to infinity. Then, when asked why it meant that the set is unbounded above, S3 said: "By definition, something that goes to infinity means that it has no bound." Other students tried to provide a more substantial response when they were asked the same question. S7, for instance, said that unbounded means that there's always a number bigger. So, there's always an n, if you go to $n+1$, it's going to be bigger than the previous n. Ok? So infinity, it's not an actual number. So you can always find an n such that the $n+1$ would be bigger than this n.

With this response, S7 seemed to be making an effort to explain the above implication by returning to a more formal definition of unbounded above. A more precise argument using formal language might look something like:

$$\lim_{n\to\infty} x_n = \infty \Rightarrow \forall M > 0, \exists N \in \mathbb{N}, \forall n \geq N, x_n > M \text{ (by definition).}$$
$$\Rightarrow \forall M > 0, \exists N \in \mathbb{N}, x_N > M.$$
$$\Rightarrow A = \{x_n : n \in \mathbb{N}\} \text{ is unbounded above (by definition).}$$

No student justified their technique in this way.

"Argue that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$" can be seen as a subtask. We identified six different techniques that participants seemed to use to solve the task. The techniques, as well as the students who seemed to use them are summarized in Table 6.13 below. Notice that one student (S9) exhibited three different techniques, which he claimed to be more or less appropriate depending on the course context (we provide examples of his discourse in the relevant sections below). In the following, we present each of these techniques and any interesting elements of how students seemed to implement and explain them.

Students	Techniques
S5	Plug in values.
S2, S6, S11, S15	Compare the growth of the numerator and the denominator.
S8, S9, S13	Think about what matters in the long run.
S9, S14	Compare the degree of the polynomial in the numerator and the denominator.
S3, S9	Use l'Hospital's Rule.
S7, S12, S14	Use Algebra of Limits.

Table 6.13 Participants' techniques for arguing that $lim_{n\to\infty} x_n = \infty$.

6.4.2.1.2.1 Plug in values.

One student (S5) convinced herself that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$ by plugging in values. To deal with the complicated nature of $A = \left\{\frac{n^p}{n+1}, n = 1, 2, \dots\right\}$, S5 explained: "I'm just going to plug in values and see where it goes." She let $p = 2$ and wrote the first few elements of A (note that she made a calculation error for the third element, which should have a numerator of 9):

$$\frac{1}{2}, \frac{4}{3}, \frac{27}{4}.$$

After checking the decimal expressions of these numbers with her calculator, S5 concluded that the sequence is increasing; and "it's going to keep on increasing. Yeah, because it's quite a big power. [-] It's just going to approach infinity." S5 quickly generalized her argument to any value of p, and when asked why looking at the three numbers above convinced her, she explained that "it's always going to have a higher numerator. And, it's always going to keep on increasing." Although S5 seemed to rely principally on her calculation of some elements of the sequence to convince herself that it should increase to infinity, she also showed evidence of reflecting on the difference between the numerator and the denominator. It is possible that with some more prompting, she may have progressed from a discourse based solely on comparative size to a discourse including also comparative growth, like the following group of students. We note that S5 was unsatisfied with her argument as a "proof," but she could not recall the way they did things in RA I (more on this in 6.4.2.1.4).

6.4.2.1.2.2 Compare the growth of the numerator and the denominator.

There were four students (S2, S6, S11, and S15) whose principal argument for $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ was that n^p grows faster than $n + 1$ for $p > 1$. Like S5, S6 and S11 both indicated that they were thinking about plugging in numbers: S11 wrote the element corresponding to $n = 10$ and $p = 2$, and S6 spoke about plugging in $n = 100$. The difference is that S6 and S11 clearly stated that the numerator is increasing faster than the denominator. Referring to n^p, S6 explained:

> This one is obviously going to increase, like the value's going to increase faster than the one that's at the bottom [i.e., $n + 1$]. So [the sequence] will, like, you know, increase

exponentially. Like this one [i.e., n^p] wins over. [-] And exponential functions, like, are unbounded.

In this explanation, S6 shows some evidence of thinking like the students in the next group: that is, she seems to be approximating the long-term behaviour of $\frac{n^p}{n+1}$ by n^p, which she refers to as an "exponential." Hence why it is obvious to her that "the value's just going to keep increasing." Notice that, like S4, S6 also seemed to use the word "increasing" to sometimes mean "increasing without bound." It is also important to note that S6 was hesitant about arguing in this way: "I might be wrong. [-] These are like little tricks I've learned." Like S5, she too wanted to recall the technique used in RA I, but could not remember (see 6.4.2.1.4).

S2 and S15 distinguished themselves from S6 and S11 by their interest in providing graphical evidence that n^p indeed increases faster than $n+1$. In the context of the interview, S2 was unable to find a graphical representation for n^p. Given that she tried graphs like $y = 3^x$, she may have been struggling with the general nature of n^p (a more appropriate graph would have been $y = x^3$). S15, on the other hand, produced the drawing in Figure 6.24 and said:

> Just looking at the graph… When it's a progression, maybe, it's a straight line. The power is like, is similar to an exponential function. So, but, at some point, it's evident… Well, if you divide something that's growing much faster than the denominator, [-] it must diverge.

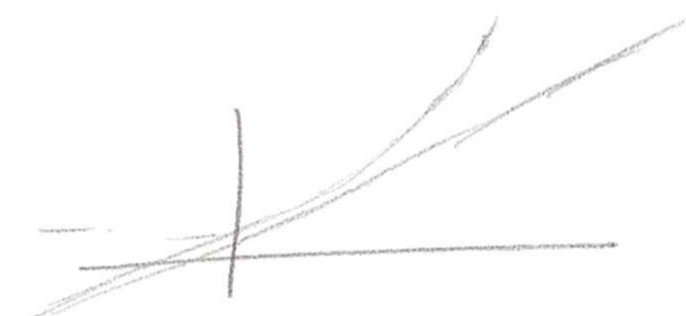

Figure 6.24 The drawing used by S15 to explain that $n^p, p > 1$ grows faster than $n + 1$.

Three other students (S3, S8, and S9) explicitly mentioned that n^p "grows," "increases," or "goes to infinity" faster than $n+1$ if $p > 1$; but they provided additional arguments to show that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$. These are described in the next three sections.

6.4.2.1.2.3 Think about what matters in the long run.

Three students (S8, S9, and S13) argued that $\lim_{n\to\infty}\frac{n^p}{n+1} = \infty$ by thinking about the approximate behaviour of the sequence as n tends towards infinity: i.e., $\frac{n^p}{n+1} \sim \frac{n^p}{n} = n^{p-1}$, and $\lim_{n\to\infty} n^{p-1} = \infty$ for $p > 1$. These students justified the approximate long-term behaviour of the sequence in similar manners. For instance, S8 said that as n goes to infinity, "the powers of n become much more important than, like, the constants." In other words, and as explained by S9, "the $+1$ on the denominator becomes, you know, not very significant. So we can skip it." Or, as S13 said: "When n is really big, then n and $n+1$, it really doesn't matter. It's very close. Because n is very very big."

One notable difference between these students' solutions is the generality with which they made their arguments. While S13 maintained a general p throughout her entire argument, both S8 and S9 mentioned examples with specified values for p. In fact, S9's entire argument was based on the consideration of specific extreme examples (i.e., with p very close to 1). He started by specifying $p = 1.5$, and then claimed that if p were smaller, "it would still work." He exemplified what he meant by choosing $p = 1.0001$ and noting that as n goes to infinity, $\frac{n^{1.0001}}{n+1} \sim n^{0.0001}$, which still goes to infinity. The generalization of S9's argument to larger values of p remained implicit in his discourse.

Thinking about what matters in the long run is the only way that S8 justified the infinite limit. S9, in contrast, spontaneously demonstrated two other techniques, in search of one that he felt would be more acceptable in RA I (see 6.4.2.1.2.4 and 6.4.2.1.2.5). Regarding this technique, S9 said: "If I had to prove it in an Analysis course, I know this wouldn't suffice. This is crap. This is zero percent." In a somewhat similar manner (we discuss important subtleties later, in 6.4.3), S13 said of the technique:

> It's not very mathematics, right? It's not exact. [-] I understand that, you know, when you want to do mathematics, you have to be exact, you have to write down the formulas. But at first I have to feel it. I have to understand it. Make sure that I believe my formulas, my solutions. Then I'm able to write the solutions.

In comparison to S9, S13 eventually demonstrated what *she* believed to be a more mathematical way of thinking about the unboundedness of $A = \left\{\frac{n^p}{n+1} : n \in \mathbb{N}\right\}$ for $p > 1$. We outline this technique in 6.4.2.1.4.

6.4.2.1.2.4 Compare the degree of the polynomials in the numerator and the denominator.

Upon looking at $\frac{n^p}{n+1}$, several students noted the degrees of the polynomials: i.e., in the denominator, the highest power of n is 1, and in the numerator, the highest power of n is $p > 1$. S9 and S14 were the only students to explicitly suggest that this observation can form a technique for arguing that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$. When reading the statement of Task 4(a), S14 said that he didn't even have to look at the condition "$p > 1$": "I knew that if it was greater than 1, then [the sequence] would diverge. I actually was just assuming that it was greater than one. Because it would be boring otherwise." S9 gave a clear description of the technique, as he had learned it in a previous Calculus course:

> You just checked the highest degree polynomial, whether it was on the numerator or the denominator. [-] If it was on [-] the numerator, the highest degree polynomial, that means [the sequence] goes to infinity. And if it's on the denominator, it goes to zero.

As alluded to above, S9 did not stop here. He continued his search for a technique that he felt would be more acceptable in RA I (see 6.4.2.1.2.5). In a similar vein, S14 also exhibited another technique, which he felt represented a more "rigorous" manner of arguing about the limit (see 6.4.2.1.2.6).

6.4.2.1.2.5 Use L'Hospital's Rule.

With l'Hospital's Rule, S9 felt as though he had a technique that would be acceptable in RA I: "Because we saw it," he explained, "[in] [t]he two last weeks. And now I guess I'm thinking of this question as if it was in the final. And we'd be allowed to use l'Hospital's Rule." This was also the main technique chosen by S3 for showing that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$. Both students noted that they had an "infinity over infinity" case before applying the rule, since, as S9 explained, "for l'Hospital's Rule to work, it has to give one of those indeterminate forms." The main difference between the two students' solutions was that S9 worked solely with the sequence and S3 decided to introduce a function. More specifically, S3 wrote $f(x) = \frac{x^p}{x+1}$, calculated

$$\lim_{x\to\infty} \frac{x^p}{x+1} = \lim_{x\to\infty} \frac{px^{p-1}}{1} = p \lim_{x\to\infty} x^{p-1},$$

and said: "So $p - 1$ is greater than 0, cause p is greater than one… So that goes to infinity. Hence proved." S9's solution was almost identical, but with $f(n) = \frac{n^p}{n+1}$ in place of $f(x)$.

6.4.2.1.2.6 Use the algebra of limits.

Three students (S7, S12, and S14) showed that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ (for $p > 1$) by rewriting the sequence in some way and then applying, more or less explicitly, certain laws about the algebra of limits. Both S7 and S14 rewrote $\lim_{n\to\infty} \frac{n^p}{n+1}$ as $\lim_{n\to\infty} \frac{n^{p-1}}{1+\frac{1}{n}}$. "You just need to take out n," explained S7: i.e.,

$$\frac{n^p}{n+1} = \frac{n}{n}\left(\frac{n^{p-1}}{1+\frac{1}{n}}\right) = \frac{n^{p-1}}{1+\frac{1}{n}}.$$

After rewriting the sequence in this way, S7 then analyzed the behaviour of each part as $n \to \infty$: i.e., $\frac{1}{n}$ "decreases to zero," 1 "is constant, so it goes to 1," and n^{p-1} "would tend to infinity"; "so infinity, it would tend to infinity." S14 argued in a similar manner. Recall that he also suggested that this would constitute a more "rigorous" way of showing the limit (so long as all the details are provided), when compared with just noticing the degrees of the polynomials in numerator and denominator.

The way in which S12 used the algebra of limits to argue that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ (for $p > 1$) was slightly different, in part because he had already shown that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ (i.e., when given Task 4, he chose to complete part (b) before part (a)). When it came time to think about $\lim_{n\to\infty} \frac{n^p}{n+1}$, he explained: "The way I want to do this is I want to represent this as being the product of two functions:" i.e.,

$$y_n = x_n \cdot n^k$$

$$\text{where } y_n = \frac{n^p}{n+1}, x_n = \frac{n}{n+1}, k = p-1, p > 1 \Rightarrow k > 0.$$

Unlike S7 and S14, S12 tried to make explicit the limit law he was going to use: "If we accept that the product [-] of a convergent function and a divergent function is also divergent to that same limit," then

$$x_n \to 1, n^k \to \infty$$
$$\Rightarrow y_n = x_n \cdot n^k \to \infty.$$

6.4.2.1.3 Use the formal definition of an unbounded set.

Two of the fifteen students we interviewed (S1 and S10) tried to show that $A = \left\{\frac{n^p}{n+1} : n \in \mathbb{N}\right\}$ is unbounded above for $p > 1$ by recalling the formal definition of a set that is unbounded above. S10 had a good memory of the definition. While writing the formal symbols – $\forall M \in \mathbb{R}$, $\exists a \in A, a > M$ – he said:

> We need to say that [-] given any real number, M, there's going to exist an element of A, um… Yeah. So that a is larger than M. [-] Cause that's just the negation of saying that A [-] has an upper bound.

After specifying what he needed to show for the given set A (i.e., $\forall M \in \mathbb{R}, \exists n \in \mathbb{N}, \frac{n^p}{n+1} > M$) and letting $M \in \mathbb{R}$, S10 further noted that if $M < 0$, "it's trivial":

> I looked at the sequence that we've used to define A, and it's clearly always greater than zero. So, I know that whenever M is less than zero, the proof is pretty trivial. [-] We can choose any n and x_n is going to be greater than M.

It seemed, therefore, that S10 understood the mathematical proof technique he was trying to implement: namely, for a fixed M, he needed to show the existence of a natural number n so that $x_n > M$. In particular, for positive values of M, he knew that he needed to find an n such that

$$\frac{n^p}{n+1} > M \text{ (*)}.$$

S10 struggled, however, to completely implement the technique because he could not determine the required algebraic manipulations. He claimed: "There's some kind of spark here that's missing."

S1 seemed to have the "spark" to which S10 referred: namely, when faced with the same inequality (*), S1 noticed that

$$\frac{n^p}{n+1} > \frac{n^p}{n+n} = \frac{n^p}{2n} = \frac{n^{p-1}}{2},$$

whereby it is sufficient to work with the inequality $\frac{n^{p-1}}{2} > M$. Nevertheless, S1 seemed to have a weaker memory of the overall proof technique. For instance, when he first saw Task 4(a), he recalled only the inequality (*) and that he needed to prove it "for any" M. Moreover, when asked if the inequality needed to hold for every n, S1 said: "It would be nice, but no. Not necessarily. It's as n goes to infinity." Given the behaviour of the majority of students (as described in 6.4.2.1.2), it is possible that S1 was facing a conflict with his choice of technique. Indeed, throughout his entire solution to Task 4(a), S1 gave hints that he was thinking about the limit of a sequence. For instance, after realizing that he needed to find only one n for which the inequality $\frac{n^{p-1}}{2} > M$ holds, he claimed that he would use "the greatest n possible, so a diverging n let's say"; and he mentioned the possibility of using l'Hospital's Rule or Squeeze Theorem. Then, after abandoning these techniques and (incorrectly) reducing the task to finding an n such that $n^{p-1} > M$, S1 concluded: "For any p greater than one… [-] Yeah, [n^{p-1} will] be greater than any bound [M]. [-] Infinity to any power alpha will be greater than [-] any number."

6.4.2.1.4 Assume A is bounded and reach a contradiction.

S13 was the only student who demonstrated two different techniques for showing that the given set, $A = \left\{\frac{n^p}{n+1} : n \in \mathbb{N}, p > 1\right\}$, is unbounded above (for a description of her other more informal technique based on arguing that $\lim_{n\to\infty} \frac{x^p}{n+1} = \infty$, see 6.4.2.1.2.3). S13 was also the only participant to argue by contradiction when faced with Task 4(a). Two other students (S5 and S6) expressed some awareness of this technique and felt it was a more acceptable way to solve the given "prove that" task than the techniques they used (as described in 6.4.2.2.1 and 6.4.2.1.2.2). S5, for instance, recalled that in RA I, one would "try to show that [the set is] bounded above, and see that we're not able to do that. [-] I know I saw it in my notes last semester, but I don't know how to do it." S13, in contrast, was able to describe how the overall technique works before implementing it:

> How am I going to prove this? Uh, let's say that it doesn't go to infinity. [-] If it doesn't go to infinity, it means that it is bounded. And if it's bounded, then I can show you that it means that the natural numbers are bounded. So it means it goes to infinity.

Note that there is some ambiguity in the way S13 was speaking here: we are not sure if "it" refers to the sequence $(x_n)_{n\in\mathbb{N}}$ or the set $A = \{x_n : n \in \mathbb{N}\}$, and this subtle distinction is important from a mathematical point of view. For instance, when S13 started her proof, she said "I want to show that it goes to infinity," and she wrote the following line:

$$\forall M > 0, \exists n \in \mathbb{N}, x_n > M.$$

She further specified: "It means that for any number that [-] I choose, I can find at least one element that is bigger than that number. It means that this is going to infinity." The line shown above is the formal definition of "$A = \{x_n : n \in \mathbb{N}\}$ is unbounded above," which is not equivalent to the formal definition that "$\lim_{n\to\infty} x_n = \infty$" (i.e., $\forall M > 0, \exists n \in \mathbb{N},\ \forall n > n, x_n > M$). It is notable, nonetheless, that S13 was the only participant to seem to (be able to) make a connection between her informal technique of showing $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ and the formal definition of "unbounded above." Had this been the case for the students who used the formal definition (see 6.4.2.1.3), then they may have realized that showing $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$ suffices to show that $A = \left\{\frac{n^p}{n+1} : n \in \mathbb{N}, p > 1\right\}$ is unbounded above. And for the students who made this realization (see 6.4.2.1.2), knowing a connection between the sequence "going to infinity" and the formal definition of unbounded above would have made their theoretical discourses more mathematical in nature.

The rest of S13's implementation of the technique essentially went as follows:

> We want to show that $\forall M > 0, \exists n \in \mathbb{N}, x_n > M$.
> Let's say it's not this: i.e., $\exists M > 0, \forall n \in \mathbb{N}, x_n \leq M$.
> Then $\frac{n^p}{n+1} \leq M\ \forall n \in \mathbb{N}$.
> $n^p \leq M(n+1)$
> $n^p \leq nM + M \leq nM + nM = 2nM$
> $n^{p-1} \leq 2M$
> So the natural numbers are bounded.
> But they are not bounded.
> So this is not true. ∎

When asked, in retrospect of this solution, how she knew that the natural numbers are not bounded, S13 said:

Because every number that you give me, I'll add one to it, and it's bigger than your number. [-] Oh! And I can also add the zero to your number. [-] This is something we take for granted. Don't we?

6.4.2.2 Choosing a Technique: Proving that the limit of a sequence is M.

Students	Techniques
S5	$\tau_{4_{b_{21}}}$: Plug in values.
S4, S6, S9, S11, S13, S15	$\tau_{4_{b_{22}}}$: Think about what matters in the long run.
S6, S9, S11	$\tau_{4_{b_{23}}}$: Use l'Hospital's Rule.
S2, S5, S7, S12, S13, S14	$\tau_{4_{b_{24}}}$: Use the algebra of limits.
S1, S10	$\tau_{4_{b_{25}}}$: Show that $\forall\epsilon > 0, \exists N \geq 1, \forall n \geq N \; \lvert x_n - M\rvert < \epsilon$. (the expected technique, denoted $\tau_{4_{b_2}}$ in 5.1.4)
S8	$\tau_{4_{b_{26}}}$: Argue that $(x_n)_{n\in\mathbb{N}}$ is increasing and $\sup\{x_n : n \in \mathbb{N}\} = M$.

Table 6.14 Participants' techniques for proving that $\boldsymbol{lim_{n\to\infty} x_n = M}$.

While trying to solve part (b) of Task 4 – prove that for $p = 1, \sup A = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$ – most participants, at some point, extracted the need to solve the following task (the notation reflects the models in 5.1.4):

$$t_{4_{b_2}} : \text{Prove that } \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Notice that we have used the words "prove that" even if most students did not choose the formal proof technique that we had expected from a successful RA I student (we could have, e.g., used the words "argue that" instead; but we wanted to highlight the fact that Task 4 explicitly said "prove"). One student (S3) identified $t_{4_{b_2}}$, but quickly abandoned the need to solve it, claiming that the limit was "obvious enough that I don't have to justify it further," and "to use the actual proof for limits, the rigorous one that we use in Analysis [-] takes a little bit long." The other fourteen participants could be categorized according to the argumentation technique(s) they demonstrated, as summarized in Table 6.14. Once again, there were students (S5, S6, S9, S11, and S13) who demonstrated more than one technique for arguing about the limiting behaviour of a sequence. This reflects a more general observation that participants tended to compare limit techniques as more or less intuitive (or rigorous), straightforward (or tedious), and/or acceptable in Calculus (or Analysis). In what follows, we present how students seemed to be choosing, implementing, and/or describing these techniques.

6.4.2.2.1 Plug in values.

Given the way in which S5 argued that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$ (see 6.4.2.1.2.1), it is perhaps not surprising that she also seemed to convince herself that that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ by plugging in values. An important difference in the case of the simpler limit is that this was not the sole technique that S5 demonstrated. Her first instinct was to use the algebra of limits (see 6.4.2.2.4). Plugging in values seemed, for her, a double check, which she performed later, while thinking about how to show that the supremum of $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$ is 1 (see 6.4.2.3.2.1). S5 made the list of numbers depicted in Figure 6.25 and, with a calculator in her hand, explained: "Well I'm plugging all the values, like five over six, six over seven, seven over eight, and now I'm at ten over eleven, which is like close to zero point nine. So now we clearly see that it's going towards one."

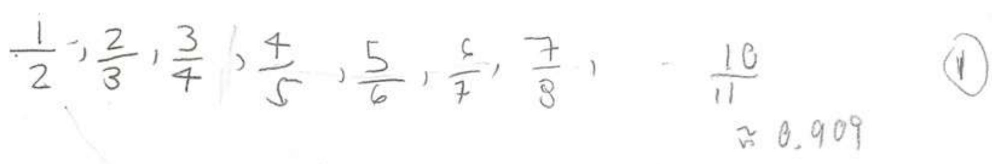

Figure 6.25 An illustration of one technique used by S5 to argue that $lim_{n\to\infty} \frac{n}{n+1} = 1$.

6.4.2.2.2 Think about what matters in the long run.

We identified six students (S4, S6, S9, S11, S13, and S15) as using the technique of thinking about what matters in the long run to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Recall that S9 and S13 had justified the limit $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$ using this technique (see 6.4.2.1.2.3), while the other four students (S4, S6, S11, and S15) showed evidence of talking, rather, about the comparative growth of the numerator and the denominator (see 6.4.2.1.1 and 6.4.2.1.2.2). For the simpler sequence, $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$, S6 said it was "obvious": "As n tends to infinity, the one here [i.e., in $n + 1$] isn't going to do much; you can just ignore it. So it just becomes n over n. That's one." In symbols, we can represent S6's argument as follows:

$$\text{As } n \to \infty,\ \frac{n}{n+1} = \frac{n}{n} = 1. \text{ Hence, } \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

The arguments provided by the other students in this group could sound quite different, but the main idea was always the same: As $n \to \infty$, the constant in the denominator of $\frac{n}{n+1}$ becomes insignificant and can be ignored. S13 simply noted that n and $n+1$ "are the same, almost, when n goes to infinity"; S4 explained that "the numerator is always going to be one less than the denominator. And, as n approaches infinity, the difference between the two will be much less. So it will be much closer to one"; in a similar manner, S11 said that as n is getting bigger, "you have like a huge number over a huge number that's just partially different. It's just going to be one"; and, finally, S15 noted that the powers in the numerator and the denominator are the same, so "the numerator will be cancelled by the denominator when we tend to infinity."

This was the only way that S4 and S15 spontaneously argued about the limit. The other four students in this group (S6, S9, S11, and S13) all provided other arguments, which we outline in the following sections. When reflecting on the different techniques to which she had been exposed, S6 said that this one was more "straightforward. It's just right there. It's just common sense." Both S6 and S9 claimed that this is the technique that was acceptable in Calculus and the technique they would use if they wanted to justify the limit to someone who does not know a lot about mathematics. "If I explained this to my parents, that's what I'd say," claimed S9. S6, S9, and S11 all suggested that a different technique would be more acceptable in RA I (see 6.4.2.2.3 and 6.4.2.2.5). In a somewhat similar sense (we discuss important subtleties later, in 6.4.3), S13 reiterated that, while she would use this technique to convince herself of the limit, she would use a different technique to "speak mathematics" to someone else (see 6.4.2.2.4).

6.4.2.2.3 Use L'Hospital's Rule.

In addition to thinking about what matters in the long run, S6, S9, and S11 also decided to use l'Hospital's Rule to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. S9 simply adapted his work for the more general limit, $\lim_{n\to\infty} \frac{n^p}{n+1}$: i.e.,

$\lim_{n\to\infty} \frac{n^p}{n+1} = \lim_{n\to\infty} \frac{pn^{p-1}}{1}$ since it's "going to infinity over infinity" for $p \geq 1$.

And for $p = 1$, it gives $1 \cdot n^0 = 1 \cdot 1 = 1$.

S6 demonstrated a comparable efficiency in carrying out the technique for the specific sequence, $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$, and needed only a few sentences to describe her work: “I think you use l’Hospital’s. So you get one over one; one. Cause it’s an infinity over infinity case.” S11, on the other hand, had some issues in recalling the appropriate technology and implementing the corresponding technique. For instance, he recalled that you “take the derivative on the bottom and the top,” but he made a mistake in his calculations: $\left(\frac{n}{n+1}\right)' = \frac{1}{2} \neq 1$. Then, upon obtaining this unexpected value for the limit, S11 concluded that l’Hospital’s Rule does not work in this case: “But like, if it’s going to zero, let’s say, and it’s not supposed to go to zero, then you would use l’Hospital’s Rule, right?”

6.4.2.2.4 Use the algebra of limits.

Recall that S7, S12, and S14 all used the algebra of limits to show that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$ (see 6.4.2.1.2.6); so it is not surprising that they used the same kind of technique to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. They each argued, in some way or another, that

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1\,.$$

In this case, however, they were not the only ones to turn to an algebraic technique: S2, S5, and S13 also chose to argue in this way.

What was particularly interesting were the different ways in which the students spoke about their choice of technique. When asked why they used this technique, S5, S12, and S14 seemed to indicate that it is *the* technique to use. For example, S5 said: “Isn’t that how you solve when we have a limit of something as n approaches infinity?” S14 added: “We’re told to do this. [-] It’s not always a bad thing to do things just as you were told. Like this is a perfectly fine way to show a limit.” In their explanations, S2 and S7 seemed to place more emphasis on the idea that the technique is based on representing the sequence in an “easier way.” S7 added that “you’re just rearranging your sequence in a way that you can use stuff that was proved in class, and that can easily be proven to get the result that you need.” S13’s way of speaking about the technique was entirely different. Recall that S13 had already exhibited what she felt was a more intuitive technique for showing the limit, as described in 6.4.2.2.2. She used the algebra of limits to exemplify what she said might be a more “beautiful” solution, which she would use if she wanted to “speak mathematics” to someone else.

The differences exemplified above are reflected in the different ways in which the students claimed to know one of the fundamental facts about limits that they used: i.e., $\lim_{n\to\infty}\frac{1}{n}=0$. Some provided an intuitive justification. S14, for example, explained that “with increasing terms on the bottom, it gets smaller and smaller, I guess. It seems difficult to explain. It’s just very intuitive to me.” Based on his previous experience, S12 said: “I know that one over infinity, I can evaluate that to be zero.” In the citation above, S7 seemed to indicate that he knew $\lim_{n\to\infty}\frac{1}{n}=0$ because it was a result proved in class, or at least a result that could be proven. S13, in contrast, was the only participant for whom knowing that $\lim_{n\to\infty}\frac{1}{n}=0$ seemed to be equivalent not only to knowing that a proof exists, but also knowing how to do the proof herself. If she were communicating her technique to a mathematician or a mathematics student, she said that she would tell them: “Either we know [the limit], or if we don’t know it, I’ll prove it to you.” When prompted by the interviewer, she argued as follows:

> We have to prove that $\forall\epsilon>0, \exists \mathrm{N}\in\mathbb{N} \ni n\geq \mathrm{N}\ \left|\frac{1}{n}\right|<\epsilon$.
>
> $\left|\frac{1}{n}\right|<\epsilon \Rightarrow n>\frac{1}{\epsilon}$.
>
> So if we put $\mathrm{N}=\left\lceil\frac{1}{\epsilon}\right\rceil$ (the smallest integer greater than $\frac{1}{\epsilon}$),
>
> then for any $n\geq \mathrm{N}\quad n>\frac{1}{\epsilon}$.
>
> So $\epsilon>\frac{1}{n}$. ■

6.4.2.2.5 Use the formal definition of sequence convergence.

Recall that only two students (S1 and S10) were compelled to show that a set is unbounded above “by definition” (see 6.4.2.1.3). They were also the only participants to spontaneously call upon the formal definition of sequence convergence to prove that $\lim_{n\to\infty}\frac{n}{n+1}=1$. This time, both students recalled the complete definition:

$$\lim_{n\to\infty}x_n=M \iff \forall\epsilon>0, \exists N\geq 1, \forall n\geq N\ |x_n-M|<\epsilon,$$

or, in this specific case:

$$\lim_{n\to\infty}\frac{n}{n+1}=1 \;\Leftrightarrow \forall\epsilon>0, \exists N\geq 1, \forall n\geq N \;\left|\frac{n}{n+1}-1\right|<\epsilon.$$

This said, the remainder of the students' solution attempts were characterized in a similar manner as when they worked with the definition of unbounded above to solve Task 4(a).

Once again, S10 was able to describe the overall technique, but was unable to completely implement it in the context of the interview. He let $\epsilon>0$ and said that he needed to find a natural number N so that for all $n\geq N$, the inequality $1-\frac{n}{n+1}<\epsilon$ would hold. He explained that he would be able to do this by "solving" the inequality $1-\frac{N}{N+1}<\epsilon$:

> I'm trying to solve for N in terms of ϵ. So that this inequality holds. I'm confident that's going to work because I'm pretty sure at the end of it, if I take any n that's greater than N, it's still going to hold. [-] But, yeah, I think that's sometimes part of the problem: Sometimes I stop myself, still a little bit too caught up in the big pictures, and sometimes it's like, it's hard for myself to just focus on the small picture.

During the interview, S10 did not attempt to make the algebraic manipulations required to solve for N in terms of ϵ. He felt it would take him too long.

In contrast, S1 exhibited a greater facility in working with inequalities, but may have lacked an understanding of the overall technique. After fixing ϵ, S1 swiftly did the following:

$$\frac{n}{n+1}-1=\frac{n}{n+1}-\frac{n+1}{n+1}=-\frac{1}{n+1}$$

$$\Rightarrow \left|\frac{n}{n+1}-1\right|<\epsilon \Leftrightarrow \frac{1}{n+1}<\epsilon.$$

Then he attempted to use the "same thing" as when he was proving the unboundedness of A. Recall that when looking for an n such that $\frac{n^p}{n+1}>M$, S1 decided that it was sufficient to find an n such that $\frac{n^p}{2n}>M$. Indeed, if this is the case, then for that same n:

$$\frac{n^p}{n+1}>\frac{n^p}{2n}>M.$$

To do the "same thing" (mathematically speaking) in the current task would require, for example, noticing that $\frac{1}{n+1} < \frac{1}{n}$, whereby it would be sufficient to find an n such that $\frac{1}{n} < \epsilon$; because then, for that n,

$$\frac{1}{n+1} < \frac{1}{n} < \epsilon.$$

This is what S1 thought at first, but he quickly changed his mind to do the "same thing" in a different way:

i.e., he decided to work with $\frac{1}{2n} < \epsilon$ instead.

Notice that finding an n such that $\frac{1}{2n} < \epsilon$ does not help with finding an n such that $\frac{1}{n+1} < \epsilon$ since $\frac{1}{2n} < \frac{1}{n+1}$. S1 did not realize this in the interview. After rearranging to get $n > \frac{1}{2\epsilon}$, S1 concluded: "So, I can find an N such that for all epsilon greater than zero… [he writes: so N s.t. $\forall \epsilon > 0$ if $n > \frac{1}{2\epsilon}$ it holds]. So we got the limit."

Among the thirteen participants who did not spontaneously use the expected formal definition technique for proving that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, there were some who did not recall the formal definition (e.g., S5, S6, S9, and S11); and there were others who recalled how to write the definition using formal symbols but did not recall the related technique (e.g., S4 and S7). For instance, after writing out what he called "the dreaded line" – $\forall \epsilon > 0, \exists N \in \mathbb{N}$, $\forall n \geq N \ \left|\frac{n}{n+1} - 1\right| < \epsilon$ – S4 explained that he "would basically isolate epsilon to have a value in terms of n. [-] Wait, I already had it isolated." S6 and S7 also seemed to think that they needed to find a value for ϵ. After performing basic algebraic manipulations to get to $\left|\frac{1}{n+1}\right| < \epsilon$, S6 deemed herself stuck: "I know I've done exercises where you find a value for epsilon. And then… No wait, nevermind, that's not it. That was with delta." At the same step, S7 defined epsilon as $\epsilon = E\left(\frac{1}{n+1}\right) + 1$, where E is the greatest integer function, and noted that "epsilon [-] had to be an integer for some reason. And then you do plus one. And once you define that, you basically end your proof." In all cases, it seemed the students did not understand the formal definition technique.

6.4.2.2.6 Argue the sequence is increasing and has supremum M.

Only one student (S8) argued that $\lim_{n\to\infty}\frac{n}{n+1}=1$ by stating that $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing and has supremum one. Surprisingly, S8 did not refer to the Monotone Sequence Theorem, or any theorem, to justify his statement. He claimed that it "makes sense because I have a mental picture of it." It is possible that S8 had built up this mental picture during his rather intuitive argumentation for "proving" that the supremum of $A=\left\{\frac{n}{n+1}: n\in\mathbb{N}\right\}$ is 1. We describe this argumentation in the next section (see 6.4.2.3.1).

6.4.2.3 Choosing a Technique: Proving that the supremum of a set A is M.

To complete part (b) of Task 4 – prove that for $p=1, \sup A = 1 = \lim_{n\to\infty}\frac{n^p}{n+1}$ – we also expected participants to try to solve the following task:

$$t_{4_{b_1}}\text{: Prove that } \sup\left\{\frac{n}{n+1}: n\in\mathbb{N}\right\}=1.$$

While solving Task 4(b), S5 was the only participant who did not spontaneously identify a need to do something more than show that $\lim_{n\to\infty}\frac{n}{n+1}=1$. When asked about the supremum, S5 said that she did not think about proving it "cause it's given. [-] Cause usually, I feel like it wouldn't be given. It would just tell you to find the sup of A." We suspect that S5 was acting based on a difference between the formulation of Task 4(b) (shown above) and the formulation of a path of activities received in RA I, which asked students to "find" the supremum of a given set. She nevertheless proceeded to try to solve $t_{4_{b_1}}$, and seemed to choose an expected technique (i.e., $\tau_{4_{b_{11}}}$), like the majority of her peers.

All the techniques we identified, as well as the students who seemed to be trying to implement them, are summarized in Table 6.15. Although some students indicated an awareness of multiple techniques, all of them chose to implement only one technique within the context of the interview. Notice that only one student (S12) used an expected "formal definition" approach (i.e., $\tau_{4_{b_{12}}}$); interestingly, S10 used the same formal definition to build a different technique (i.e., $\widetilde{\tau_{4_{b_{12}}}}$). As previously indicated, most students used (or seemed to be recalling parts of) $\tau_{4_{b_{11}}}$, which was another expected approach (the one based on the theoretical discourse we expected students to explore in Task 3). At first, it seemed that S8 was also recalling parts of this technique; but our analysis led us to identify in his solution a separate, albeit strongly related, technique, which we have denoted $\widetilde{\tau_{4_{b_{11}}}}$. In the following sections, we offer more details about these techniques and how participants chose, implemented, and described them.

Students	Techniques
S8	$\widetilde{\tau_{4_{b_{11}}}}$: Rewrite x_n as $M - y_n$ and argue that $\inf\{y_n : n \in \mathbb{N}\} = 0$.
S1, S2, S4, S5, S7, S9, S13, S14, S15	$\tau_{4_{b_{11}}}$: Argue that $(x_n)_{n\in\mathbb{N}}$ is increasing and $\lim_{n\to\infty} x_n = M$. (one of the three expected techniques, denoted $\tau_{4_{b_{11}}}$ in 5.1.4)
S10	$\widetilde{\tau_{4_{b_{12}}}}$: Argue that $a \leq M\ \forall a \in A$ and $\lim_{n\to\infty} x_n = M$.
S12	$\tau_{4_{b_{12}}}$: Argue that $a \leq M\ \forall a \in A$ and $\forall \epsilon > 0, \exists a \in A, M - \epsilon < a$. (one of the three expected techniques, denoted $\tau_{4_{b_{12}}}$ in 5.1.4)

Table 6.15 Participants' techniques for proving that $supA = sup\{x_n : n = 1, 2, \ldots\} = M$.

Notice that not all students are represented in Table 6.15. This is because some students (i.e., S3, S6, and S11) provided solutions that were difficult to classify. When faced with proving the supremum in Task 4(b), S6 claimed that "you'd have to use the definition," but was unable to state what "the definition" is. It is possible that she was trying to recall $\tau_{4_{b_{12}}}$; it is also possible that she was thinking of another technique entirely. S2, for example, had also begun her solution by saying that she needed to "prove by the definition"; and, when prompted by the interviewer, she indicated that it was the following definition that she had in mind:

$$\sup A = M \Leftrightarrow ((\forall\, a \in A, a \leq M) \wedge (\text{If } B \text{ is an upper bound for } A,\, B \geq M)).$$

In other words, S2 was compelled to use the definition of the supremum as the least upper bound (as opposed to the formal definition of the supremum used by S12). In 5.1.4, we had denoted the corresponding technique studied in RA I as $\tau_{4_{b_{13}}}$. As indicated in Table 6.15, no students, including S2, went further in implementing such a technique, or any other based on the definition of the supremum as the least upper bound.

Both S11 and S3 also presented some uncertainties about the definition (informal and/or formal) on which they were trying to base a solution. S11 seemed unsure if the supremum of a set had to be included in the set. When considering $A = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$, he said: "I'm just not sure if you can say that the supremum of A is equal to one if it actually doesn't go to one. But like it gets very close to one. Like it goes to zero point nine, nine, nine, nine, nine, …" It is possible that S11's uncertainty came from him recalling the commonly assessed technique we denoted as $\tau_{4_{b_{14}}}$: i.e., show $\sup A = M$ by arguing that M is the greatest element of A (i.e., $M \in A$ and M is an upper bound for A). S3, on the other hand, provided the following (incorrect) implications:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$
$$\Rightarrow \forall \epsilon > 0, \exists n \in \mathbb{Z}, \left|\frac{n}{n+1} - 1\right| < \epsilon$$
$$\Rightarrow \sup A = 1$$

After writing the second line, S3 explained: "I don't remember the exact [i.e., the formal] definition of a sup. But that should prove that I should get arbitrarily close to one, but I will never actually reach one." Note that while the first implication holds, the second implication requires an additional assumption: e.g., that $\forall n \in \mathbb{N}, \frac{n}{n+1} \leq 1$. While S3 and S11 may have been on the verge of developing an appropriate technique for solving the given task (S3, for example, was very close to $\widetilde{\tau_{4_{b_{12}}}}$), it seemed that they were unable to develop one in the interview due to a lack of understanding of the underlying technologies (in this case, the informal and/or formal meaning of the supremum).

6.4.2.3.1 Rewrite the elements of A and use the algebra of extrema of sets.

When starting to solve Task 4(b), S8's instinct was to consider the sequence in question, $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$, in a new form. He used long division to get $\frac{n}{n+1} = 1 - \frac{1}{n+1}$, a representation he later claimed was "much easier to deal with." Only one other student (S2) rewrote the sequence in this way; but she did not use it beyond calculating the limit. S8, in comparison, looked at the new representation of the sequence and reasoned as follows: "[1] is a positive number. So if I subtract the least large number [of $\frac{1}{n+1}$], then that would be the maximum." Although his discourse lacked some precision, it seemed S8 was employing a rule (part of what we might call "the algebra of extrema of sets"), which could be written in a mathematically precise way: e.g., If $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and $M \in \mathbb{R}$, then $\sup\{M - x_n : n \in \mathbb{N}\} = M - \inf\{x_n : n \in \mathbb{N}\}$. It is questionable if S8 would be able to describe his technique in such a general manner. To argue about the infimum of

$\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\}$, S8 seemed to call upon the same kind of technique as the students in the next section: i.e., he noted that $\frac{1}{n+1}$ is decreasing and has a limit of 0, whereby the supremum of $\left\{1 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ is $1 - \inf\left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} = 1 - 0 = 1$.

6.4.2.3.2 Argue that the elements of A form an increasing sequence converging to M.

As shown in Table 6.15, we identified nine of fifteen participants as trying to prove that $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$ by arguing that $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is an increasing sequence that converges to 1. The participants offered different kinds of justifications for how they knew the technique works.

To justify this technique, S9 referred to his experience of solving similar looking tasks in RA I:

> That was the way to go: [-] First prove that it's monotone increasing, and then prove that [-] the limit as n goes to infinity leads to something. And since it's monotone increasing – that was one of the theorems – like, that has to give the sup.

As expected, several other students (S7, S13, and S15) also referred to a theorem to justify their work. S15, for example, noted that "the sequence is increasing, but it's bounded. [-] So we can use the Monotone Convergence Theorem to say that it will converge to the supremum." As already discussed in 6.3.2.1, S13 was unique in demonstrating an interest and ability in proving the theorem. She also stood out in her solution to Task 4(b) in that she expressed her conviction in her chosen technique by referencing the theorem and then noting: "I can prove it. But I'll take it for granted now because we proved it." S9, in comparison, cited the theorem and then noted that "it's easy to visualize. I mean, to understand. It's not very abstract."

The other five participants in this group either made no mention of "the theorem" (S2, S4, S5, and S14), or did not recall its relation to the sup (S1). "The theorem says that a bounded converging sequence converges to its limit," S1 explained: "But that's not really useful for this sup. So that's why I went back to the definition." Note that such a comment might reflect the fact that the "Monotone Convergence Theorem" is often stated without reference to the supremum. In the textbook used for RA I, for example, the theorem is stated as follows: "A bounded monotone sequence converges," and the specification that the sequence converges to its supremum (or infimum) is introduced only in the proof. Whatever the case, S1 justified his chosen technique, not by “the theorem,” but “by definition.” When encouraged to explain further, it seemed S1 was not referring to a formal definition, as one might expect: he explained that

> since the supremum of A is the largest… not really value, cause it’s not really in the set, but I’ll just say it that way. It’s easier for me to explain it that way: the largest value in the set. And one is the furthest value that [the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$] goes. [-] And it increases. So that means one is the highest value of the set.

S1’s explanation seems to be based on an informal personal concept definition of the supremum as “the highest value of the set,” which is “not really in the set.” Other students seemed to be operating according to similar informal imagery. S4 and S5, for instance, emphasized that the supremum is something that the sequence increasingly goes towards, but does not surpass. S14 also seemed to understand the supremum as the “greatest possible value.” Consider, for instance, when he concluded that “the term [in $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$] is increasing and increasing. [-] and one is as it goes to infinity – you can’t have a number bigger than infinity – so then the supremum is one.”

Among the students in this group, we observed variation not only in the ways they justified their technique, but also in how they implemented it. The diverse arguments used by these participants to show that $\lim_{n\to\infty}\frac{n}{n+1}=1$ are summarized throughout 6.4.2.2. The group also demonstrated different techniques for showing that the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing, as depicted in Table 6.16, none of which completely reflected our expectations. We describe each of these techniques in the following sections.

Students	Techniques
S5	Plug in values.
S1, S2, S7, S9	Use Induction.
S15	Argue that $x_{n+1}-x_n>0$.
S14	Analyze the monotonicity of $(x_n)_{n\in\mathbb{N}}$ in parts.

Table 6.16 Participants’ techniques for showing that $(x_n)_{n\in\mathbb{N}}$ is increasing.

6.4.2.3.2.1 Plug in values.

When prompted to think about how to prove that $\sup\left\{\frac{n}{n+1}: n \in \mathbb{N}\right\} = 1$, S5 had an idea: "I think the sequence is decreasing. So it makes sense that one would be the sup, cause it's [-] the highest bound of the sequence. Cause then we just keep approaching a smaller number." When asked why she thought the sequence is decreasing, S5 said: "Oh, cause I kind of, I think I plugged in the numbers properly. I'll check." To check, S5 listed the first few elements of the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ (as depicted in Figure 6.25) and realized: "Oh no, it doesn't decrease." S5 was the only participant to exhibit this technique – plugging in values – alone to convince herself that the sequence in question is increasing (and converging to 1). It is possible that she was acting based on a partial recall of the induction proof technique chosen by the students in the next section. She did not, however, make mention of the induction proof technique.

6.4.2.3.2.2 Use Induction.

Induction was the most commonly exhibited technique for proving that $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing. Six participants (S1, S2, S4, S7, S9, and S15) were drawn to this technique, though only four of them (S1, S2, S7, and S9) implemented it on their own (i.e., without support from the interviewer). These four students started their proofs by writing out the first couple terms of the sequence:

$$x_1 = \frac{1}{2}, x_2 = \frac{2}{3}, \frac{2}{3} > \frac{1}{2}.$$

"It usually helps to see how it behaves," explained S7, "[-] to make sure that it's actually increasing." The students then continued their proofs in similar manners:

> Assume that $x_n < x_{n+1}$ for n, i.e., $\frac{n}{n+1} < \frac{n+1}{n+2}$.
>
> Prove that $x_{n+1} < x_{n+2}$, i.e., $\frac{n+1}{n+2} < \frac{n+2}{n+3}$.

At this point, S2 got stuck, while S1, S7, and S9 swiftly completed their proofs by finding equivalent inequalities: e.g.,

$$\frac{n+1}{n+2} < \frac{n+2}{n+3}$$

$$(n+1)(n+3) < (n+2)^2$$

$$n^2 + n + 3n + 3 < n^2 + 4n + 4$$

$n^2 + 4n + 3 < n^2 + 4n + 4$

$3 < 4$ ■

Recall that we had expected the successful RA I student to carry out manipulations like this to prove that $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing.

S1, S7, and S9 all gave reasons for choosing the induction proof technique. S1, for instance, seemed to be triggered by the need to prove a mathematical statement – in this case, $x_n < x_{n+1}\ \forall n \in \mathbb{N}$ – that depends on $\mathbb{N}$:

> Natural numbers. 1, 2, 3, … Induction. That's just the way it works. Well, not *it*… the way *I* work. If I see natural numbers, prove that something happens with natural numbers, I'd always start with induction. [-] I think it's the easiest way… for me.

S9 and S7 also called the technique "easy." S9 said it was the "easiest way [he] know[s] to prove that a sequence is increasing, or decreasing. Cause the other way would be to just enumerate everything and prove that every time it is smaller." In saying this, perhaps sarcastically, S9 seemed to be suggesting as an alternative the technique of checking that $x_1 < x_2, x_2 < x_3$, and so on. S7 emphasized how induction allows one to avoid this: "If you want to prove it for any values, you prove by induction. So you prove the first one. And then you prove it for $n + 1$ value. And then, the rest, you know that it's true for every one."

6.4.2.3.2.3 Show that the difference between consecutive terms is positive.

S13 was the only student who, when faced with the task of showing that a sequence $(x_n)_{n\in\mathbb{N}}$ is increasing, chose the following technique:

Calculate $x_{n+1} - x_n$ and argue that it is positive.

While looking at $A = \left\{\frac{n}{n+1} : n = 1,2, \ldots\right\}$, S13 said: "Let's see if this is increasing or decreasing." Then she wrote the following line:

$$x_{n+1} - x_n : \quad \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0\ .$$

She concluded: "So this is increasing. [-] This is obvious." The interviewer did not ask S13 to explain how her chosen technique works: e.g., if $x_{n+1} - x_n > 0\ \forall n \in \mathbb{N}$, then $x_{n+1} > x_n \forall n \in \mathbb{N}$, which means, by definition, that $(x_n)_{n\in\mathbb{N}}$ is increasing. Given the simplicity of this explanation and the fluency with which S13 implemented the technique, we suspect that she understood what she was doing.

6.4.2.3.2.4 Analyze the monotonicity of the sequence in parts.

Like S13, S14 also exhibited great fluency in a technique for arguing that a sequence like $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing. Having rewritten the sequence in the form $\left(\frac{1}{1+\frac{1}{n}}\right)_{n\in\mathbb{N}}$ in order to show its limit, S14 explained that

> as n increases, this term [$1/n$] decreases and thus the denominator decreases, which means the whole term increases. [-] And this is going really nitpicky. I'm not really thinking this deeply because I just know this stuff quickly, right?

S14 showed some fluency in what we could call "the algebra of monotonicity of sequences," which is supported by technologies such as: If $(x_n)_{n\in\mathbb{N}}$ decreases (and $x_n \neq 0\ \forall n \in \mathbb{N}$), then $\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}}$ increases. The interviewer did not prompt S14 to give such an explanation (or others) for his chosen technique. It is possible that he would consider such explanations "nitpicky," and would rather offer intuitive explanations, as he did in other cases (see, e.g., 6.4.2.2.4).

6.4.2.3.3 Argue that the elements of A form a sequence that is bounded above by M and converges to M.

When faced with the task of showing that $\sup\left\{\frac{n}{n+1}: n \in \mathbb{N}\right\} = 1$, three students (S6, S10, and S12) had the instinct to start by showing that the sequence $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is bounded above by 1. S10 provided a rather formal argument:

> Need to show: $\forall n \in \mathbb{N}, \frac{n}{n+1} \leq 1$.
>
> Let $n \in \mathbb{N}$. Then $\frac{n}{n+1} < \frac{n}{n} = 1$. ■

S6, in comparison, noticed that in the expression $\frac{n}{n+1}$, "the bottom part is bigger, so you'd get a fraction. I mean, a decimal. [-] So it's less than one for sure." Following his proof that 1 is an upper bound for $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$, S10 considered using the same technique as the students mentioned in

6.4.2.3.2: "I can show that this function is always [-] going to be increasing. And it's limit is one." He even wrote out the inequality that he needed to show – $\frac{n+1}{n+2} > \frac{n}{n+1}$ – to prove that the sequence is increasing. Then he stopped: "Actually, I'm just going to show the limit. [-] Now that I've shown the bound, I think the limit and the bound will be sufficient to show that the supremum is also one."

S10 distinguished himself from his peers in that he seemed to develop his chosen technique in the context of the interview (as opposed to, for example, recalling a technique learned in RA I). Moreover, S10 knew that his technique would work through his fluency in formal definitions. Recall that S10 chose to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ using the formal definition of sequence convergence (see 6.4.2.2.5 for the details). He claimed that, if he did this, "then it's easy to show that for any epsilon greater than zero, there's going to be an element of A [= $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$] that's in that interval [i.e., $(1-\epsilon, 1]$]." The following implications were more or less implicit in S10's solution (recall that S10 also showed evidence of being able to argue in this way throughout his solution to Task 3, as discussed in 6.3.2.4):

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \left|\frac{n}{n+1} - 1\right| < \epsilon$$

$$\Rightarrow \ \forall \epsilon > 0, \exists N \in \mathbb{N}, \left|\frac{N}{N+1} - 1\right| < \epsilon$$

$$\Rightarrow \ \forall \epsilon > 0, \exists N \in \mathbb{N}, 1 - \epsilon < \frac{N}{N+1}$$

If we also know that $\forall n \in \mathbb{N}, \frac{n}{n+1} \leq 1$, then $\forall \epsilon > 0, \exists N \in \mathbb{N}, N \in (1-\epsilon, 1]$.

So $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$.

Note that both S6 and S12 also succeeded in showing that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, either by thinking about what matters in the long run (see 6.4.2.2.2), invoking l'Hospital's Rule (see 6.4.2.2.3), or using the algebra of limits (see 6.4.2.2.4). They did not realize that this conclusion, combined with their argument that $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is bounded above by 1, could be used to prove that $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$. They decided to go back to "the definition," which is what they recalled doing in RA I for similar looking tasks.

6.4.2.3.4 Use a formal definition of the supremum.

Both S6 and S12 claimed that they wanted to use a definition to prove that $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$; but neither could recall such a definition. S12 explained that "the definitions were super helpful. [-] Being fluent in Real Analysis was like being able to remember the terms." In response to this, the interviewer chose to provide a partial definition, which S12 seemed to recognize: i.e., $\forall \epsilon > 0, \exists a \in A, a + \epsilon > \sup A$. S12 also seemed to understand the overall idea of the related technique: "So, for any epsilon we choose, we can find an x_n [$= \frac{n}{n+1}$] such that [-] this inequality holds." He proceeded to manipulate the inequality as follows:

$$\frac{n}{n+1} + \epsilon > 1$$
$$\epsilon > 1 - \frac{n}{n+1}$$
$$\epsilon > \frac{1}{n+1}$$
$$n + 1 > \frac{1}{\epsilon}$$
$$n > \frac{1}{\epsilon} - 1.$$

At this point, S12 was a bit unsure of how to conclude, though he seemed to have the necessary ingredients for doing so. He noted, for instance, that the required n would exist because "the set of natural numbers has no upper bound."

6.4.3 Reflections on the Nature of Practices, Positioning, and Activities[37]

Non-Mathematical: Identify task with technique in an automatic fashion.

VS.

Mathematical: Study the task, think of equivalent formulations or different possible techniques, and perform the assessment: "Will this work? Is this better?"

[37] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.4.1 and 6.4.2 that begin to address our research objectives: characterizing the nature (mathematical or not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

It seemed many participants had developed non-mathematical practices in the sense of identifying tasks with techniques in an automatic fashion. What was interesting in Task 4 was how this sometimes led participants to unknowingly do more to solve the task than was required. Consider, for example, the participants who automatically thought to use the by-induction approach to prove that $(x_n)_{n\in\mathbb{N}} = \left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing, and did not realize that their manipulations of the inequality $x_{n+2} > x_{n+1}$ would have sufficed. Another example is how for Task 4(b) – prove that $\sup\left\{\frac{n}{n+1} : n = 1,2,\ldots\right\} = 1 = \lim_{n\to\infty}\frac{n}{n+1}$ – most participants saw two tasks (prove a limit and prove a supremum), each of which corresponded to its own technique. S1, for instance, proved the limit using the formal definition and then proved the supremum by showing that the set forms an increasing sequence that converges to the supremum; S12 proved the limit using the algebra of limits and then proved the supremum using a formal definition; and so on. Only one student (S10) was able to challenge his reflex of seeing Task 4(b) as two tasks requiring two techniques and reduced the task to proving the limit and showing the sequence is bounded above.

As mentioned before, we think automatic behaviour can be linked to different positions (perhaps the only exception is the Learner). It is natural for the Student, who may perceive the minimal core of practices to be learned as being comprised of one-to-one correspondences between types of tasks and techniques. It can arise for the Skeptic if they do not know multiple techniques that they perceive as being appropriate for solving the given task. It can also occur for the Mathematician in Training or Enthusiast, whose careful study of the practices to be learned would allow them to quickly recognize when a given task might be solvable using those practices. In the case of Task 4(b), we also expected the identification of two tasks and two techniques given that the activities that were offered to participants in RA I suggested disconnected paths to practices for "proving a limit" and "proving a supremum." S10's ability to connect the paths seemed linked to his positioning as an Enthusiast; in particular, his personal interest of breaking things down to their basic formal definitions, which enabled him to formally represent Task 4(b), and think about how proving the limit and proving the supremum are connected.

Non-mathematical: Construct a proof
by recalling each step of a proof one has seen or done before.

Vs.

Mathematical: Implement a general proof technique.

Although most participants did not spontaneously construct the expected formal proofs to solve Task 4, we found evidence of the development of non-mathematical practices in the above sense: i.e., constructing a proof by recalling each step of a proof one has seen or done before. In reflecting on their non-formal approaches, many participants revealed that they had only learned the formal proof techniques long enough to succeed on assessments. Consider, for example, the participants who were asked by the interviewer if they could prove $\lim_{n\to\infty}\frac{n}{n+1}=1$ using the formal definition of sequence convergence. Some did not recall the definition; and if they recalled or were given the definition, it was clear that they were trying to proceed based on remembering the steps (rather than interpreting what the definition means and how it produces a proof technique). This also seemed to be the approach taken by S1, who spontaneously chose to construct formal proofs of both the limit $\lim_{n\to\infty}\frac{n}{n+1}=1$ and the unboundedness of $A=\left\{\frac{n^p}{n+1}: n\in\mathbb{N}, p>1\right\}$. S1 had a better memory of the definitions and the steps (including the tricks) than many of his peers; but he did not seem to completely understand how the steps worked or what he was trying to do overall. He made a mistake, for example, in adapting "the trick" from the proof that A is unbounded above to the proof of the limit. He was also unable to give a coherent explanation of what he was trying to accomplish in his proofs. We think this behaviour is linked to a Student positioning towards assessment activities that could be solved based on a good memory of the steps. To support students in the transition to mathematical practices in the above sense, it could be helpful to pose activities that explicitly invite them to abstract their proof solving experiences and think about the global ideas underlying the proof technique.

There were three students (S10, S12, and S13) who seemed to have learned mathematical practices in the sense of being able to implement a more general proof technique. S10 and S13 knew the formal definitions well, and all three students could use formal definitions – of (un)boundedness, of the limit, and/or of the supremum – to clearly explain what they needed to do to complete their proofs. We think this is linked to their positioning as an Enthusiast (S10), Student-Learner-Mathematician-in-Training hybrid (S12), and Mathematician in Training (S13); in particular, their initiative to think about the bigger picture underlying the assessment activities they were offered. We note that while S13 exhibited an impressive fluency in general proof techniques, S10 and S12 still presented some challenges. It seemed S10 had not learned the general algebraic techniques needed to perform the manipulations of inequalities in his proofs: e.g., rearranging $\frac{n^p}{n+1} > M$ to have n in terms of M. S12 also lacked some confidence in making his conclusions. We suspect that an explicit invitation to reflect on certain parts of the bigger picture in proof construction activities could have also been helpful for these students.

Non-Mathematical: Theoretical discourses based on established experience, inert knowledge, and/or taking one's own understanding for granted. VS. *Mathematical*: Theoretical discourses based on clarifying, questioning, and verifying one's own understanding.

When it came to proving limits of rational sequences, it seemed most participants were still operating according to the non-mathematical practices they had learned in Calculus: i.e., intuitive, informal, or procedural techniques for calculating limits, which lacked mathematical theoretical discourses for explaining when and why they are valid. This could be linked with the Student position; in particular, with trying to routinize the formal proof techniques taught in RA I just long enough to succeed on assessments. As mentioned above, many participants forgot the formal definitions and lacked an understanding of how to use them to produce a general proof technique. Hence, their only option for proving the limit of a rational sequence may have been to use the non-mathematical practices they had developed previously. It is also likely that many participants were silent Skeptics when it came to the limit proving activities in RA I: i.e., they were not convinced that they needed to learn the formal proof techniques because they already knew other simpler techniques for solving the same task.

S9 was very open about his positioning as a Skeptic and clearly exhibited why students might have been skeptical about learning the formal proof techniques. Throughout his solutions to Tasks 3 and 4, he exhibited an awareness of four different techniques for determining the limit of a rational sequence: compare the degree of the numerator and the denominator (based on his established experience from Calculus), think about what matters in the long run (based on taking his own understanding of what's significant and what's not for granted), use l'Hospital's Rule (based on inert knowledge: i.e., the acceptance of the truth of some theorem), and by definition (based on the formal definition of sequence convergence). S9's position of Skeptic was rooted not only in his awareness of these techniques, but in his perception that (a) they all equally accomplished the job of solving the same task; (b) some are easier to implement than others; and (c) some were either not at all or only eventually acceptable in RA I. It makes sense that one would question the need to replace perfectly good techniques with more complicated ones. It seems this questioning may be enhanced if one feels the simpler techniques are being temporarily forbidden for no good reason. We think that the assessment activities offered in RA I did not enable S9 (and likely many of his peers) to face, struggle with, and resolve questions about whether certain intuitive techniques are mathematically legitimate or not, and, in connection with this, whether the formal techniques for proving limits are really useful (and what they are useful for).

Indeed, asking students to prove (by definition) a simple limit like $\lim_{n\to\infty} \frac{n}{n+1} = 1$ neither invites them to work on previously learned non-mathematical practices, nor convinces them that the new formal practices are worth learning. To assist students in making the transition towards mathematical practices as depicted in the above box, it might be helpful to explicitly invite them to think about their previously developed techniques in the context of the new material. One could pose activities that invite students to give more precise formulations of the techniques they learned previously and to question if they are legitimate from a mathematical point of view: e.g., Can you prove that the technique works on a certain type of task? Indeed, some of the techniques the students used (like comparing growth or ignoring constants) can lead to legitimate mathematical techniques for certain types of tasks (like proving the limit of a rational sequence). One could also pose activities with specified sequences that would explicitly confront students with the limitations of previously developed techniques: e.g., where imprecisions or informalities in the techniques lead to erroneous conclusions, or where the more formal techniques are simply necessary.

There were two participants (S7 and S12) who showed a slight indication of supporting their algebraic approach to "proving limits" with some sort of acknowledgement of underlying theory; and there were two participants (S1 and S10) who showed signs of having been convinced of learning the formal approach to proving limits (perhaps because they had to or were interested in it). Only one participant (S13), however, consistently demonstrated an inclination towards mathematical practices in the sense of appreciating the difference between convincing oneself and "speaking mathematics" to a mathematician: i.e., trying to clarify, question, and verify one's understanding based on solid mathematical foundations. This was one of the main ways that S13 revealed her positioning as *the* Mathematician in Training.

6.5 Task 5

Find the limit

$$\lim_{n\to+\infty} \frac{ln(2 + \ a^n + arctan(x_n))}{n}$$

where $a \in \mathbb{R}$ *and* $(x_n)_{n\in N}$ *is any sequence of real numbers.*

6.5.1 Practices to be Learned[38]

We have, for $a > 1$ and large enough n,

$$\frac{\ln(a^n)}{n} \leq \frac{\ln(2 + a^n + \arctan(x_n))}{n} \leq \frac{\ln(a^n + a^n + a^n)}{n}$$

$$\frac{\ln(a^n)}{n} \leq \frac{\ln(2 + a^n + \arctan(x_n))}{n} \leq \frac{\ln(3a^n)}{n}$$

$$\frac{\ln(a^n)}{n} \leq \frac{\ln(2 + a^n + \arctan(x_n))}{n} \leq \frac{1}{n}\ln(3) + \frac{\ln(a^n)}{n}$$

The limit on the left is $\ln(a)$ and the limit on the right is also $\ln(a)$.
Thus, by the Squeeze Theorem, the limit of the sequence is $\ln(a)$. ■

Figure 6.26 An example of the kind of partial solution anticipated for Task 5.

We would expect a successful RA I student to use the Squeeze Theorem to solve Task 5. We were unsure if they would spontaneously analyze different limiting behaviours of the given sequence for different values of a. We were also unsure if they would make an explicit assumption, as shown in the first line of Figure 6.26, or if such an assumption on a would remain implicit in their solution. A successful RA I student would be able to construct the subsequent argument in the figure: i.e.,

[38] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 5. This model is synthesized in table form in Appendix F.

write the bounds, note that the limit on the left and the limit on the right are equal, and, finally, cite the Squeeze Theorem to conclude that the limit of the sequence is also equal to that same limit. To receive full marks, the student would not need to write down all justifications for each step: e.g., they would not necessarily be required to note that $\ln(x)$ is an increasing function.

$T_{5_{ST}}$: Use the Squeeze Theorem to find $\lim\limits_{n\to+\infty} b_n$. Typically: b_n is of the form $\frac{\ln(f(n))}{n}$ or $\sqrt[n]{f(n)}$ where $f(n)$ is a sum including a dominant exponential term (i.e., a^n with $a \geq 2$), and possibly a trig function (sine or cosine) whose argument is of some interesting algebraic form (e.g., containing factorials and large exponentials).
τ_5: Find y_n and z_n such that $y_n \leq b_n \leq z_n$ and $\lim\limits_{n\to+\infty} y_n = \lim\limits_{n\to+\infty} z_n$. Typically: y_n and z_n have the same form as b_n, with $f(n)$ equal to some multiple of the dominant exponential term.
θ_5: "By the Squeeze Theorem."

Table 6.17 The practice to be learned in RA I most relevant to Task 5.

We could model the practice to be learned in RA I and most relevant to Task 5 as shown in Table 6.17. Among the assessment activities we analyzed, we characterized a path where the Squeeze Theorem was used to find the limit of specified sequences. Some activity statements gave the direct instruction that the theorem should be used; others looked similar enough to these activities to judge that the theorem would be useful. Students were exposed to two types of sequences that look like the one in Task 5: one of the same form, $\frac{\ln(f(n))}{n}$, and the other of a slightly different more commonly tested form, $\sqrt[n]{f(n)}$, where f has some particular properties. To illustrate these properties, consider the following two sequences whose limits students were asked to find on past midterms:

$$\frac{\ln(5^n + \pi \cdot (-1)^n + \sin^2(n^n + (10!)^n))}{n}$$

$$\sqrt[n]{9^n - 3^n + \cos^2(n^n + (2016!)^n)}$$

As expressed in Table 6.17, and exemplified above, the sequences often contained a trig function (sine or cosine) with an interesting looking argument (e.g., containing factorials and large exponentials). They also typically contained a dominant exponential term (5^n or 9^n in the sequences shown above). Hence why we thought a successful RA I student might spontaneously construct a solution to Task 5 under the assumption that a^n is the dominant term.

In contrast, we would expect a mathematician to see the more general nature of the sequence in Task 5 as necessitating a consideration of how the sequence is defined and the construction of a solution based on different cases for a. For instance, the mathematician would recognize that the sequence is not well-defined for all real numbers a and sequences $(x_n)_{n\in\mathbb{N}}$, and they may point out that this can be avoided by requiring $a > \frac{\pi}{2} - 2$. They would also identify the importance of the different behaviours of a^n for different values of a in determining the different limiting values of the sequence: i.e., they would call upon the fact that if $a > 1$, then $a^n \to \infty$ as $n \to \infty$, if $\frac{\pi}{2} - 2 < a < 1$, then $a^n \to 0$ as $n \to \infty$, and if $a = 1$, then $a^n = 1\ \forall n \in \mathbb{N}$. Note that a successful RA I student would have engaged in at least one assignment activity where they proved that if $|a| < 1$, then $\lim\limits_{n\to\infty} a^n = 0$. What we wondered about was whether the participants of our study would be triggered to recall and use this fact to solve Task 5, or if they would ignore, overlook, or be unaware of a need to vary a. If a participant seemed to be operating in the latter sense, we planned to ask if they were making an assumption on a. We hoped to see if, in spite of a developed routine, they were indeed capable of performing the analysis needed to solve Task 5.

Another potential difference between a successful RA I student and a mathematician is that we would expect the latter to be able to discern the limit values of $\frac{\ln(2+a^n+\arctan(x_n))}{n}$ just by looking at the sequence and noting that $\arctan(x_n)$ is bounded between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Alternatively, the mathematician might write the inequalities shown below (with the ability to mathematically justify them), from which they could quickly calculate the limits using the Squeeze Theorem (i.e., they would not need to continue bounding, as in the solution in Figure 6.26):

$$\frac{\ln\left(2 + a^n - \frac{\pi}{2}\right)}{n} \leq \frac{\ln(2 + a^n + \arctan(x_n))}{n} \leq \frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n}$$

In particular, the limit of the sequence on the left, right, and therefore middle, is equal to $\ln(a)$ if $a > 1$ and 0 if $\frac{\pi}{2} - 2 < a \leq 1$. This said, the mathematician would also be able to adapt the solution shown in Figure 6.26 to when $\frac{\pi}{2} - 2 < a \leq 1$, if asked to do so. We were interested in seeing if the participants of our study would also be able to see the limit values and/or adapt their solutions to different values of a.

6.5.2 Practices Actually Learned[39]

We expected a successful RA I student to identify Task 5 with the need to use the Squeeze Theorem. Hence, it was not surprising when thirteen out of fifteen of the participants in our study identified the task in this way. What was interesting were the different ways in which they spoke about choosing a Squeeze Theorem-inspired technique; and how a couple of them, after identifying the task in this way, decided that they knew how to solve a more particular type of task. We were also surprised at the other techniques some participants chose for solving Task 5, and how this led them to reflect on the difference between "limit finding" and "limit proving" tasks. Among participants, we also observed very different ways of dealing with the generality of Task 5. Only two students realized that they needed to vary parameters to solve the task, while others struggled with the parameters or did not address them at all. We present these features of our data, related to identifying Task 5 and choosing a technique, throughout the first section (6.5.2.1) below. Then, in 6.5.2.2, we describe some key characteristics of the participants' implementation of the Squeeze Theorem; in particular, the approaches they took to bounding the given sequence and how they justified their bounds.

6.5.2.1 Identifying the Task and Choosing a Technique

There are at least three components of Task 5 that could contribute to the way in which it is interpreted and identified as belonging (or not) to a certain type of task. First and foremost, there is the actual sequence, $\frac{\ln(2+ a^n+\arctan(x_n))}{n}$, the nature of which led most students (thirteen out of fifteen) to identify Task 5 as requiring a specific technique, based on using the Squeeze Theorem (see 6.5.2.1.1). Two of such students felt, nevertheless, that the sequence made the task significantly different from the types they knew how to solve (see 6.5.2.1.4). Second, there is the way that the task is stated – "find the limit" – which did not stop students from feeling that solving the task necessitates a proof (see 6.5.2.1.2). Third, there is the dependence of the task on parameters

[39] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 5. These models are synthesized in table form in Appendix F.

($a \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ any sequence of real numbers), which the students addressed to varying degrees (see 6.5.2.1.3). In what follows, we present our interpretation of how students identified Task 5 and chose their techniques according to each of these three components.

6.5.2.1.1 It's a Squeeze Theorem task.

All but two students (S2 and S3) spontaneously identified Task 5 as being a member of the expected type:

> $T_{5_{ST}}$: Use the Squeeze Theorem to find the limit of a given sequence.

For several students, this identification was practically immediate. S11's first words upon seeing Task 5 were: "Ok well usually when I see something like that, I just… Don't ask me why cause I can't tell you why, but… Squeeze Theorem." In a similar manner, S15 said: "It looks like the Squeeze Theorem. [-] Well, it's my feeling, my intuition." Other students gave more explanation as to why the appearance of Task 5 inspired in them the instinct to use the Squeeze Theorem. S1, for example, explained that "the teacher told us 'When you see complicated things, we're doing this on purpose to confuse you. And go and do the Squeeze Theorem.'" It seemed that many students had learned to identify tasks as belonging (or not) to $T_{5_{ST}}$ based on how complicated (or "not nice," or "confusing," or "weird") the sequence looked.

Among the collection of explanations provided by other students, we found different reasons that the given sequence may have been judged as "complicated." For S4, S5, and S6, it seemed to come from the fact that the sequence is a composition of several sequences. S5 said that the problem is "big," "with a lot of values"; S6 noted that there are "three terms" in the argument of the logarithm; S4 elaborated further:

> Having practiced it a lot, you start to, you know, see patterns. [-] Like the fact that the variable, I guess, that is being pushed to a value in the limit, it is present at more than one spot. [-] And the presence of a trig function also kind of ticks off another box kind of: "Oh, let's confuse the student here."

S7, S8, S10, and S13 noted, in addition, the general nature of $(x_n)_{n\in\mathbb{N}}$. S10, for instance, claimed that he "would never be expected to deal with this term here, $\arctan x_n$, for any possible sequence of real numbers, unless that's clearly bounded by something on either side." This comment points to a certain expectation, which some students may have built up through their experiences: i.e., If

I am given a complicated function, I should be able to deal with it through bounding. This idea was also present in the way S9 justified his choice of technique:

L: "What made you, [-] with this problem, say 'Squeeze Theorem is the way to go'?"

S9: "Cause they usually looked like that. [-] You'd have something weird, but, you know, it would end up being bounded. So you could [-] get away with it by just… yeah, using it with your bounds for your Squeeze Theorem."

A few students also justified their identification of Task 5 with the Squeeze Theorem by stating that the other techniques they knew would not work. S4 said that "there's no way to take the limit of this with like the epsilon-delta definition"; S7 ruled out the technique he had used in Task 4 (see 6.4.2.3.2), noting that "proving that the sequence is decreasing or increasing, it would be…hard? Uh, yes, it would be too hard given the function. [-] It's not even monotone"; and S8 pointed out that the general nature of $(x_n)_{n\in\mathbb{N}}$ would not allow him to use l'Hospital's Rule. S8 also thought that "evaluating [the limit] straight is going to be disgusting"; i.e., he did not think he would be able to just plug in $n = \infty$ (as he could for a simpler sequence: e.g., $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} = \left(1 - \frac{1}{n+1}\right)_{n\in\mathbb{N}}$). For these students, it seemed the given sequence was too complicated to use any technique other than one based on the Squeeze Theorem. This idea was challenged by those students (S2 and S3) who did not spontaneously identify Task 5 with $T_{5_{ST}}$.

When compared to the other fourteen participants, S3 spent a considerable amount of time reading and rereading the statement of Task 5. He too noted the complicated nature of the sequence: "There's a lot of things. There's a ln. There's an exponential function. There's an arc function. [-] For the record, I have no idea what I'm going to do yet." Even after noticing that the arctangent term can be bounded above, S3's reflex was not to use the Squeeze Theorem: "So first I'm going to solve it intuitively," he explained to the interviewer. After a few moments, S3 had "the answer" written down on his page:

$$\lim_{n\to\infty} \frac{\ln(2 + a^n + \tan^{-1}(x_n))}{n} = \lim_{n\to\infty} \frac{\ln(a^n)}{n} = \ln(a).$$

The given limit is indeed equal to $\ln(a)$ if $a > 1$ (we discuss the details of how the students dealt with $a \in \mathbb{R}$ in 6.5.2.1.3 below). In a similar manner, S2 quite quickly reduced the given task to finding $\lim_{n\to\infty} \frac{\ln(a^n)}{n}$, though she was unable to see which logarithm rule would help her proceed to the same answer as S3.

It seemed that S2 and S3 had called upon a different technique for solving Task 5. To explain his work, S3 said:

> Whenever I solve with limits, it's like: What doesn't matter here? When things go to infinity, there are a lot of things that stop to matter. [-] The two stops to matter and this [the $\tan^{-1}(x_n)$ term] stops to matter, because this is never greater than pi over two [-] and this [a^n] goes to infinity.

Similarly, S2 explained that "when n is going to infinity, the a^n is going to be very big, bigger than these two [2 and $\arctan(x_n)$], so these two [-] I don't need to include them in my function." The technique S2 and S3 were applying seemed to consist of comparing the growth rates of the different terms in the sequence with the goal of determining which terms could be ignored in the calculation of the limit. We called this technique "thinking about what matters in the long run" when we discussed how students argued about the values of limits of rational sequences in solving Task 4 (see 6.4.2.1.2.3 and 6.4.2.2.2). For the limit given in Task 5, S2 and S3 decided that the constant 2 and the bounded sequence $\tan^{-1}(x_n)$ could be ignored because a^n (again, assuming that $a > 1$) increases to infinity.

Although S2 and S3 were the only ones to spontaneously choose this technique for solving Task 5, they were not the only ones to highlight that certain parts of the sequence in question "matter more" or "dominate" as n tends to infinity. One student (S12) did this when he justified why he used the Squeeze Theorem:

> I thought Squeeze Theorem because I know I have one term here that's really the dominant term, that really dictates the behaviour of the function: a^n. [-] I thought the Squeeze Theorem because I was trying to find a way to show the behaviour of this function without looking too closely at these terms [i.e., 2 and $\arctan(x_n)$].

There could be several reasons why students who had this kind of discourse did not use it to support the same kind of technique chosen by S2 and S3. In S12's case, it is possible that he was not aware of such a technique since he was not one of the seven students (i.e., S4, S6, S8, S9, S11, S13, and S15) who we identified as solving a limit in Task 4 by thinking about what matters in the long run.

It is also possible that some students' identification of the type of sequence in Task 5 with the Squeeze Theorem was so strong that they would not have thought to try to use another technique. When justifying his choice to use the Squeeze Theorem, S12 also noted that his "thought process in part comes from seeing examples in class in which the teacher will take like maybe some cos function, some trig function that has two bounds in it." He was one of two students (the other was S14) who explicitly linked the given sequence with the Squeeze Theorem, not because it was complicated, but because it could be bounded. Lastly, it is also possible that some students did not think the technique employed by S2 and S3 was appropriate for solving the given task. We discuss this further in the next section.

6.5.2.1.2 Finding vs. Proving a limit.

Most of the students we interviewed tried to find a value (or values) for the limit in Task 5 – $\lim_{n\to\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n}$ – through a technique based on the Squeeze Theorem. We might say that these students were trying to "find the limit" and "prove the limit" at the same time. In contrast, three students (S1, S3, and S14) used what we might call "intuitive" techniques to first find (or guess) a value for the limit, before attempting to construct a proof based on an "acceptable" theorem or rule. In other words, they seemed to indicate that the given task could be identified with at least two different types of tasks, each of which required its own type of technique:

T_{5_F}: Find (or guess) the limit of a given sequence using "intuitive" techniques.

T_{5_P}: Prove the limit of a given sequence using an "acceptable" theorem or rule.

We already described the intuitive technique used by S3 in 6.5.2.1.1. S1 and S14 made their guesses based on a slightly different technique. Upon seeing the task, S14 said: "Well, instantly I would say it doesn't matter what's in the log, but it can never grow faster than n. So the limit's zero. I think. That would be my guess." In a similar manner, S1 immediately wrote "$= 0$" beside the limit in the task statement and later gave the following explanation: "Well I'm pretty sure it goes to zero. [-] Cause this ln goes up really slowly, and $[1/n]$ goes down pretty quickly." It seems that both students were deducing a value for the limit by comparing the growth of the numerator and the denominator of the given sequence: $\frac{\ln(2+ a^n+\arctan(x_n))}{n}$. We identified other students as successfully using the same technique to argue that when $p > 1$, $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ (see 6.4.2.1.2.2). The difference in complexity between the two sequences may be why the intuitive technique was a bit more difficult to implement in Task 5. We could abstract the argument made by S1 and S14 as follows: Since n grows faster than $\ln n$, $\lim_{n\to\infty} \frac{\ln(f(n))}{n} = 0$ for any positive real-valued function $f(n)$. This is not true in general (consider, e.g., $f(n) = 0.1^n$ or 2^n); it is true, however, under the condition that there exist positive real numbers c and d such that for sufficiently large n, $c \leq f(n) \leq d$. For the sequence in Task 5, this means that the guess made by S1 and S14 would work if $a \in \left[\frac{\pi}{2} - 2, 1\right]$.

As mentioned above, both S1 and S14 did not stop at making a guess for the limit. They proceeded in trying to develop a proof using the Squeeze Theorem; and, in doing so, they were forced to confront potential issues with their intuitive technique. While trying to bound $\frac{\ln(2+a^n+\arctan(x_n))}{n}$, both students were brought to focus on the argument of the logarithm, $f(n) = 2 + a^n + \arctan(x_n)$, and their reasoning began to resemble S3's (as described in 6.5.2.1.1). S14, for example, explained that

> [$\arctan(x_n)$] is never more than pi over two, [2] is never more than two – never less than two either – and I would say [a^n], you know, just by the log rule, this n would come down and it would cancel with this n [in the denominator]. Because the logarithm is how quickly the exponent is going up, right?

Arguing in this way led S14 to refute his original guess for $a > 1$: "I think it would be the $\log(a)$. Or the log with some funny stuff going on in here." In a similar manner, S1 exhibited an awareness of the logarithm rule $\ln(a^n) = n\ln(a)$ and determined that as $n \to \infty$, $f(n)$ is "kind of an exponential"; but he was unsure about how the additional terms, $2 + \arctan(x_n)$, would affect the answer:

> There are these constants that kind of play a bit, but mostly an exponential. [-] So then, wait, if we were to apply the ln on that, I don't know what it does. [-] So if it were only $\ln(a^n)$, then the relationship is easy. But since there's this [i.e., $2 + \arctan(x_n)$], it kind of… it doesn't work as easily as I wish it would.

S1 and S14 seemed to be on the verge of developing the same intuitive technique as S3, but they did not know or were not confident in how it worked. More specifically, S1 and S14 did not know, as S2 and S3 did, that $2 + \arctan(x_n)$ could be ignored in the calculation of the limit.

All this said, it seems that the students' choice to not only "find the limit," but also "prove the limit," did not arise solely from doubts in their intuitive techniques. When he got stuck in constructing a proof, S14 looked back at the statement of Task 5 and seemed to question the need to continue: "I was thinking [-] how I would prove it. But it does just say to find the limit." S3 also got stuck at proving that $\lim_{n\to\infty} \frac{\ln(2 + a^n + \tan^{-1}(x_n))}{n} = \ln(a)$, at which point he asked the interviewer: "Should I continue trying to solve it? [-] For the record, I'm fully convinced that this is the answer." S3 gave some more evidence that his inclination to prove came from his experience as a student. While implementing his intuitive technique, he claimed that "a professor would not like what I'm doing right now. [-] Because they'll say: 'How did you know this? You've got to show it, that it's true.' [-] Like I do, I do need to show it." S1 said similarly about his intuitive technique:

> I wrote zero, but I can't just assume things like that. Teachers would love me if I just wrote zero and then… [-] Slowly, quickly don't mean anything in math, but… I think it goes to zero. And that's what I'm trying to prove.

S1 and S3 seemed to be acting at least in part based on their perception that their teachers would not accept their intuitive techniques. S14 seemed to be wondering if the wording of Task 5 meant that, in this case, the teacher's expectations were slightly different: i.e., If it only says "find," and not "prove," then perhaps an intuitive technique would suffice.

Some other students (S4, S9, S13, and S15) demonstrated the same intuitive techniques as S1, S3, and S14 while they embarked on implementing the Squeeze Theorem. These students also indicated that if Task 5 were posed on an exam in RA I, then they would identify it with T_{5_P} (i.e., "prove the limit"), and the intuitive techniques would not suffice. S9 seemed to indicate that he would not have experienced the same compulsion had he been interviewed during earlier experiences within the Analysis Stream. According to S9, the intuitive techniques were once the ones that his professors expected to see on exams. When considering the limit $\lim_{n\to\infty} \frac{\ln\left(2+ a^n+\frac{\pi}{2}\right)}{n}$, he said:

> If we're in a Calculus course, we can just say: forget the constant, forget the constant, it would be close to $\frac{\ln(a^n)}{n}$. [-] But if I had to prove it in Analysis, this is not good. [-] I feel

like you're not allowed to just say "This is not important." Or, if you do, it has to be like your last step, and you show like why this isn't important.

S9 seemed to abandon his intuitive technique due to the most recent didactic contract in which he had engaged, where solving a previously known task came with different rules. We might even say that S9 identified Task 5 with only one type (i.e., Find/Prove the limit.), which could require different techniques depending on the course context. Recall that S9 had exhibited similar behaviour while solving Task 4 (see, e.g., 6.4.2.1.2.3, 6.4.2.1.2.4, and 6.4.2.1.2.5).

Only one student (S15) told us that he was trying to prove the limit because intuitive techniques can sometimes lead to incorrect answers. At one point, S15 seemed to be thinking, like S2 and S3, that

$$\lim_{n\to\infty} \frac{\ln(2+a^n+\mathrm{arct}\quad(x_n))}{n} = \lim_{n\to\infty} \frac{\ln(a^n)}{n} = \lim_{n\to\infty} \frac{n\ln(a)}{n} = \ln(a).$$

He explained that a^n "will be more dominant than the other two terms in the log." This said, when the interviewer asked if he had convinced himself of the value of the limit, S15 responded: "I don't see all the details in my mind. [-] I have an intuition. And it's really hard to believe our intuitions in mathematics. Because intuition is not necessarily an assurance of success." In this case, the intuitive technique S15 seemed to be using does indeed lead to success (at least for $a > 1$, which seemed to be implicitly assumed in S15's work). S1 and S14, on the other hand, serve as examples of when "intuitions" can be wrong. Trying to produce a proof led S14 to correct his original guess, and perhaps also realize the flaw in his original intuitive technique. S1, in comparison, believed that the limit is zero until the end of the interview. Moreover, when the interviewer mentioned the possibility that the limit could be otherwise, S1 said: "If the limit is not zero, then I was not doing the right thing at all. That explains why I was not getting anywhere." It seemed that S1 was using his guess to guide his implementation of the Squeeze Theorem, more than using bounding techniques to verify or refute his guess. In fact, we question if S1's "intuition" can really be called as such. As we have alluded to before, S1's belief that the limit is zero may have stemmed from his application of what he thought was a legitimate (even if "intuitive") technique for finding the limit. Engaging in the proof may have been a formality, required for convincing his teachers that he could do it.

6.5.2.1.3 Dealing (or not) with parameters.

Task 5 depends on two parameters: $a \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$, any sequence of real numbers. The group of students we interviewed did not present much variance in the way they dealt with the general nature of $(x_n)_{n\in\mathbb{N}}$. They all either expected or knew that $\arctan(x)$ is a bounded function and, in one way or another, used this to eliminate the unknown $(x_n)_{n\in\mathbb{N}}$ when they were implementing their limit finding technique. For instance, most figured out (either on their own, or by asking the interviewer) that $\forall\ (x_n)_{n\in\mathbb{N}},\ -\frac{\pi}{2} < \arctan(x_n) < \frac{\pi}{2}$, whereby

$$\lim_{n\to\infty} \frac{\ln\left(2+\ a^n - \frac{\pi}{2}\right)}{n} < \lim_{n\to\infty} \frac{\ln(2+\ a^n + \arctan(x_n))}{n} < \lim_{n\to\infty} \frac{\ln\left(2+\ a^n + \frac{\pi}{2}\right)}{n}.$$

Task 5 can hence be reduced to thinking about limits of the form $\lim\limits_{n\to\infty} \frac{\ln(a^n+k)}{n}$, where k is a positive constant. When it came to the parameter a, our analysis led us to categorize students into three groups: those who did not spontaneously identify the task as depending on a (S1, S2, S5, S6, S9, S10, S11, and S15), those who struggled to deal with the dependence of the task on a (S3, S7, S8, S12, and S14), and those who varied a as part of their solution to the task (S4 and S13). In this section, we describe the behaviours of the students in each of these groups.

Notice that Task 5 can be identified with the following three tasks, depending not only on the behaviour of a^n as $n \to \infty$, but also on the domain of the natural logarithm:

$t_{a>1}$: Find the limit $\lim\limits_{n\to\infty} \frac{\ln(2+\ a^n+\ \arctan(x_n))}{n}$ where $a > 1$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers (Answer: $\ln(a)$).

$t_{a\in\left[\frac{\pi}{2}-2,1\right]}$: Find the limit $\lim\limits_{n\to\infty} \frac{\ln(2+\ a^n+\ \arctan(x_n))}{n}$ where $a \in \left[\frac{\pi}{2} - 2,1\right]$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers (Answer: 0).

$t_{a<\frac{\pi}{2}-2}$: Find the limit $\lim\limits_{n\to\infty} \frac{\ln(2+\ a^n+\ \arctan(x_n))}{n}$ where $a < \frac{\pi}{2} - 2$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers (Answer: the limit might not exist because the argument of the logarithm might not be positive).

Two students (S4 and S13) spontaneously identified Task 5 as representing a collection like this and exhibited no hesitation in trying to think about how the variation of a might change their solution.

Case	Argument
When $a > 1$, the limit is $\ln(a)$.	When n goes to infinity, a^n is so big that $$\lim_{n\to\infty} \frac{\ln(2 + a^n + \arctan(x_n)))}{n} = \lim_{n\to\infty} \frac{\ln(a^n)}{n} = \lim_{n\to\infty} \frac{n\ln(a)}{n} = \ln(a).$$
When $0 < a \leq 1$, the limit is 0.	When n goes to infinity, a^n goes to 0, $\arctan(x_n)$ goes to something between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and 2 remains 2. Since these are all smaller than n, $$\lim_{n\to\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n} = 0.$$

Table 6.18 The cases spontaneously considered by S13 while solving Task 5 and the intuitive arguments she sketched for "finding the limit."

S13 was unique in that her initial reading of Task 5 led her to base her entire solution on considering different cases for a. "It depends if a is less than one, or a is more than one," S13 explained before she sketched the intuitive arguments shown in Table 6.18. S13 eventually noticed that she had forgotten to consider when a is negative, which she felt would lead to a problem with the definition of the sequence, "because then it will be a ln of a negative number here, which doesn't exist." She noted that she would need to more carefully check for which values of a the sequence is well-defined; but she did not pursue this in the context of the interview. When it came to solving Task 5, S13 seemed more concerned with convincing herself of the different limiting values for the sequence than constructing a polished final solution.

Though it occurred to him a bit later than S13, S4 eventually wondered: "What if it's one of those things where it depends on the range of a?" Like S13, S4 seemed to recall that there are certain values of a for which $\lim_{n\to\infty} a^n = 0$ and he did not hesitate to consider how this might influence his solution:

> [a is] a real number. If it's negative, it's irrelevant. But if it's a fraction, it makes a world of difference. [-] So if it's a fraction between minus one and one, excluding them, then the limits [$\lim_{n\to\infty} \frac{\ln(a^n+k)}{n}$] would be going to zero. Which would mean that [-] the desired limit would also be zero [by the Squeeze Theorem]. [-] So, that would be like a part of it. But it wouldn't be all of it.

S4 spontaneously identified three cases similar to S13's: a "negative," $a \in (-1,1)$, and $a > 1$. This said, by the end of his solution, S4 had convinced himself that these different cases were "pointless." It seemed S4 did not possess the same intuitive techniques as S13 for calculating a limit like $\lim_{n\to\infty}\frac{\ln(a^n+k)}{n}$. He tried to determine its value for $a > 1$ by plugging in numbers on his calculator and thinking back to the definition of the natural logarithm. This led him to the same conclusion that S1 and S14 had made upon seeing Task 5 (as explained in 6.5.2.1.2): "I have no way of showing it, but I think that the denominator of n overpowers ln of a^n plus or minus whatever it's going to be"; hence, the limit is always zero. Since S1 had this same thought throughout his entire solution attempt for Task 5, it is perhaps not surprising that he did not comment on how the parameter a influenced his solution.

In complete opposition to S4 and S13, the most common approach taken by the participants of our study was to not deal with the parameter a. Concretely, this meant that the participants constructed (or tried to construct) a solution to Task 5 without explicitly indicating how it depends on the value of a. We identified eight students (S1, S2, S5, S6, S9, S10, S11, and S15) as falling within this category. Characteristic of their work is that it contributed to solving $t_{a>1}$, without specifying $a > 1$. Indeed, all their solution attempts depended on or implicitly included the assumption that $a^n \to \infty$, or, in the words of S6, that "a^n would win over." We predict that several, if not all, students in this group were basing their solutions on a recall of the "find the limit" tasks they had encountered in RA I, where the technique typically depended on an exponential being a dominant term in the sequence. S6, for example, thought back to her assignments from RA I and explained that "it's like automatic. [-] a^n would just be a bigger value. Cause, you know, it's like, it increases faster." In a similar manner, S11's recall of what they usually did in RA I led him to identify a^n as "the biggest number" and "the one that grows the fastest." Even S4 had the instinct that "2 gets overpowered by a^n" before realizing that this is not always the case.

It is possible that some students in this group were not fluent in the different behaviours of $(a^n)_{n\in\mathbb{N}}$ for different values of a; but this was not the case for all students in the group. S2, for example, started her solution by sketching the two graphs of a^x shown in Figure 6.27. She then continued as though she was ignoring the case on the left and gave no indication as to why. S9, in comparison, claimed that his awareness of the different behaviours of $(a^n)_{n\in\mathbb{N}}$ was not triggered by Task 5. When he was asked, at the end of his solution, if he had been making an assumption on a, he responded:

Oh. Um… […] I guess I am. I'm assuming it's greater than one. [-] If it's smaller than one, this whole uh sequence, um, goes to zero, I guess. Yeah. [-] a could have been smaller than one. And I would not have seen it if you didn't tell me.

It seems that some students were ignoring or overlooking the variance of a for reasons other than a lack of understanding of how sequences behave. In fact, when S9 was prompted further about the possibility of having $a < 0$, he exhibited some of the precise argumentation that was missing from S4's and S13's solutions: To avoid the logarithm of a negative number in the given sequence, $\left(\frac{\ln(2+a^n+\arctan(x_n))}{n}\right)_{n\in\mathbb{N}}$, he determined that we need $a^n > 2 - \frac{\pi}{2}$, or else n has to be even. S9 noted, nonetheless: "I was hoping $[a]$ wasn't negative. [-] We didn't see any examples where we had a negative value that switched like that."

Figure 6.27 Sketches made by S2 of different possible behaviours of a^n.

Two key features distinguish the remaining group of students (S3, S7, S8, S12, and S14). First, they noticed that some of their arguments were not true for certain values of a while they were working through or reflecting on their solution (without the interviewer asking a pointed question about a). Second, they struggled with what to do about the generality of a. When asked to clarify how he found $\lim_{n\to\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n} = \lim_{n\to\infty} \frac{\ln(a^n)}{n}$, S3 stopped mid-explanation: "Hold on. […] a has to be positive. Or not. Well, nevermind. Ok, assuming that there's no tricks going on, ok? So let's assume that a is equal to 10, ok?" For S7, S8, S12, and S14, the struggle with a became most evident when they were trying to find an upper bound for $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$. At this point, each of them noted, in their own way, that a is a real number. "That's tricky," claimed S12. S14 clearly explained the issue: "It's actually difficult because I don't know the deal with two. I don't know if a^n is greater than two. Because it just says that a is in the real numbers." Each student seemed to have a certain bounding technique in mind: e.g., for sufficiently large n, $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} \leq$

$\frac{\ln(a^n + a^n + a^n)}{n} = \frac{\ln(3a^n)}{n}$. "But the problem is that this inequality doesn't hold because a is any real number," explained S12: "So that's where I'm stumped." To cope with the difficulty, S7, S12, and S14 eventually decided to include in their solution the assumption on a they felt was necessary for their arguments to hold. For the above inequality, for instance, S7 and S14 decided that they needed to have $a > 1$, and they completed the task for this range of a.

There could be different reasons that these students struggled with Task 5's dependence on a. Like the students who did not address a, it seems that S7, S8, S12, and S14 had developed practices that were specified to the case where a^n is the dominant term. In other words, they knew how to solve $t_{a>1}$ and could not (immediately) see how to adapt their chosen technique for different values of a. For instance, after completing his proof, S14 explained that the inequality $\frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n} \leq \frac{\ln(3a^n)}{n}$

> would be true for large n if a was greater than one. [-] Like eventually a^n would be greater than two and it would be greater than $\frac{\pi}{2}$, that's not a problem. But I don't know for [-] a is between zero, strictly, inclusive one.

Perhaps with more time, outside of the interview context, S14 would have noticed that his bounding technique can easily be adapted for $a \in (0,1]$ (actually, for $a \in \left[\frac{\pi}{2} - 2,1\right]$). Indeed, if this is the case, the dominant term is 2, rather than a^n. Hence $\frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n} \leq \frac{\ln(2 + 2 + 2)}{n} = \frac{\ln(3 \cdot 2)}{n}$ $\to 0$ as $n \to \infty$.

It is also possible that the students in this group struggled because they did not expect to be given a poorly defined task that has multiple solutions depending on the value of a. Put differently, it seems the students were used to constructing a single solution for a given task. For example, when S8 got stuck at trying to find an upper bound that works for all a, he was convinced that "after lots of time and thought, [he] would probably find one." Similarly, after completing a proof that worked for a limited range of a, S12 concluded: "If I want to show for any value of a, I need to choose a different proof." S3, in comparison, may have realized that his solution needed to change depending on the value of a, but he seemed hesitant to go further in the interview: "Wouldn't the question here be: it depends what a is? I don't know. [-] I don't feel like I know enough about this problem yet. [-] It seems like an interesting problem. I don't know." When asked if specifying a might help him in constructing a proof, S3 added: "But I don't know what a is. It could be anything. [-] I guess I feel like if I do this, I might restrict myself. I might just give examples of something, but wouldn't generalize."

S8 responded to the interviewer's question about specifying a very differently, claiming that it "would most likely help. Cause I could find my bounds a bit easier I think." After choosing $a = 2$, S8 noticed that there is another way (other than bounding) that he could deal with a limit like $\lim_{n\to\infty} \frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n}$: "by l'Hospital's, this would probably be a lot easier actually." His solution went smoothly for $a = 2$:

$$\lim_{n\to\infty} \frac{\ln\left(2 + \ 2^n + \frac{\pi}{2}\right)}{n} = \lim_{n\to\infty} \frac{\frac{1}{2 + 2^n + \frac{\pi}{2}} \cdot \ln(2) \cdot 2^n}{1} = \lim_{n\to\infty} \frac{\ln(2) \cdot 2^n}{2^n + 2 + \frac{\pi}{2}} = \ln(2).$$

Then came the question of generalizing: "So most likely the other ones would converge to $\ln(a)$," S8 conjectured: "Probably. Cause all I did was replace a by 2." A quick check over his solution led S8 to conclude that "it would be the same argument. Yeah. Yeah, so it would be $\ln(a)$."

Once again, there could be many interconnected reasons why S8 did not realize that his argument was only valid if $a > 1$. We do not suspect that S8 was lacking knowledge of how sequences of the form $(a^n)_{n\in\mathbb{N}}$ behave. He noted, for example, that a could be "a fraction" and that a^n could be made "small" in that case. It seemed that S8 did not realize how this different behaviour affects the limit techniques he applied to get the equalities shown above. It is possible that S8 lacked some understanding of the technique used for the first equality; after all, he applied l'Hospital's Rule without mentioning the need for an indeterminate case, which does not occur for all a. It is also possible that the issue was a metacognitive one: a lack of attention to the important details of implementing a technique and a careful questioning of whether it would work for all a. For instance, S8 seemed to calculate the limit $\lim_{n\to\infty} \frac{\ln(2)\cdot 2^n}{2^n + 2 + \frac{\pi}{2}}$ (the last equality) using the technique we have previously called "thinking about what matters in the long run": in this case, he noted that only the exponential terms matter (as usual, the constants can be ignored). In quickly checking through his solution, S8 did not recognize the importance of the value of a in implementing this technique (i.e., in determining which part of the sequence matters as n tends towards infinity). While the exponential terms are indeed the only ones that "matter in the long run" if $a > 1$, this is no longer the case when $a \in \left[\frac{\pi}{2} - 2{,}1\right]$. Finally, and as alluded to above, it is possible that S8 was driven by an interest in finding *the* answer. While making the transition to Task 6, he asked the interviewer: "Was that the right answer by the way? $\ln(a)$? [-] It's just for my own, like, you know, ego. I need to know."

6.5.2.1.4 I know how to find the limit of a different type of sequence.

Thirteen out of the fifteen students we interviewed identified Task 5 as belonging to a type of task they knew how to solve. S6 and S11, in contrast, claimed to not know how to deal with the given sequence: $\left(\frac{\ln(2+\,a^n+\,\arctan(x_n))}{n}\right)_{n\in\mathbb{N}}$. Both identified the logarithm as their main stumbling block because they had not worked with it before. "So I don't know how to go about it yet," said S11; "I don't want to come up with some kind of function that doesn't work," added S6. In saying this, S6 was referring to the two "functions" – "one that's smaller" and "one that's greater" – that she needed to find to implement a technique based on the Squeeze Theorem. It appeared that these two students had developed a more restricted practice than their peers in at least one way: they knew how to solve a subset of $T_{5_{ST}}$ (i.e., Find the limit of a given sequence using the Squeeze Theorem.), which was characterized by a particular kind of sequence. The students tried to describe this kind of sequence, both verbally and in writing (as depicted in Figure 6.28). Thinking back to his experience in RA I, S11 said that "usually what we had were functions that were like, let's say… [he draws the nth root] and a constant, and e^n, and sine of whatever." In a similar manner, S6 recalled a "root," a "constant," "sine," and "exponentials." Based on our analysis of the assessment activities these students had received in RA I, it seems S6 and S11 were recalling the most commonly tested task type, which we could model as:

$$\widetilde{T_{5_{ST}}}\text{: Use the Squeeze Theorem to find}$$

$$\lim_{n\to\infty} \sqrt[n]{a\ constant + an\ exponential + sine\ of\ something}.$$

Figure 6.28 S11's (left) and S6's (right) representations of the type of task they knew how to solve.

It is interesting to note that one participant identified in Task 5 a subtask that is very close to being of type $\widetilde{T_{5_{ST}}}$. At the beginning of his solution, S10 decided to write

$$\frac{\ln(2+\ a^n+\arctan(x_n))}{n}=\ln\left((2+a^n+\arctan(x_n))^{\frac{1}{n}}\right)=\ln(\sqrt[n]{2+a^n+\arctan(x_n)})\,.$$

Then he found $\lim\limits_{n\to\infty}\sqrt[n]{2+a^n+\arctan(x_n)}$ (using the Squeeze Theorem) and made his conclusion (using the continuity of the logarithm). There may be different reasons why S6 and S11 did not spontaneously recognize within Task 5 a type that they knew how to solve. Adapting the task as S10 did requires some fluency in rules of logarithms that S6 and S11 did not demonstrate in their interviews. It also necessitates a certain metacognitive skill: i.e., the solver must take time to try to modify the given task to better fit the practices known to them. We suspect that S6 and S11 had not yet developed this skill. Nevertheless, S10's modification is not necessary for solving the task; S6 and S11 could have instead tried to solve Task 5 by adapting the technique they had for $\widetilde{T_{5_{ST}}}$.

Figure 6.29 The image accompanying S11's description of how the Squeeze Theorem works.

S6 and S11 did not demonstrate an ability to adapt their technique in the context of the interview, even if they seemed to have some of the necessary ingredients for doing so. Both students seemed to have an idea of how the Squeeze Theorem worked. For instance, after identifying Task 5 with $T_{5_{ST}}$, S11 wrote what is shown in Figure 6.29 and said: "I would try to find something here and here that converges to the same thing, that are bigger and smaller than this function. And then I could say that this function converges to that same point." Both students also took some steps towards finding lower and upper bounds for the given sequence: They had the instinct that $\arctan(x)$ would be bounded and that a^n would grow the fastest. They both also seemed to have a general technique that could lead from these initial steps to a solution. S11, for example, said that if the sequence was of the form $\sqrt[n]{2+e^n+\sin(\dots)}$, then "what we did was basically just use the biggest number in the interval": i.e.,

$$\sqrt[n]{e^n}\leq\sqrt[n]{2+e^n+\sin(\dots)}\leq\sqrt[n]{3e^n}\,.$$

Still, both students deemed themselves unable to solve Task 5.

We could make different conjectures as to why S6 and S11 were unable to go from inequalities like the ones shown above to an inequality like

$$\frac{\ln(a^n)}{n} \leq \frac{\ln(2 + a^n + \arctan(x_n))}{n} \leq \frac{\ln(3a^n)}{n}.$$

As mentioned earlier, both students claimed that they did not know how to deal with the ln because they had never worked with it before. Perhaps they foresaw the inequalities shown above and knew that, due to a lack of fluency with logarithm rules, they would not know what to do next. Another likely possibility is that the students were so used to relying on routine practices that the seemingly small adaptation was, for them, too far out of their comfort zone. For example, when S11 was on the verge of getting stuck, he said: "I feel like I'm guessing here. Which is not good." In a similar vein, S6 recalled being much more comfortable with tasks of type $\widetilde{T_{5_{ST}}}$: "It's something I knew. So I'd be able to find those two things [i.e., the bounds]. Cause I can visualize it. Like, you know, I know how it looks like. [-] I'm sure, I've seen it, so…"

6.5.2.2 Implementing a Technique: Using the Squeeze Theorem to find a limit.

As outlined in the previous section, most participants spontaneously identified Task 5 with $t_{a>1} \in T_{5_{ST}}$ (we recall the definitions below):

> $T_{5_{ST}}$: Use the Squeeze Theorem to find the limit of a given sequence.
>
> $t_{a>1}$: Find the limit $\lim\limits_{n\to\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n}$ where $a > 1$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers.

In fact, all fifteen students eventually demonstrated how they would implement a technique for solving $t_{a>1} \in T_{5_{ST}}$ in the context of the interview. The two students (S2 and S3) who did not identify Task 5 with the Squeeze Theorem were prompted by the interviewer to do so when they got stuck. In this section, we present what "using the Squeeze Theorem" to solve $t_{a>1}$ seemed to involve for the students we interviewed. Note that this means that we will be assuming $a > 1$ for the remainder of this section.

Of course, implementing a technique based on the Squeeze Theorem involved bounding for all fifteen of the students we interviewed. As mentioned above, noticing that $\arctan(x_n)$ is bounded naturally leads to a first step: i.e.,

$$\forall (x_n)_{n\in\mathbb{N}}, -\frac{\pi}{2} < \arctan(x_n) < \frac{\pi}{2}$$
$$\Rightarrow \frac{\ln\left(2+\ a^n-\frac{\pi}{2}\right)}{n} < \frac{\ln(2+\ a^n+\arctan(x_n))}{n} < \frac{\ln\left(2+\ a^n+\frac{\pi}{2}\right)}{n}.$$

Although not all students wrote this exact implication or these exact inequalities, they all seemed to (implicitly or explicitly) make an implication of this nature at some point in their solution; at the very least, their solutions always depended on such an implication. Moreover, the most significant differences in participants' implementation of the Squeeze Theorem can be most efficiently explained by assuming that all participants took this initial step. In 6.5.2.2.1, we describe the varying degrees to which students were able to mathematically justify this implication. Then we describe how students proceeded beyond the implication according to how they dealt with the upper bound $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$: either they continued to look for upper bounds (6.5.2.2.2), or they stopped and calculated $\lim\limits_{n\to\infty}\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$ (6.5.2.2.3).

6.5.2.2.1 Justifying bounds.

The implication shown above depends on several theorems about the ordering of the real numbers, as well as the fact that $\ln(x)$ is an increasing function (i.e., if $0 < x < y$, then $\ln(x) < \ln(y)$). Among the students we interviewed, none directly acknowledged the ordering theorems; but the importance of $\ln(x)$ being increasing was acknowledged by the students to varying degrees.

Nine out of the fifteen students did not note that $\ln(x)$ is an increasing function while they were solving Task 5 (S1, S3, S4, S5, S6, S7, S12, S13, and S14). Follow-up questions with some of them revealed at least two possible reasons for this. When asked how he went from bounding the argument of the logarithm to bounding the given sequence, S12 explained that "the logarithm is an increasing function. [-] That's kind of the implicit judgement there." S7, in comparison, explained:

It's just taking the natural log of the value of the number, so… [-] the inequality sign holds. [-] It's just like squaring both sides or dividing both sides. You just apply the same thing. So the… the inequality holds.

It is possible that, like S12, some students' solutions came with the implicit assumption that the logarithm is an increasing function. Alternatively, students may not have realized the need for the logarithm to be increasing. As it seemed to be the case with S7, perhaps applying the logarithm to both sides of an inequality had been learned like an algebraic rule, similar to dividing both sides by a number. Of course, it is not wrong to think about $x < y \Rightarrow \ln(x) < \ln(y)$ as a rule, so long as it is accompanied by the additional assumption that x is positive. For S7, the rule did not seem to come with a mathematical reason.

The remaining six students all chose to mention that $\ln(x)$ is an increasing function while they were solving Task 5. Two of these students (S11 and S15) did not seem to know why they were mentioning this property. For instance, during his search for bounds, S11 asked the interviewer: "How does ln behave? Is ln increasing?" After the interviewer provided a sketch of $\ln(x)$, S11 confirmed: "So yeah. It is increasing, right? So can you… Um… Cause usually what we do, well what we did in Analysis was like…" S11 seemed to be trying to call up memories of his experience in RA I, but could not quite figure out the importance of $\ln(x)$ being increasing. As discussed in 6.5.2.1.4, he abandoned the sequence in Task 5 before writing down any bounds, claiming that he did not know how to deal with $\ln(x)$. And yet, he seemed to have no difficulty in coming up with the following inequalities:

$$\sqrt[n]{a^n} \leq \sqrt[n]{2 + a^n + \arctan(x_n)} \leq \sqrt[n]{3a^n}\,.$$

One might assume that implicit in these inequalities is an implication of the form:

$$a^n < 2 + a^n + \arctan(x_n) \Rightarrow \sqrt[n]{a^n} \leq \sqrt[n]{2 + a^n + \arctan(x_n)}.$$

This is what the interviewer was thinking when she asked S11 about this implication and how he knew it is true. His response echoed S7's: "You're multiplying two sides of the equation by the square root of one n. The inequality should still hold. [-] Yeah I'm pretty sure. [-] If you take like times five and times five?" Nonetheless, S11 specified: "Our problems usually started, like, the square root was there. [-] But I'm guessing you could [-] just apply the square root." It seemed that

S11 had not thought about the above implication when coming up with his inequalities. Hence, S11 did not seem to have a mathematical reason for checking that $\ln(x)$ is increasing.

Mentioning that $\ln(x)$ is increasing seemed more purposeful for S2, S8, and S9, who seemed to rely on the property to support their bounding process. S2, for example, prefaced her attempt at bounding $\frac{\ln(2+a^n+\arctan(x_n))}{\mathrm{n}}$ by noting that "the function of $\ln(x)\ldots$ [-] This is always increasing"; and when he chose $\frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{n}$ as a lower bound, S9 said: "This is definitely the minimum it can be. [-] As the value gets bigger, ln of that value also gets bigger." We suspect that these students were making these statements as part of engaging in the think aloud norm of the interview. In other words, they were demonstrating that when determining the bounds, their thought processes involved confirming that $\ln(x)$ is an increasing function. It is questionable if these students would include this property as part of their final written polished solution.

S10 distinguished himself from all other participants of our study in that he purposefully took time at the beginning of his solution to Task 5 to think about the kinds of theorems he would need in order to implement his Squeeze Theorem inspired technique. Recall (from 6.5.2.1.4) that S10 was the only student to rewrite the given sequence in the following manner:

$$\frac{\ln(2+a^n+\arctan(x_n))}{n}=\ln\left((2+a^n+\arctan(x_n))^{\frac{1}{n}}\right)=\ln(\sqrt[n]{2+a^n+\arctan(x_n)}).$$

Then he paused:

> I'm just kind of trying to figure out what assumptions I'm going to have to have at hand for this. Because, first of all, I'm only going to be able to use the Squeeze Theorem for this if I know that this logarithm function is always increasing. [-] I know that's going to help because [-] I can show inequalities for the inner term, and then those inequalities will hold for the whole function.

S10 also noted that he would require $\sqrt[n]{x}$ to be increasing; and he kept track of both "assumptions" in writing, as part of his solution to the task. When compared to his peers, S10 seemed much more interested in thinking about and making explicit the theories underlying his work. It is unclear how he went about choosing which theories to make explicit. In some cases, it seemed the theories were left out because they were part of the practices to be learned in previous courses. For example, S10 did not spontaneously make note of the property $n\cdot\ln(b)=\ln(b^n)$, which he said he had learned in a prior course. S10 also seemed to struggle to rely on theories that he did not know to be true. For instance, when it came time to calculate the limits of his bounds (i.e., $\ln\left(\sqrt[n]{2 + a^n \pm \frac{\pi}{2}}\right)$), S10 explained that he would like to use the following implication:

$$\lim_{n\to\infty} \sqrt[n]{2 + a^n \pm \frac{\pi}{2}} = a \Rightarrow \lim_{n\to\infty} \ln\left(\sqrt[n]{2 + a^n + \frac{\pi}{2}}\right) = \ln(a).$$

But he was unsure how to justify it: "I feel that this entailment here is going to follow from continuity. From continuity of $\ln(x)$," S10 explained: "[-] But I'm not quite sure if that's the case."

6.5.2.2.2 Just keep bounding, just keep bounding, …

After coming up with inequalities of the form

$$\frac{\ln\left(2 + a^n - \frac{\pi}{2}\right)}{n} < \frac{\ln(2 + a^n + \arctan(x_n))}{n} < \frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n},$$

one group of students chose to continue bounding above (and below). They exhibited three different kinds of upper bounds, which can be exemplified as follows:

1. $\frac{\ln\left(2+ a^n+\frac{\pi}{2}\right)}{n} \leq \ln\left(2 + a^n + \frac{\pi}{2}\right)$;
2. $\frac{\ln\left(2+ a^n+\frac{\pi}{2}\right)}{n} \leq \frac{\ln(a^{n+m})}{n}$ for some real number m; and
3. $\frac{\ln\left(2 + a^n+\frac{\pi}{2}\right)}{n} \leq \frac{\ln(a^n + a^n + a^n)}{n}$ for sufficiently large n.

The upper bound in (1) was used by one student (S5) whose technique seemed to consist essentially of looking for something that is bigger. When asked why she chose to remove the denominator of the sequence, S5 explained that it was "just to make it greater. But I'm not sure if that's a bad thing to do, because maybe I would have needed it later." If we rewrite the upper bounds in (2) and (3), we see why the n is "needed":

$$\frac{\ln(a^{n+m})}{n} = \frac{(n+m)\ln(a)}{n} = \ln(a) + \frac{m\ln(a)}{n} \to \ln(a) \text{ as } n \to \infty$$

$$\frac{\ln(a^n + a^n + a^n)}{n} = \frac{\ln(3a^n)}{n} = \frac{\ln(3) + n\ln(a)}{n} = \frac{\ln(3)}{n} + \ln(a) \to \ln(a) \text{ as } n \to \infty$$

Note that removing the n also does not make sense if one thinks about bounding as a formal method of approximating the behaviour of a given sequence. Indeed, removing the n makes a fundamental change to the behaviour of the sequence in Task 5.

The upper bound in (2) was also used by one student (S12). S12 distinguished himself from S5 in that he based his upper bounding on his findings from bounding below. More specifically, S12 started his implementation of the Squeeze Theorem as follows:

$$\frac{\ln(2+a^n+\arctan(x_n))}{\mathrm{n}} \geq \frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{\mathrm{n}} \geq \frac{\ln(a^n)}{n} = \frac{n\ln(a)}{n} = \ln(a).$$

"So I know my limit here is going to equal to $\ln(a)$," he explained as he transitioned into finding upper bounds: "So I'm trying to find some function here that when I solve it I'm also going to end up with a $\ln(a)$ as my answer." S12 gravitated towards playing with the exponent of a^n and checked his attempts based on his expectation that the answer should be $\ln(a)$. His first attempt at an upper bound was $\frac{\ln(a^{2n})}{n}$, which he quickly rejected because he knew that it would lead to an answer of $2\ln(a)$. Shortly thereafter, S12 came up with the idea for the upper bound in (2): "This is only coming to me now," he said, as he wrote $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} \leq \frac{\ln(a^{n+m})}{n}$ for some value m. S12 was not immediately confident in the bound: "I don't know if I accept this," he noted. Once he carried out the algebra and saw that it leads to an answer of $\ln(a)$, however, S12 no longer expressed any doubt.

Two students (S7 and S14) chose the upper bound in (3) almost immediately. Although it was like a reflex, they both seemed understand what they were doing: namely, they were replacing each term in the argument of the logarithm by the biggest term, a^n. S7 said that he used this technique "because then it's easier to factor it out. [-] You get something nice out of it." Like S5 and S12, S7 seemed to focus on how the technique he chose allows for algebraic manipulations that lead to a conclusion; he performed these manipulations easily. Several other students demonstrated an awareness of this technique (S2, S8, S11, and even S5 and S12). This is perhaps not surprising given that this was a common technique employed in the solutions that were made available to participants while they were enrolled in RA I. Nevertheless, not all students exhibited the same level recall or confidence as S7.

S5, for example, continued her upper bounding as though she was trying to recall the technique exemplified in (3):

$$\ln\left(2 + a^n + \frac{\pi}{2}\right) \leq \ln(4 + a^n) \leq \ln(2a^n).$$

Then she stopped and said: "I'm not sure if that's valid. [-] I still don't know where I'm going with this. [-] I know my teacher would do something similar. But I'm pretty sure that it's wrong what I'm doing." Of course, S5 was correct in questioning the validity of the last inequality she wrote: it is not true for all $a > 1$ and $n \in \mathbb{N}$. One way to resolve this problem would be to realize that when one is interested in finding a limit using the Squeeze Theorem, the bounds only need *eventually* hold: that is, there need only exist an $N \in \mathbb{N}$ such that the inequalities are true $\forall n \geq N$. S7, for example, explained that "since n goes to infinity, we can assume that n is bigger than a hundred." Another way forward, demonstrated by S11, would be to check and adjust the inequalities by plugging in values. If we mimic S11's behaviour for the above inequality, we would let $a = n = 1$ to get $\ln(5) \leq \ln(2)$; and then we would change the upper bound to $\ln(5a^n)$. Although it is unclear how well S11 understood what he was doing, he seemed to have the foundations on which to build a useful approach. Indeed, for the given task, letting $a = n = 1$ is like considering an extreme case; and it turns out that $\ln(5a^n)$ is a valid upper bound for $\ln(4 + a^n)\ \forall\ a > 1, n \in \mathbb{N}$. Given that S5 had plugged in values to deal with limits and suprema in Task 4 (see, e.g., 6.4.2.1.2.1), we suspect that she would have been able to carry out a validation like S11 for Task 5. But she was unable to get unstuck in the context of the interview. It is possible that S5 did not think of trying something like plugging in values because she had become so used to solving tasks like Task 5 in an automatic fashion. "I'm still mad at myself for not getting it," she said in retrospect of her solution attempt: "[-] We did it a lot. And it's one of the easiest things. Ever." Note that S5 was not the only student to talk about how easy Task 5 should have been: S1 and S9 were also frustrated by not being able to finish the task.

In sum, the students' bounding techniques seemed to be driven more by recall or trial and error, in hopes of finding something for which the algebra worked, rather than thinking about the best upper approximation for the given sequence. For example, both bounding techniques used to obtain the upper bounds in (2) and (3) were successful because they ultimately replaced the argument of the logarithm by $c \cdot a^n$ for some scalar c. There is another way of thinking about why such a replacement leads to success, beyond the algebraic manipulations it allows. If k is a positive constant, then $a^n + k$ behaves approximately like the exponential a^n in the long run. We could formally represent this as follows:

$$\forall c > 1, \exists N \in \mathbb{N}, \text{ such that } \forall n \geq N,$$

$$a^n \leq a^n + k \leq c \cdot a^n$$

$$\Rightarrow \frac{\ln(a^n)}{n} \leq \frac{\ln(a^n + k)}{n} \leq \frac{\ln(c \cdot a^n)}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\ln(a^n+k)}{n} = \ln(a) \text{ by the Squeeze Theorem.}$$

In other words, $a^n + k$ will eventually be bounded above by $c \cdot a^n$ for any real number c bigger than one. The students' choice of $c = 3$ could, in this sense, be classified as overkill.

6.5.2.2.3 Stop bounding; calculate a limit.

Some students (S1, S3, S8, and S15) explicitly tried to keep bounding, like the students in the previous section, but were unsuccessful in doing so. With S8, the issue seemed to be mostly organizational: while he showed signs of having the same kinds of bounding techniques as those described above, it was as if he made so many attempts that he could not keep track. S1, S3, and S15, on the other hand, struggled to find an upper bound for $\frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n}$ (we note that S4, S9, S10, and S13 also did not exhibit any techniques for dealing with this kind of bound, even if they did not explicitly try to keep bounding in the interview). S15 felt that, in order to "get rid of" the constants, "I have to use a trick. [-] The problem is the addition. It's not a multiplication. Because I would like to use some rules about log to simplify the expression." Both S1 and S3 also considered simplifying the sequence using logarithm rules and verified with the interviewer that it is not valid to write $\ln\left(a^n + 2 + \frac{\pi}{2}\right) = \ln(a^n) + \ln\left(2 + \frac{\pi}{2}\right)$. Shortly after this, S1 got stuck: "That one was supposed to be easy," he claimed in retrospect: "[-] Cause I used to have those problems. [-] During tests, I'd say that one's easy and I'd start with that one." It seems these students had a weaker memory of the "trick" that had been recalled by some of the students mentioned previously (e.g., bounding each term in the argument of the logarithm by the dominant term – in this case, a^n). S15, who was keen to self-analyze, suggested that the problem may have been that he had relied too much on memorization: "I would say that maybe, yes, there's something that I did not understand really, and I'm just trying to reproduce some strategies." This said, each of S1, S3, S8, and S15 also demonstrated an awareness of other fruitful ways that they might deal with the upper bound, $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$.

Indeed, once one knows that

$$\frac{\ln\left(2+\ a^n-\frac{\pi}{2}\right)}{n}<\frac{\ln(2+\ a^n+\arctan(x_n))}{n}<\frac{\ln\left(2+\ a^n+\frac{\pi}{2}\right)}{n},$$

it is not necessary to continue bounding; all one needs to do is calculate the limits of the lower and upper bound. Recall that both S1 and S3 had intuitive techniques that they could use to do this (we describe them in 6.5.2.1.1 and 6.5.2.1.2). Some of the students who did not explicitly attempt to continue bounding also seemed to turn to these intuitive techniques to calculate $\lim_{n\to\infty}\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$. S10 did not provide much explanation beyond stating that he could "kind of intuitively see" the value of the limit. S4 concluded, like S1, that the denominator overpowers the numerator, whereby the limit is 0. S9 and S13, like S3, indicated that they could ignore the constants to find that the limit is $\ln(a)$. As discussed in 6.5.2.1.2, some of the students did not seem to see these intuitive techniques as sufficient for getting full marks in RA I. As a result, S9 chose to demonstrate what he felt would be a more Analysis-appropriate way of calculating $\lim_{n\to\infty}\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$: i.e., using l'Hospital's Rule.

When S8 and S15 were unable to continue bounding, they too turned to l'Hospital's Rule. With minimal assistance, all three students produced the following equality (recall that S8 had specified $a=2$, as described in 6.5.2.1.3):

$$\lim_{n\to\infty}\frac{\ln\left(2+\ a^n+\frac{\pi}{2}\right)}{n}=\lim_{n\to\infty}\frac{\ln(a)\cdot a^n}{a^n+2+\frac{\pi}{2}}.$$

From here, S8 and S9 were quick to calculate the limit to be $\ln(a)$. Both relied on a similar argument: As n tends to infinity, the constant terms matter less and can be ignored. In mathematical symbols:

$$\lim_{n\to\infty}\frac{\ln(a)\cdot a^n}{a^n+2+\frac{\pi}{2}}=\lim_{n\to\infty}\frac{\ln(a)\cdot a^n}{a^n}=\lim_{n\to\infty}\ln(a)=\ln(a)\,.$$

The difference with S9 is that he questioned if he was allowed to calculate the limit in this way. He had chosen to use l'Hospital's Rule in attempt to avoid using an intuitive technique; but he ended up using that intuitive technique anyways. He could not see any way around this in the context of the interview and, with much hesitation, proceeded to a conclusion. Interestingly, S15 provided an example of what proceeding in a less intuitive manner may have looked like. After successfully applying l'Hospital's Rule to find $\lim_{n\to\infty} \frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} = \lim_{n\to\infty} \frac{\ln(a)\cdot a^n}{a^n+2+\frac{\pi}{2}}$, S15 simply returned to bounding: i.e.,

$$\lim_{n\to\infty} \frac{\ln(a)\cdot a^n}{a^n+2+\frac{\pi}{2}} \leq \lim_{n\to\infty} \frac{a^n \ln a}{a^n} = \lim_{n\to\infty} \ln(a) = \ln(a).$$

6.5.3 Reflections on the Nature of Practices, Positioning, and Activities[40]

Non-Mathematical: Identify task with technique in an automatic fashion.

VS.

Mathematical: Study the task and perform assessments such as:
"Is the task well-posed? Do I need to consider different cases? Can I guess an answer?"

In the context of Task 5, it seemed most participants had learned non-mathematical practices in the sense of automatically identifying the given task with a technique. There were two interesting ways in which the identification of task with technique was automatic. First, most participants saw a task about finding the limit of a "complicated" sequence and immediately thought: "I'm going to use the Squeeze Theorem." Second, most participants saw a task about finding the limit of a sequence including an exponential term a^n and immediately assumed: "a^n grows the fastest."

We think the first automatic behaviour would be natural for students who had occupied different positions in RA I, especially given that Task 5 looks so much like a collection of assessment activities that explicitly asked students to "use the Squeeze Theorem." Moreover, the

[40] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.5.1 and 6.5.2 that begin to address our research objectives: characterizing the nature (mathematical or not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

theorem turns out to be useful in solving the given task. We note that some participants did not seem to have discourses for explaining why the Squeeze Theorem would be useful for dealing with a limit like $\lim_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$ beyond noting its "complicated" nature. This said, the most interesting distinctions in positioning seemed to arise in students' abilities to complement their choice to use the Squeeze Theorem with more intuitive techniques for guessing possible limit values. Many participants seemed unable to confidently and correctly think about a "complicated" sequence like $\lim_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$ using intuitive techniques. We think this is mainly related to the Student position; in particular, solving activities by copying the steps of professors' solutions, which did not include intuitive a priori studies of potential sequence behaviour. It could also be linked to the Enthusiast position; for instance, their perception that RA I is mainly about constructing formal arguments, where informal arguments have no place. The fact that some students tried to guess the limit of the given sequence or a very similar sequence using intuitive techniques showed some potential for transitioning to mathematical practices in the above sense (i.e., complementing proving with guessing). Only a few students (S3 – the Learner, S9 – the Skeptic, and S13 – the Mathematician in training), however, seemed to have a facility with pinpointing the potential limiting values of a sequence like $\lim_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$ just by looking at it.

We think the second automatic behaviour – assuming $a^n \to \infty$ as $n \to \infty$ – could also be linked to different kinds of positioning; again, since the sequence in Task 5 looks so much like the limits given to students in RA I, all of which had a specified to a number bigger than 2. Interesting distinctions in positioning seemed to come out more in the degree to which participants recognized and addressed their assumption on a^n. Completing an entire solution based on a tacit assumption seems strongly linked to the Student position; in particular, working based on recalling the steps, rather than critically examining why one step leads to another. In comparison, we think that the participants who recognized that they were making an assumption on a and explicitly addressed it showed that they had either occupied positions other than Student in RA I or they could spontaneously shift from studenting to learning. More specifically, they could think critically about the deductions they were making. It is important to note, nevertheless, that most participants seemed uncomfortable with dealing with a messy, poorly defined task that requires the consideration of different cases. We think that posing more activities that are purposefully ill-defined and depend on parameters (like Task 5) could support students in the transition from studenting to learning, and to practices that are more mathematical in nature. Only one participant (S13) immediately recognized the messy nature of Task 5 and confidently built a solution based on the consideration of different cases for a. This is likely connected with her positioning as a Mathematician in Training, which may have enabled her to recognize when a given task requires more study (i.e., when it is not possible to proceed based on a seemingly automatic identification of task with technique).

Non-mathematical: Construct a proof by recalling each step of a proof one has seen or done before. Vs. *Mathematical:* Implement a general proof technique.

Some participants seemed to have developed non-mathematical practices in the sense that their proof techniques comprised recalling the steps of particular proofs they had seen or done before. This was particularly evident for the participants (S6 and S11) who said they had learned how to find the limit of a sequence of a very particular form – $\sqrt[n]{c + e^n + \sin(?)}$ – and were unable to adapt the steps for finding $\lim_{n\to\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$. It also seemed to be the case for the participants (e.g., S1 and S5) who got stuck at the step $\frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{n} \leq \frac{\ln(2+a^n+\arctan(x_n))}{n} \leq \frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}$ because they were unable to figure out how to find an appropriate upper bound for $\frac{\ln(a^n+k)}{n}$, and exhibited frustration because tasks like Task 5 were always easy on exams. Even those participants (e.g., S7, S8, S12, S14) who recalled a useful technique for bounding $\frac{\ln(a^n+k)}{n}$ (i.e., $\frac{\ln(a^n+k)}{n} \leq \frac{\ln(a^n+a^n)}{n}$) seemed to get a bit stuck in the recall of steps since they were unable to abstract the technique they were using to deal with different values of a (e.g., $a \in (0,1)$). We think these behaviours are linked with a positioning of Student towards a narrow path to a practice: i.e., a path that emphasized sequences of a particular form, whose limits could be found by following the steps.

All this said, when it came to Task 5, it seemed no participant had learned a general mathematical "Squeeze Theorem" proof technique; that is, no participant clearly indicated that they were able to think about bounding in terms of formal approximation. One participant (S12) exhibited some creativity in determining an upper bound – $\frac{\ln(a^n+k)}{n} \leq \frac{\ln(a^{n+m})}{n}$ for some m – and this seemed to be based on a general proof technique: i.e., find a lower bound and its limit (in this case $\ln(a)$), and then find an upper bound with the same limit. While such a technique could be helpful, we would call it non-mathematical since it relies on the algebra working out, rather than the general idea of bounding as formal approximation. Other participants (e.g., S3, S8, S9, S13, and S15) seemed to have the potential for a more mathematical general proof technique based on combining bounding and limit finding. Most of them, however, needed to rely on intuitive techniques at some point in their proof. As mentioned above, some students were very fluent in the intuitive technique of "thinking about what matters in the long run": i.e., in approximating a sequence like $\frac{\ln(a^n+k)}{n}$ for large n. In the context of the interview, no participant showed that they could use an intuitive approximation to inform their bounding: e.g., $\forall\ c > 1, \exists N \in \mathbb{N}, \forall n \geq N$, $\frac{\ln(a^n+k)}{n} \leq \frac{\ln(c \cdot a^n)}{n}$.

Non-Mathematical: Theoretical discourses
based on established experience, inert knowledge,
and/or taking one's own understanding for granted.

VS.

Mathematical: Theoretical discourses
based on clarifying, questioning, and verifying one's own understanding.

There was some evidence in Task 5 of participants having learned non-mathematical practices in the sense of supporting them with non-mathematical theoretical discourses. For instance, in working with a sequence like $\frac{\ln(a^n+k)}{n}$, both bounding and the intuitive technique "thinking about what matters in the long run" rely on important theories about the logarithm: namely, it is an increasing continuous function. Only one participant (S10) showed an inclination towards using such theories to try to clarify, question, and verify their own thinking. This was not surprising given S10's positioning as an Enthusiast in RA I, i.e., his interest in breaking things down and trying to figure out how to properly build a formal argument. No other participant spoke about the continuity of the logarithm as being important in Task 5; and when it came to the increasing condition, participants either seemed to leave it implicit in their solution or did not really know why it was important. Indeed, some participants' responses showed that students might still be thinking that you can apply the logarithm to both sides of an inequality just like performing any other algebraic operation.

In solving Task 5, several participants also exhibited practices for finding a limit that might be deemed non-mathematical in the above sense: i.e., they seemed to be based on non-mathematical theoretical discourses. As mentioned above, some participants "thought about what matters in the long run" based on tacit assumptions (e.g., the continuity of the logarithm, which allows one to think about the limit of $\ln(f(n))$ by approximating the long term behaviour of $f(n)$). Other participants seemed to be using l'Hospital's Rule based on inert knowledge (i.e., a procedure informed by some theorem). Some also "compared the growth of the numerator and the denominator" based on taking their own understanding for granted (e.g., $\ln(f(n))$ grows slowly and always slower than n). Interestingly, in Task 5, we saw how these techniques might lead to erroneous answers. Recall, for example, how S8 used a combination of the above techniques to find that when $a = 2$, the answer is $\ln(2)$; then he quickly generalized the answer to $\ln(a)$ for any a, without carefully questioning whether the techniques still apply for different values of a. Recall also how S1 and S14 thought the limit of $\ln(f(n))\,/n$ was 0 no matter the behaviour of $f(n)$. This said, most participants who used the above techniques in Task 5 showed a potential for transitioning to mathematical practices in that they questioned the legitimacy of the intuitive techniques in constructing a proof. This seemed to be linked to different positions, the didactic contract in RA I (i.e., the expectation that students should prove), and the activities offered in the course (the solutions to which emphasized proof techniques and did not explicitly include intuitive arguments). For many participants, however, there still seemed to be some unresolved tension about the intuitive techniques; in particular, the students did not seem to know if, when, and/or why the intuitive techniques might be legitimate from a mathematical point of view. We think that inviting students to work on these intuitive techniques – e.g., to clarify, question, and verify them – in the context of complex sequences (like the one used in Task 5) might help them face and work on resolving the tension, while engaging them in developing practices that are more mathematical in nature.

6.6 Task 6

$$\textit{Let } g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a) For what values of p *is* g *continuous on* $\mathbb{R}$*?*
b) For what values of p *is* g *differentiable on* $\mathbb{R}$*?*
c) What if $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$ *?*
For what values of p *is* g *continuous on* $\mathbb{R}$*? differentiable on* $\mathbb{R}$*?*

6.6.1 Practices to be Learned[41]

Recall that we had designed our task-based interview so that the tasks would gradually increase in complexity. Task 6 is complex in and of itself: it concerns a function that is ill-defined, includes a parameter p, and comprises several interconnected parts, with part (c) requiring a particularly complex argument. This said, Task 6 might also be considered complex by a successful RA I student because it relates only partially to the practices to be learned in the course.

$$g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

b) For what p is g differentiable at $x = 0$?
First we calculate the derivative at $x = 0$:

$$g'(0) = \lim_{h\to 0} \frac{h^p \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h^{p-1} \sin\left(\frac{1}{h}\right)$$

For the limit to exist, we need $p > 1$.
For $p \leq 1$, the limit does not exist and f is not differentiable at $x = 0$. ■

Figure 6.30 An example of the kind of partial solution anticipated for Task 6(b).

We would expect a successful RA I student to be able to determine if a function that looks like g is differentiable at $x = 0$, as shown in Figure 6.30. Implicit in the solution is the definition of differentiability: i.e., g is differentiable at x_0 if and only if $\lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} = g'(x_0)$ exists. Notice that to receive full marks, students would not be required to provide arguments to support their conclusions about the existence of the limit: $\lim_{h\to 0} h^{p-1} \sin\left(\frac{1}{h}\right)$. We would expect a successful

[41] As stated at the beginning of Chapter 6, the goal of this section is to construct a model of practices to be learned in RA I most relevant for Task 6. This model is synthesized in table form in Appendix G.

RA I student to be able to provide an argument (based on the Squeeze Theorem) that $h^{p-1}\sin\left(\frac{1}{h}\right) \to 0$ as $h \to 0$ if $p > 1$ (since sine is bounded between -1 and 1). We do not know how a successful RA I student would justify the non-existence of limits since no techniques for doing so were demonstrated within the assessment activities that we analyzed.

$T_{6_{ab}}$: Determine if (or prove that) g is differentiable at $x = 0$, where g is a piecewise function with $g(0) = 0$ and, for $x \neq 0$, $g(x)$ includes the product of a monomial, $x^p, p \in \{2, 3, 4, \dots\}$, and sine or cosine of a rational function of the form $\frac{c}{r(x)}$ with $c \in \mathbb{R}$ and $r(x)$ a polynomial. Possibly also determine if (or prove that) g' is continuous at $x = 0$.
$\tau_{6_{ab}}$: Calculate $g'(0) = \lim\limits_{h\to 0}\frac{g(h)-g(0)}{h}$ and check if the limit exists. To determine if g' is continuous at $x = 0$: Check if $g'(x)$ converges to $g'(0)$ as $x \to 0$. Commonly tested was $g(x) = \begin{cases} x^p f\left(\frac{c}{r(x)}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ where $p \in \{2, 3, 4, \dots\}$, $f(x) = \sin(x)$ or $f(x) = \cos(x)$, $c \in \mathbb{R}$, and $r(x)$ is a polynomial: We have $g'(0) = \lim\limits_{h\to 0}\frac{h^p f\left(\frac{c}{r(h)}\right)}{h} = \lim\limits_{h\to 0} h^{p-1} f\left(\frac{c}{r(h)}\right) = 0$. So g is differentiable at 0. Outside 0 we have $g'(x) = x^{p-1} f\left(\frac{c}{r(x)}\right) + x^p f'\left(\frac{c}{r(x)}\right)\left(\frac{-cr'(x)}{r(x)^2}\right)$. For $x \to 0$, the first term converges to 0. If $x^p\left(\frac{-cr'(x)}{r(x)^2}\right)$ converges to 0, the second term also converges to 0 and g' is continuous at 0. If not, the second term does not have a limit and g' is not continuous at 0. ■
$\theta_{6_{ab}}$: Implicit: g is differentiable at $x_0 \Leftrightarrow \lim\limits_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h} = g'(x_0)$ exists. g' is continuous at $x_0 \Leftrightarrow \lim\limits_{x\to x_0} g'(x) = g'(x_0)$.

Table 6.19 The practice to be learned in RA I most relevant to Tasks 6(a) and (b).

We could model the practice to be learned in RA I and most relevant to Task 6(b) as shown in Table 6.19. The specification of $T_{6_{ab}}$ reflects the types of functions to which students were exposed and for which $\tau_{6_{ab}}$ might be reduced to a set of steps (as shown in the table). In the assessment activities we analyzed, the function g was piecewise with $g(0) = 0$; and for $x \neq 0$, $g(x)$ was typically equal to the product of a monomial and the cosine or sine of a rational function whose limit at 0 is infinity (a property that ensures that the existence of the derivative depends on the power of the monomial). Some examples of pieces for $x \neq 0$ include $x^2\cos\left(\frac{2013}{x}\right)$,

$x^4 \sin\left(\frac{5}{x^2+x^3}\right)$, $x^4 \sin\left(\frac{5}{x+2x^3}\right)$, and $x^s \sin\left(\frac{1}{x^t}\right)$, where s and t are general positive real numbers. Note that the last, most general, example occurred on an assignment, while the activities on exams were all specified. We wondered how a successful RA I student would deal with the parameter p in Task 6 after following this path of activities.

We did not know how a successful RA I student would deal with the request of determining if g is differentiable on all of $\mathbb{R}$ (not just at $x = 0$). It is possible that they would be drawn to using the definition of differentiability since "prove by definition" was an important type of task in RA I. There were also some activities in exams that requested students to "prove directly from the definition" that a specified function (e.g., $g(x) = |x^3|$) is differentiable on $\mathbb{R}$. We would expect a mathematician to argue that the only potential problem for the differentiability of g is at $x = 0$, since $g(x) = x^p \sin\left(\frac{1}{x}\right), x \neq 0$, is a product of differentiable functions (with the ability to state and prove the underlying theorems). We would also expect a mathematician to notice that g is ill-defined for certain non-integer values of p when $x < 0$, an issue that they would address either by altering the definition of g or by discussing how the ill-defined cases affect the solution to the task (e.g., g is not differentiable on $\mathbb{R}$ for values of p where it is not defined on $\mathbb{R}$). We were interested to see if the participants of our study would make similar arguments or observations.

$$g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

a) For what p is g continuous at $x = 0$?

For $x \neq 0$, we have $g(x) = x^p \sin\left(\frac{1}{x}\right)$.

For $x \to 0$, g converges to 0 if $p > 0$.

So g is continuous at 0 if $p > 0$. ■

Figure 6.31 An example of the kind of partial solution anticipated for Task 6(a).

In some of the activities mentioned above, students were not only asked to prove that the piecewise function g is differentiable at 0; they were also asked about the continuity of g' at 0. We include this possibility in the model in Table 6.19. If we adapt the practice to g, then we can produce a partial answer to part (a) of Task 6, as shown in Figure 6.31. Implicit in the solution is the limit definition of continuity: i.e., g is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} g(x) = g(x_0)$. Once again, students would not be required to justify their calculations of limits in their solution; so we do not know for sure which kinds of techniques and discourses they would be expected to use. It is possible that inquiring into the continuity of g (in Task 6(a)), rather than g' (as in the assessment activities we analyzed), would lead a successful RA I student to struggle in adapting their learned practice to produce the solution in Figure 6.31. We also wondered if participants might be drawn to applying other routinized practices they had learned to solve Task 6(a). For instance, in another path to a practice, a successful RA I student would have learned how to prove that simple specified functions (like polynomials or rational functions) were continuous at specified points x_0 using the ϵ-δ definition of continuity: i.e., g is continuous at $x_0 \Leftrightarrow \forall\epsilon > 0, \exists\delta,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. They would have also learned how to state this definition and/or apply and manipulate it in solving other types of tasks: e.g., Prove that if $f: \mathbb{R} \to \mathbb{R}$ is continuous and $f(a) = b$ (where a and b are specified), then there exists a $\delta > 0$ such that $f(x) \leq b + 1$ for x in the δ neighbourhood of a. Given that the $\epsilon - \delta$ definition of continuity was used more often and more explicitly in assessment activities, we thought that a successful RA I student might be compelled to use it for Task 6(a). We were interested in seeing if participants would realize the inefficiency of going in this direction.

Just like for Task 6(b), it is unclear how a successful RA I student would go about dealing with the continuity of g on all of $\mathbb{R}$. One related collection of activities concerned the general type of task: Judge the possibility of a statement about the (dis)continuity of certain combinations of (dis)continuous functions. In the case where the statement is possible, students were expected to provide an example; otherwise, they were expected to provide a short proof of the impossibility of the statement. For instance, one activity asked if the following is possible:

$$(f \text{ continuous at } x_0) \wedge (g \text{ not continuous at } x_0) \wedge (f + g \text{ continuous at } x_0).$$

The corresponding solution stated the answer to be "no" and provided a short proof: $(f + g) - f$ is continuous as a difference of continuous functions. We were not sure if the exposure to such activities would assist participants in recalling and implementing the corresponding general statements for solving other tasks: e.g., for Task 6(a), $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous when $x \neq 0$ because the product of continuous functions is continuous. As alluded to above, this is how we would expect a mathematician to argue, along with a critique of the ill-defined nature of g for $x < 0$. In addition to wondering about how participants would deal with the $x \neq 0$ case, we were also interested in knowing if they would spontaneously study where g is undefined. Such behaviour would represent a departure from the practices to be learned in RA I, which seemed to be based almost exclusively on well-defined functions.

$$g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

c) At which points is g continuous on $\mathbb{R}$?
Let g be continuous at x_0.
i.e., $\forall (x_n)_{n\in\mathbb{N}}$, if $x_n \to x_0$, then $f(x_n) \to f(x_0)$.
Let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $x_n \to x_0$ and $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $y_n \to x_0$.
Since g is continuous at x_0, we have

$$\begin{array}{l} g(x_n) \to g(x_0) \\ \| \\ 0 \to 0 \end{array}$$

and $$\begin{array}{l} g(y_n) \to g(x_0) \\ \| \\ y_n^p \sin\left(\frac{1}{y_n}\right) \to \begin{cases} x_0 \sin\left(\frac{1}{x_0}\right) & \text{if } x_0 \neq 0 \\ 0 & \text{if } x_0 = 0 \text{ and } p > 0 \end{cases} \end{array}$$

$$\begin{aligned} g \text{ continuous at } x_0 &\Leftrightarrow x_0 = 0, p > 0 \text{ or } x_0^p \sin\left(\frac{1}{x_0}\right) = 0 \\ &\Leftrightarrow x_0 = 0, p > 0 \text{ or } x_0 = \frac{1}{n\pi}, n \in \mathbb{Z}\backslash\{0\}. \blacksquare \end{aligned}$$

Figure 6.32 An example of the kind of partial solution anticipated for Task 6(c).

There is a third definition of continuity (the so-called "sequence definition of continuity") that a successful RA I student might be expected to state or use to solve certain types of tasks in final exams: namely, g is continuous at $x_0 \Leftrightarrow \forall (x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} g(x_n) = g(x_0)$. This definition turns out to be useful in solving Task 6(c). Recall that in part (c) of Task 6, the definition of g changes so that the pieces are defined on the irrationals and rationals (rather than on $\mathbb{R}\backslash\{0\}$ and $\{0\}$): i.e.,

$$g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

If asked to locate the points of continuity of this function, we expected a successful RA I student to be able to attempt a solution like the one depicted in Figure 6.32. More specifically, the student would consider a general point of continuity, x_0, and then try to use the sequence definition of continuity to deduce possible values for the point. Students would not be required to explicitly explain all deductions: e.g., the existence of a sequence of entirely rational numbers (or entirely irrational numbers) that converges to x_0, or the fact that $y_n^p \sin\left(\frac{1}{y_n}\right) \to x_0 \sin\left(\frac{1}{x_0}\right)$ as $y_n \to x_0$ for $x_0 \neq 0$. Hence, we did not know what kinds of explanations a successful RA I student would have for such deductions.

We could model the practice to be learned in RA I most relevant to Task 6(c) as shown in Table 6.20. In assessment activities, the functions to which this practice was applied were piecewise defined on the irrationals and rationals, like g in Task 6(c). But the pieces were simple rational or polynomial functions, as in

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \\ x^3 & \text{if } x \in \mathbb{Q} \end{cases}.$$

Given the increased complexity of the function in Task 6(c), it is possible that a successful RA I student would struggle to construct the solution in Figure 6.32. We expected that the generality of the function (i.e., the parameter p) could also be a source of difficulty since it requires the consideration of cases, which would be unnecessary for the simpler g shown above.

T_{6_c}: Find the points of continuity of $g(x) = \begin{cases} g_1(x) & x \in \mathbb{R}\backslash\mathbb{Q} \\ g_2(x) & x \in \mathbb{Q} \end{cases}$, where g_1 and g_2 are specified functions (typically simple rational functions and/or polynomials).
τ_{6_c}: Let g be continuous at x_0, $(x_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to x_0, and $(y_n)_{n\in\mathbb{N}}$ be a sequence of irrational numbers converging to x_0. Then $\lim_{n\to\infty} g_2(x_n) = g(x_0)$ and $\lim_{n\to\infty} g_1(y_n) = g(x_0)$. Implicit: Solve $\lim_{n\to\infty} g_2(x_n) = g(x_0) = \lim_{n\to\infty} g_1(y_n)$. Specify $g_2(x_n)$ and $g_1(y_n)$ based on the definitions of g_2 and g_1, and calculate possible limits. Consider all possible combinations of these limits being equal.
θ_{6_c}: "If g is continuous at x_0, then $\forall (x_n)_{n\in\mathbb{N}}$ that converge to x_0, we have $\lim_{n\to\infty} g(x_n) = g(x_0)$."

Table 6.20 The practice to be learned in RA I most relevant to Task 6(c).

Just like for Task 6(a) and Task 6(b), there is also a difference in the way that Task 6(c) is phrased when compared to the type of task modelled in Table 6.20. That is, Task 6(c) asks students to determine if a function is continuous and differentiable on $\mathbb{R}$, not to locate the points of continuity of the function. We were unsure if a successful RA I student would be able to adapt the solution in Figure 6.32 to answer 6(c). Nevertheless, such an adaptation seems relatively easy when compared to constructing the solution. Indeed, once one knows that g is continuous only at 0 or at $\frac{1}{n\pi}$ for non-zero integers n (assuming that g is also well-defined), then g is not continuous at all other points, whereby it is not continuous on $\mathbb{R}$. Although we would expect a mathematician to be able to make such an argument, we would also expect them to produce a different kind of solution to Task 6(c): e.g.,

> We can show that $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$ is not continuous at 1, assuming p is such that g is well-defined.
> $\exists (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $x_n \to 1$.
> Then $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n^p \sin\left(\frac{1}{x_n}\right) = 1^p \sin\left(\frac{1}{1}\right) = \sin(1) \neq 0 = g(1)$, where we have used the fact that $x^p \sin\left(\frac{1}{x}\right)$ is continuous for $x \neq 0$.
> So $x_0 = 1$ does not satisfy the sequence definition of continuity.
> Since g is not continuous at 1, it is not continuous on $\mathbb{R}$, whereby it is not differentiable on $\mathbb{R}$. ■

We were unsure about how a successful RA I student would argue about the differentiability of g in Task 6(c). As indicated above, we would expect a mathematician to easily argue that g is not differentiable on $\mathbb{R}$ after having argued that g is not continuous on $\mathbb{R}$. Moreover, we would expect their argument to be supported by a mathematical technology of the sort:

> I know that if g is differentiable on $\mathbb{R}$, then it is continuous on $\mathbb{R}$. I can prove it. So the contrapositive statement must also be true – and I can prove that also. Hence if g is not continuous on $\mathbb{R}$, it is not differentiable on $\mathbb{R}$ either.

The statements "g differentiable implies g continuous" and "$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$" were present within several practices to be learned in RA I, though not the practices to be learned most relevant to Task 6. We wondered if a successful RA I student would be able to call upon such statements in a new way: e.g., to make connections between parts (a) and (b) of Task 6, or to enable an efficient conclusion to part (c).

6.6.2 Practices Actually Learned[42]

When it came to Task 6, we expected a successful RA I student to have learned only partially related practices. In relation to (a) and (b), we expected the student to be able to use the limit definitions to study the differentiability and continuity of functions that look like g, at $x = 0$, and with p specified. We were unsure how a successful RA I student would deal with the question of differentiability and continuity on $\mathbb{R}$, or the general nature of p. Regarding (c), we expected a successful RA I student to be able to find the points of continuity of a simple and specified piecewise function with pieces defined on $\mathbb{R}\backslash\mathbb{Q}$ and $\mathbb{Q}$. We thought the student might struggle with the complexity and generality of g, or the different phrasing of the given task (i.e., Task 6(c) is about finding values of p for which a function is continuous, and then differentiable, on $\mathbb{R}$, rather than finding points of continuity).

The complexity of Task 6, in relation to the practices to be learned in RA I, is perhaps why participants' approaches to solving the task did not seem to be as routine as for some of the other tasks. Our analysis of students' task solving led us to identify many more pairs of the form $[\tau, \theta]$ than we had anticipated. To highlight this, we structure this section according to the ways in which students seemed to consider producing a technique for each part, (a), (b), and (c). Within each section, we describe how students seemed to be coming up with their techniques (in contrast with how we expected them to), and we discuss why the students may have thought the resulting technique(s) would work (or not). Note that S15 was unable to contemplate solving Task 6 due to the time constraints of the interview. Thus, the following analysis is based on fourteen out of the fifteen participants of our study.

6.6.2.1 Choosing a Technique: Part (a)

For part (a) of Task 6, we had anticipated the use of the limit definition of continuity to determine for which values of p the given function is continuous at $x = 0$: i.e., for which values of p

$$\lim_{x\to 0} g(x) = g(0) \Leftrightarrow \lim_{x\to 0} x^p \sin\left(\frac{1}{x}\right) = 0.$$

[42] As stated at the beginning of Chapter 6, the goal of this section is to construct models of the practices actually learned by the fifteen successful RA I students we interviewed in relation to Task 6. These models are synthesized in table form in Appendix G.

To our surprise, the students we interviewed considered constructing solutions based on several other technologies, including the formal $\epsilon - \delta$ definition of continuity (6.6.2.1.1), theorems such as "if g is differentiable on D, then g is continuous on D" (6.6.2.1.2), and discourses that describe what the graph of a continuous function looks like (6.6.2.1.3). Among those who we identified as (eventually) taking some version of the expected approach, we also found some interesting variations, not only in the conclusions they made, but also in how they justified their chosen techniques (6.6.2.1.4). We present our observations and inferences in more detail throughout the following sections.

6.6.2.1.1 By the formal ϵ-δ definition.

Upon seeing part (a) of Task 6, four students (S4, S5, S6, and S10) had the instinct to recall the formal ϵ-δ definition of continuity. "So immediately, before I even look at this, I need to remind myself exactly what continuity means," said S10, as he started to write the following:

$$g \text{ is continuous on } \mathbb{R} \Leftrightarrow$$
$$\forall x_0 \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall x \in (x_0 - \delta, x_0 + \delta), |g(x) - g(x_0)| < \epsilon \,.$$

Note that S10's written definition is lacking one precision: the interval $(x_0 - \delta, x_0 + \delta)$ should be closed; otherwise, it is correct. S4, S5, and S6 had much weaker memories of the definition. They all recalled, as S4 put it, that "continuity is with the delta"; but they struggled to come up with the rest of the formal statement. S6 recalled the inequalities (i.e., $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$), but not the quantifiers; and S4 recalled the quantifiers, but not their order: "It was something about for all epsilon greater than zero, there exists a delta greater than zero… […] No. I flipped the signs. It's there exists epsilon, for all delta. I think." It is possible that S4 was confusing the definition of a function being continuous at a point and its negation, both of which were used in solutions to activities given in RA I. S5 also exemplified how the students may have been struggling to remember one definition among many. After writing the line shown in Figure 6.33, she noted: "I'm not even sure what definition this is." Figure 6.33 is not exactly representative of any of the definitions to be learned in RA I, though it is very close to the definition of uniform continuity.

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall x, y \in \mathbb{R} \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Figure 6.33 S5's attempt at writing the definition needed for a task about "continuity."

It seems that these students had developed a reflex of writing the ϵ-δ definition of continuity for tasks that included a related word. S5 explicitly explained: "Whenever I see like let's say 'continuous,' 'uniformly continuous,' I just write down the theorem right away and then see if I could do something about it." In a similar manner, S10 said that he

> wasn't even looking that much at the function itself yet. [-] By writing down the definition of continuity, I was hoping to remember how to show that something is continuous. [-] I was hoping that this would sort of guide my approach.

It is likely that these students had become used to the ϵ-δ definition being helpful in solving tasks about "continuity." S4, for example, justified his choice to try to recall the definition by saying that "it's the main way of proving continuity that I can remember." Two other students (S7 and S14) also considered using the ϵ-δ definition to get unstuck while solving Task 6(a). S7 added:

> Usually, when you're not sure of something, you go back to the definition. [-] So usually just going back to this helps give you an idea of whether or not you're on the right track, if what you're saying makes sense. Here, not so much.

None of the students who considered building a technique from the ϵ-δ definition made progress with this approach. Perhaps if they had better recalled the definition and the corresponding direct proof technique, S4, S5, and S6 would have been able to see that it is not an efficient way to solve the given task. Imagine, for example, letting $x_0 \in \mathbb{R}$ and $\epsilon > 0$, and then trying to determine for which values of p there is always a δ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \epsilon$ for the given g. While it might be possible to imagine doing such a thing, the technique is impractical and unnecessarily cumbersome. This said, it seemed that S6 thought she could not move forward simply because she was out of practice: "As soon as I'm done with the material, you know, like the exam, I forget everything." S5 had an inkling that the task may have called for a different approach because of the way it is non-routinely phrased. After trying to write the formal definition, she said: "I don't even know if we needed that. Cause now the question is asking for the values of p." As alluded to above, S7 also questioned the usefulness of the formal definition. Like S5 and S6, he was unable to coherently devise another technique for solving Task 6(a); unlike S5 and S6, he recalled the ϵ-δ definition completely. We suspect, therefore, that although S7 had a good memory of formal definitions, he had not yet developed a certain level of fluency in them; for example, the level required to detect in the ϵ-δ definition of continuity the corresponding limit definition, which might have assisted him in solving the task.

6.6.2.1.2 By a sufficient condition.

We identified three sufficient conditions for continuity that students spontaneously considered using to produce a technique for part (a) of Task 6: namely,

1. g is differentiable on $\mathbb{R}$;
2. $\lim_{x \to x_0^-} g'(x) = \lim_{x \to x_0^+} g'(x) = g'(x_0)\ \forall x_0 \in \mathbb{R}$; and
3. g is uniformly continuous on $\mathbb{R}$.

S1's instinct upon seeing Task 6(a) was to contemplate the relationship between continuity and differentiability. After some thought, he concluded: "If we can derive it, then it's continuous; but it's not that it's continuous that we can derive it. Yeah. So then I'd derive this." Recall that Task 6(a) asks for the values of p such that a function g is continuous on $\mathbb{R}$. When he remembered that differentiability implies continuity, it seemed that S1 decided to solve the task by finding the values of p such that g is differentiable on $\mathbb{R}$. He did not seem to realize that this would not completely solve the task. Indeed, there are values of p (e.g., $p = 1$) for which g is continuous on $\mathbb{R}$, but not differentiable on $\mathbb{R}$. Thus, if S1 implemented his chosen technique, he would miss these values of p. Interestingly, and as exemplified in the above citation, S1 seemed to know the technology required for realizing the insufficiency of his technique. More specifically, he seemed to know that functions could be continuous, but not differentiable. He gave the typical example of the absolute value function, and even mentioned that his stats teacher had introduced him to a more exotic function that is continuous everywhere, but not differentiable anywhere.

There could be several reasons why S1 did not realize in the interview that a sufficient condition would not necessarily produce a sufficient technique. When asked why he transformed the given task into a task about differentiability, S1 said: "I hate the continuous definition. [-] So I'd say that I tried to work my way around it." Hence, it is possible that "using differentiation" was the only technique that S1 knew (or thought of) to solve the given task, besides using the formal ϵ-δ definition of continuity. In connection with this, it seems that "using differentiation" to show that a function is continuous was a common practice in RA I for some students. At least three other students we interviewed (S5, S9, and S12) considered producing a technique in the same way as

S1. When S5 got stuck using the formal definition and the interviewer asked if she knew of any other approach, S5 replied: "Could we use differentiation? [-] I know we had something else we learned, and basically that meant continuous. But I don't remember." S9, whose memory was much stronger than S5's, recounted his experience with more detail:

> I remember being in the Analysis class. And a function being continuous was a big deal for a lot of the proofs. So it was like: How do you prove that the function is continuous? So the derivative was like my best friend for those things.

During his solution to Task 2, S9 exemplified what he meant: He used differentiation rules to calculate the derivative of $f(x) = e^x - 100(x-1)(2-x)$ and concluded that it was differentiable (on $\mathbb{R}$), hence continuous (on $\mathbb{R}$), whereby he could invoke the Intermediate Value Theorem. Of course, if it were to happen that the function in Task 6(a) was also differentiable on $\mathbb{R}$ for all values of p, then the technique of "using differentiation" would be sufficient for solving the task. Since this is what S1 ultimately (and incorrectly) found, he would have no reason to consider other techniques. Briefly speaking, S1 determined that g is differentiable on $\mathbb{R}\ \forall\, p \in \mathbb{R}$ (and therefore continuous on $\mathbb{R}\ \forall\, p \in \mathbb{R}$) by computing

$$g'(x) = px^{p-1} \sin\tfrac{1}{x} + x^p \left(\cos\tfrac{1}{x}\right) \cdot \left(-\tfrac{1}{x^2}\right)$$

and noting that it is well-defined $\forall\, p \in \mathbb{R}$. This technique for thinking about the differentiability of g is presented and analyzed in more detail in 6.6.2.2.

The second sufficient condition in the list given at the beginning of this section is stronger than differentiability: If $\lim\limits_{x \to x_0^-} g'(x) = \lim\limits_{x \to x_0^+} g'(x) = g'(x_0)\ \forall x_0 \in \mathbb{R}$, then the derivative of g not only exists on $\mathbb{R}$, but is continuous on $\mathbb{R}$. If one looks for the values of p for which g' is continuous on $\mathbb{R}$ to solve Task 6(a), one would miss even more values of p (e.g., not only $p = 1$, but also $p = 2$). S11 was the only student to seemingly use this condition. It did not come to him immediately when he saw the task: "Man, if you asked me this question one week ago," he explained, "I probably would have been able to tell you what to do." Nevertheless, within the next few moments, S11 had the line $\lim\limits_{x \to x_0^-} f'(x) = \lim\limits_{x \to x_0^+} f'(x) = f'(x_0)$ written down on his page. "You do this if you want to check that it's continuous," he said. When prompted by the interviewer to show what he meant, S11 specified: "I don't know how to check if it's continuous on $\mathbb{R}$ though.

[-] Like I can check if it's continuous at zero." Although S11 did not completely do this in the context of the interview (and hence did not provide an answer for Task 6(a)), he exhibited a certain level of know-how. For instance, he found that $g'(0) = \lim_{h\to 0}\left(h^{p-1}\sin\frac{1}{h}\right)$ only exists if $g'(0) = 0$, and he described his next steps in detail: i.e., he would calculate the derivative of $g(x) = x^p \sin\left(\frac{1}{x}\right)$ $(x \neq 0)$ using Wolfram Alpha and then see if the limit of the resulting function as x tends to zero would give zero. S11, like S1, did not realize that these steps would not solve the given task.

S11's actions and discourse hint at another reason why students might turn to sufficient conditions for creating insufficient techniques. It seems that S11 was working based off a vague memory of solving a particular type of task using a particular technique, rather than an understanding of a fundamental (sufficient) condition that might produce a (partial) technique. More specifically, we think that S11 recalled writing the line "$\lim_{x\to x_0^-} f'(x) = \lim_{x\to x_0^+} f'(x) = f'(x_0)$" to solve tasks about the "continuity" of a certain function at zero. Based on our analysis of assessment activities from RA I (see Section 6.6.1), it is likely that S11 was recalling a practice that he had developed for solving tasks that asked about the continuity of f', rather than f. He did not realize, however, that there is a significant difference with the given task (i.e., it asks about the continuity of a function, not its derivative), which calls for a difference in the corresponding technique. Such an oversight indicates that S11's underlying technology was lacking in some way. When the interviewer asked what it was about Task 6(a) that made S11 choose his technique, he simply underlined the word "continuous" in the task statement.

One student (S14) considered and rejected a technique based on a sufficient condition. While solving Task 6(a), S14 was the only student to think about the theorem that says that a uniformly continuous function is also continuous. In fact, this was his first thought: "I know that there's some nice techniques to figure that out," he explained,

> but it's not like if and only if, right? So like if you have a uniformly continuous [-] function, [-] it means it's continuous. But just because it's continuous doesn't mean it's necessarily uniformly continuous. So you might have something that's continuous, but not uniformly continuous. So that doesn't help always.

Unlike S1 and S11, S14 abandoned his initial idea because he seemed to realize that thinking about uniform continuity may not give him the entire story. As mentioned previously, using a sufficient condition such as this might lead him to finding some values of p for which g is continuous on $\mathbb{R}$, but not necessarily *all* such values of p. Another significant difference between S14 and the students mentioned above is that S14 exhibited an awareness of another way of producing a technique for Task 6(a), as outlined in the following section.

6.6.2.1.3 By the way the graph looks.

Figure 6.34 S14's visual representation of continuity.

S14 was the only student to choose to solve part (a) of Task 6 through graphing. When S14 tried (and was unable to correctly) recall the formal definition of a continuous function, he described the definition verbally and visually. "The value at each point is the limit of the function approaching that point, right? So it's like this," he said, as he started to draw the sketch in Figure 6.34. He continued, while adding to the sketch: "If you have this point, the value at this point is the [-] left and right-side limit. For all points on that [-] entire domain, which is the real numbers." One might expect such a discourse to lead S14 to use the limit definition of continuity, like some of the students in the next section. To our surprise, S14 said: "Like if it's ok, I mean, I would probably turn this into a graph, if I was at home. [-] I'm not in an exam, so…"

S14 used the Desmos application on his cell phone to graph the function $x^p \sin\left(\frac{1}{x}\right)$ for different values of p, starting with positive integers. After considering $p = 1$ through $p = 8$, he noticed that

> it's switching… um… above and below, similar to a parabola does, right? [-] But, to me, there's no reason why that wouldn't be continuous. [-] I mean, like I'm not proving it. But I would say that it's fine for integer values.

The images in Figure 6.35 show what the graphs of $x^p \sin\left(\frac{1}{x}\right)$ look like in Desmos for $p = 1, 7,$ and 8 (from left to right). The two rightmost images represent the pattern that S14 seemed to be noticing when he said that the graph is "switching," "above and below." In observing such a pattern, we conjecture that S14 gained a sense of what the graphs would continue to look like if he were to keep increasing p. Based on this, and his visual meaning for continuity, S14 was able to stop and conclude that the function is continuous on $\mathbb{R}$ for (positive) integer values.

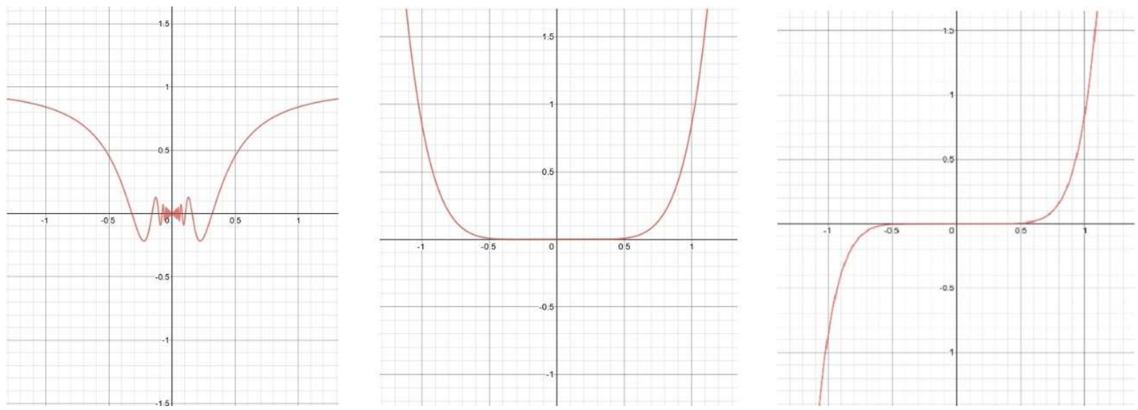

Figure 6.35 Graphs of $x^p sin\left(\frac{1}{x}\right)$ produced by Desmos for $p = 1, 7$, and 8 (from left to right).

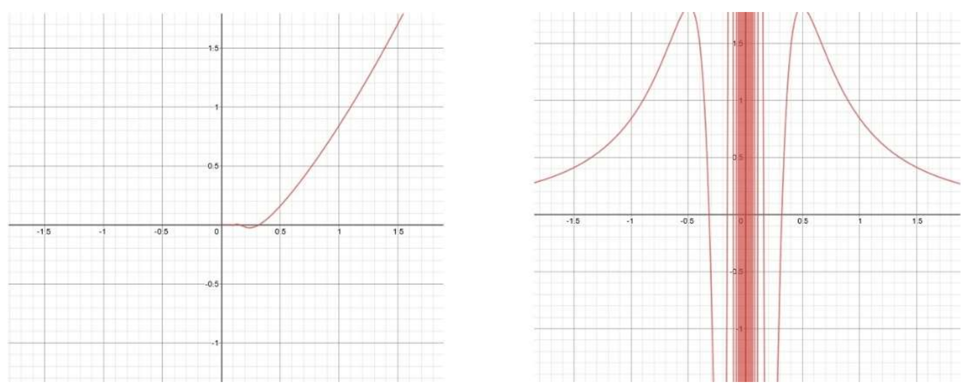

Figure 6.36 Graphs of $x^p sin\left(\frac{1}{x}\right)$ produced by Desmos for a non-integer value of p (left) and a negative value of p (right).

When S14 considered non-integer and negative values of p, he was given reasons to conclude that the function is not continuous on $\mathbb{R}$ for those values. For non-integer values of p (an example of a graph is shown on the left in Figure 6.36), S14 noted that "it doesn't look like it's defined at all for negatives"; and he verified by hand that if $p = \frac{5}{2}$, then $x^p \sin\left(\frac{1}{x}\right) = \left(\sqrt{x}\right)^5 \sin\left(\frac{1}{x}\right)$, which is "not defined for negative numbers." Upon considering a negative value for p and obtaining a graph like the one on the right in Figure 6.36, S14 exclaimed: "Woo! So, I'm not sure. This is not looking good to me because it looks like it goes up to infinity. It looks like it diverges to infinity." In both cases, S14 identified characteristics of the graphs that did not meet his visual meaning for continuity. We suspect that S14 was also relying on additional discourses, which he did not make explicit in the interview: e.g., a function g cannot be continuous on $\mathbb{R}$ if $\exists a \in \mathbb{R}$ where g is undefined or where $\lim_{x \to a} g(x) = \pm\infty$.

We can think of two interconnected reasons why S14 decided to use graphing to solve Task 6(a). First and foremost, among the approaches S14 recalled at the time of the interview, graphing seemed to be the only one that he judged as adequate for solving the task. When asked if the formal ϵ-δ definition would be helpful, S14 said: "Not really. [-] Not right now, no." As discussed in 6.6.2.1.2, S14 also explained that he "wouldn't go about trying to prove that it's uniformly continuous, because [-] it's sufficient, but not necessary for continuity." Secondly, the interview environment afforded S14 the possibility to use his graphing tool. S14 ended his solution by clarifying that he would solve the task in this way only if he was permitted access to the tool: "Otherwise, [-] I don't know. I probably would have prepared for a question like this and [-] I would have known how to attack it with [-] some conditions." In saying this, S14 seemed to be expressing a certain expectation of a mathematics course: i.e., with enough preparation, one can learn the conditions needed to solve the types of tasks that will be tested on exams. At the same time, S14 seemed to be suggesting that he was unaware of the conditions needed for tasks like Task 6(a).

It is notable that, like several of his peers, S14 did not think about (or did not recall) the limit definition of continuity. We suspect that using such a definition would have been considered by S14 as more suitable for an exam situation (in RA I), not only because it does not rely on a digital tool, but also because it might be perceived as more proof-like when compared to the approach he took. And yet, S14 was able to use graphing very efficiently to obtain a correct partial response to Task 6(a): i.e., g is continuous on $\mathbb{R}$ for $p \in \mathbb{Z}^+$. In fact, S14 was the only student to provide an answer to Task 6(a) that did not include "incorrect" values of p (i.e., values of p for which the function g *is not* continuous on $\mathbb{R}$). He nevertheless missed some values of p for which g *is* continuous on $\mathbb{R}$. One of the affordances of graphing software like Desmos is that it permits a user to quickly and easily vary parameters and observe resulting graphs. If accompanied by a curious skepticism, continuing to vary p may have led S14 to realize that his answer is incomplete. For instance, if S14 had questioned his rejection of all non-integer values of p, he may have found that for certain non-integers (in particular, for certain rational numbers), the graph is actually continuous: take, e.g., any value from the set $\left\{\frac{a}{b} : a \in \mathbb{Z}, b = 2n+1, n \in \mathbb{N}, \gcd(a,b) = 1\right\}$. Perhaps S14 would have also been led to question what happens if p is irrational, and, more generally: What does x^p mean in that case?

6.6.2.1.4 By known properties and limits.

Recall that the given function for part (a) of Task 6 is:

$$g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since g is defined as a piecewise function with two pieces, it would seem natural to find the values of p for which g is continuous in two steps:

1. determine the values of p for which g is continuous on $\mathbb{R}\backslash\{0\}$. This could be done by recalling known properties of the function $x^p \sin\left(\frac{1}{x}\right)$ (e.g., it is a product of continuous functions, so long as x^p is defined on $\mathbb{R}\backslash\{0\}$); and
2. determine the values of p for which g is continuous at $x = 0$. This could be done by finding the values of p for which $\lim_{x\to 0} g(x) = g(0) \Leftrightarrow \lim_{x\to 0} x^p \sin\left(\frac{1}{x}\right) = 0$.

The organization of this section is in line with this natural division of work to solve Task 6(a). In 6.6.2.1.4.1, we discuss how participants considered the continuity of g on $\mathbb{R}\backslash\{0\}$. Then, in 6.6.2.1.4.2, we discuss what they did to study the continuity of g at $x = 0$.

6.6.2.1.4.1 When $x \neq 0$.

We identified two students (S4 and S10) as spontaneously including in their solution a discussion of the continuity of $x^p \sin\left(\frac{1}{x}\right), x \neq 0$ by thinking about the function as a product: i.e.,

$$f(x) \cdot h(x), \text{ with } f(x) = x^p \text{ and } h(x) = \sin\left(\frac{1}{x}\right).$$

When asked why he did this, S4 carefully explained that

> by separating the $\sin\left(\frac{1}{x}\right)$ as always being continuous and isolating the x^p, which is the part that's actually important because we're asked about p, it simplifies the actual handling of the function. [-] Now I might be wrong, but I kind of think that two continuous functions multiplied together also result in a continuous function.

Although S4's discourse was a bit more uncertain than S10's, both students' technologies were essentially the same: in the words of S10, if f and h are continuous, "then the product of those two is continuous." The students were hence led to contemplate the continuity of f and h (i.e., of $\sin\left(\frac{1}{x}\right)$ and x^p).

Both S4 and S10 seemed to conclude rather quickly that $\sin\left(\frac{1}{x}\right)$ is continuous for $x \neq 0$. S4 simply said that the "sine function is always continuous," while S10 was a bit more methodical in constructing an argument:

> $\frac{1}{x}$ is continuous when $x \neq 0$.
>
> Assuming sine is continuous, then $\sin\left(\frac{1}{x}\right)$ is continuous when $x \neq 0$.

Implicit in this argument is an additional technology: i.e., if $l(x)$ is continuous on $\mathbb{R}$ and $m(x)$ is continuous on D, then $l(m(x))$ is continuous on D (that is, the composition of continuous functions is continuous). When it came to thinking about the continuity of x^p, S10 was a bit hesitant: "I don't really know what that means for all real numbers. But, I kind of want to say, well that's continuous for any value of p. I can sort of, like, I know what those curves look like, I guess." In a similar manner, S4 turned to his knowledge of graphs, sketching x^p for $p = 1, 2, 3$ (as shown in Figure 6.37) and noticing that "it's always continuous." S4 displayed some hesitation when he thought about what happens if p is negative; he nevertheless concluded that since $x \neq 0$, "it's never going to be one over zero." Despite their doubts, both students ultimately concluded that $\forall p \in \mathbb{R}, x^p \sin\left(\frac{1}{x}\right)$ is continuous whenever $x \neq 0$.

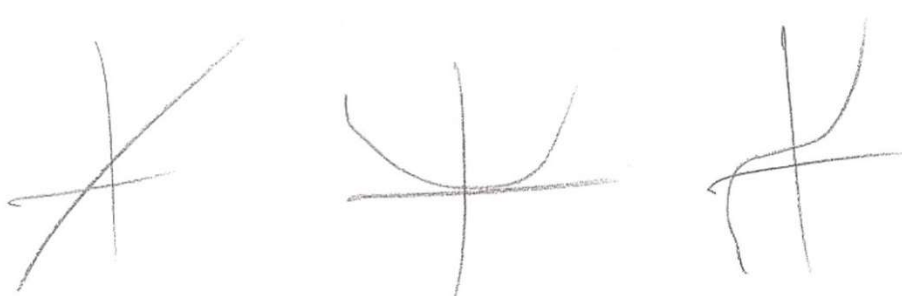

Figure 6.37 Graphs sketched by S4 to check that the function x^p $(x \neq 0)$ is always continuous (i.e., is continuous $\forall p \in \mathbb{R}$).

In comparison with S4 and S10, there was a group of students (S2, S3, S8, and S13) who focussed on $x = 0$ and said little to nothing about when $x \neq 0$. S8 explicitly noted that for the given function,

$$g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0' \end{cases}$$

"there's like a hole. [-] There's sort of like a jump. [-] So the only place where it could be discontinuous is zero." Follow-up questions confirmed that others had also identified $x = 0$ as the only potential contributor to discontinuity. "For the rest, the function is well-defined," explained S13: "[-] Oh sorry… Should I explain these things also from the beginning? [-] To me it was obvious that it's well-defined everywhere except for $x = 0$." Some students also showed they had a discourse like S4 and S10. For example, when asked for clarification on why he was focussing on the $x = 0$ case, S8 said:

> I knew $x^p \sin\left(\frac{1}{x}\right)$ would be continuous everywhere. Because sine is continuous. [-] And x^p will always be continuous, apart from if I made p negative, and then $x = 0$. But [-] it's not equal to zero, so…

S2 started to give a very similar explanation, and then highlighted another reason why the students may have been focussing on $x = 0$: "From my experience, g can only be not continuous at the point of the condition. Yes, yes. So I think I just need to prove g continuous at $x = 0$." It is possible that some students were triggered to consider the continuity of g only at $x = 0$ because they expected the discontinuities of piecewise functions to occur only where the pieces meet, an expectation that may have been reinforced by the assessment activities they received in RA I (as described in 6.6.1). Of course, this need not be the case, as exemplified by the function in Task 6.

Contrary to the conclusions outlined above, two students (S9 and S14) noticed that the function $x^p \sin\left(\frac{1}{x}\right)$ might not be well-defined, even if $x \neq 0$. As described in 6.6.2.1.3, S14 made this realization through experimental graphing. He observed that for certain non-integer values of p, the graph of g does not extend into the negative x-axis, suggesting that for such values of p and for $x < 0$, g is undefined (and therefore not continuous). S9, in comparison, spontaneously spent more time studying the expression, $x^p \sin\left(\frac{1}{x}\right)$:

I'm trying to see, like, what possible restrictions could there be? […] [-] Cause the only restriction I can see is when x is worth zero. And we have that covered. So now… [-] What would be the problem? I guess, mm, p has to be, like a whole. Like an integer.

S9 followed up this comment with an example: $(-1.5)^{1.5}$. "I don't know if you can do that," he said, as he reached for his calculator to confirm: "Yeah. It doesn't like it, eh?" Indeed, if one tries to calculate the number $(-1.5)^{1.5}$ with a calculator, one should obtain an error message. Like S14, S9 decided to rule out non-integer values of p based on this observation.

There are several reasons why students may have overlooked the discontinuities identified by S9 and S14. First, these students may have thought that they had already identified *the* major issue: i.e., the potential discontinuity of g at $x = 0$. It is perhaps because S9 had not identified such an issue that he was pushed to look for others. Second, given our analysis of assessment activities from RA I, it is likely that the students did not expect to receive a poorly defined function. Third, it is possible that some of these students had developed concept images of x^p that were restricted to integer values of p. None of the students we interviewed considered what x^p would be like if p were a general rational or irrational number.

6.6.2.1.4.2 When $x = 0$.

Seven students (S2, S3, S4, S8, S10, S12, and S13) eventually explained that to solve part (a) of Task 6 – for what values of p is $g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous on $\mathbb{R}$ – they needed to determine the values of p for which $\lim_{x \to 0} x^p \sin\left(\frac{1}{x}\right) = 0$. To produce this technique, the students used different types of technologies. Some students called upon informal concept definitions of continuity only. S4, for instance, said:

I don't remember exactly the definition of continuity. But, if continuity is that it can't have a jump, so that the graph of the function would be drawn without raising your pencil, then what I would do would be to take the limit approaching zero from both sides.

When asked why the limit needed to be zero, S2 explained similarly that "if g is continuous, it means there is no gap when $x = 0$"; or, in S3's words, the function "just goes nicely to zero." Other students accompanied informal imagery with a more formal discourse. S8, for example, said: "I know that continuity, you can also kind of write it as, like in the limit definition. Like limit as x approaches a of let's say $f(x)$ has to be equal to $f(a)$." Later in the interview, S8 added: "This seems a lot more intuitive to me than, like, the for all epsilon, there exists, blah, blah, blah… I find that a lot more difficult to deal with personally." It seems that S8 also chose his technique because he felt that the formal definition would be more difficult to use. In a similar vein, S13 noted that, when compared with epsilon-delta, the limit definition approach is "easier."

Although we have chosen to not go into detail about the students' implementation of techniques for Task 6, we feel it is important to point out that not all students succeeded in using limits to solve (a). Two of the seven students (S8 and S13) quickly and correctly concluded that to have $\lim_{x \to 0} x^p \sin\left(\frac{1}{x}\right) = 0$ they needed $p > 0$. The remaining five students either required significant time and interaction with the interviewer to settle on and carry out a technique (S3), were unable to make significant progress and did not provide an answer (S4 and S10), or provided the incorrect answer of $p \neq 0$ (S2 and S12). Evidently, by this point in the interview, the students had already been working for a significant amount of time (more than an hour and a half) and may have been losing steam. In addition to this, it is possible that some students may have still been grappling with the meaning of continuity or lacked a certain fluency in calculating limits for functions of the given type. On one extreme was S2, who seemed to be relying on her memory for the values of the limits: for example, she said that she recalled that $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ does not exist due to one of the problems she had done before and that she would use the solution to that problem to help her solve Task 6(a).

6.6.2.2 Choosing a Technique: Part (b)

Part (b) of Task 6 asked students to consider the same function:

$$g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and determine the values of p for which it is differentiable on $\mathbb{R}$. Like for part (a), we wondered if students might simply note that g is differentiable when $x \neq 0$ because it is the product of differentiable functions, and then use the definition of the derivative to focus on the $x = 0$ case: i.e., determine the values of p such that

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{h^p \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} \left(h^{p-1} \sin\left(\frac{1}{h}\right)\right) \text{ exists.}$$

If they had successfully solved part (a) using the expected approach (i.e., using the limit definition of continuity), then this would require a simple adaptation of their previous solution. In general, this did not seem to happen.

One student (S2) thought that the response to part (b) would be the same as for part (a) because she recalled that differentiability and continuity were related in some way. The remaining students tended to call upon the differentiation rules they knew and/or the definition of the derivative to produce techniques in unexpected ways. We organize our analysis according to whether the techniques seemed to be used for thinking about the differentiability of g when $x \neq 0$ (6.6.2.2.1) or when $x = 0$ (6.6.2.2.2). Note that our analysis is now based on only thirteen of the fifteen students: S10, in addition to S15, was unable to attempt part (b) of Task 6 due to a lack of time in the interview.

6.6.2.2.1 When x ≠ 0.

To find the values of p for which $x^p \sin\left(\frac{1}{x}\right)$ $(x \neq 0)$ is differentiable, at least eight students (S5, S6, S7, S8, S11, S12, S13, and S14) considered using the definition of the derivative. We identified this as the only approach taken by S5, S6, and S11. S11 recalled the definition; however, as mentioned in 6.6.2.1.2 above, he quickly chose to specify his task to studying if g is differentiable at $x = 0$ only. S5 and S6, in comparison, claimed to have a weak memory of the definition; but they were willing to try to think about the $x \neq 0$ case. Both students seemed to be driven by their experience in RA I, which, for them, included the expectation of basing a technique on the definition. Almost immediately after being presented with the task, S6 said: "No, I wouldn't know how to do this. I forgot. I don't know the definition, like the definition they gave me." She nevertheless confirmed that she would use the definition if she were taking an exam in RA I: "But if it was something general, [-] like a math test in general, like it doesn't focus on Analysis, I would try to find other ways." In a similar fashion, S5 stated that she would use the definition: "Cause we would always do that."

In hopes of observing how they would use the definition, the interviewer provided both S5 and S6 with the following:

$$g \text{ is differentiable at } x \text{ if } g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \text{ exists.}$$

The students were triggered to make the adequate substitutions, i.e.,

$$\lim_{h\to 0}\frac{(x+h)^p \sin\frac{1}{x+h} - x^p \sin\frac{1}{x}}{h}.$$

Then they attempted to simplify. "I don't think I can," S5 concluded almost immediately. When she got stuck at simplifying, S6 turned to thinking about what happens as h goes to zero. She noticed that "it's a zero over zero case. And, you'd have to use l'Hospital, I think." Although S6 did not continue past this point, her consideration of using l'Hospital's Rule brings up a complex and interesting issue of circularity. Namely, the goal of considering the above limit is to check the differentiability of a certain function; but assuming the differentiability of that function is part of the conditions required to apply l'Hospital's Rule.

For the other students mentioned above (S7, S8, S12, S13, and S14), using the definition of the derivative seemed to be judged as a more difficult alternative, which they consciously chose against. S7 and S14 attempted to make some progress in using the definition. S7's instinct was to study the limit from the left and from the right (shown below). He did not make any substantial steps before concluding: "I can't seem to see if it's the same or not."

$$\lim_{h\to 0^+}\frac{(x+h)^p \sin\left(\frac{1}{x+h}\right) - x^p \sin\frac{1}{x}}{h},\ \lim_{h\to 0^-}\frac{(x+h)^p \sin\left(\frac{1}{x+h}\right) - x^p \sin\frac{1}{x}}{h}$$

S14, in comparison, attempted to use the Binomial Theorem to simplify the limit (see below); but he could not figure out which values of p would eliminate the possibility of having infinite limits (e.g., even if $p > 1$, the last two terms in the sum would include h^{-1}).

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{(x+h)^p \sin\left(\frac{1}{x+h}\right) - x^p \sin\left(\frac{1}{x}\right)}{h} \\ &= \lim_{h\to 0}\sum_{n=0}^{p} x^n h^{p-n-1} \sin\left(\frac{1}{x+h}\right) - \lim_{h\to 0}\frac{1}{h} x^p \sin\left(\frac{1}{x}\right) \end{aligned}$$

In contrast, S8, S12, and S13 completely avoided using the definition in the context of the interview (for the $x \neq 0$ case). S13, for example, said: "I could use the definition of course. [-] But it would be harder, right?" After writing out the necessary limit, S8 said similarly that it "seems like a hard one to evaluate like this, so I'm not going to do that."

S12 abandoned his instinct to study the differentiability of g using the definition when he realized: "I can also do so kind of using Cal I." He, along with S1, S3, S9, S13, and S14, decided to differentiate $x^p \sin\left(\frac{1}{x}\right)$ to get:

$$g'(x) = px^{p-1}\sin\frac{1}{x} + x^p\left(\cos\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right).$$

Then they asked themselves, in the words of S3: "What p would make this problematic?" Or, in the words of S12: "Are there any cases in which I would have kind of violated the rules I learned in Cal I?" Most of the students determined that no values of p posed a problem. In other words, $x^p \sin\left(\frac{1}{x}\right)$ is differentiable on $\mathbb{R}\setminus\{0\}$ for all $p \in \mathbb{R}$. Recalling his solution to part (a), which included noticing that x^p is undefined for negative values of x if p is equal to certain non-integer values, S9 restricted his response to whole numbers. In either case, the students' solutions exhibited an issue of circularity similar to the one highlighted by S6's: the students were aiming to check if a function is differentiable using differentiation rules and differentiation rules only apply under the assumption that the function is differentiable. This circularity remained undetected, or at least unaddressed. S13 explained, in retrospect, that

> you don't know if you can differentiate this way or not, right? You say that if $g(x) = f(x)h(x)$, then $g'(x) =$… But if they're both differentiable. Yeah? I assume that they're both differentiable at any p. And then I did it.

Recall that for part (a) of Task 6, S13 focussed on the case $x = 0$, and was caught off guard by the interviewer's question about the $x \neq 0$ case because she felt g was obviously continuous on $\mathbb{R}\setminus\{0\}$. It is possible that when she proceeded to part (b), she felt as though she needed to do something more than simply state that $g(x) = x^p \sin\left(\frac{1}{x}\right)$ is obviously differentiable for $x \neq 0$.

Two students (S4 and S8) did not feel the need to make additional computations. S8 simply said: "I'm going to say that for all x not equal to zero it's differentiable. I can always find a derivative." S4 showed hints of developing an argument like he did for part (a) (see 6.6.2.1.4.1):

> The sine value is always differentiable. Because it is. And then, x^p… Is there any exponent that can't be differentiated? I don't think there is. Regardless of what it is, fraction or integer, negative or not. [-] Although, wait, does it apply? Cause I'm fairly certain that it applies for continuity, where the product of two [continuous] functions is always

continuous. But is that the case for differentiability? Hmm… I don't remember that. But, I would assume so.

Like the students mentioned above, S4 and S8 seemed to be judging the differentiability of g based on whether they could compute the derivative using the rules they had learned. S4, for example, further explained that in the particular case of $p = 1$, the function would become $x \sin\left(\frac{1}{x}\right)$, and "that's just product rule. So yes, that's differentiable." The difference is that S4 showed signs of producing a technique based on the differentiability of known functions and the property that the product of differentiable functions is differentiable. No other students explicitly argued in this way.

6.6.2.2.2 When $\mathbf{x = 0}$.

At first, S3, S4, and S9 seemed to completely overlook the differentiability of g at $x = 0$. However, follow-up questions by the interviewer revealed that these students may have purposefully omitted this case because they thought the derivative did not involve p. Indeed, when asked about the derivative of g at $x = 0$, S4 said that "it gives you nothing," S9 clarified that "it would stay zero," and S3 simply stated that it's "zero." In other words, these students seemed to have calculated the derivative of

$$g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

to be

$$g'(x) = \begin{cases} px^{p-1} \sin(1/\mathrm{x}) + x^p \cos(1/\mathrm{x}) \cdot (-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Although it is difficult to express with words, the technology producing this technique might be summarized as follows: To derive a piecewise function, apply differentiation rules to the expression for each piece. We predict that this is how S1 calculated the derivative of g to solve part (a) (see 6.6.2.1.2 above), which is why he concluded that g is differentiable on $\mathbb{R}$ for all $p \in \mathbb{R}$ (whereby it is also continuous on $\mathbb{R}$ for all $p \in \mathbb{R}$). Other students may have also considered this idea at some point during their solution. S7, for instance, asked the interviewer: "Would [-] g' have the same definition as g? [-] Where, at zero, it would be equal to zero?"

Some of these students seemed to not know how else to deal with the $x = 0$ case. After stating that $g'(0) = 0$, S3 quickly revoked his statement: "Oh no. The value is the limit. I don't know. [-] I don’t remember how to go about this one, when there’s like a discontinuity like this.” S4 specified that “for differentiability, I can’t remember the definition we were given. And that might be why [-] I kind of disregarded it [i.e., $x = 0$].” In response, the interviewer provided both students with the definition of the derivative. Neither made any progress with it. S4 eventually concluded: “I don’t know what to do. I’m at a loss.” S3 decided not to try because he thought “it will be very difficult. [-] […] Are the next problems easier than this?”

Some students who had trouble implementing a technique based on the definition demonstrated other ideas for solving the problem. S5, S6, and S8 all made the mistake of calculating:

$$g'(x) = \lim_{h \to 0} \frac{\left((x+h)^p \sin\left(\frac{1}{x+h}\right) - x^p \sin\left(\frac{1}{x}\right)\right)}{h}$$

$$\Rightarrow g'(0) = \lim_{h \to 0} \frac{(0+h)^p \sin\left(\frac{1}{h}\right) - 0^p \sin\left(\frac{1}{0}\right)}{h}.$$

They each noted that the $\frac{1}{0}$ poses a problem. S5 got stuck and needed help to realize her mistake, S6 decided to ignore the problem and simplify the term $0^p \sin\left(\frac{1}{0}\right)$ to 0, and S8 changed his approach: “That’s the way I saw it in Analysis. [-] This looks like a bad limit [-] And I’m not too sure where to go from here. [-] So I use the product rule.” Like other students had done before him, S8 computed

$$\left(x^p \sin\left(\frac{1}{x}\right)\right)' = x^p \left(\frac{p}{x} \sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x^2}\right).$$

The difference was why he did it: “I was essentially hoping that by plugging in $x = 0$, I’d be able to kind of identify, like ‘Oh, for this p, it doesn’t make sense.’” At least two other students (S7 and S14) seemed to consider the same technique. For instance, after making a similar computation to S8, S14 concluded:

> I don’t think it is differentiable at zero. [-] Cause if I have an x in the denominator, then that means [-] you have one over x, one over zero, and hmm…. Yeah. You’re not

> evaluating as it goes to zero, or as it goes to infinity. You're evaluating it at zero. And you can't do that. I think.

The more general technology implicit in the production of this technique might be expressed as follows:

If $g(x) = \begin{cases} h(x) & x \neq c \\ f(x) & x = c \end{cases}$, then g is differentiable at $x = c$ if and only if $h'(x)|_{x=c}$ exists.

When S14 realized that $h'(x)|_{x=c}$ never exists in this case, he concluded that g is never differentiable at $x = 0$ (and hence, never differentiable on $\mathbb{R}$). S7 and S8 seemed uneasy with such a conclusion. S8, for example, said that "it feels like you have to use limits in a sense because, um… Yeah, because of that fact that I can't plug directly in."

Many participants (S1, S2, S5, S6, S7, S8, S11, S12, and S13) eventually studied the differentiability of g at $x = 0$ by considering

$$g'(0) = \lim_{h \to 0} \left(h^{p-1} \sin\left(\frac{1}{h}\right) \right).$$

Most were assisted by the interviewer, either in deciding to try out a technique based on the definition (S1), recalling the definition (S2, S5, and S6), noticing simple errors in substitution or simplification (S1, S5, and S8), or considering the $x = 0$ case on its own (S5, S6, S7). Three students independently reached the above limit. Recall that S11 had specified his task to the $x = 0$ case, while both S12 and S13 decided to use differentiation rules to consider the differentiability of $x^p \sin\left(\frac{1}{x}\right)$. Following this, S12 said that his approach "doesn't show differentiability [-] at zero. [-] How would I show the derivative at $x = 0$? That's what I'm asking myself. [-] I'm thinking of using this definition." In a similar vein, S13 clearly separated the $x = 0$ case, noting that it "is tricky because you have to find it using the limit." Both S12 and S13 swiftly calculated $g'(0)$ as shown above and thought about which values of p would make the limit exist. S13 also noticed:

> We can use the other problem. [-] Yeah, you see, differentiable on $\mathbb{R}$ for p is like being continuous on $\mathbb{R}$ when this is p minus one. So you decrease the degree. And then it's the same thing.

6.6.2.3 Choosing a Technique: Part (c)

Part (c) of Task 6 presented students with a function that differs only slightly in its visual representation, but yields significantly different behaviours in terms of continuity and differentiability:

$$g(x) = \begin{cases} x^p \sin(1/x) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$

Based on our analysis of assessment activities from RA I, we anticipated the use of two technologies to study the continuity of g:

1. the fact that to each real number, there exists a sequence of entirely rational (or irrational) numbers that converges to it; and
2. the so-called Heine (or sequence) definition of continuity, i.e., g is continuous at $x \in \mathbb{R}$ if and only if $\forall (x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} g(x_n) = g(x)$.

To our surprise, only two students relied on sequences (6.6.2.3.2) to respond to the question: “For what values of p is g continuous on $\mathbb{R}$?” The remaining eight students who provided an answer to this question did so based on a different technology: the fact that the rationals form a dense subset of $\mathbb{R}$ (6.6.2.3.1). We note that the density of the rationals (or irrationals) is used in a proof of (1); i.e., the technologies are not necessarily unrelated from a mathematical standpoint. This relation did not come up in our interviews.

If a student found that g is not continuous on $\mathbb{R}$ for any p, we wondered if they would be able to immediately conclude that g is also not differentiable on $\mathbb{R}$ for any p. Nine students responded to the question: “For what values of p is g differentiable on $\mathbb{R}$?” Six did as we had somewhat anticipated: i.e., they were quick to conclude that the function is not differentiable because it is not continuous. What we did not anticipate was how the students explained their conclusion: rather than referring to a known theorem (i.e., “if a function is differentiable, then it is continuous”), they spoke about knowing that it is not possible to differentiate, e.g., at a point or at a jump. The three other students who considered the differentiability of g in (c) gave unexpected responses. We provide more details in 6.6.2.3.3.

As alluded to above, two students (S10 and S15) were unable to attempt Task 6(c) due to a lack of time in the interview. For similar reasons, S12 only addressed the question about continuity. In addition to this, there were three students (S5, S9, and S11) who did not attempt to solve the task because they claimed to not know how. Like some of his peers, S9 connected Task 6(c) to a more recent experience in RA II (the Real Analysis course following RA I): "We had something similar to that, and [-] you can't use [-] the Riemann integral. [-] I'm trying to see if it can relate to differentiability. [-] I don't know." In comparison, S5 and S11 seemed to recall seeing tasks like Task 6(c) in RA I; they were stuck because they could not recall the corresponding technique. S11, for example, explained that "we did an example of this in the class. I don't know how to answer this. [-] Because I completely skipped right passed it." In reflecting on her attempts at parts (a) and (b), S5 added: "If I can't do the basic ones, I wouldn't be able to do this." Both students clearly identified the change in definition as the problem. S11, for example, explained that

> the cases that I'm familiar with [-] are like x is equal to zero and not zero, and then you have the case where it's less than zero or… So you have like different functions to check from the left side and right side. But like here I actually have no idea [-] Like if I'm taking the limit, how do I check from the left side or right side?

Unlike their peers, S5, S9, and S11 could not see how to produce a technique for solving the given task in the context of the interview. They also had no gut feelings as to whether the function in question is continuous or differentiable on $\mathbb{R}$.

6.6.2.3.1 Continuity: By the density of $\mathbb{Q}$.

Several students who struggled to find an answer to parts (a) and (b) of Task 6, swiftly determined an answer to part (c). Upon seeing the task, S3 expressed his frustration, which had built up during his attempts at parts (a) and (b): "I don't like this function. [-] Sometimes there's a simple function and you feel like you should have a grip on it, but you don't. And for me, that could be quite frustrating." Nevertheless, within just a few moments of taking a closer look at the function in part (c), S3 said:

> Oh. Never continuous. This is not a continuous function. Because they're both dense: [-] There are infinitely many [-] rationals in every interval. [-] So they're not continuous. [-] Hence it wouldn't be differentiable.

Seeing that the two pieces of g were now defined on the dense subsets, $\mathbb{R}\backslash\mathbb{Q}$ and $\mathbb{Q}$, seemed to trigger in S3 an immediate conclusion: no matter the value of p, g simply could not be continuous on ℝ, whereby it certainly could not be differentiable on ℝ either. The argument spontaneously constructed by S3 was contained in the few sentences shown above.

Five students (S1, S2, S4, S6, and S14) spontaneously bolstered their argument by trying to explain that the density of ℚ leads the function to have discontinuities in the form of jumps. S6 and S1 each provided a sketch, as shown in Figure 6.38, on the left and in the middle, respectively. Note that S3 also drew a similar sketch at some point during his interactions with the interviewer (shown on the right), which may have meant that he was thinking in the same way. S6 explained that on the x-axis, "it would kind of look like a line, but it's not. [-] It's not a continuous line." S1 said similarly that "for us, it looks like a line, but there's like holes at every irrational." Both students concluded from this that there had to be jumps that make the function not continuous. S2, S4, and S14 made a similar conclusion without sketching a graph. S2 and S4 pointed out the difference between the value of the function when x is irrational (i.e., $x^p \sin\left(\frac{1}{x}\right)$) and the value of the function when x is rational (i.e., 0). S14 added that $x^p \sin\left(\frac{1}{x}\right)$ "is a function that is not often going to be zero." He also found that the only irrational numbers for which $x^p \sin\left(\frac{1}{x}\right) = 0$ are of the form $x = \frac{1}{n\pi}$:

> But, there's plenty of irrational numbers in between those that are not going to satisfy that [i.e., $x^p \sin\left(\frac{1}{x}\right) = 0$], I would say. [-] So I'd say it's not really continuous at all. [-] I think it's bouncing all over the place. [-] For any value of p.

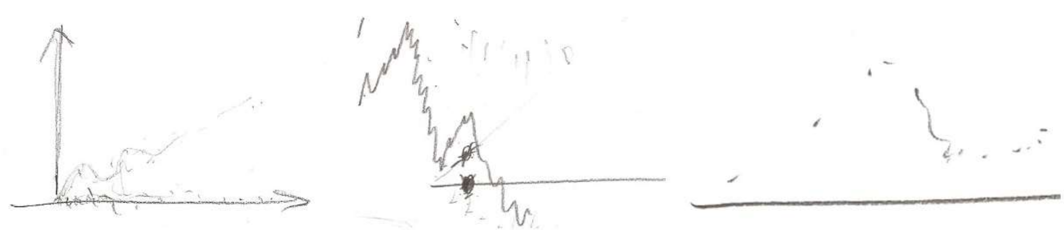

Figure 6.38 The sketches used by S1, S6, and S3 (from left to right) to argue that the function in Task 6(c) has jump discontinuities, whereby it is not continuous on ℝ for any p.

Although S8 also thought about the graph of the given function, he used it, along with the density of ℚ, to proceed in a slightly different manner. "So it's kind of clear to me that if I want this to be continuous, this [i.e., $x^p \sin\left(\frac{1}{x}\right)$] can't be any other value apart from zero," he claimed:

"Because that will fill in the line and it will be complete. That's my idea." Unlike the students mentioned above, S8 devised a plan for solving Task 6(c) that involved looking for the values of p for which $x^p \sin\left(\frac{1}{x}\right) = 0\ \forall x \in \mathbb{R}\backslash\mathbb{Q}$. He verified that there were no such values of p by thinking about different cases (i.e., $p = 0, p > 0$, and $p < 0$), as he had done for parts (a) and (b). He also thought of specific examples. For instance, he chose $p = 2$ and $x = \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$ to get $x^p \sin\left(\frac{1}{x}\right) = 2\sin\left(\frac{1}{\sqrt{2}}\right)$, and then asked himself: "Is that equal to zero? No, obviously not. So I think that kind of confirms my logic."

S7 distinguished himself from the students mentioned above in at least two ways. First, he did not express any immediate intuition about the continuity of the given function. In fact, upon receiving the task, he echoed S9: "I'm not sure I can answer this one. [-] I can tell you it's not Riemann integrable." Second, S7 attempted to produce a much more formal technique. When the interviewer asked S7 to clarify why he thought the function is not Riemann integrable, S7 provided the following (incorrect) outline of a proof (note that he used f instead of g):

$m_i = \inf\{f(x)\colon\ x \in [x_{i-1}, x_i]\} = 0\ \forall i$

$M_i = \sup\{f(x)\colon x \in [x_{i-1}, x_i]\} = x_i^p \sin\frac{1}{x_i}\ \forall i$

Since $M_i \neq m_i$, then the difference $(M_i - m_i) \geq \epsilon$

This proves the negation of the definition of Riemann integrable:

i.e., $(M_i - m_i)\Delta x_i < \epsilon$

We predict that S7 constructed this argument by trying to recall a solution to a task he had solved recently in the RA II course he had been taking at the time of his interview. What's interesting is how this led him to unexpectedly conclude: "Well I just proved it's not continuous."

It seems that in attempting to remember a technique for proving the (non-)integrability of a given function, S7 recalled something that could be useful for proving the (dis)continuity of a function: "There's no interval where you can find only rational numbers. Or an interval where you can find only irrational numbers." Hence, any interval would have values of x for which $f(x) = 0$ and $f(x) = x^p \sin\left(\frac{1}{x}\right)$. S7 attempted to use this idea to produce an argument for the discontinuity of the given function. He explained: "For every delta, $0 < |x_i - x_{i-1}| < \delta$, there would be two

values, the values of f would not match up. Like it wouldn't imply that $|f(x_i) - f(x_{i-1})| < \epsilon$."
S7 clarified that his argument was based on the negation of the definition of continuity on $\mathbb{R}$: i.e.,

$$\exists x, \exists \epsilon > 0, \forall \delta > 0, \exists y_\delta, |x - y_\delta| < \delta \wedge |f(x) - f(y_\delta)| \geq \epsilon.$$

S7 did not make his argument more precise in the context of the interview: e.g.,

> Suppose $x = \sqrt{2}$ and let $\delta > 0$.
> Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number $y_\delta \in \left(\sqrt{2} - \delta, \sqrt{2} + \delta\right)$.
> For this y_δ, we have $|x - y_\delta| < \delta \wedge |f(x) - f(y_\delta)| = \left|\sqrt{2}^p \sin\left(\frac{1}{\sqrt{2}}\right) - 0\right| = \left|\sqrt{2}^p \sin\left(\frac{1}{\sqrt{2}}\right)\right|$.
> Hence, if we take $\epsilon = \left|\sqrt{2}^p \sin\left(\frac{1}{\sqrt{2}}\right)\right| > 0$, the negation of the definition of continuity shown above has been satisfied. ■

Nevertheless, S7 did not require such precision to convince himself that the argument would be possible; and that there are no values of p for which the given function is continuous on $\mathbb{R}$.

It is possible that some students in this group had convinced themselves of a stronger conclusion: namely, g is not continuous at *any* point on $\mathbb{R}$. When asked what his answer would be if the question had been about the continuity of g at 1, 0, or any other point, S3 said: "No matter what you'll give me, my answer will be no." Two other students (S6 and S7) were asked similar questions and confirmed that they too thought g is discontinuous at every point. The difference with S6 and S7 was that they had produced more substantial arguments about the discontinuity of g, which could be leveraged by the interviewer to assist them in confronting their incorrect deduction. S7, for instance, was guided to notice that his formal argument depends on $x^p \sin\left(\frac{1}{x}\right)$ not being equal to 0; but there are some values of x for which $x^p \sin\left(\frac{1}{x}\right) = 0$. Similarly, S6 was led to realize that there might be points on her graph where the function is continuous. After the interviewer gave S6 the limit definition of g being continuous at 0 (i.e., g is continuous at 0 if and only if $\lim_{x \to 0} g(x) = g(0) = 0$), S6 noticed that "as you approach zero, it gets closer and closer to zero. [-] You do get jumps. But the overall limit doesn't really care about that. It only looks at what value you're approaching." The students described in the next section noticed on their own that g might be continuous at certain real numbers and spontaneously incorporated this exploration into their solutions.

6.6.2.3.2 Continuity: By the sequence definition of continuity.

S12 and S13 had similar intuitions about the continuity of the function

$$g(x) = \begin{cases} x^p \sin(1/x) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}.$$

Almost immediately upon receiving the task, S13 said: "Only at zero. Continuous if p is greater than zero. And differentiable if p is greater than one." S12 said similarly that he thought the function is "not continuous on any real number, probably with the exception of zero. [-] I know at $x = 0$, there's some particular property."

At first, S13 proceeded according to the practices to be learned in RA I. That is, she started by using the expected technologies to look for the points of continuity of g. Then she produced an argument for why g is not continuous at a particular point, whereby it is not continuous on $\mathbb{R}$. Her steps in solving the task could be summarized as follows:

> For any rational number, you can find a sequence of irrational numbers that goes to that rational number. i.e., $\forall x_q \in \mathbb{Q}, \exists (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $\lim\limits_{n\to\infty} x_n = x_q$
>
> If g is continuous, then $\lim\limits_{n\to\infty} g(x_n) = g(x_q)$. i.e., $\lim\limits_{n\to\infty} \left(x_n^p \sin\frac{1}{x_n}\right) = 0$.
>
> Hence, x_n^p should go to 0. (*)
>
> $\Rightarrow p$ should be greater than 0 and x_n should go to 0.
>
> It is just continuous at $x_q = 0$.
>
> For instance, if $x_q = 2$, then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $\lim\limits_{n\to\infty} x_n = x_q$ and $\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty} \left(x_n^p \sin\left(\frac{1}{x_n}\right)\right) = 2^p \sin\left(\frac{1}{2}\right) \neq g(x_q) = 0$.

Note that at step (*), S13 missed the possibility that $1/x_n$ could go to $1/(2m\pi)$ for $m \in \mathbb{N}$; otherwise, her argument is complete and correct. S12 showed signs of trying to recall a similar kind of argument. When he struggled to put his thoughts together in the same general manner as S13, he turned to considering a specific example and sketching a graph.

A unique characteristic of S12 was that he knew: "If we multiply any trig function – well, specifically sine and cos – by some other function, its behaviour tends to be bound by the positive and negative forms of that function." Hence, for $p = \frac{1}{2}$, S12 was capable of drawing a rather accurate representation of g (his sketch is compared with a computer-generated graph in Figure 6.39). To justify that g is continuous at a point like $x = 0$, S12 looked at his graph and reasoned as follows:

> Whether or not we approach it using [-] rational or irrational numbers, both of these series reach the same point. [-] Because they both converge to the same thing, I'm tempted to say that it's continuous at that location.

As suggested by the image in Figure 6.39, S12's visual representation of g also enabled him to notice that the function is continuous at points other than $x = 0$: in particular, at $x_n = \frac{1}{n\pi}, n \in \mathbb{N}$. Note that S13 did not make this same realization when she carried out her formal proof technique.

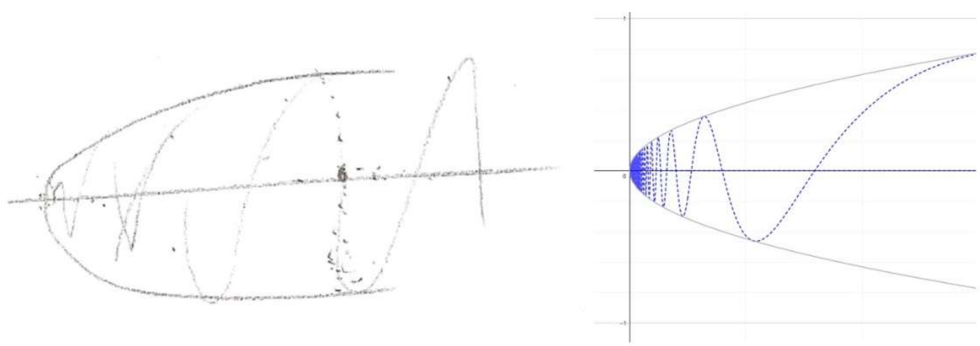

Figure 6.39 S12's drawing (left) and a computer drawing (right) of the function in Task 6(c) for $p = 1/2$.

6.6.2.3.3 Observations for Differentiability

After calling upon the density of $\mathbb{Q}$ in $\mathbb{R}$ to conclude that

$$g(x) = \begin{cases} x^p \sin(1/x) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

is not continuous on $\mathbb{R}$ for any p, six students (S1, S3, S4, S6, S7, and S14) used this conclusion to deduce that g is also not differentiable on $\mathbb{R}$ for any p. S6 explained her reasoning as follows:

> If you want it to be differentiable, you need a continuous function to start off with. [-] If you have irregularities, [-] you can't differentiate it. [-] It's something that I know. But then, I wouldn't know based off of like the definition, like, the true definition. It's just like, like spontaneous. Like, ok, you know, if it's not continuous, then it's not differentiable.

The implication "g not continuous ⇒ g not differentiable" also seemed to be "spontaneous," i.e., automatic and without mathematical theoretical underpinnings, for some of the other students. S7 said: "I just have a feeling that it's it. And I know feelings are bad. Like I shouldn't go with the guts, but… I should have something to support that, but…" S4 seemed to question the implication because, for him, it was also based solely on a feeling. He said, more specifically: "When there's a jump, as in this case, I'm like ninety-nine percent sure it's not differentiable. [-] But, I'm… Like I feel like I'm missing something. And I don't know what that is."

Most students attempted to provide additional explanations for how they knew that continuity is a necessary condition for differentiability. These explanations tended to be incomplete and/or incoherent. S14, for example, explained that for a function to be differentiable,

> you have to know the rate of change at a point. And it has to be changing from one point to the next in some kind of measurable manner. And that's how I would have my derivative at that point, would be given the fact that it's continuous at that point.

S4 returned to the definition of the derivative, i.e.,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h},$$

and said that he did not think the value would exist for the specific g in Task 6(c) "because it's jumping up and down." S3 offered prototypical examples of discontinuous functions (see Figure 6.40), and spoke about some "limits," which he decided were not equal for the function on the left and were equal for the function on the right. It seemed S3 was concluding that it is impossible to differentiate at a jump, but it is possible to differentiate at a removable discontinuity.

Figure 6.40 The prototypical examples of discontinuous functions drawn by S3 to explain why the function in Task 6(c) is not differentiable (like the function depicted on the left).

Only one participant, S1, called upon a piece of the practices to be learned in RA I to further justify his conclusion. S1 eventually decided that "not continuous implies not differentiable. And, differentiable implies continuous. Oh yeah, cause we had the logic thing," i.e.,

$$\alpha \Rightarrow \beta, \neg\beta \Rightarrow \neg\alpha.$$

S1 hesitated on concluding the equivalence of these implications until he produced the following table:

α	β	$\Rightarrow$
1	0	0
0	1	1
1	1	1
0	0	1
$\neg\alpha$	$\neg\beta$	$\neg\beta \Rightarrow \neg\alpha$
0	1	0
1	0	1
0	0	1
1	1	1

Notice that the equivalence of contrapositive statements was not automatic to S1. He was nevertheless able to check the equivalence using the truth table proof method he had mastered in RA I. He concluded that since he knew that differentiable implies continuous, he now knew that not continuous implies not differentiable.

The remaining three students who thought about the differentiability of

$$g(x) = \begin{cases} x^p \sin(1/x) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

suggested that they could use a solution they had constructed previously. S8 had a similar inkling as the students mentioned above: "Well, for something to be differentiable, it has to be continuous as well." The difference was that he seemed to want to use this idea to construct an argument resembling the one he had used to argue about the continuity of g (see 6.6.2.3.1). More specifically, he claimed that if g were to be continuous on $\mathbb{R}$, it would need to be equal to zero on $\mathbb{R}$, whereby g' would also be equal to the zero function; however, there are values of x for which $x^p\left(\frac{p}{x}\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x^2}\right)$ (the derivative of g for $x \neq 0$) is not equal zero. In a similar vein, S13 felt it would be straightforward to adapt her solution concerning the continuity of g, as described in 6.6.2.3.2. She explained (correctly) that studying the differentiability of g is almost the same as studying the continuity of g; there's just a different limit. S2, in contrast, suggested (incorrectly) that studying the differentiability of g would be the same as studying the differentiability of the function in (b): i.e.,

$$g(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

In other words, S2 did not know that changing the domains of the pieces of g would lead to significant changes in its differentiability. It is perhaps surprising that more students did not express a similar thought: for example, those who wondered if g' could be calculated simply by applying differentiation rules to each piece (while maintaining the domains).

6.6.3 Reflections on the Nature of Practices, Positioning, and Activities[43]

Non-Mathematical: Produce a technique
based on established experiences, inert knowledge,
and/or taking one's own understanding for granted.

VS.

Mathematical: Produce a technique
based on clarifying, questioning, and verifying one's own understanding; i.e., based on a combination of informal understandings and formal theory.

While solving Task 6, there were several participants who seemed to be operating according to non-mathematical practices in the above sense: that is, their techniques seemed to be supported principally by their established experiences, inert knowledge, and/or taking their own understanding for granted. Consider, for example, the participants who thought they should produce a technique for Task 6(a) and/or Task 6(b) based solely on the formal ϵ-δ definition of continuity and/or the definition of differentiability. Of course, the habit of returning to definitions

[43] As stated at the beginning of Chapter 6, the goal of this section is to offer some reflections on the results presented in Sections 6.6.1 and 6.6.2 that begin to address our research objectives: characterizing the nature (mathematical or not) of the practices actually learned by students (first objective) and conjecturing how these practices are shaped by the positions students adopt and the activities they are offered (second objective).

can be very useful in mathematics, and this seemed to be recognized by several participants. What made some participants' practices non-mathematical was (a) their expectation that tasks about "continuity" or "differentiability" should/could always be entirely solved "by definition," since that was what they remembered doing in RA I; and (b) their lack of understanding of the definitions (and how they produce techniques), which is necessary for making sound mathematical judgements about when, how, and why they can be more or less useful. Another pertinent example of non-mathematical practices in the above sense occurred when participants tried to solve Task 6(a) using sufficient conditions and did not seem to realize the insufficiency of their chosen techniques. Once again, it is important to note that using a sufficient condition to produce a partial technique could be useful in mathematics; but the participants did not seem to be thinking in this direction. They proceeded as though they would be able to solve the task completely with their chosen technique and did not stop to clarify, question, or verify what they were doing. It seemed they were progressing based mainly on their established experiences in RA I: more specifically, their use of differentiation for quickly proving continuity (and avoiding the use of the formal ϵ-δ definition), or their routinized use of an equality like $\lim_{x \to x_0^-} g'(x) = g'(x_0) = \lim_{x \to x_0^+} g'(x)$ for checking that "some function" is continuous. We think the behaviours described in this paragraph are closely linked to the Student position; in particular, the attempt at learning routinized praxis for solving particular types of tasks, without thinking too carefully about the underlying theory and how it produces techniques that might be useful in some cases, but impractical or insufficient in others.

There were several participants who showed signs of being able to shift out of the Student position and develop practices that are more mathematical in nature. Like the students mentioned above, they were inclined to argue "by definition" or use some sufficient condition; the difference was that they recognized that these approaches might not work and were open to trying to think up different (possibly incorrect) techniques. For instance, when faced with an inability to proceed based on the formal ϵ-δ definition of continuity for Task 6(a), S4 and S10 tried to produce a technique by looking at g as a product of two functions (for the $x \neq 0$ case); and then using limits (for the $x = 0$ case). Another interesting example was S14, who realized the potential problem with his instinct to use "g uniformly continuous $\Rightarrow$ g continuous," and eventually called upon his informal understanding of continuity to produce a technique based on experimental graphing.

These students demonstrated an ability to clarify and to question their own understanding; they exhibited some limitations, however, when it came to mathematically verifying their thoughts. For instance, S4 and S10 analyzed the continuity of $x^p, p \in \mathbb{R}$ based solely on their graphical imagery for the function, which seemed limited to cases where $p \in \mathbb{Z}$; and although S14 had developed a great informal understanding of continuity, he had not yet combined this with the related formal definitions and proof techniques that could validate his graphical experiments.

We note that many participants simply seemed to not know how to deal with the continuity and differentiability of a piecewise function like the one in Task 6(a) and Task 6(b); and yet, when it came to Task 6(c), they were able to use intuition to see the response: i.e., the function is not continuous on $\mathbb{R}$ and hence not differentiable on $\mathbb{R}$ for any p. These participants called upon the density of $\mathbb{Q}$ (and $\mathbb{R}\backslash\mathbb{Q}$) in $\mathbb{R}$ and their intuitions of what that would imply for the given function: i.e., its graph would have jumps, its graph would only be continuous if the two pieces were equal, or it would be possible to satisfy the negation of the formal ϵ-δ definition. Once again, we found that many participants did not quite have the habit or the tools to more carefully question or verify their understandings. What is interesting about Task 6(c) is that it seems to be an example of a task that could productively confront students with the need to go beyond intuition. There are three key reasons for this. First, the students actually had strong intuitions about the answer. Second, those intuitions were not entirely correct: recall, for example, how S3 (the Learner), S6 (a Student), and S7 (a Student-Learner) clearly indicated that they thought the given function would not be continuous at any real number, no matter the value of p. Third, increasing one's level of informal and formal understanding is necessary to confront and correct the intuitions. It is also important to note that this productive confrontation towards the development of mathematical practices is also made possible by the way in which Task 6 is stated: i.e., it asks students "to determine" the continuity and differentiability of a function "on $\mathbb{R}$" (as opposed to, for example, asking students "to show" that a function is continuous or differentiable "at some specified x value").

There were a few participants whose solutions to Task 6 seemed to indicate practices that were more mathematical in nature in the sense depicted in the above box. Most obvious were S12 and S13, who clearly recognized that they should treat the $x \neq 0$ and $x = 0$ cases differently in both Task 6(a) and Task 6(b), had no problems in calling upon the limit definitions of continuity and differentiability to think about what happens at $x = 0$, and demonstrated advanced techniques for thinking about the continuity of the function in Task 6(c), which combined informal and formal thinking. As we have noted before, we think these students' behaviours can be linked to their positioning as Mathematicians in Training; in particular, their interest in going beyond what was expected from them on the assessment activities in RA I. Nevertheless, Task 6 still seemed to present some potentially productive challenges for these students due to its complexity: e.g.,

- like most of their peers, S12 and S13 did not recognize that $g(x) = x^p \sin(1/x), x \neq 0$ is poorly defined. In fact, S13 said this was obviously a well-defined function;
- like several of their peers, S12 and S13 thought about the differentiability of $g(x) = x^p \sin(1/x), x \neq 0$ in a procedural manner, based on whether or not they could apply the differentiation rules they had learned. They did not recognize or address the circularity that is present in such an argument (i.e., to use differentiation rules, one needs to assume differentiability); and
- in Task 6(c), S13 did not naturally detect the points of continuity of the form $1/(2m\pi)$, $m \in \mathbb{N}$ while applying the highly formal technique she had learned; and although S12 was able to get a sense of all the points of continuity using an informal graphical approach, he was not yet able to write out all the formal details.

We think such behaviours highlight some key points, which can likely be linked to the kinds of activities students are given to practice in any mathematics course. For instance, the activities given to students are often well-posed and emphasize some key issue, which means that students do not learn to question the task or to develop mathematical techniques for thinking beyond the key issue. Solutions to activities are also often polished formal arguments, which means students may not be encouraged to combine formal and informal approaches.

Chapter 7: Discussion and Conclusions

The previous two chapters are dedicated to presenting and analysing the results of our doctoral study, as problematized, theorized, contextualized, and described in the first four chapters. In this chapter, we aim to discuss these results and present our main conclusions. We first discuss our results in relation to the specific research objectives underlying our current work (Section 7.1). Then we discuss our results in relation to the work that came before ours and highlight our contributions to the field (Section 7.2). We finish with some final remarks (Section 7.3), including our main conclusions, the limitations of our project, potential avenues for future work, and general reflections on the vague, but pertinent question that initially inspired us: Do students really learn to "behave like mathematicians" in the context of university mathematics coursework?

7.1 Discussion in Relation to our Research Objectives

Recall that our research aimed to gain an understanding of the nature (mathematical or otherwise) of the practices actually learned by students who are deemed successful in a first Real Analysis course in the institution of University Mathematics, and how these practices may be shaped by the positions that students adopt and the activities they are offered. Key to the formulation of these objectives was a definition of "mathematical knowledge" as the enactment of "mathematical practice": i.e., the identification of a given mathematical task as belonging (or not) to a general mathematical type of task, the choosing and implementation of an appropriate mathematical technique for solving that type of task, and the explanation, in a mathematical discourse, of how and why the technique works. Throughout our analyses, we came to realize that it is not always straightforward or even interesting to characterize a student's practice – in its entirety – as either mathematical or not. It seemed to be too much to expect students' practices to be entirely mathematical in our sense. Moreover, in our initial reflections presented throughout Chapter 6, we noticed that students' practices could be mathematical in some ways and non-mathematical in others. We have found that these different "ways" of being non-mathematical versus mathematical (highlighted in the boxes throughout Sections $6.i.3$) are each naturally linked to one of three different parts of a practice: namely,

1. the identification of a given task with a type of task and technique;
2. the implementation of a technique; or

3. the explanation or production of a technique with some theoretical discourse.

The following discussions are guided by this division of a practice into three parts.

Conceptually speaking, Section 7.1.1 is meant to mimic the first section of our literature review (Section 1.1), where we review work on Calculus students' development of non-mathematical practices. More specifically, we use the results of our study to discuss how students might still be developing non-mathematical practices in the context of a first course in Real Analysis, and we propose some possible links between the different ways in which students' practices may be non-mathematical, the activities students are offered, and the positions students adopt. In general, our results seem to be in line with previous work on Calculus. Our discussion in Section 7.1.1 aims to deepen the conversation and extend it to Real Analysis. In comparison, the discussion in Section 7.1.2 focusses on key findings that seem to diverge from the results presented in the literature on Calculus that we reviewed. Essentially, we have found that some students may be (or at least have the potential to be) developing practices that are more mathematical in nature in a first course in Real Analysis. Once again, we discuss possible links with the positions students adopt and the activities they are offered. In particular, we highlight how different positions might be productive towards the development of mathematical practices in different ways, and we reflect on how it might be possible to take advantage of this by posing activities that perturb a student's positioning. Finally, in Section 7.1.3, we take a step back and discuss the complexity of the relationships between practices, activities, and positioning, as they are established in the context of a didactic institution like University Mathematics.

7.1.1 Real Analysis Students' Development of Non-Mathematical Practices

In our study, we identified three ways in which the practices actually learned by students in a first course in Real Analysis may be *non-mathematical*:

1. the identification of a given task with a type of task and technique is *automatic*;
2. the implementation of a technique is *limited to recalling the steps of a solution one has seen or done before*; and/or
3. the explanation or production of a technique is *based solely on established experiences, inert knowledge, and/or taking one's own understanding for granted.*

Concretely, (1) may arise when a given task is immediately recognized by a student as "old" in the sense of looking very similar to some collection of tasks they solved before, thereby requiring the same technique. We saw how, as a result of automatically identifying a task with a technique, a student does not take the time to consider how the given task could be "new." For instance, they do not think about if (or realize when):

- there are other, better ways to solve the task;
- the task could be reformulated in some productive way;
- guessing an answer might help;
- the task is ill-posed; or
- the task requires the consideration of different cases or the construction of multiple solutions.

We also saw how when the choice of technique is based on quickly recognizing certain surface properties of a task statement, students may not really understand why the technique can (or cannot) be applied, or when the technique is useful (or not). This behaviour seems to be linked to activities that introduce a one-to-one correspondence between types of tasks and techniques through seemingly narrow and disjoint paths to practices. In following the paths, students master very specific, rigid, and isolated praxis blocks $[T, \tau]$, which they may not be able to generalize, apply flexibly, or connect. Eventually, the student has no reason to look at an "old" task as "new" or to be able to explain a choice of technique beyond recognizing surface similarities between the given activity and prior activities in the path. For some students, encountering a "new" task becomes an unusual and uncomfortable experience.

Implementing a technique as in (2) means that a student constructs the next step of a solution by trying to recall the next step of a solution they have seen or done before, rather than thinking about how to produce a next step based on the meaning of the objects involved in the task, possible implications of that meaning, and the overall goal. We saw how, in this case, the student's ability to make progress depends on the strength of their memory and the kind of solution image they have stored: if they forget the next step or are recalling a solution that is specified in some way, they get stuck. This behaviour seems to be linked to paths to practices where the next activity can be solved through replicating the steps of a solution to a previous activity, often beginning with a professor's polished solution. In following the paths, students are left to their own devices to fill in the thinking that remained implicit in the professor's solution, including justifications for certain steps, abstractions of key ideas, and more general problem solving or proof strategies. Instead of filling in the gaps, the students may simply try to store in their memory some image of the solution and develop techniques τ that are reduced to the steps in that image. We saw how a reference to some mathematical theory (e.g., "by contradiction," "by Rolle's Theorem," "$\forall\epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon$") may be included in the solution image; but the theory might not be understood to the point that the student can interpret it and use it to explain the technique (i.e., to explain how and why it works). In other words, the acknowledgement of mathematical theory becomes one step of the technique, rather than a tool for producing a mathematical technology.

The last way in which students' practices may be non-mathematical, (3), relates to the kinds of theoretical discourses students may have to support their techniques. We identified several kinds of non-mathematical theoretical discourses. First, a student may know a technique works based on the established experience of it working before on surface similar activities. Second, a student may know a technique works based on "some theorem," "some definition," or "some condition" that they do not know how to clearly state, that they do not understand, or that they do not know how to use to mathematically explain how and why the technique works. And third, a student may know a technique works based on some informal understanding of the underlying concepts. We saw that when a student's theoretical discourses were restricted to these kinds, they did not really know when or why the corresponding techniques were mathematically appropriate or legitimate. Such behaviour seems to be linked to paths to practices that do not necessarily confront students with the limitations of explaining their techniques in these ways. The learning of techno-theoretical blocks $[\tau, \Lambda]$ where Λ is either of the form "that's how we normally do that" or contains only a distant disconnected acknowledgement of theory seems to be linked to narrow paths to practices where later activities can be solved through the development of routines that do not require mathematical explanation beyond possibly citing a theorem, a definition, or a condition. And the learning (or perhaps maintaining) of techno-theoretical blocks $[\tau, \Lambda]$ where Λ is strictly informal seems to be linked to paths to practices that ask students to unnecessarily and temporarily replace simple familiar techniques with techniques that are much more difficult to understand (e.g., the replacement of informal techniques for "finding limits" or "determining continuity" with the formal techniques for "proving limits" or "proving continuity").

It is perhaps not surprising that the Student position could be linked to the development of non-mathematical practices in all three of the senses described above. The Student will follow the paths to practices that are laid out for them in attempt to identify what they need to learn to do well on assessments. If, in following the paths, they perceive the minimal core of practices to be learned as including principally isolated types of tasks, each with their own technique that can be implemented by routinizing the steps of some specific solution image, then they will master practices that reflect this: i.e., they will learn to automatically identify tasks with techniques, to implement techniques based on recalling steps, and to explain the choice and validity of a technique based principally on established experiences and/or inert knowledge. All this said, in our study we also saw how the Skeptic, Mathematician in Training, Enthusiast, and Learner may exhibit practices that are non-mathematical in some senses, depending on the nature of the given task. For instance, the Skeptic may not be convinced to master anything beyond the steps when they are asked to learn a technique for a type of task that they already know how to solve. The Mathematician in Training and the Enthusiast, similar to the Student, will also be dedicated to following the paths to practices laid out for them; hence, for the types of tasks that are emphasized in the paths, they may also learn practices that are automatic or routinized to some degree. And whenever it is possible, the Learner may be compelled to produce techniques based on taking their own understanding for granted, since this understanding is typically very deep, and a lack of motivation or tools may inhibit them from developing that understanding into a more clear, precise, and formal mathematical theoretical discourse.

7.1.2 The Potential for Developing Mathematical Practices in Real Analysis

As complementary to the three ways in which students' practices may be non-mathematical, we identified three ways in which the practices students actually learn in a first course in Real Analysis may be *mathematical*:

1. the identification of a given task with a type of task and technique is accomplished through *studying the properties of the given task and performing assessments of possible types of tasks and techniques;*
2. the implementation of a technique is accomplished through *pinpointing the essential characteristics of the objects involved in the task, abstracting key ideas*

from solutions seen or done before, and using general problem solving or proof strategies; and/or

3. the explanation or production of a technique is accomplished through *clarifying, questioning, and verifying one's own understanding with mathematical theory.*

In most instances, we linked the development of mathematical practices in the above senses to adopting positions other than Student. Moreover, it seemed that different kinds of positioning were productive towards mathematical practices in different ways:

- The Skeptic is characterized by critically reflecting on whether they should be learning what they are being asked to learn. Hence, if a given task looks like it might be solvable with techniques they learned before, the Skeptic will naturally take time to study the task, assess if the techniques they learned before can be applied, and question the mathematical legitimacy and necessity of both the techniques they learned before and the techniques they are being asked to learn.
- The Mathematician in Training is characterized by trying to incorporate the practices they are being asked to learn into a larger collection of practices they perceive as pertinent for participating in a related community of mathematicians. Hence, the Mathematician in Training will naturally try to abstract key ideas and strategies from the solutions they produce, and seek to clarify, question, and verify their understanding to the point of being able to "speak mathematics."
- The Enthusiast is characterized by devoting themselves to what they are being asked to learn. Hence, if they perceive the didactic contract as requiring them to develop certain kinds of practices, then these are the kinds of practices they will seek to develop. In Real Analysis, we saw how this may correspond with a natural inclination to study any given task in terms of its formal meaning, try to justify each step of a solution completely by reducing it to basic underlying assumptions, and produce techniques based on returning to formal theory.
- The Learner position is characterized by seeking to understand what they are being asked to learn. Hence, the Learner will naturally approach any given task as "new," spend time studying the task and thinking of a best possible approach, proceed from one step to the next by trying to think of the meaning of objects in relation to the

overall goal, and develop a deep personal understanding that can serve as a foundation for the development of more formal mathematical theoretical discourses.

We see the availability of the above positions in a course institution as representing a potential for the development of mathematical practices for two reasons. First, there may be students who adopt these positions in the course and, irrespective of the activities given to them, naturally develop mathematical practices in the senses outlined above. This seemed to be the case for some of the participants in our study. Second, it might be possible for students who do not initially or naturally adopt the above positions to shift into them and develop practices that are more mathematical in nature.

This second point leads to the question of whether it is possible to perturb a student's positioning. Since a student's positioning can be strongly connected to the student's personal experiences, abilities, interests, goals, habits, and so on, perturbing a student's positioning might be a challenge. For both theoretical and empirical reasons, however, we expect some perturbation would be possible. Theoretically speaking, a position is a construct that is too simple to completely represent a human being and predict their actions in every context. In other words, we would expect a given student to have in their total bank of experiences, abilities, interests, goals, habits, and so on, seeds for adopting different positions to different degrees in different circumstances. Empirically speaking, we found evidence that students may be willing and/or able to shift their positioning, at least locally, and develop practices that are more mathematical in nature. In particular, at moments where students struggled in solving a task and/or were prompted by the interviewer, we saw that:

- they could try to come up with ideas for solving a "new" task;
- they could think up different possible techniques for solving an "old" task;
- they could realize that their automatically chosen technique might not solve the task;
- they could try to adapt the steps of a solution image they had in mind; and
- they could begin to clarify and question their own experiences and understandings.

This evidence suggests to us that it may be possible to encourage shifts in positioning and the development of practices that are more mathematical in nature by posing activities that purposefully lead students to get stuck and/or include certain explicit prompts.

More specifically, we think it might be possible to support the development of mathematical practices by designing paths of activities where the development of non-mathematical practices is purposefully and explicitly confronted. Furthermore, we expect that our general characterization of the different ways in which students' practices may be non-mathematical versus mathematical (as summarized in Table 7.1 below) may be useful in informing the design of such paths. One idea would be to construct paths to practices where the next activity discourages the non-mathematical behaviours in the second column of Table 7.1, either because the activity simply cannot be solved through those behaviours or the corresponding mathematical behaviours (in the third column) are being promoted in some way. Although the design of such paths was not the focus of our current study, the detailed task-by-task results and analyses presented throughout Chapter 6 might provide some hints in this direction. For instance,

- to assist students in developing mathematical practices in the first sense depicted in Table 7.1, one might consider posing a next activity where studying the task more carefully is simply necessary (e.g., Task 3 and Task 5); or one might consider posing a next activity where a different formulation of the task may be productive or several techniques may be possible (e.g., Task 1, Task 2, and Task 4), and explicitly invite students to specify and compare these formulations and techniques;
- to assist students in developing mathematical practices in the second sense depicted in Table 7.1, one might consider posing a next activity where significant adaptations of previously constructed solutions are required (e.g., Task 1, Task 2, and Task 5); or one might consider posing a next activity that explicitly invites students to fill in all the blanks, including justifying all steps, abstracting key ideas, and/or listing useful strategies (this could have been useful, e.g., in Task 1, Task 4, and Task 5); and
- to assist students in developing mathematical practices in the third sense depicted in Table 7.1, one might consider posing a next activity where producing a technique based solely on established experiences, inert knowledge, and/or taking one's own

understanding for granted will not work; or one might consider posing a next activity that explicitly invites students to clarify the techniques they are using and construct mathematical explanations about how, when, and why they work (see, e.g., our reflections for Task 2, Task 3, Task 4, Task 5, and Task 6).

Three Parts of a Practice	Non-Mathematical	Mathematical
Identify task with type of task and technique…	…in an automatic fashion.	…by studying the properties of the task and performing assessments of possible types of tasks and techniques
Implement technique…	…by recalling each step of a solution one has seen or done before.	…by pinpointing essential characteristics of the objects involved in the task, abstracting key ideas from solutions seen or done before, and using general problem solving or proof strategies.
Explain or produce technique…	…based on established experiences, inert knowledge, and/or taking one's own understanding for granted.	…based on clarifying, questioning, and verifying one's own understanding with formal theory.

Construct paths to practices where the next activity cannot be solved solely through the non-mathematical behaviours or the mathematical behaviours are being promoted in some way.

Table 7.1 Different ways in which students' practices may be non-mathematical or mathematical.

The recognition that most students may position themselves as a Student, at least to some degree, may seem unfortunate to some professors. On the contrary, we have come to think of this position as having a lot of potential for the development of mathematical practices. Indeed, when it comes to the Student, a professor can count on two things: (1) the Student will follow the paths to practices that are being laid out for them; and (2) in trying to identify the minimal core of practices to be learned, the Student may become very skilled in developing the related non-mathematical practices. In light of the ideas proposed above, this means that the Student might be productively perturbed by even slight modifications to the next activity in a path, in any of the ways outlined above. An additional motivation, but also challenge to be considered, when it comes to designing such activities, is that when compared to participants who had occupied other positions, we found that the participants who positioned themselves strongly as a Student often lacked the tools to be able to fully engage in the "-ings" appearing in the "mathematical" column of Table 7.1: that is, they often lacked the foundations needed for doing things like "studying" the properties of a given task, "assessing" different techniques, "pinpointing" essential characteristics of objects, "abstracting" key ideas, and so on. Moreover, the tools that they were lacking included anything from general proof techniques or abilities in interpreting formal language, to basic fluency in numbers, algebra, functions, and limits.

7.1.3 Recognizing Complexity: Positioning and Activities in the Broad Institutional Context

In the previous two sections, we present key ideas in a rather straightforward manner. We acknowledge, nonetheless, that the phenomena we study are far from being straightforward. In particular, we recognize that the many links we have proposed between the activities students are offered, the positioning students adopt, and the practices students develop are formed within a broad institutional context, whose functioning has the power to condition and constrain both what students (are able to) learn and what professors (are able to) teach. More specifically, we acknowledge that the activities offered by professors and the positions adopted by students are not entirely the result of conscious choice. Rather, there is a complex web of contextual factors that may encourage certain kinds of activities and positioning.

As a particularly constraining example, consider the examination procedures that typically serve as the measuring tool of students' success in any university mathematics course. In the course we studied, a student's grade is almost entirely determined by their performance on two exams: a midterm halfway through the course and a final exam at the end. A common expectation in the institution of University Mathematics is that such exams serve as consistent and fair measuring tools. There is often a Course Examiner put in place to ensure the exam tasks remain relatively similar from year to year. Some of the participants in our study mentioned this and explicitly indicated that it influenced their positioning: i.e., it facilitated a position of Student characterized by strategic studying based on patterns in past exams. The expectation that exams should be fair may also have a major influence on the kinds of activities professors pose in their courses since this expectation also tends to mean that the examination tasks need to be "old." It is perfectly natural for a professor to want to guide their students in their preparation for exams. As a result, they may construct very narrow paths to practices (perhaps without even realizing it). In addition to this, students may feel a lot of pressure when it comes to taking exams, which may be heightened by the time constraints under which exams are typically written. Several participants in our study spoke about not having the time to think on exams, which encouraged them to develop practices that do not involve thinking: i.e., practices where the identification of a given task with a technique is automatic, and implementing a technique requires quickly reproducing some memorized steps. Such a situation discourages students from adopting positions that could be more productive to mathematical practices. Falling into the position of Student, students develop non-mathematical practices, not necessarily because they want to, but because these practices help them to deal with how the institution evaluates their success.

On top of examination procedures, the interactions between students and professors are also strongly guided by time constraints and curricular expectations, which often imply that there is a large amount of material to cover in a short amount of time. For each passing topic, professors are restricted in the number of activities they can solve in lectures, pose on assignments, and evaluate on exams, which may make it difficult to expose students to the number of activities that may be needed to support the development of mathematical practices. Students may end up being explicitly invited to engage with only small pieces of disconnected practices, which they are left to expand, deepen, and connect on their own. The issue is that, like professors, students may also feel there are simply too many pieces and too little time. Going above and beyond the activities they are given (like a Mathematician in Training might), or even studying the activities they are given in more detail (like an Enthusiast), could be a challenge for any student.

Finally, we think it is interesting and important to think about a student's participation in a given course institution in terms of their broader journey, including the many institutional memberships they had before and those they are aiming for in the future. More specifically, we think that the way in which a student positions themselves in a current course can be influenced by the ways in which they positioned themselves in previous courses, as well as their perceptions of the positions they expect to adopt in future professions. Imagine, for example, if a student's prior experiences in mathematics education encouraged them to take on a position of Student, not only because the conditions and constraints of mathematics courses made studenting seem like the way to go, but also because studenting was validated by the fact that it enabled students to achieve high grades. Then adopting a Student position in the next course would seem perfectly natural; in fact, it may be the only way in which a student knows how to position themselves. If such a situation persists over many years, then the Student position may become more and more resilient to change. Eventually, the student may come to identify "learning mathematics" with the actions that characterize a Student position and they may develop non-mathematical foundations in numerous topics. As a result, shifting positions and developing mathematical practices may become much more demanding for both the students and their professors.

In our study, we also found that there could be conflicts between the material covered in a course and the ways in which students perceive the professional institutions to which they will eventually seek membership. Recall that the Real Analysis course we studied is one of eight core courses in the Department of Mathematics, deemed mandatory for each of its four specialization programs: Actuarial Mathematics, Mathematical and Computational Finance, Statistics, and Pure and Applied Mathematics. Several participants in our study (especially, but not only, those in the Actuarial Mathematics program) did not understand how the practices they were being asked to learn in Real Analysis might contribute to their future work; and this seemed to encourage them to adopt the Student and/or Skeptic positions. We do not think that these students were simply being defiant. Their perceptions and positioning may have been encouraged by several factors. Consider, for example, the fact that the Real Analysis courses – with their focus on axioms and formal proof – are characteristically different from the other core courses, and how, for many students, the Real Analysis courses may mark the end of their journey in the Analysis Stream. In addition to this, some students seemed to have a limited perception of what mathematicians (in any field) really do; and others were confused about how the Real Analysis courses seemed to be asking them to relearn everything that they had already learned in Calculus courses many years before.

This section is not meant to be discouraging; it is an attempt to offer a more realistic view of what is a very complex situation. Moreover, the constraints highlighted above lead to pertinent questions that would need to be considered if one wants to try to support students in the development of mathematical practices. For instance, if it is possible to design paths to practices where activities progressively urge students to develop practices that are more mathematical in nature (and we think it is), then: How can professors manage to fit these into courses, given the time constraints they face, the massive amount of material they need to cover, and the pressures of ensuring that their students do well on exams? How can students be inspired to productively engage with newly constructed paths, rather than focus on the paths that have already been laid out by past final exams over many years? And how can the new paths interact with these old paths in such a way that the important principle of consistency is conserved? Also, if we assume that the conditions and constraints of any mathematics course institution are more likely to encourage studenting and the development of non-mathematical practices, then: How might we work to change this? Can individual professors support and incite students to shift their positioning? Is this an issue that needs to be considered at all levels of education, from the moment students start learning mathematics in schools? Finally: How can we deepen students' perceptions of what it means to "behave mathematically"? In particular, how can we share with students the importance of shifting their positioning and developing mathematical practices, no matter which topic they are being asked to study in the present moment or which profession they will seek in the future?

On a positive note, all the participants in our study seemed to exhibit an openness towards developing practices that are more mathematical in nature. Even those who had positioned themselves strongly as a Student expressed an awareness of their studenting and, upon getting stuck in the interview, made comments that indicated a sentiment of disappointment. Several participants noted that they had tried to understand what they were doing during the Real Analysis course, but they eventually felt forced – by the limited amount of time, the large amount of challenging material, and the pressures of exams – to try do things by rote. More generally, we believe that most students at all levels of education would be open to learn (and maybe even to enthuse, train, or be skeptical), if the right contextual conditions could be put in place.

7.2 Discussion in Relation to Existing Literature

Throughout Chapter 1, we review the large body of existing work on the transition from Calculus to Analysis, where we highlight various observations researchers have made and how these imply certain predictions about the kinds of practices students might develop in a first Real Analysis course. The goal of this section is to revisit these observations and predictions in light of our results and to specify the empirical and theoretical contributions we have made.

In Figure 7.1 (below), we recall Winsløw's (2006) model of the two transitions that students may be expected to go through as they progress further in their university mathematics coursework. We present this model in more detail in Section 1.2. Empirically speaking, our study confirmed that students may not be experiencing these two transitions in a first Real Analysis course.

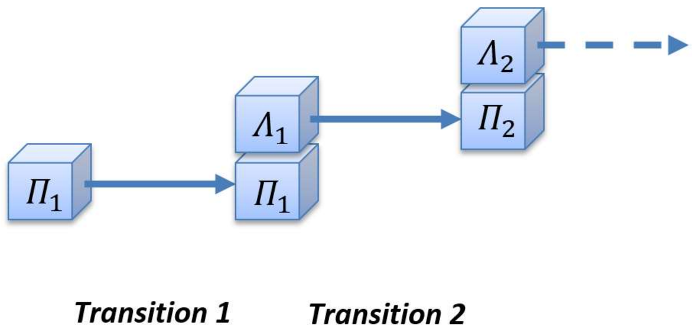

Figure 7.1 Winsløw's (2006) model of transitions in students' practices as they progress in university mathematics coursework (reproduced from Figure 1.2).

As predicted by Kondratieva and Winsløw (2018), there seems to be a disconnection between the practices students are invited to learn in Real Analysis and the practices they were invited to learn in previous Calculus courses, and students are left to their own devices to make the missing links. We also observed, as Winsløw (2016) did, that students will not necessarily (be able to) make these links on their own. In particular, students will not necessarily (be able to) use what they learn in Real Analysis to further develop the practices they learned in Calculus; for example, they will not spontaneously construct mathematical theoretical discourses that may have been missing from the previously developed practices. In our study, we found several examples of this. Some particularly strong examples could be found in relation to the topic of limits of rational sequences (as explored in Task 4). Participants seemed to have learned separate practices generated by the types of tasks "find the limit of a rational sequence" (which we could model by (Π_1, L)) and "prove the limit of a rational sequence" (which we could model by $(P, \widetilde{\Lambda_1})$, where $\widetilde{\Lambda_1}$ is the formal definition of sequence convergence). After finishing a first course in Real Analysis, the practices of the form (Π_1, L) remained unchanged; in particular, participants had not used $\widetilde{\Lambda_1}$ to work on and update the non-mathematical theoretical discourses denoted by L. In fact, many participants had also not learned the practices of the form $(P, \widetilde{\Lambda_1})$; at least not in the expected manner.

As predicted by Bergé (2008), some students do not seem to accept the invitation to prove "obvious" results: i.e., results they already know how to obtain using some simpler, procedural, informal practice. One contribution of our work is the realization that students' acceptance of the invitation to learn formal proof practices like $(P, \widetilde{\Lambda_1})$ may depend on their positioning. In general, the invitation seems to be accepted by the Student, the Mathematician in Training, and the

Enthusiast, all of whom are dedicated to learning what they were being asked to learn, for different reasons. The invitation seems to be less accepted by the Learner and extremely challenging to accept for the Skeptic, both of whom need to be convinced of the limitations of their informal understandings and/or their previously developed practices to engage seriously in developing new formal practices. Within such differences, there is nevertheless an important commonality: namely, each position, it its own way, might be linked to a lack of coordination between informal and formal practices. As predicted by Raman (2002, 2004), we saw how Real Analysis students may not be explicitly invited to engage in this coordination. As a result, the Student, the Mathematician in Training, and the Enthusiast might be inclined to favour the formal, the Learner and the Skeptic might naturally favour the informal, and most students may be left with a lack of awareness or an unresolved tension when it comes to the place of the informal and the formal in solving mathematical tasks.

Another key idea arising from our work is that just because a student accepts the invitation to learn a formal proof practice of the form $(P, \widetilde{\Lambda_1})$ or (Π_2, Λ_2) does not mean that they will learn the practice in the expected ways. As predicted by cognitively oriented work (e.g., Edwards, 1997; Edwards & Ward, 2004; Moore, 1994; Przenioslo, 2004; Weber, 2005b), we saw that by the end of a first course in Real Analysis, students may not have learned the expected mathematical practices. One of our main contributions is a characterization of the ways in which the practices students learn may be non-mathematical, and how these different ways may be linked to the activities students are offered in a course and how the students interact with those activities (as discussed in Section 7.1.1). It seems the compartmentalization of practices not only occurs at a curricular level, from one course to the next (as discussed above); it can also occur within a course, from one topic to the next, or even from one type of task to the next. In other words, students may be offered paths of activities that expose them to collections of isolated, specified, routinized praxis blocks; and the students may be left to their own devices to make connections, to generalize, and to develop the corresponding mathematical theoretical discourses. As predicted by Maciejewski and Merchant (2016), many students may maintain approaches to learning that aim at retaining the minimal core of what they need to know just long enough to perform well on assessment tasks. Evidently, such students do not spontaneously engage in the independent work that may be required for the development of practices that are more mathematical in nature. An additional observation contributed by our work is that students who occupy positions other than Student might also develop non-mathematical practices in some senses (as discussed at the end of Section 7.1.1), which tells us that they too may not (be able to) engage in all parts of the independent work that may be required to make their practices mathematical.

In sum, our study has confirmed and added to the many predictions proposed by previous work about students' development of non-mathematical practices in a first course in Real Analysis. In addition to this, our study has made what seem to be novel contributions in terms of the potential that may arise in such a course for supporting transitions to practices that are more mathematical in nature (as discussed in Section 7.1.2). We saw how some Real Analysis students may occupy positions other than Student, each of which may be productive to the development of mathematical practices in different ways. We suspect that some students who strongly occupy a position other than Student in Real Analysis may have occupied positions other than Student in past courses. This said, we also saw that some Real Analysis students may occupy a hybrid positioning including Student or be able to shift their positioning of Student locally when solving certain kinds of mathematical tasks. Accordingly, it seems that some Real Analysis students may be in the transition to practices that are more mathematical in nature, or at least have the potential for embarking on the transition. As discussed in Section 7.1.2, we predict that it is possible to offer activities that could further support students in shifting their positioning and transitioning to practices that are more mathematical in nature. Our main theoretical contributions – the notion of a path to a practice, our positioning framework, and our characterizations of non-mathematical versus mathematical practices – may turn out to be practical in this regard.

We finish this section by recalling the prediction that it may simply take time for students to develop mathematical practices (e.g., Sfard, 1991; Timmerman, 2005). In other words, the more courses students take, the more mathematical their practices will become. By this logic, we might expect the participants of our study to naturally develop practices that are more mathematical in nature after they finish a second course in Real Analysis. We agree with Sfard (1991) that it is perfectly normal for students (or even mathematicians) to linger in stages of insufficient understanding and mechanical drill. We have come to wonder, however, how long this lingering may be able to go on under the current ecological conditions of University Mathematics.

7.3 Final Remarks

7.3.1 Conclusions of the Current Study

This doctoral work had two main objectives. First, we wanted to gain an understanding of the nature of the practices developed by students in a first Real Analysis course. Second, we wanted to make conjectures about the factors that may be shaping the development of those practices.

Our study revealed that the practices developed by successful Real Analysis students could be non-mathematical in different ways and to varying degrees. During the task-based interviews we conducted, many students' spontaneous task solving behaviour suggested that they had maintained practices from Calculus or learned practices in Real Analysis that were generally non-mathematical. These practices were characterized by an automatic identification of tasks with techniques (based on superficial properties of task statements), an algorithmic implementation of techniques (based on recalling the steps of particular solution images), and non-mathematical discourses (based on established experiences, inert knowledge, or taking one's own understanding for granted). This said, there were several students who spontaneously exhibited practices that can be seen as mathematical or in transition to becoming mathematical. Some students were inclined to study the properties of task statements more closely and assess possible types of tasks and techniques. Some students implemented techniques using essential characteristics of objects involved in the task, key ideas from solutions seen or done before, and general problem solving or proof strategies. And some students explained their techniques by clarifying, questioning, and verifying their own understanding with mathematical theory.

Our qualitative analyses led us to conclude that the students' practices were shaped by both the activities they were offered and the characteristically different ways in which they interacted with those activities. As theoretical tools for thinking about this, we introduced the notion of a path to a practice and a framework of five positions that students may adopt in a university mathematics course institution: the Student, the Skeptic, the Mathematician in Training, the Enthusiast, and the Learner. We found that, irrespective of the position a student adopted, they may have been enabled and encouraged to develop practices that were not exclusively mathematical through the offering of disjoint and narrow paths to practices, where assessment activities could be solved by mimicking the steps of some previous surface-similar activity, and where it was not necessary to learn how to mathematically describe, explain, and justify an appropriate mathematical technique for a general mathematical type of task. We also found that, irrespective of the activities that were offered, each of the positions could be productive towards the development of mathematical practices in different ways. While many of the Real Analysis students we interviewed showed strong signs of occupying one position (most often, but not always, the position of Student), many also exhibited a willingness or ability to shift their position and develop practices that were more mathematical in nature. This willingness or ability became most evident when students were explicitly pushed by the interviewer to go in that direction or when students struggled in solving an interview task because it differed in some essential way from the activities in the paths to practices they had followed in their course.

This highlighted two interconnected themes to consider if one hopes to support students in the development of mathematical practices: namely, the design of paths to practices and the perturbation of students' positioning. Taking an institutional perspective on the teaching and learning of mathematics allowed us to see, and pointed to, the necessity of making such considerations while keeping in mind how mathematics courses actually function (according to examination procedures, time constraints, curricular expectations, students' experiences in previous courses, students' future goals, and so on).

7.3.2 Limitations and Future Work

To accompany the discussions and conclusions presented above, it is important to point out some of the limitations of our study, which arose mainly due to the time constraints of completing a doctoral program. One decision we made early on was to focus on what students are invited to learn, independent of their professors. This meant that we did not analyse all the activities that students may be offered in a course. For instance, we did not attend lectures. Moreover, we did not interview professors. Hence, we do not know if our results reflect their expectations or not. We also had to be selective about the types of tasks we posed in our interviews. While we covered a range of topics, the tasks we chose were not necessarily representative of all the practices students may have learned. On the other hand, by choosing a range of topics, we were not necessarily able to go in depth into every topic that we chose. With only so much time to spend on each task in the interview, we were not always able to elicit a complete view of all practices for all participants.

Finally, we were only able to analyze participants' behaviour at a particular moment in time, whereby we could only make predictions about evolution or transition on a larger scale.

Based on the current study, we can envision two main avenues for future work. The first avenue would be to continue exploring what is currently happening in the institution of University Mathematics. It seems both natural and pertinent to investigate how students' positioning and practices might continue to evolve as they progress even further in the Analysis Stream of university coursework, into graduate work, and beyond. We are wondering, for example, if it is possible for students to maintain the Student position and the development of non-mathematical practices all the way through their coursework. To fully capture evolution, it would be fitting to perform some longitudinal studies that follow students over longer periods of time. Given that graduate work is typically when students shift from coursework to research work, this might be a particularly interesting evolution to observe. We also think it would be interesting to examine how students' positions and practices might evolve within and change across different streams of coursework. Recall that our positioning framework was defined in relation to a given course institution to acknowledge the possibility that a student, especially at university, may change their positioning drastically depending on the topics covered in a course and how these topics relate to the student's personal interests and goals. A future research project could look into this using our positioning framework. For instance, one could perform task-based interviews with students who are taking several courses at the same time and try to see if students change their positioning from one course to another, not only in terms of the perceptions they share, but also in terms of the nature of the practices they develop.

A second avenue for future work would be to start engaging in design-based research, perhaps using our notion of a path to a practice, our positioning framework, and our characterizations of the ways in which students' practices may be non-mathematical or mathematical. We would like to think more deeply about how to design paths to practices that might perturb students' positioning and/or encourage the development of mathematical practices. It would also be very important to think about how such paths could be effectively integrated into a course under existing conditions and constraints. These topics could be studied systematically by testing designed paths, not only with students, but also in courses. Wherever possible, it seems it could be beneficial to perform such work in collaboration with mathematics professors, who would be able to offer unique perspectives and expertise, and who may also be interested in informing their instruction with mathematics education research.

On a more general level, recall that this thesis began with our interest in a broad, but vague question: Do students really learn to "behave like mathematicians" in the context of university mathematics coursework? Our future work will certainly be oriented by this more general interest. For instance, we would like to think more about what constitutes the key elements of "mathematical behaviour," including and in addition to "mathematical practices." We would also like to explore how this varies across domains and professions (e.g., not only in Analysis, not only in an academic setting). And we would like to continue investigating the conditions that may inhibit or nurture mathematical behaviour among students.

Taking a critical stance on our current work, we realize that the behaviours involved in studenting and in our characterization of non-mathematical practices may not be entirely "non-mathematical" in the sense that they may not be entirely absent from mathematicians' work. Indeed, there may be a time and a place where mathematicians engage in automatically applying techniques, trying to copy the steps of existing solutions, or producing potential solutions based on the established experience of something working before, some result they remember hearing about, or some personal intuition or understanding. In fact, we suspect that in most cases, becoming a mathematician in some area of expertise requires one to act like a Student in that area, at least to some degree. After all, one of the conditions of becoming a professional mathematician is first succeeding in the corresponding didactic institutions, and to succeed in such institutions one must operate according to their rules, strategies, and norms, at least to some degree. The key point we are assuming is that to truly become an expert in some area, acting like a Student is not enough: one must somehow, at some point, develop an ability to flexibly and frequently shift between studenting, learning, training, being skeptical, being enthusiastic, and so on. If a mathematician may at some point be operating according to practices that are non-mathematical (e.g., when reading, exploring, and experimenting), they are nevertheless able to shift their positioning and transform their practices to be more mathematical if they so choose (e.g., when questioning, validating, writing, and sharing). Perhaps we should expect that most students will student and that most students will develop practices that are non-mathematical in some senses. The question then becomes: Are students being given enough opportunities to go beyond this, to acquire an ability to shift their positioning, and to develop practices that are more mathematical in nature? Of course, one could argue that somehow, at some point, some students will naturally do this on their own. We would argue that we should and could be working to facilitate this for more students, before, during, and after their first course in Real Analysis.

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Appendix A: Materials for our Task-Based Interview

The Tasks

1. Is $\sqrt{8}$ rational or irrational?

2. Show that the function $f(x) = e^x - 100(x-1)(2-x)$ has 2 zeros.

3. Let $A = \{x_n \in \mathbb{R} : n = 1, 2, \dots\}$.
   a) Under what conditions is $\lim_{n\to\infty} x_n = \sup A$?
   b) Give an example of A where the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$.

4. Let $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \dots\right\}$.
   a) Prove that for any $p > 1$, A is unbounded above.
   b) Prove that for $p = 1$, $\sup A = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$.

5. Find the limit

$$\lim_{n\to+\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n}$$

   where $a \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers.

6. a) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

   For what values of p is g continuous on $\mathbb{R}$?

   b) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

   For what values of p is g differentiable on $\mathbb{R}$?

   c) What if $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$?

   For what values of p is g continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?

The Protocol

<u>Introduction (~ 10 minutes)</u>

- **Soliciting Consent:** I will hand the participant a copy of the consent form and give them time to read through it and ask questions.

- **Brief Explanation of the Interview:** If the consent form is signed, I will give a short explanation of how the interview will proceed:

  > "During the interview, I'm going to ask you to solve some problems. While you're working, I'll ask that you think aloud so that I can keep track of what you are thinking. For the purposes of my research, it is more important that you tell me what you are doing and why you are doing it; it is less important that you find the correct answer. Ok?"

- **Beginning the Interview:** I will ask if the participant agrees to begin the interview. Once they agree, I will start the audio recording. I will pass them the sheet of paper with the first problem on it, and I will say: "Here is the first problem."

<u>Interview Procedure: General Considerations</u>

- **Long Silences:** It is important that the participant says what they are thinking. If at any point in the interview, they are completely silent, even if they are doing things on paper without giving verbal explanations, I may say something like:

  > "Remember to tell me what you're thinking."

- **Incomprehensible Comments:** It is also important that I understand what the participant is saying. Thus, at any point in the interview, I may ask a clarification-type question like:

  > "When you say ______, what do you mean?"

- **Momentary Silence or Incomprehension:** If the participant is silent or incomprehensible only for a moment, I will let them finish their thoughts before intervening. I will keep notes of when something was unclear or unexplained, or when the participant paused and seemed to be thinking. Later, I may say something like:

  > "What were you thinking when_________?"

- **Questions About What They Can Do:** If the participant asks about the approaches they are allowed to use, I will say something like:

  > "You can use any approach you want to solve the problem."

- **Questions About What They Should Do:** If the participant asks if it is a good idea to use a particular approach, I will respond honestly by saying something like:

  > "Yes, that is a way I would go about solving this problem."
  >
  > "I didn't think about solving the problem in that way. So I'm not sure if that approach will work or not."

"That approach will not be helpful in this case. Can you think of another way to solve the problem?"

- **Questions About the Correctness of their Solution:** If the participant asks if their solution is correct or not, I will remind them that the goal of the interview is not to see if they can get to the correct solutions and that I am more interested in seeing how they think about solving the problem. I will also mention that we can discuss the solutions to the problems after the interview has been completed.

- **Questions About My Opinion:** The participant may ask me what I think. I will respond by telling them that I can certainly share my opinion, but I do not want to influence their answer and I also want to make sure that we are able to go through the whole interview. I will suggest that we talk about it after the interview has been completed.

- **Six Anticipated Scenarios for a given Problem:** Interventions will occur depending on the kind of scenario that arises. Below is a summary of the six kinds of scenarios that have been anticipated and the kinds of interventions that may occur in each scenario:

    1. **Immediately Stuck (IS):** If the participant is unable to spontaneously start solving a problem, they will be provided with a series of hints that get progressively more directive, to see if they can get unstuck. If they cannot, I will suggest that we move to the next problem and inform them that if there is time later, they may come back to this problem if they want. Once all the other problems have been considered, the participant will be given the opportunity to go back to problems where they got stuck (time permitting).

    2. **Off Track (OT):** If the participant seems to be approaching a problem in a way that will not lead to a solution, and they persist in using that approach for a period of time, but are not making substantial progress, I will ask them to clarify the approach they are taking, why they chose it, and what they are hoping will happen. If their response seems to indicate that they are indeed using an approach that will not be successful, I may choose to explain this to them and I will ask if they know of any other approach.

    3. **Stuck During (SD):** If the participant starts solving a problem in a promising way, but at some point stops making progress (either they verbally express that they do not know how to proceed, or they spend a significant amount of time on figuring out the next step), they will be provided with a series of hints that get progressively more directive, to see if they can get unstuck. If they cannot, I will consider asking some follow-up questions about any partially unexpected or meaningful production they have made (see Scenarios 5 and 6). Then I will suggest that we move to the next problem. Again, once all the other problems have been considered, participants will be given the opportunity to go back to problems where they got stuck (time permitting).

4. **Unexplained or Unclear Thinking (UT):** If the participant seems to have skipped an explanation of part of their thinking about a problem or gives an explanation that is incomprehensible in some way, they may be interrupted with a question aimed at getting them to reveal or clarify their thinking immediately. If the participant is in a good flow of thinking, however, it will always be preferable to take a note and make a decision afterwards as to whether it is a priority to ask the thinking to be explained or clarified as a follow-up question.

5. **Unexpected Production (UP):** If the participant seems to accept as a solution something that is very different from what would have been expected in RA I (based on our analysis of assessment activities), they will be asked some questions regarding the acceptability of their solution in three different contexts: their personal work, the RA I course, and a community of mathematicians. For example: Would their solution receive full marks if it were to be graded by their professor in their RA I course? Would their solution be accepted by mathematicians?

6. **Meaningful Production (MP):** If the participant produces a meaningful solution to a problem (i.e., one where they have exhibited an appropriate RA I approach, at least in part), several kinds of follow-up questions may be posed. Some questions may seek to clarify unexplained or unclear thinking (see Scenario 4). It is particularly important that I solicit the reasons the participant has for their overall approach, and possibly also some of the specific steps they took. The participant may be asked if they thought about other ways to approach the problem; and if so, they will be asked to compare the approaches and explain why they chose the one they did. I may also inquire into the acceptability of the participant's solution in different contexts (as in Scenario 5): e.g., How would the participant's solution differ depending on whether they were trying to convince themselves, their RA I professor, or a community of mathematicians? More generally, I may pose questions about how the participant perceives mathematics, RA I, themselves in relation to these, etc.

- **Non-Evaluative Atmosphere:** I will do my best to pose follow-up questions in such a way that participants do not feel like they are being evaluated. I want them to know that I am interested in what they think; not whether they can formulate "the right" response.

- **Progressing Through all the Problems:** The participant will be given as much time as possible to solve a problem, including overcoming any blockages, and I may have many follow-up questions I want to ask. However, priority will be given to observing the participant formulate at least one solution approach for every problem and a reason for approaching the problem in that way. I will constantly have my eye on the time and, when deemed necessary, suggest that we move on to the next problem. Once the participant agrees to move on, I will provide them with the next problem. I will make note of any follow-up questions that I am unable to ask about a particular problem and decide if these questions are still important to ask at the end of the interview (time permitting).

- **Additional Questioning at the End of the Interview:** Depending on what happens throughout the interview, I will ask some additional follow-up questions at the very end (time permitting). For instance, if the participant seems to find some problems particularly easy or challenging, fun or boring, relieving or stressful, etc., I will ask them to explain how they feel about the different problems and why. Similarly, if the participant seems quite comfortable (or, in contrast, quite anxious) throughout the entire interview, I will mention this and ask if they thought this resulted from their experience in RA I. If a participant got stuck on a problem, I will ask how they might go about getting unstuck (e.g., if the problem had been posed on an assignment in RA I). If a participant said they would use the internet in some way, I may ask if they want to show me what they mean using my computer. Finally, I will ask the participant to compare Calculus and Analysis and describe their transition from one into the other.

- **Ending the Interview:** The participant will have been informed in the consent form that the interview will last approximately two hours. I will plan to end the interview after two hours have passed or all follow-up questions have been asked (whichever comes first). I will express my deep appreciation to the participant and stop the audio recording.

- **Flexibility:** This document is a guideline. Depending on what actually happens in each interview, the interventions and questions may vary quite a bit. I will allow myself the flexibility to ask questions that seem interesting in the moment.

Interview Procedure: Problem-Specific Considerations

On the pages that follow are summaries of the kinds of situations that have been anticipated for each problem and examples of the kinds of interventions that may occur. The actual interventions that occur will depend on the participant's actual behaviour while solving the problems, the interventions that have already occurred (e.g., we may not carry out an intervention if we think it will bring about something that has already been addressed), the time remaining in the interview (the planned duration is approximately two hours), and our priority of observing the participant formulate at least one solution approach for every problem and a reason for approaching the problem in that way.

Problem 1: Is $\sqrt{8}$ rational or irrational?

UT	If just claims $\sqrt{8}$ is rational or irrational (and jumps into a proof): “(Wait. Before you start your proof,) what made you say that the number is rational/irrational?”
UP	If completed explanation is, e.g., it is irrational because: there’s a root, it is a root of a number that is not a perfect square or a product of perfect squares, it is 2 times an irrational number: “So you are convinced that $\sqrt{8}$ is rational/irrational?” … “Do you think your explanation would be enough to get you full marks in RA I?” … “Do you think it would convince a mathematician?” If mentions the need to give something else: “Can you show me what you mean?”
IS	If no decision on rational or irrational: “What if the question was about $\sqrt{2}$ instead of $\sqrt{8}$?” If decides irrational and wants to prove: H1: “During your course, you may have seen the proof that $\sqrt{2}$ is irrational?” … “Could that help you here?” H2: “The proof that $\sqrt{2}$ is irrational starts something like this: Suppose $\sqrt{2} = m/n$, …” Write $\sqrt{2} = \frac{m}{n}$ on their paper.
SD	If stuck finding common factors of m and n: “Would it make a difference if it had been $\sqrt{2}$ instead of $\sqrt{8}$?”
MP	If clarifications not already made: • Squaring both sides or isolating for $\sqrt{2}$: “Why did you do this?” • Making a conclusion like “m^2 is divisible by 2, whereby m also is” or “m^2 is even, whereby m is even” or “$\frac{r}{2}$ is rational” or “$\sqrt{2}$ is irrational”: “How do you know this?” About general approach: “Why did you answer the question in this way?” “Did you need to do the proof to convince yourself that $\sqrt{8}$ is irrational?”

Problem 2: Show that the function $f(x) = e^x - 100(x-1)(2-x)$ has 2 zeros.

OT	If lets $f(x) = 0$, starts to rearrange, and is trying to solve for x (e.g., adding lns): "In most cases you can't actually solve for x. In this case, you can't. Is there another way that you can show that this function has 2 zeros?"
IS	H1: "Would it help if I told you that the zeros are in $[0, \infty)$?" Write the interval. H2: "What if I said they were in [0,10]?" Write the interval.
UT	If plugs in values for x without explaining choice: "Why did you plug in (or are you plugging in) *these* values for x?"
SD	If continuing to plug in x values and not getting desired result: "What are you expecting will happen?" If mentions trying to find where the sign of f changes: "Could you study the sign of this function in a different way?" If unable to find sign changes: "If you found the sign changes you're looking for, what would you do next?"
UP	If explains (solely through a numerical or graphical approach) without citing the Intermediate Value Theorem (IVT) or Rolle's Theorem (RT): "Are you convinced that the function has (exactly) 2 zeros now?" … "Do you think this would convince other mathematicians?" … "Would it convince your professor in RA I?"
MP	If clarifications not already made: • Citing the IVT/RT without checking assumptions: "Why did you mention the IVT/RT?" • Finding where the quadratic function is negative by finding roots and testing numbers in between them: "Why did you use that approach to figure out where the quadratic function is negative?" About general approach (unless already discussed, e.g., at explaining choice of x values and expectations for what should happen): "What is it about this problem that made you choose the approach you did?"
Necessary follow-up: "How would your solution to this problem change if the question had been to show that f has *exactly* two zeros?"	
MP	If RA I approach: "Would you have been convinced if I had just shown you a graph of this function?" … "Do you think graphing is an appropriate approach for a mathematician?" … "Would it have been appropriate in the RA I course?"

Problem 3: Let $A = \{x_n \in \mathbb{R} : n = 1, 2, \ldots\}$.

a) Under what conditions is $\lim_{n\to\infty} x_n = \sup A$.

IS	H1: "Would it help to think about what these things mean? The limit and the supremum?" Point to limit and sup. H2: "What if the question was to find an example where this is true?"
UT	If just states conditions (e.g., x_n increasing and bounded above) (and jumps into a proof): "(Before you go on.) What made you choose *those* conditions?"
UP	If solely cites a theorem (e.g., Monotone Convergence) or gives intuitive explanations: "So you are convinced that those conditions work?" … "Do you think your explanation would convince mathematicians?" … "What about in RA I? Would your answer get full marks?" "Do you think you could prove the limit under the conditions you gave?"
SD	If stuck starting a proof: H1: "Do you remember how to prove a limit using the definition?" H2: "To show that the limit of x_n is equal to this value, say alpha [write α], you could start by writing something like let $\varepsilon > 0$ [write Let $\varepsilon > 0$]. Then [write and say We want $N \in \mathbb{N}$ st $\forall n \geq N \ldots$]." … "Can you complete the proof?" If stuck at finding N in the definition of convergence: H1: "What do you know about the supremum?" H2: "Can you use this?" Write $\alpha = \sup A \Rightarrow \forall \varepsilon > 0, \exists\ a \in A, \alpha - \varepsilon < a$.
MP	If clarifications not already made: • Using a conclusion of a non-verbalized fact (e.g., the supremum is an upper bound, the above implication, the sequence is increasing): "How do you know this?" • Choosing both increasing and bounded above: "Why did you say that the sequence *must* be bounded above?" About proof approach: "If a student taking RA I now asked you to explain why these conditions work, what would you say?" … "Do you think that explanation would be enough to convince a mathematician?"

Problem 3: Let $A = \{x_n \in \mathbb{R} : n = 1, 2, \dots\}$.

b) Give an example of A where the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$.

<table>
<tr><td>IS</td><td>H1: "During the RA I course, maybe you were asked to come up with examples that satisfy certain properties?" … "Do you remember how you approached those kinds of problems?"
H2: "Would trial and error work?"</td></tr>
<tr><td>UT</td><td>If just states a sequence (e.g., $x_n = 1/n$), (before jumping into checking it):
"(Wait. Before checking if that sequence works.) Why did you choose that sequence?"</td></tr>
<tr><td>SD</td><td>If stuck finding the supremum of the example set:
"Would it help to write out some elements of the set?"</td></tr>
<tr><td>UP</td><td>If gives solely intuitive explanations for the supremum:
"So you are convinced that the supremum is __?" … "Would your answer be any different if this were an assignment in RA I?" … "What if you were tasked with convincing a mathematician of the supremum? Would that change the way you respond to the question?"</td></tr>
<tr><td>MP</td><td>If clarifications not already made:
• Stating the limit of an example sequence:
"How did you know the value of this limit?"
"What is it about this limit that makes it easy to know that it exists?"
• Stating the value of the supremum:
"Why do you think that is the supremum of A?"
About example-finding approach:
"Why did you go about finding your example in this way?"</td></tr>
<tr><td colspan="2">Follow-up if example is given, but not related back to conditions determined in part (a):
"Does this example satisfy the conditions you came up with in part (a)?"</td></tr>
</table>

Problem 4: Let $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \dots\right\}$.

a) Prove that for any $p > 1$, A is unbounded above.

<table>
<tr><td>IS</td><td colspan="2">H1: “Would it help to specify p?”
H2: “Do you remember the definition of an unbounded above set?”
H3: “The definition is [say and write]: $\forall M > 0, \exists a \in A, a > M$.”</td></tr>
<tr><td>UP</td><td colspan="2">If gives solely intuitive explanations for why A is unbounded above:
“So you are convinced that A is unbounded above?” … “Do you think a mathematician would consider your explanation a proof?” … “Would your RA I professor give you full marks for your explanation?”
If no: “Can you show me what you mean?” … “Do you think you could prove it?”</td></tr>
<tr><td></td><td>If definition approach:</td><td>If Problem 3 approach:</td></tr>
<tr><td>SD</td><td>If stuck finding n in the definition of unbounded:
“Is it true that $\frac{n^p}{n+1} > \frac{n^p}{n+n}$?”
Write the inequality. …
“Can you use this?”</td><td>If stuck manipulating the inequality:
“Are the following inequalities true?” Write the inequalities:
$\left(\frac{n+1}{n}\right)^p > \frac{n+1}{n} > \frac{n+2}{n+1}$. …
“Can you use this?</td></tr>
<tr><td rowspan="2">MP</td><td>If clarifications not already made:
• Using inequality like above:
“How do you know this inequality?”
• Stating n exists such that $n^{p-1} > 2M$:
“What made you say this n exists?”
• Stating n exists due to the unboundedness of $\mathbb{N}$:
“How do you know that the natural numbers are unbounded?”
About general approach:
“What about this problem made you approach it in this way?”</td><td>If clarifications not already made:
• Using unexplained inequalities:
“Why did you write this?”
• Concluding that $1 > 0$:
“How do you know this?”
• Going from a true equivalent inequality to the conclusion:
“How did you go from here to here?”
• Stating the value of a limit:
“How do you know the value of that limit?”
• Going from infinite supremum to unbounded:
“How did you go from here to here?”
About general approaches:
“Why did you use the approach you did to show the sequence is increasing?”
“Why did you use the approach you did to show the inequality?”
“Why did you use the approach you did to show the limit of the sequence?”</td></tr>
<tr><td colspan="2">If RA I approach:
“Could you have convinced yourself that A is unbounded without this proof?” … “What about a mathematician?”</td></tr>
<tr><td colspan="3">Potential follow-up at the end of the interview, if Problem 3 is solved for this case, but not used:
“Can you think of a different approach for this problem given the conclusions you made in the other problem?” Show Problem 3.</td></tr>
</table>

Problem 4: Let $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \dots\right\}$.

b) Prove that for $p = 1$, $\sup A = 1 = \lim_{n\to\infty} \frac{n^p}{n+1}$.

<table>
<tr><td>IS</td><td colspan="2">H1: “What is this limit if you make $p = 1$?”
H2: “Do you remember how to prove that the supremum of a set is a number like 1?”
H3: “For example, you could show that [say and write]: $(a \leq 1 \forall a \in A)\ \wedge\ (\forall \varepsilon > 0, \exists a \in A,$ …” … “Can you complete the proof?”</td></tr>
<tr><td>UP</td><td colspan="2">If just concludes the limit and/or uses intuition to talk about the supremum:
“So you are convinced of these equalities?” … “Do you think your response would be accepted in RA I?” … “What about in a community of mathematicians?”</td></tr>
<tr><td></td><td>If definition approach:</td><td>If Problem 3 approach:</td></tr>
<tr><td>SD</td><td>If stuck starting to prove the limit:
“Remember that it starts something like this: You let $\varepsilon > 0$ and you want $N \in \mathbb{N}$ so that $\forall n \geq N, \dots$” Write Let $\varepsilon > 0$. Want $N \in \mathbb{N}$ st $\forall n \geq N, \dots$
If stuck starting to prove the supremum:
“You could show something like $(a \leq 1 \forall a \in A)\ \wedge\ (\forall \varepsilon > 0, \exists a \in A, \dots$” … “Can you complete the proof?”</td><td>If stuck starting to show increasing:
“Remember that a sequence x_n is increasing if $x_{n+1} > x_n\ \forall n \in \mathbb{N}$.” Write the inequality.</td></tr>
<tr><td rowspan="2">MP</td><td>If clarifications not already made:
• Stating that N exists (due to the unboundedness of $\mathbb{N}$):
“How do you know that (the natural numbers are unbounded?)”
• Asserting existence of N without checking the inequality for all $n \geq N$:
“How do you know that the inequality also holds for all n bigger than N?” Circle $\forall n \geq N$ if it was written.
About general approaches:
“Why did you prove the limit (or the supremum) in this way?”</td><td>If clarifications not already made:
• Concluding that $1 > 0$:
“How do you know this?”
• Going from a true equivalent inequality to the conclusion:
“How did you go from here to here?”
• Stating the value of a limit:
“How do you know the value of that limit?”
About general approaches:
“Why did you prove the supremum in this way?”
“Why did you use the approach you did to show the sequence is increasing?”</td></tr>
<tr><td colspan="2">If RA I approach:
“Do you have to prove the limit and/or the supremum like this to convince yourself that they are true?” … “Do you think you’d have to include the proofs to convince a mathematician” … “What about your RA I professor? Would they give you full marks if you used a different approach?”</td></tr>
<tr><td colspan="3">Follow-up if did this problem and part (b) of Problem 3 in different ways:
“Why did you do something differently here to show the limit and/or the supremum when compared to part (b) of the previous problem?” Show Problem 3.
Potential follow-up at the end of the interview, if Problem 3 is solved for this case, but not used:
“Can you think of a different approach for solving this problem given the conclusions you made in the other problem?” Show Problem 3.</td></tr>
</table>

Problem 5: Find the limit

$$\lim_{n \to +\infty} \frac{\ln(2 + a^n + \arctan(x_n))}{n}$$

where $a \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{N}}$ is any sequence of real numbers.

<table>
<tr><td>IS</td><td>H1: "Would it help to specify a?"
H2: "Would it help to think about how each of the terms in the argument of the logarithm behave?"</td></tr>
<tr><td>UP</td><td>If gives solely intuitive explanation for the value:
"You seem convinced of the value of the limit. Am I right?" … "Would this explanation be enough to get you full marks in RA I?" … "Do you think it would convince a mathematician of the limit?"</td></tr>
<tr><td>MP</td><td>If clarifications not already made:
• Avoiding the sequence x_n
"Does your argument work for all sequences x_n?"
• Assuming a particular range or value of a:
"Does your argument work for all values of a?" … "What if $a \neq 2$? OR "What if $a < 0$?" OR "What if $a = \frac{1}{2}$?"
• Choosing different ranges for a:
"What made you choose these different scenarios for the value of a?"
• Writing an inequality that is only eventually true:
"What made you choose this inequality?"
• Going from bounding the argument to bounding the sequence:
"How did you go from this step to this step?"
• Stating the value of a limit:
"How do you know the value of that limit?"
• Concluding without stating the ST:
"How did you go from this step to your conclusion?"
About general approach:
"Why did you choose this approach to find this particular limit?"</td></tr>
<tr><td colspan="2">Potential follow-up at the end of the interview:
"How is finding this limit different from finding the limits in some of the other problems?"</td></tr>
</table>

Problem 6: a) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

For what values of p **is** g **continuous on** $\mathbb{R}$**?**

IS	H1: "Do you remember how to show that a function is continuous?" H2: "One way to show g is continuous at a is to show that $\lim_{x \to a} g(x) = g(a)$." Write the limit. H3: "Would it be easier to show that g is continuous at 0?"
SD	If stuck dealing with p: "Would it help to specify p?" "What if $p = 1$?" … "What if $p = 0$?" … "What if $p = -1$?"
UT	If states $g(x)$ is continuous when $x \neq 0$ without reason: "Why did you say that g is continuous when $x \neq 0$?" … "How do you know that these functions are continuous?" … "How do you know that the product of continuous functions is continuous?"
MP	If clarifications not already made: • Focussing on $x = 0$ (ignoring $x \neq 0$): "Why did you focus on when $x = 0$?" • Stating the value of a limit (or concluding a limit does not exist): "How do you know that this limit equals this value (or that this limit does not exist)?"
UP	If gives solely intuitive explanations for the value of a limit (or a limit's non-existence): "So, you are convinced that this limit equals this value (or does not exist)?" … "How would you convince a mathematician that this limit is this value (or does not exist?)" … "If you were asked on an assignment in RA I to show that this limit equals this value (or that this limit does not exist), how would you respond?"
MP	About general approach: "Why did you take this approach to solve this problem?" "Why did you calculate *that* limit?" If proved all limit behaviours: "Do you have to do these proofs to convince yourself that the limits equal these values or do not exist?" … "Do you think a mathematician would need to see these proofs to be convinced?" … "What about your RA I professor? Would they need to see the proofs to give you full marks?"
Follow-up if response is reached: "How would your answer change if this [point at the definition for $g(0)$] was equal to something other than 0?"	

Problem 6: b) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

For what values of p is g differentiable on $\mathbb{R}$?

<table>
<tr><td colspan="2">If previous problem is solved, point out that this is the same function.</td></tr>
<tr><td>IS</td><td>H1: “Do you remember how to show that a function is differentiable?”
H2: “g is differentiable at a if the limit [write limit] $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exists.”
H3: “Would it be easier to show that g is differentiable at 0?”</td></tr>
<tr><td>SD</td><td>If stuck dealing with p:
If previous problem solved: “Can you use a similar approach here to deal with this p as you used in the previous problem?”
If previous problem unsolved:
“Would it help to specify p?”
“What if $p = 2$?” … “What if $p = 1$?” … “What if $p = 0$?”</td></tr>
<tr><td>UT</td><td>If states $g(x)$ is differentiable when $x \neq 0$ without a reason:
“What made you say that g is differentiable when $x \neq 0$?” … “How do you know that these functions are differentiable?” … “How do you know that the product of differentiable functions is differentiable?”</td></tr>
<tr><td>MP</td><td>It is likely that the previous solution will be used to make conclusions here. If for some reason clarifications are not made previously:
• Focussing on $x = 0$ (ignoring $x \neq 0$):
“Why did you focus on when $x = 0$?”
• Conclusion about a limit:
“How do you know that that limit exists (or does not exist)?”</td></tr>
<tr><td>UP</td><td>If gives solely intuitive explanations for the (non-)existence of a limit:
“So, you are convinced that this limit exists (or does not exist)?” … “Would you have to show more work in order to get full marks in RA I if you were asked to show that this limit exists (or does not exist)?” … “Do you think you would need extra work to convince a mathematician?”</td></tr>
<tr><td>MP</td><td>About general approach:
“Why did you take this approach to solve this problem?”
“Why did you calculate *that* limit?”
If proved all limit behaviours:
“Do you have to do these proofs to convince yourself that the limit exists (or does not exist)?” … “Do you think these proofs would be necessary to convince a mathematician?” … “What about your RA I professor? Would they need to see the proofs to give you full marks?”</td></tr>
<tr><td colspan="2">Follow-up if response is reached:
“How would your answer change if this [point at the definition for $g(0)$] was equal to something other than 0?”
Potential follow-up at the end of the interview, if responses are reached for both parts (a) and (b) of Problem 6:
“Looking at the responses to these two problems [show Problem 6(a) and Problem 6(b)], is there a way that you could use your response to one problem to help in responding to the other problem?”</td></tr>
</table>

Problem 6: c) Let $g(x) = \begin{cases} x^p \sin\frac{1}{x} & x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$

For what values of p is g continuous on $\mathbb{R}$? differentiable on $\mathbb{R}$?

<table>
<tr><td colspan="2">If one of the previous two problems is solved: "What if we changed the domain of the function?"</td></tr>
<tr><td>IS/
SD</td><td>H1: "Would it help to specify p? Say $p = 1$. Is the function continuous on $\mathbb{R}$?"
H2: "Is the function continuous at 1?
H3: "To have continuity at $x = a$ we need $\lim_{x\to a} g(x) = g(a)$, right?" Write the limit. "What is the definition of this limit in terms of sequences?"
H4: "Is it true that $\lim_{n\to\infty} g(x_n) = g(1)$ for all sequences $(x_n)_{n\in\mathrm{N}}$ that converge to 1?" Write the corresponding mathematics.
H5: "Could you use the fact that for any real number, there is a sequence of irrational numbers converging to that number?"
Write: $\forall r \in \mathbb{R}, \exists\, x_n \in \mathbb{R}\setminus\mathbb{Q}$ such that $x_n \to r$ as $n \to \infty$.</td></tr>
<tr><td>SD</td><td>If stuck on differentiability:
"Could you use your conclusion about continuity?"</td></tr>
<tr><td>MP</td><td>About general approach for finding where g is (not) continuous:
"Why did you use this approach for finding where g is (not) continuous?"
If clarifications not already made:
• Using a specific instance of the theorem that for every real number there is a sequence of rationals or irrationals converging to that number:
"How do you know that?" ... "Could you prove it [the specific instance]?"
• Using continuity of functions to compute the limit of a sequence (i.e., distributing the limit in $\lim_{n\to\infty} x_n^p \sin\frac{1}{x_n}$):
"How did you go from here to here?"
• Not describing connection between sequential definition of limits and continuity (e.g., why discontinuity follows from finding a sequence $(x_n)_{n\in\mathrm{N}}$ that converges to 1 such that $\lim_{n\to\infty} g(x_n) \neq g(1)$):
E.g., "How did you go from this to the conclusion that g is not continuous at 1?"
• Using not continuous implies not differentiable:
"How do you know that?"</td></tr>
<tr><td colspan="2">Follow-up if partially solved:
"Is this function continuous at 0?"
"Is it the only value where the function is continuous?"</td></tr>
</table>

The Consent Form

INFORMATION AND CONSENT FORM

Study Title: An exploration of the knowledge to be learned in RA I and II[44]
Researcher: Laura Broley
Researcher's Contact Information: l_brole@live.concordia.ca
Faculty Supervisor: Nadia Hardy
Faculty Supervisor's Contact Information: nadia.hardy@concordia.ca

You are being invited to participate in the research study mentioned above. This form provides information about what participating would mean. Please read it carefully before deciding if you want to participate or not. If there is anything you do not understand, or if you want more information, please ask the researcher.

A. PURPOSE

The purpose of the research is to gain a better understanding of the kinds of knowledge students learn and the transitions they face as they progress in their undergraduate mathematics degree, from Calculus to Real Analysis I, from Real Analysis I to Real Analysis II, and so on.

B. PROCEDURES

If you participate, you will be asked to engage in a task-based interview that takes approximately 2 hours. During the interview, you will be asked to solve some mathematics problems like the ones you encountered in Real Analysis I or II. While you solve the problems, you will be asked to "think aloud": i.e., you will be asked to describe what you are doing and why. No preparation is required.

C. RISKS AND BENEFITS

You will not encounter risks by participating in this project. Your refusal or acceptance to participate will have no impact on your grade in any course, and your teachers will not be informed of what goes on in your interview. Engaging in the interview could be beneficial in that it may help you to review some important mathematics concepts.

[44] This was the temporary title given to the study at the time when the interviews were first taking place. The research (and its title) evolved as we continued our data collection and analysis. In particular, the consent form reflects the fact that we had initially hoped to interview successful RA I students *and* successful RA II students. Due to the time constraints of completing a doctoral program, we had to narrow our focus to RA I.

D. CONFIDENTIALITY

We will gather the following information as part of this research:

- your name and contact information;
- if you agree, your final grade in Real Analysis I or II (whichever you took last semester);
- the signed consent form;
- the audio recording and transcripts of the 2-hour interview;
- anything you write down while solving problems during the 2-hour interview.

We will protect the information you provide by storing it in a safe place. The information gathered from you will be identified by a code. Solely the researcher, Laura Broley, will have a list that links the code to your name. We will only use the coded information for the purposes of the research described in this form. When the results of the research are published, it will not be possible to identify you.

Any non-published information will be destroyed five years after the end of the study.

F. CONDITIONS OF PARTICIPATION

You do not have to participate in this research. If you do participate in the interview, you can stop it at any time. After the interview, you have two full calendar days to withdraw your information from the research. If you decide that you do not want us to use your information, your choice will be respected. There are no negative consequences for not participating, stopping in the middle, or asking us not to use your information.

G. PARTICIPANT'S DECLARATION

I have read and understood this form. I have had the chance to ask questions and any questions have been answered. I agree to participate in this research under the conditions described.

NAME (please print) ____

SIGNATURE ____

DATE ____

If you have questions about the scientific or scholarly aspects of this research, please contact the researcher. Their contact information is on the first page of this consent form. You may also contact their faculty supervisor.

If you have concerns about ethical issues in this research, please contact the Manager, Research Ethics, Concordia University, 514.848.2424 ex. 7481 or oor.ethics@concordia.ca.

Appendix B: Models for Task 1

A Model of Practices to be Learned

T_1: Prove that a given real number c involving a root is not a rational number.

τ_{11}: Assume $c = m/n$ where m and n are integers without common factors. Show m and n have a common factor.

Commonly tested was $c = \sqrt{p}$ for a specified prime number p:

Assume $\sqrt{p} = \frac{m}{n}$ where m and n are integers without common factors. Then $p = m^2/n^2$ or $pn^2 = m^2$. So m^2 is divisible by p and m is also divisible by p. $m = pk$. We obtain $pn^2 = p^2k^2$ or $n^2 = pk^2$. So n^2 is divisible by p and n is also divisible by p. We showed that m and n have a common factor p. Contradiction. Thus, $\sqrt{p}$ is not a rational number. ■

τ_{12}: Assume $c = m/n$ where m and n are natural numbers. Use algebra to show that a known irrational number is equal to a rational number. Then apply τ_{11} to the irrational number.

Commonly tested was $c = \sqrt{p} + \sqrt{q}$ for specified distinct prime numbers p and q:

Assume $\sqrt{p} + \sqrt{q} = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Then $p + 2\sqrt{pq} + q = \frac{m^2}{n^2}$, or $\sqrt{pq} = \frac{m^2/n^2 - p - q}{2}$. Thus, we showed that $\sqrt{pq}$ is a rational number. Let $\sqrt{pq} = r/s$ where r, s are natural numbers without common divisors. Then $pq = r^2/s^2$ or $pqs^2 = r^2$ and r^2 is divisible by p. Thus, r is divisible by p and $r = pk$ where k is a natural number. We have $pqs^2 = p^2k^2$ or $qs^2 = pk^2$. Thus, s^2 is divisible by p and then s is also divisible by p, which contradicts the assumption that they have no common divisors. This contradiction proves that $\sqrt{p} + \sqrt{q}$ is not a rational number. ■

θ_1: "By contradiction." Use the Table Method to prove the principle of contradiction, i.e., $[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$.

α	β	$\gamma \wedge \neg\gamma$	$\alpha \wedge \neg\beta$	$(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)$	$\alpha \Rightarrow \beta$	$[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$
0	1	0	0	1	1	1
0	0	0	0	1	1	1
1	1	0	0	1	1	1
1	0	0	1	0	0	1

Thus, the statement is true. ■

Λ_1: (m is an integer and m^2 is even) $\Rightarrow$ (m is even).

Proof, using the principle of contradiction: i.e., $[(\alpha \wedge \neg\beta) \Rightarrow (\gamma \wedge \neg\gamma)] \Rightarrow (\alpha \Rightarrow \beta)$

Let $\alpha =$ "m^2 is even" and $\beta =$ "m is even."
We assume α and $\neg\beta$. i.e., m^2 is even and m is odd.
If m is odd, then $m = 2k + 1$ and

$$m^2 = 4k^2 + 4k + 1 = 2(2k + 2)k + 1$$

So m^2 is odd. If we set $\gamma =$ "m^2 is even," we have γ and $\neg\gamma$. We have a contradiction. ■

Proof, using the principle of contraposition: i.e., $(\alpha \Rightarrow \beta) \Leftrightarrow (\neg\beta \Rightarrow \neg\alpha)$

Let $\alpha =$ "m^2 is even" and $\beta =$ "m is even."
Then $\neg\beta =$ "m is odd" and $\neg\alpha =$ "m^2 is odd."
If m is odd, then $m = 2k + 1$ and

$$m^2 = 4k^2 + 4k + 1 = 2(2k + 2)k + 1$$

So m^2 is odd. We proved $\neg\alpha$. The theorem has been proved by contraposition. ■

$\widetilde{T_1}$: Judge the truth of statements about the rationality or irrationality of products and sums of rational and irrational numbers.
$\widetilde{\tau_1}$: If judged false, give a simple counterexample. If judged true, prove it. E.g., using τ_{12}.

Models of Practices Actually Learned

t_1: Prove that $\sqrt{8} \notin \mathbb{Q}$. The practices for solving t_1:

Type of Task	Technique	Considered by	Example(s) of Discourse
Prove that $\sqrt{c} \notin \mathbb{Q}$ using the method from class; i.e., the same proof we used for $\sqrt{2}$.	τ_{11} In this case: Assume $\sqrt{8} = m/n$ where m and n are integers without common factors. Show m and n have a common factor; in particular, show m and n are even.	S1*, S2**, S4*, S5*, S6*, S8, S10, S11*, S12, S14**, S15*	It's the method from class. I'd remember how we did $\sqrt{2}$ and do it exactly the same way. If m is even and n is even, then they share a common factor, and the assumption doesn't hold. So by contradiction, it would be irrational. We're assuming that it's rational. It has to be irrational if it's not rational. (S1) To verify whether it is a rational or an irrational, it is about a fraction of two integers. (S4) It was such an easy common question. I would just proceed on following the steps mostly. (S5) My thought process is based on a methodology I've seen, that I just approach blindly for any root of any number. It's a way of proving that the teacher has used that I use to regurgitate back to him. I'm not thinking as I'm going. I'm recalling a solution I've seen. (S12) You could have given me $\sqrt{4}$ and I would have started out this way. (S10) It's out of habit. (At the step $8n^2 = m^2$:) I want to write $8\|m^2 \Rightarrow 8\|m$. But 8 is not a prime number. You need a prime number for you to be able to say that m is also divisible by 8 (see the next table). (S11)
Prove that $qr \notin \mathbb{Q}$ for $q \neq 0$ rational and r an irrational for which I know how to prove $r \notin \mathbb{Q}$.	τ_{12} In this case: Assume $\sqrt{8} = 2\sqrt{2} = \frac{m}{n}$, $m, n \in \mathbb{N}$. Then $\sqrt{2} = m/(2n)$ is rational. Show $\sqrt{2} \notin \mathbb{Q}$ using τ_{11}.	S7, S10, S14	(When compared to the approach in the above row:) This is a lot easier and more convincing. But both work. (S10) (When compared to the approach in the above row:) As someone who leans towards the pure math side of things, I prefer a more elegant and creative proof. Rather than just going: I saw how to do this problem in class and I'm just going to robot proof it to get through. (S14) I try to remember visually what we learned in class. I remember there were a lot of examples in sample exams where you had to add two numbers and prove that they were irrational. I tried to apply it with what I knew about multiplication. If you divide a rational by a rational, you get a rational. (S7)

	Show that $r \notin \mathbb{Q}$. In this case: $\sqrt{8} = 2\sqrt{2}$. Show $\sqrt{2} \notin \mathbb{Q}$ using τ_{11}.	S2, S3, S7*, S9*, S13, S15**	(When compared to the approach in the first row:) Maybe this is simpler. But I don't remember all the details of the result about the multiplication of a rational and an irrational. And usually teachers emphasize more the other approach. (S15) Can you tell me if this is right or wrong? I know that any rational times a rational is rational. But I forgot some other knowledge I learned last semester. I'm not sure if a rational times an irrational is irrational. (She suggests to change her approach to the one in the first row.) (S2) When we simplify $\sqrt{8}$ to $2\sqrt{2}$, I remember proving that $\sqrt{2}$ was irrational. I tried to recreate that. If you multiply an irrational number with another irrational number, it doesn't necessarily mean it's going to be irrational. And if you multiply an irrational with a rational, I guess the same thing applies. (S9) A rational multiplied by an irrational is irrational. That can be proved by contradiction. (When seeking a proof, he demonstrates the approach in the second row). (S7) I'm taking as a given: If $a \in \mathbb{Q}$ and $b \in \mathbb{R}\backslash\mathbb{Q}$, then $ab \in \mathbb{R}\backslash\mathbb{Q}$. (When asked how he knows this, he tries and succeeds at sketching a proof). (S3) We know that any rational number multiplied by an irrational number, unless it's zero, it's an irrational. This is very easy to prove. Let's say qr is rational, where r is irrational and q is rational. So $qr = \frac{a}{b} \Rightarrow r = \frac{a}{bq}$; we assume $q \neq 0$ so we can write this. But we cannot write irrationals like this. So the first assumption is wrong. $qr \notin \mathbb{Q}$. ■ (S13)

* indicates that the student was unable to fully implement the technique (i.e., they got stuck). ** indicates that the student hinted at the technique but did not try to (fully) implement it.

The practice actually learned by S11 in RA I, which could not be used to solve t_1:

Type of Task	Technique	Discourse
Prove that $\sqrt{p} \notin \mathbb{Q}$ for p prime.	Say $\sqrt{p}$ is rational. Then you can write it as $\sqrt{p} = \frac{m}{n}$, where m and n have no common factors. Then $pn^2 = m^2$. So $p\|m^2 \Rightarrow p\|m$. $\Rightarrow pn^2 = p^2k^2$ $\Leftrightarrow n^2 = pk^2$. So $p\|n^2 \Rightarrow p\|n$. Contradiction to m and n having no common factors. ■	"Honestly, [I chose this route] out of habit. We saw these problems in class. I literally just finished the final, so… [-] I can't tell you why [it works]. But I know that's how it's done. [-] Unfortunately. [-] Because I like understanding sometimes what I'm doing. But like, I don't know, I guess if you assume that it is [rational]… If you assume the contrary and then you can show a contradiction, like it works that way."

The additional practice developed by S3 when trying to solve t_1 without using $\sqrt{2} \notin \mathbb{Q}$:

Type of Task	Technique	Discourse
Prove that $\sqrt{c} \notin \mathbb{Q}$ using the method from class; i.e., the same proof we used for $\sqrt{2}$.	Let $\sqrt{8} = \frac{m}{n} \Leftrightarrow 8n^2 = m^2$. $\Rightarrow 2\|m^2 \Rightarrow 2\|m \Rightarrow m = 2k$ $\Rightarrow 8n^2 = 4k^2 \Rightarrow 2n^2 = k^2$. But then, this means that $\sqrt{2} \in \mathbb{Q}$. And we know that's not true. ■	I was trying to prove that $\sqrt{8} \notin \mathbb{Q}$ without using the fact that $\sqrt{2} \notin \mathbb{Q}$. During the proof, I was trying to show that n is also even. But then I noticed that I could use $\sqrt{2} \notin \mathbb{Q}$. There might be a way to show $\sqrt{8} \notin \mathbb{Q}$ without using $\sqrt{2} \notin \mathbb{Q}$. I do not know it yet.

$\widehat{t_1}$: Convince someone that $\sqrt{8} \notin \mathbb{Q}$. The practices for solving $\widehat{t_1}$:

Type of Task	Technique	Considered by	Example(s) of Discourse
Argue that $\sqrt{c} \notin \mathbb{Q}$.	Simplify the number as much as possible and find that the square root symbol cannot be eliminated.	S2, S5, S9, S15	When you look at $\sqrt{8}$, you know it's irrational. I would feel that it's irrational. A square root is usually irrational. And when you write $\sqrt{8}$ as $2\sqrt{2}$, it still has a square root of two. (S5) I just know that $\sqrt{8} = 2\sqrt{2}$ is irrational. The proof is necessary to get full marks. (S2, S9) It's kind of evident that $\sqrt{8} = 2\sqrt{2} \notin \mathbb{Q}$. But we must be formal. (S15)
	Note that c is not a square number.	S4, S6, S12, S14	It's an instinct. I wouldn't know if it's right or wrong. (S6) It's something I don't require proof for. It's an intuition I've built up through experience. (S12) It's a theorem I've seen proved in some course: the square root of a non-perfect square is irrational. (S14)
	Let $\sqrt{c} = \frac{m}{n}$ (i.e., $\sqrt{c} \in \mathbb{Q}$). Then $c \cdot n \cdot n = m \cdot m$. Write c as its prime factorization. Notice that one factor will occur an odd number of times on the left and an even number of times on the right. This is not possible.	S12	The equality is not possible by the uniqueness of prime factorization.
Argue that $r \notin \mathbb{Q}$.	Type the number on a calculator and press the fraction button to find that the number cannot be expressed as a fraction.	S11	A rational number is something that you can write in m/n form. If you can figure out that you can't, then that's pretty much it. You don't need to see the whole proof. But if you tell me to prove it, then I would prove it.

	Show that the number can be rewritten as a power of a known irrational number.	S4	Any power of an irrational number is also irrational.
	Simplify the number until you get a known rational number times a known irrational number.	S1, S3, S7, S8, S10, S13	I know that an irrational times a rational stays irrational. But I don't know how to prove it. (S1) A rational multiplied by an irrational is irrational. That can be proved by contradiction. (S7) We know that any rational number multiplied by an irrational number, unless it's zero, it's an irrational. This is very easy to prove. (S13) (For the statement $qr \notin \mathbb{Q}$ for $q \in \mathbb{Q}\backslash\{0\}$ and $r \in \mathbb{R}\backslash\mathbb{Q}$: • S7 and S10 proved it in the specific case $q = 2, r = \sqrt{2}$; and • S3 and S13 proved it in general. See the above table corresponding to the practices for solving t_1.)

Students' solutions relied on a divisibility statement. Either:
(1) If $m \in \mathbb{Z}$, p is prime, and $p|m^2$, then $p|m$; or (2) If $m \in \mathbb{Z}$ and m^2 is even, then m is even.

The discourses for justifying the statement:

Student(s)	Statement	Discourse
S11	(1)	I remember my professor mentioning something. Isn't it a theorem?
S8	(2)	Maybe the proof is by induction: You could start with 2 and note that $2^2 = 4$ is even… I can't remember the proof. But I remember seeing the proof in class. So I know I can use it.
S10	(2)	I remember doing the proof of the statement enough in the recent past to have confidence that it is true. I would still like to prove it to make my solution more complete.
S2, S7	(2)	Because an even times an even is even.
S12	(2)	Either m is odd or m is even. If m is odd, m^2 is odd. If m is even, m^2 is even. That's a pattern I've seen enough to recognize and identify. So, going backwards, we have m^2 is even, then we know m is even.
S15	(2)	If m is odd, $m = 2k + 1$ for $k \in \mathbb{Z}$ and $m^2 = 2(2k^2 + 2k) + 1$ is odd. If m is even, $m = 2k$ for $k \in \mathbb{Z}$ and $m^2 = 2(2k^2)$ is even. Hence, if m^2 is even, m is also even. ■
S13, S14	(2)	Λ_1: Assume m^2 is even and m is odd. Then $m^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1$ is odd. Contradiction. ■
S3	(1)	If a prime divides a product, it must divide one of the factors. Otherwise, you contradict the Fundamental Theorem of Numbers.

S1, S4, S5, S6, and S9 are not included because they did not recall or state a divisibility statement like (1) or (2), and hence they were not prompted to provide a justification.

Appendix C: Models for Task 2

A Model of Practices to be Learned

<table>
<tr><td colspan="3">$\dot{T_2}$: Prove that $g(x)$ has exactly n fixed points on a domain D. $\Rightarrow$ (let $h(x) = x$)
$\widetilde{T_2}$: Prove that $g(x) = h(x)$ has exactly n solutions on a domain D. $\Leftrightarrow$
$\widehat{T_2}$: Prove that g and h intersect exactly n times on a domain D $\Rightarrow$ (let $f(x) = g(x) - h(x)$)</td></tr>
<tr><td colspan="3">T_2: Prove that a function $f(x)$ has exactly n zeros on a domain D.
Typically: $n \in \mathbb{N}$ is small (e.g., $1, 2, 3$, or 4) and D is a specified interval $[a, b]$ with $a, b \in \mathbb{Z}$.
$T_2 = T_{2_a} \wedge T_{2_b}$</td></tr>
<tr><td colspan="3">T_{2_a}: Prove that $f(x)$ has at least n zeros on D.</td></tr>
<tr><td colspan="3">τ_{2_a}: Find n sign changes of $f(x)$ on D.
Typically: calculate $f(a)$ and $f(b)$, and maybe $f(c)$ for c equal to integers in (a, b) or midpoints between integers.</td></tr>
<tr><td colspan="3">θ_{2_a}: “By the Intermediate Value Theorem.”</td></tr>
<tr><td colspan="3">T_{2_b}: Prove that $f(x)$ has at most n zeros on D.</td></tr>
<tr><td>$\tau_{2_{b_1}}$: Show that f' is strictly positive (or negative) on n intervals I_i that form a partition of D.
Illustrated for $n = 1$:
Show that f' is strictly positive (or negative) on D.</td><td>$\tau_{2_{b_2}}$: Assume that f has $n+1$ zeros and derive a contradiction. More specifically, argue that f' has n zeros, f'' has $n-1$ zeros, …, and f^n has 1 zero; and show f^n has no zeros.
Illustrated for $n = 2$:
Assume that f has 3 zeros, whereby f'' has 1. Show that f'' has no zeros.</td><td>$\tau_{2_{b_3}}$: Illustrated for f a polynomial:
Note that the degree (or order) of f is n.</td></tr>
<tr><td>$\theta_{2_{b_1}}$: “If $f' > 0$ (or < 0) on an interval I, then f is strictly increasing (or decreasing) on I and can cross the line $y = 0$ at at most one point.”</td><td>$\theta_{2_{b_2}}$: “By Rolle’s Theorem and by contradiction.”</td><td>$\theta_{2_{b_3}}$: “If f is a polynomial of order n, then f has at most n zeros.”</td></tr>
</table>

Models of Practices Actually Learned

t_2: Prove that $f(x) = e^x - 100(x-1)(2-x)$ has exactly two zeros.

Students' identification of t_2 with a type of task:

Student(s)	Type of Task
S1, S2, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15	Prove that a function has zeros using specific theorems (e.g., "Intermediate Value Theorem," "Mean Value Theorem," "Rolle's Theorem," "the theorem from class") or tools (e.g., the derivative).
S1, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14	T_2: Prove that a function $f(x)$ has exactly n zeros. $\Leftrightarrow T_{2_a}$: Prove that $f(x)$ has at least n zeros. $\wedge\ T_{2_b}$: Prove that $f(x)$ has at most n zeros.
S2	T_2: Prove that a function $f(x)$ has exactly n zeros. $\Leftrightarrow T_2'$: Prove that $f'(x)$ has exactly $n-1$ zeros. $\Leftrightarrow {T_{2_a}}'$: Prove that $f'(x)$ has at least $n-1$ zeros. $\wedge\ {T_{2_b}}'$: Prove that $f'(x)$ has at most $n-1$ zeros.
S3	T_2: Prove that a function $f(x) = g(x) - h(x)$ has exactly n zeros. $\Leftrightarrow \widetilde{T_2}$: Prove that $g(x) = h(x)$ has exactly n solutions. $\Leftrightarrow \widehat{T_2}$: Prove that the graphs of $g(x)$ and $h(x)$ intersect exactly n times.

t_{2_a}: Prove that $f(x) = e^x - 100(x-1)(2-x)$ has at least two zeros on $\mathbb{R}$.

The technique and discourses for solving t_{2_a}:

Technique	Considered by	Example(s) of Discourse
τ_{2_a}: Find two sign changes of $f(x)$ on $\mathbb{R}$. Cite the theorem used in class.	S1*, S4*, S5, S6, S7, S8, S9, S10*, S11*, S12, S13, S14*	Honestly, that's the only way I could think of to solve this problem. (S5) It's just having repeated it so often, on assignments, in class, … (S4) If e^x wasn't there, it'd be pretty easy to find the two zeros. But now we have to do the non-high school way. I know the Intermediate Value Theorem (IVT) is applicable in this situation. Why do I know? Well, I'm cheating. Cause I know that that's how we used to solve it. Cause we did it in class. (He does not mention the continuity condition.) (S1) (Referring to the statement of Task 2:) "Show that it has zeros" is IVT for sure. Like the fact that, ok, if it's a continuous function, and, you know, so you plug in some values, you get a negative, then positive, then negative. So it must cross the x-axis at some point. So it does have a zero. (S6) I understand the IVT works like that. If I find a value of $f(x)$ that's positive and one that's negative, I knew that I could find a zero. That's why I felt like using that. (He mentions the continuity condition and that f is continuous because it's an exponential and a polynomial, which are continuous.) (S8) We could find when does zero equal $f(x)$. And it would be hard because we have the e^x. It's not as simple as just isolating the x. So in this case, we have to use one of those theorems we saw in RA I. (He checks the continuity condition by calculating the derivative of $f(x)$.) (S9)

* indicates that the student was unable to fully implement the technique (i.e., they got stuck).

$[T_{2_a}, \tau_{2_a}]$: [Prove that $f(x)$ has at least n zeros., Find n sign changes of $f(x)$.]

Students' implementation of τ_{2_a} can be seen as solving a subtask:
Find two sign changes of $f(x) = e^x - 100(x-1)(2-x)$ on $\mathbb{R}$.

The practices for solving this task:

Type of Task	Technique	Considered by	Example(s) of Discourse
Find n sign changes of $f(x)$ on $\mathbb{R}$.	Calculate $\lim_{x\to\pm\infty} f(x)$.	S1, S7, S9*, S10*, S13	It's usually an easy way to start, to get an idea of your function. (S7)
	Plug in x values "at random." Maybe try different kinds of x values, or x values for which $f(x)$ is simple to calculate.	S1, S4, S5, S6, S7, S8, S9, S10, S11, S13, S14	I know there's a better way. But I forgot the better way. (S1) I haven't done something like this in a little while, so… I don't know. (S14)
	Pay attention to how the value of $f(x)$ is changing as x changes.	S4, S5, S6	If f is a continuous function and I see that $f(x)$ is getting closer to zero as I change x, then I'm getting closer to finding a sign change. (S6)
	Study the sign of $f'(x)$ for particular x.	S10, S11*	(After finding positive values for $f(x)$:) If you take the derivative at different points, you can see if the slope of the tangent line is negative. Then you know the function is decreasing at that point. And then you might want to check the intervals around that. (S11)
Find n sign changes of $f(x) = g(x) - h(x)$ on $\mathbb{R}$.	Compare the growth of $g(x)$ and $h(x)$.	S8, S11*, S14*	If g grows much quicker than h, then the sign changes will occur for small values of x. In this case, $g(x) = e^x$ grows much quicker than any polynomial $h(x)$. (S8)
	Compare the signs of $g(x)$ and $h(x)$.	S4, S5, S9, S12	If g is always positive, then sign changes will occur where h is negative. In this case, $g(x) = e^x$ is always positive. We need to look for where $h(x) = -100(x-1)(2-x)$ is negative: • Expanding it out to $100x^2 - 300x + 200$ shows that x needs to be positive. (S4) • (Looking at the expression for h:) If x is more than 1 and less than 2, the negative signs won't cancel out. (S5) • This is an upward-facing parabola with roots at 1 and 2, and minimum (or most negative point) at 1.5. (S9, S12)
	Graph g and h.	S7	Sign changes will occur where the graphs of g and h cross. In this case, the graphs are known: It is just e^x and a downward-facing parabola with roots at 1 and 2, and maximum at 1.5.

* indicates that the student hinted at the technique but did not implement it. S2, S3, and S15 are not included for analysis purposes because they were not solving the same task and the techniques they exhibited for finding sign changes are all represented in the table.

The practice actually learned by S11 in RA I, which could not be used to solve the task:
Find two sign changes of $f(x) = e^x - 100(x-1)(2-x)$ on ℝ.

Type of Task	Technique	Discourse
Find n sign changes of $f(x)$ on $[a,b]$, $a,b \in \mathbb{Z}$.	Calculate $f(a)$, $f(a+1)$, …, $f(b-1)$, $f(b)$ and count the number of sign changes.	S11: "If you just tell me an interval, I can just plug in values at that point. [-] Like if you tell me, ok, like it's not this function, it's another function, and you tell me, ok, it's zero to five [he is writing [0,5]]. Then at that point you can just plug in the values and just see how many zeros you have. No?" L: "Ok. Which values do you mean? Do you mean like plug in zero and five?" S11: "Yeah. Exactly. Zero, one, two, three, four, five. In this function. And you'll see which one alternates between negative and positive. And you'll figure out how many zeros you have. Yeah, the hard part is not having the interval."

The practice learned by S10 before RA I, which he no longer wanted to use to solve the task:
Find two sign changes of $f(x) = e^x - 100(x-1)(2-x)$ on ℝ.

Type of Task	Technique	Discourse
Find n sign changes of $f(x)$ on $[a,b]$.	Set up a computer program (e.g., in Mathematica) that checks the sign of $f(i)$ for i in a list of values in $[a,b]$ and counts the number of times the sign changes.	Suppose the function is continuous and the program can check however many values for i we specify. Then it's a powerful and easy method of computation that should be sufficient for solving the task.

t_{2_b}: Prove that $f(x) = e^x - 100(x-1)(2-x)$ has at most two zeros on ℝ.

The techniques and discourses for solving t_{2_b}:

Technique	Considered by	Example(s) of Discourse
A variation of $\tau_{2_{b_3}}$: Note that the degree of the polynomial in f is two.	S6**, S7**, S11, S15**	The function has a polynomial of degree two. So I know it has at most two zeros. Isn't that a theorem? (S11) You go based off the highest degree of the polynomial you get. That's the max number of zeros you can get. But usually we would only get some easy polynomial. The fact that there's an e^x is something I'm not used to. (S6) I think it's the Fundamental Theorem of Algebra. Also: the polynomial has two zeros and the exponential is strictly positive (so maybe adding e^x does not change the number of zeros). But I want to use a theorem from RA I (like Rolle's Theorem) because it seems more reliable than just saying this function has two zeros because we have the polynomial. I hesitate on the method because it's far in my head at this point. (S15) I have the feeling that e^x would just shift the graph of the polynomial, which has two zeros. But I am not sure. (S7)

A variation of $\tau_{2_{b_2}}$: Show that f' has exactly one zero.	S1, S7, S13	There's a theorem that says that if the derivative has exactly one zero, then the original has at most two zeros. (S1) It's by Rolle's Theorem. (S7) If f has two zeros, then f' has one zero. (S13)
A variation of $\tau_{2_{b_1}}$: Find x_m such that $f'(x_m) = 0$ (the minimum of f). Argue that $f'(x) < 0$ when $x < x_m$ (i.e., on $(-\infty, x_m)$) and $f'(x) > 0$ when $x > x_m$ (i.e., on (x_m, ∞)).	S12*, S14*	If $f' > 0$ (or < 0) on an interval I, then f is strictly increasing (or decreasing) on I and can cross the line $y = 0$ at at most one point. So $f' < 0$ on $(-\infty, x_m)$ and $f' > 0$ on (x_m, ∞) means that f is decreasing up to x_m and then increasing; it crosses the x-axis at most twice. E.g., 
Picture the graph of f using information about f, f', and f''.	S8, S9, S10**	There's clearly a very straightforward way to do this if I give myself rules of Calculus and kind of a visual representation of this function. (S10) $f'' > 0$ means that f' is always increasing ⇒ f' crosses the x-axis once ⇒ f has one min or max. So the graph of f looks like one of the two graphs below: (S8)  $f'' > 0$ means the slope of the tangent is always increasing ⇒ f could go from the positives into the negatives; however if it goes back into the positives at some point (like in the graph on the left above), it can't go back into the negatives, because that would mean that the slope would have to start decreasing. (S9)

* indicates that the student was unable to fully implement the technique (i.e., they got stuck). ** indicates that the student hinted at the technique but did not choose to implement it. S4 and S5 are not included because they did not clearly consider a technique. S5 said she did not know how to solve the task. S4 recalled needing to use the derivative but did not know how. S2 and S3 are not included because they solved a different task (see the table about students' identification of t_2 with a type of task at the beginning of this section). S2's approach resembled that of S1, S7, and S13. S3's approach resembled that of S8 and S9.

Appendix D: Models for Task 3

A Model of Practices to be Learned

$T_{3_{a_1}}$: Show that a specified recursively defined sequence is convergent.	$T_{3_{a_2}}$: Prove that the supremum (or infimum) of a specified set, often of the form $A = \{x_n : n = 1, 2, \ldots\}$, is M (or m)
$\tau_{3_{a_1}}$: Show that the sequence is increasing (or decreasing) and bounded above (or below).	$\tau_{3_{a_2}}$: Show that $(x_n)_{n \in \mathbb{N}}$ is increasing (or decreasing) and converges to M (or m).

θ_3: "By the Monotone Sequence Theorem (MST)."

Λ_3 : If $(x_n)_{n \in \mathbb{N}}$ is an increasing (or decreasing) bounded above (or below) sequence and $A = \{x_n : n \in \mathbb{N}\}$, then $\lim\limits_{n \to \infty} x_n = \sup A$ (or $\inf A$).

Proof: Let $(x_n)_{n \in \mathbb{N}}$ be increasing and bounded above.
Then $A = \{x_1, x_2, x_3, \ldots\}$ is non-empty and bounded above.
So by the Completeness Axiom, $\sup A = M$ is a finite real number.
We will show that $x_n \to M$.
We have to show that $\forall \epsilon > 0, \exists N \geq 1, \forall n \geq N, |x_n - M| < \epsilon$.
Let us take an $\epsilon > 0$.
Then we can find an element of A, say x_N, such that $x_N > M - \epsilon$.
Since $(x_n)_{n \in \mathbb{N}}$ is increasing, $\forall n \geq N, x_n \geq x_N > M - \epsilon$.
On the other hand, all x_n's satisfy $x_n \leq M$.
Thus, for $n \geq N$, we have $M - \epsilon < x_n < M + \epsilon$.
We proved that $x_n \to M$. ■

(The proof for $(x_n)_{n \in \mathbb{N}}$ decreasing and bounded below is essentially the same.)

T_{3_b}: Give examples of sequences that satisfy certain properties.

τ_{3_b}: No technique illustrated.
Write down the example and some indication that the properties are satisfied.

Models of Practices Actually Learned

t_{3_a}: Under what conditions is $\lim\limits_{n \to \infty} x_n = \sup A$, where $A = \{x_n \in \mathbb{R} : n = 1, 2, \ldots\}$?

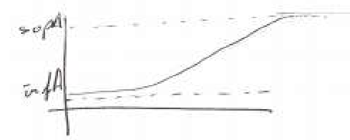
Students' identification of t_{3_a} with a type of task:

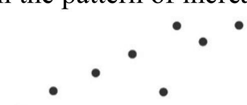
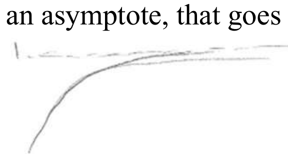
Student(s)	Type of Task
S2, S4, S5, S6, S11	I do not know how to solve this type of task.
S2*, S3, S4*, S6*, S7, S9, S11*, S14, S15	Pose some conditions so that a certain property holds. (Unclear if students aimed for sufficient and/or necessary conditions.)

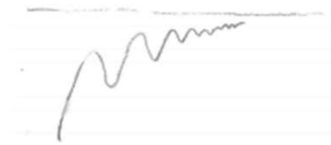
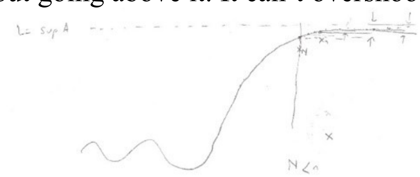
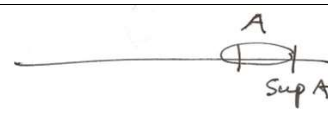
S1, S8, S12, S13	Devise sufficient and necessary conditions for a certain property.
S10	Determine the formal conditions that characterize a certain property.

* indicates that the student required significant prompting from the interviewer.

The conditions and discourses for solving t_{3_a}:

<table>
<tr><th>Condition</th><th>Considered by</th><th>Example(s) of Discourse</th></tr>
<tr><td>$(x_n)_{n\in\mathbb{N}}$ increasing.</td><td>S1*, S2, S3, S4, S7, S8*, S9, S12*, S13*, S14, S15</td><td>I know the Theorem of Convergence (θ_3). (S15)
If the sequence does not increase, say it's decreasing or cyclical, it might not go to the sup. (S2, S3, S7, S8, S15)
It doesn't say I should prove it. (S2)
E.g., $x_n = \frac{n}{n+1}$. I know that showing an example is not enough. I need to prove it. But I don't want to prove it. It would take too much energy. And I need to go open a book to see the rigorous definition of a sup. (S3)
An increasing sequence means it will crawl towards the supremum / grow continuously until it reaches its maximal value. (S3, S4, S9)
The intuition behind it can be explained by a graph: (S14)


I am picturing sequences. But I would try to prove it. (He tries to prove $(x_n)_{n\in\mathbb{N}}$ increasing $\Leftrightarrow \lim_{n\to\infty} x_n = \sup\{x_n \in \mathbb{R} : n = 1, 2, ...\}$. He is unable to construct a coherent proof. When trying to prove the "$\Leftarrow$" direction, he changes his condition as shown in another row of this table, below.) (S1)
I can prove it. ($\Rightarrow$) $\sup\{A\} = M \Rightarrow \forall\epsilon > 0, \exists n_0 \ni M - \epsilon < x_{n_0}$.
$\{x_n\}$ is increasing $\Rightarrow\ \forall n > n_0,\ M - \epsilon < x_n \Leftrightarrow M - x_n < \epsilon$.
It means that if these situations exist, then $(x_n)_{n\in\mathbb{N}}$ goes to $\sup A$. (When trying to prove the "$\Leftarrow$" direction, she changes her condition as shown in another row of this table, below.) (S13)</td></tr>
<tr><td>$(x_n)_{n\in\mathbb{N}}$ eventually increasing.</td><td>S13</td><td rowspan="2">There could be breaks in the pattern of increasing: (S8, S13) e.g.,

(Having already showed "$\Rightarrow$" as above, for "$\Leftarrow$":) I would try to show that if $(x_n)_{n\in\mathbb{N}}$ is not eventually increasing, then $(x_n)_{n\in\mathbb{N}}$ does not converge to the supremum. The idea is to find a subsequence that is smaller than the neighbours, that converges to a number smaller than the supremum. (S13)</td></tr>
<tr><td>$(x_n)_{n\in\mathbb{N}}$ not exactly increasing.</td><td>S8</td></tr>
<tr><td>No x_n can be greater than the sup / asymptote.</td><td>S1, S12</td><td rowspan="2">Any function that has an asymptote, that goes like this: (S6)

The sequence could drop down and come back up. So, it doesn't have to be increasing. (S1)</td></tr>
<tr><td>$(x_n)_{n\in\mathbb{N}}$ has an asymptote above</td><td>S6, S12</td></tr>
</table>

<table>
<tr><td>that it approaches from below.</td><td></td><td>The magnitude of oscillation of the sequence could decrease gradually and it could still reach the supremum: e.g.,

Saying it's increasing seems like an arbitrary choice. Asymptote means that as this function continues along infinity, it will be getting closer and closer to the asymptote. For it to be the supremum, it can never go beyond it. So it can have any kind of behaviour at the beginning. But eventually it needs to flatten out around the asymptote without going above it. It can't overshoot. E.g., (S12)
</td></tr>
<tr><td>A or $(x_n)_{n\in\mathbb{N}}$ is bounded above.</td><td>S1*, S3, S7*, S8*, S9, S11, S13*, S15</td><td rowspan="2">If it's not bounded above, there's no sup. That's what I recall. (S9)
If it's not bounded above (or the limit does not exist, or the sequence is not convergent), it will go to infinity. But $\sup A$ is a bound / a finite real number. It cannot be infinite. (S2, S3, S11, S15)
Actually: It does not need to be bounded above. $\sup A$ could be infinite. (S1, S7, S8, S13).
E.g., $x_n = n$. (S7, S8)
But I would need to check my notes. (S7)
I needed the bounded condition to do my proof because I needed the limit to exist (see S13's proof in the row corresponding to the "increasing" condition, above). But $\sup A$ means all the elements of A are less than it. So I need to decide the ∞ case separately. (S13)
Obviously if $\lim_{n\to\infty} x_n = \infty$, then the sequence is going to the "greatest number." I.e., $\lim_{n\to\infty} x_n = \sup A$. (S1)</td></tr>
<tr><td>$\lim_{n\to\infty} x_n$ exists / $(x_n)_{n\in\mathbb{N}}$ is convergent.</td><td>S2, S11, S14</td></tr>
<tr><td>$\forall\epsilon > 0, \exists\, N \in \mathbb{N}$, $\sup A - \epsilon < x_N \le \sup A$.</td><td>S10*, S11**</td><td rowspan="3">

For any epsilon greater than zero, there's going to have to be an element of A that's greater than the supremum minus epsilon, but it's also less than or equal to the supremum. And I think the second condition is that the supremum is an upper bound of the set. I'm wondering if we have to hold the N constant and then say that $\sup A - \epsilon < x_n \le \sup A$ is actually true for all natural numbers larger and equal to it. I would need this if the sequence doesn't have a limit; but it does have a limit. (While solving Task 4, he seems to accept adding this other condition.) (S10)</td></tr>
<tr><td>$\forall n \in \mathbb{N}$, $x_n \le \sup A$.</td><td>S10</td></tr>
<tr><td>$\forall\epsilon > 0,\ \exists N \in \mathbb{N}$, $\forall n \ge N, \sup A - \epsilon < x_n \le \sup A$.</td><td>S10, S11**, S12**</td></tr>
<tr><td>All subsequences of $(x_n)_{n\in\mathbb{N}}$ go to $\sup A$.</td><td>S4*</td><td>It's a theorem from class. It works, but it's not practical to check every subsequence. My professor said it's useful for showing that a sequence does not converge.</td></tr>
</table>

* indicates that the student did not choose the condition. ** indicates that the student seemed to be considering the condition but could not formulate it exactly.

t_{3_b}: Give an example of $A = \{x_n \in \mathbb{R} : n = 1, 2, \dots\}$ where the limit $\lim_{n\to\infty} x_n$ exists but does not equal $\sup A$.

The examples and discourses for solving t_{3_b}:

Example	Considered by	Example(s) of Discourse
$\sin(n)$	S2	(She does not know if the limit exists and says she would use Google to check.) Do I need to find an example? Because I don't think I can.
$\frac{1}{n} + \sup A$	S5	It's one example that we used a lot. But that's not what we're asking for here. We want something that does not equal $\sup A$. I honestly don't know how to do it.
$0, \frac{1}{2}, 2, 0, \frac{1}{4}, 2, \dots$	S11	(He realizes the example does not work because the limit does not exist.) I know the theory around this, I just don't… I'm used to numerical problems.
$-n$	S4	The sequence is decreasing towards infinity, but the sup is -1.
$.\frac{1}{n-3}, n = 4, 5, \dots$	S13	I can graph the function to see that the supremum is 1 and the limit is zero. I could prove it also. But I see it, I feel it, I understand it.
$\frac{1}{n}$	S1, S6, S7, S9, S10, S12, S14, S15	$A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. $\frac{1}{n} \le 1, \forall n \in \mathbb{N}$ and $1 \in A$. So 1 is the maximum of A. Thus, 1 is $\sup A$. But $\lim_{n\to\infty} \frac{1}{n} \left(= \frac{1}{\infty}\right) = 0$. (S10, S12) For $n = 1$, you get 1. And then it decreases. And the limit of x_n as n tends to infinity is just zero. So your sup would be one. But the limit of x_n would be zero. (S7) When asked to "give an example," we don't prove it. (S1, S7) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ It's getting smaller and smaller. The supremum is 1, but the limit is 0. (S6, S9). You'd have to prove using the definition, which I wasn't comfortable with. (S6) Maybe you're going to think I went fast on this. I'm not going to do the definition. That's the whole RA I course. Every statement, you had to back it up. I remember enjoying limits in Calculus. It was no longer enjoyable in Analysis when we had to use like epsilon and stuff. That's what I didn't like about the RA I course: it's like relearning stuff. (S9)

S3 and S8 are not included because we did not present their discourses for the examples they considered: $\frac{\sin(n)}{n}$ and $\sin\left(\frac{1}{n}\right)$, respectively.

Appendix E: Models for Task 4

A Model of Practices to be Learned

T_{4_a}: Prove that $\sup A = \infty$, where $A = \left\{\frac{r(n)}{s(n)} : n = 1, 2, \ldots\right\}$ and $r(n)$ and $s(n)$ are specified polynomials with $\deg(r(n)) > \deg(s(n))$.
τ_{4_a}: For $M > 0$, show that there exists an n such that $\frac{r(n)}{s(n)} > M$. Trick: find a sequence smaller than $\frac{r(n)}{s(n)}$ that will make the inequality easier to manipulate. Manipulate the inequality until finding something of the form $n > f(M)$. Conclude the existence of n from the unboundedness of $\mathbb{N}$. Commonly tested was $\frac{r(n)}{s(n)} = \frac{n^{p+1}+c}{n^p+d}$, where $c, d \in \mathbb{R}$ and $p \in \mathbb{N}$: For any $M > 0$, we can find an n such that $\frac{n^{p+1}+c}{n^p+d} > M$. Since $\frac{n^{p+1}+c}{n^p+d} > \frac{n^{p+1}}{2n^p}$ for large enough n, it is enough to find n such that $\frac{n}{2} > M$ or $n > 2M$. Its existence follows from the unboundedness of $\mathbb{N}$. ■
θ_{4_a}: Implicit: $\sup A = \infty \Leftrightarrow A$ is not bounded above $\Leftrightarrow \neg(\exists M, \forall a \in A, a \leq M)$ $\Leftrightarrow \forall M, \exists a \in A, a > M$ $\Leftrightarrow \forall M > 0, \exists a \in A, a > M$

$T_{4_{b_1}}$: Prove that $\sup A = M$, where $A = \{x_n : n = 1, 2, \ldots\}$ and x_n is a specified sequence.			
$\tau_{4_{b_{11}}}$: List elements of A to check if the sequence seems to be increasing. If so, show $x_{n+1} > x_n$ $\forall n \in \mathbb{N}$. Argue that $x_n \to M$ as $n \to \infty$.	$\tau_{4_{b_{12}}}$: Argue that $x_n \leq M$ $\forall n \in \mathbb{N}$ so that M is an upper bound for A. Show that for an arbitrary $\epsilon > 0$ one can find an $x_n \in A$ such that $x_n > M - \epsilon$. Manipulate the inequality to find $n > f(\epsilon)$. Conclude the existence of n since $\mathbb{N}$ is unbounded.	$\tau_{4_{b_{13}}}$: Argue that $x_n \leq M$ $\forall n \in \mathbb{N}$ so that M is an upper bound for A. Assume there is a lower number $M - \epsilon$ ($\epsilon > 0$) that is also an upper bound. Then $x_n \leq M - \epsilon$ $\forall n \in \mathbb{N}$. Manipulate the inequality to find $n \leq f(\epsilon)$ $\forall n \in \mathbb{N}$. Conclude that this contradicts the unboundedness of $\mathbb{N}$.	$\tau_{4_{b_{14}}}$: Argue that $M \in A$ and $x_n \leq M$ $\forall n \in \mathbb{N}$ (i.e., M is an upper bound for A).

$\theta_{4_{b_{11}}}$: "We proved a theorem that an increasing bounded above sequence converges to its supremum."	$\theta_{4_{b_{12}}}$: Implicit: $\sup A = M \Leftrightarrow (\forall a \in A, a \leq M) \wedge (\forall \epsilon > 0, \exists a \in A, a > M - \epsilon)$	$\theta_{4_{b_{13}}}$: Implicit: $\sup A = M \Leftrightarrow M$ is a least upper bound for $A \Leftrightarrow (\forall a \in A, a \leq M) \wedge [(a \leq B\ \forall a \in A) \Rightarrow M \leq B)]$ $\therefore$ if $\forall a \in A, a \leq M$, $\neg(\sup A = M) \Rightarrow \exists B < M, (a \leq B, \forall a \in A)$	$\theta_{4_{b_{14}}}$: "If A has a greatest element (or a maximum), then this greatest element is the supremum."

$T_{4_{b_2}}$: Prove by the definition: $\lim_{n\to\infty} x_n = L$, where $(x_n)_{n\in\mathbb{N}}$ is a simple specified sequence (typically of rational or polynomial form), and L is its specified limit.
$\tau_{4_{b_2}}$: Need to show $\forall\epsilon > 0, \exists N \geq 1, \forall n \geq N, \lvert x_n - L\rvert < \epsilon$. Fix $\epsilon > 0$ and write the specified inequality $\lvert x_n - L\rvert < \epsilon$. Manipulate the inequality to find $n > f(\epsilon)$. Sometimes a smaller sequence can be used to make the inequalities easier to manipulate. Choose $N = E\big(f(\epsilon)\big) + 1$ (E is the greatest integer function) or assert the existence of $N > f(\epsilon)$ by citing the Archimedean Property, or the unboundedness of $\mathbb{N}$. Conclude that if $n \geq N$, then $n > f(\epsilon)$ and the inequality holds.
$\theta_{4_{b_2}}$: Implicit: By definition, $\lim_{n\to\infty} x_n = L \Leftrightarrow \forall\epsilon > 0, \exists N \geq 1, \forall n \geq N, \lvert x_n - L\rvert < \epsilon$.

Models of Practices Actually Learned

t_{4_a}: Prove that for any $p > 1$, $A = \left\{\frac{n^p}{n+1} : n = 1, 2, \ldots\right\}$ is unbounded above.

Type of task: Prove that $A = \{x_n : n \in \mathbb{N}\}$ is unbounded above.

The techniques and discourses for solving this type of task:

Technique	Considered by	Example(s) of Discourse
Show that $(x_n)_{n\in\mathbb{N}}$ increases by induction.	S4, S8*	(In the case $x_n = \frac{n^p}{n+1}$:) If $p > 1$, the numerator will consistently increase. And the bigger p is, the bigger it will increase by. So it's going to be increasing for all n. So if I can show it's increasing by induction, it would show that A is unbounded above. It's not sufficient to say "it's obvious." You need to show it. (S4) Usually when you want to prove something for all $n \in \mathbb{N}$, you have to do induction. But this seemed longer to me. Arguing that $\lim_{n\to\infty} x_n = \infty$ is less rigorous; but it's quicker and easier. (S8)

Argue that $\lim_{n\to\infty} x_n = \infty$.	S2, S3, S5, S6, S7, S8, S9, S11, S12, S13, S14, S15	By definition, something that goes to infinity means that it has no bound. (S3) Unbounded means that if you go to $n+1$, it's going to be bigger than the previous n. Infinity is not an actual number. So you can always find an n such that the $n+1$ would be bigger. (S7)
τ_{4_a}: For any $M \in \mathbb{R}$, show that $\exists n \in \mathbb{N}, x_n > M$.	S1, S10	(In the case $x_n = \frac{n^p}{n+1}$:) We need to show that $\frac{n^p}{n+1} > M$ for any M as n goes to infinity. The trick: $\frac{n^p}{n+1} > \frac{n^p}{2n} > M$. (He incorrectly reduces the problem to dealing with:) $n^{p-1} > M$. For $p > 1, n^{p-1}$ will be greater than any bound. Infinity to any power will be greater than any number. (S1) We need to show that given any real number M, there exists an element a of A so that $a > M$. That's just the negation of saying that A has an upper bound. (In the case $x_n = \frac{n^p}{n+1}$:) The sequence is positive. So for $M < 0$, the proof is trivial: we can choose any n and x_n will be greater than M. For $M > 0$, we need to find n such that $\frac{n^p}{n+1} > M$. I'm missing the spark to do this. (S10)
Assume $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, x_n \leq M$. Derive a contradiction.	S5*, S6*, S13	It was what we did in class: we would try to show that the set is bounded above and see that we're not able to do that. I don't know how to do it. (S5) We want to show that $\forall M > 0, \exists n \in \mathbb{N}, x_n > M$ (it means it goes to infinity). Let's say it's not this: i.e., $\exists M > 0, \forall n \in \mathbb{N}, x_n \leq M$ (it means it is bounded). Then we can show that $\mathbb{N}$ is bounded. (She correctly carries out the steps.) This means our assumption is not true because $\mathbb{N}$ is unbounded. The fact that $\mathbb{N}$ is unbounded is something that we take for granted, right? But the idea is that if you give me any number, I can just add one to it and give you a larger number. (S13)

* indicates that the student hinted at the technique but did not implement it.

Students' implementation of the technique "argue that $\lim_{n\to\infty} x_n = \infty$" can be seen as solving a subtask: Argue that $\lim_{n\to\infty} \frac{n^p}{n+1} = \infty$ for $p > 1$.

The techniques and discourses for solving this task:

Technique	Considered by	Example(s) of Discourse
Plug in values: For $p = 2, n = 1, 2, 3$, we get: $\frac{1}{2}, \frac{4}{3}, \frac{9}{4}$.	S5	I see that the sequence is increasing, that the numerator is always bigger than the denominator, and since it's a big power, it will approach infinity. This is not a proof, but I don't remember how to prove it.
Compare the growth of the numerator and the denominator: n^p grows faster than $n+1$ for $p > 1$.	S2, S6, S11, S15	If the numerator grows faster than the denominator, then the numerator wins. In this case, the sequence will increase exponentially like n^p. I might be wrong. These are little tricks I've learned. (S6)

		It's evident that if you divide something that's growing much faster than the denominator, it must diverge. (S15)
Think about what matters in the long run: For large n, $\frac{n^p}{n+1} \sim \frac{n^p}{n} = n^{p-1} \to \infty$ for $p > 1$.	S8, S9, S13	As $n \to \infty$, constants (like $+1$ in the denominator) become insignificant and can be ignored. (S8, S9, S13) If I had to prove it in RA I, this would not suffice. (He considers two other techniques, as shown in the next two rows.) (S9) It's not very "mathematics." It's not very exact. But first I have to feel it, understand it, and believe it. (She can then prove it, as shown in the previous table.) (S13)
Compare the degree of the polynomials in the numerator and the denominator: The degree of n^p is p. The degree of $n+1$ is 1. So for $p > 1$, the degree of the numerator is greater than the degree of the denominator.	S9, S14	In a Calculus course, you just checked for the highest degree polynomial: if it was on the numerator, the sequence goes to infinity, and if it was on the denominator, the sequence goes to zero. (S9) I just know it. Maybe this is not rigorous. (He considers another technique, as shown in the last row.) (S14)
Use l'Hospital's Rule: $\lim_{x\to\infty} \frac{x^p}{x+1} = \lim_{x\to\infty} \frac{px^{p-1}}{1} = p \lim_{x\to\infty} x^{p-1} = \infty$ for $p > 1$. (S9 worked with the sequence.)	S3, S9	L'Hospital's Rule applies because it's an ∞/∞ case. (S3, S9) This would suffice in RA I because we saw it in the last two weeks of the course. (S9)
Use algebra of limits: $\lim_{n\to\infty} \frac{n^p}{n+1} = \lim_{n\to\infty} \frac{n^{p-1}}{1+\frac{1}{n}} = \frac{\infty}{1+0} = \infty$ S12's approach was different: $y_n = x_n \cdot n^k$, where $y_n = \frac{n^p}{n+1}, x_n = \frac{n}{n+1}$, $k = p-1, p > 1 \Rightarrow k > 0$. $x_n \to 1, n^k \to \infty \Rightarrow y_n = x_n \cdot n^k \to \infty$.	S7, S12, S14	As long as I show all the steps, this would be a more rigorous approach (compared to the approach two rows up). (S14) I am rewriting the function as a product. We need to accept that the product of a convergent function and a divergent function is also divergent to that same limit. (S12)

t_{4_b}: Prove that $\sup\left\{\frac{n}{n+1} : n = 1, 2, \dots\right\} = 1 = \lim_{n\to\infty} \frac{n}{n+1}$.

Students' identification of t_{4_b} with a type of task:

Student(s)	Type of Task
S3	There is nothing to show because the limit is obvious, and it implies the sup.
S5	Prove that $\lim_{n\to\infty} x_n = M$. Since the supremum is given, we do not have to prove it. Usually we are asked to "Find the supremum."
S1, S2, S4, S6, S7, S8, S9, S11, S12, S13, S14, S15	T_{4_b}: Prove that $\sup\{x_n : n \in \mathbb{N}\} = M = \lim_{n\to\infty} x_n$. $\Leftrightarrow T_{4_{b_1}}$: Prove that $\sup\{x_n : n \in \mathbb{N}\} = M$. $\wedge$ Prove that $\lim_{n\to\infty} x_n = M$.
S10	T_{4_b}: Prove that $\sup\{x_n : n \in \mathbb{N}\} = M = \lim_{n\to\infty} x_n$. $\Leftrightarrow$ Prove that $x_n \le M\ \forall n \in \mathbb{N}$. $\wedge\ T_{4_{b_2}}$: Prove by the definition: $\lim_{n\to\infty} x_n = M$.

$t_{4_{b_2}}$: Prove that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. The techniques and discourses for solving $t_{4_{b_2}}$:

Technique	Considered by	Example(s) of Discourse
Plug in values: $\frac{1}{2}, \frac{2}{3}, \ldots, \frac{10}{11} \sim 0.909$	S5	We clearly see that it's going towards one. (This is like a check after she uses another technique, as indicated below.)
Think about what matters in the long run: For large n, $\frac{n}{n+1} \sim \frac{n}{n} = 1$.	S4, S6, S9, S11, S13, S15	As $n \to \infty$, constants (like $+1$ in the denominator) become insignificant and can be ignored. (S4, S6, S9, S11, S13, S15) This is straightforward common sense and was acceptable in Calculus; but not in RA I. (They consider another technique, as shown in the next row.) (S6, S9) This is how I would convince myself. But I would use a different approach if I wanted to "speak mathematics" to someone else (see two rows down). (S13)
Use l'Hospital's Rule: $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1} = 1$	S6, S9, S11	L'Hospital's Rule applies because it's an ∞/∞ case. (S6, S9) This would suffice in RA I because we learned it. (S9) (After making a mistake in his calculation of the derivative:) I guess you use l'Hospital's if it's going to zero, and it's not supposed to go to zero, right? (S11)
Use algebra of limits: $\lim_{n\to\infty} \frac{n}{n+1}$ $= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$ $= \frac{1}{1+0}$ $= 1$	S2, S5, S7, S12, S13, S14	This is how you solve when you have a limit as $n \to \infty$. (S5) It's how we're told to do it, and it's a perfectly fine way to show a limit. $\lim_{n\to\infty} \frac{1}{n} = 0$ is intuitive: with increasing terms on the bottom, the whole term gets smaller and smaller. (S14) I know that I can evaluate $\frac{1}{\infty} = 0$ from Calculus. (S12) You're rearranging the sequence so that it's easier. (About $\lim_{n\to\infty} \frac{1}{n} = 0$:) Can I say that I just know it? (S2) You're rearranging the sequence so that you can use things that were proved in class and can be easily proved. E.g., $\lim_{n\to\infty} \frac{1}{n} = 0$. (S7) This is how I would speak mathematics to someone else. It is perhaps a more beautiful solution (when compared to the solution two rows up). I would say that either we know $\lim_{n\to\infty} \frac{1}{n} = 0$, or if we don't know it, I will prove it for you: We have to prove that $\forall \epsilon > 0, \exists \mathrm{N} \in \mathbb{N} \ni n \geq \mathrm{N} \left\lvert\frac{1}{n}\right\rvert < \epsilon$. $\left\lvert\frac{1}{n}\right\rvert < \epsilon \Rightarrow n > \frac{1}{\epsilon}$. So if we put $\mathrm{N} = \left\lceil\frac{1}{\epsilon}\right\rceil$ (the smallest integer greater than $\frac{1}{\epsilon}$), then for any $n \geq \mathrm{N}$, $n > \frac{1}{\epsilon}$. So $\epsilon > \frac{1}{n}$. ■ (S13)
$\tau_{4_{b_2}}$, by the definition: Show that $\forall \epsilon > 0$, $\exists N \geq 1, \forall n \geq N$ $\left\lvert\frac{n}{n+1} - 1\right\rvert < \epsilon$.	S1, S5*, S6*, S7*, S9*, S10, S11*	I don't remember the definition. (S5, S6, S9, S11) You basically need to isolate ϵ to have a value in terms of n… Wait, I already had it isolated. (S4) (After manipulating the inequality to $\left\lvert\frac{1}{n+1}\right\rvert < \epsilon$:) I've done exercises where you find a value for epsilon… Wait, that was with the delta. (S6)

		(After manipulating the inequality to $\left\|\frac{1}{n+1}\right\| < \epsilon$:) You just define $\epsilon = E\left(\frac{1}{n+1}\right) + 1$. Epsilon had to be an integer for some reason. And then you do plus one. And once you define that, you basically end your proof. (S7) (After manipulating the inequality to $\left\|\frac{1}{n+1}\right\| < \epsilon$, he says he will use the same trick as for the proof that A is unbounded above, as shown in the first table of this section:) $\frac{1}{2n} < \epsilon \Leftrightarrow \frac{1}{2\epsilon} < n$. I can find an N so that for all epsilon greater than zero, if $n > \frac{1}{2\epsilon}$, it holds. We got the limit. (S1) We let $\epsilon > 0$ and we need to find $N \in \mathbb{N}$ such that for all $n \geq N$ the inequality $1 - \frac{n}{n+1} < \epsilon$ would hold. I know I should let $n = N$ and rearrange the inequality to solve for N in terms of ϵ. But I cannot see how to do this right now. I'm pretty sure at the end of it, if I take any n that's greater than that N, the inequality is still going to hold. (S10)
Argue that the sequence is increasing and has supremum equal to the desired limit: $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} = \left(1 - \frac{1}{n+1}\right)_{n\in\mathbb{N}}$ is increasing and has supremum 1.	S8	This makes sense. I have a mental picture of it.

* indicates that the student considered the technique upon request of the interviewer. S3 is not included because he felt the limit was so obvious that it did not require justification.

$t_{4_{b_1}}$: Prove that $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$.

$\in T_{4_{b_1}}$: Prove that $\sup A = M$, where $A = \{x_n : n = 1, 2, \dots\}$ and x_n is a specified sequence.

The techniques and discourses for solving $T_{4_{b_1}}$:

Technique	Considered by	Example(s) of Discourse
Rewrite x_n as $M - y_n$ and argue that $\inf\{y_n : n \in \mathbb{N}\} = 0$.	S8	If I subtract the least large number of $\{y_n : n \in \mathbb{N}\}$, I will get the max of $\{x_n : n \in \mathbb{N}\}$.
$\tau_{4_{b_{11}}}$: Argue that $(x_n)_{n\in\mathbb{N}}$ is increasing and $\lim\limits_{n\to\infty} x_n = M$.	S1, S2, S4, S5, S7, S9, S13, S14, S15	This was the way to go. It's easy to understand and to visualize. It's not very abstract. (S9) It's by a theorem. (S7, S9, S13, S15) It's "by definition": the supremum is the highest value in the set that is not really in the set, and the limit is the furthest that it goes. (S1) I can prove it. But I'll take it for granted now because we proved it in Task 3. (S13)

Argue that $a \leq M \ \forall a \in A$ and $\lim_{n\to\infty} x_n = M$.	S10	$\lim_{n\to\infty} x_n = M$ $\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \lvert x_n - M \rvert < \epsilon$ $\Rightarrow \ \forall \epsilon > 0, \exists N \in \mathbb{N}, M - \epsilon < x_N$ If we also know that $\forall n \in \mathbb{N}, x_n \leq M$, then $\forall \epsilon > 0, \exists N \in \mathbb{N}, x_N \in (M - \epsilon, M]$. So $\sup\{x_n : n \in \mathbb{N}\} = M$.
$\tau_{4_{b_{12}}}$: Argue that $a \leq M \ \forall a \in A$ and $\forall \epsilon > 0, \exists a \in A, M - \epsilon < a$.	S12	The definitions were super helpful. Being fluent in Analysis was like being able to remember the terms. (Upon receiving the second part of the definition, which he did not remember:) It means that for any epsilon we choose, we can find an x_n such that the inequality is satisfied. (In the case $x_n = \frac{n}{n+1}$, he manipulates $\frac{n}{n+1} + \epsilon > 1$ to get n in terms of ϵ. He is unsure how to conclude. He thinks the required n exists:) because $\mathbb{N}$ has no upper bound.

S3, S6, and S11 are not included because they did not clearly exhibit practices for solving $T_{4_{b_1}}$. S3 and S6 wanted to use a formal definition but did not know the definition and/or did not know how to use it. S11 was unsure if the supremum had to be included in the set.

Students' implementation of $\tau_{4_{b_{11}}}$ includes solving a subtask: Show $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ is increasing.

The techniques and discourses for solving this task:

Technique	Considered by	Example(s) of Discourse
Plug in values: $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$	S5	The interviewer did not inquire into why the students used these techniques.
Argue that $x_{n+1} - x_n > 0$: $\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{n^2+2n+1-n^2-2n}{(n+1)(n+2)}$ $= \frac{1}{(n+1)(n+2)} > 0$	S13	
Analyze the monotonicity of $(x_n)_{n\in\mathbb{N}}$ in parts: $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}} = \left(\frac{1}{1+\frac{1}{n}}\right)_{n\in\mathbb{N}}$ As n increases, $1/n$ decreases, so the denominator decreases, which means the whole term increases.	S14	
Use Induction: $x_1 = \frac{1}{2}, x_2 = \frac{2}{3}, \frac{2}{3} > \frac{1}{2}$ Assume $\frac{n}{n+1} < \frac{n+1}{n+2}$. Prove $\frac{n+1}{n+2} < \frac{n+2}{n+3}$. $(n+1)(n+3) < (n+2)^2$ $n^2 + 4n + 3 < n^2 + 4n + 4$ $3 < 4$ ■	S1, S2, S4, S7, S9, S15	It's an easy technique. (S1, S7, S9) The other way would be to enumerate everything and prove that every time it is smaller. (S9) If I see natural numbers, prove that something happens with natural numbers, I'd always start with induction. (S1) If you want to prove it for any values, you prove by induction. So you prove the first one. And then you prove it for $n + 1$. And then, the rest, you know that it's true for every one. (S7)

Appendix F: Models for Task 5

A Model of Practices to be Learned

$T_{5_{ST}}$: Use the Squeeze Theorem to find $\lim_{n\to+\infty} b_n$. Typically: b_n is of the form $\frac{\ln(f(n))}{n}$ or $\sqrt[n]{f(n)}$ where $f(n)$ is a sum including a dominant exponential term (i.e., a^n with $a \geq 2$), and possibly a trig function (sine or cosine) whose argument is of some interesting algebraic form (e.g., containing factorials and large exponentials).
τ_5: Find y_n and z_n such that $y_n \leq b_n \leq z_n$ and $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} z_n$. Typically: y_n and z_n have the same form as b_n, with $f(n)$ equal to some multiple of the dominant exponential term.
θ_5: "By the Squeeze Theorem."

Models of Practices Actually Learned

t_5: Find $\lim_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$ where $a \in \mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ is any sequence of real numbers.

Students' identification of t_5 with a type of task:

Student(s)	Type of Task
S2	Find the limit of a given sequence.
S1, S3, S14	T_{5_F}: Find (or guess) the limit of a given sequence using an intuitive technique. $\wedge$ T_{5_P}: Prove the limit of a given sequence using an acceptable theorem or rule.
S1, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15	$T_{5_{ST}}$: Use the Squeeze Theorem to find the limit of a given sequence.
S1, S2, S5, S6, S9, S10, S11, S15	Find the limit of a sequence containing a^n under the tacit assumption that $a^n \to \infty$ as $n \to \infty$.
S3, S7, S8, S12, S14	$T_{5_{a>1}}$: Find the limit of a sequence containing a^n under the explicit assumption that $a^n \to \infty$ as $n \to \infty$.
S4, S13	$T_{5_{a\in\mathbb{R}}}$: Find the limit of a sequence containing a^n depending on the value of a: $a > 1$, $0 < a < 1$, or a negative.
S6, S11	I know how to solve a different type of task: $\widetilde{T_{5_{ST}}}$: Use the Squeeze Theorem to find $\lim_{n\to+\infty} \sqrt[n]{a\ constant + an\ exponential + sine\ of\ something}$.
S10	Find $\lim_{n\to+\infty} \frac{\ln(f(n))}{n}$. $\Leftrightarrow$ Find $\lim_{n\to+\infty} \ln(\sqrt[n]{f(n)})$. $\Leftrightarrow$ Find $\ln(\lim_{n\to+\infty} \sqrt[n]{f(n)})$. (He feels this follows from the continuity of $\ln(x)$.)

$t_{5_{ST}}$: Use the Squeeze Theorem to find $\lim\limits_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$.

The technique and discourses for solving $t_{5_{ST}}$:

Technique	Considered by	Example(s) of Discourse
τ_5: Let the given sequence be b_n. Find y_n and z_n such that $y_n \le b_n \le z_n$ and $\lim\limits_{n\to+\infty} y_n = \lim\limits_{n\to+\infty} z_n$.	S1, S4, S5, S6, S7, S8, S9, S10, S11, S12, S13, S14, S15	Usually when I see something like that… Don't ask me why cause I can't tell you why, but… Squeeze Theorem. (S11) It looks like the Squeeze Theorem. It's my feeling, my intuition. (S15) The sequences usually looked like that. You'd have something weird, but it would end up being bounded. (S9) The teacher told us: "When you see complicated things, we're doing this on purpose to confuse you. And go and do the Squeeze Theorem." (S1) Having practiced it a lot, you start to see patterns. Like the fact that the variable that is being pushed to a value in the limit, it is present at more than one spot. And the presence of a trig function also kind of ticks off another box: "Oh, let's confuse the student here." Also: There's no way to take the limit of this sequence with the epsilon-delta definition. (S4) Proving that the sequence is decreasing or increasing, it would be too hard given the function. It's not even monotone. (He is comparing his approach to this task with the one he had used for $\left(\frac{n}{n+1}\right)_{n\in\mathbb{N}}$ in Task 4.) (S7) Evaluating the limit straight is going to be disgusting. (S8) I would never be expected to deal with the term arctan x_n for any possible sequence of real numbers unless that's clearly bounded by something on either side. (S10) I thought "Squeeze Theorem" because I know I have one term here that's really the dominant term, that really dictates the behaviour of the function: a^n. I thought "Squeeze Theorem" because I was trying to find a way to show the behaviour of this function without looking too closely at the terms 2 and $\arctan(x_n)$. My thought process in part comes from seeing examples in class in which the teacher will take like maybe some cos function, some trig function that has two bounds in it. (S12)

$t_{5_{ST,a>1}}$: Use the Squeeze Theorem to find $\lim\limits_{n\to+\infty} \frac{\ln(2+a^n+\arctan(x_n))}{n}$ under the assumption that $a^n \to \infty$ as $n \to \infty$.

A first possible step in solving $t_{5_{ST,a>1}}$:

$$\forall (x_n)_{n\in\mathbb{N}}, -\frac{\pi}{2} < \arctan(x_n) < \frac{\pi}{2} \Rightarrow \frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{n} < \frac{\ln(2+a^n+\arctan(x_n))}{n} < \frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n}.$$

Students had different tools they could use to address this step (see the following two tables).

The techniques and discourses for bounding a sequence:

Technique	Considered by	Example of Discourse
Find something smaller and something bigger: E.g., $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} \le \ln\left(2+a^n+\frac{\pi}{2}\right)$	S5*	I got rid of the n just to make the upper bound greater. I'm not sure if that's bad. I maybe would have needed the n later.
Find one bound and its limit. Then try to find the other bound so it has the same limit: $\frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{n} \ge \frac{\ln(a^n)}{n} = \ln(a)$ So we want to find an upper bound that converges to $\ln(a)$. E.g., $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} \le \frac{\ln(a^{n+m})}{n}$ for some m $= \frac{(n+m)\ln(a)}{n}$ $= \ln(a) + \frac{m\ln(a)}{n}$ $\to \ln(a)$	S12	(After seeing the lower bound:) I know my limit is going to equal $\ln(a)$. So I'm trying to find some function for the upper bound that when I solve it I'm also going to end up with $\ln(a)$ as my answer. The fact that $\ln(x)$ is increasing is implicit in my solution.
Eliminate or replace terms in the sequence based on what is dominant and what is not: $\frac{\ln\left(2+a^n-\frac{\pi}{2}\right)}{n} \ge \frac{\ln(a^n)}{n} = \ln(a)$ $\frac{\ln\left(2+a^n+\frac{\pi}{2}\right)}{n} \le \frac{\ln(3a^n)}{n}$ $= \frac{\ln(3)+\ln(a^n)}{n}$ $= \frac{\ln(3)}{n} + \ln(a)$ $\to \ln(a)$ S15 used this technique after applying l'Hospital's Rule (as in the next table): $\frac{\ln(a)\cdot a^n}{a^n+2+\frac{\pi}{2}} \le \frac{\ln(a)\cdot a^n}{a^n} = \ln(a)$	S2**, S5**, S6**, S7, S8*, S11**, S12, S14, S15	I use this approach because then it's easier to factor out. You get something nice out of it. a^n is bigger than 2 and $\arctan(x_n)$. You can take the natural log of both sides. It's just like squaring both sides or dividing both sides; you just apply the same thing. So the inequality holds. It holds for n bigger than a certain number, let's say a hundred. Since n goes to infinity, we can assume that n is bigger than a hundred. (S7)

* indicates that the student did not fully implement the technique (i.e., they got stuck). ** indicates that the student hinted at the technique but did not implement it. S1, S3, S4, S9, S10, and S13 are not included because they did not exhibit a technique for finding an upper bound of $\frac{\ln(a^n+k)}{n}$. S1, S3, and S15 explicitly tried to find such an upper bound and were unable to do so.

The techniques and discourses for finding the limit of a sequence:

Technique	Considered by	Example(s) of Discourse
Compare the growth of the numerator and the denominator: n grows faster than $\ln(f(n))$ $\therefore \lim_{n\to+\infty} \frac{\ln(f(n))}{n} = 0$	S1, S4, S14	Log goes up really slowly. $1/n$ goes down really quickly. But "slowly" and "quickly" don't mean anything in math. My teachers would not like this. I need to prove the limit of the sequence. (S1) It doesn't matter what's in the log: it can never grow faster than n. Maybe this is enough to solve the task

		since I'm asked to "find" the limit, not to "prove" the limit. (S14)
Think about what matters in the long run: E.g., For large n, $\frac{\ln(a^n+k)}{n} \sim \frac{\ln(a^n)}{n}$ $= \ln(a)$ S8 used this technique after applying l'Hospital's Rule (as in the next row): For large n, $\frac{\ln(a)\cdot a^n}{a^n + 2 + \frac{\pi}{2}} \sim \frac{\ln(a)\cdot a^n}{a^n}$ $= \ln(a)$	S1*, S2*, S3, S8, S9, S13, S14*, S15	(After applying the approach in the above row and starting to use the Squeeze Theorem:) The argument of the log (i.e., $2 + a^n + \arctan(x_n)$) is going to behave kind of like an exponential. But I'm not sure if I can just ignore 2 and $\arctan(x_n)$. (S1, S14) If I have a term that is big (e.g., a^n), much bigger than the others (e.g., 2 and $\arctan(x_n)$), then I don't need to include the others in my function. (S2) Whenever I'm solving limits, I always think about what matters. When things go to infinity, a lot of things stop to matter. Constants and bounded terms stop to matter when there's a term (like a^n) that goes to infinity. Even if this convinces me, teachers would not like this. I need to prove it. (S3) As n tends to infinity, a^n is so big that the constant terms matter less and can be ignored. (S9, S13) We can ignore constants if we're in a Calculus course. But in Analysis, you can't just say "this is not important." Or if you do, it needs to be your last step, and you need to show why it's not important. (He considers another technique, as shown in the next row.) (S9) This is not a proof. It is how I convince myself. (S13) I have an intuition. But it's hard to believe our intuitions in mathematics. Because intuition is not necessarily an assurance of success. (S15)
Use l'Hospital's Rule: $\lim_{n\to\infty} \frac{\ln\left(2 + a^n + \frac{\pi}{2}\right)}{n}$ $= \lim_{n\to\infty} \frac{\ln(a)\cdot a^n}{a^n + 2 + \frac{\pi}{2}}$	S8, S9, S15	I was trying to use l'Hospital's Rule because it was acceptable in RA I. But to deal with $\lim_{n\to\infty} \frac{\ln(a)\cdot a^n}{a^n + 2 + \frac{\pi}{2}}$, I need to use the same kind of argument as before (i.e., thinking about what matters in the long run, as in the above row). I can't see how to find the limit without this (i.e., using only techniques that were acceptable in RA I). (S9)

* indicates that the student was unable to fully implement the technique (i.e., they got stuck).

Appendix G: Models for Task 6

A Model of Practices to be Learned

$T_{6_{ab}}$: Determine if (or prove that) g is differentiable at $x=0$, where g is a piecewise function with $g(0)=0$ and, for $x \neq 0$, $g(x)$ includes the product of a monomial, $x^p, p \in \{2,3,4,\dots\}$, and sine or cosine of a rational function of the form $\frac{c}{r(x)}$ with $c \in \mathbb{R}$ and $r(x)$ a polynomial.

Possibly also determine if (or prove that) g' is continuous at $x=0$.

$\tau_{6_{ab}}$: Calculate $g'(0) = \lim\limits_{h\to 0} \frac{g(h)-g(0)}{h}$ and check if the limit exists.
To determine if g' is continuous at $x=0$: Check if $g'(x)$ converges to $g'(0)$ as $x \to 0$.

Commonly tested was $g(x) = \begin{cases} x^p f\left(\frac{c}{r(x)}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

where $p \in \{2,3,4,\dots\}$, $f(x)=\sin(x)$ or $f(x)=\cos(x)$, $c \in \mathbb{R}$, and $r(x)$ is a polynomial:

We have $g'(0) = \lim\limits_{h\to 0} \frac{h^p f\left(\frac{c}{r(h)}\right)}{h} = \lim\limits_{h\to 0} h^{p-1} f\left(\frac{c}{r(h)}\right) = 0$.
So g is differentiable at 0.
Outside 0 we have $g'(x) = x^{p-1} f\left(\frac{c}{r(x)}\right) + x^p f'\left(\frac{c}{r(x)}\right)\left(\frac{-cr'(x)}{r(x)^2}\right)$.
For $x \to 0$, the first term converges to 0. If $x^p\left(\frac{-cr'(x)}{r(x)^2}\right)$ converges to 0, the second term also converges to 0 and g' is continuous at 0. If not, the second term does not have a limit and g' is not continuous at 0. ■

$\theta_{6_{ab}}$: Implicit: g is differentiable at $x_0 \Leftrightarrow \lim\limits_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h} = g'(x_0)$ exists.
g' is continuous at $x_0 \Leftrightarrow \lim\limits_{x\to x_0} g'(x) = g'(x_0)$.

T_{6_c}: Find the points of continuity of $g(x) = \begin{cases} g_1(x) & x \in \mathbb{R}\setminus\mathbb{Q} \\ g_2(x) & x \in \mathbb{Q} \end{cases}$, where g_1 and g_2 are specified functions (typically simple rational functions and/or polynomials).

τ_{6_c}: Let g be continuous at x_0, $(x_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to x_0, and $(y_n)_{n\in\mathbb{N}}$ be a sequence of irrational numbers converging to x_0. Then $\lim\limits_{n\to\infty} g_2(x_n) = g(x_0)$ and $\lim\limits_{n\to\infty} g_1(y_n) = g(x_0)$. Implicit: Solve $\lim\limits_{n\to\infty} g_2(x_n) = g(x_0) = \lim\limits_{n\to\infty} g_1(y_n)$.
Specify $g_2(x_n)$ and $g_1(y_n)$ based on the definitions of g_2 and g_1, and calculate possible limits. Consider all possible combinations of these limits being equal.

θ_{6_c}: "If g is continuous at x_0, then $\forall (x_n)_{n\in\mathbb{N}}$ that converge to x_0, we have $\lim\limits_{n\to\infty} g(x_n) = g(x_0)$."

Models of Practices Actually Learned

t_{6_a}: For what values of p is $g(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous on $\mathbb{R}$?

The techniques and discourses for solving t_{6_a}:

Technique	Considered by	Example(s) of Discourse
Use the formal ϵ-δ definition of continuity: i.e., determine the values of p such that $\forall x_0 \in \mathbb{R}$ $\forall \epsilon > 0, \exists \delta > 0,$ $\lvert x_0 - y\rvert \leq \delta \Rightarrow$ $\lvert f(x_0) - f(y)\rvert < \epsilon$	S4, S5, S6, S7*, S10*, S14*	Continuity is with the delta. (S4, S5, S6) I'm not sure if it's $\forall \epsilon, \exists \delta$ or $\forall \delta, \exists \epsilon$. Using the $\epsilon - \delta$ definition is the main way of proving continuity that I remember. (S4) I remember $\lvert x - x_0\rvert < \delta$, $\lvert f(x) - f(x_0)\rvert < \epsilon$. But I wouldn't know how to do it. After the exam, I forget everything. (S6) Whenever I see "continuous," "uniformly continuous," I just write down the theorem right away and then see if I can do something about it: $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x, y \in \mathbb{R}, \lvert x - y\rvert < \delta \Rightarrow \lvert f(x) - f(y)\rvert < \epsilon$ I'm not even sure what definition that is. I'm not sure if we need the definition because the question is asking for values of p. (S5) g is continuous on $\mathbb{R} \Leftrightarrow \forall x_0 \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0,$ $\forall x \in (x_0 - \delta, x_0 + \delta), \lvert g(x) - g(x_0)\rvert < \epsilon$. By writing down the definition of continuity, I was hoping to remember how to show that something is continuous. I was hoping that this would sort of guide my approach. (He chooses a different approach, as indicated later in this table.) (S10) Usually, when you're not sure of something, you go back to the definition. It can help give you an idea of if you're on the right track, if what you're saying makes sense. Here, not so much. (S7)
Use differentiation: i.e., determine the values of p such that g is differentiable on $\mathbb{R}$.	S1, S5, S9, S12**	Could we use differentiation? I know that we had something else (other than the formal definition) that we learned, and basically that meant continuous. But I don't remember. (S5) If we can derive it, then it's continuous; it's not that it's continuous that we can derive it. I hate the formal definition. (S1) The derivative was like my best friend in RA I for proving that a function is continuous. (S9)
Calculate $g'(x)$ and determine the values of p such that $\lim_{x \to x_0^-} g'(x)$ $= g'(x_0)$ $= \lim_{x \to x_0^+} g'(x)$	S11***	You do this if you want to check that it's continuous.
Use uniform continuity: i.e., determine the values of p such that g is uniformly continuous on $\mathbb{R}$.	S14*	I know there are nice techniques to figure out uniform continuity. But it's not if and only if. If you have a uniformly continuous function, it means it's continuous. But just because it's continuous doesn't mean it's necessarily uniformly continuous. So you might have something that's continuous, but not uniformly continuous. So that doesn't help always.

Use a graphing tool: i.e., determine the values of p such that the graph of g is smooth on $\mathbb{R}$ using the tool (e.g., Desmos) to observe the graph for various values of p.	S14	Continuity means that the value at each point is the limit of the function approaching that point. So it's like this:  This is not a proof. This is what I would do if I was at home. Otherwise, I probably would have prepared for a question like this and I would have known how to attack it with some conditions.
Determine the values of p such that g is well-defined on $\mathbb{R}$: i.e., such that $g(x)$ can be calculated on my calculator $\forall x \in \mathbb{R}$.	S9	I'm trying to see the possible restrictions. E.g., you cannot do $(-1.5)^{1.5}$ because my calculator does not like it.
Determine the values of p such that g is an algebraic combination of continuous functions: e.g., in this case, when $x \neq 0$, $g(x) = f(x) \cdot h(x)$ with $f(x) = x^p$ and $h(x) = \sin\left(\frac{1}{x}\right)$.	S4**, S10**	By separating the $\sin\left(\frac{1}{x}\right)$ as always being continuous and isolating the x^p, which is the part that's important because we're asked about p, it simplifies the handling of the function. I might be wrong, but I kind of think that two continuous functions multiplied together also result in a continuous function. In graphing x^p for $p = 1, 2, 3$, I see that x^p is always continuous (i.e., $\forall p$). (S4) If f and h are continuous, then the product of those two is continuous. Since sine is continuous and $1/x$ is continuous for $x \neq 0$, $\sin(1/x)$ is continuous for $x \neq 0$. I am hesitant about the continuity of x^p because I don't know what x^p means for all real numbers p. I want to say that x^p is continuous because I know the graphs. (S10)
An adaptation of τ_{6_a}: Determine the values of p such that $\lim_{x \to x_0} g(x) = g(x_0)$ $\forall x_0 \in \mathbb{R}$.	S2***, S3***, S4***, S8***, S10***, S12***, S13***	Continuity means that it cannot have a gap at $x = 0$. I think $x^p \sin\left(\frac{1}{x}\right)$ is continuous. And from my experience, g can only be not continuous at the point of the condition. (S2) The function should just go nicely to 0. (S3) I don't remember exactly the definition of continuity. But, if continuity is that it can't have a jump, so that the graph of the function would be drawn without raising your pencil, then what I would do would be to take the limit approaching zero from both sides. (S4) Continuity means that the graph can be drawn without lifting your pencil. It also means, like in the limit definition: the limit as x approaches a of let's say $f(x)$ has to be equal to $f(a)$. This seems a lot more intuitive to me than the for all epsilon, there exists, blah, blah, blah… I find the formal definition a lot more difficult to deal with personally. Also: there's like a hole, a jump. So the only place where g could be discontinuous is zero. I know $x^p \sin\left(\frac{1}{x}\right)$ is continuous everywhere. (S8)

		To me it was obvious that g is well-defined everywhere except for $x = 0$. Using the limit is easier than using the formal definition. (S13)

* indicates that the student explicitly decided against the technique. ** indicates that the student specified $x \neq 0$. *** indicates that the student specified $x = 0$. S15 is not included because he did not attempt to solve the task due to the time constraints of the interview.

t_{6_b}: For what values of p is $g(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable on $\mathbb{R}$?

The techniques and discourses for solving t_{6_b}:

Technique	Considered by	Example(s) of Discourse
Determine the values of p such that g is continuous on $\mathbb{R}$.	S2	I remember that differentiability and continuity are related in some way.
Determine the values of p such that the pieces of g can each be differentiated using differentiation rules. Or: Calculate the derivative of each piece of g using differentiation rules and determine the values of p such that the resulting functions are well-defined on their corresponding domains.	S1, S3, S4, S8**, S7, S9, S12**, S13**, S14	(For $x \neq 0$:) If you can use the product rule, then it's differentiable. The derivative of 0 is just 0. I can't remember the definition we were given. (S4) I'm looking for values of p that would make the derivative (of each piece) problematic. I don't remember how to go about this kind of problem, when there's like a discontinuity like this. (S3) For $x \neq 0$, I can always find (i.e., calculate) a derivative. (S8) I can do this using Calculus: i.e., look for cases where I would have violated the differentiation rules I learned. (S12) (For $x \neq 0$, after using the product rule:) You don't know if you can differentiate this way or not. You say that if $g(x) = f(x)h(x)$, then $g'(x) = \ldots$ But if they're both differentiable. I assume that they're both differentiable at any p. And then I did it. (S13)
Calculate the derivative of g on $\mathbb{R}\backslash\{0\}$ and determine the values of p such that the resulting function is defined at 0.	S7***, S8***, S14***	(After calculating the derivative of g on $\mathbb{R}\backslash\{0\}$:) I was hoping that by plugging in $x = 0$, I'd be able to kind of identify the values of p where it doesn't make sense. It feels like you have to use limits in a sense because of that fact that I can't plug directly in. (S8) (In relation to $g'(0)$:) You're not evaluating as it goes to zero, or as it goes to infinity. You're evaluating it at zero. (S14)
τ_{6_b}, use the definition: i.e., determine the values of p such that $\forall x_0 \in \mathbb{R}$, $g'(x_0) = \lim_{h\to 0} \frac{g(x_0+h)-g(x_0)}{h}$ exists.	S5, S6, S7*, S8*, S11***, S12***, S13***, S14*	I forgot the definition they gave me in RA I. If I was not in RA I, I would try to think of a different way. (She does not show what she means.) (S6) We would always use the definition in RA I. (S5) This is the way I saw it in RA I. (In relation to $g'(x) = \lim_{h\to 0} \frac{\left((x+h)^p \sin\left(\frac{1}{x+h}\right) - x^p \sin\left(\frac{1}{x}\right)\right)}{h}$ $\Rightarrow g'(0) = \lim_{h\to 0} \frac{(0+h)^p \sin\left(\frac{1}{h}\right) - 0^p \sin\left(\frac{1}{0}\right)}{h}$:) For $x \neq 0$, the limit seems like a hard one to evaluate. For $x = 0$, it looks like a bad limit. So I'm not going to do that. (S8)

		The definition is that the limit from the left and the limit from the right should be equal. (In relation to $g'(x)$ shown above:) I cannot see if the limits are equal in this case. (S7) (In relation to $g'(x)$ shown above:) I cannot see how to simplify the limit. (S14) For $x \neq 0$, I could use the definition of course. But it would be harder. (After writing the limit in the definition of differentiability for $x = 0$:) We can use part (a) (i.e., her work using the limit definition of continuity for $x = 0$). Differentiable on $\mathbb{R}$ for p is like being continuous on $\mathbb{R}$ for p minus one. So you decrease the degree. And then it's the same thing. (S13)
Determine the values of p such that g is an algebraic combination of differentiable functions.	S4**	I'm fairly certain that it applies for continuity, where the product of two continuous functions is always continuous. But is that the case for differentiability? I don't remember that. But, I would assume so.

* indicates that the student explicitly decided against the technique. ** indicates that the student specified $x \neq 0$. *** indicates that the student specified $x = 0$. S10 and S15 are not included because they did not attempt to solve the task due to the time constraints of the interview.

t_{6_c}: For what values of p is $g(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$ continuous/differentiable on $\mathbb{R}$?

The techniques and discourses for arguing that g is not continuous on $\mathbb{R}$ for any value of p:

Technique	Considered by	Example(s) of Discourse
Argue that g has jump discontinuities.	S1, S2, S3, S4, S6, S14	It's because of the density of $\mathbb{Q}$ in $\mathbb{R}$: • The x-axis looks like a straight line, but it's not. It's not a continuous line. At every irrational, there's a hole. So there has to be jumps. (S1, S3, S6) e.g.,  • In every interval, there's a rational and an irrational. And $x^p \sin\left(\frac{1}{x}\right)$ is different from 0. (S2, S4) • $x^p \sin\left(\frac{1}{x}\right)$ is not often going to be zero. The only irrational numbers for which $x^p \sin\left(\frac{1}{x}\right) = 0$ are of the form $x = \frac{1}{n\pi}$. But there's plenty of irrational numbers in between those that are not going to satisfy $x^p \sin\left(\frac{1}{x}\right) = 0$. The function is bouncing all over the place. I think that the function is not continuous at any point. (S3, S6)
Argue that there are no values of p such that $x^p \sin\left(\frac{1}{x}\right) = 0$.	S8	It's kind of clear to me that if I want the function to be continuous, $x^p \sin\left(\frac{1}{x}\right)$ can't be any other value apart from zero because that will fill in the line ($y = 0$) and it will be complete.

Use the negation of the formal ϵ-δ definition of continuity: i.e., $\exists x \in \mathbb{R}, \exists \epsilon > 0, \forall \delta > 0, \exists y_\delta, \|x - y_\delta\| < \delta \wedge \|f(x) - f(y_\delta)\| \geq \epsilon$	S7	(After showing that g is not Riemann integrable:) Actually, I just proved it's not continuous. There's no interval where you can find only rational numbers or only irrational numbers. Thus, for every delta, $0 < \|x_i - x_{i-1}\| < \delta$, there would be two values, the values of f would not match up. Like it wouldn't imply that $\|f(x_i) - f(x_{i-1})\| < \epsilon$. I think that the function is not continuous at any point.
τ_{6_c}: Use the sequence definition of continuity to find the points where g is continuous.	S12, S13	(After struggling to argue more formally using the definition:) If we multiply sine and cos by some other function, its behaviour tends to be bound by the positive and negative forms of that function. For example, for $p = 1/2$, the graph of g looks like:  At $x = 0$ and $1/n\pi$ $n \in \mathbb{N}$: Whether we approach it using rational numbers or irrational numbers, both of these series reach the same point. Because they both converge to the same thing, I'm tempted to say that it's continuous at that location. (S12) For any rational, you can find a sequence of irrational numbers that goes to it. i.e., $\forall x_q \in \mathbb{Q}, \exists (x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $\lim\limits_{n\to\infty} x_n = x_q$. If g is continuous, then $\lim\limits_{n\to\infty} g(x_n) = g(x_q)$. i.e., $\lim\limits_{n\to\infty}\left(x_n^p \sin\frac{1}{x_n}\right) = 0$. Hence, x_n^p should go to 0. So g is only continuous at $x = 0$ for $p > 0$. (S13)
Use the sequence definition of continuity to show one point where g is not continuous.	S13	(In continuing the argument directly above:) For instance, if $x_q = 2$, then there is a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\backslash\mathbb{Q}$ such that $\lim\limits_{n\to\infty} x_n = x_q$, and: $\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty}\left(x_n^p \sin\left(\frac{1}{x_n}\right)\right) = 2^p \sin\left(\frac{1}{2}\right) \neq g(x_q) = 0.$

S10 and S15 are not included because they did not attempt to solve the task due to the time constraints of the interview. S5, S9, and S11 are not included because they said that they did not know how to solve the task. S11 recalled similar tasks from RA I and said that he skipped over them. S9 connected the task with a type of task seen more recently, in RA II (i.e., showing that a function is or is not Riemann integrable). Some of the students in the table (not only S7) may have been making the same connection as S9.

t_{6_c}: For what values of p is $g(x) = \begin{cases} x^p \sin\left(\frac{1}{x}\right) & x \in \mathbb{R}\backslash\mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$ continuous/differentiable on $\mathbb{R}$?

The techniques and discourses for arguing that g is not differentiable on $\mathbb{R}$ for any p:

Technique	Considered by	Example(s) of Discourse
Use the fact that g is not continuous on $\mathbb{R}$ for any p.	S1, S3, S4, S6, S7, S14	If you want it to be differentiable, you need a continuous function to start off with. If you have irregularities, you can't differentiate it. It's something that I know. But I wouldn't know based off of like the true definition. (S6)

		(To support the statement "not continuous"⇒ "not differentiable":) I just have a feeling. And I know feelings are bad. I should have something to support that, but I don't. (S7) When there's a jump, as in this case, I'm like ninety-nine percent sure it's not differentiable. But I feel like I'm missing something. $g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$ would not exist because it's jumping up and down. (S4)  You can't differentiate at a jump because the limits are not equal (as in the sketch on the left). In comparison: the limits are equal in the sketch on the right. (S3) For a function to be differentiable, you have to know the rate of change at a point. It has to be changing from one point to the next in some kind of measurable manner. (S14) I know that differentiable implies continuous. I remember the logic thing: $\alpha \Rightarrow \beta, \neg\beta \Rightarrow \neg\alpha$. I can check if these are equivalent: $\begin{array}{ccc} \alpha & \beta & \Rightarrow \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ \neg\alpha & \neg\beta & \neg\beta \Rightarrow \neg\alpha \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}$ Thus, not continuous implies not differentiable. (S1)
Assume g is differentiable on ℝ and show this is not possible for any p.	S8	For something to be differentiable, it has to be continuous as well. If g were continuous on ℝ, it would need to be equal to zero on ℝ (see S8's argument from the previous table). Then g' would also be equal to the zero function. However, there are values of x for which $x^p\left(\frac{p}{x}\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x^2}\right)$ (the derivative of g for $x \neq 0$) is not equal to zero.
Mimic the argument for showing g is not continuous on ℝ for any p.	S13	Studying the differentiability of g is almost the same as studying the continuity of g; there's just a different limit. (She concludes that she would use the same kind of argument as in the previous table.)

S10, S12, and S15 are not included because they did not attempt to solve the task due to the time constraints of the interview. S5, S9, and S11 are not included because they said that they did not know how to solve the task (see the note for the previous table). S2 is not included because she said very quickly that she thought that she could solve the task using the same technique as for Task 6(b).