Heuristic conjectures for moments of cubic L-functions over function fields

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ABSTRACT

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Let $L_q(s, \chi)$ be the Dirichlet L-function associated to χ, a cubic Dirichlet character with conductor of degree d over the polynomial ring $\mathbb{F}_q[T]$. Following similar work by Keating and Snaith for moments of Riemann ζ-function, Conrey, Farmer, Keating, Rubinstein, and Snaith [Con+05] introduced a framework for proposing conjectural formulae for integral moments of general L-functions with the help of random matrix theory.

In this thesis we review the heuristic found in [Con+05] and apply their work in order to propose moments for $L_q(s, \chi)$, cubic L-functions over function fields. We find asymptotic formulae when $q \equiv 1 \pmod 3$, the Kummer case, and when $q \equiv 2 \pmod 3$, the non-Kummer case. Moreover, while the authors of [Con+05] provide only the framework for proposing (k, k)-moments of primitive L-functions, we extend their work following the work of David, Lalín, and Nam to propose (k, l)-moments of cubic L-functions where $k \geq l \geq 1$ [DLN]. Furthermore, we provide explicit computations that elucidate the combinatorics of leading order moments and find a general form as well.

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List of symbols and notation

$\mathbb{F}_q$	Finite field with cardinality q
$\mathbb{F}_q[T]$	Polynomial ring over $\mathbb{F}_q$
$\mathbb{F}_q(T)$	Quotient field of $\mathbb{F}_q[T]$
P	Prime polynomial, i.e. monic irreducible polynomial in $\mathbb{F}_q[T]$
$\mathcal{M}_q$	Set of monic polynomials in $\mathbb{F}_q[T]$
$\mathcal{M}_{q,d}$	Set of monic polynomials in $\mathbb{F}_q[T]$ with degree exactly d
$\mathcal{M}_{q,\leq d}$	Set of monic polynomials in $\mathbb{F}_q[T]$ with degree at most d
$\mathcal{H}_q$	Set of monic square-free polynomials in $\mathbb{F}_q[T]$
$\mathcal{H}_{q,d}$	Set of monic square-free polynomials in $\mathbb{F}_q[T]$ with degree exactly d
$\mathcal{H}_{q,\leq d}$	Set of monic square-free polynomials in $\mathbb{F}_q[T]$ with degree at most d
$\zeta_q(s)$	Zeta function of $\mathbb{F}_q[T]$
$\mathcal{Z}_q(u)$	$\zeta_q(s)$ with substitution $u = q^{-s}$
$\mathcal{Z}_C(u)$	Zeta function of the curve C
$L_q(s, \chi)$	L-function over $\mathbb{F}_q[T]$ associated to the Dirichlet character χ
$\mathcal{L}_q(u, \chi)$	$L_q(s, \chi)$ with the substitution $u = q^{-s}$
$\mathcal{L}_C(u, \chi)$	L-function associated to the curve C with non-principal character χ
$Z_L(s, \chi)$	Adjusted $L_q(s, \chi)$ used for constructing the (k, l)-th moment of $L_q(s, \chi)$
$Z_{(k)}$	Product of k different $Z_L(s, \chi)$ each evaluated at a point slight off the critical line
$\bar{Z}_{(l)}$	Product of l different $Z_L(s, \bar{\chi})$ each evaluated at a point slight off the critical line

Contents

Chapter 1

Introduction

Precise asymptotic formulae for the k-th moments of the Riemann ζ-function,

$$M_k(T) = \frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} \mathrm{d}t,$$

have long been investigated, dating back to the first contributions made by Hardy and Littlewood to establish the $k = 1$ case [HL16]. While it is believed that

$$M_k(T) \sim g_k a_k \left(\log T\right)^{k^2}, \tag{1.1}$$

where g_k and a_k are positive constants, only low-level cases have been proven. There is also a similar interest in studying the asymptotic behavior of moments of L-functions in general as well. More recently however, significant advancements in the problem of moments have come from considerations in random matrix theory. In 1972, Montgomery—with contributions from Dyson—noticed that the pair correlation of spacing between zeroes of the Riemann ζ-function coincides with the pair correlations of eigenvalues of large random Hermitian matrices. Specifically, this meant that the short-range (meaning local) statistics of the zeroes that are scaled to have unit mean spacing coincide with the statistics of eigenvalues that are similarly spaced. Extensive numerical calculations by Odlyzko later supported Montgomery's conjecture (as a result, this is sometimes called the Montgomery-Odlyzko law or it is referred to as the GUE conjecture) [Con01]. Furthermore, Rudnick and Sarnak were able to show analogous results when considering other general L-functions [RS96].

This allowed Keating and Snaith—in a substantial development—to conjecture all k-th moments of the Riemann ζ-function by considering the moments of the characteristic polynomial of unitary matrices as a model for the moments of ζ [KS00]. Let U be a $N \times N$ unitary matrix with eigenvalues $e^{i\theta_n}$. They proved that the k-th moment of the characteristic polynomial of U averaged over the circular unitary ensemble, the group $U(N)$, with respect to the Haar measure is

$$\left\langle \left| \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right) \right|^{2k} \right\rangle_{U(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(2k+j)}{(\Gamma(j+k))^2}. \tag{1.2}$$

Because the short-range correlations of ζ can be represented by random matrix models, Keating and Snaith conjectured that the moments of ζ are asymptotically split into a product of two terms; one corresponding to short-range correlations, g_k, and the other coming from long-range deviations that take the form as a product over the primes, a_k. Since (1.2) is a polynomial in N of degree k^2, they conjectured that

$$g_k = \lim_{N\to\infty} \frac{1}{N^{k^2}} \left\langle \left| \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right) \right|^{2k} \right\rangle_{U(N)} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{1.3}$$

It turned out that Keating and Snaith's hypothesized g_k matched the results established for $k = 1$ by Hardy and Littlewood, $k = 2$ by Ingham, and conjectured for $k = 3$ by Conrey and Ghosh, $k = 4$ by Conrey and Gonek, all using number-theoretic methods.

Inevitably, the work in [KS00] meant it there was a plausible that one could also conjecture all moments of general L-functions if one would make the appropriate choice of random matrix family as model. Consequently, Conrey, Farmer, Keating, Rubenstein, and Snaith developed a heuristic framework to conjecture such integral moments [Con+05].

In this thesis, we study the heuristic developed by the authors of [Con+05] to propose conjectural formulae for moments of cubic Dirichlet L-functions over function fields, $L_q(s, \chi)$, evaluated on the critical line. A cubic L-function is the L-function associated to a cubic Dirichlet character χ of $\mathbb{F}_q[T]$ and is denoted by the series

$$L_q(s,\chi) = \sum_{f \in \mathcal{M}_q} \frac{\chi(f)}{|f|^s}, \tag{1.4}$$

where $\mathcal{M}_q$ is the set of all monic polynomials in $\mathbb{F}_q[T]$, $|f| = q^{\deg(f)}$, and $s \in \mathbb{C}$. Alternatively, because we consider function field L-functions, we can think of $L_q(s,\chi)$ from a geometric perspective. If we define an algebraic curve with certain properties whose function field is the field of fractions of $\mathbb{F}_q[T]$, it happens that the Weil conjectures provide us with a natural definition of a cubic L-function that matches our previous definition. (We say more about the specifics of this construction in Chapter 3, Section 2.)

As there is a deep analogy between number fields and function fields, the study of L-functions in the context of the latter can be an interesting if not useful area of research. In fact, structural similarities notwithstanding, results that are still elusive in the setting of number fields like the Riemann hypothesis and the Hilbet-Pólya conjecture (the suggestion that the zeroes of ζ have a spectral interpretation) have been proven for function fields. This makes explicit that the analogy we are working with is not a superficial one, but rather, one that can be generative for our understanding of its number field counterpart.

We take up our problem on the asymptotics of moments given Katz and Sarnak's work in showing that almost all curves over finite fields satisfy the Montgomery-Odlyzko law when the cardinality of the finite field is sent to infinity [KS99]. In other words, function field L-functions do indeed also have random matrix models. This problem on moments has been considered previously a number of times by others, usually in the quadratic case [AK14]. While the literature on the quadratic twists of L-functions is relatively large, there is little study of cubic twists more generally. As lamented by David, Lalín, and Nam [DLN], this is because cubic twists are highly 'chaotic objects' due to the cubic Gauss sums that accompany cubic twists of L-functions. This further exacerbates the already tortuous asymptotic behavior found when working with moments of L-functions.

Our goal is to find conjectural formulae for the (k,l)-th moment of $L_q(\frac{1}{2},\chi)$, that is,

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^k L_q\left(\tfrac{1}{2},\bar{\chi}\right)^l\right\rangle_d := \frac{1}{\#\mathcal{N}(d)} \sum_{\chi\in\mathcal{N}(d)} L_q\left(\tfrac{1}{2},\chi\right)^k L_q\left(\tfrac{1}{2},\bar{\chi}\right)^l, \tag{1.5}$$

where $\mathcal{N}(d)$ is the set of primitive cubic characters with conductor of degree d. After reviewing the necessary rudiments and 'grammar' for working with the polynomial ring over finite fields in Chapter 2 and upon presenting the background on Dirichlet characters in the opening of Chapter 3, our first goal is to build our definition of a cubic L-function and obtain the functional equation and approximate functional equation of $L_q(s,\chi)$. From there, we prove a series of results related to $\mathcal{N}(d)$.

In Chapter 4, show our main result and make explicit calculations using our conjectural formulae. We then make our 'random matrix theory computation' by computing a large determinant to determine the form of $g_{k,l}$, which is the analogue of (1.3), corresponding to the asymptotics of (1.5). We present the finalized form of our conjecture in Chapter 5.

Chapter 2

Polynomials over finite fields

This chapter is a partial review of the first two chapters of Rosen's *Number theory in function fields* [Ros02]. We review the necessary rudiments on polynomials over finite fields but in the interest of brevity we avoid covering material that is not directly relevant to our needs.

In all that follows let q be an odd prime power. We let $\mathbb{F}_q[T]$ denote the set of polynomials over the finite field $\mathbb{F}_q$. Elements in $\mathbb{F}_q[T]$ have the form $f = \alpha_0 T^n + \alpha_1 T^{n-1} + \cdots + \alpha_n$, where $\alpha_i \in \mathbb{F}_q$ for $i = 0, \ldots, n$. We call n the degree of f if $\alpha_0 \neq 0$ and denote it by writing $\deg(f) = n$. If $\alpha_0 = 1$, then we say f is a monic polynomial. We call α_0 the sign of f and denote it by $\operatorname{sgn}(f) = \alpha_0$. Now, we can introduce the notion of size in $\mathbb{F}_q[T]$.

Definition 2.0.1. *Let $f \in \mathbb{F}_q[T]$. If $f \neq 0$, then set $|f| = q^{deg(f)}$. If $f = 0$, then $|f| = 0$.*

Although we do not prove it here, it is not hard to see that $\mathbb{F}_q[T]$ is a Euclidean domain which means that it is also a unique factorization domain. Therefore, like the integers, the set of polynomials over $\mathbb{F}_q$ has a notion of an irreducible element. We can actually write any polynomial in $\mathbb{F}_q[T]$ as a product of primes and a unit. This leads us to now describe the unit group of $\mathbb{F}_q[T]$ in the proposition below.

Proposition 2.0.1. *The group of units in $\mathbb{F}_q[T]$ is $\mathbb{F}_q^\times$. In particular, it is a finite cyclic group with order $q-1$.*

Proof. Suppose that $g \in \mathbb{F}_q[T]$ is a unit. Then there exists an f such that $fg = 1$. Therefore, $0 = \deg(1) = \deg(f) + \deg(g)$ so $\deg(f) = \deg(g) = 0$. The only units in $\mathbb{F}_q[T]$ are constants. Finally, since the unit group of $\mathbb{F}_q[T]$ is a finite subgroup of the multiplicative group of a field, it is cyclic. $\square$

Now we define what an irreducible polynomial is in the definition below in order to state the unique factorization property of elements in $\mathbb{F}_q[T]$

Definition 2.0.2. *We say that $f \in \mathbb{F}_q[T]$ and $f \neq 0$ is irreducible if it cannot be written as a product of two polynomials, each of positive degree.*

Proposition 2.0.2 (Unique factorization property)**.** *Let $f \neq 0$. Then f can be written uniquely in the form*

$$f = \alpha P_1^{e_1} \cdots P_t^{e_t}$$

where α is a unit and P_i is a monic irreducible polynomial and $P_i \neq P_j$ for any $i \neq j$ and e_i is a nonnegative integer.

From now on, *prime* will be used in place where we mean 'monic irreducible polynomial.'

In the introduction, we merely hinted that there is a deep analogy between number fields and function fields. This also extends to the polynomial ring $\mathbb{F}_q[T]$, which makes a natural analogue of the integers. Rosen enumerates some of these similarities in [Ros02]. Consequently, this allows us to define an analogue of ζ-function over $\mathbb{F}_q[T]$.

Definition 2.0.3. *The ζ-function of $\mathbb{F}_q[T]$ is denoted by $\zeta_q(s)$ and is defined by the infinite series*

$$\zeta_q(s) = \sum_{f \in \mathcal{M}_q} \frac{1}{|f|^s},$$

where $\mathcal{M}_q = \{$monic polynomials of $\mathbb{F}_q[T]\}$.

Unlike the ζ-function over the integers, ζ_q can be viewed in a closed form, since there are exactly q^d monic polynomials of degree d in $\mathbb{F}_q[T]$. Therefore, we note

$$\sum_{f \in \mathcal{M}_{q,\leq d}} |f|^{-s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \frac{q^3}{q^{3s}} + \cdots + \frac{q^d}{q^{ds}},$$

where $\mathcal{M}_{q,\leq d}$ denotes the set of all monic polynomials with degree at most d. The above means that

$$\zeta_q(s) = \frac{1}{1 - q^{1-s}}, \tag{2.1}$$

which makes it explicit that ζ_q is initially defined for $\Re(s) > 1$ and has a simple pole at $s = 1$. Like the ζ-function over the integers, ζ_q has a meromorphic continuation. This leads us to define ξ_q, the $\mathbb{F}_q[T]$ analogue of ξ over the integers.

Definition 2.0.4 (ξ-function corresponding to ζ_q)**.**

$$\xi_q(s) = q^{-s}(1 - q^{-s})^{-1}\zeta_q(s).$$

This, it's clear, satisfies the functional equation $\xi_q(1 - s) = \xi_q(s)$.

Moreover, like the original Riemann ζ-function, we have a Euler product over the primes. This leads us to write the following:

Definition 2.0.5 (Euler product of ζ_q)**.**

$$\zeta_q(s) = \prod_P \left(1 - \frac{1}{|P|^s}\right)^{-1}$$

Since the derivation of the Euler product is nearly identical to the methods required to obtain the Euler product of the Riemann ζ-function, we omit it.

Chapter 3

Cubic L-functions over function fields

3.1 Dirichlet characters

This section introduces Dirichlet characters in general before a discussion on cubic characters. Most of the material in this section can be found in the fourth chapter of Rosen's *Number theory in function fields* [Ros02].

Definition 3.1.1. *Let $m \in \mathbb{F}_q[T]$ with positive degree. We say that $\chi \colon \mathbb{F}_q[T] \to \mathbb{C}$ is a Dirichlet character modulo m if*

1. $\chi(a + bm) = \chi(a)$ *for all* $a, b \in \mathbb{F}_q[T]$,
2. $\chi(ab) = \chi(a)\chi(b)$ *for all* $a, b \in \mathbb{F}_q[T]$,
3. $\chi(a) \neq 0$ *if and only if* $(a, m) = 1$.

We refer to m as the modulus of the character χ. Dirichlet characters also have a notion of sign; we say that χ is *even* if $\chi(a) = 1$ for all $a \in \mathbb{F}_q^\times$, and *odd* otherwise. It can be shown that a Dirichlet character modulo m induces a homomorphism from $(\mathbb{F}_q[T]/(m))^\times$, the unit group of $(\mathbb{F}_q[T]/(m))$, to $\mathbb{C}^\times$, the unit group of $\mathbb{C}$. Moreover, each such induced homomorphism uniquely corresponds to a Dirichlet character modulo m. It then immediately follows that the image of any Dirichlet character is a complex root of unity or 0. We know this because χ

can be alternatively understood as a homomorphism from $(\mathbb{F}_q[T]/(m))^\times$ to $\mathbb{C}^\times$, every torsion element in $(\mathbb{F}_q[T]/(m))^\times$ is mapped to the torsion elements of $\mathbb{C}^\times$. And since $(\mathbb{F}_q[T]/(m))^\times$ is a finite group, every element is torsion. Furthermore, the only torsion elements of $\mathbb{C}^\times$ are the complex roots of unity. The simplest example of a character is the *principal character.*

Definition 3.1.2. *The principal character χ_0 modulo m is defined as*

$$\chi_0(a) = \begin{cases} 1 & \text{if } (m,a) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Because of the nature of residue classes, it follows that a character mod m can give rise to other characters modulo a proper divisor of m. If χ is a character mod m and ϕ is a character mod n where n is a proper divisor of m and also $\chi(f) = \phi(f)$ whenever $(f,m) = 1$, then we say that χ is *induced* by ϕ. Naturally, to avoid redundancy, this leads us to introduce the notion of a character that is not induced by other characters.

Definition 3.1.3. *We say that a Dirichlet character modulo m, χ, is **primitive** if it cannot be induced by a character of smaller modulus.*

Definition 3.1.4. *The smallest modulus for which χ a Dirichlet character is primitive is called the **conductor** of χ.*

The properties of characters should have made it relatively evident that the set of Dirichlet characters form a multiplicative group. Let X_m denote the set of Dirichlet characters modulo m. If $\chi, \phi \in X_m$, then their product is defined as $\chi\phi(f) = \chi(f)\phi(f)$ and is also in X_m. The identity element is χ_0, the principal character. We denote the inverse of a character χ as χ^{-1}. Furthermore, $\chi^{-1}(f) = \chi(f)^{-1}$ if $(f,m) = 1$ and $\chi^{-1}(f) = 0$ otherwise. In addition, we also define $\bar{\chi}(f) = \overline{\chi(f)}$, where $\overline{\chi(f)}$ is the complex conjugate of $\chi(f)$. Because χ is either 0 or a root of unity, it follows that $\bar{\chi} = \chi^{-1}$.

Definition 3.1.5. *χ is a **cubic Dirichlet character** with conductor m if $\chi : (\mathbb{F}_q[T]/(m))^\times \to \mu_3$ and $\chi^3 = \chi_0$, where $\mu_3 \subset \mathbb{C}^\times$ is the set of cubic roots of unity.*

If χ is a cubic character, we can immediately deduce that $\bar{\chi} = \chi^2$. Every Dirichlet character discussed from now on is a primitive cubic character unless otherwise stated. We also importantly remark that the set of primitive cubic characters will vary depending on the cardinality of the finite field $\mathbb{F}_q$. (We will show how different the sets are in Section 3 of this chapter). If $q \equiv 1 \pmod 3$, then $3 \mid q-1$ and so $\mathbb{F}_q$ contains the third roots of unity. Now if $q \equiv 2 \pmod 3$ or equivalently $q \equiv -1 \pmod 3$, $\mathbb{F}_q$ does not have the third roots of unity since $3 \nmid q-1$. In line with Kummer theory, we refer to the case when $q \equiv 1 \pmod 3$ as the Kummer case and the case when $q \equiv 2 \pmod 3$ as the non-Kummer case.

For $q \equiv 1 \pmod 3$ we fix an isomorphism λ from the cubic roots of unity in $\mathbb{F}_q^\times$ and the cubic roots of unity in μ_3.

An example of a non-principal cubic character is the cubic residue symbol, which we introduce now.

Definition 3.1.6 (Cubic residue symbol)**.** *Let P be a prime. If $P \nmid a$, let $\left(\frac{a}{P}\right)_3$ be the unique element of $\mathbb{F}_q^\times$ such that*

$$a^{\frac{|P|-1}{3}} \equiv \left(\frac{a}{P}\right)_3 \pmod P.$$

*If $P \mid a$, define $\left(\frac{a}{P}\right)_3 = 0$. We call $\left(\frac{a}{P}\right)_3$ the **cubic residue symbol**.*

The cubic residue symbol over $\mathbb{F}_q[T]$ has similar properties to its analogue over the integers and can be found in Rosen's text (specifically, Proposition 3.4 in [Ros02]). Moreover, like the integers, we also have a reciprocity property for our residue symbol.

Theorem 3.1.1 (Cubic reciprocity law)**.** *Let $f, g \in \mathbb{F}_q[T]$ be relatively prime monic non-zero polynomials. Then,*

$$\left(\frac{f}{g}\right)_3 = (-1)^{\frac{q-1}{3}deg(f)deg(g)} \left(\frac{g}{f}\right)_3$$

Proof. This is Theorem 3.5 in Rosen's text [Ros02] with $d = 3$. □

We note also, that if also $q \equiv 1 \pmod 6$ in addition to the conditions of Theorem 3.1.1, we have perfect reciprocity. This means $\left(\frac{a}{b}\right)_3 = \left(\frac{b}{a}\right)_3$ or, in other words, $\chi_b(a) = \chi_a(b)$.

3.2 Cubic L-functions over function fields

We review the work of David, Florea, and Lalín [DFL19] in this section in order to formally define cubic L-functions over function fields. From our geometric definition, we obtain the functional and approximate functional equation—both of which are necessary in Chapter 4.

Let C be a curve over $\mathbb{F}_q[T]$ whose function field is a cyclic cubic extension of $\mathbb{F}_q[T]$. The Weil conjectures allow us to write the ζ-function of C as

$$\mathcal{Z}_C(u) = \frac{\mathcal{P}_C(u)}{(1-u)(1-qu)},$$

where

$$\mathcal{P}_C(u) = \prod_{j=1}^{g} \left(1 - \sqrt{q}ue^{2\pi i\theta_j}\right) \prod_{j=1}^{g} \left(1 - \sqrt{q}ue^{-2\pi i\theta_j}\right), \tag{3.1}$$

for some eigenangles θ_j, $j = 1, \ldots, g$, and where g is the genus of the curve C. We can write $\mathcal{P}_C(u)$ in terms of the L-functions of the two cubic Dirichlet characters χ and $\bar{\chi}$. Therefore $\mathcal{P}_C(u) = \mathcal{L}_C(u,\chi)\mathcal{L}_C(u,\bar{\chi})$. Now we define

$$\mathcal{L}_q(u,\chi) = \sum_{f \in \mathcal{M}_q} \chi(u)u^{\deg(f)}. \tag{3.2}$$

We also note that $\mathcal{L}_q(q^{-s},\chi) = L_q(s,\chi)$, where

$$L_q(s,\chi) = \sum_{f \in \mathcal{M}_q} \frac{\chi(f)}{|f|^s}$$

is the Dirichlet L-function corresponding to a cubic character χ (as previously defined in the introduction). Like the ζ-function, our L-function has an Euler product over the primes

$$\mathcal{L}_q(u,\chi) = \prod_{P} \left(1 - \chi(P)u^{\deg(P)}\right)^{-1},$$

and is finite sum. This follows from the orthogonality relations of χ [Ros02].

Because working with $\mathcal{L}_C(u,\chi)$ means we are working from a geometric point of view, if we consider its relationship to $\mathcal{L}_q(u,\chi)$ we have to adjust for the fact that $\mathcal{L}_q(u,\chi)$ is missing information on the ramification at infinity. Therefore we write

$$\mathcal{L}_C(u,\chi) = \begin{cases} \mathcal{L}_q(u,\chi) & \text{if } \chi \text{ is odd,} \\ \frac{\mathcal{L}_q(u,\chi)}{1-u} & \text{if } \chi \text{ is even.} \end{cases} \tag{3.3}$$

Let h be the conductor of χ appearing above, then by the Riemann-Hurwitz formula,

$$\deg(h) = g + 2 - \begin{cases} 1 & \text{if } \chi \text{ is odd,} \\ 0 & \text{if } \chi \text{ is even.} \end{cases} \tag{3.4}$$

Lemma 3.2.1 (Functional equation)**.** *Let χ be a primitive cubic character with modulus h. If χ is odd, then $\mathcal{L}_q(u,\chi)$ satisfies the functional equation*

$$\mathcal{L}_q(u,\chi) = \omega(\chi)\varphi(u,\chi)\mathcal{L}_q\left(\frac{1}{qu},\bar{\chi}\right), \tag{3.5}$$

where the sign of the functional equation is

$$\omega(\chi) = q^{-(deg(h)-1)/2} \sum_{f \in \mathcal{M}_{q,deg(h)-1}} \chi(f), \tag{3.6}$$

and where $\varphi(u,\chi) = (\sqrt{q}u)^{deg(h)-1}$.

If χ is even, then $\mathcal{L}_q(u,\chi)$ satisfies the functional equation

$$\mathcal{L}_q(u,\chi) = \omega(\chi)\varphi(u,\chi)\mathcal{L}_q\left(\frac{1}{qu},\bar{\chi}\right), \tag{3.7}$$

where the sign of the functional equation is

$$\omega(\chi) = q^{-(deg(h)-2)/2} \sum_{f \in \mathcal{M}_{q,deg(h)-1}} \chi(f), \tag{3.8}$$

and where $\varphi(u,\chi) = (\sqrt{q}u)^{deg(h)-2} \frac{1-u}{1-\frac{1}{qu}}$.

Remark. This proof can be found in [DFL19] but we show it here for sake of completeness.

Proof. Suppose χ is odd. Then by (3.4), $g = \deg(h) - 1$. Moreover, this means that $\mathcal{L}_C(u,\chi) = \mathcal{L}_q(u,\chi)$. This means we can write, if we take into account (3.1) as well;

$$\begin{aligned}
\mathcal{L}_C(u,\chi) &= \mathcal{L}_q(u,\chi) && (3.9)\\
&= \prod_{j=1}^{\deg(h)-1} \left(1 - \sqrt{q}ue^{2\pi i\theta_j}\right) && (3.10)\\
&= (\sqrt{q}u)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} \left((\sqrt{q}u)^{-1} - e^{2\pi i\theta_j}\right) && (3.11)\\
&= (\sqrt{q}u)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} -e^{2\pi i\theta_j} \prod_{j=1}^{\deg(h)-1} \left(1 - \frac{e^{2\pi i\theta_j}}{\sqrt{q}u}\right) && (3.12)\\
&= (\sqrt{q}u)^{\deg(h)-1}(-1)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j} \prod_{j=1}^{\deg(h)-1} \left(1 - \frac{e^{-2\pi i\theta_j}}{\sqrt{q}u}\right) && (3.13)\\
&= (\sqrt{q}u)^{\deg(h)-1}(-1)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j} \overline{\mathcal{L}_C}\left(\frac{1}{qu}\right). && (3.14)
\end{aligned}$$

Now, recall that

$$\sum_{n=0}^{\deg(h)-1} u^n \sum_{f \in \mathcal{M}_{q,n}} \chi(f) = \prod_{j=1}^{\deg(h)-1} \left(1 - \sqrt{q}ue^{2\pi i\theta_j}\right).$$

We compare coefficients and find that

$$\sum_{f \in \mathcal{M}_{q,\deg(h)-1}} \chi(f) = (q)^{(\deg(h)-1)/2}(-1)^{\deg(h)-1} \prod_{j=1}^{\deg(h)-1} e^{2\pi i\theta_j}.$$

which means that

$$\omega(\chi) = q^{-(\deg(h)-1)/2} \sum_{f \in \mathcal{M}_{q,\deg(h)-1}} \chi(f).$$

which shows (3.5) and (3.6).

Now suppose that χ is even. We take into account (3.1), (3.3) and (3.4) in order to write

$$\begin{aligned}
\mathcal{L}_q(u,\chi) &= (1-u)\mathcal{L}_C(u,\chi) && (3.15)\\
&= (1-u)\prod_{j=1}^{\deg(h)-2}\left(1-\sqrt{q}ue^{2\pi i\theta_j}\right) && (3.16)\\
&= (1-u)(\sqrt{q}u)^{\deg(h)-2}\prod_{j=1}^{\deg(h)-2}\left((\sqrt{q}u)^{-1}-e^{2\pi i\theta_j}\right) && (3.17)\\
&= (1-u)(\sqrt{q}u)^{\deg(h)-2}(-1)^{\deg(h)-2}\prod_{j=1}^{\deg(h)-2}e^{2\pi i\theta_j}\prod_{j=1}^{\deg(h)-2}\left(1-\frac{e^{-2\pi i\theta_j}}{\sqrt{q}u}\right) && (3.18)\\
&= \left(\frac{1-u}{1-\frac{1}{qu}}\right)(\sqrt{q}u)^{\deg(h)-2}(-1)^{\deg(h)-2}\prod_{j=1}^{\deg(h)-2}e^{2\pi i\theta_j}\overline{\mathcal{L}_q}\left(\frac{1}{qu},\bar{\chi}\right). && (3.19)
\end{aligned}$$

Knowing that

$$\sum_{n=0}^{\deg(h)-1} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) = (1-u)\prod_{j=1}^{\deg(h)-2}\left(1-\sqrt{q}ue^{2\pi i\theta_j}\right),$$

we compare the coefficients of $u^{\deg(h)-1}$ and see

$$\sum_{f\in\mathcal{M}_{q,\deg(h)-1}} \chi(f) = (q)^{(\deg(h)-2)/2}(-1)^{\deg(h)-1}\prod_{j=1}^{\deg(h)-2} e^{2\pi i\theta_j}.$$

And it follows that

$$\omega(\chi) = q^{-(\deg(h)-2)/2} \sum_{f \in \mathcal{M}_{q,\deg(h)-1}} \chi(f),$$

which proves (3.7) and (3.8). $\square$

Although it is not necessarily clear from the form of the sign of the functional equation presented in Lemma 3.2.1, $|\omega(\chi)| = 1$ and $\omega(\bar{\chi}) = \overline{\omega(\chi)}$. Since it is not necessarily critical for our purposes that we show this, we leave it to the reader to consult Corollary 2.4 in [DFL19] which writes the sign in terms of Gauss sums. Now we move on to showing a result that allows us to separate our cubic L-functions into multiple shorter sums. This result is necessary in order to formulate our proposed moments in Chapter 4.

Proposition 3.2.2 (Approximate functional equation)**.** *Let χ be a primitive cubic character and let A denote a parameter that can be chosen later. If χ is odd, then*

$$\mathcal{L}_q\left(u, \chi\right) = \sum_{f \in \mathcal{M}_{q, \leq A}} \chi(f) u^{deg(f)} + \omega(\chi)\varphi(u,\chi) \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \bar{\chi}(f) u^{deg(f)},$$

where $\omega(\chi)$ and $\varphi(u,\chi)$ are the same as in the functional equation of $\mathcal{L}_q(u,\chi)$.

If χ is even, then

$$\begin{aligned}\mathcal{L}_q\left(u, \chi\right) = &\sum_{f \in \mathcal{M}_{q, \leq A}} \chi(f) u^{deg(f)} + \omega(\chi)\varphi(u,\chi) \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \bar{\chi}(f) (qu)^{deg(f)} \\ &+ \frac{1}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q, \leq A+1}} \chi(f) u^{deg(f)} + \frac{\omega(\chi)}{1-\sqrt{q}} \sum_{f \in \mathcal{M}_{q, \leq g-A}} \bar{\chi}(f) (qu)^{deg(f)},\end{aligned}$$

where $\omega(\chi)$ and $\varphi(u,\chi)$ are the same as in the functional equation of $\mathcal{L}_q(u,\chi)$.

Proof. If χ is odd, Lemma 3.2.1 tells us the functional equation is

$$\mathcal{L}_q(u,\chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}_q\left(\frac{1}{qu}, \chi\right).$$

Since we can write $\mathcal{L}_q(u,\chi) = \sum_{d=0}^{\deg(h)} u^d \sum_{f \in \mathcal{M}_{q,d}} \chi(f)$, the functional equation above means that

$$\sum_{f \in \mathcal{M}_{q,n}} \chi(f) = \omega(\chi) q^{n-\frac{g}{2}} \sum_{f \in \mathcal{M}_{q,g-n}} \bar{\chi}(f). \tag{3.20}$$

Now we can write $\mathcal{L}_q(u,\chi)$ as sum of two shorter sums, splitting it at our parameter A and use (2.23) to arrive at

$$\begin{aligned}
\mathcal{L}_q(u) &= \sum_{n=0}^{A} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) + \sum_{n=A+1}^{g} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) \\
&= \sum_{n=0}^{A} u^n \sum_{f\in\mathcal{M}_{q,n}} \chi(f) + \omega(\chi)(\sqrt{q}u)^g \sum_{n=A+1}^{g} (qu)^n \sum_{f\in\mathcal{M}_{q,g-n}} \bar{\chi}(f) \\
&= \sum_{f\in\mathcal{M}_{q,A}} \chi(f) u^{\deg(f)} + \omega(\chi)(\sqrt{q}u)^g \sum_{f\in\mathcal{M}_{q,g-A-1}} \bar{\chi}(f) u^{\deg(f)},
\end{aligned}$$

which is the desired result.

Suppose χ is even. Then we can write $\mathcal{L}_q(u,\chi)$ in the form

$$\begin{aligned}
\mathcal{L}_q(u,\chi) &= \sum_{n=0}^{g+1} a_n u^n, \\
a_n &= \sum_{f\in\mathcal{M}_{q,n}} \chi(f).
\end{aligned}$$

And recall we can also write

$$\mathcal{L}_C(u) = \prod_{j=1}^{g} \left(1 - \sqrt{q}ue^{2\pi i\theta_j}\right) = \sum_{n=0}^{g} b_n u^n.$$

Then we use the functional equation

$$\begin{aligned}
\sum_{n=0}^{g} b_n u^n &= \mathcal{L}_C(u,\chi) \\
&= \omega(\chi)(\sqrt{q}u)^g \mathcal{L}_C\left(\frac{1}{qu}, \bar{\chi}\right) \\
&= \omega(\chi)(\sqrt{q}u)^g \sum_{n=0}^{g} \overline{b_n} q^{-n} u^{-n} \\
&= \omega(\chi) \sum_{n=0}^{g} \overline{b_n} q^{g/2-n} u^{g-n} \\
&= \omega(\chi) \sum_{m=0}^{g} \overline{b_{g-m}} q^{m-g/2} u^m.
\end{aligned}$$

which means that $b_n = \omega(\chi)\overline{b_{g-n}}q^{n-g/2}$. Again, like in the case when χ is odd, we write $\mathcal{L}_C(u,\chi)$ as the sum of two shorter terms and replace b_n in the second sum to find

$$\mathcal{L}_C(u,\chi) = \sum_{n=0}^{g} b_n u^n + \omega(\chi) \sum_{n=0}^{g-A-1} \overline{b_n} q^{-n} u^{-n}.$$

Moreover, since χ is even, it means that $\mathcal{L}_q(u,\chi) = (1-u)\mathcal{L}_C(u)$ and it follows that $a_n = b_n - b_{n-1}$ for $n = 0, \dots, g$. Also this means $a_{g+1} = -b_g$. Therefore

$$b_n = a_0 + \cdots + a_n \tag{3.21}$$

for $n = 0, \dots, g$. Therefore we can write $\mathcal{L}_q(u,\chi)$ as

$$\mathcal{L}_q(u,\chi) = \sum_{n=0}^{A} b_n u^{\deg(f)} (1-u) + \omega(\chi) \sum_{n=0}^{g-A-1} \overline{b_n} (qu)^{\deg(f)} (1-u),$$

Now, using (3.21) for b_n and b_{n+1}, subtracting the two equations and using the functional equation for b_n we get

$$\overline{a_0} + \cdots + \overline{a_{g-n-1}} = \frac{1}{q-1} a_{n+1} \overline{\omega(\chi)} q^{g/2-n} + \frac{\overline{a_{g-n}}}{q-1},$$

which implies

$$a_0 + \cdots + a_{g-n-1} = \frac{1}{q-1} \overline{a_{n+1}} \omega(\chi) q^{g/2-n} + \frac{a_{g-n}}{q-1}.$$

If we use the above equations with $n = g - A - 1$ and $n = A$ we get, after some work,

$$\mathcal{L}_q(u,\chi) = \sum_{f \in \mathcal{M}_{q,\le A}} \chi(f) u^{\deg(f)} + \omega(\chi)\varphi(u,\chi) \sum_{f \in \mathcal{M}_{q,\le g-A-1}} \bar{\chi}(f)(qu)^{\deg(f)}$$
$$+ \frac{a_{A+1}}{(1-\sqrt{q})} u^{A+1} + \omega(\chi) \frac{\overline{a_{g-A}}}{(1-\sqrt{q})} u^{g-A}.$$

The result follows. $\square$

3.3 Expected value of cubic character sums

In this section, we prove a series of results in order to obtain the expected value for a cubic character sum that is necessary for the construction of our (k,l)-th moments in Chapter 4. After some preliminary work, we count the number of primitive cubic characters with conductor degree d before averaging a character sum over said family. A reminder that

$$\mathcal{N}(d) := \{\chi \text{ primitive cubic character} \mid \deg(\mathrm{cond}(\chi)) = d\}.$$

First we recall Perron's formula, which we will use throughout this section.

Lemma 3.3.1 (Perron's formula). *Suppose that the generating series*

$$\mathcal{A}(u) = \sum_{f \in \mathcal{M}_q} a(f) u^{deg(f)}$$

is absolutely convergent in the disk $|u| \leq r < 1$. Then

$$\sum_{f \in \mathcal{M}_{q,n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n} \frac{\mathrm{d}u}{u}$$

and

$$\sum_{f \in \mathcal{M}_{q,\leq n}} a(f) = \frac{1}{2\pi i} \oint_{|u|=r} \frac{\mathcal{A}(u)}{u^n(1-u)} \frac{\mathrm{d}u}{u},$$

where $\oint$ signifies the integral over the circle oriented counterclockwise.

3.3.1 Kummer setting

Lemma 3.3.2. *Suppose that $q \equiv 1 \pmod 3$ and that*

$$\mathcal{G}_{1,f}(u) = \prod_{P \nmid f} \left(1 + 2u^{deg(P)}\right),$$

where $f \in \mathbb{F}_q[T]$. Then $\mathcal{G}_{1,f}(u)$ has a double pole at $u = q^{-1}$ and

$$\mathcal{G}_{1,f}(u) = \frac{\mathcal{F}_1(u)}{(1-qu)^2} \prod_{P|f} \left(1 + 2u^{deg(P)}\right)^{-1},$$

where

$$\mathcal{F}_1(u) = \prod_{P} \left(1 - 3u^{2deg(P)} + 2u^{3deg(P)}\right). \tag{3.22}$$

Proof. We see, by some manipulation:

$$\begin{aligned}
\mathcal{G}_{1,f}(u) &= \prod_{P\nmid f} \left(1 + 2u^{\deg(P)}\right) \\
&= \prod_{P|f} \left(1 + 2u^{\deg(P)}\right)^{-1} \prod_{P} \left(1 + 2u^{\deg(P)}\right) \\
&= \prod_{P|f} \left(1 + 2u^{\deg(P)}\right)^{-1} \prod_{P} \left(1 + 2u^{\deg(P)}\right) \prod_{P} \left(1 - u^{\deg(P)}\right)^{-2} \prod_{P} \left(1 - u^{\deg(P)}\right)^{2} \\
&= \frac{1}{(1-qu)^2} \prod_{P|f} \left(1 + 2u^{\deg(P)}\right)^{-1} \prod_{P} \left(1 - 3u^{2\deg(P)} + 2u^{3\deg(P)}\right).
\end{aligned}$$

Next, we call $\mathcal{F}_1(u)$ the Euler product over all primes to arrive at the result.

Pulling out the zeta factors makes it clear that $G(u)$ has a double pole at $u = q^{-1}$ or equivalently $s = 1$. We also see that the rest converges since $\prod_{P|f} \left(1 + 2u^{\deg(P)}\right)^{-1}$ is a finite product and $\prod_{P} \left(1 - 3u^{2\deg(P)} + 2u^{3\deg(P)}\right)$ converges since $\sum_P |-3u^{2\deg(P)} + 2u^{3\deg(P)}|$ clearly converges. Moreover, $\mathcal{G}_{1,f}(u)$ is analytic for $|u| < q^{-1}$. $\square$

Corollary 3.3.3. *Suppose that $q \equiv 1 \pmod 3$ and that*

$$\mathcal{G}_1(u) = \prod_{P} \left(1 + 2u^{deg(P)}\right).$$

Then $\mathcal{G}_1(u)$ has a double pole at $u = q^{-1}$ and

$$\mathcal{G}_1(u) = \frac{\mathcal{F}_1(u)}{(1-qu)^2},$$

where $\mathcal{F}_1(u)$ is exactly the same as in Lemma 3.3.2.

Proof. This is clear from the proof of Lemma 3.3.2. $\square$

Lemma 3.3.4. *Suppose that $q \equiv 1 \pmod 3$, then*

$$\#\mathcal{N}(d) = C_{1,1} d q^d + C_{1,2} q^d + O\left(q^{(1/2+\epsilon)d}\right),$$

where

$$\begin{aligned} C_{1,1} &= -\mathcal{F}_1(q^{-1}), \\ C_{1,2} &= -\mathcal{F}_1(q^{-1}) + \frac{1}{q}\mathcal{F}_1'(q^{-1}), \end{aligned}$$

with $\mathcal{F}_1(u)$ is given as in Lemma 3.3.2.

Proof. Let χ_F be a primitive cubic Dirichlet character with conductor F having prime factorization $F = P_1^{e_1} \cdots P_t^{e_t}$. Multiplicativity of characters implies that $\chi_F = \chi_{P_1}^{e_1} \cdots \chi_{P_t}^{e_t}$. Since χ is a primitive cubic character, then all of the e_j's are either 1 or 2 (where $1 \leq j \leq t$). Therefore the conductors of primitive cubic characters are square-free monic polynomials in $\mathbb{F}_q[T]$. Let $a(F)$ be the number of characters of conductor F, then we have

$$a(F) = \begin{cases} 2^{\omega(F)} & \text{if } F \in \mathcal{H}_q, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{H}_q$ is the set of square-free monic polynomials in $\mathbb{F}_q[T]$. Then the generating series becomes

$$\sum_{F \in \mathcal{M}_q} a(F) u^{\deg(F)} = \prod_P \left(1 + 2u^{\deg(P)}\right) = \mathcal{G}_1(u).$$

Recognizing that the generating series is the same as given in Corollary 3.3.3, we apply Perron's formula by shifting the contour from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\epsilon)}$ for some $\epsilon > 0$ and pick up the double pole at $u = q^{-1}$. This means

$$
\begin{aligned}
\#\mathcal{N}(d) &= \frac{1}{2\pi i} \oint\limits_{|u|=q^{-2}} \frac{\mathcal{G}_1(u)}{u^{d+1}} \, \mathrm{d}u \\
&= \lim_{u \to q^{-1}} \frac{\mathrm{d}}{\mathrm{d}u} \left((u - q^{-1})^2 \frac{\mathcal{F}_1(u)}{(1-qu)^2 u^{d+1}} \right) \\
&= -\mathcal{F}_1(q^{-1}) d q^d + \left(-\mathcal{F}_1(q^{-1}) + \frac{1}{q} \mathcal{F}_1'(q^{-1}) \right) q^d + O\left(q^{(1/2+\epsilon)d} \right),
\end{aligned}
$$

which is the desired result. $\square$

Lemma 3.3.5. *Suppose that $q \equiv 1 \pmod 3$ and let $f = f_1 \cdots f_{k+l}$ where $k \geq l \geq 1$. Also suppose that χ is a primitive cubic character. If $f = \cube$, then*

$$
\langle \chi(f_1) \cdots \chi(f_{k+l}) \rangle_d = a_{1,f},
$$

where

$$
a_{1,f} = \prod_{P|f} \left(1 + 2q^{-deg(P)}\right)^{-1} = \prod_{P|f} \left(1 + \frac{2}{|P|}\right)^{-1}.
$$

Remark. The following 'proof' for Lemma 3.3.5 is technically incomplete because we do not consider the character sum when f is not a cube. The simple answer is that the recipe in [Con+05] dictates that we keep the cubic terms only, however, this answer does not adequately describe the asymptotic behavior of our character sum. Our intent in this small digression is to provide some justification. It would be natural to apply the function field version of the Pólya-Vinogradov inequality as a way of dealing with non-cubic terms as they seemingly contribute to the error term we will eventually get when conjecturing our moments in Chapter 4. (See [AK14] for the quadratic case of our problem of moments for an example of this). While this strategy works for low-order moments, when we start to consider higher order moments, Pólya-Vinogradov fails because these character sums begin to balloon in size at higher moments—shifting pieces that Pólya-Vinogradov would send to error term into the main term in ways that are not yet fully understood. Therefore, writing a 'complete' proof where we use the Pólya-Vinogradov inequality to send non-cubic terms to zero is misleading as it fails to illustrate the strange asymptotic behavior that these character sums take on as we consider higher order moments. See [DFL19] for more details.

Proof. We define

$$\langle \chi(f_1)\cdots\chi(f_{k+l})\rangle_d = \lim_{d\to\infty} \frac{1}{\#\mathcal{N}(d)} \sum_{\chi\in\mathcal{N}(d)} \chi(f_1)\cdots\chi(f_{k+l}),$$

where the sum is over all primitive cubic characters with conductor degree d. Since Dirichlet characters are multiplicative we have $\chi(f_1)\cdots\chi(f_{k+l}) = \chi(f_1\cdots f_{k+l}) = \chi(f)$.

Suppose that $f = \cube$. We write $f = m^3$ and since $\chi(f) = \chi(m^3) = \chi(m)^3 = 1$ for $(\mathrm{cond}(\chi), m) = 1$ and $\chi(m^3) = 0$ for $(\mathrm{cond}(\chi), m) > 1$, we have

$$\sideset{}{^*}\sum_{\substack{\chi^3=\chi_0\\ \deg(\mathrm{cond}(\chi))=d}} \chi\left(f = \cube\right) = \sideset{}{^*}\sum_{\substack{\chi^3=\chi_0\\ \deg(\mathrm{cond}(\chi))=d\\ (\mathrm{cond}(\chi),f)=1}} 1 = \sum_{\substack{\deg(F)=d\\ (F,f)=1}} a(F) = \frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{\mathcal{G}_{1,f}(u)}{u^{d+1}}\,\mathrm{d}u,$$

where $a(F)$ is the same as in Lemma 3.3.4.

Then we apply Perron's formula and find that the generating series is the same as the one given in Lemma 3.2.1. Therefore, we know that there is a double pole at $u = q^{-1}$ and so we shift our contour from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\epsilon)}$ for some $\epsilon > 0$ and we find that

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{|u|=q^{-2}} \frac{\mathcal{G}_{1,f}(u)}{u^{d+1}}\,\mathrm{d}u &= \lim_{u\to q^{-1}} \frac{\mathrm{d}}{\mathrm{d}u}\left((u-q^{-1})^2 \frac{\mathcal{F}_1(u)}{(1-qu)^2u^{d+1}} a_{1,f}\right) + O\left(q^{(1/2+\epsilon)d}\right)\\
&= \#\mathcal{N}(d)\prod_{P|f}\left(1+2q^{-\deg(P)}\right)^{-1}\\
&\quad + q^{d-1}\mathcal{F}_1(q^{-1})\prod_{P|f}\left(1+2q^{-\deg(P)}\right)^{-1}\left(-\sum_{P|f}\frac{2\deg(P)q^{1-\deg(P)}}{1+2q^{-\deg(P)}}\right)\\
&\quad + O\left(q^{(1/2+\epsilon)d}\right).
\end{aligned}$$

Dividing by $\#\mathcal{N}(d)$ means that the first term just becomes $a_{1,f}$ and the second and third terms go to 0 when we send $d \to \infty$, which is the desired result for $f = \cube$. □

Remark. It should be noted that if one were to be a bit more particular about the error term in the proof of Lemma 3.3.5, the additional power savings would result in the addition of another term so that the average value of our character sum is $a_{1,f} + Kd^{-1}$ where K is another factor involving a derivative of an Euler product. However—and this will become clear in Chapter 4—neglecting this second term is out of necessity because multiplicativity is an essential property that the heuristic framework from [Con+05] relies on.

3.3.2 Non-Kummer setting

Lemma 3.3.6. *Suppose* $q \equiv 2 \pmod 3$ *and that*

$$\mathcal{G}_{2,f}(u) = \prod_{\substack{P \nmid f \\ 2 \mid deg(P)}} \left(1 + 2u^{deg(P)}\right),$$

where $f \in \mathbb{F}_q[T]$*. Then* $\mathcal{G}_2(u)$ *has a simple poles at* $u = \pm q^{-1}$ *and*

$$\mathcal{G}_{2,f}(u) = \frac{\mathcal{F}_2(u)}{1 - q^2u^2} \prod_{\substack{P \mid f \\ 2 \mid deg(P)}} \left(1 + 2u^{deg(P)}\right)^{-1},$$

where

$$\mathcal{F}_2(u) = \prod_{\substack{P \\ 2 \mid deg(P)}} \left(1 - 3u^{2deg(P)} + 2u^{3deg(P)}\right) \prod_{\substack{P \\ 2 \nmid deg(P)}} \left(1 - u^{2deg(P)}\right).$$

is analytic for $|u| > q^{-1/2-\epsilon}$*, where* $\epsilon > 0$*.*

Proof. First, recall that

$$\mathcal{Z}_{q^2}(u^2) = \frac{1}{1 - q^2u} = \prod_{\substack{P \\ 2 \mid \deg(P)}} \left(1 - u^{\deg(P)}\right)^{-2} \prod_{\substack{P \\ 2 \nmid \deg(P)}} \left(1 - u^{2\deg(P)}\right)^{-1}.$$

We see, after some manipulation that

$$\begin{aligned}
\mathcal{G}_{2,f}(u) &= \prod_{\substack{P\nmid f\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right)\\
&= \prod_{\substack{P|f\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right)^{-1} \prod_{\substack{P\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right)\\
&= \frac{1}{1-q^2u^2} \prod_{\substack{P|f\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right)^{-1} \prod_{\substack{P\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right) \prod_{\substack{P\\ 2|\deg(P)}} \left(1-u^{\deg(P)}\right)^2\\
&\quad\times \prod_{\substack{P\\ 2\nmid\deg(P)}} \left(1-u^{2\deg(P)}\right).
\end{aligned}$$

If we let

$$\begin{aligned}
\mathcal{F}_2(u) &= \prod_{\substack{P\\ 2|\deg(P)}} \left(1+2u^{\deg(P)}\right) \prod_{\substack{P\\ 2|\deg(P)}} \left(1-u^{\deg(P)}\right)^2 \prod_{\substack{P\\ 2\nmid\deg(P)}} \left(1-u^{2\deg(P)}\right)\\
&= \prod_{\substack{P\\ 2|\deg(P)}} \left(1-3u^{2\deg(P)}+2u^{3\deg(P)}\right) \prod_{\substack{P\\ 2\nmid\deg(P)}} \left(1-u^{2\deg(P)}\right),
\end{aligned}$$

the result is proven. Moreover, analyticity is obvious. □

Corollary 3.3.7. *Suppose $q \equiv 2 \pmod 3$ and that*

$$\mathcal{G}_2(u) = \prod_{2|\,deg(P)} \left(1+2u^{deg(P)}\right),$$

where $f \in \mathbb{F}_q[T]$. Then $\mathcal{G}_2(u)$ has a simple poles at $u = \pm q^{-1}$ and

$$\mathcal{G}_2(u) = \frac{\mathcal{F}_2(u)}{1-q^2u^2}.$$

where $\mathcal{F}_2(u)$ is the same as in Lemma 3.3.6.

Lemma 3.3.8. *Suppose that $q \equiv 2 \pmod 3$, then*

$$\#\mathcal{N}(d) = \begin{cases} C_{2,1} q^d + O(q^{1/2+\epsilon}) & \text{if } 2 \mid d, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$C_{2,1} = \left(\frac{\mathcal{F}_2(q^{-1})}{2} + (-1)^d \frac{\mathcal{F}_2(-q^{-1})}{2} \right).$$

Proof. Suppose χ_F is a cubic Dirichlet character with conductor $F \in \mathbb{F}_q[T]$ having prime factorization $F = P_1^{e_1} \cdots P_t^{e_t}$ where the P_j's are distinct primes in $\mathbb{F}_q[T]$ for $1 \leq j \leq t$. It follows that $\chi_F = \chi_{P_1}^{e_1} \cdots \chi_{P_t}^{e_t}$ since characters are multiplicative. Since χ_F is a primitive cubic character, it is primitive if and only if all the e_j's are either 1 or 2 for every prime. Furthermore, since $q \equiv 2 \pmod 3$, this means that χ_P exists only for primes of even degree since $3 \nmid q^{\deg(P)} - 1$. Therefore, the conductors of primitive cubic characters are square-free monic polynomials supported on primes whose degree is even.

Let $a(F)$ denote the number of primitive square-free monic cubic characters with conductor F with primes of even degree. Then, the generating series of $a(F)$ is

$$\sum_{F \in \mathcal{M}_q} a(F) u^{\deg(F)} = \prod_{2 \mid \deg(P)} \left(1 + 2u^{\deg(P)}\right) = \mathcal{G}_2(u),$$

where $\mathcal{G}_2(u)$ is exactly the same as in Corollary 3.3.7. Therefore, we know that there are two simple poles at $u = \pm q^{-1}$ and that $\mathcal{G}_2(u)$ is analytic for $|u| < q^{-1}$. We apply Perron's formula and find

$$\#\mathcal{N}(d) = \sum_{F \in \mathcal{M}_q} a(F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\mathcal{G}_2(u)}{u^{d+1}} \, \mathrm{d}u.$$

We then shift the contour from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\epsilon)}$, where $\epsilon > 0$, and pick up the residue of the simple poles at $u = \pm q^{-1}$. After a simple computation we find

$$\#\mathcal{N}(d) = \left(\frac{\mathcal{F}_2(q^{-1})}{2} + (-1)^d \frac{\mathcal{F}_2(-q^{-1})}{2} \right) q^d + O\left(q^{(1/2+\epsilon)d}\right),$$

for d even which proves the result. $\square$

Lemma 3.3.9. *Suppose $q \equiv 2 \pmod 3$ and let $f = f_1 \cdots f_{k+l}$ where $k \geq l \geq 1$. Also suppose that χ is a primitive cubic character. If $f = \cube$, then*

$$\langle \chi(f_1) \cdots \chi(f_{k+l}) \rangle_d = a_{2,f}$$

where

$$a_{2,f} = \prod_{\substack{P|f \\ 2|deg(P)}} \left(1 + 2q^{deg(P)}\right)^{-1}.$$

Remark. The same remark for Lemma 3.3.5 applies here in the context of Lemma 3.3.9.

Proof. Recall that

$$\langle \chi(f_1) \cdots \chi(f_{k+l}) \rangle_d = \lim_{d \to \infty} \frac{1}{\#\mathcal{N}(d)} \sum_{\chi \in \mathcal{N}(d)} \chi(f_1) \cdots \chi(f_{k+l}),$$

where the sum is over all primitive cubic characters with conductor degree d. By multiplicativity, we have that $\chi(f_1) \cdots \chi(f_{k+l}) = \chi(f_1 \cdots f_{k+l}) = \chi(f)$.

Suppose that $f = \cube$, where $f = m^3$. Now, if $(h, m) = 1$ then $\chi(f) = 1$. Also, if $(h, m) > 1$ then $\chi(f) = 0$. Therefore, the sum becomes

$$\sideset{}{^*}\sum_{\substack{\chi^3 = \chi_0 \\ \deg(\operatorname{cond}(\chi)) = d}} \chi(f) = \sideset{}{^*}\sum_{\substack{\chi^3 = \chi_0 \\ \deg(\operatorname{cond}(\chi)) = d}} 1 = \sideset{}{^*}\sum_{\substack{\deg(F) = d \\ (F, f) = 1}} a(F) = \frac{1}{2\pi i} \oint_{|u| = q^{-2}} \frac{\mathcal{G}_{2,f}(u)}{u^{d+1}} \mathrm{d}u.$$

Next, we apply Perron's formula and recognize that the generating series of our sum is the same as the one in Lemma 3.3.6, that is, $\mathcal{G}_{2,f}(u) = \prod_{\substack{P \nmid f \\ 2|\deg(P)}} \left(1 + 2u^{\deg(P)}\right)$. Therefore, we know that $\mathcal{G}_{2,f}(u)$ has two simple poles at $u = \pm q^{-1}$ and is analytic for $|u| \leq q^{-1}$. So we shift the contour from $|u| = q^{-2}$ to $|u| = q^{-(1/2+\epsilon)}$, where $\epsilon > 0$, and we find that

$$\begin{aligned}
\frac{1}{\#\mathcal{N}(d)} \sum_h{}^{*} \chi(f) &= \prod_{\substack{P|f \\ 2|\deg(P)}} \left(1 + 2q^{-\deg(P)}\right)^{-1} + O\left(\frac{|h|^{1/2}}{\mathcal{N}_2(d)}\right) \\
&= \prod_{\substack{P|f \\ 2|\deg(P)}} \left(1 + 2q^{\deg(P)}\right)^{-1} + O\left(q^{-d/2}\right),
\end{aligned}$$

which means taking the limit as $d \to \infty$ sends the error term to 0. $\square$

Chapter 4

(k, l)-th moment of cubic L-functions

In this chapter, we apply the heuristic developed by Conrey, Farmer, Keating, Rubenstein, and Snaith [Con+05] in order to propose conjectural formulae for the (k, l)-th moment of L-functions summed over the family of cubic characters on the half-line.

First, a reminder on notation. We denote

$$\left\langle L_q\left(s, \chi\right)^k L_q\left(s, \bar{\chi}\right)^l \right\rangle_d := \frac{1}{\#\mathcal{N}(d)} \sum_{\chi \in \mathcal{N}(d)} L_q\left(s, \chi\right)^k L_q\left(s, \bar{\chi}\right)^l ,$$

as the (k, l)-th moment of $L_q(s, \chi)$. As defined in Chapter 3, $\mathcal{N}(d)$ is the set of primitive cubic characters having conductor degree d.

4.1 Shifted moments

As in [Con+05], we consider and work with *shifted moments* of our product of L-functions. By evaluating each L-function at a point slightly off the critical line, we are able to arrive at a conjecture by solving the combinatorics of the shifts and sending each shift to zero.

The *recipe*, as the authors in [Con+05] call it, makes use of the approximate functional equation in deriving the conjectured moments. By working with a product of shifted L-functions in the form of their approximate functional equations, we are able to pick the terms that survive because the coefficients of our cubic L-functions have an approximate orthogonality relation when averaged over the family of cubic characters. The approximate functional equation allows us to exploit this. First, we note a small, but important lemma.

Lemma 4.1.1. *Let χ denote an cubic Dirichlet character with conductor degree d. If*

$$\varphi(s,\chi) = |cond(\chi)|^{\frac{1}{2}-s} X(s) = q^{d\left(\frac{1}{2}-s\right)} X(s),$$

where

$$X(s) = \begin{cases} q^{-\frac{1}{2}+s}, & \text{if } \chi \text{ is odd,} \\ q^{-1+2s}\frac{1-q^{-s}}{1-q^{s-1}} & \text{if } \chi \text{ is even,} \end{cases}$$

then

$$\varphi(s,\chi)^{\frac{1}{2}} = \varphi(1-s,\chi)^{-\frac{1}{2}},$$

and

$$\varphi(s,\chi)\varphi(1-s,\chi) = 1.$$

Note that the factor $\varphi(s,\chi)$ is the same factor that is present in the functional equation (Lemma 3.2.1). We also work with a slightly adjusted L-function we denote by Z_L.

Definition 4.1.1.

$$Z_L(s,\chi) = \varphi(s,\chi)^{-1/2} L_q(s,\chi).$$

We also work with a product of shifted Z_L functions so we also define the following.

Definition 4.1.2. *Suppose that $|\Re\alpha_j| < \frac{1}{2}$ for $j = 1,\ldots,k+l$ where k,l are positive integers. Then we define*

$$Z_{(k)}(s;\alpha_1,\ldots,\alpha_k) = Z_L(s+\alpha_1,\chi)\cdots Z_L(s+\alpha_k,\chi),$$

and

$$\bar{Z}_{(l)}(s;\alpha_{k+1},\ldots,\alpha_{k+l}) = Z_L(s-\alpha_{k+1},\bar{\chi})\cdots Z_L(s-\alpha_{k+l},\bar{\chi}).$$

Remark. In Definition 4.1.2, we have resorted to using misleading notation. Note that $\bar{Z}_{(l)}$ is not actually a conjugate, even though the notation erroneously suggests so.

If we use the approximate functional equation (Proposition 3.2.2), noting the substitution $u = q^{-s}$, and Lemma 4.1.1, we can write

$$Z_L(s, \chi) = \varphi(s, \chi)^{-\frac{1}{2}} \sum_{f \in \mathcal{M}_{q, \leq A}} \frac{\chi(f)}{|f|^s} + \omega(\chi) \varphi(s, \chi)^{-\frac{1}{2}} \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \frac{\bar{\chi}(f)}{|f|^{1-s}}. \tag{4.1}$$

In arriving at the conjecture, we almost exclusively use the adjusted approximate functional equation (4.1) in place of our original definition. If χ is odd, we use the same approximate functional equation in Proposition 3.4.2 and if χ is even, we truncate the approximate functional equation at the first two terms so it looks exactly like (4.1). This is in line with the heuristic framework that the authors in [Con+05] describe. Again, we draw attention to the fact that we do not specify what the parameter A is in (4.1). As it turns out, defining this parameter is not essential for our purposes because we are required to extend these sums over all monic polynomials at some later point.

4.2 (k, l)-th moment of cubic L-functions

Similar to Chapter 3, we present the Kummer and non-Kummer cases separately. We will see that the difference in the conjectural formulae lies only in the arithmetic factor that takes the shape of an Euler product over the primes. The work to arrive at the conjecture is the same so we present it once when we derive the conjecture in the Kummer case and simply state the conjecture in the non-Kummer setting for brevity.

4.2.1 Kummer setting

We first begin by writing the product $Z_{(k)} \times \bar{Z}_{(l)} := Z_{(k)}\left(\frac{1}{2}; \alpha_1, \ldots, \alpha_k\right) \bar{Z}_{(l)}\left(\frac{1}{2}; \alpha_{k+1}, \ldots, \alpha_{k+l}\right)$, so we have a $k+l$ product of shifted L-functions. Explicitly, this is

$$
\begin{aligned}
Z_{(k)}\times\bar{Z}_{(l)} =& \left(\varphi\left(\frac{1}{2}+\alpha_1,\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le A}}\frac{\chi(f)}{|f|^{\frac{1}{2}+\alpha_1}}+\omega(\chi)\varphi\left(\frac{1}{2}-\alpha_1,\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le g-A-1}}\frac{\bar{\chi}(f)}{|f|^{\frac{1}{2}-\alpha_1}}\right)\\
&\times\cdots\times\left(\varphi\left(\frac{1}{2}+\alpha_k,\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le A}}\frac{\chi(f)}{|f|^{\frac{1}{2}+\alpha_k}}+\omega(\chi)\varphi\left(\frac{1}{2}-\alpha_k,\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le g-A-1}}\frac{\bar{\chi}(f)}{|f|^{\frac{1}{2}-\alpha_k}}\right)\\
&\times\left(\varphi\left(\frac{1}{2}-\alpha_{k+1},\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le A}}\frac{\bar{\chi}(f)}{|f|^{\frac{1}{2}+\alpha_{k+1}}}+\overline{\omega(\chi)}\varphi\left(\frac{1}{2}+\alpha_{k+1},\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le g-A-1}}\frac{\chi(f)}{|f|^{\frac{1}{2}-\alpha_{k+1}}}\right)\\
&\times\cdots\times\left(\varphi\left(\frac{1}{2}-\alpha_{k+l},\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le A}}\frac{\bar{\chi}(f)}{|f|^{\frac{1}{2}+\alpha_{k+l}}}+\overline{\omega(\chi)}\varphi\left(\frac{1}{2}+\alpha_{k+l},\chi\right)^{-\frac{1}{2}}\sum_{f\in\mathcal{M}_{q,\le g-A-1}}\frac{\chi(f)}{|f|^{\frac{1}{2}-\alpha_{k+l}}}\right).
\end{aligned}
\tag{4.2}
$$

because $\omega(\bar{\chi})=\overline{\omega(\chi)}$ and (3.6).

Although the product in (4.2) has 2^{k+l} terms, the recipe dictates that we keep the terms that do not oscillate. This means that any term that contains $\omega(\chi)$ or $\overline{\omega(\chi)}$ is discarded. If we are constructing a general term of (4.2), it is apparent that while there are no restrictions on how many times we can choose the principal term of the approximate functional equation, there are limitations on how many times we can choose the dual term of the approximate functional equation. If we choose any j dual terms from $Z_{(k)}$, we must also choose j dual terms from $\bar{Z}_{(l)}$ to eliminate $\omega(\chi)$. This means $0\le j\le l$. Therefore one example of a term from $Z_{(k)}\times\bar{Z}_{(l)}$ looks like:

$$
\begin{aligned}
&\left(\omega(\chi)\overline{\omega(\chi)}\right)^l\prod_{j=1}^{k-l}\varphi\left(\frac{1}{2}+\alpha_j,\chi\right)^{-\frac{1}{2}}\prod_{j=1}^{l}\varphi\left(\frac{1}{2}-\alpha_{k-l+j},\chi\right)^{-\frac{1}{2}}\prod_{j=1}^{l}\varphi\left(\frac{1}{2}+\alpha_{k+j},\chi\right)^{-\frac{1}{2}}\\
&\qquad\times\sum_{f_1}\frac{\chi(f_1)}{|f_1|^{\frac{1}{2}+\alpha_1}}\cdots\sum_{f_k}\frac{\bar{\chi}(f_k)}{|f_k|^{\frac{1}{2}-\alpha_k}}\sum_{f_{k+1}}\frac{\chi(f_1)}{|f_{k+1}|^{\frac{1}{2}-\alpha_{k+1}}}\cdots\sum_{f_{k+l}}\frac{\chi(f_{k+l})}{|f_{k+l}|^{\frac{1}{2}-\alpha_{k+l}}}
\end{aligned}
$$

where the first $k-l$ terms are chosen from the principal term of the Z_L—i.e., equation (4.1)—and the next $2l$ terms are chosen from the dual term of Z_L. We note that we have not forgotten about the degree of monic polynomials, f_j, but knowing that later on in the recipe [Con+05] that we need to extend the sum for all monic polynomials, we just do it here for simplicity. It follows that there are $\binom{k+l}{l}$ ways of constructing valid (i.e., terms that have no power of $\omega(\chi)$) terms of $Z_{(k)} \times \bar{Z}_{(l)}$. Therefore, we end up with the following:

$$Z_{(k)} \times \bar{Z}_{(l)} = \sum_{\sigma \in \Xi} \prod_{j=1}^{k+l} \varphi\left(\tfrac{1}{2} + \epsilon_j \alpha_{\sigma(j)}, \chi\right)^{-\frac{1}{2}} \sum_{f_1, \ldots, f_{k+l}} \frac{\chi\left(f_1 \cdots f_k\right) \overline{\chi\left(f_{k+1} \cdots f_{k+l}\right)}}{\prod_{j=1}^{k+l} |f_j|^{\frac{1}{2} + \epsilon_j \alpha_{\sigma(j)}}},$$

where $\epsilon_j = 1$ for $1 \leq j \leq k$ and $\epsilon_j = -1$ for $k+1 \leq j \leq k+l$ and where Ξ is the set of permutations in ${S_{kl}}/{S_k \times S_l}$.

Again, working with the recipe, we move to replace the average of the characters over the entire family. We note that

$$\begin{aligned} \chi\left(f_1 \cdots f_k\right) \overline{\chi\left(f_{k+1} \cdots f_{k+l}\right)} &= \chi\left(f_1 \cdots f_k\right) \chi\left(f_{k+1}^2 \cdots f_{k+l}^2\right) \\ &= \chi\left(f_1 \cdots f_k f_{k+1}^2 \cdots f_{k+l}^2\right) \end{aligned}$$

which means by Lemma 3.3.5 we have

$$\left\langle \chi\left(f_1 \cdots f_k f_{k+1}^2 \cdots f_{k+l}^2\right) \right\rangle_d = \prod_{P|f} \left(1 + \frac{2}{|P|}\right)^{-1}, \tag{4.3}$$

where $f = f_1 \cdots f_k f_{k+1}^2 \cdots f_{k+l}^2 = \cube$.

We therefore predict, using the heuristic from [Con+05], that

$$\left\langle Z_{(k)} \times \bar{Z}_{(l)} \right\rangle_d = \left\langle M_{k,l}\left(\alpha_1, \ldots, \alpha_{k+l}\right) \right\rangle_d,$$

where

$$M_{k,l}\left(\alpha_1, \ldots, \alpha_{k+l}\right) = \sum_{\sigma \in \Xi} \prod_{j=1}^{k+l} \varphi\left(\tfrac{1}{2} + \epsilon_j \alpha_{\sigma(j)}, \chi\right)^{-\frac{1}{2}} R_{k,l}\left(\epsilon_1 \alpha_{\sigma(1)}, \ldots, \epsilon_{k+l} \alpha_{\sigma(k+l)}\right),$$

and

$$R_{k,l}\left(\epsilon_1 \alpha_{\sigma(1)}, \ldots, \epsilon_{k+l} \alpha_{\sigma(k+l)}\right) = \sum_{\substack{f_1, \ldots, f_{k+1} \text{ monic} \\ f_1 \cdots f_k f_{k+1}^2 \cdots f_{k+l}^2 = f^3}} \frac{a_{2,f^3}}{\prod_{j=1}^{k+l} |f_j|^{\frac{1}{2} + \epsilon_j \alpha_{\sigma(j)}}},$$

where

$$a_{2,f^3} = \prod_{P|f^3} \left(1 + \frac{2}{|P|}\right)^{-1}.$$

Now, we search for the poles in $R_{k,l}$. First we define

$$\psi(x) = \sum_{\substack{f_i \text{ monic} \\ f_1 \cdots f_k f_{k+1}^2 \cdots f_{k+l}^2 = x}} \frac{1}{|f_1|^{\frac{1}{2}+\alpha_{\sigma(1)}} \cdots |f_{k+l}|^{\frac{1}{2}+\alpha_{\sigma(k+l)}}},$$

so we can write

$$\begin{aligned} R_{k,l}\left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+l)}\right) &= \sum_{f \text{ monic}} a_{2,f^3} \psi(f^3) \\ &= \prod_P \left(1 + \sum_{j=1}^{\infty} \left(1 + \frac{2}{|P|}\right)^{-1} \psi(P^{3j})\right), \end{aligned}$$

where

$$\psi(P^{3j}) = \sum_{\substack{e_1, \ldots, e_{k+l} \geq 0 \\ e_1 + \cdots + e_k + 2e_{k+1} + \cdots + 2e_{k+l} = 3j}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}},$$

since we have $f_i = P^{e_i}$ for $i = 1, \ldots, k+l$ and because $a_{2,P^{3j}} = (1 + 2|P|^{-1})^{-1}$. Then we have

$$\begin{aligned} R_{k,l}\left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k+l)}\right) &= \prod_P \left(1 + \sum_{j=1}^{\infty} \left(1 + \frac{2}{|P|}\right)^{-1} \psi(P^{3j})\right) \\ &= \prod_P \left(1 + \left(1 + \frac{2}{|P|}\right)^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1, \ldots, e_{k+l} \geq 0 \\ e_1 + \cdots + e_k + 2e_{k+1} + \cdots + 2e_{k+l} = 3j}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}}\right) \\ &= \prod_P R_{k,l,P}, \end{aligned}$$

where

$$R_{k,l,P} = 1 + \left(1 + \frac{2}{|P|}\right)^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1,\ldots,e_{k+l} \geq 0 \\ e_1+\cdots+e_k+2e_{k+1}+\cdots+2e_{k+l}=3j}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}}.$$

Moreover, since $(1+2|P|^{-1})^{-1} = \sum_{m=0}^{\infty} \left(\frac{-2}{|P|}\right)^m$, we can write

$$\begin{aligned}
R_{k,l,P} &= 1 + \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \sum_{\substack{e_1,\ldots,e_{k+l} \geq 0 \\ e_1+\cdots+e_k+2e_{k+1}+\cdots+2e_{k+l}=3j}} \prod_{i=1}^{k+l} \frac{(-2)^m}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)+m}} \\
&= 1 + \sum_{\substack{e_1,\ldots,e_{k+l} \geq 0 \\ e_1+\cdots+e_k+2e_{k+1}+\cdots+2e_{k+l}=3}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}} + (\text{lower order terms}) \\
&= 1 + \sum_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq k+l}} \frac{1}{|P|^{1+\alpha_{\sigma(i)}-\alpha_{\sigma(j)}}} + (\text{lower order terms})
\end{aligned}$$

We remark that the only terms that contain poles are the ones where every shift $\alpha_i = 0$ for $1 \leq i \leq k+l$, $j=1$, $m=0$ and where all but two of the e_i's are nonzero (i.e., $e_x = e_y = 1$ where $1 \leq x \leq k$ and $k+1 \leq y \leq k+l$). Because of that, we factor out an appropriate number of ζ-functions, $\zeta_q(1+\alpha_i-\alpha_j)$ and we are left with

$$R_{k,l}\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right) = A\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right) \prod_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq k+l}} \zeta_q(1+\alpha_i-\alpha_j),$$

where

$$A(\alpha_1,\ldots,\alpha_{k+l}) = \prod_{\substack{P \text{ monic} \\ \text{irreducible}}} \left(R_{k,l,P} \prod_{\substack{1 \leq i \leq k \\ k+1 \leq j \leq k+l}} \left(1 - \frac{1}{|P|^{1+\alpha_i-\alpha_j}}\right)\right).$$

Note that we can manipulate $R_{k,l,P}$ into

$$R_{k,l,P} = 1 + \left(1+\frac{2}{|P|}\right)^{-1} \sum_{j=1}^{\infty} \sum_{\substack{e_1,\dots,e_{k+l}\geq 0 \\ e_1+\cdots+e_k+2e_{k+1}+\cdots+2e_{k+l}=3j}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}}$$

$$= \left(1+\frac{2}{|P|}\right)^{-1} \left(\frac{2}{|P|} + \sum_{j=0}^{\infty} \sum_{\substack{e_1,\dots,e_{k+l}\geq 0 \\ e_1+\cdots+e_k+2e_{k+1}+\cdots+2e_{k+l}=3j}} \prod_{i=1}^{k+l} \frac{1}{|P|^{e_i\left(\frac{1}{2}+\alpha_{\sigma(i)}\right)}} \right)$$

$$= \left(1+\frac{2}{|P|}\right)^{-1} \left(\frac{2}{|P|} + \frac{1}{3} \left(\prod_{j=1}^{k+l} \left(1+\frac{1}{|P|^{\left(\frac{1}{2}+\alpha_{\sigma(j)}\right)}}\right)^{-1} + \prod_{j=1}^{k+l} \left(1-\frac{1}{|P|^{\left(\frac{1}{2}-\alpha_{\sigma(j)}\right)}}\right)^{-1} \right)\right)$$

This leaves us with

$$M_{k,l}\left(\alpha_1,\dots,\alpha_{k+l}\right) = \sum_{\sigma\in\Xi} \prod_{j=1}^{k+l} \varphi\left(\tfrac{1}{2}+\epsilon_j\alpha_{\sigma(j)},\chi\right)^{-\frac{1}{2}} A\left(\epsilon_1\alpha_{\sigma(1)},\dots,\epsilon_{k+l}\alpha_{\sigma(k+l)}\right) \prod_{\substack{1\leq i\leq k \\ k+1\leq j\leq k+l}} \zeta_q\left(1+\alpha_i-\alpha_j\right).$$

And now the conjecture becomes the following if we keep in mind Lemma 4.1.1:

$$\left\langle Z_L\left(\frac{1}{2}+\alpha_1,\chi\right)\cdots Z_L\left(\frac{1}{2}+\alpha_k,\chi\right) Z_L\left(\frac{1}{2}-\alpha_{k+1},\bar{\chi}\right)\cdots Z_L\left(\frac{1}{2}-\alpha_{k+l},\bar{\chi}\right)\right\rangle_d$$
$$= \left\langle \sum_{\sigma\in\Xi}\prod_{j=1}^{k+l} X\left(\frac{1}{2}+\epsilon_j\sigma(\alpha_j)\right)^{-\frac{1}{2}} R\left(\epsilon_1\alpha_{\sigma(1)},\dots,\epsilon_{k+l}\alpha_{\sigma(k+l)}\right) \times q^{\frac{d}{2}\sum_{j=1}^{k+l}\epsilon_j\alpha_{\sigma(j)}} \right\rangle_d.$$

We now turn to writing the contour integral representation of the conjecture with use of the following lemma adapted from [Con+05].

Lemma 4.2.1. *Suppose $F(a;b) = F(a_1,\dots,a_k;b_1,\dots,b_k)$ is a function of $k+l$ variables, which is symmetric with respect to the first k variables and also symmetric with respect to the second set of l variables. Suppose also that F is regular near $(0,\dots,0)$. Suppose further that $f(s)$ has a simple pole of residue 1 at $s=0$ but is otherwise analytic in a neighborhood* *about $s = 0$. Let*

$$K\left(a_1, \ldots, a_k; b_1, \ldots, b_k\right) = F\left(a_1, \ldots; \ldots, b_k\right) \prod_{i=1}^{k} \prod_{j=1}^{k} f(a_i - b_j) \tag{4.4}$$

If for all $1 \leq i, j \leq k$, $\alpha_i - \alpha_{k+j}$ is contained in a region of analyticity of $f(s)$ then

$$\sum_{\sigma \in \Xi} K\left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}; \alpha_{\sigma(k+1)}, \ldots, \alpha_{\sigma(2k)}\right) =$$
$$\frac{(-1)^{\frac{(k+l)(k+l-1)}{2}}}{k!l!} \frac{1}{(2\pi i)^{k+l}} \oint \cdots \oint \frac{K\left(z_1, \ldots, z_k; z_{k+1}, \ldots, z_{k+l}\right) \Delta\left(z_1, \ldots, z_{k+l}\right)^2}{\prod_{i=1}^{k+l} \prod_{j=1}^{k+l} (z_i - \alpha_j)} \mathrm{d}z_1 \cdots \mathrm{d}z_{k+l} \tag{4.5}$$

where one integrates about small circles enclosing the α_j's, and where Ξ is the set of permutations such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l)$.

Recall that since

$$Z_L(s, \chi) = \varphi(s, \chi)^{-1/2} L_q(s, \chi),$$

we have

$$\begin{aligned}
&\left\langle Z_L\left(\frac{1}{2} + \alpha_1, \chi\right) \cdots Z_L\left(\frac{1}{2} + \alpha_k, \chi\right) Z_L\left(\frac{1}{2} - \alpha_{k+1}, \bar{\chi}\right) \cdots Z_L\left(\frac{1}{2} - \alpha_{k+l}, \bar{\chi}\right)\right\rangle_d \\
&\qquad = \left\langle \prod_{j=1}^{k+l} \varphi\left(\frac{1}{2} + \epsilon_j \alpha_j, \chi\right)^{-\frac{1}{2}} L_q\left(\frac{1}{2} + \alpha_1, \chi\right) \cdots L_q\left(\frac{1}{2} + \alpha_k, \chi\right)\right. \\
&\qquad\qquad \left. \times L_q\left(\frac{1}{2} - \alpha_{k+1}, \bar{\chi}\right) \cdots L_q\left(\frac{1}{2} - \alpha_{k+l}, \bar{\chi}\right)\right\rangle_d.
\end{aligned} \tag{4.6}$$

Furthermore, knowing that $\varphi(s, \chi) = q^{d\left(\frac{1}{2} - s\right)} X(s)$, we know that

$$\prod_{j=1}^{k+l} \varphi\left(\frac{1}{2} + \epsilon_j \alpha_j, \chi\right)^{-\frac{1}{2}} = q^{\frac{d}{2}\left(-\sum_{i=1}^{k} \alpha_i + \sum_{j=k+1}^{k+l} \alpha_j\right)} \prod_{j=1}^{k+l} X\left(\frac{1}{2} + \epsilon_j \alpha_j\right)^{-\frac{1}{2}}. \tag{4.7}$$

Therefore, using (4.6) and (4.7), we can write

$$\begin{aligned}&\left\langle L_q\left(\frac{1}{2}+\alpha_1,\chi\right)\cdots L_q\left(\frac{1}{2}+\alpha_k,\chi\right)L_q\left(\frac{1}{2}+\alpha_{k+1},\bar{\chi}\right)\cdots L_q\left(\frac{1}{2}+\alpha_{k+l},\bar{\chi}\right)\right\rangle_d\\&\quad=\left\langle\prod_{j=1}^{k+l}X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{\frac{1}{2}}q^{-\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}\sum_{\sigma\in\Xi}\prod_{j=1}^{k+l}X\left(\frac{1}{2}+\epsilon_j\alpha_{\sigma(j)}\right)^{\frac{1}{2}}\right.\\&\qquad\left.\times A\left(\epsilon_1\alpha_{\sigma(1)},\ldots,\epsilon_{k+l}\alpha_{\sigma(k+l)})\right)q^{\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_{\sigma(i)}-\sum_{j=k+1}^{k+l}\alpha_{\sigma(j)}\right)}\prod_{\substack{1\le i\le k\\k+1\le j\le k+l}}\zeta_q\left(1+\alpha_i-\alpha_j\right)\right\rangle_d.\end{aligned}$$

In order to use Lemma 4.2.1, we take out a factor of $(\log q)$ to get

$$\begin{aligned}&\left\langle L_q\left(\frac{1}{2}+\alpha_1,\chi\right)\cdots L_q\left(\frac{1}{2}+\alpha_k,\chi\right)L_q\left(\frac{1}{2}+\alpha_{k+1},\bar{\chi}\right)\cdots L_q\left(\frac{1}{2}+\alpha_{k+l},\bar{\chi}\right)\right\rangle_d\\&\quad=\left\langle\frac{\prod_{j=1}^{k+l}X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{\frac{1}{2}}q^{-\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}}{(\log q)^{kl}}\sum_{\sigma\in\Xi}\prod_{j=1}^{k+l}X\left(\frac{1}{2}+\epsilon_j\alpha_{\sigma(j)}\right)^{\frac{1}{2}}\right.\\&\left.\times A\left(\epsilon_1\alpha_{\sigma(1)},\ldots,\epsilon_{k+l}\alpha_{\sigma(k+l)}\right)q^{\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}\prod_{\substack{1\le i\le k\\k+1\le j\le k+l}}\zeta_q\left(1+\alpha_i-\alpha_j\right)(\log q)\right\rangle_d.\end{aligned}\tag{4.8}$$

If we let

$$\begin{aligned}F\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right)=\prod_{j=1}^{k+l}X\left(\frac{1}{2}+\epsilon_j\alpha_{\sigma(j)}\right)^{-\frac{1}{2}}A\left(\epsilon_1\alpha_{\sigma(1)},\ldots,\epsilon_{k+l}\alpha_{\sigma(k+l)}\right)&\\\times q^{\frac{d}{2}\left(\sum_{i=1}^{k+l}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)},&\end{aligned}$$

and also

$$f(s)=\zeta_q(1+s)\log q.$$

This implies

$$f(\alpha_i-\alpha_j)=\zeta_q(1+\alpha_i-\alpha_j)\log q,$$

and

$$K\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right)=F\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right)\prod_{1\le i<j\le k+l} f(\alpha_i-\alpha_j).$$

Therefore (4.8) becomes

$$\left\langle \frac{\prod_{j=1}^{k+l} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{\frac{1}{2}} q^{-\frac{d}{2}\left(\sum_{i=1}^{k+l}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}}{(\log q)^{kl}} \sum_{\sigma\in\Xi} K\left(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(k+l)}\right)\right\rangle_d .$$

Next, we replace $\sum_{\sigma\in\Xi} K(\alpha_1,\ldots,\alpha_{k+l})$ with the $k+l$ contour integral (and the additional factors) from the lemma and get

$$\left\langle \frac{\prod_{j=1}^{k+l} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{\frac{1}{2}} q^{-\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}}{(\log q)^{kl}} \right.$$
$$\left. \times \frac{(-1)^{(k+l)(k+l-1)}}{k!l!}\frac{1}{(2\pi i)^{k+l}}\oint\cdots\oint K\left(z_1,\ldots,z_{k+l}\right)\frac{\Delta\left(z_1,\ldots,z_{k+l}\right)^2}{\prod_{i=1}^{k+l}\prod_{j=1}^{k+l}(z_i-\alpha_j)}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}\right\rangle_d .$$

After substituting $K(z_1,\ldots,z_{k+l})=F(z_1,\ldots,z_{k+l})\prod_{1\le i<j\le k+l} f(z_i-z_j)$ and some rearranging and cancelling we find

$$\left\langle \prod_{j=1}^{k+l} X\left(\frac{1}{2}+\epsilon_j\alpha_j\right)^{\frac{1}{2}} q^{-\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)}\frac{(-1)^{(k+l)(k+l-1)}}{k!l!}\frac{1}{(2\pi i)^{k+l}}\right.$$
$$\left.\times\oint\cdots\oint F(z_1,\ldots,z_{k+l})\prod_{1\le i<j\le k+l}\zeta_q(1+z_i-z_j)\frac{\Delta\left(z_1,\ldots,z_{k+l}\right)^2}{\prod_{i=1}^{k+l}\prod_{j=1}^{k+l}(z_i-\alpha_j)}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}\right\rangle_d . \tag{4.9}$$

Moreover, let

$$\begin{aligned} G\left(z_1,\ldots,z_{k+l}\right) &= F\left(z_1,\ldots,z_{k+l}\right)\\ &=\prod_{j=1}^{k+l} X\left(\frac{1}{2}+\epsilon_j z_j\right)^{-\frac{1}{2}} A\left(z_1,\ldots,z_{k+l}\right). \end{aligned}$$

Then (4.8) becomes

$$\left\langle \prod_{j=1}^{k+l} X\left(\frac{1}{2}+\alpha_j\right)^{\frac{1}{2}} q^{-\frac{d}{2}\left(\sum_{i=1}^{k}\alpha_i-\sum_{j=k+1}^{k+l}\alpha_j\right)} \right.$$
$$\times \frac{(-1)^{(k+l)(k+l-1)}}{k!l!}\frac{1}{(2\pi i)^{k+l}}\oint\cdots\oint G(z_1,\ldots,z_{k+l})q^{\frac{d}{2}\left(\sum_{i=1}^{k}z_i-\sum_{j=k+1}^{k+l}z_j\right)}$$
$$\left. \times \prod_{1\le i<j\le k+l}\zeta_q(1+z_i-z_j)\frac{\Delta\left(z_1,\ldots,z_{k+l}\right)^2}{\prod_{i=1}^{k+l}\prod_{j=1}^{k+l}(z_i-\alpha_j)}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}\right\rangle_d.$$

Finally, we send all of the α_i's to 0 in order to arrive the following:

Conjecture 4.2.2 (Conjecture in Kummer setting)**.** *Suppose that $q\equiv 1 \pmod 3$ is an odd prime power, where q is the cardinality of the finite field $\mathbb{F}_q$. Also let*

$$X(s)=\begin{cases} q^{-1/2+s} & \text{if } \chi \text{ odd,}\\ q^{-1+2s}\frac{1-q^{-s}}{1-q^{s-1}} & \text{if } \chi \text{ even.}\end{cases}$$

Then,

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^k L_q\left(\tfrac{1}{2},\bar{\chi}\right)^l\right\rangle_d = Q_{k,l}\left(d\right),$$

where k,l are non-zero integers and $Q_{k,l}$ is a polynomial given by the $(k+l)$-fold residues

$$Q_{k,l}(x)=\frac{(-1)^{(k+l)(k+l-1)/2}}{k!l!}\frac{1}{(2\pi i)^{k+l}}\oint\cdots\oint G(z_1,\ldots,z_{k+l})$$
$$\times\frac{\Delta\left(z_1,\ldots,z_{k+l}\right)^2}{\prod_{i=1}^{k+l}z_i^{k+l}}q^{\frac{x}{2}\left(\sum_{i=1}^{k}z_i-\sum_{j=k+1}^{k+l}z_j\right)}\prod_{1\le i<j\le k+l}\zeta_q(1+z_i-z_j)\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l},$$

where $\Delta(z_1,\ldots,z_{k+l})$ is the Vandermonde determinant given by

$$\Delta(z_1,\ldots,z_{k+l})=\prod_{1\le i<j<k+l}(z_j-z_i),$$

and

$$G\left(z_1,\ldots,z_{k+l}\right)=A\left(z_1,\ldots,z_{k+l}\right)\prod_{j=1}^{k}X\left(\tfrac{1}{2}+z_j\right)^{-\frac{1}{2}}\prod_{j=k+1}^{k+l}X\left(\tfrac{1}{2}-z_j\right)^{-\frac{1}{2}}$$

where also, $A(z_1,\ldots,z_{k+l})$ is an Euler product, absolutely convergent for $|\Re z_j|<\frac{1}{2}$, given by

$$\begin{aligned}A(z_1,\ldots,z_{k+l})=\prod_{P}\prod_{\substack{1\le i\le k\\ k+1\le j\le k+l}}&\left(1+\frac{1}{|P|^{(1+z_i-z_j)}}\right)\\ &\times\left(\frac{2}{|P|}+\frac{1}{3}\left(\prod_{j=1}^{k+l}\left(1+\frac{1}{|P|^{(1/2+z_j)}}\right)^{-1}+\prod_{j=1}^{k+l}\left(1-\frac{1}{|P|^{(1/2-z_j)}}\right)^{-1}\right)\right)\\ &\times\left(1+\frac{2}{|P|}\right)^{-1}.\end{aligned}$$

4.2.2 Non-Kummer setting

When $q\equiv 2 \pmod 3$, we have $\langle\chi(f_1)\cdots\chi(f_{k+l})\rangle=a_{2,f^3}$ where a_{2,f^3} is as described in Lemma 3.3.9. The work that follows is the same with the exception that the Euler product is over primes P of even degree instead of being over all primes.

Conjecture 4.2.3 (Conjecture in non-Kummer setting)**.** *Suppose $q\equiv 2(3)$ is an odd prime power, where q is the cardinality of the finite field $\mathbb{F}_q$. Then the statements in Conjecture 4.2.2 apply with the only difference being that the Euler product $A(z_1,\ldots,z_{k+l})$ is given by*

$$\begin{aligned}A(z_1,\ldots,z_{k+l})=\prod_{\substack{P\\ 2|deg(P)}}\prod_{\substack{1\le i\le k\\ k+1\le j\le k+l}}&\left(1+\frac{1}{|P|^{(1+z_i-z_j)}}\right)\\ &\times\left(\frac{2}{|P|}+\frac{1}{3}\left(\prod_{j=1}^{k+l}\left(1+\frac{1}{|P|^{(1/2+z_j)}}\right)^{-1}+\prod_{j=1}^{k+l}\left(1-\frac{1}{|P|^{(1/2-z_j)}}\right)^{-1}\right)\right)\\ &\times\left(1+\frac{2}{|P|}\right)^{-1}.\end{aligned}$$

4.3 Explicit computation of low-level moments

In this section, we compute some low-level moments using Conjecture 4.2.2 and Conjecture 4.2.3 (the Kummer and non-Kummer setting respectively). In the interest of brevity, we show nearly all of the work involved in the computation of the contour integrals for the first moment but omit the work for the higher-level moments.

As partial derivatives appear in these computations, we denote the j-th variable partial derivative of the Euler product $A(z_1, \ldots, z_{k+l})$ as $A_j(z_1, \ldots, z_{k+l})$. We repeat indices in the subscript for higher order partial derivatives. For example, this means

$$\frac{\partial^6 A}{\partial^2 z_1 \partial z_2 \partial^3 z_3}(z_1, z_2, z_3)\Big|_{z_1=z_2=z_3=0} = A_{1,1,2,3,3,3}(0,0,0).$$

Remark. It went otherwise unmentioned before but we draw the reader's attention to an important point that comes up in the context of computing our moments. It follows from our conjecture (based on the contour integral) that it makes sense computationally to treat the cases where χ is odd and even separately, then to find their sum. In Chapter 3, we found the expected value of $\chi(f_1)\cdots\chi(f_k)\overline{\chi(f_{k+1})\cdots\chi(f_{k+l})}$ over $\mathcal{N}(d)$, the family of primitive cubic characters (Lemma 3.3.5 and Lemma 3.3.9). This, of course, is the expected value for our character sum over all characters—both odd and even. Using this expected value en route to writing our conjecture (Conjecture 4.2.2 and Conjecture 4.2.3) means that while we can still consider the odd and even cases separately we have to make an adjustment to the expected value we used in (4.3). It turns out that in the non-Kummer case, there are only even characters so we do not need to make any adjustment when $q \equiv 2 \pmod 3$. In the Kummer case, or when $q \equiv 1 \pmod 3$, it can be shown that two-thirds of $\mathcal{N}(d)$ is odd and one-thirds is even. Consequently, we need only to include the factor $\frac{1}{3}$ or $\frac{2}{3}$ in $Q_{k,l}(x)$ depending on the appropriate circumstances. The reader can consult [DFL19] for more specific details on the matter.

$(1,1)$**-moment**

When $k=l=1$ and χ odd, we have

$$Q_{1,1}(x)=\frac{(-1)^{2(1)/2}}{(2\pi i)^2}\oint\oint\frac{G(z_1,z_2)\Delta\left(z_1,z_2\right)^2\zeta_q(1+z_1-z_2)}{z_1^2z_2^2}q^{\frac{x}{2}(z_1-z_2)}\,\mathrm{d}z_1\mathrm{d}z_2$$

We recall that

$$G\left(z_1,z_2\right)=X\left(\tfrac{1}{2}+z_1\right)^{-\frac{1}{2}}X\left(\tfrac{1}{2}-z_2\right)^{-\frac{1}{2}}A\left(z_1,z_2\right)$$

where $A(z_1,z_2)$ is the Euler product defined in Conjecture 4.2.2 or Conjecture 4.2.3. Since χ is odd, $X(s)=q^{-1/2+s}$. This means that

$$G\left(z_1,z_2\right)=q^{-\frac{z_1}{2}}q^{\frac{z_2}{2}}A\left(z_1,z_2\right)$$

Moreover, recall that $\Delta(z_1,z_2)^2=(z_2-z_1)^2$ so our polynomial becomes

$$Q_{1,1}(x)=\frac{-1}{(2\pi i)^2}\oint\oint\frac{A(z_1,z_2)q^{\frac{x}{2}(z_1-z_2)}\left(z_2-z_1\right)^2\zeta_q(1+z_1-z_2)}{\left(z_1z_2\right)^2}q^{\frac{-z_1+z_2}{2}}\,\mathrm{d}z_1\mathrm{d}z_2$$

Since we compute the contour integral as an iterated integral, we can rearrange the terms in the above expression to get

$$Q_{1,1}(x)=\frac{-1}{(2\pi i)}\oint\frac{q^{-\frac{x}{2}z_2}q^{\frac{z_2}{2}}}{z_2^2} \times\left[\frac{1}{(2\pi i)}\oint\frac{A(z_1,z_2)\left(z_2-z_1\right)^2\zeta_q(1+z_1-z_2)}{z_1^2}q^{\frac{-z_1}{2}}q^{\frac{x}{2}z_1}\,\mathrm{d}z_1\right]\mathrm{d}z_2$$

and let the integrand of the first contour integral be denoted by $I_1(z_1)$. It's clear that $I_1(z_1)$ has a double pole at $z_1=0$. Now we note the series expansions of the factors of $I_1(z_1)$ are

$$\frac{A\left(z_1,z_2\right)}{z_1^2}=\frac{A\left(0,z_2\right)}{z_1^2}+\frac{A_1\left(0,z_2\right)}{z_1}+\frac{1}{2}A_{1,1}\left(0,z_2\right)+O\left(z_1\right),$$

$$q^{\frac{x}{2}z_1} = 1 + \left(\frac{x\log q}{2}\right) z_1 + \left(\frac{x\log q}{2}\right)^2 \frac{z_1^2}{2} + O\left(z_1^3\right),$$
$$q^{\frac{-z_1}{2}} = 1 - \left(\frac{\log q}{2}\right) z_1 + \left(\frac{\log q}{2}\right)^2 \frac{z_1^2}{2} + O\left(z_1^3\right),$$
$$\zeta_q(1+z_1-z_2) = \zeta_q(1-z_2) + \frac{\partial\zeta_q}{\partial z_1}(1-z_2)z_1 + \frac{1}{2}\frac{\partial^2\zeta_q}{\partial z_1^2}(1-z_2)z_1^2 + O(z_1^3).$$

Therefore, we have

$$\begin{aligned}
I_1(z_1) = &\left(\frac{A\left(0,z_2\right)}{z_1^2} + \frac{A_1\left(0,z_2\right)}{z_1} + \frac{1}{2}A_{1,1}\left(0,z_2\right) + \cdots\right) \\
&\times\left(1 + \left(\frac{x\log q}{2}\right) z_1 + \left(\frac{x\log q}{2}\right)^2 \frac{z_1^2}{2} + \cdots\right)\left(z_2 - z_1\right)^2 \\
&\times\left(1 - \left(\frac{\log q}{2}\right) z_1 + \left(\frac{\log q}{2}\right)^2 \frac{z_1^2}{2} + \cdots\right) \\
&\times\left(\zeta_q(1-z_2) + \frac{\partial\zeta_q}{\partial z_1}(1-z_2)z_1 + \frac{1}{2}\frac{\partial^2\zeta_q}{\partial z_1^2}(1-z_2)z_1^2 + \cdots\right),
\end{aligned}$$

and we collect the coefficient of the $1/z_1$ term and by the Cauchy residue theorem we have

$$\begin{aligned}
R_1(z_2) := \frac{1}{2\pi i}\oint I_1\,\mathrm{d}z_1 &= \operatorname*{Res}_{z_1=0} I_1 \\
&= \frac{A\left(0,z_2\right)\zeta_q(1-z_2)(\log q)(x-1)z_2^2}{2} \\
&\quad + A\left(0,z_2\right)\frac{\partial\zeta_q}{\partial z_1}(1-z_2)z_2^2 \\
&\quad + \zeta_q(1-z_2)\left(-2A\left(0,z_2\right)z_2 + A_1\left(0,z_2\right)z_2^2,\right)
\end{aligned}$$

where we let the $R_1(z_2)$ be the value of the computed integral. Therefore we find

$$Q_{1,1}(x) = \frac{-1}{(2\pi i)}\oint \frac{q^{\frac{x}{2}z_2}q^{\frac{z_2}{2}}}{z_2^2}R_1(z_2)\,\mathrm{d}z_2.$$

Again we let integrand of the above contour integral be denoted by $I_2(z_2)$ and we remark that it too has a double pole at $z_2 = 0$. By the residue theorem, we know that $Q_{1,1}(x) =$

$\mathrm{Res}_{z_2=0} I_2(z_2)$. With the help of the computer algebra system MAPLE, we find

$$Q_{1,1}(x) = (A(0,0)\log q)\, x + A(0,0)\log q + A_1(0,0) - A_2(0,0) + 2\gamma A(0,0). \tag{4.10}$$

When χ is even, $X(s) = q^{-1+2s}\frac{1-q^{-s}}{1-q^{s-1}}$. Therefore we have

$$Q^*_{1,1}(x) = \frac{1}{(2\pi i)^2}\oint\oint \frac{A(z_1,z_2)q^{-\frac{x}{2}(z_1-z_2)}\Delta\left(z_1,z_2\right)^2\zeta(1+z_1-z_2)}{\left(z_1z_2\right)^2} \times q^{-z_1+z_2}\left(\frac{1-q^{-\frac{1}{2}-z_1}}{1-q^{-\frac{1}{2}+z_1}}\right)^{-\frac{1}{2}}\left(\frac{1-q^{-\frac{1}{2}+z_2}}{1-q^{-\frac{1}{2}+z_2}}\right)^{-\frac{1}{2}} \mathrm{d}z_1\mathrm{d}z_2,$$

and with the help of a computer algebra system, we compute and find

$$Q^*_{1,1}(x) = (A(0,0)\log q)\, x + \frac{1}{\sqrt{q}-1}\Bigg(\sqrt{q}\left((2\gamma - 2\log q)\, A(0,0) + A_1(0,0) - A_2(0,0)\right) - 2\gamma A(0,0) - A_1(0,0) + A_2(0,0)\Bigg). \tag{4.11}$$

This leads us to conjecture that

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)L_q\left(\tfrac{1}{2},\overline{\chi}\right)\right\rangle_d \sim \begin{cases} \frac{2}{3}Q_{1,1}(d) + \frac{1}{3}Q^*_{1,1}(d) & \text{if } q\equiv 1 \pmod 3,\\ Q^*_{1,1}(d) & \text{if } q\equiv 2 \pmod 3,\end{cases}$$

that is,

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)L_q\left(\tfrac{1}{2},\overline{\chi}\right)\right\rangle_d \sim (A(0,0)\log q)d.$$

$(2,1)$**-moment**

After computing

$$Q_{2,1}(x) = \frac{(-1)^{\frac{3(2)}{2}}}{(2!1!)(2\pi i)^3} \oint \oint \oint \frac{A(z_1,z_2,z_3) q^{\frac{x}{2}(z_1+z_2-z_3)} \Delta(z_1,z_2,z_3)^2}{(z_1 z_2 z_3)^3} \times \zeta_q(1+z_1-z_3)\zeta_q(1+z_2-z_3) q^{\frac{-z_1-z_2+z_3}{2}} \,\mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3,$$

and

$$Q^*_{2,1}(x) = \frac{(-1)^{\frac{3(2)}{2}}}{(2!1!)(2\pi i)^3} \oint \oint \oint \frac{A(z_1,z_2,z_3) q^{\frac{x}{2}(z_1+z_2-z_3)} \Delta(z_1,z_2,z_3)^2}{(z_1 z_2 z_3)^3} \times \zeta_q(1+z_1-z_3)\zeta_q(1+z_2-z_3) q^{-z_1-z_2+z_3} \times \left(\frac{1-q^{-\frac{1}{2}-z_1}}{1-q^{-\frac{1}{2}+z_1}}\right)^{-\frac{1}{2}} \left(\frac{1-q^{-\frac{1}{2}-z_2}}{1-q^{-\frac{1}{2}+z_2}}\right)^{-\frac{1}{2}} \left(\frac{1-q^{-\frac{1}{2}+z_3}}{1-q^{-\frac{1}{2}-z_3}}\right)^{-\frac{1}{2}} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3,$$

with the help of a computer algebra system, we conjecture for $k = 2$, $l = 1$ that

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\overline{\chi}\right)\right\rangle_d \sim \begin{cases} \frac{2}{3}Q_{2,1}(d) + \frac{1}{3}Q^*_{2,1}(d) & \text{if } q \equiv 1 \pmod 3, \\ Q^*_{2,1}(d) & \text{if } q \equiv 2 \pmod 3, \end{cases}$$

where

$$Q_{2,1}(x) = \left(\frac{1}{2}A(0,0,0)\log^2 q\right) x^2 + O(x),$$

$$Q^*_{2,1}(x) = \left(\frac{1}{2}A(0,0,0)\log^2 q\right) x^2 + O(x),$$

so

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\overline{\chi}\right)\right\rangle_d \sim \left(\frac{1}{2}A(0,0,0)\log^2 q\right) d^2.$$

$(2,2)$**-moment**

Once we compute

$$Q_{2,2}(x) = \frac{(-1)^{\frac{4(3)}{2}}}{(2!2!)(2\pi i)^4} \oint\oint\oint \frac{A(z_1,z_2,z_3,z_4) q^{\frac{x}{2}(z_1+z_2-z_3-z_4)} \Delta\left(z_1,z_2,z_3,z_4\right)^2}{\left(z_1 z_2 z_3 z_4\right)^4}$$
$$\times \zeta_q(1+z_1-z_3)\zeta_q(1+z_1-z_4)\zeta_q(1+z_2-z_3)\zeta_q(1+z_2-z_4) q^{\frac{-z_1-z_2+z_3+z_4}{2}} \,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4,$$

and

$$Q^*_{2,2}(x) = \frac{(-1)^{\frac{4(3)}{2}}}{(2\pi i)^4} \oint\oint\oint \frac{A(z_1,z_2,z_3,z_4) q^{\frac{x}{2}(z_1+z_2-z_3-z_4)} \Delta\left(z_1,z_2,z_3,z_4\right)^2}{\left(z_1 z_2 z_3 z_4\right)^4}$$
$$\times \zeta_q(1+z_1-z_3)\zeta_q(1+z_1-z_4)\zeta_q(1+z_2-z_3)\zeta_q(1+z_2-z_4)$$
$$\times q^{-z_1-z_2+z_3+z_4} \left(\frac{1-q^{-\frac{1}{2}-z_1}}{1-q^{-\frac{1}{2}+z_1}}\right)^{-\frac{1}{2}} \left(\frac{1-q^{-\frac{1}{2}-z_2}}{1-q^{-\frac{1}{2}+z_2}}\right)^{-\frac{1}{2}}$$
$$\times \left(\frac{1-q^{-\frac{1}{2}+z_3}}{1-q^{-\frac{1}{2}-z_3}}\right)^{-\frac{1}{2}} \left(\frac{1-q^{-\frac{1}{2}+z_4}}{1-q^{-\frac{1}{2}-z_4}}\right)^{-\frac{1}{2}} \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4,$$

with the help of a computer algebra system, we conjecture for $k=l=2$ that

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\overline{\chi}\right)^2\right\rangle_d \sim \begin{cases} \frac{2}{3}Q_{2,2}(d) + \frac{1}{3}Q^*_{2,2}(d) & \text{if } q \equiv 1 \pmod 3, \\ Q^*_{2,2}(d) & \text{if } q \equiv 2 \pmod 3, \end{cases}$$

where

$$Q_{2,2}(x) = \left(\frac{1}{12}A(0,0,0,0)\log^4 q\right)x^4 + O\left(x^3\right),$$

$$Q^*_{2,2}(x) = \left(\frac{1}{12}A(0,0,0)\log^4 q\right)x^4 + O\left(x^3\right).$$

Therefore we conjecture

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\overline{\chi}\right)\right\rangle_d \sim \left(\frac{1}{12}A(0,0,0,0)\log^4 q\right) d^4.$$

Remark. More complete data from these computations can be found in the appendix.

4.4 Combinatorics of $g_{k,l}$

From the explicit computations shown in the previous section, it is evident that the leading coefficient of $Q_{k,l}(x)$ has a distinct form as a product of a arithmetic factor over the primes, $\log^{kl} q$, and a combinatorial factor. Because $Q_{k,l}(x)$ is a polynomial in x of degree kl, we can avoid the tedious parts of computing these iterated contour integrals by dividng $Q_{k,l}(x)$ by x^{kl} and taking the limit as $x \to \infty$. This idea is taken up in the following lemma.

Lemma 4.4.1. *Let $Q_{k,l}(x)$ be defined as in Conjecture 4.2.2 and Conjecture 4.2.3. Then,*

$$g_{k,l} = \lim_{x\to\infty} \frac{Q_{k,l}(x)}{x^{kl}\log^{kl} q} = \frac{A(0,\ldots,0)}{2^{kl}(2\pi i)^{k+l}} \oint \cdots \oint \frac{\Delta\left(z_1,\ldots,z_{k+l}\right)\Delta\left(z_1,\ldots,z_k\right)\Delta\left(z_{k+1},\ldots,z_{k+l}\right)}{\left(z_1\cdots z_{k+l}\right)^{k+l}} \times e^{\sum_{i=1}^{k} z_i - \sum_{j=k+1}^{k+l} z_j}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}$$

Proof. Let $z_i = z_i/(\log q)x/2$, then we can rearrange the the polynomial $Q_{k,l}(x)$ as follows:

$$\begin{aligned}
Q_{k,l}(x) &= \frac{(-1)^{(k+l)(k+l-1)/2}}{(k!l!)(2\pi i)^{k+l}} \oint \cdots \oint G\left(z_1,\ldots,z_{k+l}\right) \prod_{1\le i<j\le k+l} \zeta_q\left(1+z_i-z_j\right) \\
&\quad \times \frac{\Delta\left(z_1,\ldots,z_{k+l}\right)^2}{\left(z_1\cdots z_{k+l}\right)^{k+l}} e^{\frac{x\log q}{2}\left(\sum_{i=1}^{k} z_i - \sum_{j=k+1}^{k+l} z_j\right)}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l} \\
&= \frac{(-1)^{(k+l)(k+l-1)/2}}{(k!l!)(2\pi i)^{k+l}} \oint \cdots \oint G\left(\frac{z_1}{(\log q)x/2},\ldots,\frac{z_{k+l}}{(\log q)x/2}\right) \\
&\quad \times \prod_{1\le i<j\le k+l} \zeta_q\left(1+\frac{z_i}{(\log q)x/2}-\frac{z_j}{(\log q)x/2}\right) \frac{\Delta\left(\frac{z_1}{(\log q)x/2},\ldots,\frac{z_{k+l}}{(\log q)x/2}\right)^2}{\left(\frac{z_1}{(\log q)x/2}\cdots\frac{z_{k+l}}{(\log q)x/2}\right)^{k+l}} \\
&\quad \times e^{\frac{(\log q)x}{2}\left(\sum_{i=1}^{k}\frac{z_i}{(\log q)x/2} - \sum_{j=k+1}^{k+l}\frac{z_j}{(\log q)x/2}\right)} \frac{2^{kl}}{x^{kl}(\log^{kl} q)}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l} \\
&= \frac{(-1)^{(k+l)(k+l-1)/2}}{(k!l!)(2\pi i)^{k+l}} \oint \cdots \oint G\left(\frac{z_1}{(\log q)x/2},\ldots,\frac{z_{k+l}}{(\log q)x/2}\right) \\
&\quad \times \left(1+\frac{2\gamma}{x(\log q)}\left(l\sum_{i=1}^{k} z_i - k\sum_{j=1}^{k} z_j\right) + O\left(x^{-2}\right)\right) \\
&\quad \times \frac{\Delta\left(z_1,\ldots,z_{k+l}\right)\Delta\left(z_1,\ldots,z_k\right)\Delta\left(z_{k+1},\ldots,z_{k+l}\right)}{\left(z_1\cdots z_{k+l}\right)^{k+l}} e^{\sum_{i=1}^{k} z_i - \sum_{j=k+1}^{k+l} z_j}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}.
\end{aligned}$$

This is because we see that

$$
\begin{aligned}
\prod_{i=1}^{k+l}\left(\frac{z_i}{(\log q)x/2}\right)^{k+l} &= \left(\frac{2}{x(\log q)}\right)^{(k+l)(k+l)}\prod_{i=1}^{k+l} z_i^{k+l},\\
\mathrm{d}z_1\cdots\mathrm{d}z_{k+l} &= \left(\frac{2}{x(\log q)}\right)^{(k+l)}\mathrm{d}z_1\cdots\mathrm{d}z_{k+l},\\
\Delta\left(\tfrac{z_1}{(\log q)x/2},\dots,\tfrac{z_{k+l}}{(\log q)x/2}\right)^2 &= \left(\frac{2}{x(\log q)}\cdots\left(\frac{2}{x(\log q)}\right)^{(k+l-1)}\right)^2\Delta\left(z_1,\dots,z_{k+l}\right)^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{2}{x(\log q)}\cdots\left(\frac{2}{x(\log q)}\right)^{(k+l-1)}\right)^2\times\left(\frac{x(\log q)}{2}\right)^{(k+l)(k+l)}\times\left(\frac{2}{x(\log q)}\right)^{(k+l)}\\
&\qquad\qquad=\left(\frac{x(\log q)}{2}\right)^{(k+l)(k+l)-2(k+l)(k+l-1)-(k+l)}=1.
\end{aligned}
$$

Moreover, since we have

$$
\begin{aligned}
\zeta_q\left(1+\tfrac{z_i}{(\log q)x/2}-\tfrac{z_j}{(\log q)x/2}\right) &= \frac{(\log q)x/2}{z_i-z_j}+\gamma+c_1\left(\tfrac{z_i}{(\log q)x/2}-\tfrac{z_j}{(\log q)x/2}\right)+\cdots\\
&= \frac{x(\log q)}{2}\left(\frac{1}{z_i-z_j}+\frac{2\gamma}{x(\log q)}+O(x^{-2})\right),
\end{aligned}
$$

and if we note that

$$
\Delta\left(z_1,\dots,z_{k+l}\right)^2=\Delta\left(z_1,\dots,z_{k+l}\right)\Delta\left(z_1,\dots,z_k\right)\Delta\left(z_{k+1},\dots,z_{k+l}\right)\prod_{1\le i<j\le k+l}(z_i-z_j).
$$

This means taking into account

$$
\begin{aligned}
&\prod_{1\le i<j\le k+l}(z_i-z_j)\prod_{1\le i<j\le k+l}\zeta_q\left(1+\tfrac{z_i}{(\log q)x/2}-\tfrac{z_j}{(\log q)x/2}\right)\\
&\qquad\qquad=\frac{x^{kl}}{2^{kl}}\left(1+\frac{2\gamma}{x(\log q)}\left(l\sum_{i=1}^{k}z_i-k\sum_{j=1}^{k}z_j\right)+O\left(x^{-2}\right)\right),
\end{aligned}
$$

leads to the result above (4.16).

From here, it's clear that

$$\lim_{x\to\infty}\frac{Q_{k,l}(x)}{x^{kl}\log^{kl}q}=\frac{A(0,\dots,0)}{(2\pi i)^{k+l}}\oint\cdots\oint\frac{\Delta\left(z_1,\dots,z_{k+l}\right)\Delta\left(z_1,\dots,z_k\right)\Delta\left(z_{k+1},\dots,z_{k+l}\right)}{\left(z_1\cdots z_{k+l}\right)^{k+l}}$$
$$\times\, e^{\sum_{i=1}^{k}z_i-\sum_{j=k+1}^{k+l}z_j}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l},$$

if we recall that

$$G\left(\frac{z_1}{(\log q)x/2},\dots,\frac{z_{k+l}}{(\log q)x/2}\right)=A\left(\frac{z_1}{(\log q)x/2},\dots,\frac{z_{k+l}}{(\log q)x/2}\right)$$
$$\times\prod_{j=1}^{k}X\left(\frac{1}{2}+\frac{z_j}{(\log q)x/2}\right)^{-\frac{1}{2}}\prod_{j=k+1}^{k+l}X\left(\frac{1}{2}-\frac{z_j}{(\log q)x/2}\right)^{-\frac{1}{2}},$$

and realize that sending $x\to\infty$ means that $G\to A$ since the products of $X(s)$ go to 1 as x goes to infinity and also that the terms in $O(x^{-1})$ vanish as well. $\square$

Lemma 4.4.1 makes explicit that the combinatorics of the $k+l$ shifts that we constructed in the first section of Chapter 4 determine the factor that is present in the leading coefficient of $Q_{k,l}(x)$ and is represented as a $(k+l)$-fold contour integral. Before trying to find a general form, we compute this factor for low-level moments for the purposes of demystification as well as to show that they match the explicit computations made in the previous section.

$(1,1)$**-moment**

$$\begin{aligned}
g_{1,1} &= \lim_{x\to\infty} \frac{Q_{1,2}(x)}{xA(0,0)\log q} = \frac{1}{2^{1(1)}(1!1!)(2\pi i)^2} \oint\oint \frac{\Delta(z_1,z_2)\,\Delta(z_1)\,\Delta(z_2)}{(z_1z_2)^2} e^{z_1-z_2}\,\mathrm{d}z_1\mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)^2} \oint\oint \frac{(z_2-z_1)}{(z_1z_2)^2} e^{z_1-z_2}\,\mathrm{d}z_1\mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)^2} \oint \frac{e^{-z_2}}{z_2^2} \left(\oint \frac{(z_2-z_1)e^{z_1}}{z_1^2}\,\mathrm{d}z_1 \right) \mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)^2} \oint \frac{e^{-z_2}}{z_2^2} \left((2\pi i) \operatorname*{Res}_{z_1=0} \left[\frac{(z_2-z_1)e^{z_1}}{z_1^2} \right] \right) \mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)} \oint \frac{e^{-z_2}}{z_2^2} \left(\lim_{z_1\to 0} \frac{\mathrm{d}}{\mathrm{d}z_1} \left[z_1^2 \frac{z_2 e^{z_1}}{z_1^2} \right] - \lim_{z_1\to 0} \left[z_1 \frac{z_1 e^{z_1}}{z_1^2} \right] \right) \mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)} \oint \frac{e^{-z_2}}{z_2^2} (z_2-1)\mathrm{d}z_2 \\
&= \frac{1}{2(2\pi i)} \left((2\pi i) \operatorname*{Res}_{z_2=0} \left[\frac{e^{-z_2}}{z_2^2}(z_2-1) \right] \right) \\
&= \frac{1}{2} \left(\lim_{z_2\to 0} \left[z_2 \frac{e^{-z_2}}{z_2} \right] - \lim_{z_2\to 0} \frac{\mathrm{d}}{\mathrm{d}z_2} \left[z_2^2 \frac{e^{-z_2}}{z_2^2} \right] \right) \\
&= \frac{1}{2}(1+1) \\
&= 1
\end{aligned}$$

$(2,1)$**-moment**

$$\begin{aligned}
g_{2,1} &= \lim_{x\to\infty} \frac{Q_{2,1}(x)}{x^2A(0,0,0)\log^2 q} \\
&= \frac{(-1)^{3(2)/2}}{2^{2(1)}(2!1!)(2\pi i)^3} \oint\oint\oint \frac{\Delta(z_1,z_2,z_3)\,\Delta(z_1,z_2)\,\Delta(z_3)}{(z_1z_2,z_3)^3} e^{z_1+z_2-z_3}\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \frac{-1}{2^{2(1)}(2!1!)(2\pi i)^3} \oint\oint\oint \frac{(z_3-z_2)(z_3-z_1)(z_2-z_1)^2}{(z_1z_2z_3)^3} e^{z_1+z_2-z_3}\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \frac{-1}{8(2\pi i)^3} \oint \frac{e^{-z_3}}{z_3^3} \oint \frac{e^{z_2}}{z_2^3} \oint \left(\frac{(z_3-z_2)(z_3-z_1)(z_2-z_1)^2 e^{z_1}}{z_1^3} \right) \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \frac{-1}{8(2\pi i)^2} \oint \frac{e^{-z_3}}{z_3^3} \oint \frac{e^{z_2}}{z_2^3} \left(-\frac{(z_2^2z_3 - 2z_2(z_2+2z_3) + 2z_2 + 2z_3)(z_2-z_3)}{2} \right) \mathrm{d}z_2\mathrm{d}z_3 \\
&= \frac{-1}{8(2\pi i)} \oint \frac{e^{-z_3}}{z_3^3} \oint \frac{e^{z_2}}{z_2^3} \left(z_3^2 - \frac{4z_3^2 - 2z_3}{2} + z_3 - 2 \right) \mathrm{d}z_3 \\
&= \frac{-1}{8}(-4) = \frac{1}{2}
\end{aligned}$$

$(2,2)$**-moment**

$$
\begin{aligned}
g_{2,2} &= \lim_{x\to\infty} \frac{Q_{2,1}(x)}{x^4 A(0,0,0)\log^4 q} \\
&= \frac{(-1)^{4(3)/2}}{2^{2(2)}(2!2!)(2\pi i)^4} \oint\oint\oint\oint \frac{\Delta\left(z_1,z_2,z_3,z_4\right)\Delta\left(z_1,z_2\right)\Delta\left(z_3,z_4\right)}{\left(z_1z_2z_3z_4\right)^4} e^{z_1+z_2-z_3-z_4}\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4 \\
&= \frac{1}{2^{2(2)}(2!2!)(2\pi i)^4} \oint\oint\oint\oint \frac{(z_4-z_3)^2(z_4-z_2)(z_4-z_1)(z_3-z_2)(z_3-z_1)(z_2-z_1)^2}{\left(z_1z_2z_3z_4\right)^4} \\
&\qquad \times e^{z_1+z_2-z_3-z_4}\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4 \\
&= \frac{1}{64}\left(\frac{16}{3}\right) \\
&= \frac{1}{12}
\end{aligned}
$$

In order to find the general form of $g_{k,l}$, we follow [Con+05] and rewrite the Vandermonde determinant as a sum over permutations;

$$
\begin{aligned}
\Delta\left(z_1,\ldots,z_{k+l}\right) &= \sum_{\sigma} \operatorname{sgn}(\sigma) z_1^{\sigma(0)} z_2^{\sigma(1)} \cdots z_k^{\sigma(k-1)} \cdots z_{k+l}^{\sigma(k+l-1)}, \\
\Delta\left(z_1,\ldots,z_k\right) &= \sum_{\tau} \operatorname{sgn}(\tau) z_1^{\tau(0)} \cdots z_k^{\tau(k-1)}, \\
\Delta\left(z_{k+1},\ldots,z_{k+l}\right) &= \sum_{\rho} \operatorname{sgn}(\rho) z_{k+1}^{\rho(0)} \cdots z_{k+l}^{\rho(l-1)},
\end{aligned}
$$

where σ are permutations of $\{0,1,\ldots,k+l-1\}$, τ are permutations of $\{0,1,\ldots,k-1\}$, and ρ are permutations of $\{0,1,\ldots,l-1\}$.

Then by Lemma 4.4.1:

$$\lim_{x\to\infty}\frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}(\log^{kl}q)}=\frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}k!l!(2\pi i)^{k+l}}\oint\cdots\oint e^{\sum_{i=1}^{k}z_i-\sum_{j=k+1}^{k+l}z_j}$$
$$\times\left(\sum_{\sigma}\operatorname{sgn}(\sigma)z_1^{\sigma(0)}z_2^{\sigma(1)}\cdots z_k^{\sigma(k-1)}\cdots z_{k+l}^{\sigma(k+l-1)}\right)$$
$$\times\left(\sum_{\tau}\operatorname{sgn}(\tau)z_1^{\tau(0)}\cdots z_k^{\tau(k-1)}\right)$$
$$\times\left(\sum_{\rho}\operatorname{sgn}(\rho)z_{k+1}^{\rho(0)}\cdots z_{k+l}^{\rho(l-1)}\right)$$
$$\times z_1^{-k-l}\cdots z_{k+l}^{-k-l}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}.$$

Because the integrand is symmetric with respect to $z_1,\ldots,z_k$ and also symmetric with respect to $z_{k+1},\ldots,z_{k+l}$, in each term of the sum over τ we permute the variables so that z_j appears with the exponent $j-1$, for $j=1,\ldots,k$. This redefines the permutations of σ and the additional sign cancels $\operatorname{sgn}(\tau)$. The same is done for the sum over ρ and we are left with $k!l!$ copies of the sum. Therefore:

$$\lim_{x\to\infty}\frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl}q}=\frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}(2\pi i)^{k+l}}\oint\cdots\oint e^{\sum_{i=1}^{k}z_i-\sum_{j=k+1}^{k+l}z_j}$$
$$\times\sum_{\sigma}\operatorname{sgn}(\sigma)z_1^{-(k+l-\sigma(0))}z_2^{-(k+l-\sigma(1)-1)}\cdots z_k^{-(k+l-\sigma(k-1)-(k-1))}$$
$$\times z_{k+1}^{-(k+l-\sigma(k))}z_{k+2}^{-(k+l-\sigma(k)-1)}\cdots z_{k+l}^{-(k+l-\sigma(k+1-1)-(l-1))}\,\mathrm{d}z_1\cdots\mathrm{d}z_{k+l}.$$

Because

$$\frac{1}{\Gamma(z)}=\frac{1}{(2\pi i)}\oint_C(-t)^{-z}e^{-z}\,\mathrm{d}z,$$

where C is the path of integration that starts at $+\infty$ on the real line circling the origin counterclockwise and returns to its starting point. We can rewrite our integral as:

$$\begin{aligned}\lim_{x\to\infty}\frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl}q} &= \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}}\sum_{\sigma}\operatorname{sgn}(\sigma)\Bigg(\Gamma(k+l-\sigma(0))\Gamma(k+l-\sigma(1)-1)\cdots\\ &\qquad \times\Gamma(k+l-\sigma(k-1)-(k-1))\\ &\qquad \times(-1)^{\sigma(k)}\Gamma(k+l-\sigma(k))(-1)^{\sigma(k+)+1}\Gamma(k+l-\sigma(k)-1)\cdots\\ &\qquad \times(-1)^{\sigma(k+l-1)+l-1}\Gamma(k+l-\sigma(k+1-1)-(l-1))\Bigg)\\ &= \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}}\\ &\times\begin{vmatrix} \frac{1}{\Gamma(k+l)} & \frac{1}{\Gamma(k+l-1)} & \cdots & \frac{1}{\Gamma(l+1)} & \frac{1}{\Gamma(k+l)} & \frac{-1}{\Gamma(k+l-1)} & \cdots & \frac{(-1)^{l-1}}{\Gamma(k+1)}\\ \frac{1}{\Gamma(k+l-1)} & \frac{1}{\Gamma(k+l-2)} & \cdots & \frac{1}{\Gamma(l)} & \frac{-1}{\Gamma(k+l-1)} & \frac{1}{\Gamma(k+l-2)} & \cdots & \frac{(-1)^{l}}{\Gamma(k)}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{1}{\Gamma(2-k)} & \frac{-1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{(-1)^{k+2l-2}}{\Gamma(2-l)} \end{vmatrix}.\end{aligned}$$

Then we multiply by the first row by $(k+l-1)!$ to get

$$\lim_{x\to\infty}\frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl}q} = \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}}\times\frac{1}{(k+l-1)!}$$
$$\times\begin{vmatrix} \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1}(k-1)! & \binom{k+l-1}{0} & -\binom{k+l-1}{1} & \cdots & (-1)^{l-1}\binom{k+l-1}{l-1}(l-1)!\\ \frac{1}{\Gamma(k+l-1)} & \frac{1}{\Gamma(k+l-2)} & \cdots & \frac{1}{\Gamma(l)} & \frac{-1}{\Gamma(k+l-1)} & \frac{1}{\Gamma(k+l-2)} & \cdots & \frac{(-1)^{l}}{\Gamma(k)}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{1}{\Gamma(2-k)} & \frac{-1}{\Gamma(1)} & \frac{1}{\Gamma(0)} & \cdots & \frac{(-1)^{k+2l-2}}{\Gamma(2-l)} \end{vmatrix},$$

and then the second row by $(k+l-2)!$ and the third row by $(k+l-3)!$ and so on which results in finding

$$\lim_{x\to\infty}\frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl}q} = \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}}\times\frac{1}{\prod_{j=0}^{k+l-1}j!}$$

$$\times \begin{vmatrix} \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1}(k-1)! & \binom{k+l-1}{0} & -\binom{k+l-1}{1} & \cdots & (-1)^{l-1}\binom{k+l-1}{l-1}(l-1)! \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1}(k-1)! & -\binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & (-1)^{l}\binom{k+l-2}{l-1}(l-1)! \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1}(k-1)! & (-1)^{k+l-1}\binom{0}{0} & (-1)^{k+l}\binom{0}{1} & \cdots & (-1)^{k+2l-2}\binom{0}{l-1}(l-1)! \end{vmatrix}.$$

We divide the first column by 0!, the second by 1! up until the k-th column which we divide by $(k-1)!$. Then we divide the $(k+1)$-th column by 0!, the $(k+2)$-th column by 1! up until the$(k+l)$-th column, which we divide by $(l-1)!$ to get the following:

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!}$$
$$\times \begin{vmatrix} \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} & \binom{k+l-1}{0} & -\binom{k+l-1}{1} & \cdots & (-1)^{l-1}\binom{k+l-1}{l-1} \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1} & -\binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & (-1)^{l}\binom{k+l-2}{l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} & (-1)^{k+l-1}\binom{0}{0} & (-1)^{k+l}\binom{0}{1} & \cdots & (-1)^{k+2l-2}\binom{0}{l-1} \end{vmatrix}.$$

Then, we swap the rows of the determinant and multiply columns $k+1$ to $k+l-1$ by $(-1)^{k+l-1}$,

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!} \times (-1)^{(k+l)(k+l-1)/2} \times (-1)^{(k+l-1)l}$$
$$\times \begin{vmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} & \binom{0}{0} & -\binom{0}{1} & \cdots & (-1)^{l-1}\binom{0}{l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1} & (-1)^{k+l}\binom{k+l-2}{0} & (-1)^{k+l-1}\binom{k+l-2}{1} & \cdots & (-1)^{k+2l-3}\binom{k+l-2}{l-1} \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} & (-1)^{k+l-1}\binom{k+l-1}{0} & (-1)^{k+l}\binom{k+l-1}{1} & \cdots & (-1)^{k+2l-2}\binom{k+l-1}{l-1} \end{vmatrix}.$$

If we take the first k columns of the above matrix, we can write a $k+l \times k+l$ matrix

$$M_k = \begin{vmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k+l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k+l-1} \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k+l-1} \end{vmatrix},$$

which has the same first k rows of our previous matrix. We also note that M_k is a lower triangular matrix and $\det M_k = 1$. Again, taking the last l columns of our matrix (i.e., columns $k+1$ to $k+l$), and continuing it in order to create a $k+l \times k+l$ matrix,

$$M_l = \begin{vmatrix} -\binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k+l-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{k+l}\binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & (-1)^{l}\binom{k+l-2}{k+l-1} \\ (-1)^{k+l-1}\binom{k+l-1}{0} & (-1)^{k+l}\binom{k+l-1}{1} & \cdots & (-1)^{k+l-1}\binom{k+l-1}{k+l-1} \end{vmatrix}.$$

We also remark that M_l happens to be the inverse of M_k. We multiply our expression by $\det M_k$ since it does not alter the value and find:

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{(-1)^{(k+l)(k+l-1)/2}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!} \times (-1)^{(k+l)(k+l-1)/2} \times (-1)^{(k+l-1)l}$$
$$\times \begin{vmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k+l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k+l-1} \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k+l-1} \end{vmatrix}$$
$$\times \begin{vmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} & \binom{0}{0} & -\binom{0}{1} & \cdots & (-1)^{l-1}\binom{0}{l-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1} & (-1)^{k+l}\binom{k+l-2}{0} & (-1)^{k+l-1}\binom{k+l-2}{1} & \cdots & (-1)^{k+2l-3}\binom{k+l-2}{l-1} \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} & (-1)^{k+l-1}\binom{k+l-1}{0} & (-1)^{k+l}\binom{k+l-1}{1} & \cdots & (-1)^{k+2l-2}\binom{k+l-1}{l-1} \end{vmatrix}$$

$$= \frac{(-1)^{kl}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!} \times \begin{vmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} & 1 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{k-1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{l-1}{0} & \binom{l-1}{1} & \cdots & \binom{l-1}{k-1} & 0 & 0 & \cdots & 1 \\ \binom{l}{0} & \binom{l}{1} & \cdots & \binom{l}{k-1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1} & 0 & 0 & \cdots & 0 \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

Because the bottom right block of our determinant is 0 and the top right is the identity, our determinant is simply the bottom left $k \times k$ block. Therefore we write

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{(-1)^{kl}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!} \times \begin{vmatrix} \binom{l}{0} & \binom{l}{1} & \cdots & \binom{l}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k+l-2}{0} & \binom{k+l-2}{1} & \cdots & \binom{k+l-2}{k-1} \\ \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} \end{vmatrix}.$$

We reverse the order of the rows and see

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{(-1)^{k(k-1)/2}}{2^{kl}} \times \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!} \times \begin{vmatrix} \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2}{0} & \binom{2}{1} & \cdots & \binom{2}{k-1} \\ \binom{l}{0} & \binom{l}{1} & \cdots & \binom{l}{k-1} \end{vmatrix}.$$

Finally we notice that the matrix in the above expression can be decomposed as

$$\begin{pmatrix} \binom{k+l-1}{0} & \binom{k+l-1}{1} & \cdots & \binom{k+l-1}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{2}{0} & \binom{2}{1} & \cdots & \binom{2}{k-1} \\ \binom{l}{0} & \binom{l}{1} & \cdots & \binom{l}{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} \binom{k-1}{0} & \binom{k-1}{1} & \cdots & \binom{k-1}{k-1} \\ \binom{k-2}{0} & \binom{k-2}{1} & \cdots & \binom{k-2}{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{k-1} \end{pmatrix} \begin{pmatrix} \binom{l}{0} & \binom{l}{1} & \cdots & \binom{l}{k-1} \\ \binom{l}{-1} & \binom{l}{0} & \cdots & \binom{l}{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{l}{1-l} & \binom{l}{2-l} & \cdots & \binom{0}{0} \end{pmatrix}.$$

Now because the first matrix on the right hand side is zero in the lower right triangle with determinant $(-1)^{k(k-1)/2}$ while the other matrix on the right hand side is upper triangular with determinant 1, we therefore have

$$\lim_{x\to\infty} \frac{Q_{k,l}(x)}{A(0,\ldots,0)x^{kl}\log^{kl} q} = \frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!}.$$

So we see that, if we let $g_{k,l} = (kl)!\frac{\prod_{j=0}^{k-1} j! \prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!}$, we have,

$$\left\langle L\left(\tfrac{1}{2},\chi\right) L\left(\tfrac{1}{2},\bar{\chi}\right) \right\rangle_d \sim \frac{g_{k,l}}{(kl)!} A\left(0,\ldots,0\right) \log^{kl} q.$$

Remark. The factor $(kl)!$ is added in order for $g_{k,l}$ to be an integer. This is standard normalization.

Chapter 5

Conjectural formulae for moments

Conjecture 5.0.1. *Suppose that χ is a cubic Dirichlet character in $\mathbb{F}_q[T]$ with conductor of degree d and also suppose that $L_q(s,\chi)$ is its associated L-function. If $k \geq l \geq 1$, then*

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^k L_q\left(\tfrac{1}{2},\bar{\chi}\right)^l\right\rangle_d \sim \left(\frac{g_{k,l}a_{k,l}}{(kl)!}\log^{kl} q\right) d^{kl} + c_{kl-1}d^{kl-1} + \cdots + c_0,$$

where

$$g_{k,l} = (kl)!\frac{\prod_{j=0}^{k-1} j!\prod_{j=0}^{l-1} j!}{\prod_{j=0}^{k+l-1} j!},$$

and $a_{k,l} = A(0,\ldots,0)$, given that

$$A(z_1,\ldots,z_{k+l}) = \begin{cases} \prod_P R_{k,l}\left(1+\frac{2}{|P|}\right)^{-1} & \text{if } q \equiv 1 \pmod 3, \\ \prod_{\substack{P \\ 2|deg(P)}} R_{k,l}\left(1+\frac{2}{|P|}\right)^{-1} & \text{if } q \equiv 2 \pmod 3, \end{cases}$$

and

$$R_{k,l} = \prod_{\substack{1\leq i\leq k \\ k+1\leq j\leq k+l}}\left(1+\frac{1}{|P|^{(1+z_i-z_j)}}\right)\left(\frac{2}{|P|}+\frac{1}{3}\left(\prod_{j=1}^{k+l}\left(1+\frac{1}{|P|^{(1/2+z_j)}}\right)^{-1}+\prod_{j=1}^{k+l}\left(1-\frac{1}{|P|^{(1/2-z_j)}}\right)^{-1}\right)\right).$$

Appendix A

Explicit computations

This section contains more 'complete' data from the explicit computations of contour integrals presented in Chapter 4, Section 3. These computations result in a polynomial in d of degree kl. Although it is our intention to present every coefficient, it becomes clear that the higher order moments are much more computationally tedious and produce larger coefficients. Moreover, the information that is gained from the presentation of this data is rather minimal.

Recall that if $q \equiv 1 \pmod 3$, then two-thirds of $\mathcal{N}(d)$ correspond to χ odd and $\mathcal{N}(d)$ correspond to χ even. And if $q \equiv 2 \pmod 3$, then $\mathcal{N}(d)$ is composed of only even characters. Therefore let

$$C_i = \begin{cases} \frac{2}{3}c_i + \frac{1}{3}c_i^* & \text{if } q \equiv 1 \pmod 3, \\ c_i^* & \text{if } q \equiv 2 \pmod 3, \end{cases}$$

where

$$C_{kl}x^{kl} + C_{kl-1}x^{kl-1} + \cdots + C_0,$$

where C_i, c_i, c_i^* are the i-th power coefficients of a polynomial in d of degree kl.

A.1 $(1,1)$-moment

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)L_q\left(\tfrac{1}{2},\bar{\chi}\right)\right\rangle_x \sim C_1x+C_0 \tag{A.1}$$

where

$$\begin{aligned}
c_1 &= A(0,0)\log q\\
c_0 &= A(0,0)\log q + A_1(0,0) - A_2(0,0) + 2\gamma A(0,0)\\
c_1^* &= A(0,0)\log q\\
c_0^* &= \frac{1}{\sqrt{q}-1}\big(\sqrt{q}\left(\left(2\gamma-2\log q\right)A(0,0)+A_1(0,0)-A_2(0,0)\right)-2\gamma A(0,0)\\
&\qquad\qquad 4-A_1(0,0)+A_2(0,0)\big)
\end{aligned}$$

A.2 $(2,1)$-moment

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\bar{\chi}\right)\right\rangle_x \sim C_2x^2 + C_1x + C_0 \tag{A.2}$$

where

$$c_2 = \frac{1}{2}A(0,0,0)\log^2 q$$

$$c_1 = \frac{\log q}{2}\Big((6\gamma - 2\log q)A(0,0,0) + A_1(0,0,0) \\ + A_2(0,0,0) - 2A_3(0,0,0)\Big)$$

$$c_0 = \frac{12\gamma^2 - 12\gamma\log q + 2\log^2 q - 12\gamma_1}{4}A(0,0,0) + \frac{6\gamma - 2\log q}{4}A_1(0,0,0) \\ + \frac{6\gamma - 2\log q}{4}A_2(0,0,0) + \frac{-12\gamma + 4\log q}{4}A_3(0,0,0) + A_{1,2}(0,0,0) - \frac{1}{2}A_{1,3}(0,0,0) \\ - \frac{1}{4}A_{2,2}(0,0,0) - \frac{1}{2}A_{2,3}(0,0,0) + \frac{1}{2}A_{3,3}(0,0,0) - \frac{1}{4}A_{1,1}(0,0,0)$$

$$c_2^* = \frac{1}{2}A(0,0,0)\log^2 q$$

$$c_1^* = -\frac{\log q}{2(\sqrt{q}-1)^9}\Bigg(\Big((504\gamma - 112\log q)A(0,0,0) + 84A_1(0,0,0) + 84A_2(0,0,0) \\ - 168A_3(0,0,0)\Big)q^{3/2} + \Big((756\gamma - 280\log q)A(0,0,0) + 126A_1(0,0,0) + 126A_2(0,0,0) \\ - 252A_3(0,0,0)\Big)q^{5/2} + \Big((216\gamma - 112\log q)A(0,0,0) + 36A_1(0,0,0) + 36A_2(0,0,0) \\ - 72A_3(0,0,0)\Big)q^{7/2} + \Big((6\gamma - 4\log q)A(0,0,0) + A_1(0,0,0) + A_2(0,0,0) \\ - 2A_3(0,0,0)\Big)q^{9/2} + \Big((54\gamma - 4\log q)\sqrt{q} + (32q^4 + 224q^3 + 224q^2 + 32q)\log q \\ - 54\gamma\left(q + \tfrac{1}{3}\right)\left(q^3 + 9q^2 + 11q + \tfrac{1}{3}\right)\Big)A(0,0,0) + 9\Big(A_1(0,0,0) + A_2(0,0,0) \\ - 2A_3(0,0,0)\Big)\left(\tfrac{1}{9} - \sqrt{q} + 4q + 14q^2 + q^4 + \tfrac{28}{3}q^3\right)\Bigg)$$

$$c_0^* = \frac{1}{4(\sqrt{q}-1)^{12}}\Bigg(\Big((-144\gamma^2 + 264\gamma\log q - 80\log^2 q + 144\gamma_1)A(0,0,0)$$

$$
\begin{aligned}
&+(-72\gamma+44\log q)A_1(0,0,0)+(-72\gamma+44\log q)A_2(0,0,0)+(144\gamma-88\log q)A_3(0,0,0)\\
&+12A_{1,1}(0,0,0)-48A_{1,2}(0,0,0)+24A_{1,3}(0,0,0)+12A_{2,2}(0,0,0)+24A_{2,3}(0,0,0)\\
&-24A_{3,3}(0,0,0)\Big)q^{11/2}+\Big((-2640\gamma^2+1320\gamma\log q-80\log^2 q+2640\gamma_1)A(0,0,0)\\
&+(-1320\gamma+220\log q)A_1(0,0,0)+(-1320\gamma+220\log q)A_2(0,0,0)\\
&+(2640\gamma-440\log q)A_3(0,0,0)+220A_{1,1}(0,0,0)-880A_{1,2}(0,0,0)+440A_{1,3}(0,0,0)\\
&+220A_{2,2}(0,0,0)+440A_{2,3}(0,0,0)-440A_{3,3}(0,0,0)\Big)q^{3/2}\\
&+\Big((-9504\gamma^2+7920\gamma\log q-960\log^2 q+9504\gamma_1)A(0,0,0)\\
&+(-4752\gamma+1320\log q)A_1(0,0,0)+(-4752\gamma+1320\log q)A_2(0,0,0)\\
&+(9504\gamma-2640\log q)A_3(0,0,0)+792A_{1,1}(0,0,0)-3168A_{1,2}(0,0,0)+1584A_{1,3}(0,0,0)\\
&+792A_{2,2}(0,0,0)+1584A_{2,3}(0,0,0)-1584A_{3,3}(0,0,0)\Big)q^{5/2}\\
&+\Big((-9504\gamma^2+11088\gamma\log q-2016\log^2 q+9504\gamma_1)A(0,0,0)+(-4752\gamma+1848\log q)A_1(0,0,0)\\
&+(-4752\gamma+1848\log q)A_2(0,0,0)+(9504\gamma-3696\log q)A_3(0,0,0)+792A_{1,1}(0,0,0)\\
&-3168A_{1,2}(0,0,0)+1584A_{1,3}(0,0,0)+792A_{2,2}(0,0,0)\\
&+1584A_{2,3}(0,0,0)-1584A_{3,3}(0,0,0)\Big)q^{7/2}\\
&+\Big((-2640\gamma^2+3960\gamma\log q-960\log^2 q+2640\gamma_1)A(0,0,0)\\
&+(-1320\gamma+660\log q)A_1(0,0,0)+(-1320\gamma+660\log q)A_2(0,0,0)\\
&+(2640\gamma-1320\log q)A_3(0,0,0)+220A_{1,1}(0,0,0)-880A_{1,2}(0,0,0)+440A_{1,3}(0,0,0)\\
&+220A_{2,2}(0,0,0)+440A_{2,3}(0,0,0)-440A_{3,3}(0,0,0)\Big)q^{9/2}\\
&+\Big((-144\gamma^2+24\gamma\log q+144\gamma_1)\sqrt{q}+8q(q+1)(q^4+44q^3+166q^2+44q+1)\log^2 q\\
&-24q\gamma\log q(q^5+55q^4+330q^3+462q^2+165q+11)\\
&-12(q^2+6q+1)(q^4+60q^3+134q^2+60q+1)(-\gamma^2+\gamma_1)\Big)A(0,0,0)\\
&+((-72\gamma+4\sqrt{q}\log q-4q\log q(q^5+55q^4+330q^3+462q^2+165q+11)\\
&+6\gamma(q^2+6q+1)(q^4+60q^3+134q^2+60q+1))A_1(0,0,0)\\
&+((-72\gamma+4\log q)\sqrt{q}-4q\log q(q^5+55q^4+330q^3+462q^2+165q+11)
\end{aligned}
$$

$$+ 6\gamma(q^2 + 6q + 1)(q^4 + 60q^3 + 134q^2 + 60q + 1))A_2(0, 0, 0)$$

$$+ ((144\gamma - 8\log q)\sqrt{q} + 8q\log q(q^5 + 55q^4 + 330q^3 + 462q^2 + 165q + 11)$$

$$- 12\gamma(q^2 + 6q + 1)(q^4 + 60q^3 + 134q^2 + 60q + 1))A_3(0, 0, 0)$$

$$- (-12\sqrt{q} + (q^2 + 6q + 1)(q^4 + 60q^3 + 134q^2 + 60q + 1))(A_{2,2}(0, 0, 0) + 2A_{1,3}(0, 0, 0)$$

$$+ A_{1,1}(0, 0, 0) - 4A_{1,2}(0, 0, 0) + 2A_{2,3}(0, 0, 0) - 2A_{3,3}(0, 0, 0))\Big)$$

A.3 $(2,2)$-moment

$$\left\langle L_q\left(\tfrac{1}{2},\chi\right)^2 L_q\left(\tfrac{1}{2},\bar{\chi}\right)^2\right\rangle_x \sim C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 \tag{A.3}$$

where

$$c_4 = \frac{1}{12}A(0,0,0,0)\log^4 q$$

$$\begin{aligned} c_3 = -\frac{\log^3 q}{6}\bigg(&2A(0,0,0,0)\log q - 8\gamma A(0,0,0,0)\\ &- A_2(0,0,0,0) + A_3(0,0,0,0) + A_4(0,0,0,0) - A_1(0,0,0,0)\bigg)\end{aligned}$$

$$\begin{aligned} c_2 = \frac{\log^2 q}{2}\bigg(&(14\gamma^2 - 8\gamma\log q + \log^2 q - 4\gamma_1)A(0,0,0,0) + (4\gamma - \log q)A_1(0,0,0,0)\\ &+ (4\gamma - \log q)A_2(0,0,0,0) + (-4\gamma + \log q)A_3(0,0,0,0) + (-4\gamma + \log q)A_4(0,0,0,0)\\ &+ A_{1,2}(0,0,0,0) - \frac{1}{2}A_{1,3}(0,0,0,0) - \frac{1}{2}A_{1,4}(0,0,0,0) - \frac{1}{2}A_{2,3}(0,0,0,0) - \frac{1}{2}A_{2,4}(0,0,0,0)\\ &+ A_{3,4}(0,0,0,0)\bigg)\end{aligned}$$

$$\begin{aligned} c_1 = -7\log q\Bigg(\bigg(&\frac{\log^3 q}{21} - \frac{4\gamma\log^2 q}{7} + \left(2\gamma^2 + \frac{4\gamma_1}{7}\right)\log q + \frac{20\gamma\gamma_1}{7} - \frac{12\gamma^3}{7}\\ &+ \frac{2\gamma_2}{7}\bigg)A(0,0,0,0) + \left(\frac{4\gamma\log q}{7} + \frac{2\gamma_1}{7} - \gamma^2 - \frac{\log^2 q}{14}\right)A_1(0,0,0,0)\\ &+ \left(\frac{4\gamma\log q}{7} + \frac{2\gamma_1}{7} - \gamma^2 - \frac{\log^2 q}{14}\right)A_2(0,0,0,0)\\ &+ \left(-\frac{4\gamma\log q}{7} - \frac{2\gamma_1}{7} + \gamma^2 + \frac{\log^2 q}{14}\right)A_3(0,0,0,0)\\ &+ \left(-\frac{4\gamma\log q}{7} - \frac{2\gamma_1}{7} + \gamma^2 + \frac{\log^2 q}{14}\right)A_4(0,0,0,0)\\ &+ \left(-\frac{4\gamma}{7} + \frac{\log q}{7}\right)A_{1,2}(0,0,0,0) + \left(\frac{2\gamma}{7} - \frac{\log q}{14}\right)A_{1,3}(0,0,0,0)\\ &+ \left(\frac{2\gamma}{7} - \frac{\log q}{14}\right)A_{1,4}(0,0,0,0) + \left(\frac{2\gamma}{7} - \frac{\log q}{14}\right)A_{2,3}(0,0,0,0)\\ &+ \left(\frac{2\gamma}{7} - \frac{\log q}{14}\right)A_{2,4}(0,0,0,0) + \left(-\frac{4\gamma}{7} + \frac{\log q}{7}\right)A_{3,4}(0,0,0,0)\end{aligned}$$

$$+\frac{1}{84}A_{1,1,1}(0,0,0,0)-\frac{1}{28}A_{1,1,2}(0,0,0,0)-\frac{1}{28}A_{1,2,2}(0,0,0,0)+\frac{1}{14}A_{1,2,3}(0,0,0,0)$$
$$+\frac{1}{14}A_{1,2,4}(0,0,0,0)-\frac{1}{14}A_{1,3,4}(0,0,0,0)+\frac{1}{84}A_{2,2,2}(0,0,0,0)-\frac{1}{14}A_{2,3,4}(0,0,0,0)$$
$$\left.-\frac{1}{84}A_{3,3,3}(0,0,0,0)+\frac{1}{28}A_{3,3,4}(0,0,0,0)+\frac{1}{28}A_{3,4,4}(0,0,0,0)-\frac{1}{84}A_{4,4,4}(0,0,0,0)\right)$$

$$c_0=\frac{1}{24}\Big(A_{2,2,2,4}(0,0,0,0)-2A_{3,4,4,4}(0,0,0,0)-3A_{1,3,3,4}(0,0,0,0)-3A_{2,3,3,4}(0,0,0,0)$$
$$-2A_{3,3,3,4}(0,0,0,0)+6A_{1,2,3,3}(0,0,0,0)-3A_{1,1,2,4}(0,0,0,0)-3A_{1,2,2,4}(0,0,0,0)$$
$$+A_{1,1,1,4}(0,0,0,0)+A_{2,3,3,3}(0,0,0,0)+A_{1,3,3,3}(0,0,0,0)+A_{2,4,4,4}(0,0,0,0)$$
$$+A_{1,1,1,3}(0,0,0,0)-3A_{1,2,2,3}(0,0,0,0)+A_{2,2,2,3}(0,0,0,0)-3A_{1,1,2,3}(0,0,0,0)$$
$$+6A_{2,2,3,4}(0,0,0,0)+6A_{1,1,3,4}(0,0,0,0)-2A_{1,2,2,2}(0,0,0,0)-2A_{1,1,1,2}(0,0,0,0)$$
$$+6A_{1,1,2,2}(0,0,0,0)-3A_{1,1,3,3}(0,0,0,0)-3A_{2,2,3,3}(0,0,0,0)+6A_{1,2,4,4}(0,0,0,0)$$
$$-3A_{2,3,4,4}(0,0,0,0)+A_{1,4,4,4}(0,0,0,0)-3A_{1,3,4,4}(0,0,0,0)+6A_{3,3,4,4}(0,0,0,0)$$
$$-3A_{1,1,4,4}(0,0,0,0)-3A_{2,2,4,4}(0,0,0,0)+(-4\log^3 q+48\gamma\log^2 q+(-168\gamma^2+48\gamma_1)\log q$$
$$+144\gamma^3-240\gamma\gamma_1-24\gamma_2)A_2(0,0,0,0)+(-6\log q+24\gamma)A_{1,1,2}(0,0,0,0)$$
$$+(-6\log q+24\gamma)A_{1,2,2}(0,0,0,0)+(-48\gamma+12\log q)A_{1,2,3}(0,0,0,0)$$
$$+(168\gamma^2-96\gamma\log q+12\log^2 q-48\gamma_1)A_{3,4}(0,0,0,0)+(2\log q-8\gamma)A_{1,1,1}(0,0,0,0)$$
$$+(-84\gamma^2+48\gamma\log q-6\log^2 q+24\gamma_1)A_{1,4}(0,0,0,0)$$
$$+(-84\gamma^2+48\gamma\log q-6\log^2 q+24\gamma_1)A_{1,3}(0,0,0,0)+(2\log^4 q-32\gamma\log^3 q$$
$$+(168\gamma^2-48\gamma_1)\log^2 q+(-288\gamma^3+480\gamma\gamma_1+48\gamma_2)\log q+144\gamma^4-576\gamma^2\gamma_1+48\gamma\gamma_2$$
$$+336\gamma_1^2+80\gamma_3)A(0,0,0,0)+(-4\log^3 q+48\gamma\log^2 q+(-168\gamma^2+48\gamma_1)\log q+144\gamma^3$$
$$-240\gamma\gamma_1-24\gamma_2)A_1(0,0,0,0)+(-48\gamma+12\log q)A_{1,2,4}(0,0,0,0)$$
$$+(-12\log q+48\gamma)A_{1,3,4}(0,0,0,0)+(2\log q-8\gamma)A_{2,2,2}(0,0,0,0)$$
$$+(-12\log q+48\gamma)A_{2,3,4}(0,0,0,0)+(4\log^3 q-48\gamma\log^2 q+(168\gamma^2-48\gamma_1)\log q$$
$$-144\gamma^3+240\gamma\gamma_1+24\gamma_2)A_3(0,0,0,0)+(4\log^3 q-48\gamma\log^2 q$$
$$+(168\gamma^2-48\gamma_1)\log q-144\gamma^3+240\gamma\gamma_1+24\gamma_2)A_4(0,0,0,0)+(168\gamma^2-96\gamma\log q$$
$$+12\log^2 q-48\gamma_1)A_{1,2}(0,0,0,0)+(-84\gamma^2+48\gamma\log q-6\log^2 q+24\gamma_1)A_{2,3}(0,0,0,0)$$

$$+ (-84\gamma^2 + 48\gamma \log q - 6\log^2 q + 24\gamma_1)A_{2,4}(0,0,0,0) + (-2\log q + 8\gamma)A_{3,3,3}(0,0,0,0)$$

$$+ (-24\gamma + 6\log q)A_{3,3,4}(0,0,0,0) + (-24\gamma + 6\log q)A_{3,4,4}(0,0,0,0)$$

$$+ (-2\log q + 8\gamma)A_{4,4,4}(0,0,0,0)\Big)$$

$$c_4^* = \frac{1}{12}A(0,0,0,0)\log^4 q$$

$$c_3^* = \frac{25\log^3 q}{6(\sqrt{q}-1)^{25}}\Bigg(\bigg(\left(-\tfrac{3496\log q}{25} - 736\gamma\right)A(0,0,0,0) - 92A_1(0,0,0,0) - 92A_2(0,0,0,0)$$

$$+ 92A_3(0,0,0,0) + 92A_4(0,0,0,0)\bigg)q^{3/2} + \bigg(\left(-\tfrac{1069776\log q}{25} - 1426368\gamma\right)A(0,0,0,0)$$

$$- 178296A_1(0,0,0,0) - 178296A_2(0,0,0,0) + 178296A_3(0,0,0,0)$$

$$+ 178296A_4(0,0,0,0)\bigg)q^{11/2} + \bigg(\left(\tfrac{416024\log q}{25} - 1664096\gamma\right)A(0,0,0,0)$$

$$- 208012A_1(0,0,0,0) - 208012A_2(0,0,0,0) + 208012A_3(0,0,0,0) + 208012A_4(0,0,0,0)\bigg)q^{13/2}$$

$$+ \bigg(\left(-\tfrac{5230016\log q}{25} + \tfrac{1307504\gamma}{25}\right)A(0,0,0,0)$$

$$- \tfrac{653752}{5}A_1(0,0,0,0) - \tfrac{653752}{5}A_2(0,0,0,0) + \tfrac{653752}{5}A_3(0,0,0,0) + \tfrac{653752}{5}A_4(0,0,0,0)\bigg)q^{15/2}$$

$$+ \bigg(\left(\tfrac{778734\log q}{25} - 346104\gamma\right)A(0,0,0,0)$$

$$- 43263A_1(0,0,0,0) - 43263A_2(0,0,0,0) + 43263A_3(0,0,0,0)$$

$$+ 43263A_4(0,0,0,0)\bigg)q^{17/2} + \bigg(\left(\tfrac{184184\log q}{25} - 56672\gamma\right)A(0,0,0,0)$$

$$- 7084A_1(0,0,0,0) - 7084A_2(0,0,0,0) + 7084A_3(0,0,0,0)$$

$$+ 7084A_4(0,0,0,0)\bigg)q^{19/2} + \bigg(\left(\tfrac{17204\log q}{25} - 4048\gamma\right)A(0,0,0,0)$$

$$- 506A_1(0,0,0,0) - 506A_2(0,0,0,0) + 506A_3(0,0,0,0)$$

$$+ 506A_4(0,0,0,0)\bigg)q^{21/2} + \bigg(\left(\tfrac{504\log q}{25} - 96\gamma\right)A(0,0,0,0)$$

$$- 12A_1(0,0,0,0) - 12A_2(0,0,0,0) + 12A_3(0,0,0,0)$$

$$+ 12A_4(0,0,0,0)\bigg)q^{23/2} + \bigg(\left(\tfrac{2\log q}{25} - \tfrac{8\gamma}{25}\right)A(0,0,0,0)$$

$$- \tfrac{1}{25}A_1(0,0,0,0) - \tfrac{1}{25}A_2(0,0,0,0) + \tfrac{1}{25}A_3(0,0,0,0)$$

$$+ \tfrac{1}{25}A_4(0,0,0,0)\bigg)q^{25/2} + \bigg(\left(-\tfrac{63756\log q}{25} - \tfrac{85008\gamma}{5}\right)A(0,0,0,0)$$

$$
\begin{aligned}
&- \tfrac{10626}{5} A_1(0,0,0,0) - \tfrac{10626}{5} A_2(0,0,0,0) + \tfrac{10626}{5} A_3(0,0,0,0) \\
&+ \tfrac{10626}{5} A_4(0,0,0,0)\Big) q^{5/2} + \Big(\left(-\tfrac{423016\log q}{25} - 153824\gamma\right) A(0,0,0,0) \\
&- 19228 A_1(0,0,0,0) - 19228 A_2(0,0,0,0) + 19228 A_3(0,0,0,0) \\
&+ 19228 A_4(0,0,0,0)\Big) q^{7/2} + \Big(\left(-\tfrac{1144066\log q}{25} - 653752\gamma\right) A(0,0,0,0) \\
&- 81719 A_1(0,0,0,0) - 81719 A_2(0,0,0,0) + 81719 A_3(0,0,0,0) \\
&+ 81719 A_4(0,0,0,0)\Big) q^{9/2} + \Big(\left(-\tfrac{46\log q}{25} - 8\gamma\right)\sqrt{q} - \tfrac{46(q-1)\log q}{25}\Big(q^{11} + 77q^{10} + 1463q^9 \\
&+ 10659q^8 + 35530q^7 + 58786q^6 + 49742q^5 + 21318q^4 + 4389q^3 + 385q^2 + 11q + \tfrac{1}{23}\Big) \\
&+ 8\gamma\Big(q^2 + 2q + \tfrac{1}{5}\Big)\Big(q^{10} + 90q^9 + 1945q^8 + 15320q^7 + 50690q^6 + 73852q^5 + 50170q^4 \\
&+ 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5}\Big)\Big) A(0,0,0,0) + \Big(\tfrac{1}{25} + \tfrac{653752}{5} q^5 + 7084q^3 - \sqrt{q} + 81719q^8 \\
&+ 208012q^6 + 19228q^9 + 12q + 506q^2 + \tfrac{10626}{5} q^{10} + 178296q^7 + 43263q^4 + 92q^{11} + q^{12}\Big) \\
&\times \Big(A_1(0,0,0,0) + A_2(0,0,0,0) - A_3(0,0,0,0) - A_4(0,0,0,0)\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
c_2^* = \frac{\log^2 q}{4(\sqrt{q}-1)^{25}} &\Big(\Big(\left(124807200\gamma^2 + 8558208\gamma\log q - 9152528\log^2 q - 35659200\gamma_1\right) A(0,0,0,0) \\
&+ (-35659200\gamma - 1069776\log q)\, A_3(0,0,0,0) + (-35659200\gamma - 1069776\log q)\, A_4(0,0,0,0) \\
&+ (35659200\gamma + 9984576\log q)\, A_1(0,0,0,0) + (35659200\gamma - 7845024\log q)\, A_2(0,0,0,0) \\
&+ 8914800 A_{1,2}(0,0,0,0) - 4457400 A_{1,3}(0,0,0,0) - 4457400 A_{1,4}(0,0,0,0) \\
&- 4457400 A_{2,3}(0,0,0,0) - 4457400 A_{2,4}(0,0,0,0) + 8914800 A_{3,4}(0,0,0,0)\Big) q^{11/2} \\
&+ \Big(\left(145608400\gamma^2 - 3328192\gamma\log q - 10816624\log^2 q - 41602400\gamma_1\right) A(0,0,0,0) \\
&+ (-41602400\gamma + 416024\log q)\, A_3(0,0,0,0) + (-41602400\gamma + 416024\log q)\, A_4(0,0,0,0) \\
&+ (41602400\gamma + 9984576\log q)\, A_1(0,0,0,0) + (41602400\gamma - 10816624\log q)\, A_2(0,0,0,0) \\
&+ 10400600 A_{1,2}(0,0,0,0) - 5200300 A_{1,3}(0,0,0,0) - 5200300 A_{1,4}(0,0,0,0) \\
&- 5200300 A_{2,3}(0,0,0,0) - 5200300 A_{2,4}(0,0,0,0) + 10400600 A_{3,4}(0,0,0,0)\Big) q^{13/2} \\
&+ \Big(\left(91525280\gamma^2 - 10460032\gamma\log q - 6537520\log^2 q - 26150080\gamma_1\right) A(0,0,0,0) \\
&+ (-26150080\gamma + 1307504\log q)\, A_3(0,0,0,0) + (-26150080\gamma + 1307504\log q)\, A_4(0,0,0,0)
\end{aligned}
$$

$$
\begin{aligned}
&+ (26150080\gamma + 5230016 \log q)\, A_1(0,0,0,0) + (26150080\gamma - 7845024 \log q)\, A_2(0,0,0,0) \\
&+ 6537520 A_{1,2}(0,0,0,0) - 3268760 A_{1,3}(0,0,0,0) - 3268760 A_{1,4}(0,0,0,0) \\
&- 3268760 A_{2,3}(0,0,0,0) - 3268760 A_{2,4}(0,0,0,0) + 6537520 A_{3,4}(0,0,0,0)\Big) q^{15/2} \\
&+ \Big(\left(30284100\gamma^2 - 6229872\gamma \log q - 1961256 \log^2 - 8652600\gamma_1\right) A(0,0,0,0) \\
&+ (-8652600\gamma + 778734 \log q)\, A_3(0,0,0,0) + (-8652600\gamma + 778734 \log q)\, A_4(0,0,0,0) \\
&+ (8652600\gamma + 1384416 \log q)\, A_1(0,0,0,0) + (8652600\gamma - 2941884 \log q)\, A_2(0,0,0,0) \\
&+ 2163150 A_{1,2}(0,0,0,0) - 1081575 A_{1,3}(0,0,0,0) - 1081575 A_{1,4}(0,0,0,0) \\
&- 1081575 A_{2,3}(0,0,0,0) - 1081575 A_{2,4}(0,0,0,0) + 2163150 A_{3,4}(0,0,0,0)\Big) q^{17/2} \\
&+ \Big(\left(4958800\gamma^2 - 1473472\gamma \log q - 269192 \log^2 q - 1416800\gamma_1\right) A(0,0,0,0) \\
&+ (-1416800\gamma + 184184 \log q)\, A_3(0,0,0,0) + (-1416800\gamma + 184184 \log q)\, A_4(0,0,0,0) \\
&+ (1416800\gamma + 170016 \log q)\, A_1(0,0,0,0) + (1416800\gamma - 538384 \log q)\, A_2(0,0,0,0) \\
&+ 354200 A_{1,2}(0,0,0,0) - 177100 A_{1,3}(0,0,0,0) - 177100 A_{1,4}(0,0,0,0) \\
&- 177100 A_{2,3}(0,0,0,0) - 177100 A_{2,4}(0,0,0,0) + 354200 A_{3,4}(0,0,0,0)\Big) q^{19/2} \\
&+ \Big(\left(354200\gamma^2 - 137632\gamma \log q - 14168 \log^2 q - 101200\gamma_1\right) A(0,0,0,0) \\
&+ (-101200\gamma + 17204 \log q)\, A_3(0,0,0,0) + (-101200\gamma + 17204 \log q)\, A_4(0,0,0,0) \\
&+ (101200\gamma + 8096 \log q)\, A_1(0,0,0,0) + (101200\gamma - 42504 \log q)\, A_2(0,0,0,0) \\
&+ 25300 A_{1,2}(0,0,0,0) - 12650 A_{1,3}(0,0,0,0) - 12650 A_{1,4}(0,0,0,0) \\
&- 12650 A_{2,3}(0,0,0,0) - 12650 A_{2,4}(0,0,0,0) + 25300 A_{3,4}(0,0,0,0)\Big) q^{21/2} \\
&+ \Big(+ \left((8400\gamma^2 - 4032\gamma \log q - 184 \log^2 q - 2400\gamma_1\right) A(0,0,0,0) \\
&+ (-2400\gamma + 504 \log q)\, A_3(0,0,0,0) + (-2400\gamma + 504 \log q)\, A_4(0,0,0,0) \\
&+ (2400\gamma + 96 \log q)\, A_1(0,0,0,0) + (2400\gamma - 1104 \log q)\, A_2(0,0,0,0) \\
&+ 600 A_{1,2}(0,0,0,0) - 300 A_{1,3}(0,0,0,0) - 300 A_{1,4}(0,0,0,0) \\
&- 300 A_{2,3}(0,0,0,0) - 300 A_{2,4}(0,0,0,0) + 600 A_{3,4}(0,0,0,0)\Big) q^{13/2} \\
&+ \Big(\left(64400\gamma^2 + 27968\gamma \log q - 2024 \log^2 q - 18400\gamma_1\right) A(0,0,0,0)
\end{aligned}
$$

$$
\begin{aligned}
&+ (-18400\gamma - 3496\log q)\, A_3(0,0,0,0) + (-18400\gamma - 3496\log q)\, A_4(0,0,0,0) \\
&+ (18400\gamma + 8096\log q)\, A_1(0,0,0,0) + (18400\gamma - 1104\log q)\, A_2(0,0,0,0) \\
&+ 4600A_{1,2}(0,0,0,0) - 2300A_{1,3}(0,0,0,0) - 2300A_{1,4}(0,0,0,0) \\
&- 2300A_{2,3}(0,0,0,0) - 2300A_{2,4}(0,0,0,0) + 4600A_{3,4}(0,0,0,0)\Big) q^{3/2} \\
&+ \Big(\left(1487640\gamma^2 + 510048\gamma\log q - 70840\log^2 q - 425040\gamma_1\right) A(0,0,0,0) \\
&+ (-425040\gamma - 63756\log q)\, A_3(0,0,0,0) + (-425040\gamma - 63756\log q)\, A_4(0,0,0,0) \\
&+ (425040\gamma + 170016\log q)\, A_1(0,0,0,0) + (425040\gamma - 42504\log q)\, A_2(0,0,0,0) \\
&+ 106260A_{1,2}(0,0,0,0) - 53130A_{1,3}(0,0,0,0) - 53130A_{1,4}(0,0,0,0) \\
&- 53130A_{2,3}(0,0,0,0) - 53130A_{2,4}(0,0,0,0) + 106260A_{3,4}(0,0,0,0)\Big) q^{5/2} \\
&+ \Big(\left(13459600\gamma^2 + 3384128\gamma\log q - 807576\log^2 q - 3845600\gamma_1\right) A(0,0,0,0) \\
&+ (-3845600\gamma - 423016\log q)\, A_3(0,0,0,0) + (-3845600\gamma - 423016\log q)\, A_4(0,0,0,0) \\
&+ (3845600\gamma + 1384416\log q)\, A_1(0,0,0,0) + (3845600\gamma - 538384\log q)\, A_2(0,0,0,0) \\
&+ 961400A_{1,2}(0,0,0,0) - 480700A_{1,3}(0,0,0,0) - 480700A_{1,4}(0,0,0,0) \\
&- 480700A_{2,3}(0,0,0,0) - 480700A_{2,4}(0,0,0,0) + 961400A_{3,4}(0,0,0,0)\Big) q^{7/2} \\
&+ \Big(\left(57203300\gamma^2 + 9152528\gamma\log q - 3922512\log^2 q - 16343800\gamma_1\right) A(0,0,0,0) \\
&+ (-16343800\gamma - 1144066\log q)\, A_3(0,0,0,0) + (-16343800\gamma - 1144066\log q)\, A_4(0,0,0,0) \\
&+ (16343800\gamma + 5230016\log q)\, A_1(0,0,0,0) + (16343800\gamma - 2941884\log q)\, A_2(0,0,0,0) \\
&+ 4085950A_{1,2}(0,0,0,0) - 2042975A_{1,3}(0,0,0,0) - 2042975A_{1,4}(0,0,0,0) \\
&- 2042975A_{2,3}(0,0,0,0) - 2042975A_{2,4}(0,0,0,0) + 4085950A_{3,4}(0,0,0,0)\Big) q^{9/2} \\
&+ \Big(\left(28\gamma^2 - 16\gamma\log q - 8\gamma_1\right) A(0,0,0,0) + (-8\gamma + 2\log q)\, A_3(0,0,0,0) \\
&+ (-8\gamma + 2\log q)\, A_4(0,0,0,0) + 8\gamma A_1(0,0,0,0) + (8\gamma - 4\log q)\, A_2(0,0,0,0) + 2A_{1,2}(0,0,0,0) \\
&- A_{1,3}(0,0,0,0) - A_{1,4}(0,0,0,0) - A_{2,3}(0,0,0,0) - A_{2,4}(0,0,0,0) + 2A_{3,4}(0,0,0,0)\Big) q^{25/2} \\
&+ \Big(\sqrt{q}(700\gamma^2 + 368\gamma\log q - 8\log^2 q - 200\gamma_1) + 8q\log^2 q(q^{11} + 253q^{10} + 8855q^9 \\
&+ 100947q^8 + 490314q^7 + 1144066q^6 + 1352078q^5 + 817190q^4 + 245157q^3
\end{aligned}
$$

$$+ 33649q^2 + 1771q + 23) + 368\gamma(q-1)\log q\big(q^{11} + 77q^{10} + 1463q^9 + 10659q^8$$

$$+ 35530q^7 + 58786q^6 + 49742q^5 + 21318q^4 + 4389q^3 + 385q^2 + 11q + \tfrac{1}{23}\big)$$

$$+ 200\left(-\tfrac{7\gamma^2}{2} + \gamma_1\right)\left(q^2 + 2q + \tfrac{1}{5}\right)\left(q^{10} + 90q^9 + 1945q^8 + 15320q^7\right.$$

$$\left.+ 50690q^6 + 73852q^5 + 50170q^4 + 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5}\right)\Big)A(0,0,0,0)$$

$$+ \Big(\sqrt{q}(-200\gamma - 46\log q) - 46(q-1)\log q(q^{11} + 77q^{10} + 1463q^9 + 10659q^8 + 35530q^7$$

$$+ 58786q^6 + 49742q^5 + 21318q^4 + 4389q^3 + 385q^2 + 11q + \tfrac{1}{23})$$

$$+ 200\gamma\left(q^2 + 2q + \tfrac{1}{5}\right)\left(q^{10} + 90q^9 + 1945q^8 + 15320q^7 + 50690q^6 + 73852q^5 + 50170q^4\right.$$

$$\left.+ 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5}\right)\Big)A_3(0,0,0,0)$$

$$+ \Big(\sqrt{q}(-200\gamma - 46\log q) - 46(q-1)\log q\big(q^{11} + 77q^{10} + 1463q^9 + 10659q^8 + 35530q^7$$

$$+ 58786q^6 + 49742q^5 + 21318q^4 + 4389q^3 + 385q^2 + 11q + \tfrac{1}{23}\big)$$

$$+ 200\gamma\left(q^2 + 2q + \tfrac{1}{5}\right)\left(q^{10} + 90q^9 + 1945q^8 + 15320q^7\right.$$

$$\left.+ 50690q^6 + 73852q^5 + 50170q^4 + 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5}\right)\Big)A_4(0,0,0,0)$$

$$+ \Big(\sqrt{q}(200\gamma + 96\log q) - 4\log q(q^4 + 28q^3 + 70q^2 + 28q + 1)\big(q^8 + 248q^7 + 361q^6$$

$$+ 16072q^5 + 25670q^4 + 16072q^3 + 3612q^2 + 248q + 1\big)$$

$$- 200\gamma(q^2 + 2q + \tfrac{1}{5})(q^{10} + 90q^9 + 1945q^8 + 15320q^7 + 50690q^6$$

$$+ 73852q^5 + 50170q^4 + 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5})\Big)A_1(0,0,0,0)$$

$$+ \Big(\sqrt{q}(200\gamma - 4\log q) + 96\log q(q^2 + 3q)(q + \tfrac{1}{3})(q^2 + 6q + 1)(q + 1)(q^4 + 60q^3 + 134q^2$$

$$+ 60q + 1)(q^2 + 14q + 1) - 200\gamma(q^2 + 2q + \tfrac{1}{5})(q^{10} + 90q^9 + 1945q^8 + 15320q^7 + 50690q^6$$

$$+ 73852q^5 + 50170q^4 + 15640q^3 + 1949q^2 + 58q + \tfrac{1}{5})\Big)A_2(0,0,0,0)$$

$$- 50(A_{1,2}(0,0,0,0) - \tfrac{1}{2}A_{1,3}(0,0,0,0) - \tfrac{1}{2}A_{1,4}(0,0,0,0) - \tfrac{1}{2}A_{2,3}(0,0,0,0)$$

$$- \tfrac{1}{2}A_{2,4}(0,0,0,0) + A_{3,4}(0,0,0,0))(q^{12} + 92q^{11} + \tfrac{10626}{5}q^{10} + 19228q^9 + 81719q^8 + 178296q^7$$

$$+ 208012q^6 + \tfrac{653752}{5}q^5 + 43263q^4 + 7084q^3 + 506q^2 + 12q - \sqrt{q} + \tfrac{1}{25})\Big)$$

$$c_1^* = \frac{\log q}{12(-q^{3/2} - 3\sqrt{q} + 3q + 1)^4(\sqrt{q}-1)^{12}(q - 2\sqrt{q} + 1)^4}\Big(\Big((-67396\log^3 q - 779520\gamma\log^2 q$$

$$
\begin{aligned}
&+ (10572240\gamma^2 - 3020640\gamma_1)\log q - 28998144\gamma^3 + 48330240\gamma\gamma_1 + 4833024\gamma_2)A(0,0,0,0) \\
&+ (-16915584\gamma^2 + 3020640\gamma\log q - 97440\log^2 q + 4833024\gamma_1)A_1(0,0,0,0) \\
&+ (-16915584\gamma^2 + 3020640\gamma\log q - 97440\log^2 q + 4833024\gamma_1)A_2(0,0,0,0) \\
&+ (16915584\gamma^2 - 3020640\gamma\log q + 97440\log^2 q - 4833024\gamma_1)A_3(0,0,0,0) \\
&+ (16915584\gamma^2 - 3020640\gamma\log q + 97440\log^2 q - 4833024\gamma_1)A_4(0,0,0,0) \\
&+ (-9666048\gamma + 755160\log q)A_{12}(0,0,0,0) + (4833024\gamma - 377580\log q)A_{1,3}(0,0,0,0) \\
&+ (4833024\gamma - 377580\log q)A_{1,4}(0,0,0,0) + (4833024\gamma - 377580\log q)A_{2,3}(0,0,0,0) \\
&+ (4833024\gamma - 377580\log q)A_{2,4}(0,0,0,0) + (-9666048\gamma + 755160\log q)A_{3,4}(0,0,0,0) \\
&+ 201376A_{1,1,1}(0,0,0,0) - 604128A_{1,1,2}(0,0,0,0) - 604128A_{1,2,2}(0,0,0,0) \\
&+ 1208256A_{1,2,3}(0,0,0,0) + 1208256A_{1,2,4}(0,0,0,0) - 1208256A_{1,3,4}(0,0,0,0) \\
&+ 201376A_{2,2,2}(0,0,0,0) - 1208256A_{2,3,4}(0,0,0,0) - 201376A_{3,3,3}(0,0,0,0) \\
&+ 604128A_{3,3,4}(0,0,0,0) + 604128A_{3,4,4}(0,0,0,0) - 201376A_{4,4,4}(0,0,0,0)\Big)q^{5/2} \\
&+ \Big((-2688\gamma^2 + 96\log q\gamma + 768\gamma_1)\sqrt{q} + 24q(q+1)(q^2+14q+1)(q^4+44q^3+166q^2 \\
&+ 44q+1)(q^8+376q^7+4380q^6+15944q^5+24134q^4+15944q^3+4380q^2+376q+1)\log q^2 \\
&- 96q\gamma(q^{15}+465q^{14}+31465q^{13}+736281q^{12}+7888725q^11+44352165q^{10} \\
&+ 141120525q^9+265182525q^8+300540195q^7+206253075q^6+84672315q^5 \\
&+ 20160075q^4+2629575q^3+169911q^2+4495q+31)\log q + 84(q^{16}+496q^{15} \\
&+ 35960q^{14}+906192q^{13}+10518300q^{12}+64512240q^{11}+225792840q^{10} \\
&+ 471435600q^9+601080390q^8+471435600q^7+225792840q^6+64512240q^5 \\
&+ 10518300q^4+906192q^3+35960q^2+496q+1)(\gamma^2 - \tfrac{2\gamma_1}{7})\Big)A_2(0,0,0,0) \\
&+ \Big((-665028\log q^3 - 27361152\gamma\log^2 q + (247390416\gamma^2 - 70682976\gamma_1)\log q \\
&- 484683264\gamma^3 + 807805440\gamma\gamma_1 + 80780544\gamma_2)A(0,0,0,0) + (-282731904\gamma^2 \\
&+ 70682976\gamma\log q - 3420144\log q^2 + 80780544\gamma_1)A_1(0,0,0,0) + (-282731904\gamma^2 \\
&+ 70682976\log q\gamma - 3420144\log q^2 + 80780544\gamma_1)A_2(0,0,0,0) + (282731904\gamma^2
\end{aligned}
$$

$$- 70682976 \log q\gamma + 3420144 \log^2 q - 80780544\gamma_1)A_3(0,0,0,0) + (282731904\gamma^2$$

$$- 70682976\gamma \log q + 3420144 \log^2 q - 80780544\gamma_1)A_4(0,0,0,0)$$

$$+ (-161561088\gamma + 17670744 \log q)A_{1,2}(0,0,0,0)$$

$$+ (80780544\gamma - 8835372 \log q)A_{1,3}(0,0,0,0) + (80780544\gamma - 8835372 \log q)A_{1,4}(0,0,0,0)$$

$$+ (80780544\gamma - 8835372 \log q)A_{2,3}(0,0,0,0) + (80780544\gamma - 8835372 \log q)A_{2,4}(0,0,0,0)$$

$$+ (-161561088\gamma + 17670744 \log q)A_{3,4}(0,0,0,0)$$

$$+ 3365856A_{1,1,1}(0,0,0,0) - 10097568A_{1,1,2}(0,0,0,0) - 10097568A_{1,2,2}A(0,0,0,0)$$

$$+ 20195136A_{1,2,3}(0,0,0,0) + 20195136A_{1,2,4}(0,0,0,0) - 20195136A_{1,3,4}(0,0,0,0)$$

$$+ 3365856A_{2,2,2}(0,0,0,0) - 201951361_{2,3,4}(0,0,0,0) - 3365856A_{3,3,3}(0,0,0,0)$$

$$+ 10097568A_{3,3,4}(0,0,0,0) + 10097568A_{3,4,4}(0,0,0,0) - 3365856A_{4,4,4}(0,0,0,0)\Big)q^{7/2}$$

$$+ \Big((51370020 \log^3 q - 390873600\gamma \log^2 q + (883537200\gamma^2 - 252439200\gamma_1) \log q$$

$$- 484683264\gamma^3 + 807805440\gamma\gamma_1 + 80780544\gamma_2)A(0,0,0,0)$$

$$+ (-282731904\gamma^2 + 252439200\gamma \log q - 48859200 \log^2 q + 80780544\gamma_1)A_1(0,0,0,0)$$

$$+ (-282731904\gamma^2 + 252439200\gamma \log q - 48859200 \log^2 q + 80780544\gamma_1)A_2(0,0,0,0)$$

$$+ (282731904\gamma^2 - 252439200\gamma \log q + 48859200 \log^2 q - 80780544\gamma_1)A_3(0,0,0,0)$$

$$+ (282731904\gamma^2 - 252439200\gamma \log q + 48859200 \log^2 q - 80780544\gamma_1)A_4(0,0,0,0)$$

$$+ (-161561088\gamma + 63109800 \log q)A_{1,2}(0,0,0,0)$$

$$+ (80780544\gamma - 31554900 \log q)A_{1,3}(0,0,0,0)$$

$$+ (80780544\gamma - 31554900 \log q)A_{1,4}(0,0,0,0)$$

$$+ (80780544\gamma - 31554900 \log q)A_{2,3}(0,0,0,0)$$

$$+ (80780544\gamma - 31554900 \log q)A_{2,4}(0,0,0,0)$$

$$+ (-161561088\gamma + 63109800 \log q)A_{3,4}(0,0,0,0)$$

$$+ 3365856A_{1,1,1}(0,0,0,0) - 10097568A_{1,1,2}(0,0,0,0) - 10097568A_{1,2,2}(0,0,0,0)$$

$$+ 20195136A_{1,2,3}(0,0,0,0) + 20195136A_{1,2,4}(0,0,0,0) - 20195136A_{1,3,4}(0,0,0,0)$$

$$+ 3365856A_{2,2,2}(0,0,0,0) - 20195136A_{2,3,4}(0,0,0,0) - 3365856A_{3,3,3}(0,0,0,0)$$

$$+ 10097568A_{3,3,4}(0,0,0,0) + 10097568A_{3,4,4}(0,0,0,0) - 3365856A_{4,4,4}(0,0,0,0)\Big)q^{25/2}$$

$$+ \Big((-1476\log^3 q - 5760\gamma\log^2 q + (156240\gamma^2 - 44640\gamma_1) - 714240\gamma^3 + 1190400\gamma\gamma_1$$

$$+ 119040\gamma_2)A(0,0,0,0) + (-416640\gamma^2 + 44640\gamma - 720\log^2 q + 119040\gamma_1)A_1(0,0,0,0)$$

$$+ (-416640\gamma^2 + 44640\gamma - 720\log^2 q + 119040\gamma_1)A_2(0,0,0,0)$$

$$+ (416640\gamma^2 - 44640\gamma + 720\log^2 q - 119040\gamma_1)A_3(0,0,0,0)$$

$$+ (416640\gamma^2 - 44640\gamma + 720\log^2 q - 119040\gamma_1)A_4(0,0,0,0)$$

$$+ (-238080\gamma + 11160)A_{1,2}(0,0,0,0) + (119040\gamma - 5580)A_{1,3}(0,0,0,0)$$

$$+ (119040\gamma - 5580)A_{1,4}(0,0,0,0) + (119040\gamma - 5580)A_{2,3}(0,0,0,0)$$

$$+ (119040\gamma - 5580)A_{2,4}(0,0,0,0) + (-238080\gamma + 11160)A_{3,4}(0,0,0,0) + 4960A_{1,1,1}(0,0,0,0)$$

$$- 14880A_{1,1,2}(0,0,0,0) - 14880A_{1,2,2}(0,0,0,0) + 29760A_{1,2,3}(0,0,0,0) + 29760A_{1,2,4}(0,0,0,0)$$

$$- 29760A_{1,3,4}(0,0,0,0) + 4960A_{2,2,2}(0,0,0,0) - 29760A_{2,3,4}(0,0,0,0) - 4960A_{3,3,3}(0,0,0,0)$$

$$+ 14880A_{3,3,4}(0,0,0,0) + 14880A_{3,4,4}(0,0,0,0) - 4960A_{4,4,4}(0,0,0,0)\Big)q^{3/2}$$

$$+ \Big((932\log^3 q - 5760\gamma\log^2 q + (10416\gamma^2 - 2976\gamma_1)\log q - 4608\gamma^3$$

$$+ 7680\gamma\gamma_1 + 768\gamma_2)A(0,0,0,0)$$

$$+ (-2688\gamma^2 + 2976\gamma - 720\log^2 q + 768\gamma_1)A_1(0,0,0,0)$$

$$+ (-2688\gamma^2 + 2976\gamma - 720\log^2 q + 768\gamma_1)A_2(0,0,0,0)$$

$$+ (2688\gamma^2 - 2976\gamma + 720\log^2 q - 768\gamma_1)A_3(0,0,0,0)$$

$$+ (2688\gamma^2 - 2976\gamma + 720\log^2 q - 768\gamma_1)A_4(0,0,0,0)$$

$$+ (-1536\gamma + 744)A_{1,2}(0,0,0,0) + (768\gamma - 372)A_{1,3}(0,0,0,0)$$

$$+ (768\gamma - 372)A_{1,4}(0,0,0,0) + (768\gamma - 372)A_{2,3}(0,0,0,0)$$

$$+ (768\gamma - 372)A_{2,4}(0,0,0,0) + (-1536\gamma + 744)A_{3,4}(0,0,0,0)$$

$$+ 32A_{1,1,1}(0,0,0,0) - 96A_{1,1,2}(0,0,0,0) - 96A_{1,2,2}(0,0,0,0) + 192A_{1,2,3}(0,0,0,0)$$

$$+ 192A_{1,2,4}(0,0,0,0) - 192A_{1,3,4}(0,0,0,0) + 32A_{2,2,2}(0,0,0,0) - 192A_{2,3,4}(0,0,0,0)$$

$$- 32A_{3,3,3}(0,0,0,0) + 96A_{3,3,4}(0,0,0,0) + 96A_{3,4,4}(0,0,0,0) - 32A_{4,4,4}(0,0,0,0)\Big)q^{31/2}$$

$$
\begin{aligned}
&+ \Big((-1536\gamma + 24)\sqrt{q} - 24q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} \\
&+ 44352165q^{10} + 141120525q^{9} + 265182525q^{8} + 300540195q^{7} + 206253075q^{6} \\
&+ 84672315q^{5} + 20160075q^{4} + 2629575q^{3} + 169911q^{2} + 4495q + 31) + 48\gamma(q^{16} \\
&+ 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12} + 64512240q^{11} + 225792840q^{10} \\
&+ 471435600q^{9} + 601080390q^{8} + 471435600q^{7} + 225792840q^{6} + 64512240q^{5} \\
&+ 10518300q^{4} + 906192q^{3} + 35960q^{2} + 496q + 1))A_{3,4}(0,0,0,0) \\
&+ \Big((768\gamma - 12)\sqrt{q} + 12q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} \\
&+ 44352165q^{10} + 141120525q^{9} + 265182525q^{8} + 300540195q^{7} + 206253075q^{6} \\
&+ 84672315q^{5} + 20160075q^{4} + 2629575q^{3} + 169911q^{2} + 4495q + 31) - 24\gamma(q^{16} \\
&+ 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12} + 64512240q^{11} + 225792840q^{10} \\
&+ 471435600q^{9} + 601080390q^{8} + 471435600q^{7} + 225792840q^{6} + 64512240q^{5} \\
&+ 10518300q^{4} + 906192q^{3} + 35960q^{2} + 496q + 1)\Big)A_{2,4}(0,0,0,0) \\
&+ \Big((3880548\log^3 q - 27361152\gamma\log^2 q + (57090096\gamma^2 - 16311456\gamma_1) \\
&- 28998144\gamma^3 + 48330240\gamma\gamma_1 + 4833024\gamma_2)A(0,0,0,0) + (-16915584\gamma^2 + 16311456\gamma \\
&- 3420144\log^2 q + 4833024\gamma_1)A_1(0,0,0,0) + (-16915584\gamma^2 + 16311456\gamma - 3420144\log^2 q \\
&+ 4833024\gamma_1)A_2(0,0,0,0) + (16915584\gamma^2 - 16311456\gamma + 3420144\log^2 q \\
&- 4833024\gamma_1)A_3(0,0,0,0) + (16915584\gamma^2 - 16311456\gamma + 3420144\log^2 q \\
&- 4833024\gamma_1)A_4(0,0,0,0) + (-9666048\gamma + 4077864)A_{1,2}(0,0,0,0) \\
&+ (4833024\gamma - 2038932)A_{1,3}(0,0,0,0) + (4833024\gamma - 2038932)A_{1,4}(0,0,0,0) \\
&+ (4833024\gamma - 2038932)A_{2,3}(0,0,0,0) + (4833024\gamma - 2038932)A_{2,4}(0,0,0,0) \\
&+ (-9666048\gamma + 4077864)A_{3,4}(0,0,0,0) + 201376A_{1,1,1}(0,0,0,0) - 604128A_{1,1,2}(0,0,0,0) \\
&- 604128A_{1,2,2}(0,0,0,0) + 1208256A_{1,2,3}(0,0,0,0) + 1208256A_{1,2,4}(0,0,0,0) \\
&- 1208256A_{1,3,4}(0,0,0,0) + 201376A_{2,2,2}(0,0,0,0) - 1208256A_{2,3,4}(0,0,0,0) \\
&- 201376A_{3,3,3}(0,0,0,0) + 604128A_{3,3,4}(0,0,0,0) + 604128A_{3,4,4}(0,0,0,0)
\end{aligned}
$$

$$
\begin{aligned}
&- 201376A_{4,4,4}(0,0,0,0)\Big)q^{27/2}\\
&+ \Big((97288620\log^3 q - 2746972800\gamma\log^2 q + (14902327440\gamma^2 - 4257807840\gamma_1)\\
&- 18579525120\gamma^3 + 30965875200\gamma\gamma_1 + 3096587520\gamma_2)A(0,0,0,0)\\
&+ (-10838056320\gamma^2 + 4257807840\gamma - 343371600\log^2 q + 3096587520\gamma_1)A_1(0,0,0,0)\\
&+ (-10838056320\gamma^2 + 4257807840\gamma - 343371600\log^2 q + 3096587520\gamma_1)A_2(0,0,0,0)\\
&+ (10838056320\gamma^2 - 4257807840\gamma + 343371600\log^2 q - 3096587520\gamma_1)A_3(0,0,0,0)\\
&+ (10838056320\gamma^2 - 4257807840\gamma + 343371600\log^2 q - 3096587520\gamma_1)A_4(0,0,0,0)\\
&+ (-6193175040\gamma + 1064451960)A_{1,2}(0,0,0,0) + (3096587520\gamma - 532225980)A_{1,3}(0,0,0,0)\\
&+ (3096587520\gamma - 532225980)A_{1,4}(0,0,0,0) + (3096587520\gamma - 532225980)A_{2,3}(0,0,0,0)\\
&+ (3096587520\gamma - 532225980)A_{2,4}(0,0,0,0) + (-6193175040\gamma + 1064451960)A_{3,4}(0,0,0,0)\\
&+ 129024480A_{1,1,1}(0,0,0,0) - 387073440A_{1,1,2}(0,0,0,0) - 387073440A_{1,2,2}(0,0,0,0)\\
&+ 774146880A_{1,2,3}(0,0,0,0) + 774146880A_{1,2,4}(0,0,0,0) - 774146880A_{1,3,4}(0,0,0,0)\\
&+ 129024480A_{2,2,2}(0,0,0,0) - 774146880A_{2,3,4}(0,0,0,0) - 129024480A_{3,3,3}(0,0,0,0)\\
&+ 387073440A_{3,3,4}(0,0,0,0) + 387073440A_{3,4,4}(0,0,0,0) - 129024480A_{4,4,4}(0,0,0,0)\Big)q^{11/2}\\
&+ \Big((4275180\log^3 q - 390873600\gamma\log^2 q + (2650611600\gamma^2 - 757317600\gamma_1) - 4039027200\gamma^3\\
&+ 6731712000\gamma\gamma_1 + 673171200\gamma_2)A(0,0,0,0)\\
&+ (-2356099200\gamma^2 + 757317600\gamma - 48859200\log^2 q + 673171200\gamma_1)A_1(0,0,0,0)\\
&+ (-2356099200\gamma^2 + 757317600\gamma - 48859200\log^2 q + 673171200\gamma_1)A_2(0,0,0,0)\\
&+ (2356099200\gamma^2 - 757317600\gamma + 48859200\log^2 q - 673171200\gamma_1)A_3(0,0,0,0)\\
&+ (2356099200\gamma^2 - 757317600\gamma + 48859200\log^2 q - 673171200\gamma_1)A_4(0,0,0,0)\\
&+ (-1346342400\gamma + 189329400)A_{1,2}(0,0,0,0) + (673171200\gamma - 94664700)A_{1,3}(0,0,0,0)\\
&+ (673171200\gamma - 94664700)A_{1,4}(0,0,0,0) + (673171200\gamma - 94664700)A_{2,3}(0,0,0,0)\\
&+ (673171200\gamma - 94664700)A_{2,4}(0,0,0,0) + (-1346342400\gamma + 189329400)A_{3,4}(0,0,0,0)\\
&+ 28048800A_{1,1,1}(0,0,0,0) - 84146400A_{1,1,2}(0,0,0,0) - 84146400A_{1,2,2}(0,0,0,0)
\end{aligned}
$$

$$
\begin{aligned}
&+ 168292800A_{1,2,3}(0,0,0,0) + 168292800A_{1,2,4}(0,0,0,0) - 168292800A_{1,3,4}(0,0,0,0)\\
&+ 28048800A_{2,2,2}(0,0,0,0) - 168292800A_{2,3,4}(0,0,0,0) - 28048800A_{3,3,3}(0,0,0,0)\\
&+ 84146400A_{3,3,4}(0,0,0,0) + 84146400A_{3,4,4}(0,0,0,0) - 28048800A_{4,4,4}(0,0,0,0)\Big)q^{9/2}\\
&+ \Big((-2688\gamma^2 + 96\gamma + 768\gamma_1)\sqrt{q} + 24q(q+1)(q^2+14q+1)(q^4+44q^3+166q^2+44q+1)\\
&\times (q^8 + 376q^7 + 4380q^6 + 15944q^5 + 24134q^4 + 15944q^3 + 4380q^2 + 376q + 1)\log^2 q\\
&- 96q\gamma(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10} + 141120525q^9\\
&+ 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4 + 2629575q^3\\
&+ 169911q^2 + 4495q + 31) + 84(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12}\\
&+ 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8 + 471435600q^7 + 225792840q^6\\
&+ 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1)(\gamma^2 - \tfrac{2\gamma_1}{7})\Big)A_1(0,0,0,0)\\
&+ \Big((2688\gamma^2 - 96\gamma - 768\gamma_1)\sqrt{q} - 24q(q+1)(q^2+14q+1)(q^4+44q^3+166q^2+44q+1)\\
&\times (q^8 + 376q^7 + 4380q^6 + 15944q^5 + 24134q^4 + 15944q^3 + 4380q^2 + 376q + 1)\log^2 q\\
&+ 96q\gamma(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10} + 141120525q^9\\
&+ 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4 + 2629575q^3\\
&+ 169911q^2 + 4495q + 31) - 84(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12}\\
&+ 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8 + 471435600q^7 + 225792840q^6\\
&+ 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1)(\gamma^2 - \tfrac{2\gamma_1}{7})\Big)A_3(0,0,0,0)\\
&+ \Big((768\gamma - 12)\sqrt{q} + 12q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11}\\
&+ 44352165q^{10} + 141120525q^9 + 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5\\
&+ 20160075q^4 + 2629575q^3 + 169911q^2 + 4495q + 31) - 24\gamma(q^{16} + 496q^{15} + 35960q^{14}\\
&+ 906192q^{13} + 10518300q^{12} + 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8\\
&+ 471435600q^7 + 225792840q^6 + 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2\\
&+ 496q + 1)\Big)A_{2,3}(0,0,0,0)\\
&+ \Big((571765740\log^3 q - 10488441600\gamma\log^2 q + (47416496400\gamma^2 - 13547570400\gamma_1)
\end{aligned}
$$

$$
\begin{aligned}
&- 50021798400\gamma^3 + 83369664000\gamma\gamma_1 + 8336966400\gamma_2)A(0,0,0,0) \\
&+ (-29179382400\gamma^2 + 13547570400\gamma - 1311055200\log^2 q + 8336966400\gamma_1)A_1(0,0,0,0) \\
&+ (-29179382400\gamma^2 + 13547570400\gamma - 1311055200\log^2 q + 8336966400\gamma_1)A_2(0,0,0,0) \\
&+ (29179382400\gamma^2 - 13547570400\gamma + 1311055200\log^2 q - 8336966400\gamma_1)A_3(0,0,0,0) \\
&+ (29179382400\gamma^2 - 13547570400\gamma + 1311055200\log^2 q - 8336966400\gamma_1)A_4(0,0,0,0) \\
&+ (-16673932800\gamma + 3386892600)A_{1,2}(0,0,0,0) + (8336966400\gamma - 1693446300)A_{1,3}(0,0,0,0) \\
&+ (8336966400\gamma - 1693446300)A_{1,4}(0,0,0,0) + (8336966400\gamma - 1693446300)A_{2,3}(0,0,0,0) \\
&+ (8336966400\gamma - 1693446300)A_{2,4}(0,0,0,0) + (-16673932800\gamma + 3386892600)A_{3,4}(0,0,0,0) \\
&+ 347373600A_{1,1,1}(0,0,0,0) - 1042120800A_{1,1,2}(0,0,0,0) - 1042120800A_{1,2,2}(0,0,0,0) \\
&+ 2084241600A_{1,2,3}(0,0,0,0) + 2084241600A_{1,2,4}(0,0,0,0) - 2084241600A_{1,3,4}(0,0,0,0) \\
&+ 347373600A_{2,2,2}(0,0,0,0) - 2084241600A_{2,3,4}(0,0,0,0) - 347373600A_{3,3,3}(0,0,0,0) \\
&+ 1042120800A_{3,3,4}(0,0,0,0) + 1042120800A_{3,4,4}(0,0,0,0) - 347373600A_{4,4,4}(0,0,0,0)\Big)q^{13/2} \\
&+ \Big((-4\log^3 q + (336\gamma^2 - 96\gamma_1) - 4608\gamma^3 + 7680\gamma\gamma_1 + 768\gamma_2)\sqrt{q} - 32q(q^{15} + 819/2q^{14} \\
&+ 24157q^{13} + 973791/2q^{12} + 4427865q^{11} + 41490735/2q^{10} + 53716845q^9 + 159109515/2q^8 \\
&+ 67863915q^7 + 65202585/2q^6 + 8194095q^5 + 1690845/2q^4 - 16965q^3 - 16443/2q^2 \\
&- 377q - 7/2)\log^3 q + 192q\gamma(q+1)(q^2 + 14q + 1)(q^4 + 44q^3 + 166q^2 + 44q + 1)(q^8 + 376q^7 \\
&+ 4380q^6 + 15944q^5 + 24134q^4 + 15944q^3 + 4380q^2 + 376q + 1)\log^2 q - 336q(q^{15} + 465q^{14} \\
&+ 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10} + 141120525q^9 + 265182525q^8 \\
&+ 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4 + 2629575q^3 + 169911q^2 \\
&+ 4495q + 31)(\gamma^2 - \tfrac{2\gamma_1}{7}) + 144(\gamma^3 - \tfrac{5\gamma\gamma_1}{3} - \tfrac{\gamma_2}{6})(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} \\
&+ 10518300q^{12} + 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8 + 471435600q^7 \\
&+ 225792840q^6 + 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1)\Big)A(0,0,0,0) \\
&+ \Big((768\gamma - 12)\sqrt{q} + 12q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10} \\
&+ 141120525q^9 + 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4
\end{aligned}
$$

$$+ 2629575q^3 + 169911q^2 + 4495q + 31) - 24\gamma(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13}$$

$$+ 10518300q^{12} + 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8 + 471435600q^7$$

$$+ 225792840q^6 + 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1)\Big)A_{1,4}(0,0,0,0)$$

$$+ \Big((1621890540\log^3 q - 22993891200\gamma\log^2 q + (89101328400\gamma^2 - 25457522400\gamma_1)$$

$$- 81464071680\gamma^3 + 135773452800\gamma\gamma_1 + 13577345280\gamma_2)A(0,0,0,0)$$

$$+ (-47520708480\gamma^2 + 25457522400\gamma - 2874236400\log^2 q + 13577345280\gamma_1)A_1(0,0,0,0)$$

$$+ (-47520708480\gamma^2 + 25457522400\gamma - 2874236400\log^2 q + 13577345280\gamma_1)A_2(0,0,0,0)$$

$$+ (47520708480\gamma^2 - 25457522400\gamma + 2874236400\log^2 q - 13577345280\gamma_1)A_3(0,0,0,0)$$

$$+ (47520708480\gamma^2 - 25457522400\gamma + 2874236400\log^2 q - 13577345280\gamma_1)A_4(0,0,0,0)$$

$$+ (-27154690560\gamma + 6364380600)A_{1,2}(0,0,0,0)$$

$$+ (13577345280\gamma - 3182190300)A_{1,3}(0,0,0,0)$$

$$+ (13577345280\gamma - 3182190300)A_{1,4}(0,0,0,0)$$

$$+ (13577345280\gamma - 3182190300)A_{2,3}(0,0,0,0)$$

$$+ (13577345280\gamma - 3182190300)A_{2,4}(0,0,0,0)$$

$$+ (-27154690560\gamma + 6364380600)A_{3,4}(0,0,0,0)$$

$$+ 565722720A_{1,1,1}(0,0,0,0) - 1697168160A_{1,1,2}(0,0,0,0) - 1697168160A_{1,2,2}(0,0,0,0)$$

$$+ 3394336320A_{1,2,3}(0,0,0,0) + 3394336320A_{1,2,4}(0,0,0,0) - 3394336320A_{1,3,4}(0,0,0,0)$$

$$+ 565722720A_{2,2,2}(0,0,0,0) - 3394336320A_{2,3,4}(0,0,0,0) - 565722720A_{3,3,3}(0,0,0,0)$$

$$+ 1697168160A_{3,3,4}(0,0,0,0) + 1697168160A_{3,4,4}(0,0,0,0) - 565722720A_{4,4,4}(0,0,0,0)\Big)q^{15/2}$$

$$+ \Big((768\gamma - 12)\sqrt{q} + 12q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10}$$

$$+ 141120525q^9 + 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4$$

$$+ 2629575q^3 + 169911q^2 + 4495q + 31) - 24\gamma(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13}$$

$$+ 10518300q^{12} + 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8$$

$$+ 471435600q^7 + 225792840q^6 + 64512240q^5 + 10518300q^4$$

$$
\begin{aligned}
&+ 906192q^3 + 35960q^2 + 496q + 1)\Big) A_{1,3}(0,0,0,0) \\
&+ \Big((2688\gamma^2 - 96\gamma - 768\gamma_1)\sqrt{q} - 24q(q+1)(q^2+14q+1)(q^4+44q^3+166q^2+44q+1) \\
&\times (q^8 + 376q^7 + 4380q^6 + 15944q^5 + 24134q^4 + 15944q^3 + 4380q^2 + 376q + 1)\log^2 q \\
&+ 96q\gamma(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11} + 44352165q^{10} + 141120525q^9 \\
&+ 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5 + 20160075q^4 + 2629575q^3 \\
&+ 169911q^2 + 4495q + 31) - 84(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12} \\
&+ 64512240q^{11} + 225792840q^{10} + 471435600q^9 + 601080390q^8 + 471435600q^7 + 225792840q^6 \\
&+ 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1)(\gamma^2 - \tfrac{2\gamma_1}{7})\Big) A_4(0,0,0,0) \\
&+ \Big((331405620\log^3 q - 2746972800\gamma\log^2 q + (6773785200\gamma^2 - 1935367200\gamma_1) \\
&- 4039027200\gamma^3 + 6731712000\gamma\gamma_1 + 673171200\gamma_2) A(0,0,0,0) \\
&+ (-2356099200\gamma^2 + 1935367200\gamma - 343371600\log^2 q + 673171200\gamma_1) A_1(0,0,0,0) \\
&+ (-2356099200\gamma^2 + 1935367200\gamma - 343371600\log^2 q + 673171200\gamma_1) A_2(0,0,0,0) \\
&+ (2356099200\gamma^2 - 1935367200\gamma + 343371600\log^2 q - 673171200\gamma_1) A_3(0,0,0,0) \\
&+ (2356099200\gamma^2 - 1935367200\log q\gamma + 343371600\log^2 q - 673171200\gamma_1) A_4(0,0,0,0) \\
&+ (-1346342400\gamma + 483841800\log q) A_{1,2}(0,0,0,0) \\
&+ (673171200\gamma - 241920900\log q) A_{1,3}(0,0,0,0) \\
&+ (673171200\gamma - 241920900\log q) A_{1,4}(0,0,0,0) \\
&+ (673171200\gamma - 241920900\log q) A_{2,3}(0,0,0,0) \\
&+ (673171200\gamma - 241920900\log q) A_{2,4}(0,0,0,0) \\
&+ (-1346342400\gamma + 483841800\log q) A_{3,4}(0,0,0,0) \\
&+ 28048800A_{1,1,1}(0,0,0,0) - 84146400A_{1,1,2}(0,0,0,0) - 84146400A_{1,2,2}(0,0,0,0) \\
&+ 168292800A_{1,2,3}(0,0,0,0) + 168292800A_{1,2,4}(0,0,0,0) - 168292800A_{1,3,4}(0,0,0,0) \\
&+ 28048800A_{2,2,2}(0,0,0,0) - 168292800A_{2,3,4}(0,0,0,0) - 28048800A_{3,3,3}(0,0,0,0) \\
&+ 84146400A_{3,3,4}(0,0,0,0) + 84146400A_{3,4,4}(0,0,0,0) - 28048800A_{4,4,4}(0,0,0,0)\Big) q^{23/2}
\end{aligned}
$$

$$+ \Big((-1536\gamma + 24\log q)\sqrt{q} - 24q(q^{15} + 465q^{14} + 31465q^{13} + 736281q^{12} + 7888725q^{11}$$
$$+ 44352165q^{10} + 141120525q^9 + 265182525q^8 + 300540195q^7 + 206253075q^6 + 84672315q^5$$
$$+ 20160075q^4 + 2629575q^3 + 169911q^2 + 4495q + 31)\log q$$
$$+ 48\gamma(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12} + 64512240q^{11} + 225792840q^{10}$$
$$+ 471435600q^9 + 601080390q^8 + 471435600q^7 + 225792840q^6$$
$$+ 64512240q^5 + 10518300q^4 + 906192q^3 + 35960q^2 + 496q + 1) \Big) A_{1,2}(0,0,0,0)$$
$$+ \Big((1147173300\log^3 q - 10488441600\gamma\log^2 q + (28449897840\gamma^2 - 8128542240\gamma_1)\log q$$
$$- 18579525120\gamma^3 + 30965875200\gamma\gamma_1 + 3096587520\gamma_2)A(0,0,0,0)$$
$$+ (-10838056320\gamma^2 + 8128542240\log q\gamma - 1311055200\log^2 q + 3096587520\gamma_1)A_1(0,0,0,0)$$
$$+ (-10838056320\gamma^2 + 8128542240\log q\gamma - 1311055200\log^2 q + 3096587520\gamma_1)A_2(0,0,0,0)$$
$$+ (10838056320\gamma^2 - 8128542240\log q\gamma + 1311055200\log^2 q - 3096587520\gamma_1)A_3(0,0,0,0)$$
$$+ (10838056320\gamma^2 - 8128542240\log q\gamma + 1311055200\log^2 q - 3096587520\gamma_1)A_4(0,0,0,0)$$
$$+ (-6193175040\gamma + 2032135560\log q)A_{1,2}(0,0,0,0)$$
$$+ (3096587520\gamma - 1016067780\log q)A_{1,3}(0,0,0,0)$$
$$+ (3096587520\gamma - 1016067780\log q)A_{1,4}(0,0,0,0)$$
$$+ (3096587520\gamma - 1016067780\log q)A_{2,3}(0,0,0,0)$$
$$+ (3096587520\gamma - 1016067780\log q)A_{2,4}(0,0,0,0)$$
$$+ (-6193175040\gamma + 2032135560\log q)A_{3,4}(0,0,0,0)$$
$$+ 129024480A_{1,1,1}(0,0,0,0) - 387073440A_{1,1,2}(0,0,0,0) - 387073440A_{1,2,2}(0,0,0,0)$$
$$+ 774146880A_{1,2,3}(0,0,0,0) + 774146880A_{1,2,4}(0,0,0,0) - 774146880A_{1,3,4}(0,0,0,0)$$
$$+ 129024480A_{2,2,2}(0,0,0,0) - 774146880A_{2,3,4}(0,0,0,0) - 129024480A_{3,3,3}(0,0,0,0)$$
$$+ 387073440A_{3,3,4}(0,0,0,0) + 387073440A_{3,4,4}(0,0,0,0) - 129024480A_{4,4,4}(0,0,0,0) \Big) q^{21/2}$$
$$+ \Big((2240839860\log^3 q - 22993891200\gamma\log^2 q + (69301033200\gamma^2 - 19800295200\gamma_1)\log q$$
$$- 50021798400\gamma^3 + 83369664000\gamma\gamma_1 + 8336966400\gamma_2)A(0,0,0,0)$$

$$+ (-29179382400\gamma^2 + 19800295200 \log q\gamma - 2874236400 \log^2 q + 8336966400\gamma_1)A_1(0,0,0,0)$$
$$+ (-29179382400\gamma^2 + 19800295200 \log q\gamma - 2874236400 \log^2 q + 8336966400\gamma_1)A_2(0,0,0,0)$$
$$+ (29179382400\gamma^2 - 19800295200 \log q\gamma + 2874236400 \log^2 q - 8336966400\gamma_1)A_3(0,0,0,0)$$
$$+ (29179382400\gamma^2 - 19800295200 \log q\gamma + 2874236400 \log^2 q - 8336966400\gamma_1)A_4(0,0,0,0)$$
$$+ (-16673932800\gamma + 4950073800 \log q)A_{1,2}(0,0,0,0)$$
$$+ (8336966400\gamma - 2475036900 \log q)A_{1,3}(0,0,0,0)$$
$$+ (8336966400\gamma - 2475036900 \log q)A_{1,4}(0,0,0,0)$$
$$+ (8336966400\gamma - 2475036900 \log q)A_{2,3}(0,0,0,0)$$
$$+ (8336966400\gamma - 2475036900 \log q)A_{2,4}(0,0,0,0)$$
$$+ (-16673932800\gamma + 4950073800 \log q)A_{3,4}(0,0,0,0)$$
$$+ 347373600A_{1,1,1}(0,0,0,0) - 1042120800A_{1,1,2}(0,0,0,0) - 1042120800A_{1,2,2}(0,0,0,0)$$
$$+ 2084241600A_{1,2,3}(0,0,0,0) + 2084241600A_{1,2,4}(0,0,0,0) - 2084241600A_{1,3,4}(0,0,0,0)$$
$$+ 347373600A_{2,2,2}(0,0,0,0) - 2084241600A_{2,3,4}(0,0,0,0) - 347373600A_{3,3,3}(0,0,0,0)$$
$$+ 1042120800A_{3,3,4}(0,0,0,0) + 1042120800A_{3,4,4}(0,0,0,0) - 347373600A_{4,4,4}(0,0,0,0)\Big)q^{19/2}$$
$$+ \Big((2520659700 \log^3 q - 29782563840\gamma \log^2 q + (100981505520\gamma^2 - 28851858720\gamma_1) \log q$$
$$- 81464071680\gamma^3 + 135773452800\gamma\gamma_1 + 13577345280\gamma_2)A(0,0,0,0)$$
$$+ (-47520708480\gamma^2 + 28851858720 \log q\gamma - 3722820480 \log^2 q + 13577345280\gamma_1)A_1(0,0,0,0)$$
$$+ (-47520708480\gamma^2 + 28851858720 \log q\gamma - 3722820480 \log^2 q + 13577345280\gamma_1)A_2(0,0,0,0)$$
$$+ (47520708480\gamma^2 - 28851858720 \log q\gamma + 3722820480 \log^2 q - 13577345280\gamma_1)A_3(0,0,0,0)$$
$$+ (47520708480\gamma^2 - 28851858720 \log q\gamma + 3722820480 \log^2 q - 13577345280\gamma_1)A_4(0,0,0,0)$$
$$+ (-27154690560\gamma + 7212964680 \log q)A_{1,2}(0,0,0,0)$$
$$+ (13577345280\gamma - 3606482340 \log q)A_{1,3}(0,0,0,0)$$
$$+ (13577345280\gamma - 3606482340 \log q)A_{1,4}(0,0,0,0)$$
$$+ (13577345280\gamma - 3606482340 \log q)A_{2,3}(0,0,0,0)$$
$$+ (13577345280\gamma - 3606482340 \log q)A_{2,4}(0,0,0,0)$$

$$
\begin{aligned}
&+ (-27154690560\gamma + 7212964680 \log q) A_{3,4}(0,0,0,0) \\
&+ 565722720 A_{1,1,1}(0,0,0,0) - 1697168160 A_{1,1,2}(0,0,0,0) - 1697168160 A_{1,2,2}(0,0,0,0) \\
&+ 3394336320 A_{1,2,3}(0,0,0,0) + 3394336320 A_{1,2,4}(0,0,0,0) - 3394336320 A_{1,3,4}(0,0,0,0) \\
&+ 565722720 A_{2,2,2}(0,0,0,0) - 3394336320 A_{2,3,4}(0,0,0,0) - 565722720 A_{3,3,3}(0,0,0,0) \\
&+ 1697168160 A_{3,3,4}(0,0,0,0) + 1697168160 A_{3,4,4}(0,0,0,0) - 565722720 A_{4,4,4}(0,0,0,0)\Big) q^{17/2} \\
&+ \Big((118436 \log^3 q - 779520\gamma \log^2 q + (1510320\gamma^2 - 431520\gamma_1) \log q \\
&- 714240\gamma^3 + 1190400\gamma\gamma_1 + 119040\gamma_2) A(0,0,0,0) \\
&+ (-416640\gamma^2 + 431520 \log q\gamma - 97440 \log^2 q + 119040\gamma_1) A_1(0,0,0,0) \\
&+ (-416640\gamma^2 + 431520 \log q\gamma - 97440 \log^2 q + 119040\gamma_1) A_2(0,0,0,0) \\
&+ (416640\gamma^2 - 431520 \log q\gamma + 97440 \log^2 q - 119040\gamma_1) A_3(0,0,0,0) \\
&+ (416640\gamma^2 - 431520 \log q\gamma + 97440 \log^2 q - 119040\gamma_1) A_4(0,0,0,0) \\
&+ (-238080\gamma + 107880 \log q) A_{1,2}(0,0,0,0) \\
&+ (119040\gamma - 53940 \log q) A_{1,3}(0,0,0,0) \\
&+ (119040\gamma - 53940 \log q) A_{1,4}(0,0,0,0) \\
&+ (119040\gamma - 53940 \log q) A_{2,3}(0,0,0,0) \\
&+ (119040\gamma - 53940 \log q) A_{2,4}(0,0,0,0) \\
&+ (-238080\gamma + 107880 \log q) A_{3,4}(0,0,0,0) \\
&+ 4960 A_{1,1,1}(0,0,0,0) - 14880 A_{1,1,2}(0,0,0,0) - 14880 A_{1,2,2}(0,0,0,0) \\
&+ 29760 A_{1,2,3}(0,0,0,0) + 29760 A_{1,2,4}(0,0,0,0) - 29760 A_{1,3,4}(0,0,0,0) \\
&+ 4960 A_{2,2,2}(0,0,0,0) - 29760 A_{2,3,4}(0,0,0,0) - 4960 A_{3,3,3}(0,0,0,0) \\
&+ 14880 A_{3,3,4}(0,0,0,0) + 14880 A_{3,4,4}(0,0,0,0) - 4960 A_{4,4,4}(0,0,0,0)\Big) q^{29/2} \\
&- \Big(q^{16} + 496q^{15} + 35960q^{14} + 906192q^{13} + 10518300q^{12} + 64512240q^{11} + 225792840q^{10} \\
&+ 471435600q^{9} + 601080390q^{8} + 471435600q^{7} + 225792840q^{6} + 64512240q^{5} \\
&+ 10518300q^{4} + 906192q^{3} + 35960q^{2} + 496q - 32\sqrt{q} + 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big(- A_{3,3,3}(0,0,0,0) + A_{2,2,2}(0,0,0,0) - 3A_{1,1,2}(0,0,0,0) + 6A_{1,2,3}(0,0,0,0) \\
&+ A_{1,1,1}(0,0,0,0) - 3A_{1,2,2}(0,0,0,0) + 6A_{1,2,4}(0,0,0,0) - 6A_{1,3,4}(0,0,0,0) \\
&- 6A_{2,3,4}(0,0,0,0) + 3A_{3,3,4}(0,0,0,0) + 3A_{3,4,4}(0,0,0,0) - A_{4,4,4}(0,0,0,0)\Big)\Big)
\end{aligned}
$$

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