Portfolio Allocation with Temporary Price Impact

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Abstract

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This thesis analyzes an investor's portfolio choice and liquidity premium in the presence of an illiquid stock. Illiquidity is modelled by means of convex transaction costs which an investor has to pay for trading a stock. An investor is assumed to trade in the presence of a stochastic endowment which is used to generate long-term trading demands. I find that the endowment generates long-term trading incentives only if there exist correlations between the endowment and stocks returns. These incentives result in the liquidity premium which makes only a fraction of a percent of the risk premium. The portfolio choice and the conditional liquidity premium can be determined in a closed form in the case of the absence of these correlations and the main economic intuition behind the findings is confirmed.

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1. Introduction

The relation between liquidity and prices of financial assets has been studied extensively. Individual investors are concerned about liquidity due to its effect on their investment's return in the way that it costs them more to buy illiquid securities while they receive less when selling them. This is because a market maker typically increases the bid-ask spread, especially for large trades, when the market is less than perfectly liquid. As a result, an investor could be more hesitant to invest in these illiquid assets and tends to allocate their wealth into more liquid ones. The standard definition of illiquidity refers to the difficulty of trading a large number of financial assets quickly. There are different kinds of cost contributing to market illiquidity and there exist also various measures of liquidity. Therefore, the way each cost category is measured may have dissimilar influences on evaluating the portfolio choices of investors.

One of the most popular ways of modeling illiquidity is through price impact, which measures how an incoming order to purchase or sell a security affects its price. Empirical observation suggests that large trades put pressure on prices: large buys (sells) can push them up (down). But why are these changes typically followed by rapid reversals? This is among several puzzles regarding market liquidity. This phenomenon appeared in a systemic intraday event – the Flash Crash – in the U.S. financial markets on May 6, 2010. The Commodity Futures Trading Commission – Securities Exchange Commission (CFTC-SEC (2010)) joint report describes the Flash Crash as follows: "On May 6, a wide variety of broad market indices and products displayed similar behavior – a severe price decline immediately followed by a rapid recovery during a 20-minute period" (p. 17). This event demonstrates that during financial crises, the price impact can be significantly nonlinear (Kirilenko et al. (2017)).

According to Foucault et al. (2013), liquidity suppliers require a larger bid-ask spread to compensate for three kinds of costs: adverse selection costs, order-processing costs, and inventory holding costs. Adverse selection costs result from a transaction with better-informed investors. Liquidity suppliers then need to re-estimate the value of the traded asset, and thus the price impact of this market order is *permanent*. If, however, it is the real cost of processing orders or the inventory holding risk that causes the market illiquidity, then the price impact of a market order is *temporary* and should wear off over time. They also note that although both order-processing costs and inventory holding costs generate price reversals, their speeds are different: prices are reversed immediately with the former while gradually with the latter. Thus, the relative importance of order-processing costs versus inventory holding costs determines the speed of price reversals.

The literature on the permanent impact of a market order on asset prices is vast. Bagehot (1971) is among the first to introduce an asymmetric information model in which a market maker's behaviors are considered in a scenario with two indistinguishable kinds of traders: liquidity-motivated (uninformed) and information-motivated (informed) traders. The market maker sets a bid-ask spread so that his gains from uninformed traders exceed his losses to the informed. Popular models of trading with asymmetric information have then been developed by Kyle (1985); Glosten and Milgrom (1985); Easley and O'Hara (1987); Grossman and Miller (1988); Hasbrouck (1991), and many others.

The existence of temporary price impact was already shown by Kraus and Stoll (1972) when they study large institutional trades. The theoretical analysis of the impact of inventory holding costs on the bid-ask spread goes back to Stoll (1978) who defines the cost as an amount to compensate a dealer for transactions that tend to cause his portfolio to deviate from the target. Amihud and Mendelson (1980) consider a dynamic model in which a market maker faces constraints on his long and short inventory positions. They predict that the spread widens as the market maker's position strays from his preferred inventory level. Ho and Stoll (1981) also analyze the optimal dynamic pricing policy for a single dealer in the presence of inventory holding risk and predict that the dealer will control his inventory position by adjusting his price quotes. Ho and Stoll (1983) analyze the behavior of competing dealers when they have different inventory positions in centralized markets. Biais (1993) extends the analysis to fragmented markets and compares them to centralized markets. In a similar fashion, Yin (2005) introduces liquidity traders' costs of search into Biais's model.

Huang and Stoll (1997) decompose the bid-ask spread into three components – order processing, adverse information, and inventory – and find that on average, the order processing costs, the inventory costs, and the adverse selection costs account for 61.8%, 28.7%, and 9.6%, respectively, of the traded spread. They interestingly find that for large trades, the adverse selection component is even less announced, probably because information typically already leaks out before the trades. Many other papers have used actual inventory data to empirically study the behavior of market maker in stock markets (see, e.g., Madhavan and Smidt (1993); Hasbrouck and Sofianos (1993); Naik and Yadav (2003); Hendershott and Seasholes (2007); Comerton-Forde et al. (2010); Hendershott and Menkveld (2014)). Although a few studies find a weak relation between asset prices and inventory (see, e.g., Madhavan and Smidt (1991); Hasbrouck and Sofianos (1993)), most of them support the inventory mechanism which predicts a strong transitory price impact of a market trade.

The importance of price impact in financial asset prices and market liquidity makes it critical for an investor when considering portfolio allocation policies. Merton (1969) develops frictionless models and recommends strategies with infinite trading volume. Portfolio rules under permanent price impact have then been extensively studied in which permanent price impact is modeled by assuming that an investor's aggregate position affects price dynamics (see, e.g., Cuoco and Cvitanić (1998); Bank and Baum (2004); Huberman and Stanzl (2004); Bank and Kramkov (2015a, 2015b)). Linear price impact models, in which transaction costs (a measure for temporary price impact) are fixed or linear in the number of traded shares or dollar value, result in policies with the order flow proportional to the distance from the current position to the target (see, e.g., Guasoni and Weber (2017); Moreau et al. (2017)). However, empirical evidence suggests that transitory price impact is nonlinear (see, e.g., Lillo et al. (2003); Robert et al. (2012)). The convexity of transaction costs is intuitive since it becomes more difficult for a market maker to trade more shares in a short time. In the literature on nonlinear price impact, attention has mainly focused on the problems of optimal liquidation strategy (see, e.g., Almgren (2003); Vath et al. (2007); Schied et al. (2010)), optimal trade execution (see, e.g., Almgren and Chriss (2001); Gatheral and Schied (2011, 2013); Guasoni and Weber (2020)), dynamic portfolio management (see, e.g., Gârleanu and Pedersen (2013, 2016); Grinold (2018)), and hedging problems (see, e.g., Rogers and Singh (2010); Bank et al. (2017)).

The literature on portfolio selection problem with multiple risky assets in the presence of transaction costs is limited. The majority of them study the optimal consumption and investment when all risky assets subject to fixed or proportional transaction costs. For example, Akian et al. (1996) consider the optimal consumption and investment strategy for an investor who has constant relative risk aversion (CRRA) preference in the presence of proportional transaction costs, assuming that asset returns are uncorrelated. Liu (2004) studies an optimal intertemporal consumption and investment problem with fixed and proportional transaction costs for a constant absolute risk aversion (CARA) investor when asset returns are uncorrelated. Muthuraman and Kumar (2006) provide a computational study of a similar problem for a CRRA investor while assuming that transaction costs are proportional to the dollar value of the transaction and there are correlations between the price processes. Lynch and Tan (2010) numerically solve the decision problem of a multiperiod CRRA investor in the presence of proportional transaction costs when asset returns are predictable. Chen and Dai (2013) provide a thorough theoretical characterization of the optimal strategy on multiple correlated assets with proportional transaction costs for a risk-averse (CARA or CRRA) investor.

It is, however, reasonable to think of the case when there are only some risky assets subject to transaction costs while others are liquid. If so then, how does the illiquidity of these stocks affect an investor's optimal investment policies? Isaenko (2006) provides numerical and approximate solutions for this question when studying an optimization problem for a CARA investor who maximizes expected utility from her terminal wealth, assuming that only one risky asset is subject to nonlinear transaction costs. The thesis is an extension of his analysis in several ways. First, the optimization problem is considered for an investor who has intermediate consumptions with an infinite horizon. In addition, the value function is conjectured similar to that in Isaenko (2020). This, together with other assumptions, enables me to obtain explicit solutions in many cases. Moreover, a non-tradable risky endowment flow is used to generate long-term trading demands. Lo et al. (2004) suggest the use of the endowment for the effect of transaction costs on asset prices to exist.

A portfolio choice problem with convex price impact is studied for a price-taker investor who has CARA preference and trades in the market with one riskless bond, $(n-1)$ liquid stocks, and one illiquid stock. For the sake of tractability, I ignore the permanent price impact entirely and examine only the transitory price impact on the investor's portfolio allocation decisions. This can be done by assuming that when the investor trades an illiquid stock, she is required to pay transaction costs which are convex in the number of traded shares. The investor invests in both types of stocks to diversify. I assume that the illiquid stock can experience a temporary price impact but liquid stocks do not.

Given the presence of transaction costs in only one risky asset, it is of interest to understand the influence of this illiquid stock's position on other liquid stocks' optimal allocations, and thus the effects of transaction costs on an investor's optimal consumption and investment strategies. Isaenko (2006) numerically finds that optimal allocations to liquid stocks depend on the illiquid stock's allocation, and the trading rate of this stock is affected by the presence of liquid stocks even when returns of all stocks are uncorrelated. Different from his findings, the closed-form solutions of optimal policies indicate the independence between allocations of the illiquid stock and liquid stocks. Besides, the liquid stocks' optimal investment policies are the same regardless of whether transaction costs are present or not. This implies that transaction costs subjected by the illiquid stock do not affect other liquid stocks' optimal trading policies. The optimal allocation to each liquid stock then relies only on their risk premium, the volatility of each stock returns, the risk-free interest rate, and the investor's risk tolerance. The separability of the optimal allocation for each stock makes the computation of the optimal investment policies for a large number of stocks become easier.

Although the case of uncorrelated stock returns is commonly recommended for efficient diversification and for tractability of analysis, it is practically oversimplified since the returns among stocks are typically positively correlated. Therefore, the case of multiple correlated risky assets is also considered. It is shown that the dependence of liquid stocks' target allocations on the illiquid stock's position is affected by correlations among asset returns. Specifically, the optimal allocations to liquid stocks are negatively related to the number of illiquid stock shares. The longer the position that an investor takes in the illiquid risky asset, the less wealth is allocated to liquid ones. The strength of this relationship increases with the strength of correlations between stock returns and decreases with the number of liquid stocks in the portfolio. Since the impact of transaction costs on an investor's optimal allocations to liquid stocks only exists when asset returns are correlated, it suggests another advantage of investing in independent stocks in addition to the benefit from diversification. In the case of uncorrelated risky assets, when a stock becomes illiquid, there is no need to rebalance her optimal allocations to liquid stocks, assuming that the investor's expectation on these stocks' returns remains unchanged. By contrast, if asset returns are correlated, the investor needs to readjust not only the illiquid stock's position but also those of liquid stocks.

Another crucial objective of the thesis is to examine how the holding of the illiquid stock affects the risk premium that an investor requires for her investment. This goal can be done by using the liquidity premium. Amihud and Mendelson (1986) are among the first to document the illiquidity premium for the equity market (see also, e.g., Brennan and Subrahmanyam (1996); Acharya and Pedersen (2005); Lee (2011)). The behavior of liquidity premium is typically analyzed in a general equilibrium framework or a partial equilibrium framework. The former has been developed based on asset supply and demand whose changes affect the asset price in equilibrium. For example, Vayanos (1998) assumes that trading demand is generated by lifetime consumption smoothing. Lo et al. (2004) consider heterogeneous agents who trade to hedge their nontraded risk exposure. On the other hand, the partial equilibrium framework is introduced by Constantinides (1986) to determine the impact of transaction costs on asset's mean return (see also, e.g., He and Mamaysky (2005); Jang et al. (2007)). In this framework, the liquidity premium is defined as the difference between expected returns of two assets – with and without transaction costs – that leaves an investor's expected utility unchanged. For tractability, I analyze the liquidity premium in a partial equilibrium framework and define it as the additional return to compensate an investor for holding the illiquid stock.

Lo et al. (2004), however, point out a limitation of the partial equilibrium framework, stating that partial equilibrium models tend to undervalue the impact of transaction costs on asset returns because they ignore the price effect of market-clearing motive. Indeed, Constantinides (1986) only finds a second-order effect of transaction costs on an investor's expected utility, and hence the liquidity premium is very small. Because transaction costs prevent the investor from trading frequently, it is important to generate long-term trading incentives so that the effect of transaction costs on asset prices and thus on the liquidity premium is better reflected. The long-term trading demands are typically achieved by using either a short-term unpredictable endowment, or a time-varying risk premium[1], or both. For mathematical tractability, in the light of Lo et al. (2004), and Isaenko (2020), I include a non-tradable cumulative risky endowment in the economic setting while considering a constant risk premium. This stochastic endowment may induce an investor to trade continuously in the long run and thus is expected to reveal a significant effect of transaction costs on an investor's optimal portfolio selection.

The market without transaction costs is used as a benchmark in comparison with the illiquid market to determine the liquidity premium. The presence of illiquid stock in the investor's portfolio makes her allocation to this stock a state variable. This implies that the utility function and, therefore, the conditional liquidity premium will depend on this allocation. The conditional liquidity premium then can be modeled as a function of the illiquid stock's position and the endowment volatility. Especially in the absence of correlations between the endowment flow and risky assets, a closed-form formula of the liquidity premium can be written in terms of the

[1] See, e.g., Campbell and Viceira (1999); Campbell et al. (2004); Liu (2010) in which the expected excess return of a risky asset follows a mean-reverting process.

illiquidity coefficient α whose magnitude defines the level of illiquidity. The explicit functions and numerical results show that there is a positive relationship between the level of illiquidity (α) and the liquidity premium, indicating that when the illiquidity of a stock becomes more significant, an investor demands a higher premium to compensate for holding this stock. However, in the steady state (as time t approaches infinity), the conditional liquidity premium becomes very small and even disappears in the case when there is no correlation between the shocks to the endowment and stocks returns. In this case, the investor tends to stop trading the illiquid stock and the long-term optimal investment strategies are the same as those in the absence of transaction costs, suggesting that transaction costs have no effect on the investor's portfolio choices in the long run. If the endowment and stocks are correlated, there exist long-term trading incentives. However, these trading needs seem to be very weak, leading to a negligible liquidity premium. These findings cast a shadow over the efficiency of the use of a non-tradable risky endowment in generating dynamical trading demands in the partial equilibrium framework. Despite the insignificant effect of transaction costs in the long term, they still play an important role in an investor's short-term decisions on portfolio selection. Therefore, for an investor who cares about long-term investment, transaction costs might not be of concern; but for those who have a short-term investment horizon, transaction costs should be considered.

One critical contribution of the thesis is to derive in closed-form the optimal consumption and investment policies and the conditional liquidity premium in many cases (when innovations to stocks and the endowment are uncorrelated). In particular, they enhance the understanding of how transaction costs affect an investor's optimal portfolio strategies. The explicit solutions also provide a better insight of the relationship between fundamental parameters (the coefficient of risk aversion, correlations between stock returns, the risk-free interest rate, the volatility of stock returns) and the optimal allocation to each risky asset. In addition, a closed-form function of the conditional liquidity premium allows me to transparently analyze the impact of these fundamental parameters on the liquidity premium and obtain some interesting results. Specifically, an increase in the coefficient of risk aversion, in correlations between stock returns, or in the volatility of the illiquid stock return decreases the investor's allocation to this illiquid asset, which in turns reduces its effects on the investor's expected utility, resulting in a smaller liquidity premium.

The rest of the thesis is presented as follows. Section 2 and Section 3 describe the basic models, including the economic setting and the portfolio selection by an investor. The solutions to the optimization problem can be found in Section 4 in which Subsection 4.1 describes the trading behaviors of an investor in the market with only three stocks, while Subsection 4.2 does it when there is a higher number of stocks trading in the economy. Section 5 summarizes all of my findings. Appendix A presents the Hamilton-Jacobi-Bellman (HJB) equations of the optimization problem mentioned in Section 3 in the economies with and without transaction costs. Finally, details of solutions in Section 4 are provided in Appendix B and Appendix C.

2. Economic Setting

A Markov economy is considered with an infinite horizon. Uncertainty in the model is driven by a standard $(n+2)$-dimensional Brownian motion $W = (W_1, \dots, W_n; W_0; W_Y)$.

A portfolio of $(n+1)$ securities including a riskless bond and n stocks is assumed. The price dynamics of the riskless bond B_t are:

$$dB_t = B_t r dt, \tag{1}$$

where r denotes a constant interest rate. The other risky assets are assumed to have no dividend for simplicity and have price dynamics given by:

$$\begin{aligned} dS_{1t} &= (RP_1 + rS_1 + \Delta_{l1})dt + \sigma_1 dW_{1t}, \\ dS_{it} &= (RP_i + rS_i)dt + \sigma_i dW_{it}, \qquad i = 2, \dots, n, \end{aligned} \tag{2}$$

where conditional dollar risk premium, RP_i, and σ_i of each stock are constant. It is set that only stock 1 is illiquid and Δ_{l1} is its liquidity premium. The covariance matrix of the stock returns can be degenerate but just in the circumstance that the degeneracy is caused by the existence of the illiquid stock. [2]

It is assumed that the stock prices have normal distribution. The combination of the normal distribution of the dollar returns with CARA utility function will allow to separate state variables and find a closed form solution in most of the cases to be considered. This is contrary to the case

[2] According to Isaenko (2006), the presence of illiquid stocks may result in a substantial change in correlations between stocks when a majority of investors exercise the same allocation strategies. Therefore, I do not rule out the possibility that the change in correlations could make the covariance matrix to be degenerate if it contains rows or columns which are proportionally interrelated.

where the stock *rates* of returns are assumed to be normal and the utility function is CRRA-type (or CARA-type). The latter will make the trading strategy to depend on the stock price and the investor's wealth. The separation of variables becomes impossible and a portfolio selection problem can be only solved numerically. Given a large number of stocks, this solution has been shown to be very difficult to manage and explain (see, e.g., Isaenko (2006)).

Following Isaenko (2006), the temporary price impact is modeled as the transaction costs $\alpha|u_1|^2dt$ that an investor pays for the trading of stock 1 within the time interval dt, where u_1 is instantaneous trading rate and the illiquidity coefficient α is positive (a higher α indicates the more illiquidity of stock 1). The term $\alpha|u_1|^2$ results from the temporary price impact and does not allow the trading speed to be infinite due to its convex shape (suggesting that the higher the $|u_1|$, the more costly it is for an investor to conduct a transaction). The change in the number of illiquid stock shares is given by:

$$dN_{1t} = u_{1t}dt, \tag{3}$$

This process suggests that an investor can only purchase or sell shares of stock 1 at a finite rate and implies that the trading rate u_1 becomes her control variable.

Similar to Lo et al. (2004) and Isaenko (2020), I assume that an investor is endowed with a stream of nontraded risky income with cumulative cash flow $I_t = \int_0^t Y_s\, dW_{0s}$, where W_0 is a Brownian motion which has correlation coefficient ρ_{0i} with shock W_i. The endowment is assumed to have a normal distribution. The endowment volatility Y is mean-reverting and follows the dynamic process:

$$dY_t = \kappa_Y(\bar{Y} - Y_t)dt + \sigma_Y dW_{Yt}, \tag{4}$$

where $\kappa_Y, \bar{Y},$ and σ_Y are constant and κ_Y, σ_Y are positive. The parameter κ_Y needs to be positive to ensure stability around the long-term value $\bar{Y}$. The correlation coefficients of W_Y with W_i and W_0, are constant and given by ρ_{Yi} and ρ_{0Y}, respectively. Clearly, an endowment can be negative due to, for example, a negative shock in business enterprises. A cumulative endowment is typically modeled without a diffusion term. However, this term is kept here due to its significance for motivating an investor's trading and the drift term is dropped for the purpose of tractability. The term dW_0 specifies the nontraded risk, and the endowment volatility Y_t gives an investor's exposure to this nontraded risk at time t. Since process Y is time-varying, the investor has inducement to continuously trade in stocks (including the illiquid one) to hedge her nontraded risk as it changes over time. It follows that trading of investors is propelled by the process Y. The presence of this trading demand is essential to better analyze how transaction costs – which discourage the investor from trading dynamically in the long run – affect an investor's optimal investment policies because their effects only matter when the investor actually trade.

3. The Investor's Problem

A price-taker investor is assumed to have a CARA preference and obtain her utility from intermediate consumption c with time discounting. The investor's problem is then to choose consumption (c) and investment $(u_1, N_2, \dots, N_n)$ strategies to maximize her expected utility function:

$$\max_{c,u_1,N_2,\dots,N_n \in R^n} E_0 \int_0^{\infty} \left(-\frac{1}{\gamma} \exp(-\tau t - \gamma c_t) \right) dt, \tag{5}$$

where $\gamma > 0$ is a coefficient of absolute risk aversion, τ is a time discount rate. Solving an infinite horizon with intermediate consumption is often easier than solving an otherwise similar problem in finite time (see, e.g., Merton (1969)). According to Brandt (2010), an infinite horizon problem only needs to be solved for a steady-state policy (as t approaches ∞), whereas a finite time problem requires the optimal policy to be found for each period.

Let X_t denotes the investor's wealth at time t. Given the intermediate consumption and the transaction costs in the time interval dt are $c_t dt$ and $\alpha |u_1|^2 dt$, respectively, the dynamic budget constraint that an investor faces can be derived as:

$$dX_t = \left(rX_t + N_{1t}(RP_1 + \Delta_{l1}) + \sum_{k=2}^{n} N_{kt} RP_k - c_t - \alpha |u_{1t}|^{1+\epsilon} \right) dt + Y_t dW_{0t} \tag{6}$$

$$+ \sum_{k=1}^{n} N_{kt} \sigma_k dW_{kt}.$$

The dynamic programming approach is applied to solve this optimization problem. Equations (3), (4), and (6) suggest that the state variables should include processes X (an investor's wealth), Y (an endowment volatility), and N_1 (an investor's allocation to the illiquid stock 1).

The value function at time t is defined as:

$$V(t,X,Y,N_1) = \max_{c,u_1,N_2,\dots,N_n \in R^n} E\left(\int_t^\infty \left(-\frac{1}{\gamma}\exp(-\tau s - \gamma c_s)\right) ds \,\middle|X_t = X, N_{1t} = N_1, Y_t = Y\right), \tag{7}$$

Similar to Isaenko (2020), it is conjectured to have the following form:

$$V(t,X,Y,N_1) = -\frac{1}{\gamma}\exp[-\tau t - \gamma r X + g(Y,N_1)].$$

where $g(Y,N_1)$ is assumed to have a quadratic form (8). It is also assumed that $V(t,X,Y,N_1)$ is monotonically increasing in X and continuously differentiable up to the second order in X and Y, and up to the first order in N_1 and t.

The value function of an investor must satisfy the HJB equation (A–1) shown in Appendix A, which then can be reduced to a system of six linear-quadratic equations to simultaneously solve for six coefficients in the function $g(Y,N_1)$. The next two propositions summarize the value function and the optimal controls in the market with and without transaction costs.

Proposition 1 *Assume that the set of equations (A–10) to (A–15) has a solution, then the functions* $\hat{u}_1(Y,N_1)$, $\hat{c}(X,Y,N_1)$, $\widehat{N}(Y,N_1)$ and $V(t,X,Y,N_1) = -\frac{1}{\gamma}\exp[-\tau t - \gamma r X + g(Y,N_1)]$, *where*:

$$g = A_{0C} + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 + (B_{0C} + B_{1C}N_1)Y + \frac{1}{2}C_{0C}Y^2, \tag{8}$$

$$\hat{u}_1 = -\frac{A_{1C} + A_{2C}N_1 + B_{1C}Y}{2\alpha\gamma r}, \tag{9}$$

$$\hat{c} = -\frac{1}{\gamma}\left[\ln(r) + A_{0C} + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 + (B_{0C} + B_{1C}N_1)Y + \frac{1}{2}C_{0C}Y^2\right] + rX, \tag{10}$$

$$\widehat{N} = COV^{-1} \times B, \tag{11}$$

$where\ (COV)_{i,j} = \sigma_{i+1}\sigma_{j+1}\rho_{i+1,j+1}, \qquad i,j = 1,\dots,n-1,$

$$B_i = \frac{(\sigma_Y \sigma_{i+1} \rho_{Y,i+1} B_{0C} + RP_{i+1})}{\gamma r} + \frac{(\sigma_Y \sigma_{i+1} \rho_{Y,i+1} B_{1C} - \gamma r \sigma_1 \sigma_{i+1} \rho_{1,i+1})}{\gamma r} N_1 + \frac{(\sigma_Y \sigma_{i+1} \rho_{Y,i+1} C_{0C} - \gamma r \sigma_{i+1} \rho_{0,i+1})}{\gamma r} Y \qquad i = 1, \dots, n-1$$

and all coefficients in equation (8) are constant with $A_{2C} > 0$[3], are the optimal controls and the value function of an investor.

Equations (9), (10), and (11) are derived from the first order conditions of the HJB equation (A–1) with respect to c, u_1, and N_k $(k = 2, \dots, n)$, respectively. It follows that the optimal strategies of an investor in the illiquid market are specified by these equations. In addition, the following corollary presents the allocation to the illiquid stock 1 (N_1) at time t:

Corollary 1

$$N_{1t} = -\frac{a}{b} + \frac{c\bar{Y}}{(b-\kappa_Y)}\left(\frac{\kappa_Y}{b} - 1\right) + \left[N_{10} + \frac{a}{b} - \frac{c}{(b-\kappa_Y)}\left(\frac{\kappa_Y \bar{Y}}{b} - Y_0\right)\right] e^{-bt} - \frac{c}{(b-\kappa_Y)}(Y_0 - \bar{Y}) e^{-\kappa_Y t} - \frac{c\sigma_Y}{(b-\kappa_Y)} \int_0^t \left(e^{-\kappa_Y (t-s)} - e^{-b(t-s)}\right) dW_{Ys}, \tag{12}$$

where $a = \frac{A_{1C}}{2\alpha\gamma r}, b = \frac{A_{2C}}{2\alpha\gamma r}, c = \frac{B_{1C}}{2\alpha\gamma r}$.

It follows that the illiquid stock's allocation is a Gaussian process and its stationary limit is given by:

$$\lim_{t\to\infty} N_{1t} = -\frac{a}{b} + \frac{c\bar{Y}}{(b-\kappa_Y)}\left(\frac{\kappa_Y}{b} - 1\right). \tag{13}$$

The next proposition addresses an investor's optimization problem in the economy without transaction costs. In this market, because all securities are assumed to be liquid, an investor's allocation to stock 1, N_1, should be excluded from the list of state variables. Thus, the investor's problem is to choose consumption (c) and investment ($N_1, N_2, \dots, N_n$) strategies to maximize her expected utility function. The state variables include processes X and Y. The value function and the optimal controls in this liquid market are presented as follows:

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[3] Following Isaenko (2020), this condition should hold so that $\lim_{N_1 \to \pm\infty} V(t, X, Y, N_1) = -\infty$.

Proposition 2 *Assume that the set of equations (A–26) to (A–28) has a solution, then the functions* $V(t,X,Y) = -\frac{1}{\gamma}\exp[-\tau t - \gamma r X + g(Y)]$, $\hat{c}(X,Y)$ *and* $\hat{N}(Y)$ *where*:

$$g = A_0 + B_0 Y + \frac{1}{2} C_0 Y^2, \tag{14}$$

$$\hat{c} = -\frac{1}{\gamma}\left[\ln(r) + A_0 + B_0 Y + \frac{1}{2} C_0 Y^2\right] + rX, \tag{15}$$

$$\hat{N} = COV^{-1} \times B, \tag{16}$$

$$where\ (COV)_{i,j} = \sigma_i \sigma_j \rho_{i,j}, \qquad i,j = 1, \dots, n,$$

$$B_i = \frac{\sigma_Y \sigma_i \rho_{Y,i} C_0 - \gamma r \sigma_i \rho_{0,i}}{\gamma r} Y + \frac{\sigma_Y \sigma_i \rho_{Y,i} B_0 + RP_i}{\gamma r}, \qquad i = 1, \dots, n$$

and all coefficients in equation (14) are constant, are the value function and the optimal controls of an investor.

Equations (15), and (16) are derived from the first order conditions of the HJB equation (A–19) with respect to c and N_k $(k = 1, \dots, n)$, respectively. It follows that an investor's optimal policies in the economy without transaction costs are determined by these equations. More details of proof for Proposition 1, Corollary 1, and Proposition 2 are discussed in Appendix A.

Because n could be large, two cases will be examined: when there are only three stocks in the stock market ($n = 3$), and when there are multiple stocks ($n > 3$). The former analysis is simple enough to determine the solutions for the optimization problem, and thus provides a better insight of the topic. In particular, I am able to estimate not only the effect of the number of illiquid stock shares held by an investor on her utility function and on the liquidity premium, but also the impact of correlations between stocks returns on this effect. On the other hand, the analysis of multiple stocks ($n > 3$) shows how the findings change when there are arbitrary numbers of liquid stocks whose returns are correlated. Unfortunately, as n increases, it becomes more difficult to figure out the covariance matrix (COV^{-1}) in equations (11) and (16). Hence, for simplicity, it is assumed that while stocks returns are correlated, other correlation coefficients between the endowment and stocks are zero.

4. Optimization Solutions

This section presents a few cases to illustrate how temporary transaction costs affect an investor's optimal portfolio choice. The economy with no trading costs is considered as a benchmark case to determine the conditional liquidity premium.

4.1. Three-Stock Market

In this subsection, I examine a portfolio of four securities including a riskless bond and three risky stocks. The optimization problem is solved in three different cases. Firstly, all stocks are assumed to be independent. Secondly, it is assumed that there are constant positive correlations between stocks returns, given that other correlation coefficients are zero. Finally, the optimization problem is considered when all shocks in the economy are correlated. The detailed analysis is shown in Appendix B.

4.1.1. Independent stocks

This subsection represents the findings with the assumption that the covariance matrix of stocks returns is diagonal, and all other correlation coefficients between any pair of Brownian motions are zero. In this case, the closed-form solutions are available for the optimization problem, which in turn allows me to derive a closed-form function of liquidity premium in terms of the number of illiquid stock shares (N_1). In particular:

The optimal allocations in the economy without transaction costs are:

$$\widehat{N}_k = \frac{RP_k}{\gamma r \sigma_k^2} \qquad k = 1, 2, 3, \tag{17}$$

The last result implies that the allocations to stocks in the portfolio are proportional to their risk premium. On the contrary, the higher the coefficient of risk aversion (a bigger γ) and the more volatile the stock returns (a higher σ_k), then the smaller the optimal allocation to each stock.

The optimal strategies in the economy with transaction costs are:

$$\hat{u}_1 = -\frac{A_{1C} + A_{2C} N_1}{2\alpha\gamma r}, \tag{18}$$

$$\widehat{N}_k = \frac{RP_k}{\gamma r \sigma_k^2} \qquad k = 2, 3, \tag{19}$$

where A_{1C} *and* A_{2C} *are given by formulas* $(B-20)$ *and* $(B-21)$, *respectively.*

It follows that both the optimal trading rate of illiquid stock and optimal investment policies of liquid stocks are independent of the endowment volatility. In addition, the trading rate is negatively related to the illiquid stock's position which, otherwise, has no impact on the liquid stocks' optimal allocations. Liu (2004), while studying the effect of proportional transaction costs, also states that the optimal stock trading strategy is separable in individual stocks since their optimal positions are independent of each other. Thus, given that the only difference across the two markets is the illiquidity of stock 1, if the investor's expectation on liquid stocks returns remains unchanged, then no adjustment is needed for these stocks' positions. This explains why their optimal positions are similar in both economies with and without transaction costs, implying that transaction costs subjected by the illiquid stock have no effect on other liquid stock's optimal trading policies. Their optimal allocations then rely only on the risk premium, the volatility of returns, the risk-free interest rate, and the investor's risk aversion.

As a market becomes illiquid, expectation of an investor on the illiquid stock's returns changes. More specifically, her expected utility would reduce due to the illiquidity. A higher risk premium would then be required to compensate for this reduction. Therefore, the risk premium in the economy with transaction costs should be increased by the liquidity premium. Thus, after arriving at the solutions for optimal portfolio choice of the investor in the economy with and without transaction costs, I set her utility functions in both economies to be the same to determine the liquidity premium. The conditional liquidity premium function is then equal to:

$$\Delta_{l1} = -RP_1 + \frac{\left(\alpha r + \sqrt{\alpha r\left(\alpha r + 2\gamma\sigma_1^2\right)}\right) \times \left[\sqrt{2\alpha\gamma^2 r N_1^2\left[\alpha r + \sqrt{\alpha r\left(\alpha r + 2\gamma\sigma_1^2\right)}\right] + \frac{2\alpha\gamma}{r}\left(\frac{RP_1}{\sigma_1}\right)^2} - 2\alpha\gamma r N_1\right]}{2\alpha\gamma}, \tag{20}$$

It follows that N_1 should be different than $\frac{RP_1}{\gamma r \sigma_1^2}$ for the liquidity premium to be positive and well-defined, where $\frac{RP_1}{\gamma r \sigma_1^2}$ is the optimal allocation to stock 1 in the economy with no transaction costs. If $N_1 = \frac{RP_1}{\gamma r \sigma_1^2}$, then the liquidity premium is equal to zero and the investor stops trading stock 1 since $u_1 = 0$.

Furthermore, equation (20) describes how the liquidity premium depends on the number of illiquid stock shares held by an investor (this dependence is illustrated in Panel A Figure 1). It is shown that the conditional risk premium decreases with N_1 when $N_1 < \frac{RP_1}{\gamma r \sigma_1^2}$. But if $N_1 > \frac{RP_1}{\gamma r \sigma_1^2}$, then a higher allocation to the illiquid stock (N_1) results in a higher premium that an investor requires to compensate for holding this stock.

For a better understanding of these findings, the relation between the trading rate and the number of illiquid stock shares should be taken into consideration. Following equation (18), u_1 is negatively related to N_1 and it is equal to zero when $N_1 = \frac{RP_1}{\gamma r \sigma_1^2}$. Moreover, if N_1 is smaller than $\frac{RP_1}{\gamma r \sigma_1^2}$, then u_1 is positive; and u_1 becomes negative when N_1 gets larger than $\frac{RP_1}{\gamma r \sigma_1^2}$. When N_1 is smaller than $\frac{RP_1}{\gamma r \sigma_1^2}$, since u_1 is positive, an investor keeps buying the illiquid stock until $u_1 = 0$. Because this positive trading rate decreases with N_1, the conditional liquidity premium also decreases as N_1 increases. This is because the lower the $|u_1|$, the less costly it is to trade for an investor. By contrast, if N_1 is larger than $\frac{RP_1}{\gamma r \sigma_1^2}$, then u_1 becomes negative, indicating that the investor is selling the illiquid stock. In this case, the higher the N_1, the higher the $|u_1|$ and the more costly it is for the investor to trade. The conditional liquidity premium then increases with N_1.

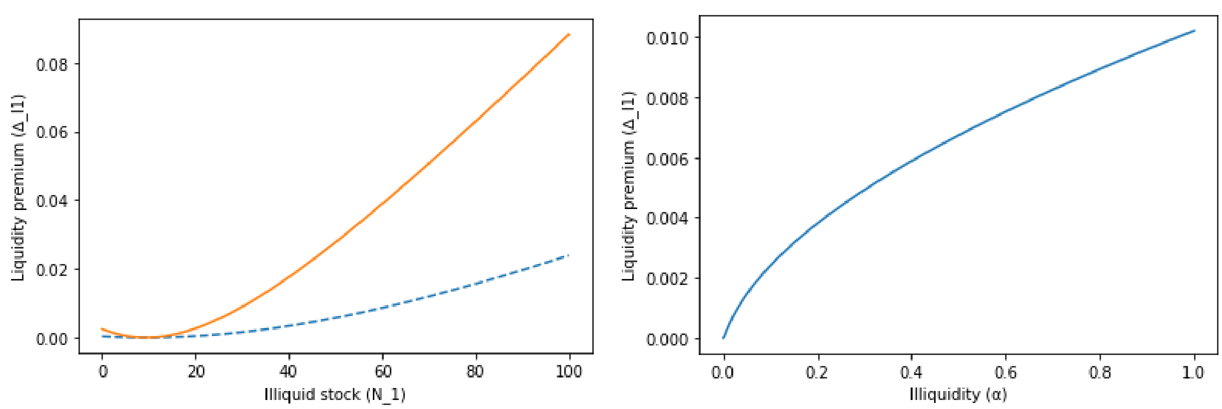
It is shown in equation (20) that the expected value of the conditional liquidity premium Δ_{l1} relies on the expectation on N_1 which can be derived from equation (12). To provide a better insight of the impacts of fundamental parameters on the number of illiquid stock shares held by the investor, the initial allocation to the illiquid stock 1 (N_{10}) is set to be zero. This assumption should not have a crucial effect on the final results. Indeed, since the process N_1 is stationary, the stationary limit of liquidity premium does not depend on the initial values. With $N_{10} = 0$, then $N_{1t} = \frac{(RP_1 + \Delta_{l1})(1 - e^{-bt})}{\gamma r \sigma_1^2}$. The expected value of liquidity premium is given by:

$$\Delta_{l1} = -RP_1 + RP_1 \sqrt{\frac{\gamma \sigma_1^2}{e^{-2bt}\left[\alpha r - \sqrt{\alpha r(\alpha r + 2\gamma \sigma_1^2)}\right] + \gamma \sigma_1^2}}, \tag{21}$$

$where: b = \frac{A_{2C}}{2\alpha \gamma r}$.

The last result implies that an investor requires a higher liquidity premium for a higher level of illiquidity (this finding is illustrated in Panel B Figure 1). In addition, a higher coefficient of risk aversion or a lager volatility of the illiquid stock's return results in a lower liquidity premium. It is intuitive since an increase in the coefficient of risk aversion or in the volatility of the illiquid stock's return decreases an investor's allocation to this stock (this relation can be seen from the process N_{1t}), which in turns reduces its effects on the investor's expected utility, leading to a smaller liquidity premium.

However, it is noticeable that in the steady state (as time t approaches infinity), the conditional liquidity premium is zero ($\lim_{t\to\infty} \Delta_{l1} = 0$). The liquidity premium disappears in the long run because the trading rate (u_1) is independent of the endowment volatility (Y), which is expected to generate long-term trading needs, and depends only on the position in the illiquid stock (N_1). In the short term, since N_1 is not at the optimal position yet, the investor is away from the state that maximizes her expected utility. She continues to trade with $|u_1| \neq 0$ to achieve this optimal level. The liquidity premium, therefore, could be relatively high (as illustrated in Figure 1). When N_1 reaches the long-term target allocation which is similar to that in the absence of transaction costs, the trading rate is equal to zero, and the investor stops trading this stock. Consequently, her utility function in the presence of transaction costs converges to the utility function when the costs are absent and the liquidity premium becomes zero. This finding suggests that the effect of transaction costs on the investor's optimal portfolio choices does not exist in the long run.



Panel A Panel B

Figure 1: Panel A and Panel B show the liquidity premium as a function of the number of illiquid stock shares N_1 and the illiquidity coefficient α, respectively, when $RP_1 = 0.06,\ r = 0.02,\ \gamma = 2,\ \sigma_1 = 0.4,\ t = 1$. In Panel A, the solid line presents the results when $\alpha = 0.05$ while the dashed line shows the findings when $\alpha = 0.001$.

Figure 1.A and Figure 1.B describe how the liquidity premium is affected by the allocation to stock 1 (as shown in equation (20)) and by the illiquidity coefficient α (as shown in equation (21)), respectively. Because the liquidity premium becomes zero in the long run, I set time $t = 1$ to get a better view of these relations in the short run. The initial stock price is assumed to be 1$ so that the dollar returns RP_1 could be related to the rate of returns. Therefore, RP_1 is set to be close to the historical average risk premium in the US stock market. In Panel A Figure 1, I consider the cases when $\alpha = 0.001$ and $\alpha = 0.05$ representing the market with a weak (dashed line) and very strong (solid line) illiquidity of stock 1, respectively. Following Isaenko (2006), I set the coefficient of risk aversion γ at 2 and the volatility of stock return at 0.4. The risk-free interest rate is set at 2% similar to Isaenko (2020). This calibration is used onwards unless specified otherwise. These graphs verify the results that have been analyzed in this section in a visual manner. In particular, Panel A shows that when N_1 is small, the liquidity premium slightly decreases with N_1. However, when N_1 becomes larger, the liquidity premium rises as N_1 increases. Moreover, when the illiquidity of stock 1 is weaker, the liquidity premium is accordingly smaller (dashed line). This is also shown in Panel B which illustrates the positive relation between the level of illiquidity and the liquidity premium.

4.1.2. Correlated Stocks

This subsection illustrates a case when all Brownian motions W_i $(i = 1, 2, 3)$ have the same constant positive correlation coefficients while the correlations between other Brownian motions are zero. Thus, $\rho_{ik} = \rho > 0$, while $\rho_{Yi} = 0$, $\rho_{0i} = 0 \; and \; \rho_{0Y} = 0$. The choice of these coefficients is for tractability of the results which allow me to better study the impact of correlations between returns of risky assets on their optimal allocations, and on the liquidity premium. For simplicity, these stocks are assumed to have the same characteristics, namely volatility (σ) and risk premium (RP).

The optimal allocations in the economy without transaction costs are:

$$\widehat{N}_k = \frac{RP}{\gamma r \sigma^2 (1 + 2\rho)} \qquad k = 1, 2, 3, \tag{22}$$

This formula shows that correlations among asset returns are a critical determinant in an investor's portfolio allocation decision. The stronger the correlations, the lower the optimal allocations to these stocks. The comovement of stock returns increases the overall risk of the portfolio, and thus discourages the investor from investing in these risky assets.

The optimal strategies in the economy with transaction costs are:

$$\hat{u}_1 = -\frac{A_{1C} + A_{2C}N_1}{2\alpha\gamma r}, \tag{23}$$

$$\widehat{N}_k = \frac{RP - \gamma r\sigma^2 \rho N_1}{\gamma r\sigma^2(1+\rho)} \qquad k = 2, 3, \tag{24}$$

where A_{1C} *and* A_{2C} *are determined by formulas* $(B\text{–}39)$ *and* $(B\text{–}40)$, *respectively.*

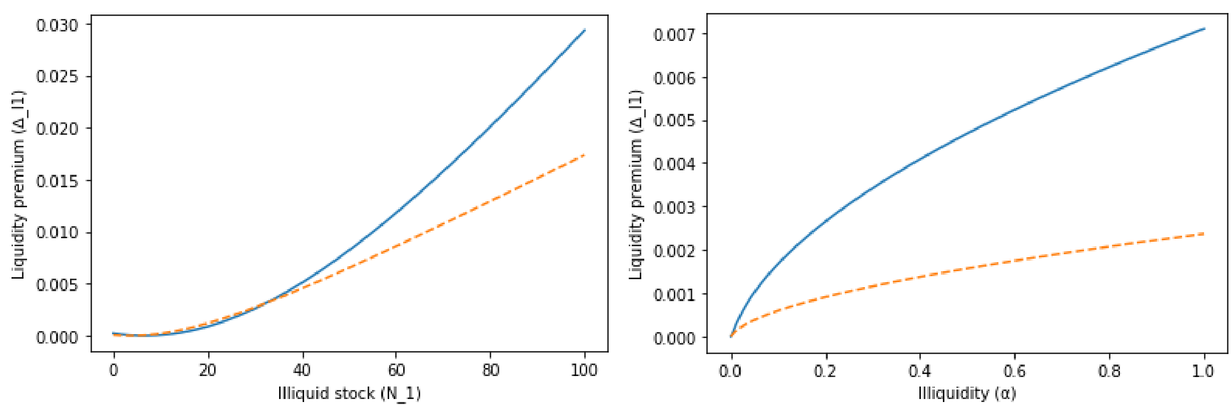
Both the optimal trading rate of stock 1 and the optimal investment policies of liquid stocks are independent of the endowment volatility while are influenced by the illiquid stock's position. Specifically, the optimal allocations to liquid stocks are negatively related to the number of illiquid stock shares, suggesting that the higher the illiquid stock's position, the less of wealth that the investor allocates to other liquid stocks. This is explained by their positive correlations. In addition, the magnitude of this relationship is affected by the strength of the correlations between stocks returns. When the correlations between stocks returns are stronger, the coefficient multiplying process N_1 becomes larger. Thus, in the case of correlated stocks, transaction costs have impacts on an investor's consumption and investment strategies.

The conditional liquidity premium is derived as a function of the number of illiquid stock shares N_1:

$$\Delta_{l1} = \frac{RP(\rho - 1)}{(1+\rho)} + \left\{\sqrt{N_1^2\left(r^2 + \frac{A_{2C}}{2\alpha\gamma}\right) + \frac{RP^2(1-\rho)}{2\alpha\gamma r\sigma^2(1+2\rho)(1+\rho)}} - rN_1\right\} \times \left(\frac{A_{2C}}{\gamma r} + 2\alpha r\right). \tag{25}$$

It follows that N_1 should be different than $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$ for the liquidity premium to be positive and well-defined, where $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$ is the optimal allocation to stock 1 in the economy without transaction costs. If $N_1 = \frac{RP}{\gamma r\sigma^2(1+2\rho)}$, then the investor stops trading since $u_1 = 0$, and the liquidity premium becomes zero.

Similar to the case of independent stocks, if $N_1 < \frac{RP}{\gamma r\sigma^2(1+2\rho)}$, then the conditional liquidity premium Δ_{l1} slightly decreases as N_1 increases. By contrast, if $N_1 > \frac{RP}{\gamma r\sigma^2(1+2\rho)}$, then the liquidity premium Δ_{l1} increases with N_1. These findings are demonstrated in Panel A Figure 2. The reason behind these results lies in the dependence of the trading rate u_1 on the process N_1. Following equation (23), u_1 is negatively related to N_1. In addition, u_1 is positive when N_1 is smaller than $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$; u_1 equals to zero when $N_1 = \frac{RP}{\gamma r\sigma^2(1+2\rho)}$; and u_1 becomes negative when N_1 gets larger than $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$. When N_1 is smaller than $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$, because the $|u_1|$ decreases as N_1 increases, the conditional liquidity premium also decreases as a result. This is because the lower the absolute value of trading rate $|u_1|$, the less costly it is to trade for an investor. But if N_1 is larger than $\frac{RP}{\gamma r\sigma^2(1+2\rho)}$, the $|u_1|$ increases with N_1, and thus it becomes more costly for the investor to trade this illiquid stock. The conditional liquidity premium then increases with N_1.



Panel A   Panel B

Figure 2: Panel A and Panel B describe the liquidity premium as a function of the number of illiquid stock shares N_1 and the illiquidity coefficient α, respectively, with the parameters: $RP = 0.06, r = 0.02, \gamma = 2, \sigma = 0.4, t = 1$, when $\rho = 0.2$ (solid line) and $\rho = 0.8$ (dashed line).

Panel A and Panel B in Figure 2 demonstrate how the liquidity premium is influenced by N_1 and by α, respectively, when the correlations between stocks returns are weak ($\rho = 0.2$) and strong ($\rho = 0.8$), assuming the previously-mentioned calibration. In Figure 2.A, I only consider one case of a weak level of illiquidity with $\alpha = 0.001$ because the focus here is to understand how the correlations between stocks returns affect the relation between the illiquid stock's position (N_1) and the liquidity premium (Δ_{l1}). Figure 2.B presents the results of the liquidity premium when the level of illiquidity (α) changes. As previously analyzed, Panel A shows that if N_1 is large

$\left(> \frac{RP}{\gamma r\sigma^2(1+2\rho)}\right)$, then the longer the position that an investor is taking in the illiquid stock 1, the higher the liquidity premium she demands for holding this stock. On the other hand, Panel B indicates that as stock 1 becomes more illiquid, it is more costly for an investor to trade in this stock, resulting in a higher liquidity premium. Moreover, it could be noticed that if the correlations between stocks returns are strong, then the liquidity premium becomes relatively small. A strong comovement in stocks returns discourages an investor from investing in risky assets (including the illiquid stock) due to the increasing risk of the portfolio. A smaller weight of the illiquid stock in the portfolio reduces its influence on the investor's expected utility function. Therefore, the liquidity premium is smaller than that in the case of weakly correlated stocks.

However, the stationary limits of the conditional liquidity premium and the trading rate of stock 1 are zero, suggesting that the investor will stop trading the illiquid stock in the long run. The long-term allocation to the illiquid stock is then similar to that in the market without transaction costs. When time t is small, N_1 is away from its optimal position, transaction costs still have effect on the investor's portfolio allocations. The investor then continuously rebalances her allocations to liquid stocks and the illiquid stock until achieving the long-term optimal strategies which are the same as those in the absence of transaction costs. As a consequence, her utility function in the presence of the illiquid stock converges to the utility function in the absence of the transaction costs and the liquidity premium becomes zero. This again implies that transaction costs have no effect on the investor's steady-state optimal investment policies.

So far, two cases have been considered: one with all independent stocks and one with correlated stocks. In both cases, I assume that all other correlation coefficients $(\rho_{Yi}, \rho_{0i}, and\ \rho_{0Y})$ are zero. The common point in both instances is that $B_{1C} = 0$. Consequently, the optimal trading rate and portfolio choices are independent of an endowment shock (Y), and thus the stationary limit of liquidity premium equals to zero. To put it differently, the approach of a short-term unpredictable endowment in the partial equilibrium framework does not work well in these cases since investors seem to have no incentive to trade in the illiquid stock in the long run. Therefore, it is important to examine the conditions for $B_{1C} \neq 0$ and to study how the liquidity premium and the optimal portfolio allocation change in this situation. I simplify this task by further assuming that $\rho_{Yi} = \rho_Y, \rho_{0i} = \rho_0$. Following equation (B–49), if $\rho_Y = \rho_0 = 0$, then $B_{1C} = 0$. Therefore, either the condition of $\rho_Y \neq 0$ or $\rho_0 \neq 0$ or both should hold for B_{1C} to be different from zero. The next section will analyze more closely the case when $\rho_Y \neq 0$ and $\rho_0 \neq 0$.

4.1.3. Numerical Example

In both previous cases, it is assumed that there is no correlation between stocks and the endowment. In this section, I relax this assumption making the shocks to stock returns and the endowment are negatively correlated. This choice comes from the intuition that if shocks to the endowment are negatively correlated with innovations to stock returns, the stocks are potentially attractive because they give a good hedge against unfavorable shocks in the endowment. The expected utility loss (gain) resulted from a drop (rise) in the endowment flow then can be balanced by the financial gain (loss) from stock returns. Unfortunately, I cannot find closed-form solution for the value function, thus the case has to be solved numerically. The optimal policies is determined in an analytical manner before a numerical calibration is applied.

The optimal allocations in the economy without transaction costs are:

$$\widehat{N}_k = \frac{\sigma_Y \rho_Y C_0 - \gamma r \rho_0}{\gamma r \sigma (1+2\rho)} Y + \frac{RP + \sigma \sigma_Y \rho_Y B_0}{\gamma r \sigma^2 (1+2\rho)} \qquad k = 1, 2, 3, \tag{26}$$

where B_0 and C_0 can be found by solving equations (B–51) and (B–52).

It follows that in the absence of transaction costs, the allocations to stocks are dependent on the endowment volatility (Y).

The optimal strategies in the economy with transaction costs are:

$$\hat{u}_1 = -\frac{A_{1C} + A_{2C} N_1 + B_{1C} Y}{2\alpha\gamma r}, \tag{27}$$

$$\widehat{N}_k = \frac{\sigma_Y \rho_Y B_{1C} - \gamma r \sigma \rho}{\gamma r \sigma (1+\rho)} N_1 + \frac{\sigma_Y \rho_Y C_{0C} - \gamma r \rho_0}{\gamma r \sigma (1+\rho)} Y + \frac{\sigma_Y \sigma \rho_Y B_{0C} + RP}{\gamma r \sigma^2 (1+\rho)} \qquad k = 2, 3, \tag{28}$$

where A_{2C}, B_{1C} and C_{0C} can be found by solving equations (B–46), (B–48) and (B–49).

In the presence of transaction costs, both liquid stocks' optimal allocations and the optimal trading rate of stock 1 depend on the number of illiquid stock shares (N_1) and on the endowment volatility (Y).

A numerical calibration is then applied to better estimate the dependence of optimal policies on the illiquid stock's position (N_1) and on the endowment volatility (Y), using the following parameters: $RP = 0.06,\ r = 0.02,\ \gamma = 2,\ \tau = 0.05,\ \sigma = 0.4,\ \bar{Y} = 1,\ \kappa_Y = 1,\ \sigma_Y = 0.4,\ \rho_Y = -0.4,\ \rho_0 = -0.4,\ \rho_{0Y} = 0.4,\ \rho\ = 0.2,\ \alpha = 0.001$. The time discount rate τ is set at 0.05 similar to Isaenko (2020). $\bar{Y}, \kappa_Y$, and σ_Y are chosen to create dynamic trading incentives for an investor. ρ_Y, ρ_0 and ρ_{0Y} are selected with an assumption that there are moderate correlations between the endowment and stocks returns. The correlations between stock returns are set at 0.2 (weak correlations) so that the illiquid stock has more effect on an investor's expected utility, and thus the liquidity premium is higher. The illiquidity coefficient is set at 0.001 indicating a weak level of illiquidity of stock 1.

The optimal allocations in the economy without transaction costs are:

$$\widehat{N}_k = 0.7106 \times Y + 6.7286 \qquad k = 1, 2, 3. \tag{29}$$

The optimal strategies in the economy with transaction costs are:

$$\hat{u}_1 = 12.0272 - 1.7175 \times N_1 + 0.7744 \times Y, \tag{30}$$

$$\widehat{N}_k = -0.1662 \times N_1 + 0.8287 \times Y + 7.8465 \qquad k = 2, 3. \tag{31}$$

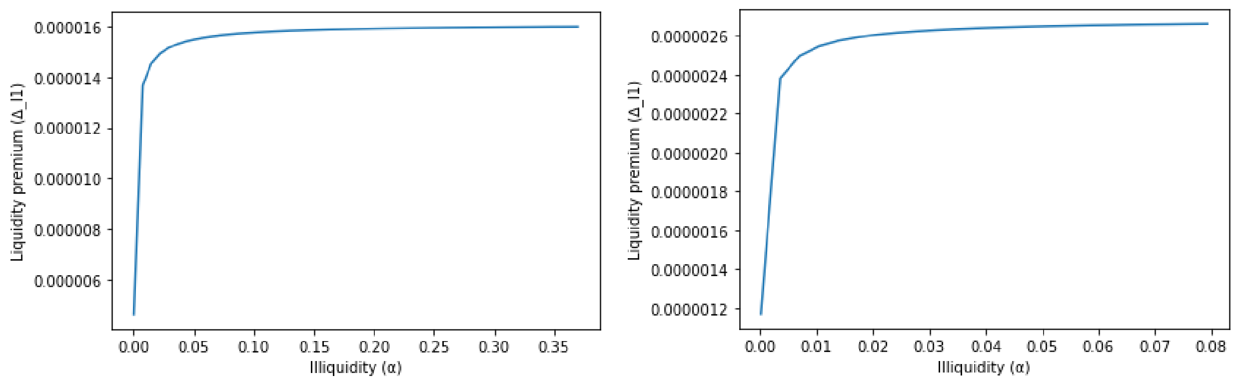
It follows that an increase in the endowment volatility lifts the target allocation of risky assets and raises the trading rate. If an investor exposes more to the nontraded risk in the endowment, then she has incentives to invest more in risky assets due to their negative correlations.

The liquidity premium is then found by setting the utility functions of an investor in both the economies with and without transaction costs to be the same. The quadratic equation of the conditional liquidity premium in terms of two state variables N_1 and Y is:

$$\begin{aligned} -165.8748\,\Delta_{l1}^2 - \Delta_{l1}(0.0230N_1 + 0.0175\mathrm{Y} + 14.7589) + 6.8699 \times 10^{-5}N_1^2 \\ - 6.1951 \times 10^{-5}N_1 Y - 0.0009N_1 + 1.795 \times 10^{-5}Y^2 + 0.0004\mathrm{Y} + 0.003 = 0 \end{aligned} \tag{32}$$

This equation allows me to estimate the stationary limit ($t \to \infty$) of the expected liquidity premium by simulating the processes of N_1 and Y. In particular, the time period $[0, t]$ is first partitioned into equal intervals. The time t is set at 100 which is large enough to find a stationary value of liquidity premium. The length of each discretization interval is chosen to be 0.1. In other words, this problem is numerically solved based on 1000 periods. Next, I simulate a path for Brownian motion W, from which I figure out the value of N_1 as following equation (12) and Y from equation (A–18). This process is repeated 100,000 times for a set of Δ_{l1}, and then by calculating the average of this set, I get a long-term expected value of liquidity premium, which is 9.655×10^{-6}.[4]

Because the relation between α and Δ_{l1} is also of interest, the previous calibration is run through again for higher values of α. The results of liquidity premium Δ_{l1} are shown in Panel A Figure 3. In addition, Panel B Figure 3 displays the findings in the case of strongly correlated stocks with $\rho = 0.8$. It is shown that as the illiquidity of stock 1 becomes stronger, an investor requires a higher liquidity premium. However, the size of this premium also depends on how strong the correlations between stocks returns are. The liquidity premium in the case of strongly correlated stocks is smaller than it is when stocks are weakly correlated. These results are consistent with the findings from the last two subsections.



Panel A

Panel B

Figure 3: Panel A and Panel B describe the relation between the illiquidity coefficient α and the conditional liquidity premium Δ_{l1} when $\rho = 0.2$ and $\rho = 0.8$, respectively, with the parameters as follows: $RP = 0.06,\ \sigma = 0.4,\ \rho_Y = -0.4,\ \rho_0 = -0.4,\ \rho_{0Y} = 0.4,\ r = 0.02,\ \gamma = 2,\ \bar{Y} = 1,$ $\sigma_Y = 0.4,\ \kappa_Y = 1,\ \tau = 0.05,\ t = 100.$

There are several points should be noticed from the results here. First, the conditional liquidity premiums (Δ_{l1}) are very small even when α is relatively big. The liquidity premium is typically very small in the partial equilibrium approach, especially in the case when the investment opportunity set is constant (risk-free interest rate, risk premium and volatility). See, for example,

[4] The coefficients in function $g(Y)$ and $g(N_1, Y)$ are found to be the following: $A_0 = 0.9721$, $B_0 = -4.5022 \times 10^{-3}$, $C_0 = 5.2005 \times 10^{-4}$, $A_{0C} = 0.9755$, $A_{1C} = -9.6218 \times 10^{-4}$, $A_{2C} = 1.3740 \times 10^{-4}$, $B_{0C} = -4.0769 \times 10^{-3}$, $B_{1C} = -6.1951 \times 10^{-5}$, and $C_{0C} = 5.5595 \times 10^{-4}$.

Constantinides (1986). According to Lo et al. (2004), partial equilibrium models tend to understate the impact of transaction costs on asset returns because they ignore the mechanism of market-clearing motive. Besides, Jang et al. (2007) indicate that the assumption of a constant investment opportunity set is likely to underestimate the impact of transaction costs on expected return because transaction costs prevent an investor from trading frequently. Thus, they suggest that the investment opportunity set is a critical consideration in generating a greater effect of transaction costs. In this regard, this result is not a surprise. Second, the liquidity premium exists in the long run because of non-zero correlations between stock returns and endowment shocks. However, its stationary expected value is negligible, suggesting that the long-term trading is very weak. Since a short-term unpredictable endowment is utilized to generate long-term trading incentives, a very small stationary liquidity premium implies that this approach seems to have very limited applications in the partial equilibrium framework.

In the last three cases, I have analyzed the relation between the illiquidity of a stock caused by temporary price impact and the corresponding liquidity premium, how the correlations between returns of stocks affect this relation, and how liquidity premium behaves in the long run if all the shocks in the economy are correlated. These analyses have been done for a market with three stocks. The next section provides more general solutions for arbitrary number of stocks in the portfolio.

4.2. Multiple-Stock Market

In this section, I examine a generalized case in an economy with arbitrary number of liquid stocks. Unfortunately, the closed-form solutions are not available when n becomes larger. Therefore, I set $\rho_Y = \rho_0 = \rho_{0Y} = 0$ for tractability of results. The detailed analysis is presented in Appendix C.

The market where all stocks are independent is considered. The results of this analysis are not shown in detail since they are similar to the findings in the case of three independent stocks. These findings suggest that no matter how many stocks are available in the economy, the optimization problem can be solved in a closed form with similar solutions. Thus, the number of liquid stocks in the market has no impact on the liquidity premium and on the optimal allocation to each liquid stock.

The case of correlated stocks are then studied, assuming that any pairs of stocks returns have the same positive correlation coefficient ρ, and all stocks have the same characteristics (namely, risk premium and return volatility).

The optimal allocations in the economy without transaction costs are:

$$\widehat{N}_k = \frac{RP}{\gamma r \sigma^2[1 + (n-1)\rho]} \qquad k = 1, \dots, n, \tag{33}$$

If there are more positively correlated stocks in the portfolio, then the optimal allocation to each stock becomes smaller because the comovement of these stocks increases the overall risk of the portfolio. In general, the optimal trading policy for each stock relies on the risk premium, the coefficient of risk aversion, the risk-free interest rate, the volatility of stock return, the correlations between stock returns, and the number of stocks available in the market.

The optimal strategies in the economy with transaction costs are:

$$\hat{u}_1 = -\frac{A_{1C} + A_{2C}N_1}{2\alpha\gamma r}, \tag{34}$$

$$\widehat{N}_k = \frac{RP}{\gamma r \sigma^2[1 + (n-2)\rho]} - \frac{\rho}{[1 + (n-2)\rho]} N_1 \qquad k = 2, \dots, n, \tag{35}$$

where A_{1C} and A_{2C} are determined by formulas $(C\!-16)$ and $(C\!-17)$, respectively.

It follows that the optimal allocations to liquid stocks are negatively related to the number of illiquid stock shares. The strength of this dependence relies on the number of stocks in the portfolio and on the correlations between these stocks returns. If the number of liquid stocks available in the market increases or the correlations between stocks returns become weaker, then the optimal allocation to each liquid stock is less affected by the illiquid stock's position. As a result, the transaction costs have less effect on the investor's optimal investment policies.

The conditional liquidity premium is given by:

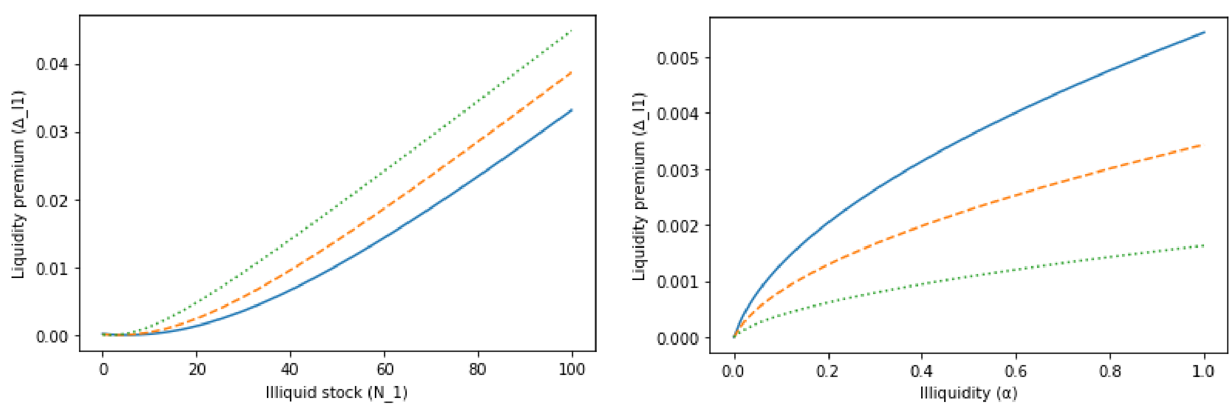
$$\Delta_{l1} = \frac{(\rho-1)RP}{[1+(n-2)\rho]} + \left\{\sqrt{\left(r^2 + \frac{A_{2C}}{2\alpha\gamma}\right)N_1^2 + \frac{RP^2(1-\rho)}{2\alpha\gamma r\sigma^2[1+(n-1)\rho][1+(n-2)\rho]}} - rN_1\right\} \times \left(\frac{A_{2C}}{\gamma r} + 2\alpha r\right). \tag{36}$$

The last formula shows that liquidity premium is not always a monotonically increasing function of N_1. Similar to the findings in Subsection 4.1.1 and Subsection 4.1.2, the condition of $N_1 > \frac{RP}{\gamma r\sigma^2[1+(n-1)\rho]}$ should hold for the liquidity premium to be positive and increasing with N_1.

Moreover, based on equation (12), one can arrive at the expected value of the number of illiquid stock shares N_1 at time t:

$$N_{1t} = \left\{ \frac{RP}{\gamma r \sigma^2 [1 + (n-1)\rho]} + \frac{[1 + (n-2)\rho]\Delta_{l1}}{\gamma r \sigma^2 (1-\rho)[1 + (n-1)\rho]} \right\} \times (1 - e^{-bt}) \tag{37}$$

Substituting this formula into equation (36), one can arrive at $\lim_{t \to \infty} \Delta_{l1} = 0$, indicating that the stationary expected value of the liquidity premium is zero. This result is consistent with the finding in the three-stock market in which an investor stops trading the illiquid stock in the long run ($u_1 = 0$), the long-term target policies are the same as those in the absence of transaction costs. The investor's utility function in the presence of transaction costs will eventually be the same as that when these costs are absent, and the conditional liquidity premium becomes zero. Thus, transaction costs have no effect on the investor's long-term optimal portfolio allocation choices. This suggests an implication that the steady-state optimal allocation can be computed separately for each stock even if their returns are correlated since transaction costs do not influence the investor's optimal choices in the long run. It then becomes feasible to compute the long-term target investment strategies for a large number of risky assets.



Panel A Panel B

Figure 4: Panel A and Panel B describe the liquidity premium as a function of the number of illiquid stock shares N_1 and the illiquidity coefficient α, respectively, with the parameters: $RP = 0.06,\ r = 0.02,\ \gamma = 2,\ \sigma = 0.4,\ \rho = 0.2,\ t = 1$, when $n = 5$ (solid line), $n = 10$ (dashed line), and $n = 25$ (dotted line).

Figure 4.A and 4.B illustrate the relation between Δ_{l1} and N_1, and between Δ_{l1} and α, respectively, with different numbers of stocks available in the market ($n = 5, 10, 25$). It is shown in Figure 4.A that with the same N_1, the liquidity premium becomes bigger if n increases. Because an investor faces a budget constraint, a constant proportion of illiquid stock in the portfolio affects her ability to trade other liquid stocks, given that there is an increasing number of liquid stocks available in the market. This reduces the investor's expected utility, and therefore a higher liquidity premium is required to compensate for keeping the illiquid stock's position unchanged. Similarly, Figure 4.B demonstrates a decrease in the influence of the level of illiquidity (α) on the liquidity premium as there are more stocks available in the market. An increasing number of available liquid stocks reduces the weight of the illiquid stock in the portfolio and its influence on the investor's utility function. A smaller effect of the illiquid stock on the expected utility function leads to a smaller liquidity premium.

5. Conclusion

The thesis studies the optimization problem of an investor trading in the economy with multiple liquid stocks and one illiquid stock. In the short term, transaction costs have no effect on liquid stocks' optimal trading strategies in the case of independent stocks. Their impact on optimal investment polices does exist when stocks returns are correlated. The strength of this impact depends on how strong the correlations between stocks returns are, and on the number of liquid stocks in the market. If asset returns are strongly correlated, then the illiquid stock's position significantly influences the investor's decision on optimal allocations to liquid stocks. This influence, however, becomes less substantial when the illiquid stock accounts for only a small proportion in the portfolio. In addition, the liquidity premium – measuring the additional return to compensate an investor for holding the illiquid stock – decreases as the level of illiquidity reduces, the correlations between stocks returns are stronger, or there is an increasing number of liquid stocks in the portfolio.

Nevertheless, transaction costs do not have significant effect on the optimal portfolio allocation in the long run. The stationary liquidity premium is very small and even disappears because the investor stops trading the illiquid stock in the case when there is no correlation between the endowment and stocks. Only in the presence of these correlations do the long-term trading demands exist, yet they seem to be very weak, leading to a negligible liquidity premium. The long-term target policies are very close, and similar in some cases, to those in the absence of transaction costs. The non-tradable risky endowment is used to generate long-term trading needs with the intuition that an investor facing with uncertainty about future risky income has incentives to dynamically rebalance her consumption and investment strategies. However, it has very limited applications in this partial equilibrium analysis. For further research, one can consider modelling time-varying investment opportunities, for example, by assuming that the risk premium of the illiquid stock follows a mean-reverting process (see, e.g., Campbell and Viceira (1999); Campbell et al. (2004); Liu (2010)).

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Appendix A

This appendix shows the HJB equations for the indirect utility function of an investor in the economy with and without transaction costs. In the economy without trading costs, the investor can trade one liquid bond and n liquid stocks; while in the market with transaction costs, I assume that stock 1 is illiquid and $(n-1)$ remaining stocks are liquid.

Proof of Proposition 1. If stock returns are correlated, the value function $V(t, X, Y, N_1)$ of an investor solves the following PDE:

$$\max_{c,u_1,N_2,\dots,N_n \in R^n} \left\{ \frac{1}{2}\left[(N_1\sigma_1)^2 + \sum_{k=2}^{n}(N_k\sigma_k)^2 + 2N_1\sigma_1 \sum_{k=2}^{n} N_k\sigma_k\rho_{1k} + \sum_{k,j=2,k\neq j}^{n} N_k N_j \sigma_k \sigma_j \rho_{kj} + Y^2 \right.\right.$$
$$\left. + 2Y\sum_{k=1}^{n} N_k\sigma_k\rho_{0k} \right] V_{XX} + \frac{1}{2}\sigma_Y^2 V_{YY} + \sigma_Y\left[Y\rho_{0Y} + \sum_{k=1}^{n} N_k\sigma_k\rho_{Yk} \right] V_{XY} + u_1 V_{N_1}$$
$$+ \kappa_Y(\bar{Y} - Y)V_Y + \left[rX + N_1(RP_1 + \Delta_{l1}) + \sum_{k=2}^{n} N_k RP_k - c - \alpha u_1^2 \right] V_X + V_t \tag{A–1}$$
$$\left. - \frac{1}{\gamma}\exp\left(-\tau t - \gamma c\right) \right\} = 0.$$

- F.O.C with respect to c:

$$\hat{c} = -\frac{1}{\gamma}[\tau t + \ln(V_X)], \tag{A–2}$$

- F.O.C with respect to u_1:

$$\hat{u}_1 = \frac{V_{N_1}}{2\alpha V_X}, \tag{A–3}$$

- F.O.C with respect to N_k:

$$\widehat{N} = COV^{-1} \times B, \tag{A–4}$$

where:

$(COV)_{i,j} = \sigma_{i+1}\sigma_{j+1}\rho_{i+1,j+1}\,,\qquad i,j = 1,\dots,n-1,$

$$B_i = -\frac{[N_1\sigma_1\sigma_{i+1}\rho_{1,i+1} + Y\sigma_{i+1}\rho_{0,i+1}]V_{XX} + \sigma_Y\sigma_{i+1}\rho_{Y,i+1}V_{XY} + RP_{i+1}V_X}{V_{XX}}, \qquad i = 1, \dots, n-1.$$

I conjecture that $V(t, X, Y, N_1) = -\frac{1}{\gamma}e^{-\tau t - \gamma r X + g(Y, N_1)}$ and find the PDE for $g(Y, N_1)$:

$$\begin{aligned}
&-r[\ln(r) + g] - \tau + r \\
&\quad + \frac{1}{2}(\gamma r)^2\left[(N_1\sigma_1)^2 + \sum_{k=2}^{n}\left(\widehat{N}_k\sigma_k\right)^2 + 2N_1\sigma_1\sum_{k=2}^{n}\widehat{N}_k\sigma_k\rho_{1k} + \sum_{k,j=2,k\neq j}^{n}\widehat{N}_k\widehat{N}_j\sigma_k\sigma_j\rho_{kj}\right. \\
&\quad \left. + \; Y^2 + 2Y\sum_{k=1}^{n}\widehat{N}_k\sigma_k\rho_{0k}\right] + \frac{1}{2}\sigma_Y^2(g_{YY} + g_Y^2) - \gamma\sigma_Y r\left[Y\rho_{0Y} + \sum_{k=1}^{n}\widehat{N}_k\sigma_k\rho_{Yk}\right]g_Y \qquad \text{(A–5)} \\
&\quad + \hat{u}_1 g_{N_1} + \kappa_Y(\bar{Y} - Y)g_Y - r\gamma\left[N_1(RP_1 + \Delta_{l1}) + \sum_{k=2}^{n}\widehat{N}_k RP_k - \alpha\hat{u}_1^2\right] = 0,
\end{aligned}$$

in which

$$\hat{u}_1 = -\frac{g_{N_1}}{2\alpha\gamma r}, \qquad \text{(A–6)}$$

$$\widehat{N} = COV^{-1} \times B, \qquad \text{(A–7)}$$

where:

$$B_i = \frac{-\gamma r[N_1\sigma_1\sigma_{i+1}\rho_{1,i+1} + Y\sigma_{i+1}\rho_{0,i+1}] + \sigma_Y\sigma_{i+1}\rho_{Y,i+1}g_Y + RP_{i+1}}{\gamma r} \qquad i = 1, \dots, n-1.$$

PDE (A–5) then is expressed into a system of linear-quadratic equations by assuming that function $g(Y, N_1)$ has a quadratic form (8). Subsequently, equations (9), (10), and (11) follow from equations (A–2), (A–6) and (A–7), respectively.

It follows from equation (11) that the optimal allocation to each liquid stock $\widehat{N}_k$ can be represented as a linear function of two state variables N_1 and Y:

$$\widehat{N}_k = G_{2k} + G_{3k}N_1 + G_{4k}Y, \qquad \text{(A–8)}$$

where coefficients G_{2k}, G_{3k}, G_{4k} can be found from equation (11).

By replacing function $g(Y, N_1)$ with quadratic form (8), equation (A–5) becomes:

$$-r\left[\ln(r) + A_{0C} + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 + (B_{0C} + B_{1C}N_1)Y + \frac{1}{2}C_{0C}Y^2\right] - \tau + r$$

$$+ \frac{1}{2}(\gamma r)^2\left[(N_1\sigma_1)^2 + \sum_{k=2}^{n}\left(\widehat{N}_k\sigma_k\right)^2 + 2N_1\sigma_1\sum_{k=2}^{n}\widehat{N}_k\sigma_k\rho_{1k} + \sum_{k,j=2,k\neq j}^{n}\widehat{N}_k\widehat{N}_j\sigma_k\sigma_j\rho_{kj}\right.$$

$$\left. + Y^2 + 2YN_1\sigma_1\rho_{01} + 2Y\sum_{k=2}^{n}\widehat{N}_k\sigma_k\rho_{0k}\right] + \frac{1}{2}\sigma_Y^2[C_{0C} + (B_{0C} + B_{1C}N_1 + C_{0C}Y)^2]$$

$$-\gamma\sigma_Y r\left[Y\rho_{0Y} + N_1\sigma_1\rho_{Y1} + \sum_{k=2}^{n}\widehat{N}_k\sigma_k\rho_{Yk}\right](B_{0C} + B_{1C}N_1 + C_{0C}Y) \quad \text{(A–9)}$$

$$+ \kappa_Y(\bar{Y} - Y)(B_{0C} + B_{1C}N_1 + C_{0C}Y) - r\gamma\left[N_1(RP_1 + \Delta_{l1}) + \sum_{k=2}^{n}\widehat{N}_k RP_k\right]$$

$$-\frac{1}{4\alpha\gamma r}(A_{1C} + A_{2C}N_1 + B_{1C}Y)^2 = 0,$$

Equation (A–9) implies the following set of six equations which are solved for the roots of six unknown variables A_{0C}, A_{1C}, A_{2C}, B_{0C}, B_{1C} and C_{0C} in the function $g(Y, N_1)$:

$$-rA_{1C} + (\gamma r)^2\left\{\sum_{k=2}^{n}\sigma_k^2 G_{2k}G_{3k} + \sigma_1\sum_{k=2}^{n}(\sigma_k\rho_{1k}G_{2k}) + \sum_{k,j=2,k\neq j}^{n}\sigma_k\sigma_j\rho_{kj}\left(G_{2k}G_{3j} + G_{2j}G_{3k}\right)\right\}$$

$$-\gamma\sigma_Y rB_{0C}\sum_{k=2}^{n}\sigma_k\rho_{Yk}G_{3k} - \gamma\sigma_Y rB_{1C}\sum_{k=2}^{n}\sigma_k\rho_{Yk}G_{2k} + \sigma_Y^2 B_{0C}B_{1C} \quad \text{(A–10)}$$

$$-\gamma\sigma_Y r\sigma_1\rho_{Y1}B_{0C} + \kappa_Y\bar{Y}B_{1C} - r\gamma(RP_1 + \Delta_{l1}) - r\gamma\sum_{k=2}^{n}RP_k G_{3k} - \frac{A_{1C}A_{2C}}{2\alpha\gamma r} = 0,$$

$$-\frac{r}{2}A_{2C} + \frac{1}{2}(\gamma r\sigma_1)^2 + \frac{1}{2}(\gamma r)^2\sum_{k=2}^{n}\sigma_k^2 G_{3k}^2 + (\gamma r)^2\sigma_1\sum_{k=2}^{n}\sigma_k\rho_{1k}G_{3k}$$

$$+ (\gamma r)^2\sum_{k,j=2,k\neq j}^{n}\sigma_k\sigma_j\rho_{kj}G_{3k}G_{3j} - \gamma\sigma_Y rB_{1C}\sum_{k=2}^{n}\sigma_k\rho_{Yk}G_{3k} + \frac{1}{2}\sigma_Y^2 B_{1C}^2 \quad \text{(A–11)}$$

$$-\gamma\sigma_Y r\sigma_1\rho_{Y1}B_{1C} - \frac{A_{2C}^2}{4\alpha\gamma r} = 0,$$

$$-rB_{0C} + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 \, G_{2k} G_{4k} + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k \sigma_j \rho_{kj} \big(G_{2k} G_{4j} + G_{2j} G_{4k} \big)$$

$$+ (\gamma r)^2 \sum_{k=2}^{n} \sigma_k \rho_{0k} G_{2k} + \sigma_Y^2 B_{0C} C_{0C} - \gamma \sigma_Y r \rho_{0Y} B_{0C} - \gamma \sigma_Y r C_{0C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{2k} \qquad \text{(A–12)}$$

$$- \gamma \sigma_Y r B_{0C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{4k} - \kappa_Y B_{0C} + \kappa_Y \bar{Y} C_{0C} - r\gamma \sum_{k=2}^{n} RP_k G_{4k} - \frac{A_{1C} B_{1C}}{2\alpha\gamma r} = 0,$$

$$-\frac{r}{2} C_{0C} + \frac{1}{2} (\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 \, G_{4k}^2 + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k \sigma_j \rho_{kj} G_{4k} G_{4j} + \frac{1}{2} (\gamma r)^2 + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k \rho_{0k} G_{4k}$$

$$+ \frac{1}{2} \sigma_Y^2 C_{0C}^2 - \gamma \sigma_Y r \rho_{0Y} C_{0C} - \gamma \sigma_Y r C_{0C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{4k} - \kappa_Y C_{0C} - \frac{B_{1C}^2}{4\alpha\gamma r} = 0, \qquad \text{(A–13)}$$

$$-rB_{1C} + (\gamma r)^2 \sigma_1 \rho_{01} + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 \, G_{3k} G_{4k} + (\gamma r)^2 \sigma_1 \sum_{k=2}^{n} \sigma_k \rho_{1k} G_{4k} + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k \rho_{0k} G_{3k}$$

$$+ (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k \sigma_j \rho_{kj} \big(G_{3k} G_{4j} + G_{3j} G_{4k} \big) + \sigma_Y^2 B_{1C} C_{0C} - \gamma \sigma_Y r \rho_{0Y} B_{1C} \qquad \text{(A–14)}$$

$$- \gamma \sigma_Y r \sigma_1 \rho_{Y1} C_{0C} - \gamma \sigma_Y r C_{0C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{3k} - \gamma \sigma_Y r B_{1C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{4k} - \kappa_Y B_{1C}$$

$$- \frac{A_{2C} B_{1C}}{2\alpha\gamma r} = 0,$$

$$-r[\ln(r) + A_{0C}] - \tau + r + \frac{1}{2} (\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 G_{2k}^2 + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k \sigma_j \rho_{kj} G_{2k} G_{2j} + \frac{1}{2} \sigma_Y^2 C_{0C} \qquad \text{(A–15)}$$

$$+ \frac{1}{2} \sigma_Y^2 B_{0C}^2 - \gamma \sigma_Y r B_{0C} \sum_{k=2}^{n} \sigma_k \rho_{Yk} G_{2k} + \kappa_Y \bar{Y} B_{0C} - r\gamma \sum_{k=2}^{n} RP_k G_{2k} - \frac{A_{1C}^2}{4\alpha\gamma r} = 0.$$

Proof of Corollary 1.

Following Isaenko (2020), since $dN_{1t} = u_{1t}dt$ and $\hat{u}_1 = -\frac{A_{1C}+A_{2C}N_1+B_{1C}Y}{2\alpha\gamma r}$, one can write:

$$N_{1t} = N_{10}e^{-bt} - \frac{a}{b}(1-e^{-bt}) - c\int_0^t e^{-b(t-s)}Y_s ds, \tag{A–16}$$

where $\frac{A_{1C}}{2\alpha\gamma r} = a$, $\frac{A_{2C}}{2\alpha\gamma r} = b$, and $\frac{B_{1C}}{2\alpha\gamma r} = c$.

By using Ito's lemma, process Y becomes:

$$e^{bt}Y_t = Y_0 + \frac{\kappa_Y \bar{Y}}{b}(e^{bt}-1) + (b-\kappa_Y)\int_0^t e^{bs}Y_s ds + \sigma_Y\int_0^t e^{bs}dW_{Y_s}, \tag{A–17}$$

I isolate the term $\int_0^t e^{bs}Y_s ds$ from this equation, and then arrive at equation (12) by using also:

$$Y_t = Y_0 e^{-\kappa_Y t} + \bar{Y}(1-e^{-\kappa_Y t}) + \sigma_Y\int_0^t e^{-\kappa_Y(t-s)}dW_{Y_s}. \tag{A–18}$$

Proof of Proposition 2. If stock returns are correlated, the value function $V(t,X,Y)$ of an investor solves the following PDE:

$$\begin{aligned}\max_{c,N_1,N_2,\dots,N_n\in R^n}\Bigg\{&\frac{1}{2}\left[\sum_{k=1}^{n}(N_k\sigma_k)^2 + \sum_{k,j=1,k\neq j}^{n} N_kN_j\sigma_k\sigma_j\rho_{kj} + Y^2 + 2Y\sum_{k=1}^{n}N_k\sigma_k\rho_{0k}\right]V_{XX} + \frac{1}{2}\sigma_Y^2V_{YY}\\ &+\sigma_Y\left[Y\rho_{0Y} + \sum_{k=1}^{n}N_k\sigma_k\rho_{Yk}\right]V_{XY} + \kappa_Y(\bar{Y}-Y)V_Y + \left[rX + \sum_{k=1}^{n}N_kRP_k - c\right]V_X\\ &+V_t - \frac{1}{\gamma}\exp(-\tau t - \gamma c)\Bigg\} = 0,\end{aligned} \tag{A–19}$$

- F.O.C with respect to c:

$$\hat{c} = -\frac{1}{\gamma}[\tau t + \ln(V_X)], \tag{A–20}$$

- F.O.C with respect to N_k:

$$\hat{N} = COV^{-1} \times B, \tag{A–21}$$

where:

$$(COV)_{i,j} = \sigma_i \sigma_j \rho_{i,j}\,, \qquad i,j = 1, \dots, n,$$

$$B_i = -\frac{Y\sigma_i \rho_{0,i} V_{XX} + \sigma_Y \sigma_i \rho_{Y,i} V_{XY} + RP_i V_X}{V_{XX}}, \qquad i = 1, \dots, n.$$

I conjecture that $V(t,X,Y) = -\frac{1}{\gamma} e^{-\tau t - \gamma r X + g(Y)}$. Thus, the function $g(Y)$ solves the following PDE:

$$r[\ln(r) + g] + \tau - r - \frac{1}{2}(\gamma r)^2 \left[\sum_{k=1}^{n} \left(\widehat{N}_k \sigma_k\right)^2 + \sum_{k,j=1,k\neq j}^{n} \widehat{N}_k \widehat{N}_j \sigma_k \sigma_j \rho_{kj} + Y^2 + 2Y \sum_{k=1}^{n} \widehat{N}_k \sigma_k \rho_{0k} \right]$$
$$- \frac{\sigma_Y^2}{2}\left(g_{YY} + g_Y^2\right) + \gamma r \sigma_Y \left[Y\rho_{0Y} + \sum_{k=1}^{n} \widehat{N}_k \sigma_k \rho_{Yk} \right] g_Y - \kappa_Y(\bar{Y} - Y) g_Y \qquad \text{(A–22)}$$
$$+ \gamma r \sum_{k=1}^{n} \widehat{N}_k RP_k = 0$$

in which

$$\widehat{N} = COV^{-1} \times B \qquad \text{(A–23)}$$

where

$$B_i = \frac{-\gamma r Y \sigma_i \rho_{0,i} + \sigma_Y \sigma_i \rho_{Y,i} g_Y + RP_i}{\gamma r}, \qquad i = 1, \dots, n.$$

By replacing the function $g(Y)$ with quadratic form (14), equation (A–22) becomes:

$$-r\left[\ln(r) + \left(A_0 + B_0 Y + \frac{1}{2} C_0 Y^2\right)\right] - \tau + r + \frac{\sigma_Y^2}{2}\left[C_0 + (B_0 + C_0 Y)^2\right] - \gamma r \sum_{k=1}^{n} \widehat{N}_k RP_k$$
$$+ \frac{1}{2}(\gamma r)^2 \left[\sum_{k=1}^{n} \left(\widehat{N}_k \sigma_k\right)^2 + \sum_{k,j=1,k\neq j}^{n} \widehat{N}_k \widehat{N}_j \sigma_k \sigma_j \rho_{kj} + Y^2 + 2Y \sum_{k=1}^{n} \widehat{N}_k \sigma_k \rho_{0k} \right] \qquad \text{(A–24)}$$
$$- \gamma r \sigma_Y \left[Y\rho_{0Y} + \sum_{k=1}^{n} \widehat{N}_k \sigma_k \rho_{Yk} \right] (B_0 + C_0 Y) + \kappa_Y(\bar{Y} - Y)(B_0 + C_0 Y) = 0,$$

In addition, equations (15) and (16) follow from equations (A–20) and (A–23), respectively.

It follows from equation (16) that each optimal allocation $\widehat{N}_k$ can be written as a linear function of variable Y:

$$\widehat{N}_k = G_{0k} + G_{1k} Y, \qquad \text{(A–25)}$$

where coefficients G_{0k}, G_{1k} can be found from equation (16).

PDE (A–24) can be reduced to the following system of three equations which are solved to figure out the roots of three unknown variables A_0, B_0 and C_0 in the function $g(Y)$:

$$-rB_0 + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 G_{0k}G_{1k} + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k\sigma_j\rho_{kj}\left(G_{0k}G_{1j} + G_{0j}G_{1k}\right)$$
$$+ (\gamma r)^2 \sum_{k=2}^{n} \sigma_k\rho_{0k}G_{0k} + \sigma_Y^2 B_0C_0 - \gamma\sigma_Y r\rho_{0Y}B_0 - \gamma\sigma_Y rC_0 \sum_{k=2}^{n} \sigma_k\rho_{Yk}G_{0k} \tag{A–26}$$
$$- \gamma\sigma_Y rB_0 \sum_{k=2}^{n} \sigma_k\rho_{Yk}G_{1k} - \kappa_Y B_0 + \kappa_Y \bar{Y} C_0 - r\gamma \sum_{k=2}^{n} RP_k G_{1k} = 0,$$

$$-\frac{r}{2}C_0 + \frac{1}{2}(\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 G_{1k}^2 + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k\sigma_j\rho_{kj}G_{1k}G_{1j} + \frac{1}{2}(\gamma r)^2 + (\gamma r)^2 \sum_{k=2}^{n} \sigma_k\rho_{0k}G_{1k}$$
$$+\frac{1}{2}\sigma_Y^2 C_0^2 - \gamma\sigma_Y r\rho_{0Y}C_0 - \gamma\sigma_Y rC_0 \sum_{k=2}^{n} \sigma_k\rho_{Yk}G_{1k} - \kappa_Y C_0 = 0, \tag{A–27}$$

$$-r[\ln(r) + A_0] - \tau + r + \frac{1}{2}(\gamma r)^2 \sum_{k=2}^{n} \sigma_k^2 G_{0k}^2 + (\gamma r)^2 \sum_{k,j=2,k\neq j}^{n} \sigma_k\sigma_j\rho_{kj}G_{0k}G_{0j} + \frac{1}{2}\sigma_Y^2(C_0 + B_0^2)$$
$$- \gamma\sigma_Y rB_0 \sum_{k=2}^{n} \sigma_k\rho_{Yk}G_{0k} + \kappa_Y\bar{Y}B_0 - r\gamma \sum_{k=2}^{n} RP_k G_{0k} = 0. \tag{A–28}$$

Moreover, I am also interested in determining the conditional expectation value of liquidity premium. Thus, I set the utility functions of an investor in the markets with and without transaction costs to be equal, and then solve this equation to obtain the liquidity premium as a function of N_1 and Y. In particular, by assuming the wealth is the same in both markets, I arrive at:

$$U_{maxC} = U_{max} \iff -\frac{1}{\gamma}\exp[-\tau t - \gamma rX + g(Y, N_1)] = -\frac{1}{\gamma}\exp[-\tau t - \gamma rX + g(Y)],$$
$$\Rightarrow g(Y, N_1) = g(Y),$$

Thus,

$$(A_{0C} - A_0) + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 + (B_{0C} - B_0)Y + B_{1C}N_1Y + \frac{1}{2}(C_{0C} - C_0)Y^2 = 0, \tag{A–29}$$

where the processes N_1 and Y are given by equations (12) and (A–18), respectively.

Appendix B

In this appendix, I solve the optimization problem of an investor trading one liquid bond and three stocks. I first determine the optimal allocations to liquid stocks in the portfolio. In particular:

In the economy without transaction costs

The PDE (A–24) for the market with three liquid stocks becomes:

$$-r\left[\ln(r) + \left(A_0 + B_0 Y + \frac{1}{2}C_0 Y^2\right)\right] - \tau + r + \frac{\sigma_Y^2}{2}\left[C_0 + (B_0 + C_0 Y)^2\right] - \gamma r \sum_{k=1}^{3} N_k RP_k$$
$$+\frac{1}{2}(\gamma r)^2\left[\sum_{k=1}^{3}(N_k\sigma_k)^2 + \sum_{k,j=1,k\neq j}^{3} N_k N_j \sigma_k \sigma_j \rho_{kj} + Y^2 + 2Y\sum_{k=1}^{3} N_k \sigma_k \rho_{0k}\right] \qquad \text{(B–1)}$$
$$-\gamma r \sigma_Y\left[Y\rho_{0Y} + \sum_{k=1}^{3} N_k \sigma_k \rho_{Yk}\right](B_0 + C_0 Y) + \kappa_Y(\bar{Y} - Y)(B_0 + C_0 Y) = 0,$$

It follows that each optimal allocation $\widehat{N}_k$ $(k = 1, 2, 3)$ is given by:

$$\widehat{N}_1 = \left[\frac{(\rho_{12} - \rho_{13}\rho_{23})\rho_{02} + (\rho_{13} - \rho_{12}\rho_{23})\rho_{03} - (1 - \rho_{23}^2)\rho_{01}}{\sigma_1(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)}\right.$$
$$\left. - \frac{\sigma_Y[(\rho_{12} - \rho_{13}\rho_{23})\rho_{Y2} + (\rho_{13} - \rho_{12}\rho_{23})\rho_{Y3} - (1 - \rho_{23}^2)\rho_{Y1}]}{\gamma r \sigma_1(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} C_0\right] Y \qquad \text{(B–2)}$$
$$- \left[\frac{(\rho_{12} - \rho_{13}\rho_{23})RP_2\sigma_1\sigma_3 + (\rho_{13} - \rho_{12}\rho_{23})RP_3\sigma_1\sigma_2 - (1 - \rho_{23}^2)RP_1\sigma_2\sigma_3}{\gamma r \sigma_1^2 \sigma_2 \sigma_3(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)}\right]$$
$$- \frac{\sigma_Y[\rho_{Y2}(\rho_{12} - \rho_{13}\rho_{23}) + \rho_{Y3}(\rho_{13} - \rho_{12}\rho_{23}) - \rho_{Y1}(1 - \rho_{23}^2)]}{\gamma r \sigma_1(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} B_0,$$

$$\widehat{N}_2 = \left\{\frac{\rho_{01}(\rho_{12} - \rho_{13}\rho_{23}) + \rho_{03}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{02}(1 - \rho_{13}^2)}{\sigma_2(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)}\right.$$
$$\left. - \frac{\sigma_Y[\rho_{Y1}(\rho_{12} - \rho_{13}\rho_{23}) + \rho_{Y3}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{Y2}(1 - \rho_{13}^2)]}{\gamma r \sigma_2(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} C_0\right\} Y \qquad \text{(B–3)}$$
$$- \frac{B_0\sigma_Y[\rho_{Y1}(\rho_{12} - \rho_{13}\rho_{23}) + \rho_{Y3}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{Y2}(1 - \rho_{13}^2)]}{\gamma r \sigma_2(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)}$$
$$- \frac{[RP_1\sigma_2\sigma_3(\rho_{12} - \rho_{13}\rho_{23}) + RP_3\sigma_1\sigma_2(\rho_{23} - \rho_{12}\rho_{13}) - RP_2\sigma_1\sigma_3(1 - \rho_{13}^2)]}{\gamma r \sigma_1 \sigma_2^2 \sigma_3(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)},$$

(B–4)

$$\widehat{N}_3 = \left\{\frac{\rho_{01}(\rho_{13} - \rho_{12}\rho_{23}) + \rho_{02}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{03}(1 - \rho_{12}^2)}{\sigma_3(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} - \frac{\sigma_Y[\rho_{Y1}(\rho_{13} - \rho_{12}\rho_{23}) + \rho_{Y2}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{Y3}(1 - \rho_{12}^2)]}{\gamma r\sigma_3(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} C_0\right\} Y - \frac{B_0\sigma_Y[\rho_{Y1}(\rho_{13} - \rho_{12}\rho_{23}) + \rho_{Y2}(\rho_{23} - \rho_{12}\rho_{13}) - \rho_{Y3}(1 - \rho_{12}^2)]}{\gamma r\sigma_3(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)} - \frac{[RP_1\sigma_2\sigma_3(\rho_{13} - \rho_{12}\rho_{23}) + RP_2\sigma_1\sigma_3(\rho_{23} - \rho_{12}\rho_{13}) - RP_3\sigma_1\sigma_2(1 - \rho_{12}^2)]}{\gamma r\sigma_1\sigma_2\sigma_3^2(1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2)}. \tag{B–4}$$

<u>*In the economy with transaction costs*</u>

The PDE (A–9) for the market with one illiquid stock and two liquid stocks becomes:

$$-r\left[\ln(r) + A_{0C} + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 + (B_{0C} + B_{1C}N_1)Y + \frac{1}{2}C_{0C}Y^2\right] - \tau + r + \frac{1}{2}(\gamma r)^2\left[(N_1\sigma_1)^2 + \sum_{k=2}^{3}(N_k\sigma_k)^2 + 2N_1\sigma_1\sum_{k=2}^{3}N_k\sigma_k\rho_{1k} + 2N_2N_3\sigma_2\sigma_3\rho_{23} + Y^2 2YN_1\sigma_1\rho_{01} + 2Y\sum_{k=2}^{3}N_k\sigma_k\rho_{0k}\right] + \frac{1}{2}\sigma_Y^2[C_{0C} + (B_{0C} + B_{1C}N_1 + C_{0C}Y)^2] - \gamma r\sigma_Y\left[Y\rho_{0Y} + N_1\sigma_1\rho_{Y1} + \sum_{k=2}^{3}\widehat{N}_k\sigma_k\rho_{Yk}\right](B_{0C} + B_{1C}N_1 + C_{0C}Y) + \kappa_Y(\bar{Y} - Y)(B_{0C} + B_{1C}N_1 + C_{0C}Y) - r\gamma\left[N_1(RP_1 + \Delta_{l1}) + \sum_{k=2}^{3}\widehat{N}_k RP_k\right] - \frac{1}{4\alpha\gamma r}(A_{1C} + A_{2C}N_1 + B_{1C}Y)^2 = 0, \tag{B–5}$$

It follows that each optimal allocation $\widehat{N}_k$ $(k = 2, 3)$ is given by:

$$\widehat{N}_k = \frac{\gamma r\sigma_1(\rho_{1j}\rho_{23} - \rho_{1k}) - \sigma_Y(\rho_{Yj}\rho_{23} - \rho_{Yk})B_{1C}}{\gamma r\sigma_k(1 - \rho_{23}^2)}N_1 + \frac{\gamma r(\rho_{0j}\rho_{23} - \rho_{0k}) - \sigma_Y(\rho_{Yj}\rho_{23} - \rho_{Yk})C_{0C}}{\gamma r\sigma_k(1 - \rho_{23}^2)}Y - \frac{\sigma_Y\sigma_2\sigma_3(\rho_{Yj}\rho_{23} - \rho_{Yk})B_{0C} + (RP_j\sigma_k\rho_{23} - RP_k\sigma_j)}{\gamma r\sigma_k^2\sigma_j(1 - \rho_{23}^2)} \qquad k, j = 2,3; j \neq k. \tag{B–6}$$

Therefore, equations (17), (22), and (26) follow from equations (B–2), (B–3), and (B–4); while equations (19), (24), and (28) come from equation (B–6).

Next, I will determine the unknown variables $A_{0C}, A_{1C}, A_{2C}, B_{0C}, B_{1C}, C_{0C}$ for the function $g(Y, N_1)$ and A_0, B_0, C_0 for the function $g(Y)$ in three cases shown in Section 4.1 as follows:

Case 1 – Independent Stocks. In this case, I assume that all correlation coefficients are zero.

<u>*In the economy without transaction cost*</u>:

The system of three equations (A–26) to (A–28) becomes:

$$-(r+\kappa_Y)B_0 + \sigma_Y^2 B_0 C_0 + \kappa_Y \bar{Y} C_0 = 0, \tag{B–7}$$

$$\sigma_Y^2 C_0^2 - (r+2\kappa_Y)C_0 + (\gamma r)^2 = 0, \tag{B–8}$$

$$-r[\ln(r) + A_0] - \tau + r - \frac{1}{2}\sum_{k=1}^{3}\left(\frac{RP_k}{\sigma_k}\right)^2 + \frac{1}{2}\sigma_Y^2 B_0^2 + \kappa_Y \bar{Y} B_0 + \frac{1}{2}\sigma_Y^2 C_0 = 0. \tag{B–9}$$

Solving these three equations, I arrive at the formulas for A_0, B_0 and C_0:

$$A_0 = 1 - \ln(r) - \frac{\tau}{r} - \frac{1}{2r}\sum_{k=1}^{3}\left(\frac{RP_k}{\sigma_k}\right)^2 + \frac{1}{2r}\sigma_Y^2 B_0^2 + \frac{1}{2r}\sigma_Y^2 C_0 + \frac{\kappa_Y \bar{Y} B_0}{r}, \tag{B–10}$$

where

$$B_0 = \frac{\kappa_Y \bar{Y}(r+2\kappa_Y) \pm \kappa_Y \bar{Y}\sqrt{(r+2\kappa_Y)^2 - 4\sigma_Y^2(\gamma r)^2}}{\sigma_Y^2 r \mp \sigma_Y^2\sqrt{(r+2\kappa_Y)^2 - 4\sigma_Y^2(\gamma r)^2}}, \tag{B–11}$$

$$C_0 = \frac{(r+2\kappa_Y) \pm \sqrt{(r+2\kappa_Y)^2 - 4(\gamma r \sigma_Y)^2}}{2\sigma_Y^2}. \tag{B–12}$$

<u>*In the economy with transaction cost*</u>:

The system of six equations (A–10) to (A–15) becomes:

$$-rA_{1C} + \sigma_Y^2 B_{0C} B_{1C} + \kappa_Y \bar{Y} B_{1C} - r\gamma(RP_1 + \Delta_{l1}) - \frac{1}{2\alpha\gamma r}A_{1C}A_{2C} = 0, \tag{B–13}$$

$$-rA_{2C} + (\gamma r \sigma_1)^2 + \sigma_Y^2 B_{1C}^2 - \frac{A_{2C}^2}{2\alpha\gamma r} = 0, \tag{B–14}$$

$$-rB_{0C}+\sigma_Y^2 B_{0C}C_{0C}+\kappa_Y\bar{Y}C_{0C}-\kappa_Y B_{0C}-\frac{A_{1C}B_{1C}}{2\alpha\gamma r}=0, \tag{B–15}$$

$$-rC_{0C}+(\gamma r)^2+\sigma_Y^2 C_{0C}^2-2\kappa_Y C_{0C}-\frac{B_{1C}^2}{2\alpha\gamma r}=0, \tag{B–16}$$

$$-rB_{1C}+\sigma_Y^2 B_{1C}C_{0C}-\kappa_Y B_{1C}-\frac{A_{2C}B_{1C}}{2\alpha\gamma r}=0, \tag{B–17}$$

$$-r\ln(r)-rA_{0C}-\tau+r-\frac{1}{2}\sum_{k=2}^{3}\left(\frac{RP_k}{\sigma_k}\right)^2+\frac{1}{2}\sigma_Y^2 C_{0C}+\frac{1}{2}\sigma_Y^2 B_{0C}^2+\kappa_Y\bar{Y}B_{0C}-\frac{A_{1C}^2}{4\alpha\gamma r}=0. \tag{B–18}$$

Solving these six equations, I find that $B_{1C}=0$, and the formulas for $A_{0C}, A_{1C}, A_{2C}, B_{0C}, C_{0C}$ are given by:

$$A_{0C}=1-\ln(r)-\frac{\tau}{r}-\frac{1}{2r}\sum_{k=2}^{3}\left(\frac{RP_k}{\sigma_k}\right)^2+\frac{1}{2r}\sigma_Y^2 C_{0C}+\frac{1}{2r}\sigma_Y^2 B_{0C}^2+\frac{\kappa_Y\bar{Y}B_{0C}}{r}-\frac{A_{1C}^2}{4\alpha\gamma r^2}, \tag{B–19}$$

$$A_{1C}=-\frac{2\alpha\gamma r(RP_1+\Delta_{l1})}{\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}}, \tag{B–20}$$

$$A_{2C}=-\alpha\gamma r^2+\gamma r\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}, \tag{B–21}$$

$$B_{0C}=\frac{\kappa_Y\bar{Y}\left[(r+2\kappa_Y)\pm\sqrt{(r+2\kappa_Y)^2-4(\gamma r\sigma_Y)^2}\right]}{\sigma_Y^2\left[r\mp\sqrt{(r+2\kappa_Y)^2-4(\gamma r\sigma_Y)^2}\right]}, \tag{B–22}$$

$$C_{0C}=\frac{(r+2\kappa_Y)\pm\sqrt{(r+2\kappa_Y)^2-4(\gamma r\sigma_Y)^2}}{2\sigma_Y^2}. \tag{B–23}$$

The conditional liquidity premium

With $B_0=B_{0C}, C_0=C_{0C}, and\ B_{1C}=0$, equation (A–29) becomes:

$$A_{0C}+A_{1C}N_1+\frac{1}{2}A_{2C}N_1^2=A_0, \tag{B–24}$$

By replacing $A_{0C}, A_{1C}, A_{2C}, and\ A_0$ with the formulas (B–19), (B–20), (B–21) and (B–10), respectively, in equation (B–24), I arrive at equation (20).

It follows that the conditional liquidity premium Δ_{l1} is well-defined since α, γ, r are all positive constants, and therefore, the term under the radical is positive. Furthermore, for $\Delta_{l1} > 0$:

$$\frac{\left(\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)\times\left[\sqrt{2\alpha\gamma^2 rN_1^2\left[\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right]+\frac{2\alpha\gamma}{r}\left(\frac{RP_1}{\sigma_1}\right)^2}-2\alpha\gamma rN_1\right]}{2\alpha\gamma} > RP_1$$

Thus, the two conditions of $N_1 \neq -\frac{2\alpha\gamma rRP_1}{\left[\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right]\times\gamma r\left[\alpha r-\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right]}$ and $\gamma r\left[\alpha r - \sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right] < 0$ should be satisfied for the last inequation to occur. To put it differently, the conditions of $2\gamma\alpha r\sigma_1^2 > 0\ and\ N_1 \neq \frac{RP_1}{\gamma r\sigma_1^2}$ should hold. Because α, γ, r, and σ_1 are all positive constants, only the latter condition is needed for $\Delta_{l1} > 0$.

Moreover, to figure out the conditions for a positive relation between Δ_{l1} and N_1, I solve the problem of finding the minimum value of Δ_{l1} with respect to N_1. I find that Δ_{l1} gets the minimum value of 0 when $N_1 = \frac{RP_1}{\gamma r\sigma_1^2}$. This finding indicates that if N_1 is larger than $\frac{RP_1}{\gamma r\sigma_1^2}$, then Δ_{l1} increases with N_1.

The expected value of the liquidity premium can be estimated by replacing N_1 with the process N_{1t} in equation (20). In particular, following equation (12), I find that $N_{1t} = N_{10}e^{-bt} + \frac{A_1}{A_2}e^{-bt} - \frac{A_1}{A_2}$. Substitute this formula into equation (20), one arrives at:

$$(RP_1 + \Delta_{l1}) = \frac{-4\alpha\gamma r\sigma_1^2\left(\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)N_{10}e^{-2bt} + \sigma_1^2\sqrt{\varphi}}{4\alpha\gamma\sigma_1^2 - 4\alpha\left(\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)(e^{-2bt}-1)}, \quad \text{(B–25)}$$

where

$$\varphi = 16(\alpha\gamma r)^2\left(\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)^2 N_{10}^2 e^{-2bt} + 16\gamma(\alpha\gamma r)^2\sigma_1^2\left(\alpha r + \sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)N_{10}^2 e^{-2bt} + \frac{8\alpha RP_1^2\left(\alpha r+\sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)^2}{r\sigma_1^2}\times\left[\gamma - \left(\alpha r + \sqrt{\alpha r(\alpha r+2\gamma\sigma_1^2)}\right)\frac{(e^{-2bt}-1)}{\sigma_1^2}\right].$$

Equation (21) follows the last result with $N_{10} = 0$.

Case 2 – Correlated Stocks. This part presents the case when all correlations between Brownian motions W_i $(i = 1, 2, 3)$ are the same while the correlations between other Brownian motions are zero. Thus, $\rho_{ik} = \rho > 0$ while $\rho_{Yi} = 0, \rho_{0i} = 0\ and\ \rho_{0Y} = 0$.

In the economy without transaction cost:

The system of three equations (A–26) to (A–28) becomes:

$$-(r+\kappa_Y)B_0+\sigma_Y^2 B_0 C_0+\kappa_Y \bar{Y} C_0=0, \tag{B–26}$$

$$\sigma_Y^2 C_0^2-(r+2\kappa_Y)C_0+(\gamma r)^2=0, \tag{B–27}$$

$$-2r[\ln(r)+A_0]-2\tau+2r+\sigma_Y^2 C_0+\sigma_Y^2 B_0^2+2\kappa_Y \bar{Y} B_0-\frac{3RP^2}{\sigma^2(1+2\rho)}=0. \tag{B–28}$$

Solving these three equations, the formulas for A_0, B_0 and C_0 are given by:

$$A_0=1-\ln(r)-\frac{\tau}{r}+\frac{1}{2r}\sigma_Y^2 B_0^2+\frac{1}{2r}\sigma_Y^2 C_0+\frac{\kappa_Y \bar{Y} B_0}{r}-\frac{3}{2r}\left(\frac{RP}{\sigma(1+2\rho)}\right)^2, \tag{B–29}$$

$$B_0=\frac{\kappa_Y \bar{Y}(r+2\kappa_Y)\pm\kappa_Y \bar{Y}\sqrt{(r+2\kappa_Y)^2-4\sigma_Y^2(\gamma r)^2}}{\sigma_Y^2 r\mp\sigma_Y^2\sqrt{(r+2\kappa_Y)^2-4\sigma_Y^2(\gamma r)^2}}, \tag{B–30}$$

$$C_0=\frac{(r+2\kappa_Y)\pm\sqrt{(r+2\kappa_Y)^2-4(\gamma r\sigma_Y)^2}}{2\sigma_Y^2}. \tag{B–31}$$

In the economy with transaction cost:

The system of six equations (A–10) to (A–15) becomes:

$$-rA_{1C}+\sigma_Y^2 B_{0C}B_{1C}+\kappa_Y \bar{Y} B_{1C}-r\gamma(RP+\Delta_{l1})-\frac{A_{1C}A_{2C}}{2\alpha\gamma r}+\frac{2\gamma r\rho RP}{1+\rho}=0, \tag{B–32}$$

$$-rA_{2C}+(\gamma r\sigma)^2+\sigma_Y^2 B_{1C}^2-\frac{A_{2C}^2}{2\alpha\gamma r}-2(\gamma r\sigma)^2\frac{\rho^2}{1+\rho}=0, \tag{B–33}$$

$$-rB_{0C}+\sigma_Y^2 B_{0C}C_{0C}+\kappa_Y \bar{Y} C_{0C}-\kappa_Y B_{0C}-\frac{A_{1C}B_{1C}}{2\alpha\gamma r}=0, \tag{B–34}$$

$$-rC_{0C}+(\gamma r)^2+\sigma_Y^2 C_{0C}^2-2\kappa_Y C_{0C}-\frac{B_{1C}^2}{2\alpha\gamma r}=0, \tag{B–35}$$

$$-rB_{1C}+\sigma_Y^2 B_{1C}C_{0C}-\kappa_Y B_{1C}-\frac{A_{2C}B_{1C}}{2\alpha\gamma r}=0, \tag{B–36}$$

$$-r\ln(r)-rA_{0C}-\tau+r+\frac{1}{2}\sigma_Y^2 C_{0C}+\frac{1}{2}\sigma_Y^2 B_{0C}^2+\kappa_Y \bar{Y} B_{0C}-\frac{A_{1C}^2}{4\alpha\gamma r}-\frac{RP^2}{\sigma^2(1+\rho)}=0. \tag{B–37}$$

Solving these six equations, I find that $B_{1C} = 0$, and arrive at the formulas for other variables as follows:

$$A_{0C} = 1 - \ln(r) - \frac{\tau}{r} + \frac{1}{2r}\sigma_Y^2 C_{0C} + \frac{1}{2r}\sigma_Y^2 B_{0C}^2 + \frac{\kappa_Y \bar{Y} B_{0C}}{r} - \frac{A_{1C}^2}{4\alpha\gamma r^2} - \frac{RP^2}{r\sigma^2(1+\rho)}, \tag{B–38}$$

$$A_{1C} = \frac{2\alpha\gamma r RP(\rho - 1) - 2\alpha\gamma r \Delta_{l1}(1+\rho)}{(1+\rho)\left[\alpha r + \sqrt{(\alpha r)^2 + 2\alpha\gamma r\sigma^2\left[1 - 2\frac{\rho^2}{(1+\rho)}\right]}\right]}, \tag{B–39}$$

$$A_{2C} = -\alpha\gamma r^2 + \gamma r\sqrt{(\alpha r)^2 + 2\alpha\gamma r\sigma^2\left[1 - 2\frac{\rho^2}{(1+\rho)}\right]}, \tag{B–40}$$

$$B_{0C} = \frac{\kappa_Y \bar{Y}\left[(r + 2\kappa_Y) \pm \sqrt{(r + 2\kappa_Y)^2 - 4(\gamma r\sigma_Y)^2}\right]}{\sigma_Y^2\left[r \mp \sqrt{(r + 2\kappa_Y)^2 - 4(\gamma r\sigma_Y)^2}\right]}, \tag{B–41}$$

$$C_{0C} = \frac{(r + 2\kappa_Y) \pm \sqrt{(r + 2\kappa_Y)^2 - 4(\gamma r\sigma_Y)^2}}{2\sigma_Y^2}. \tag{B–42}$$

<u>*The conditional liquidity premium*</u>

With $B_0 = B_{0C}, C_0 = C_{0C}, and\ B_{1C} = 0$, equation (A–29) becomes:

$$A_{0C} + A_{1C}N_1 + \frac{1}{2}A_{2C}N_1^2 = A_0, \tag{B–43}$$

By replacing $A_{0C}, A_{1C}, A_{2C}, and\ A_0$ with the formulas (B–38), (B–39), (B–40) and (B–29), respectively, in the last equation, I arrive at equation (25).

It follows that Δ_{l1} is well-defined because α, γ, r, ρ are all positive constants and A_{2C} is also positive, and thus, the term under the radical is positive. Furthermore, similar to Case 1, the condition of $N_1 > \frac{RP}{\gamma r\sigma^2(1+2\rho)}$ should hold for $\Delta_{l1} > 0$ and for Δ_{l1} to increase with N_1.

The expected value of the conditional liquidity premium can be estimated by replacing N_1 with the process N_{1t} in equation (25). Following equation (12), I find that if $N_{10} = 0$, then $N_{1t} = \frac{RP(1-\rho)+\Delta_{l1}(1+\rho)}{\gamma r\sigma^2(1+2\rho)(1-\rho)}(1 - e^{-bt})$. Substituting this process into equation (25), I obtain the conditional liquidity premium as a function of the illiquidity coefficient α:

$$\Delta_{l1} = \frac{\left[\frac{xy}{2\alpha\gamma r^2} - xv - yu - A_{2C}uv\right] - \sqrt{\varphi_1}}{2yv + A_{2C}v^2 - \frac{y^2}{2\alpha\gamma r^2}}, \tag{B–44}$$

where

$x = \frac{2\alpha\gamma r RP(\rho-1)}{(1+\rho)\left[\alpha r + \sqrt{(\alpha r)^2 + 2\alpha\gamma r\sigma^2\left[1-2\frac{\rho^2}{(1+\rho)}\right]}\right]}, y = \frac{-2\alpha\gamma r(1+\rho)}{(1+\rho)\left[\alpha r + \sqrt{(\alpha r)^2 + 2\alpha\gamma r\sigma^2\left[1-2\frac{\rho^2}{(1+\rho)}\right]}\right]}, u = \frac{RP(1-e^{-bt})}{\gamma r\sigma^2(1+2\rho)},$

$v = \frac{(1+\rho)(1-e^{-bt})}{\gamma r\sigma^2(1+2\rho)(1-\rho)}, \varphi_1 = \left(\frac{xy}{2\alpha\gamma r^2} - xv - yu - A_{2C}uv\right)^2 - \left(4yv + 2A_{2C}v^2 - \frac{y^2}{\alpha\gamma r^2}\right)\left(xu + \frac{1}{2}A_{2C}u^2 + \frac{RP^2(1-\rho)}{2r\sigma^2(1+2\rho)(1+\rho)} - \frac{x^2}{4\alpha\gamma r^2}\right).$

Case 3 – Numerical Example. This example illustrate the findings using the following parameters: $r = 0.02,\ \bar{Y} = 1,\ RP_1 = RP_2 = RP_3 = 0.06,\ \sigma_Y = 0.4,\ \sigma_1 = \sigma_2 = \sigma_3 = 0.4,\ \tau = 0.05,\ \kappa_Y = 1,\ \rho_{Y1} = \rho_{Y2} = \rho_{Y3} = -0.4,\ \rho_{01} = \rho_{02} = \rho_{03} = -0.4,\ \rho_{0Y} = 0.4,\ \gamma = 2,\ \rho_{12} = \rho_{13} = \rho_{23} = 0.2$

When the three stocks have the same characteristics (RP and σ) and $\rho_{Yi} = \rho_Y \neq 0, \rho_{0i} = \rho_0 \neq 0$, the six equations (A–10) to (A–15) for the economy with transaction costs and the three equations (A–26) to (A–28) for the market without costs become:

For the economy with transaction costs

$$-rA_{1C} - \frac{A_{1C}A_{2C}}{2\alpha\gamma r} + B_{1C}\left[\kappa_Y\bar{Y} - \frac{2\sigma_Y\rho_Y RP}{\sigma(1+\rho)}\right] - \frac{\gamma r\sigma\sigma_Y\rho_Y(1-\rho)}{1+\rho}B_{0C} + \sigma_Y^2\left[1 - \frac{2\rho_Y^2}{(1+\rho)}\right]B_{0C}B_{1C} - r\gamma(RP + \Delta_{l1}) + \frac{2\gamma r\rho RP}{(1+\rho)} = 0, \tag{B–45}$$

$$-rA_{2C} - \frac{A_{2C}^2}{2\alpha\gamma r} + \sigma_Y^2\left[1 - \frac{2\rho_Y^2}{(1+\rho)}\right]B_{1C}^2 - \frac{2\gamma r\sigma\sigma_Y\rho_Y(1-\rho)}{1+\rho}B_{1C} + (\gamma r\sigma)^2\left[1 - \frac{2\rho^2}{(1+\rho)}\right] = 0, \tag{B–46}$$

$$\left[-r - \gamma r\sigma_Y\rho_{0Y} - \kappa_Y + \frac{2\gamma r\sigma_Y\rho_0\rho_Y}{(1+\rho)}\right]B_{0C} + \left[\kappa_Y\bar{Y} - \frac{2\sigma_Y\rho_Y RP}{\sigma(1+\rho)}\right]C_{0C} + \sigma_Y^2\left[1 - \frac{2\rho_Y^2}{(1+\rho)}\right]B_{0C}C_{0C} - \frac{A_{1C}B_{1C}}{2\alpha\gamma r} + \frac{2\gamma r\rho_0 RP}{\sigma(1+\rho)} = 0, \tag{B–47}$$

$$\left[-\frac{r}{2} - \gamma r\sigma_Y\rho_{0Y} - \kappa_Y + \frac{2\gamma r\sigma_Y\rho_0\rho_Y}{(1+\rho)}\right]C_{0C} + \frac{1}{2}\sigma_Y^2\left[1 - \frac{2\rho_Y^2}{(1+\rho)}\right]C_{0C}^2 - \frac{B_{1C}^2}{4\alpha\gamma r} + \frac{1}{2}(\gamma r)^2\left[1 - \frac{2\rho_0^2}{(1+\rho)}\right] = 0, \tag{B–48}$$

$$\left[-r-\gamma r\sigma_Y\rho_{0Y}-\kappa_Y+\frac{2\gamma r\sigma_Y\rho_0\rho_Y}{(1+\rho)}\right]B_{1C}-\frac{\gamma r\sigma\sigma_Y\rho_Y(1-\rho)}{1+\rho}C_{0C}+\sigma_Y^2\left[1-\frac{2\rho_Y^2}{(1+\rho)}\right]B_{1C}C_{0C}-\frac{A_{2C}B_{1C}}{2\alpha\gamma r}+\frac{(\gamma r)^2\sigma\rho_0(1-\rho)}{1+\rho}=0, \quad \text{(B–49)}$$

$$-2rA_{0C}+\sigma_Y^2C_{0C}+\sigma_Y^2\left[1-\frac{2\rho_Y^2}{(1+\rho)}\right]B_{0C}^2+2\left[\kappa_Y\bar{Y}-\frac{2\sigma_Y\rho_Y RP}{\sigma(1+\rho)}\right]B_{0C}-\frac{A_{1C}^2}{2\alpha\gamma r}-2r\ln(r)-2\tau+2r-\frac{2RP^2}{\sigma^2(1+\rho)}=0. \quad \text{(B–50)}$$

For the economy without transaction costs:

$$\left[-r-\gamma r\sigma_Y\rho_{0Y}-\kappa_Y+\frac{3\gamma r\rho_0\sigma_Y\rho_Y}{(1+2\rho)}\right]B_0+\left[\kappa_Y\bar{Y}-\frac{3\sigma_Y\rho_Y RP}{\sigma(1+2\rho)}\right]C_0+\sigma_Y^2\left[1-\frac{3\rho_Y^2}{(1+2\rho)}\right]B_0C_0+\frac{3\gamma r\rho_0 RP}{\sigma(1+2\rho)}=0, \quad \text{(B–51)}$$

$$\left[-r-2\gamma r\sigma_Y\rho_{0Y}-2\kappa_Y+\frac{6\gamma r\rho_0\sigma_Y\rho_Y}{(1+2\rho)}\right]C_0+\sigma_Y^2\left[1-\frac{3\rho_Y^2}{(1+2\rho)}\right]C_0^2+(\gamma r)^2\left[1-\frac{3\rho_0^2}{(1+2\rho)}\right]=0, \quad \text{(B–52)}$$

$$-2rA_0+\sigma_Y^2C_0+\sigma_Y^2\left[1-\frac{3\rho_Y^2}{(1+2\rho)}\right]B_0^2+2\left[\kappa_Y\bar{Y}-\frac{3\sigma_Y\rho_Y RP}{\sigma(1+2\rho)}\right]B_0-2r\ln(r)-2\tau+2r-\frac{3RP^2}{\sigma^2(1+2\rho)}=0. \quad \text{(B–53)}$$

The last system of nine equations needs solving numerically with detailed results shown in Subsection 4.1.3. Equation (32) can be achieved by using also equation (A–29).

Appendix C

In this Appendix, I present the solutions of optimization problem when there are arbitrary number of stocks available in the market. Unfortunately, a large number of stocks prevents me from finding out a closed-form optimal allocation to each liquid stock $\widehat{N}_k$. Therefore, I first simplify this problem by making an assumption that all stocks have the same positive correlation coefficient (ρ). The allocations to liquid stocks in the economy with and without transaction costs are then given by:

<u>*In the economy without transaction cost*</u>

$$\widehat{N}_k = \left[\frac{\{\rho \sum_{i=1}^{n} \rho_{0i} - [1+(n-1)\rho]\rho_{0k}\}}{\sigma_k(1-\rho)[1+(n-1)\rho]} - \frac{C_0\sigma_Y\{\rho \sum_{i=1}^{n} \rho_{Yi} - [1+(n-1)\rho]\rho_{Yk}\}}{\gamma r \sigma_k(1-\rho)[1+(n-1)\rho]}\right]Y$$
$$- \left[\frac{B_0\sigma_Y\{\rho \sum_{i=1}^{n} \rho_{Yi} - [1+(n-1)\rho]\rho_{Yk}\}}{\gamma r \sigma_k(1-\rho)[1+(n-1)\rho]}\right. \tag{C–1}$$
$$\left. + \frac{\{\rho \sum_{i=1}^{n}\left(RP_i \times \prod_{j\neq i,j=1}^{n}\sigma_j\right) - [1+(n-1)\rho]RP_k \times \prod_{j\neq k,j=1}^{n}\sigma_j\}}{\gamma r \sigma_k(1-\rho)[1+(n-1)\rho]\prod_{i=1}^{n}\sigma_i}\right] \quad k = 1, \dots, n.$$

<u>*In the economy with transaction cost*</u>

$$\widehat{N}_k$$
$$= \frac{\gamma r \sigma_1 \rho(\rho-1) - B_{1C}\sigma_Y\{\rho \sum_{i=2}^{n} \rho_{Yi} - [1+(n-2)\rho]\rho_{Yk}\}}{\gamma r \sigma_k(1-\rho)[1+(n-2)\rho]} N_1$$
$$+ \frac{\gamma r\{\rho \sum_{i=2}^{n} \rho_{0i} - [1+(n-2)\rho]\rho_{0k}\} - C_{0C}\sigma_Y\{\rho \sum_{i=2}^{n} \rho_{Yi} - [1+(n-2)\rho]\rho_{Yk}\}}{\gamma r \sigma_k(1-\rho)[1+(n-2)\rho]} Y \tag{C–2}$$
$$- \left[\frac{B_{0C}\sigma_Y\{\rho \sum_{i=2}^{n} \rho_{Yi} - [1+(n-2)\rho]\rho_{Yk}\}}{\gamma r \sigma_k(1-\rho)[1+(n-2)\rho]}\right.$$
$$\left. + \frac{\{\rho \sum_{i,j=2}^{n}\left(RP_i \times \prod_{j\neq i}^{n}\sigma_j\right) - [1+(n-2)\rho]RP_k \times \prod_{j\neq k,j=2}^{n}\sigma_j\}}{\gamma r \sigma_k(1-\rho)[1+(n-2)\rho]\prod_{i=2}^{n}\sigma_i}\right] \quad k = 2, \dots n.$$

However, the task of finding the unknown variables for the function $g(Y, N_1)$ and for the function $g(Y)$ still needs solving numerically. Therefore, I only consider two cases which enable me to arrive at closed-form solutions: one is when all stocks are independent and other correlation coefficients are zero; the other is when all stocks have the same characteristics and same positive correlation coefficients while assuming other correlation coefficients are zero. The results of the former case are similar to those in Example 1 – Appendix B. Thus, I do not show detailed results of this case.

In the second case, I assume $\rho_{ik} = \rho > 0$ while $\rho_{Yi} = 0, \rho_{0i} = 0 \; and \; \rho_{0Y} = 0$. It follows that equation (33) results from equation (C–1) and equation (35) is derived from equation (C–2). The solutions for unknown variables $A_{0C}, A_{1C}, A_{2C}, B_{0C}, B_{1C}, C_{0C}$ in the function $g(Y, N_1)$ and A_0, B_0, C_0 in the function $g(Y)$ are presented as follows:

In the economy without transaction cost:

The system of three equations (A–26) to (A–28) becomes:

$$-rB_0 + \sigma_Y^2 B_0 C_0 + \kappa_Y \bar{Y} C_0 - \kappa_Y B_0 = 0, \tag{C–3}$$

$$-rC_0 + (\gamma r)^2 + \sigma_Y^2 C_0^2 - 2\kappa_Y C_0 = 0, \tag{C–4}$$

$$-r[\ln(r) + A_0] - \tau + r + \frac{1}{2}\sigma_Y^2 C_0 + \frac{1}{2}\sigma_Y^2 B_0^2 + \kappa_Y \bar{Y} B_0 - \frac{nRP^2}{2\sigma^2[1 + (n-1)\rho]} = 0. \tag{C–5}$$

Solving the last three equations, I receive the formulas for A_0, B_0 and C_0:

$$A_0 = 1 - \ln(r) - \frac{\tau}{r} + \frac{1}{2r}\sigma_Y^2 B_0^2 + \frac{1}{2r}\sigma_Y^2 C_0 + \frac{\kappa_Y \bar{Y} B_0}{r} - \frac{nRP^2}{2r\sigma^2[1 + (n-1)\rho]}, \tag{C–6}$$

$$B_0 = \frac{\kappa_Y \bar{Y}(r + 2\kappa_Y) \pm \kappa_Y \bar{Y}\sqrt{(r + 2\kappa_Y)^2 - 4\sigma_Y^2(\gamma r)^2}}{\sigma_Y^2 r \mp \sigma_Y^2\sqrt{(r + 2\kappa_Y)^2 - 4\sigma_Y^2(\gamma r)^2}}, \tag{C–7}$$

$$C_0 = \frac{(r + 2\kappa_Y) \pm \sqrt{(r + 2\kappa_Y)^2 - 4(\gamma r \sigma_Y)^2}}{2\sigma_Y^2}. \tag{C–8}$$

In the economy with transaction cost

The system of six equations (A–10) to (A–15) becomes:

$$-rA_{1C} + \frac{\gamma r \rho RP(n-1)}{[1 + (n-2)\rho]} + \sigma_Y^2 B_{0C} B_{1C} + \kappa_Y \bar{Y} B_{1C} - \gamma r(RP_1 + \Delta_{l1}) - \frac{A_{1C} A_{2C}}{2\alpha\gamma r} = 0, \tag{C–9}$$

$$-\frac{r}{2}A_{2C} + \frac{1}{2}(\gamma r \sigma_1)^2 - \frac{(\gamma r)^2 \sigma^2 \rho^2 (n-1)}{2[1 + (n-2)\rho]} + \frac{1}{2}\sigma_Y^2 B_{1C}^2 - \frac{A_{2C}^2}{4\alpha\gamma r} = 0, \tag{C–10}$$

$$-rB_{0C} + \sigma_Y^2 B_{0C} C_{0C} + \kappa_Y \bar{Y} C_{0C} - \kappa_Y B_{0C} - \frac{A_{1C} B_{1C}}{2\alpha\gamma r} = 0, \tag{C–11}$$

$$-\frac{r}{2}C_{0C}+\frac{1}{2}(\gamma r)^2+\frac{1}{2}\sigma_Y^2C_{0C}^2-\kappa_Y C_{0C}-\frac{B_{1C}^2}{4\alpha\gamma r}=0, \tag{C–12}$$

$$-rB_{1C}+\sigma_Y^2B_{1C}C_{0C}-\kappa_Y B_{1C}-\frac{A_{2C}B_{1C}}{2\alpha\gamma r}=0, \tag{C–13}$$

$$-r[\ln(r)+A_{0C}]-\tau+r+\frac{1}{2}\sigma_Y^2C_{0C}+\frac{1}{2}\sigma_Y^2B_{0C}^2+\ \kappa_Y\bar{Y}B_{0C}-\frac{RP^2(n-1)}{2\sigma^2[1+(n-2)\rho]}-\frac{A_{1C}^2}{4\alpha\gamma r}=0. \tag{C–14}$$

Solving these six equations, I find that $B_{1C}=0$, and the roots for $A_{0C}, A_{1C}, A_{2C}, B_{0C}, C_{0C}$ are given by:

$$A_{0C}=1-\ln(r)-\frac{\tau}{r}+\frac{1}{2r}\sigma_Y^2C_{0C}+\frac{1}{2r}\sigma_Y^2B_{0C}^2+\frac{\kappa_Y\bar{Y}B_{0C}}{r}-\frac{A_{1C}^2}{4\alpha\gamma r^2}-\frac{RP^2(n-1)}{2r\sigma^2[1+(n-2)\rho]}, \tag{C–15}$$

$$A_{1C}=\frac{2\alpha\gamma r(\rho-1)RP-2\alpha\gamma r[1+(n-2)\rho]\Delta_{l1}}{[1+(n-2)\rho]\left[\alpha r+\sqrt{(\alpha r)^2+2\alpha\gamma r\sigma^2\left[1-\frac{\rho^2(n-1)}{[1+(n-2)\rho]}\right]}\right]}, \tag{C–16}$$

$$A_{2C}=\gamma r\left[\sqrt{(\alpha r)^2+2\alpha\gamma r\sigma^2\frac{(1-\rho)[1+(n-1)\rho]}{[1+(n-2)\rho]}}-\alpha r\right], \tag{C–17}$$

$$B_{0C}=\frac{\kappa_Y\bar{Y}(r+2\kappa_Y)\pm\kappa_Y\bar{Y}\sqrt{(r+2\kappa_Y)^2-4\sigma_Y^2(\gamma r)^2}}{\sigma_Y^2 r\mp\sigma_Y^2\sqrt{(r+2\kappa_Y)^2-4\sigma_Y^2(\gamma r)^2}}, \tag{C–18}$$

$$C_{0C}=\frac{(r+2\kappa_Y)\pm\sqrt{(r+2\kappa_Y)^2-4(\gamma r\sigma_Y)^2}}{2\sigma_Y^2}. \tag{C–19}$$

<u>*The conditional liquidity premium*</u>

With $B_0=B_{0C}, C_0=C_{0C}, and\ B_{1C}=0$, equation (A–29) becomes:

$$A_{0C}+A_{1C}N_1+\frac{1}{2}A_{2C}N_1^2=A_0, \tag{C–20}$$

By replacing $A_{0C}, A_{1C}, A_{2C}, and\ A_0$ with the formulas (C–15), (C–16), (C–17) and (C–6), respectively, into the last equation, I arrive at equation (36).

It follows that Δ_{l1} is well-defined because α, γ, r, ρ are all positive constants; A_{2C} is also positive, and thus the term under the radical is positive. Furthermore, similar to Case 1 and Case

2 in Appendix B, the condition of $N_1 > \frac{RP}{\gamma r\sigma^2[1+(n-1)\rho]}$ should hold for $\Delta_{l1} > 0$ and for Δ_{l1} to increase with N_1.

Following equation (12), I arrive at the process N_{1t} shown in equation (37) when $N_{10} = 0$. By substituting this formula for N_1 in equation (36), I obtain the conditional liquidity premium as a function of the illiquidity coefficient α:

$$\Delta_{l1} = \frac{\left[\frac{xy}{2\alpha\gamma r^2} - xv - yu - A_{2C}uv\right] - \sqrt{\varphi_2}}{2yv + A_{2C}v^2 - \frac{y^2}{2\alpha\gamma r^2}}, \tag{C–21}$$

where

$x = \frac{2\alpha\gamma rRP(\rho-1)}{[1+(n-2)\rho]\left[\alpha r+\sqrt{(\alpha r)^2+2\alpha\gamma r\sigma^2\left[1-\frac{\rho^2(n-1)}{[1+(n-2)\rho]}\right]}\right]}$, $y = \frac{-2\alpha\gamma r[1+(n-2)\rho]}{[1+(n-2)\rho]\left[\alpha r+\sqrt{(\alpha r)^2+2\alpha\gamma r\sigma^2\left[1-\frac{\rho^2(n-1)}{[1+(n-2)\rho]}\right]}\right]}$, $u = \frac{RP(1-e^{-bt})}{\gamma r\sigma^2[1+(n-1)\rho]}$, $v = \frac{[1+(n-2)\rho](1-e^{-bt})}{\gamma r\sigma^2(1-\rho)[1+(n-1)\rho]}$, $\varphi_2 = \left(\frac{xy}{2\alpha\gamma r^2} - xv - yu - A_{2C}uv\right)^2 - \left(4yv + 2A_{2C}v^2 - \frac{y^2}{\alpha\gamma r^2}\right)\left(xu + \frac{1}{2}A_{2C}u^2 + \frac{RP^2(1-\rho)}{2r\sigma^2[1+(n-1)\rho][1+(n-2)\rho]} - \frac{x^2}{4\alpha\gamma r^2}\right)$.