A submodular representation for hub network design problems with profits and single assignments

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Abstract

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Maryam Hemmati

Hub network design problems (HNDPs) lie at the heart of network design planning in transportation and telecommunication systems. They constitute a challenging class of optimization problems that focus on the design of a hub network. In this work, we study a class of HNDPs, named hub network design problems with profits and single assignments, which forces each node to be assigned to exactly one hub facility.

We propose three different combinatorial representations for maximizing the total profit defined as the difference between the perceived revenues from routing a set of commodities minus the setup cost for designing a hub network, considering the single allocation assumption. We investigate whether the objective function of each representation satisfies the submodular property or not. One representation satisfies submodularity, and we use it to present an approximation algorithm with polynomial running time. We obtain worst-case bounds on the approximations' quality and analyze some special cases where these worst-case bounds are sharper.

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”Since the building of the universe is perfect and is created by the wisdom creator, nothing arises in the universe in which one cannot see the sense of some maximum or minimum. ”

(L. Euler)

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Chapter 1

Introduction

In the broad areas of facility location and network design, *Hub-and-spoke* refers to a distribution architecture that has been widely used to route flows efficiently, especially when the problem involves routing commodities or information between several origins and destinations. The main feature of hub-and-spoke networks is in the transshipment, consolidation and sorting points, called hub facilities, instead of establishing direct connections for a large number of origins and destinations. Flows with the same origin but different destinations are consolidated when routed to the hub sites. Then, the flow from different origin but the same destination gets combined (i.e. consolidated, sorted, labelled, and assembled) in the hubs and rerouted to their destination directly or possibly via another hub. This reduces the total set-up cost, improves the handling of the commodities by centralizing the network and results in economies of scale due to the reduction of costs through consolidation of flows.

Most Hub Location Problems HLPs belongs to the class of $\mathscr{N}\mathscr{P}$-hard problems. Their fundamental difficulty originates from the interrelation between two levels of decisions. The first level of decision is the location decisions that determine where to open hubs, whereas the second level is the network design decisions that activate links in the network and define paths for routing the commodities. In this work, we concentrate on discrete location decision; that is, a predefined set of possible sites to open hubs is given. It is mostly assumed that Discrete HLPs are typically modelled using a graph (i.e., a set of nodes and edges or arcs). More formality on the definitions will be provided in the following chapters.

The existing literature in this field focuses mainly on the classic version of HLPs called *Hub Node Location Problem (HNLP)*. In HNLPs, one can benefit from using links that are commonly known as hub links or *hub arcs* between chosen hubs without incurring set-up costs. Therefore, on HNLPs, hubs are fully interconnected. Moreover, when we assume that each non-hub site must be connected to at least one hub, the final selection of hubs and links defines a feasible solution in that all commodities may be routed in such a network.

However, in various applications, there are set-up costs for installing hub arcs (i.e., telecommunication or pipeline networks). Therefore, other variants within HLPs have been developed. In particular, Hub Arc Location Problems (HALPs) neglect the assumption that hubs are fully connected and incorporate the rest of the above assumptions with activating hub arcs and the corresponding set-up costs of the both set of hub nodes and hub arcs. In some cases, a cardinality constraint is enforced on the number of hub arcs that may be opened. All these modifications make HALPs optimality conditions differ from HNLPs.

HLPs are applied the most in the network design planning of transportation and telecommunication industries. Depending on the applications and assumptions, the objectives of HNLPs and HALPs aim to minimize the total cost or maximize the profit obtained by serving demand nodes. In the maximization case, the profit considers not only the transportation cost for serving (routing) flows, but it may consider additional set-up costs of installing facilities like hub nodes and hub arcs. In some cases, it is possible to ignore the set-up costs and reflect it by restricting the problem to serve all demands or ignore installing a facility along with making benefits from the discount factor. Generally, these hypotheses make the total costs of the network compensated by the total captured profit. However, for some problems, there might be other factors on the decisions of the origin/destination

(O/D) nodes and their associated commodities that significantly impact the operational costs, which are not negligible.

Several applications stem from HLPS in transportation systems. These include passenger transportation, airfreight, postal services, express package delivery, trucking and shipping, public transportation, and rapid transit systems. Demand in these models appears in the form of passengers or packages that go through hub nodes such as transit airports in the airline industry. Hub arcs in such applications could be physical or non-physical links. Examples of physical links include railways transportation, which could be associated set-up costs for them; on the other hand, non-physical links appear in the airline industry or maritime industry, and there is mostly no set-up cost related to these type of links. It is good to mention that sometimes, access arcs may need the installation of physical links such as installing fibre optic links in telecommunication networks. The economy of scale of consolidating flow through such a network is not only enabled from routing flows between hub nodes but also between O/D nodes.

Applications of HLPs in telecommunication networks arise in the configuration and design of various data networks. Demand usually corresponds to electronic data that needs to be routed on physical links such as co-axial cables, fibre optic links, or non-physical links (i.e. air) like satellite channels and microwaves. The hardware such as concentrators, multiplexers and switches corresponds to hub facilities. The data transmission through such a network resulted in economies of scale and increased network utilization. Large set-up costs of hub nodes and communication links have given the motivation to apply hub-and-spoke architecture in telecommunication networks.

Broadly speaking, there are different strategic decisions on how to route the flows between O/D nodes (i.e. how to allocate the nodes to the hubs). Each of the non-hub nodes (i.e. spokes) might be allocated either to a single hub or multiple hubs. In some cases, we might route a commodity from a path that results in gaining the maximum profit (minimum cost). Therefore, there might be multiple options to link O/D nodes to the set of hub nodes to route a specific commodity, which results in assigning each non-hub node to several hubs (multiple allocations). On the other side, in some cases, there is a need to install physical links between non-hub nodes and hubs, such as installing fibre optic links in telecommunication networks, which incurs additional set-up cost. One might also aim to make the handling and operation easier through the network with single allocation. Therefore, it is more efficient to design the network in a way, so that are assigned each non-hub node of the network to exactly one hub (single allocation).

In this work, we study a class of problems in hub location denoted as the Hub Network Design Problem with Profits and Single Assignments (HNDPSA). The objective of HNDPSA is to maximize the total profit by measuring the trade-off between the revenue from serving a set of commodities and the total cost of designing the network. We propose three representations for modelling the HNDPSA. In all representations, the strategic decisions consist of (i) determining the nodes to open as hubs and which hub arcs should be activated, and (ii) how to assign the nodes to the opened hub nodes with a single allocation strategy (i.e., which path should be chosen to route a commodity) to maximize the objective function.

We overview a property named 'submodularity' of the objective function, which we formally define in the next chapter. We aim to obtain a representation for the HNDPSA such that the objective function satisfies submodularity. After finding counterexamples for two different representations, we progressively reached a representation where the objective function is submodular. We then propose a greedy heuristic algorithm to approximate the optimal solution value. We focus on the mathematical insights and prove the proposed algorithm runs in polynomial time. We show the quality of the approximation achieved by the designed algorithm using theoretical aspects.

The contribution of this research is four-fold. First, we state three representations for the

HNDPSA as the maximization of a set function subject to linear constraints. To the best of our knowledge, this is the first work that studies HNDPSA using set functions. Second, we prove the objective function of one of the representations satisfies submodularity. Third, we use the submodular representation to obtain the worst-case performance results of a greedy heuristic for the HNDPSA when the objective function is also non-decreasing. In particular, when the profit function is non-decreasing, we propose a $(1 - 1/e)$-approximation algorithm. HLPs mostly have been studied when the intermediate nodes between O/D pairs are hub nodes. Fourth, we incorporate this assumption when considering O/D nodes are hub nodes as well.

The remainder of this thesis is organized as follows. In Chapter 2, we present the main features and assumptions of HNDPSA and review the most relevant literature to the HLPs and submodularity. In Chapter 3, we propose three representations for which we either provide counterexamples for disproving submodularity or mathematically prove it. Chapter 4, we provide a polynomial-time greedy heuristic, study its complexity, and present its theoretical worst-case bound. Finally, in Chapter 5, we present the research conclusions and provide directions for future studies.

Chapter 2

Preliminaries and Literature Review

In this chapter, we start by introducing the basic features of HLPs, including their assumptions, properties, and widely employed objectives. We also introduce the mathematical definition of a submodular function with a brief literature review. Then, we focus on the literature review of HLPs along with the single allocation HLPs.

2.1 Features and assumptions

The primary recognizing features of HLPs can be outlined as follows:

I. Demand is associated with flows between O/D pairs.

II. The intermediate nodes on the O/D path should be hub facilities that act as transshipment and consolidation points.

III. The objective of HPLs could be cost-based (or service-based), which relies on the scope of the concrete application (locating hubs and selecting links of routing flows).

IV. Routing through hubs results in a benefit.

We denote the discount factor parameter by α $(0 \leq \alpha \leq 1)$, which is used to represent reduced unit flow costs on hub arcs due to economies of scale. Each O/D path's length determines the unit flow cost between its O/D nodes. Therefore, each O/D path may contain a $collection$ leg from the non-hub origin to the first hub, possibly a $transfer$ leg between hubs, and a $distribution$ leg from the last hub to the non-hub destination. Note that not all O/D paths contain a transfer leg and might be connected through one hub node.

Four common assumptions underline most HLPs:

1. Flows have to be routed through hub nodes, which could be either one hub or, at most, two hubs.

2. For routing each flow, the distance should satisfy the triangle inequality.

3. The O/D nodes of a given flow could be either a hub node or a non-hub node.

4. The discount factor α for the arcs connecting hubs should be constant regardless of the amount of flow routed on each hub arc.

The above assumptions describe the structure of O/D paths. We should note that in HNLPs, the set-up cost of installing hub arcs is equal to zero, and this results in a complete subgraph on the set of hub nodes with no extra cost. A consequence of the assumption of 1 is that the direct linking between two non-hub nodes is not allowed. However, two nodes may be linked directly if at least one of them is a hub node. Therefore, in the solution network, commodities pass through at most three types of arcs, $access\ arcs$, $hub\ arcs$ and $bridge\ arcs$. Access arcs connect non-hub nodes to hub nodes. Both hub arcs and bridge arcs connect two hub nodes. However, hub arcs have an associated set-up cost and, therefore, benefits from a discount factor α that results in economies of scale. In contrast, bridge arcs connect two hub nodes without having a reduced unit flow cost parameter α. Although the discount factor α reduces the unit transportation cost between two hub nodes, in some cases of HALPs, activating a hub arc is much more costly than saving from the discount of applying α. Therefore, for those cases, routing through a bridge arc is more beneficial than paying the set-up cost of activating a hub arc. As a result, in HNLPs, each O/D path includes no intermediate hub, a single intermediate hub node and no intermediate hub arc, two hub nodes, and a single intermediate hub arc. Whereas in HALPs, each O/D path may include no intermediate hub nodes, single hub nodes and no intermediate hub arcs or two intermediate hub nodes. An intermediate arc between these hubs could be either a hub arc or an access arcs. Note that if O/D nodes are hubs, the arcs between them and the intermediate hub nodes (or the direct arc between them) could be either hub arcs or access arcs.

There are three possible allocation strategies for assigning non-hub nodes to the selected hubs: multiple assignments, single assignment, and r-allocation. In the multiple assignments strategy, we first select the hub nodes, and then we find the shortest path connecting each O/D pair. This may result in assigning non-hub nodes to more than one hub. This pattern simplifies the routing decision and creates flexibility in designing the hub network. However, it may significantly increase network design costs since we might activate many access links. Hence, multiple assignments are more applicable when there is no set-up cost for access links such as air freight and passenger travels, public transportation, and rapid transit systems. There are no physical links in these cases, or there may exist physical infrastructure (i.e., road or highway); therefore, there would be no set-up cost for access arcs.

O/D nodes must be linked to exactly one hub node in the single assignment strategy, which makes the problem more challenging. Commodities with the same origin (or destination) must be routed via the same access arcs. Thus, in the single assignment version, we either need to install physical links (access arcs) such as installing fibre optic links in telecommunication or attempting to take care of the operation through the network.

The r-allocation problem is a cross-over that takes advantage of both single assignment and multiple assignment strategies. In this case, each non-hub node can be connected to at most r hub nodes. It has more flexibility for choosing the cheaper path and, at the same time, it controls the number of opened access arcs in the designing of the network.

2.2 Submodular set functions

Submodularity is a property of set functions. We present a brief review of combinatorial optimization problems and the definition of submodular set functions. Then we present a brief literature review on submodular functions.

Definition 2.1. *(see Nemhauser and Wolsey, 1988) Let $N = \{1, ..., n\}$ be a finite set and let $c = (c_1, ..., c_n)$ be an n-dimension vector. For $F \subseteq N$, define $c(F) = \sum_{j \in F} c_j$. Suppose we are given a collection of subsets $\mathscr{F}$ on N. A combinatorial optimization problem is*

$$\max\{c(F) : \; F \in \mathscr{F}\}. \tag{1}$$

The concept of submodularity in a discrete space is close to that of convexity in the continuous space. Accordingly, Frank et al. (1982) results show a polynomial-time algorithm exists to find the minimum of a submodular function. Surprisingly, a submodular function somewhere also acts similar to concave functions (see Lovász, 1983). In continuous space, one way to maximize a concave function (minimize a convex function) or approximate the maximum value of the function is to use the gradient vector of the function as the ascent (decent) direction. The derivative of a function f shows the slope of the tangent line. In the case of a concave function, this slope decreases when the input of the function increases. Here submodularity is deeply akin to concavity. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a concave function, then $f'(x)$ is non-increasing for $x \in \mathbb{R}$. Accordingly, we want to evaluate the sensitivity to change of a discrete function. Let $\partial z_S(j) = z(S \cup \{j\}) - z(S)$ for $j \in N$ is defining the *discrete derivative* for the function z (see Stobbe and Krause, 2010). Then, z is submodular if for each $S \subseteq T \subseteq N$ and $j \in N \setminus T$, the following is satisfied $\partial z_S(j) \geq \partial z_T(j)$, which means the discrete derivative should be non-increasing.

Supermodularity is the inverse of submodularity and is similar to convexity in the sense that the derivative of the function is nondecreasing. Thus, for a supermodular function, the discrete derivative should be nondecreasing (see Fujishige, 1984).

Now we present the definition of a submodular function (see Nemhauser et al., 1978).

Definition 2.2. *Let N be a finite set and $z : 2^N \to \mathbb{R}$ be a real-valued function. The function z is submodular if it satisfies $z(S) + z(T) \geq z(S \cup T) + z(S \cap T)$ for all $S, T \subseteq N$.*

In what follows, we denote the discrete derivative via $\rho_j(S) = z(S \cup \{j\}) - z(S)$ for $j \in N$, which is also known as the incremental value for a submodular function. The following lemma provides another equivalent inequality for the definition of submodularity via the concept of discrete derivative.

Lemma 2.1. *(see Nemhauser et al., 1978) The following statements are equivalent and define a submodular set function.*

(a) $z(S) + z(T) \geq z(S \cup T) + z(S \cap T), \quad \forall S, T \subseteq N$.

(b) $\rho_i(S) = z(S \cup \{i\}) - z(S) \geq z(T \cup \{i\}) - z(T) = \rho_i(T), \quad \forall S \subset T \subset N, i \in N \setminus T$

Lemma 2.1 part (b) shows that the increments of adding a single element to a smaller subset are more than adding the same element to a larger set. Grötschel et al. (1981) provided a polynomial-time algorithm to minimize a submodular function, whereas maximizing a submodular function is in the class of $\mathscr{N}\mathscr{P}$-hard problems.

Maximization of a submodular set function has started to be attractive for researchers with the seminal work of Nemhauser et al. (1978), where they maximize a combinatorial representation for the k-median location problem to approximate an optimal solution using greedy heuristics and LP-relaxations. Nemhauser et al. (1978) presented a worst-case performance result for the specific structured greedy algorithm for a nondecreasing submodular function z, and they obtained a sharper bound when $z(\emptyset) = 0$. They show their results for k-median location problem as an example that we explain below.

Let N be the set of facilities and I be the set of customers. Consider a non-negative matrix $C_{|I| \times |N|} = (c_{ij})$ such that $i \in I$ and $j \in N$. For each nonempty $S \subseteq N$, let

$z(S) = \sum_{i \in I} \max_{j \in S} c_{ij}$, be the objective function and $z(\emptyset) = 0$. The problem is to find an S such that $S \leq K$ and K is an integer $(K \leq n)$ such that $z(S)$ is maximum; i.e.,

$$\max_{S \subseteq N} \{z(S) : |S| \leq K, z(S) \textbf{ is submodular}\}. \tag{2}$$

The proposed greedy heuristic in Nemhauser et al. (1978) firstly solves the problem for $K = 1$ and then iterates on the K and approximates the value of the z. When cardinality of S reaches to an integer p ($|S| = p$), and while the number of opened facilities does not violate the cardinality constraint, the iteration $p+1$ is determined by adding an element j^* to S (if possible) such that $z(S \cup \{j^*\}) = \max_{j \notin S} z(S \cup j)$ and $z(S \cup j^*) \geq z(S)$, which results in the following bound:

$$\frac{\textbf{value of greedy approximation}}{\textbf{value of the optimal solution}} \geq 1 - \left(\frac{K-1}{K}\right)^K \geq \frac{e-1}{e} \tag{3}$$

Later Nemhauser and Wolsey (1981) proposed a cutting plane method and a branch-and-bound algorithm to obtain the optimal solution while maximizing a submodular function of p-Median Problems (p-MPs). Wolsey (1982) analyzed a greedy algorithm for the submodular set covering problem. Fujito (2000) minimized a linear function subject to nondecreasing submodular constraints. Later, Sviridenko (2004) studied the maximization problem of a non-negative submodular set function subject to a knapsack constraint and obtained the same worst-case bound in which the solution achieved by an algorithm is at most $1 - \frac{1}{e}$ times of the optimal solution. For the problem of maximizing a monotone submodular function subject to an arbitrary matroid, Calinescu et al. (2011) presented a randomized algorithm with worst-case bound. Kulik et al. (2009, 2013) provided a non-exact algorithm for maximizing a nondecreasing and non-negative submodular function subjected to multiple linear and knapsack constraints. Contreras and Fernández (2014) studied the minimization of a supermodular function subject to two cardinality constraints for HLPs with multiple allocations. Ljubić and Moreno (2018) proposed a solution for a family of competitive facility location problems by combining two cutting plane methods, which include outer-approximation cuts and submodular cuts. Ortiz-Astorquiza et al. (2017) provided a combinatorial representation for the multi-level facility location problem that satisfies submodularity, then Ortiz-Astorquiza et al. (2019) proposed a mixed-integer programming formulation and an exact algorithm based on a Benders reformulation to compare the quality of the bound of the submodular representation. Recall that there might be different combinatorial representations for an optimization problem, some may satisfy submodularity, and some may not. Therefore, we should note that submodularity is a property of a function of a representation of a given problem. Some of the first papers developed a solution method for Multi-level Uncapacitated Facility Location Problem (MUFLP) using the representations of Ro and Tcha (1984) and Tcha and Lee (1984). However, submodularity was proved just for single level of UFLP by Nemhauser et al. (1978). It was assumed submodularity extends directly from the single level cases to multi-level cases. Later, Barros and Labbé (1994) provided a counter-example that shows the corresponding representation of MUFLP does not satisfy submodularity. However, Ortiz-Astorquiza et al. (2015) proposed another ground set and combinatorial representation that satisfies the submodularity for MUFLP.

We study the submodularity property over different combinatorial representations (Definition 2.1) for the HNDPSA. We prove the submodularity of the objective functions considering the following definitions and lemmas (Nemhauser et al., 1978; Ortiz-Astorquiza et al., 2015, see);

Definition 2.3. *Let N be a finite set and z a real-valued function on the subsets of N.*

(a) z is submodular if $\rho_i(S) \geq \rho_i(T), \ \forall \ S \subseteq T \subseteq N$ and $i \in N \setminus T$

(b) z is non-decreasing if $\rho_i(S) \geq 0, \ \forall \ S \subseteq N$ and $i \in N$.

Lemma 2.2. *Let d_j be the weight of $j \in N$. Then, the linear set function $f(S) =$*

$-\sum_{j \in S} d_j$ *is submodular function.*

Lemma 2.3. *A positive linear combination of submodular functions is submodular.*

2.3 A review on HLPs

The interest for studying HLPs has grown with the seminal work of O'Kelly (1986) for models in a continuous space. Later, O'Kelly (1987) provided a quadratic integer model, and, Campbell (1994) proposed the first mixed-integer linear programming formulation for HLPs. For the last three decades, HLPs have been widely used to solve different problems in numerous industries. One of the first papers by O'Kelly and Lao (1991) formulated HLPs for an air freight network. Jaillet et al. (1996) proposed a formulation for capacitated airline networks. Ernst and Krishnamoorthy (1996) involved hub location for postal delivery systems. Yaman and Carello (2005) studied HLPs in telecommunication and computer systems. Nickel et al. (2001) and Maheo et al. (2019) considered HLPs in public transportation.

Most of the papers on HLPs focuses on the discrete space. For studying HLPs in the continuous space, we refer the reader to O'Kelly (1986), Aykin (1988), O'Kelly and Miller (1991), Klincewicz (1998), Wieberneit (2008), Campbell (1990, 2013), and Mahmassani et al. (2013).

HLPs are closely similar to Facility Location Problems (FLPs). As a result, HLPs have also been studied as: p-hub median problems, analogous to a p-median in the FLPs, which is when the number p of hubs to be located is given, and the goal is to minimize the average distance from non-hub nodes t those p facilities (Campbell, 1996; Ernst and Krishnamoorthy, 1998a; García et al., 2012);

p-hub center problems, similar to the p-center problems in FLPs, which is to select exactly p hubs and allocate non-hub nodes to the determined hubs in a way to minimize the longest distance between non-hub nodes and hub nodes(Campbell, 1994; Kara and Tansel, 2000; Ernst et al., 2009);

In Hub covering problems, like the covering problem in FLPs, the number of hubs (facilities) is not predetermined, and the goal is to minimize the total cost of the opening hub and transportation such that each O/D pair should be served (Kara and Tansel, 2003; Calık et al., 2009).

All these problems can be classified into capacitated and uncapacitated problems analogous to FLPs. The capacity limitations could be on the cardinality of hub nodes (Ernst and Krishnamoorthy, 1999) or hub arcs (Labbé and Yaman, 2004; Campbell et al., 2005 a,b). Depending on the application, there may exist other assumptions for HLPs such as hub congestion (Elhedhli and Hu, 2005), stochasticity (Contreras et al., 2011a), (Rostami et al., 2018), and timing considerations (Kara and Tansel, 2001), (Alumur et al., 2012).

A detailed review of HLPs, their classification, and solution approaches can be found in the following surveys. O'Kelly and Miller (1994) and Campbell (1994c) gave one of the first surveys for HLPs that classifies basic models and topological structures for network design. Klincewicz (1998) presented a survey HLPs for telecommunication, and Bryan and O'kelly (1999) provided a review for the use of HLP in air transportation networks. Campbell et al. (2001) again investigated HLPs and brought a comprehensive survey that concludes locating hubs is a crucial decision in HLPs. Alumur and Kara (2008) prepared a survey on the progress of HLPs in the literature and classified the works based on their assumptions. Campbell and O'Kelly (2012) again provided a survey investigating new approaches and directions of HLP. Farahani et al. (2013) studied several classes of HLPs and reviewed their solution methods. Contreras and O'Kelly (2019) wrote about a concise overview of the main developments and most recent trends in hub location research. Also, they talked about the most successful integer programming formulations and efficient algorithms for HLPs.

Now, we provide a brief review of the single allocation version of HLPs. O'Kelly (1992) proposed the first mathematical model for the *uncapacitated single allocation hub location problem (USAHLP)* as a quadratic integer programming formulation. Campbell (1994) presented the first mixed-integer linear program for the *single allocation p-hub median problem (SAPHMP)*, minimizing the total flow costs between all pairs of nodes subject to the given number p of the hub. Skorin-Kapov et al. (1996) and Ernst and Krishnamoorthy (1996) proposed a path based Mixed-Integer Linear Programming (MILP) formulation for the USAHLP. Ernst and Krishnamoorthy (1999) studied the *capacitated single allocation hub location problem (CSAHLP)*. Contreras et al. (2009) obtained a tight upper and lower bounds for capacitated hub location problem with a single assignment (CLPSA) by applying the Lagrangian and relaxation method. Contreras et al. (2011b) solve CLPSA via a branch-and-price algorithm for Ernst and Krishnamoorthy (1999) formulation using the bounds achieved from Lagrangian relaxation method in Contreras et al. (2009). Correia et al. (2010) extended Ernst and Krishnamoorthy (1999) formulation for CSAHLP by adding hub capacities to the set of decision variables. Rostami et al. (2018) provided a new linearization of the quadratic formulation for hub location problem with single allocation. Meier and Clausen (2018) solved instances with up to 200 nodes taken from the Australian Post (AP) data set using the mentioned linearization for both the capacitated and uncapacitated versions of the SAHLP. For multiple allocation version of HLPs, we refer the interested readers to Campbell (1996) and Ernst and Krishnamoorthy (1998b).

Many works have also proposed several heuristics algorithms to solve single allocation HLPs due to their intrinsic difficulty. O'Kelly (1987) showed that USAHLP is $\mathscr{NP}$-hard, and developed two enumeration-based heuristics. In the first heuristic, each non-hub is allocated to its nearest hub. In the second one, each non-hub gets linked to one of the first two nearest hubs. Klincewicz (1992) proposed a tabu search heuristic and a randomize greedy heuristic for single allocation HLPs. Skorin-Kapov and Skorin-Kapov (1994) propose tabu search heuristic for the Uncapacitated Single Allocation p-Hub Median Problem (USApHMP). Ernst and Krishnamoorthy (1996) presented a new linear programming (LP) formulation for the problem, proposed simulated annealing heuristic to obtain upper bounds for the problem, and then applied the branch-and-bound method to obtain the exact solution. Campbell (1996) developed two greedy exchange heuristics and proved that solving multiple allocations p-hub median problem provides a lower bound for the single allocation SAPHMP. Abdinnour-Helm (2001) provided abound for USApHMP via simulated annealing method. Chen (2007) improved the bound for USAHLP by providing a hybrid heuristic based on the simulated annealing method. Silva and Cunha (2017) proposed an efficient multistart tabu search heuristic and a two-stage integrated tabu search heuristic, respectively, to solve USAHLP. Calık et al. (2009) provides a tabu search heuristic for the single allocation hub covering problems for incomplete hub networks. Jabalameli et al. (2012) proposed an efficient simulated annealing-based heuristic to solve a single allocation hub covering problems. Saboury et al. (2013) developed two-hybrid heuristics which incorporate a Variable Neighborhood Search (VNS) algorithm into the framework of simulated annealing and tabu search for solving hub location problem for a complete backbone graph. Abyazi-Sani and Ghanbari (2016) improved the existing bound via an efficient tabu search-based heuristic for solving the USAHLP. Later, Silva and Cunha (2017) developed another tabu search heuristic for the uncapacitated single allocation p-hub covering problem. Dai et al. (2017) proposed a heuristic algorithm called the general contraction method (GCM) to solve large-scale instances of the single allocation HLPs. Ghaffarinasab et al. (2018) developed Simulated annealing based algorithms to solve both single and multiple allocations p-hub maximal covering problems. Lüer-Villagra et al. (2019) reformulated single allocation p-hub median problem as a general piecewise-linear function and solved it with a metaheuristic.

Chapter 3

Problem Representation

In this chapter, we first present fundamental definitions and assumptions that we use throughout our study of HNDPSA. Then, we propose three different combinatorial representations for HNDPSAs. For the first two representations, we provide counterexamples to disprove the submodularity of the objective functions. For the last one, we directly prove that the objective function is submodular.

3.1 Fundamentals

Let $G = (N, E)$ be a complete undirected graph without loops, where N is the set of nodes, and E is the set of edges. We define K as a set of commodities, where each commodity $k \in K$ is defined as a pair of nodes (i.e., k = $\{i, j\}$), and a positive amount of flow W_k that needs to be routed between the endpoints of commodity k. Also, let $d_{ij} \geq 0$ be the distance or transportation cost, from node i to node j. From here on, we will assume that d is a distance function, and therefore is non-negative, symmetric ($d_{ij} = d_{ji}\ \forall i, j \in N$), and satisfies the triangle inequality.

Moreover, for each node $i \in N$, we define f_i to be a fixed set-up cost that is paid for installing a hub at node i. Similarly, for each $e \in E$, g_e denotes the fixed set-up cost of activating a hub edge. Hub edges $e = \{m, n\} \in E$ have a unit flow cost of αd_{mn}. Recall that $0 \leq \alpha \leq 1$ represents a discount factor to account for economies of scale. We should note that flows have directions, but we work over an undirected graph.

For a given path p of G, let l_p be the number of nodes in p. From now on, we refer to l_p as the length of path p. Also, r-length path is a path of length r. For each commodity $k \in K$, let $p_k := i \longrightarrow m_1 \longrightarrow \ldots \longrightarrow m_{l_{p_k}-2} \longrightarrow j$ be a directed path with origin i and destination j and passing through hub nodes $m_1, m_2, \ldots m_{l_p-2}$. In this study, we focus on a set of paths, denoted as P, that have the following characteristics. Each path is divided in three legs: a collection leg, a transfer leg and a distribution leg. The collection and transfer legs may contain either no edges or exactly one access edge. The transfer leg is composed only of hub nodes and may contain a combination of hub and bridge edges with at most $l_p - 2$ edges between the first and the last hub nodes.

Figure 3.1 shows all possible path configurations for routing commodity k. Black squares represent the set of hub nodes and black circles correspond to the set of non-hub nodes.

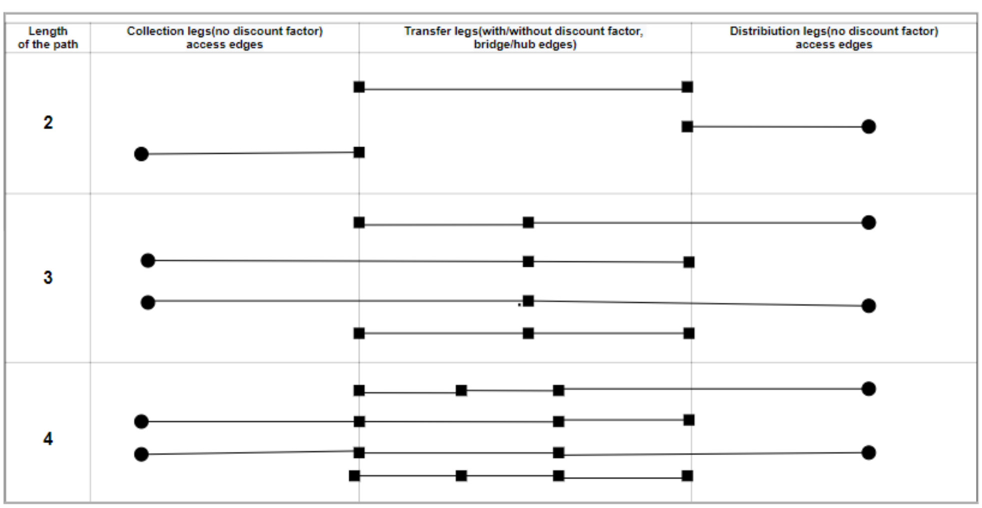


Figure 3.1: The set of possible paths for routing a commodity

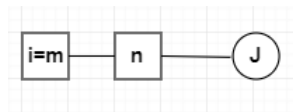
In a feasible solution of the HNDPSA, besides the set of hub nodes, we have a set of access edges A, a set of hub edges H, and a set of the bridge edges B. Recall that a bridge edge is an edge between two hub nodes without benefiting from a discount factor α. For commodity k, let C_{p_k} be the unit transportation cost for routing k through p_k, and T_k be the total revenue received from serving the demand of commodity k. For routing commodity k, we define the profit as $h_k = T_k - W_k C_{p_k}$. We assume that commodities cannot be split through different paths. Therefore, not every commodity must be served.

Now, we propose three different combinatorial representations for the HNDPSA, namely: an access edge-based representation, a star-based representation, and a path-based representation. For the first two representations, we construct counterexamples that show that the corresponding objective function does not satisfy the submodular property. Note that in disproving submodularity to simplifying the counterexamples for both of access edge-base and star-based representations, we consider the set-up cost for hub edges as zero and we neglect the cardinality constraint for hub edges, which is a particular case of the most general version. The set of constraints includes the single assignment rule and the cardinality constraint on the number of hub nodes based on the previous paragraph. For the third representation, we prove submodularity of the corresponding objective function using Definition 1 and Lemma 2.1. The set of constraints includes the single assignment rule and opening at most p hub nodes and q hub edges ($q \leq p(p-1)/2$).

3.2 An access edge-based representation

Consider the undirected graph $G = (N, E)$. Let $U = N \cup E$ be a finite set that includes both the set of nodes N and the set of edges E of graph G. For each nonempty subset $(S, Q) \subseteq U$, where $S \subseteq N$ and $Q \subseteq E$, we assume that S corresponds to the set of hub nodes and Q corresponds to the set of access edges. Let $\overline{\mathrm{S}} = N \setminus S$ be the set of non-hub nodes. For each node $i \in N$, let $R_i(Q) := \{e = \{i, j\} \in Q\}$ be the set of adjacent edges in Q to node i. Moreover, let $Q_s = \{i \in S : \exists e = \{i, j\} \in Q\}$ be the set of hub endpoints of Q. We define a set of all feasible paths for routing commodity $k = \{i, j\}$

such as $\overline{\mathrm{P_k}} := \{p_k = i \longrightarrow m \longrightarrow n \longrightarrow j : (m,n) \in S \times S,\ \textit{if}\ i \in \overline{\mathrm{S}}, \textit{then}\ \{i,m\} \in Q,\ \textit{if}\ j \in \overline{\mathrm{S}}, \textit{then}\ \{n,j\} \in Q\}$. Here we assume that the path $p_k := i \longrightarrow m \longrightarrow n \longrightarrow j$ is not necessarily a simple path. Recall that a simple path contains a sequence of disjoint nodes. For example, Figure 3.2 shows that in $p_k = i(= m) \longrightarrow m \longrightarrow n \longrightarrow j$, i and m are not disjoint.



(a) node i and m are not disjoint

Figure 3.2: $i(= m) \longrightarrow m \longrightarrow n \longrightarrow j$ is not a simple path.

We define $c_{p_k} := C_{p_k} W_k$ as the total transportation cost of commodity k where commodity k is routed through path p_k. Consider the following definitions for the cost functions:

$$f(S,Q) = -\sum_{i \in S} f_i \tag{1}$$

$$c_k(S,Q) = \begin{cases} \min\limits_{p_k \in \overline{\mathrm{P_k}}} c_{p_k} & \overline{\mathrm{P_k}} \neq \emptyset \\ \\ \infty & \text{otherwise.} \end{cases} \tag{2}$$

In particular, for a nonempty set of nodes $S \subseteq N$, $f(S,Q)$ corresponds to the total set-up cost of hub nodes and for the set of access edges $Q \subseteq E$, $c_k(S,Q)$ represents the transportation cost of commodity $k \in K$.

Recall that $h_k = T_k - c_{p_k}$ denotes the profit of routing commodity k. Let $T_k(S,Q)$ be the amount of revenue from hub end-points of the given set Q. Consequently, we define

$h(S,Q) = \sum_{k\in K} T_k(S,Q) - \sum_{k\in K} c_k(S,Q)$. Therefore, we want to maximize

$$z(S,Q) = h(S,Q) + f(S,Q) \tag{3}$$

Moreover, for a given subset $M \subseteq N$, we define the *cut* of M as $\delta(M) := \{\{i,j\}; i \in M, j \in N \setminus M\}$.

The HNDPSA can be represented as the problem of finding a set of hub nodes $S \subseteq N$ and the set of access edges $Q \subseteq E$ such that $z(S,Q)$ is maximized, i.e.,

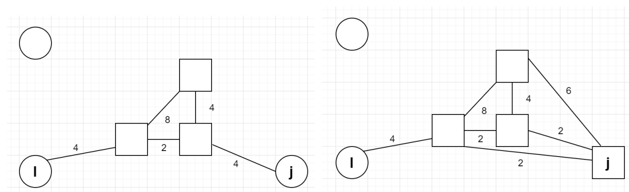
$$\max_{(S,Q)\subseteq U} \{z(S,Q) : |S| \leq p,\ \ Q_s = S,\ \ |R_i(Q)| = 1\ \forall i \in \overline{\mathrm{S}},\ \ Q \subseteq \delta(S)\} \tag{4}$$

The first constraint guarantees that the cardinality of S is less than p ($p \leq |N|$). The second constraint states that the set of hub endpoints of Q is equal to the set of hub nodes S. The third set of constraints guarantees that each non-hub node $i \in \overline{\mathrm{S}}$ is connected to exactly one hub node. The last constraint states that the set of access edges is in the cut of set S, or in other words, the chosen edges in set Q have one endpoint in S and the other one in $\overline{\mathrm{S}}$.

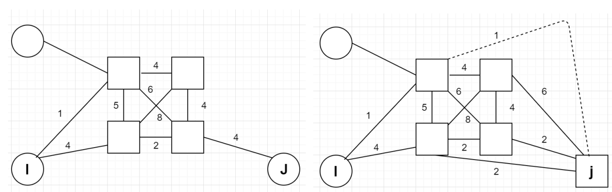
Now we construct a counter example which shows that the objective function z does not satisfy submodularity.

Example 1. *Assume that we want to route commodity $k = \{i,j\}$. Consider the set of hub nodes and access edge (S,Q) and (T,R) shown in Figure 3.3 such that $S \subseteq T$ and $Q \subseteq R$. Consider the set-up cost of hub nodes for S and $S \cup \{j\}$ is equal to $\sum_{i\in S} f_i = 5$ and $\sum_{i\in S\cup\{j\}} f_i = 6$. Moreover, the set-up cost of hub nodes for T and $T \cup \{j\}$ is equal to $\sum_{i\in T} f_i = 11$ and $\sum_{i\in T\cup\{j\}} f_i = 10$ and assume that $\alpha = 0.5$. Let $T_k = 8$ for each $k \in K$.*

Here we are adding a hub node j as an single element to both sets (S,Q) and (T,R). $(S,Q) \cup \{j\}$ and $(T,R) \cup \{j\}$ are also shown in Figure 3.3. Therefore, $\rho_j(S,Q) = z((S,Q) \cup \{j\}) - z(S,Q) = (8-6-6) - (8-10-5)$ and $\rho_j(T,R) = z((T,R) \cup \{j\}) - z(T,R) = (8-2-11) - (8-10-10)$. Note that in each network of Figure 3.3, *we choose the minimum cost path between i and j. $\rho_j(S,Q) = 3 \le \rho_j(T,R) = 7$ and it is in contradiction with the definition of submodularity.*



(a) (S,Q), $(S,Q) \cup \{j\}$

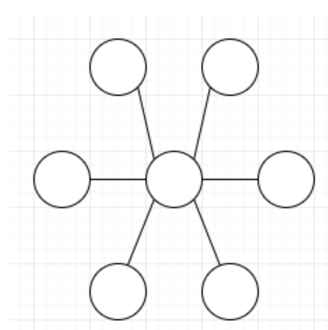


(b) (T,R), $(T,R) \cup \{j\}$

Figure 3.3: A counterexample for the access edge-based representation.

3.3 A star-based representation

Now, we propose the star-based representation for the HNDPSA. A *star* is a particular case of a tree with n nodes where a single node has degree $n-1$ called *root* and $n-1$ nodes with degree 1 called *ends*. Figure 3.4 shows a star with 7 nodes.



(a)

Figure 3.4: A star with 7 nodes

Again consider the undirected graph $G = (N, E)$. We denote the set of all stars over the graph G by S that takes its nodes and edges from G. Moreover, let $U = N \cup S$ be the ground set. For each nonempty subset $(T, Q) \subseteq U$ (Note that for different representations, we might use the same letters for defining different sets.), we assume that $T \subseteq N$ corresponds to set of hub nodes and $Q \subseteq S$. For $s \in Q \subseteq S$, let $R_s(Q)$ be the root of star s and $R(Q) = \bigcup_{s \in Q} R_s(Q)$. Furthermore, let $E_s(Q)$ be the set of ends of star $s \in Q$ and $E(Q) = \bigcup_{s \in Q} E_s(Q)$. We define a set of all feasible paths for routing commodity $k = \{i, j\}$ such as $\overline{\mathrm{P_k}} := \{p_k = i \longrightarrow m \longrightarrow n \longrightarrow j : (m, n) \in R(Q) \times R(Q)\}$. Let $f(T, Q) = -\sum_{i \in T} f_i$ denotes the set-up cost for the set of hub nodes, and recall that $c_{p_k} = C_{p_k} W_k$. Let

$$c_k(T, Q) = \begin{cases} \min\limits_{p_k \in \overline{\mathrm{P_k}}} c_{p_k} & \overline{\mathrm{P_k}} \neq \emptyset \\ \\ \infty & \text{otherwise} \end{cases} \tag{5}$$

denotes the transportation cost for commodity k. As in the access edge-based representation we define $h(T, Q) = \sum_{k \in K} T_k(T, Q) - \sum_{k \in K} c_k(T, Q)$ such that $T_k(T, Q)$ denote the amount of revenue gained from routing flows through hub roots of the subset of stars T. We define the objective function over the subset $(T, Q) \subseteq U$ as follows.

$$z(T, Q) = h(T, Q) + f(T, Q) \tag{6}$$

Then, the HNDPSA can be represented as the problem of finding the set of hub nodes $T \subseteq N$ and the set of stars $Q \subseteq S$ to maximize $z(S, R)$, i.e.,

$$\max_{(T,Q) \subseteq U} \{z(T, Q) : |T| \leq p, \;\; |\delta(i)| = 1 \; \forall i \in E(Q) \setminus E(Q) \cap R(Q), T = R(Q)\} \tag{7}$$

The first constraint restricts the cardinality of T to at most p ($p \leq |N|$) nodes. The second set of constraints guarantees the single allocation assumption. Recall that for any given subset of $M \subseteq N$, $\delta(M) = \{\{i,j\}; i \in M, j \in N \setminus M\}$ is the cut of set M. Therefore, the second set of constraints ensures that the number of incidence edges of end nodes of each star, which is not a root in any other stars is exactly one. The last constraint states that the set of chosen hub nodes is equal to the set of roots of chosen stars.

The following counter example proves that the objective function of the star set based representation does not satisfy submodularity. We will show that for commodity $k = \{i,j\}$ and $(T,Q) \subseteq (T',Q')$, the inequality $\rho(T,Q) \geq \rho(T',Q')$ over the function z is not satisfied.

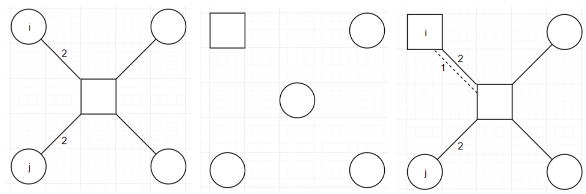
Example 2. *Assume that we have a graph with five nodes and we want to route commodity* $k = \{i,j\}$*. Consider the set of hub nodes and stars* (T,Q) *and* (T',Q') *shown in Figure 3.5 such that* $(T,Q) \subseteq (T',Q')$*. We assume that the set-up cost of hub nodes for* T *and* $T \cup \{R\{s\}\}$ *is* $\sum_{j \in T} f_j = 5$ *and* $\sum_{j \in T \cup \{R(\{s\})\}} f_j = 10$*. Moreover, the set-up cost of hub nodes for* T' *and* $T' \cup \{R\{s\}\}$ *is* $\sum_{j \in T'} f_j = 10$ *and* $\sum_{j \in T' \cup \{R(\{s\})\}} f_j = 15$*. We assume that the amount of revenue of routing commodity* k *is* $T_k = 8$*.*

We add star s *that is shown in Figure 3.5, which is a single root with zero ends, to both set* (T,Q) *and* (T',Q')*, and consider* $\alpha = 0.5$*. Now we calculate the value of* ρ *by adding star* s*.*

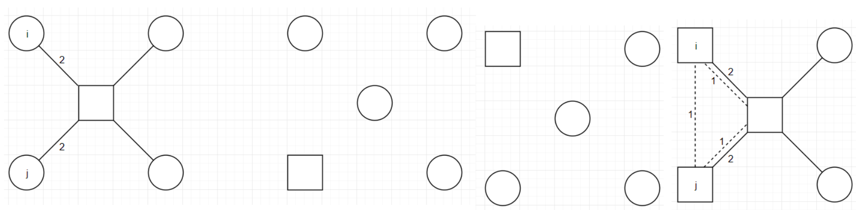
$$\rho_s(T,Q) = (8 - 3 - 10) - (8 - 4 - 9) = 0 \tag{8}$$

$$\rho_s(T',Q') = (8 - 1 - 15) - (8 - 3 - 14) = 1 \tag{9}$$

Here we have $\rho_s(T,Q) = 0 \leq \rho_s(T',Q') = 1$ *which is in contradiction with the definition of submodularity.*



(a) (T, Q) (b) s (c) $(T, Q) \cup \{s\}$



(d) (T', Q') (e) s (f) $(T, Q') \cup \{s\}$

Figure 3.5: A counterexample for the star-based representation.

3.4 A path-based representation

In this section, we propose path-based representation for the HNDPSA. Consider the graph $G = (N, E)$. Let P be the set of feasible paths in G, excluding the paths with edges between non-hub nodes. Let $S \subseteq N$ be a set of hub nodes, $S' \subseteq N$ be a set of activated non-hub nodes, $H \subseteq E$ be a set of hub edges, $A \subseteq E$ be a set of access edges, and $R \subseteq P$ be a set of paths. Also, let $U = N \cup E \cup P$ be a finite set that defines the ground set such that any subset of U is of the form of $(S, S', H, A, R) \subseteq U$. Note that here since we serve just the most profitable commodities, there could be some nodes in the original graph that are neither in the set S, nor in the S'.

Let $R_k \subseteq P$ be the subset of paths for commodity $k \in K$ such that $R = \bigcup_{k=1}^{|K|} R_k$. We denote the set of nodes of the paths in R by $N(R)$, and the set of edges of R by $E(R)$. In addition, let $N_s(R)$ be the set of hub nodes and $N_{s'}(R)$ be the set of non-hub nodes in the paths of R. Likewise, let $E_h(R)$ be the set of hub edges, and $E_a(R)$ be the set of access edges of paths in R. We define f and g as the set-up cost functions of the set of selected nodes and the set of selected edges, respectively as follows:

$$f(S, S') = -\sum_{j \in S} f_j - \sum_{j \in Q} f_j^{'}, \tag{10}$$

$$g(H, A) = -\sum_{e \in H} g_e - \sum_{e \in A} g_e^{'}. \tag{11}$$

Also, we define h as the the profit function associated with serving commodities $k \in K$ as:

$$h(R) = \sum_{k \in K} h_k(R) = \sum_{k \in K} \left(\max_{p_k \in R_k} \{T_k - W_k C_{p_k}\} \right)^+, \tag{12}$$

where $(x)^+ = \max\{x, 0\}$. Moreover, we define the total profit as

$$z(S, S', H, A, R) = h(R) + g(H, A) + f(S, S'), \tag{13}$$

where we consider that $z(\emptyset) = 0$.

The HNDPSA can now be stated as the problem of selecting a set of nodes, edges and paths $(S, H, A, R) \subseteq U$ such that $z(S, H, A, R)$ is maximum, i.e.,

$$\begin{aligned} \max_{(S,S',H,A,R) \subseteq U} \{ z(S, S', H, A, R) : |S| \leq p, |H| \leq q, N_s(R) = S, N_{s'}(R) = S', \\ E_h(R) = H, E_a(R) = A, S \cap S' = \emptyset, H \cap A = \emptyset, |\delta(\{i\})| = 1, \forall i \in S' \}. \end{aligned} \tag{14}$$

This is the path-based combinatorial representation for the HNDPSA. The first two constraints ensure that the cardinality constraints on the set of hub nodes and hub edges. The third, fourth, fifth and sixth constraints guarantee that the set of selected hub nodes, non-hub nodes, hub edges, and access edges should be exactly the same with the nodes and edges of the chosen paths R, respectively. The last set of constraints guarantees that for each non-hub node $i \in S'$, we select exactly one access edge incident to i. We note that these constraints only affect the selected non-hub nodes since we are maximizing the profit and we are not forced to serve all commodities.

3.4.1 Submodular properties of HNDPSA for the path-based representation

Now, we prove that the objective function of the proposed path-based representation satisfies the submodularity property.

Proposition 3.1. *(Submodularity of the objective function)*

(a) $h(R)$ *is submodular and non-decreasing.*

(b) $z(S, S', H, A, R)$ *is submodular.*

Proof. (a) Let $(S, S', H, A, R) \subseteq (T, T', L, B, Q) \subseteq U$, with $S \subseteq T \subseteq N$, $S' \subseteq T' \subseteq N$, $H \subseteq L \subseteq E$, $A \subseteq B \subseteq E$, $R \subseteq Q \subseteq T$. Without loss of generality, we assume that $q \in T \setminus Q$ is a path with endpoints of an arbitrary commodity $k = \{i, j\}$. For $R_k \neq \emptyset$:

$$\begin{aligned} h_k(R_k \cup \{q\}) - h_k(R_k) &= \max_{p_k \in R_k \cup \{q\}} h_{p_k} - \max_{p_k \in R_k} h_{p_k} \\ &= \max\{0, h_q - \max_{p_k \in R_k} h_{p_k}\} \\ &\geq \max\{0, h_q - \max_{p_k \in Q_k} h_{p_k}\} \\ &= \max_{p_k \in Q_k \cup \{q\}} h_{p_k} - \max_{p_k \in Q_k} h_{p_k} \\ &= h_k(Q_k \cup \{q\}) - h_k(Q_k) \end{aligned} \tag{15}$$

where the inequality follows from $\max_{p_k \in R_k} h_p \leq \max_{p_k \in Q_k} h_p$. Moreover, the inequality $\max_{p \in Q} h_p - \max_{p \in R} h_p \geq 0 \quad \forall Q \subseteq T$ shows that h_k is non-decreasing.

As a consequence of Lemma 2.3, with taking the summation overall commodities, we obtain that $h(R)$ is a submodular and non-decreasing function.

(b) Consider again the sets S, T, H and L. By adding node $i \in N \setminus T$ to set S, and edge $e' \in E \setminus L$ to set H, we have:

$$
\begin{aligned}
& -\sum_{j \in S \cup \{i\}} f_j + \sum_{j \in S} f_j \\
& = -f_i \geq -f_i \\
& = -\sum_{j \in T \cup \{i\}} f_j + \sum_{j \in T} f_j
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
& -\sum_{e \in H \cup \{e'\}} g_e + \sum_{e \in H} g_e \\
& = -g_{e'} \geq -g_{e'} \\
& = -\sum_{e \in L \cup \{e'\}} g_e + \sum_{e \in L} g_e
\end{aligned}
\tag{17}
$$

With the similar proof f'_j and g'_e are submodular over $S' \subseteq T' \subseteq N$ and $A \subseteq B \subseteq E$. Therefore, according to Lemma 2.2 f and g satisfy submodularity over $(S, S', H, A, R) \subseteq (T, T', L, B, Q) \subseteq U$. Then, we have zero difference in the value of the function if we add a single element from the sets T or E, and finally for the function g, we can have a difference in the value of the g by just adding an element from set E. As a result, by lemma 2.3, z is a submodular function over the set U. □

Chapter 4

Worst-Case Bounds for a Greedy Heuristic

In this section, we propose a greedy heuristic for the HNDPSA based on the representation 14 with considering all the set-up cost functions equal to zero. Moreover, we provide the time complexity of the greedy algorithm and present the worst-case bounds.

4.1 A Greedy Heuristic for the path-based representation of HNDPSA

Let $q \in P$ and $M_q = q \cup N_s(q) \cup E_h(q) \cup N_{s'}(q) \cup E_a(q) \subseteq U$. The following proposition states that we can evaluate a submodular function's decremental value by adding a subset to the solution set rather than a single element (see Ortiz-Astorquiza et al., 2017).

Proposition 4.1. *For* $(S, S', H, A, R) \subseteq (T, T', L, B, Q) \subseteq U$ *and any subset* $M \subseteq U \setminus (T, T', L, B, Q)$, $\rho_M(S, S', H, A, R) \geq \rho_M(T, T', L, B, Q)$.

Let $(S^t, S'^t, H^t, A^t, R^t)$ be the solution at iteration t such that S^t is a set of hub nodes, S'^t is a set of non-hub nodes, H^t is a set of hub edges, A^t is a set of access edges, R^t is a set of paths.

The idea of algorithm 1 is to select a subset of commodities to be served that gives the most profit. To do so, we choose a feasible path q and its corresponding subset M_q that gives the maximum improvement at each iteration. We then add M_q to the solution set. Note that we must respect the capacity of the network (at most p hub nodes an q hub edges) on the number of hub nodes at each iteration.

Algorithm 1 Greedy Heuristic for the HNDPSA

Set $(S, S', H, A, R)^0 \leftarrow \emptyset$, ${M_q}^0 \leftarrow \emptyset$, $U^0 = (N^0, E^0, P^0) \leftarrow U$, $S' \leftarrow \emptyset$, $A \leftarrow \emptyset$, $\rho_0 \leftarrow 0$ and $t \leftarrow 1$
while $t \leq |K|$ **do**
  Select $q^t \subseteq P^{t-1}$ such that $\rho_{{M_q}^t} = \max_{M_q \in U^{t-1}} \rho_{M_q}(S, S', H, A, R)^{t-1}$, $N_s(q^t) \notin S'$ and $E_h(q^t) \notin A$.
  Set $\rho_{t-1} \leftarrow \rho_{{M_q}^t}$
  if $\rho_{t-1} \leq 0$ **then**
    Stop with $(S, S', H, A, R)^{t-1}$ as the greedy solution
  else
    Set $S' \leftarrow N_{s'}(q^t)$ and $A \leftarrow E_a(q^*)$
    $(S, S', H, A, R)^t \leftarrow (S, S', H, A, R)^{t-1} \cup {M_q}^t$
    $U^t \leftarrow U^{t-1} \setminus \{q^t\}$
  end if

  for $i \in N(q^t)$ such that $i \in S'$ **do**
    for $e \in E$ such that $i \in e$ **do**
      if $e \notin E(q^t)$ **then**
        $U^t \leftarrow U^t \setminus e$
      end if
    end for
  end for
  $t \leftarrow t + 1$
end while
Set $(S, S', H, A, R)^t$ as the greedy solution.
STOP

Moreover, if a node is selected as a hub (non-hub), we keep it as a hub (non-hub) during all the remaining iterations. Also, we handle the single assignment assumption such that for the non-hub nodes of the chosen path q^t at iteration t, we remove all their adjacent edges in the original graph G except $E_a(q^t)$. We continue iterating until either we cannot further improve the solution or we serve all commodities. After choosing the best path at each iteration, we update the hub nodes, non-hubs nodes, hub edges, and access edges at each iteration. Our greedy algorithm is summarized in Algorithm 1.

Proposition 4.2. *The complexity of Algorithm 1 is* $O(|K|^2|N|^2)$.

Proof. The set ${M_q}^t$ at iteration t can be found efficiently by solving the series of longest-path problems on an auxiliary acyclic directed graph for each commodity $k = \{i, j\} \in K$ that is not served yet. We define auxiliary directed acyclic graphs $G_k(V_k, E_k)$ as a multi-layer graph for $U^{t-1} = (N^{t-1}, E^{t-1}, P^{t-1})$ at iteration t for commodity k as follows.

For constructing $G_k(V_k, E_k)$, we define the first layer as node i and the last layer as node j. If any node of the original graph G is not in the set S'^{t-1}, we copy it twice between i, j as two intermediate layers of hub nodes since we might pass through more than one hub node for routing a commodity (see Figure 4.1.).

Assume that the amount of each commodity is equal to one (i.e., $W_k = 1$).For each $e \in E_k$ we define the weight as follows. From i to the node $\{h_{1m}\}_{m=1}^n$ of the first layer, we have $w_{ih_{1m}} = -d_{ih_{1m}}$. From $\{h_{1m}\}_{m=1}^n$ of the first layer to $\{h_{2m'}\}_{m'=1}^n$ of the second layer we have $w_{h_{1m}h_{2m'}} = -\alpha d_{h_{1m}h_{2m'}}$. And finally, from $\{h_{2m}\}_{m=1}^n$ of the second layer to j, we have $w_{h_{2m}j} = -d_{h_{2m}j}$. We should note that having an arc between two nodes of the same layer is forbidden. The output of $G_k(V_k, E_k)$ is shown in the Figure 4.1.

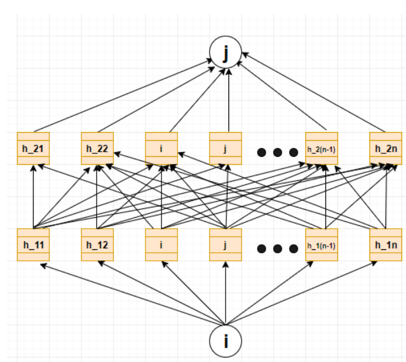


Figure 4.1: $G_k(V_k, E_k)$ designed for serving commodity k={i,j}

We consider the following restrictions in designing $G_k(V_k, E_k)$.

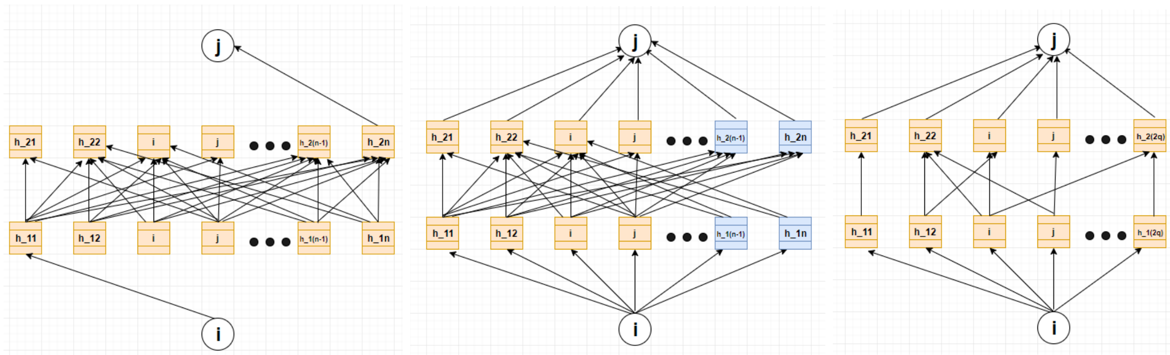
I: if either i or j are already in the set S'^{t-1}, we draw an arc just between them and their associated hubs (Figure 4.1, part (a)).

II: if the cardinality of the set of opened hubs has reached $p-1$, we draw arcs from the first intermediate layer of hubs just to the ones in the second layers, which already are in the set S (Figure 4.1, part (b)).

III: if the cardinality of the set of opened hub arcs has reached q between two intermediate layers, we will draw nodes that already in S and hub arcs between them in H(Figure 4.1, part (c)).

Note that case II and case III are considered to avoid violating cardinality constraints on the hub nodes and hub edges. This operation is taking $O(|K||N|^2)$ for each commodity since for each auxiliary graph, we enumerate at most $O(|N|)$ number of nodes and at most, $O(|N|^2)$ number of arcs for the set of commodities with cardinality $|K|$. We then chose the best path q^t and its corresponding subset ${M_q}^t$ via solving a series of the longest path problem for auxiliary graphs using topological ordering algorithm in $O(|N|^2)$ time. Then, we store the best path for serving each commodity in a set, and a the end, we choose the commodity among this set that gives the most profit. We do this at most for $|K|$ iterations. Therefore, the Algorithm 1 runs in $O(|K|^2|N|^2)$. □

Figure 4.2 shows graph $G_k(V_k, E_k)$ for three different cases I, II and III that can happen for commodity $k = \{i, j\}$. Note that in the case I, the origin and the destination nodes are already in the set S'. In the case II, we show the intermediate hub layers that are already in S by orange and the ones that can be opened in the current iteration with blue. We do not draw any arc from the blue nodes to the blue nodes (i.e., we cannot open two new hubs). For the case III, we should not open a new hub arc, and we use the ones that are already opened and we just draw their associated hub nodes.



(a) I (b) II (c) III

Figure 4.2: $G_k(V_k, E_k)$ for commodity k={i,j} with restrictions.

4.2 A worst-case bound for the greedy heuristic

Here we present a worst-case bound for Algorithm 1 using the following propositions. (see Nemhauser et al., 1978).

Let z^G denote the value of the solution constructed by the greedy heuristic.

$$z^G = z(\emptyset) + \rho_0 + ... + \rho_{t-1} \qquad t \leq |K|. \tag{1}$$

Let z^* be the value of an optimal solution to problem 14. We may assume $t \geq 1$ to exclude the trivial problem with $z^* = z^G = z(\emptyset)$.

Let $C(\theta)$ be the class of submodular set functions satisfying $\rho_{M_q}(S, S', H, A, R) \geq -\theta$, for all $(S, S', H, A, R) \subset U$ and $M_q \in U \setminus (S, S', H, A, R)$.

Proposition 4.3. *For all* $(S, S', H, A, R), (T, T', L, B, Q) \subseteq U$ *with* $N_s(R) = S$, $N_{s'}(R) = S'$, $E_a(R) = A$, $E_h(R) = H$, *and* $N_s(Q) = P$, $N_{s'}(Q) = T'$, $E_a(Q) = B$, $E_h(Q) = L$. *Then*

$$z(T, T', L, B, Q) \leq z(S, S', H, A, R) + \sum_{q \in Q \setminus R} \rho_{M_q}(S, S', H, A, R) + |Q \setminus R|\theta. \tag{2}$$

Proof. Let $(S, S', H, A, R), (T, T', L, B, Q) \subseteq U$ with $|R \setminus Q| = \alpha$ and $|Q \setminus R| = \beta$ such that $N_s(R) = S$, $N_{s'}(R) = S'$, $E_a(R) = A$ and $E_h(R) = H$, and also $N_s(Q) = P$, $N_{s'}(Q) = T'$, $E_a(Q) = B$ and $E_h(Q) = L$. Consider the set M_q with $q \in Q \setminus R$ and similarly for $q' \in R \setminus Q$ consider $M_{q'}$, as defined before. Then

$$
\begin{aligned}
& z((S, S', H, A, R) \cup (T, T', L, B, Q)) - z(S, S', H, A, R) = \\
& z((S, S', H, A, R) \cup M_{q_1}) - z(S, S', H, A, R) \\
& + z((S, S', H, A, R) \cup M_{q_1} \cup M_{q_2}) - z((S, S', H, A, R) \cup M_{q_1}) + ... \\
& + z((S, S', H, A, R) \cup M_{q_1} \cup M_{q_2} \cup ... \cup M_{q_\alpha}) - z((S, S', H, A, R) \cup M_{q_1} \cup M_{q_2} \cup \\
& ... \cup M_{q_{\alpha-1}}) = \sum_{i=1}^{\alpha} \rho_{M_{q_i}}((S, S', H, A, R) \cup M_{q_1} \cup M_{q_2} \cup ... \cup M_{q_{i-1}}) \\
& \leq \sum_{t=1}^{\alpha} \rho_{M_{q_t}}(S, S', H, A, R) = \sum_{q \in Q \setminus R} \rho_{M_q}(S, S', H, A, R),
\end{aligned}
\tag{3}
$$

which shows

$$
\begin{aligned}
& z((S, S', H, A, R) \cup (T, T', L, B, Q)) - z(S, S', H, A, R) \leq \\
& \sum_{q \in Q \setminus R} \rho_{M_q}(S, S', H, A, R)
\end{aligned}
\tag{4}
$$

and we obtain similarly

$$
\begin{aligned}
& z((S, S', H, A, R) \cup (T, T', L, B, Q)) - z(T, T', L, B, Q) \geq \\
& \sum_{q' \in R \setminus Q} \rho_{M_{q'}}((T, T', L, B, Q) \cup (S, S', H, A, R) \setminus M_{q'}).
\end{aligned}
\tag{5}
$$

Subtracting the above inequalities results in the following:

$$z(T,T',L,B,Q) \le z(S,S',H,A,R) + \sum_{q\in Q\setminus R} \rho_{M_q}(S,S',H,A,R) \tag{6}$$

$$- \sum_{q'\in R\setminus Q} \rho_{M_{q'}}((T,T',L,B,Q) \cup (S,S',H,A,R) \setminus M_{q'}).$$

Then,

$$z(T,T',L,B,Q) \le z(S,S',H,A,R) + \sum_{q\in Q\setminus R} \rho_{M_q} z(S,S',H,A,R) + \beta\theta \tag{7}$$

for $\rho \ge -\theta$. □

Proposition 4.4. *Suppose $z \in C(\theta), \theta \ge 0$, and the greedy heuristic stops after k^* iterations, then the corresponding $\{\rho_i\}_{i=0}^{k^*-1}$ satisfy*

$$z^* \le z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + |K|\rho_t + t\theta, \qquad t = 0, ..., k^* - 1 \tag{8}$$

and also

$$z^* \le z(\emptyset) + \sum_{i=0}^{k^*-1} \rho_i + k^*\theta, \qquad if \quad k^* < |K| \tag{9}$$

Proof. From 4.3 we have

$$z(T,T',L,B,Q) \le z(S,S',H,A,R) + \sum_{q\in Q\setminus R} \rho_{M_q}(S,S',H,A,R) + |R \setminus Q|\theta. \tag{10}$$

Let (T,T',L,B,Q) be the optimal solution set of the problem 14 and (S,S',H,A,R) be the set $(S,S',H,A,R)^t$ constructed in t-th iteration of the greedy heuristic. We can show

$$z^* = z(T,T',L,B,Q), \qquad \rho_{M_q}(S,S',H,A,R) \le \rho_t, \quad \rho_t \ge 0, \tag{11}$$

$$|(S,S',H,A,R)^t - (T,T',L,B,Q)| \leq t, \quad \theta \geq 0, \tag{12}$$

$$|(T,T',L,B,Q) - (S,S',H,A,R)^t| \leq |K|,$$

and

$$z(S,S',H,A,R)^t = z(\emptyset) + \sum_{i=0}^{t-1} \rho_i, \tag{13}$$

we obtain

$$z^* \leq z(\emptyset) + \sum_{i=0}^{t-1} \rho_i + |K|\rho_t + t\theta \qquad t = 0, ..., k^* - 1 \tag{14}$$

If $k^* \leq |K|$, taking $(S,S',H,A,R) = (S,S',H,A,R)^{k^*}$ yields

$$z^* \leq z(\emptyset) + \sum_{i=0}^{k^*-1} \rho_i + k^*\theta \tag{15}$$

as $\rho_{k^*} \leq 0$ □

Proposition 4.5. *For nondecreasing z, if the greedy heuristic is applied to problem 14, and the greedy heuristic stops after $k^* < |K|$ steps, the greedy solution is optimal.*

Proof. The proof is trivial. □

Proposition 4.6. *By applying greedy heuristic to problem 14, we obtain*

$$\frac{z^* - z^G}{z^* - z(\emptyset)} \leq \frac{|K| - 1}{|K|} \tag{16}$$

Proof. For $t = 0$, the inequality 8 yields $z - z(\emptyset) \leq |K|\rho_0 \leq |K|(z^G - z(\emptyset))$ or, equivalently,

$$\frac{z^* - z^G}{z^* - z(\emptyset)} \leq \frac{|K| - 1}{|K|} \tag{17}$$

□

Nemhauser et al. (1978) also proved that it is possible to make the bound of Proposition 4.6 tighter only for very large values θ.

Proposition 4.7. *If the greedy heuristic terminates after t^* iterations, then*

$$\frac{z^* - z^G}{z^* - z(\emptyset) + |K|\theta} \leq \frac{t^*}{|K|}\left(\frac{|K|-1}{|K|}\right)^{t^*} \leq \left(\frac{|K|-1}{|K|}\right)^{|K|} \leq \frac{1}{e} \tag{18}$$

and considering $z(\emptyset)$=0

$$\frac{z^* - z^G}{z^* + |K|\theta} \leq \frac{t^*}{|K|}\left(\frac{|K|-1}{|K|}\right)^{t^*} \leq \left(\frac{|K|-1}{|K|}\right)^{|K|} \leq \frac{1}{e} \tag{19}$$

$$1 - \left(\frac{|K|-1}{|K|}\right)^{|K|} \geq \frac{e-1}{e} \tag{20}$$

According to the bound Nemhauser et al. (1978) proved for the k-median problem, they achieved the value of the worst-case bound on the ground set they defined for the problem. The following proposition shows that we can provide a submodular representation for HN-SPSA defined on the set of commodities K as the ground set since we achieved the bound over this set's cardinality.

Proposition 4.8. *(see Nemhauser et al., 1978) A submodular function v on a set of subset of E is given. Let $\{Q_j\}, j \in N$ be a collection of subsets of E. If*

(a) v is nondecreasing or

(b) The $\{Q_j\}$ are disjoint,

Then $z(S) = v(\bigcup_{j \in S} Q_j)$ is a submodular set function on the set of subset of N.

According to proposition 4.8, let $\{Q_k\}, k \in K$ denote the set of paths between the endpoint of commodity k. We already proved that the function z is non-decreasing when the total set-up costs are considered zero (i.e. h is non-decreasing). Using Proposition 4.8 we can show a function $v(S) := z(\bigcup_{k \in S} Q_k), S \subseteq K$ is a submodular function on the set of subsets of K.

Proposition 4.8 clarifies that we can have a submodular presentation for HNDPSA considering the set of commodities K as a ground set. Recalling Nemhauser et al. (1978), the cardinality of commodities will help us for the theoretical bound we are finding for the proposed greedy heuristics.

Chapter 5

Conclusion

This thesis addressed an important class of problems in hub facility location called hub network design problem with profits and single assignments (HNDPSA). It contributed to the current literature by providing combinatorial representation for HNDPSA for the first time and proving whether the objective function satisfies the submodular property. These contributions show the power of combinatorial optimization problems and submodular property to cover the different classes of both facility location and hub location problems. Also, the potential of obtaining the theoretical worst-case bound over a greedy heuristic algorithm to approximate the optimal solution motivated us to study HNDPSA.

In Chapter 2, we addressed the basic definitions and assumptions of HNDPSA. We first presented the basic assumptions that hold throughout the thesis. We discussed three different versions of the hub location problems: single allocation, multiple allocation and r-allocation assumptions that can be hold for HLPs. We also pointed out the differences and similarities between them. We then brought the definition of the combinatorial optimization problem and submodular property with its fundamental features. We reviewed the main references related to submodularity. Then, we provided a brief review of the HLPs and single allocation assumption over it.

In Chapter 3, we introduced fundamental definitions along with three different combinatorial representations for HNDPSA. We provided counterexamples that showed the objective function of the first two representations did not satisfy submodularity. We proved that the objective function of the third representation satisfied the submodularity property.

Finally, in Chapter 4, we provided a greedy heuristic algorithm to approximate the optimal solution based on the third representation. We obtained the worst-case performance results for the algorithm by considering set-up cost functions equal to zero. We also proved that the algorithm runs in polynomial time, and we proposed an approximation ratio. We presented a sharper bound by considering zero value for the objective function when the solution set is empty.

Beyond the contributions mentioned, our results have a strong potential for extension and more investigation, such as providing the mixed-integer formulation of HNDPSA based on this combinatorial representation. Another important step towards the field's growth is to compare our greedy heuristic results with the optimal solution by running experiments to see the quality of the approximation and the power of the theoretical bound.

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