# Quantization of Calogero-Painlevé system and Multi-particle quantum Painlevé equations $II - VI$

Fatane Mobasheramini

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By: **Ms. Fatane Mobasheramini**

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*Dr. Ronald Stern*

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*Dr. Marco Bertola*

Approved by ______________________
*Dr. Cody Hyndman, Chair*
*Department of Mathematics and Statistics*

April 13, 2021 ______________________
*Dr. Pascale Sicotte, Dean*
*Faculty of Art and Science*

# Abstract


**Quantization of Calogero-Painlevé system**
**and Multi-particle quantum Painlevé equations $II - VI$**



**Fatane Mobasheramini, Ph.D.**
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In this dissertation, we implement canonical quantization within the framework of the so–called Calogero-Painlevé correspondence for isomonodromic systems. The classical systems possess a group of symmetries and in the quantum version, we implement the (quantum) Hamiltonian reduction using the Harish-Chandra homomorphism. This allows reducing the matrix operators to Weyl–invariant operators on the space of eigenvalues. We then consider the scalar quantum Painlevé equations as Hamiltonian systems and generalize them to multi–particle systems; this allows us to formulate the multi–particle quantum time–dependent Hamiltonians for the Schrödinger equation $\hbar\partial_t\Psi = H_J\Psi$, $J = II, \ldots, VI$.

We then generalize certain integral representations of solutions of quantum Painlevé equations to the multi-particle case. These integral representations are in the form of special $\beta$ ensembles of eigenvalues and can be constructed for all the Painlevé equations except the first one. They play the role, in the quantum world, of rational solutions in the classical world.

These special solutions exist only for particular values of the quantum Hamiltonian reduction parameter (or coupling constant) $\kappa$. We elucidate the special values of the corresponding parameters appearing in the quantized Calogero–Painlevé equations $II - VI$.

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*"The knowledge of anything, since all things have causes, is not acquired or complete unless it is known by its causes."*

Avicenna (1020 CE)

# Contents

# Chapter 1

# Introduction

The six Painlevé equations were discovered by [33, 14, 15] as a result of the search for second–order nonlinear ordinary differential equations in the complex plane with Painlevé property which implies the only movable singularities are the poles (a very rare property of nonlinear equations). These equations are only solvable in terms of special functions, and it has been proven that they admit exact solutions such as hypergeometric solutions, algebraic or rational solutions. Ever since their discovery, these equations were extensively studied by mathematicians and physicists, and still, they are of great importance because of their applications in different areas. These equations can be regarded as completely integrable equations due to the fact that they arise as reductions of the soliton equations [12], which are solvable by inverse scattering. Their other applications include quantum gravity and string theory [5, 16, 6], topological field theories [7], random matrices [35, 13], $\beta$–models [4], and stochastic growth processes [25].

Historically, the Hamiltonian structure of the Painlevé equations was studied by [27, 8, 32] as they all can be written as a time-dependent Hamiltonian system

$$\ddot{q} = -V(q;t).$$

For some potential function $V$ of dependent and independent variables.
This perspective makes the study of Painlevé equations as a classical integrable system slightly complicated. However, these equations admit a natural generalization in terms of a canonical transformation to the case of multi-particle with an interaction of Calogero type [26] (rational, trigonometric, or elliptic). This work which is finalized by K. Takasaki [34] yields a new system of multi-component Hamiltonian operators named Calogero-Painlevé correspondence. This system was obtained by applying a change of variable to Painlevé equations that converts them to differential equations of the form

$$\ddot{q} = -\partial_q V(q,t).$$

Here, we list the Hamiltonian operators in this system:

$$
\begin{aligned}
\tilde{H}_I =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}-2q_j^3-tq_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}\\
\tilde{H}_{II} =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}-\frac{1}{2}\Big(q_j^2+\frac{t}{2}\Big)^2-\alpha q_j\right)+g_4^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}\\
\tilde{H}_{III} =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}-\frac{\alpha}{4}e^{q_j}+\frac{\beta t}{4}e^{-q_j}-\frac{\gamma}{8}e^{2q_j}+\frac{\delta t^2}{8}e^{-2q_j}\right)+g_4^2\sum_{j\neq k}\frac{1}{\sinh^2\big((q_j-q_k)/2\big)}\\
\tilde{H}_{IV} =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}-\frac{1}{2}\Big(\frac{q_j}{2}\Big)^6-2t\Big(\frac{q_j}{2}\Big)^4-2(t^2-\alpha)\Big(\frac{q_j}{2}\Big)^2+\beta\Big(\frac{q_j}{2}\Big)^{-2}\right.\\
&\left.+g_4^2\sum_{j\neq k}\left(\frac{1}{(q_j-q_k)^2}+\frac{1}{(q_j+q_k)^2}\right)\right)\\
\tilde{H}_V =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}-\frac{\alpha}{\sinh^2(q_j/2)}-\frac{\beta}{\cosh^2(q_j/2)}+\frac{\gamma t}{2}\cosh(q_j)+\frac{\delta t}{8}\cosh(2q_j)\right)\\
&+g_4^2\sum_{j\neq k}\left(\frac{1}{\sinh^2\big((q_j-q_k)/2\big)}+\frac{1}{\sinh^2\big((q_j+q_k)/2\big)}\right)\\
\tilde{H}_{VI} =&\sum_{j=1}^{n}\left(\frac{p_j^2}{2}+\sum_{l=0}^{3}g_l^2\wp(q_j+\omega_l)\right)+g_4^2\sum_{j\neq k}\Big(\wp(q_j-q_k)+\wp(q_j+q_k)\Big)
\end{aligned}
\tag{1.1.1}
$$

where $\wp$ is the Weierstrass function, $\alpha$, $\beta$, $\gamma$, $\delta$ are arbitrary constants, $g_0,\dots,g_4$ are the coupling constants, and the coefficients $\omega_l$, $l=0,...,3$ take the following values

$$(\omega_0,\omega_1,\omega_2,\omega_3)=(0,1/2,-(1+\tau)/2,\tau/2)$$

for $\tau$ being the modular parameter that is considered to play the role of the independent time variable $t$.
The integrability of the system (1.1.1) is proved in [3] as the isomonodromic formulation in terms of a $2N\times 2N$ Lax pair matrices, where $N$ is the number of particles.

This means there are $2\times 2$ block (of $N$ components) matrices * $A$ and $B$, such that the system

$$
\begin{cases}
\hbar\dfrac{\partial}{\partial z}\Phi(z;t)=A(z;q,p,t)\Phi(z;t)\\[2ex]
\hbar\dfrac{\partial}{\partial t}\Phi(z;t)=B(z;q,p,t)\Phi(z;t)
\end{cases}
$$

is compatible and hence has a joint fundamental solution $\Psi(z;t)$. This implies that the

*Each of the components of the matrices $A$ and $B$ are $N\times N$ matrices themselves.

matrices $A, B$ satisfy the zero-curvature equation:

$$\hbar\partial_t A - \hbar\partial_z B + [A, B] = 0$$

for $\hbar$ a formal parameter in $\mathbb{C}$. Through this isomonodromic formulation, a system of multi-component Hamiltonian operators corresponding to the Calogero-Painlevé equations is obtained. These equations are presented in chapter 3 together with the full detail and the result of the work of [3].

On a parallel track, the integrability of the classical Calogero-Painlevé equations has brought an ever-increasing curiosity about their quantization. As a result, the discussion about the integrability of such a system and their corresponding solutions has appeared in the work of mathematicians and mathematical physicists such as H. Nagoya [31], K. Okamoto [32], A. Zabrodin and A. Zotov [37].

In [37], Zabrodin and A. Zotov show that Calogero-like Painlevé equations I-VI can be represented in the form of the non-stationary Schrödinger equation in imaginary time.

The statement of their theory proceeds as follows. For all the six Calogero-Painlevé equations I-VI in the classic form with the standard Hamiltonian operators $H(p, q, t)$ , there is a system of linear problems

$$\begin{cases} \partial_z \Psi = U(z, t, q, \dot{q})\Psi \\ \\ \partial_t \Psi = V(z, t, q, \dot{q})\Psi \end{cases}$$

where $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, and $U$, and $V$ satisfy the zero-curvature equation

$$\partial_t U - \partial_z V + [U, V] = 0$$

and the function $\Psi = e^{\int H(\dot{q}, q, t')dt'}\psi_1$ satisfies the non-stationary standard Schrödinger equation

$$\partial_t \Psi = H\Psi$$

with

$$H = \Big(\frac{1}{2}\partial_z^2 + V(z, t)\Big). \tag{1.1.2}$$

The equation (1.1.2) should be viewed as a natural quantization of the classical Calogero-like Hamiltonian operator

$$H(p, q, t) = \frac{p^2}{2} + V(q; t).$$

By extending the Calogero-Painlevé correspondence to the auxilary linear problems associated to the Painlevé equations, the authors of this paper [37] formulate the Hamiltonian system that has the interpretation as "quantized" Hamiltonian system in one particle case. One of the motivations of this thesis is to provide the Hamiltonian system corresponding to the quantum Calogero-Painlevé correspondence (a canonical quantization of the Hamiltonians in [3]) in the case of $N$ particles.

The logic that we follow is similar to [3]: we start from the formulation of the multi-particle Hamiltonians in loc. cit. where the classical position and momenta are non-commutative symbols (i.e. matrices) with a canonical Poisson bracket. The canonical quantization is then of the form

$$q_{ij} \longrightarrow q_{ij} \quad , \quad p_{ij} \longrightarrow \hbar \frac{\partial}{\partial q_{ji}}$$

in terms of the canonical coordinates $p$ and $q$.

The construction of the Hamiltonians is subordinated to the requirement that the resulting non-commutative isomonodromic equations that express the compatibility of the matrices remain the same. This implies a certain ordering of the operators in the Hamiltonians which is not the same as the one presented in [3] (where the matrices were classical). Through these computations, we adopt the definition of the Harish-Chandra homomorphism that plays an important key in the theory of quantum Hamiltonian reduction. As a result of this quantization, we obtain a multi-particle quantized Hamiltonian system that satisfies the Schrödinger equation of a similar form to (1.1.2).

Further in this project, we compare the result of this quantization to what H. Nagoya [31] introduces as quantum Painlevé Hamiltonian system for a single particle wave function with coordinate $z$.

We generalize these Hamiltonian operators to the case of $N$ particles with coordinates $z_\rho$, $\rho = 1, ..., N$.

We also extend the integral representation of solutions in [31] to the multi-component integral solution of the Schrödiner equation for $N$-particle quantum Painlevé equations. These integral representations are defined only for quantum Painlevé equation $II-VI$; these solutions should be understood as the quantum counterpart of rational solutions and it is well known that the first Painlevé equation does not admit rational solutions, which heuristically explains the absence of integral representations for solutions thereof. Intriguingly, these integral representations are presented as some type of $\beta$ integrals that appear in other areas such as conformal field theory [22], $\beta$-ensembles [2], the theory of orthogonal polynomials, and hypergeometric functions.

Finally, we show that under some constraints on the parameters of the Hamiltonian operators obtained in chapter 3 and chapter 4, these generalized Nagoya integrals provide solutions for the quantization of the multi-particle Hamiltonian systems described in the first part.

# Outline

- The abstract theories and fundamental definitions required to understand this work will be presented in chapter 2.

- In chapter 3 we first present the isomonodromic formulation (3.1.6) for Calogero-Painlevé system. Moreover, we introduce the quantization that later on, will be applied to the Hamiltonian operators corresponding to the Calogero-Painlevé correspondence. Furthermore, the definition of the Harich-Chandra homomorphism [10] will be presented to help us simplify the quantization in terms of the eigenvalues of the matrix operators. At the end of this chapter, we state the quantized Hamiltonian system for Calogero-Painlevé correspondence.

- At the beginning of the chapter 4, the single-particle Hamiltonian system corresponding to the quantum Painlevé equations will be presented. This system that is defined by H. Nagoya [31], satisfies the Schrödinger equation $\hbar\partial_t\Psi = H\Psi$ with $\Psi$ an integral solution. We succeed in generalizing this system to $N$ particles and construct the new Hamiltonian system with integral solutions corresponding to each equation.

- In chapter 5, we conclude that the two systems from the previous chapters are equal under some constraints on the parameters of the Hamiltonian operators from both systems.

- Finally, in chapter 6, we provide a summary of the works that have been done in the same stream, and we discuss the connections, overlapping topics, and comparisons between these works and our project.

# Chapter 2

# Abstract theory

## 2.1 Painlevé equations

### 2.1.1 Singularities of ODEs

In general, there are two types of singular points for ordinary differential equations (ODEs): fixed and movable. Consider the $n$th order equation in the complex plain

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)y = 0 \tag{2.1.1}$$

If all the coefficients $p_0(x), \ldots, p_{n-1}(x)$ are analytic in the neighbourhood of a point $x_0$, and $x_0$ is a regular point of the ODE, then there exists a unique solution of the equation (2.1.1) such that this solution, and its first $(n-1)$ derivatives take arbitrarily assigned values at $x = x_0$. This solution is expressible as a Taylor series in $(x - x_0)$, convergent at least within the circle centered at $x_0$ and passing through the singular point of the coefficients lying close to $x_0$. Hence, the singular points of the unique solution can be located only at singularities of the coefficients, which are fixed singularities as they only depend on the equation, not the particular solution.

This well-known consideration shows that solutions of linear ODEs possess only fixed singularities and they are located (at most) only at the points where the coefficients of the equation have a singularity (typically the singularities of the coefficients are poles). However, nonlinear ODEs do not have this property. The solution of a nonlinear ordinary differential equation can have both movable and fixed singularities. The "movable" singularities are those that depend on the representative in the general solution (i.e. their location depends on the initial data).

### 2.1.2 Second order ODEs

Consider ordinary differential equations of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = F\left(\frac{\mathrm{d}y}{\mathrm{d}x}, y, x\right) \tag{2.1.2}$$

where $F$ is a rational function of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $y$, and an analytic function of $x$.

Picard posed the problem of finding equations of the form (2.1.2) with solutions whose only poles are the movable singularities, i.e., the location of singularities of any of the solutions other than poles are independent of the particular solution chosen and depends only on the equation.

This problem was solved by Painlevé and this property of ODEs is called the "Painlevé property". Ordinary differential equations possessing this property are called "Painlevé type". Painlevé and his students (and then followers) showed there are only fifty types of equations of the form (2.1.2) having the Painlevé property. Within a few years of research, they found out that all but six of these equations are genuinely “transcendental”, while the rest are integrable in terms of the previously known functions or could reduce to one of the remaining six nonlinear ODEs.

These fifty equations are generalizable by the Möbius transformation

$$Y(X) = \frac{a(x)y + b(x)}{c(x)y + d(x)}, \quad X = \phi(x) \tag{2.1.3}$$

where $a(x)$, $b(x)$, $c(x)$, $d(x)$, and $\phi(x)$ are analytic in $x$.

The new six nonlinear ordinary differential equations with solutions having singularities only at poles, define new transcendental functions (i.e. analytical functions which are, generically, not expressible in terms of the algebraic operations such as addition, subtraction, multiplication, division, raising to a power, and extracting a root, and generally, not obtainable from applying any of these operations on the solutions of linear differential equations with rational coefficients). They are called Painlevé equations I-VI and are defined as

$$
\begin{aligned}
P_I : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& 6y^2 + x,\\
P_{II} : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& 2y^3 + xy + \alpha,\\
P_{III} : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y^2}{4x^2}\left(\alpha + \frac{\beta x}{y^2} + \gamma y + \frac{\delta x^2}{4y^3}\right),\\
P_{IV} : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& \frac{1}{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2-\alpha)y + \frac{\beta}{y},\\
P_V : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& \left\{\frac{1}{2y} + \frac{1}{y-1}\right\}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{(y-1)^2}{x^2}\left(\alpha\frac{\beta}{y^2} + \frac{\gamma x}{(y-1)^2} + \frac{\delta x^2(y+1)}{(y-1)^3}\right),\\
P_{VI} : \frac{\mathrm{d}^2y}{\mathrm{d}x^2} =& \frac{1}{2}\left\{\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right\}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \left\{\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right\}\frac{\mathrm{d}y}{\mathrm{d}x} +\\
&+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left\{\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right\},
\end{aligned}
\tag{2.1.4}
$$

where $\alpha$, $\beta$, $\gamma$, and $\delta$ are arbitrary constants. In the above list, the third equation is slightly modified and the original one can be obtained by a simple change of variable $(x, y) \to (x^2, xy)$. The solutions to these equations are called Painlevé transcendents.

Later we show that the Painlevé equations are irreducible in general, but for special values of the parameters the equations $P_{II} - P_{VI}$ have rational solutions and admit one-parameter families of solutions expressible in terms of the classical transcendental functions:

- Painlevé II     Airy functions
- Painlevé III     Bessel functions
- Painlevé IV     Weber-Hermite functions
- Painlevé V     Whittaker functions
- Painlevé VI     hypergeometric functions

### 2.1.3 Physical application

Painlevé equations were discovered from a mathematical point of view. Nonetheless, they later have been proven to have several applications in physics such as: spin-spin correlation function of the two-dimensional Ising model [36], quantum gravity [5], the asymptotic behaviour of solutions of the mKdV equation [1], the one-particle reduced density matrix of the one-dimensional Bose gas [20], general relativity, and nonlinear optics.

## 2.2 Calogero-Painlevé correspondence

The Painlevé equations were the result of the research by Painlevé himself, Gambier and Fuchs [14] who gave the general form of the sixth Painlevé equation. To do so, Fuchs proposed two different approaches:

- isomonodromic deformations: In this approach, $P_{VI}$ is interpreted as a differential equation describing isomonodromic deformations of a linear ordinary differential equation on the Riemann sphere.
- elliptic integrals: In this approach, a new expression of $P_{VI}$ is derived in terms of the Weierstrass $\wp$-function.

Painlevé took the second approach. Later, Okamoto [32] uncovered the Hamiltonian aspects of the six equations and discovered the underlying affine Weyl group symmetries of $P_{VI}$.

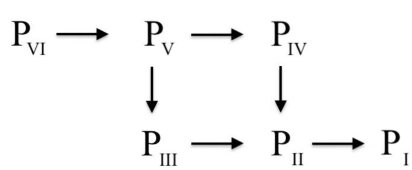
Manin [28] revived the almost forgotten work of Fuchs and Painlevé after almost ninety years. Manin's remarkable idea is to use the elliptic modulus $\tau$, rather than $t$, as an independent variable. The outcome is a Hamiltonian system with a Hamiltonian in the normal form

$$H = \frac{P^2}{2} + V(q)$$

where $V$ is the potential function which is a linear combination of the Weierstrass $\wp$-function and its shift by three half periods. This is a nonautonomous system because the Hamiltonian depends on the "time" $\tau$ through the $\tau$-dependence of the $\wp$-function.

Levin and Olshanetsky [26] pointed out that Manin's equation resembles the so called Calogero–Moser systems (we study this system in more detail in the next section). More precisely, the Hamiltonian operator $H$ is identical to the rank-one elliptic model of Inozemtsev's integrable generalization of the Calogero-Moser system [19]. Levin and Olshanetsky called this relation the "Painlevé–Calogero correspondence."

Later, K. Takasaki [34] shows that this correspondence can be extended to the other Painlevé equations and it comes from the important principle of degeneration *cascade* of the six Painlevé equations:




Takasaki shows that the Painlevé side of this correspondence for all the six equations is a multi-dimensional extension of the Painlevé equations and therefore, they are called the multi-component version of $P_I$-$P_{VI}$. The Hamiltonian system generating these equations was introduced in the previous chapter in equation (1.1.1).

The crucial problem proposed by Takasaki is whether one can find an isomonodromic description of the multi-component Calogero-Painlevé correspondence.

### 2.2.1 Isomonodromic description

In [3], the authors answer Takasaki's question by introducing the isomonodromic formulation of the six Calogero-Painlevé equations in Takasaki's list.

**Theorem 2.2.1.** *All the Hamiltonian systems in* (1.1.1) *have an isomonodromic formulation in terms of* $2N \times 2N$ *Lax pair, where* $N$ *is the number of particles.*

In each of the systems corresponding to this isomonodromic description, the dynamical variables appear as the eigenvalues of an $N \times N$ non-commutative matrix $\mathbf{q}$. The Hamiltonian operators in [3] are obtained by Kazhdan-Kostant-Sternberg [24] Hamiltonian reduction on the Lax system for the matrix Painlevé equations introduced by Kawakami [23], except for $P_{II}$ which possesses Lax system closer to the standard Flaschka-Newell Lax pair [12]. One important result of the isomonodromic formulation is that all the equations satisfy the Painlevé property, i.e., the solutions $\mathbf{q}(\mathbf{t})$ have only movable poles when considered as functions of the complex time $t$.

Another important result of this paper is that the space of the initial data of each equation can be identified with a suitable manifold of monodromy data, which is an algebraic variety.

This work is explained in more detail in chapter 3.

### 2.2.2 Hamiltonian reduction

In this section, we provide the definitions and the basic idea of Hamiltonian reduction that plays a key role in the construction of the isomonodromic description in chapter 3. First, we need to review some definitions related to Poisson manifolds [9].

**Definition 2.2.2.** *Let* $A$ *be a commutative algebra over a field* $F$*. Then* $A$ *is a Poisson algebra if it is equipped with a Lie bracket* $\{,\}$ *satisfying the Leibniz's law*

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$

**Definition 2.2.3.** $M$ *is a* $2N$*-dimensional Poisson manifold if its structure algebra* $C^\infty(M)$ *is equipped with a Poisson bracket.*

**Definition 2.2.4.** *A morphism of Poisson manifolds* $M$ *and* $N$*, is a map* $\phi : M \to N$ *that induces a homomorphism of Poisson algebras* $C^\infty(M) \to C^\infty(N)$*. Precisely, it is a map that preserves structure.*

If $M$ is smooth, then a Poisson structure on $M$ is defined by a Poisson bivector $\Pi$. In particular, when $M$ is symplectic with a closed and nondegenerate 2-form $\omega$, then it is Poisson with $\Pi = \omega^{-1}$.

For any Poisson manifold $M$, there is a homomorphism

$$v : C^\infty(M) \to V_\Pi(M) \tag{2.2.1}$$

from the Lie algebra of functions on $M$ to the Lie algebras of the vector fields on $M$ preserving the Poisson structure.

**Example 2.2.5.** *Let $M = T^*X$ where $X$ is a smooth manifold. A 1-form $\eta$ on $T^*X$ is defined as follows: Let $\pi : T^*X \to X$ be the projection map. Then given $v \in T_{(x,p)}(T^*X)$, then the action of the one–form $\eta$ on $v$ is computed by first projecting it into the tangent bundle using $d\pi : T(T^*X) \to TX$, and then applying the two–form on $X$ on the projection. Therefore, if $x_i$'s are local coordinates on $X$, and $p_i$'s are the linear coordinates in the fibers of $T^*X$ with respect to the basis $dx_i$, then $\eta = \sum_i p_i dx_i$.*
*Now if we suppose $\omega = d\eta$, then $\omega$ is a symplectic structure on $M$ and in local coordinates we have*

$$\omega = \sum_i dp_i \wedge dx_i.$$

Let $M$ be a Poisson manifold and $G$ a Lie group acting on $M$ by Poisson automorphisms. Let $\mathfrak{g}$ be the Lie algebra of $G$, hence there is a homomorphism of Lie algebras $\phi : \mathfrak{g} \to V_\Pi(M)$.

**Definition 2.2.6.** *A **moment map** (or momentum map) is $G$-equivariant map $\mu : M \to \mathfrak{g}^*$, if the pullback map $\mu^* : \mathfrak{g} \to C^\infty(M)$ satisfies the equation*

$$v\left(\mu^*(a)\right) = \phi(a) \tag{2.2.2}$$

*where $v$ is given by* (2.2.1).

**Theorem 2.2.7.** *Let $M$ be a Poisson manifold with the action of the Lie group $G$ preserving the Poisson structure. Then the algebra of $G$-invariants $C^\infty(M)^G$ is a Poisson algebra.*

**Definition 2.2.8.** *The manifold*

$$M//G := \mu^{-1}(0)/G \tag{2.2.3}$$

*which is obtained by, first, restricting to a fixed point and then quotienting by the group action, is called the Hamiltonian reduction of $M$ with respect to $G$.*

**Remark 2.2.9.** *Both $C^\infty(M//G)$ and $M//G$ are Poisson manifolds.*

**Corollary 2.2.10.** *In the setting of Definition 2.2.8 if $M$ is symplectic, then so is $M//G$.*

#### 2.2.2.1 KKS reduction and Calogero-Moser system

**Definition 2.2.11.** *Let $G$ be a Lie group with $\mathfrak{g}$ the Lie algebra on $G$, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. Also, let $ad^*$ denote the representation of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$ induced by the coadjoint representation of the Lie group $G$ that is defined by*

$$Ad^* : G \to Aut(\mathfrak{g}^*). \tag{2.2.4}$$

*Then for $\nu \in \mathfrak{g}^*$ the* ***coadjoint orbit*** $\mathcal{O}_\nu$ *is a submanifold of $\mathfrak{g}^*$ that carries a natural symplectic structure. In fact, on each $\mathcal{O}_\nu$ there is a closed, non–degenerate two-form $\omega$ inherited from $\mathfrak{g}$ so that*

$$\omega(ad_X^* v, ad_Y^* v) := \langle v, [X, Y] \rangle, \quad v \in \mathcal{O}_\nu, \quad X, Y \in \mathfrak{g} \tag{2.2.5}$$

where $\langle , \rangle$ represents the angular bracket.

**Definition 2.2.12.** *Suppose $M$ is a Poisson manifold with a Hamiltonian action of the Lie group $G$ and moment map $\mu : M \to \mathfrak{g}^*$. Let us choose a closed coadjoint orbit $\mathcal{O}$ of $G$. Then $\mu^{-1}(\mathcal{O})/G$ is called the Hamiltonian reduction on $M$ with respect to $G$. We denote this scheme by $R(M, G, \mathcal{O})$.*

**Remark 2.2.13.** *Hamiltonian reduction can be defined along any closed $G$-invariant subsets of $\mathfrak{g}^*$.*

**Example 2.2.14.** *This example is quite central to the development of the theory of (classical) Calogero-Moser systems (and their generalization to Painlevé case). Suppose $M = T^* Mat_N(\mathbb{C})$, and $G = PGL_N(\mathbb{C})$ acting on $M$ (so $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$).*

*Using a trace form we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $M$ with*

$$Mat_N(\mathbb{C}) \oplus Mat_N(\mathbb{C}).$$

*By conjugations:*

$$g \cdot (X, Y) = (gXg^{-1}, gYg^{-1}).$$

*Given a point $p = (X, Y) \in M$ and an element $L \in \mathfrak{g}$ the corresponding vector field $\mathbb{L} \in T_p M$ is then given by the formula*

$$\mathbb{L}(X, Y) = ([L, X], [L, Y]). \tag{2.2.6}$$

*To find the moment map we need to find the Hamiltonian, $H_L$ that generates the above vector field. Since the symplectic form $\omega$ on $M$ is $\mathrm{Tr}(\mathrm{d}Y \wedge \mathrm{d}X)$ and its corresponding symplectic potential is $\theta = \mathrm{Tr}(Y \mathrm{d}X)$, we deduce that the Hamiltonian associated to the element $L$ is*

$$H_L(X, Y) = \mathrm{Tr}\Big(L[X, Y]\Big) = \langle L, \mu(X, Y) \rangle \,. \tag{2.2.7}$$

*Then equation (2.2.7) implies that the moment map is given by $\mu(X, Y) = [X, Y]$. To see this we need to verify that the above Hamiltonian generates the equation (2.2.6). Indeed*

$$\{Y_{ba}, H_L\} = -\frac{\partial}{\partial X_{ab}} H_L = -\frac{\partial}{\partial X_{ab}} \sum_{b,c} X_{bc}[Y, L]_{cb} = [L, Y]_{ba} \tag{2.2.8}$$

*and similarly*

$$\{X_{ba}, H_L\} = \frac{\partial}{\partial Y_{ab}} H_L = \frac{\partial}{\partial Y_{ab}} \sum_{b,c} Y_{bc}[L, X]_{cb} = [L, X]_{ba}. \tag{2.2.9}$$

*The zero–level set of the moment map $\mu^{-1}(0)$ is then defined by the equation $[X,Y]=0$ and it is a commuting scheme, denoted by Comm(N). Hence $M//G = Comm(N)/G$ with the ring of functions being $A = \mathbb{C}[Comm(N)]^G$.*

Concerning the above definition of the quotient $\mathrm{Comm}(N)/G$ the following theorem is proven in [10]:

**Theorem 2.2.15.** *The quotient $Comm(N)/G$ is reduced and it is isomorphic to $\mathbb{C}^{2N}/S_N$, where $S_N$ denotes the symmetric group. Therefore*

$$A = \mathbb{C}[Comm(N)]^G = \mathbb{C}[x_1,\ldots,x_N,y_1,\ldots,y_N]^{S_N}. \tag{2.2.10}$$

*Accordingly, the Poisson algebra is induced from the standard symplectic structure on $\mathbb{C}^{2N}$.*

A more general Hamiltonian reduction can be defined on a different level set of the moment map; one can fix $\mu_0 \in \mathfrak{g}^*$ and consider $\mu^{-1}(\mu_0)$; the stabilizer in $G$ of $\mu_0$ acts on this level set and one can construct the quotient. Denoting the stabilizer by $G_{\mu_0}$ then the "point" symplectic reduction is the statement that the quotient

$$J_{\mu_0} := \mu^{-1}(\mu_0)/G_{\mu_0}$$

is naturally a symplectic manifold.

An equivalent, but different, construction is called the "coadjoint orbit" symplectic reduction. In this case one considers the co-adjoint orbit $\mathcal{O}_{\mu_0}$ passing through the same value $\mu_0$ and quotients $\mu^{-1}(\mathcal{O}_{\mu_0})$ by the whole Lie group $G$.

The proof can be found in Thm. 1.2.4 of [29]; here it is sufficient to make some simple dimensional considerations. If $\dim M = m$, $\dim G = s$ and $\mu_0$ is a regular value then

$$\dim \mathcal{O}_{\mu_0} + \dim G_{\mu_0} = \dim G. \tag{2.2.11}$$

On the other hand, the dimension of a fixed level set is

$$\dim \mu^{-1}(\mu_0) = \dim M - \dim G \tag{2.2.12}$$

and therefore the dimension of the inverse of the whole orbit is

$$\dim \mu^{-1}(\mathcal{O}_{\mu_0}) = \dim M + \dim \mathcal{O}_{\mu_0} - \dim G = \dim M - \dim G_{\mu_0}. \tag{2.2.13}$$

Then the dimension of the quotient of $\mu^{-1}(\mu_0)$ by $G_{\mu_0}$ and that of $\mu^{-1}(\mathcal{O}_{\mu_0})$ by $G$ are equal:

$$\dim \mu^{-1}(\mu_0) - \dim G_{\mu_0} = \dim M - \dim G - \dim G_{\mu_0} = \dim \mu^{-1}(\mathcal{O}_{\mu_0}) - \dim G. \tag{2.2.14}$$

It is this point of view which is more amenable to a quantum description (later on).

Now consider $M$ and $G$ as defined in Example 2.2.14, and $\mathcal{O}$ be the orbit of the matrix proportional to $\mathrm{diag}(-1,-1,\ldots,-1,N-1)$. This can be described more invariantly as the set of traceless matrices $T$ in $\mathfrak{sl}_N$ such that $T+1$ has rank 1. In the original paper [24] the matrix is presented as

$$\mu_0 = c \begin{bmatrix} 0 & 1 & \dots & & 1 \\ 1 & 0 & 1 & \dots & 1 \\ & & \ddots & & \\ 1 & \dots & & 1 & 0 \end{bmatrix}.$$

**Definition 2.2.16.** *[24] The scheme $\mathcal{C}_N := R(M, G, \mathcal{O})$ is called the Calogero-Moser space.*

Thus, $\mathcal{C}_N$ is the space of conjugacy classes of pairs of $N \times N$ matrices $(X, Y)$ such that the matrix $XY - YX + 1$ has rank 1.

**Corollary 2.2.17.** *The action of $G$ on $\mu^{-1}(\mathcal{O})$ is free, hence $\mathcal{C}_N$ is a smooth symplectic variety of dimension $2N$.*

### 2.2.3 Hamiltonians and integrable systems

In a classical mechanical system, the phase space is a Poisson manifold $M$ that is usually symplectic and equal to $T^*X$, where $X$ is another manifold that is called configuration space. The dynamics of such systems are defined by Hamiltonians $H \in C^\infty(M)$ whose flow is that attached to the vector field $v(H)$. If $q_i$'s are the coordinates on $M$, then the Hamiltonian equations are defined as

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \{H, q_i\}$$

where $t$ is time.

If $M$ is symplectic, then Darboux theorem assures us that there are coordinates $q_i$, $p_i$ locally defined on $M$ for which the symplcetic form is $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$. These coordinate are called canonical coordinates with the property

$$\{p_i, q_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = \{q_i, q_j\} = 0.$$

In these coordinates the Hamiltonian equations are written as

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}.$$

**Conservation laws and symmetries.** On a 2-dimensional manifold $M$, the conservation of energy says that the Hamiltonian is constant along the trajectories of the system. So they can be explicitly solved. However, for $2N$-dimensional symplectic manifold with $N > 1$ this becomes complicated. Hence, we take advantage of the symmetries in classical mechanics.

If a classical system has a symmetry, then that symmetry can be used to reduce the order of the system and that, basically, is the Hamiltonian reduction.

**The scheme.** Let a Lie group $G$ act freely on $M$ preserving the Hamiltonian $H$, and let $\mu : M \to \mathfrak{g}^*$ be a moment map. We also assume the vector field $v(H)$ is transversal to the $G$-orbits.

If $y = y(t)$ is a solution of Hamiltonian equations then $\mu(y(t))$ is constant. Therefore, the Hamiltonian flow descends to the symplectic manifold $R(M, G, \mathcal{O}) = \mu^{-1}(\mathcal{O})/G$. Here $\mathcal{O}$ runs over orbits of the coadjoint representation of $G$, with the same Hamiltonian. These manifolds have smaller dimensions than the dimension of $M$.

On the other hand, knowing $y_*(t)$ in $R(M, G, \mathcal{O})$, an image of a trajectory, we can find $y(t)$ explicitly.

**Definition 2.2.18.** *An **integrable system** on a symplectic manifold $M$ of dimension $2N$ is a collection of smooth functions $H_1, \ldots, H_N$ on $M$ such that they are in involution:*

$$\{H_i, H_j\} = 0, \quad \forall 1 \leq i, j \leq N$$

*and the differentials $dH_i$ are linearly independent on a dense open set in $M$.*

Knowing the Hamiltonian flow given by $H$ on $M$, and assuming that $H$ can be included in an integrable system $H = H_1, \ldots, H_N$ ($H_1, \ldots, H_N$ are the first integrals of the flow), one can use the involution and use $H_N$ to reduce the order of the system from $2N$ to $2N - 2$. Repeating this scheme, eventually we reduce the $2N$-dimensional system to a 2-dimensinal one which can be integrated in quadratures, therefore, so does the flow of the original manifold.

**How to construct an integrable system by Hamiltonian reduction:** Let $M$ be a symplectic manifold, and let $H_1, \ldots, H_N$ be smooth functions on $M$ such that $\{H_i, H_j\} = 0$ and $\mathrm{d}H_i$'s are linearly independent everywhere. Suppose $G$ is a structure preserving Lie group acting on $M$, with the moment map $\mu : M \to \mathfrak{g}^*$. Moreover, let $\mathcal{O}$ be a coadjoint orbit of $G$. Consider the assumption on the action of $G$ on $\mu^{-1}(\mathcal{O})$ such that the symplectic manifold $R(M, G, \mathcal{O})$ carries a collection of functions $H_1, \ldots, H_N$ with the property $\{H_i, H_j\} = 0$ and $\mathrm{d}H_i$'s linearly independent everywhere. One assures that for $N = \frac{1}{2}\mathrm{dim}R(M, G, \mathcal{O})$ the collection $H_1, \ldots, H_N$ is an integrable system on $M$. Therefore, the Hamiltonian flow can be solved in quadratures.

**Example 2.2.19.** *The integrable system called "Calogero-Moser system" is an example of a system constructed by the Hamiltonian reduction, by Kazhdan, Kostant, Sternberg [24].*
*In this case $M = T^*Mat_N(\mathbb{C})$, regarded as the set of pairs of matrices $(X, Y)$ with the usual symplectic from $\omega = \mathrm{Tr}(dY \wedge dX)$. Let*

$$H_i = Tr(Y^i), \quad i = 1, \ldots, N. \tag{2.2.15}$$

*One can easily show that $\{H_i, H_j\} = 0$ (where $\{,\}$ is the Poisson bracket), but since there are few of them, they do not form an integrable system.*

*Now, let $G = PGL_N(\mathbb{C})$ acting on $M$ by conjugation, and let $\mathcal{O}$ be the coadjoint orbit of $G$ consisting of the traceless matrices $T$ such that $T+1$ has rank 1 (i.e., the orbit of the matrix diag$(-1,-1,\ldots,-1,N-1)$).*

*Then by Hamiltonian reduction, the system $H_1,\ldots,H_N$ reduces to a system of functions on the Calogero-Moser space $\mathcal{C}_N = R(M,G,\mathcal{O})$, such that they are in involution. Since the space $\mathcal{C}_N$ is $2N$-dimensional, $H_1,\ldots,H_N$ form an integrable system on this space. This system is called the (rational) Calogero-Moser system.*

*On $\mathcal{C}_N$, the flow corresponding to Hamiltonians $H_i$ from the equation* (2.2.15)*, can be described by the formula*

$$g_t^{(i)}(X,Y) = (X + iY^{i-1}t, Y). \tag{2.2.16}$$

*This is a consequence of the set up for $\mathcal{C}_N$, as it is in the space of matrices, and in general, the motion of a free particle on such spaces is defined by $g_t(X,Y) = (X+2Yt,Y)$.*

To be able to represent the explicit Hamiltonians, we need to understand the coordinates on $\mathcal{C}_N$.

**$\mathcal{C}_N$, coordinates and Hamiltonians.** To introduce the coordinates on $\mathcal{C}_N$, we restrict our attention to the open set $U_N \subset \mathcal{C}_N$ consisiting of conjugacy classes of pairs $(X,Y)$ for which the matrix $X$ is diagonalizable with distinct eigenvalues. This set is dense in $\mathcal{C}_N$. A point $P \in U_N$ can be represented by $(X,Y)$, with $X = \text{diag}(x_1,\ldots,x_N)$, $x_i \neq x_j$. In this case, the entries of $T := XY - YX$ are $(x_i - x_j)y_{ij}$, which means the diagonal entries are zero. Since $T+1$ has rank 1, its entries $\kappa_{ij}$ have the form $a_i b_j$ for some numbers $a_i$, $b_j$. Also, since $\kappa_{ii} = 1$, then $b_j = a_j^{-1}$, so $\kappa_{ij} = a_i a_j^{-1}$. By conjugating $(X,Y)$ by the matrix $\text{diag}(a_1,\ldots,a_N)$ we can reduce the situation to the case $a_i = 1$, so $\kappa_{ij} = 1$, therefore the entries of the matrix $T+1$ have the form $1 - \delta_{ij}$.

This representation for $P$ is unique up to the action of the symmetry group $S_N$. So, we get

$$(x_i - x_j)y_{ij} = 1, \qquad i \neq j.$$

Therefore

$$y_{ij} = \frac{1}{x_i - x_j}, \qquad \text{if} \quad i \neq j.$$

As a result we have the following proposition:

**Proposition 2.2.20.** *Let $\mathbb{C}^N_{reg}$ be the open set of $(x_1,\ldots,x_N) \in \mathbb{C}^N$ such that $x_i \neq x_j$, for $i \neq j$. Then there exists an isomorphism of algebraic varieties*

$$\xi : T^*(\mathbb{C}^N_{reg}/S_N) \to U_N$$

*given by the formula $(x_1,\ldots,x_N,p_1,\ldots,p_N) \to (X,Y)$, where $X = \text{diag}(x_1,\ldots,x_N)$, and $Y = Y(x,p) := (y_{ij})$,*

$$y_{ij} = \frac{1}{x_i - x_j}, \quad i \neq j, \quad y_{ii} = p_i.$$

The Hamiltonians of the Calogero-Moser system in the defined coordinates have the following form

$$H = \mathrm{Tr}(Y(x,p)^2) = \sum_i p_i^2 - \sum_{\substack{i,j \\ i\neq j}} \frac{1}{(x_i - x_j)^2}. \tag{2.2.17}$$

The reduction procedure that was explained, guarantees the complete integrability of $H$, and gives an explicit formula for the first integral:

$$H_i = \mathrm{Tr}\left(Y(x,p)^i\right)$$

Furthermore, the explicit form of the solution is provided by this procedure; assume $x(t)$, $p(t)$ is the solution with the initial condition $x(0)$, $p(0)$, and let $(X_0, Y_0) = \xi(x(0), p(0))$. Then $x_i(t)$ are the eigenvalues of the matrix $X_t := X_0 + 2tY_0$ and $p_i(t) = x_i'(t)$.

### 2.2.4 Quantum version of the Hamiltonian reduction

The quantized version of the Hamiltonian reduction is very similar to the classical version. First, we need to quantize the concept of the moment map.

In the quantum setting one cannot expect to "fix" the value of the moment map since in general, the moment operators do not commute: hence the "fixed moment" description of the Hamiltonian reduction is inadequate. The point of view that best translates is that of "co-adjoint orbit reduction". However, in the co-adjoint orbit point of view this is possible because of the following considerations. A coadjoint orbit is characterized by the value of the Casimir elements of the universal enveloping algebra $U(\mathfrak{g})$; these, by definition, are commuting elements. When quantizing a phase space that carries a Hamiltonian $G$ action, the action of the Lie algebra $\mathfrak{g}$ is represented by, typically, quantum (differential) operators. The Casimirs commute by definition and hence there is a common joint eigenspace in the Hilbert space where the quantum system is set up. In other words, the quantum version of the Hamiltonian reduction should be thought of as a Hilbert space carrying a representation of the group $G$ and with fixed values of the Casimirs of the group.

In algebraic terms the above ideas are translated in the following setup:

**Definition 2.2.21.** *Let $\mathfrak{g}$ be a Lie algebra, and $A$ be an associative algebra so that there exists a Lie algebra map*

$$\phi : \mathfrak{g} \to Der(A)$$

*where $Der(A)$ is the Lie algebra of derivations of $A$. Then the associative algebra homomorphism $\mu : U(\mathfrak{g}) \to A$ is quantum moment map for $(A, \phi)$ if for any $L \in \mathfrak{g}$ and $a \in A$:*

$$[\mu(L), a] = \phi(L)a.$$

In the above definition $[,]$ indicates the commutator operator.

**Remark 2.2.22.** *Suppose that $A$ is a deformation quantization of a Poisson algebra $A_0$ equipped with a $\mathfrak{g}$-action $\phi_0$ and a classical moment map $\mu_0$. Suppose that $\phi = \phi_0 mod\hbar$. A quantization of $\mu_0$ is a quantum moment map $\mu : U(\mathfrak{g}) \to A[\hbar^{-1}]$ such that for $a \in \mathfrak{g}$ we have*

$$\mu(a) = \hbar^{-1}\mu_0(a) + O(1). \tag{2.2.18}$$

**Hamiltonian reduction.** Considering $A$, $\mathfrak{g}$, and the quantum moment map $\mu$ as in the previous definition, the space of all elements $a \in A$ for which $[\mu(L), a] = 0$ for all $L \in \mathfrak{g}$, is a subalgebra of $A$ denoted by $A^{\mathfrak{g}}$. To see this consider $a, a' \in A^{\mathfrak{g}}$. Then, for all $L \in \mathfrak{g}$

$$[\mu(L), aa'] = [\mu(L), a]a' + a[\mu(L), a'] = 0. \tag{2.2.19}$$

Then for $J \subset A$ the left ideal generated by $\mu(\mathfrak{g})$, $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a 2-sided ideal of $A^{\mathfrak{g}}$ whose elements have the form $a = \sum_i a_i\mu(L_i)$, $a_i \in A$, and $L_i \in \mathfrak{g}$.

Indeed, suppose $c \in A^{\mathfrak{g}}$, $a \in J^{\mathfrak{g}}$ where $a = \sum_i a_i\mu(L_i)$ for $a_i \in A$, $L_i \in \mathfrak{g}$. Then:

$$\begin{aligned} ac =& \sum_i a_i\mu(L_i)c = \sum_i a_i c\mu(L_i) \in J^{\mathfrak{g}}, \\ ca =& c\sum_i a_i\mu(L_i) = \sum_i ca_i\mu(L_i) \in J^{\mathfrak{g}}, \end{aligned}$$

Moreover, for $d = \sum_i d_i\mu(L_i) \in J^{\mathfrak{g}}$, with $d_i \in A$, it is obvious that $a - d \in J^{\mathfrak{g}}$.

Then algebra $A//\mathfrak{g} := A^{\mathfrak{g}}/J^{\mathfrak{g}}$ defines a quantum Hamiltonian reduction of $A$ with respect to the quantum moment map $\mu$.

Computing a quantum Hamiltonian reduction is usually difficult. For example, consider $A = D(\mathfrak{g})$, namely, the algebra of differential operators on a reductive Lie algebra $\mathfrak{g}$ acting on $A$ by the adjoint action on itself. To describe $A//\mathfrak{g}$ we need to construct the Harish-Chandra homomorphism [17] $\mathrm{HC} : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h})^W$, with $\mathfrak{h}$ a Cartan subalgebra, and $D(\mathfrak{h})^W$ indicating the Weyl-invariant differential operators. This is done through the following steps [10]:

- Consider the classic Harish-Chandra isomorphism $\zeta : \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$ which provides an action of $D(\mathfrak{g})^{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{h}]^W$ that is given by Weyl-invariant differential operators,

- To avoid the poles on the reflection hyperplane, take the homomorphism $\mathrm{HC}' : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}_{\mathrm{reg}})^W$, with $\mathfrak{h}_{\mathrm{reg}}$ the complement of $\mathfrak{h}$ or the set of its regular points (this homomorphism is called the radial part homomorphism),

- by twisting the map $HC'$ by $\delta = \prod_{\alpha>0}(\alpha, x), x \in \mathfrak{h}$ and $\alpha$ running over the positive roots of $\mathfrak{g}$,
$$\mathrm{HC}(D) := \delta \circ \mathrm{HC}'(D) \circ \delta^{-1} \in D(\mathfrak{h}_{\mathrm{reg}})^W$$
the poles will disappear.

**Theorem 2.2.23.** *The homomorphism $HC : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h})^W$ defines an isomorphism $D(\mathfrak{g})//\mathfrak{g} = D(\mathfrak{h})^W$.*

Similar to the Hamiltonian reduction along a closed orbit (or any closed $G$-invariant subset) of $\mathfrak{g}^*$, the quantum analog of the reduction must be constructed with respect to a 2-sided ideal $I \subset U(\mathfrak{g})$: Let $\mu : U(\mathfrak{g}) \to A$ be the quantum moment map, and $J(I) = A\mu(I) \subset A$, so $J(I)^{\mathfrak{g}}$ is a 2-sided ideal. As previously was explained, the algebra

$$R(A, \mathfrak{g}, I) := A^{\mathfrak{g}}/J(I)^{\mathfrak{g}}$$

is the Hamiltonian reduction with respect to the ideal $I$.

The quantization of Kazhdan-Kostant-Sternberg Hamiltonian reduction plays a key role in the next chapter of this dissertation. Therefore, as an example of the application of the Harish-Chandra homomorphism in quantum Hamiltonian reduction, we look at the quantization of the Kazhdan-Kostant-Sternberg construction of the Calogero-Moser space.

**Example 2.2.24.** *Let $\mathfrak{g} = \mathfrak{gl}_N$, $A = D(\mathfrak{g})$ as described above. Let $\kappa$ be a complex number and $\mathbb{V}_\kappa$ be the representation of $\mathfrak{sl}_N$ on the space of functions of the form*

$$(x_1. \dots .x_N)^\kappa f(x_1, \dots, x_N)$$

*where $(x_1, \dots, x_N)$ are the coordinates in $C^N$, and $f$ is a Laurent polynomial of homogeneity degree 0. Under the natural projection map $\mathfrak{g} \to \mathfrak{sl}_N$ and a pullback to $\mathfrak{g}$, we regard $\mathbb{V}_\kappa$ as a $\mathfrak{g}$-module. Let $I_\kappa$ be the annihilator of $\mathbb{V}_\kappa$ in $U(\mathfrak{g})$. Then the homomorphism*

$$HC_\kappa : D(\mathfrak{g})^{\mathfrak{g}} \to R(A, \mathfrak{g}, I_\kappa)$$

is called the deformed Harish-Chandra homomorphism.

The algebra $R(A, \mathfrak{g}, I_\kappa)$ acts naturally on the space $E_\kappa$ (space of $\mathfrak{g}$-equivariant functions with values in $\mathbb{V}_\kappa$) on the neighbourhood of $\mathfrak{h}_{\text{reg}}$ in $\mathfrak{g}_{\text{reg}}$, the set of matrices with different eigenvalues. An equivariant function on such a neighbourhood with values in $\mathbb{V}_\kappa$ is completely determined by its values in $\mathfrak{h}_{\text{reg}}$, and the only restriction for these values is that they lie in $\mathbb{V}_\kappa[0]$ the zero-weight subspace of $\mathbb{V}_\kappa$.

Note that the space $\mathbb{V}_\kappa[0]$ is 1-dimensional spanned by the function $(x_1, \dots, x_N)^\kappa$. Therefore, $E_\kappa$ is isomorphic to $\mathbb{C}[\mathfrak{h}_{\text{reg}}]$.

The algebra $R(A, \mathfrak{g}, I_\kappa)$ with the above description, is a **quantization of the Calogero-Moser space**.

In the following section we introduce the quantum Hamiltonian operator corresponding to this space.

### 2.2.5 Quantum integrable system

The phase space in quantum mechanical systems is a noncommutative algebra $A$; in our case, concretely in the next chapter, it will be the algebra of operator of multiplication by a matrix $Q$ and differentiation with respect to its entries. The Hamiltonians $H$ are operators defined on an Hilbert space $\mathcal{H}$ with a representation of the noncommutative algebra $A$. For a given Hamiltonian $H$ the Schrödinger equation describes the dynamics of such systems for wave function $\psi(t) \in \mathcal{H}$

$$\hbar\dot{\psi} = H\psi$$

where $\hbar$ is the Planck constant which we will consider as a formal parameter*
In general, assume that the algebra $A$ is a deformation quantization of the phase space $M$, and the Hamiltonian $H \in A$ is a deformation quantization of a classical Hamiltonian $\widetilde{H} \in \widetilde{A} = C^\infty(M)$. Considering these, the quantum mechanics can be regarded as a deformation of classical mechanics.
Supposing $M = T^*X$ with $X$ a Riemannian manifold, we have

$$\widetilde{H} = \frac{p^2}{2} + U(x) \tag{2.2.20}$$

where $x \in X$, $p$ represents the momentum and $U(x)$ is a function on $X$ that defines the potential. Under the necessary assumptions, the quantization operator takes the form

$$H = \hbar^2\frac{\partial_x^2}{2} + U(x) \tag{2.2.21}$$

which converts the Schrödinger equation to

$$\hbar\frac{\partial\psi}{\partial t} = H(x, \hbar\frac{\partial}{\partial x}, t)\psi. \tag{2.2.22}$$

**Construction of quantum integrable system by quantum reduction.** We recall the counterpart construction from the classical integrable system: Let $\tilde{A}$ be the function algebra on a symplectic manifold $M$, and suppose $\tilde{\mu} : M \to \mathfrak{g}^*$ be the classical moment map.

Now suppose the functions $\tilde{H}_1, \dots, \tilde{H}_N$ that are in involution, reduce to a classical integrable system $R(M, G, \mathcal{O})$.
Now suppose $A$ is a deformation quantization of $\tilde{A}$, and $\mu : \mathfrak{g} \to \hbar^{-1}A$ defines the quantum moment map.
Also, suppose $I \subset U(\mathfrak{g}_\mathfrak{h})$ is an ideal which is the deformation of $\tilde{I}$ an ideal of $\tilde{A}$ that vanishes on the closed orbit $\mathcal{O}$. Assume $H_1, \dots, H_N$ is a commuting system of $\mathfrak{g}$-invariants which is a quantization of the system $\tilde{H}_1, \dots, \tilde{H}_N$. Suppose the algebra $R(A, \mathfrak{g}, I)$ is a quantization of

*To be precise, $\hbar$ in this Thesis will denote the Planck's constant mutliplied by the imaginary unit. This is only for convenience reason in the writing of the formulæ. We will still call it "Planck" constand without further note.

the symplectic manifold $R(M, G, \mathcal{O})$. Then the system $H_1, \ldots, H_N$ descends to commuting elements in $R(A, \mathfrak{g}, I)$, that quantize $\tilde{H}_1, \ldots, \tilde{H}_N$. Hence, they are a quantum integrable system that is called the quantum reduction of the system $H_1, \ldots, H_N$.

**Continuation of example** (2.2.24)**:** Consider again $\mathfrak{g} = \mathfrak{gl}_N$ and $M = T^*\mathfrak{g} = \{(X,Y)|X, Y \in \mathfrak{g}\}$. Recall that the reduced phase space is the space of matrices $X, Y$ whose commutator is a rank–one perturbation of the identity, modulo conjugations. In this case the orbit is of dimension $2N$ and it is uniquely determined by the value of just one Casimir function, for example the fundamental one $\mathrm{Tr}(\mu^2)$, with $\mu(X,Y) = [X,Y]$ (all other Casimirs are then uniquely determined by the rank-one condition). In this case the commuting Hamiltonians are $H_j = \mathrm{Tr}(Y^j)$, $j = 1, \ldots N$, with the Calogero Hamiltonian being $H_2$ (see definition (2.2.17)).

The quantization proceeds as follows: one replaces the algebra of functions of $X, Y$ by the algebra of differential operators on $\mathfrak{g}$. Denoting now by $Q$ the point of $\mathfrak{g}$ (i.e. a matrix) then $X \to \mathbf{q}$ (the matrix of multiplication operators by the entries of $Q$) and $Y \to \mathbf{p}$, where $\mathbf{p}_{ab} = \hbar \frac{\partial}{\partial Q_{ba}}$.

The Hamiltonians are correspondingly quantized simply by replacing $Y$ by $\mathbf{p}$ and become the differential operators with constant coefficients given by $\widehat{H}_i = \mathrm{Tr}(\mathbf{p}^i)$, $i = 1, \ldots, N$. The group $\mathbb{P}GL_N$ acts on the functions of $Q$ by $g\Psi(Q) := \Psi(gQg^{-1})$ and hence the Lie algebra acts by sending the elementary matrix $E_{\sigma\rho} \in \mathfrak{sl}_N$ to the operator

$$\phi(E_{\sigma\rho}) = \mathbb{E}_{\sigma\rho} = \sum_{\ell=1}^{N} \left( Q_{\sigma\ell} \frac{\partial}{\partial Q_{\rho\ell}} - Q_{\ell\rho} \frac{\partial}{\partial Q_{\ell\sigma}} \right) = \frac{1}{\hbar} \sum_{\ell=1}^{N} \Big( \mathbf{q}_{\sigma\ell}\mathbf{p}_{\ell\rho} - \mathbf{q}_{\ell\rho}\mathbf{p}_{\sigma\ell} \Big) = \tag{2.2.23}$$

$$= \frac{1}{\hbar}[\mathbf{q}, \mathbf{p}]_{\sigma\rho} + \delta_{\rho\sigma}. \tag{2.2.24}$$

Hence for $L \in \mathfrak{sl}_N$ we have the simple formula for the quantum operator that is formally the same as the classical one:

$$\hbar \widehat{H}_L = \mathrm{Tr}\Big(L[\mathbf{q}, \mathbf{p}]\Big),$$

on account that $\mathrm{Tr}L = 0$.

This is a homomorphism of Lie algebras, namely

$$[\widehat{H}_L, \widehat{H}_K] = \hbar \widehat{H}_{[L,K]}, \qquad \forall L, K \in \mathfrak{g}, \tag{2.2.25}$$

which can thus be extended to the universal enveloping algebra $U(\mathfrak{g})$. In this case the algebra $A^{\mathfrak{g}}$ is simply the algebra of differential operators that are invariant under conjugation action. In practical terms, this is the associative algebra generated by the differential operators of the form

$$\mathrm{Tr}\Big(\mathbf{q}^{\ell_1}\mathbf{p}^{\ell_2}\mathbf{q}^{\ell_3}\cdots\Big),$$

which are clearly conjugation invariant.

To construct the ideal $I$ which is appropriate to the quantization of the Kazhdan–Kostant-Sternberg scheme, Etingof has shown [10] that we should proceed as follows; define a representation of $\mathfrak{g}$ (called the "Cherednik representation") as in the Example 2.2.24, of which we now flesh out the details. The $\mathfrak{g}$ module $\mathbb{V}_\kappa$ consists of rational functions of $\xi_1, \dots \xi_N$ of homogeneity degree zero, multiplied by the factor $\prod \xi_j^\kappa$, with $\kappa$ a formal parameter (hence these are now functions of homogeneity $\kappa N$). The representation, $\gamma : \mathfrak{g} = \mathfrak{sl}_N \to End(\mathbb{V}_\kappa)$ is the natural infinitesimal action of $SL_N$ on the variables $\vec{\xi}$ and can be pulled–back to the reductive algebra $\mathfrak{gl}_N$; in concrete terms this means that the matrix unit $\mathbb{E}_{ab} \in \mathfrak{gl}_N$ is sent to the operator (of homogeneity degree zero)

$$\gamma(\mathbb{E}_{ab}) = \xi_a \frac{\partial}{\partial \xi_b} - \xi_b \frac{\partial}{\partial \xi_a}. \tag{2.2.26}$$

This representation is then extended to the universal enveloping algebra $U(\mathfrak{g})$. The kernel of this representation is the ideal

$$I_\kappa := \mathrm{Ker}(\gamma).$$

Then the quantization of the Hamiltonians $\widehat{H}_j = \mathrm{Tr}(\mathbf{p}^j)$ consists in considering them as acting on the image in $A$ of the quotient $(U(\mathfrak{g})/I_\kappa)^{\mathfrak{g}}$ which can be shown to be equal to $A^{\mathfrak{g}}/\phi(I_\kappa)^{\mathfrak{g}}$.

Again, in practical terms it is then more expedient to consider directly the action of $U(\mathfrak{g})$ on the representation space $\mathbb{V}_\kappa$; then it is shown in [10] that the action of the $\mathfrak{g}$–invariant operators on $A^{\mathfrak{g}}/\phi(I_\kappa)^{\mathfrak{g}}$ is isomorphic to an action of the operators in the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices on the zero–weight subspace

$$\mathbb{V}_\kappa[0] := \bigcap_{H \in \mathfrak{h}} \mathrm{Ker}(\phi(H)). \tag{2.2.27}$$

These Hamiltonians descend to a quantum integrable system in the algebra $R(A, \mathfrak{g}, I)$ because they commute.

We will show how to perform practically this reduction and obtain simple differential operators on $\mathfrak{h}$ in the next chapter. For example, if we apply the above scheme, the Calogero-Moser Hamiltonian reduces to

$$H = \sum_{i=1}^{N} \partial_i^2 - \sum_{\substack{i,j \\ i \neq j}} \frac{\kappa(\kappa+1)}{(x_i - x_j)^2} \tag{2.2.28}$$

and the higher Hamiltonians $\widehat{H}_j = \mathrm{Tr}(\mathbf{p}^j)$ become differential operators of the general form

$$\sum_{\ell=1}^{N} \partial_{x_\ell}^j + \text{ lower order terms.} \tag{2.2.29}$$

In the lower order terms, in general, there are expressions that exhibit a pole along the mirrors of the fundamental reflections of the Weyl group, which in our case simply means the diagonals $x_i = x_j,\ i \neq j$.

In Chapter 3 of this dissertation, we explicitly compute the Hamiltonian reduction of all possible operators up to order 2 with terms of the form $\mathrm{Tr}(\mathbf{q}^a \mathbf{p}^2), \quad a = 0, 1, \ldots$. The logic of the computation can be extended *in principle* to operators of higher order but we did not pursue the issue because it was not instrumental to our goals. To our knowledge, there is no explicit formula for this reduction, or the Harish–Chandra homomorphism in the literature, even at the level of the second-order operators that we have computed for this thesis.

# Chapter 3

# Isomonodromic system and the quantization

## 3.1 Isomonodromic Formulation

In [3] the authors provided the answer to a conjecture postulated by K. Takasaki in [34]. Takasaki considered the de-autonomization of the Calogero systems proposed by Inozemtsev [19] by observing that for Painlevé VI the rank-one Inozemtsev system reduces to the autonomous version of the Hamiltonian form of Painlevé VI. This leads to the postulation of the deautonomized version and the conjecture that these deautonomized Hamiltonians should be describing the isomonodromic deformations of an appropriate system. For this reason Takasaki coined the term "Painlevé–Calogero correspondence".

The core idea of [3] is as follows; they consider a (complexified) phase space consisting of pairs of $N \times N$ matrices $\mathbf{p}, \mathbf{q}$, identified with the cotangent bundle of $X = \mathrm{Mat}_{N\times N}(\mathbb{C})$. The starting point of these computations is a Lax system of type

$$\begin{cases} \dfrac{\partial}{\partial z}\Phi(z;t) = A(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Phi(z;t), \\ \\ \dfrac{\partial}{\partial t}\Phi(z;t) = B(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Phi(z;t). \end{cases} \tag{3.1.1}$$

In the above representation, $A$ and $B$ are $2 \times 2$ block matrices with blocks of arbitrary size $N$. These matrices depend rationally on parameter $z \in \mathbb{CP}^1$, called the spectral parameter, and they are polynomials in matrices $\mathbf{q}, \mathbf{p}$ of size $N \times N$. The matrices $\mathbf{q}, \mathbf{p}$ depend on $t$ both implicitly and explicitly.

The compatibility condition on the system (3.1.1) results in the zero-curvature equation represented by

$$\partial_t A - \partial_z B + [A, B] = 0. \tag{3.1.2}$$

**Remark 3.1.1.** *The original work of [3] consists also of a type of Lax system for $P_{III}$ in* *terms of* $\{\mathbf{q}, \mathbf{q}^{-1}, \mathbf{p}\}$*. This exceptional case is excluded from the computations in this project.*

Considering the set up as above, the (complex) symplectic form is then

$$\omega = \mathrm{Tr}(\ \mathbf{p} \wedge \mathbf{q}) = \sum_{i,j} p_{ij} \wedge q_{ji} \quad \Leftrightarrow \{p_{ab}, q_{cd}\} = \delta_{ad}\delta_{bc}. \tag{3.1.3}$$

Consider a Hamiltonian $H(\mathbf{p}, \mathbf{q})$ which is conjugation invariant

$$H(\mathbf{p}, \mathbf{q}) = H(C\mathbf{p}C^{-1}, C\mathbf{q}C^{-1})$$

with $C \in \mathrm{GL}_N(\mathbb{C})$, then Noether's theorem guarantees the conservation of the associated momentum $M = [\mathbf{p}, \mathbf{q}]$. This allows us to fix a particular value of the momentum $M$ and investigate the reduced system on the leaf of this value. If the momentum $M$ is fixed to be of the form

$$\boldsymbol{M = [\mathbf{p}, \mathbf{q}] = ig(1 - v^T v)}, \qquad \text{with } \boldsymbol{v := (1, \ldots, 1)} \tag{3.1.4}$$

then one can apply a theorem used in the classical theory of Calogero system and due to Kazhdan, Kostant, and Sternberg [24] which allows us to diagonalize $\mathbf{q} = CXC^{-1}$ with $X = \mathrm{diag}(x_1, \ldots, x_N)$ and in such a way that the matrix $Y = C^{-1}\mathbf{p}C$ is of the form

$$Y = \mathrm{diag}(y_1, \ldots, y_N) + \left[\frac{ig}{x_j - x_k}\right]_{j,k=1}^{N}. \tag{3.1.5}$$

The variables $y_j$'s are the momenta conjugated to the eigenvalues $x_j$ in the reduced system. Then the first result of [3] was that all the Calogero–Painlevé systems of [34] are the reduction on the particular value of the momentum (3.1.4) of a list of conjugation-invariant hamiltonian systems:

$$\begin{aligned}
\widetilde{H}_I &= Tr\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - \frac{t\mathbf{q}}{4}\right)\\
\widetilde{H}_{II} &= Tr\left(\frac{\mathbf{p}^2}{2} - \frac{1}{2}(\mathbf{q}^2 + \frac{t}{2})^2 - \theta\mathbf{q}\right)\\
t\widetilde{H}_{III} &= Tr\left(\mathbf{p}^2\mathbf{q}^2 - \Big(\mathbf{q}^2 + (\theta_0-\theta_1)\mathbf{q} - t\Big)\mathbf{p} - \theta_1\mathbf{q}\right)\\
\widetilde{H}_{IV} &= Tr\left(\mathbf{pq}\Big(\mathbf{p}-\mathbf{q}-t\Big) + \theta_0\mathbf{p} - (\theta_0+\theta_1)\mathbf{q}\right) \qquad (3.1.6)\\
t\widetilde{H}_{V} &= Tr\left(\mathbf{p}(\mathbf{p}+t)\mathbf{q}(\mathbf{q}-1) + (\theta_0-\theta_2)\mathbf{pq} + \theta_2\mathbf{p} + (\theta_0+\theta_1)t\mathbf{q}\right)\\
t(t-1)\widetilde{H}_{VI} &= Tr\Bigg(\mathbf{qpqpq} - t\mathbf{pq}^2\mathbf{p} + t\mathbf{pqp} - \mathbf{pqpq} - \theta\mathbf{qpq} + t(\theta_0+\theta_1)\mathbf{pq} +\\
&\quad + (\theta_0+\theta_t)\mathbf{pq} - \theta_0 t\mathbf{p} - \frac{1}{4}(k^2-\theta^2)\mathbf{q}\Bigg).
\end{aligned}$$

where $\theta_0$, $\theta_1$, $\theta_2$, $\theta_t$, $k$, are arbitrary parameters in $\mathbb{C}$ and for the case $VI$, $\theta = \theta_0+\theta_1+\theta_t$. These Hamiltonians should be thought of as non–commutative polynomials in $\mathbf{p}, \mathbf{q}$ generalizing the Okamoto Hamiltonians for the six Painlevé equations.

They showed that these Hamiltonians describe the isomonodromic deformations of a ODE in the $z$–plane for a matrix $\Phi(z)$ of size $2N \times 2N$, which reduces, for $N = 1$ to the classical Lax pair formulation for Painlevé equations (see, e.g. [21]).

**Painlevé VI.** We start the explanations corresponding to the details of these computations by the Painlevé VI case; the starting point is the Lax system given as

$$\begin{cases}
\dfrac{\partial\Phi}{\partial z} = \left(\dfrac{A_0}{z} + \dfrac{A_1}{z-1} + \dfrac{A_t}{z-t}\right)\Phi = \mathbf{A}(z)\Phi,\\[2ex]
\dfrac{\partial\Phi}{\partial t} = -\left(\dfrac{A_t}{z-t} + B\right)\Phi = \mathbf{B}(z)\Phi,
\end{cases} \qquad (3.1.7)$$

where the matrices are explicitly given by

$$A_0 := \begin{bmatrix} -1-\theta_t & \frac{\mathbf{q}}{t}-1 \\ 0 & 0 \end{bmatrix},\ A_1 := \begin{bmatrix} -\mathbf{qp}+\frac{1}{2}(k+\theta) & 1 \\ (\theta-\mathbf{qp})\mathbf{qp}+\frac{1}{4}(k^2-\theta^2) & \mathbf{qp}+\frac{1}{2}(k-\theta) \end{bmatrix},$$

$$A_t := \begin{bmatrix} \mathbf{qp}-\theta_0 & -\frac{\mathbf{q}}{t} \\ t(-\theta_0+\mathbf{pq})\mathbf{p} & -\mathbf{pq} \end{bmatrix},\ B := \begin{bmatrix} \frac{t\Big([\mathbf{q},\mathbf{p}]_+-\theta_0\Big)+\theta\mathbf{q}-[\mathbf{qp},\mathbf{q}]_+}{t(t-1)} & 0 \\ -\theta_0\mathbf{p}+\mathbf{pqp} & 0 \end{bmatrix}. \tag{3.1.8}$$

The expression $[X,Y]_+$ stands here for the anti-commutator of the noncommutative symbols $X,Y$, namely $[X,Y]_+ = XY+YX$. The partitioning is in $N\times N$ block, and all scalars are automatically considered mutliple of the identity matrix of size $N$. The isomonodromic equations consist in the "zero-curvature" equations for the pair

$$\frac{\partial \mathbf{A}(z)}{\partial t} - \frac{\partial \mathbf{B}(z)}{\partial z} + [\mathbf{A}(z),\mathbf{B}(z)] = 0 \tag{3.1.9}$$

and, with some elementary algebra, they become the following evolutionary system for the operators $\mathbf{p},\mathbf{q}$:

$$\begin{cases} \dot{\mathbf{q}} &= \mathcal{A}(\mathbf{q},\mathbf{p}) \\ \dot{\mathbf{p}} &= \mathcal{B}(\mathbf{q},\mathbf{p}), \end{cases} \tag{3.1.10}$$

where the non-commutative polynomials $\mathcal{A},\mathcal{B}$ are given by

$$\begin{aligned} t(t-1)\mathcal{A}(\mathbf{q},\mathbf{p}) :=& -\theta_0 t+(\theta_0+\theta_t)\mathbf{q}+(\theta_0+\theta_1)t\mathbf{q}-\theta\mathbf{q}^2-2\mathbf{qpq}+t[\mathbf{p},\mathbf{q}]_+ \\ & -[t\mathbf{p},\mathbf{q}^2]_+ + [\mathbf{qpq},\mathbf{q}]_+ \end{aligned}$$

$$\begin{aligned} t(t-1)\mathcal{B}(\mathbf{q},\mathbf{p}) :=& \frac{1}{4}(k^2-\theta^2)-(\theta_0+\theta_t)\mathbf{p}-(\theta_0+\theta_1)t\mathbf{p}+\theta[\mathbf{q},\mathbf{p}]_+-t\mathbf{p}^2+ \\ & +t[\mathbf{q},\mathbf{p}^2]_+ + \mathbf{p}(2\mathbf{q}-\mathbf{q}^2)\mathbf{p}-[\mathbf{q},\mathbf{pqp}]_+. \end{aligned}$$

The key observation, which is the initial thrust of our project, is the following: in the above derivation of the zero curvature equations the symbols $\mathbf{p},\mathbf{q}$ can be taken in some arbitrary non-commutative algebra. In the case of [3] where $\mathbf{p},\mathbf{q}$ are matrices, the above equations turn out, by inspection, to be Hamiltonian equations with respect to symplectic structure

(3.1.3) with a Hamiltonian given by

$$t(t-1)H = \mathrm{Tr}\Big(\mathbf{qpqpq} - t\mathbf{pq}^2\mathbf{p} + t\mathbf{pqp} - \mathbf{pqpq} - \theta\mathbf{qpq} + t(\theta_0+\theta_1)\mathbf{pq} + (\theta_0+\theta_t)\mathbf{pq} - \\ - \theta_0 t\mathbf{p} - \frac{1}{4}(k^2-\theta^2)\mathbf{q}\Big). \tag{3.1.11}$$

Analogous considerations apply to each of the other cases, and the result of these computation is summarized bellow.

**Painlevé V.** The initial consideration of this computation is the Lax system

$$\begin{cases} \dfrac{\partial\Phi}{\partial z} = \Big(-tE + \dfrac{A_0}{z} + \dfrac{A_1}{z-1}\Big)\Phi, \\ \dfrac{\partial\Phi}{\partial t} = B\Phi, \end{cases} \tag{3.1.12}$$

where the matrices are given by

$$E := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_0 := \begin{bmatrix} Z_1 - Z_1\mathbf{q} & Z_1Z_2 \\ t - t\mathbf{q} & tZ_2 \end{bmatrix},\ A_1 := \begin{bmatrix} S_1 & S_1S_2 \\ t\mathbf{q} & t\mathbf{q}S_2 \end{bmatrix}, \\ B := \begin{bmatrix} 0 & \dfrac{1}{t}\left(Z_1Z_2 + S_1S_2\right) \\ 1 & -z + \dfrac{t\mathbf{q} + [\mathbf{p},\mathbf{q}] + 1 - \theta_0 - 2\theta_1 - \theta_2}{t} \end{bmatrix} \tag{3.1.13}$$

where

$$Z_1 := \mathbf{qp} + \theta_0 + \theta_1, \qquad Z_2 := \frac{\mathbf{q}^2\mathbf{p} - \mathbf{qp} + (\theta_0+\theta_1)\mathbf{q} - \theta_1}{t},$$
$$S_1 := \mathbf{qpq} - \mathbf{pq} + (\theta_0+\theta_1)\mathbf{q} + \theta_2, \quad S_2 := -\frac{\mathbf{qp} - \mathbf{p} + \theta_0 + \theta_1}{t}.$$

The resulting equations of motion are obtained as the following

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{[\mathbf{p},\mathbf{q}^2]_+ - [\mathbf{p},\mathbf{q}]_+ + t(\mathbf{q}^2 - \mathbf{q}) + (\theta_0-\theta_2)\mathbf{q} + \theta_2}{t} \\ \dot{\mathbf{p}} = \dfrac{-[\mathbf{p}^2,\mathbf{q}]_+ + \mathbf{p}^2 - t([\mathbf{p},\mathbf{q}]_+ + \theta_0 + \theta_1) + (\theta_2 - \theta_0 + t)\mathbf{p}}{t} \end{cases} \tag{3.1.14}$$

which results in the Hamiltonian operator

$$t\widetilde{H}_V = Tr\Big(\mathbf{p}(\mathbf{p}+t)\mathbf{q}(\mathbf{q}-1)+(\theta_0-\theta_2)\mathbf{p}\mathbf{q}+\theta_2\mathbf{p}+(\theta_0+\theta_1)t\mathbf{q}\Big) \tag{3.1.15}$$

**Painlevé IV.** The computation starts with the Lax system

$$\begin{cases} \dfrac{\partial\Phi}{\partial z} = \begin{bmatrix} -\dfrac{\mathbf{pq}}{z} & \mathbf{qp}+\theta_0+\theta_1-\dfrac{\mathbf{pqp}+\theta_0\mathbf{p}}{z} \\ 1+\dfrac{\mathbf{q}}{z} & -z+t+\dfrac{\mathbf{qp}+\theta_0}{z} \end{bmatrix}\Phi, \\ \dfrac{\partial\Phi}{\partial t} = \begin{bmatrix} 0 & -\mathbf{qp}-\theta_0-\theta_1 \\ -1 & z-\mathbf{q}-t \end{bmatrix}\Phi, \end{cases} \tag{3.1.16}$$

with the following equations of motion for $\mathbf{p}$ and $\mathbf{q}$:

$$\begin{cases} \dot{\mathbf{q}} = [\mathbf{p},\mathbf{q}]_+ - \mathbf{q}^2 - t\mathbf{q}+\theta_0 \\ \dot{\mathbf{p}} = [\mathbf{p},\mathbf{q}]_+ - \mathbf{p}^2 + t\mathbf{p}\theta_0+\theta_1 \end{cases} \tag{3.1.17}$$

and the Hamiltonian operator

$$\widetilde{H}_{IV} = Tr\Big(\mathbf{pq}\big(\mathbf{p}-\mathbf{q}-t\big)+\theta_0\mathbf{p}-(\theta_0+\theta_1)\mathbf{q}\Big) \tag{3.1.18}$$

**Painlevé III.** The Lax system to start with is given by

$$\begin{cases} \dfrac{\partial\Phi}{\partial z} = \begin{bmatrix} \dfrac{\mathbf{qp}+\theta_1}{z-1} & t-\dfrac{\mathbf{qpq}+\theta_1\mathbf{q}}{z-1} \\ -\dfrac{\mathbf{p}-1}{z}+\dfrac{\mathbf{p}}{z-1} & \dfrac{\theta_0}{z}-\dfrac{\mathbf{pq}}{z-1} \end{bmatrix}\Phi, \\ \dfrac{\partial\Phi}{\partial t} = \begin{bmatrix} \dfrac{\mathbf{pq}-\theta_0}{t} & z \\ \dfrac{1}{t} & -\dfrac{\mathbf{qp}+\theta_1}{t} \end{bmatrix}\Phi, \end{cases} \tag{3.1.19}$$

which yields the following equations of motion:

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{[\mathbf{p},\mathbf{q}^2]_+ - \mathbf{q}^2 + (\theta_1-\theta_0)\mathbf{q} + t}{t} \\ \\ \dot{\mathbf{p}} = \dfrac{-[\mathbf{p}^2,\mathbf{q}]_+ + [\mathbf{p},\mathbf{q}]_+ - (\theta_1-\theta_0)\mathbf{p} + \theta_1}{t} \end{cases} \tag{3.1.20}$$

resulting in the Hamiltonian operator

$$t\widetilde{H}_{III} = Tr\Big(\mathbf{p}^2\mathbf{q}^2 - \big(\mathbf{q}^2 + (\theta_0-\theta_1)\mathbf{q} - t\big)\mathbf{p} - \theta_1\mathbf{q}\Big) \tag{3.1.21}$$

**Remark 3.1.2.** *The Hamiltonian* (3.1.21) *is one of the Hamiltonian operators corresponding to the Calogero-Painlevé III equation. Depending on the choice of the spectral type we either can obtain the above operator or one of the below operators:*

$$\textit{Type D7} \quad : \quad tH = Tr(\mathbf{pqpq} - \theta\mathbf{pq} + t\mathbf{p} + \mathbf{q}),$$

$$\textit{Type D8} \quad : \quad tH = Tr(\mathbf{pqpq} + \mathbf{pq} - \mathbf{q} - t\mathbf{q}^{-1}).$$

**Painlevé II.** The Lax system to begin with is given by

$$\begin{cases} \dfrac{\partial\Phi}{\partial z} = \begin{bmatrix} i\dfrac{z^2}{2} + i\mathbf{q}^2 + i\dfrac{t}{2} & z\mathbf{q} - i\mathbf{p} - \dfrac{\theta}{z} \\ \\ z\mathbf{q} + i\mathbf{p} - \dfrac{\theta}{z} & -i\dfrac{z^2}{2} - i\mathbf{q}^2 - i\dfrac{t}{2} \end{bmatrix}\Phi, \\ \\ \dfrac{\partial\Phi}{\partial t} = \begin{bmatrix} i\dfrac{z}{2} & \mathbf{q} \\ \\ \mathbf{q} & -i\dfrac{z}{2} \end{bmatrix}\Phi, \end{cases} \tag{3.1.22}$$

resulting in the following equations of motion:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p} \\ \\ \dot{\mathbf{p}} = 2\mathbf{q}^3 + t\mathbf{q} + \theta \end{cases} \tag{3.1.23}$$

and the Hamiltonian operator

$$\widetilde{H}_{II} = Tr\left(\frac{\mathbf{p}^2}{2} - \frac{1}{2}(\mathbf{q}^2 + \frac{t}{2})^2 - \theta\mathbf{q}\right) \tag{3.1.24}$$

**Painlevé I.** For the first Painlevé equation, the Lax system to initiate the computation is given by

$$\begin{cases} \dfrac{\partial \Phi}{\partial z} &= \begin{bmatrix} \mathbf{p} & z-\mathbf{q} \\ z^2 + z\mathbf{q} + \mathbf{q}^2 + \dfrac{t}{2} & -\mathbf{p} \end{bmatrix} \Phi, \\ \dfrac{\partial \Phi}{\partial t} &= \begin{bmatrix} 0 & \dfrac{1}{2} \\ \dfrac{z}{2} + \mathbf{q} & 0 \end{bmatrix} \Phi, \end{cases} \tag{3.1.25}$$

which gives the following equations of motion:

$$\begin{cases} \dot{\mathbf{q}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= \dfrac{3}{2}\mathbf{q}^2 + \dfrac{t}{4} \end{cases} \tag{3.1.26}$$

resulting in the Hamiltonian operator

$$\widetilde{H}_I = Tr\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - \frac{t\mathbf{q}}{4}\right) \tag{3.1.27}$$

## 3.2 Quantization

In view of the considerations above, we want to consider the canonical quantization of the symplectic structure (3.1.3). The main logic is that we keep equations (3.1.10), (3.1.14), (3.1.17), (3.1.20), (3.1.23), and (3.1.26), and seek a Hamiltonian formulation with *quantum* Hamiltonians. The canonical quantization in "Schrödinger" representation amounts to considering the entries of $\mathbf{q}$ as multiplication operators and the entries of $\mathbf{p}$ as corresponding differential operators as follows:

$$q_{ij} \longrightarrow q_{ij} \quad \text{and} \quad p_{ij} \longrightarrow \hbar \frac{\partial}{\partial q_{ji}}. \tag{3.2.1}$$

The effect of this canonical quantization is that we cannot simply take the expressions (3.1.6) as Hamiltonians generating the relevant equations of motions like (3.1.10), (3.1.14), (3.1.17), (3.1.20), (3.1.23), and (3.1.26), because there are issues of normal ordering. To explain the issue we point out that in the classical case the expressions $\mathrm{Tr}(\mathbf{pq})$ and $\mathrm{Tr}(\mathbf{qp})$ coincide, but if $\mathbf{p}, \mathbf{q}$ are quantum operators (3.2.1) then these two expressions differ. This should explain why the *quantum* version of the Hamiltonians (3.1.6) will be slightly different due to the fact that the correct scheme depends on the non-commutativity of the traces in this case. Note that the commutation relations that lead to these Hamiltonian operators read the following equations for each of the Calogero-Painlevé equations:

$$\hbar\dot{\mathbf{q}} = [H_J, \mathbf{q}] \quad \text{and} \quad \hbar\dot{\mathbf{p}} = [H_J, \mathbf{p}] \qquad J \in (I, \dots, VI) \tag{3.2.2}$$

Therefore the correct Hamiltonian operators to which we apply the quantization, are the following operators

$$\begin{aligned}
t\widetilde{H}_{III} &= Tr\left(\frac{\mathbf{p}^2\mathbf{q}^2 + \mathbf{q}^2\mathbf{p}^2}{2} - \frac{\mathbf{q}^2\mathbf{p} + \mathbf{p}\mathbf{q}^2}{2} - (\theta_0 - \theta_1)\mathbf{q}\mathbf{p} + t\mathbf{p} - \theta_1\mathbf{q}\right),\\
\widetilde{H}_{IV} &= Tr\left(\mathbf{p}\mathbf{q}\mathbf{p} - \frac{\mathbf{p}\mathbf{q}^2 + \mathbf{q}^2\mathbf{p}}{2} - t\mathbf{p}\mathbf{q} + \theta_0\mathbf{p} - (\theta_0 + \theta_1)\mathbf{q}\right),\\
t\widetilde{H}_{V} &= Tr\left(\frac{\mathbf{p}^2\mathbf{q}^2 + \mathbf{q}^2\mathbf{p}^2}{2} - \frac{\mathbf{p}^2\mathbf{q} + \mathbf{q}\mathbf{p}^2}{2} + \frac{t(\mathbf{p}\mathbf{q}^2 + \mathbf{q}^2\mathbf{p})}{2} + (\theta_0 - \theta_2 - t)\mathbf{p}\mathbf{q} + \right.\\
&\qquad \left. + \theta_2\mathbf{p} + (\theta_0 + \theta_1)t\mathbf{q}\right),\\
t(t-1)\widetilde{H}_{VI} &= Tr\left(\mathbf{q}\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q} - t\mathbf{p}\mathbf{q}^2\mathbf{p} + t\mathbf{p}\mathbf{q}\mathbf{p} - \frac{\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q} + \mathbf{q}\mathbf{p}\mathbf{q}\mathbf{p}}{2} - \theta\mathbf{q}\mathbf{p}\mathbf{q} + t(\theta_0 + \theta_1)\mathbf{p}\mathbf{q} + \right.\\
&\qquad \left. + (\theta_0 + \theta_t)\mathbf{p}\mathbf{q} - \theta_0 t\mathbf{p} - \frac{1}{4}(k^2 - \theta^2)\mathbf{q}\right).
\end{aligned} \tag{3.2.3}$$

whereas the Painlevé I and Painlevé II Hamiltonians remain formally the same. Note that we use $\mathbf{p}, \mathbf{q}$ here and below to denote the *quantum* operators, without further notice.

**Example 3.2.1.** *We compute the both quantum and classical commutator of* $\mathbf{p}$ *and* $Tr(\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q})$ *to show the origin of the difference in the quantum Hamiltonians* (3.2.3)*. We start with the classical computation where* $\{p_{ij}, q_{k\ell}\} = \delta_{i\ell}\delta_{jk}$*:*

$$\left\{\mathbf{p}, Tr\big(\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q}\big)\right\} = 2\mathbf{p}\mathbf{q}\mathbf{p}. \tag{3.2.4}$$

*In computing this we have used also the cyclicity of the trace.*

*Vice versa, considering the quantized operators, the same expression yields*

$$\begin{aligned}
[\mathbf{p}, Tr(\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q})] =& \sum_{i,j}\sum_{\alpha,\beta,\mu,\nu}\left[p_{ij}, p_{\alpha\beta}q_{\beta\mu}p_{\mu\nu}q_{\nu\alpha}\right] = \\
=& \sum_{i,j}\sum_{\alpha,\beta,\mu,\nu} p_{\alpha\beta}\Big(p_{ij}q_{\beta\mu}p_{\mu\nu}q_{\nu\alpha} - q_{\beta\mu}p_{\mu\nu}q_{\nu\alpha}p_{ij}\Big)
\end{aligned} \tag{3.2.5}$$

*we add and subtract* $q_{\beta\mu}p_{ij}p_{\mu\nu}q_{\nu\alpha}$ *to the expression inside the bracket, combining the terms and using the commutation relation* $[p_{ij}, q_{k\ell}] = \hbar\delta_{i\ell}\delta_{jk}$*:*

$$\begin{aligned}
[\mathbf{p}, Tr(\mathbf{p}\mathbf{q}\mathbf{p}\mathbf{q})] =& \hbar\sum_{i,j}\sum_{\alpha,\beta,\mu,\nu} p_{\alpha\beta}\left(\delta_{i\mu}\delta_{j\beta}p_{\mu\nu}q_{\nu\alpha} + \delta_{i\alpha}\delta_{j\nu}q_{\beta\mu}p_{\mu\nu}\right) = \\
=& \hbar\mathbf{p}^2\mathbf{q} + \hbar\mathbf{p}\mathbf{q}\mathbf{p} = 2\hbar\mathbf{p}\mathbf{q}\mathbf{p} + \hbar^2\mathbf{p}.
\end{aligned} \tag{3.2.6}$$

*Since the desired term in the equation of motion is* $2\hbar\mathbf{pqp}$ *we need to replace the term* $Tr(\mathbf{pqpq})$ *in the Hamiltonian with a "symmetrized" version* $\frac{1}{2}Tr(\mathbf{pqpq}+\mathbf{qpqp})$. *Indeed, one then similarly computes:*

$$[\mathbf{p}, \frac{1}{2}Tr(\mathbf{pqpq}+\mathbf{qpqp})] = 2\hbar\mathbf{pqp} \tag{3.2.7}$$

*which is one of the terms that appears in the expression of* $\mathcal{A}$ *in equation of motion* (3.1.11).

### 3.2.1 Quantization in radial form

Having established the correct context of the quantization of the isomonodromic equations, we proceed now to the quantum version of the Kazhdan-Kostant-Sternberg reduction (3.1.4). The first step is to express the quantum Hamiltonians (3.2.3) in terms of the eigenvalues. The fact that this is at all possible is simply a consequence of the invariance of the Hamiltonians under conjugations. To do so, we need to use the Harish-Chandra homomorphism [10].

**Definition 3.2.2.** *Let* $\mathcal{M}$ *be the manifold consisting of the diagonalizable matrices and denote*

$$\mathcal{D}(\mathcal{M})^G \quad \textit{and} \quad \mathcal{D}(Diag(GL_n))^W$$

*the adjoint-invariant subset of differential operators over* $\mathcal{M}$ *and the Weyl-invariant subset of differential operators over diagonal matrices, respectively.*

*The canonical isomorphism*

$$\mathcal{H}_c : \mathcal{D}(\mathcal{M})^G \longrightarrow \mathcal{D}(Diag(GL_n))^W \tag{3.2.8}$$

*is called the Harish-Chandra map.*

This means the following; for a character function $\Psi(Q)$ (i.e. $\Psi(Q) = \Psi(GQG^{-1})$, $G \in GL_n$) and $\mathcal{L}$ a differential operator invariant under the adjoint map we have

$$(\mathcal{L}\Psi)|_{Diag} = \mathcal{H}_c(\mathcal{L})(\Psi|_{Diag})$$

where $\mathcal{H}_c(\mathcal{L})$ is a differential operator on the eigenvalues.

Our goal now is to make this isomorphism completely explicit and subsequently express all Hamiltonians in (3.2.3) as differential operators acting on the eigenvalues when applied to character functions or pseudo-character functions, namely $\Psi(\mathbf{q}) = \Psi(G\mathbf{q}G^{-1})\mathrm{e}^{\theta(G,\mathbf{q})}$, with $\theta$ an appropriate cocycle.

**Explicit construction of the Harish-Chandra isomorphism.** To make the Harish-Chandra homomorphism (3.2.8) explicit, we write the matrix $Q = Z + M$ with $Z$ diagonal and $M$ off-diagonal: we then act with an infinitesimal conjugation up to order two in $M$ to diagonalize it. Concretely this means the following; we conjugate the matrix $Q = Z + M$ by a matrix of the form $G := e^{A^{(1)}+A^{(2)}}$ where $A^{(1)}$ is assumed to be of first order in the entries of $M$ and $A^{(2)}$ of second order and both are off–diagonal matrices.

We then impose that the conjugation of $Q$ by $G$ is diagonal up to order 2. Then

$$e^{ad_{A^{(1)}+A^{(2)}}}(Z+M) = Z+M+[A^{(1)}+A^{(2)}, Z+M] + \frac{1}{2}[A^{(1)},[A^{(1)},Z]] + \mathcal{O}(3) = \\ = Z+M+[A^{(1)},Z]+[A^{(1)},M]+[A^{(2)},Z]+\frac{1}{2}[A^{(1)},[A^{(1)},Z]]+\mathcal{O}(3) \tag{3.2.9}$$

where $\mathcal{O}(3)$ denotes terms of order 3 or higher in the entries of $M$. We need to impose that the result is a diagonal matrix up to the indicated order. Separating the equations according to their order in $M$ we obtain

$$[Z, A^{(1)}] = M \qquad \text{at order 1,} \tag{3.2.10}$$

$$[Z, A^{(2)}] = [A^{(1)}, M] + \frac{1}{2}[A^{(1)},[A^{(1)},Z]] \qquad \text{at order 2.} \tag{3.2.11}$$

The matrices $A^{(1,2)}$ are off-diagonal, and the $ad_Z$ operator is invertible on the subspace of off-diagonal matrices, so that we can solve the two equations above to obtain

$$A^{(1)}_{ab} = \frac{M_{ab}}{z_a - z_b} \qquad A^{(2)}_{ab} = -\frac{1}{2}\frac{[A^{(1)},[A^{(1)},Z]]_{ab}}{z_a - z_b} = \frac{1}{2}\frac{[A^{(1)},M]_{ab}}{z_a-z_b} = \frac{M_{ac}M_{cb}}{(z_a-z_c)(z_a-z_b)}. \tag{3.2.12}$$

Substituting (3.2.12) into (3.2.9) we obtain a diagonal matrix $\widetilde{Z}$ which is a shift of the marix $Z$ as follows

$$\widetilde{Z} = Z + \frac{1}{2}[A^{(1)}, M]_D + \mathcal{O}(3) = A^{(2)} + \frac{1}{2}\mathrm{diag}\left(\sum_c \frac{M_{*c}}{z_* - z_c}M_{c*} - M_{*c}\frac{M_{c*}}{z_c - z_*}\right) + \mathcal{O}(3) = \\ = Z + \mathrm{diag}\left(\sum_d \frac{M_{*d}M_{d*}}{z_* - z_d}\right). \tag{3.2.13}$$

We now show how to use the above diagonalization to second order (3.2.13) to construct the Harish–Chandra homomorphism; we anticipate that the reason why we expand up to order 2 is that all the operators we consider are at most quadratic in the momenta $\mathbf{p}$ and hence translate to differential operators of order 2. If we had to consider the Harish-Chandra homomorphism for operators of higher order, we would have to perform the above diagonalization up to the corresponding order.

**Space of radial functions.** When considering the quantum version of the Kazhdan-Kostant-Sternberg reduction, as explained in chapter 2, the choice of the special value of the momentum $M$ (3.1.4) is replaced by the requirement that the quantum operators act on specific representations. We start with a general discussion on equivariant functions.

Let $\mathbb{V}$ be a vector space carrying a representation $\gamma$ of $G = \mathrm{GL}_N$ and let $\Psi : \mathrm{Mat}_{N\times N}(\mathbb{C}) =$

$Lie(G) \to \mathbb{V}$ an $\gamma$*–equivariant* function in the sense that

$$\Psi(Q) = \gamma(g^{-1})\Psi(gQg^{-1}) \tag{3.2.14}$$

where $g \in \mathrm{GL}_N(\mathbb{C})$ and $\gamma : \mathrm{GL}_N(\mathbb{C}) \to Aut(\mathbb{V})$ is a representation. Let us denote

$$H_\gamma := \left\{\Psi : \mathrm{Mat}_{N\times N}(\mathbb{C}) \to V, \quad \gamma\text{–equivariant}\right\}. \tag{3.2.15}$$

For simplicity we denote by the same symbol $\gamma$ the representation of $SL_N$, the corresponding representation of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_N$ as well as its natural extension to the universal enveloping algebra $U(\mathfrak{g})$. We recall that the zero–weight subspace of the $SL_N$–representation $\mathbb{V}$ is

$$\mathbb{V}(0) := \bigcap_{H\in\mathfrak{h}} \mathrm{Ker}\gamma(H) \tag{3.2.16}$$

where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}_N$ (traceless diagonal matrices).

**Lemma 3.2.3.** *Let $\mathcal{D} \subset \mathrm{Mat}_{N\times N}(\mathbb{C})$ consist of the subspace of diagonal matrices. Then any equivariant function $\Psi$ restricts to a function from $\mathcal{D}$ to $\mathbb{V}(0)$.*

*Proof.* Consider a matrix $g = \mathrm{e}^{\epsilon H}$ with $H \in \mathfrak{h}$; then $\Psi(Q) = \gamma(\mathrm{e}^{-\epsilon H})\Psi(\mathrm{e}^{\epsilon H}Q\mathrm{e}^{-\epsilon H})$. Restricting $Q = Z \in \mathcal{D}$ we have $\Psi(Z) = \gamma(\mathrm{e}^{-\epsilon H})\Psi(Z)$. We now take the derivative with respect to $\epsilon$ at $\epsilon = 0$ and we obtain $\gamma(-H)\Psi(Z) = 0$. Since $H$ is arbitrary in $\mathfrak{h}$ it follows that $\Psi(Z) \in \mathbb{V}(0)$. ■

Following [10], the quantum Hamiltonian reduction that correspond to the Kazdan-Kostant-Sternberg orbit, consists in taking a particular representation $\gamma$ of $\mathfrak{sl}_N$; the main feature of the $\mathfrak{g}$–module (which we denote by $\mathbb{V}_\kappa$) is that the zero weight space $\mathbb{V}_\kappa(0)$ is unidimensional. We denote with $\mathbf{c}$ a spanning element. Specifically, $\mathbb{V}_\kappa$ consists of the space of functions of the form

$$F(\xi_1, \ldots, \xi_N) = \left(\prod_{j=1}^N \xi_j\right)^\kappa f(\vec{\xi}) \tag{3.2.17}$$

where $f(\vec{\xi})$ is a rational function with zero degree of homogeneity. The representation of the Lie algebra $\mathfrak{sl}_N$ is then the one obtained by restriction of the following $\mathfrak{gl}_N$ representation

$$\gamma(\mathbb{E}_{ab}) = \xi_a \frac{\partial}{\partial \xi_b}, \quad a, b = 1, \ldots, N. \tag{3.2.18}$$

It is easy then to see that $\mathbb{V}_\kappa(0) = \mathbb{C}\{\prod_{j=1}^N \xi_j^\kappa\}$.

Keeping this in mind we illustrate the type of computations needed to compute the extended Harish–Chandra homomorphism in the following example.

**Example 3.2.4.** *To illustrate the type of computations necessary, we consider the quantum radial reduction of the operator* $\mathrm{Tr}(\mathbf{q}^{\mathrm{k}}\mathbf{p}^2)$*. Using the form of the quantum operators* $\mathbf{p}, \mathbf{q}$ *we* *obtain*

$$Tr(\mathbf{q}^k\mathbf{p}^2)\Psi(Q) = \left(\sum_{\rho,\sigma,\tau}(q^k)_{\rho\sigma}p_{\sigma\tau}p_{\tau\rho}\right)\Psi(Q) = \left(\hbar^2\sum_{\rho,\sigma,\tau}(q^k)_{\rho\sigma}\partial_{q_{\tau\sigma}}\partial_{q_{\rho\tau}}\right)\Psi(Q). \tag{3.2.19}$$

*Since the function* $\Psi$ *is* $\gamma$*–equivariant and the operator is* Ad*–invariant we can write* $\Psi(Q) = \gamma(g^{-1})\Psi(Z)$ *where* $g$ *is the matrix diagonalizing* $Q$ *and* $Z$ *is the diagonal matrix of its eigenvalues (this can be done on the set of diagonalizable matrices* $Q$ *whose complement of non–diagonalizable matrices is of zero measure and hence inessential to our considerations). We then consider matrices of the form* $Q = Z + M$ *with* $Z$ *diagonal and* $M$ *off–diagonal and its diagonalization up to order* 2 *as in* (3.2.9)*. We then need to perform the derivatives and, at the end of the computation, restrict them to the locus of diagonal matrices* $Q = Z$*. Using equation* (3.2.13) *and the matrices* $A^{(1,2)}$ *introduced in* (3.2.11) *we can continue the above computation by noticing that the terms involving the multiplication operator* $\mathbf{q}$ *can be directly evaluated at* $Q = Z$ *setting* $M = 0$*:*

$$\begin{aligned} Tr(\mathbf{q}^k\mathbf{p}^2)\Psi(Q) &= \left(\hbar^2\sum_{\rho,\sigma,\tau}\delta_{\rho\sigma}z_\sigma^k\partial_{q_{\tau\sigma}}\partial_{q_{\rho\tau}}\right)\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z}) = \\ &= \left(\hbar^2\sum_{\sigma,\tau}z_\sigma^k\partial_{q_{\tau\sigma}}\partial_{q_{\sigma\tau}}\right)\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z}). \end{aligned} \tag{3.2.20}$$

*The second order operator* $\sum_{\rho,\sigma} z_\sigma\partial_{q_{\sigma\rho}}\partial_{q_{\rho\sigma}}$ *written in terms of* $Z, M$ *becomes the operator* $\sum_\rho z_\rho^k\partial_{z_\rho}^2 + \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} z_\rho^k\partial_{M_{\sigma\rho}}\partial_{M_{\rho\sigma}}$*; the part involving the derivatives with respect to* $z_\rho$ *can be directly evaluated at* $Q = Z$ *while we postpone the evaluation of the part involving the derivatives in* $M_{\rho,\sigma}$*:*

$$(3.2.20) = \hbar^2\sum_\sigma z_\sigma^k\partial_{z_\sigma}^2\Psi(Z) + \underbrace{\hbar^2\sum_{\substack{\sigma,\tau\\ \sigma\neq\tau}} z_\sigma^k\partial_{M_{\tau\sigma}}\partial_{M_{\sigma\tau}}\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z})}_{*} \tag{3.2.21}$$

*where* $\widetilde{Z} = Z + diag\left(\sum_d \frac{M_{\star d}M_{d\star}}{z_\star - z_d}\right)$ *as in* (3.2.13)*. Consider now the term marked with an asterisk: since* $\widetilde{Z} - Z$ *is a quadratic expression in the entries of* $M$*, if we differentiate once* $\Psi$ *or* $\gamma$ *by* $M_{\rho\sigma}$*, by the chain rule there will be a multiplication by entries of* $M$ *in the result. Thus, subsequent evaluation at* $M = 0$ *will eliminate these terms. Therefore we need to* *consider the second order operator acting on $\gamma$ or $\Psi$ separately. When acting on $\Psi$ we have*

$$\begin{aligned}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} z_\sigma^k \partial_{M_{\sigma\rho}}\partial_{M_{\rho\sigma}}\Psi(\widetilde{Z})\Big|_{M=0} &= \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} z_\sigma^k \frac{\partial}{\partial M_{\sigma\rho}}\left(\frac{M_{\sigma\rho}\partial_{z_\sigma}\Psi}{z_\sigma - z_\rho} + \frac{M_{\sigma\rho}\partial_{z_\rho}\Psi}{z_\rho - z_\sigma}\right)\Big|_{M=0} = \\ &= \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} z_\sigma^k \left(\frac{\partial_{z_\sigma} - \partial_{z_\rho}}{z_\sigma - z_\rho}\right)\Psi(Z).\end{aligned} \tag{3.2.22}$$

*For the computation of the second term involving the representation $\gamma$ we note that*

$$\begin{aligned}&\partial_{M_{\sigma\tau}} A^{(1)} = \sum_{\substack{\sigma,\tau\\ \sigma\neq\tau}} \frac{E_{\sigma\tau}}{z_\sigma - z_\tau} \quad , \quad \partial_{M_{\tau\sigma}}\partial_{M_{\sigma\tau}} A^{(1)} = 0 \\ &\partial_{M_{\sigma\tau}} A^{(2)} = \sum_{\substack{\sigma,\tau,\nu\\ \nu\neq\sigma\neq\tau}} \frac{M_{\tau\nu}E_{\sigma\nu}}{(z_\sigma - z_\tau)(z_\sigma - z_\nu)} + \sum_{\substack{\sigma,\tau,\mu\\ \mu\neq\sigma\neq\tau}} \frac{M_{\mu\sigma}E_{\mu\tau}}{(z_\mu - z_\sigma)(z_\mu - z_\tau)} \quad , \quad \partial_{M_{\tau\sigma}}\partial_{M_{\sigma\tau}} A^{(2)} = 0.\end{aligned} \tag{3.2.23}$$

*Therefore the action of these differential operators on the group element $\mathrm{e}^{-A^{(1)}-A^{(2)}}$ gives (retaining the terms up to order 2 in $M$ in the expansion of the exponential, since all higher order terms will give zero contribution when evaluated at $M = 0$)*

$$\begin{aligned}\partial_{M_{\sigma\tau}} e^{-A^{(1)}-A^{(2)}} &= -\partial_{M_{\sigma\tau}} A^{(1)} - \partial_{M_{\sigma\tau}} A^{(2)} + \frac{[\partial_{M_{\sigma\tau}} A^{(1)}, A^{(1)}]_+}{2} \\ \partial_{M_{\tau\sigma}}\partial_{M_{\sigma\tau}} e^{-A^{(1)}-A^{(2)}}\Big|_{M=0} &= -\frac{[E_{\tau\sigma}, E_{\sigma\tau}]_+}{2(z_\sigma - z_\tau)^2}.\end{aligned} \tag{3.2.24}$$

*Here $E_{\tau\sigma}$ denote the elementary matrices. Note also that $E_{\tau\sigma}E_{\sigma\tau} = E_{\tau\tau}$ are diagonal matrices and $[E_{\tau\sigma}, E_{\sigma\tau}]_+ = E_{\tau\tau} + E_{\sigma\sigma}$. Hence, the equation* (3.2.21) *yields*

$$\begin{aligned}Tr(\mathbf{q}^k\mathbf{p}^2)\Psi(Q) =& \hbar^2 \sum_\sigma z_\sigma^k \partial_{z_\sigma}^2 \Psi(Z) + \\ &+ \hbar^2 \sum_{\substack{\sigma,\tau\\ \sigma\neq\tau}} z_\sigma^k \left( -\frac{\gamma([E_{\tau\sigma}, E_{\sigma\tau}]_+)}{2(z_\sigma - z_\tau)^2}\Psi(\widetilde{Z}) + \left(\frac{\partial_{z_\sigma} - \partial_{z_\tau}}{z_\sigma - z_\tau}\right)\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z})\right)\Big|_{M=0}.\end{aligned} \tag{3.2.25}$$

*To complete the computation, we recall that under the assumption for the representation space $\mathbb{V}_\kappa$ (see* (3.2.18)*) we easily see that $\gamma(E_{\sigma\tau}E_{\tau\sigma}) = \kappa(\kappa+1)\mathrm{Id}_{\mathbb{V}_\kappa(0)}$. Recall also (Lemma 3.2.3) that $\Psi$ evaluated on diagonal matrices takes values in the zero weight space $\mathbb{V}_\kappa(0)$. Therefore we conclude that $\gamma([\mathbb{E}_{\tau\sigma}\mathbb{E}_{\sigma\tau}]_+)$ reduces simply to the multiplication by $2\kappa(\kappa+1)$.*

*This leads finally to the result*

$$Tr(\mathbf{q}^k\mathbf{p}^2)\Psi(Q) = \Bigg(\hbar^2\sum_\sigma z_\sigma^k\partial_{z_\sigma}^2 - \hbar^2\kappa(\kappa+1)\sum_{\substack{\sigma,\tau\\ \sigma\neq\tau}}\frac{z_\sigma^k}{(z_\sigma-z_\tau)^2} + \hbar^2\sum_{\substack{\sigma,\tau\\ \sigma\neq\tau}}\frac{z_\sigma^k\partial_{z_\sigma} - z_\tau^k\partial_{z_\tau}}{z_\sigma - z_\tau}\Bigg)\Psi(Z) - \\ - \hbar^2\sum_{j=1}^{k-1}\sum_\sigma z_\sigma^j\sum_\tau z_\tau^{k-j-1}\partial_{z_\tau}\Psi(Z) - \hbar^2(N-k)\sum_\tau z_\tau^{k-1}\partial_{z_\tau}\Psi(Z). \tag{3.2.26}$$

Throughout the following section, we use this method to obtain the quantized Calogero-Painlevé I-VI Hamiltonian operators.

## 3.3 Quantized Calogero-Painlevé I-VI

We apply the quantum Hamiltonian KKS reduction explained in the previous sections to the list of Hamiltonians (3.2.3).

Moreover, note that unlike the classical case, in the quantum case we have non-commutative operator-valued matrices $\mathbf{p}$ and $\mathbf{q}$ so that $\mathrm{Tr}(\mathbf{pq})$ is not equal to $\mathrm{Tr}(\mathbf{qp})$. For example, we have the following

$$\begin{aligned} \mathrm{Tr}(\mathbf{pq}) =& \mathrm{Tr}(\mathbf{qp}) + \hbar N^2 \\ \mathrm{Tr}(\mathbf{pqp}) =& \mathrm{Tr}(\mathbf{qp}^2) + \hbar N\mathrm{Tr}(\mathbf{p}) \\ \mathrm{Tr}(\mathbf{qpq}) =& \mathrm{Tr}(\mathbf{q}^2\mathbf{p}) + \hbar N\mathrm{Tr}(\mathbf{q}) \\ \mathrm{Tr}(\mathbf{pq}^2) =& \mathrm{Tr}(\mathbf{q}^2\mathbf{p}) + 2\hbar N\mathrm{Tr}(\mathbf{q}) \\ \mathrm{Tr}(\mathbf{p}^2\mathbf{q}^2) =& \mathrm{Tr}(\mathbf{q}^2\mathbf{p}^2) + 2\hbar N\mathrm{Tr}(\mathbf{qp}) + 2\hbar\mathrm{Tr}(\mathbf{q})\mathrm{Tr}(\mathbf{p}) + \hbar^2 N(1+N^2)\ . \end{aligned} \tag{3.3.1}$$

**Calogero-Painlevé I.** We start from the Hamiltonian operator corresponding to Calogero-Painlevé $I$ and we apply it to the $\gamma$–equivariant wave function $\Psi(Q)$, and then we apply the quantization (3.2.1) to the result.

$$\widetilde{H}_I\Psi(Q) = \mathrm{Tr}\left(\frac{\mathbf{p}^2}{2} - \frac{\mathbf{q}^3}{2} - \frac{t\mathbf{q}}{4}\right)\Psi(Q) = \left(\frac{1}{2}\sum_{\rho,\sigma}p_{\rho\sigma}p_{\sigma\rho} - \frac{1}{2}\sum_{\rho,\sigma,\tau}q_{\rho\sigma}q_{\sigma\tau}q_{\tau\rho} - \frac{t}{4}\sum_{\rho,\sigma}\delta_{\rho\sigma}q_{\rho\sigma}\right)\Psi(Q) \tag{3.3.2}$$

applying the quantization yields

$$\widetilde{H}_I\Psi(Q) = \underbrace{\frac{\hbar^2}{2}\sum_{\rho,\sigma}\partial_{q_{\sigma\rho}}\partial_{\rho\sigma}\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z})}_{*} - \frac{1}{2}\sum_{\rho,\sigma,\tau}\delta_{\rho\sigma}z_\sigma\delta_{\sigma\tau}z_\tau\delta_{\tau\rho}z_\rho\gamma(e^{-A^{(1)}-A^{(2)}})\Psi(\widetilde{Z}) - \\ - \frac{t}{4}\sum_\rho z_\rho\Psi(Z) \tag{3.3.3}$$

where $\widetilde{Z} = Z + \mathrm{diag}\Bigg( \sum_d \frac{M_{\star d} M_{d \star}}{z_\star - z_d} \Bigg)$ as in (3.2.13). Note that the terms involving only the multiplication operator $\mathbf{q}$ can be directly evaluated at $Q = Z$ setting $M = 0$. The second order operator $\sum_{\rho,\sigma} \partial_{q_{\sigma\rho}} \partial_{q_{\rho\sigma}}$ in terms of $Z, M$ becomes the operator $\sum\limits_{\rho} \partial^2_{z_\rho} + \sum\limits_{\substack{\rho,\sigma \\ \rho \neq \sigma}} \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}}$ as illustrated in the example of the operator $\mathrm{Tr}(\mathbf{q}^k \mathbf{p}^2)$; according to the general result of the Example 3.2.4, we substitute the following for the term $(*)$

$$\sum_{\rho,\sigma} \partial_{q_{\sigma\rho}} \partial_{\rho\sigma} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) = \\ = \Bigg( \hbar^2 \sum_{\sigma} \partial^2_{z_\sigma} - \hbar^2 \kappa(\kappa+1) \sum_{\substack{\sigma,\tau \\ \sigma \neq \tau}} \frac{1}{(z_\sigma - z_\tau)^2} + \hbar^2 \sum_{\substack{\sigma,\tau \\ \sigma \neq \tau}} \frac{\partial_{z_\sigma} - \partial_{z_\tau}}{z_\sigma - z_\tau} \Bigg) \Psi(Z) \tag{3.3.4}$$

Therefore, the quantized Hamiltonian operator corresponding to Calogero-Painlevé I will be the following

$$\widetilde{H}_I = \frac{\hbar^2}{2} \sum_{\substack{\rho,\sigma \\ \rho \neq \sigma}} \frac{\partial_{z_\sigma} - \partial_{z_\rho}}{z_\sigma - z_\rho} - \frac{\hbar^2 \kappa(\kappa+1)}{2} \sum_{\substack{\rho,\sigma \\ \rho \neq \sigma}} \frac{1}{(z_\sigma - z_\rho)^2} + \frac{\hbar^2}{2} \sum_{\rho} \partial^2_{z_\rho} - \sum_{\rho} \Bigg( \frac{z_\rho^3}{2} + \frac{t z_\rho}{4} \Bigg) \tag{3.3.5}$$

**Calogero-Painlevé II.** Similar to the previous case, we apply the Hamiltonian operator corresponding to Calogero-Painlevé $II$ to the $\gamma$–equivariant wave function $\Psi(Q)$, and then we apply the quantization (3.2.1) to the result:

$$\begin{aligned} \widetilde{H}_{II} \Psi(Q) =& \mathrm{Tr}\Big( \frac{\mathbf{p}^2}{2} - \frac{1}{2} (\mathbf{q}^2 + \frac{t}{2})^2 - \theta \mathbf{q} \Big) \Psi(Q) = \\ =& \Bigg( \frac{\hbar^2}{2} \sum_{\rho,\sigma} \partial_{q_{\sigma\rho}} \partial_{q_{\rho\sigma}} - \frac{1}{2} \sum_{\rho,\sigma} \Big( (\delta^2_{\sigma\rho} q^2_{\rho\sigma}) + \frac{t}{2} \Big)^2 - \theta \delta_{\sigma\rho} q_{\rho\sigma} \Bigg) \Psi(Q). \end{aligned} \tag{3.3.6}$$

Following the same logic used in the example (3.2.4) we can continue the computation

$$\widetilde{H}_{II} \Psi(Q) = \frac{\hbar^2}{2} \sum_{\rho,\sigma} \partial_{q_{\sigma\rho}} \partial_{q_{\rho\sigma}} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) - \frac{1}{2} \sum_{\rho} (z_\rho^2 + \frac{t}{2})^2 \Psi(Z) - \theta \sum_{\rho} z_\rho \Psi(Z). \tag{3.3.7}$$

The term including the derivatives with respect to $\mathbf{q}$ will result in differential operators with respect to $z$ and $M$, hence

$$\begin{aligned} (3.3.7) =& \frac{\hbar^2}{2} \sum_{\rho} \partial^2_{z_\rho} \Psi(Z) + \underbrace{\frac{\hbar^2}{2} \sum_{\substack{\rho,\sigma \\ \rho \neq \sigma}} \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \Big( \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) \Big)}_{*} - \frac{1}{2} \sum_{\rho} (z_\rho^2 + \frac{t}{2})^2 \Psi(Z) - \\ & - \theta \sum_{\rho} z_\rho \Psi(Z). \end{aligned} \tag{3.3.8}$$

The term indicated by the asterisk is dealt with in complete analogy to the similarly marked term in (3.2.21). We thus obtain

$$\begin{aligned}
&\widetilde{H}_{II}\Psi(Q) = \\
&=\frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho}\Psi(Z) + \frac{\hbar^2}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\left(-\frac{1}{2}\gamma\left(\frac{[E_{\rho\sigma},E_{\sigma\rho}]_+}{(z_\sigma - z_\rho)^2}\right) + \gamma(e^{-A^{(1)}-A^{(2)}})\left(\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho - z_\sigma}\right)\right)\Psi(\widetilde{Z}) - \\
&\quad -\frac{1}{2}\sum_\rho (z_\rho^2 + \frac{t}{2})^2\Psi(Z) - \theta\sum_\rho z_\rho \Psi(Z).
\end{aligned} \tag{3.3.9}$$

Hence, putting these all together, we obtain

$$\begin{aligned}
\widetilde{H}_{II}\Psi(Q) =& \frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho}\Psi(Z) - \frac{\hbar^2\kappa(\kappa+1)}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{1}{(z_\rho - z_\sigma)^2}\Psi(Z) + \frac{\hbar^2}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho - z_\sigma}\Psi(Z) - \\
&-\frac{1}{2}\sum_\rho (z_\rho^2 + \frac{t}{2})^2\Psi(Z) - \theta\sum_\rho z_\rho\Psi(Z).
\end{aligned} \tag{3.3.10}$$

The equation takes a more convenient form if we apply to the wave function a gauge transformation of the form

$$\Psi(Z) = \exp\left[-\frac{1}{\hbar}\sum_\alpha\left(\frac{z_\alpha^3}{3} + \frac{t}{2}z_\alpha\right)\right]\Phi(Z). \tag{3.3.11}$$

As a result, the Schrödinger equation $\hbar\partial_t\Psi(Z) = \widetilde{H}_{II}\Psi(Z)$ is transformed into the one with the new Hamiltonian

$$\begin{aligned}
\hat{H}_{II} =& \frac{\hbar^2}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{z_\sigma}-\partial_{z_\rho}}{z_\sigma - z_\rho} - \frac{\hbar^2\kappa(\kappa+1)}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{1}{(z_\sigma - z_\rho)^2} + \frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho} - \hbar\sum_\rho\left(z_\rho^2 + \frac{t}{2}\right)\partial_{z_\rho} + \\
&+\left(\frac{1}{2} - \theta - \hbar N\right)\sum_\rho z_\rho.
\end{aligned} \tag{3.3.12}$$

**Calogero-Painlevé III.** In this section, we apply the Hamiltonian operator corresponding to Calogero-Painlevé $III$ to the $\gamma$–equivariant wave function $\Psi(Q)$, and then we apply the quantization (3.2.1) to the result:

$$t\widetilde{H}_{III}\Psi(Q) = Tr\left(\frac{\mathbf{p}^2\mathbf{q}^2 + \mathbf{q}^2\mathbf{p}^2}{2} - \frac{\mathbf{q}^2\mathbf{p} + \mathbf{p}\mathbf{q}^2}{2} - (\theta_0 - \theta_1)\mathbf{q}\mathbf{p} + t\mathbf{p} - \theta_1\mathbf{q}\right)\Psi(Q) \tag{3.3.13}$$

by applying the quantization and permuting the operators $\mathbf{p}$ and $\mathbf{q}$ we obtain

$$
\begin{aligned}
t\widetilde{H}_{III}\Psi(Q) =& \Big(\hbar^2 \sum_{\rho,\sigma,\tau,\eta} q_{\rho\sigma}q_{\sigma\tau}\partial_{q_{\eta\tau}}\partial_{q_{\rho\eta}} + \hbar^2 N \sum_{\rho,\sigma} q_{\sigma\rho}\partial_{q_{\sigma\rho}} + \hbar^2 \sum_{\rho,\sigma} \delta_{\rho\sigma}q_{\rho\sigma} \sum_{\tau} \partial_{q_{\tau\tau}} - \\
&- \hbar \sum_{\rho,\sigma,\tau} q_{\rho\sigma}q_{\sigma\tau}\partial_{q_{\rho\tau}} - (\hbar N + \theta_1) \sum_{\rho,\sigma} \delta_{\rho\sigma}q_{\rho\sigma} - (\theta_0 - \theta_1)\hbar \sum_{\rho\sigma} q_{\rho\sigma}\partial_{q_{\rho\sigma}} + \\
&+ t\hbar \sum_{\rho} \partial_{q_{\rho\rho}} + \frac{\hbar^2 N(1+N^2)}{2}\Big)\Psi(Q).
\end{aligned} \tag{3.3.14}
$$

Following the same argument as the example (3.2.4) and combining the similar terms the quantized Hamiltonian operator for Calogero-Painlevé III is obtained as following

$$
\begin{aligned}
t\widetilde{H}_{III} =& \hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{z_\sigma^2\partial_{z_\sigma} - z_\rho^2\partial_{z_\rho}}{z_\sigma - z_\rho} + \sum_{\rho} \Big(\hbar^2 z_\rho^2 \partial_{z_\rho}^2 - \hbar\Big(z_\rho^2 + (-2\hbar + \theta_0 - \theta_1)z_\rho - t\Big)\partial_{z_\rho} - \\
&- (\hbar N + \theta_1)z_\rho\Big) - \frac{\hbar^2\kappa(\kappa+1)}{2} \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{z_\rho^2 + z_\sigma^2}{(z_\sigma - z_\rho)^2} + \frac{\hbar^2 N(1+N^2)}{2}.
\end{aligned} \tag{3.3.15}
$$

**Calogero-Painlevé IV.** Similar to the previous cases, we apply the Hamiltonian operator corresponding to Calogero-Painlevé $IV$ to the $\gamma$–equivariant wave function $\Psi(Q)$, and then we apply the quantization (3.2.1) to the result:

$$
H_{IV}\Psi(Q) = Tr\Big(\mathbf{pqp} - \frac{\mathbf{pq}^2 + \mathbf{q}^2\mathbf{p}}{2} - t\mathbf{pq} + \theta_0\mathbf{p} - (\theta_0 + \theta_1)\mathbf{q}\Big)\Psi(Q) \tag{3.3.16}
$$

by commuting the operators and combining the similar terms we obtain

$$
\begin{aligned}
H_{IV}\Psi(Q) =& \Big(\sum_{\rho,\sigma,\tau} q_{\sigma\tau}p_{\rho\sigma}p_{\tau\rho} - \sum_{\rho,\sigma,\tau} q_{\rho\sigma}q_{\sigma\tau}p_{\tau\rho} - t\sum_{\rho,\sigma} q_{\sigma\rho}p_{\rho\sigma} + (\theta_0 + \hbar N)\sum_{\rho,\sigma} \delta_{\rho\sigma}p_{\rho\sigma} - \\
&- (\theta_0 + \theta_1 + \hbar N)\sum_{\rho,\sigma} \delta_{\rho\sigma}q_{\rho\sigma} - t\hbar N^2\Big)\Psi(Q)
\end{aligned} \tag{3.3.17}
$$

we apply the quantization and the same procedure as in the example (3.2.4) is considered, therefore the computation continues as the following

$$
\begin{aligned}
H_{IV}\Psi(Q) =& \Big(\hbar^2 \sum_{\rho,\sigma,\tau} q_{\sigma\tau}\partial_{q_{\sigma\rho}}\partial_{q_{\rho\tau}} - \hbar \sum_{\rho,\sigma,\tau} q_{\rho\sigma}q_{\sigma\tau}\partial_{q_{\rho\tau}} - t\hbar\sum_{\rho,\sigma} q_{\sigma\rho}\partial_{q_{\sigma\rho}} + \hbar(\theta_0 + \hbar N)\sum_{\rho} \partial_{q_{\rho\rho}} - \\
&- (\theta_0 + \theta_1 + \hbar N)\sum_{\rho} q_{\rho\rho} - t\hbar N^2\Big)\Psi(Q).
\end{aligned} \tag{3.3.18}
$$

Note that, according to equations (3.2.23), the first derivatives with respect to $\mathbf{q}$ will vanish as we reduce to the eigenvalues and put $M = 0$, the multiplication operator $\mathbf{q}$ will be substituted by $z$ directly, and the second derivatives follow the same pattern as the example (3.2.4); putting all these together results in the quantized Calogero-Painlevé IV to be the following operator

$$\begin{aligned}\widetilde{H}_{IV} =&\hbar^2 \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} \frac{z_\sigma \partial_{z_\sigma} - z_\rho \partial_{z_\rho}}{z_\sigma - z_\rho} + \sum_{\rho} \left( \hbar^2 z_\rho \partial^2_{z_\rho} - \hbar\Big(z_\rho^2 + t z_\rho - \theta_0 - \hbar\Big)\partial_{z_\rho} \right) \\ &- \hbar^2 \kappa(\kappa+1) \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} \frac{z_\rho}{(z_\sigma - z_\rho)^2} - (\hbar N + \theta_0 + \theta_1) \sum_\rho z_\rho - t\hbar N^2.\end{aligned} \tag{3.3.19}$$

**Calogero-Painlevé V.** To start the computation of the quantum Calogero-Painlevé V, we apply the Hamiltonian operator corresponding to this equation in the list (3.2.3) to the $\gamma$-equivariant wave function $\Psi(Q)$, and then we apply the quantization. The computation reads the following procedure:

$$\begin{aligned}&t\widetilde{H}_V \Psi(Q) = \\ &Tr\left( \frac{\mathbf{p}^2\mathbf{q}^2 + \mathbf{q}^2\mathbf{p}^2}{2} - \frac{\mathbf{p}^2\mathbf{q} + \mathbf{q}\mathbf{p}^2}{2} + \frac{t(\mathbf{p}\mathbf{q}^2 + \mathbf{q}^2\mathbf{p})}{2} + (\theta_0 - \theta_2 - t)\mathbf{p}\mathbf{q} + \theta_2 \mathbf{p} + (\theta_0 + \theta_1) t q \right)\Psi(Q)\end{aligned} \tag{3.3.20}$$

using the proper permutations of the multiplication operator $\mathbf{q}$ and the differential operator $\mathbf{p}$ and applying the quantization (3.2.1) we obtain

$$\begin{aligned}(3.3.20) =&\Bigg( \hbar^2 \sum_{\rho,\sigma,\tau,\eta} q_{\rho\sigma} q_{\sigma\tau} \partial_{q_{\eta\tau}} \partial_{q_{\rho\eta}} + \hbar^2 \sum_{\rho,\sigma} \delta_{\rho\sigma} q_{\rho\sigma} \sum_{\tau\eta} \delta_{\tau\eta} \partial_{q_{\tau\eta}} - \hbar^2 \sum_{\rho,\sigma,\tau} q_{\rho\sigma} \partial_{q_{\tau\sigma}} \partial_{q_{\rho\tau}} + \\ &+ t\hbar \sum_{\rho,\sigma,\tau} q_{\rho\sigma} q_{\sigma\tau} \partial_{q_{\rho\tau}} + \Big(\hbar^2 N + (\theta_0 - \theta_2 - t)\hbar\Big) \sum_{\rho,\sigma} q_{\rho\sigma} \partial_{q_{\rho\sigma}} + \\ &+ \Big(\theta_2 \hbar - \hbar^2 N\Big) \sum_{\rho,\sigma} \delta_{\rho\sigma} \partial_{q_{\rho\sigma}} + \Big(\hbar N t + (\theta_0 + \theta_1) t\Big) \sum_{\rho\sigma} \delta_{\rho\sigma} q_{\rho\sigma} + \frac{\hbar^2 N(1 + N^2)}{2} + \\ &+ (\theta_0 - \theta_2 - t)\hbar N^2 \Bigg) \Psi(Q).\end{aligned} \tag{3.3.21}$$

Applying the same logic as in the example (3.2.4) for multiplication operator $\mathbf{q}$ and the first and second derivative terms, the computation continues as follows

$$
\begin{aligned}
(3.3.21) =&\hbar^2 \sum_{\rho} z_\rho^2 \partial_{z_\rho}^2 \Psi(Z) + \hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho^2 \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \gamma(e^{-A^{(1)}-A^{(2)}}) \Psi(\widetilde{Z}) + \hbar^2 \sum_{\rho} z_\rho \sum_{\sigma} \partial_{z_\sigma} \Psi(Z) - \\
&- \hbar^2 \sum_{\rho} z_\rho \partial_{z_\rho}^2 \Psi(Z) - \hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \gamma(e^{-A^{(1)}-A^{(2)}}) \Psi(\widetilde{Z}) + t\hbar \sum_{\rho} z_\rho^2 \partial_{z_\rho} \Psi(Z) + \\
&+ \Big(\hbar^2 N + (\theta_0 - \theta_2 - t)\hbar\Big) \sum_{\rho} z_\rho \partial_{z_\rho} \Psi(Z) + \hbar(\theta_2 - \hbar N) \sum_{\rho} \partial_{z_\rho} \Psi(Z) + \\
&+ t\Big(\hbar N + \theta_0 + \theta_1\Big) \sum_{\rho} z_\rho \Psi(Z) + \left( (\theta_0 - \theta_2 - t) N^2 \hbar + \frac{N\hbar^2(1+N^2)}{2} \right) \Psi(Z).
\end{aligned}
\tag{3.3.22}
$$

Finally, applying the result of the computations for the example (3.2.4), combining similar terms, and rearranging, we obtain the quantum Calogero-Painlevé V Hamiltonian operator to be the following

$$
\begin{aligned}
t\widetilde{H}_V =&\hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{z_\sigma(z_\sigma - 1)\partial_{z_\sigma} - z_\rho(z_\rho - 1)\partial_{z_\rho}}{z_\sigma - z_\rho} + \hbar^2 \sum_{\rho} z_\rho(z_\rho - 1)\partial_{z_\rho}^2 - \hbar^2 \kappa(\kappa+1) \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{z_\rho(z_\rho - 1)}{(z_\sigma - z_\rho)^2} \\
&+ \hbar \sum_{\rho} \left( t z_\rho^2 + \Big(2\hbar + (\theta_0 - \theta_2 - t)\Big) z_\rho + \theta_2 - \hbar \right) \partial_{z_\rho} + t(\hbar N + \theta_0 + \theta_1) \sum_{\rho} z_\rho + \\
&+ \left( (\theta_0 - \theta_2 - t) N^2 \hbar + \frac{N\hbar^2(1+N^2)}{2} \right).
\end{aligned}
\tag{3.3.23}
$$

**Calogero-Painlevé VI.** In order to compute the quantized Hamiltonian operator corresponding to the sixth Calogero-Painlevé equation, similar to the previous cases, we apply the Hamiltonian operator in the list (3.2.3) assiciated to Calogero-Painlevé VI to the $\gamma$-equivariant wave function $\Psi(Q)$, so the computation starts with the following equation

$$
\begin{aligned}
t(t-1)\widetilde{H}_{VI}\Psi(Q) =&Tr\Bigg( \mathbf{qpqpq} - t\mathbf{pq}^2\mathbf{p} + t\mathbf{pqp} - \frac{\mathbf{pqpq} + \mathbf{qpqp}}{2} - \theta\mathbf{qpq} + t(\theta_0 + \theta_1)\mathbf{pq} + \\
&+ (\theta_0 + \theta_t)\mathbf{pq} - \theta_0 t\mathbf{p} - \frac{1}{4}(k^2 - \theta^2)\mathbf{q} \Bigg) \Psi(Q).
\end{aligned}
\tag{3.3.24}
$$

Then using the permutation relations and applying the quantization (3.2.1) one obtains

$$
\begin{aligned}
(3.3.24) = &\Bigg(\hbar^2 \sum_{\rho,\sigma,\tau,\eta,\nu} q_{\rho\sigma} q_{\tau\eta} q_{\nu\rho} \partial_{q_{\tau\sigma}} \partial_{q_{\nu\eta}} + \hbar^2 \sum_{\rho,\sigma} \delta_{\rho\sigma} q_{\rho\sigma} \sum_{\tau\eta} q_{\tau\eta} \partial_{q_{\eta\tau}} + \\
&+ \hbar(2\hbar N - \theta) \sum_{\rho,\sigma,\tau} q_{\rho\sigma} q_{\tau\rho} \partial_{q_{\tau\sigma}} + \left(\hbar^2 N^2 - \theta\hbar N - \frac{1}{4}(k^2 - \theta^2)\right) \sum_{\rho,\sigma} \delta_{\rho\sigma} q_{\rho\sigma} - \\
&- \hbar^2 (t+1) \sum_{\rho,\sigma,\tau,\eta} q_{\sigma\tau} q_{\tau\eta} \partial_{q_{\sigma\rho}} \partial_{q_{\rho\sigma}} - t\hbar^2 \sum_{\rho,\sigma} \delta_{\rho\sigma} q_{\rho\sigma} \sum_{\tau,\eta} \delta_{\tau\eta} \partial_{q_{\tau\eta}} - \\
&- \hbar \left(\hbar N(t-1) - t(\theta_0 + \theta_1) - (\theta_0 + \theta_t)\right) \sum_{\rho,\sigma} q_{\rho\sigma} \partial_{q_{\rho\sigma}} + t\hbar^2 \sum_{\rho,\sigma,\tau} q_{\sigma\tau} \partial_{q_{\sigma\rho}} \partial_{q_{\rho\tau}} + \\
&+ t\hbar(\hbar N - \theta_0) \sum_{\rho,\sigma} \delta_{\rho\sigma} \partial_{q_{\rho\sigma}} + \hbar N^2 \left(t(\theta_0 + \theta_1) + (\theta_0 + \theta_t) - \frac{\hbar N}{2}\right)\Bigg) \Psi(Q).
\end{aligned}
\tag{3.3.25}
$$

We apply the reduction instructions for **q** terms and the partial derivative term, the computation continues as

$$
\begin{aligned}
(3.3.25) = &\hbar^2 \sum_{\rho} z_\rho^3 \partial_{z_\rho}^2 \Psi(Z) + \hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho^2 z_\sigma \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) + \\
&+ \hbar^2 \sum_{\rho} z_\rho \sum_{\sigma} z_\sigma \partial_{z_\sigma} \Psi(Z) + \hbar(2\hbar N - \theta) \sum_{\rho} z_\rho^2 \partial_{z_\rho} \Psi(Z) + \\
&+ \left(\hbar^2 N^2 - \theta\hbar N - \frac{1}{4}(k^2 - \theta^2)\right) \sum_{\rho} z_\rho \Psi(Z) - t\hbar^2 \sum_{\rho} z_\rho^2 \partial_{z_\rho} \Psi(Z) - \\
&- t\hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho^2 \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) - t\hbar^2 \sum_{\rho} z_\rho \sum_{\sigma} \partial_{z_\sigma} \Psi(Z) - \\
&- \hbar \left(t\hbar N + \hbar N - t(\theta_0 + \theta_1) - (\theta_0 + \theta_t)\right) \sum_{\rho} z_\rho \partial_{z_\rho} \Psi(Z) + t\hbar^2 \sum_{\rho} z_\rho \partial_{z_\rho}^2 \Psi(Z) + \\
&+ t\hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho \partial_{M_{\sigma\rho}} \partial_{\rho\sigma} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) + t\hbar(\hbar N - \theta_0) \sum_{\rho} \partial_{z_\rho} \Psi(Z) - \\
&- \hbar^2 \sum_{\rho} z_\rho^2 \partial_{z_\rho}^2 \Psi(Z) - \hbar^2 \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} z_\rho z_\sigma \partial_{M_{\sigma\rho}} \partial_{M_{\rho\sigma}} \gamma(e^{-A^{(1)} - A^{(2)}}) \Psi(\widetilde{Z}) + \\
&+ \hbar N^2 \left(t(\theta_0 + \theta_1) + (\theta_0 + \theta_t) - \frac{\hbar N}{2}\right) \Psi(Z).
\end{aligned}
\tag{3.3.26}
$$

The rest of the computations follows the instruction of the example (3.2.4), and the

Hamiltonian operator corresponding to the quantum Calogero-Painlevé VI is obtained as

$$
\begin{aligned}
t(t-1)\widetilde{H}_{VI} = \\
=&\hbar^2 \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} \frac{z_\sigma(z_\sigma-1)(z_\sigma-t)\partial_{z_\sigma} - z_\rho(z_\rho-1)(z_\rho-t)\partial_{z_\rho}}{z_\sigma-z_\rho} + \hbar^2\sum_\rho z_\rho(z_\rho-1)(z_\rho-t)\partial^2_{z_\rho} \\
&+\hbar\sum_\rho\Big((3\hbar-\theta)z_\rho^2 + \big(-\hbar(1+t)+t(\theta_0+\theta_1)+\theta_0+\theta_t\big)z_\rho \\
&+t(\hbar-\theta_0)\Big)\partial_{z_\rho} - \hbar^2\kappa(\kappa+1)\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{z_\rho^2 z_\sigma - tz_\rho^2 + tz_\rho - z_\rho z_\sigma}{(z_\sigma-z_\rho)^2} \\
&+\Big(N^2\hbar^2 - \theta N\hbar - \frac{1}{4}(k^2-\theta^2)\Big)\sum_\rho z_\rho - \frac{N^3\hbar^2}{2} + t\hbar N^2(\theta_0+\theta_1) + \hbar N^2(\theta_0+\theta_t)
\end{aligned}
\tag{3.3.27}
$$

To have a symmetric representation for the second term of the third line, we use the following equality

$$
\begin{aligned}
\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{z_\rho^2 z_\sigma - tz_\rho^2 + tz_\rho - z_\rho z_\sigma}{(z_\sigma-z_\rho)^2} = &\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{z_\rho(z_\rho-1)(z_\rho-t)+z_\sigma(z_\sigma-1)(z_\sigma-t)}{(z_\sigma-z_\rho)^2} + \\
&+(N-1)\hbar^2\sum_\rho z_\rho - \frac{N(N-1)\hbar^2}{2}
\end{aligned}
\tag{3.3.28}
$$

which is obtained by adding and subtracting the symmetric compensate of each monomial in the numerator.

Therefore, the final representation of the Hamiltonian operator for the sixth quantum Calogero-Painlevé equation is the following operator

$$
\begin{aligned}
t(t-1)\widetilde{H}_{VI} = \\
=&\hbar^2 \sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}} \frac{z_\sigma(z_\sigma-1)(z_\sigma-t)\partial_{z_\sigma} - z_\rho(z_\rho-1)(z_\rho-t)\partial_{z_\rho}}{z_\sigma-z_\rho} + \hbar^2\sum_\rho z_\rho(z_\rho-1)(z_\rho-t)\partial^2_{z_\rho} + \\
&+\hbar\sum_\rho\Big((3\hbar-\theta)z_\rho^2 + \big(-\hbar(1+t)+t(\theta_0+\theta_1)+\theta_0+\theta_t\big)z_\rho + \\
&+t(\hbar-\theta_0)\Big)\partial_{z_\rho} - \frac{\hbar^2\kappa(\kappa+1)}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{z_\rho(z_\rho-1)(z_\rho-t)+z_\sigma(z_\sigma-1)(z_\sigma-t)}{(z_\sigma-z_\rho)^2} + \\
&+\Big(N^2\hbar^2 - \theta N\hbar - \frac{1}{4}(k^2-\theta^2) + (N-1)\kappa(\kappa+1)\hbar^2\Big)\sum_\rho z_\rho - \\
&-\frac{N^3\hbar^2}{2} + t\hbar N^2(\theta_0+\theta_1) + \hbar N^2(\theta_0+\theta_t) - \frac{\hbar^2 N(N-1)\kappa(\kappa+1)}{2}.
\end{aligned}
\tag{3.3.29}
$$

# Chapter 4

# Integral solution of the multi-particle quantum Painlevé equations

## 4.1 Quantum Painlevé system

Nagoya in [31], introduced the Hamiltonian operators corresponding to the quantum Painlevé equations in the case of a single particle, satisfying the Schrödinger equation

$$\hbar\frac{\partial}{\partial t}\Phi(z,t) = H_J(z,\hbar\frac{\partial}{\partial z},t)\Phi(z,t) \qquad J = II, III, IV, V, VI \tag{4.1.1}$$

where Hamiltonian operators $H_J$ are obtained from the polynomial Hamiltonian operators of the Painlevé equations by substituting the operators $z$, $\hbar\frac{\partial}{\partial z}$ into the canonical coordinates. These operators are defined as

$$\begin{aligned}
H_{II} =& \frac{1}{2}(\hbar\partial_z)^2 - (z^2 + \frac{t}{2})\hbar\partial_z + az \\
tH_{III} =& z^2(\hbar\partial_z)^2 - (z^2 + bz + t)\hbar\partial_z + az^* \\
H_{IV} =& z(\hbar\partial_z)^2 - (z^2 + tz + b)\hbar\partial_z + a(z + t) \\
tH_V =& z(z-1)(\hbar\partial_z)^2 + \Big(tz^2 - (b + c + t)z + b\Big)\hbar\partial_z \\
& + a(b + c - a + \hbar + t - tz) \\
t(t-1)H_{VI} =& z(z-1)(z-t)(\hbar\partial_z)^2 - \Big((a+b)(z-1)(z-t) + cz(z-t) \\
& + dz(z-1)\Big)\hbar\partial_z + (b + c + d + \hbar)a(z-t)
\end{aligned} \tag{4.1.2}$$

He showed that the quantum Hamiltonians (4.1.2) admit special solutions in integral form when the parameters take certain specific values: in fact, the wave function $\Phi$ in Eq.

*The coefficient of $z$ in [31] is shifted by the parameter $b$, however, based on our generalization, the generated coefficient of $z$ is $a$ itself.

(4.1.1) can be taken in the following form

$$\Phi_m^J(z,t) = \int_\Gamma \prod_{1\le i<j\le m} (u_i - u_j)^{2\hbar} \prod_i (z-u_i)\Theta_J(u_i,t)du_i. \tag{4.1.3}$$

Here $\Gamma = \prod_j \gamma_j$ is a cartesian product of "admissible" contours $u_j \in \gamma_j$ that can be chosen on the Riemann surface of the "master" function $\Theta_J$. By this we mean that

- $\Theta_J(u_j)$ is single valued along $\gamma_j$;
- $\int_{\gamma_j} u_j^k \Theta_J(u_j)\mathrm{d}u_j$ is a convergent integral and not identically zero (as an expression in $k \in \mathbb{N}$);
- The contours are pairwise non-intersecting if $\hbar \notin \frac{1}{2}\mathbb{N}$

For example for $J = II$ the contours can be taken as contours starting from infinity along one of the three directions $\arg(u_j) = \frac{2\pi}{3}k$, $k = 0,1,2$ and ending at infinity along any of the remaining ones. We could also take a circle, but then the Cauchy theorem would imply that the integral $\int u^\tau \Theta_{II}(u)\mathrm{d}u$ is zero. The requirement that the different $\gamma_j$'s do not intersect is due to the fact that if $\hbar \notin \frac{1}{2}\mathbb{N}$ then the power of the Vandermonde term in the integrand (4.1.3) yields a non-single valued function.

The master functions $\Theta_J(u_i,t)$ $(J = II, III, IV, V, VI)$ are the weight functions defined below:

$$\begin{aligned} \Theta_{II} &= \exp\Big(-(u_i t + \frac{2}{3}u_i^3)\Big) \\ \Theta_{III} &= u_i^{-b-1}\exp\Big(\frac{t}{u_i} - u_i\Big) \\ \Theta_{IV} &= u_i^{-b-1}\exp\Big(-(u_i t + \frac{u_i^2}{2})\Big) \\ \Theta_V &= u_i^{-b-1}(1-u_i)^{-c-1}\exp(u_i t) \\ \Theta_{VI} &= u_i^{-a-b-1}(1-u_i)^{-c-1}(t-u_i)^{-d}. \end{aligned} \tag{4.1.4}$$

With the positions (4.1.4) and formula (4.1.3) the functions $\Phi_m^J$ satisfy the Schrödinger equations (4.1.1) provided that the parameters $a, b, c, d$ satisfy the following condition

$$\begin{cases} a = m\hbar \quad \text{and} \quad b + c + d = (m-1)\hbar & J = VI \\ a = m\hbar & J = II, III, IV, V \end{cases} \tag{4.1.5}$$

We want to generalize this result of Nagoya's to our multi-particle quantum Hamiltonians (3.3.12, 3.3.15,3.3.19, 3.3.23, 3.3.27) by providing a multi–particle extension of the integral formulæ (4.1.3).

## 4.2 Integral representations for the quantum Painlevé–Calogero Schrödinger wave functions

We now explain the general approach behind the extension of Nagoya's formulæ. We start from an Ansatz of the form

$$\Psi(z_\rho;t) = \int_\Gamma \prod_{1\le i<j\le m} (u_i-u_j)^{2\hbar} \prod_{\rho=1}^{N}\prod_{i=1}^{m} (z_\rho - u_i)\Theta_J(u_i,t)du_i \tag{4.2.1}$$

Observe that, similarly to Nagoya's result, these are polynomials in the $z_\rho$'s of total degree $Nm$ and of degree $m$ in each of the variables $z_\rho$. With the Ansatz (4.2.1) in place we verify, by a direct calculation on a case-by-case basis, that they satisfy a multi-variate generalization of (4.1.1) and identify the corresponding Hamiltonian operator. The result of these computations, whose details are reported in the following sections, are summarized in the following table (for readability, the range of greek indices is assumed to be $1,\dots,N$ without explicit mention):

**Quantum Calogero–Painlevé II**

$$H_{II} = \frac{\hbar}{2}\sum_{\substack{\rho,\sigma\\ \rho\ne\sigma}} \frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho - z_\sigma} + \frac{\hbar^2}{2}\sum_\rho \partial_{z_\rho}^2 - \hbar\sum_\rho (z_\rho^2+\frac{t}{2})\partial_{z_\rho} + m\hbar\sum_\rho z_\rho \tag{4.2.2}$$

**Quantum Calogero–Painlevé III**

$$tH_{III} = \hbar\sum_{\substack{\rho,\sigma\\ \rho\ne\sigma}} \frac{z_\rho^2\partial_{z_\rho}-z_\sigma^2\partial_{z_\sigma}}{z_\rho - z_\sigma} + \sum_\rho \Big(\hbar^2 z_\rho^2\partial_{z_\rho}^2 - \hbar(z_\rho^2+(b+N-1)z_\rho+t)\partial_{z_\rho} + m\hbar z_\rho\Big) \tag{4.2.3}$$

**Quantum Calogero–Painlevé IV**

$$H_{IV} = \hbar\sum_{\substack{\rho,\sigma\\ \rho\ne\sigma}} \frac{z_\rho\partial_{z_\rho}-z_\sigma\partial_{z_\sigma}}{z_\rho - z_\sigma} + \sum_\rho \Big(\hbar^2 z_\rho\partial_{z_\rho}^2 - \hbar\Big(z_\rho^2+tz_\rho+b\Big)\partial_{z_\rho} + m\hbar z_\rho\Big) + \hbar Nmt \tag{4.2.4}$$

**Quantum Calogero–Painlevé V**

$$\begin{aligned} tH_V =& \hbar\sum_{\substack{\rho,\sigma\\ \rho\ne\sigma}} \frac{z_\rho(z_\rho-1)\partial_{z_\rho}-z_\sigma(z_\sigma-1)\partial_{z_\sigma}}{z_\rho - z_\sigma} + \hbar Nm\Big(b+c+t-\hbar(m-1)-N+1\Big)+ \\ &+\sum_\rho \Big(\hbar^2 z_\rho(z_\rho-1)\partial_{z_\rho}^2 + \hbar\Big(tz_\rho^2-(b+c+t)z_\rho+b\Big)\partial_{z_\rho} - m\hbar t z_\rho\Big) \end{aligned} \tag{4.2.5}$$

**Quantum Calogero–Painlevé VI**

$$
\begin{aligned}
t(t-1)H_{VI} = & \\
=&\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{z_\rho(z_\rho-1)(z_\rho-t)\partial_{z_\rho}-z_\sigma(z_\sigma-1)(z_\sigma-t)\partial_{z_\sigma}}{z_\rho-z_\sigma}+\sum_\rho \hbar^2 z_\rho(z_\rho-1)(z_\rho-t)\partial^2_{z_\rho}-\\
&-\hbar\sum_\rho\left((a+b)(z_\rho-1)(z_\rho-t)+cz_\rho(z_\rho-t)+(d+N-1)z_\rho(z_\rho-1)\right)\partial_{z_\rho}-\\
&-\hbar m(N-1-\hbar m)\sum_\rho z_\rho-\hbar mN(\hbar m+1-N)t
\end{aligned}
\tag{4.2.6}
$$

We claim that the generalized wave functions (4.2.1) are indeed solutions to the quantum Calogero Hamiltonians (3.3.12, 3.3.15, 3.3.19, 3.3.23, 3.3.27) constructed by canonical quantization of the non-commutative Hamiltonians of the classical isomonodromic noncommutative equations of [3]. The identification requires to choose the parameters in a special way. The details of these result are presented in the next chapter.

**Planck constant $\hbar = 1$.** According to the result that will be presented in the next chapter, when $\hbar = 1$ and some of the other parameters including $\kappa$ take some specific values, our claim regarding the generalized wave functions being solutions to the Schrödinger equation for quantum Calogero-Painlevé system holds true. (For details see chapter (5))

We now briefly comment on the value of the Planck constant $\hbar = 1$: observing the original integral representation of Nagoya (4.1.3) and the generalized one (4.2.1), we see that for $\hbar = 1$ the integrand contains the square of the Vandermonde determinant of the variables $u_j$. This type of expression is very familiar in the context of Random Matrices [30]: it is the Jacobian of the change of variables from the Lebesgue measure on Hermitean matrices (or normal matrices) to the unitary-radial coordinates. Specifically, if $M$ is a Hermitean matrix of size $m \times m$ and we write it at $M = VDV$ with $D = \mathrm{diag}(u_1, \ldots, u_m)$ and $V \in U(m, \mathbb{C})$, then the Lebesgue measure (up to inessential multiplicative constant) is

$$
\mathrm{d}M = \Delta(u)^2 \mathrm{d}V \prod_j \mathrm{d}u_j \ , \tag{4.2.7}
$$

where $\Delta(u) = \prod_{i<j}(u_i - u_j)$. This expression is also valid if the $u_j$'s are allowed to take complex values along specified curves (but now $\mathrm{d}M$ is the measure on $m \times m$ normal matrices). This allows us to rewrite the integral formulæ as matrix integrals; for example for Painlevé II we have

$$
\Phi_m^{II}(\vec{z}, t) = \int \prod_\rho \det(z_\rho - M)\mathrm{e}^{-\mathrm{Tr}\left(\frac{2}{3}M^3+tM\right)}\mathrm{d}M, \tag{4.2.8}
$$

which expresses the wave function as the expectation value of the product of characteristic polynomials. Similar expressions hold for the other cases. Therefore it appears that the class of generalized Nagoya solutions (4.2.1) and the wave–functions of the quantum Calogero–Painlevé Hamiltonians intersect on the class of solutions that are related to matrix integrals of the form (4.2.8) only (and their similar expressions for the other master functions (4.1.4)).

During the coming sections, we provide the detailed computations of generalizing the quantum Hamiltonian operators (4.1.2) to several particles.

The starting point of all the following computations is to take the integral representation (4.2.1)

$$\Psi(z_\rho;t) = \int_\Gamma \prod_{1\le i<j\le m} (u_i-u_j)^{2\hbar} \prod_{\rho,i} (z_\rho - u_i)\Theta_J(u_i,t)du_i \tag{4.2.9}$$

and apply to it the direct sum of the second–order parts in the quantum Hamiltonians (4.1.2). For the sake of simplicity of the notation in the following computations, we denote by $\Big\langle \Psi(u_1,...,u_m)\Big\rangle$ the un-normalized expectation value as follows:

$$\Big\langle \Psi(u_1,...,u_m)\Big\rangle := \int_\Gamma \prod_{1\le i<j\le m} (u_i-u_j)^{2\hbar} \prod_i \Theta_J(u_i,t)\Psi(u_1,...,u_m)du_i. \tag{4.2.10}$$

### 4.2.1 Quantum Painlevé II

Define $P(\vec{z}) = \prod_{\rho,i}(\vec{z}_\rho - u_i)$ and $\Delta = \prod_{1\le i<j\le m}(u_i-u_j)$, then

$$\begin{aligned} \sum_\rho \partial^2_{\vec{z}_\rho} P =& \sum_\rho P \sum_{i\ne j} \frac{1}{(\vec{z}_\rho - u_i)(\vec{z}_\rho - u_j)} = \\ =& \sum_\rho P \sum_{i\ne j} \Big( \frac{1}{(\vec{z}_\rho - u_i)(u_i-u_j)} - \frac{1}{(\vec{z}_\rho - u_j)(u_i-u_j)} \Big) \end{aligned} \tag{4.2.11}$$

This yields

$$\begin{aligned} \hbar^2 \sum_\rho \partial^2_{\vec{z}_\rho}\Psi =& \hbar^2 \Big\langle P \sum_\rho \sum_{i\ne j} \Big( \frac{1}{(\vec{z}_\rho - u_i)(u_i-u_j)} - \frac{1}{(\vec{z}_\rho - u_j)(u_i-u_j)} \Big) \Big\rangle = \\ =& 2\hbar^2 \Big\langle P \sum_\rho \sum_{i\ne j} \frac{1}{(\vec{z}_\rho - u_i)(u_i-u_j)} \Big\rangle = \\ =& \hbar \int \sum_i \partial_{u_i}(\Delta^{2\hbar}) \sum_\rho \frac{1}{\vec{z}_\rho - u_i} P \prod_k \Theta(u_k) du_i. \end{aligned} \tag{4.2.12}$$

We now use integration by parts in the integrand and obtain:

$$\begin{aligned} =& -\hbar \int \Delta^{2\hbar} \sum_\rho \sum_i \partial_{u_i} \left( \frac{1}{\vec{z}_\rho - u_i} P \prod_k \Theta(u_k) \right) du_i = \\ =& -\hbar \int \Delta^{2\hbar} \sum_\rho \sum_i \left( \frac{P}{(\vec{z}_\rho - u_i)^2} + \frac{P_{u_i}}{\vec{z}_\rho - u_i} - \frac{2u_i^2 + t}{\vec{z}_\rho - u_i} P \right) \prod_k \Theta(u_k) du_i. \end{aligned} \tag{4.2.13}$$

After simplifying we obtain

$$\frac{\hbar^2}{2}\sum_{\rho}\partial^2_{\vec{z}_\rho}\Psi = -\frac{\hbar}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{\vec{z}_\rho}-\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}\Psi + \hbar\sum_{\rho}(\vec{z}_\rho^2+\frac{t}{2})\partial_{\vec{z}_\rho}\Psi - \hbar m\sum_{\rho}\vec{z}_\rho\Psi + \hbar N\partial_t\Psi. \tag{4.2.14}$$

Rearranging the terms appropriately, we obtain the Schrödinger equation with the Hamiltonian (4.2.2):

$$H_{II} = \frac{\hbar}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{\vec{z}_\rho}-\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma} + \frac{\hbar^2}{2}\sum_{\rho}\partial^2_{\vec{z}_\rho} - \hbar\sum_{\rho}(\vec{z}_\rho^2+\frac{t}{2})\partial_{\vec{z}_\rho} + m\hbar\sum_{\rho}\vec{z}_\rho. \tag{4.2.15}$$

### 4.2.2 Quantum Painlevé III

The initial set up matches the previous case, except that we need to consider the direct sum of the operators $\vec{z}_\rho^2\partial^2_{\vec{z}_\rho}$ acting on the wave function. To this end we observe that:

$$\begin{aligned}\sum_{\rho}\vec{z}_\rho^2\partial^2_{\vec{z}_\rho}P(\vec{z}) &= \sum_{\rho}P\sum_{i\neq j}\frac{\vec{z}_\rho^2}{(\vec{z}_\rho-u_i)(\vec{z}_\rho-u_j)} = \\ &= \sum_{\rho}P\sum_{i\neq j}\Big(1+\frac{u_i^2}{(\vec{z}_\rho-u_i)(u_i-u_j)}-\frac{u_j^2}{(\vec{z}_\rho-u_j)(u_i-u_j)}\Big).\end{aligned} \tag{4.2.16}$$

This yields

$$\begin{aligned}\hbar^2\sum_{\rho}\vec{z}_\rho^2\partial^2_{\vec{z}_\rho}\Psi =& \hbar^2\Big\langle P\sum_{\rho}\sum_{i\neq j}\Big(1+\frac{u_i^2}{(\vec{z}_\rho-u_i)(u_i-u_j)}-\frac{u_j^2}{(\vec{z}_\rho-u_j)(u_i-u_j)}\Big)\Big\rangle = \\ =& \hbar^2 Nm(m-1)\Psi + \hbar Nm\Psi + \hbar N^2 m\Psi - \hbar N\sum_{\sigma}\vec{z}_\sigma\partial_{\vec{z}_\sigma}\Psi - \hbar\sum_{\rho}\vec{z}_\rho\sum_{\sigma}\partial_{\vec{z}_\sigma}\Psi + \\ &+ \hbar\sum_{\rho}(\vec{z}_\rho^2+b\vec{z}_\rho+t)\partial_{\vec{z}_\rho}\Psi - \hbar bNm\Psi - \hbar Nm\Psi + \hbar\sum_{\rho}\partial_{\vec{z}_\rho}\Psi - \hbar m\sum_{\rho}\vec{z}_\rho\Psi - \\ &- \hbar N\Big\langle\sum_{i}u_iP\Big\rangle - \hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho^2\partial_{\vec{z}_\rho}-\vec{z}_\sigma^2\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}\Psi + \hbar\sum_{\rho}\vec{z}_\rho\sum_{\sigma}\partial_{\vec{z}_\sigma}\Psi - \hbar\sum_{\rho}\vec{z}_\rho\partial_{\vec{z}_\rho}\Psi + \\ &+ \hbar(N-1)\sum_{\sigma}\vec{z}_\sigma\partial_{\vec{z}_\sigma}\Psi.\end{aligned} \tag{4.2.17}$$

Rearranging the terms we obtain the following expression:

$$\begin{aligned}
\hbar^2 \sum_\rho \vec{z}_\rho^2 \partial^2_{\vec{z}_\rho} \Psi = & - \hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{\vec{z}_\rho^2 \partial_{\vec{z}_\rho} - \vec{z}_\sigma^2 \partial_{\vec{z}_\sigma}}{\vec{z}_\rho - \vec{z}_\sigma} \Psi + \hbar \sum_\rho (\vec{z}_\rho^2 + b\vec{z}_\rho + t)\partial_{\vec{z}_\rho} \Psi - m\hbar \sum_\rho \vec{z}_\rho \Psi + \\
& + t\hbar N \partial_t \Psi + \hbar^2 N m(m-1)\Psi + \hbar N^2 m \Psi - bNm\hbar\Psi - \hbar \sum_\rho \vec{z}_\rho \partial_{\vec{z}_\rho} \Psi - \\
& - \hbar N \left\langle \sum_i (\frac{t}{u_i} + u_i) P \right\rangle .
\end{aligned} \tag{4.2.18}$$

In order to handle the remaining expectation value we need to derive some further identities: consider the Euler differential operator

$$\mathbb{E} := \sum_i u_i \partial_{u_i}. \tag{4.2.19}$$

Applying this operator $\mathbb{E}$ to the integrand of $\Phi_m^{III}$,

$$\prod_{1\le i<j\le m} (u_i - u_j)^{2\hbar} \prod_{\rho,i} (\vec{z}_\rho - u_i) u_i^{-b-1} e^{(\frac{t}{u_i} - u_i)} \tag{4.2.20}$$

gives the following expression:

$$\begin{aligned}
& \sum_i u_i \partial_{u_i} \left( \prod_{1\le i<j\le m} (u_i - u_j)^{2\hbar} \prod_{\rho,i} (\vec{z}_\rho - u_i) u_i^{-b-1} e^{(\frac{t}{u_i} - u_i)} \right) = \\
& = \sum_i u_i \Bigg[ \prod_{i<j} (u_i - u_j)^{2\hbar} \left( (2\hbar) \sum_{i\neq j} \frac{1}{u_i - u_j} \right) \prod_{\rho,i} (\vec{z}_\rho - u_i) u_i^{-b-1} e^{(\frac{t}{u_i} - u_i)} + \\
& + \prod_{i<j} (u_i - u_j)^{2\hbar} \prod_{\rho,i} (\vec{z}_\rho - u_i) \left( \sum_\rho \frac{-1}{\vec{z}_\rho - u_i} \right) u_i^{-b-1} e^{(\frac{t}{u_i} - u_i)} + \\
& + \prod_{i<j} (u_i - u_j)^{2\hbar} \prod_{\rho,i} (\vec{z}_\rho - u_i) u_i^{-b-1} e^{(\frac{t}{u_i} - u_i)} \left( \frac{-b-1}{u_i} - \frac{t}{u_i^2} - 1 \right) \Bigg]
\end{aligned} \tag{4.2.21}$$

$$\begin{aligned}
& = \sum_i u_i \left[ 2\hbar \sum_{i\neq j} \frac{1}{u_i - u_j} - \sum_\rho \frac{1}{\vec{z}_\rho - u_i} - \frac{b+1}{u_i} - \frac{t}{u_i^2} - 1 \right] \Delta^{2\hbar} P \prod_i \Theta(u_i) \\
& = \Delta^{2\hbar} \left[ \hbar m(m-1)P + NmP - \sum_\rho \vec{z}_\rho \partial_\rho P - (b+1)mP - \sum_i (\frac{t}{u_i} + u_i) P \right] \prod_i \Theta(u_i).
\end{aligned} \tag{4.2.22}$$

Then we observe that

$$\int \sum_j u_j \partial_{u_j} \left( \Delta^{2\hbar} P \prod_i \Theta(u_i) \mathrm{d}u_i \right) = \int \sum_j \partial_{u_j} \left( u_j \Delta^{2\hbar} P \prod_i \Theta(u_i) \mathrm{d}u_i \right) - m\Psi(\vec{z}) \tag{4.2.23}$$

and the first integral is zero because it is a divergence of a vector. Therefore,

$$\underbrace{\int \mathbb{E}\left(\Delta^{2\hbar} P \prod_i \Theta(u_i)\mathrm{d}u_i\right)}_{=-m\Psi} = \hbar m(m-1)\Psi + Nm\Psi - \sum_\rho \vec{z}_\rho \partial_{\vec{z}_\rho}\Psi - (b+1)m\Psi - \\ - \left\langle \sum_i (\frac{t}{u_i} + u_i)P \right\rangle \tag{4.2.24}$$

$\Longrightarrow$

$$\left\langle \sum_i (\frac{t}{u_i} + u_i)P \right\rangle = \hbar m(m-1)\Psi + Nm\Psi - \sum_\rho \vec{z}_\rho \partial_{\vec{z}_\rho}\Psi - b\Psi. \tag{4.2.25}$$

Substituting the left side into the equation (4.2.18) results in the following conclusion

$$t\hbar N \partial_t \Psi = \sum_\rho \left(\hbar^2 \vec{z}_\rho^2 \partial_{z_\rho}^2 - \hbar(\vec{z}_\rho^2 + (b+N-1)\vec{z}_\rho + t)\partial_{\vec{z}_\rho} + m\hbar \vec{z}_\rho\right)\Psi + \hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{\vec{z}_\rho^2 \partial_{\vec{z}_\rho} - \vec{z}_\sigma^2 \partial_{\vec{z}_\sigma}}{\vec{z}_\rho - \vec{z}_\sigma}\Psi \tag{4.2.26}$$

hence, the general Hamiltonian operator for quantum Painlevé III equation is given by

$$tH_{III} = \hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{\vec{z}_\rho^2 \partial_{\vec{z}_\rho} - \vec{z}_\sigma^2 \partial_{\vec{z}_\sigma}}{\vec{z}_\rho - \vec{z}_\sigma} + \sum_\rho \left(\hbar^2 \vec{z}_\rho^2 \partial_{\vec{z}_\rho}^2 - \hbar(\vec{z}_\rho^2 + (b+N-1)\vec{z}_\rho + t)\partial_{\vec{z}_\rho} + m\hbar \vec{z}_\rho\right). \tag{4.2.27}$$

### 4.2.3 Quantum Painlevé IV

We follow the same general scheme and consider the sum of the $z_\rho \partial_{z\rho}^2$ term applied to $P$:

$$\begin{aligned} \sum_\rho \vec{z}_\rho \partial_{\vec{z}_\rho}^2 P =& \sum_\rho \vec{z}_\rho P \sum_{i\neq j} \frac{1}{(\vec{z}_\rho - u_i)(\vec{z}_\rho - u_j)} = \\ =& \sum_\rho P \sum_{i\neq j} \Big( \frac{u_i}{(\vec{z}_\rho - u_i)(u_i - u_j)} - \frac{u_j}{(\vec{z}_\rho - u_j)(u_i - u_j)} \Big). \end{aligned} \tag{4.2.28}$$

This yields

$$\begin{aligned} \hbar^2 \sum_\rho \vec{z}_\rho \partial_{\vec{z}_\rho}^2 \Psi =& \hbar^2 \left\langle P \sum_\rho \sum_{i\neq j} \Big( \frac{u_i}{(\vec{z}_\rho - u_i)(u_i - u_j)} - \frac{u_j}{(\vec{z}_\rho - u_j)(u_i - u_j)} \Big) \right\rangle \\ =& 2\hbar^2 \left\langle P \sum_\rho \sum_{i\neq j} \frac{u_i}{(\vec{z}_\rho - u_i)(u_i - u_j)} \right\rangle \\ =& \hbar \int \sum_i \partial_{u_i}(\Delta^{2\hbar}) \sum_\rho \frac{u_i}{\vec{z}_\rho - u_i} P \prod_k \Theta(u_k) du_i \\ =& -\hbar \int \Delta^{2\hbar} \sum_\rho \sum_i \partial_{u_i} \left( (-1 + \frac{\vec{z}_\rho}{\vec{z}_\rho - u_i}) P \prod_k \Theta(u_k) \right) du_i \end{aligned} \tag{4.2.29}$$

which gives

$$\hbar^2\sum_\rho \vec{z}_\rho\partial^2_{\vec{z}_\rho}\Phi = -\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho\partial_{\vec{z}_\rho}-\vec{z}_\sigma\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}\Phi+\hbar\sum_\rho(\vec{z}^2_\rho+t\vec{z}_\rho+b)\partial_{\vec{z}_\rho}\Phi- \\ -\hbar NmtΦ+\hbar N\partial_t\Phi-\hbar m\sum_\rho\vec{z}_\rho\Phi. \tag{4.2.30}$$

Rearranging the terms in the above expression, the general Hamiltonian operator for quantum Painlevé IV equation is given by

$$H_{IV}=\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho\partial_{\vec{z}_\rho}-\vec{z}_\sigma\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}+\sum_\rho\left(\hbar^2\vec{z}_\rho\partial^2_{\vec{z}_\rho}-\hbar(\vec{z}^2_\rho+t\vec{z}_\rho+b)\partial_{\vec{z}_\rho}+m\hbar\vec{z}_\rho\right)+\hbar Nmt. \tag{4.2.31}$$

### 4.2.4 Quantum Painlevé V

We start from the same set up as previous operators

$$\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)\partial^2_{\vec{z}_\rho}P=\sum_\rho P\sum_{i\neq j}\frac{\vec{z}_\rho(\vec{z}_\rho-1)}{(\vec{z}_\rho-u_i)(\vec{z}_\rho-u_j)}= \\ =\sum_\rho P\sum_{i\neq j}\left(1+\frac{u_i(u_i-1)}{(\vec{z}_\rho-u_i)(u_i-u_j)}-\frac{u_j(u_j-1)}{(\vec{z}_\rho-u_j)(u_i-u_j)}\right) \tag{4.2.32}$$

This yields

$$\hbar^2\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)\partial^2_{\vec{z}_\rho}\Psi=\hbar^2\left\langle P\sum_\rho\sum_{i\neq j}\left(1+\frac{u_i(u_i-1)}{(\vec{z}_\rho-u_i)(u_i-u_j)}-\frac{u_j(u_j-1)}{(\vec{z}_\rho-u_j)(u_i-u_j)}\right)\right\rangle \tag{4.2.33}$$

$$=\hbar^2Nm(m-1)\Psi+2\hbar^2\left\langle P\sum_\rho\sum_{i\neq j}\frac{u_i(u_i-1)}{(\vec{z}_\rho-u_i)(u_i-u_j)}\right\rangle \\ =\hbar^2Nm(m-1)\Psi+\hbar\int\sum_i\partial_{u_i}(\Delta^{2\hbar})\sum_\rho\frac{u_i(u_i-1)}{\vec{z}_\rho-u_i}P\prod_k\Theta(u_k)du_i \tag{4.2.34}$$

$$\hbar^2\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)\partial^2_{\vec{z}_\rho}\Psi=-\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho(\vec{z}_\rho-1)\partial_{\vec{z}_\rho}-\vec{z}_\sigma(\vec{z}_\sigma-1)\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}\Psi- \\ -\hbar\sum_\rho(t\vec{z}^2_\rho-(b+c+t)\vec{z}_\rho+b)\partial_{\vec{z}_\rho}\Psi-\hbar Nm(b+c+t)\Psi+ \\ +m\hbar t\sum_\rho\vec{z}_\rho\Psi+\hbar Nt\partial_t\Psi+\hbar^2Nm(m-1)\Psi+\hbar Nm(N-1)\Psi \tag{4.2.35}$$

Therefore, the general Hamiltonian operator for quantum Painlevé V equation is given by

$$
\begin{aligned}
tH_V =&\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho(\vec{z}_\rho-1)\partial_{\vec{z}_\rho}-\vec{z}_\sigma(\vec{z}_\sigma-1)\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}+\sum_\rho\Big(\hbar^2\vec{z}_\rho(\vec{z}_\rho-1)\partial^2_{\vec{z}_\rho}+\\
&+\hbar\Big(t\vec{z}_\rho^2-(b+c+t)\vec{z}_\rho+b\Big)\partial_{\vec{z}_\rho}-m\hbar t\vec{z}_\rho\Big)+\hbar Nm(b+c+t)-\\
&-\hbar^2Nm(m-1)-\hbar Nm(N-1)
\end{aligned}
\tag{4.2.36}
$$

### 4.2.5 Quantum Painlevé VI

The Laplacian operator applied to $P$ yields

$$
\begin{aligned}
\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)\partial^2_{\vec{z}_\rho}P &= \sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)P\sum_{i\neq j}\frac{1}{(\vec{z}_\rho-u_i)(\vec{z}_\rho-u_j)}\\
&=P\sum_\rho\sum_{i\neq j}\frac{\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)}{(\vec{z}_\rho-u_i)(\vec{z}_\rho-u_j)}\\
&=P\sum_\rho\sum_{i\neq j}\Bigg(\Big(\vec{z}_\rho-(t+1)+u_i+u_j\Big)+\frac{u_i(u_i-1)(u_i-t)}{(u_i-u_j)(\vec{z}_\rho-u_i)}-\\
&\qquad-\frac{u_j(u_j-1)(u_j-t)}{(u_i-u_j)(\vec{z}_\rho-u_j)}\Bigg).
\end{aligned}
\tag{4.2.37}
$$

This yields

$$
\begin{aligned}
\hbar^2\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)\partial^2_{\vec{z}_\rho}\Psi =&\hbar^2\Bigg\langle P\sum_\rho\sum_{i\neq j}\Bigg(\Big(\vec{z}_\rho-(t+1)+u_i+u_j\Big)+\frac{u_i(u_i-1)(u_i-t)}{(u_i-u_j)(\vec{z}_\rho-u_i)}-\\
&-\frac{u_j(u_j-1)(u_j-t)}{(u_i-u_j)(\vec{z}_\rho-u_j)}\Bigg)\Bigg\rangle.
\end{aligned}
\tag{4.2.38}
$$

Upon rearranging of the terms we obtain the equation

$$
\begin{aligned}
\hbar^2\sum_\rho\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)\partial^2_{\vec{z}_\rho}\Psi =& -\hbar\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\vec{z}_\rho(\vec{z}_\rho-1)(\vec{z}_\rho-t)\partial_{\vec{z}_\rho}-\vec{z}_\sigma(\vec{z}_\sigma-1)(\vec{z}_\sigma-t)\partial_{\vec{z}_\sigma}}{\vec{z}_\rho-\vec{z}_\sigma}\Psi+\\
&+t(t-1)\hbar\partial_t\Psi+\hbar\sum_\rho\Big((a+b)(\vec{z}_\rho-1)(\vec{z}_\rho-t)+c\vec{z}_\rho(\vec{z}_\rho-t)+d\vec{z}_\rho(\vec{z}_\rho-1)\Big)\partial_{\vec{z}_\rho}\Psi-\\
&-\hbar\sum_\rho\vec{z}_\rho^2\partial_{\vec{z}_\rho}\Psi+\hbar\sum_\rho\vec{z}_\rho\partial_{\vec{z}_\rho}\Psi+(2\hbar Nm-\hbar m-\hbar^2m^2)\sum_\rho\vec{z}_\rho\Psi+\\
&+(\hbar^2Nm-\hbar tN^2m-\hbar N^2m+m^2\hbar^2Nt-d\hbar Nmt+\hbar Nmt+(b+d)\hbar Nm)\Psi+\\
&+(\hbar N^2-\hbar^2N)\Big\langle\sum_i u_iP\Big\rangle+t(t-1)\hbar N\Big\langle\sum_i\frac{d}{t-u_i}P\Big\rangle.
\end{aligned}
\tag{4.2.39}
$$

Now, instead of applying the Euler operator, we consider the operator

$$\mathbb{L} := \sum_i \partial_{u_i}\Big(u_i(1-u_i)\Big). \tag{4.2.40}$$

We apply $\mathbb{L}$ to the integrand

$$J(\vec{u},\vec{z}) := \prod_{1\le i<j\le m}(u_i-u_j)^{2\hbar}\prod_{\rho,i}(\vec{z}_\rho-u_i)u_i^{-a-b-1}(1-u_i)^{-c-1}(t-u_i)^{-d}. \tag{4.2.41}$$

After some calculations we obtain

$$\begin{aligned}
\mathbb{L}J(\vec{u},\vec{z}) = \sum_i \Bigg[&(1-2u_i)\prod_{i<j}(u_i-u_j)^{2\hbar}\prod_{\rho,i}(\vec{z}_\rho-u_i)u_i^{-a-b-1}(1-u_i)^{-c-1}(t-u_i)^{-d}+\\
&+u_i(1-u_i)\prod_{i<j}(u_i-u_j)^{2\hbar}\Big(2\hbar\sum_{i\neq j}\frac{1}{u_i-u_j}\Big)\prod_{\rho,i}(\vec{z}_\rho-u_i)u_i^{-a-b-1}(1-u_i)^{-c-1}(t-u_i)^{-d}+\\
&+u_i(1-u_i)\prod_{i<j}(u_i-u_j)^{2\hbar}\prod_{\rho,i}(\vec{z}_\rho-u_i)\Big(\sum_\rho\frac{-1}{\vec{z}_\rho-u_i}\Big)\prod_i u_i^{-a-b-1}(1-u_i)^{-c-1}(t-u_i)^{-d}+\\
&+u_i(1-u_i)\prod_{i<j}(u_i-u_j)^{2\hbar}\prod_{\rho,i}\frac{(\vec{z}_\rho-u_i)}{u_i^{a+b+1}(1-u_i)^{c+1}(t-u_i)^d}\Big(\frac{-a-b-1}{u_i}+\frac{c+1}{1-u_i}+\frac{d}{t-u_i}\Big)\Bigg]
\end{aligned} \tag{4.2.42}$$

$$\begin{aligned}
=&\Delta^{2\hbar}\Bigg[mP-2\sum_i u_iP-2\hbar(m-1)\sum_i u_iP-m\sum_\rho\vec{z}_\rho P-N\sum_i u_iP+NmP-\\
&-\sum_\rho\vec{z}_\rho(1-\vec{z}_\rho)\partial_{\vec{z}_\rho}P+(-a-b-d-1)mP+tdmP+(a+b+c+d+2)\sum_i u_iP-\\
&-t(t-1)\sum_i\frac{d}{t-u_i}P\Bigg]\prod_i\Theta(u_i).
\end{aligned} \tag{4.2.43}$$

Integrating (4.2.43), one obtains zero beacause the integrand $\mathbb{L}J$ can be viewed as the divergence of a vector field. Therefore,

$$\begin{aligned}
t(t-1)\Big\langle\sum_i\frac{d}{t-u_i}P\Big\rangle =&(-\hbar m+Nm+(-b-d)m+tdm)\Psi+\\
&+(-2\hbar(m-1)-N+a+b+c+d)\Big\langle\sum_i u_iP\Big\rangle-\\
&-m\sum_\rho\vec{z}_\rho\Psi-\sum_\rho\vec{z}_\rho(1-\vec{z}_\rho)\partial_{\vec{z}_\rho}\Psi
\end{aligned} \tag{4.2.44}$$

Substituting the LHS into the equation (4.2.39) yields the following result:

$$\hbar^2 \sum_{\rho} \vec{z}_\rho(\vec{z}_\rho - 1)(\vec{z}_\rho - t)\partial^2_{\vec{z}_\rho}\Psi = -\hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{\vec{z}_\rho(\vec{z}_\rho - 1)(\vec{z}_\rho - t)\partial_{\vec{z}_\rho} - \vec{z}_\sigma(\vec{z}_\sigma - 1)(\vec{z}_\sigma - t)\partial_{\vec{z}_\sigma}}{\vec{z}_\rho - \vec{z}_\sigma}\Psi +$$
$$+ t(t-1)\hbar\partial_t\Psi + \hbar\sum_{\rho}\Big((a+b)(\vec{z}_\rho - 1)(\vec{z}_\rho - t) + c\vec{z}_\rho(\vec{z}_\rho - t) +$$
$$+ d\vec{z}_\rho(\vec{z}_\rho - 1)\Big)\partial_{\vec{z}_\rho}\Psi + \hbar(1-N)\sum_{\rho}\vec{z}_\rho(1-\vec{z}_\rho)\partial_{\vec{z}_\rho}\Psi +$$
$$+ \hbar m(N - 1 - \hbar m)\sum_{\rho}\vec{z}_\rho\Psi + \hbar m N(\hbar m + 1 - N)t\Psi \tag{4.2.45}$$

Therefore, the general Hamiltonian operator for quantum Painlevé VI equation is given by

$$t(t-1)H_{VI} =$$
$$= \hbar \sum_{\substack{\rho,\sigma \\ \rho\neq\sigma}} \frac{\vec{z}_\rho(\vec{z}_\rho - 1)(\vec{z}_\rho - t)\partial_{\vec{z}_\rho} - \vec{z}_\sigma(\vec{z}_\sigma - 1)(\vec{z}_\sigma - t)\partial_{\vec{z}_\sigma}}{\vec{z}_\rho - \vec{z}_\sigma} + \sum_{\rho}\hbar^2\vec{z}_\rho(\vec{z}_\rho - 1)(\vec{z}_\rho - t)\partial^2_{\vec{z}_\rho} -$$
$$- \hbar\sum_{\rho}\Big((a+b)(\vec{z}_\rho - 1)(\vec{z}_\rho - t) + c\vec{z}_\rho(\vec{z}_\rho - t) + (d + N - 1)\vec{z}_\rho(\vec{z}_\rho - 1)\Big)\partial_{\vec{z}_\rho} -$$
$$- \hbar m(N - 1 - \hbar m)\sum_{\rho}\vec{z}_\rho - \hbar m N(\hbar m + 1 - N)t \tag{4.2.46}$$

**Remark 4.2.1.** *All these operators reduce to the Hamiltonian operators in* (4.1.2) *for* $N = 1$.

**Remark 4.2.2.** *The reason why quantum Painlevé I is excluded from the calculations in this section is that no transcendental function is proved to be close to expressions for the solution of Painlevé I. Therefore, there is no integral representation for the wave function satisfying the Schrödinger equation for quantum Painlevé I.*

In the next chapter, we proceed with a comparison between the Hamiltonian system obtained in Chapter 3 as a quantization of the Calogero-Painlevé system, and the generalized Hamiltonian operators that we computed in the current chapter.

# Chapter 5

# Results

## 5.1 The equivalence of the two systems

We now claim that the generalized wave functions (4.2.1) are indeed solutions to the quantum Calogero Hamiltonians (3.3.12, 3.3.15, 3.3.19, 3.3.23, 3.3.27) constructed by canonical quantization of the non-commutative Hamiltonians of the classical isomonodromic noncommutative equations of [3]. The identification requires to choose the parameters in a special way. Consider for example (3.3.12) and (4.2.2):

$$\begin{aligned}\tilde{H}_{II} =& \frac{\hbar^2}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma} - \frac{\hbar^2\kappa(\kappa+1)}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{1}{(z_\sigma-z_\rho)^2} + \frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho} - \hbar\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\partial_{z_\rho} + \\ &+\left(\frac{1}{2}-\theta-\hbar N\right)\sum_\rho z_\rho,\\ H_{II} =& \frac{\hbar}{2}\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma} + \frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho} - \hbar\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\partial_{z_\rho} + m\hbar\sum_\rho z_\rho.\end{aligned}$$

The first observation is that in the two Hamiltonians the second and first-order differential parts have different powers of $\hbar$; they coincide only for $\hbar = 1$ (we exclude the trivial case $\hbar = 0$). We commented on this in Section 4.2.

Then there is the Calogero-like potential term in (3.3.12) which is absent in (4.2.2); this forces us to choose $\kappa = 0$. These values mean that the $GL_n$ representation in $\mathbb{V}$ is the trivial one in the quantization scheme. These constraints determine the value of $\theta$ and it turns out to be $\theta = \frac{1}{2} - N - m$.

The other observation for Calogero-Painlevé II is summarized in the following Lemma:

**Lemma 5.1.1.** *Considering $\Delta = \prod_{1\leq\alpha<\beta\leq N}(z_\alpha - z_\beta)$ be a Vandermonde polynomial in $\vec{z}$, then the action of quantum Hamiltonian operator $H_{II}$ on the generalized wave functions, is* *equivalent to the action of* $\Delta^{-R}\widetilde{H}_{II}\Delta^R$ *on the integral representation of the wave function:*

$$\Delta^{-R}\widetilde{H}_{II}\Delta^R\Psi(Z) = H_{II}\Psi(Z) \tag{5.1.1}$$

*for* $R$ *and the scalar* $\kappa$ *determined as each of the following pairs*

$$\left(R = \frac{1}{\hbar} - 1, \qquad \kappa = \frac{1}{\hbar} - 1\right), \qquad \left(R = \frac{1}{\hbar} - 1, \qquad \kappa = -\frac{1}{\hbar}\right), \qquad \hbar \neq 0 \tag{5.1.2}$$

*Under this circumstances, the parameter* $\theta$ *is determined by*

$$\theta = \hbar(1-m) + N(1-2\hbar) - \frac{1}{2}. \tag{5.1.3}$$

*Proof.* Considering $\Delta = \prod_{1\le\alpha<\beta\le N}(z_\alpha - z_\beta)$, and the Hamiltonians $\widetilde{H}_{II}$ and $H_{II}$ as (3.3.12) and (4.2.2), the direct computation of (5.1.1) gives

$$\begin{aligned}
&\left(\hbar\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho - z_\sigma} + \frac{\hbar^2}{2}\sum_\rho \partial^2_{z_\rho} - \hbar\sum_\rho\left(z_\rho^2 + \frac{t}{2}\right)\partial_{z_\rho} + m\hbar\sum_\rho z_\rho\right)\Psi(Z) - \\
&- \Delta^{-R}\left(\hbar^2\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho - z_\sigma} - \hbar^2\kappa(\kappa+1)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\sigma - z_\rho)^2} + \frac{\hbar^2}{2}\sum_\rho\partial^2_{z_\rho} - \hbar\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\partial_{z_\rho} + \right.\\
&\qquad \left. + \left(\frac{1}{2} - \theta - \hbar N\right)\sum_\rho z_\rho\right)\Delta^R\Psi(Z) = 0.
\end{aligned} \tag{5.1.4}$$

Note that

$$\begin{aligned}
\partial_{z_\rho}\left(\Delta^R\right) &= R\Delta^R\sum_{\alpha\neq\rho}\frac{1}{z_\rho - z_\alpha},\\
\partial^2_{z_\rho}\left(\Delta^R\right) &= R^2\Delta^R\left(\sum_{\alpha\neq\rho}\frac{1}{z_\rho - z_\alpha}\right)^2 - R\Delta^R\sum_{\alpha\neq\rho}\frac{1}{(z_\rho - z_\alpha)^2}.
\end{aligned} \tag{5.1.5}$$

Therefore, the equation (5.1.4) continues

$$\begin{aligned}
&\Bigg(\hbar\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma}+\frac{\hbar^2}{2}\sum_\rho\partial_{z_\rho}^2-\hbar\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\partial_{z_\rho}+m\hbar\sum_\rho z_\rho\Bigg)\Psi(Z)-\\
&-\Bigg(\hbar^2\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma}+\hbar^2R\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{z_\rho-z_\sigma}\Bigg(\sum_{\alpha\neq\rho}\frac{1}{z_\rho-z_\alpha}-\sum_{\alpha\neq\sigma}\frac{1}{z_\sigma-z_\alpha}\Bigg)-\\
&-\hbar^2\kappa(\kappa+1)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\sigma-z_\rho)^2}+\frac{\hbar^2}{2}\sum_\rho\partial_{z_\rho}^2+\hbar^2R\sum_\rho\sum_{\alpha\neq\rho}\frac{\partial_{z_\rho}}{z_\rho-z_\sigma}+\\
&+\frac{\hbar^2}{2}\sum_\rho\Bigg(R^2\Bigg(\sum_{\alpha\neq\rho}\frac{1}{z_\rho-z_\alpha}\Bigg)^2-R\sum_{\alpha\neq\rho}\frac{1}{(z_\rho-z_\alpha)^2}\Bigg)-\hbar\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\partial_{z_\rho}-\\
&-\hbar R\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\sum_{\alpha\neq\rho}\frac{1}{z_\rho-z_\alpha}+\left(\frac{1}{2}-\theta-\hbar N\right)\sum_\rho z_\rho\Bigg)\Psi(Z)=0.
\end{aligned}\tag{5.1.6}$$

By straightforward manipulation of indices one can show that the following relations are true:

$$\begin{aligned}
\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma}=&2\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}}{z_\rho-z_\sigma},\\
\sum_\rho\left(z_\rho^2+\frac{t}{2}\right)\sum_{\alpha\neq\rho}\frac{1}{z_\rho-z_\alpha}=&(N-1)\sum_\rho z_\rho.
\end{aligned}\tag{5.1.7}$$

Also, one can prove (done in the next Lemma)

$$\begin{aligned}
\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\frac{1}{z_\rho-z_\sigma}\Bigg(\sum_{\alpha\neq\rho}\frac{1}{z_\rho-z_\sigma}-\sum_{\alpha\neq\sigma}\frac{1}{z_\sigma-z_\alpha}\Bigg)=&2\sum_{\substack{\rho,\sigma\\ \rho\neq\sigma}}\sum_{\alpha\neq\rho,\sigma}\frac{1}{(z_\rho-z_\sigma)(z_\alpha-z_\rho)}+2\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\rho-z_\sigma)^2}\\
=&2\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\rho-z_\sigma)^2}
\end{aligned}\tag{5.1.8}$$

putting all together, and simplifying the similar terms, equation (5.1.6) yields

$$\begin{aligned}
(5.1.6)=\left(\hbar-\hbar^2-\hbar^2R\right)&\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{\partial_{z_\rho}-\partial_{z_\sigma}}{z_\rho-z_\sigma}\Psi(Z)+\hbar^2\kappa(\kappa+1)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\sigma-z_\rho)^2}\Psi(Z)-\\
&-2\hbar^2R\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{1}{(z_\sigma-z_\rho)^2}\Psi(Z)-R(R-1)\hbar^2\sum_\rho\sum_{\alpha\neq\rho}\frac{1}{(z_\rho-z_\alpha)^2}\Psi(Z)+\\
&+\left(m\hbar-\hbar R(N-1)-\frac{1}{2}+\theta+\hbar N\right)\Psi(Z)=0.
\end{aligned}\tag{5.1.9}$$

To have the equation (5.1.9) satisfied, we need to have the following relations to be true for $R$, $\kappa$, and the parameter $\theta$:

- We either get

$$\begin{aligned} R &= \frac{1}{\hbar} - 1, \\ \kappa &= \frac{1}{\hbar} - 1, \\ \theta &= \hbar(1-m) + N(1-2\hbar) - \frac{1}{2} \end{aligned} \tag{5.1.10}$$

- or

$$\begin{aligned} R &= \frac{1}{\hbar} - 1, \\ \kappa &= -\frac{1}{\hbar}, \\ \theta &= \hbar(1-m) + N(1-2\hbar) - \frac{1}{2}. \end{aligned} \tag{5.1.11}$$

■

**Lemma 5.1.2.** *For an arbitrary $a$, and $z_\alpha$, $\alpha = 1, ..., N$ the following equality holds true:*

$$\sum_\alpha \sum_{\beta\neq\alpha} \sum_{\tau\neq\alpha,\beta} \frac{z_\alpha^a}{(z_\alpha - z_\tau)(z_\alpha - z_\beta)} = \begin{cases} 0 & a = 0,1 \\ \frac{N(N-1)(N-2)}{3} & a = 2 \\ (N-1)(N-2)\sum_\alpha z_\alpha & a = 3 \end{cases} \tag{5.1.12}$$

*Proof.* Consider the term $P(z) = \prod_{\alpha=1}^N (z - z_\alpha)$. We compute the following residue

$$\operatorname*{res}_{z=\infty} z^a \left(\frac{P'(z)}{P(z)}\right)^3 dz. \tag{5.1.13}$$

This can be done in two ways; first note that we can express $\frac{P'}{P}$ as the sum of Logarithmic derivatives

$$\frac{P'(z)}{P(z)} = \sum_\alpha \frac{\mathrm{d}}{\mathrm{d}z} \ln(z - z_\alpha) = \sum_\alpha \frac{1}{z - z_\alpha} \tag{5.1.14}$$

using the geometric series expansion one obtains

$$\begin{aligned} \sum_\alpha \frac{1}{z - z_\alpha} =& \frac{1}{z} \sum_\alpha \left(1 + \frac{z_\alpha}{z} + \frac{z_\alpha^2}{z^2} + \mathcal{O}(3)\right) \\ =& \frac{N}{z} + \frac{\sum_\alpha z_\alpha}{z^2} + \mathcal{O}(3) \end{aligned} \tag{5.1.15}$$

Hence, for the computation of the residue, we have

$$\operatorname*{res}_{z=\infty} z^a \left(\frac{P'(z)}{P(z)}\right)^3 dz = \operatorname*{res}_{z=\infty} z^a \left(\frac{N^3}{z^3} + \frac{3N^2\sum_\alpha z_\alpha}{z^4} + \frac{3N\left(\sum_\alpha z_\alpha\right)^2}{z^5} + \mathcal{O}(6)\right) \tag{5.1.16}$$

depending on the value of $a$, the final answer of this computation varies, concerning the proof of this Lemma, we only compute the result for $a = 0, 1, 2, 3$.

Computing the residue at infinity one obtains

$$\operatorname*{res}_{z=\infty} z^a \left(\frac{P'(z)}{P(z)}\right)^3 dz = \begin{cases} 0 & a = 0, 1 \\ -N^3 & a = 2 \\ -3N^2 \sum_\alpha z_\alpha & a = 3 \end{cases} \tag{5.1.17}$$

from complex analysis, we know that the sum of residues at poles plus the residue at infinity is zero, since the poles of $z^a \left(\frac{P'}{P}\right)^3$ are located at $z_\alpha$'s, we have

$$\sum_\alpha \operatorname*{res}_{z=z_\alpha} z^a \left(\frac{P'}{P}\right)^3 \mathrm{d}z + \operatorname*{res}_{z=\infty} z^a \left(\frac{P'}{P}\right)^3 \mathrm{d}z = 0 \tag{5.1.18}$$

To compute the the first term in equation (5.1.18), we proceed via a different approach; we first compute the exponent which results in the following expression:

$$\begin{aligned}
&\sum_\alpha \operatorname*{res}_{z=z_\alpha} z^a \left(\frac{P'}{P}\right)^3 \mathrm{d}z = \sum_\alpha \operatorname*{res}_{z=z_\alpha} z^a \left(\sum_\tau \frac{1}{z - z_\tau}\right)^3 \\
&= \sum_\alpha \operatorname*{res}_{z=z_\alpha} z^a \left(\sum_\tau \frac{1}{(z-z_\tau)^3} + 3\sum_{\substack{\zeta,\tau \\ \zeta\neq\tau}} \frac{1}{(z-z_\zeta)(z-z_\tau)^2} + \sum_{\substack{\zeta,\tau,\beta \\ \zeta\neq\tau,\beta \\ \tau\neq\beta}} \frac{1}{(z-z_\zeta)(z-z_\tau)(z-z_\beta)}\right).
\end{aligned} \tag{5.1.19}$$

Applying Cauchy's residue theorem for the poles at $z_\alpha$'s with different orders, one gets

$$\begin{aligned}(5.1.19) = &\sum_{\alpha}\left(\frac{a(a-1)}{2}z_\alpha^{a-2} + 3\sum_{\tau\neq\alpha}\frac{z_\alpha^a}{(z_\alpha - z_\tau)^2} + 3\sum_{\beta\neq\alpha}\left(\frac{a z_\alpha^{a-1}}{z_\beta - z_\alpha} - \frac{z_\alpha^a}{(z_\beta - z_\alpha)^2}\right) + \right.\\ &\left. + 3\sum_{\substack{\beta,\rho\\ \beta\neq\rho\\ \beta\neq\alpha,\rho\neq\alpha}}\frac{z_\alpha^a}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)}\right)\\ = &\sum_{\alpha}\left(\frac{a(a-1)}{2}z_\alpha^{a-2} + 3\sum_{\beta\neq\alpha}\frac{a z_\alpha^{a-1}}{z_\alpha - z_\beta} + 3\sum_{\substack{\beta,\rho\\ \beta\neq\rho\\ \beta\neq\alpha,\rho\neq\alpha}}\frac{z_\alpha^a}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)}\right).\end{aligned} \tag{5.1.20}$$

One concludes

$$\begin{aligned}\sum_{\alpha}\operatorname*{res}_{z=z_\alpha} z^a\left(\frac{P'}{P}\right)^3 \mathrm{d}z = &\sum_{\alpha}\left(\frac{a(a-1)}{2}z_\alpha^{a-2} + 3\sum_{\beta\neq\alpha}\frac{a z_\alpha^{a-1}}{z_\alpha - z_\beta} + 3\sum_{\substack{\beta,\rho\\ \beta\neq\rho\\ \beta\neq\alpha,\rho\neq\alpha}}\frac{z_\alpha^a}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)}\right)\\ = &-\operatorname*{res}_{z=\infty} z^a\left(\frac{P'}{P}\right)^3 \mathrm{d}z.\end{aligned} \tag{5.1.21}$$

From equation (5.1.17) we have the following observations:

- For $a = 0$:

$$\sum_{\substack{\beta,\rho\\ \beta\neq\rho\\ \beta\neq\alpha,\rho\neq\alpha}}\frac{1}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} = 0. \tag{5.1.22}$$

- For $a = 1$:

$$3\sum_{\substack{\alpha,\beta\\ \beta\neq\alpha}}\frac{1}{z_\alpha - z_\beta} + 3\sum_{\substack{\alpha,\beta,\rho\\ \alpha\neq\beta,\rho\\ \beta\neq\rho,\rho\neq\alpha}}\frac{z_\alpha}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} = 0. \tag{5.1.23}$$

  By a simple manipulation of indices or using the fact that the function $\dfrac{1}{z_\alpha - z_\beta}$ is anti-symmetric, one gets the first term of the equation (5.1.23) to be zero, therefore:

$$\sum_{\substack{\alpha,\beta,\rho\\ \alpha\neq\beta,\rho\\ \beta\neq\rho,\rho\neq\alpha}}\frac{z_\alpha}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} = 0. \tag{5.1.24}$$

- For $a = 2$:

$$N + 6 \sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\alpha}{z_\alpha - z_\beta} + 3 \sum_{\substack{\alpha,\beta,\rho \\ \alpha\neq\beta,\rho \\ \beta\neq\rho,\rho\neq\alpha}} \frac{z_\alpha^2}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} = N^3. \tag{5.1.25}$$

For the second term of the equation (5.1.25) consider the following; let $S = \sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\alpha}{z_\alpha - z_\beta}$, by adding and subtracting $z_\beta$ in the numerator we get

$$\sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\alpha}{z_\alpha - z_\beta} = \sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} (1) + \underbrace{\sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\beta}{z_\alpha - z_\beta}}_{=-S} \tag{5.1.26}$$

hence

$$2S = N(N-1) \qquad \Longrightarrow \qquad S = \frac{N(N-1)}{2}. \tag{5.1.27}$$

Therefore, the equation (5.1.25) gives

$$\begin{aligned} \sum_{\substack{\alpha,\beta,\rho \\ \alpha\neq\beta,\rho \\ \beta\neq\rho,\rho\neq\alpha}} \frac{z_\alpha^2}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} &= \frac{N^3 - N - 3N(N-1)}{3} \\ &= \frac{N(N-1)(N-2)}{3}. \end{aligned} \tag{5.1.28}$$

- For $a = 3$:

$$3\sum_\alpha z_\alpha + 9 \sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\alpha^2}{z_\alpha - z_\beta} + 3 \sum_{\substack{\alpha,\beta,\rho \\ \alpha\neq\beta,\rho \\ \beta\neq\rho,\rho\neq\alpha}} \frac{z_\alpha^3}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} = 3N^2 \sum_\alpha z_\alpha \tag{5.1.29}$$

with the same analogy as the previous case, for the second term we get

$$\sum_{\substack{\alpha,\beta \\ \beta\neq\alpha}} \frac{z_\alpha^2}{z_\alpha - z_\beta} = (N-1)\sum_\alpha z_\alpha. \tag{5.1.30}$$

Hence, the equation (5.1.29) gives

$$\begin{aligned} \sum_{\substack{\alpha,\beta,\rho \\ \alpha\neq\beta,\rho \\ \beta\neq\rho,\rho\neq\alpha}} \frac{z_\alpha^3}{(z_\alpha - z_\beta)(z_\alpha - z_\rho)} &= \left(N^2 - 3N + 2\right) \sum_\alpha z_\alpha \\ &= (N-1)(N-2)\sum_\alpha z_\alpha. \end{aligned} \tag{5.1.31}$$

■

The Lemma 5.1.1 is actually the manifestation of a more general result which is contained in the following theorem:

**Theorem 5.1.3.** *Define the two sequences of differential operators*

$$
\begin{aligned}
H_a &:= \hbar^2 \sum_{\rho} z_\rho^a \partial_{z_\rho}^2 + 2\hbar \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho^a \partial_{z_\rho} - z_\sigma^a \partial_{z_\sigma}}{z_\rho - z_\sigma} \\
\widetilde{H}_a &:= \hbar^2 \sum_{\rho} z_\rho^a \partial_{z_\rho}^2 + 2\hbar^2 \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho^a \partial_{z_\rho} - z_\sigma^a \partial_{z_\sigma}}{z_\rho - z_\sigma} - \hbar^2 \kappa(\kappa+1) \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho^a + z_\sigma^a}{(z_\rho - z_\sigma)^2} + \\
&\quad + 2(1-\hbar^2) \begin{cases} 0 & a = 0,1 \\ \dfrac{N(N-1)(N-2)}{3} & a = 2 \\ (N-1)(N-2)\sum_\rho z_\rho & a = 3 \;. \end{cases}
\end{aligned}
\tag{5.1.32}
$$

*Then we have the following identity*

$$
H_a = \Delta^{-R} \circ \widetilde{H}_a \circ \Delta^R \tag{5.1.33}
$$

*provided that* $R = \dfrac{1}{\hbar} - 1$ *and* $\kappa = \dfrac{1}{\hbar} - 1$ *or* $\kappa = -\dfrac{1}{\hbar}$.

*Proof.* Define

$$
\begin{aligned}
\mathcal{L}_a &:= \sum_{\rho} z_\rho^a \partial_{z_\rho}^2 \\
\mathcal{M}_a &:= \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho^a \partial_{z_\rho} - z_\sigma^a \partial_{z_\sigma}}{z_\rho - z_\sigma}.
\end{aligned}
\tag{5.1.34}
$$

To start the proof of this theorem we apply both operations $H_a$ and $\widetilde{H}_a$ on a wave function $\Psi(z)$.

Note that, following the instruction of computations for the Lemma 5.1.1, and using the equations in (5.1.8), the action of the operators $\Delta^{-R}\mathcal{L}_a\Delta^R$ on and $\Delta^{-R}\mathcal{M}_a\Delta^R$ on $\Psi(Z)$ gives

$$\begin{aligned}\Delta^{-R}\mathcal{L}_a\Delta^R\Psi(Z) =&\mathcal{L}_a\Psi(Z) + 2R\mathcal{M}_a\Psi(Z) + R(R-1)\sum_{\rho} z_\rho^a \sum_{\alpha\neq\rho} \frac{1}{(z_\rho - z_\alpha)^2}\Psi(Z)+\\ &+ 2R^2\sum_{\rho} z_\rho^a \sum_{\substack{\alpha,\beta\\ \alpha\neq\beta\\ \rho\neq\alpha,\beta}} \frac{1}{(z_\rho - z_\alpha)(z_\rho - z_\beta)}\Psi(Z),\\ \Delta^{-R}\mathcal{M}_a\Delta^R\Psi(Z) =&\mathcal{M}_a\Psi(Z) + 2R\sum_{\rho}\sum_{\substack{\alpha,\sigma\\ \alpha\neq\sigma\\ \rho\neq\alpha,\sigma}} \frac{z_\rho^a}{(z_\rho - z_\sigma)(z_\rho - z_\alpha)}\Psi(Z)+\\ &+ 4R\sum_{\substack{\rho,\sigma\\ \rho<\sigma}} \frac{z_\rho^a}{(z_\rho - z_\sigma)^2}\Psi(Z).\end{aligned} \tag{5.1.35}$$

Therefore, the equation (5.1.33) yields

$$\begin{aligned}&2\hbar(\hbar R + \hbar - 1)\mathcal{M}_a\Psi(Z) + \hbar^2\left(R(R+1) - \kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}} \frac{z_\rho^a}{(z_\rho - z_\sigma)^2}\Psi(Z)+\\ &+ 2\hbar^2 R(R+2)\sum_{\rho}\sum_{\substack{\alpha,\sigma\\ \alpha\neq\sigma\\ \rho\neq\alpha,\sigma}} \frac{z_\rho^a}{(z_\rho - z_\sigma)(z_\rho - z_\alpha)}\Psi(Z).\end{aligned} \tag{5.1.36}$$

To obtain zero for the above computation, first we must have the following pair of equalities for the first two terms to vanish

$$\left(\kappa = \frac{1}{\hbar} - 1, \qquad R = \frac{1}{\hbar} - 1\right) \qquad \text{or} \qquad \left(\kappa = -\frac{1}{\hbar}, \qquad R = \frac{1}{\hbar} - 1\right). \tag{5.1.37}$$

In order to discuss the different determinations for the remaining terms in equation (5.1.36) depending on the value of $a$, we recall the result of the Lemma 5.1.12. ■

**Note 5.1.4.** *For $\hbar = 1$ the values of $R$, $\kappa$, and $\theta$ reduce to result of the first observation.*

Using Theorem 5.1.3 in the various cases of the Hamiltonian Calogero–Painlevé operators and direct computations, allows us to express the Hamiltonians (4.2.2–4.2.6) as special cases of (3.3.12–3.3.27), and identify the parameters $\theta_0$, $\theta_1$, $\theta_2$, $\theta_t$, $\theta$, $k$, $a$, $b$, $c$ and $d$.

The result of these computations results in two observations. Similar to the case of the Calogero-Painlvé II, one observation forces the values of $\hbar$ to be equal to 1 and $\kappa = 0$ in order for the mentioned Hamiltonians to represent the same systems. The second observation, which is the general case including the special case of the first observation, determines the value of parameters for arbitrary non-zero $\hbar$, and results in the value of $\kappa$ to be an expression in terms of $\hbar$.

The details of the case by case study of Calogero-Painlevé III-VI is stated in the following sections:

### 5.1.1 Calogero-Painlevé III

Similar to the case of second Calogero-Painlevé, the first result of the comparison between (3.3.15) and (4.2.27) forces $\hbar = 1$ and $\kappa = 0$, that determines the value of other parameters as

$$\theta_0 = b - m + 1 \qquad\qquad \theta_1 = -N - m. \tag{5.1.38}$$

The result of the second observation is the following Lemma:

**Lemma 5.1.5.** *Considering $\Delta = \prod_{1\le\alpha<\beta\le N}(z_\alpha - z_\beta)$ be a Vandermonde polynomial in $\vec{z}$, then the action of quantum Hamiltonian operator $tH_{III}$ on the generalized wave functions, is equivalent to the action of $\Delta^{-R}t\widetilde{H}_{III}\Delta^R$ on the integral representation of the wave function:*

$$tH_{III}\Psi(Z) = \Delta^{-R}t\widetilde{H}_{III}\Delta^R\Psi(Z) \tag{5.1.39}$$

*for $R$ and the scalar $\kappa$ determined as*

$$\kappa = \frac{1}{\hbar} - 1 \quad or \quad \kappa = -\frac{1}{\hbar}, \quad R = \frac{1}{\hbar} - 1, \qquad \hbar \neq 0. \tag{5.1.40}$$

*Under this circumstances, the parameter $\theta_0$ and $\theta_1$ are determined by*

$$\theta_0 = b + \hbar(1-m), \qquad \theta_1 = -\hbar(m+1) - N + 1. \tag{5.1.41}$$

*Proof.* The proof of this Lemma is a conclusion of Theorem 5.1.3 for $a = 2$. We have:

$$\begin{aligned} tH_{III} =& tH_2 - \hbar\sum_\rho \left(z_\rho^2 + (b+N-1)z_\rho + t\right)\partial_{z_\rho} + m\hbar\sum_\rho z_\rho, \\ t\widetilde{H}_{III} =& t\widetilde{H}_2 - \hbar\sum_\rho \left(z_\rho^2 + (-2\hbar+\theta_0-\theta_1)z_\rho + t\right)\partial_{z_\rho} - (\hbar N + \theta_1)\sum_\rho z_\rho + \widetilde{\Theta} \end{aligned} \tag{5.1.42}$$

where $\Theta$ is a constant. from the computations of the proof of theorem (5.1.3), we obtain

$$\begin{aligned} \left(\Delta^{-R}t\widetilde{H}_{III}\Delta^R - tH_{III}\right)\Psi(Z) =& 2\hbar(\hbar R + \hbar - 1)\mathcal{M}_2\Psi(Z) + \\ &+ \hbar^2\left(R(R+1) - \kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{z_\rho^2}{(z_\rho - z_\sigma)^2}\Psi(Z) - \\ &- \hbar\sum_\rho\left(-2\hbar + \theta_0 - \theta_1 - b - N + 1\right)z_\rho\partial_{z_\rho}\Psi(Z) + \\ &+ \left(\hbar(N-m) - \theta_1 - \hbar R(N-1)\right)\sum_\rho z_\rho\Psi(Z) + \Theta\Psi(Z). \end{aligned} \tag{5.1.43}$$

Where $\Theta$ is a constant that can be easily disregarded as a simple gauge transformation can be applied to omit it.

Equation 5.1.43 will be equal to zero if we have the following relations for the parameters:

$$\begin{aligned} \kappa =&\frac{1}{\hbar}-1 \quad \text{or} \quad \kappa=-\frac{1}{\hbar}, \quad R=\frac{1}{\hbar}-1, \qquad \hbar\neq 0 \\ \theta_0 =&b+\hbar(1-m), \quad \theta_1=-\hbar(m+1)-N+1. \end{aligned} \tag{5.1.44}$$

■

### 5.1.2 Calogero-Painlevé IV

In the case of Calogero-Pinlevé IV, the constraints of the values of $\hbar = 1$ and $\kappa = 0$, which comes from the comparison between the first-order differential parts in (3.3.19) and (4.2.31), and the Calogero-like potential term in (3.3.19) which does not appear in (4.2.31), determines the value of the other parameters by

$$\theta_0 = -b-1 \qquad\qquad \theta_1 = b-m. \tag{5.1.45}$$

The second observation, reads the same relations for $R$ and $\kappa$ as in the equation (5.1.40) according to the following Lemma:

**Lemma 5.1.6.** *Considering $\Delta = \prod_{1\le\alpha<\beta\le N}(z_\alpha - z_\beta)$ be a Vandermonde polynomial in $\vec{z}$, then the action of quantum Hamiltonian operator $H_{IV}$ on the generalized wave functions, is equivalent to the action of $\Delta^{-R}\widetilde{H}_{IV}\Delta^R$ on the integral representation of the wave function:*

$$H_{IV}\Psi(Z) = \Delta^{-R}\widetilde{H}_{IV}\Delta^R\Psi(Z) \tag{5.1.46}$$

*for $R$ and the scalar $\kappa$ determined as*

$$\kappa = \frac{1}{\hbar}-1 \quad \mathit{or} \quad \kappa=-\frac{1}{\hbar}, \quad R=\frac{1}{\hbar}-1, \qquad \hbar\neq 0. \tag{5.1.47}$$

*Under this circumstances, the parameter $\theta_0$ and $\theta_1$ are determined by*

$$\theta_0 = -b-\hbar, \qquad \theta_1 = b+1-N-m\hbar. \tag{5.1.48}$$

*Proof.* The proof of this Lemma is also a conclusion of Theorem 5.1.3 for $a = 1$. We recall the equation (5.1.36):

$$\begin{aligned} H_{IV} =&H_1 - \hbar\sum_\rho\left(z_\rho^2+tz_\rho+b\right)\partial_{z_\rho} + m\hbar\sum_\rho z_\rho + mN\hbar t, \\ \widetilde{H}_{IV} =&\widetilde{H}_1 - \hbar\sum_\rho\left(z_\rho^2+tz_\rho-\theta_0-\hbar\right)\partial_{z_\rho} - (\hbar N+\theta_0+\theta_1)\sum_\rho z_\rho - tN^2\hbar \end{aligned} \tag{5.1.49}$$

from the computations of the proof of theorem (5.1.3), we obtain

$$\begin{aligned}&\left(\Delta^{-R}\widetilde{H}_{IV}\Delta^{R}-H_{IV}\right)\Psi(Z)=\\&2\hbar(\hbar R+\hbar-1)\mathcal{M}_1\Psi(Z)+\hbar^2\left(R(R+1)-\kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}}\frac{z_\rho}{(z_\rho-z_\sigma)^2}\Psi(Z)-\\&-\left(\hbar N+\theta_0+\theta_1+\hbar R(N-1)+\hbar m\right)\sum_{\rho}z_\rho\Psi(Z)+\hbar\left(\theta_0+\hbar+b\right)\partial_{z_\rho}\Psi(Z)+\mathcal{T}(\hbar,t)\Psi(Z)\end{aligned} \tag{5.1.50}$$

where $\mathcal{T}(\hbar,t)$ is a constant that can always be gauged away from the operators. According to the result of the Lemma 5.1.12, this expression vanishes for the following determinations of $R$, $\kappa$, $theta_0$, and $\theta_1$:

$$\begin{aligned}\kappa=&\frac{1}{\hbar}-1\quad\text{or}\quad\kappa=-\frac{1}{\hbar},\quad R=\frac{1}{\hbar}-1,\\\theta_0=&-b-\hbar,\qquad\theta_1=b+1-N-m\hbar.\end{aligned} \tag{5.1.51}$$

■

### 5.1.3 Calogero-Painlevé V

For Calogero-Painlevé V, to have the relation $t\widetilde{H}_V\Psi(Z)=tH_V\Psi(Z)$ satisfied, we have to consider $\hbar=1$ and $\kappa=0$ which yields

$$\theta_0=-c-1\qquad\qquad\theta_1=-N-m+c+1\qquad\qquad\theta_2=b+1. \tag{5.1.52}$$

However, to have the equation

$$tH_V\Psi(Z)=\Delta^{-R}t\widetilde{H}_V\Delta^{R}\Psi(Z) \tag{5.1.53}$$

to be true, we prove that the constant $\hbar$ remains arbitrary (and non-zero). The statement of these computations is the following Lemma:

**Lemma 5.1.7.** *Considering $\Delta=\prod_{1\leq\alpha<\beta\leq N}(z_\alpha-z_\beta)$ be a Vandermonde polynomial in $\vec{z}$, then the action of quantum Hamiltonian operator $tH_V$ on the generalized wave functions, is equivalent to the action of $\Delta^{-R}t\widetilde{H}_V\Delta^{R}$ on the integral representation of the wave function for the values of $R$ and the scalar $\kappa$ to be*

$$\kappa=\frac{1}{\hbar}-1\quad\textit{or}\quad\kappa=-\frac{1}{\hbar},\quad R=\frac{1}{\hbar}-1,\qquad\hbar\neq0. \tag{5.1.54}$$

*Subsequently, the parameters $\theta_0$ and $\theta_1$, and $\theta_2$ are determined by*

$$\theta_0=c-\hbar,\qquad\qquad\theta_1=c+1-N-m\hbar,\qquad\qquad\theta_2=b+\hbar. \tag{5.1.55}$$

*Proof.* According to the result of the theorem (5.1.3) and the instructions of the proof of the Lemma (5.1.5) and (5.1.6), we have the following:

$$\begin{aligned} tH_V =& t(H_2 - H_1) + \hbar \sum_\rho \left( t z_\rho^2 - (b+c+t) z_\rho + b \right) \partial_{z_\rho} - m\hbar t \sum_\rho z_\rho + \tau(t) + \eta, \\ t\widetilde{H}_V =& t(\widetilde{H}_2 - \widetilde{H}_1) + \hbar \sum_\rho \left( t z_\rho^2 + (2\hbar + \theta_0 - \theta_2 - t) z_\rho + \theta_2 - \hbar \right) \partial_{z_\rho} + t(\hbar N + \theta_0 + \theta_1) \sum_\rho z_\rho + \\ & + \widetilde{\tau}(t) + \widetilde{\eta}. \end{aligned} \tag{5.1.56}$$

where $\tau(t)$ and $\widetilde{\tau}(t)$ are constant functions of $t$, and $\eta$ and $\widetilde{\eta}$ are arbitrary constants. Therefore, one gets

$$\begin{aligned} &\left( \Delta^{-R} t\widetilde{H}_V \Delta^R - tH_V \right) \Psi(Z) = \\ &2\hbar(\hbar R + \hbar - 1)\mathcal{M}_2 \Psi(Z) + \hbar^2 \left( R(R+1) - \kappa(\kappa+1) \right) \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho^2}{(z_\rho - z_\sigma)^2} \Psi(Z) - \\ &- 2\hbar(\hbar R + \hbar - 1)\mathcal{M}_1 \Psi(Z) + \hbar^2 \left( R(R+1) - \kappa(\kappa+1) \right) \sum_{\substack{\rho,\sigma \\ \rho<\sigma}} \frac{z_\rho}{(z_\rho - z_\sigma)^2} \Psi(Z) + \\ &+ \hbar \sum_\rho \left( 2\hbar + \theta_0 - \theta_2 + b + c \right) z_\rho \partial_{z_\rho} \Psi(Z) + \hbar \sum_\rho \left( \theta_2 - \hbar - b \right) \partial_{z_\rho} \Psi(Z) + \\ &+ t \left( N\hbar + \theta_0 + \theta_1 + m\hbar + R\hbar(N-1) \right) \sum_\rho z_\rho \Psi(Z) + \mathcal{T}(t)\Psi(Z) + \Theta\Psi(Z) \end{aligned} \tag{5.1.57}$$

For $\mathcal{T}(t)$ a constant function of $t$, and $\Theta$ a constant, both of which can be gauged away. If the parameters of the equation 5.1.57 take the following values, then the equation 5.1.53 is satisfied:

$$\begin{aligned} \kappa =& \frac{1}{\hbar} - 1 \quad \text{or} \quad \kappa = -\frac{1}{\hbar}, \quad R = \frac{1}{\hbar} - 1, \qquad \hbar \neq 0 \\ \theta_0 =& -c - \hbar, \quad \theta_1 = c + 1 - N - m\hbar, \quad \theta_2 = b + \hbar. \end{aligned} \tag{5.1.58}$$

■

### 5.1.4 Calogero-Painlevé VI

Finally, for Calogero-Painlevé VI with the conditions of $\hbar = 1$ and $\kappa = 0$ we have the same Hamiltonian operators from equations (3.3.29) and (4.2.46), with the following determination of the other parameters:

$$\begin{aligned} \theta_0 = a + b + 1 \qquad\qquad \theta_1 = c + 1 \qquad\qquad \theta_t = d + N \\ k = \pm\sqrt{(\theta - 2N)^2 + 4m(N - 1 - m)} \end{aligned} \tag{5.1.59}$$

The second observation in this case results in the following Lemma:

**Lemma 5.1.8.** *Considering* $\Delta = \prod_{1 \le \alpha < \beta \le N} (z_\alpha - z_\beta)$ *be a Vandermonde polynomial in* $\vec{z}$, *then the action of quantum Hamiltonian operator $H_{VI}$ on the generalized wave functions, is equivalent to the action of $\Delta^{-R}\widetilde{H}_{VI}\Delta^R$ on the integral representation of the wave function*

$$\Delta^{-R}\widetilde{H}_{VI}\Delta^R\Psi(Z) = H_{VI}\Psi(Z) \tag{5.1.60}$$

*for the values of $R$ and the scalar $\kappa$ to be the same as*

$$\kappa = \frac{1}{\hbar} - 1 \quad \text{or} \quad \kappa = -\frac{1}{\hbar}, \quad R = \frac{1}{\hbar} - 1, \qquad \hbar \neq 0 \tag{5.1.61}$$

*and the other parameters determined by*

$$\begin{gathered} \theta_0 = a + b + \hbar, \qquad \theta_1 = c + \hbar, \qquad \theta_t = d + N + \hbar - 1, \\ k = \pm\sqrt{(\theta - 2N\hbar)^2 + 4\hbar m(N - 1 - \hbar m) + 4(N-1)(1-\hbar) + N(N-1)(1-\hbar)(3\hbar - \theta)}. \end{gathered} \tag{5.1.62}$$

*Proof.* Adapting the definitions 5.1.34, we have the following definition for the operators $H_{VI}$ and $\widetilde{H}_{VI}$:

$$\begin{aligned} t(t-1)H_{VI} &= t(t-1)(H_3 - (1+t)H_2 + tH_1) - \hbar\sum_\rho\Big((a+b+c+d+N-1)z_\rho^2 + t(a+b) - \\ &\quad - ((1+t)(a+b) + tc + d + N - 1)\, z_\rho\Big)\partial_{z_\rho} - \hbar m(N-1-\hbar m)\sum_\rho z_\rho + \tau(t) + \eta, \\ t(t-1)\widetilde{H}_{VI} &= t(t-1)(\widetilde{H}_3 - (1+t)\widetilde{H}_2 + t\widetilde{H}_1) + \hbar\sum_\rho\Big((3\hbar - \theta)z_\rho^2 + t(\hbar - \theta_0) + \\ &\quad + (-\hbar(1+t) + t(\theta_0 + \theta_1) + \theta_0 + \theta_t)\, z_\rho\Big)\partial_{z_\rho} + \Big(N^2\hbar^2 - \theta N\hbar - \frac{1}{4}(k^2 - \theta^2) + \\ &\quad + \kappa(\kappa+1)(N-1)\hbar^2\Big)\sum_\rho z_\rho + \widetilde{\tau}(t) + \widetilde{\eta} \end{aligned} \tag{5.1.63}$$

where $\tau(t)$ and $\widetilde{\tau}(t)$ are constant functions of $t$, and $\eta$ and $\widetilde{\eta}$ are constants. By substituting the equation 5.1.63 into the equation 5.1.60 we obtain:

$$
\begin{aligned}
&(\Delta^{-R}\widetilde{H}_{VI}\Delta^{R} - H_{VI})\Psi(Z) = \\
&2\hbar(\hbar R + \hbar - 1)\mathcal{M}_3\Psi(Z) + \hbar^2\left(R(R+1) - \kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}} \frac{z_\rho^3}{(z_\rho - z_\sigma)^2}\Psi(Z) - \\
&- 2(1+t)\hbar(\hbar R + \hbar - 1)\mathcal{M}_2\Psi(Z) - (1+t)\hbar^2\left(R(R+1) - \kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}} \frac{z_\rho^2}{(z_\rho - z_\sigma)^2}\Psi(Z) + \\
&+ 2t\hbar(\hbar R + \hbar - 1)\mathcal{M}_1\Psi(Z) + t\hbar^2\left(R(R+1) - \kappa(\kappa+1)\right)\sum_{\substack{\rho,\sigma\\ \rho<\sigma}} \frac{z_\rho}{(z_\rho - z_\sigma)^2}\Psi(Z) + \\
&+ \hbar\sum_{\rho}\Bigg((3\hbar - \theta + a + b + c + d + N - 1)z_\rho^2 + t(\hbar N - \theta_0 + a + b) + \\
&+ \Big(-\hbar(1+t) + t(\theta_0 + \theta_1) + \theta_0 + \theta_t - (1+t)(a+b) - tc - d - N + 1\Big)z_\rho\Bigg)\partial_{z_\rho}\Psi(Z) + \\
&+ \Bigg(R\hbar(3\hbar - \theta)N(N-1) + N^2\hbar^2 - \theta\hbar N - \frac{1}{4}(k^2 - \theta^2) + \kappa(\kappa+1)(N-1)\hbar^2 + \\
&+ \hbar m(N - 1 - \hbar m)\Bigg)\sum_{\rho} z_\rho\Psi(Z) + \mathcal{T}(t)\Psi(Z) + \Theta\Psi(Z)
\end{aligned}
\tag{5.1.64}
$$

where $\mathcal{T}(t) = \tilde{\tau}(t) - \tau(t)$, and $\Theta = \tilde{\eta} - \eta$.

Equation 5.1.64 will be equal to zero if the parameters in the equation take the following values:

$$\kappa = \frac{1}{\hbar} - 1 \quad \text{or} \quad \kappa = -\frac{1}{\hbar}, \quad R = \frac{1}{\hbar} - 1, \qquad \hbar \neq 0. \tag{5.1.65}$$

This determines the value of the rest of the parameters

$$
\begin{aligned}
&\theta_0 = a + b + \hbar, \quad \theta_1 = c + \hbar, \quad \theta_t = d + N + \hbar - 1, \\
&k = \pm\sqrt{(\theta - 2N\hbar)^2 + 4\hbar m(N - 1 - \hbar m) + 4(N-1)(1-\hbar) + N(N-1)(1-\hbar)(3\hbar - \theta)}
\end{aligned}
\tag{5.1.66}
$$

where $\theta = \theta_0 + \theta_1 + \theta_t$. Note that $\mathcal{T}(t)$ and $\Theta$ can be omitted by choice of suitable transformations. ■

**Remark 5.1.9.** *In all the cases the parameters obtained from the second observation reduce to the corresponding ones from the first observation when* $\hbar = 1$.

# Chapter 6

# Conclusion and Related Works

## 6.1 Concluding notes

In recent years, there have been different approaches to "quantum Painlevé equations" using as starting point, the linear differential equation of rank 2 classically associated to the Painlevé equations, or the theory of topological recursion associated to semiclassical spectral curves.

In [37], the authors start from the system of linear equations (Lax system) associated to the Painlevé equations I-VI written in the form

$$\begin{cases} \partial_z \Psi = U(z,t)\Psi \\ \\ \partial_t \Psi = V(z,t)\Psi \end{cases} . \tag{6.1.1}$$

where $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, and the complex-valued matrices $U$, and $V$ are considered in general to have the forms

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad V = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.1.2}$$

Since the matrices $U$ and $V$ are traceless, the zero-curvature equation of the form

$$\partial_z V - \partial_t U + [V, U] = 0 \tag{6.1.3}$$

yields

$$\begin{cases} a_t = A_z + bC - cB = 0 \\ b_t - B_z + 2aB - 2bA = 0 \\ c_t - C_z + 2cA - 2aC = 0. \end{cases} \tag{6.1.4}$$

After applying a suitable change of variables and gauge transformation, they obtain a pair of compatible PDEs for a scalar wave function $\psi$ (obtained from the $(1,1)$ entry of the matrix $\Psi$)

$$\begin{cases} \left(\frac{1}{2}\partial_z^2 - \frac{1}{2}(\partial_z \log b)\partial_z + W(z,t)\right)\psi = 0 \\ \\ \partial_t \psi = \left(\frac{1}{2}\partial_z^2 + \mathcal{U}(z,t)\right)\psi \end{cases} \tag{6.1.5}$$

where $W$ and $\mathcal{U}$ are the potentials that are described explicitly in terms of the entries of the matrices $U, V$.

The first equation in (6.1.5) has apparent singularities but otherwise exhibits the same (generalized) monodromy associated to (6.1.1) (in $\mathbb{P}\mathrm{SL}_2$), while the second equation describes the isomonodromic deformation of the former and is presented in the form of a non-stationary Schrödinger equation with imaginary time.

The Hamiltonian operators corresponding to each Painlevé equation I-VI from the second equation of the system (6.1.5) are, in particular, a natural quantization of those corresponding to the Calogero-like Painlevé equations (those obtained from the first equation of the system (6.1.5)). These operators that are called the quantum Calogero-Painlevé Hamiltonian system, are obtained in single variable representation.

The question then arises as to whether a similar description is possible in the multi-particle case; the naïve approach of considering the matrices as $2 \times 2$ blocks does not lead to equations of the same type as (6.1.5). The quantization of the Hamiltonians from [3] does not seem to be the direct analogue of (6.1.5) because it is an equation where the wave equation plays the role rather of the (scalar) "quantum tau function".

Also, in [11], the authors use the topological recursion on spectral curves of different genera which results in wave functions that satisfy a family of partial differential equations. In fact, these PDEs are the quantization of the original spectral curves. As an application of their theorem, they introduce a system of PDEs corresponding to Painlevé transcendents whose assigned Hamiltonian system has significant similarities to the system of Hamiltonian operators that we introduced in this paper for the quantum Calogero-Painlevé system.

Finally, we comment on the possible relationship with equations of the Knizhnik-Zamolodchikov (KZ) type. For the single–particle case, in [31], H. Nagoya provides a representation-theoretic correspondence between the Schrödinger equation for quantum Painlevé VI (single-particle) and the Knizhnik-Zamolodchikov (KZ) equation, and between the Schrödinger equation for quantum Painlevé II-V(single-particle) and the confluent KZ equations that are defined in [22]. We also mention the work of J. Harnad [18] where the author proposed a quantization of the Schlesinger system; this is a generalization of the sixth Painlevé equation to include more than four Fuchsian singularities in the corresponding Lax pair. It is reasonable that our current work corresponds to a reduction thereof for the case of Painlevé VI.

As an example, we mention the case of quantum Painlevé VI and the KZ equation. To do so, we briefly review the definition of the confluent KZ equation and the usual KZ equation.

**Verma modules.** [22] Set $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{g}[z] = \mathfrak{g} \otimes \mathbb{C}[z]$. Suppose $e, f, h$ to be the standatd basis of $\mathfrak{g}$, For non-negative integer $r$, denote by $\mathfrak{g}_{(r)}$ and $\mathfrak{g}'_{(r)}$ the truncated Lie algebra $\mathfrak{g}_{(r)} = \mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$ and $\mathfrak{g}'_{(r)} = z\mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$. For an $(r+1)$-tuple $\gamma = (\gamma_0, \dots, \gamma_{r-1}, \gamma_r)$, a confluent Verma module $M(\gamma)$ of Poincaré rank $r$ is a cyclic $\mathfrak{g}_{(r)}$-module generated by $1_\gamma$ such that

$$(e \otimes z^p)1_\gamma = 0, \quad (h \otimes z^p)1_\gamma = \gamma_p 1_\gamma \qquad (0 \le p \le r).$$

For the Lie subalgebra $\mathfrak{g}'_{(r)} = z\mathfrak{g}[z]/z^{r+1}\mathfrak{g}[z]$, a confluent Verma module $M'(\gamma)$ of Poincaré rank $r$ with parameters $\gamma = (\gamma_0, \dots, \gamma_{r-1}, \gamma_r)$ is a cyclic $\mathfrak{g}'_{(r)} \oplus \mathbb{C}(h \otimes z^0)$-module generated by $1_\gamma$ such that

$$(e \otimes z^p)1_\gamma = 0, \quad (h \otimes z^p)1_\gamma = \gamma_p 1_\gamma \qquad (0 \le p \le r), \quad (h \otimes z^0)1_\gamma = 0,$$

and $e \otimes z^r$ and $f \otimes z^r$ act as zero operators on $M'(\gamma)$.

Let differential operators $D_k$ $(0 \le k \le r-1)$ be defined as

$$D_k = \sum_{p=1}^{r-k} p\gamma_{k+p} \frac{\partial}{\partial \gamma_p}$$

acting on $M(\gamma)$ as

$$D_k(x \otimes z^p) = p(x \otimes z^{p+k}) \qquad (x \in \mathfrak{g}, 0 \le p \le r), \qquad D_k(1_\gamma) = 0$$

Here $x \otimes z^p$ is regarded as an operator on $M(\gamma)$.

Let $z_1, \dots, z_n$ be distinct points in $\mathbb{C}$ and let $r_1, \dots, r_n, r_\infty$ be non-negative integers. Set $\mathfrak{a} = \left(\oplus_{i=1}^n \mathfrak{g}^{(i)}\right) \oplus \mathfrak{g}^{(\infty)}$, where $\mathfrak{g}^{(i)} = \mathfrak{g}_{(r_i)}$ $(i = 1, \dots, n)$ and $\mathfrak{g}^{(\infty)} = \mathfrak{g}'_{(r_\infty)}$. Consider now a family of $\mathfrak{a}$-modules

$$M(\gamma) = M^{(1)} \otimes \cdots \otimes M^{(n)} \otimes M^{(\infty)},$$

parametrized by $\gamma = (\gamma^{(1)}, \dots, \gamma^{(n)}, \gamma^{(\infty)})$, where

$$\begin{aligned} M^{(i)} =& M(\gamma^{(i)}), \qquad \gamma^{(i)} = \left(\gamma_0^{(i)}, \dots, \gamma_{r_i}^{(i)}\right), \\ M^{(\infty)} =& M'(\gamma^{(\infty)}), \qquad \gamma^{(\infty)} = \left(\gamma_1^{(\infty)}, \dots, \gamma_{r_\infty}^{(\infty)}\right). \end{aligned}$$

Set $1_\gamma = 1_{\gamma^{(1)}} \otimes \cdots \otimes 1_\gamma = 1_{\gamma^{(n)}} \otimes 1_\gamma = 1_{\gamma^{(\infty)}}$.

The confluent KZ equations in [22] are differential systems for unknown functions $\Psi(z, y)$ taking values in $M(\gamma)$ with respect to the following differential operators

$$\begin{aligned} &\frac{\partial}{\partial_{z_i}} \quad (i = 1, \dots, n), \\ &D_k^{(i)} \quad (i = 1, \dots, n), \quad (k = 0, \dots, r_i - 1), \\ &D_k^{(\infty)} \quad (k = 1, \dots, r_\infty - 1). \end{aligned}$$

If $r_i = 0$ and $r_\infty = 0$, then the confluent KZ equations are equal to the usual KZ equations. It was shown in [22] that the confluent KZ equations have integral formulas of confluent hypergeometric type of solutions.

**Case of PVI.** Let $n = 3, r_i = 0, z_1 = 0, z_2 = t, z_3 = 1$ and $\gamma_0^{(i)} \notin \mathbb{Z}$ $(1 \leq i \leq 3)$. Then $M = M(\gamma_0^{(1)}) \otimes M(\gamma_0^{(2)}) \otimes M(\gamma_0^{(3)})$ and the KZ equation for an unknown function $\Psi(t)$ taking values in $M$ is defined by

$$\kappa \frac{\partial \Psi(t)}{\partial t} = \left( \frac{\Omega^{(1,2)}}{t} + \frac{\Omega^{(2,3)}}{t-1} \right) \Psi(t) \tag{6.1.6}$$

Here $\kappa$ is a complex parameter and $\Omega^{(i,j)}$ are the Casimir operators:

$$\Omega^{(1,2)} = e^{(1)} f^{(2)} + f^{(1)} e^{(2)} + \frac{1}{2} h^{(1)} h^{(2)}, \qquad \Omega^{(2,3)} = e^{(2)} f^{(3)} + f^{(2)} e^{(3)} + \frac{1}{2} h^{(2)} h^{(3)},$$

where $x^{(i)} : M \to M$ is the linear operator acting as $x$ on $i$th tensor factor and as identities on the others. (for the rest of notes $x \otimes z^0$ is abbreviated to $x$ for $x = e, f, h$.)

Let $W_m$ $(m \in \mathbb{Z}_{\geq 0})$ be the space of singular vectors of the weight $\sum_i^3 \gamma_0^{(i)} - 2m$ in $M$, namely

$$W_m = \left\{ v \in M \middle| \sum_{i=1}^{3} e^{(i)}(v) = 0, \sum_{i=1}^{3} h^{(i)}(v) = \left( \sum_{i=1}^{3} \gamma_0^{(i)} - 2m \right) v \right\}$$

In order to write down a basis of $W_m$, the authors take the differential realizations $\mathbb{C}[x_i] (1 \leq i \leq 3)$ of $\mathfrak{g}$, that is the act of the basis $e, f, h$ on $\mathbb{C}[x_i]$ as follows:

$$e = \frac{\partial}{\partial x_i}, \qquad h = -2x_i \frac{\partial}{\partial x_i} + \gamma_0^{(i)}, \qquad f = -x_i^2 \frac{\partial}{\partial x_i} + \gamma_0^{(i)} x_i.$$

Note that if $\gamma_0^{(i)} \notin \mathbb{Z}$, then $\mathbb{C}[x_i]$ are isomorphic to Verma modules $M(\gamma_0^{(i)})$. So the assumption is $M(\gamma_0^{(i)}) = \mathbb{C}[x_i]$.

The space of singular vectors $W_m$ can be written by

$$W_m = \bigoplus_{i=0}^{m} \mathbb{C}(x_1 - x_2)^i (x_1 - x_3)^{m-i}$$

The Hamiltonian $\frac{\Omega^{(1,2)}}{t} + \frac{\Omega^{(2,3)}}{t-1}$ is denoted by $H_{\mathrm{KZ}}$, now let $\widetilde{H}_{\mathrm{KZ}}(m)$ be defined as

$$\widetilde{H}_{\mathrm{KZ}}(m) = \hbar^2 \left( H_{\mathrm{KZ}} \frac{\lambda_1 \lambda_2}{t} - \frac{\lambda_2(\lambda_3 - m)}{t-1} \right).$$

The linear isomorphism $T_m : W_m \to \bigoplus_{i=0}^{m} \mathbb{C}$ $(m \in \mathbb{Z}_{\geq 0})$ is defined as

$$T_m \left( (x_1 - x_2)^i (x_1 - x_3)^{m-i} \right) = x^i \qquad (0 \leq i \leq m).$$

The result of these scheme is the following theorem:

**Theorem 6.1.1.** *For* $\gamma_0^{(i)} \notin \mathbb{Z}(1 \leq i \leq 3)$ *and* $m \in \mathbb{Z}_{\geq 0}$*, the action of* $H_{KZ}$ *on the space of singular vectors of weight* $\sum_{i-1}^{3} \gamma_0^{(i)} - 2m$ *is equivalent to the action of the quantized Hamiltonian* $H_{VI}$ *(as in* (4.1.2)*) on the subspace* $\bigoplus_{i=0}^{m} \mathbb{C}x^i$ *with* $a = m\hbar$ *or* $b+c+d = (m-1)\hbar$*. In particular*

$$T_m \circ \widetilde{H}_{KZ}(m) = \left( H_{VI}\Big(x, \hbar\frac{\partial}{\partial x}, a, b, c, d, t\Big) + \frac{a(b+c+d+\hbar)}{t-1} \right) \circ T_m$$

*defines linear maps from* $W_m$ *to* $\bigoplus_{i=0}^{m} \mathbb{C}x^i$ *with some relations between the parameters.*

As is written above (and similarly for the rest of the operators in (4.1.2)), the correspondences are proved directly by showing relations between the integral representations for the solutions to the quantum Painlevé equations and solutions to the (confluent) KZ equations. In this case, our work in this project should play a similar role for $\mathfrak{sl}_n$ special solutions of the KZ equations. We plan to address this possible relationship in further researches and our future works.

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