# STATIC AND DYNAMIC THEORETICAL STUDIES ON IMPROVING MATCHING DESIGN

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A Thesis
In
The Department
Of
Economics

Presented In Partial Fulfillment Of The Requirements
For The Degree Of
Doctor Of Philosophy (Economics)
Concordia University
Montréal, Québec, Canada

July 2021

CONCORDIA UNIVERSITY
SCHOOL OF GRADUATE STUDIES

This is to certify that the thesis prepared

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Entitled: **Static and Dynamic Theoretical Studies on Improving Matching Design**

and submitted in partial fulfillment of the requirements for the degree of

**Doctor Of Philosophy (Economics)**

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# Abstract

**STATIC AND DYNAMIC THEORETICAL STUDIES ON IMPROVING MATCHING DESIGN**

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**Concordia University, 2021**

This thesis consists of three independent papers on market design and matching theory. Each paper addresses a different matching model and environment, and together they represent a significant range of real-life matching problems which have not received enough attention.

In the first paper, we consider a new matching model to assign agents and objects on two sides of the market to each other. The new feature is that agents have consecutive acceptance intervals which are based on an exogenously given commonly known ranking of the objects. Each agent finds acceptable a consecutive set of objects with respect to this objective common ranking of the objects. Each agent has an individual preference ranking of the objects in her acceptance interval, which is determined independently of the common ranking of the objects. The main objective is to find new matching rules (algorithms) which are simpler and more efficient than the complicated conventional general algorithms for achieving a maximum matching which is Pareto-optimal, exploiting the special structure of consecutive acceptance intervals which are a common feature of many real-life matching problems.

Our main algorithm, the Block Serial Dictatorship Rule, starts with finding an ordering of agents based only on the acceptance interval structure and thus it is preference profile independent. This ordering is then used as a basis for a Serial Dictatorship which always finds a maximum Pareto-optimal matching, regardless of the agents' preferences, for the solvable interval profiles that we characterize in the paper. These rules are also group strategy-proof.

In the second paper, I consider a matching model with minimum quotas for one side of the market. The main objective is to find algorithms which respect minimum quotas and find matchings which are both nonwasteful and fair if there exists such a matching. Otherwise, the algorithms find either fair or nonwasteful matchings. My algorithms, $CNWF$ and $FCNW$ (constrained nonwasteful fair and fair constrained nonwasteful), start with finding the range of possible matchings when there are minimum quotas. Then, using an innovative graph, they select the matchings which are both fair and nonwasteful, and if there do not exist such matchings, $CNWF$ selects a constrained nonwasteful matching with a maximum degree of fairness, and $FCNW$ selects a fair matching with a maximum degree of nonwastefulness.

Furthermore, I show that my algorithms are applicable to the case where there are different types of agents, which is a key factor for matching markets that are concerned with diversity. Compared to the existing algorithms my algorithms are unified and more

intuitive.

In the third paper, I consider a novel matching model in a dynamic environment. I define a dynamic environment in which the market is open for more than one period. At the beginning of each period new agents enter the market and the matched agents leave at the end. My model is motivated by couple match-making but the results apply to other similar matching markets as well. The main objective is to find an appropriate gender-neutral algorithm with nice properties. I introduce a new algorithm which is based on the $DA$ (Deferred Acceptance) algorithm and whose structure provides an opportunity to find two-sided optimal matchings, considering the requirements and characteristics of this dynamic marriage problem.

The novel structure of my algorithm, $DM$ (Dynamic Marriage), allows both sides to make offers simultaneously and selects a matching which is optimal for both sides in a realistic dynamic setup whenever such a matching exists, and otherwise the algorithm finds a matching without favouring either side. This property makes the matching fair in the sense that it gives both sides a fair chance. I also study the dynamic strategy-proofness of the algorithm, as well as its stability and efficiency properties.

Compared to previous algorithms that apply to the marriage problem in a static or dynamic environment, my algorithm is more realistic since it allows for realistic dynamic preferences and for real-life marriage considerations. Furthermore, it is more integrated regarding the optimality of the two sides than other algorithms and avoids some of the common issues of dynamic algorithms.

# Acknowledgments

First and foremost, I want to specially thank my supervisor, Dr. Szilvia Pápai for her help, support and valuable guides. As a supervisor, she is so knowledgeable, professional, responsible, careful and kind who shaped my academic character and taught me how to be a mechanism design researcher. I could not ask for a better supervisor.

Besides my supervisor, I would like to thank the rest of my thesis committee for kindly spend their time on reviewing this thesis and attending the defence meeting: Dr. David Cantala, Dr. Chun Wang, Dr. Huan Xie, Dr. Dipjyoti Majumdar and Dr. Danielle Gauvreau.

My sincere gratitude also goes to my Graduate Program Directors, Dr. Effrosyni Diamantoudi, Dr. Szilvia Pápai, Dr. Damba Lkhagvasuren and Dr. Christian Sigouin for their support and trust.

Furthermore, I would also like to thank the department of Economics and its chair, Dr. Jorgen Hansen and the administrative staff of the department: Mrs. Elise Melancon, Ms. Sandra Topisirovic, Ms. Émilie Martel and Ms. Kelly Routly for their insightful help and support.

I would like to use the opportunity to deeply thank Concordia University, FRQSC and CIREQ. Concordia offered me the opportunity to join their great academic team as a graduate student and provided me with financial support in terms of scholarships and several awards. FRQSC financially sponsored my second paper with a significant scholarship. CIREQ not only provided many occasions to enrich my academic knowledge by holding valuable conferences but also offered financial support to attend international conferences. All these supports helped me to concentrate on my studies and experience a great academic environment.

In addition, I want to thank my professors whom I attended their courses or worked as their $TA$ and had a very positively significant role during my studies: Dr. Szilvia Pápai, Dr. Tatyana Koreshkova, Dr. Dipjyoti Majumdar, Dr. Huan Xie, Dr. Moshe Lander, Dr. Prosper Dovonon, Dr. Bryan Campbell and Dr. Greg LeBlanc. I have learned a lot from them and they have supported me on several occasions.

I also want to cherish the memory of Dr. Babak Farzad, my supervisor during my Masters' program. He is not with us anymore but I will be thankful to him forever. He guided me toward game theory and consequently mechanism design.

Furthermore, my sincere thanks go to my lovely soul mate Mahmood Mastani Joybari for his heartfelt love, unconditional support, encouraging spirit and tender behaviour. I am so lucky for having him always by my side.

Last but not least, I would like to thank my family and friends specially my mom and dad who are the best for their unlimited love and support throughout my life. Although they currently are living far from me, they have supported me spiritually during my studies and writing this thesis. I never could be here without them.

# Contribution of Authors

Chapter 1 of the thesis is a joint work with my supervisor, Professor Szilvia Pápai. Furthermore, I have benefited from her valuable comments and guidance throughout all parts of my thesis.

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# Chapter 1

# Introduction

This thesis consists of three independent studies on Matching Theory. Each study has its own introduction, therefore here I provide a general overview of the fundamental and common concepts as well as the structure of the chapters.

## 1.1 Market Design

Traditionally markets have been simply decentralized markets, a place where supply and demand meet. Nowadays a new branch of economic science, Market Design, highlights that markets not only are places to cover demand by supply but also that the structure of the market and detailed rules have a significant role in the operation of many markets and in reaching the desired results.

Market design is a practical methodology for the creation of markets with certain properties. This methodology helps not only to analyze existing markets but also to create desirable markets from scratch or fix the broken ones.

The ultimate goal of market design is to study the markets and recognize their differences, working processes and requirements. This deep understanding will help market designers to define the rules and procedures that make different kinds of markets operate well. Relying on these studies, they should be able to provide thickness and safety to bring together enough number of participants who feel safe to share required information. In addition, there should be adequate time and means for transactions. Eventually all of these lead to finding practical solutions for real-life matching problems.

Overall, market designers have the responsibility to make sure that all details and parts of a market function together well. This is why market design studies often include a detailed description of the market's unique and distinguishing features which are collectively reflected as the model of each separate study.

Within this context, I focus on three separate market design projects, three studies which form the main body of this thesis. My work deals with matching problems and combines tools and concepts from different fields, including economic engineering, matching theory, mechanism design, graph theory, axiomatic resource allocation and game theory. I define special matching models and consider their applicable aspects in the design of matching mechanisms when there are consecutive acceptance intervals, in the presence of minimum quotas and also in dynamic environments. Furthermore, I aim for desirable properties and study the characteristics of the proposed mechanisms, such as efficiency, Pareto-optimality, maximality, stability, fairness, strategy-proofness and nonwastefulness.

## 1.2 Matching Theory

Many real-world problems are concerned with the allocation of resources. How efficiently and fairly people are matched to each other or to goods is a crucial concern in all societies. A new approach to address the issues of supply and demand in markets which work under their own special rules is called Matching Theory. Matching theory is the quantitative model of choice for studying markets where one needs to be chosen or selected, based on certain attributes, and simply paying for a good or service (or a person) is not possible due to legal, ethical or other considerations. There are lots of such matching problem examples in the real world, such as matching students to schools or dormitories, organ transplantation, so patients who need a transplant are matched to potential organ donors (kidney exchange is a well-studied case in this category), settling refugees in host countries, and matching workers and firms to one another.

Up to now market designers and matching theorists have made a significant improvement in societies by modeling and analyzing different real-life matching problems. Based on these models, they have introduced matching mechanisms which have been used in many cases. For instance, New York City, Boston and other U.S. cities have designed their school choice programs, medical communities in some countries such as Japan have reorganized their hiring procedure, and systematic kidney exchange programs have been designed to maximize the number of patients who receive required organs as well as minimize the number of wasted organs.

In the matching theory literature generally there are two main types of matching models: two-sided and one-sided. Two-sided models consist of agents on both sides of the market who have their own preferences over the other side. A very famous example is the marriage model with agents on both sides, namely men and women, who have preferences over each other. The fact is that in both models there are two sides involved in the market, but in one-sided models agents on one side are matched to indivisible objects on the other side, or more accurately agents are assigned objects. One important aspect of these matching models is that agents have their own preferences over the agents or objects on the other side, while objects don't have preferences over the agents, but often the objects have some kind of priorities over the agents on the opposite side. The agents' preferences have been formed either through a data gathering process or are based on personal taste and experiences. By contrast, the objects' priorities could be imposed by regulators or by law. For instance, when assigning students to dormitory rooms, senior students may have higher priorities than others based on the school's regulations. The reason for calling such a model one-sided is that only the agent side is considered to be active in the sense that they may have incentives to misrepresent their preferences that are private information, while the objects may not have such incentives, or the priorities can simply be verified if they are imposed by law.

In matching models all players are assumed to be rational and agents, as opposed to objects, are considered to be strategic who have the power to decide for themselves based on their own personal preferences, which have not been imposed by laws, for example, or given exogenously in general. Generally speaking, agents are often humans, but they could be countries, universities, or any kind of organization. A good example is matching medical students to medical residency positions at hospitals. Hospitals and medical residency programs are the agents in this application, and they have their own decision-making procedure which is based on their policies. These policies and the corresponding procedures form the hospitals' preferences.

I study both types of the matching models in this thesis: one-sided matching models in chapter 3 and two-sided matching models in chapters 4 and 5. For either type of matching model several well-known matching mechanisms have been defined in the literature, and I rely on three of these important mechanisms in my analysis: Serial Dictatorships (SD), Top trading cycles (TTC) and Deferred Acceptance (DA). I have used SD and TTC in chapter 3 and DA in chapters 4 and 5.

### 1.2.1 Serial Dictatorships (SD)

First introduced by Satterthwaite and Sonnenschein (1981, [66]), a Serial Dictatorship (SD) mechanism (also known as a priority mechanism) specifies an arbitrary ordering of the agents on one side of the market and all objects on the other side are marked "available". Then it lets the first agent in the ordering, the first "dictator," receive her favorite good based on her stated preferences and removes her choice from the available set of goods. Continuing this way the next agent, the second dictator, receives her favorite good among the remaining objects, and so forth, until all agents or all objects are matched. It can be said that the serial dictatorship mechanism is very easy to implement:

- Fix a priority function $\pi$ which specifies a permutation of agents randomly, or by using some existing priority ordering such as seniority.
- Let applicants choose their objects according to the permutation: for all agents, $i = 1$ to $n$, set the match of agent $i$, $\mu(i)$, equal to her favorite remaining object.

Serial dictatorships have many applications and have been used in many real-life matching problems, such as the allocation of offices to professors, the school choice system in New York, housing allocation at Columbia and Harvard, etc. In addition, serial dictatorship has several good properties such as Pareto-efficiency and strategy-proofness, which will be discussed later in Chapter 2.

### 1.2.2 Top Trading Cycles (TTC)

The Top Trading Cycles (TTC) rule, which is an algorithm for trading indivisible objects without using money, is attributed to David Gale by Shapley and Scarf (1974, [67]). The basic TTC algorithm, also known as Gale's Top Trading Cycle rule, is illustrated by the following housing market situation. Assume that there are $n$ students living in student dormitories. Each student currently lives in a single house and has her own preferences over all houses. Therefore, there are some students who prefer the houses assigned to other students to their current endowment. This may lead to mutually beneficial exchanges. For example, if student 1 prefers the house allocated to student 2 and vice versa, both of them will benefit by exchanging their houses. The goal is to find a core-stable allocation, a reallocation of houses to students, such that all mutually beneficial exchanges have been realized (i.e., no group of students can together improve their situation by exchanging their houses).

The algorithm works as follows:

- Ask each agent (student) $i$ indicate his "top" (most preferred) house.
- Draw an arrow from each agent $i$ to the agent who holds the top house of $i$.
- Note that there must be at least one cycle in the graph (this might be a cycle of length 1, if some agent $i$ currently holds his own top house).
- Implement the trade indicated by such cycle (i.e., reallocate each house to the agent pointing to it), and remove all the agents involved in trade from the graph.
- If there are remaining agents, go back to round 1 and repeat the same procedure in the reduced market as in round 1, except that now each students points to the student who is endowed with her top house among the houses that are still in the market.
- The algorithm must terminate, since in each iteration we remove at least one agent. It stops when all students who were given an initial endowment have been removed from the market with an assigned house.

### 1.2.3 Deferred Acceptance (DA)

First introduced by Gale and Shapley (1962, [25]), the Deferred Acceptance (DA) mechanism produces a matching for each preference profile. It is an iterated procedure which is based on agents' preferences on both sides of the market. It can be applied both to one-to-one matching models, such as marriage markets, and to many-to-one matching models, such as college admission problems. Focusing on marriage markets there are two alternative DA mechanisms: either the men propose or the women propose. The two mechanisms are symmetric in the two sides of a one-to-one matching market (it is presented here using the language of marriage, but can be applied to any matching problems with the same properties).

The DA mechanism involves a number of rounds (or iterations):

- In the first round, each man proposes to his most preferred acceptable woman (if a man finds all women unacceptable he remains single). Each woman who received at least one offer replies "maybe" to her suitor she most prefers (temporarily holds the offer from the most preferred man among those who proposed to her and are acceptable) and "no" to all other suitors, if there are any. She is then provisionally "engaged" to the suitor she most prefers so far, and that suitor is likewise provisionally engaged to her.
- In each subsequent round, round $k$ ($k \geq 2$), each unengaged man whose offer has been rejected in the previous round proposes to the most preferred acceptable woman to

whom he has not yet proposed, regardless of whether the woman is already engaged (if there is no such woman he remains single). Each woman who received at least one offer in this round replies "maybe" if she is currently not engaged or if she prefers this man over her current provisional partner (in this case, she rejects her current provisional partner who becomes unengaged). Therefore, she temporarily holds the offer from the most preferred man among those men who proposed to her in this (or any previous) step and are acceptable to her. She also rejects all other offers, if there are any.

- This process is repeated until everyone is engaged and there is no man whose offer is rejected in a given round.

The provisional nature of engagements preserves the right of an already-engaged woman to "trade up".

In the final matching of a DA mechanism, each woman is matched to the man whose offer she was holding temporarily when the algorithm stopped (if any). Note that the final acceptance was deferred until the end. Thus, each man is matched to the woman he was temporarily matched to when the algorithm stopped (if any).

Gale and Shapley (1962, [25]) showed that the DA mechanism is fair [1]. In addition, Dubins and Freedman (1981, [18]) and Roth (1982, [58]) proved that this mechanism is strategy-proof for the proposing side. It has been proved that the DA is optimal for the proposing side. Therefore, there are men-optimal and women-optimal matchings, but these are not Pareto-optimal when considering one side only. However, based on Gale and Shapley (1962, [25]) the DA allocation Pareto-dominates all other fair allocations at each preference profile. Moreover, when considering agents on both sides of the market the DA is Pareto-optimal.

In many-to-one matching problems, such as college admissions, the outcome of the student-proposing DA algorithm is also called the student-optimal stable matching (SOSM). Kesten (2010, [40]) provided theoretical evidence showing that the outcome of SOSM may cause large welfare losses when considering the students' welfare only. He introduced an efficiency-adjusted DA mechanism (EADAM) and proved that this algorithm selects the fair and Pareto-optimal matching when it exists and it is nonwasteful, but EADA is not strategy-proof. I discuss the properties of matching rules in Chapter 2.

Here is a simple example of how DA mechanism works:

Let us assume that there are two men $(m_1, m_2)$ and two women $(w_1, w_2)$ in the market with the following preferences:

$w_1 \succ_{m_1} w_2$ and $w_1 \succ_{m_2} w_2$, while $m_1 \succ_{w_1} m_2$ and $m_1 \succ_{w_2} m_2$.

- Round 1: $m_1$ and $m_2$ both propose to $w_1$, who is their more preferred woman. $w_1$ accepts the better offer she received, $m_1$, and rejects the offer from $m_2$. $w_2$ has received no offer, so she does nothing at this round.

[1] The definition of fairness in the matching theory literature is different from the common one. I define and review all relevant properties in Chapter 2.

- Round 2: $m_2$, who has been rejected by $w_1$, now proposes to his next preferred woman, $w_2$, while $w_1$ is engaged to $m_1$. $w_2$ has only received this one offer, so she she accepts the proposal.
- There are no rejections and the algorithm stops. The final matching is $\mu(m_1) = w_1$ and $\mu(m_2) = w_2$.

## 1.3 Historical Background of Matching Theory

Market design is partially based on matching theory which has been originally derived from R.J. Herrnstein's (1961, [30]) classic study known as pigeons' key peck. He formulated his famous matching law based on the results of his experiment with pigeons. Pigeons were presented with two buttons, each of which led to varying rates of food reward. The pigeons tended to peck the button that yielded the greater food reward more often than the other button, and the ratio of their rates to the two buttons matched the ratio of their rates of reward on the two buttons. Although Gale and Shapley (1962, [25]) published their study on college admission and marriage models early on, this field of science has started to develop mainly since the 1990's. Market design pioneer scientists started this development by using matching theory combined with empirical experiments and analysis, as well as other mathematical tools such as Graph Theory.

Matching theory has been a very active research field over the past several decades, and many formal models have been developed to describe and analyze matching markets. New mechanisms have been proposed and practical aspects of matching design have been considered in order to help real-world matching markets function more effectively. One of the most significant early works is Roth and Sotomayor (1992, [64]). They studied two-sided matching markets through a game-theoretic perspective. Later Roth and Peranson (1999, [63]) worked together to redesign the matching market for American physicians. Their algorithm is a modified DA algorithm that can be viewed as a process in which applicants offer to come to residency programs. They worked on many-to-one matching models (in which firms employ multiple workers, but workers seek just one job) and their mechanism is currently used for matching medical interns to residency programs. Balinski and Sönmez (1999, [9]) who studied the student placement problem were also among the first scholars who have worked on matching markets.

There are some examples of early matching models with money involved, such as Shapley and Shubik (1971 [68]), and Kelso and Crawford (1982, [37]), as well as Roth and Sotomayor (1992, [64]) and Hatfield and Milgrom (2005, [29]). They worked on models in which firms employ multiple workers, and wages are explicitly allowed to vary.

Shapley and Scarf (1974, [67]) studied the one-sided matching model of "housing markets". Their work resulted in of the classic mechanisms in matching theory, the Top Trading Cycle mechanism. Continuing with TTC mechanisms, Abdulkadiroğlu and Sönmez (1999, [2]) introduced a hybrid house allocation model with existing tenants. Pápai (2000, [51]) worked on assignments by hierarchical exchange and introduced assignment rules which generalize Gale's top trading cycle procedure.

Abdulkadiroğlu and Sönmez (2003, [3]) also proposed centralized mechanisms to replace the poorly functioning existing mechanisms for student placement. Their work was the first paper which applied matching theory to the practical design of public school choice systems. Their work resulted in adopting a new school choice mechanism by several cities in U.S., and has continued to be an active area of research for marker designers whose goal is to design improved mechanisms and to study the performance of mechanisms in use. One year later, Kesten (2004, [38]) studied the properties of a well-working student placement mechanism. He argued that equity (fairness), efficiency and non-manipulability are the most important properties of a good matching mechanism. It is well-known that there is an impossibility if one requires the co-existence of all three of these properties. In fact, fairness (formally identical to core-stability) already cannot be reconciled with efficiency when considering one side of the market, which is already discussed in the early work of Gale and Shapley (1962, [25]). There is now a large literature analyzing the resulting tradeoffs among incompatible properties for matching mechanisms.

Roth, Sönmez and Ünver (2004, [65]) introduced a kidney exchange model. According to their model, there are a number of pairs of patients and their donors who may be incompatible with them. For each patient a subset of donors can feasibly donate a kidney and the patient has strict preferences over these donors and his own donor (who may or may not be compatible with him). In addition to ranking all compatible donors, each patient also ranks a "wait list option" which represents trading his donor's kidney with a priority in the wait list.

Roth, A. E. (2015, [62]) published another paper on match-making and market design where he targeted a general audience by focusing on the basic questions of "Who Gets What – and Why". Matching theory is a subject that attracted much attention in recent years. As a mile stone in the development process of this subject, 2012 has a special place since the Nobel Memorial Prize in Economic Sciences was awarded to Alvin Roth at Stanford University and Lloyd Shapley at UCLA who are pioneers in matching theory and market design. In addition, Parag Pathak won the John Bates Clark medal (a.k.a. "baby Nobel") in 2018 for his research on school assignment mechanisms.

Although market design and matching theory are very young fields of science, they have been applied to many real-life allocation problems such as job market assignments and school choice problems among many other potential applications and have provided successful solutions for them. Without any doubt, matching theory has played a positive role in making the world a fairer place by by applying formal mathematical approaches to these resource allocation problems. It is not just house allocation or marriage markets that this subject has impacted, as the symbolic names of matching models might suggest. Rather, matching theory has made a difference in school choice systems, university admissions, medical residency allocations and other entry-level labor markets, and saved lives by affecting organ exchange programs, refugee settlement and adoption, among many others.

## 1.4 Goals and Motivations

My studies, which address some important matching problems that have been rarely, if ever, considered in the matching theory literature, are motivated by many real-life applications[2]. One of the main objectives of writing this thesis is to study different matching markets which cover a variety of day-to-day life matching problems which have been addressed very little in the literature up to now. My objective is also to increase the range of my knowledge about various matching markets, the models which represent them and their applications.

As a result of these studies, my goal is to introduce new matching mechanisms which represent special matching markets which perform better in some ways and can be a substitute or an alternative to existing or already studied mechanisms. I aim to introduce novel mechanisms that avoid unrealistic assumptions and overcome some weaknesses of the already existing mechanisms.

My ultimate objective is to provide a strong integrated theoretical foundation which contributes to a more flourishing society by introducing new tools which ensure that all crucial resources, including human resources, are used fairly and efficiently.

## 1.5 Thesis Outline

### 1.5.1 Chapter 2: Properties of Matching Rules

In Chapter 2 I briefly review the standard properties of matching rules which will be used throughout this thesis. The properties will also be defined more precisely in each chapter according to the topic of each study and its requirements.

### 1.5.2 Chapter 3: Maximum Matching with Consecutive Acceptance Intervals

In Chapter 3 we consider a novel model to match agents and objects on two sides of the market to each other, namely, a matching model with consecutive acceptance intervals according to a given commonly known ranking of the objects. Each agent is associated with a consecutive set of objects with respect to this objective common ranking of the objects, which represents the set of acceptable objects for the agent, that is, the agent's consecutive acceptance interval. Each agent has an individual preference ranking of the objects in her acceptance interval, which is determined independently of the common ranking of the objects. The main objective is to find new matching rules (algorithms) which are simpler and more efficient than the complicated conventional general algorithms for achieving a maximum matching which is Pareto-optimal, exploiting the special structure of consecutive acceptance intervals which are a common feature of many real-life matching problems.

We present first the Common-Ranking Successive Rules which are greedy algorithms and find a maximum matching in this model. This set of rules is extended to the class of Successive Rules, which find all maximum matchings for the general case of arbitrary acceptance intervals. To ensure that matchings are not just of maximum cardinality but

[2] I will discuss these applications in detail in Chapters 3 to 5.

are also efficient, we propose two further classes of matching mechanisms which are maximal, Pareto-optimal and group strategy-proof. One is based on the TTC rules and the other one is a sequential dictatorship.

Since these two maximal and Pareto-optimal mechanisms are based on fairly complex algorithms, we also focus on serial dictatorships which are simple and transparent. Since not all consecutive interval profiles allow for the existence of a maximal serial dictatorship, we establish first this impossibility result and then characterize the set of interval profiles for which a maximal serial dictatorship exists. We also identify all permutations of the agents that work for serial dictatorships. Each class of matching rules that we propose are group strategy-proof, since the acceptance intervals are common knowledge and cannot be manipulated.

### 1.5.3 Chapter 4: Matching with Minimum Quotas

The main focus of Chapter 4 is matching agents on two sides of the market to each other when there are minimum quota restrictions for one side. Using the school choice model, this means that some schools have a minimum quota requirement, which puts a distributional constraint on the matching problem. This is a common feature of many real-life matching markets. Standard matching mechanisms, such as the DA, cannot be used, as the minimum quotas cannot be violated.

The principal objective of this study is to find algorithms which respect minimum quotas and find matchings which are both nonwasteful and fair if there exists such a matching. Otherwise, the algorithms find fair or nonwasteful matchings. I introduce an innovative graph for my algorithms which greatly assists in finding a solution to these matching problem and clearly represents information to effectively help the matching procedure. Compared to the existing algorithms for providing a matching which is fair, nonwasteful, or if possible both, my algorithms are integrated and more intuitive.

These algorithms, $CNWF$ and $FCNW$ (constrained nonwasteful fair and fair constrained nonwasteful), start with finding the range of possible matchings in the presence of minimum quotas. Then, using the representative graph, they select matchings which are both fair and nonwasteful, and if there do not exist such matchings then $CNWF$ selects a constrained nonwasteful matching with the maximum degree of fairness, and $FCNW$ selects a fair matching with the maximum degree of nonwastefulness.

I also show that these algorithms work properly when there are different types of students, which is hugely important for any matching market that is concerned with diversity.

### 1.5.4 Chapter 5: Dynamic Marriage Markets

Chapter 5 of the thesis focuses on dynamic matching. I consider a model that matches agents on two sides of the market to each other. Agents on both sides have their preferences and the market environment is dynamic. This means that while the market operates for more than one period, at the beginning of each period new agents enter the market and the matched ones leave it at the end of the period. In my model each agent has an individual preference ranking over the other side's agents who are in her acceptance set.

The main objective is to find an algorithm which, considering the dynamic features of the model, finds matchings which are optimal for both sides if there exists such a matching, and if not then it finds matchings which are fair for both sides in the sense that it does not favour any side. I introduce a new algorithm, Dynamic Marriage ($DM$), which is based on the $DA$ algorithm. The structure of the $DM$ algorithm provides the opportunity to find two-sided optimal matchings, considering the requirements and characteristics of a dynamic environment. I present my model based on the marriage problem (couple match-making), but it can also be used in other similar matching markets. Compared to existing algorithms for the marriage problem in static or dynamic environments, my algorithm is more realistic since it includes real-life marriage considerations and it is based on realistic assumptions. Furthermore, it is more integrated regarding the optimality of the two sides and avoids some of the issues of already existing algorithms, such as using a pre-defined list of agents for the proposing procedure.

The $DM$ algorithm is designed for a dynamic marriage market environment. The structure of the algorithm allows both sides to make offers simultaneously and selects a matching which is optimal for both sides, considering the dynamic structure, if such a matching exists. Otherwise the algorithm finds a matching which lies somewhere in-between the optimal solutions for the two sides. This property makes the matching fair in terms of giving both sides a fair chance, and fairness is a very important concept to be considered in marriage markets, whether static or dynamic. This is why my algorithm is set up to allow both sides to propose, alongside the dynamic nature of the model. The novelty of my study is not only that the DM algorithm has both sides making offers simultaneously in a dynamic setting, but additionally it also allows agents on either side to stay loyal to their chosen mates and wait multiple periods until their proposal might be accepted.

Within this framework I study the dynamic strategy-proofness of the algorithm and analyze its Pareto-optimality and stability. Furthermore, I discuss methods to increase the matching size in each period.

### 1.5.5 Chapter 6: Conclusion

In the last chapter I provide a brief conclusion of the three main studies of the thesis, presented in Chapters 3, 4 and 5, and discuss some extensions and further questions related to my findings.

# Chapter 2

# Properties of Matching Rules

In this chapter I review the general definitions of properties of matching rules that will be used in this thesis. All of these properties are standard properties that have been studied extensively in various models. In later chapters the properties used for each specific topic have been adopted based on the models' requirements.

A matching market is generally given by $(I, J, \succ_I, \succ_J)$, where $I$ is the set of agents $i$ on one side of the market, $J$ is the set of agents (or objects) $j$ on the other side, $\succ_I$ and $\succ_J$ are agents' preference/priority orderings on two sides of the market, respectively.

## 2.1 Matching

*A matching is a mapping $\mu : I \to J$ that satisfies:*

- $\mu(i) \in J \cup \{i\}$ *for all* $i \in I$.
- $\mu(j) \in I \cup \{j\}$ *for all* $j \in J$,
- *for any* $i \in I$ *and* $j \in J$, *we have* $\mu(i) = j$ *if and only if* $i = \mu(j)$ *or* $i \in \mu(j)$.

*Here* $\mu(i) = i$ *indicates that the agent remains unmatched.*

*A* **matching rule** *is a function* $f$ *that assigns a matching* $\mu$ *to each preference profile,* $\succ$. *Formally, a matching rule (or mechanism) is* $f : \mathcal{R} \to \mathcal{M}$, *mapping from the set of preference profiles* $\mathcal{R}$ *to the set of matchings* $\mathcal{M}$.

## 2.2 Individual Rationality

Based on a common definition in economics, a rational individual is an individual who has or can access the information and knowledge about her options, their alternatives, the respective consequences of her actions and the involved risks. Knowing all these, she chooses the best possible option which will provide her either the optimal result or the one with the highest expected value in case of any uncertainty.

In matching theory, a matching $\mu$ is **individually rational** if all agents prefer their current match under $\mu$ to being unmatched, i.e., for each agent $i$, $\mu(i) \succ_i i$.

We also say that a **matching** $\mu$ is **blocked** by an agent $i$ if $i \succ_i \mu(i)$. Then *a* **matching** *$\mu$ is* **individually rational** *if it is not blocked by any agent.*

*A* **matching rule** *$f$ is* **individually rational** *if it assigns an individually rational matching to each preference profile.*

## 2.3 Stability

A pair of agents $(i, j)$ from the opposite sides of the market is called a blocking pair to matching $\mu$ if $i$ and $j$ prefer each other to their current match under $\mu$.

*A* **matching** *$\mu$ is* **stable** *if there is no blocking pair $(i, j)$ who prefer each other to their current match under $\mu$, and each agent is individually rational.*

*A* **matching rule** *$f$ is* **stable** *if $f(R)$ is stable for each preference profile $R$, i.e., it assigns a stable matching to each preference profile.*

## 2.4 Justified Envy

In one-sided matchings problems, if an agent $i$ is assigned an object such that another agent $i'$ who has a higher priority for this object prefers it to her current assigned object, then $i'$ has justified envy for $i$. This is called justified envy since the envy is justified based on the priorities.

**Agent** $i'$ **justifiably envies** *agent $i$ who is assigned an object $k$ under $\mu$ if $k \succ_{i'} \mu(i')$ and $i' \succ_k i$.*

Similarly, in two-sided matching problems, if agent $i'$ has been matched to $\mu(i')$ under matching $\mu$ but $i'$ prefers another agent $j$ to $\mu(i')$ while $j$ also prefers $i'$ to her current match, $i$, then the pair of agents, $(i', j)$ (as discussed before) is called a blocking pair. Blocking pairs are formally identical to justified envy instances, and eliminating these blocking pairs together with individual rationality results in a stable matchings.

## 2.5 Fairness

*A matching $\mu$ is* **fair (*F*)** *if there is no agent who justifiably envies another agent.*

Fairness is the result of eliminating all justified envy instances for all different profiles of preferences or priorities.

Since eliminating blocking pairs together with individual rationality means that we have a stable matching, fairness combined with individual rationality is formally identical to stability. The only difference is in the interpretation of these notions. Fairness applies to one-sided matching models, while the concept of stability is more relevant for two-sided matching.

*A* **matching rule** *$f$ is* **fair** *if $f(R)$ is fair for each preference profile $R$.*

## 2.6 Pareto-optimality

When a matching is Pareto-optimal there is no way to improve anyone's outcome unless someone else gets hurt.

*A* **matching** *$\mu$ is* **Pareto-optimal** *if there is no other matching $\mu'$ that makes at least one agent better off without making any other agent worse off.*

That is, matching $\mu$ is Pareto-optimal if there is no other matching $\mu'$ such that, for all agents $i$, $\mu' \succeq_i \mu$, while $\mu' \succ_j \mu$ for some agents $j$. A Pareto-optimal matching is not Pareto-dominated in the sense that there is another matching which makes none of the agents worse off and at least one agent better off.

*A* **matching rule** *$f$ is* **Pareto-optimal** *if it specifies a Pareto-optimal matching for each preference profile.*

First Gale and Shapley (1962, [25]) and then Balinski and Sönmez (1999, [9]) showed that it is impossible to have a matching rule which finds matchings which satisfy both fairness and Pareto-optimality for the agents' side at each preference profile. In fact, there exist preference profiles for which none of the fair matchings are Pareto-optimal (when considering one side of the market only).

Due to this incompatibility problem, lots of matching theorists have studied the issue from different angles. They have introduced modified definitions of fairness/stability or efficiency which are compatible weakened versions of these properties. Kesten (2004, [38]), Cantala and Pápai (2014, [13]), Morrill (2013, [50]) and Pápai (2013, [53]) are some of the studies inspired by the incompatibility between fairness and efficiency which shed light on the trade-offs. Inspired by the same issue, Ergin (2002, [22]) and Kesten (2006, [39]) chose the structure of priorities as the focus of their studies and have identified priority structures for which Pareto-optimality and fairness are compatible.

## 2.7 Strategy-proofness

A matching rule is strategy-proof if it is not manipulable. This means no agent can make herself better off by misreporting her preferences and thus truth-telling is a weakly dominant strategy for all agents. A matching is therefore strategy-proof if it is a weakly dominant strategy for every player to report her true preferences. Every agent fares best (or at least not worse) by being truthful, regardless of what preferences the others report. Strategy-proofness is also known as dominant strategy incentive compatibility which means that it cannot be manipulated by any agent at any preference profile.

*A* **matching rule** *$f$ is* **strategy-proof** *if there do not exist $i$, $\succ$ and $\hat{\succ}_i$ such that $f_i(\hat{\succ}_i, \succ_{-i}) \succ_i f_i(\succ)$.*

An even more demanding incentive property of matching rules than strategy-proofness is group strategy-proofness, which requires that no group of agents can collude to misreport their preferences in a way that makes at least one member of the group better off, without making any other member of the group worse off. Note that this is the stronger notion of the two standard notions of group strategy-proofness.

We will say that a group of agents $S \subset N$ can manipulate matching rule $f$ at profile $\succ$ if there exists $\hat{\succ}_S$ such that for all $i \in S$, $f_i(\hat{\succ}_S, \succ_{-S}) \succeq_i f_i(\succ)$ and there exists $j \in S$ such that $f_j(\hat{\succ}_S, \succ_{-S}) \succ_j f_j(\succ)$.

*A* **matching rule** *$f$ is* ***group strategy-proof (GSP)*** *if there is no group of agents that can manipulate $f(\succ)$ at any preference profile $\succ$.*

It is clear that group strategy-proofness of a matching rule implies strategy-proofness of the rule, but not vice versa.

Roth (1982, [58]) proved that there is no stable matching rule that is strategy-proof for both sides of the market. On the other hand, if only one side of the market is active then the DA rule is stable (i.e., fair and individually rational) and strategy-proof for the proposing side of the market if the proposing agents have unit capacity. Moreover, Pareto-optimality and group strategy-proofness are compatible with each other, and the TTC rule satisfies both.

## 2.8 Nonbossiness

*A* **matching rule** *$f$ is* **nonbossy** *if for all $i$, $\succ$, and all $\hat{\succ}_i$, $f_i(\succ) = f_i(\hat{\succ}_i, \succ_{-i})$ implies that $f(\succ) = f(\hat{\succ}_i, \succ_{-i})$.*

If there is such an agent, then this agent is **bossy** at preference profile $R$, since she can change somebody else's assignment without changing her own.

Nonbossiness combined with strategy-proofness is equivalent to group strategy-proofness in the static models that I study.

## 2.9 Maximum Cardinality

*A* **matching** *is* **maximum (cardinality)** *if there is no other matching which matches more agents to each other.*

Consequently, a maximum matching should contain the highest number of possible matches. Note that the cardinality of a matching does not depend on the preference profile, and thus maximum matchings can be determined based on the set of acceptable objects only.

*A* **matching rule** *$f$ is a* ***maximal*** *rule if it specifies a maximum matching for each preference profile.*

## 2.10 Nonwastefulness

If there is an unmatched agent $j$ whom agent $i$ prefers to her current match under $\mu$ and $j$ also prefers $i$ to being unmatched then $i$ justifiably claims this possible match. Similarly, if there is an empty slot at a hospital or university (when assigning doctors to hospitals or students to universities) that $i$ prefers to her current match under $\mu$ and $i$ is acceptable to the hospital or university then it is wasteful not to use this slot and $i$ justifiably claims it.

**Agent** $i$ **justifiably claims** *a possible match with unmatched agent $j$ (or an empty slot at $j$) under a feasible matching $\mu$ if $j \succ_i \mu(i)$ and $i \succ_j \varnothing$ for any $j$, while $\mu(j) = \varnothing$ or $|\mu(j)|$ is less than $j$'s capacity.*

*A* **feasible matching** *$\mu$ is* **nonwasteful** *if there is no agent who justifiably claims a possible match with an unmatched agent $j$ (or an empty slot at $j$).*

# Chapter 3

# Maximum Matching with Consecutive Acceptance Intervals

## 3.1 Introduction

Matching markets, such as school choice, entry-level labor markets and kidney exchange, have been explored by numerous theoretical and applied papers. The purpose of this study is to define and analyze a model in which agents on one side of the market rank only a subset of the objects on the other side of the market, and these subsets are consecutive, where consecutiveness is determined by an objectively given ranking of the objects. However, each agent independently ranks the objects in these consecutive acceptance intervals according to their own preferences.

The study of consecutive acceptance intervals is motivated by several real-life applications. For instance, employers that offer an internship may choose applicants for interviews from a given applicant pool, where applicants have been ranked according to some known criterion, such as the number of years of experience, level of education or grades. Employers choose a range of applicants according to this criterion to interview, and rank the interviewed applicants based on the interviews. If this is a buyer's market, which could be a realistic assumption since paid internships may be highly desirable and relatively rare, then applicants could be modeled as objects to be assigned to internships. Or consider adoptions arranged by an agency between a fixed pool of adoptive parents and children for adoption. Eligible families that are ready to adopt a child may only be able to or willing to adopt a child from a certain age group, for example. After meeting the children from the selected age interval, the families rank the children according to their personal preferences.

Another application involves prospective students who are choosing universities to apply to. Each prospective student chooses a certain range of universities on the basis of a published ranking of universities, a range that the student deems compatible with her background and records, and after campus visits ranks the selected universities according to the information that the student has obtained. Here the universities' preferences may simply be expressed by a cutoff score for students, and if these are consistent with the publicly known ranking of universities, this may dictate the choice of acceptance intervals for students. Our model is less relevant for this application, however, since in this study we only consider one-to-one matching, as a first exploration of consecutive acceptance models.

This paper addresses an open question raised by Al Roth on economic design engineering (Roth, 2002 [61]), where he states that some properties of the medical residency market in the US are related to the fact that applicants apply for a small fraction of the jobs only, and that both applicants and residency programs only list each other in their preferences if they had completed an interview. This is an important feature of many markets, not just the US medical residency market, and Al Roth has suggested that this fact may have important consequences. He concluded that better theory is needed to deal with this issue, and this study aims to provide some theoretical foundations by constructing a new model with an explicit structure of acceptance sets.

The main objective of this paper is to find more straightforward algorithms than the complicated general algorithms to achieve Pareto-optimal maximum matchings, while taking incentives into consideration as well. To this end, we assume consecutive acceptance intervals, which makes our model unique and at the same time realistic in some settings, as already discussed. To the best of our knowledge our paper is the first one that studies matching markets with consecutive acceptance intervals. Another assumption that is used by most of our results is that the acceptance sets allow for a matching that assigns an object to each agent. This is a realistic assumption in both of our main applications. Internship positions may be in short supply, which would imply that the applicant pool is large enough to fill each internship position. Qualified adoptive parents are also likely to be able to adopt a child typically, especially if they are flexible enough in terms of their requirements for the child.

Maximum matchings have been considered in general and there are some well-known classical results. Philip Hall in his paper titled "On Representatives of Subsets" (Hall, 1935 [27]) describes the general requirement to have a maximum matching which saturates one side of the market, known as Hall's Theorem. Other examples include the Hopcroft and Karp algorithm (Hopcroft and Karp, 1973 [31]) which finds a maximum matching by iteratively increasing the size of the matching using augmenting paths, and the Ford-Fulkerson algorithm (Cormen et al. 2001 [14]) which is based on the max flow in a graph.

Taking into account preferences, Irving at al. (2006, [32]) proposed "rank-maximal" matchings, which maximize first-choice matches for one side of the market and, subject to this, it maximizes second-choice matches, and so on. For the refugee resettlement application Andersson and Ehlers (2020, [5]) proposes an algorithm that starts with a maximum matching and tries to improve the optimality through trade steps while maintaining stability. Afacan et al. (2020, [4]) have introduced two classes of mechanisms, the Efficient Assignment Maximizing and the Fair Assignment Maximizing mechanisms, which first enumerate all feasible matchings and then eliminate non-maximum matchings. In their setting, as well as in Andersson and Ehlers (2020, [5]), the model is two-sided and they also study stability, which is not possible in our one-sided model.

The incentive properties of maximal rules have also been studied. These are mostly negative findings. For example, neither of the two proposed mechanisms are strategy-proof in Afacan et al. (2020, [4]), and thus they study the Nash equilibria of the revelation games corresponding to their mechanisms. The closest paper to ours studying incentives of maximal matching rules is Krysta et al. (2014, [7]) who show that in a housing allocation problem there is no deterministic matching rule which is maximal and strategy-proof. This is in contrast to our maximal matching rules which are all group strategy-proof. The difference is due to our assumption that the consecutive acceptance intervals are common knowledge and are independent of the preferences, which is a natural assumption in some settings with consecutive acceptance intervals. For example, in the case of internship matching the employers may insist on a minimal level of experience or a minimum GPA (grade point average) which would be established in advance, and commonly known upper bounds are also plausible, especially regarding the level of experience (e.g., the applicant should have graduated no more than three years prior to the application). Such requirements would be included in the internship application materials and are likely to be fixed rules that don't change from one year to the next. In the case of adoptions, adoptive parents may not be allowed to adopt an infant, depending on the parents' age, so a lower bound on the adopted child's age might be required by law, although upper bounds are more likely to be based on preferences and less easy to verify in advance.

Our hope is that this paper will serve as a basis for further research which will generalize the newly designed algorithms to extend our findings to more complex matching models in which the acceptable set of matches is relatively small, thus eventually leading to better theoretical and practical understanding of the questions raised by Al Roth about the phenomenon of small acceptance sets in matching markets.

## 3.2 A Matching Model with Consecutive Acceptance Intervals

Let $I = \{1, \ldots, n\}$ be the set of $n$ agents and let $O = \{o_1, \ldots, o_m\}$ be the set of $m$ objects. The objects are ordered according to the *common ranking* $(o_1, \ldots, o_m)$, an exogenously given ranking of the objects. Each agent $i \in I$ has a set of acceptable objects $A_i \subseteq O$, the consecutive *acceptance interval,* which is assumed to be a primitive of the model. Acceptance intervals are consecutive with respect to the common ranking: for each agent $i \in I$, $A_i = \{o_{x_i}, o_{x_i+1}, \ldots, o_{x_i+k_i}\}$ such that $x_i \in \{1, \ldots, m\}$, $k_i \in \{0, \ldots, m-1\}$, and $k_i + 1$ is the number of objects in $A_i$, with $x_i + k_i \leq m$. We will also use the notation $N_o$ for $o \in O$ to indicate the set of agents who have object $o$ in their acceptance interval. Each agent is to be matched to at most one object and the assigned object has to be in her acceptance interval. The interval profile $A = (A_i)_{i \in I}$ is the fixed profile of consecutive acceptance intervals which is assumed to be common knowledge. The set of interval profiles is denoted by $\mathcal{A}$.

Agent $i \in I$ has strict preferences $\succ_i$ over objects in her acceptance interval. Individual preference rankings are independent of the "objective" common ranking and are private information. A preference profile $\succ = (\succ_i)_{i \in I}$ specifies strict preferences for each agent over the objects in their respective acceptance intervals. Let $\mathcal{R}$ denote the set of preference profiles. We also use the following additional notation to indicate various preferences. Weak preferences are denoted by $\succeq_i$, but given that preferences are assumed to be strict, $\mu(i) \succeq_i \nu(i)$ means that either $\mu(i) \succ_i \nu(i)$ or $\mu(i) = \nu(i)$. In addition, $\succ_{-i}$ denotes strict preferences for all agents except for $i$, and $\succ_S$ indicates strict preferences for a set of agents $S \subseteq I$.

A feasible matching $\mu : I \to O \cup \{\varnothing\}$ is a function such that either $\mu(i) = o$, where $o \in A_i$, or $\mu(i) = \varnothing$ which means that agent $i$ remains unmatched. Each object is assigned to at most one agent, so if $\mu(i), \mu(j) \in O$ for a pair of agents $i$ and $j$ then $\mu(i) \neq \mu(j)$. We will write matchings, when unambiguous, in the order of the agents. For example, matching $\mu = (a, d, b, \varnothing)$ for a four-agent problem means that agents 1, 2 and 3 are assigned objects $a$, $d$ and $b$ respectively, and agent 4 is unmatched. Let $\mathcal{M}$ denote the set of feasible matchings. A matching rule is a function $f$ that assigns a feasible matching $\mu$ to each preference profile $\succ \in \mathcal{R}$, that is, $f : \mathcal{R} \to \mathcal{M}$. Let $f_i(\succ)$ denote agent $i$'s assignment at preference profile $\succ$ when the matching rule is $f$.

## 3.3 Properties of Matching Rules

The properties of matching rules that we study are standard properties that have been studied extensively in various models.

Let $|\mu|$ denote the number of agents who are assigned an object in matching $\mu$. A feasible matching is a *maximum* matching if there is no other feasible matching which assigns objects to more agents: $\mu \in \mathcal{M}$ is a maximum matching if there is no $\mu' \in \mathcal{M}$ such that $|\mu'| > |\mu|$. Note that whether a matching is a maximum matching does not depend on the preference profile and can be determined solely based on the interval profile $A$.

**Property 1:** *A* **matching rule** *is* **maximal** *if it assigns a maximum matching to each preference profile.*

**Property 2: A matching** $\mu$ *is* **Pareto-optimal** *if it is not Pareto-dominated, that is, there is no other matching* $\nu$ *such that, for all* $i \in I$, $\nu(i) \succeq_i \mu(i)$, *while* $\nu(j) \succ_j \mu(j)$ *for some* $j \in I$.

**A matching rule** *is* **Pareto-optimal** *if it assigns a Pareto-optimal matching to each preference profile.*

The following example demonstrates that in our model not all Pareto-optimal matchings are maximum matchings, and not all maximum matchings are Pareto-optimal.

**Example 1** *Independence of Pareto-optimal and maximum matchings.*

Consider the preference profile shown below. Agent 1's acceptance interval is $\{a\}$, and agent 2's and 3's is $\{a, b, c\}$.

| $\succ_1$ | $\succ_2$ | $\succ_3$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| | $c$ | $a$ |
| | $b$ | $c$ |

The matching $\mu = (\varnothing, a, b)$ is Pareto-optimal, but it is not a maximum matching, while the matching $\mu' = (a, b, c)$ is a maximum matching but it is not Pareto-optimal. The unique maximum Pareto-optimal matching at this preference profile is $\mu'' = (a, c, b)$. ▲

**Property 3: A matching rule** $f$ *is* **strategy-proof** *if there do not exist* $i \in I$, $\succ \in \mathcal{R}$ *and alternative preferences* $\hat{\succ}_i$ *for* $i$ *such that* $f_i(\hat{\succ}_i, \succ_{-i}) \succ_i f_i(\succ)$.

A matching rule is *strategy-proof* if it is a weakly dominant strategy for every player to report her true preferences. A stronger incentive property than strategy-proofness is *group strategy-proofness,* which requires that no group of agents can collude to misreport their preferences in a way that makes at least one member of the group better off, without making any other member of the group worse off. Note that this is the stronger notion of the two standard notions of group strategy-proofness. We will say that a group of agents $S \subseteq I$ can *manipulate* matching rule $f$ at profile $\succ$ if there exist alternative preferences $\hat{\succ}_S$ for $S$ such that for all $i \in S$, $f_i(\hat{\succ}_S, \succ_{-S}) \succeq_i f_i(\succ)$, and there exists $j \in S$ such that $f_j(\hat{\succ}_S, \succ_{-S}) \succ_j f_j(\succ)$.

**Property 4: A matching rule** $f$ *is* **group strategy-proof** *if there is no preference profile* $\succ \in R$ *and a group of agents* $S \subseteq I$ *who can manipulate* $f$ *at preference profile* $\succ$.

Group strategy-proofness of a matching rule implies strategy-proofness of the rule, but not vice versa. An additional well-known property of matching rules is nonbossiness. A matching rule $f$ is *nonbossy* if for all $\succ \in \mathcal{R}$, $i \in I$, and $\tilde{\succ}_i$, $f_i(\succ) = f_i(\hat{\succ}_i, \succ_{-i})$ implies that$f(\succ) = f(\hat{\succ}_i, \succ_{-i})$. In our model strategy-proofness and nonbossiness together imply group strategy-proofness (Pápai, 2000 [51]).

## 3.4 Maximum Matchings

The main criterion that we study in this paper is maximizing the size of the matching, that is, matching as many agents to objects as possible. While maximum matchings are not straightforward to find in a general matching model (i.e., in an arbitrary bipartite graph)[1], in our model the bipartite graph that describes feasible matchings has a consecutive structure due to the consecutive acceptance intervals, and this makes it possible to identify a maximum matching in one round using a greedy algorithm, and hence does not require the use of augmenting paths. We define next the Common-Ranking Successive Rules which identify a maximum matching for an arbitrary consecutive interval profile.

### 3.4.1 Common-Ranking Successive Rules

Let $\Pi$ be the set of permutations of $I$. A permutation $\pi \in \Pi$ is used as a tie-breaker for the Common-Ranking Successive Rule $f$. Given the common ranking $(o_1, \ldots, o_m)$ of objects, consider first $N_{o_1}$, the set of agents who have object $o_1$ in their acceptance interval. If $N_{o_1} \neq \emptyset$, assign $o_1$ to the agent in $N_{o_1}$ with the shortest acceptance interval (i.e., the least number of objects in the agent's acceptance interval compared to all other agents in $N_{o_1}$). If there is more than one such agent, break the tie according to the fixed permutation $\pi$ by assigning $o_1$ to the first agent in $\pi$ who meets the above criterion. Remove object $o_1$ together with the agent to whom it was assigned, and repeat the procedure in the reduced problem iteratively for $o_2, o_3$, and so on, until $o_m$ is reached. If there is an object that is

[1] See, for example, the Ford–Fulkerson algorithm which computes the max flow (Cormen et al., 2001 [14]). This method requires an initial matching which is then improved upon by searching for augmenting paths. An augmenting path for a matching $\mu$ is a path in the bipartite graph with an odd number of edges $e_1, e_2, \cdots, e_m$ such that $e_{odd} \notin \mu$ and $e_{even} \in \mu$.

not in any remaining agent's acceptance interval then the object remains unassigned, and the procedure moves on to the next object in the common ranking.[2]

This determines a matching for a given interval profile, independently of the preferences of the agents. Thus, a Common-Ranking Successive Rule can be written as $f^\pi : \mathcal{A} \to \mathcal{M}$, a function of the interval profile $A$, where each rule $f^\pi$ is associated with the tie-breaking permutation $\pi \in \Pi$. As we state next, all Common-Ranking Successive Rules are maximal rules. Therefore, a straightforward greedy algorithm is enough to identify a maximum matching for each consecutive acceptance interval profile, without the use of improvement steps.

**Theorem 1** *The Common-Ranking Successive Rules are maximal.*

Theorem 1 is proved in Appendix $A$.

Once $\pi$ is fixed, the corresponding Common-Ranking Successive Rule identifies a maximum matching, but there may be multiple maximum matchings for a given fixed interval profile which may not be reached, even if we allow the agent permutation $\pi$ to vary. This is because, for example, if two agents $i$ and $j$ with $k_i < k_j$, where these are the respective indices of the last-ranked objects according to the common ranking in $i$'s and $j$'s acceptance intervals, both have two consecutive objects, say $o_x$ and $o_{x+1}$ in their acceptance intervals, then if $i$ is assigned $o_x$ and $j$ is assigned $o_{x+1}$ in a maximum matching, there is also a maximum matching such that these assignments are swapped and $i$ is assigned $o_{x+1}$ and $j$ is assigned $o_x$, but the latter maximum matching cannot be reached by any Common-Ranking Successive rule.

One may conjecture that if we use different orderings of the objects and apply a similar sequential rule with a fixed permutation $\pi$ then we can identify all maximum matchings. While this is true, an arbitrary ordering of the objects may lead to a matching at some interval profiles which is not a maximum matching, which we demonstrate in Example 2. Therefore, in order to characterize all maximum matchings, we now introduce a more general class of rules than Common-Ranking Successive Rules, called Successive Rules, which ensure that the output is a maximum matching. These rules use different object orderings which, together with the different tie-breaking rules $\pi$, allow for reaching all maximum matchings for a given interval profile. Since we no longer use the common ranking of objects for the following result, it applies in general to arbitrary acceptance sets, not just to problems with a consecutive acceptance interval.

However, since we know that finding maximum matchings in an arbitrary bipartite graph is a complex problem which has been solved already, in order to simplify this task and provide accessible results, we assume for the rest of this paper that the interval profile $A$ is such that each agent $i \in I$ can be assigned an object, and thus $n \leq m$. This is not necessarily a crucial assumption since our results could be generalized by accounting for the omitted cases, but we believe that the increased technical detail that would be necessary without this assumption would only distract from the clear intuition behind our results, why

[2]Note that we could define a symmetric set of rules which work the same way as the Common-Ranking Successive Rules, except that they are based on the reverse order of the common ranking, starting from $o_m$. The results hold for these rules as well.

the assumption itself is quite realistic in our applications, as explained in the Introduction.

**Definition 1: I-block**

*An I-block at interval profile $A$ consists of a set of $k$ agents and a set of $k$ objects such that the $k$ agents only have these $k$ objects in total in their acceptance intervals at $A$.*

Note that it is not required that all $k$ agents in an I-block have all $k$ objects in their acceptance intervals. We call this an I-block because the set of agents in an I-block can only be assigned the objects in the I-block. Note that agents who are not in an I-block may also have objects in an I-block in their acceptance intervals, but they also have some other acceptable objects.

**Definition 2: Minimal I-block for an object**

*The minimal I-block for an object at interval profile $A$ is the smallest I-block containing the object.*

Note that a minimal I-block for an object always exists if there is an I-block which contains the object. Moreover, it is unique, given our assumption that a maximum matching $\mu$ is such that $|\mu| = n$.

**Example 2** *I-blocks and minimal I-blocks.*

Consider the two interval profiles below.[3]

| $A_1$ | $A_2$ | $A_3$ |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $b$ | $b$ |
| | $c$ | $c$ |

| $A_1$ | $A_2$ | $A_3$ |
|---|---|---|
| $\boxed{\text{a}}$ | $a$ | $a$ |
| | $b$ | $b$ |
| | $c$ | $c$ |

Both interval profiles form an I-block with the three agents and three objects. The first one is a minimal I-block for all three objects. The second one is a minimal I-block for objects $b$ and $c$. The minimal I-block for object $a$ in the second interval profile is the I-block consisting of agent 1 and object $a$. ▲

Let $\Sigma$ denote the set of permutations of $O$, and denote the common ranking of objects by $\bar{\sigma} \in \Sigma$.

### 3.4.2 Successive Rules

The Successive Rules work the same way as the Common-Ranking Successive Rules, with two exceptions: (1) an arbitrary ordering of the objects $\sigma \in \Sigma$ may be used instead of the common ranking $\bar{\sigma}$, and (2) if an object is in an I-block then it cannot be assigned to an

[3] In all the examples, the interval profiles indicate all the acceptable objects for agents, in the order of the common ranking when relevant. Interval profiles do not indicate preferences over objects.

agent who is not in the minimal I-block for this object; instead, the object is assigned to the first agent according to $\pi$ who is in the minimal I-block for this object.

Note that I-blocks need to be considered iteratively in the successively reduced problem where agents who have already been assigned an object have been removed together with their assigned objects. Each Successive Rule can be written as $\hat{f}^{\sigma\pi} : \mathcal{A} \to \mathcal{M}$, which yields a matching for each object ordering $\sigma \in \Sigma$ and tie-breaking agent permutation $\pi \in \Pi$.

**Example 3** *A Successive Rule.*

Consider the following interval profile with four agents and four objects:

| $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|
| $a$ | $a$ | | |
| $b$ | $b$ | | |
| | $c$ | $c$ | |
| | | $d$ | $d$ |

Let the tie-breaking permutation be $\pi = (1, 2, 3, 4)$. The common ranking is $\bar{\sigma} = (a, b, c, d)$ and the Common-Ranking Successive Rule $f^{\bar{\sigma}\pi}$ yields matching $\bar{\mu} = (a, b, c, d)$. Now consider the Successive Rule $f^{\sigma\pi}$ with object ordering $\sigma = (b, c, a, d)$. Note that without considering I-blocks the resulting matching would have been $\nu = (b, c, d, \varnothing)$, since after assigning $b$ to agent 1, $c$ would have been assigned to agent 2 based on the tie-breaking. However, given that agents 3 and 4 and objects $c$ and $d$ form the minimal I-block for $c$, $c$ cannot be assigned to 2 and it is thus assigned to 3 who is the only agent in the minimal I-block for $c$ who can be assigned $c$. Thus, the resulting matching is $\mu = (b, a, c, d)$. Given that $|\nu| < |\bar{\mu}| = |\mu|$, $\nu$ is not a maximum matching but $\mu$ is. ▲

The example demonstrates that using the common ranking to guarantee a maximum matching is important, as already explained above, and thus Theorem 1 relies on the special structure of consecutive acceptance intervals and would not hold for arbitrary acceptance sets. If we use a different ordering of the objects from the common ranking then we need to take special care to ensure that the resulting matching is a maximum matching. In fact, it is interesting to note that the class of Common-Ranking Successive Rules is a special case and a subset of the class of Successive Rules $(f^{\sigma\pi})_{\sigma\in\Sigma,\pi\in\Pi}$, because when the common ranking $\bar{\sigma}$ is used as the object ordering by a Successive Rule, if an object is in an I-block then the agent who is to be assigned the object (regardless of the tie-breaking rule) is always in the minimal I-block for the object. Hence the rule simplifies to the Common-Ranking Successive Rule and $\hat{f}^{\bar{\sigma}\pi} = f^{\pi}$.

**Theorem 2** *The Successive Rules are maximal. Moreover, a matching $\mu$ is a maximum matching for interval profile $A$, given the tie-breaker agent permutation $\pi \in \Pi$, if and only if there exists an object ordering $\sigma \in \Sigma$ such that the Successive Rule $\hat{f}^{\sigma\pi}$ yields $\mu$ at $A$, that is, $\hat{f}^{\sigma\pi}(A) = \mu$.*

Theorem 2 is proved in Appendix $B$.

Theorem 2 not only says that Successive Rules are maximal but also provides a characterization of maximum matchings. This theorem, unlike Theorem 1, does not rely on the special structure of consecutive acceptance intervals. It parallels the well-known characterization of Pareto-optimal matchings in assignment problems, given in terms of the existence of an agent permutation used by a serial dictatorship. As opposed to the existence of an agent ordering for Pareto-optimal matchings, maximum matchings are identified by the existence of an object ordering, since the tie-breaker agent ordering can be fixed exogenously.

## 3.5 Maximal Trading Rules and Sequential Dictatorships

While the maximum cardinality and Pareto-optimality of a matching are independent properties, as shown by Example 1, for any preference profile there exists at least one matching which is both maximum and Pareto-optimal. This is easy to see since starting from an arbitrary maximum matching which is not Pareto-optimal we can make Pareto-improvements until the matching becomes Pareto-optimal, without losing its maximum cardinality property. Next we introduce the Maximum Endowment Hierarchical Exchange Rules, which are a subset of the hierarchical exchange rules of Pápai (2000, [51]) and are based on the principle of trading objects such that the initial endowments for trading include a maximum matching, which guarantees a maximum matching after all the trading takes place.

Let us recall that hierarchical exchange rules are generalized top trading cycle (TTC) rules (Shapley and Scarf, 1974 [67]) and are based on the well-known original TTC algorithm attributed to David Gale. For hierarchical exchange rules objects are endowed to agents without necessarily endowing them in a one-to-one fashion, and each agent points to the agent who is endowed with his favorite object, which could be the agent herself. Since $I$ is finite, there exists at least one cycle of pointing agents, and the agents in these cycles are assigned the object for which they are pointing. The assigned agents and objects are removed from the market, after which the new endowments of objects that are left behind are determined. Each object is endowed to the highest priority agent who is still in the market, and we repeat the procedure iteratively in the reduced market. For further details and formal definitions see Pápai (2000, [51]).

Given a fixed interval profile $A$, each hierarchical exchange rule is associated with a system of trading priorities which determines which agent is endowed with which object in each round of the procedure.[4] In our setup the system of trading priorities only allows to endow agents with objects that are in their acceptance interval, which imposes a natural feasibility requirement on the initial endowments and on inheritance in later rounds of the procedure. This is a slight departure from the original definition, since in Pápai (2000, [51]) all objects were assumed to be acceptable to agents.

### 3.5.1 Maximum Endowment Hierarchical Exchange Rules

Each Maximum Endowment Hierarchical Exchange Rule is associated with a system of trading priorities, in addition to an agent ordering $\pi \in \Pi$ used for tie-breaking and to an object ordering $\sigma \in \Sigma$, to determine a maximum matching using the corresponding

[4] A formal description of inheritance trees is given by Pápai (2000, [51]), and a more concise and general definition of trading rights can be found in Pycia and Ünver (2017, [55]), which they call a *structure of control rights*.

Successive Rule. In order to ensure maximum matchings, a restriction on endowments is imposed in Step 1, while Step 2 is the usual trading with iterative rounds.

**Step 1:** Given the Successive Rule $\hat{f}^{\sigma\pi}$, find the resulting maximum matching for interval profile $A$, that is, determine $\hat{f}^{\sigma\pi}(A)$. Let this matching be part of the initial endowments. The rest of the trading priorities are fixed arbitrarily.

**Step 2:** The iterative procedure of trading objects as in hierarchical exchange rules, given the system of trading priorities determined in Step 1.

**Theorem 3** *Maximum Endowment Hierarchical Exchange Rules are maximal, Pareto-optimal and group strategy-proof.*

**Proof:** In Step 1 of a Maximum Endowment Hierarchical Exchange Rule we use a Successive Rule to find the initial endowments for agents together with arbitrarily fixing the rest of the system of trading priorities. This step is independent of the preferences, so agents cannot manipulate it. Then in Step 2 we run the hierarchical exchange rule based on this system of trading priorities, which is shown to be individually rational, Pareto-optimal and group strategy-proof by Pápai (2000, [51]). Finally, these rules are maximal since a maximum matching is part of the initial endowments by Theorem 2, given Step 1, and the individual rationality of hierarchical exchange rules ensures that the final matching is also a maximum matching at each preference profile. ■

The following example demonstrates that the results in Theorem 3 cannot be extended to the Trading Cycle rules of Pycia and Ünver (2017, [55]). As the example shows, this is because if not all objects are acceptable to agents, the presence of brokers may lead to a violation of individual rationality.

**Example 4** *Maximum Endowment Trading Cycle rules are not maximal.*

Consider the preference profile shown below. Agent 1's and 2's acceptance interval is $\{a, b, c\}$, and agent 3's is $\{a\}$. Let the initial endowments be given by the maximum matching $(b, c, a)$. Assume that agent 3 is a broker.

| $\succ_1$ | $\succ_2$ | $\succ_3$ |
|---|---|---|
| $a$ | $a$ | $a$ |
| $b$ | $c$ | $\varnothing$ |
| $c$ | $b$ | $b$ |
| $\varnothing$ | $\varnothing$ | $c$ |

Here both agents 1 and 2 point to 3, and since agent 3 is a broker 3 points to 1 for object $b$. Agents 1 and 3 trade their endowed objects and the final matching is $\mu = (a, c, \varnothing)$, since $b$ is not in 3's acceptance interval. Clearly, $\mu$ is not a maximum matching. ▲

Maximum Endowment Hierarchical Exchange Rules are not the only rules which are maximal, Pareto-optimal and group strategy-proof. Although a maximum endowment is sufficient to guarantee the maximality of the rule, it is not necessary. We can define, for example, sequential dictatorships which have the same properties. The well-known and straightforward serial dictatorships allocate objects based on an ordering of the agents, by assigning each agent sequentially. A sequential dictatorship as defined in Pápai (2001, [52]) is similar to a serial dictatorship, except that it only fixes the first agent in the ordering, and subsequent agents in the ordering are determined based on the assignments of preceding agents in the ordering. Before introducing similar sequential dictatorships that produce maximal matchings in our setting, let us recall the definition of serial dictatorships.

**Serial dictatorships (Satterthwaite and Sonnenschein, 1981 [66]; Svensson, 1994 [69]):**

Each serial dictatorship is associated with a fixed agent ordering $\pi \in \Pi$. A serial dictatorship assigns the first agent in $\pi$ her first choice among all the objects, the second agent in $\pi$ her first choice among all the remaining objects, and so forth.

### 3.5.2 Maximal Sequential Dictatorships

Each Maximal Sequential Dictatorship is associated with a fixed agent ordering $\pi \in \Pi$. Each agent is assigned her favorite object among the remaining objects according to $\pi$, just like in a serial dictatorship, unless the favorite object of the next agent according to $\pi$, say agent $i$, is in an I-block, given the remaining agents and objects, and $i$ is not in the minimal I-block for this object. In this case the agents in the minimal I-block for this object become the agents to be assigned objects next. Using $\pi$ as a tie-breaker among these agents in the minimal I-block if needed, these agents receive their assignments subject to minimal I-blocks as well. Once the assignments to agents in the minimal I-block are determined, we return to agent $i$ and continue iteratively according to $\pi$ in a similar manner. Note that I-blocks need to be considered iteratively in the successively reduced problem where agents who have already been assigned an object have been removed together with their assigned objects.

**Theorem 4** *Maximal Sequential Dictatorships are maximal, Pareto-optimal and group strategy-proof.*

**Proof:** Since sequential dictatorships in the case of full acceptance sets are hierarchical exchange rules (see Pápai, 2000 [51]), Maximal Sequential Dictatorships in our setting are also Pareto-optimal and group strategy-proof, which can be verified similarly. To see that these rules are maximal, suppose by contradiction that there exist a preference profile $\succ \in \mathcal{R}$ and an agent permutation $\pi \in \Pi$ such that the Maximal Sequential Dictatorship associated with $\pi$ assigns a matching $\mu$ to $\succ$ which is not a maximum matching. Let agent $i$ be the first agent according to $\pi$ such that if the assignment of all previous agents before $i$ according to $\pi$ remained the same but agent $i$ was not assigned her favorite object $o$ among the remaining objects (i.e., $\mu(i) = o$), then it is still possible to have a maximum matching, but once we assign $o$ to $i$ the matching can no longer be a maximum matching. This implies that when it is agent $i$'s turn to obtain her assignment at preference profile $\succ$ there is a minimal I-block for object $o$, and agent $i$ is not in the minimal I-block for $o$. Then the agents in the minimal I-block are assigned their objects first, and given that in any I-block the number of agents is the same as the number of objects, the minimal I-block requirement ensures that each agent in the minimal I-block obtains an object. This means that object $o$ is assigned to an agent other than $i$, which is a contradiction. Hence Maximal Sequential Dictatorships are maximal. ■

## 3.6 Maximal serial dictatorships: An Impossibility Result

So far we have identified two classes of matching rules which are maximal, Pareto-optimal and group strategy-proof, namely Maximum Endowment Hierarchical Exchange Rules and Maximal Sequential Dictatorships. Both of these classes of rules are fairly complex, since both involve the identification of I-blocks in an iterative manner. Even the subset of Maximum Endowment Hierarchical Exchange Rules which start with the maximum endowment based on a Common-Ranking Successive Rule require iterative trading rounds. We now want to find rules that have similar nice properties but are simpler, and we focus here on serial dictatorships that produce maximum matchings. Serial dictatorships are also a subset of Pápai's (2000, [51]) hierarchical exchange rules, namely, they are associated with the same fixed trading priority ordering being assigned to each object (although in this case the priorities don't lead to any trading). These rules are vastly simpler than the matching rules proposed in the previous section, and thus even though less equitable they are of particular interest. Unfortunately our first result indicates that, depending on the interval profile, serial dictatorships may not always produce a maximum matching regardless of which agent ordering we use. This is in line with our understanding that most well-known matching rules are not maximal.

**Theorem 5** *If there are at least four agents and four objects, there may not exist a maximal serial dictatorship for an arbitrary interval profile.*

**Proof:** We demonstrate this result by showing an example of an interval profile with four agents and four objects for which no agent permutation guarantees that the corresponding serial dictatorship yields a maximum matching at each preference profile. The example can be extended to an arbitrary number of agents and objects by adding more agents and objects such that the acceptance intervals for additional agents don't include any of the four objects in the example.

Let the interval profile be the following:

| $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|
| $a$ | $a$ | | |
| $b$ | $b$ | $b$ | |
| $c$ | $c$ | $c$ | $c$ |
| | | $d$ | $d$ |

First note that $\mu = (a, b, c, d)$ is a maximum matching at this interval profile, so each agent is assigned an object in a maximum matching. If the agent permutation for the serial dictatorship makes agent 1 or 2 last, then consider a preference profile where agents 1 and 2 both rank $a$ first, and agents 3 and 4 rank $b$ and $c$ first, respectively. Then if agent 1 is last then 1 remains unmatched, and if agent 2 is last then 2 remains unmatched. Similarly, if the agent permutation for the serial dictatorship makes agent 3 or 4 last, then consider a preference profile where agents 1 and 2 rank $b$ and $c$ first, respectively, and agents 3 and 4 both rank $d$ first. Then if agent 3 is last then 3 remains unmatched, and if agent 4 is last then 4 remains unmatched. Therefore, there is no agent permutation that can guarantee a maximum matching at this interval profile. ■

We know that if each agent finds all objects acceptable (i.e., $A_i = O$ for all agents $i \in I$), then all permutations of agents for a serial dictatorship result in a maximum matching. Thus, one interesting question for our more restrictive setting is the following: for which interval profiles would some permutations of agents give a maximum matching? Moreover, if we can identify the set of interval profiles for which such an agent permutation exists, can we characterize all the agent permutations which produce a maximum matching? We address these questions next.

## 3.7 Block Serial Dictatorships

Now we introduce another class of matching rules, the Block Serial Dictatorships, and we prove that these rules yield Pareto-optimal maximum matchings for a large subset of consecutive interval profiles. We call these interval profiles *solvable*. Block Serial Dictatorships are serial dictatorships with a restricted set of agent permutations, and thus they are Pareto-optimal and group strategy-proof.

**Definition 3: O-block**

*An O-block at interval profile $A$ consists of a set of $k$ agents and a set of $k$ objects such that each of the $k$ agents has all of the $k$ objects in her acceptance interval and no other agents have any of these $k$ objects in their acceptance interval at $A$. We will refer to agents in an O-block as block agents.*

An O-block is important because the set of objects in an O-block can only be assigned to the agents in this O-block. Since $k$ objects are available to $k$ agents, the agents in an O-block would not remain unmatched even if they were to choose their objects last and no other objects were left. O-blocks have the property that a serial dictatorship restricted to an O-block with any arbitrary agent ordering always matches each of the $k$ agents to one of the $k$ objects, that is, a matching within an O-block is always a maximum matching, since each agent in an O-block finds each object in an O-block acceptable.

**Definition 4: Fully flexible agent**

*A fully flexible agent at interval profile $A$ is an agent who has more objects acceptable than the number of agents overlapping with her in terms of their acceptance intervals. Formally, agent $i$ is a flexible agent if*

$$|A_i| > |\{j \in I \setminus \{i\} : A_i \cap A_j \neq \emptyset\}|.$$

Fully flexible agents are flexible in the sense that if each of the other agents were to choose their favorite object before a fully flexible agent, a flexible agent would still have at least one object left in her acceptance interval, regardless of the choices of the other agents, since there are fewer competing agents than acceptable objects for a fully flexible agent. A special case of a fully flexible agent is an agent who has at least as many objects in her acceptance interval as the total number of agents.

We will refer to the set of agents whose acceptance interval overlaps with $i$'s acceptance interval the *competing agents for* $i$ at a given interval profile. We will also apply this terminology to subsets of $A_i$ for agent $i$.

**Definition 5: Flexible agent**

*A flexible agent at interval profile $A$ is an agent who has at least one subset of her acceptable objects for which the number of competing agents for $i$ is less than the number of objects in this subset. Formally, agent $i$ is a flexible agent if there exists $\bar{A}_i \subseteq A_i$ such that*

$$|\bar{A}_i| > |\{j \in I \setminus \{i\} : \bar{A}_i \cap A_j \neq \emptyset\}|.$$

Flexible agents are flexible in the sense that if each of the other agents were to choose their favorite object before a flexible agent, a flexible agent would still have at least one object left in her acceptance interval, namely in a subset $\bar{A}_i$ which satisfies the above inequality, regardless of the choices of the other agents, since there exists a subset of acceptable objects for these agents for which there are fewer competing agents than acceptable objects for $i$.

Note that both fully flexible agents and block agents are flexible agents, so both of these are special cases of the more general definition of flexible agents. Fully flexible agents are clearly flexible agents since the definition of a flexible agent holds for the fully flexible agent's entire set of acceptable objects. Block agents are flexible agents because for the $k$ objects in the O-block there are $k-1$ competing agents, so the $k$ objects in the block satisfy the definition of a flexible agent.

The definitions of an O-block, fully flexible agents and flexible agents are all based on the interval profile only (as opposed to the preferences), and we will apply them not only to the entire problem with all agents and objects, but also to reduced problems with fewer agents (and objects) after some of the agents are removed. It is an important property of flexible agents that they remain flexible agents after the removal of other agents, and new flexible agents may also emerge in such a reduced problem.

Using these above definitions we can now define interval profiles which are solvable. We will say that such interval profiles have an *iterated block structure* (a more suitable name than "iterated flexible structure").

**Definition 6: Iterated block structure and solvable interval profiles**

*An interval profile $A$ has an iterated block structure if all agents can be removed using the following iterated block procedure:*

- Given the interval profile, there is at least one flexible agent.

- Remove all flexible agents.
- Repeat iteratively based on the interval profile of the reduced problem.

If all agents can be removed this way then the interval profile has an iterated block structure and we call it a solvable interval profile for short.

We have introduced O-blocks and fully flexible agents, less general notions of flexible agents, since these are easier to identify and make the iterated block procedure more intuitive, and can also speed up the procedure at the same time. These weaker notions may suffice to identify some solvable interval profiles, however we need the full strength of flexible agents to determine whether some interval profiles are solvable.

**Example 5** *A solvable interval profile.*

Here is a solvable interval profile with seven agents and seven objects:

| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | | | | | |
| $b$ | $b$ | $b$ | | | | |
| | $c$ | $c$ | | | | |
| | | $d$ | $d$ | $d$ | | |
| | | $e$ | $e$ | $e$ | $e$ | |
| | | | | | $f$ | $f$ |
| | | | | | $g$ | $g$ |

**Step 1:** Agents 2 is a fully flexible agent, since she has three acceptable objects and two competitors. Agents 6 and 7 together with objects $f$ and $g$ form an O-block. Remove agents 2, 6 and 7.

| $A_1$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|
| $a$ | | | |
| $b$ | $b$ | | |
| | $c$ | | |
| | $d$ | $d$ | $d$ |
| | $e$ | $e$ | $e$ |

**Step 2:** Agent 1 forms an O-block with object $a$. Agent 3 is now a fully flexible agent, since she has four acceptable objects and three competitors left (in fact, only four agents are left and she has four acceptable objects). Remove agents 1 and 3.

**Step 3:** Agents 4 and 5 remain only, who together with objects $d$ and $e$ form an O-block. Remove these two remaining agents.

The interval profile is solvable since we could remove each agent following the iterated block structure procedure. In this example we could remove all agents either as fully flexible or block agents. ▲

An example of an interval profile which is not solvable can be found in the proof of Theorem 5. Using this interval profile, we demonstrate next that having a larger acceptance interval for agents is not necessarily better in terms of the acceptance interval being solvable.

**Example 6** *A solvable interval profile may become not solvable when some of the acceptance intervals become larger.*

| $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|
| $a$ | $a$ | | |
| $b$ | $b$ | | |
| $c$ | $c$ | $c$ | $c$ |
| | | $d$ | $d$ |

| $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|
| $a$ | $a$ | | |
| $b$ | $b$ | $b$ | |
| $c$ | $c$ | $c$ | $c$ |
| | | $d$ | $d$ |

The first interval profile is solvable. Agents 1 and 2 form an O-block together with objects $a$ and $b$. After removing agents 1 and 2, agents 3 and 4 together with objects $c$ and $d$ form an O-block, and thus we can remove all agents using the iterated block procedure.

The second interval profile is the one used in the proof of Theorem 5 and is not solvable. This interval profile doesn't have any flexible agents. It is interesting to note that the change is minimal between the two interval profiles: only agent 3 has one more acceptable object, $b$, which turns a solvable interval profile into a not solvable one. ▲

We state next our main result, which is on the existence of maximal serial dictatorships. It provides a characterization based on the iterated block structure of interval profiles.

**Theorem 6** *There exists a serial dictatorship which is maximal if and only if the interval profile is solvable.*

Theorem 6 is proved in Appendix $C$.

Now we define serial dictatorships which are maximal for solvable interval profiles, following the proof of Theorem 6. The class of Block Serial Dictatorships is specific to the interval profile and is not defined unless the interval profile is solvable.

### Block Serial Dictatorships

Block Serial Dictatorships are serial dictatorships for solvable interval profiles with the following agent permutations $\pi \in \Pi$. Fix a solvable interval profile $A \in \mathcal{A}$. Apply the iterated block procedure to $A$ and keep track of when the agents were removed as flexible agents. Use the reverse ordering of when agents were removed during the iterated block procedure. Thus, flexible agents removed in step $t$ are subsequent in $\pi$ to flexible agents removed in step $t'$, where $t < t'$. Flexible agents who were removed in the same step of the iterated block procedure are interchangeable in $\pi$. In particular, block agents within the same O-block are interchangeable in $\pi$, and agents in different O-blocks are also interchangeable if they were removed in the same step of the iterated block procedure.

In order to find all agent permutations $\pi \in \Pi$ for Block Serial Dictatorships for an interval profile $A$, we need to identify each flexible agent in each step of the procedure and remove them simultaneously. The resulting collection of all agent permutations for a specific solvable interval profile constitutes the class of Block Serial Dictatorships for the fixed interval profile $A$. Note that the agent permutations are independent of the preferences and only depend on the interval profile.

Next we illustrate Block Serial Dictatorships using previous examples of solvable interval profiles. We indicate interchangeable agents in brackets.

**Example 7** *Agent permutations for Block Serial Dictatorships.*

For the solvable interval profile in Example 6 the Block Serial Dictatorships are given by the set of permutations $([3,4],[1,2])$. This means the four permutations of $(3,4,1,2)$, $(3,4,2,1)$, $(4,3,1,2)$, and $(4,3,2,1)$, since 3 and 4 are interchangeable, and so are 1 and 2. Note that agents 3 and 4 always have to precede agents 1 and 2, otherwise agent 1 or 2 might be assigned object $c$ which would not lead to a maximum assignment.

For the interval profile in Example 5, the Block Serial Dictatorships are serial dictatorships with agent permutations $([4,5],\ [1,3],\ [2,6,7])$. ▲

We show next that for solvable interval profiles only Block Serial Dictatorships lead to a maximum assignment at each preference profile. We remark that two agents who have no common acceptable object can clearly go in an arbitrary order, such as agent 1 versus agents 4, 5, 6 and 7 in Example 5, but we don't need to include these additional permutations in the class of Block Serial Dictatorships since they lead to equivalent outcomes. By contrast, the interchangeability of agents indicated in brackets need to be included in the class of permutations in general because they can lead to different assignments.

Two matching rules are *outcome equivalent* if they assign the same matching to each preference profile.

**Theorem 7** *Given a solvable interval profile $A \in \mathcal{A}$, a serial dictatorship is maximal if and only if it is outcome equivalent to a Block Serial Dictatorship for $A$.*

Theorem 7 is proved in appendix $D$.

Our last theorem below is stated for completeness.

**Theorem 8** *Given a solvable interval profile, Block Serial Dictatorships are maximal, Pareto-optimal and group strategy-proof.*

Theorem 8 requires no proof, since the maximality property of Block Serial Dictatorships follows from Theorem 7, and it is well-known that serial dictatorships are Pareto-optimal and group strategy-proof (see, e.g., Svensson, 1999 [70]).

## 3.8 Conclusion

In this paper we introduced a novel matching model in which one side of the market has consecutive acceptance intervals according to a given commonly known ranking of the objects on the other side of the market. We presented the Common-Ranking Successive Rules which provide a straightforward method to find a maximum matching in this model. We extended this set of rules to Successive Rules, which find all maximum matchings, for the general case where we no longer assume the consecutiveness of acceptance intervals.

We proposed two classes of matching rules which are maximal, Pareto-optimal and group strategy-proof. One carries out top trading cycles starting from a maximum endowment, while the other one is a sequential dictatorship which allows agents to choose their favorite objects in a given order, but assuring maximality requires a complex way to identify who is going to be next to choose.

Both of these are complex classes of algorithms, and thus we turned next to studying serial dictatorships, which are simple greedy algorithms. Since not all interval profiles allow for the existence of a maximal serial dictatorship, we characterize the set of interval profiles for which a maximal serial dictatorship exists, and we identify all such serial dictatorships. Each class of matching rules that we introduce and study are group strategy-proof, since we assume that the acceptance intervals are common knowledge and cannot be manipulated.

The model with consecutive acceptance intervals describes realistic scenarios of matching markets where agents may not be familiar with all items on the other side of the market, and information acquisition about all items is either too costly or not feasible. Our assumption that we study those consecutive interval profiles for which it is feasible to assign an object to each agent is also plausible in applications, and it has the advantage that it allowed us to present clean and intuitive results in this first study of maximal matching rules for problems with consecutive acceptance intervals. While this assumption can be relaxed by modifying some of our definitions to add more scope and generality, we believe that our analysis provides a good theoretical foundation for market designers who are interested in reaching maximal and Pareto-optimal matchings via algorithms with strong incentive properties.

For future research, this model can be extended to the two-sided case where both sides of the market have agents, and thus both sides rank each other. In this case the core stability of this two-sided matching model would also be interesting to study, and the compatibility of maximum Pareto-optimal matchings and core stability could be analyzed with the aim of designing simple and efficient algorithms that provide desirable matching outcomes. Furthermore, both one-sided and two-sided models can be studied when one side of the market has multiple capacity, such as when employers are looking for more than one employee, and thus all previous questions could be addressed in the framework of a many-to-one matching model with consecutive acceptance intervals.

## 3.9 Appendices: Omitted Proofs

### 3.9.1 Appendix $A$

**Theorem 1** *The Common-Ranking Successive Rules are maximal.*

**Proof:** Fix the tie-breaker agent permutation $\pi \in \Pi$. Suppose that the Common-Ranking Successive Rule $f^\pi$ does not yield a maximum matching, given the fixed interval profile $A$. Let $f^\pi(A) = \mu$. Since $\mu$ is not a maximum matching, there exists a matching $\tilde{\mu}$ such that $|\tilde{\mu}| > |\mu|$. Without loss of generality, let $\tilde{\mu}$ be such that for all $i, j \in I$ and $o_x \in O$, if $\mu_i = \varnothing$, $\tilde{\mu}(i) = o_x$ and $\mu(j) = o_x$, then $\tilde{\mu}_j \neq \varnothing$. Order agents according to when they receive their assignment when $f^\pi$ is applied to $A$, ranking the agents who are not assigned an object in $\mu$ last. Denote this permutation of $I$ by $\tilde{\pi}$.

Then there exists agent $i \in I$ and object $o_x \in A_i$ such that $\mu(i) = \varnothing$ and $\tilde{\mu}(i) = o_x$. Given that $\mu = f^{\pi}(A)$, there is no agent-object pair that could be matched in addition to the assignments made in $\mu$, and thus there exists $j \in I$ such that $\mu(j) = o_x$. Without loss of generality, choose $i$ such that $j$ is the first agent with $\mu(j) \neq \tilde{\mu}(j)$ according to $\tilde{\pi}$.

Since $o_x \in A_i \cap A_j$ and $\mu(j) = o_x$, it follows that the last object according to the common ranking in $A_j$ either precedes the last object in $A_i$, or the last object is the same in $A_i$ and $A_j$ (in the latter case $\pi(j) < \pi(i)$). If $A_j \subseteq A_i$, then giving object $o_x$ to agent $i$ instead of agent $j$ does not allow for a higher cardinality matching than $\mu$, contrary to our assumption. Therefore, since the last object in $A_j$ is either the same as the last object in $A_i$ or precedes it, there exists at least one object $o_y \in A_j$ which precedes $o_x$ in the common ranking and $\tilde{\mu}(j) = o_y$. Then, given that $\mu = f^{\pi}(A)$ and $o_y \in A_j$, it must be the case that $o_y$ is assigned to an agent in $\mu$, otherwise it would have been assigned to $j$, since $y$ precedes $x$ in the common ranking. Thus, there exists $h \in I$ such that $\mu(h) = o_y$. Since $o_y$ precedes $o_x$ in the common ranking, $\tilde{\pi}_h < \tilde{\pi}_j$. Given that $\mu(h) \neq \tilde{\mu}(h)$, this contradicts our assumption on the choice of $j$. Therefore, $\mu$ is a maximum matching.

### 3.9.2 Appendix $B$

**Theorem 2** *The Successive Rules are maximal. Moreover, a matching $\mu$ is a maximum matching for interval profile $A$, given the tie-breaker agent permutation $\pi \in \Pi$, if and only if there exists an object ordering $\sigma \in \Sigma$ such that the Successive Rule $\hat{f}^{\sigma\pi}$ yields $\mu$ at $A$, that is, $\hat{f}^{\sigma\pi}(A) = \mu$.*

**Proof:** We first show that the Successive Rules $(\hat{f}^{\sigma\pi})_{\sigma\in\Sigma,\pi\in\Pi}$ are maximal. Suppose by contradiction that this is not the case. Then there exist an interval profile $A \in \mathcal{A}$, an agent permutation $\pi \in \Pi$ and an object ordering $\sigma \in \Sigma$ such that $\hat{f}^{\sigma\pi}(A)$ is not a maximum matching. Let $\hat{f}^{\sigma\pi}(A) = \mu$, for ease of notation. Let object $o \in O$ be the first object in $\sigma$ such that if the assignment of all previous objects before $o$ in $\sigma$ remained the same but agent $j$ was not assigned $o$, given that $\mu(j) = o$, then it would still be possible to have a maximum matching. Let $I^j$ be the set of agents who receive their assignments after $j$ when $\hat{f}^{\sigma\pi}$ is applied to $A$. Let $\bar{O}^j \subset O$ denote the set of objects assigned to agents in $I \setminus I^j$ by $\mu$. Then there exists $\tilde{I} \subseteq I^j$ such that $|\cup_{i\in\tilde{I}} A_i \cap \bar{O}^j| = |\tilde{I}|$, that is, there exists a subset $\tilde{I}$ of the agents who receive their assignment after $j$ such that the total number of remaining acceptable objects to agents in $\tilde{I}$, after all the agents before $j$ got their assignments is the same as the number of agents in $\tilde{I}$. Moreover, since a maximum matching is possible if $o$ is not assigned to $j$, while assigning $o$ to $j$ guarantees that the matching won't be a maximum matching, there is a feasible way to match each agent in $\tilde{I}$ to a remaining object (i.e., an object that is not in $\bar{O}^j$), and $o \in A_i$ for some $i \in \tilde{I}$. This means that $\tilde{I}$ and the objects assigned to the agents in $\tilde{I}$ in this feasible matching form in I-block when all the agents before $j$ have been assigned, and since $o$ is assigned to one of these agents, the minimal I-block for $o$ exists and is a subset of this I-block. Since agent $j$ is not in the I-block, $j$ is not in the minimal I-block for $o$ either, and thus the Successive Rule $\hat{f}^{\sigma\pi}$ cannot assign object $o$ to $j$ at interval profile $A$. This is a contradiction and thus the Successive Rules $(\hat{f}^{\sigma\pi})_{\sigma\in\Sigma,\pi\in\Pi}$ are maximal.

To show the second statement in the theorem, fix an interval profile $A \in \mathcal{A}$, an agent permutation $\pi \in \Pi$ and a maximum matching $\mu$ at $A$. We will show that we can construct an object ordering $\sigma \in \Sigma$ such that $\hat{f}^{\sigma\pi}(A) = \mu$. First identify agent $i \in I$ who has the shortest acceptance interval (i.e., among all agents, $i$ has the fewest objects in $A_i$). If this agent is not unique, break ties according to $\pi$. Let the first object in $\sigma$ be agent $i$'s object in $\mu$: $\sigma_1 = \mu(i)$. Remove $i \in I$ and $\sigma_1 \in O$ and repeat the same step in the reduced problem to find $\sigma_2$. And so on, we can construct the entire object ordering $\sigma \in \Sigma$ by repeating this step iteratively. Given this construction of $\sigma$, and noting that there is no minimal I-block for any object such that when an agent is assigned to this object by $\hat{f}^{\sigma\pi}$ that agent is not in the minimal I-block for the object, since $\mu$ is a maximum matching, it is easy to check that $f^{\sigma\pi}(A) = \mu$.

### 3.9.3 Appendix *C*

**Theorem 6** *There exists a serial dictatorship which is maximal if and only if the interval profile is solvable.*

**Proof:**

1. *If the interval profile is solvable then there exists a serial dictatorship which is maximal.*

   Fix a solvable interval profile $A \in \mathcal{A}$. Construct an agent permutation $\pi \in \Pi$ following the iterated block procedure applied to $A$ by letting removed flexible agents in the first step be last in $\pi$. If several flexible agents are removed in the first step they can be ordered arbitrarily. Continue iteratively in the reduced problem, making sure that flexible agents are ordered following the reversed order of the steps in which they were removed. This way we get a particular permutation $\pi$ of the set of agents $I$, since the solvable interval profile $A$ has an iterated block structure, which means that we can remove and hence order all agents in this manner.

   Now we show that the serial dictatorship based on $\pi$ leads to a maximum matching. By the definition of a flexible agent, regardless of how agents choose objects before the next agent chooses, there is always at least one object still available to a flexible agent, hence a reverse ordering of the flexible agents compared to the order in which they appear in the iterative block procedure guarantees a maximum matching. The ordering of flexible agents that are removed in the same step doesn't matter, since regardless of their ordering there is always at least one acceptable object remaining for each agent when it is her turn to receive an object. In particular, it is clear that each agent who is in an O-block gets an object no matter how the block agents in it are ordered, and if there are multiple O-blocks in the same step of the iterative block procedure then the definition of an O-block implies that there is no overlap regarding the objects in different O-blocks, so any arbitrary ordering of block agents in $\pi$ works, as long as the block agents were removed in the same step. This proves that the constructed $\pi$ leads to a maximum matching.

2. *If there exists a serial dictatorship which is maximal then the interval profile is solvable.*

   Suppose, by contradiction, that there exists a serial dictatorship with agent permutation $\pi$ which is maximal for an interval profile $A \in \mathcal{A}$, but the interval profile is not solvable. First apply the iterative block procedure to $A$ and remove iteratively all the flexible agents. Since $A$ is not solvable, there is an iteration such that the reduced problem with the remaining agents does not have any flexible agents (which may be the entire problem). Let the interval profile for the remaining problem be denoted by $\tilde{A}$ and let the remaining set of agents be denoted by $\tilde{I}$. $\tilde{A}$ is not solvable since there are no flexible agents remaining. We will show that there is no agent permutation for $\tilde{A}$ that leads to a maximum matching at every preference profile, which will then imply that there is no agent permutation for $A$ that leads to a maximum matching at every preference profile.

Fix an agent $i \in \tilde{I}$, and let this agent be the last one in $\tilde{\pi}$, which is a permutation of $\tilde{I}$. Since $i$ is not a (fully) flexible agent, $|\tilde{A}_i| \leq |\{j \in \tilde{I} \setminus \{i\} : A_i \cap A_j \neq \varnothing\}|$. Now consider only the objects in $\tilde{A}_i$ together with the agents in $\hat{I} \equiv \{j \in \tilde{I} \setminus \{i\} : \tilde{A}_i \cap A_j \neq \emptyset\}$. Note that this even further reduced profile with agents $\hat{I}$ and objects $\tilde{A}_i$ is a consecutive interval profile, since removing agents does not change the consecutiveness of the acceptance intervals for the remaining objects.

Given the common ranking $(o_1, \ldots, o_{\tilde{m}})$ of objects in $\tilde{A}_i$, consider first $N_{o_1} \cap \hat{I}$, the set of agents in $\hat{I}$ who have object $o_1$ in their acceptance interval. Let an agent in $N_{o_1}$ with the shortest acceptance interval rank $o_1$ first, breaking ties arbitrarily. Remove object $o_1$ together with the agent who ranks it first, and repeat the procedure in the reduced problem iteratively for $o_2, o_3$, and so on, until $o_{\tilde{m}}$ is reached. If for some object $o_t$ there is no agent remaining who has the object in her acceptance interval then the object remains unassigned, and the procedure moves on to the next object in the common ranking. Note that agents with their removed objects that they rank first, respectively, form a maximum matching for $\tilde{A}$ when we remove agent $i$, given that the Common-Ranking Successive Rule is maximal by Theorem 1.

We now show that each object $\tilde{A}_i$ is matched to an agent in $\hat{I}$ this way (but it is possible that not all agents in $\hat{I}$ are matched). Note first that for all $o \in \tilde{A}_i$, $N_o \cap \hat{I} \neq \emptyset$, since otherwise agent $i$ and $o$ would form an O-block in $\tilde{A}$, which is a contradiction. Suppose that some object $o \in \tilde{A}_i$ is unmatched. Since at least one agent $j \in \hat{I}$ has $o$ in her acceptance interval and the matching is a maximum matching, $j$ is matched to another object $o' \in \tilde{A}_i$. Then $o$ and $o'$ together are in at least two different agents' acceptance interval among agents in $\hat{I}$, otherwise agent $i$ would be a flexible agent for $\tilde{A}$ since objects $o$ and $o'$ in $\tilde{A}_i$ would only have one competing agent ($j$) in $\tilde{A}$, which is a contradiction. Therefore, there exists agent $j' \in \hat{I} \setminus \{j\}$ who has $o'$ in her acceptance interval but is matched to a different object, $o'' \in \tilde{A}_i$, given that we have a maximum matching. Since the sets of agents and objects keep expanding in each iteration, while $\tilde{I}$ and $\tilde{A}_i$ are finite, the iterative repetition of this argument leads to a contradiction. Therefore, it is possible to match each object $\tilde{A}_i$ to an agent in $\hat{I}$. Given that each matched agent in $\hat{I}$ ranks her assigned object first, applying a serial dictatorship to such a preference profile where agent $i$ is last in $\tilde{\pi}$ (a permutation of $\tilde{I}$) would not leave any object unassigned in $i$'s acceptance interval and agent $i$ would remain unmatched.

The above argument holds for an arbitrary agent $i \in \tilde{I}$, and thus no matter which agent is last in $\tilde{\pi}$ among the agents in $\tilde{I}$, we can find preferences for the remaining agents in the reduced problem for which the serial dictatorships based on $\tilde{\pi}$ does not lead to a maximum matching within the reduced problem. This proves that the serial dictatorship based on $\pi$ cannot be maximal when the interval profile is unsolvable.

### 3.9.4 Appendix $D$

**Theorem 7** *Given a solvable interval profile $A \in \mathcal{A}$, a serial dictatorship is maximal if and only if it is outcome equivalent to a Block Serial Dictatorship for $A$.*

**Proof:** Given the definition of Block Serial Dictatorships, Part 1 of the proof of Theorem 6 proves that a serial dictatorship applied to a solvable interval is maximal if it is a Block Serial Dictatorship. We need to prove the converse, that is, we need to show that if a serial dictatorship is maximal then it is outcome equivalent to a Block Serial Dictatorship.

Let $A \in \mathcal{A}$ be a solvable interval profile. Fix a maximal serial dictatorship $\pi \in \Pi$. Suppose that $\pi$ is not a Block Serial Dictatorship for $A$. Then there exist some pairs of agents $i, j \in I$ such that agent $i$ is a flexible agent for the first time in step $t$ of the iterated block procedure at $A$, agent $j$ is a flexible agent for the first time in step $t'$ of the iterated block procedure at $A$, $t < t'$ and $i$ precedes $j$ in $\pi$. Then, if $A_i \cap A_j = \emptyset$ for all such reversed agent pairs $i$ and $j$, then there exists a Block Serial Dictatorship $\pi'$ which is outcome equivalent to the maximal serial dictatorship $\pi$. If there exists a reversed pair $i, j$ such that $A_i \cap A_j \neq \emptyset$ then agent $j$ is not a flexible agent when all the agents preceding $j$ in $\pi$, including agent $i$, are still in the problem. This means that for all $\bar{A}_j \subseteq A_j$, $|\bar{A}_j| \leq |\{l \in I \setminus \{j\} : \bar{A}_j \cap A_l \neq \emptyset\}|$. Thus, there is a way for the preceding agents to choose their favorite objects, following the agent permutation $\pi$, such that $j$ does not have any acceptable objects left unassigned when it is $j$'s turn to receive an assignment. This means that the serial dictatorship $\pi$ is not maximal, which is a contradiction.

# Chapter 4

# Matching with Minimum Quotas

## 4.1 Introduction

The purpose of this study is defining a set of matching rules for the situations where the agents on both sides of the market have preferences/priorities over the other side agents and one side has minimum quotas. For instance, some places need at least a minimum number of participants to be able to run a business or open a course, such as professional workshop which needs a minimum number of registered people to be held or projects within companies which cannot be run unless a minimum number of employees are assigned to that project.

Standard matching rules and classic matching theory results do not apply in settings with minimum quotas (see Fragiadakis et al., 2015 [24]). When minimum quotas are binding, which is quite common in college admissions, course allocations, school choice, medical residency allocation, etc., regular matching institutions cannot be easily adopted without making serious compromises in terms of other important criteria, such as efficiency and stability of the matching outcomes. Typically, artificially low maximum quotas are exposed by designers to deal with such problems and this can significantly harm the efficiency, with some of the desired resources (such as school seats and internships) remaining unused. Such designs are far from being optimal, and the proposed algorithms satisfying minimum quotas either result in substantially limited fairness and efficiency given the priorities over agents or depend upon some predefined artificial priority ordering of agents which makes the allocation severely unfair and lopsided in many settings. I will use more sophisticated and innovative designs to suggest optimal ways of making practical allocation decisions. Although compromises are necessary to a certain extent, trade-offs can only be exactly described and hence minimized by thinking outside the box, as has been demonstrated by some of the preliminary findings of Fragiadakis and Troyan (2017, [23]) in a similar setting.

Ehlers et al. (2014, [20]) show that a fair and nonwasteful matching may not exist when we have the minimum quotas. Therefore, most of the existing literature have tried to deal with fairness and nonwastefulness separately. Up to now, all the algorithms proposed by papers working on minimum quotas work on some kind of weakened version of restricted fairness which not only looks at fairness beyond the minimum restrictions (they accept unfairness if making it fair violates the minimum constraints), but also adds some other constraints to consider a matching rule, fair. Besides, they do not use a unified framework to find fair or nonwasteful matchings. For example, Fragiadakis et al. (2015, [24]) provided two different classes of mechanism: 1) $ESDA$ (Extended Seat $DA$) to find strategy-proof fair matchings. 2) $MSDA$ (Multistage $DA$) to find strategy-proof nonwasteful matchings. They used a Precedence List ($PL$) of students and showed that $MSDA$ is $PL$-fair (fair with respect to $PL$). Both $ESDA$ and $MSDA$ algorithms may not find a matching which is simultaneously fair and nonwasteful even if such a matching exist.

Fragiadakis and Troyan (2017, [23]) introduced another algorithm, $DQDA$ (Dynamic Quotas $DA$) which reduces the capacity of schools to ensure that the minimum quotas for specific types will be satisfied after considering the preferences. Tomoeda (2017, [72]) showed that there always exists a fair and non-wasteful assignment if for each type of students, schools have common priority rankings over a certain number of bottom students.

I will propose innovative new matching rules that minimize the negative features and allow for constrained optimal matching outcomes, aiming for the best possible algorithm which compromises to achieve efficiency and fairness. Moreover, I will introduce a unified framework, based on the novel idea of constructing a representative directed graph that summarizes all pertinent information on preferences, in order to compare and contrast different matching rules, both already known and new ones that I will propose. The objectives are twofold: 1) provide a unified framework in which to analyze and compare all the relevant algorithms, 2) propose alternative new algorithms which are better in some ways. All this allows market designers to choose from an array of matching rules according to the unique criteria and objectives that apply in their specific real-life settings.

Using a unified framework, I identify algorithms which find a fair and nonwasteful matching whenever it exists, unlike $ESDA$ and $MSDA$ (Fragiadakis et al., 2015 [24]). Furthermore, my algorithms are more intuitive than the existing algorithms such as $ESDA$ and $MSDA$. In my method, I start by finding the range of possible assignments with and without the minimum quotas. Based on these assignments I construct a representative directed graph with feasible matched pairs as vertices. This will serve as a tool to find a sequence of matched pairs which does not violate the preferences of the agents on one side or the other side's priorities and, if possible, both. This graph contains two types of arcs, one pertaining to the preferences of one side of the market, and the other to the other side's priorities. The graph provides the basis for selecting a feasible matching which satisfies minimum quotas and allows to select either a fair or a nonwasteful matching. In addition, both criteria will be automatically satisfied whenever possible.

While existing applicable algorithms only partially deal with the issue of fairness versus nonwastefulness, I am able to find intuitive solutions, taking advantage of the clear structure of the graph, which exactly summarizes the information needed for making matching decisions in the presence of binding minimum quotas. This will provide an ideal framework to explore the trade-offs and the general compatibility of the nonwastefulness and fairness notions.

I study the structure of the graph regarding different preferences of agents and minimum quotas in order to gain a thorough understanding of the structure of the problem, which will allow me to create new mechanisms for matching in the presence of minimum quotas.

Based on my findings I prove that the graph can find fair and nonwasteful matchings, if such matchings exist. Furthermore, I characterize the specifics of the model and the preferences in terms of its representative graph, for which a fair and nonwasteful rule exists. This will constitute a substantial improvement over current results and will integrate and unify the existing algorithms.

Furthermore, in multicultural countries like Canada, respecting diversity is a crucial aim in many situations. Diversity is very important to keep the country as a united whole, bringing tons of benefits regarding different aspects. Therefore, to uphold diversity in terms of gender, race, etc. at educational institutions, many impose maximum and minimum quotas separately for different groups of students. Quotas that need to be filled minimally with specific types of agents who need protection (e.g. students from minority communities) are aimed at providing help and guaranteeing the balance of different types of agents at the same time. However, fulfilling these objectives may not be fair for all agents, such as for majority students, and may introduce inefficiencies even for those who are to be protected. Nevertheless, having minimum quotas is one of the most common constraints which matching models face in real life.

This project investigates how the normative principle of diversity and distributional constraints may shape institutions in matching allocation problems. It deals with the design of institutions whose outcomes meet desirable normative principles such as efficiency, equity, and stability, in addition to diversity, the focus of my study. Specifically, I will investigate the existence and capacity of matching rules to realize a diversity goal with the introduction of minimum quotas for different types of agents, where the diversified matching outcome is measured in terms of its type composition with respect to gender balance, the participation of religious or ethnic minorities, age equality, disability, etc. The main objective is to apply fundamentals of the matching literature framework to a diversity setting and analyze the implications of diversity requirements for the design and functioning of the matching allocation rules in practical applications, where the use of minimum quotas is both necessary and desirable to achieve the distributional goals. The diversity objectives will be considered together with efficiency, and fairness, and I will study and analyze algorithms that I propose in a unified setting.

Therefore, I also generalize my results to settings with multiple agent types, for which I will use a family of representative directed graphs, and all this will give us a new set of algorithms to choose from and a much better intuitive idea of what is possible for practical matching designers whose objective is to increase diversity and satisfy type-specific distributional constraints.

This study will hopefully serve as a foundation for future research which will generalize the newly designed algorithms to extend my findings to more complex matching models with minimum quotas.

I review the existing literature in Section 2. In Section 3 I introduce the main ideas and the model and its properties, with the aim of properly defining the model and its structure. In Section 4 I define the properties of the matching rules. Section 5 discusses the properties of graphs. In Section 6 I continue with defining my algorithms, $CNWF$ and $FCNW$. In Section 7 I introduce the theorems and results of my study. In Section 8 I show the relation between $ESDA$ algorithm and my algorithms. In Section 9 I show the extension of the algorithm to different types of the agents. Finally in Section 10 I conclude my research and results.

## 4.2 Literature Review

There exist lots of real-life matching situations which need to consider minimum quotas. However, there do not exist much work done in this field. On the other hand, a lot of work has been done on matching under constraints or having maximum constraints.

The first paper on this is Roth (1986, [60]), introducing the rural hospitals theorem. Abdulkadiroğlu et al. (2005, [1]) studied New York City school admission system. In order to keep the diversity of ability levels, New York City schools require Educational Option (*Ed Opt*) through accepting students from different ranges of test scores.

In the presence of minimum quotas, some algorithms have used an imposed artificial cap on the maximum capacities with an $DA$ algorithm (Artificial Cap $DA$ ($ACDA$)). For example, the $JRMP$ algorithm was used first by the Japanese government (2008), which reduces the capacity of hospitals to make sure a feasible matching with respect to the minimum quotas exists. The main limitation is that since the reduction does not consider the preferences, lots of seats will be left empty. Consequently, $ACDA$ is strongly wasteful. Kojima (2012, [42]) illustrated this problem that some types of affirmative action quotas not only do not help minorities who are supposed to be helped but also may actually hurt them. That is why, Hafalir et al. (2013, [26]) tried to address the issue by using the minority reserves. Some studies can be considered as a generalized version of Hafalir et al. (2013, [26]). For instance, Kominers and Sönmez (2016, [43]) whose work allows for slot-specific priorities and Ehlers et al. (2014, [20]) model which can be used if there are more than two types of students. Ehlers et al. (2014, [20]) studied diversity constraints in school choice and introduced a trading cycles-style mechanism similar in spirit to Erdil and Ergin (2008, [21]). Although their mechanism is not strategy-proof, they proved that there is no feasible mechanism that is strategy-proof, fair for same types and constrained nonwasteful. In addition, there is no feasible mechanism that is strategy-proof and fair across types. They introduced the impossibility theorem which is mentioned earlier. Based on this theorem a fair and nonwasteful matching may not exist when we have minimum quotas. Therefore, most of the existing literature have tried to deal with fairness and nonwastefulness separately.

Fragiadakis et al. (2015, [24]) considered a model similar to the diversity constraints model of Ehlers et al. (2014, [20]) but provided two different classes of mechanisms:

- $ESDA$ (Extended Seat $DA$) which divides each school with total capacity $Q$ into two schools, one with the minimum quotas ($l$) as the total capacity and the other (extended school) with $Q - l$ as the total capacity. They rank each extended school just after the original one and do not assign more than $n - \sum l_c$ students to all extended schools, letting the standard $DA$ assign students to schools ($n$ here is the number of students). They fix an ordering of the schools and let the schools accept students one by one in this order until $Q - l$ students have been accepted across all of the extended schools. They show that $ESDA$ is strategy-proof and fair.

- $MSDA$ (Multistage $DA$) which uses a pre-defined ranking of students (Precedence List ($PL$)). At each stage of $DA$ they reserve the $\sum l_c$ of students who have the lowest rank according to $PL$ and let the others be assigned to schools using a $DA$ algorithm. They prove $MSDA$ is strategy-proof, nonwasteful and $PL - fair$ (fair with respect to $PL$).

Fragiadakis and Troyan (2017, [23]) introduced another algorithm, Dynamic Quotas $DA$ ($DQDA$), which reduces the capacity of schools to ensure that the minimum quotas for specific types will be satisfied after considering the preferences. They showed $DQDA$ eliminates justified envies among same types and it is strategy-proof. Clearly because it considers the preferences probably it is nonwasteful or at least less wasteful than $ACDA$. Furthermore, Tomoeda (2017, [72]) studied matching with minimum quotas when there is more than one type of agents. He investigated the structure of schools' priority rankings which guarantees stability and shows that there always exists a fair and nonwasteful assignment if for each type of students, schools have common priority rankings over a certain number of bottom students. Type-specific minimum quotas have been studied in the matching problems between students and supervisors by Kawagoe and Matsubae (2017, [36]). Kamada and Kojima (2018, [35]) focused on feasible, individually rational, and fair matchings and characterized the class of constraints on individual schools under which there exists a student-optimal fair matching. Yahiro, et al. (2018, [75]) introduced a new type of distributional constraints called ratio constraints and developed a novel mechanism called Quota Reduction Deferred Acceptance, which repeatedly applies the standard $DA$ by sequentially reducing artificially introduced maximum quotas. Westkamp (2013, [74]) studied complex (maximum) quota constraints in the German university admissions system. An experimental analysis of the same system was conducted by Braun et al. (2014, [12]).

Furthermore, there are other studies carried out on matching with minimum quotas, including some in computer science. First of all, the medical residency market was studied by Roth (1984, [59]) followed by Martinez et al. (2000, [49]), Hatfield and Milgrom (2005, [29]) and Biró et al. (2010, [10]). Although Biró et al. have not introduced any explicit mechanism, they studied the difficulty of finding stable matchings in the presence of minimum quotas of the Hungarian college admission system. In addition, Kamada and Kojima (2015, [34]) analyzed the method used in Japan: capping the number of residents who can be assigned to a given region. Hamada et al. (2016, [28]) studied matching with minimum quotas in the hospital residency problem.

Li (2017, [47]) defined obvious strategy-proofness. He looked for a strategy such that every outcome under this strategy is (weakly) better than every outcome under other strategies. Given this definition, very few mechanisms will be obviously strategy-proof. Therefore, Troyan and Morrill (2020, [73]) introduced a new approach to define obvious manipulations which relax strategy-proofness instead of strengthening it.

## 4.3 The Model

I consider a many-to-one matching model which matches students to schools when both sides of the market have preferences/priorities. Schools have minimum quotas and all agents on one side are acceptable to the agents on the other side. I consider the minimum quotas as one of the feasibility constraints.

A market is given by $(S, C, p, q, \succ_S, \succ_C)$.

It consists of:

- $n$ students; $S = \{s_1, s_2, \ldots, s_n\}$ and $m$ schools; $C = \{c_1, c_2, \ldots, c_m\}$.
- $q = \{q_{c_1}, q_{c_2}, \ldots, q_{c_m}\}$ is the list of maximum capacities for each school.
- $c_j$ can be assigned to maximum $q_{c_j}$ students where $j \in \{1, 2, \ldots, m\}$.
- $s_i$ can only be assigned to one school where $i \in \{1, 2, \ldots, n\}$.
- $p = \{p_{c_1}, p_{c_2}, \ldots, p_{c_m}\}$ is the list of minimum quotas for each school.
- $p_c \geq 0$, $q_c > 0$, $p_c \leq q_c$ and $n \geq q_c + \sum_{c' \neq c} p_{c'}$ for all $c \in C$.
- $\sum_{c \in C} p_c < n < \sum_{c \in C} q_c$.
- $\succ_S$ (Students' preferences): Each student $s$ has a strict preference relation $\succ_s$ over $C$.
- $\succ_C$ (Schools' priorities): Each school $c$ has a strict priority ordering $\succ_c$ over $S$.

## 4.4 Properties of Matching Rules

The properties of matching rules which will be used in this chapter are introduced in this section. All these properties are standard properties that have been studied extensively in various models.

**Definition:** *A matching is a mapping $\mu : S \rightarrow C$ that satisfies;*

1. *$\mu(s) \in C$ for all $s \in S$,*
2. *$\mu(c) \subseteq S$ for all $c \in C$,*
3. *for any $s \in S$ and $c \in C$, if student $s$ has been assigned to school $c$ in $\mu$, we have $\mu(s) = c$ if and only if $s \in \mu(c)$.*

**Definition:** *A matching $\mu$ is feasible if $p_c \leq |\mu(c)| \leq q_c$ for all $c \in C$ under $\mu$.*

In other words, a feasible matching is a matching which respects schools' capacities and minimum quotas.

**Definition:** *A matching rule is a function that assigns a matching to each preference profile. Formally, a matching rule is $f : R \rightarrow M$*

**Property 1:** *Student $s$ justifiably claims an empty seat at school $c$ under a matching $\mu$ if $c \succ_s \mu(s)$ and $|\mu(c)| < q_c$.*

**Property 2:** *A matching $\mu$ is* **nonwasteful** *if there is no student with a justifiable claim for an empty seat under $\mu$.*

- A feasible matching is nonwasteful if no student prefers a school with one or more empty seats to her assignment and whenever a student $s$ justifiably claims an empty seat at school $c$, we have $|\mu(\mu(s))| = p_{\mu(s)}$. This means that in a feasible nonwasteful matching justifiable claims are allowed only if the student is needed to satisfy a minimum quota.

- A feasible nonwasteful matching is called a **Constrained Nonwasteful ($CNW$)** matching.

**Property 3:** *Student $s$ justifiably envies student $s'$ at school $c$ under matching $\mu$ if $c \succ_s \mu(s)$ and $s \succ_c s'$ for some $s' \in \mu(c)$.*

**Property 4:** *A matching $\mu$ is* **fair ($F$)** *if there is no student who justifiably envies another student.*

**Definition 1:** $F\&CNW$ matching
*I call a matching which is both fair and constrained nonwasteful an $F\&CNW$ matching.*

**Property 5:** *A matching rule $f$ is* **strategy-proof ($SP$)** *if there do not exist $i, \succ$ and $\hat{\succ}_i$ such that $f_i(\hat{\succ}_i, \succ_{-i}) \succ_i f_i(\succ)$.*

I present the impossibility theorem which shows that a feasible, fair and nonwasteful assignment does not exist for some preferences when there are minimum quota restrictions.

**Impossibility Theorem (Ehlers et al., 2014 [20]):** In the presence of minimum quotas, the set of feasible assignments that are both fair and nonwasteful may be empty.

This can be seen from the following simple example:

**Example 1**

Students' preferences:

| $s_1$ | $s_2$ |
|---|---|
| $c_2$ | $c_3$ |
| $c_3$ | $c_2$ |
| $c_1$ | $c_1$ |

Schools' priorities and capacities:

| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| | $s_2$ | $s_2$ | $s_1$ |
| | $s_1$ | $s_1$ | $s_2$ |
| $p$ | 1 | 0 | 0 |
| $q$ | 1 | 1 | 1 |

School $c_1$ has a minimum quota which needs to be filled. Therefore, there are only 4 feasible matchings:

1. $\mu$: If student $s_1$ is assigned to $c_1$ then $s_2$ can be assigned to her top choice, $c_3$. $\mu$ is constrained nonwasteful but it is not fair since $c_3 \succ_{s_1} c_1$ and $s_1 \succ_{c_3} s_2$.

2. $\mu'$: If student $s_2$ is assigned to $c_1$ then $s_1$ can be assigned to her top choice, $c_2$. $\mu'$ is constrained nonwasteful but it is not fair since $c_2 \succ_{s_2} c_1$ and $s_2 \succ_{c_2} s_1$.

3. $\nu$: If student $s_2$ is assigned to $c_1$ then $s_1$ can be assigned to $c_3$. $\nu$ is fair but it is not constrained nonwasteful since $c_2 \succ_{s_1} c_3$ and there is an empty seat at $c_2$. Although $s_1$ could be assigned the empty seat at $c_2$ without violating the minimum quota requirements, this would make the assignment unfair.

4. $\nu'$: If student $s_1$ is assigned to $c_1$ then $s_2$ can be assigned to $c_2$. $\nu'$ is fair but it is not constrained nonwasteful since $c_3 \succ_{s_2} c_2$ and there is an empty seat at $c_3$. Although $s_2$ could be assigned the empty seat at $c_3$ without violating the minimum quota requirements, this would make the assignment unfair.

It can be seen that in this case there is no matching which is $F\&CNW$. ▲

## 4.5 Properties of Graphs

In this section I review some definitions related to graph theory and introduce some new definitions that will be used later.

**Definition: Directed graph**
A digraph or a directed graph, is a graph which has a non-empty set of vertices and a set of directed arcs or edges which is a set of ordered pairs of distinct vertices.

**Definition: In-degree and Out-degree**
The in-degree of a vertex is the number of arcs aimed to it and out-degree is the number of directed edges (arcs) which leave this vertex.

**Definition: Planar graph**
Planar graph is a graph that can be drawn on the plane in such a way that its edges (arcs) intersect only at their end or start points.

**Definition: Chromatic number**
The Chromatic number of a graph is the smallest number of colors needed to color its vertices so that no two adjacent vertices (vertices sharing one arc) share the same color.

**Definition: Topological order**
Topological sort or topological ordering of a directed graph is a linear ordering of its vertices such that for every directed edge $uv$ from vertex $u$ to vertex $v$, $u$ comes before $v$ in the ordering.

**Definition: Directed acyclic graph**
A directed acyclic graph ($DAG$) is a directed graph with no directed cycles. A directed graph is a $DAG$ if and only if it can be topologically ordered, by arranging the vertices as a linear ordering that is consistent with all edges directions[1].

**Definition: Dangling vertex**
A dangling vertex is a vertex with the out-degree of zero.

Based on the dangling vertex definition, I introduce two new concepts:

[1]There are algorithms which in linear time find a topological ordering of a given $DAG$. Kahn's algorithm (1962, [33]) is one of them.

**Definition 2:** Two-connected dangling vertex

*A two-connected dangling vertex is a vertex with out-degree zero and in-degree two.*

**Definition 3:** One-connected dangling vertex

*A one-connected dangling vertex is a vertex with out-degree zero and in-degree one*

## 4.6 Two New Mechanisms: $CNWF$ and $FCNW$

Based on Ehlers et al. (2014, [20]), we know that there is no feasible mechanism that is strategy-proof, fair and nonwasteful.

Extended Seats Deferred Acceptance ($ESDA$) and Multi Stage Deferred Acceptance ($MSDA$), proposed by Fragiadakis et al. (2015, [24]), are $SP$ algorithms that provide fair and efficient (nonwasteful) matchings, respectively, but not both. That is, even if an $F\&CNW$ matching exists, $ESDA$ and $MSDA$ may not choose it. A further shortcoming of these two rules is that there is no clear relationship between them. In addition, $MSDA$ uses a predefined list of students which indeed weakens the fairness.

The main focus of my work is to fill this gap by providing an innovative and unified design framework which allows for pinpointing the trade-offs and lets the designer prioritize over the criteria of fairness and nonwastefulness, while also automatically selecting a fair and nonwasteful solution when it exists. My algorithms, $CNWF$ (Constrained Nonwasteful Fair) and $FCNW$ (Fair Constrained Nonwasteful) are not $SP$ but find the matchings which are both fair and $CNW$ whenever they exist and if such a matching does not exist, they select the constrained nonwasteful or fair matchings respectively, depending on the designer's objectives. We can even find a matching which compromises between fairness and nonwastefulness (with different degrees of $CNW$ and fairness).

### 4.6.1 $CNWF$ and $FCNW$ Algorithm

Let $\pi$ be a permutation of $S$ which I use as a tie-breaker.

- Step 1:

  Start from using the schools maximum capacities with a standard $DA$ to find a matching $\mu$ regardless of the minimum quotas.

- Step 2:

  Let $um^1$ be the number of seats that need to be filled in order to satisfy the minimum quotas in $\mu$ found in Step 1.[2] If $um^1 = 0$, $\mu$ satisfies the minimum quotas, and the standard $DA$ matching $\mu$ becomes the final matching. If $um^1 > 0$, use

[2] $um^1$ is the number of unsatisfied minimum quotas after Step 1.

$q'_{c_j} = max\{p_{c_j}, |\mu(c_j)|^1 - um^1\}$ as the schools' capacities and find the matching $\mu'$ using a standard $DA$. Here, $q'_{c_j}$ is the max of $p_{c_j}$ which is school $c_j$'s minimum quota and $|\mu(c_j)|^1$ which is the number of students matched to school $c_j$ in Step 1.

- Step 3:

  Let student $s_i$ be assigned to school $c_j$ under $\mu$ and $c_{j'}$ under $\mu'$. Then, student $s_i$ can be matched to one of the schools in the range of $(c_j \dots c_{j'})$ according to her preferences. Therefore, we will have a set of potential matched pairs where each pair consists of a student and one of the schools in her range of $(c_j \dots c_{j'})$.

  If a student has only one school in her range assign that student to the school and remove her from the game provided that the match is feasible. Furthermore, if only one student has a school in her range, assign the student to this school and remove both of them from the market only if either the school has an unsatisfied minimum quota or it is the best remaining choice for the student provided that the match is feasible. Update the unsatisfied minimum quotas and maximum capacities.

- Step 4:

  Draw the following directed graph of all possible matches (I call this graph a $CNWF$ and $FCNW$ representative graph or simply a $CNWF$ and $FCNW$ graph):

  - Vertices: the school-student pairs found in Step 3 are the vertices.[3]

  - Arcs: We have two sets of directed arcs:
    1. $NW$-chain: One set of arcs which shows the student's preferences. For each student $s_i$ there is an arc which goes from vertex $(s_i, c_j)$ to vertex $(s_i, c_{j'})$ if $c_{j'} \succ_{s_i} c_j$ and $c_j$ is $c'_j$ 's immediate neighbor in $s_i$'s preference rankings.
    2. $F$-chain: The other set of arcs which indicates the schools' priorities. For each school $c_j$ there is an arc which goes from vertex $(s_i, c_j)$ to vertex $(s_{i'}, c_j)$ if $s_{i'} \succ_{c_j} s_i$ and $s_i$ is $s'_i$ 's immediate neighbor in $c_j$'s preference rankings

- Step 5:
  Check the graph:

  - Round $k$:

    * Case 1: When we consider arcs in both NW-chains and F-chains[4], if there is no cycle then there is an $F\&CNW$ matching. If there are some cycles but

[3] Write down all the possible matches for each student $s_i$ according to her preferences (if student $s_i$ prefers $c_j$ to $c'_j$, $c_j$ comes before $c'_j$). This list must indicate schools' ranking according to the students' preferences. For example; $s_2 : c_2, c_3$ means $c_2$ and $c_3$ are two possible matches for student $s_2$ while $s_2$ prefers $c_2$ to $c_3$.

[4] Note that just one set cannot have a cycle due to transitivity

there is at least one vertex with out-degree zero (a dangling vertex), there may be an $F\&CNW$ matching.
Start from the end points (top vertices) of the $F$-chains. Choose a vertex with out-degree zero and assign the student in the pair to the school in the pair (if the match is feasible).

* Case 2: If all vertices are in some cycles so that there is no vertex with out-degree zero then there is no $F\&CNW$ matching. Then the two algorithms are as follows.

  1. $FCNW$ : Start from the end points (top vertices) of the $F$-chains. If taking the student-school pairs of the endpoints as final assignments satisfies the minimum quotas, assign all students in the pairs to the schools in the pairs. Otherwise, choose a vertex with the smallest out-degree and make the corresponding assignment.
  2. $CNWF$ : Start from the end points (top vertices) of the $NW$-chains. If taking the student-school pairs of the endpoints as final assignments satisfies the minimum quotas, assign all students in the pairs to the schools in the pairs. Otherwise, choose a vertex with the smallest out-degree and make the corresponding assignment.

* If there is more than one vertex with the same out-degree assign all of the students in the vertices to the schools in them as long as feasibility is not violated. If you need to choose between them, then choose the one with the highest in-degree[5]. In the case of having the same in-degree, the vertex which includes a school with the highest minimum quota has the higher rank. Otherwise, choose a vertex based on $\pi$.

* All the vertices which include the already matched students will be removed from the market as well as the schools which have reached their maximum capacity.

* Update the remaining unsatisfied minimum quotas and the maximum capacities after each round.

– Iterate round $k$.

- The procedure terminates when all students are assigned, or all capacities are filled.

To further clarify how the algorithms work, here is a simple example.

[5]Except in the case of same zero out-degree. In this case one-connected dangling vertex is prioritized over a two-connected dangling vertex since it is the only choice for the school in the pair (provided that the match is feasible).

**Example 2**

Schools' priorities:

| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| | $s_5$ | $s_3$ | $s_3$ |
| | $s_3$ | $s_4$ | $s_4$ |
| | $s_1$ | $s_1$ | $s_2$ |
| | $s_2$ | $s_2$ | $s_5$ |
| | $s_4$ | $s_5$ | $s_1$ |
| $p$ | 1 | 1 | 1 |
| $q$ | 2 | 3 | 1 |

Students' preferences:

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|
| $c_2$ | $c_2$ | $c_1$ | $c_2$ | $c_1$ |
| $c_1$ | $c_1$ | $c_2$ | $c_3$ | $c_2$ |
| $c_3$ | $c_3$ | $c_3$ | $c_1$ | $c_3$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

$CNWF$ and $FCNW$:

- Step 1:

| $Rounds\backslash Schools$ | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| 1 | $\underline{s_5}, \underline{s_3}$ | $\underline{s_1}, \underline{s_2}, \underline{s_4}$ | |
| $q$ | 2 | 3 | 1 |

- Step 2: $um^1 = 1 > 0$

| $Rounds\backslash Schools$ | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| 1 | $\underline{s_5}, s_3$ | $s_1, s_2, \underline{s_4}$ | |
| 2 | $\underline{s_5}, s_2, s_1$ | $\underline{s_3}, s_4$ | |
| 3 | $\underline{s_5}$ | $\underline{s_3}$ | $s_2, s_1, \underline{s_4}$ |
| $q'$ | 1 | 1 | 1 |

- Step 3:

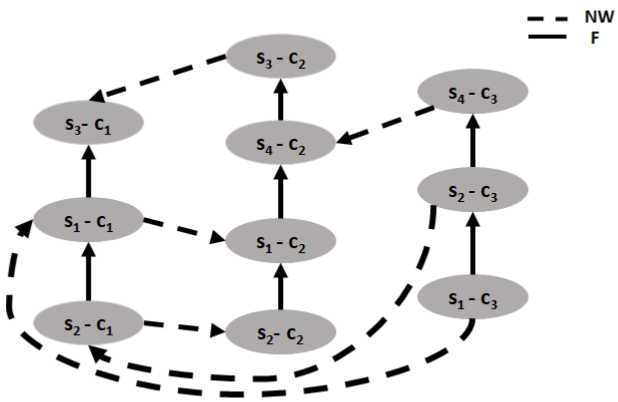
| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| | $s_5$ | $s_3$ | $\cancel{s_3}$ |
| | $s_3$ | $s_4$ | $s_4$ |
| | $s_1$ | $s_1$ | $s_2$ |
| | $s_2$ | $s_2$ | $\cancel{s_5}$ |
| | $\cancel{s_4}$ | $\cancel{s_5}$ | $s_1$ |
| $p$ | 1 | 1 | 1 |
| $q$ | 2 | 3 | 1 |

| $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|
| $c_2$ | $c_2$ | $c_1$ | $c_2$ | $c_1$ |
| $c_1$ | $c_1$ | $c_2$ | $c_3$ | $c_2$ |
| $c_3$ | $c_3$ | $c_3$ | $c_1$ | $c_3$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

$s_5$ only has one school in her range of possible matches. Therefore, $(s_5, c_1)$ is the first assignment. We can assign them to each other and remove $s_5$ from the market.

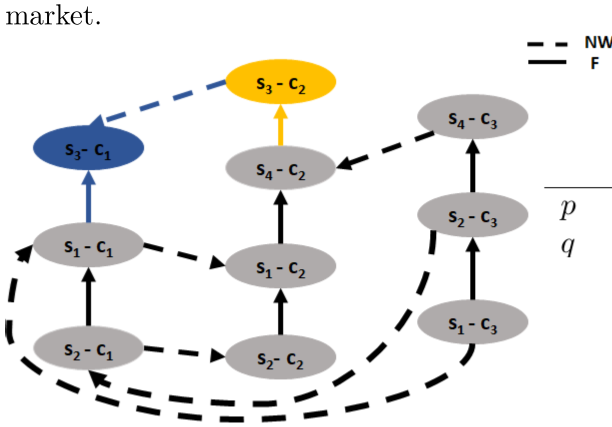
- Step 4:

  The $CNWF$ and $FCNW$ graph:



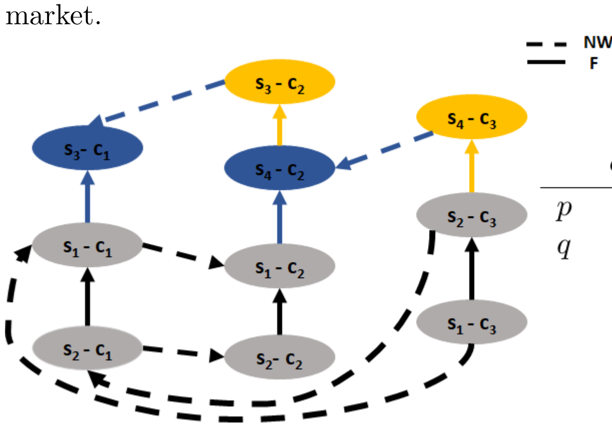

- Step 5:

  – Round 1: $(s_3, c_1)$ has out-degree zero. Assign $s_3$ to $c_1$ and remove $s_3$ from the market.



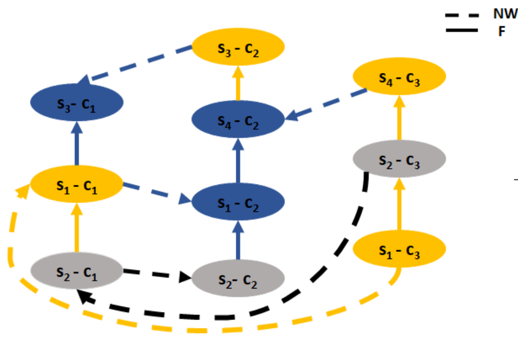

| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| $p$ | 0 | 1 | 1 |
| $q$ | 0 | 2 | 1 |

  – Round 2: $(s_4, c_2)$ has out-degree zero. Assign $s_4$ to $c_2$ and remove $s_4$ from the market.




| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| $p$ | 0 | 0 | 1 |
| $q$ | 0 | 1 | 1 |

  – Round 3: $(s_1, c_2)$ has out-degree zero. Assign $s_1$ to $c_2$. Finally, assign $s_2$ to $c_3$.




| | $c_1$ | $c_2$ | $c_3$ |
|---|---|---|---|
| $p$ | 0 | 0 | 1 |
| $q$ | 0 | 0 | 1 |

– The final matching using $CNWF$ and $FCNW$ is $\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_3, s_5 & s_1, s_4 & s_2 \end{pmatrix}$, which is F&CNW.▲

### 4.6.2 Methods to Speed up the Algorithms

There are some steps which can help increase the speed of the algorithms.

Let $\pi$ be a permutation of students which I use as a tie-breaker and $r^{s_i}_{c_j}$ be the rank of $s_i$ for $c_j$;

1. Minimum quotas restrictions may decrease the maximum capacity of schools. Then, before running the algorithms we can modify the maximum capacities by replacing them with real capacities.

**Definition 4:** Real capacity ($req$)

*The real capacity of a school $c_j$ is:*

$$req_{c_j} = min\{q_{c_j}, n - \sum_{j \neq j'} p_{c_{j'}}\}$$

2. At the beginning of Step 1, we can remove certain matches.

**Definition 5:** Certain Match

*For any student $s_i$ and school $c_j$ such that $c_j \succ_{s_i} c_{j'}, \forall c_{j'} \in C, j' \neq j$;*

- *If $r^{s_i}_{c_j} \leq p_{c_j}$ then $(s_i, c_j)$ is a certain match.*
- *If $r^{s_i}_{c_j} \leq req_{c_j}$ and $r^{s_i}_{c_j} \leq n - \sum_{j \neq j'} p_{c_{j'}}$ then $(s_i, c_j)$ is a certain match.*[6]

[6] If there is more than one student with the same situation, we choose students based on $\pi$ as long as the feasibility condition is satisfied (use $n'$ instead of $n$ which shows the number of students remaining after each assignment).

- *One-connected dangling vertex of an $F$-chain is a certain match (a student who only has one possible school to match).*

3. After each step, we can eliminate the forbidden matches, which are implied by fairness.

**Definition 6:** Forbidden match

*If a student $s_i$ is matched to school $c_j$, the following matches are forbidden:*

- $\forall c_{j'}, j' \neq j$ *and* $\forall s_{i'}, i' \neq i$ *such that* $c_{j'} \succ_{s_i} c_j$ *and* $s_i \succ_{c_{j'}} s_{i'}$, $(s_{i'}, c_{j'})$ *is a forbidden match.*
- $\forall s_{i'}$ *such that* $s_{i'} \succ_{c_j} s_i$, $s_{i'}$ *cannot be matched to any school which she ranks lower than* $c_j$ ($\forall s_{i'}, c_{j'} : s_{i'} \succ_{c_j} s_i$ *if* $r^{c_{j'}}_{s_{i'}} < r^{c_j}_{s_{i'}}$, $(s_{i'}, c_{j'})$ *is a forbidden match where* $r^{c_j}_{s_{i'}}$ *is rank of* $c_j$ *for* $s_i$*).*

4. After each step, we can eliminate the infeasible matches.

$CNWF$ and $FCNW$ find the range of the possible matchings based on $DA$ algorithm. That means, students propose according to their preferences and schools accept them based on their priorities provided that the minimum quotas and real capacities are not violated. Therefore, when we want to find an $F\&CNW$, $CNW$ or $F$ matching, $CNWF$ and $FCNW$ are not just assigning students to schools with respect to students' preferences and schools' priorities with the aim of finding the $F$ or $CNW$ matchings. Minimum quotas are imposing some restrictions on the matching problem, making some matchings infeasible.

**Definition 7:** Infeasible Match

*There are some fair, $CNW$ or $F\&CNW$ matchings which can be selected if using a standard DA algorithm; however, they are not feasible due to the minimum quotas and school capacities.*

- Let $um^t$ be the sum of unsatisfied minimum quotas and $n^t$ be the number of unassigned students at Step $t$. If assigning $s_i$ to $c_j$ makes $um^t > n^t$ then $(s_i, c_j)$ is an infeasible match.
- Let $|\mu(c_j)^t|$ be the number of assigned students to $c_j$ in Step $t$ and $|\mu(c_j)^t| = req_{c_j}$. Assigning another student, $s_i$, to $c_j$ is not feasible. Then, $(s_i, c_j)$ is an infeasible match.

The following is an alternative way of describing fair assignments in terms of the $CNWF$ and $FCNW$ graph.

**Definition 8:** Superiors' priority

*A matching $\mu$ satisfies the Superiors' priority if in all $NW$-chains which include one of the vertices in $\mu$, none of the vertices superior to each of the $\mu$ vertices (assignments) is superior to another vertex of $\mu$ in an $F$-chain.*

A matching which satisfies the superior's priority is fair. Therefore, if we are looking for fair matchings by going backward through the $F$-chains (backward trips starting from the top vertex of the chains and moving toward the vertices at the bottom of the chains), the found feasible matching $\mu$ is fair if the superiors' priority is not violated. This means that even if the end points (top vertices) of the graphs' chains satisfy the minimum quotas, we may have more than one fair matching by conducting the backward trips through $F$-chains provided that the superior's priority is satisfied. Removing forbidden matches satisfies the superiors' priority. We can simply show that there is always a fair matching.

In the case of $CNW$ matchings, moving backward through the $NW$-chains may lead us to more $CNW$ matchings as long as the matching is feasible (minimum quotas and school capacities are respected).

### 4.6.3 Observations

1. A one-connected dangling vertex is part of an $NW$-chain. It cannot be part of a $F$-chain because with out-degree zero and in-degree one, the vertex includes a student which has only one possible school to match which is a certain match and cannot be part of the graph.

2. The maximum out-degree of each vertex is two. The maximum in-degree of each vertex is two. These are due to the fact that each vertex can be on at most one $F$-chain and at most one $NW$-chain which result from the students' preference ordering and the schools' priority orderings.

3. The graph shows the existence of a fair and $CNW$ matching (Theorem 2 will prove that if there is at least one vertex with out-degree zero at each round of Step 5 then, there exists at least one $F\&CNW$ matching).

4. There may exist more than one $F\&CNW$ matching if there is more than one vertex with out-degree zero in some rounds.

5. Removing infeasible matches satisfies constrained nonwastefulness.

   The only reasons why we cannot have an $NW$ match are the minimum quotas or maximum capacities, therefore if we respect the feasibility constraints we can always find a $CNW$ match. I defined infeasible match as a match that violates schools capacities or minimum quotas. Therefore, if we remove such a vertex[7] from the graph after each round (after each assignment), the final matching will be $CNW$.

6. Removing forbidden matches satisfies fairness. I defined a forbidden match as a match that causes justify envies. Therefore, removing the vertices which include forbidden matches from the graph will eliminate justify envies and satisfies fairness.

[7]The vertex that includes the infeasible match.

7. In the case of ties (having more than one vertex with the same out-degree and in-degree which includes schools with the same amount of unsatisfied minimum quotas), if we use different tie-breakers (a permutation of students which is used to select the assignments in case of ties), we can reach all $F\&CNW$, fair or $CNW$ matchings whenever they exist.

## 4.7 Theorems and Results

**Theorem 1** *$CNWF$ and $FCNW$ select a feasible matching at each profile.*

**Proof:** A feasible matching is a matching which satisfies the minimum quotas and total capacities of schools. Therefore, based on the construction of the $CNWF$ and $FCNW$, the algorithms select possible matchings separately for maximum and minimum capacities of schools, allowing to choose the feasible matchings. In the basic algorithms we update the schools' capacities and unsatisfied minimum quotas after each round to make sure that the feasibility constraints are not violated while in the speedy version of the algorithms eliminating the infeasible matches at each round guarantees selecting only the feasible matchings. ■

**Theorem 2** *If there is at least one vertex with out-degree zero after the assignment in each round then there is at least one matching which is $F\&CNW$.*

**Proof:** Each vertex includes a matched school-student pair. Out-degree zero for the vertex means that the student has the highest priority for the school among the remaining students and also she prefers this school over the remaining schools. Such a matching is clearly $F\&CNW$. ■

**Corollary to Theorem 2** *$CNWF$ and $FCNW$ always select an $F\&CNW$ matching whenever such a matching exists.*

**Proof:** The procedure of the algorithms carries on by selecting the vertices with the lowest out-degree. This means the vertices which have the out-degree zero will be the first ones to be selected. If there is at least one vertex with out-degree zero at each step then there is at least one $F\&CNW$ matching which the algorithms will select. ■

**Theorem 3** *$CNWF$ and $FCNW$ select a $CNW$ or fair matchings respectively whenever there is no $F\&CNW$ matching.*

**Proof:** Based on the construction of the algorithms, if there is no $F\&CNW$ matching, $FCNW$ and $CNWF$ find fair and $CNW$ matchings using the end points (top vertices) of $F$-chains or $NW$-chains or through the backward trips respecting the superiors' priority and feasibility constrains. ■

$CNWF$ and $FCNW$ eliminate some fair or $CNW$ matchings because they use a range of possible matches which may not include some feasible matches. Theorem 4 proves that the matching selected by my algorithms maximizes the property that it cannot satisfy.

Thus, the ranges do not eliminate the feasible matchings that are of interest.

I define the maximum degree of nonwastefulness as the minimum number of feasible claims for empty seats and the maximum degree of fairness as the minimum number of students with justified envy.

**Theorem 4** *$FCNW$ selects fair matchings with the maximum degree of nonwastefulness. $CNWF$ selects $CNW$ matchings with the maximum degree of fairness.*

Theorem 4 is proved in Appendix $A$.

### 4.7.1 Properties of the $CNWF$ and $FCNW$ Graph

The $CNWF$ and $FCNW$ graph has some interesting properties. The following theorems focus on these properties.

**Theorem 5** *The maximum chromatic number of the $CNWF$ and $FCNW$ graph is three.*

Theorem 4 is proved in Appendix $B$.

**Theorem 6** *The $CNWF$ and $FCNW$ graph is planar.*

Theorem 5 is proved in Appendix $C$.

Therefore, the $CNWF$ and $FCNW$ graph is planar with chromatic number three. Moreover, if there is an $F\&CNW$ matching then the $CNWF$ and $FCNW$ graph has some extra properties. Before dealing with these unique properties let me define some new concepts.

Recall that if there is at least one vertex with out-degree zero at each round, then there exists at least one $F\&CNW$ matching. However, as already mentioned, an $F\&CNW$ matching may exist even if the $CNWF$ and $FCNW$ graph contains some cycles. To cover such cases and clarify them, I introduce some new definitions.

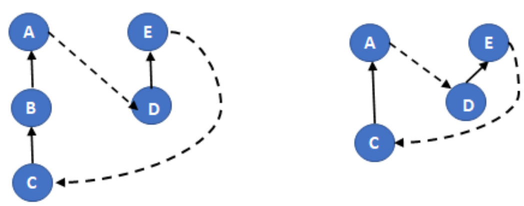
**Definition 9:** Removable vertex

*A vertex is called removable if it includes a student who has been already assigned or a school with full capacity. A vertex including a pair whose matching is not possible due to the unsatisfied minimum quotas is also removable.*

**Definition 10:** Cycle breaker vertex

*A removable vertex is called* **cycle breaker** *if it is part of a cycle such that its incoming edge in the cycle is in an $F$-chain and its outgoing edge in the cycle is in an $NW$-chain or vice versa*[8]*. Therefore, removing this vertex and consequently the corresponding edges will break the cycle.*

If the incoming and outgoing edge of a removable vertex are both from the same chain,[9] removing this vertex cannot break the cycle, it just makes the cycle shorter since it does not change the ordering of the vertices in the chain to which it belongs. In the following example, removing vertex $B$ from cycle ($a$) only transforms the cycle to a shorter one, cycle ($b$), while removing any other vertex will break the cycle.




cycle (a)          cycle (b)

**Definition 11:** Breakable cycle

*A cycle is called breakable if it contains at least one cycle breaker vertex.*

Note that not all the cycles are breakable since they may not contain a removable vertex which is also a cycle breaker and we know that a cycle breaker vertex is needed to break a cycle while a cycle breaker vertex itself needs to be a removable vertex in the first place.

**Definition 12:** Semi-topological order

*A semi-topological sort or semi-topological ordering of a directed graph is a linear ordering of its vertices such that for each directed edge $uv$ from vertex $u$ to vertex $v$, $u$ comes before $v$ in the ordering. If there is any breakable cycle, all vertices in this cycle will appear in the ordering as a whole. Let us call this subset of vertices a cyclic clique. Therefore, all vertices outside this cycle which are the starting point of a directed arc aiming to one of the vertices in the breakable cycle will come before the cyclic clique and all vertices outside the cycle which are the end point of a directed arc leaving one of the vertices in the breakable cycle will come after the cyclic clique.*

[8](for example; $\dashrightarrow \circ \rightarrow$)
[9](for example; $\rightarrow \circ \rightarrow$)

**Definition 13:** Directed semi-acyclic graph ($DSAG$)

*A directed graph is a $DSAG$ if and only if it can be semi-topologically ordered.*

I conclude all general and specific properties of the $CNWF$ and $FCNW$ graph when there is an $F\&CNW$ matching in the next theorem.

**Theorem 7** *A matching is $F\&CNW$ if and only if the $CNWF$ and $FCNW$ representative graph of its corresponding matching problem is:*

*-planar directed semi-acyclic graph,*
*-the maximum chromatic number of the graph is three,*[10]
*-there is at least one vertex with out-degree zero.*[11]

Theorem 6 is proved in Appendix $D$.

## 4.8 $ESDA$ and the $FCNW$ and $CNWF$ Graph

In this section I discuss the relation between my algorithms and $ESDA$ algorithm and show how we can get the $ESDA$ results using the $FCNW$ and $CNWF$ graph.

Recall that $ESDA$ divides each school into two schools. The first one with its minimum quota as the capacity and the second one (the extended school) with the difference between the school's capacity and its minimum quota as total capacity. It ranks each extended school just under the original one and then assigns students to schools using a $DA$ algorithm but does not let more than $n - \sum p_{c_j}$ be assigned to extended schools ($n$ is the total number of students and $p_{c_j}$ is $c_j$'s minimum quota). They fix an ordering of the schools and let the schools accept students one by one in this order until $n - \sum p_{c_j}$ students have been accepted across all of the extended schools. We also know that $ESDA$ finds fair matchings.

$FCNW$ starts from the top vertices of $F$-chains and finds $F\&CNW$ matchings when they exist, otherwise it finds fair matchings. If there is a unique fair matching then $FCNW$ gives us the $ESDA$ result. In case of more than one fair matching, I proved that $FCNW$ algorithm always chooses the $F\&CNW$ matchings whenever they exist and if not, then it chooses a fair matching with the maximum degree of nonwastefulness but $ESDA$ may not do that (it depends on the fixed ordering of the schools). Therefore, to get the $ESDA$ results, we need to change the way $FCNW$ selects the matches after the $FCNW$ and $CNWF$ graph is drawn (the first 4 steps stay the same). Then, starting from the end points (top vertices) of $F$-chains, $FCNW$ assigns students to school in these top vertices based on the $ESDA$ fixed ordering of schools until $n' = \sum p'_{c_j}$ where $n'$ is the number of remaining students and $\sum p'_{c_j}$ is the total number of unsatisfied minimum quotas. To do this, after each assignment, $n' - \sum p'_{c_j}$ must be updated to make sure there are enough students to satisfy the remaining minimum quotas. When $n' = \sum p'_{c_j}$, $FCNW$ selects the

[10]It can be two in easily recognizable cases such as when there are a small number of students and schools involved in the matching problem.

[11]If there is more than one vertex with out-degree zero in some rounds, there may be more than one $F\&CNW$ matching.

next top vertex of the $F$-chains which includes the first next school in the schools' ordering which has an unsatisfied minimum quota and assigns the student in this vertex to the school and it keeps going on in the same way until all students are assigned. This will give us the $ESDA$ results.

## 4.9 Minimum Quotas with Different Types

As explained earlier, respecting diversity is a very important issue arising in many societies. Diversity is crucial to unite a country. It also provides significant socio-economic benefits. That is why increasing diversity is one of the main focuses of this study. My algorithms, $FCNW$ and $CNWF$, generally deal with the diversity issue through imposed minimum quotas by finding matchings which are $F\&CNW$, fair or $CNW$. Furthermore, the framework of my algorithms, make it possible to define these minimum quotas separately for different groups of agents (students) such as minority and majority students. The ultimate goal in this case is providing help and fair chance for minority students, while simultaneously guaranteeing the balance of different groups of students. I make it possible by defining quotas that need to be filled minimally with specific groups of agents (students who need protection such as minority communities) and introducing algorithms which select fair and nonwasteful matchings. The different groups of agents are formally called different types.

In this section I show that my algorithms, $FCNW$ and $CNWF$, will also find $F\&CNW$, fair and $CNW$ matchings when one side of the market (the student side) includes different types of agents. First, let us review an algorithm proposed in the literature which deals with minimum quotas when there are different types.

Dynamic quotas deferred acceptance ($DQDA$) (Fragiadakis and Troyan, 2017, [23]): decreases the schools' capacities one by one through steps until the minimum quotas are satisfied. It chooses a school-type pair $(s;\theta)$ and lowers the type ceiling at $c$ and the capacity of $c$ by exactly one seat; the ceilings and capacities of the remaining schools are unchanged. $DQDA$ Pareto dominates $ACDA$, is $SP$ and is fair among the same type.

### 4.9.1 The Model of Minimum Quotas with Different Types

Let me first add the definition of type to my model with minimum quotas:

The finite set of types for students is $\theta = \{\theta_1, \ldots, \theta_r\}$, and each student is of exactly one type. The function $\rho : S \to \theta$ gives the type of each student, and $S_\theta$ is the set of students of type $\theta$. Types are fixed and common knowledge (i.e., types cannot be misreported). In addition to a capacity, each school $c_j \in C$, has a maximum capacity for each type $\theta$; $\{q_{c_1}^{\theta_1}, \ldots, q_{c_1}^{\theta_r}, \ldots, q_{c_m}^{\theta_1}, \ldots, q_{c_m}^{\theta_r}\}$ and a type-specific minimum quota for the number of students of each type $\theta$ who must be assigned to it, $\{p_{c_1}^{\theta_1}, \ldots, p_{c_1}^{\theta_r}, \ldots, p_{c_m}^{\theta_1}, \ldots, p_{c_m}^{\theta_r}\}$, where $p_{c_m}^{\theta_r}$ is the minimum number of students of type $\theta_r$ that must be assigned to school $c_m$.

The easiest case is when the maximum capacities of a school for different types add up to the school's total capacity. In this case, each school can be simply divided into type-specific schools, i.e., into $r$ schools and everything else remains the same. It is like having

$r$ different markets with one type of students in each. Then all my results will easily apply. In next section I show that even if it is not the case, my algorithms still apply to models with type-specific minimum quotas.

### 4.9.2 $FCNW$ and $CNWF$ Algorithms:

Each student despite her type will be considered as an independent agent and will be assigned based on the $FCNW$ and $CNWF$ algorithms' procedure. The thing to note is that the minimum number of each type which are needed to be filled must be considered as another feasibility constraint. In Step 1 the schools' maximum capacities for different types are considered as the schools' capacities separately. For example, if there are two types of students, $\theta_1$ and $\theta_2$ and school $c_j$ has one capacity q for each type, $q_{c_j}^{\theta_1} = 1$ and $q_{c_j}^{\theta_2} = 1$, then in Step 1, one student of type $\theta_1$ and one student of type $\theta_2$ will be assigned to $c_j$. In Step 2 we need to define the number of unsatisfied minimum quotas for each type at the end of Step 1 and use it to define the schools capacities in this step. In Step 3 the range of possible matches will be defined for each student of each type individually based on the two previous steps and the rest of the procedure is the same as the case without types. Appendix $E$ presents an example which shows how the algorithms work with different types.

**Theorem 8** *In the presence of minimum quotas with different types, $FCNW$ and $CNWF$ select a feasible $F\&CNW$ matching if it exists and if not then $FCNW$ selects fair matchings with a maximum degree of nonwastefulness and $CNWF$ selects $CNW$ matchings with a maximum degree of fairness at each profile.*

**Proof:** A feasible matching is a matching which satisfies the minimum quotas and total capacities of schools for different types. Therefore, based on the construction of the $CNWF$ and $FCNW$, the algorithms select possible matchings separately for maximum and minimum capacities of schools, allowing to choose the feasible matchings. Eliminating the infeasible matches at each round guarantees selecting only the feasible matchings. Also, if there is no $F\&CNW$ matchings, $FCNW$ and $CNWF$, find fair and $CNW$ matchings respectively using the top vertices of $F$ or $NW$-chains or through the backward trips as long as the superiors' priority is not violated. $CNWF$ and $FCNW$ eliminate some fair or $CNW$ matchings because they use a range of possible matches which may not include some feasible matches. The proof of Theorem 4 in Appendix $A$ shows that the matching selected by my algorithms maximizes the property that it cannot satisfy. ■

**Theorem 9** *In the presence of minimum quotas with different types, if there is at least one vertex with out-degree zero at each round (after the assignment in each round), there is at least one matching which is $F\&CNW$.*

The proof is similar to the proof of Theorem 2.

## 4.10 Conclusion

In this chapter I introduced a matching model where one side of the market has minimum quotas and strict priority over the other side, while the agents on the other side have strict preferences. I have studied the models with minimum quotas in-depth, without and with type-specific quotas, to provide a fundamental understanding of relevant cases and to aid practical market designers. I elaborated the idea, defined the model and studied the properties of the model.

Furthermore, I introduced two algorithms, $CNWF$ and $FCNW$, which find $F\&CNW$ (whenever they exist), $CNW$ (with the maximum degree of fairness) or fair (with the maximum degree of nonwastefulness) matchings.

I also discussed very useful and promising properties of the representative graph of my algorithms. Chromatic number, planar graph and acyclic directed graph have interesting properties with great potential for further research. These properties can be used to make it easier to recognize the existence of a $F\&CNW$ matching by analyzing the graph or getting the outcomes of other proposed algorithms.

This research addresses an important issue of real-world matching problems, namely, respecting minimum quotas. Within the same framework, I introduced algorithms which always select $F\&CNW$ matchings when they exist and otherwise select fair or $CNW$ matchings. My algorithms also work with different types of agents. I have introduced the $CNWF$ and $FCNW$ algorithms in a single framework which provide alternatives to three different and unrelated algorithms, $ESDA$, $MSDA$ and $DQDA$.

My overall aim is to lay down an integrated theoretical foundation for building a more prosperous society by ensuring that resources (including human resources) are used fairly and efficiently.

## 4.11 Appendices

### 4.11.1 Appendix A

**Theorem 3** *$FCNW$ selects fair matchings with the maximum degree of nonwastefulness. $CNWF$ selects $CNW$ matchings with the maximum degree of fairness.*

**Proof:** Using unsatisfied minimum quotas to define the capacity of schools for the second step of the algorithm eliminates some fair or $CNW$ matchings. This may prevent some students from being matched in the second step. Therefore, all schools below the one she has been matched to in the first step will be included in her range of possible matches. This means the top possible choice for the student does not change (the top vertex of $NW$-chain does not change for any student). Step 2 shortens the range of possible matches for each student but only less preferred schools from the bottom of her preference ordering will be eliminated since in this step each student will be assigned to her best choice which still lets the minimum quotas be filled. Therefore, all the remaining possible matches for students are better options than the eliminated ones. In other words, the assignments by $CNWF$ and $FCNW$ are as good as the removed matchings. Consequently, there would be fewer claims and fewer justified envy instances.

When no $F\&CNW$ matching exists, we can find fair or $CNW$ matchings. Consider the $FCNW$ algorithm. Since we start from the top vertices of $F$-chains, all schools are assigned to their top choices unless that match has been infeasible. Obviously, the number of justified envy instances will be zero and the matching is fair.

I show that not only it is fair but also it has the highest degree of constrained nonwastefulness. When we start from the top points of the $F$-chains, we chose vertices with the minimum out-degree and in the case of a tie, the ones with the highest in-degree are prioritized over others. This means that we chose vertices which are as high as possible on the $NW$-chains too, provided that feasibility is not violated. Therefore, the number of justifiable claims for empty seats will be minimized. It works for both the basic algorithm and the speedy version. The same logic works for the $CNWF$ algorithm.

Eliminating the forbidden and infeasible matches maximizes the degree of fairness and nonwastefulness, respectively.

### 4.11.2 Appendix B

**Theorem 4** *The maximum chromatic number of $CNWF$ and $FCNW$ graph is three.*

**Proof:** Each vertex in a $CNWF$ and $FCNW$ graph includes a student-school pair. The maximum degree of each vertex is four (the maximum out-degree and in-degree are both two) which means any vertex and its immediate neighbors can be independently colored [12] with two different colors. Each vertex is part of an $NW$ and/or an $F$-chain. First, consider only $F$-chains which are based on schools' preferences. These chains are separated from each other; therefore, they can easily be colored with two colors (for example we can color the first vertex in each chain blue and continue coloring the next vertices alternately red, blue, red and so forth). Now, we add the $NW$-chains which are based on students' preferences and connect the $F$-chains. $NW$-chains can be independently colored by two colors in the same way as $F$-chains but when we put all of them together, two colors may not be enough to color the whole graph. I show that only one more color is enough to properly color the entire graph.

Consider an arbitrary $NW$-chain. If its vertices are already colored properly, we are done. If an edge on this $NW$-chain connects two vertices with the same color, we simply can change the color of either of them to the third color. Let us call the vertex which needs the third color a conflicting vertex and color it green[13]. The only problem remains when we are not able to color the conflicting vertex with the third color because one of its immediate neighbors have been already changed to green.

Assume vertex $(s_i, c_j)$ is red while its two adjacent vertices on $F$-chain are blue and green. Now we put this vertex also in an $NW$-chain where it has only one adjacent neighbor which is red. We cannot color $(s_i, c_j)$ with any of our three colors since in that case it will have the same color with one of its immediate neighbors. Remember we started coloring $F$-chains with two original colors, blue and red. Then we added green to avoid neighbors

[12] Use the minimum number of colors to color the vertices of a graph in such a way that no immediate neighbors have the same color.

[13] If the chain is connecting more than two vertices with the same color, we only need to alternately change the color of the vertices to green.

from having the same color when they became the end points of $NW$-chains' arcs. That means the green neighbor of $(s_i, c_j)$ in the $F$-chain, let us call it $(s_{i'}, c_j)$, was originally blue[14] and it has been forced to turn green because of the color of its adjacent neighbors including $(s_i, c_j)$. Now that $(s_i, c_j)$ itself has to change color, we can simply make it green and turn its green neighbor $(s_{i'}, c_j)$ to red. Note that we cannot turn $(s_{i'}, c_j)$ back to blue, since it has originally been blue and has turned to green because of the two adjacent blue neighbors in an $NW$-chain and a red adjacent neighbor $(s_i, c_j)$ in the $F$ chain. Now, if $(s_{i'}, c_j)$ has another adjacent neighbor in the $F$-chain which is red, we can make that one blue and alternately change the next ones in the $F$-chain to red, blue, red and so forth. Even in the worst case scenario, with recoloring of some parts, we will have a proper 3-colored graph.

### 4.11.3 Appendix C

**Theorem 5** *The $CNWF$ and $FCNW$ graph is planar.*

There are plenty of theorems regarding the planar graphs which can be used to check the planarity of $CNWF$ and $FCNW$ graph[15].

1. A corollary of Euler's formula: If $G$ is a connected planar simple graph with $e$ edges and $v$ vertices, with $v \geq 3$, and if $G$ has no circuits of length 3, then $e \leq 2v - 4$.
2. Five Color Theorem (Heawood Theorem): Every planar graph can be colored with at most 5 colors.
3. Kuratowski's and Wagner's theorem: A finite graph is planar if and only if it does not contain a subgraph that is a subdivision [16] of the complete graph $K_5$ or the complete bipartite graph $K_{3,3}$.

A corollary of Euler's formula provides a way to check the planarity of graphs and five color theorem sets an upper bond for chromatic number of a planar graph. I use Kuratowski's and Wagner's theorem to easily prove that $CNWF$ and $FCNW$ graph is planar. First, let us officially define the $K_5$ and $K_{3,3}$ graphs.

**Definition:** $k_5$
A complete $K_5$ graph is a simple undirected graph with five vertices in which every pair of the distinct vertices is connected by a unique edge.

**Definition:** $K_{3,3}$
A bipartite $(3, 3)$ graph is a graph whose vertices can be divided into two disjoint and independent sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$. A complete bipartite graph $K_{3,3}$ is a bipartite graph with three vertices in each set of its

[14] We alternately colored $F$-chains red, blue and so forth. Therefore if $(s_i, c_j)$ is red, its adjacent neighbors in $F$-chain $(s_{i'}, c_j)$ both must have originally been blue.

[15] There exist fast algorithms like Planarity testing which for a graph with $n$ vertices, determine in time $O(n)$ (linear time) whether the graph may be planar or not.

[16] A subdivision of a graph results from inserting vertices into edges (for example, changing an edge $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ to $\bullet\!-\!\bullet\!-\!\bullet$) zero or more times.

vertices where every vertex of the first set is connected to every vertex of the second set.

**Proof:**

1. No subgraph of the $CNWF$ and $FCNW$ graph can be a subdivision of $K_5$.

   For any $K_5$ graph we need five vertices which are connected to each other. That means that the degree of each of them should be four. Let us take an arbitrary vertex $(s_i, c_j)$ with degree four and assume it is in a subgraph of $CNWF$ and $FCNW$ which is a subdivision of $K_5$. Two of the other vertices in this subdivision should be the neighbors of $(s_i, c_j)$ in its $F$-chain and the other two are in its $NW$-chain with different $F$-chains. There is no way for these two vertices to be connected to each other since the only way for this to happen is having a cycle in a student's preferences which is impossible since preferences are transitive. With the same logic, for $(s_i, c_j)$'s immediate neighbors in its $F$-chain to be connected there must be a cycle in the school's priorities which never happens due to the strict priorities.

2. No subgraph of the $CNWF$ and $FCNW$ graph can be a subdivision of $K_{3,3}$.

   We know any complete bipartite graph needs two sets of vertices where the vertices inside each set are not connected to each other but each of them is connected to all the vertices in the other set. Therefore, for a $CNWF$ and $FCNW$ graph to be a subdivision of a $K_{3,3}$ graph, we need two disjoint sets of vertices. This means the three vertices in each of these sets should belong to different $F/NW$-chains. We also need all of them to be connected to all the three vertices in the other set. Let us take two disjoint sets of vertices in a subgraph of our $CNWF$ and $FCNW$ graph. We call one set $V$ and the other $U$. Each of the sets $V$ and $U$ has three disjoint vertices.

   Consider an arbitrary vertex, $v \in V$. Vertex $v$ should be connected to three vertices in $U$ and based on the properties of our graph, we know that a vertex can be connected to another vertex only if both of them belong to the same $F$ or $NW$-chain. Assume $u, y, w$ are three disjoint vertices in $U$ which are connected to $v$. Therefore, they should belong to the same $F$-chain or $NW$-chain as $v$. There is no way that the three of them could be connected to $v$ without contradicting the disjointness constraint for the vertices in each set. For instance, if $u$ is in the same $F$-chain as $v$, then $y$ can be in $v$'s $NW$-chain. $w$ is either on $v$'s $F$-chain or $NW$-chain which means it belongs to the same chain as $u$ or $y$ since each vertex can belong at most to one $F$-chain and at most to one $NW$-chain. This contradicts disjointness of $u, y$ and $w$.

### 4.11.4 Appendix $D$

**Theorem 6** *A matching is $F\&CNW$ if and only if the $CNWF$ and $FCNW$ representative graph of its corresponding matching problem is:*

*-planar directed semi-acyclic graph,*
*-the maximum chromatic number of the graph is three,*
*-there is at least one vertex with out-degree zero.*

**Proof:** I have already proved that a $CNWF$ and $FCNW$ graph is planar and its chromatic number is three. In addition, we know if all vertices are part of a cycle then there is no $F\&CNW$ matching. That means whenever there is an $F\&CNW$ matching, there is at least one vertex with out-degree zero at each step. Therefore, to prove this theorem we only need to prove:

1. If a matching is $F\&CNW$ then the $CNWF$ and $FCNW$ representative graph of its corresponding matching problem is directed semi-acyclic.

   Whenever a matching is $F\&CNW$, there is at least one vertex with out-degree zero at each step which goes to the top of the topological ordering of vertices. If no vertex is in a cycle and since always in next steps there are new vertices with out-degree zero, then it is possible to topologically order all vertices of the graph. If there are some cycles, still there should be at least one vertex with out-degree zero which could appear at the top of the ordering (the vertices which are the starting point of the arcs aiming to this vertex come before it if their out-degree is one). When the algorithm is working after assigning the student in this vertex to the school in it, there should be at least one new vertex with out-degree zero; otherwise, there is no $F\&CNW$ matching. If there is a new vertex with out-degree zero, it simply goes before the vertex already on the top of the topological ordering. If there is no new vertex with out-degree zero, the only way for having an $F\&CNW$ matching is that the remaining cycle must be breakable. In that case we can write the vertices in the cycle as a whole on the topological ordering, so the graph is directed semi-acyclic.

2. If the planar, 3-colorable $CNWF$ and $FCNW$ representative graph of a matching problem is directed semi-acyclic and there is at least one vertex with out-degree zero, then the corresponding matching selected by the $CNWF$ and $FCNW$ algorithms is $F\&CNW$.

   When a graph is directed semi-acyclic then the graph is either acyclic (if there is no cycle then the matching is $F\&CNW$) or if it contains a cycle, that cycle includes at least one cycle breaker vertex and at least one vertex including the student of the cycle breaker vertex is ranked higher than the other vertices of the cycle. That means when we start assigning students to schools from the top of the chains, we have already assigned the student of the cycle breaker vertex or the capacity of its school has been filled. Either way the cycle has been broken, providing new vertices with out-degree zero to go on with the algorithm. This procedure is iterated until all students are assigned. We also know based on Theorem 2 that $CNWF$ and $FCNW$ always select an $F\&CNW$ matching whenever such a matching exists.

### 4.11.5 Appendix $E$

**Example 3** *$FCNW$ and $CNWF$ algorithms with different types.*

There are four schools, $C = \{c_1, c_2, c_3, c_4\}$, and three students, $S = \{{s_1}^l, {s_1}^h, {s_2}^h\}$ of two types, $\theta = \{l, h\}$.

Schools' priorities:

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| | ${s_1}^h$ | ${s_1}^l$ | ${s_1}^l$ | ${s_1}^h$ |
| | ${s_2}^h$ | ${s_1}^h$ | ${s_2}^h$ | ${s_2}^h$ |
| | ${s_1}^l$ | ${s_2}^h$ | ${s_1}^h$ | ${s_1}^l$ |
| $p^l$ | 0 | 0 | 0 | 0 |
| $p^h$ | 0 | 0 | 0 | 1 |
| $q^l$ | 1 | 1 | 1 | 1 |
| $q^h$ | 1 | 1 | 1 | 2 |
| $Q$ | 1 | 1 | 1 | 2 |

Students' preferences:

| ${s_1}^l$ | ${s_1}^h$ | ${s_2}^h$ |
|---|---|---|
| $c_2$ | $c_2$ | $c_3$ |
| $c_3$ | $c_1$ | $c_4$ |
| $c_1$ | $c_4$ | $c_1$ |
| $c_4$ | $c_3$ | $c_2$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ |

Using $DQDA$ the final matching is $\nu = \left( \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ - & {s_1}^l & {s_2}^h & {s_1}^h \end{array} \right)$.

$FCNW$ **and** $CNWF$**:**

- Step 1:

| $Rounds \backslash Schools$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| 1 | $-$ | ${s_1}^h, \underline{{s_1}^l}$ | $\underline{{s_2}^h}$ | $-$ |
| 2 | $\underline{{s_1}^h}$ | $\underline{{s_1}^l}$ | $\underline{{s_2}^h}$ | $-$ |
| $q^l$ | 1 | 1 | 1 | 1 |
| $q^h$ | 1 | 1 | 1 | 2 |
| $Q$ | 1 | 1 | 1 | 2 |

- Step 2:

  $um^l = 0 \Rightarrow q^l_{c_1} = q^l_{c_2} = q^l_{c_3} = q^l_{c_4} = 0$. Then we do not need to include type $l$ quotas in Step 2.

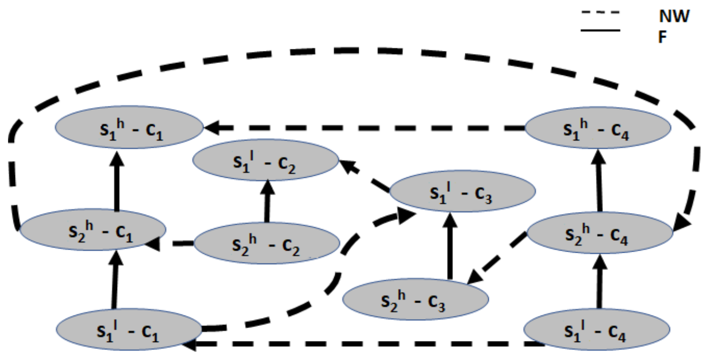
  $um^h = 1 > 0:$

| $Rounds \backslash Schools$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| 2 | $-$ | $-$ | $-$ | $\underline{{s_1}^h}, {s_2}^h$ |
| $q'^h$ | 0 | 0 | 0 | 1 |

- Step 3:

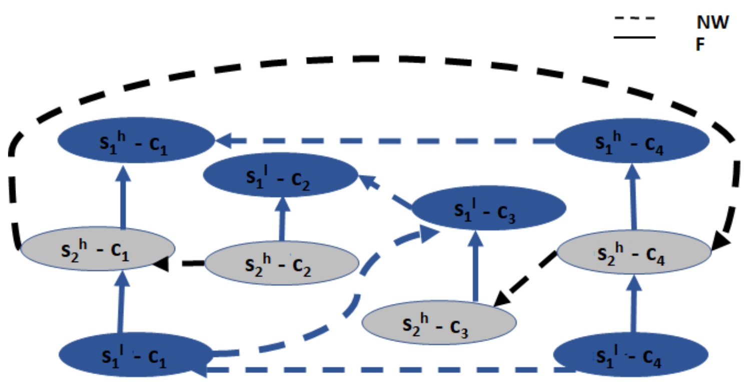
| ${s_1}^l$ | ${s_1}^h$ | ${s_2}^h$ |
|---|---|---|
| $c_2$ | $c_2$ | $c_3$ |
| $c_3$ | $c_1$ | $c_4$ |
| $c_1$ | $c_4$ | $c_1$ |
| $c_4$ | $c_3$ | $c_2$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ |

- Step 4:

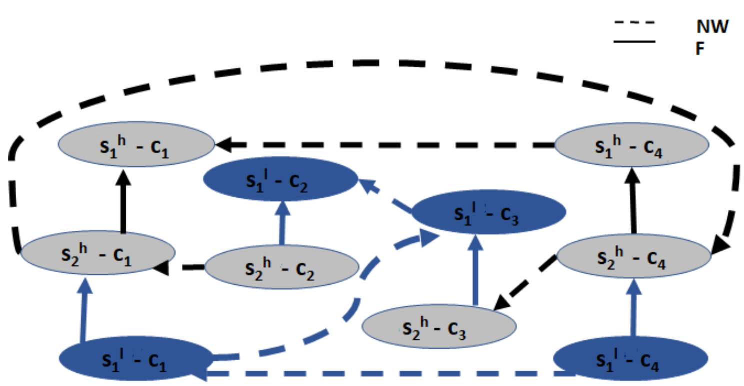



- Step 5:

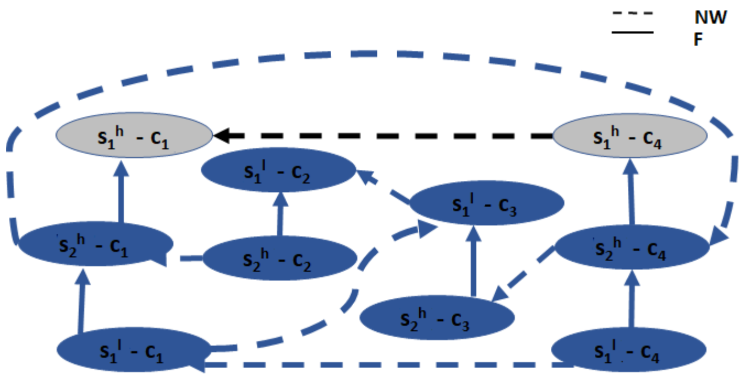
There are two vertices with out degree zero. If we start with assigning ${s_1}^h$ to $c_1$, and then ${s_1}^l$ to $c_2$, the next vertex with out degree zero is $({s_2}^h, c_3)$ which is not a feasible assignment, due to the minimum quotas restrictions, ${s_2}^h$ must be matched to $c_4$. The final matching is $\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ {s_1}^h & {s_1}^l & - & {s_2}^h \end{pmatrix}$.




If we start with assigning ${s_1}^l$ to $c_2$ then in the next round there are two vertices with out degree zero, $({s_2}^h, c_3)$ and $({s_1}^h, c_1)$.



If we continue with selecting $({s_1}^h, c_1)$ and then $({s_2}^h, c_3)$, we will end up with the same matching $\mu$. On the other hand, if we first select $({s_2}^h, c_3)$, then $({s_1}^h, c_1)$ is not feasible and the final matching is $\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ - & {s_1}^l & {s_2}^h & {s_1}^h \end{pmatrix}$.




both matchings $\mu$ and $\mu'$ are $FCNW$. ▲

# Chapter 5

# Dynamic Marriage Markets

## 5.1 Introduction

While the majority of the matching literature deals with the static matching problems, there are lots of situations where the matching markets are actually operating in a dynamic environment. For instance, in a kidney exchange market, at any time, there are new patients entering the patients' list on one side of the market. On the other side, new kidneys may be available while the matched patients and kidneys leave the market. There is a similar situation in matching teachers (a new period begins at the beginning of a school year) or medical staff to positions. Another ongoing situation is the refugee problem which also can be considered a dynamic matching problem. Furthermore, dynamic matching could be easily applied to marriage markets. New men and women enter the matching market at the beginning of each period and they leave after finding their match. The purpose of this paper is to define a matching rule with desirable properties when the agents on both sides of the market have preferences over the other side in a dynamic environment.

Furthermore, it considers how each agent can select some of the agents on the other side as her special options by giving them extra waiting time. While the agent's preferences only shows the ranking of the other side agents for an agent, my model defines a way to reflect the fact that some options are much more preferred to others. This aspect rarely, if ever, has been discussed in the literature while it is highly realistic especially in a marriage market. Assume one agent is so important for another agent that not only has he ranked her as his top choice but also he is willing to risk being unmatched for a while and wait for her. In marriage market and many other similar matching problems, people want to be able to wait if they think somebody or something is worth it. The structure of my model and the dynamic environment of having more than one period make the waiting possible. It is also realistic to assume that each agent only accepts a subset of available agents. In addition, we cannot assume that an agent stays in the market forever after entering it if she does not find her match. Since new agents enter the market in each period and the situation changes over time, especially in a marriage market, agents must be able to update their status at the beginning of the new period if they are unmatched. They need to have some flexibility to change their mind over time about some of the previously declared statuses, such as staying in or out of the market or expanding the acceptance sets, as it definitely happens in real cases.

I propose an innovative new matching rule which finds a stable matching in a dynamic environment with realistic features of the model. This stable matching at the same time is more fair for both genders (two sides of the market) since it does not favour either side, minimizes the unrealistic assumptions and allows for both sides optimal matching outcomes. If a two-sided optimal matching does not exist, it aims for the best compromises possible to increase fairness in the sense of fair opportunities[1]. I also study the strategy-proofness of my algorithm and address the issue of maximum cardinality.

My intention is to make the model as realistic as possible which makes it somewhat complicated. However, for preserving the real aspects of the model, some complications are inevitable. I characterize the dynamic specifics of the model, and study real life considerations in marriage markets. This constitutes a substantial improvement over current results in the sense that my model leads to a realistic matching in marriage markets and avoids some constraints or unrealistic assumptions. Some already existing algorithms take the set of available agents on one side (Pereyra, 2013 [54] and Liu, 2020 [48]) or both sides (Kurino, 2020 [45]) fixed and only define multiple periods as the dynamic aspect of their model. Others fix the number of periods (Du and Livne, 2014 [17]) or agents' preferences (Pereyra, 2013 [54]). There are models which assume that everybody knows who will join the market in the future (Doval, 2020 [16]) or all agents can change their match in each period (Kurino, 2020 [45], Kotowski, 2019 [44] and Damiano and Lam, 2005 [15]). None of these assumptions are realistic for a marriage market and my model makes more realistic assumptions. Regarding the two-sided offering algorithms, my algorithm avoids complicated aspects of some existing algorithms such as making many offers by one agent at the same time (Kuvalekar, 2014 [46]). It also modifies some issues of other algorithms, since it lets the two sides make offers at the same time instead of only one side (Gale and Shapley, 1962 [25]) or one-by-one offering algorithms which are based on a pre-defined list of agents (Romero-Medina, 2005 [56] and Dworczak, 2021 [19]). If there is no two-sided optimal matching, my model does not favour one side nor uses other methods such as allowing men and women to propose alternatively or finding a median stable matching (Teo and Sethuraman, 1998 [71]). Instead it provides the possibility to compromise and choose a matching which is not optimal for either side and thus it is more fair because it gives both sides a fair chance. I also provide some ways to increase the number of matched pairs in each period. These methods make it possible for practical matching designers whose objective is maximizing the number of matches to get matchings with higher number of matches.

I introduce new dynamic concepts and prove that my matching rule is dynamically strategy-proof and stable. I also discuss dynamic optimality and show that my algorithm is dynamically Pareto-optimal.

This paper aims to provide a foundation for future research which will generalize the newly designed algorithm to other matching markets and extend my findings to more complex cases.

The structure of this chapter is as follows: In Section 3 I discuss the related literature. In Section 4 I introduce the model and its properties. In Section 5 I review some matching

[1] The meaning of fairness in this study is more like the common day to day meaning rather than fairness in matching theory.

properties related to this study. Section 6 is dedicated to my new algorithm, Dynamic Marriage ($DM$) algorithm and new definitions related to it. In Section 7 I discuss the findings of my study and present theorems. In Section 8 I provide some ways to increase the number of matches in each period. Finally I conclude my study in Section 9.

## 5.2 Literature Review

In real life markets we can easily find lots of situations with multiple periods or repeated matching problems. Nevertheless, these kinds of markets have only started to receive some attention recently. Dynamic matching is generally a new topic in the matching literature and there are only a few papers on it. Furthermore, although the characteristics of these markets are different, all of them must have a main common feature; the next or previous periods have influence on the present period's matches. Otherwise they could be modeled as separate single period matchings.

One of the most recent papers on dynamic matching is "Stability in Repeated Matching Markets" by Liu (2020, [48]). He considers a fixed set of agents (hospitals) and calls them long-lived players. Hospitals are matched to a new generation of agents, short-lived players (medical students), in every period. In his setting, to make more students available for rural hospitals, urban hospitals need to decrease their hiring capacity. In this case, if a hospital does not respect the recommendation from the matching clearing house, it will be prohibited from future participation in the market. Therefore, the dynamic feature of the model has been used as a motivation or punishment tool to enforce more stable outcomes.

In another work, Arnosti and Shi (2020, [8]) consider a continuum of agents and a new object which must be immediately assigned after entering the market. They defined different types of lotteries and compared the results using different types of waiting lists in term of welfare and some other properties.

Kurino (2020, [45]) studies a dynamic marriage model. In his model there is no entry and exit of agents (women and men) who can change their partners at each period. In each period $t$, each agent has a utility function which defines the preferred outcome for her. He defines dynamic group stability for a matching $\mu$ when there is no group deviation: there is no group of agents who are better off by deviating from $\mu$ and choosing another matching $\mu'$. He also assumes that when a group of agents deviate from a matching $\mu$, the agents inside this group can only be matched with each other and the ones outside the group whose ex-partner is inside the group became unmatched.

There are three other recent studies on the dynamic context. The first one is "A Perfectly Robust Approach to Multiperiod Matching Problems" by Kotowski (2019, [44]). Kotowski's model lets agent change their assignment over multiple periods. Each agent has preferences over a sequence of assignments over time. The history and future of assignments can affect the current period preferences. The second one is "Static versus dynamic deferred acceptance in school choice: Theory and experiment" by Klijn, Pais and Vorsatz (2019, [41]). Their work compares the static and a dynamic student proposing $DA$ by using an experiment. The last one is "Dynamically Stable Matching" by Doval (2019, [16]). She defines a dynamic stability concept based on the assumption that only agents available in the same period are allowed to form a blocking pair. Furthermore, she assumes that preferences are common knowledge and that every agent knows who will be available in the market in future periods.

Andersson et al. (2018, [6]) studies dynamic refugee matching markets. They propose a specific matching mechanism, Dynamic Order Mechanisms and show that any matching selected by this mechanism is Pareto-efficient and satisfies envy bounded by a single asylum seeker. In other words, envy between localities (such as states) is bounded by a single asylum seeker, i.e., whenever some locality $m$ envies some locality $m'$, the envy can be "eliminated" by removing a single asylum seeker either from the set of asylum seekers matched to locality $m$ or from the ones matched to locality $m'$.

Bloch and Cantala (2017, [11]) worked on assigning objects to queuing agents in a dynamic context. They considered a constant size waiting list of agents while the objects arrive over time. Whenever a new object is available it is offered to the agents in the waiting list starting with the agent at the top of the list. If she rejects the object, it is offered to the next agent in the fixed sequence and if the object is rejected by all the agents, it will be wasted. They showed that using a lottery to offer the object decreases and even minimizes the waste while with both private and common values all agents prefer first-come first-served to lottery.

Du and Livne (2014, [17]) studies a two-period matching market. In their model, agents can decide to be matched in period 1 and leave the market or wait until period 2 when new agents enter the market. They showed that there is a stable matching for agents who are present in period 2. They also proved under some restrictive assumptions (such as a large number of agents in period 1 and a small number of new arrivals), that on average at least one quarter of all agents present in period 1 prefer to be matched before period 2, provided that they anticipate others are going to wait until the next period.

Pereyra (2013, [54]) studies "A dynamic school choice model". In his model the $DA$ algorithm matches teachers to schools. From one period to another, they can remain in their current positions or apply for a more preferred one. In order to overcome the issue of respecting the teachers' improvement, he has moved each teacher who has been assigned to a school in the previous period to the top of the schools' priority list (which was originally based on teachers' grades from an evaluating test). This made the process manipulable. Therefore, he assumes that teachers cannot change their preferences from one period to another. He also assumes that the school positions are fixed, so actually one side of the market is not changing.

Damiano and Lam (2005, [15]) have presented "Stability in Dynamic Matching Markets". In their paper, they assume that each agent can be rematched at the end of each period and the payoff of the first period for each agent is equal to the sum of that period's payoff and the discounted payoffs of the next periods.

In addition, there are some studies that concentrate on two-sided offering. One of the early ones is Teo and Sethuraman (1998, [71]). They showed that when the total number of distinct stable marriage solutions, $l$, is odd, there is a stable marriage solution in which every person is assigned to a partner who is the "median" partner among all their possible mates. Their study shows a way to find a matching which compromises between the men-optimal and women-optimal matchings. It is neither men-optimal nor women-optimal, but more fair since it does not favour any side.

Romero-Medina (2005, [56]) introduced an algorithm, called the Equitable Algorithm which uses a fixed ordering of agents to compromise between both sides' ideal matchings. In his algorithm, agents propose based on a fixed ordering. At step $k$ each person who receives the proposal accepts it if the offer is among her $k$ best choices and/or is better than the proposal she has accepted in previous steps. In case of a rejection, the rejected agent proposes to his second most preferred choice among her $k$ first choices.

Dworczak (2021, [19]) presents a class of algorithms, called *DACC* (Deferred Acceptance with Compensation Chains) in which both sides of the market are allowed to make offers in an arbitrary order. Agents make their offers one at a time according to a pre-defined arbitrary order. Based on his work, when all agents are allowed to propose, it is possible that an agent rejects an offer from another agent from the opposite side but proposes to her later on. As a consequence, the agent might withdraw an offer he made to another agent. He uses a compensative system in a way that whenever some $i$ deceives $j$, he compensates agent $j$ by letting her make an offer in the current round irrespective of the pre-defined order.

Kuvalekar (2014, [46]) introduces an algorithm where both sides make proposals in each round to a set of their top agents. In each round $k$, agents expand their acceptable sets according to their preferences to their $k$ top-ranked agents. The agents can only be matched when they both list each other as mutually acceptable. If the agent is matched, she proposes to only those agents that are better than her current match. Romero-Medina and Kuvalekar later completed this study (2021, [57]).

## 5.3 The Model

It is a model to match agents to each other on two sides of a dynamic market. In my model, the market runs for unlimited periods and at the beginning of each period new agents enter the market while the matched agents leave the market at the end of each period. Only the list of available agents on both sides of the market at the beginning of each period is public knowledge. Each agent accepts a subset of available agents on the other side and has preferences over the agents in her acceptance set. However, if one agent stays unmatched at the end of one period, her acceptance set may expand in the next period. It is realistic to assume that agents expand their acceptance set in fear of staying unmatched. Nevertheless, no agent can kick out of her acceptance set an unmatched agent from the previous period. Although the agents' preferences change at the beginning of each period, given arriving new agents on the other side or their expanding of the acceptance set, the preference ordering over the unmatched agents from the previous period is the same. Furthermore, if any agent adds other agents already present in the previous period to her acceptance set, she should rank them lower than the agents who were acceptable in the previous period. Each agent can only be assigned to one agent. These assumptions are all realistic in marriage markets.

Formally, the marriage market is given by $(M, W, T, L, \succ_M, \succ_W, POr)$.

It consists of:

- $M^t = \{m_1, m_2, \ldots, m_{n^t}\}$ is a set of $n^t$ men on side 1 [2] at the beginning of period $t$.
- $W^t = \{w_1, w_2, \ldots, w_{m^t}\}$ is a set of $m^t$ women on the other side at the beginning of period $t$.
- ${M_i}^t = \{w_j \in W^t\colon\ w_j \succ_{m_i} \varnothing\}$ is the set of acceptable women for man $m_i$ in period $t$, where $i \in \{1, 2, \ldots, n^t\}$ and $j \in \{1, 2, \ldots, m^t\}$.
- ${W_j}^t = \{m_i \in M^t\colon\ m_i \succ_{w_j} \varnothing\}$ is the set of acceptable men for woman $w_j$ in period $t$, where $i \in \{1, 2, \ldots, n^t\}$ and $j \in \{1, 2, \ldots, m^t\}$.
- $T$ discrete periods of running time.
  - $t \in T$ is a number assigned to each period, i.e., $t = 1$ is the first period of the market operation.
  - $T_{m_i} \in T$ is the total number of consecutive periods that agent $m_i$ is present in the market.
  - $t_{m_i} \in T_{m_i}$ is the number of each period for agent $m_i$ after joining the market; $t_{m_i} = 1$ is the first period that $m_i$ enters the market and it increases as he moves unmatched to the next periods.
- $L$ is the loyalty set.
  - ${L_{m_i}}^t = \{\{w_j : w_j \in {M_i}^t\} \subset {M_i}^t\}$ is the loyalty set of man $m_i$ at the beginning of period $t$.
  - ${L_{w_j}}^t = \{\{m_i : m_i \in {W_j}^t\} \subset {W_j}^t\}$ is the loyalty set of woman $w_j$ at the beginning of period $t$..

  A loyalty set consists of subsets of consecutive top members of an agent's preference ordering whom she/he is loyal to. That means the agent is ready to wait for them until the next period, provided that they are unmatched too.

- Each agent $m_i$ has a strict preference relation $\succ_{m_i}$ over ${M_i}^t$.
- Each agent $w_j$ has a strict preference relation $\succ_{w_j}$ over ${W_j}^t$.
- $POr^t$ (Proposing Order[3]) is a dynamic ordering of all agents at the beginning of period $t$ which will be used in case of a dead end[4]. It changes from one period to the other and also will be updated whenever it is going to be used based on the current situation of agents.[5]

[2] Through this chapter, I consider side 1 as men and refer to a member of this side (a man) as a he and side 2 as women and refer to its members (women) as a she.

[3] I use $POr$ as an abbreviation for proposing order instead of $PO$ to avoid the confusion with Pareto-Optimal ($PO$).

[4] No rejection, no possible match.

[5] I explain in detail how $POr$ is updated in Section 6.

Agents' preferences, acceptance and loyalty sets are private knowledge. Nobody knows who will enter the market in the future. At the beginning of each period, only the list of available agents on both sides of the market is public knowledge.

The main dynamic aspects of my model are:

1. Time: The market runs for an infinite number of periods.
2. Agents: At the beginning of each period new agents enter the market and matched ones leave it.
3. Preferences/Acceptance sets: The preferences/acceptance sets of agents can change/expand at each period due to arriving new agents and due to the fear of remaining unmatched.
4. Loyalty sets: Agents have the possibility to wait for the agents on the other side whom they value much higher than others.
5. $POr^t$: Not only will $POr^t$ change at the beginning of each period but also it will be updated at any time based on the current situation of the agents.

## 5.4 Properties of Matching Rules

The properties of matching rules that will be used in this chapter are introduced in this section. All of these properties are standard properties that have been studied extensively in various models.

**Property 1:** A **matching** $\mu$ is **individually rational** if all agents prefer their current mate under $\mu$ to being unmatched.

An **agent** $m_i$ is **individually rational** if he prefers his current mate under $\mu$ to being unmatched.

**Property 2:** A matching $\mu$ is **stable** if there is no blocking pair, $(m_i, w_j)$ who prefer each other to their current mate under $\mu$, and each agent is individually rational.

**Property 3:** Agent $m_i \in M^t$ justifiably envies agent $m_{i'} \in M^t$ who is matched to agent $w_j \in W^t$ under an assignment $\mu$ if $w_j \succ_{m_i} \mu(m_i)$ and $m_i \succ_{w_j} m_{i'}$. Therefore, $(m_i, w_j)$ is a blocking pair in period $t$.

**Property 4:** A matching $\mu$ is **fair** if there is no agent who justifiably envies another agent.

**Property 5:** A matching $\mu$ is **men/women-optimal** if it is the best matching for all men/women among all stable matchings.

**Property 6:** A matching rule $f$ is **strategy-proof** if there do not exist $m_i, \succ$ and $\hat{\succ}_{m_i}$ such that $f_{m_i}(\hat{\succ}_{m_i}, \succ_{-m_i}) \succ_{m_i} f_{m_i}(\succ)$.

**Property 7:** A **matching** $\mu$ is **Pareto-optimal** if it is not Pareto-dominated in the sense that there is another matching which makes none of the agents worse off and at least makes one agent better off.

A **matching rule** $f$ is **Pareto-optimal** if it specifies a Pareto-optimal matching for each preference profile.

**Property 8:** A matching $\mu$ is **maximum (cardinality)** if there is no other matching $\mu'$ such that $|\mu'| > |\mu|$.

## 5.5 Dynamic Marriage ($DM$) Algorithm

The agents' preferences, loyalty and acceptance sets are private information. Only the list of available agents at the beginning of each period is common knowledge.

Before introducing my algorithm, I provide some new definitions which will be needed in the algorithm's process:

**Definition 1:** Loyalty Set ($L$)

$L_{m_i} = \{\{w_j, \ldots\} \subset {M_i}^t\}$ *is called the loyalty set of agent* $m_i$ *if it includes subsets of consecutive top members of* $m_i$*'s preference ordering whom* $m_i$ *is loyal to and is ready to wait for until the next period, provided that they are unmatched too.*

Each member of $m_i$'s loyalty set is a subset of his acceptance set. The first member should start from the top agent in his preference ordering and it is not possible to skip agents.

**Definition 2:** Loyalty Loop ($LL$)

*If a member of* $L_{m_i}$ *consists of more than one agent, I call it a loyalty loop* ($LL_{m_i}$).

$LL_{m_i} \in L_{m_i} = \{w_j, w_{j'}, w_{j''}, \ldots\}$ *means that* $m_i$ *is willing to wait for the most preferred member of this loop (let's say* $w_j$*) until the next period if* $w_j$ *is also unmatched and* $m_i$ *is not matched to the next agents in the loop* ($w_{j'}, w_{j''}, \cdots$).

Therefore, loyalty loops are non-singleton members of loyalty sets.

**Definition 3:** Loyalty Loop Agents

*For every* $LL_{m_i} = \{w_j, w'_j, \cdots\}$*, the agents* $w_j, w'_j, \cdots$ *are called Loyalty Loop Agents.*

If $L_{m_i} = \{\{w_j\}\}$, the loyalty set of $m_i$ includes only one subset of his acceptance set, $M_i$. This subset itself includes only one agent, $w_j$. The meaning of this loyalty set is that whenever $m_i$ proposes to $w_j$, if $w_j$ does not accept him, he will wait for her until the next period in the hope that she will include him in her acceptance set in the coming period (provided that $w_j$ remains unmatched too).

If $L_{m_i} = \{\{w_j, w_j', \cdots\}\}$, the loyalty set of $m_i$ includes only one subset of his acceptance set, $M_i$. This subset itself includes more than one agent and should start from the top of $m_i$'s preference ordering. This means that $w_j$ is the top-ranked agent for $m_i$ and $w_j, w_{j'}, \cdots$ are consecutive agents on $m_i$'s preference ordering. Such a member of the loyalty set is called a loyalty loop. $LL_{m_i \in L_{m_i}} = \{w_j, w_j', \cdots\}$ means whenever $m_i$ offers to $w_j$ and $w_j$ does not accept him, he will not wait for her but offers to the next agent in the loop, $w_j'$. If she rejects or does not accept him, then he goes to the next in the loop without waiting for $w_j'$ until he gets rejected or not being accepted by all the agents in the loop, then he goes back to the top of the loop and reoffers (one offer per round) to the first agent who has not rejected him (the agent who does not accept him and he prefers her to all the other loop agents who have not rejected him) and waits for her until the next period. An agent may have more than one $LL$.

Let me clarify the definition of the loyalty set and loyalty loop and their difference with a simple example:

**Example -1**

Assume that $m_i$ has the following preference ordering over a set of five available women all of whom he accepts:

$w_2 \succ_{m_i} w_4 \succ_{m_i} w_5 \succ_{m_i} w_1 \succ_{m_i} w_3$

Let us assume that $L_{m_i} = \{\{w_2\}, \{w_4, w_5, w_1\}\}$ is the loyalty set of $m_i$. First of all, agents $w_2, w_4, w_5$ and $w_1$ are consecutive top members of $m_i$'s preference ordering. So, as mentioned, all agents in $m_i$'s loyalty set must be top consecutive members of his acceptance set starting with the top one who is $w_2$ in this case. $m_i$'s loyalty set consists of two members, $\{w_2\}$ and $\{w_4, w_5, w_1\}$, which are subsets of her acceptance set. Furthermore, $\{w_4, w_5, w_1\}$ is a loyalty loop since it consists of more than one agent; $LL_{m_i} = \{w_4, w_5, w_1\}$.

When the algorithm starts, $m_i$ proposes to his top-ranked woman, $w_2$. If $w_2$ rejects him in favour of a more preferred man, he will propose to his next most preferred woman, $w_4$. But if $w_2$ does not accept him simply because he is not in her acceptance set, then he will wait for her until the next period, hoping that $w_2$ will include him in her acceptance set in the next period. Admittedly, if $w_2$ does not include $m_i$ in her acceptance set in period 1, he will wait for her until the next period provided that $w_2$ is unmatched too (she has not received any offer from her acceptable men) in hope of finding his way to her acceptance set. However, if $w_2$ receives offers from her acceptable men, she will reject $m_i$. Now let us say $w_2$ has $m_i$ in her acceptance set but she receives a better offer in period 1. Then she rejects $m_i$ in favour of the more preferred man. Then $m_i$ will propose to his next preferred woman, $w_4$. If $w_4$ rejects him, he will propose to $w_5$. He also will propose to $w_5$ if he is not acceptable to $w_4$ (he does not wait for $w_4$). If he is not in $w_5$'s acceptance set or $w_5$ rejects him in favour of a more preferred man, then he will not wait for her either and will propose to $w_1$ who is the last member of his loyalty loop. If $w_1$ does not accept him or rejects him then he will go back to the top of her loyalty loop, $w_4$. If $w_4$ has not rejected him before (otherwise he goes to the next one in the loop who has not rejected him yet) and will wait for her until the next period, provided that $w_4$ is still unmatched. The difference between $\{w_2\}$ and $\{w_4, w_5, w_1\}$ is that $m_i$ moves to the next subset (here from $\{w_2\}$ to $\{w_4, w_5, w_1\}$) only if he has been rejected by all members of the subset, but inside the loyalty loop he moves between agents $w_4, w_5$ and $w_1$ if he is rejected or not acceptable. If $w_4$ rejects him, he is not in $w_5$'s acceptance set, and $w_1$ rejects him too, then he will wait for the top member of the loop who has not rejected him, $w_5$. ▲

**Remark:** As mentioned before, if one agent stays unmatched at the end of one period, her acceptance set may expand in the next period (she cannot exclude any unmatched agents who were in her acceptable set in the previous period). The agents' preferences change at the beginning of each period, given arriving new agents on the other side or expanding the acceptance set. But their preferences over unmatched agents from the previous period do not change. If any agent adds other agents already present in the previous period to her acceptance set, she should rank them lower than the other agents in the previous period who were acceptable.

The next four definitions, 4 to 7, are needed to define the *POr* list, which is given in Definition 8.

**Definition 4:** Super Loyal Agent

*An agent $m_i$ is called super loyal to agent $w_j$ in step $t$ if $w_j \in L_{m_i}$ for all the periods that:*

1. *$w_j$ has been in the market such that $T_{w_j} > 1$ and;*
2. *$m_i$ has been in the market such that $T_{m_i} > 1$.*

Agent $m_i$ is super loyal to agent $w_j$ if he includes her in his loyalty set in every period in which they are both in the market, provided that both of them are in the market for more than one period.

**Definition 5:** $TD_{m_i}^{t_{m_i}}$ (Tolerance Degree)

*At period $t$, $TD_{m_i}^{t_{m_i}}$ is the number of agents in $L_{m_i}^{t_{m_i}}$.*

In any period $t$, the Tolerance Degree of agent $m_i$ is the number of agents from the other side of the market that he includes in his loyalty set and is willing to wait for until the next period if they are unmatched too.

**Definition 6:** $TTD_{m_i}^{t_{m_i}}$ (Total Tolerance Degree)

$TTD_{m_i}^{t_{m_i}} = \sum_1^{t_{m_i}} TD_{m_i}^{t_{m_i}}$.

The Total Tolerance Degree for agent $m_i$ is the sum of all $TD$s for all the periods he has been in the market.

**Definition 7:** Waiting agent

*Agent $m_i$ is called a waiting agent if he has made an offer to one of the agents in his loyalty set and has been waiting for her to accept him or reject him in favour of a more preferred agent in subsequent periods.*

**Definition 8:** $POr^t$ (Proposing Order)

*$POr^t$ is a dynamic ordering of all available agents at the beginning of period $t$ based on the number of periods they have been in the market and their $TTD$ and $TD$, but it will be updated whenever it is going to be used.*

- *Original $POr^t$: Generally at the beginning of period $t$, in the original $POr$[6]:*
  - *The most present agent in the market (agent $m_i$ who has the highest $t_{m_i}$) goes to the top of the list and the more recent agents go after her/him from the older to the new comers.*
  - *In case of a tie, the one with the highest $TTD$ comes first, if they have the same $TTD$, the one with the highest $TD$ for that period goes first.*
  - *If there is still a tie then we need to use an arbitrary tie-breaker.*
- *Updated $POr^t$: Since the nature of my model is dynamic, $POr$ is dynamic too. That means that not only does $POr$ change at the beginning of each period but also whenever we want to use it we need to update it based on the current situation of all remaining agents:*
  - *If the top agent of $POr$ is a waiting agent, then the agent who has proposed to her jumps before her on the list.*
  - *If there is no such an agent then the agent who is after the waiting agent, will go first.*
  - *The new waiting agents go before the ones transferred from previous periods.*
  - *An agent who is waiting on a super loyal offer which is an offer made by a super loyal agent to the one she/he is super loyal to, goes after agents who are waiting on a loyal offer which is an offer made by a loyal agent to the one she/he is loyal to.*

**Remark:** In case of a dead end at period $t$ (no rejection, new offer or matching is possible), we take the original $POr^t$ as a reference. The agent at the top of the $POr^t$ is the first one who has to change her/his proposal and make a new offer to her/his next preferred agent unless she/he is waiting on a reactivated offer from the previous period or waiting

[6] Original $POr^t$ is the $POr$ defined at the beginning of each period $t$ based on the seniority, $TTD$ and $TD$.

on a new offer she/he has made to one of the agents in her/his loyalty set. In this case the $POr$ will be updated and the agent who has made an offer to the waiting agent will jump in front of her/him. If there is no such an agent then simply the one who is after the waiting agent in the original $POr$ will jump before her/him. Note that as mentioned in the $POr$ definition, the new waiting agents go before the ones transferred from previous periods. That is because if an agent has been waiting for the other one until the next period specially for more than one period, it is not fair to make her/him deviate from that offer. In addition, an agent who is waiting on a super loyal offer goes after agents who are waiting on a loyal offer.

At the beginning of each period, agents report their loyalty set, based on their acceptable agents in the same way that they report their preferences. The loyalty sets are private knowledge and could be completely different from the previous period. Agents have the flexibility to change their mind regarding whether they want to wait for somebody any more. This is a realistic assumption since people naturally become inpatient when period after period they remain unmatched, and also because of new arrivals who might be more interesting. That is the way a human mind normally works.

### 5.5.1 Dynamic Marriage ($DM$) Algorithm

1. Agents on both sides of the market propose at the same time to their most preferred agent.
2. Agents receiving more than one offer keep the most preferred one and reject others.
3. If $w_j \in {M_i}^t$ and $m_i \in {W_j}^t$, when $w_j$ rejects $m_i$, the rejected agent makes an offer to his next preferred agent based on his preference ordering, since he has been rejected in favour of a more preferred agent and he has no chance with $w_j$.
4. If $w_j \in {M_i}^t$ but $m_i \notin {W_j}^t$, whenever $m_i$ proposes to $w_j$;
   (a) If $w_j \notin L_{m_i}$ then $m_i$ will be automatically rejected by $w_j$ and he will propose to his next preferred agent on his preference ordering.
   (b) If $w_j \in L_{m_i}$ then $m_i$'s offer will stay active until $w_j$ rejects him in favour of another agent. Then $m_i$ will propose to his next preferred agent.
   (c) If $w_j \in L_{m_i}$ and $w_j$ does not reject $m_i$ in favour of another agent, it means that $w_j$ has not received any offer from one of her acceptable agents and she will remain unmatched at the end of this period together with $m_i$ who wants to wait for her until the next period, hoping that she will include him in her acceptance set in the next period. So, $m_i$ will not make an offer to his next preferred agent at this period and his offer to $w_j$ will stay valid until the next period.
   (d) If $w_j \in LL_{m_i \in L_{m_i}}$ and $w_j$ does not reject $m_i$ in favour of another agent, it means $w_j$ has not received any offer from one of her acceptable agents and she will remain unmatched at the end of this period. But $m_i$ will not wait for her unless he gets rejected or is unacceptable by other members of the loop. In that case $m_i$ will be unmatched too and waits for her until the next period provided that $w_j$ is the most preferred mate in $m_i$'s loyalty loop who has not rejected him.

5. If $m_i$ enters a new period with a passive offer to $w_j$, all new arrivals and new added agents to $m_i$'s acceptance set must be ranked under $w_j$, since she is an important agent who is worth waiting for. Agents can update their loyalty set at each period. As mentioned before, all members of loyalty set must be the top consecutive agents of $m_i$'s acceptance set. Here there is a special case which must be considered carefully.

   Let us consider a case when $m_i$ has been waiting on a passive offer to $w_j$ moved to period $t+1$ from period $t$. This means $m_i$ was not acceptable to $w_j$ at period $t$ but $w_j$ was in loyalty set of $m_i$, then $m_i$ waited for her until $t+1$. Now at $t+1$ he does not want to include her in his loyalty set again simply because if $w_j$ does not include him in her acceptance set in $t+1$ or if does and then rejects him in favour of a more preferred man, then $m_i$ will not want to wait for her for another period and will want to propose to his next preferred woman. Then assume some new agents enter the market and $m_i$ wants to add them to his acceptance set. As mentioned before, they must be ranked under $w_j$ because $m_i$ is waiting on a passive offer to $w_j$, actually, $w_j$ is the top agent in $m_i$ preference ordering. While $m_i$ does not want to wait for $w_j$ any more (if $w_j$ does not accept him or if she rejects him), he wants to put other women of his acceptance set in his loyalty set without including $w_j$, the rule that says a loyalty set must start with the top member of preference ordering (which is $w_j$ here) will be violated. To cover this case, I add a condition to the loyalty set definition as follows:

   *The agents in $m_i$'s loyalty set are the top consecutive members of his preference ordering. This means that $m_i$'s loyalty set must start from his most preferred women based on his preference ordering unless he is waiting on a passive offer to this woman. In this case, if he does not want to include her in his loyalty set for another period, then the loyalty set will start from his second preferred woman according to his preference ordering.*

6. Agent $m_i$ will be matched to $w_j$ whenever they both have a proposal from the other one.

7. In case of a dead end (there are no mutual proposals and no rejection or new offer), based on the $POr^t$ list, agents change their offers and propose to their next preferred agent. If the agent at the top of $POr$ who needs to withdraw her current proposal and propose to her next preferred mate does not have any agent remained in her acceptance set and she is not waiting on any loyal proposal, she has to withdraw her current proposal and stay unmatched until the next period.

**Important remarks:**

1. Each agent who is matched is out of the market for good so they do not risk to end up with a less preferred match. If an agent comes back to the market she is considered a new agent.

2. Using $DM$ algorithm we will automatically have the matching which is optimal for both sides if such a matching exists. But this two-sided optimal matching is different from the outcome of the $DA$ algorithm. The difference comes from the fact that $DM$ algorithm operates in a dynamic environment and allows agents to wait for other

agents through the periods. It also allows both sides to propose at the same time.

3. In case of a dead end, each side which the top agent of $POr$ belongs to will lose its optimality. If after the top agent in $POr$ has changed her/his offer, still there is a dead end and the next agent in $POr$ is from the other side, we will end up with mixed agents of both sides who lose their optimality unlike men-optimal and women-optimal matchings where all agents of the opposite side lose their optimal matches.

### 5.5.2 Example 2

I provide this example to clarify the procedure of the $DM$ algorithm. For the sake of simplicity, I only show a two-period matching market. Furthermore, to show all details, I explain the example in separate rounds, although all the rounds could be done in one table.

Period 1:

${L_{m_1}}^1 = \{\{w_2\}, \{w_1, w_3\}\}$, ${L_{m_4}}^1 = \{\{w_4\}\}$

$t_{m_i} = t_{w_j} = 1$ for all $m_i \in M^1$ and $w_j \in W^1$ but, $TD^1_{M_1} = 3$ while $TD^1_{M_4} = 1$. Therefore, based on the $POr$ definition, $m_1$ comes before $m_4$ in the $POr$ list. The ordering of the remaining agents in $POr$ is arbitrary:

$POr^1 = m_1, m_4, \cdots$

Men preferences:

| $m_1$ | $m_2$ | $m_3$ | $m_4$ |
|---|---|---|---|
| $w_2$ | $w_2$ | $w_2$ | $w_4$ |
| $w_1$ | $w_4$ | $w_3$ | $w_1$ |
| $w_3$ | $w_3$ | | |
| $w_4$ | $w_1$ | | |

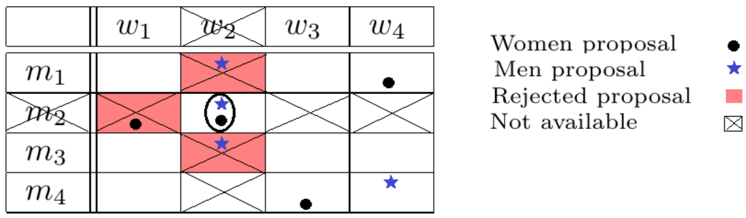
Women preferences:

| $w_1$ | $w_2$ | $w_3$ | $w_4$ |
|---|---|---|---|
| $m_2$ | $m_2$ | $m_4$ | $m_1$ |
| $m_4$ | $m_3$ | $m_3$ | $m_2$ |
| $m_3$ | $m_1$ | $m_1$ | |
| $m_1$ | $m_4$ | $m_2$ | |

*DM* **Algorithm:**

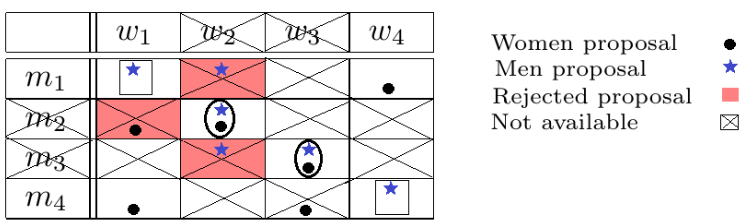
Round 1:

Two sides propose at the same time to their most preferred agent. Agents receiving more than one offer, reject the less preferred ones. At the end of this round, agents $m_2$ and $w_2$ are matched since they both have proposed to each other. As a result, $w_2$ rejects $m_1$ and $m_3$. Also, $m_2$ rejects $w_1$.



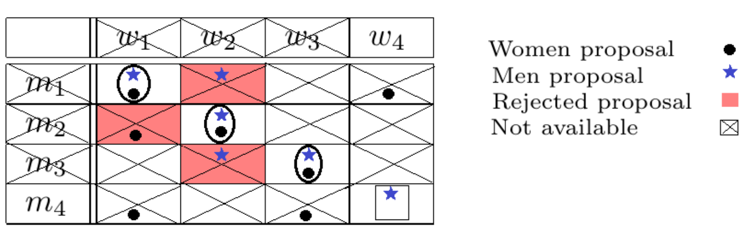

Round 2:

Agent $m_4$ has proposed to $w_4$ which is in his loyalty set. Since $m_4 \notin W_4{}^1$, and $w_4$ has not received any more preferred proposal, this offer will stay active which means $m_4$ is waiting for $w_4$. Rejected agents, $m_1$, $m_3$ and $w_1$ propose to their next agent in their preference ordering. Although $\{w_2\} \in L_{m_1}$ but since $w_2$ has rejected $m_1$ in favour of a more preferred one, he will propose to his next preferred agent. Agents receiving more than one offer reject the less preferred ones. Since $m_4$ is waiting for $w_4$ and $m_4 \notin L_{w_3}$ then $w_3$ does not wait for $m_4$ and withdraws her offer to $m_4$. She proposes to $m_3$. At the end of this round, $m_3$ and $w_3$ are matched and will leave the market while $m_1$ is waiting for $w_1$ and $m_4$ is waiting for $w_4$.




Round 3:

$m_1 \notin L_{w_4}$ and $m_1$ himself is waiting for $w_1$, so $w_4$ withdraws her proposal to $m_1$ and since her next preferred agent, $m_2$, is already matched she cannot make a new proposal. This means $w_4$ will leave this period unmatched. $m_4 \notin L_{w_1}$ and $m_4$ himself is waiting for $w_4$, so $w_1$ withdraws her proposal to $m_4$ and proposes to $m_1$ ($m_3$ is already matched). At the end of this round, $m_1$ is matched to $w_1$.




This is the end of period 1.

Matched pairs: $(m_2, w_2), (m_3, w_3), (m_1, w_1)$.

Unmatched agents: $w_4$ and $m_4$ (waiting for $w_4$).

Period 2:

New agents, $m_5, m_6, w_5, w_6$ enter the market.
There is no $L_{m_i}$ or $LL_{m_i}$. Technically all $L_{m_i}$ and $LL_{m_i}$ sets are the empty sets.

Although $m_4$ and $w_4$ have been present in the market for the same number of periods, $t_{m_4} = t_{w_4} = 2$, $m_4$ comes before $w_4$ in $POr$ since $TTD^2_{m_4} = 1$ while $TTD^2_{w_4} = 0$. The ordering of the rest agents in $POr$ is arbitrary:

$POr^2 = m_4, w_4, \cdots$

Men preferences:

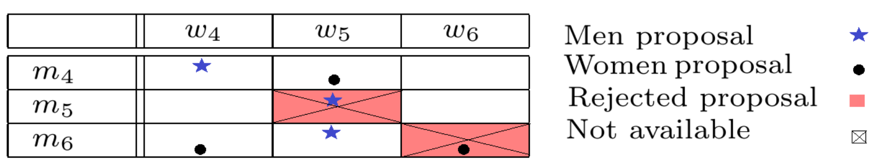
| $m_4$ | $m_5$ | $m_6$ |
|---|---|---|
| $w_4$ | $w_5$ | $w_5$ |
| $w_6$ | $w_4$ | $w_4$ |
| $w_5$ | $w_6$ | |

Women preferences:

| $w_4$ | $w_5$ | $w_6$ |
|---|---|---|
| $m_6$ | $m_4$ | $m_6$ |
| $m_4$ | $m_6$ | $m_5$ |
| $m_5$ | | |

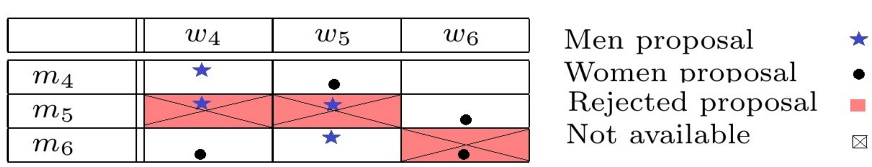
Round 1:

Two sides propose at the same time to their most preferred agent. Agents receiving more than one offer, reject the less preferred agents. At the end of this round there is no possible match.



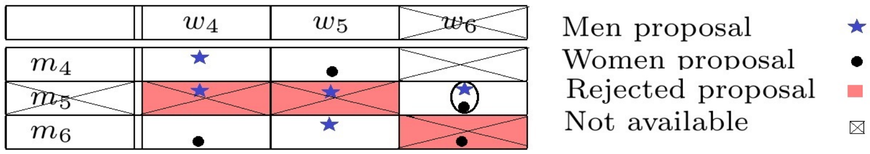

Round 2:

Rejected agents, $m_5$ and $w_6$ propose to their next preferred mates. Agents receiving more than one offer keep the most preferred one and reject others.




Round 3:

Rejected agent, $m_5$ proposes to his next preferred agent, $w_6$. At the end of this round, $m_5$ is matched to $w_6$.

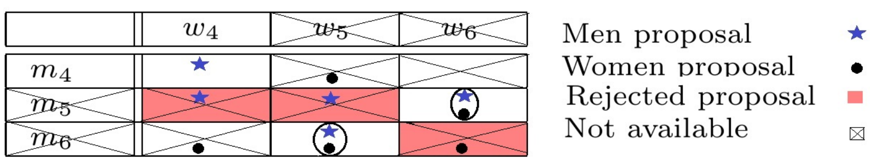



Round 4:

There is no possible match. We have a dead end and the $POr$ list is required. As explained before, the $POr$ list is dynamic and whenever it is going to be used, it will be updated based on the current situation of the market. Although $m_4$ was at the top of the $POr$ list at the beginning of period 2 but because he is waiting on a reactivated offer from period 1 so he will not be the first agent in the list after updating $POr$. Based on the $POr$ definition, $w_5$ who has proposed to waiting agent $w_4$ (the one waiting on a reactivated offer) moves to the top of the $POr$ list:

Updated $POr^2 = w_5, m_4, w_4, \cdots$

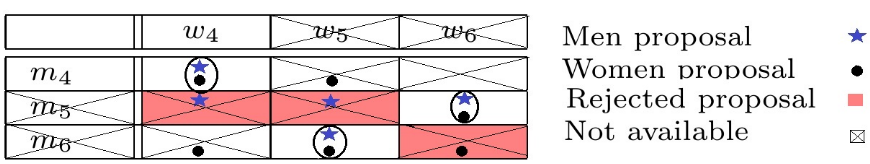
Therefore, $w_5$ has to withdraw her offer to $m_4$ and propose to her next preferred agent, $m_6$.




At the end of this round, agent $m_6$ is matched to $w_5$.

Round 5:

Rejected agent $w_4$ will propose to the last remaining agent in her preference ordering, $m_4$. Agent $m_4$ is matched to $w_4$.




This is the end of period 2 and the end of this matching problem.

Matched pairs: $(m_2, w_2), (m_3, w_3), (m_1, w_1), (m_4, w_4), (m_5, w_6), (m_6, w_5)$. ▲

## 5.6 Theorems and Results

In order to be able to state the results of this study, I will first introduce some new concepts:

**Definition 9:** Inter-periods blocking pair

*An inter-periods blocking pair consists of a pair of agents $m_i$ and $w_j$ which prefer each other to their current match under $\mu$ but they have left the market in different periods, provided that there were both in the market at least in period $t$ when one of them who is matched first leaves the market. Furthermore, in one of the periods when they were both in the market, at least the one who leaves the market first, starts to include the other one in her/his acceptance set.*

Since in my model no agent knows who will join the market in the future and every agent who gets matched leaves the market, it is not possible for $m_i$ and $w_j$ to form a blocking pair if they have never met in the market. Furthermore, it is not logical that an agent $m_i$ forms a blocking pair with another agent, $w_j$, whom he has never considered an acceptable mate while she was present in the market when he got matched. Also, let us recall that if $m_i \in {W_j}^t$, then $m_i \in {W_j}^{t'}$ $\forall\ t' > t$ as long as $m_i$ is in the market since no agent can kick out another agent from her/his acceptance set. Therefore, if $m_i$ is the agent who leaves the market first, he should include the other agent, $w_j$, in his acceptance set at least in the last period that they are both in the market which is the period that $m_i$ gets matched and leaves the market, i.e., there exists $t_{m_i} \in T_{m_i} : w_j \in {M_i}^{t'_{m_i}}$ $\forall\ t'_{m_i} \geq t_{m_i}$ or/and there exists $t_{w_j} \in T_{w_j} : m_i \in {W_j}^{t'_{w_j}}$ $\forall\ t'_{w_j} \geq t_{w_j}$.

**Definition 10:** Dynamic Individual Rationality

*A matching $\mu$ is Dynamically Individual Rational if all agents prefer their match under $\mu$ in period $t$ to staying unmatched until the next period, $t+1$.*

**Definition 11:** Dynamic Stability

*A matching $\mu$ is dynamically stable if:*

1. *It is dynamically individual rational.*
2. *There is no blocking pair of agents $(m_i, w_j)$ who leave the market at the same period $t$ and prefer each other to their current match under $\mu$.*
3. *There is no inter-periods blocking pair $(m_i, w_j)$ who leave the market in different periods and prefer each other to their current match under $\mu$ unless one of them has been waiting on a passive offer to a more preferred agent outside the pair or did not include the other one in her/his acceptance set when the other one has been matched and the other one also has not waited for her/him.*

**Theorem -1** *$DM$ is dynamically stable.*

Theorem -1 is proved in Appendix $A$.

**Definition 12:** Dynamic two-sided optimality

*A matching $\mu$ is dynamically two-sided optimal if it is the best dynamically stable matching for both sides.*

The following simple example is a well known example which shows that matchings which are optimal for both sides may not exist.

**Example 1**

Men preferences:

| $m_1$ | $m_2$ |
|---|---|
| $w_1$ | $w_2$ |
| $w_2$ | $w_1$ |

Women preferences:

| $w_1$ | $w_2$ |
|---|---|
| $m_2$ | $m_1$ |
| $m_1$ | $m_2$ |

Optimal matchings:

1. Men-optimal (men proposing): $(m_1, w_1)$, $(m_2, w_2)$.
2. Women-optimal (women proposing): $(m_2, w_1)$, $(m_1, w_2)$. ▲

**Theorem 0** *$DM$ finds dynamically two-sided optimal matchings if such a matching exist.*

Theorem 0 is proved in Appendix $B$.

We know that a matching which is optimal for both sides only exists if there is only one stable matching, in which case the unique stable matching is both men-optimal and women-optimal.

If there is no two-sided optimal matching, based on $POr$, $DM$ finds matchings which are fair for both sides since it does not favour either side.

**Definition 13:** Dynamic Strategy-proofness

*A matching rule is Dynamically Strategy-proof if no agent by misreporting her preferences at some period can get better results than what she could get using the dynamic features of the mechanism.*

**Theorem 1** *$DM$ is dynamically strategy-proof.*

Theorem 1 is proved in Appendix $C$.

**Definition 14:** Dynamic Pareto-optimality

*In a dynamic market a matching $\mu$ is Pareto-optimal if no agent can be made better off in period $t$ without making somebody else worse off in some period $t'' : t'' \geq t$.*

**Theorem 2** *$DM$ is dynamically Pareto-optimal.*

Theorem 2 is proved in Appendix $D$.

Let me recall the dynamic aspects of my model which are referred in Theorem 1 and Theorem 2.

1. Time which includes infinite number of periods.

2. Agents whose new comers enter the market at the beginning of each period and matched ones leave it.

3. Preferences/acceptance sets of agents which can change/expand at each period due to arriving new agents and due to the fear of remaining unmatched.

4. Loyalty sets which give agents the possibility of waiting for other agents on the other side whom they value much higher than others.

5. $POr^t$ which changes at the beginning of each period and also is updated at any time based on the current situation of the agents.

## 5.7 Increasing the Matching Size in each Period

In case of marriage, maximizing the number of matched pairs in each period does not make sense since it deals with major life decisions. Therefore, taking some measures to put pressure on people for accepting mates whom they do not really want to does not seem right. If we use the algorithm for other cases, maximizing the number of matched pairs may be desirable.

Generally there are three reasons for staying unmatched in the $DM$ algorithm:

1. Small acceptance set.

2. Being less preferred by others.

3. High $TD$ and $TTD$ (agents are waiting for others for long).

Since $DM$ respects the agents' preferences on both sides of the market, it could be possible to increase the number of matches by motivating the agents to be less restricted when they are reporting their acceptance and loyalty sets.

There are restrictions which can be set up to motivate agents to become more generous about their acceptance set. These restrictions help to increase the number of matched pairs based on the preferences:

1. Agents remaining from previous period, go to the top of the $POr$. In case of ties, the ones with highest $TTD$ and $TD$ go first even if they are waiting on a reactivated offer. Therefore, in case of a dead end, they are the ones who have to change their offer and propose to their next preferred agent.

2. Time limit:

   - We can put a time limit for each agent to attend the market. When time is limited for each agent, they would expand their acceptance set in fear of being unmatched at the end of their time limit. For example each agent can only attend the market 3 times but the market runs forever. We can also motivate agents more to report generous acceptance sets by adding some conditions to the time limits. For example, if an agent is still unmatched after her time limit, she will leave the market forever, unless she has included all available agents in her acceptance set at least for half of her periods in the market, and an empty $L$ set for the same amount of periods. The number of allowed periods for each of these restriction can vary based on the market situation and the designers goals. For example, when making as many matches as possible is a crucial goal, time limits will be tight.
   - The number of periods that the market operates is limited and at the last period, all agents are acceptable to each other.

These restrictions can be set up in a way to increase or even maximize the number of matchings.

## 5.8 Conclusion

In this project I introduced a matching model when both sides of the market have preferences over the other side and the market runs over multiple time periods. I have studied the different aspects of the model in-depth, to provide a fundamental understanding of relevant situations and to aid practical market designers. I have elaborated the idea, defined a novel model and studied its properties.

Furthermore, I introduced an algorithm, $DM$, which finds a matching which is dynamically optimal for both sides whenever it exists. Otherwise it selects a matching which is fair since it does not favour either side. In addition, I provided some restrictions to increase the number of matches at each period and I showed that the $DM$ algorithm is fair in the sense of not favouring either side, dynamically stable, strategy-proof and Pareto-optimal.

My study addresses an important and overlooked issue of real-life matching situation in a dynamic environment. My goal is to provide a strong theoretical foundation for building a more prosperous society by ensuring that the human capital have been treated fairly and the mechanism of matching resources is stable and efficient.

## 5.9 Appendices

### 5.9.1 Appendix A

**Theorem -1** *$DM$ is dynamically stable.*

**Proof:**

- $DM$ is dynamically individual rational:

  Each agent will be matched to one of her acceptable agents. Therefore, whoever she has been matched to is better than being unmatched. Furthermore, if she had somebody who is worth to wait for (although he did not accept her at the current period), she could put him in her loyalty set an take her chance to be matched to him in the next periods. If she has not done, then she prefers to be matched in the current period rather than waiting unmatched until the next periods.

- There is no blocking pair of agents $(m_i, w_j)$ who leave the market in the same period $t$ and prefer each other to their current match under $\mu$:

  Assume $(m_i, w_j)$ is a blocking pair of agents who has left the market in the same period and prefer each other to their match under $\mu$. This simply means that they both have been acceptable to each other but they have not been matched to each other. Therefore:

  - There is a $w_{j'}$ whom $m_i$ has been matched to while $w_j \succ_{m_i} w_{j'}$.
  - $DM$ matches a man to a woman when both propose to each other at the same time.
  - While $m_i \in {W_j}^t$ and $w_j \succ_{m_i} w_{j'}$, then he has proposed to $w_j$ before proposing to $w_{j'}$.
  - If he has not been matched to $w_j$ that means $w_j$ has rejected him in favor of a more preferred agent.

  These contradict $(m_i, w_j)$ being a blocking pair.

- There is no inter-periods blocking pair $(m_i; w_j)$ who leave the market in different periods and prefer each other to their match under $\mu$ unless one of them has been waiting on a passive offer to a more preferred agent outside the pair or did not include the other one in her/his acceptance set when the other one has been matched and the other one also has not waited for her/him:

  Assume $(m_i, w_j)$ is an inter-period blocking pair of agents who prefer each other to their match under $\mu$ and $m_i$ is the agent who leaves the market first[7] in period $t$.

  - $DM$ matches a man to a woman when both propose to each other at the same time.
  - There is a $w_{j'}$ whom $m_i$ has been matched to while $w_j \succ_{m_i} w_{j'}$.

[7] This means $m_i$ is the one who has been matched first.

- Based on the definition of inter-periods blocking pair, at least in the $m_i$'s last period, period $t$, both $m_i$ and $w_j$ were present in the market and $w_j$ was acceptable by $m_i$.
- $m_i$ has left the market in period $t$. That means $m_i$ has been matched to $w_{j'}$ in that period. Therefore, $w_j$ and $w_{j'}$ have been both in his acceptance set in period $t$.
- If $w_j \succ_{m_i} w_{j'}$, he has proposed to $w_j$ before proposing to $w_{j'}$.
- Therefore, if he has not been matched to $w_j$ that means:
  * $m_i \in {W_j}^t$ but $w_j$ has rejected him in favor of a more preferred agent. In this case they both leave the market in the same period and it is covered in blocking pair definition.
  * $w_j$ did not include $m_i$ in her acceptance set in period $t$ when $m_i$ has been matched and $m_i$ has not waited for her (did not include her in his loyalty set).
  * $w_j$ has included $m_i$ in her acceptance set but she was waiting on a passive offer when $m_i$ got matched and $m_i$ itself did not wait for her.

These contradict $(m_i, w_j)$ being an inter-period blocking pair.

### 5.9.2 Appendix B

**Theorem 0** *DM finds dynamically two-sided optimal matchings if such a matching exists.*

**Proof:** *DM* matches a man to a woman when both of them propose to each other at the same time. Furthermore, we know a two-sided optimal only exist if there is a unique stable matching. Therefore, automatically if there is a two-sided optimal, *DM* will find it.

### 5.9.3 Appendix C

**Theorem 1** *DM is dynamically strategy-proof.*

**Proof:** We know that *DA* is strategy-proof for the proposing side. *DM* is based on both sides proposing at the same time, therefore its results is different from the *DA* outcome even if it is used in one-period matching problem. On the other hand, since *DM* is a dynamic matching algorithm and static strategy-proofness does not apply to it, I show that *DM* is dynamically strategy-proof.

In static models, *DA* is not strategy-proof for both sides. The agents on non-proposing side can make themselves better off by manipulating the mechanism through shortening their acceptance set. However, in the dynamic setup of my model it is not possible for any agent to manipulate the algorithm and get better results than what she could get using the options that the model and corresponding algorithm provide her.

Generally, if any agent $w_j$ gets rejected by $m_i$ because $m_i$ has received a proposal from a more preferred mate, no matter how $w_j$ reports her preferences, she will not be matched to $m_i$. In other cases, shortening the acceptance set may improve the outcomes for $w_j$ since it causes the automatic rejection of unacceptable agents. These rejected agents make new proposals and as a result the mate that $w_j$ wanted more may get rejected from his more preferred choice and he will propose to $w_j$. The same goal can be achieved easily by defining a loyalty set including $m_i$. If $m_i$ rejects $w_j$ in favor of a preferred agent, as mentioned before no manipulation can help $w_j$. Otherwise, as long as $w_j$ has not been rejected by $m_i$, she is not accepting any other proposals (receiving proposals from others is automatically rejected) and she is not forced to change her proposal. ■

Let me recall Example 3 for further clarification. In a static men-optimal matching, both women $w_1$ and $w_2$ can manipulate the mechanism by reporting their second choice unacceptable. Men can do the same manipulation in the static women-optimal matching. Now, let me solve this matching problem using the $DM$ algorithm:

**Example 2**

If all agents report their true preferences then the results depend on the $POr$ since we will have a dead end in Round 1. If either $w_1$ or $w_2$ is at the top of the $POr$[8] then women will end up with their second choice. The same happens for men if either of them is the first one who has to change his proposal based on the $POr$. Therefore, even without misreporting, there is an equal chance for both sides to have their optimal matching due to this fact that both sides are allowed to propose at the same time. Now, assume $w_1$ wants to increase her chance of getting $m_2$ by misreporting her preferences as follows:

| ${w_1}'$ | $w_2$ |
|---|---|
| $m_2$ | $m_1$ |
| | $m_2$ |

The men's preference ordering is the same as in Example 3.

$DM$ **algorithm:**

Period 1: $m_1$ and $m_2$ propose to their first choices, $w_1$ and $w_2$ respectively. At the same time, $w_1$ and $w_2$ propose to $m_2$ and $m_1$ respectively. The offer received by $w_1$ came from an unacceptable man but since she has not received any preferred proposal, she can not reject this proposal. Nevertheless, since $m_1$ has not included $w_1$ in his loyalty set, then his proposal to $w_1$ will be rejected automatically. Rejected $m_1$ proposes to $w_2$. Now $w_2$ who has received two proposals accepts the preferred one, $m_1$ and rejects $m_2$. Rejected $m_2$ proposes to $w_1$ and gets matched to her.

The fact is that she did not need to manipulate. She could easily have the same result only by including $m_2$ in her loyalty set. The manipulation is unnecessary especially since based on $POr$ rule if an agent is at the top of this list and has to change her proposal but does not have any other agent left in her acceptance set, then she has to leave the period unmatched. Therefore, misreporting her preferences by shortening her acceptance set may even cause her staying unmatched while this situation will not happen if she includes $m_2$ in her loyalty set. That is because if $w_1$ includes $m_2$ in her loyalty set, although $w_1$ is at the top of the $POr$[9], since she is waiting on a loyal offer, then the agent proposing to her, $m_1$,

[8] $POr$ here is a random ordering of all agents since there is no seniority, $TTD$ or $TD$.
[9] $TD_{w_j} = 1$ while it is zero for all other agents.

jumps before her in the updated $POr$ list. Therefore, $m_1$ should withdraw his proposal to $w_1$ and propose to $w_2$. Then, $w_2$ who now has two offers accepts the preferred one, $m_1$, and rejects $m_2$. Rejected $m_2$ proposes to his next preferred woman, $w_1$ and gets matched to her. Therefore, $w_1$ gets her best choice in period 1 which is the same result as misreporting the preferences. ▲

The structure of the $DM$ algorithm which allows simultaneous proposals and possibility of waiting, makes manipulating unnecessary for agents.

### 5.9.4 Appendix D

**Theorem 2** *$DM$ is dynamically Pareto-optimal.*

**Proof:** $DM$ matches agents to each other when both propose to each other at the same time. An agent proposes to her next preferred agent if she has been rejected in favor of a more preferred agent or she is not acceptable by the other one and she did not want to wait for him. Therefore, when an agent gets matched, she has be assigned to her best possible option regarding her preferences and loyalty set.

Two agents can be matched to each other only if they have met in the market and at least in period $t$ when they are matched to each other they both should be included in each other's acceptance sets. Assume $w_j$ has been assigned to $m_i$ in period $t$ under $\mu$ and there exists another agent $m_{i'}$: $m_{i'} \succ_{w_j} m_i$. If $m_{i'}$ has left the market before period $t$ then:

- $m_{i'} \in {W_j}^{t'}$ for some $t' < t$, then either he has rejected $w_j$ in favour of a more preferred agent or did not accept $w_j$ in $t'$.

- $w_j$ did not accept $m_{i'}$ in $t'$ or has been waiting on an offer to a more preferred agent and $m_{i'}$ has not waited for her. This means that $m_{i'}$ did not want to wait for her until period $t$.

Anyway, changing the assignment of $w_j$ in period $t$ and assigning $m_{i'}$ to $w_j$ does not make sense since $m_{i'}$ is already matched and whoever is matched is out of the market. Moreover, no body knows who will join the market in the future, then changing the assignment of $w_j$ and assigning her to an agent who has not joined the market in period $t$ is not possible too. Therefore, if $w_j$ has been matched to $m_i$ in period $t$ and there is an agent $m_{i'}$: $m_{i'} \succ_{w_j} m_i$ then $m_{i'}$ should be in ${W_j}^t$. In this case, if we assign $m_{i'}$ to $w_j$, we have made her better off. On the other hand, since $m_{i'} \succ_{w_j} m_i$, based on $DM$ procedure, $w_j$ has proposed to $m_{i'}$ before proposing to $m_i$. Therefore, if $w_j$ has not been matched to $m_{i'}$ that means one of the following scenarios applies:

- $m_{i'}$ has rejected $w_j$ in favor of a more preferred agent in period $t$, then assigning $m_{i'}$ to $w_j$ makes $m_{i'}$ worse off.

- $w_j \notin {M_{i'}}^t$ and $m_{i'}$ has not been matched in period $t$ but $w_j$ did not want to wait for him. Thus, $w_j$ prefers to be matched to $m_i$ in period $t$ instead of waiting for $m_{i'}$. It means for being matched to $m_{i'}$ she needs to stay unmatched in period $t$ and this makes her worse off.

- $w_j \in {M_{i'}}^t$ but $m_{i'}$ is waiting on an offer to a more preferred agent in period $t$ while $w_j$ does not want to wait for him. If $m_{i'}$ leaves period $t$ unmatched but does not get matched to the mate that he was waiting for and instead gets matched in period $t'' : t'' > t$ to $w_{j'}$ while $w_j \succ_{m_{i'}} w_{j'}$ then assigning $m_{i'}$ to $w_j$ in $t$ makes both of them better off but $m_{i'}$ is the best possible mate for $w_{j'}$ in period $t''$. Therefore, assigning $w_j$ to $m_{i'}$ in period $t$ makes $w_{j'}$ worse off in period $t'' : t'' > t$.

- $w_j \in {M_{i'}}^t$ but $m_{i'}$ is waiting on an offer to a more preferred agent in period $t$ while $w_j$ does not want to wait for him. If $m_{i'}$ leaves period $t$ unmatched and gets matched in period $t'' : t'' > t$ to an agent $w_{j'}$ whom he prefers to $w_j$, then assigning $m_{i'}$ to $w_j$ in period $t$ makes both $m_{i'}$ and $w_{j'}$ worse off.

# Chapter 6

# Conclusions and Future Work

## 6.1 Conclusions

In the first study, Chapter 3, we introduced a novel matching model when one side of the market has consecutive acceptance intervals according to a given commonly known ranking of the objects on the other side of the market. We presented the Common-Ranking Successive Rules which find a maximum matching and extended this set of rules to Successive Rules, which find all maximum matchings. We proposed two classes of matching rules which are maximal, Pareto-optimal and group strategy-proof. One carries out trades in top trading cycles starting from a maximum endowment, while the other one is a sequential dictatorship which allows agents to choose their favorite objects in a given order.

Both of these are complex algorithms, and thus we also studied serial dictatorships which are simple greedy algorithms. Since not all interval profiles allow for the existence of a maximal serial dictatorship, we characterize the set of interval profiles for which a maximal serial dictatorship exists, and we identify all such serial dictatorships. Each class of matching rules that we introduce and study are group strategy-proof, since the acceptance intervals are common knowledge and cannot be manipulated.

The model with consecutive acceptance intervals describes realistic scenarios of matching markets where agents may not be familiar with all items on the other side of the market, and information acquisition about all items is either too costly or not feasible. Our assumption that we study those consecutive interval profiles for which it is feasible to assign an object to each agent is also plausible in applications, and it has the advantage that it allowed us to present clean results in this first study of maximal matching rules for problems with consecutive acceptance intervals. While this assumption can be relaxed by modifying some of our definitions, we believe that our analysis provides a good theoretical foundation for market designers who are interested in reaching maximum and Pareto-optimal matchings via algorithms with strong incentive properties.

In the second study, Chapter 4, I introduced a matching model with minimum quotas. In this model one side of the market has minimum quota requirements which need to be filled, and agents on both sides have strict priorities/preferences over the agents on the other side.

To find the desired outcome in matching problems with minimum quotas, I introduced two algorithms, $CNWF$ and $FCNW$, which select matchings that are both fair and nonwasteful (whenever they exist), $CNW$ (with the maximum degree of fairness) or fair (with the maximum degree of nonwastefulness) matchings.

In addition, I represented the matching problems with minimum quotas using a novel graph, the $FCNW$ and $CNWF$ graph, and I studied the properties of this representative graph, such as its chromatic number and whether it is a planar graph or an acyclic directed graph. These properties can be used as a starting point for further research on the graph and its corresponding model with minimum quotas.

Furthermore, I studied models with minimum quotas when there are different types of agents on one side of the market (the students' side) and I showed that my algorithms also apply to these cases.

My research in Chapter 4 is focused on matching when respecting minimum quotas which is an important issue of real-life matching problems. Within the same framework, I introduced algorithms which select $F\&CNW$, fair or $CNW$ matchings with or without type-specific quotas. Overall, I have introduced two new algorithms, $FCNW$ and $CNWF$, in a unified framework which serve as an alternative for three different unrelated algorithms, $ESDA$, $MSDA$ and $DQDA$. I studied the models of matching with minimum quotas in-depth to provide a fundamental understanding of relevant matching problems and to provide useful tools for practical market designers.

In third study, Chapter 5, I introduced a matching model when both sides of the market have preferences over the other side and the market runs for more than one period. I added other realistic dynamic aspects to the model and studied these different aspects of the model in-depth to provide a fundamental realistic framework for relevant cases and to aid practical market designers who work on dynamic markets. As a first step, I elaborated the idea of a dynamic matching model. Then I defined my new model in detail, considered the required realistic assumptions and studied the properties of my model.

Furthermore, I introduced an algorithm, Dynamic Marriage ($DM$), which selects a matching which is dynamically optimal for both sides of the market whenever it exists. Otherwise, based on the $POr$ list it selects a matching which is fair for both sides since it does not favour either side and I showed that my algorithm is also dynamically stable, dynamically strategy-proof and dynamically Pareto-optimal. In addition, I provided some tips which can be used to increase the number of matches in each period.

My study in chapter 5 is focused on the match-making problem in a dynamic environment which is an important but rarely considered issue of real-life matching cases. My goal is to provide a strong theoretical foundation in order to make sure that the agents involved in matching problems are treated equitably and the matching models reflect the real-world matching problems as much as possible.

My overall aim with these studies is to analyze real-life matching problems which have not been addressed yet or not in depth, to enrich our knowledge of different matching markets, and as a result to lay down an integrated theoretical foundation for building a more prosperous society by ensuring that all resources are used fairly and efficiently.

## 6.2 Extensions and Further Questions

Chapter 3:

1. Instead of objects on one side of the market we have agents on both sides who have preferences over each other. Here the two-sided stability notion applies.

2. Objects have strict or weak priorities over agents. Can stability and efficiency be reconciled?

3. Objects have capacities (or quotas), i.e., many-to-one matching. All previous questions regarding maximum, Pareto optimal, and stable matching rules can be considered.

4. Weighted two-sided matching with weights on the edges (weighted connections). How can we maximize the weight (instead of the cardinality) of a matching using a simple matching rule?

5. What if we relax the assumption that a maximum matching assigns an object to each agent? How much more complicated would our results become?

Chapter 4

1. What if there are some overlaps? Some agents belong to more than one type.

2. What if not all of the agents are acceptable by the other side? Each student/school or both have an acceptance set.

Chapter 5:

1. Make it possible for an agent to weight the agents in her loyalty set.

2. Make it possible for an agent to withdraw from a loyal offer if she changes her mind.

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