Inference Procedures for Copula-Based Models of Bivariate Dependence

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ABSTRACT

Inference Procedures for Copula-Based Models of Bivariate Dependence

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In this thesis, we develop inference procedures for copula-based models of bivariate dependence. We first investigate the distribution of Kendall's functions for joint survivors since Kendall's functions are important for identifying Archimedean copula models. We then provide two estimators for the generator of an Archimedean copula. We also propose a plug-in estimator for Kendall's tau and a maximum pseudo-likelihood estimation for the copula model parameters in cases where the survival times are subject to bivariate random censoring. These tests are based on the usual (univariate) Kaplan-Meier estimator and a recently proposed estimator for the bivariate case.

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Chapter 1

Introduction

1.1 Motivation

The need for bivariate survival analysis arises in domains where we must consider two dependent lifetimes. Such problems occur for example in insurance where pension contracts with reversion clauses depend on two random lifetimes (for instance, that of the policyholder and that of the spouse). Although some may assume independence for simplicity and use the results from the univariate case, such an assumption may be unrealistic: the spouses share a common environment, their survivals are therefore related. In this thesis, we use copulas to model bivariate dependence.

1.2 Background on copulas

A copula is a useful technique for modeling a joint distribution in terms of its marginals. Analytically, $C : [0,1]^d \to [0,1]$ is a d-dimensional copula if

(1) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots u_d) = 0$,

(2) $C(1, \ldots, 1, u, 1, \ldots 1) = u$,

(3) and C is *d-nondecreasing*, i.e, for each hyperrectangle $\prod_{i=1}^{n} [x_i, y_i] \subseteq [0,1]^d$:

$$\int_{\prod_{i=1}^{n}[x_i,y_i]} \mathrm{d}C(\mathbf{u}) \geq 0, \quad \text{with} \quad \mathbf{u} = (u_1, \ldots, u_d). \tag{1.1}$$

In the bivariate case ($d = 2$), the d-nondecreasingness property (1.1) becomes, for $0 \leq v_{11} \leq v_{12} \leq 1$ and $0 \leq v_{21} \leq v_{22} \leq 1$:

$$C(v_{12}, v_{22}) + C(v_{11}, v_{21}) \geq C(v_{12}, v_{21}) + C(v_{11}, v_{22}) \tag{1.2}$$

Consider a pair (X_1, X_2) of real valued random variables with joint distribution

$$F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2),$$

and marginal distribution functions

$$F_1(x_1) = \mathbb{P}(X_1 \leq x_1)$$

and

$$F_2(x_2) = \mathbb{P}(X_2 \leq x_2).$$

Sklar's theorem (Sklar, 1959) states that the joint distribution F, if continuous, can be expressed as a unique copula:

$$F(x_1, x_2) = \mathbb{P}\left(X_1 \leq x_1, X_2 \leq x_2\right) = C\left(F_1(x_1), F_2(x_2)\right) \quad \text{for} \quad x_1, x_2 \in \mathbb{R}. \tag{1.3}$$

When X_1 and X_2 are independent, their copula is

$$C(u_1, u_2) = \prod(u_1, u_2) = u_1 u_2. \tag{1.4}$$

The converse also holds. (1.4) is the *independence copula*.

Copulas are invariant under monotonic transformations on the marginals: the copula of the pair (X_1, X_2) is the same as that of $(g_1(X_1), g_2(X_2))$ for g_1 and g_2 increasing transformations (Genest and Favre, 2007).

A particular class of copulas is the Archimedean family. They are copulas such that $C(u_1, u_2) = \varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)$, where $\varphi : [0,1] \to [0, \infty)$ is a continuous, decreasing convex function such that $\varphi(1) = 0$. The information about the copula is contained in a univariate function φ. For example, the Clayton copula is given by $C(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}$, with $\theta \in (0, \infty)$, and has generator $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1), t \in (0, 1]$.

The distribution function K of $C(U_1, U_2)$ is known as the *Kendall function* of the copula:

$$K(t) := \mathbb{P}\left(C(U_1, U_2) \leq t\right) \quad \text{for} \quad t \in [0, 1]. \tag{1.5}$$

This function is related to the (population) Kendall tau τ, the probability of concordance minus the probability of discordance, i.e

$$\tau = \mathbb{P}\left((X_{11} - X_{12})(X_{21} - X_{22}) > 0\right) - \mathbb{P}\left((X_{11} - X_{12})(X_{21} - X_{22}) < 0\right), \tag{1.6}$$

by the relation

$$\tau = 3 - 4\int_0^1 K(t)\mathrm{d}t = 4\int_0^1 tK(\mathrm{d}t) - 1 \tag{1.7}$$

Genest and Rivest (1993) established that the Kendall function of an Archimedean copula $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2))$ is

$$K(t) = t - \frac{\varphi(t)}{\varphi'(t)} \text{ for } t \in (0,1). \tag{1.8}$$

Ghoudi, Khoudraji, and Rivest (1998) and Capéraà, Fougères, and Genest (2000) provide Kendall function formulas for various other dependence models (Archimax copulas).

A natural non-parametric estimator for C is the *empirical copula* (Deheuvels, 1979)

$$C_n(u_1,u_2) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\left\{\widehat{U}_{1i} \le u_1, \widehat{U}_{2i} \le u_2\right\} \tag{1.9}$$

where the pseudo-observations are $\widehat{U}_{ji} = \frac{n}{n+1}F_{nj}(X_{ji})$ and F_{nj} is the empirical distribution function of F_j, $j \in \{1,2\}, i \in \{1,\ldots,n\}$. The empirical copula is consistent and has a weak limit under some regularity conditions (Fermanian, Radulovic, and Wegkamp, 2004). Many rank-based statistics available in the literature are functions of the empirical copula: they can be written in the form

$$\int J(u_1,u_2)\mathrm{d}C_n(u_1,u_2) \tag{1.10}$$

for some suitably regular score function J. For example, $J(u_1,u_2) = u_1u_2$ gives Spearman's ρ and $J(u_1,u_2) = C_n(u_1,u_2)$ yields the sample Kendall tau, τ_n (see Equation (4.12) below for the equivalent formula with the population Kendall τ). Genest and Verret (2005) review methods taking advantage of this link between the empirical copula and statistical measures of dependence to provide the most powerful independence testing procedures. Deheuvels (1981) and Genest and Rémillard (2004) treat other general tests of independence based on the empirical copula process $\sqrt{n}(C_n - C)$.

Taking advantage of the invariance property of copulas and the fact that rank statistics are functions of data that exhibit the same property, Genest and Rivest (1993) advise us to draw inference from rank statistics. They develop a parameter estimation procedure for Archimedean copulas from the link between Kendall's τ and the Kendall function (in Equation (1.7)). Such an estimate is shown to be consistent and its asymptotic variance formula to be tractable.

Another direction for copula inference is the maximum likelihood maximization. Shih and Louis (1995) develop two-stage parametric and two-stage semi-parametric estimation procedures for the association parameter in copula models for bivariate survival data, in complete data as well as censored data. The two-stage semi-parametric likelihood maximization method optimizes the full likelihood function

$$\begin{aligned} L^*(\theta) = \prod_{i=1}^{n} \Big\{ \big(c_\theta(\bar{F}^0_{n1}(Z_{1i}), \bar{F}^0_{n2}(Z_{2i}))\big)^{\delta_{1i}\delta_{2i}} \left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_1}\Big|_{(u_1,u_2)=\left(\bar{F}^0_{n1}(Z_{1i}),\bar{F}^0_{n2}(Z_{2i})\right)} \right)^{\delta_{1i}(1-\delta_{2i})} \\ \left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_2}\Big|_{(u_1,u_2)=\left(\bar{F}^0_{n1}(Z_{1i}),\bar{F}^0_{n2}(Z_{2i})\right)} \right)^{(1-\delta_{1i})\delta_{2i}} \big(C_\theta(\bar{F}^0_{n1}(Z_{1i}), \bar{F}^0_{n2}(Z_{2i}))\big)^{(1-\delta_{1i})(1-\delta_{2i})} \Big\} \end{aligned} \tag{1.11}$$

where

- the observations $\{(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})\}_{i=1}^{i=n}$ are realizations of $Z_j = \min(X_j, Y_j)$, $\delta_j = \mathbb{1}\{X_j, Y_j\}$, from the non-negative real-valued random variables of interest X_j and the censoring random variables Y_j (non-negative and real-valued too), for $j \in \{1,2\}$;
- $C_\theta(.,.)$ is the copula family indexed by the parameter of interest θ;
- $c_\theta(.,.)$ is the density of the copula $C_\theta(.,.)$;
- $\bar{F}^0_{nj}(Z_{ji})$ is the univariate Kaplan-Meier estimate at Z_{ji}.

In conformity with the rank-based approach to copula inference, Genest, Ghoudi, and Rivest (1995) suggest maximizing, in the case of complete data, the pseudo-likelihood

$$L(\theta) = \prod_{i=1}^{n} c_\theta(\tilde{F}_{n1}(X_{1i}), \tilde{F}_{n2}(X_{2i})), \tag{1.12}$$

where

$$\tilde{F}_{nj}(x_{ji}) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbb{1}\{X_{ji} \le x_j\} = \frac{n}{n+1} \times \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_{ji} \le x_j\} \quad \text{for} \quad j \in \{1,2\} \tag{1.13}$$

are rescaled versions of the usual empirical distribution functions. The rescaling aims at avoiding numerical issues that may arise when the copula has arguments (values of the empirical distribution) that are too close to the boundaries of $[0,1]$.

Let θ_n be the parametric estimate of θ obtained under the null hypothesis. Denote $K_\theta(t) := \mathbb{P}\left(C_\theta(F_1(X_1), F_2(X_2)) \leq t\right)$ and $K_{\theta_n}(t) := \mathbb{P}\left(C_{\theta_n}(F_1(X_1), F_2(X_2)) \leq t\right)$. Estimate the Kendall distribution by the empirical distribution

$$K_n(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\widehat{V}_i \leq t\} \tag{1.14}$$

of the pseudo-observations

$$\widehat{V}_i = \frac{1}{n+1}\sum_{j=1}^{n} \mathbb{1}\{X_{1i} \leq x_{1j}, X_{2i} \leq x_{2j}\}, i = 1 \ldots n. \tag{1.15}$$

Wang and Wells (2000b) used the functional

$$S_{\xi n} = \int_{\xi}^{1} |K_n(t) - K_{\theta_n}(t)|^2 \, \mathrm{d}t \quad \text{where} \quad \xi \quad \text{is an arbitrary cutoff point in} \quad (0,1) \tag{1.16}$$

to develop a goodness-of-fit test for right censored data with an Archimedean copula structure. Although the limiting distribution of this statistic is available, it is hard to derive analytically. A bootstrap method was proposed as an alternative in their paper, but it proved to be ineffective.

Genest, Quessy, and Rémillard (2006) have since offered two improvements on the statistic $S_{\xi n}$:

$$S_n = n\int_0^1 |K_n(t) - K_{\theta_n}(t)|^2 K_{\theta_n}(\mathrm{d}t) \tag{1.17}$$

and

$$T_n = \sup_{t\in[0,1]} \left| \sqrt{n}\left(K_n(t) - K_{\theta_n}(t)\right)\right|. \tag{1.18}$$

There is currently a significant literature that offers an overview of survival analysis. We would like to suggest some readings for the reader interested in the field.

Genest and Favre (2007) provide an easy access to rank-based inferential methods about copulas. Two noteworthy monographs on copulas are Nelsen (2006) and Joe (2014). Nelsen (2006) is an introduction to copula theory and focuses mainly on bivariate cases, though multivariate cases are sketched out at the end of the chapters. It presents properties of copulas, methods for constructing them and the important role they play in modeling dependence. Joe (2014) is a comprehensive reference work on copula modelling. The book provides an overview of the different constructions that give rise to copulas. Known parametric models are presented with their dependence and tail properties. As a book driven mainly by applications, it also provides inference, diagnostics and model selection for copula applications.

1.3 Outline of the thesis

This thesis tackles several problems with regards to inference using copulas, in the cases of complete and right censored data. It is organized as follows:

- In Chapter 2, we derive the distribution for the joint survival function of two standard uniform random variables with (joint) distribution given by an Archimedean copula. This is achieved by observing that $C^*(U_1,U_2)$ has the same distribution as $1-\varphi^{-1}(WT)-\varphi^{-1}((1-W)T)+\varphi^{-1}(T)$ where W and T are independent. W has a uniform distribution on $(0,1)$ and T is from some density $g(.)$. T is linked to the copula by the fact that $C(U_1,U_2)$ has the same distribution as $\varphi^{-1}(T)$. Note that in the univariate case, if X is a continuous random variable with cumulative distribution function $F(.)$, then both $F(X)$ and $1-F(X)$ are uniformly distributed over $[0,1]$. This result is known as the *probability integral transformation* (PIT). Extension of the PIT to 2 or more dimensions is not at all straightforward. Genest and Rivest (1993) proved an analogue of the PIT for $C(F_1(X_1),\ldots,F_d(X_d))$ in the case of an Archimedean copula. We complement their result for $C^*(F_1(X_1),F_2(X_2))$ in the bivariate case.

- In Chapter 3, we develop two estimators, φ_{n1} and φ_{n2}, for the generator φ of an Archimedean copula. This is made possible by taking advantage of the relationship between the Kendall function and the generator (in Equation (1.8) above) and by making two assumptions about φ. φ_{n1} is obtained as a solution to a differential equation whereas φ_{n2} is solution to an integral equation. The limiting behaviour of those estimators is also given, by making use mainly of the limiting distribution of the Kendall process established in Barbe, Genest, Ghoudi, and Rémillard (1996).

- Chapter 4 is dedicated to presenting a plug-in estimator (τ_n) for the Kendall tau (τ) in the case of right censored data. This plug-in estimator for τ is based on a new multivariate Kaplan-Meier estimator developed by Sen and Stute (2013) and presented in the first part of the chapter. The asymptotic variance of τ_n is obtained using its linearization form. The performance of τ_n is compared to Wang and Wells (2000a)'s plug-in estimator, both on simulated and real-world data (kidney patients' data).

- In Chapter 5, a likelihood maximization procedure for copula parameters under right censoring is introduced. The maximum likelihood estimator thus obtained is also based on the multivariate Kaplan-Meier estimator of Sen and Stute (2013). The asymptotic variance of the estimator is established through linearization. The estimator is compared to that of Shih and Louis (1995) on simulated data as well as on a Canadian life insurance and kidney patients' datasets.

- Chapter 6 recapitulates the main findings of the thesis and discusses further research that may be considered to complement this work.

Chapter 2

Distribution of the joint survival function of an Archimedean copula

2.1 Introduction

In survival analysis, interest often bears on the chances of an individual living beyond some time. If the lifetime is denoted by the real-valued non-negative random variable X of distribution function F, the probability of surviving beyond the threshold x is the *survival function* (other names: *survivor function*, *reliability function*) given by $\overline{F}(x) := \mathbb{P}(X > x) = 1 - F(x)$.

Consider a pair (X_1, X_2) of real valued random variables with joint distribution $F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2)$, and marginal distribution functions $F_1(x_1) = \mathbb{P}(X_1 \leq x_1)$ and $F_2(x_2) = \mathbb{P}(X_2 \leq x_2)$.

By Sklar's theorem (Sklar, 1959), the joint distribution F, if continuous, can be expressed as a unique copula C on the unit square:

$$F(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = C\left(\underbrace{F_1(x_1)}_{:=u_1}, \underbrace{F_2(x_2)}_{:=u_2}\right) = C(u_1, u_2) \tag{2.1}$$

When considered as random variables, the distribution function $F(X_1, X_2)$ and the copula $C(U_1, U_2)$ (where U_1 and U_2 are two uniform random variables with joint distribution C) have the same distribution function, which is known as the **Kendall distribution function** of the pair (X_1, X_2):

$$K(z) := \mathbb{P}\left(C(U_1, U_2) \leq z\right), \quad \text{for } z \in [0, 1]. \tag{2.2}$$

The **joint survival function** of two standard uniform random variables U_1, U_2 with (joint) distribution the Archimedean copula C is

$$C^*(U_1,U_2) = 1 - U_1 - U_2 + C(U_1,U_2). \tag{2.3}$$

It is important to note that C^* is not a copula: $C^*(\frac{1}{2},0) = \frac{1}{2} \neq 0$. It is linked to the **survival copula** $\bar{C}(u_1,u_2) := u_1 + u_2 - 1 + C(1-u_1, 1-u_2)$ by

$$\bar{C}(u_1,u_2) = C^*(1-u_1, 1-u_2). \tag{2.4}$$

The survival copula $\bar{C}$ is the copula bonding the marginal survival functions. It is also the copula associated with the random vector $(1-U_1, 1-U_2)$.

The relation (2.3) between C^* and C comes from

$$\begin{aligned} \mathbb{P}(U_1 > u_1, U_2 > u_2) &= \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2) + \mathbb{P}(U_1 > u_1) + \mathbb{P}(U_2 > u_2) - 1 && (2.5) \\ &= 1 - \mathbb{P}(U_1 \leq u_1) - \mathbb{P}(U_2 \leq u_2) + \mathbb{P}(U_1 \leq u_1, U_2 \leq u_2). && (2.6) \end{aligned}$$

Chakak and Ezzerg (2000) provided a general formula for the distribution of a copula $C(U_1,U_2)$:

$$K(z) = z + \int_z^1 \left(\int_0^{\psi(z,u_1)} c(u_1,u_2) \mathrm{d}u_2 \right) \mathrm{d}u_1, \;\; 0 < z \leq 1, \tag{2.7}$$

where $c(u_1,u_2) = \frac{\partial^2}{\partial u_1 \partial u_2} C(u_1,u_2)$ is the copula density of the pair $(X_1,X_2) \sim C$ and $\psi(z,u_1) = \inf\{u_2 : C(z,u_2) \geq z\}$ for $0 < z \leq u_1 \leq 1$ is the quantile function of the copula C. However, for numerical applications, the quantile function ψ is not always available.

A particular class of copulas is the **Archimedean** family. They are copulas such that $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$, where $\varphi : [0,1] \to [0,\infty)$ is a continuous, decreasing convex function such that $\varphi(1) = 0$. The information about the bivariate distribution function in this family is contained in a univariate function φ, called the **generator** of the copula.

TABLE 2.1: Examples of Archimedean copulas

name	**bivariate copula** $C_\theta(u_1,u_2)$	**generator** $\varphi_\theta(t)$ (for $t \in (0,1]$)	**parameter** θ
Independence	$u_1 u_2$	$-\log(t)$	
Clayton	$\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}$	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\theta \in (0,\infty)$
Gumbel	$\exp\left[-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta\right)^{\frac{1}{\theta}}\right]$	$(-\log(t))^\theta$	$\theta \in [1,\infty)$
Ali-Mikhail-Haq	$\frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)}$	$\log\left(\frac{1-\theta(1-t)}{t}\right)$	$\theta \in [-1,1]$

Genest and Rivest (1993) established that the Kendall distribution function of an Archimedean copula $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2))$ is

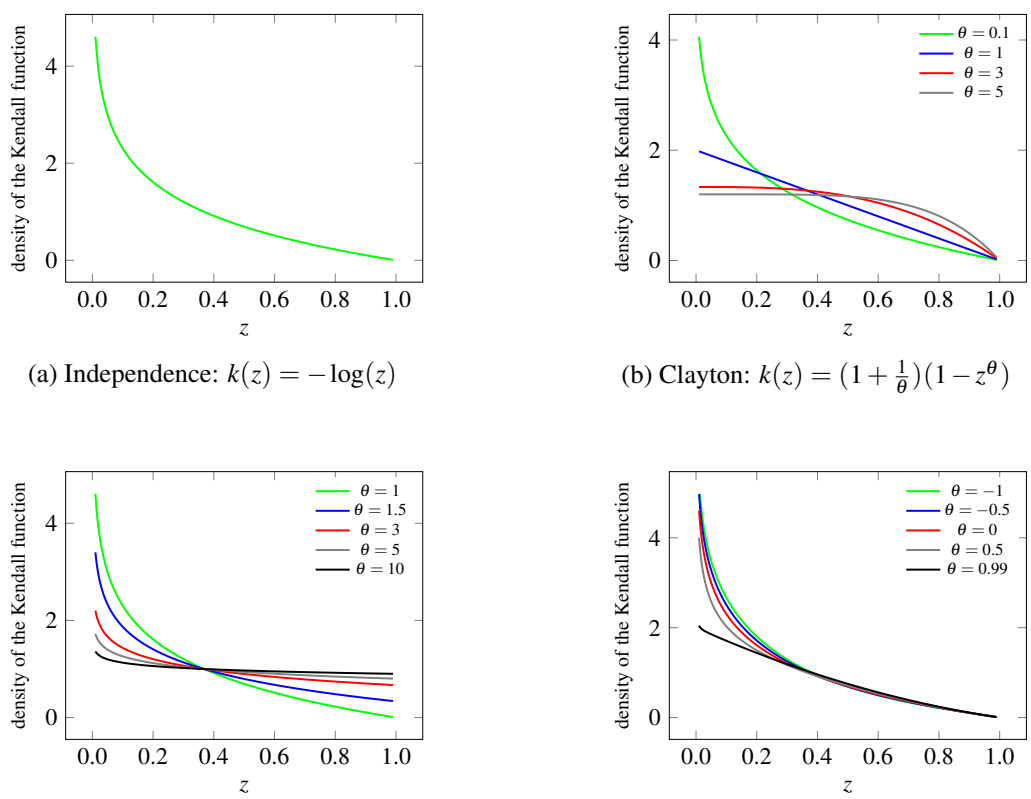
$$K(z) = z - \frac{\varphi(z)}{\varphi'(z)} \text{ for } z \in (0,1). \tag{2.8}$$

As

$$\varphi(z) \propto \exp\left[\int_{z_0}^{z} \frac{\mathrm{d}v}{v-K(v)}\right] \text{ for an arbitrary constant } 0 < z_0 < 1, \tag{2.9}$$

K is important for identifying C in the case of Archimedean copulas.

Figures (2.1a) – (2.1d) display the density $k(z) = \frac{\mathrm{d}}{\mathrm{d}z}K(z)$ of the Kendall function $K(z)$ for some Archimedean copulas.



(a) Independence: $k(z) = -\log(z)$

(b) Clayton: $k(z) = (1+\frac{1}{\theta})(1-z^{\theta})$

(c) Gumbel: $k(z) = \frac{1}{\theta}(-\log(z)+\theta-1)$

(d) Ali-Mikhail-Haq: $k(z) = \frac{(1-\theta+\theta z(2-z))(-\log(z)+\theta-1)}{1-\theta}$

FIGURE 2.1: Density $k(z)$, $z \in (0,1)$, of the Kendall distribution for some Archimedean copulas

Though the Kendall function for copulas has been known for decades, there is still no equivalent result for the survival distribution of copulas. The goal of this chapter is to obtain the density function of the joint survival function C^*.

In Section 2.2, we present our result for the bivariate case, i.e., for $C^*(U_1,U_2)$, as well as examples from the models mentioned in Table 2.1. In Section 2.3.1, plots of densities of C^* from these models and comparison with simulated values of C^* are provided. In Section 2.3.2, we present an alternative representation that may be useful in simulating $C^*(U_1,...,U_d)$ for general $d > 1$. However, the formula for the density does not seem to be tractable when $d > 2$.

Our result complements Equation (2.8) of Genest and Rivest (1993) and can be used for copula-based goodness of-fit procedures (see Genest et al., 2006). Another use for obtaining such a result would be the derivation of the weak convergence of the corresponding empirical process (supplementing the Kendall process result of Barbe et al., 1996), namely the empirical process of

$$W_{i,n} = \frac{1}{n}\sum_{j=1}^{n} \mathbb{1}\{U_{1j} > U_{1i}, U_{2j} > U_{2i}\},\ 1 \le i \le n,$$

where $(U_{1i},U_{2i}),\ 1 \le i \le n$ are independent and identically distributed (i.i.d.) with distribution $C(u_1,u_2)$. Note that the limiting covariance of $\sqrt{n}\left(\frac{1}{n}\sum_{j=1}^{n}\mathbb{1}\{W_{i,n\le z}\} - K^*(z)\right)$, $0 \le z \le 1$, will be of the form $K^*(z_1 \wedge z_2) + D(z_1,z_2)$ ($D(.,.)$ to be worked out), where $K^*(z) = \int_0^z k^*(w)\mathrm{d}w$.

2.2 Density of the survival function of a copula

Theorem 2.1. *Let U_1 and U_2 be two uniform random variables with joint distribution the Archimedean copula $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2))$. The density of the survival distribution $C^*(U_1,U_2)$ is*

$$k^*(z) = 2\int_0^{\frac{1}{2}} g\left(\psi_a^{-1}(z)\right)\left(\frac{\partial}{\partial z}\psi_a^{-1}(z)\right)\mathrm{d}a \quad \textit{for} \quad z \in [0,1], \tag{2.10}$$

where

$$g(t) = -\frac{t\varphi''\left(\varphi^{-1}(t)\right)}{\left(\varphi'\left(\varphi^{-1}(t)\right)\right)^3} \tag{2.11}$$

and

$$\psi_a(b) = \mathbb{P}\left(\frac{\varphi(U_1)}{a} \le b, \frac{\varphi(U_2)}{1-a} \le b\right) \tag{2.12}$$

is the distribution function of $\max\left(\frac{\varphi(U_1)}{a}, \frac{\varphi(U_2)}{1-a}\right)$ *for a given value* $0 < a < 1$.

The proof of the theorem is available in Appendix A.2.

Corollary 2.1.1. *$C^*(U_1,U_2)$ has the same distribution as that of $1-\varphi^{-1}(WT)-\varphi^{-1}((1-W)T)+\varphi^{-1}(T)$ where W and T are independent, W has Uniform (0,1) distribution and T has density $g(.)$, whereas $C(U_1,U_2)$ has the same distribution as that of $\varphi^{-1}(T)$.*

However, the quantile function $\psi_a^{-1}(z)$ of the distribution of $(\varphi(U_1),\varphi(U_2))$ (see Equation A.25 in the appendices) may not have a closed form expression, hindering numeric applications.

2.2.1 Examples

Let us illustrate our formula (2.10) for the density of the survival distribution through some examples.

Independence copula

The independence copula $C(u_1,u_2) = u_1u_2$, where $u_1,u_2 \in [0,1]$, has generator $\varphi(t) = -\log(t), t \in (0,1]$.

We calculate:

$$\varphi'(t) = -\frac{1}{t}, \quad \varphi''(t) = \frac{1}{t^2}, \quad \varphi^{-1}(t) = \exp(-t), \quad \text{for} \quad t \in (0,1];$$

$$g(t) = -\frac{t\frac{1}{(\exp(-t))^2}}{\left(-\frac{1}{\exp(-t)}\right)^3} = t\exp(-t), \text{ for } t \geq 0;$$

$$\psi_a(b) = 1-\exp(-ab)-\exp(-(1-a)b)+\exp(-b) = (1-\exp(-ab))\left(1-\exp\left(-(1-a)b\right)\right),$$
$$\text{for} \quad a \in [0,1], b \in [0,\infty).$$

The density of $C^*(U_1,U_2)$ is

$$k^*(z) = 2\int_0^{\frac{1}{2}} \psi_a^{-1}(z)\exp\left(-\psi_a^{-1}(z)\right)\left(\frac{\partial}{\partial z}\psi_a^{-1}(z)\right)\mathrm{d}a \quad \text{for} \quad z \in [0,1]. \tag{2.13}$$

Clayton copula

The Clayton copula

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}}, \quad \text{for} \quad \theta \in (0,\infty), u_1 \in [0,1], u_2 \in [0,1]$$

has generator

$$\varphi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1) \text{ with } t \in (0,1].$$

Then:

$$\begin{aligned}
\varphi'_\theta(t) &= -t^{-\theta-1}, \quad \varphi''_\theta(t) = (\theta+1)t^{-\theta-2}, \quad \varphi_\theta^{-1}(t) = (\theta t+1)^{-\frac{1}{\theta}} \text{ for } t \in (0,1]; \\
g_\theta(t) &= -\frac{t(\theta+1)\left((\theta t+1)^{-\frac{1}{\theta}}\right)^{-\theta-2}}{\left(-\left((\theta t+1)^{-\frac{1}{\theta}}\right)^{-\theta-1}\right)^3} = (\theta+1)t(\theta t+1)^{-2-\frac{1}{\theta}} \text{ for } t \geq 0; \\
\psi_{a,\theta}(b) &= 1 - (\theta ab+1)^{-\frac{1}{\theta}} - (\theta(1-a)b+1)^{-\frac{1}{\theta}} + (\theta b+1)^{-\frac{1}{\theta}} \text{ for } a \in [0,1], b \in [0,\infty).
\end{aligned}$$

The density of $C^*(U_1,U_2)$ is

$$k^*(z) = 2(\theta+1)\int_0^{\frac{1}{2}} \psi_{a,\theta}^{-1}(z)\left(\theta\psi_{a,\theta}^{-1}(z)+1\right)^{-2-\frac{1}{\theta}}\left(\frac{\partial}{\partial z}\psi_{a,\theta}^{-1}(z)\right)\mathrm{d}a \quad \text{for} \quad z \in [0,1]. \tag{2.14}$$

Gumbel copula

The Gumbel copula is

$$C_\theta(u_1,u_2) = \exp\left[-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta\right)^{\frac{1}{\theta}}\right] \quad \text{for} \quad \theta \in [1,\infty).$$

Its generator is given by

$$\varphi_\theta(t) = (-\log(t))^\theta \text{ where } t \in [0,1]$$

Then, for $t \in (0,1]$,

$$\begin{aligned}
\varphi'_\theta(t) &= -\frac{\theta}{t}(-\log(t))^{\theta-1}, \quad \varphi''_\theta(t) = \frac{\theta}{t^2}(-\log(t))^{\theta-2}(-\log(t)+\theta-1), \quad \varphi_\theta^{-1}(t) = \exp(-t^{\frac{1}{\theta}}); \\
g_\theta(t) &= -\frac{t\frac{\theta}{\left(\exp(-t^{\frac{1}{\theta}})\right)^2}\left(-\log\left(\exp(-t^{\frac{1}{\theta}})\right)\right)^{\theta-2}\left(-\log(\exp(-t^{\frac{1}{\theta}})\right)+\theta-1)}{\left(-\frac{\theta}{\exp(-t^{\frac{1}{\theta}})}\left(-\log\left(\exp(-t^{\frac{1}{\theta}})\right)\right)^{\theta-1}\right)^3} \\
&= \theta^{-2}t^{\frac{1}{\theta}-1}\exp(-t^{\frac{1}{\theta}})(t^{\frac{1}{\theta}}+\theta-1) \text{ where } t \geq 0; \\
\psi_{a,\theta}(b) &= 1 - \exp\left(-(ab)^{\frac{1}{\theta}}\right) - \exp\left(-((1-a)b)^{\frac{1}{\theta}}\right) + \exp(-b^{\frac{1}{\theta}}) \text{ for } a \in [0,1], b \in [0,\infty).
\end{aligned}$$

The density of $C^*(U_1,U_2)$ is

$$k^*(z) = 2\theta^{-2}\int_0^{\frac{1}{2}} \left(\psi_{a,\theta}^{-1}(z)\right)^{\frac{1}{\theta}-1} \exp\left(-\left(\psi_{a,\theta}^{-1}(z)\right)^{\frac{1}{\theta}}\right)\left(\left(\psi_{a,\theta}^{-1}(z)\right)^{\frac{1}{\theta}}+\theta-1\right)\left(\frac{\partial}{\partial z}\psi_{a,\theta}^{-1}(z)\right)\mathrm{d}a, \tag{2.15}$$
$$\text{for} \quad \theta\in(1,\infty), \quad z\in[0,1].$$

Ali-Mikhail-Haq copula

The Ali-Mikhail-Haq copula

$$C(u_1,u_2) = \frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)} \quad \text{for} \quad \theta\in[-1,1], u_1\in[0,1], u_2\in[0,1]$$

has generator

$$\varphi_\theta(t) = \log\left(\frac{1-\theta(1-t)}{t}\right) \text{ with } t\in[0,1].$$

Then:

$$\varphi_\theta'(t) = -\frac{1-\theta}{t\,(1-\theta(1-t))}, \quad \varphi_\theta''(t) = \frac{(1-\theta)\,(1-\theta+\theta t(2-t))}{t^2\,(1-\theta(1-t))^2}, \quad \varphi_\theta^{-1}(t) = \frac{1-\theta}{\exp(t)-\theta};$$
$$g_\theta(t) = -\frac{t\frac{(1-\theta)\left(1-\theta+\theta\frac{1-\theta}{\exp(t)-\theta}(2-\frac{1-\theta}{\exp(t)-\theta})\right)}{\left(\frac{1-\theta}{\exp(t)-\theta}\right)^2\left(1-\theta(1-\frac{1-\theta}{\exp(t)-\theta})\right)^2}}{\left(\frac{\theta-1}{\frac{1-\theta}{\exp(t)-\theta}\left(1-\theta(1-\frac{1-\theta}{\exp(t)-\theta})\right)}\right)^3} = \frac{(1-\theta)t(\exp(2t)-\theta)\exp(t)}{(\exp(t)-\theta)^4};$$
$$\psi_{a,\theta}(b) = 1-\frac{1-\theta}{\exp(ab)-\theta}-\frac{1-\theta}{\exp\left((1-a)b\right)-\theta}+\frac{1-\theta}{\exp(b)-\theta}.$$

The density of $C^*(U_1,U_2)$ is

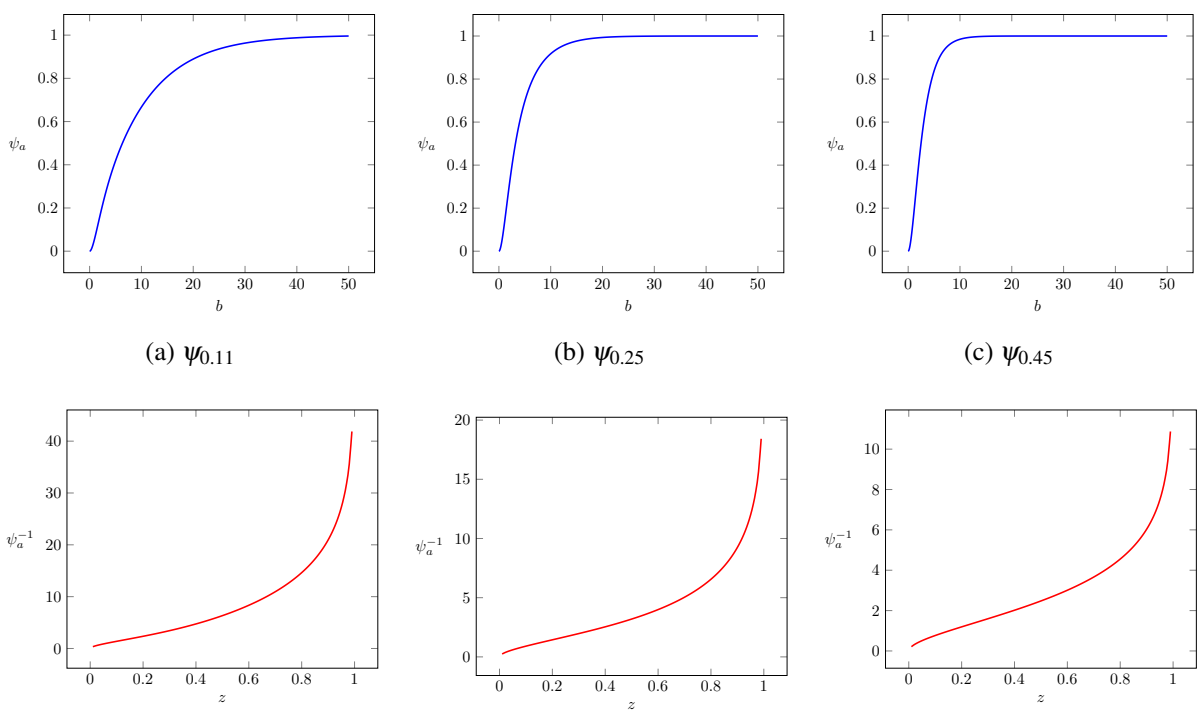
$$k^*(z) = 2(1-\theta)\int_0^{\frac{1}{2}} \frac{\psi_{a,\theta}^{-1}(z)(\exp\left(2\psi_{a,\theta}^{-1}(z)\right)-\theta)\exp\left(\psi_{a,\theta}^{-1}(z)\right)}{\left(\exp\left(\psi_{a,\theta}^{-1}(z)\right)-\theta\right)^4}\left(\frac{\partial}{\partial z}\psi_{a,\theta}^{-1}(z)\right)\mathrm{d}a \tag{2.16}$$
$$\text{for } \theta\in[-1,1), \quad z\in[0,1].$$

2.3 Illustration

In this section, we give the results of our method.

2.3.1 Illustration of our method

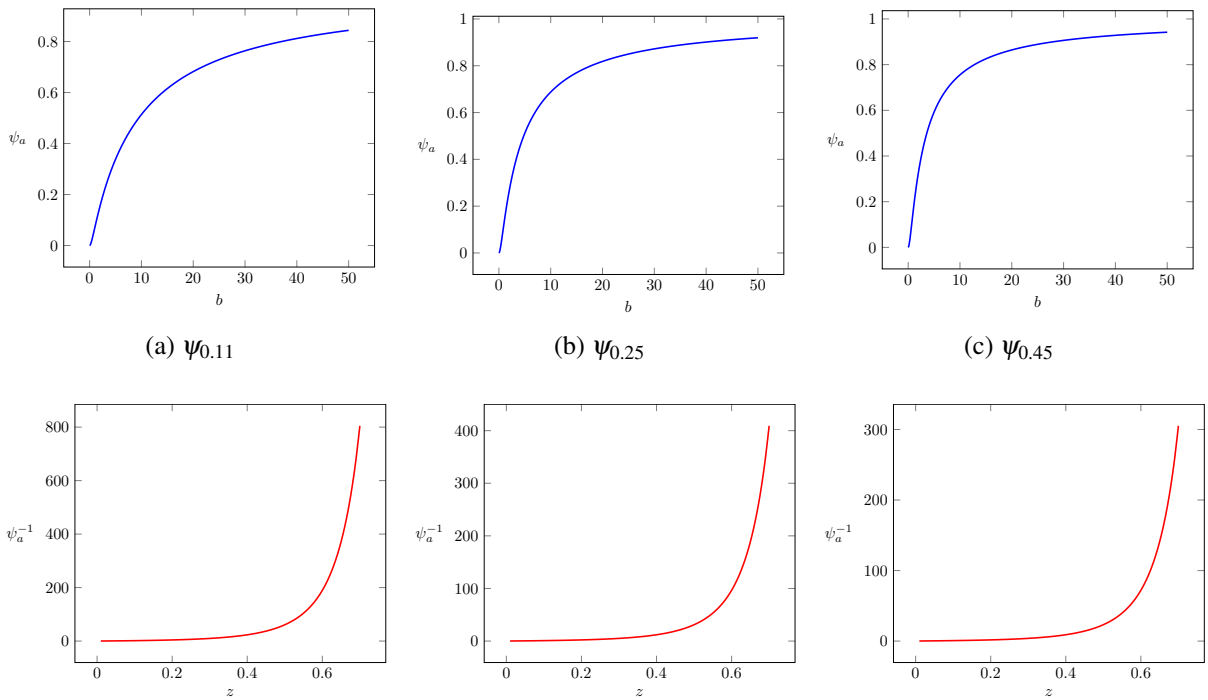
Figures (2.2) and (2.3) below (as well as Figures A.1 and A.2 in Appendix A.1) show graphs of the distribution function ψ_a and the numerically determined quantile function ($\widehat{\psi}_a^{-1}$) of $B = \varphi(U_1) + \varphi(U_2)$ for some values of $a = \frac{\varphi(u_1)}{\varphi(u_1)+\varphi(u_2)}$:



(a) $\psi_{0.11}$ (b) $\psi_{0.25}$ (c) $\psi_{0.45}$

(d) $\widehat{\psi}_{0.11}^{-1}$ (e) $\widehat{\psi}_{0.25}^{-1}$ (f) $\widehat{\psi}_{0.45}^{-1}$

FIGURE 2.2: Distribution function ψ_a and its estimated inverse $\widehat{\psi}_a^{-1}$ for $a = 0.11$, $a = 0.25$, and $a = 0.45$ in the case of the independence copula



(a) $\psi_{0.11}$ (b) $\psi_{0.25}$ (c) $\psi_{0.45}$

(d) $\widehat{\psi}_{0.11}^{-1}$ (e) $\widehat{\psi}_{0.25}^{-1}$ (f) $\widehat{\psi}_{0.45}^{-1}$

FIGURE 2.3: Distribution function ψ_a and its estimated inverse $\widehat{\psi}_a^{-1}$ for $a = 0.11$, $a = 0.25$, and $a = 0.45$ in the case of the Clayton copula with $\theta = 1$

For the independence copula,

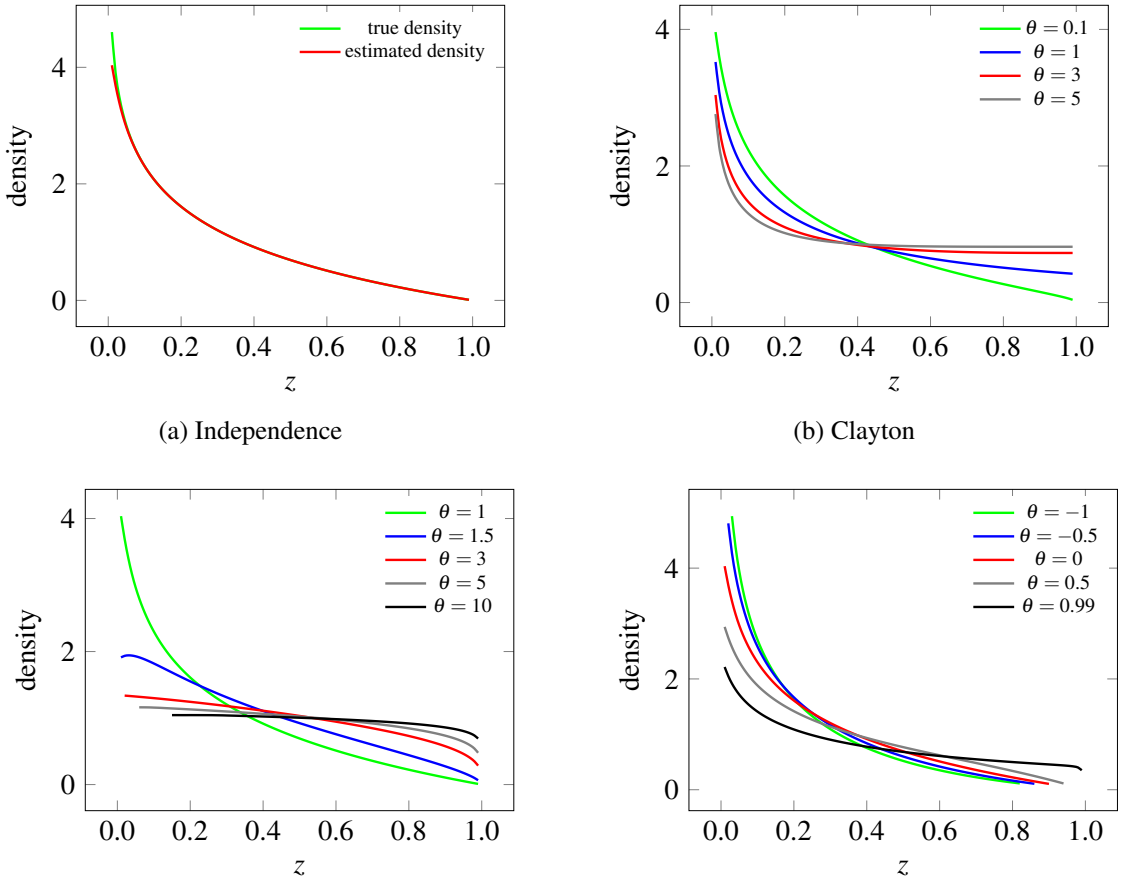
$$C^*(U_1, U_2) = 1 - U_1 - U_2 + U_1 U_2 \tag{2.17}$$
$$= (1 - U_1)(1 - U_2) \tag{2.18}$$
$$= \exp\left(-\log(1 - U_1) - \log(1 - U_2)\right) \tag{2.19}$$

As U_1 and U_2 are uniformly distributed on $[0,1]$, it follows that $\log(1-U_1) + \log(1-U_2) \sim \Gamma(2,1)$ (gamma distribution of shape 2 and scale 1) and $C^*(U_1, U_2)$ has density

$$(-\log(z)) \exp\left(-(-\log(z))\right) \frac{1}{z} = -\log(z), \text{ for } z \in (0,1]. \tag{2.20}$$

Knowing the density of C^* , we can evaluate in this case how our method performs. The estimated density is satisfactorily close to the true density (Figure 2.4a).

Figures (2.4b) to (2.4d) show the density of $C^*(U_1, U_2)$ (from Theorem 2.1) for some Archimedean copulas:



(a) Independence

(b) Clayton

(c) Gumbel

(d) Ali-Mikhail-Haq

FIGURE 2.4: Density of C^* for some Archimedean copulas

2.3.2 Simulating values of C^*

With $U \sim \mathscr{U}[0,1], V \sim \mathscr{U}[0,1], (U_1,U_2) \sim C$ and (U_1',U_2') an independent copy of (U_1,U_2),

$$C^* = C^*(U_1,U_2) = \mathbb{P}(U_1' \geq U_1, U_2' \geq U_2 | U_1,U_2) = \mathbb{P}(\varphi(U_1') \leq \varphi(U_1), \varphi(U_2') \leq \varphi(U_2) | U_1,U_2) \tag{2.21}$$

Taking advantage of the calculations in Appendix A.2, we have

$$C^* = \mathbb{P}(\varphi(U_1) \leq WT, \varphi(U_2) \leq (1-W)T | W,T) = 1 - \varphi^{-1}(WT) - \varphi^{-1}((1-W)T) + \varphi^{-1}(T), \tag{2.22}$$

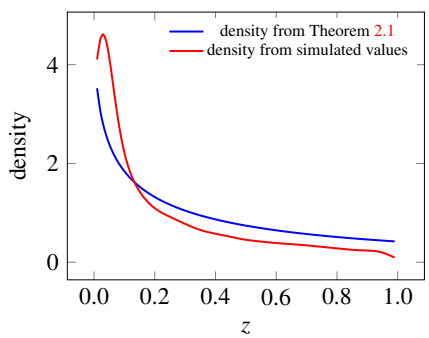
where

$$W = \frac{\varphi(U_1)}{\varphi(U_1)+\varphi(U_2)} \sim \mathscr{U}[0,1] \text{ and } T = \varphi(U_1)+\varphi(U_2) \sim G \text{ are independent.}$$

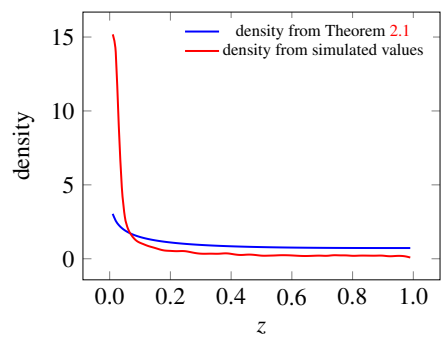
T can be simulated by observing that $\varphi^{-1}(T) = S$ has density $k(s) = \frac{\varphi(s)\varphi''(s)}{(\varphi'(s))^2}$ for $0 \leq s \leq 1$, which is of course the density of the Kendall distribution function $K(s)$; see page 7 and Figures (2.1a) – (2.1d), page 9. For example, the density of S is $\frac{\theta+1}{\theta}(1-s^\theta)$ (with $\theta \in (0,\infty)$) for a Clayton copula and $-\frac{1}{\theta}\log(s) + (1-\frac{1}{\theta})$ (with $\theta \in [1,\infty)$) for a Gumbel copula.

Then, from the simulated values of W and T, we can compute the values of C^*.

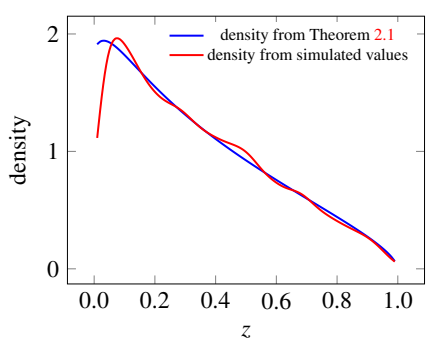
Figures (2.5a) to (2.5d) compare the densities obtained from the application of Theorem 2.1 to those obtained from simulated values of $C^*(U_1,U_2)$ for some Archimedean copulas:



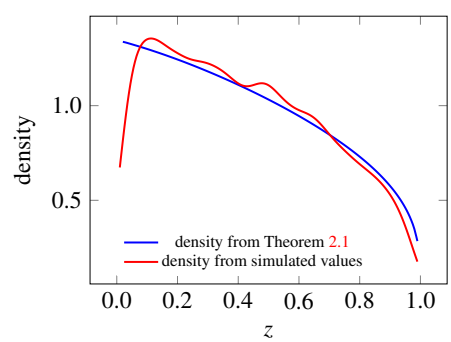
(a) Clayton copula with $\theta = 1$



(b) Clayton copula with $\theta = 3$



(c) Gumbel copula with $\theta = 1.5$



(d) Gumbel copula with $\theta = 3$

FIGURE 2.5: Comparison of the density from application of Theorem 2.1 and the density simulated using Corollary 2.1.1

Remark 2.3.1. *Let $d \geq 2$ be an integer, $(U_1, \cdots, U_d)$ a d-tuple of standard uniform random variables whose joint distribution is given by a multivariate Archimedean copula C with generator φ.*

Let $T_j = \varphi(U_j), j = 1, ..., d$. Then $(T_1, ..., T_d)$ has joint survival function $\varphi^{-1}(t_1 + ... + t_d)$.

With $T = T_1 + ... + T_d$, $C(U_1, ..., U_d)$ is distributed as $\varphi^{-1}(T)$ and $C^*(U_1, ..., U_d)$ is distributed as $1 + \sum_{j=1}^{d}(-1)^j \varphi^{-1}(T_1 + ... + T_j)$.

However, a formula for the densities or simulation of C or C^* does not seem to be tractable.

2.4 Conclusion

Inspired by the importance of Kendall functions for identifying Archimedean copula models, we decided to examine the distribution of the survival functions from Archimedean copulas in this chapter. We were able to derive a formula for the density. This was achieved by observing that $C^*(U_1, U_2)$ has the same distribution as $1 - \varphi^{-1}(WT) - \varphi^{-1}((1-W)T) + \varphi^{-1}(T)$ where W and T are independent, W has a uniform distribution on $(0,1)$ and T is from some density $g(.)$. Moreover, the approach used allows a representation that is useful in simulating the values of the survival distribution function of copulas. But the difficulty in finding an analytical form of quantile functions in general limits the reach of the formula.

Chapter 3

Estimation of the generator of an Archimedean copula

3.1 Introduction

Because they allow us to model the dependence between different random variables without consideration of the marginal distributions, copulas are useful tools in multi-dimensional statistics. In the case of Archimedean copulas, of the form $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2))$, the Kendall function

$$K(t) = t - \frac{\varphi(t)}{\varphi'(t)} \quad \text{for} \quad t \in (0,1). \tag{3.1}$$

provided in Genest and Rivest (1993) allows some inference methods.

Using a stochastic representation of Archimedean copulas discovered by McNeil and Nešlehová (2009), Genest, Nešlehová, and Ziegel (2011) provide a non-parametric estimator for φ^{-1}. From the sample $\{W_1,\ldots,W_n\}$ taking the distinct values $0 < w_1 < \ldots < w_m < 1$ and the values $0 < r_1 < \ldots < r_m < \infty$, the estimator is

$$\psi_n(t) = \sum_{k=1}^{m} \max\left\{\left(1-\frac{t}{r_k}\right)^{d-1};0\right\} p_k, \tag{3.2}$$

with the weights

$$p_k = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{W_i = w_{m-k+1}\}} \tag{3.3}$$

and

d is the dimension of the copula.

The w_i's and r_i's are obtained by solving the system of equations

$$w_1 = 0, \tag{3.4}$$

$$w_2 = \left(1 - \frac{r_{m-1}}{r_m}\right)^{d-1} p_m, \tag{3.5}$$

$$w_3 = \left(1 - \frac{r_{m-2}}{r_m}\right)^{d-1} p_m + \left(1 - \frac{r_{m-2}}{r_{m-1}}\right)^{d-1} p_{m-1}, \tag{3.6}$$

$$\vdots$$

$$w_m = \left(1 - \frac{r_1}{r_m}\right)^{d-1} p_m + \ldots + \left(1 - \frac{r_1}{r_2}\right)^{d-1} p_2 \tag{3.7}$$

Even if ψ_n gives a unique estimator for the copula, it is not an estimator of φ^{-1} *per se*, since φ^{-1} and $c\varphi^{-1}$, with $c > 0$, both generate the same Archimedean copula. It is also computationally hard. Moreover, its limiting behaviour is not available.

In this chapter, two non-parametric methods for properly estimating the generator of an Archimedean copula and their asymptotic variances are provided. The derivation of the estimators relies on assumptions about the generator. The assumptions are presented in Section 3.2. Sections 3.3 and 3.4 present the estimators and their asymptotic distributions. The last section (3.5) of the chapter illustrates the performance of the estimators.

3.2 Assumptions

From the relation (3.1) above, one can derive

$$\begin{aligned} K(t) &= t - \frac{\varphi(t)}{\varphi'(t)} \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\left(-\log\left(\varphi(t)\right)\right) &= -\frac{\varphi'(t)}{\varphi(t)} = \frac{1}{K(t) - t}. \end{aligned} \tag{3.8}$$

For some $\varepsilon \in (0,1)$:

$$\int_w^{1-\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}\left(-\log\left(\varphi(t)\right)\right)\mathrm{d}t = -\log\left(\varphi(t)\right)\Big|_w^{1-\varepsilon} \tag{3.9}$$

$$= -\log\left(\varphi(1-\varepsilon)\right) + \log\left(\varphi(w)\right) \tag{3.10}$$

$$= \log\left(\frac{\varphi(w)}{\varphi(1-\varepsilon)}\right) \tag{3.11}$$

Then:

$$\exp\left(\int_w^{1-\varepsilon} \frac{\mathrm{d}t}{K(t)-t}\right) = \exp\left(\int_w^{1-\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t}\left(-\log\left(\varphi(t)\right)\right)\mathrm{d}t\right) = \frac{\varphi(w)}{\varphi(1-\varepsilon)}$$

$$\implies \varphi(w) = \frac{\varphi(1-\varepsilon)}{\varepsilon}\varepsilon\exp\left(\int_w^{1-\varepsilon} \frac{\mathrm{d}t}{K(t)-t}\right) \tag{3.12}$$

We make the following assumptions about the generator of the copula in order to derive the estimators.

Assume that

Assumption 3.2.1.

$$-\varphi'(1) = 1.$$

That assumption is equivalent to:

$$\frac{\varphi(1-\varepsilon)}{\varepsilon} \to 1 \quad \text{as} \quad \varepsilon \to 0.$$

Assume also that

Assumption 3.2.2.

$$\lim_{\varepsilon\downarrow 0} \frac{\sigma(1-\varepsilon)}{\varepsilon} = \sigma_0 > 0$$

where

$$\sigma^2(w) = \Gamma(w,w) = \lim_{n\to\infty} \mathbb{V}\left(\sqrt{n}\left(K_n(w) - K(w)\right)\right) \quad \textit{for} \quad 0 < w < 1$$

is the asymptotic variance of the Kendall process (Barbe et al., 1996) and $K_n(.)$ is the empirical distribution of the pseudo-sample of variables

$$V_{i,n} = \frac{1}{n-1}\sum_{j=1, j\neq i}^{n} \mathbb{1}\{X_{1j} \leq X_{1i}, X_{2j} \leq X_{2i}\}, \textit{ for } i = 1,\ldots,n,$$

based on a sample $\{(X_{11},X_{21}),\ldots,(X_{1n},X_{2n})\}$, $n \geq 2$.

For Archimedean copulas (see Barbe et al., 1996), the asymptotic variance of the Kendall process is given by

$$\Gamma(w,w) = K(w)\,(1-K(w)) + k(w)\,[k(w)R(t,t) - 2t\,(1-K(w))]\,, \tag{3.13}$$

where

$$R(t,t) = \mathbb{E}\left\{\varphi^{-1}\left(\max\left(\varphi(t)U_2^{(1)},\varphi(t)U_3^{(1)}\right) + \max\left(\varphi(t)U_2^{(2)},\varphi(t)U_3^{(2)}\right)\right)\right\} - t^2, \tag{3.14}$$

with $U_2 = \left(U_2^{(1)},U_2^{(2)}\right)$ and $U_3 = \left(U_3^{(1)},U_3^{(2)}\right)$ two independent copies of a random vector uniformly distributed in $[0,1]^2$. This variance seems not derivable except for the independence copula.

Let us check the Assumptions 3.2.1 and 3.2.2 on some Archimedean copulas. Due to the difficulty in obtaining an analytic form of the variance 3.13 for copulas other than the independent one, 3.2.2 is examined only for the independence case.

Independence copula

The independence copula $C(u_1,u_2) = u_1u_2$, where $u_1,u_2 \in [0,1]$, admits the generator $\varphi(t) = -\log(t), t \in (0,1]$.

We have

$$\varphi'(t) = -\frac{1}{t}$$

and

$$-\varphi'(1) = 1.$$

The limiting standard deviation of the Kendall process is $\sigma(t) = \sqrt{t^2 - t - t\log t}$.

So that

$$\lim_{\varepsilon_n \downarrow 0} \frac{\sigma(1-\varepsilon_n)}{\varepsilon_n} = \lim_{\varepsilon_n \downarrow 0} \frac{\sqrt{(1-\varepsilon_n)(-\varepsilon_n - \log(1-\varepsilon_n))}}{\varepsilon_n} = \frac{1}{\sqrt{2}}$$

Clayton copula

The Clayton copula

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}} \quad \text{for} \quad \theta \in (0,\infty), u_1 \in [0,1], u_2 \in [0,1]$$

has generator

$$\varphi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1) \text{ with } t \in (0,1].$$

Then:

$$\varphi'_\theta(t) = -t^{-\theta-1}$$

and

$$-\varphi'_\theta(1) = 1.$$

Gumbel copula

The Gumbel copula is

$$C_\theta(u_1,u_2) = \exp\left[-\left((-\log(u_1))^\theta + (-\log(u_2))^\theta\right)^{\frac{1}{\theta}}\right] \quad \text{for} \quad \theta \in [1,\infty).$$

It admits the generator given by

$$\varphi_\theta(t) = (-\log(t))^\theta \text{ where } t \in [0,1]$$

Then

$$\varphi'_\theta(t) = -\frac{\theta}{t}(-\log(t))^{\theta-1}$$

and

$$-\varphi'_\theta(1) = 0 \quad \text{if} \quad \theta > 1.$$

Ali-Mikhail-Haq copula

The Ali-Mikhail-Haq copula

$$C(u_1,u_2) = \frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)} \quad \text{for} \quad \theta \in [-1,1], u_1 \in [0,1], u_2 \in [0,1]$$

admits the generator

$$\varphi_\theta(t) = \frac{1}{1-\theta}\log\left(\frac{1-\theta(1-t)}{t}\right) \text{ with } t \in [0,1].$$

Then:

$$\varphi'_\theta(t) = -\frac{1}{t\,(1-\theta(1-t))}$$

and

$$-\varphi'_\theta(1) = 1$$

Frank copula

The Frank copula

$$C(u_1,u_2) = -\frac{1}{\theta}\log\left[1+\frac{(\exp(-\theta u_1)-1)\,(\exp(-\theta u_2)-1)}{\exp(-\theta)-1}\right], \text{ for } \theta \neq 0, u_1 \in [0,1], u_2 \in [0,1],$$

admits the generator

$$\varphi(t) = \frac{1-\exp(\theta)}{\theta}\log\left(\frac{\exp(-\theta t)-1}{\exp(-\theta)-1}\right) \text{ when } \theta > 0.$$

Then,

$$\varphi'(t) = -\frac{(1-\exp(\theta))\exp(-\theta t)}{\exp(-\theta t)-1}$$

and

$$-\varphi'(1) = 1.$$

3.3 An estimator from a differential equation point of view

In light of Equation (3.12), we propose this estimator for φ:

$$\varphi_{n1}(w) = \varepsilon_n \exp\left(\int_w^{1-\varepsilon_n} \frac{\mathrm{d}t}{K_n(t)-t}\right) \quad \text{for} \quad 0 < w < 1, \tag{3.15}$$

where

- $\varepsilon_n \to 0$ as $n \to \infty$
- $K_n(.)$ is the empirical distribution of the pseudo-sample $\{V_{i,n}\}_{i=1}^{i=n}$, with

$$V_{i,n} = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} \mathbb{1}\{X_{1j} \leq X_{1i}, X_{2j} \leq X_{2i}\}, \text{ for } i = 1, \ldots, n, \tag{3.16}$$

based on a random sample $\{(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})\}$, $n \geq 2$.

Remark 3.3.1. *In (3.15), we have the limit:*

$$\begin{aligned}
\textit{For} \quad 0 < w < 1: \quad \exp\left(\int_w^{1-\varepsilon_n} \frac{\mathrm{d}t}{K_n(t) - t}\right) &\to \exp\left(\int_w^1 \frac{\mathrm{d}t}{K(t) - t}\right) \quad \textit{as} \quad n \to 0 && (3.17)\\
&= \exp\left(-\int_w^1 \frac{\varphi'(t)}{\varphi(t)} \mathrm{d}t\right) && (3.18)\\
&= \exp\left(-\left[\log\left|\varphi(t)\right|\right]_{t=w}^{t=1}\right) && (3.19)\\
&= \exp\left(-\left[\log\left|\underbrace{\varphi(1)}_{=0}\right| - \log\left|\varphi(w)\right|\right]\right) && (3.20)\\
&= +\infty && (3.21)
\end{aligned}$$

Thus, $\varphi_{n1}(w)$ does not vanish as $n \to \infty$.

From ordered distinct values $v_{(1)},\dots,v_{(m)}$ of the pseudo-sample $\{V_{i,n}\}_{i=1}^{i=n}$, we compute $\varphi_{n1}(w)$ as follows:

- For $0 \le w < 1-\varepsilon_n$, define $v_{(k)} := \max_{\text{s.t } v_{(j)} \le w} v_{(j)}, j = 1,\dots,m$ and $v_{(l)} := 1-\varepsilon_n$.

  Then

$$\varphi_{n1}(w) = \varepsilon_n \exp\left(\int_w^{1-\varepsilon_n} \frac{\mathrm{d}t}{K_n(t)-t}\right) \tag{3.22}$$

$$= \varepsilon_n \exp\left(\int_{v_{(k)}}^{v_{(l)}} \frac{\mathrm{d}t}{K_n(t)-t}\right) \tag{3.23}$$

$$= \varepsilon_n \exp\left(\sum_{j=k}^{l-1} \int_{v_{(j)}}^{v_{(j+1)}} \frac{\mathrm{d}t}{K_n(t)-t}\right) \tag{3.24}$$

$$= \varepsilon_n \exp\left(\sum_{j=k}^{l-1} \int_{v_{(j)}}^{v_{(j+1)}} \frac{\mathrm{d}t}{K_n(v_{(j)})-t}\right) \tag{3.25}$$

$$= \varepsilon_n \exp\left(-\sum_{j=k}^{l-1} \int_{v_{(j)}}^{v_{(j+1)}} \frac{-\mathrm{d}t}{K_n(v_{(j)})-t}\right) \tag{3.26}$$

$$= \varepsilon_n \exp\left(-\sum_{j=k}^{l-1} \Big(\log(|K_n(v_{(j)})-v_{(j+1)}|) - \log(|K_n(v_{(j)})-v_{(j)}|)\Big)\right) \tag{3.27}$$

$$= \varepsilon_n \exp\left(\sum_{j=k}^{l-1} \log\left|\frac{K_n(v_{(j)})-v_{(j)})}{K_n(v_{(j)})-v_{(j+1)})}\right|\right) \tag{3.28}$$

$$= \varepsilon_n \exp\left(\log\left(\prod_{j=k}^{l-1}\left|\frac{K_n(v_{(j)})-v_{(j)})}{K_n(v_{(j)})-v_{(j+1)})}\right|\right)\right) \tag{3.29}$$

$$= \varepsilon_n \prod_{j=k}^{l-1}\left|\frac{K_n(v_{(j)})-v_{(j)})}{K_n(v_{(j)})-v_{(j+1)})}\right| \tag{3.30}$$

- For $1-\varepsilon_n < w \le 1$,

$$\varphi_{n1}(w) = 0. \tag{3.31}$$

We turn now to the asymptotic behaviour of the estimator.

3.3.1 Asymptotic distribution

Barbe et al. (1996) showed that, under some regularity conditions, the Kendall process $\sqrt{n}\left(K_n(t)-K(t)\right)$ converges in distribution to a continuous, centered Gaussian process. The required regularity conditions are shown to be met with Archimedean copulas such as independence, Clayton, Franck, Gumbel and Ali-Mikhail-Haq copulas.

Theorem 3.1. *Under the Assumptions 3.2.1 and 3.2.2, we have the limiting distribution of our estimator φ_{n1} in (3.15):*

If $\frac{\log \varepsilon_n}{\sqrt{n}} \to 0$ *as* $n \to \infty$*, then, for* $0 < w < 1$,

$$\frac{\sqrt{n}}{\log \varepsilon_n}\left(\varphi_{n1}(w)-\varphi(w)\right) \to \sigma_0 \varphi(w) Z \quad \textit{in distribution, as} \quad n \to \infty,$$

where Z is a standard Gaussian random variable and $\sigma_0 = \lim_{\varepsilon \downarrow 0} \frac{\sigma(1-\varepsilon)}{\varepsilon}$, with σ^2 being the limiting variance of the Kendall process (Barbe et al., 1996).

The reader may refer to Appendix B.1 for a proof of this theorem.

3.4 An estimator from an integral equation point of view

From the relation (3.1), one can also derive

$$\begin{aligned} K(t) &= t - \frac{\varphi(t)}{\varphi'(t)} \\ \Longrightarrow \quad -\varphi'(t) &= \frac{1}{K(t)-t}\varphi(t) = \frac{1}{K(t)-t}\int_t^1 \left(-\varphi'(v)\right) \mathrm{d}v \end{aligned} \tag{3.32}$$

With $h(t) := -\varphi'(t)$,

$$\begin{aligned} h(t) &= \frac{1}{K(t)-t}\int_t^1 h(v)\mathrm{d}v \\ &= \frac{1}{K(t)-t}\int_{K^{-1}(t)}^1 h\left(K(w)\right) K(\mathrm{d}w) \end{aligned} \tag{3.33}$$

Denote by Z_i the jump points of $K_n(.)$, the empirical distribution of the pseudo-sample $\{V_{i,n}\}_{i=1}^{i=n}$ based on a sample $\{(X_{11},X_{21}),\ldots,(X_{1n},X_{2n})\}$: $V_{i,n} = \frac{1}{n-1}\sum_{j=1,j\neq i}^{n} \mathbb{1}\left\{X_{1j} \leq X_{1i}, X_{2j} \leq X_{2i}\right\}, i = 1,\ldots,n, , n \geq 2$.

At $t = \frac{i}{n}$, Equation (3.33) becomes:

$$h\left(\frac{i}{n}\right) = \frac{1}{K_n\left(\frac{i}{n}\right) - \frac{i}{n}} \frac{1}{n} \int_{Z_i}^{1} h\left(K_n(w)\right) K_n(\mathrm{d}w) \tag{3.34}$$

$$= \frac{1}{K_n\left(\frac{i}{n}\right) - \frac{i}{n}} \frac{1}{n} \sum_{j=i+1}^{n} h\left(\frac{j}{n}\right) \text{ for } 1 \leq i \leq n-1, \tag{3.35}$$

By defining

$$\mathbf{K} := \begin{bmatrix} 0 & \frac{1}{nK_n(\frac{1}{n})-1} & \frac{1}{nK_n(\frac{1}{n})-1} & \dots & \frac{1}{nK_n(\frac{1}{n})-1} & \dots & \frac{1}{nK_n(\frac{1}{n})-1} & \frac{1}{nK_n(\frac{1}{n})-1} \\ 0 & 0 & \frac{1}{nK_n(\frac{2}{n})-2} & \dots & \frac{1}{nK_n(\frac{2}{n})-2} & \dots & \frac{1}{nK_n(\frac{2}{n})-2} & \frac{1}{nK_n(\frac{2}{n})-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \frac{1}{nK_n(\frac{n-1}{n})-(n-1)} \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix}_{n \times n}$$

and

$$\mathbf{h} := \begin{bmatrix} h\left(\frac{1}{n}\right) \\ h\left(\frac{2}{n}\right) \\ \vdots \\ h\left(\frac{n-1}{n}\right) \\ h(1) \end{bmatrix}_{n \times 1},$$

Equation (3.35) can be rewritten in the following matrix notation:

$$\mathbf{h} = \mathbf{K}\mathbf{h} \tag{3.36}$$

Thus, $\mathbf{h}$ is an eigenvector of $\mathbf{K}$ associated with the eigenvalue 1, similarly to the mass-shifting estimator (see Equation 4.36).

Recursively,

- $h_n(1) = 1$ (from Assumption (3.2.1))
- for $i = n-1, \dots, 1$:

$$h_n\left(\frac{i}{n}\right) = \frac{1}{nK_n\left(\frac{i}{n}\right) - i} \sum_{j=i+1}^{n} h_n\left(\frac{j}{n}\right) \tag{3.37}$$

Or

- $h_n(1) = 1$
- for $i = n-1, \ldots, 1$:

$$h_n\left(\frac{i}{n}\right) = \frac{1}{nK_n\left(\frac{i}{n}\right) - i} \prod_{j=i+1}^{n-1} \left(\frac{1}{nK_n\left(\frac{j}{n}\right) - j} + 1 \right) \tag{3.38}$$

This estimator is thus a product-limit estimator, a property that exhibits the Kaplan-Meier estimator for the survivor function too (see Equation 4.21).

For $\frac{i}{n} \leq t < \frac{i+1}{n} (i = 1, \ldots, n-1)$, we interpolate the value of $h_n(t)$ as a step-function:

$$h_n(t) = h_n\left(\frac{i}{n}\right). \tag{3.39}$$

Then, we suggest the estimator

$$\varphi_{i,n2}(t) = \int_t^1 h_n(s)\mathrm{d}s = \frac{1}{n} \sum_{j=i}^{n} h_n\left(\frac{j}{n}\right). \tag{3.40}$$

3.4.1 Asymptotic distribution

With basic calculations, it is possible to derive the limiting distribution of φ_{n2} in Theorem 3.2 below. The reader may refer to Appendix B.2 for the algebra details.

Theorem 3.2. *Under the Assumption 3.2.2 and denoting the weak limits* $\sqrt{n}\left(\varphi_{n2}(w) - \varphi(w)\right) \to \varphi_0(w)$ *and* $\sqrt{n}\left(K_n(w) - K(w)\right) \to K_0(w)$ *(when* $n \to \infty$*), we have*

$$\varphi_0(t) = -\int_t^1 \left(\frac{\varphi'(w)}{\varphi(w)}\right)^2 \varphi(w)K_0(w)\mathrm{d}w - \int_t^1 \frac{\varphi'(w)}{\varphi(w)} \varphi_0(w)\mathrm{d}w. \tag{3.41}$$

The covariance function $S(s,t) := cov\left(\varphi_0(s), \varphi_0(t)\right)$ *satisfies the equation*

$$
\begin{aligned}
S(s,t) = &\int_s^1 \int_t^1 g_\varphi(w)\Gamma(w,v)g_\varphi(v)\mathrm{d}w\mathrm{d}v \\
&+ \int_s^1 \int_t^1 g_\varphi(w)\overline{G}_\varphi(v)A(w,v)g_\varphi(v)\mathrm{d}w\mathrm{d}v \\
&+ \int_s^1 \int_t^1 \overline{G}_\varphi(w)g_\varphi(v)A(w,v)g_\varphi(v)\mathrm{d}w\mathrm{d}v \\
&+ \int_s^1 \int_t^1 \overline{G}_\varphi(w)\overline{G}_\varphi(v)S(w,v)g_\varphi(v)\mathrm{d}w\mathrm{d}v,
\end{aligned}
$$

where

$$
g_\varphi := \left(\frac{\varphi'}{\varphi}\right)^2 \varphi, \quad \overline{G}_\varphi := \frac{\varphi'}{\varphi}, \quad A(w,v) := cov\left(K_0(w), \varphi_0(v)\right), \quad \Gamma(w,v) = cov\left(K_0(w), K_0(v)\right).
$$

Further, $A(s,t)$ *satisfies*

$$
A(s,t) = -\int_t^1 g_\varphi(w)\Gamma(s,w)\mathrm{d}w - \int_t^1 \overline{G}_\varphi(w)A(s,w)\mathrm{d}w.
$$

3.5 Illustration

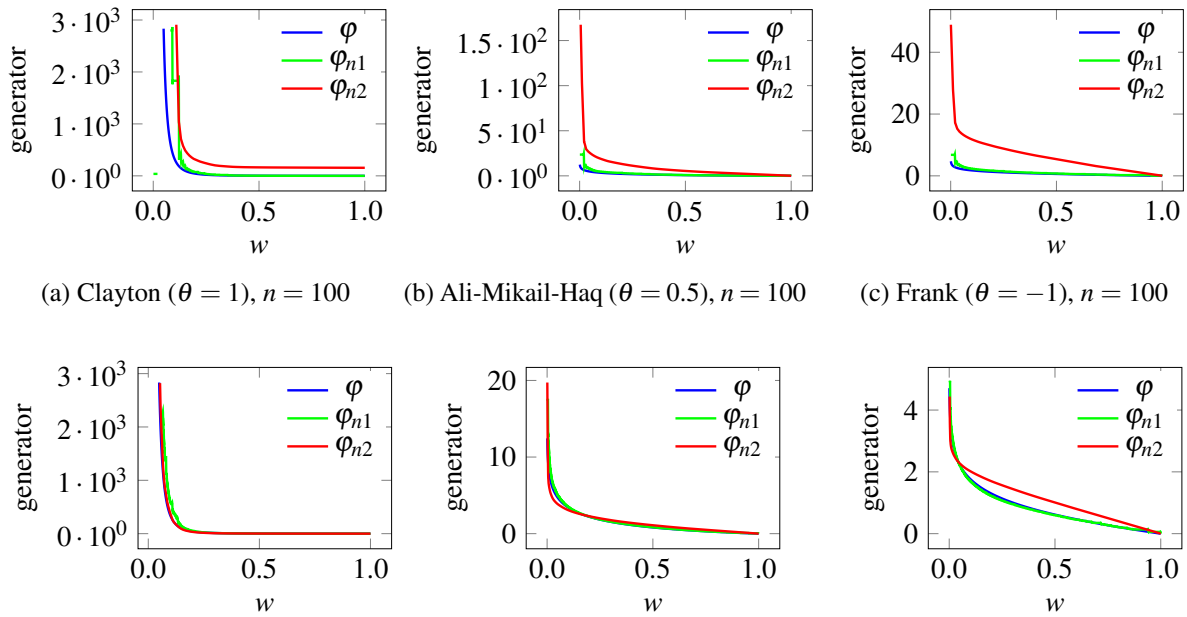
We simulate n values of two random variables X_1 and X_2 from a Weibull distribution of shape $\alpha = 2$ and scale $\beta = 2$, with density

$$
f(x) = \frac{x}{2}\exp\left(-\left(\frac{x}{2}\right)^2\right) \text{ for } x \geq 0, \tag{3.42}
$$

and joint distribution an Archimedean copula.

Using the programming language R (R Core Team), the functions *mvdc* and *rMvdc* (in the package *copula*) allow us to build a copula object with specified marginal distributions and to simulate observations from it.

Figures (3.1a) - (3.1f) show the performance of our two estimators for three Archimedean copulas, with sample size of $n = 100$ and $n = 500$. We choose $\varepsilon_n = \frac{1}{n}$. The x-axis values are from 0.001 to 1 with a step of 0.001.



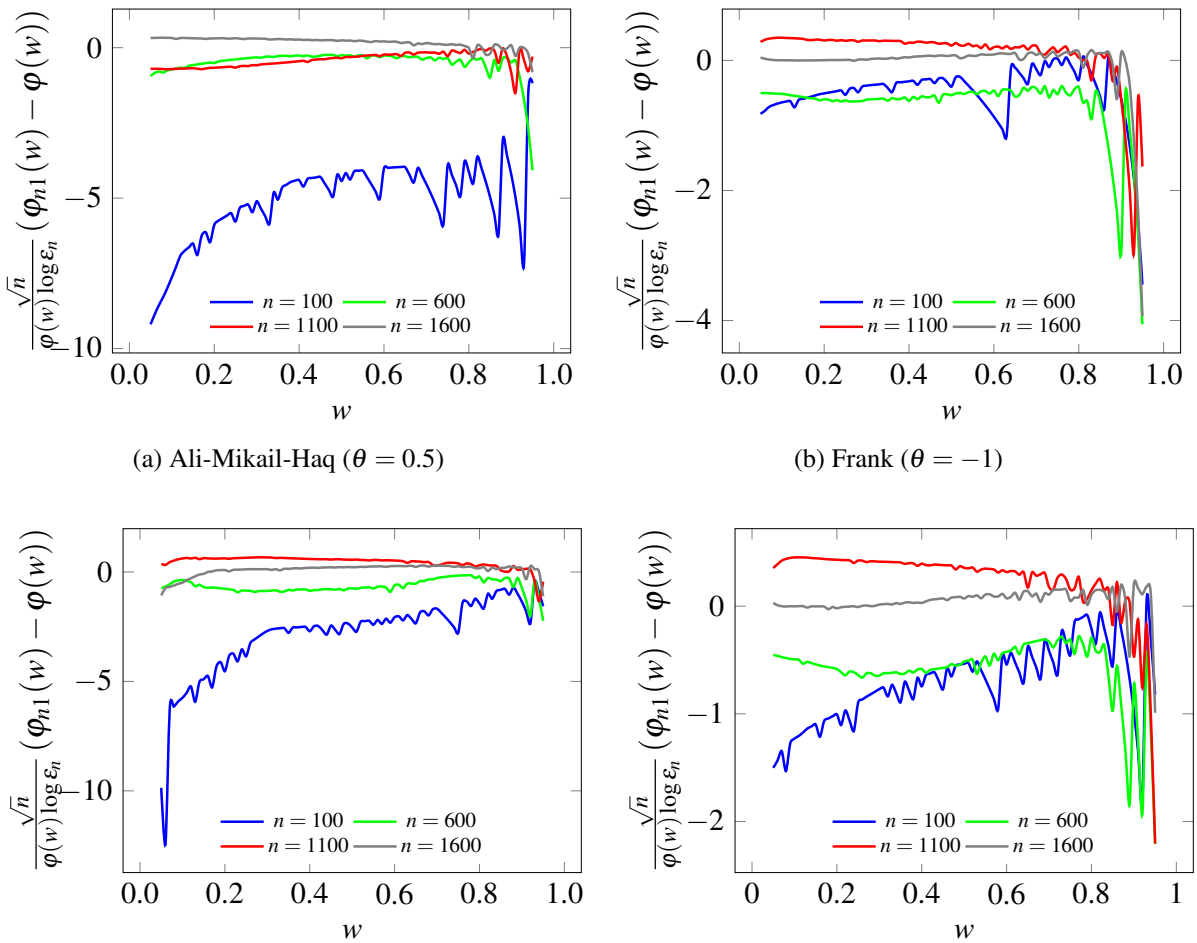
(a) Clayton ($\theta = 1$), $n = 100$ (b) Ali-Mikail-Haq ($\theta = 0.5$), $n = 100$ (c) Frank ($\theta = -1$), $n = 100$

(d) Clayton ($\theta = 1$), $n = 500$ (e) Ali-Mikhail-Haq ($\theta = 0.5$), $n = 500$ (f) Frank ($\theta = -1$), $n = 500$

FIGURE 3.1: Estimation of the generator φ

The estimates improve as the sample size increases. The estimation is less accurate for small values of t (near 0). The integral estimator (φ_{n2}) is slower to approach the true generator than the differential estimator (φ_{n1}).

Though the difficulty of obtaining an analytic formula for the variance of the Kendall process limits the derivation of σ_0, it is possible to have an idea of the convergence in Theorem 3.1. Figures (3.2a) - (3.2a) show the convergence of $\frac{\sqrt{n}}{\varphi(w)\log\varepsilon_n}\left(\varphi_{n1}(w) - \varphi(w)\right)$, with $\varepsilon_n = \frac{1}{n}$, for some Archimedean copulas. As expected from Theorem 3.1, that quantity approaches a flat line due to asymptotic degeneracy, as n grows. However, the convergence is particularly hard for values of w close to the boundary 1.



(a) Ali-Mikail-Haq ($\theta = 0.5$)

(b) Frank ($\theta = -1$)

(c) Clayton ($\theta = 1$)

(d) Independence

FIGURE 3.2: Convergence of $\frac{\sqrt{n}}{\varphi(w)\log\varepsilon_n}(\varphi_{n1}(w)-\varphi(w))$ (from Theorem 3.1)

3.6 Conclusion

Taking advantage of the formula of the Kendall distribution of Archimedean copulas, we have been able to derive two non-parametric estimators for the generator φ of a bivariate Archimedean copula. The derivation was made possible by admitting two assumptions about the generator and the limiting distribution of the Kendall process. The first estimator, φ_{n1}, was obtained from a differential equation point of view and the second one,φ_{n2}, from an integral equation standpoint. Asymptotic behaviours for both were established: φ_{n2} is faster in its convergence (in $\sqrt{n}$) compared to φ_{n1} (convergence in $\frac{\sqrt{n}}{|\log\varepsilon_n|}$).

Chapter 4

A plug-in estimator for Kendall's tau under right censoring

4.1 Introduction

Kendall's tau is a non-parametric criterion for measuring the dependence between random variables. Let (X_{11}, X_{21}) and (X_{12}, X_{22}) be two independent and identically distributed (iid) copies of a vector (X_1, X_2) of non-negative real-valued random variables. The population Kendall's tau is the probability of concordance minus the probability of discordance. When X_1 and X_2 are continuous, it is

$$\tau = \tau_{(X_1,X_2)} = \mathbb{P}\left((X_{11} - X_{12})(X_{21} - X_{22}) > 0\right) - \mathbb{P}\left((X_{11} - X_{12})(X_{21} - X_{22}) < 0\right). \tag{4.1}$$

In absence of censoring, τ is estimated by the U-statistics

$$\hat{\tau} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \operatorname{sign}\left((X_{1i} - X_{1j})(X_{2i} - X_{2j})\right) \tag{4.2}$$

where $\{(X_{1i}, X_{2i})\}_{i=1}^{i=n}$ is a sample from (X_1, X_2).

$\hat{\tau}$ is an unbiased estimator of τ. Moreover, $\sqrt{n}(\hat{\tau} - \tau)$ is asymptotically normal (Hoeffding, 1948).

In this chapter, we propose an estimator for τ in the presence of right-censored data.

Let

- $F(x_1, x_2) = \mathbb{P}(X_1 \le x_1, X_2 \le x_2)$ be the joint distribution function of (X_1, X_2),
- $\overline{F}(x_1, x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2)$ be the joint survival function,

- $F_1(x_1) = \mathbb{P}(X_1 \leq x_1)$ be the (marginal) distribution of X_1,
- $F_2(x_2) = \mathbb{P}(X_2 \leq x_2)$ be the (marginal) distribution of X_2.

As (X_{11}, X_{21}) and (X_{12}, X_{22}) are iid copies of the same vector (X_1, X_2),

$$\begin{aligned}
&\mathbb{P}\left((X_{11}-X_{12})(X_{21}-X_{22})>0\right)\\
&=\mathbb{P}\left(X_{11}-X_{12}>0, X_{21}-X_{22}>0\right)+\mathbb{P}\left(X_{11}-X_{12}<0, X_{21}-X_{22}<0\right) && (4.3)\\
&=\mathbb{P}\left(X_{11}>X_{12}, X_{21}>X_{22}\right)+\mathbb{P}\left(X_{11}<X_{12}, X_{21}<X_{22}\right) && (4.4)\\
&=2\mathbb{P}\left(X_{11}>X_{12}, X_{21}>X_{22}\right). && (4.5)
\end{aligned}$$

Integrating over the distribution of (X_{12}, X_{22}):

$$\begin{aligned}
\mathbb{P}\left(X_{11}>X_{12}, X_{21}>X_{22}\right) &= \int\int \mathbb{P}\left(X_{11}>x_1, X_{21}>x_2\right) F(\mathrm{d}x_1, \mathrm{d}x_2) && (4.6)\\
&= \int\int \bar{F}(x_1,x_2) F(\mathrm{d}x_1, \mathrm{d}x_2). && (4.7)
\end{aligned}$$

Then:

$$\mathbb{P}\left((X_{11}-X_{12})(X_{21}-X_{22})>0\right) = 2\int\int \bar{F}(x_1,x_2) F(\mathrm{d}x_1, \mathrm{d}x_2) \tag{4.8}$$

From the continuity of X_1 and X_2,

$$\mathbb{P}\left((X_{11}-X_{12})(X_{21}-X_{22})<0\right) = 1-\mathbb{P}\left((X_{11}-X_{12})(X_{21}-X_{22})>0\right). \tag{4.9}$$

Therefore, Equation (4.1) can be written as

$$\begin{aligned}
\tau_{(X_1,X_2)} =& 2\int\int \bar{F}(x_1,x_2) F(\mathrm{d}x_1, \mathrm{d}x_2) - \left(1 - 2\int\int \bar{F}(x_1,x_2) F(\mathrm{d}x_1, \mathrm{d}x_2)\right) && (4.10)\\
=& 4\int\int \bar{F}(x_1,x_2) F(\mathrm{d}x_1, \mathrm{d}x_2) - 1. && (4.11)
\end{aligned}$$

Denoting by C the copula representation for the distribution of (X_1, X_2), the Kendall tau coefficient is

$$\tau_{(X_1,X_2)} = \tau_C = 4\int_0^1\int_0^1 C(u_1,u_2)\mathrm{d}C(u_1,u_2) - 1 = 4\mathbb{E}\left(C(U_1,U_2)\right) - 1. \tag{4.12}$$

In terms of the distribution function K of $C(U_1,U_2)$,

$$\tau_C = 4\int_0^1 tK(\mathrm{d}t) - 1 = 3 - 4\int_0^1 K(t)\mathrm{d}t. \tag{4.13}$$

Genest and Rivest (1993) proved that the Kendall distribution function K of an Archimedean copula, i.e a copula that can be written as $C(u_1,u_2) = \varphi^{-1}(\varphi(u_1)+\varphi(u_2))$ with a convex non-increasing function $\varphi : [0,1] \to [0,\infty)$ such that $\varphi(1) = 0$, is

$$K(t) = t - \frac{\varphi(t)}{\varphi'(t)} \text{ for } t \in (0,1). \tag{4.14}$$

This chapter presents an estimator for the Kendall tau based on a novel estimator for the bivariate survival function developed by Sen and Stute (2013). Before introducing the Kendall tau estimator, it is useful to present that survival function estimator, called *mass-shifting estimator* in this thesis.

4.2 Mass-shifting estimator

Since the late 1980s, the estimation of the bivariate survival function $\overline{F}(x_1,x_2) = \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2)$ of a pair of censored non-negative random variables has received some attention in survival analysis.

Dabrowska (1988) generalized the product integral representation of the univariate case to the bivariate case. She also established the almost sure consistency of the estimator. However, the estimator gives negative masses at some points (Pruit, 1991).

Prentice and Cai (1992) provide an estimator for the bivariate survival function based on the marginal survivor functions and the covariance function of counting process martingales.

Gill, Laan, and Wellner (1995) examine properties of three estimators (Dabrowska's product integral representation, the Volterra estimator and Prentice and Cai's estimator). They showed the efficiency of the Dabrowska and Prentice-Cai estimators for the bivariate censoring model under independence. Bootstrap validity for estimating the variance of Dabrowska's estimator is also provided.

Lopez (2012) develops a nonparametric estimator of the bivariate survival function using an additional mass at (∞,∞), to make the estimator a proper survival function.

4.2.1 Univariate case: the Kaplan-Meier estimator

Let X be a non-negative real-valued random variable with (unknown) distribution function F, right censored by another non-negative real-valued random variable Y with (unknown) distribution G. Suppose further that X and Y are independent. Define $Z := \min(X,Y)$ and $\delta := \mathbb{1}\{X \leq Y\}$.

We observe a sample: $(Z_i, \delta_i), i = 1, ..., n$.

How would we estimate the tail probability $\overline{F}(x) = \mathbb{P}(X \geq x)$?

Note that censored observations have density

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}(\delta = 0, Z \leq z) = \overline{F}(z)G(\mathrm{d}z), \quad \text{where} \quad \overline{F}(z) = \mathbb{P}(X \geq z) \tag{4.15}$$

and uncensored observations have density

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}(\delta = 1, Z \leq z) = \overline{G}(z)F(\mathrm{d}z), \text{ where } \overline{G}(z) = \mathbb{P}(Y \geq z) \tag{4.16}$$

So that the likelihood of a sample $(Z_i, \delta_i), i = 1, ..., n$ is:

$$\prod_{i=1}^{n} \left\{\overline{F}(z_i)G(\mathrm{d}z_i)\right\}^{1-\delta_i} \left\{\overline{G}(z_i)F(\mathrm{d}z_i)\right\}^{\delta_i} \propto \prod_{i=1}^{n} \overline{F}(z_i)^{1-\delta_i} F(\mathrm{d}z_i)^{\delta_i} \tag{4.17}$$

Order the observed durations z_i's as $z_{1:n}$ (minimum), $z_{2:n}$, ..., $z_{n:n}$ (maximum) and denote by p_i the mass of the observation $z_{i:n}$.

Then

$$\prod_{i=1}^{n} \overline{F}(z_i)^{1-\delta_i} F(\mathrm{d}z_i)^{\delta_i} = \prod_{i=1}^{n} (\sum_{j=i}^{n} p_j)^{1-\delta_i} p_i^{\delta_i} \tag{4.18}$$

The likelihood is maximized with

$$p_1 = \frac{\delta_1}{n}, \tag{4.19}$$

$$p_i = \frac{\delta_i}{n-i+1} \prod_{j=1}^{i-1} \left(1 - \frac{\delta_i}{n-j+1}\right) \quad \text{for} \quad i = 2, \ldots, n \tag{4.20}$$

Those weights define the Kaplan-Meier estimator for the survival function $\overline{F}$:

$$\overline{F}_n^{KM}(x) = \sum_{i=1}^{n} p_i \mathbb{1}\{Z_{i:n} \geq x\} = \prod_{i=1}^{n} \left(1 - \frac{\delta_i}{n-i+1}\right)^{\mathbb{1}\{Z_{i:n} \leq x\}} \tag{4.21}$$

4.2.2 Bivariate case: the mass-shifting estimator

The bivariate survival function $\overline{F}(x_1,x_2) = \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2)$ is a solution to the integral equation

$$\overline{F}(\mathbf{x}) = \int_{\mathbf{x} \leq \mathbf{t} < \infty} \overline{F}(\mathbf{t}) \frac{F(\mathrm{d}\mathbf{t})}{\overline{F}(\mathbf{t})}, \text{ with } \overline{F}(\mathbf{0}) = 1 \tag{4.22}$$

where, and from now on, bold cases denote (bivariate) vectors and inequalities involving vectors are considered componentwise.

The problem here is that, unlike the univariate case, a bivariate survival function is not identifiable through its hazard function $\frac{F(\mathrm{d}\mathbf{t})}{\overline{F}(\mathbf{t})}$. Let $\overline{F}_1 = 1 - F_1$ and $\overline{F}_2 = 1 - F_2$ be univariate survival functions on the positive real line with densities f_1 and f_2 respectively. Define $\overline{F}$ as $\overline{F}(x_1,x_2) := \overline{F}_1(x_1)\overline{F}_2(x_2)$ (independence case). Then the hazard function of $\overline{F}$ is

$$h(x_1,x_2) = \underbrace{\frac{f_1(x_1)}{\overline{F}_1(x_1)}}_{h_1(x_1)} \underbrace{\frac{f_2(x_2)}{\overline{F}_2(x_2)}}_{h_2(x_2)} \tag{4.23}$$

Now, note that for any real number $a > 0$: $h(x_1,x_2) = (ah_1(x_1))\left(a^{-1}h_2(x_2)\right)$, i.e there exists a whole collection of different distributions on the plane with the same hazard function h.

To avoid this identifiability issue, Sen and Stute (2013) consider the mass-shifted survival function:

$$\overline{F}_\varepsilon(\mathbf{x}) = \begin{cases} (1-\varepsilon)\overline{F}(\mathbf{x}) + \varepsilon & \text{if} \quad \mathbf{x} \in \mathbb{R}^2 \\ \varepsilon & \text{if} \quad \mathbf{x} = \mathbf{x}_\infty \end{cases} \tag{4.24}$$

where $0 \leq \varepsilon \leq 1$ and $\mathbf{x}_\infty$ is a vector, possibly (∞,∞), which exceeds all $\mathbf{x}$ componentwise in the support of F.

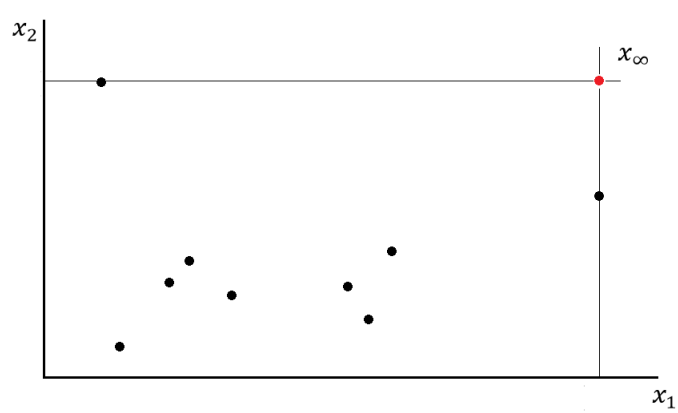


FIGURE 4.1: Infinity point $\mathbf{x}_\infty$

$\overline{F}_\varepsilon(\mathbf{x})$ is then solution to the integral equation:

$$\overline{F}_\varepsilon(\mathbf{x}) = \int_{\mathbf{x}\leq\mathbf{t}<\mathbf{x}_\infty} \overline{F}_\varepsilon(\mathbf{t}) \frac{(1-\varepsilon)F(\mathrm{d}\mathbf{t})}{(1-\varepsilon)\overline{F}(\mathbf{t})+\varepsilon} + \overline{F}_\varepsilon(\mathbf{x}_\infty), \text{ with } \overline{F}_\varepsilon(\mathbf{0}) = 1 \tag{4.25}$$

For right censored $\mathbf{X} = (X_1, X_2) \sim F$, we can only observe realizations of $(\delta_1, \delta_2, Z_1, Z_2)$, where $\delta_j = \mathbb{1}\{X_j \leq Y_j\}$ and $Z_j = \min(X_j, Y_j)$, for $j = 1,2$, with Y_1, Y_2 being the censoring variables. As in the univariate case, an important assumption is the independence of X_j and Y_j, $j = 1,2$. Under that assumption,

$$\frac{F(\mathrm{d}\mathbf{x})}{\overline{F}(\mathrm{d}\mathbf{x})} = \frac{F(\mathrm{d}\mathbf{x})\overline{G}(\mathbf{x})}{\overline{F}(\mathbf{x})\overline{G}(\mathbf{x})} = \frac{H^{11}(\mathrm{d}\mathbf{x})}{\overline{H}(\mathbf{x})} \tag{4.26}$$

where $\overline{G}(y_1, y_2) = \mathbb{P}(Y_1 \geq y_1, Y_2 \geq y_2)$, $H^{11}(\mathbf{x}) = H^{11}(x_1, x_2) = \mathbb{P}(Z_1 \leq x_1, Z_2 \leq x_2, \delta_1 = 1, \delta_2 = 1)$ and $\overline{H}(x_1, x_2) = \mathbb{P}(Z_1 \geq x_1, Z_2 \geq x_2)$.

The joint survival function under (right) censoring $\overline{F}^0_\varepsilon(\mathbf{x})$ is then solution to the integral equation

$$\overline{F}^0_\varepsilon(\mathbf{x}) = \int_{\mathbf{x}\leq\mathbf{t}<\mathbf{x}_\infty} \overline{F}^0_\varepsilon(\mathbf{t}) \frac{(1-\varepsilon)H^{11}(\mathrm{d}\mathbf{t})}{(1-\varepsilon)\overline{H}(\mathbf{t})+\varepsilon} + \overline{F}^0_\varepsilon(\mathbf{x}_\infty), \text{ with } \overline{F}^0_\varepsilon(\mathbf{0}) = 1 \tag{4.27}$$

Conjecture 4.2.1. *Equations (4.27) and (4.25) are equivalent, i.e, they have the same solutions.*

Empirically, given a sample $(\delta_{1i}, \delta_{2i}, Z_{1i}, Z_{2i}), i = 1, \ldots, n$, Equation (4.27) becomes, with $\varepsilon = \frac{1}{n+1}$,

$$\overline{F}^0_{\varepsilon,n}(\mathbf{x}) = \int_{\mathbf{x}\leq\mathbf{t}<\mathbf{x}_\infty} \overline{F}^0_{\varepsilon,n}(\mathbf{t}) \frac{H^{11}_n(\mathrm{d}\mathbf{t})}{\overline{H}_n(\mathbf{t})+\frac{1}{n}} + \frac{1}{n+1}, \text{ with } \overline{F}^0_{\varepsilon,n}(\mathbf{0}) = 1 \tag{4.28}$$

where

$$H^{11}_n(\mathbf{t}) = H^{11}_n(t_1, t_2) = \frac{1}{n}\sum_{i=1}^n \delta_{1i}\delta_{2i}\mathbb{1}\{Z_{1i} \leq t_1, Z_{2i} \leq t_2\} \tag{4.29}$$

$$\overline{H}_n(\mathbf{t}) = \overline{H}_n(t_1, t_2) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{Z_{1i} \geq t_1, Z_{2i} \geq t_2\} \tag{4.30}$$

Define

$$a_{ik} = \begin{cases} 1 & \text{if} \quad Z_{1k} \geq Z_{1i} \text{ and } Z_{2k} \geq Z_{2i} \\ 0 & \text{otherwise} \end{cases} \tag{4.31}$$

and extend the definition to $a_{i,n+1} = 1$ for $1 \leq i \leq n$ or $i = n+1$

Denote by p_i the mass given to the observation $(Z_{1i}, Z_{2i}), i = 1, \ldots, n$ by the estimator $\overline{F}^0_{\varepsilon,n}$ and $p_{n+1} = \frac{1}{n+1}$ the mass of $\mathbf{x}_\infty$. And

$$\overline{F}^0_i := \overline{F}^0_{\varepsilon,n}(Z_{1i}, Z_{2i}) = \sum_{k=1}^{n} a_{ik} p_k + \frac{1}{n+1} \tag{4.32}$$

Let

$$b_i := \frac{\Delta H_n^{11}(Z_{1i}, Z_{2i})}{\overline{H}_n(Z_{1i}, Z_{2i}) + \frac{1}{n}} = \frac{\delta_{1i}\delta_{2i}}{n\overline{H}_n(Z_{1i}, Z_{2i}) + 1} \tag{4.33}$$

With $\mathbf{x} = (x_1, x_2) = (Z_{1i}, Z_{2i})$, Equation (4.28) becomes:

$$\sum_{k=1}^{n} a_{ik} p_k = \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_k a_{kl} \right) p_l \tag{4.34}$$

In matrix notation:

$$\mathbf{Ap} = \mathbf{ABAp}, \sum_{i=1}^{n} p_i = 1, \tag{4.35}$$

where

$$\mathbf{A} = ((a_{ik}))_{1 \leq i,k \leq n}, \mathbf{B} = \mathrm{diag}(b_1, \ldots, b_n).$$

With $\overline{\mathbf{F}}^{\mathbf{0}} = (\overline{F}^0_1, \ldots, \overline{F}^0_n)$, Equation (4.35) may be written as

$$\overline{\mathbf{F}}^{\mathbf{0}} = \mathbf{AB}\overline{\mathbf{F}}^{\mathbf{0}}, \sum_{i=1}^{n} b_i \overline{F}^0_i = 1. \tag{4.36}$$

By ordering the pairs (Z_{1i}, Z_{2i}) in the increasing order of the first coordinate and with a convention for the ties, $\mathbf{A}$ becomes an upper-triangular matrix because

$$a_{ik} = \begin{cases} 0 & \text{if} \quad k < i \\ 1 & \text{if} \quad k = i \\ 0 \text{ or } 1 & \text{if} \quad k > i \end{cases} \tag{4.37}$$

Thus, $\mathbf{A}$ is invertible and Equation (4.35) is equivalent to

$$\mathbf{p} = \mathbf{BAp}, \sum_{i=1}^{n+1} p_i = 1. \tag{4.38}$$

Sen and Stute (2013) showed that the equations (4.35) and (4.36) admit a unique solution:

$$p_i = c_i \underbrace{\left(1 + \sum_{k=i+1}^{n} a_{ik}c_k + \sum_{l=i+1}^{n}\sum_{k=i+1}^{n} a_{ik}a_{kl}c_kc_l + \ldots + a_{i,i+1}a_{i+1,i+2}\ldots a_{n-1,n}c_{i+1}c_{i+2}\ldots c_n\right)}_{:=d_i} p_{n+1} \tag{4.39}$$

where

$$c_i = \frac{b_i}{1-b_i} \tag{4.40}$$

and the updated mass at $\mathbf{x}_\infty$ is

$$p_{n+1} = \left[1 + \sum_{i=1}^{n} c_i d_i\right]^{-1}. \tag{4.41}$$

Our *estimation path* is as follows:

- Our target: $\overline{F}$
- What we got through mass-shifting: $\overline{F}_\varepsilon$
- Considering censoring, we had: $\overline{F}^0_\varepsilon$
- Our estimator: $\overline{F}^0_{\varepsilon,n}$

How do we make sure that we have a consistent estimator? By triangle inequality:

$$|\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}(\mathbf{x})| \leq |\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0_\varepsilon(\mathbf{x})| + \underbrace{|\overline{F}^0_\varepsilon(\mathbf{x}) - \overline{F}_\varepsilon(\mathbf{x})|}_{=o(1) \text{ as } \varepsilon\to 0} + \underbrace{|\overline{F}_\varepsilon(\mathbf{x}) - \overline{F}(\mathbf{x})|}_{=\varepsilon\left(1-\overline{F}(\mathbf{x})\right)=O(\varepsilon)} \tag{4.42}$$

The part $\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0_\varepsilon(\mathbf{x})$ is handled by **linearization**, through the **influence function** (Sen and Stute, 2013), that is by writing

$$\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0_\varepsilon(\mathbf{x}) = L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n}) + o_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \tag{4.43}$$

where $L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n})$ is solution to

$$\begin{aligned} L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n}) - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{t} \neq \infty\} L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} &= \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{t} \neq \infty\} \sqrt{n} \left(\frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})} - \overline{H}_n(\mathbf{t}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \right) \\ - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}, \mathbf{t} \neq \infty\} L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} &= \int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}, \mathbf{t} \neq \infty\} \sqrt{n} \left(\frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})} - \overline{H}_n(\mathbf{t}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \right) \end{aligned} \tag{4.44}$$

4.2.3 The influence function and its covariance

The influence function is a measure of the effect of a small perturbation at some point on an estimator. In addition to being a measure of robustness, the influence function is useful for calculating the asymptotic variance: under some conditions, the asymptotic variance of the estimator equals the integral of the square of the influence function (Hampel, Rousseeuw, Ronchetti, and Stahel, 1986).

Sen and Stute (2013) show that $\sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right)$ has influence function $L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n})$ satisfying

$$\begin{aligned} L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n}) - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} &= \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} \left[\frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})} - \overline{H}_n(\mathbf{t}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \right] \\ L_{\mathbf{0}}(\overline{F}^0_{\varepsilon,n}) &= 0 \end{aligned} \tag{4.45}$$

Equation (4.45) may be rewritten as:

$$L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n}) - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} = \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} \mathrm{d}\alpha_n(\mathbf{t}) \tag{4.46}$$

$$- \int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}\} L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} = \int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}\} \mathrm{d}\alpha_n(\mathbf{t}), \tag{4.47}$$

where

$$\mathrm{d}\alpha_n(\mathbf{t}) := \left[\frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})} - \overline{H}_n(\mathbf{t}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \right], \ \mathbf{t} \geq \mathbf{0}. \tag{4.48}$$

Our goal is to obtain an estimator for the asymptotic covariance function, denoted by

$$v(\mathbf{x}, \mathbf{y}) := \mathbb{E}\left[L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n}) L_{\mathbf{y}}(\overline{F}^0_{\varepsilon,n}) \right], \ \mathbf{x}, \mathbf{y} \in [0, \infty) \times [0, \infty). \tag{4.49}$$

For convenience, rewrite the above equations (4.46) and (4.47) for a point $\mathbf{y}$:

$$L_{\mathbf{y}}(\overline{F}^0_{\varepsilon,n}) - \int \mathbb{1}\{\mathbf{s} \geq \mathbf{y}\} L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})} = \int \mathbb{1}\{\mathbf{s} \geq \mathbf{y}\} \left[\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{G}(\mathbf{s})} - \overline{H}_n(\mathbf{s}) \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \right], \tag{4.50}$$

$$-\int \mathbb{1}\{\mathbf{s} \geq \mathbf{0}\} L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})} = \int \mathbb{1}\{\mathbf{s} \geq \mathbf{0}\} \left[\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{G}(\mathbf{s})} - \overline{H}_n(\mathbf{s}) \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \right]. \tag{4.51}$$

Multiplying only the left-hand sides of Equations (4.46) and (4.50) and taking expectation we obtain

$$\begin{aligned} & v(\mathbf{x},\mathbf{y}) - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} v(\mathbf{t},\mathbf{y}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} - \int \mathbb{1}\{\mathbf{s} \geq \mathbf{y}\} v(\mathbf{x},\mathbf{s}) \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})} \\ & + \int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{s} \geq \mathbf{y}\} v(\mathbf{t},\mathbf{s}) \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})} \frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})} \\ = & v(\mathbf{x},\mathbf{y}) - \int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} v(\mathbf{t},\mathbf{y}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} - \int \mathbb{1}\{\mathbf{s} \geq \mathbf{y}\} v(\mathbf{x},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \\ & + \int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{s} \geq \mathbf{y}\} v(\mathbf{t},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}, \ \mathbf{x} \neq \mathbf{0}, \ \mathbf{y} \neq \mathbf{0}. \end{aligned} \tag{4.52}$$

Similarly, the multiplications of the left-hand sides Equation (4.46) $\times$ Equation (4.51), Equation (4.47) $\times$ Equation (4.50), Equation (4.47) $\times$ Equation (4.51) give, respectively,

$$-\int \mathbb{1}\{\mathbf{s} \geq \mathbf{0}\} v(\mathbf{x},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \quad + \int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{s} \geq 0\} v(\mathbf{t},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}, \ \mathbf{x} \neq \mathbf{0}; \tag{4.53}$$

$$-\int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}\} v(\mathbf{t},\mathbf{y}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \quad + \int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{y}\} v(\mathbf{t},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}, \ \mathbf{y} \neq \mathbf{0}; \tag{4.54}$$

$$\int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}\} v(\mathbf{t},\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \tag{4.55}$$

On the other hand, the right-hand side of Equation (4.45) is

$$\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}\} \mathrm{d}\alpha_n(\mathbf{t}) = n^{-1} \sum_{i=1}^{n} \left[\frac{\delta_i \mathbb{1}\{\mathbf{Z}_i \geq \mathbf{x}\}}{\overline{G}(\mathbf{Z}_i)} - \int \mathbb{1}\{\mathbf{Z}_i \geq \mathbf{t} \geq \mathbf{x}\} \frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})} \right] \text{ for all } \mathbf{x} \geq \mathbf{0}. \tag{4.56}$$

So, multiplying for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2_+ := [0,\infty) \times [0,\infty)$ and taking expectation,

$$
\begin{aligned}
&n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\vee\mathbf{y}\}\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})}-\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\}\overline{F}^0(\mathbf{t}\vee\mathbf{y})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}-\int \mathbb{1}\{\mathbf{s}\geq \mathbf{y}\}\overline{F}^0(\mathbf{s}\vee\mathbf{x})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&\quad+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{y}\}\overline{H}(\mathbf{t}\vee\mathbf{s})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\Bigg]\\
&=n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\vee\mathbf{y}\}\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})}+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{y}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee s)\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\Bigg]\\
&=n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\vee\mathbf{y}\}\overline{F}^2(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\\
&\quad+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{y}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\Bigg].
\end{aligned}
\tag{4.57}
$$

Note that the second term in Equation (4.57) is identically zero in dimension 1.

Now equate the expressions in Equations (4.52) – (4.55) with the corresponding terms from Equation (4.57); for instance, Equation (4.52) is equated with Equation (4.57) for $\mathbf{x}\neq\mathbf{0},\ \mathbf{y}\neq\mathbf{0}$, whereas Equation (4.53) is equated with Equation (4.57) for $\mathbf{x}\neq\mathbf{0},\ \mathbf{y}=\mathbf{0}$, and so on. This gives

$$
\begin{aligned}
&v(\mathbf{x},\mathbf{y})-\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\}v(\mathbf{t},\mathbf{y})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}-\int \mathbb{1}\{\mathbf{s}\geq \mathbf{y}\}v(\mathbf{x},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&\quad+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{y}\}v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&=n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\vee\mathbf{y}\}\overline{F}^2(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\\
&\quad+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{y}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&-\int \mathbb{1}\{\mathbf{s}\geq \mathbf{0}\}v(\mathbf{x},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq 0\}v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&=n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x}\}\overline{F}^2(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\\
&\quad+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{x},\mathbf{s}\geq \mathbf{0}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\Bigg]
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
&-\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0}\}v(\mathbf{t},\mathbf{y})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0},\mathbf{s}\geq \mathbf{y}\}v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&=n^{-1}\left[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{y}\}\overline{F}^2(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\right.\\
&\left.+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0},\mathbf{s}\geq \mathbf{y}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\right]
\end{aligned}
\tag{4.59}
$$

and

$$
\begin{aligned}
&\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0},\mathbf{s}\geq \mathbf{0}\}v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\\
&=n^{-1}\left[\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0}\}\overline{F}^2(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\right.\\
&\left.+\int\int \mathbb{1}\{\mathbf{t}\geq \mathbf{0},\mathbf{s}\geq \mathbf{0}\}\left(1-\mathbb{1}\{\mathbf{t}\geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\}\right)\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}^2(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\right].
\end{aligned}
\tag{4.60}
$$

Sample covariance function

We now obtain the empirical version of Equations (4.58)–(4.60) by replacing every function by its sample version. Replace $\overline{F}^0(\cdot),\overline{H}(\cdot),\overline{H}^{11}(\cdot)$ respectively by $\overline{F}^0_{\varepsilon,n}(\cdot),\overline{H}_n(\cdot),\overline{H}^{11}_n(\cdot)$ and denote the corresponding solutions to Equations (4.58)–(4.60) by $v_n(\cdot,\cdot)$. Further, specialize only to $\mathbf{x}=\mathbf{Z}_i,\ \mathbf{y}=\mathbf{Z}_j$ and denote

$$v_{ij}=v_n(\mathbf{Z}_i,\mathbf{Z}_j),\ 1\leq i,j\leq n.$$

Recalling the definitions of $a_{ij},\ b_i$ from Equation (4.36), we have, for $1\leq i,j\leq n$,

$$\overline{H}_n(\mathbf{Z}_i\vee\mathbf{Z}_j)=n^{-1}\sum_{k=1}^{n}a_{ik}a_{jk} \tag{4.61}$$

$$\int \mathbb{1}\{\mathbf{t}=\mathbf{Z}_i\}\frac{H^{11}_n(\mathrm{d}\mathbf{t})}{\overline{H}^2_n(\mathbf{t})}=nb_i^2 \tag{4.62}$$

Thus, we obtain the following sample versions of Equations (4.58)–(4.60):

$$
\begin{aligned}
& v_{ij} - \sum_{k=1}^{n} a_{ik} b_k v_{kj} - \sum_{l=1}^{n} a_{jl} b_l v_{il} + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} a_{jl} b_k b_l v_{kl} \\
& = n^{-1} \left[\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x} \vee \mathbf{y}\} \left(\overline{F}^0_{\varepsilon,n}(\mathbf{t})\right)^2 \frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{H}_n^2(\mathbf{t})} \right. \\
& \left. + \int\int \mathbb{1}\{\mathbf{t} \geq \mathbf{x}, \mathbf{s} \geq \mathbf{y}\} \left(1 - \mathbb{1}\{\mathbf{t} \geq \mathbf{s}\}\right) \left(1 - \mathbb{1}\{\mathbf{t} < \mathbf{s}\}\right) \overline{H}_n(\mathbf{t} \vee \mathbf{s}) \overline{F}^0_{\varepsilon,n}(\mathbf{t}) \overline{F}^0_{\varepsilon,n}(\mathbf{s}) \frac{H_n^{11}(\mathrm{d}\mathbf{t})}{\overline{H}_n^2(\mathbf{t})} \frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n^2(\mathbf{s})} \right] \\
& = n^{-1} \left[n \sum_{k=1}^{n} a_{ik} a_{jk} (\overline{F}^0_k)^2 b_k^2 + n^2 \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} a_{jl} (1 - a_{kl})(1 - a_{lk}) \left(n^{-1} \sum_{r=1}^{n} a_{kr} a_{lr} \right) \overline{F}^0_k \overline{F}^0_l b_k^2 b_l^2 \right] \\
& = \sum_{k=1}^{n} a_{ik} a_{jk} (\overline{F}^0_k)^2 b_k^2 + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} a_{jl} d_{kl} \overline{F}^0_k \overline{F}^0_l b_k^2 b_l^2, \qquad (4.63)
\end{aligned}
$$

where

$$
d_{kl} = (1 - a_{kl})(1 - a_{lk}) \sum_{r=1}^{n} a_{kr} a_{lr} (\equiv 0 \text{ in 1 dimension}), \ 1 \leq k, l \leq n. \qquad (4.64)
$$

Similarly,

$$
-\sum_{k=1}^{n} v_{ik} b_k + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} v_{kl} b_k b_l = \sum_{k=1}^{n} a_{ik} (\overline{F}^0_k)^2 b_k^2 + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik} d_{kl} \overline{F}^0_k \overline{F}^0_l b_k^2 b_l^2; \qquad (4.65)
$$

$$
-\sum_{l=1}^{n} v_{lj} b_l + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{jl} v_{kl} b_k b_l = \sum_{l=1}^{n} a_{jl} (\overline{F}^0_l)^2 b_l^2 + \sum_{k=1}^{n}\sum_{l=1}^{n} a_{jl} d_{kl} \overline{F}^0_k \overline{F}^0_l b_k^2 b_l^2; \qquad (4.66)
$$

$$
\sum_{k=1}^{n}\sum_{l=1}^{n} b_k v_{kl} b_l = \sum_{l=1}^{n} (\overline{F}^0_l)^2 b_l^2 + \sum_{k=1}^{n}\sum_{l=1}^{n} d_{kl} \overline{F}^0_k \overline{F}^0_l b_k^2 b_l^2. \qquad (4.67)
$$

Matrix equation

Define the matrices

$$
\mathbf{V} = ((v_{ij}))_{1 \leq i,j \leq n}, \ \mathbf{D} = ((d_{ij}))_{1 \leq i,j \leq n}, \ \overline{\mathbb{F}}^{\mathbf{0}} = \mathrm{diag}(\overline{F}^0_1, \ldots, \overline{F}^0_n).
$$

Then Equations (4.63) – (4.67) can be written as

$$
\begin{bmatrix} \mathbf{I} - \mathbf{AB} \\ -\mathbf{b}^T \end{bmatrix}_{(n+1)\times n} \mathbf{V} \begin{bmatrix} (\mathbf{I} - \mathbf{AB})^T & -\mathbf{b} \end{bmatrix}_{n \times (n+1)} = \begin{bmatrix} \mathbf{AB}\overline{\mathbb{F}}^{\mathbf{0}} \\ \mathbf{b}^T \overline{\mathbb{F}}^{\mathbf{0}} \end{bmatrix} (\mathbf{I} + \mathbf{BDB}) \begin{bmatrix} \overline{\mathbb{F}}^{\mathbf{0}} \mathbf{BA}^T & \overline{\mathbb{F}}^{\mathbf{0}} \mathbf{b} \end{bmatrix}. \qquad (4.68)
$$

Thus, we have the asymptotic covariance matrix $\mathbf{V}$ of the mass-shifting estimator by solving the Equation (4.68).

The Kaplan-Meier estimator presented above will be used in the following section to build a plug-in estimator for the Kendall tau in the case of bivariate right-censored data.

4.3 Estimation of Kendall's τ

Some estimators for Kendall's tau in the presence of censoring have been suggested in the literature. Brown, Hollander, and Korwar (1974) proposed an estimator based on the marginal Kaplan-Meier estimators. Even though the estimator accounts for some configurations of censoring, it ignores joint information for pairs of doubly censored observations (Wang and Wells, 2000a). Weier and Basu (1980) proposed to impute censored observations by their expected values obtained by using Kaplan-Meier estimates. These estimators rely mainly on the marginal survivor estimates and so miss some information about the joint distribution.

Wang and Wells (2000a) investigated the properties of plug-in estimators based on formula (4.11). They showed that such functional estimators have estimable bounds on the bias in estimation.

In light of Equation (4.11), we suggest the plug-in estimator

$$\tau_n = 4 \int \overline{F}^0_{\varepsilon,n}(\mathbf{x}) F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - 1 \tag{4.69}$$

for the Kendall tau in presence of right censoring

$$\tau = 4 \int \overline{F}^0(\mathbf{x}) F^0(\mathrm{d}\mathbf{x}) - 1, \tag{4.70}$$

where

- F^0 is the joint distribution function of (X_1, X_2) under right censoring,
- $\overline{F}^0_{\varepsilon,n}$ is the mass-shifting estimator for the survival function of (X_1, X_2) under right censoring (see Section 4.2.2),
- $\mathbf{x} = (x_1, x_2)$

Remark 4.3.1. *The estimator (4.69) is computed as*

$$\tau_n = 4 \sum_{i=1}^{n} \overline{F}^0_i p_i - 1.$$

As the values of the mass-shifting estimator $\overline{F}_i^0$ and its weights p_i (see Section 4.2.2) are obtained through vector and matrix basic operations, the estimation is relatively easy in programming languages like R.

Lopez (2012) used a similar estimator based on an estimator of the survivor that puts an excess mass at an additional point. However, his paper did not provide an asymptotic variance for the estimator.

In the next section, we will estimate the asymptotic variance of $\sqrt{n}(\tau_n - \tau)$ using its linearization.

4.3.1 Asymptotic variance of τ_n

Let us recall that (X_1, X_2) is a pair of censored non-negative random variables with joint distribution function F^0 (and survivor $\overline{F}^0$) and $\overline{F}^0_{\varepsilon,n}$ is our estimator for the survival function. The censoring variables are (Y_1, Y_2), with joint distribution G and joint survival function $\overline{G}$.

The observations are then $\{(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})\}_{i=1}^{i=n}$, realizations of $(Z_1, Z_2, \delta_1, \delta_2)$, where $Z_1 = \min(X_1, Y_1)$, $Z_2 = \min(X_2, Y_2)$, $\delta_1 = \mathbb{1}\{X_1 \leq Y_1\}$ and $\delta_2 = \mathbb{1}\{X_2 \leq Y_2\}$.

Define also $H^{11}(\mathbf{x}) := H^{11}(x_1, x_2) = \mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \delta_1 = 1, \delta_2 = 1)$ and $\overline{H}(x_1, x_2) := \mathbb{P}(X_1 \geq x_1, X_2 \geq x_2)$.

As previously decided, cases in bold font will continue to denote two-component vectors and inequalities between vectors will be considered componentwise.

Let us rewrite:

$$
\begin{aligned}
\frac{1}{4}\sqrt{n}(\tau_n - \tau) &= \sqrt{n}\left(\int \overline{F}^0_{\varepsilon,n}(\mathbf{x}) F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - \int \overline{F}^0(\mathbf{x}) F^0(\mathrm{d}\mathbf{x})\right) \\
&= \sqrt{n}\left(\int \overline{F}^0_{\varepsilon,n}(\mathbf{x}) F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - \int \overline{F}^0_{\varepsilon,n}(\mathbf{x}) F^0(\mathrm{d}\mathbf{x}) + \int \overline{F}^0_{\varepsilon,n}(\mathbf{x}) F^0(\mathrm{d}\mathbf{x}) - \int \overline{F}^0(\mathbf{x}) F^0(\mathrm{d}\mathbf{x})\right) \\
&= \sqrt{n}\int \overline{F}^0_{\varepsilon,n}(\mathbf{x})\left(F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - F^0(\mathrm{d}\mathbf{x})\right) + \int \sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right) F^0(\mathrm{d}\mathbf{x}) \\
&= \sqrt{n}\int \left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x}) + \overline{F}^0(\mathbf{x})\right)\left(F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - F^0(\mathrm{d}\mathbf{x})\right) \\
&\quad + \int \sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right) F^0(\mathrm{d}\mathbf{x}) \\
&= \sqrt{n}\int \left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right)\left(F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - F^0(\mathrm{d}\mathbf{x})\right) && (4.71) \\
&\quad + \sqrt{n}\int \overline{F}^0(\mathbf{x})\left(F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - F^0(\mathrm{d}\mathbf{x})\right) && (4.72) \\
&\quad + \int \sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right) F^0(\mathrm{d}\mathbf{x}). && (4.73)
\end{aligned}
$$

As

$$\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x}) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \tag{4.74}$$

and

$$\int \left(\overline{F}^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - \overline{F}^0(\mathrm{d}\mathbf{x})\right) = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \tag{4.75}$$

the term (4.71) is negligible. And we can focus on the linearization of (4.72) and (4.73).

From Equation (3.9) of Jin, Sen, and Stute (2013), we know that

$$\sqrt{n}\int \overline{F}^0(\mathbf{x})\left(F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - F^0(\mathrm{d}\mathbf{x})\right) = \sqrt{n}\left(\int \overline{F}^0(\mathbf{x})F^0_{\varepsilon,n}(\mathrm{d}\mathbf{x}) - \int \overline{F}^0(\mathbf{x})F^0(\mathrm{d}\mathbf{x})\right)$$

has the influence function $L_{\overline{F}^0}(F^0_{\varepsilon,n})$ given by

$$L_{\overline{F}^0}(F^0_{\varepsilon,n}) = \int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t}) + \int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t}), \tag{4.76}$$

where

$$\alpha_n(\mathrm{d}\mathbf{t}) = \frac{H^{11}_n(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})} - \overline{H}_n(\mathbf{t})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}. \tag{4.77}$$

Moreover,

$$\int \sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{x}) - \overline{F}^0(\mathbf{x})\right)F^0(\mathrm{d}\mathbf{x})$$

has the influence function

$$\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t}).$$

So that the influence function of $\frac{1}{4}\sqrt{n}(\tau_n - \tau)$ is

$$L_\tau(\overline{F}^0_{\varepsilon,n}) = 2\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t}) + \int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t}). \tag{4.78}$$

Rewrite for $\mathbf{s}$:

$$L_\tau(\overline{F}^0_{\varepsilon,n}) = 2\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s}) + \int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s}). \tag{4.79}$$

Then, the asymptotic variance of $\frac{1}{4}\sqrt{n}(\tau_n - \tau)$ is:

$$\begin{aligned}\frac{1}{16}\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right) &= \mathbb{E}\left\{\left(2\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t})\right)\left(2\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s})\right.\right.\\ &\qquad \left.\left.+\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right)\right\}\\ &= 4\mathbb{E}\left(\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s})\right)\\ &\quad +4\mathbb{E}\left(\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right)\\ &\quad +\mathbb{E}\left(\int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right).\end{aligned}$$

Taking advantage of the Fubini-Lebesgue theorem, we can rewrite each of these expectation-terms in an easily estimable form. This is performed in Appendix C.1. We can thus derive a formula for the asymptotic variance of the Kendall tau estimator:

Theorem 4.1. *The asymptotic variance of $\sqrt{n}(\tau_n-\tau)$, $\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right)$, is given by*

$$\begin{aligned}&\frac{1}{16}\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right)\\ &= 4\int\int \overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}+4\int w_{\mathbf{t}}F^0(\mathrm{d}\mathbf{t})\\ &\quad +n^{-1}\left[\int \overline{F}^4(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})^2}+\int\int(1-\mathbb{1}\{\mathbf{t}\geq\mathbf{s}\})(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\})\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^2(\mathbf{t})\overline{F}^2(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\right],\end{aligned}$$

with

- $$v(\mathbf{t},\mathbf{s}) = \mathbb{E}\left(L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\right)$$
- *$w_{\mathbf{t}}$ is the solution to the inhomogeneous Volterra equation of the second kind:*

$$w_{\mathbf{t}}-\int \mathbb{1}\{\mathbf{u}\geq\mathbf{t}\}w_{\mathbf{u}}\frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} = \mathbb{E}\left(\int \mathbb{1}\{\mathbf{u}\geq\mathbf{t}\}\alpha_n(\mathrm{d}\mathbf{u})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right),\quad \textit{with condition}\quad w_{\mathbf{0}}=0. \tag{4.80}$$

4.3.2 Estimation of the variance of τ_n

Estimation of $w_{\mathbf{t}}$

Just as with Equation (4.57), the right-hand side of Equation (4.80) can be written as:

$$\begin{aligned}
&\mathbb{E}\left(\int\int \mathbb{1}\{\mathbf{u}\geq\mathbf{t}\}\overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{u})\alpha_n(\mathrm{d}\mathbf{s})\right)\\
&=n^{-1}\left[\int \mathbb{1}\{\mathbf{u}\geq\mathbf{t}\}\left(\overline{F}^0(\mathbf{u})\right)^3\frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})}\right.\\
&\quad\left.+\int\int \mathbb{1}\{\mathbf{u}\geq\mathbf{t}\}\left(1-\mathbb{1}\{\mathbf{u}\geq\mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{u}<\mathbf{s}\}\right)\overline{H}(\mathbf{u}\vee\mathbf{s})\left(\overline{F}^0(\mathbf{u})\right)^2\left(\overline{F}^0(\mathbf{s})\right)^3\frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\right]
\end{aligned} \tag{4.81}$$

Replace $\overline{F}^0(.),\overline{H}(.),H^{11}(.)$ by their respective sample counterparts $\overline{F}^0_{\varepsilon,n}(.),\overline{H}_n(.),H^{11}_n(.)$ and put $\mathbf{u}=\mathbf{Z}_i$, $\mathbf{s}=\mathbf{Z}_j$.

Then:

$$\overline{H}_n(\mathbf{u}\vee\mathbf{s})=\overline{H}_n(\mathbf{Z}_i\vee\mathbf{Z}_j)=n^{-1}\sum_{i=1}^{n}a_{ik}a_{jk} \tag{4.82}$$

and

$$\int \mathbb{1}\{\mathbf{u}=\mathbf{Z}_i\}\frac{H^{11}_n(\mathrm{d}\mathbf{u})}{\overline{H}^2_n(\mathbf{u})}=nb_i^2, \tag{4.83}$$

where

$$a_{ik}=\begin{cases}1 & \text{if} \quad Z_{1k}\geq Z_{1i} \text{ and } Z_{2k}\geq Z_{2i}\\ 0 & \text{otherwise}\end{cases} \tag{4.84}$$

and

$$b_i:=\frac{\Delta H^{11}_n(Z_{1i},Z_{2i})}{\overline{H}_n(Z_{1i},Z_{2i})+\frac{1}{n}}=\frac{\delta_{1i}\delta_{2i}}{n\overline{H}_n(Z_{1i},Z_{2i})+1} \tag{4.85}$$

Equation (4.80) is now equivalent to:

$$
\begin{aligned}
& w_{\mathbf{t}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} w_{\mathbf{u}} \frac{H_n^{11}(\mathrm{d}\mathbf{u})}{\overline{H}_n(\mathbf{u})} \\
& = n^{-1} \Bigg[\int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \left(\overline{F}^0(\mathbf{u})\right)^3 \frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})} \\
& \quad + \int \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \left(1 - \mathbb{1}\{\mathbf{u} \geq \mathbf{s}\}\right) \left(1 - \mathbb{1}\{\mathbf{u} < \mathbf{s}\}\right) \overline{H}(\mathbf{u} \vee \mathbf{s}) \left(\overline{F}^0(\mathbf{u})\right)^2 \left(\overline{F}^0(\mathbf{s})\right)^3 \frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})} \frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n^2(\mathbf{s})} \Bigg]
\end{aligned}
\tag{4.86}
$$

$$
\begin{aligned}
& - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{0}\} w_{\mathbf{u}} \frac{H_n^{11}(\mathrm{d}\mathbf{u})}{\overline{H}_n(\mathbf{u})} \\
& = n^{-1} \Bigg[\int \mathbb{1}\{\mathbf{u} \geq \mathbf{0}\} \left(\overline{F}^0_{\varepsilon,n}(\mathbf{u})\right)^3 \frac{H_n^{11}(\mathrm{d}\mathbf{u})}{\overline{H}_n^2(\mathbf{u})} \\
& \quad + \int \int \mathbb{1}\{\mathbf{u} \geq \mathbf{0}\} \left(1 - \mathbb{1}\{\mathbf{u} \geq \mathbf{s}\}\right) \left(1 - \mathbb{1}\{\mathbf{u} < \mathbf{s}\}\right) \overline{H}_n(\mathbf{u} \vee \mathbf{s}) \left(\overline{F}^0_{\varepsilon,n}(\mathbf{u})\right)^2 \left(\overline{F}^0_{\varepsilon,n}(\mathbf{s})\right)^3 \frac{H_n^{11}(\mathrm{d}\mathbf{u})}{\overline{H}_n^2(\mathbf{u})} \frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n^2(\mathbf{s})} \Bigg]
\end{aligned}
\tag{4.87}
$$

Let

- $\widehat{\mathbf{W}} = (\hat{w}_{\mathbf{Z_1}}, \ldots, \hat{w}_{\mathbf{Z_n}})$,
- $\mathbf{I}$ be the $n \times n$ identity matrix,
- $(\mathbf{A})_{ij} = a_{ij} = \mathbb{1}\{Z_{j1} \geq Z_{1i}, Z_{j2} \geq Z_{2i}\}$ for $1 \leq i, j \leq n$,
- $\mathbf{b} = (b_1, \ldots, b_n)$,
- $\overline{\mathbb{F}}^{\mathbf{0}} = (\overline{F}^0_1, \ldots, \overline{F}^0_n) = \left(\overline{F}^0_{\varepsilon,n}(Z_{11}, Z_{12}), \ldots, \overline{F}^0_{\varepsilon,n}(Z_{n1}, Z_{n2})\right)$,
- $\overline{\mathbb{F}}^{\mathbf{0}} = \mathrm{diag}(\overline{F}^0_1, \ldots, \overline{F}^0_n)$,
- $(\mathbf{D})_{ij} = d_{ij} = (1 - a_{ij})(1 - a_{ji}) \sum_{r=1}^n a_{ir} a_{jr}$.

$\widehat{\mathbf{W}}$ is the solution to the equation in matrix notation:

$$
\begin{bmatrix} \mathbf{I} - \mathbf{AB} \\ -\mathbf{b}^T \end{bmatrix}_{(n+1)\times n} \widehat{\mathbf{W}} = \begin{bmatrix} \mathbf{AB}\overline{\mathbb{F}}^{\mathbf{0}} \\ \mathbf{b}^T \overline{\mathbb{F}}^{\mathbf{0}} \end{bmatrix} (\mathbf{I} + \mathbf{BDB}) \overline{\mathbb{F}}^{\mathbf{0}} \mathbf{B} \overline{\mathbf{F}}^0. \tag{4.88}
$$

Variance estimation

With the estimator of W above (in 4.88), we can provide an estimator for $\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right)$ as:

$$\widehat{\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right)} = 16\left((\bar{\mathbf{F}}^{\mathbf{0}})^{\mathbf{T}}\mathbf{BVB}\bar{\mathbf{F}}^{\mathbf{0}} + 2(\bar{\mathbf{F}}^{\mathbf{0}})^{\mathbf{T}}\mathbf{B}\widehat{\mathbf{W}} + (\bar{\mathbf{F}}^{\mathbf{0}})^{\mathbf{T}}\mathbf{B}\bar{\mathbb{F}}^{\mathbf{0}}(\mathbf{I}+\mathbf{BDB})\bar{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\bar{\mathbf{F}}^{\mathbf{0}}\right). \quad (4.89)$$

4.3.3 Illustration using simulated data

We simulate two random variables X_1 and X_2 from a Weibull distribution with density

$$f(x) = \frac{x}{2}\exp\left(-\left(\frac{x}{2}\right)^2\right) \text{ for } x \geq 0 \quad \text{(shape=scale=2)}$$

and linked by a Clayton copula

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}} \text{ for } 0 \leq u_1,u_2 \leq 1.$$

The Kendall tau coefficient of the couple (X_1,X_2) is given by

$$\tau = \frac{\theta}{\theta+2}. \quad (4.90)$$

X_1 and X_2 are both censored by an exponential random variable with rate parameter λ (mean = $\frac{1}{\lambda}$).

In Table 4.1, we compare the asymptotic variance $\widehat{\sigma^2_{\tau_n}} = \frac{1}{n}\widehat{\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right)}$ of τ_n from the variance estimator (4.89) calculated on a sample of n values, to the mean squared error (MSE) of the estimator τ_n, $\mathbb{E}\left[(\tau_n-\tau)^2\right]$. To calculate the MSE, we simulated $M = 100$ samples, each one of size n. We also give the decomposition of the MSE in bias and variance components:

$$\mathbb{E}\left[(\tau_n-\tau)^2\right] = \underbrace{\mathbb{E}\left[(\tau_n-\mathbb{E}(\tau_n))^2\right]}_{\text{variance}} + \underbrace{(\mathbb{E}(\tau_n)-\tau)^2}_{\text{bias}^2}$$

By trying different combinations of the values of the sample size, the parameter θ of the copula and the rate λ of the censoring variable, we can evaluate the effects of the sample size, the dependence between X_1 and X_2 and the censoring on the variance.

The performances of τ_n are contrasted with that of Wang and Wells (2000a)'s estimator, denoted γ. γ is obtained by plugging Dabrowska (1988)'s estimator for the bivariate survivor in Equation (4.11) and correcting for possible ties in the data.

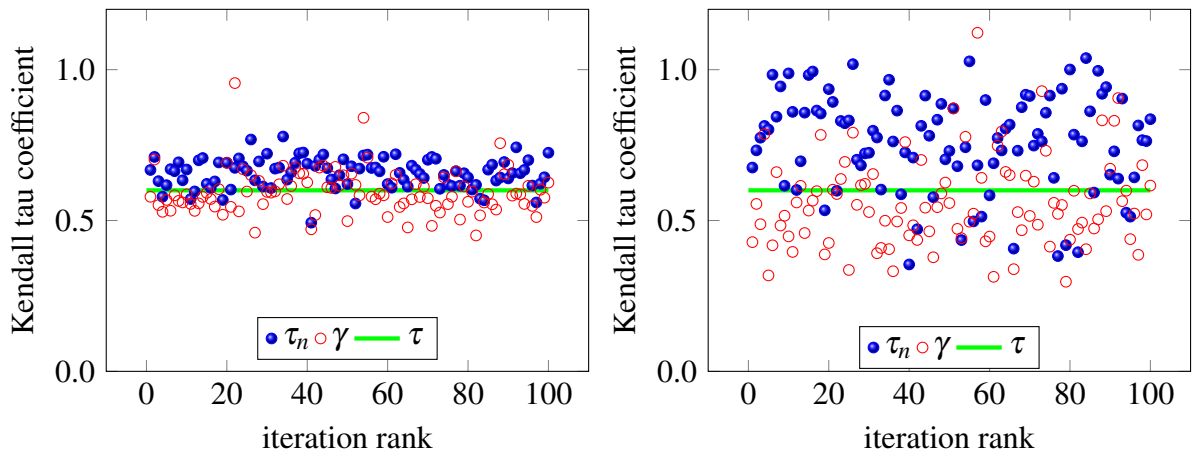
With the programming language R (R Core Team), the functions *mvdc* and *rMvdc* (in the package *copula*) allow us to build a copula object with specified marginal distributions and to simulate observations from it. Dabrowska (1988)'s estimator is available through the function *KM2* of the package *mhazard*.

TABLE 4.1: Asymptotic variance of τ_n: simulation results

Model		Criterion	$n = 400$		$n = 500$	
			τ_{400}	γ	τ_{500}	γ
$\theta = 1$	$\lambda = 1$ (79% of censoring)	$\widehat{\sigma^2_{\tau_n}}$	$2.32025 \cdot 10^{-05}$		$1.31420 \cdot 10^{-05}$	
		MSE	0.01421	0.00524	0.00809	0.00373
		bias^2	0.00674	0.00012	0.00353	0.00016
		variance	0.00747	0.00512	0.00456	0.00357
	$\lambda = 2$ (94% of censoring)	$\widehat{\sigma^2_{\tau_n}}$	0.00068		0.00235	
		MSE	0.10017	0.03301	0.08442	0.02103
		bias^2	0.07487	0.00003	0.06544	0.00085
		variance	0.02530	0.03299	0.01897	0.02018
$\theta = 3$	$\lambda = 1$ (77% of censoring)	$\widehat{\sigma^2_{\tau_n}}$	$2.79125 \cdot 10^{-05}$		$1.59180 \cdot 10^{-05}$	
		MSE	0.00939	0.00697	0.00560	0.00532
		bias^2	0.00585	0.00001	0.00323	0.00004
		variance	0.00354	0.00696	0.00238	0.00528
	$\lambda = 2$ (93% of censoring)	$\widehat{\sigma^2_{\tau_n}}$	0.00078		0.00384	
		MSE	0.04369	0.05602	0.05151	0.02370
		bias^2	0.01657	0.00002	0.02582	0.00182
		variance	0.02712	0.05601	0.02569	0.02188

Wang and Wells (2000a)'s plug-in estimator (γ) performs better on the bias aspect, although its variability is comparable to our plug-in estimator's (τ_n). The difference in the bias diminishes as the correlation between the variables grows.

Censoring noticeably increases the variability of the estimates. The increase is both in the variance and the bias components of the MSE. Figures (4.2a) - (4.2b) illustrate that change. The effect of the dependence in (X_1, X_2) on the variability is much more subtle.



(a) $\lambda = 1$ (b) $\lambda = 2$

FIGURE 4.2: Estimates of the Kendall tau for models with Clayton copula ($\theta = 3$) and censoring parameter λ

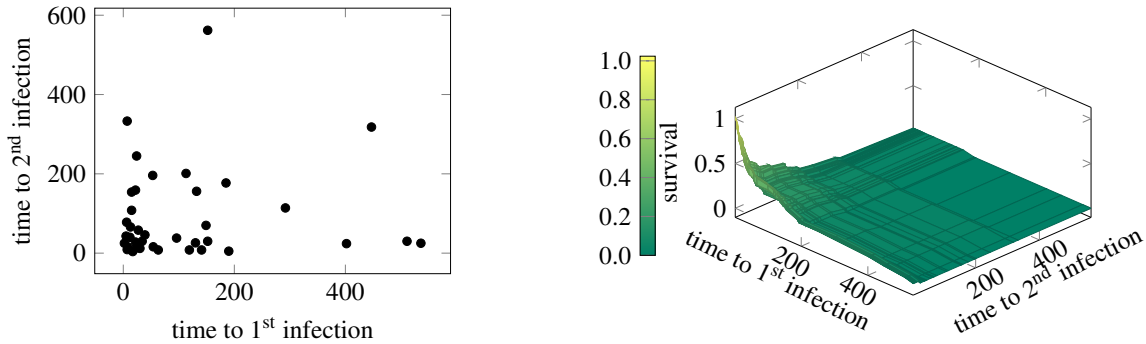
4.3.4 Illustration using kidney patients' infection data

We use the data in McGilchrist and Aisbett (1991) for estimating Kendall's tau. The data are provided in the package *SurvCorr* of the software R (R Core Team).

The data are from a study of the recurrence time of infection in kidney patients who use portable dialysis machines. The catheter is removed when an infection occurs. Once the infection is clear, the catheter is reinserted.

Two recurrence times (in days), from an infection until the next one, are recorded. The median times to infection are 46 (days) and 39 (days) for the 1st and 2nd infections respectively. Figure 4.3a shows the dispersion of the two time-to-events.

Right censoring happens at the end of the study or when the catheter is removed for some other reason. 38 patients were observed, 6 first recurrence times are censored, 12 second recurrence times are censored and 3 observations are doubly censored.



(a) Time to infections

(b) Estimated survival function $\overline{F}^0_{\varepsilon,n}$

FIGURE 4.3: Kidney patients' data

The Kendall tau estimate using (4.69) is $\tau_n = 0.359$. Wang and Wells (2000a)'s estimate is $\gamma = 0.218$, Brown et al. (1974)'s is 0.209. Our variance estimate is $5.947 \cdot 10^{-5}$, that is a standard deviation of 0.008. Wang and Wells (2000a) obtained a standard deviation of 0.072 through bootstrap.

4.4 Conclusion

The need for an efficient multivariate Kaplan-Meier estimator guided Sen and Stute (2013) to develop a new estimator for the joint survival distribution function for right-censored data. The estimator, referred to as the *mass-shifting estimator* throughout this thesis, is described at the beginning of this chapter. It inspires the plug-in estimator for Kendall's tau (τ_n) presented in the second part of the chapter. The asymptotic distribution of τ_n was provided, using the influence function approach. The estimator was illustrated and compared to other available estimators (like Wang and Wells, 2000a's γ) using both simulated and real-life data. On simulated data, we noticed that censoring noticeably increases the mean squared error (MSE) of our Kendall's tau estimator τ_n. The increase occurs both in variance and bias components of the MSE. The effect of the dependence between the time-to-event variables on the variance is much more elusive.

Chapter 5

Copula likelihood maximization under right censoring

5.1 Introduction

Consider a pair of right censored random variables $(X_1, X_2) \sim (F_1, F_2)$ with some continuous bivariate copula dependence C_θ with density c_θ. Genest et al. (1995) provided a semiparametric estimator for the pseudo-likelihood in two steps:

- first estimate the marginal distributions by the rescaled empirical distributions

$$\tilde{F}_j(x) = \frac{1}{n+1}\sum_{i=1}^{n} \mathbb{1}\{X_{ji} \leq x\}, j = 1,2 \text{ from an uncensored sample} \{(X_{1i}, X_{2i})\}_{i=1}^{i=n}$$

- then take

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \sum_{i=1}^{n} \underbrace{\log\left(c_\theta\left(\tilde{F}_1(X_{1i}), \tilde{F}_2(X_{2i})\right)\right)}_{:=l(\theta, F_1(X_1), F_2(X_2))} \tag{5.1}$$

Under regularity conditions, the authors showed that the estimator $\hat{\theta}$ is consistent and that $\sqrt{n}(\hat{\theta} - \theta)$ is asymptotically normal with variance

$$v^2 = \frac{\mathbb{V}\{l_\theta\left(\theta, F_1(X_1), F_2(X_2)\right\} + W_1(X_1) + W_2(X_2))}{\mathbb{E}\left\{l_\theta^2\left(\theta, F_1(X_1), F_2(X_2)\right)\right\}}, \tag{5.2}$$

where

$$W_i(X_i) = \int \mathbb{1}\{F_i(X_i) \leq u_i\} l_{\theta,i}(\theta, u_1, u_2) c_\theta(u_1, u_2) \mathrm{d}u_1 \mathrm{d}u_2 \quad \text{for} \quad j = 1,2 \tag{5.3}$$

and the indices $\theta, 1, 2$ on the log-likelihood l denote its partial derivatives with respect to θ, u_1,u_2 respectively.

A consistent estimator for the variance v^2 is provided by using pseudo-observations and the empirical copula function. The estimator $\hat{\theta}$ is also showed to be efficient under independence.

The goal of this chapter is to establish similar results in the case of censored data: instead of the sample $\{(X_{1i}, X_{2i})\}_{i=1}^{i=n}$, we have access to $\{(\delta_{1i}, \delta_{2i}, Z_{1i}, Z_{2i})\}_{i=1}^{i=n}$, where $\delta_{ji} = \mathbb{1}\{X_{ji} \leq Y_{ji}\}$ and $Z_{j,i} = \min(X_{ji}, Y_{ji})$, for $i = 1, \ldots, n$, $j = 1, 2$, with Y_1, Y_2 being the censoring variables. We want to estimate θ by maximizing the pseudo-likelihood

$$L(\theta) = \prod_{i=1}^{n} c_\theta \left(F_{n1}^0(Z_{1i}), F_{n2}^0(Z_{2i})\right)^{p_i} \tag{5.4}$$

where

- $c_\theta(u_1, u_2)$ is the density of some copula family with parameter θ,
- $F_{n1}^0(Z_{j1})$ and $F_{n2}^0(Z_{2j})$ are the estimated marginal distributions obtained by using the Kaplan-Meier weights (section 4.2.1),
- the p_i's are the weights of the mass-shifting estimator (section 4.2.2).

As

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, L(\theta) = \underset{\theta}{\operatorname{argmax}}\, \log\left(L(\theta)\right), \tag{5.5}$$

we will work with the log-pseudo-likelihood

$$\log\left(L(\theta)\right) = \sum_{i=1}^{n} p_i \log\left(c_\theta\left(F_{n1}^0(Z_{1i}), F_{n2}^0(Z_{2i})\right)\right). \tag{5.6}$$

Remark 5.1.1. *First of all, note that* $\log\left(L(\theta)\right)$ *in Equation (5.6) is of the form*

$$\log\left(L(\theta)\right) = \int \log\left(c_\theta\left(F_{n1}^0(x_1), F_{n2}^0(x_2)\right) \overline{F}_n^0(\mathrm{d}x_1, \mathrm{d}x_2)\right) \tag{5.7}$$

where $\overline{F}_n^0(.,.)$ *is the bivariate Kaplan-Meier estimator of Sen and Stute (2013) introduced in the previous chapter. It follows that* $\log\left(L(\theta)\right)$ *is a consistent estimator of the expected log-pseudo-likelihood for complete data, viz.,*

$$\log\left(L(\theta)\right) \to \int \log\left(c_\theta\left(F_1(x_1), F_2(x_2)\right) F(\mathrm{d}x_1, \mathrm{d}x_2)\right) \quad as \quad n \to \infty. \tag{5.8}$$

This is the motivation behind our proposal to use $\log(L(\theta))$.

Let us call the pseudo-log likelihood in Equation 5.6 the compact *log-likelihood. Shih and Louis (1995) considered the* full *log-likelihood in this case, namely,*

$$\begin{aligned}\log(L^*(\theta)) = \sum_{i=1}^{n} \Big\{ & \delta_{1i}\delta_{2i}\log\left(c_\theta(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i}))\right) \\ & + \delta_{1i}(1-\delta_{2i})\log\left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_1}\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)} \\ & + (1-\delta_{1i})\,\delta_{2i}\log\left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_2}\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)} \\ & + (1-\delta_{1i})(1-\delta_{2i})\log\left(C_\theta(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i}))\right)\Big\}\end{aligned} \tag{5.9}$$

Note then that for d-dimensional data, the full log-likelihood *(under d-dimensional censoring) will have* 2^d *terms, whereas the* compact log-likelihood *will have one term in any dimension.*

Moreover, we present a comparison between the two likelihoods based on simulated data in Section 5.6, and it appears to show that the compact likelihood *outperforms the* full likelihood *under strong dependence.*

$\hat{\theta}$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log(L(\theta))\Big|_{\theta=\hat{\theta}} = 0. \tag{5.10}$$

Taylor expansion of the score function $\frac{\mathrm{d}}{\mathrm{d}\theta}\log(L(\theta))$ at the true value θ_0 of the parameter θ:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\log(L(\theta)) = \frac{\mathrm{d}}{\mathrm{d}\theta}\log(L(\theta))\Big|_{\theta=\theta_0} + \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log(L(\theta))\Big|_{\theta=\theta_0}(\theta-\theta_0)+\dots \tag{5.11}$$

Then, from Equation (5.10) above:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}\theta}\log(L(\theta))\Big|_{\theta=\theta_0}}_{:=A_n} + \underbrace{\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log(L(\theta))\Big|_{\theta=\theta_0}}_{:=-B_n}(\hat{\theta}-\theta_0) \simeq 0 \tag{5.12}$$

Thus:

$$\sqrt{n}(\hat{\theta}-\theta_0) \simeq \frac{\sqrt{n}A_n}{B_n} \tag{5.13}$$

In the next two sections, we investigate the limits of the quantities A_n and B_n. Once those limits are obtained, the limit of our maximum likelihood estimator is readily available (Section 5.4). The estimation of its limiting variance is presented in Section 5.5. Sections 5.6, 5.7 and 5.8 show the performance of our estimator compared to Shih and Louis (1995)'s, on simulated data and real-world datasets.

5.2 Asymptotic behaviour of A_n

$$A_n = \frac{\mathrm{d}}{\mathrm{d}\theta}\log\left(L(\theta)\right)\Big|_{\theta=\theta_0} = \sum_{i=1}^{n} p_i \underbrace{\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left(c_\theta\left(F_{n1}^0(Z_{1i}), F_{n2}^0(Z_{2i})\right)\right)\Big|_{\theta=\theta_0}}_{:=\phi\left(\overline{F}_{n1}^0(Z_1),\overline{F}_{n2}^0(Z_2);\theta_0\right)} \tag{5.14}$$

$$= \int\int \phi\left(\overline{F}_{n1}^0(Z_1), \overline{F}_{n2}^0(Z_2); \theta_0\right) F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \tag{5.15}$$

A_n has limit

$$A = \int\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) F^0(\mathrm{d}z_1, \mathrm{d}z_2) \tag{5.16}$$

$$= \int\int \frac{\mathrm{d}}{\mathrm{d}\theta}\log\left(c_\theta\left(F_1^0(Z_1), F_2^0(Z_2)\right)\right)\Big|_{\theta=\theta_0} F^0(\mathrm{d}z_1, \mathrm{d}z_2) \tag{5.17}$$

$$= \int\int \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}c_\theta\left(F_1^0(Z_1), F_2^0(Z_2)\right)}{c_\theta\left(F_1^0(Z_1), F_2^0(Z_2)\right)}\Bigg|_{\theta=\theta_0} c_{\theta_0}\left(F_1^0(Z_1), F_2^0(Z_2)\right)\mathrm{d}z_1\mathrm{d}z_2 \tag{5.18}$$

$$= \int\int \frac{\mathrm{d}}{\mathrm{d}\theta}c_\theta\left(F_1^0(Z_1), F_2^0(Z_2)\right)\Big|_{\theta=\theta_0} \mathrm{d}z_1\mathrm{d}z_2 \tag{5.19}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\underbrace{\int\int c_\theta\left(F_1^0(Z_1), F_2^0(Z_2)\right)\mathrm{d}z_1\mathrm{d}z_2}_{=1}\Bigg|_{\theta=\theta_0} \tag{5.20}$$

$$= 0. \tag{5.21}$$

We claim that:

Claim 5.2.1.

$$\sqrt{n}A_n \to \mathcal{N}(0, \sigma^2) \text{ in distribution as } n \to +\infty.$$

We will estimate the variance σ^2 using the linearization of $\sqrt{n}A_n$.

By Taylor expansion at $\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)$:

$$\phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \theta_0\right) \tag{5.22}$$
$$\simeq \phi(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0) \tag{5.23}$$
$$+ \left(\overline{F}_{n1}^0(z_1) - \overline{F}_1^0(z_1)\right) \frac{\partial}{\partial u_1} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} \tag{5.24}$$
$$+ \left(\overline{F}_{n2}^0(z_2) - \overline{F}_2^0(z_2)\right) \frac{\partial}{\partial u_2} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} \tag{5.25}$$

So that:

$$\begin{aligned}
&\sqrt{n} A_n \\
&= \sqrt{n} \int\int \phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \theta_0\right) F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&= \sqrt{n} \int\int \left[\phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \theta_0\right) - \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) + \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right)\right] F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&= \sqrt{n} \int\int \left[\phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \theta_0\right) - \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right)\right] F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&\quad + \sqrt{n} \int\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&\simeq \sqrt{n} \int\int \left(\overline{F}_{n1}^0(z_1) - \overline{F}_1^0(z_1)\right) \frac{\partial}{\partial u_1} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&\quad + \sqrt{n} \int\int \left(\overline{F}_{n2}^0(z_2) - \overline{F}_2^0(z_2)\right) \frac{\partial}{\partial u_2} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) \\
&\quad + \sqrt{n} \int\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \left(F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) - F^0(\mathrm{d}z_1, \mathrm{d}z_2) + F^0(\mathrm{d}z_1, \mathrm{d}z_2)\right) \\
&= \sqrt{n} \int\int \left(\overline{F}_{n1}^0(z_1) - \overline{F}_1^0(z_1)\right) \frac{\partial}{\partial u_1} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} \left(F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) - F^0(\mathrm{d}z_1, \mathrm{d}z_2) + F^0(\mathrm{d}z_1, \mathrm{d}z_2)\right) \\
&\quad + \sqrt{n} \int\int \left(\overline{F}_{n2}^0(z_2) - \overline{F}_2^0(z_2)\right) \frac{\partial}{\partial u_2} \phi(u_1, u_2; \theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} \left(F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) - F^0(\mathrm{d}z_1, \mathrm{d}z_2) + F^0(\mathrm{d}z_1, \mathrm{d}z_2)\right) \\
&\quad + \sqrt{n} \int\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \left(F_{\varepsilon,n}^0(\mathrm{d}z_1, \mathrm{d}z_2) - F^0(\mathrm{d}z_1, \mathrm{d}z_2)\right) \\
&\quad + \sqrt{n} \underbrace{\int\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) F^0(\mathrm{d}z_1, \mathrm{d}z_2)}_{=A=0}
\end{aligned}$$

$$
\begin{aligned}
&\sqrt{n}A_n \\
&\simeq \sqrt{n}\int\int \left(\overline{F}^0_{n1}(z_1)-\overline{F}^0_1(z_1)\right)\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\left(F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right) && (5.26)\\
&\quad+\sqrt{n}\int\int \left(\overline{F}^0_{n1}(z_1)-\overline{F}^0_1(z_1)\right)\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}z_1,\mathrm{d}z_2) && (5.27)\\
&\quad+\sqrt{n}\int\int \left(\overline{F}^0_{n2}(z_2)-\overline{F}^0_2(z_2)\right)\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\left(F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right) && (5.28)\\
&\quad+\sqrt{n}\int\int \left(\overline{F}^0_{n2}(z_2)-\overline{F}^0_2(z_2)\right)\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}z_1,\mathrm{d}z_2) && (5.29)\\
&\quad+\sqrt{n}\int\int \phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)\left(F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right) && (5.30)
\end{aligned}
$$

As

$$
\overline{F}^0_{\varepsilon,n}(\mathbf{z})-\overline{F}^0(\mathbf{z})=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) \tag{5.31}
$$

and

$$
\int\int \frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\left(F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right)=O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right), \tag{5.32}
$$

the term (5.26) is negligible. A similar argument holds for (5.28) as well.

Recalling that $\overline{F}^0_{n1}(z_1)=\overline{F}^0_{\varepsilon,n}(z_1,0)$ and $\overline{F}^0_1(z_1)=\overline{F}^0(z_1,0)$, we observe that (5.27) has linearization

$$
\int L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z}) \tag{5.33}
$$

where

- $\mathbf{z_1^0}=(z_1,0)$, $\mathbf{z}=(z_1,z_2)$
- $L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n})$, linearization of $\overline{F}^0_{\varepsilon,n}$ at $\mathbf{x}$, is solution to

$$
\begin{aligned}
L_{\mathbf{x}}(\overline{F}^0_{\varepsilon,n})-\int \mathbb{1}\{\mathbf{t}\geq\mathbf{x}\}L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})}&=\int \mathbb{1}\{\mathbf{t}\geq\mathbf{x}\}\left(\frac{H^{11}_n(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})}-\overline{H}_n(\mathbf{t})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\right)\\
-\int \mathbb{1}\{\mathbf{t}\geq\mathbf{0}\}L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})}&=\int \mathbb{1}\{\mathbf{t}\geq\mathbf{0}\}\left(\frac{H^{11}_n(\mathrm{d}\mathbf{t})}{\overline{G}(\mathbf{t})}-\overline{H}_n(\mathbf{t})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\right)
\end{aligned} \tag{5.34}
$$

Similarly, (5.29) has linearization

$$\int L_{\mathbf{z_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z}), \tag{5.35}$$

where $\mathbf{z_2^0}=(0,z_2)$.

From Equation (3.9) of Jin et al. (2013), we know that the functional (5.30), which can be rewritten as

$$\begin{aligned}&\sqrt{n}\int\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)\left(F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right)\\&=\sqrt{n}\left\{\int\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)F^0_{\varepsilon,n}(\mathrm{d}z_1,\mathrm{d}z_2)-\int\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)F^0(\mathrm{d}z_1,\mathrm{d}z_2)\right\},\end{aligned} \tag{5.36}$$

has linearization

$$\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\bar{F}^0(\mathbf{z})}+\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z}). \tag{5.37}$$

We conclude that $\sqrt{n}A_n$ has linearization

$$\begin{aligned}&\int\left(L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\\&+\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\bar{F}^0(\mathbf{z})}+\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z})\end{aligned} \tag{5.38}$$

Rewrite that linearization for $\mathbf{s_1^0}=(s_1,0)$, $\mathbf{s_2^0}=(0,s_2)$ and $\mathbf{s}=(s_1,s_2)$:

$$\begin{aligned}&\int\left(L_{\mathbf{s_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{s_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2)\right)}F^0(\mathrm{d}\mathbf{s})\\&+\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\bar{F}^0(\mathbf{s})}+\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\end{aligned} \tag{5.39}$$

Multiplying (5.38) $\times$ (5.39) (as in Appendix D.1) and taking expectation, the asymptotic variance of $\sqrt{n}A_n$ obtains:

$$
\begin{aligned}
\sigma^2 =& \mathbb{E}\Bigg\{\int\int\left(L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\times \\
&\left(L_{\mathbf{s_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{s_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2)\right)}F^0(\mathrm{d}\mathbf{z})F^0(\mathrm{d}\mathbf{s})\Bigg\} \\
&+2\mathbb{E}\Bigg\{\int\left(L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\times \\
&\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\bar{F}^0(\mathbf{s})}\Bigg\} \\
&+2\mathbb{E}\Bigg\{\int\left(L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z}) \\
&\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\Bigg\} \\
&+\mathbb{E}\Bigg\{\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\bar{F}^0(\mathbf{z})}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\bar{F}^0(\mathbf{s})}\Bigg\} \\
&+2\mathbb{E}\Bigg\{\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\bar{F}^0(\mathbf{z})}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\Bigg\} \\
&+\mathbb{E}\Bigg\{\int\phi\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z})\int\phi\left(\bar{F}^0_1(s_1),\bar{F}_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\Bigg\}
\end{aligned}
\tag{5.40}
$$

We can work out each of these expectations to get expressions that will be more easily estimated from data. This task is carried out in Appendix D.2. From that, we obtain that:

Theorem 5.1. *The limiting variance of* $\sqrt{n}A_n$ *is*

$$\begin{aligned}
\sigma^2 = &\int\int v(\mathbf{z},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2)\right)}\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2)\right)} \\
&\qquad \overline{F}^0(\mathbf{z})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \\
&+2\int\int\left(v(\mathbf{z_1^0},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+v(\mathbf{z_2^0},\mathbf{s})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2)\right)}\times \\
&\qquad \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\overline{F}^0(\mathbf{z})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \\
&+2\int(r_{\mathbf{z_1^0}}+r_{\mathbf{z_2^0}})\overline{F}^0(\mathbf{z})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})} \\
&+\int\int v(\mathbf{z},\mathbf{s})\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})} \\
&+2\int e_{\mathbf{z}}\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}+\mathbb{E}\left(\int\int\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z})\alpha_n(\mathrm{d}\mathbf{s})\right)
\end{aligned} \tag{5.41}$$

5.3 Asymptotic behaviour of B_n

B_n was defined as

$$B_n = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log\left(L(\theta)\right)\Big|_{\theta=\theta_0} = -\sum_{i=1}^{n}p_i\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log\left(c_\theta\left(F_{n1}^0(Z_{1i}),F_{n2}^0(Z_{2i})\right)\right)\Big|_{\theta=\theta_0}.$$

Then:

Proposition 5.1. *As* $n\to\infty$*, in probability,*

$$\begin{aligned}
B_n \to B &= -\mathbb{E}\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\log\left(c_\theta\left(F_1(Z_1),F_2(Z_2)\right)\right)\Big|_{\theta=\theta_0}\right) \\
&= \mathbb{E}\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left(c_\theta\left(F_1(Z_1),F_2(Z_2)\right)\right)\Big|_{\theta=\theta_0}\right)^2\right\} \\
&= \mathbb{E}\left\{\phi^2\left(\overline{F}_1^0(Z_1),\overline{F}_2^0(Z_2);\theta_0\right)\right\}
\end{aligned} \tag{5.42}$$

5.4 Asymptotic behaviour of $\hat{\theta}$

From the results of the previous two sections, we have the asymptotic behaviour of the maximum likelihood estimator:

Theorem 5.2.

$$\sqrt{n}(\hat{\theta}-\theta_0) \to \mathcal{N}(0,v^2), \text{ where } v^2 = \frac{\sigma^2}{B}, \text{ with } \sigma^2 \text{ given by (5.41) and } B \text{ by (5.42).}$$

Proof. The result comes from the approximation (5.13) and Slutsky's theorem. □

5.5 Variance estimation

We turn now to the estimation of the variance v^2 above.

Rewrite Equation (D.79) as:

$$\begin{aligned} r_{\mathbf{z_0^1}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{z_1^0}\} r_{\mathbf{u}} \frac{\overline{H}^{11}(\mathrm{d}\mathbf{u})}{\overline{H}(\mathbf{u})} &= o_{\mathbf{z_1^0}} \\ -\int \mathbb{1}\{\mathbf{u} \geq \mathbf{0}\} r_{\mathbf{u}} \frac{\overline{H}^{11}(\mathrm{d}\mathbf{u})}{\overline{H}(\mathbf{u})} &= o_{\mathbf{0}} \end{aligned} \tag{5.43}$$

Just as with Equation (4.57) in Section 4.2.3:

$$o_{\mathbf{z_1^0}} = \mathbb{E}\left\{\int\int \mathbb{1}\{\mathbf{u} \geq \mathbf{z_1^0}\} \frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2)\right)} \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{u})\alpha_n(\mathrm{d}\mathbf{s})\right\} \tag{5.44}$$

$$\begin{aligned} &= n^{-1}\Bigg[\int \mathbb{1}\{\mathbf{u} \geq \mathbf{z_1^0}\} \frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2)\right)} \phi\left(\overline{F}_1^0(u_1),\overline{F}_2^0(u_2);\theta_0\right)\left(\overline{F}^0(\mathbf{u})\right)^2 \frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})} \\ &\quad + \int\int \mathbb{1}\{\mathbf{u} \geq \mathbf{z_1^0}\} \frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2)\right)} \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right) \\ &\qquad \times \left(1-\mathbb{1}\{\mathbf{u} \geq \mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{u} < \mathbf{s}\}\right)\overline{H}(\mathbf{u}\vee\mathbf{s})\overline{F}^0(\mathbf{u})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{u})}{\overline{H}^2(\mathbf{u})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})}\Bigg] \end{aligned} \tag{5.45}$$

Let us denote by $\widehat{\mathbf{r_1}} = \left(\hat{r}_{(Z_{11},0)},\ldots,\hat{r}_{(Z_{1,n},0)}\right)$ the vector of estimates of $r_{\mathbf{z_0^1}}$.

$\widehat{\mathbf{r_1}}$ is then the solution to the equation

$$\begin{bmatrix} \mathbf{I}-\mathbf{AB} \\ -\mathbf{b}^T \end{bmatrix}_{(n+1)\times n} \widehat{\mathbf{r_1}} = \begin{bmatrix} \mathbf{AB}\overline{\mathbb{F}}^{\mathbf{0}} \\ \mathbf{b}^T\overline{\mathbb{F}}^{\mathbf{0}} \end{bmatrix} (\mathbf{I}+\mathbf{BDB})\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi(\Phi_1^*)^T \tag{5.46}$$

where

- $\mathbf{I}$ is the $n \times n$ identity matrix
- $(\mathbf{A})_{ij} = a_{ij} = \mathbb{1}\{Z_{j1} \geq Z_{1i}, Z_{2j} \geq Z_{2i}\}$
- $(\mathbf{b})_i = b_i = \frac{\Delta H_n^{11}(Z_{1i},Z_{2i})}{\overline{H}_n(Z_{1i},Z_{2i})+\frac{1}{n}} = \frac{\delta_{1i}\delta_{2i}}{n\overline{H}_n(Z_{1i},Z_{2i})+1}$
- $\mathbf{B} = \text{diag}(b_1, \ldots, b_n)$
- $\overline{\mathbb{F}}^{\mathbf{0}} = \text{diag}(\overline{F}_1^0, \ldots, \overline{F}_n^0)$
- $(\mathbf{D})_{ij} = d_{ij} = (1-a_{ij})(1-a_{ji})\sum_{r=1}^n a_{ir}a_{jr}$
- $(\Phi)_i = \phi\left(\overline{F}_1^0(Z_{1i}), \overline{F}_2^0(Z_{2i}); \theta_0\right)$
- $(\Phi_1^*)_i = \frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(Z_{1i}),\overline{F}_2^0(Z_{2i})\right)}$

and

$$\widehat{\mathbf{r_2}} = \left(\hat{r}_{(0,Z_{12})}, \ldots, \hat{r}_{(0,Z_{n2})}\right)$$

is the solution to the equation

$$\begin{bmatrix} \mathbf{I}-\mathbf{AB} \\ -\mathbf{b}^T \end{bmatrix}_{(n+1)\times n} \widehat{\mathbf{r_2}} = \begin{bmatrix} \mathbf{AB}\overline{\mathbb{F}}^{\mathbf{0}} \\ \mathbf{b}^T\overline{\mathbb{F}}^{\mathbf{0}} \end{bmatrix} (\mathbf{I}+\mathbf{BDB})\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi(\Phi_2^*)^T, \tag{5.47}$$

where

- $(\Phi_2^*)_i = \frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}_1^0(Z_{1i}),\overline{F}_2^0(Z_{2i})\right)}$.

Similarly, from Equation (D.81), $\hat{\mathbf{e}} = (\hat{e}_{\mathbf{Z}_1}, \ldots, \hat{e}_{\mathbf{Z}_n})$ is the solution to the equation

$$\begin{bmatrix} \mathbf{I}-\mathbf{AB} \\ -\mathbf{b}^T \end{bmatrix}_{(n+1)\times n} \hat{\mathbf{e}} = \begin{bmatrix} \mathbf{AB}\overline{\mathbb{F}}^0 \\ \mathbf{b}^T\overline{\mathbb{F}}^0 \end{bmatrix} (\mathbf{I}+\mathbf{BDB})\overline{\mathbb{F}}^0\mathbf{B}\Phi\Phi^T. \tag{5.48}$$

Replacing terms in Equation (5.41) with their sample counterparts, we have an estimate for σ^2, the asymptotic variance of $\sqrt{n}A_n$:

$$
\begin{aligned}
\widehat{\sigma^2} = &\int\int v_n(\mathbf{z},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2)\right)}\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2)\right)} \\
&\qquad \overline{F}^0_{\varepsilon,n}(\mathbf{z})\overline{F}^0_{\varepsilon,n}(\mathbf{s})\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})} \\
&+2\int\int\left(v_n(\mathbf{z_1^0},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})+v_n(\mathbf{z_2^0},\mathbf{s})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2)\right)}\times \\
&\quad \phi\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2);\hat{\theta}\right)\overline{F}^0_{\varepsilon,n}(\mathbf{z})\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})} \\
&+2\int(\hat{r}_{\mathbf{z_1^0}}+\hat{r}_{\mathbf{z_2^0}})\overline{F}^0_{\varepsilon,n}(\mathbf{z})\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})} \\
&+\int\int v_n(\mathbf{z},\mathbf{s})\phi\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2);\hat{\theta}\right)\phi\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2);\hat{\theta}\right)\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})} \\
&+2\int\hat{e}_{\mathbf{z}}\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}+n^{-1}\Bigg[\int\phi\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2);\hat{\theta}\right)\left(\overline{F}^0_{\varepsilon,n}(\mathbf{z})\right)^2\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}^2_n(\mathbf{z})} \\
&+\int\int\phi\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2);\hat{\theta}\right)\phi\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2);\hat{\theta}\right)(1-\mathbb{1}\{\mathbf{z}\geq\mathbf{s}\})(1-\mathbb{1}\{\mathbf{z}<\mathbf{s}\}) \\
&\qquad \times\overline{H}_n(\mathbf{z}\vee\mathbf{s})\overline{F}^0_{\varepsilon,n}(\mathbf{z})\overline{F}^0_{\varepsilon,n}(\mathbf{s})\frac{H^{11}_n(\mathrm{d}\mathbf{z})}{\overline{H}^2_n(\mathbf{z})}\frac{H^{11}_n(\mathrm{d}\mathbf{s})}{\overline{H}^2_n(\mathbf{s})}\Bigg]
\end{aligned}
\tag{5.49}
$$

where

- $v_n(\mathbf{z},\mathbf{s})$ is the sample covariance of $L_{\mathbf{z}}(\overline{F}^0_{\varepsilon,n})$ (see section 4.2.3).

Each summand in Equation (5.49) is rewritten in matrix notation in Appendix D.3. This operation yields that:

$$
\begin{aligned}
\widehat{\sigma^2} = &(\Phi^*_1)^T\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{BVB}\overline{\mathbb{F}}^{\mathbf{0}}\Phi^*_2+2\mathbf{b}^T\mathbf{V}^*\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b}+2(\widehat{\mathbf{r}_1}+\widehat{\mathbf{r}_2})^T\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b} \\
&+\Phi^T\mathbf{BVB}\Phi+2\mathbf{b}^T\hat{\mathbf{e}}+\Phi^T\mathbf{B}\overline{\mathbb{F}}^{\mathbf{0}}(\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b}+\mathbf{BDB}\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi),
\end{aligned}
\tag{5.50}
$$

where, for $i,j=1,\ldots,n$:

- $(\Phi^*_1)_i=\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(Z_{1i}),\overline{F}^0_2(Z_{2i})\right)}$;

- $(\Phi_2^*)_i = \frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\bar{F}_1^0(Z_{1i}),\bar{F}_2^0(Z_{2i})\right)}$
- $\bar{\mathbb{F}}^{\mathbf{0}} = \mathrm{diag}(\bar{F}_1^0,\ldots,\bar{F}_n^0)$;
- $\mathbf{B} = \mathrm{diag}(b_1,\ldots,b_n)$, with $b_i = \frac{\Delta H_n^{11}(Z_{1i},Z_{2i})}{\bar{H}_n(Z_{1i},Z_{2i})+\frac{1}{n}} = \frac{\delta_{1i}\delta_{2i}}{n\bar{H}_n(Z_{1i},Z_{2i})+1}$;
- $\mathbf{b} = (b_1,\ldots,b_n)$;
- $(\mathbf{V})_{ij} = v_{ij} = v_n(\mathbf{Z}_i,\mathbf{Z}_j)$: sample covariance of $L_{\mathbf{t}}(\bar{F}_{\varepsilon,n}^0)$ at $(\mathbf{Z}_i,\mathbf{Z}_j)$;
- $$\begin{aligned}(\mathbf{V}^*)_{ij} =& v_n\left((Z_{1i},0),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\bar{F}_{n1}^0(Z_{1i}),\bar{F}_{n2}^0(Z_{2i})\right)}\\ &+ v_n\left((0,Z_{2i}),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\bar{F}_{n1}^0(Z_{1i}),\bar{F}_{n2}^0(Z_{2i})\right)};\end{aligned}$$
- $(\mathbf{D})_{ij} = d_{ij} = (1-a_{ij})(1-a_{ji})\sum_{r=1}^n a_{ir}a_{jr}$, with $a_{ij} = \mathbb{1}\{Z_{j1}\geq Z_{1i}, Z_{2j}\geq Z_{2i}\}$;
- $\widehat{\mathbf{r}_p}$ is the solution to the equation

$$\begin{bmatrix}\mathbf{I}-\mathbf{AB}\\ -\mathbf{b}^T\end{bmatrix}_{(n+1)\times n}\widehat{\mathbf{r}_p} = \begin{bmatrix}\mathbf{AB}\bar{\mathbb{F}}^{\mathbf{0}}\\ \mathbf{b}^T\bar{\mathbb{F}}^{\mathbf{0}}\end{bmatrix}(\mathbf{I}+\mathbf{BDB})\bar{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi(\Phi_p^*)^T,\ \text{for } p\in\{1,2\}, \tag{5.51}$$

  with $\mathbf{I}$ being the $n\times n$ identity matrix and $(\mathbf{A})_{ij} = a_{ij} = \mathbb{1}\{Z_{1j}\geq Z_{1i}, Z_{2j}\geq Z_{2i}\}$;
- $\hat{\mathbf{e}}$ is the solution to the equation

$$\begin{bmatrix}\mathbf{I}-\mathbf{AB}\\ -\mathbf{b}^T\end{bmatrix}_{(n+1)\times n}\hat{\mathbf{e}} = \begin{bmatrix}\mathbf{AB}\bar{\mathbb{F}}^{0}\\ \mathbf{b}^T\bar{\mathbb{F}}^{0}\end{bmatrix}(\mathbf{I}+\mathbf{BDB})\bar{\mathbb{F}}^{0}\mathbf{B}\Phi\Phi^T; \tag{5.52}$$

- $(\Phi)_i = \phi\left(\bar{F}_1^0(Z_{1i}),\bar{F}_2^0(Z_{2i});\hat{\theta}\right)$.

We estimate the variance of $\sqrt{n}(\hat{\theta}-\theta_0)$ by

$$\widehat{v^2} = \frac{\widehat{\sigma^2}}{\sum_{i=1}^n p_i\phi^2\left(\bar{F}_1^0(Z_{1i}),\bar{F}_2^0(Z_{2i});\hat{\theta}\right)}. \tag{5.53}$$

5.6 Illustration using simulated data

To evaluate the performance of our maximum pseudo-likelihood estimator using two dependence structures for two independent real-valued random variables X_1 and X_2, we explore two scenarios:

(1) X_1 and X_2 are linked by a Clayton copula:

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \quad (u_1,u_2) \in [0,1]^2, \theta \in (0,\infty)$$

and have a marginal Weibull distribution of scale and shape 2, that is with density

$$f_X(x) = \frac{x}{2}\exp\left(-\left(\frac{x}{2}\right)^2\right), \quad x \geq 0.$$

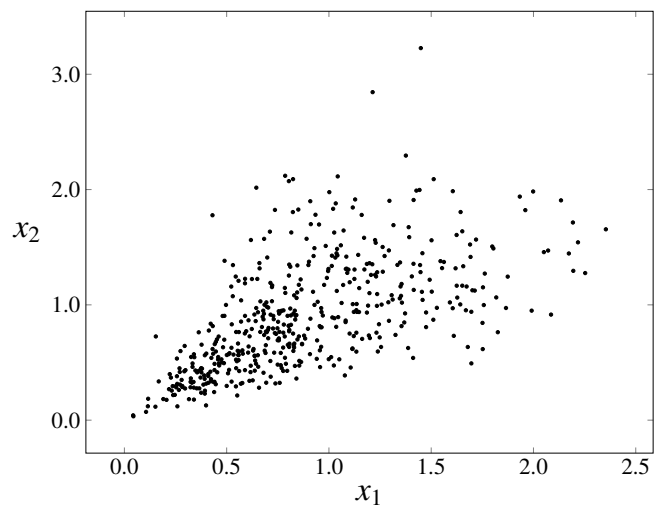


FIGURE 5.1: Clayton copula with $\theta = 2\left(\tau = \frac{1}{2}\right)$, $n = 500$

The censoring variables are both chosen from an exponential distribution of rate λ (mean = $\frac{1}{\lambda}$), with density

$$f_Y(y) = \lambda \exp(-\lambda y), \quad y \geq 0.$$

From a sample $\{(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})\}_{i=1}^{i=n}$, we can evaluate the log-pseudo-likelihood based on the mass-shifting estimator (*compact maximum log-likelihood function*) as (see Appendix D.4 for the derivation)

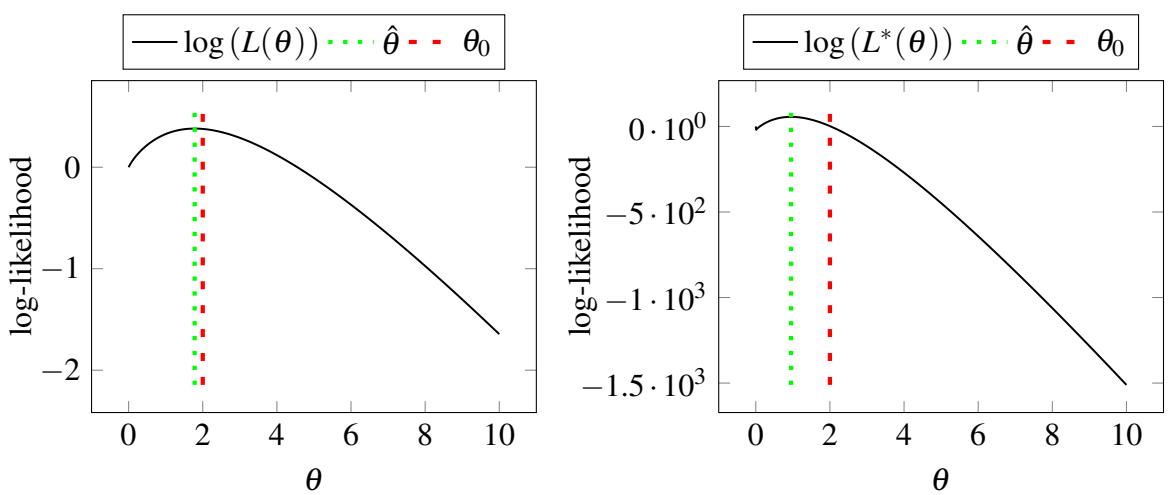
$$\log\left(L(\theta)\right) = \sum_{i=1}^{n} p_i \left(\log(\theta+1) - (\theta+1)\log\left(F_{n1}^0(Z_{1i})F_{n2}^0(Z_{2i})\right) - \left(2+\frac{1}{\theta}\right)\log\left(F_{n1}^0(Z_{1i})^{-\theta} + F_{n2}^0(Z_{2i})^{-\theta} - 1\right)\right), \tag{5.54}$$

where we recall that

- p_i is the weight given by the mass-shifting estimator (section 4.2.2)
- $1 - F_{nj}^0(Z_{ji}) = \overline{F}_{nj}^0(Z_{ji})$, $j \in \{1,2\}$, is the univariate Kaplan-Meier estimate at the point Z_{ji}.

In the approach developed by Shih and Louis (1995), the log-likelihood function (*full log-likelihood function*) is (see Appendix D.4 for the derivation):

$$\log\left(L^*(\theta)\right) = \sum_{i=1}^{n} \left\{ \delta_{1i}\delta_{2i}\log(\theta+1) - (\theta+1)\left(\delta_{1i}\log\left(\overline{F}_{n1}^0(Z_{1i})\right) + \delta_{2i}\log\left(\overline{F}_{n2}^0(Z_{2i})\right)\right) - \left(\frac{1}{\theta} + \delta_{1i} + \delta_{2i}\right)\log\left(\left(\overline{F}_{n1}^0(Z_1)\right)^{-\theta} + \left(\overline{F}_{n2}^0(Z_2)\right)^{-\theta} - 1\right)\right\} \tag{5.55}$$



(a) compact log-likelihood (5.54)

(b) full log-likelihood (5.55)

FIGURE 5.2: Log-likelihood functions for a model with a Clayton copula dependence structure of parameter $\theta = 2$ and 9% censoring ($\lambda = 0.05$), $n = 500$ observations

(2) X_1 and X_2 are linked by an Ali-Mikhail-Haq copula

$$C_\theta(u_1,u_2) = \frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)}, \quad (u_1,u_2) \in [0,1]^2, \theta \in [-1,1].$$

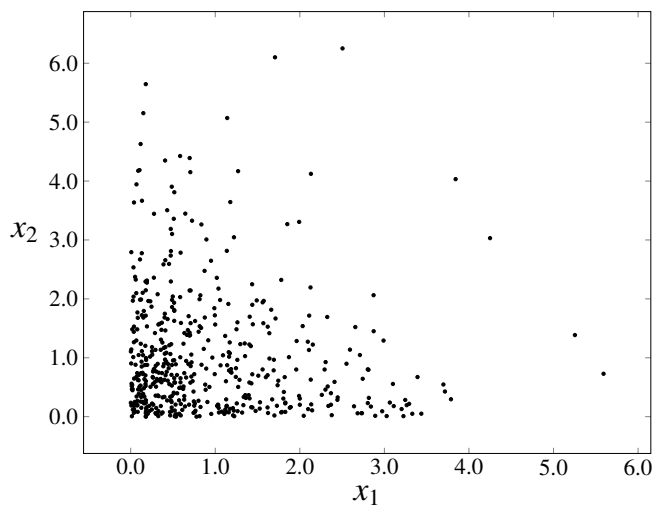


FIGURE 5.3: Ali-Mikhail-Haq copula with $\theta = -\frac{1}{2}(\tau \approx -0.1)$, $n = 500$

The censoring variables are random variables uniformly distributed in $[0,a]$ (where $a > 0$), with density $f_Y(y) = \frac{1}{a}\mathbb{1}_{[0,a]}(y)$.

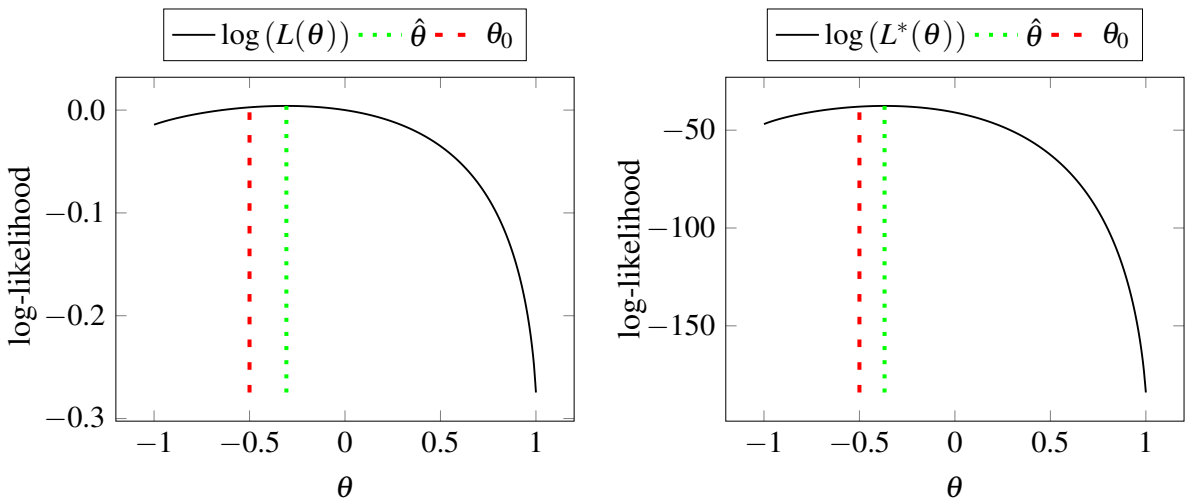
The compact log-likelihood is then

$$\begin{aligned}\log(L(\theta)) = \sum_{i=1}^{n} p_i \{ & -3\log\left(1-\theta(1-F_{n1}^0(Z_{1i}))(1-F_{n2}^0(Z_{2i})\right) \\ & +\log\left(1-2\theta+\theta^2+\theta(1-\theta)\left(F_{n1}^0(Z_{1i})+F_{n2}^0(Z_{2i})\right)+\theta(\theta+1)F_{n1}^0(Z_{1i})F_{n2}^0(Z_{2i})\right)\},\end{aligned} \tag{5.56}$$

and the full log-likelihood is

$$\begin{aligned}\log(L^*(\theta)) = \sum_{i=1}^{n} \{ & \delta_{1i}\delta_{2i}\log\left(1-2\theta+\theta^2+\theta(1-\theta)(\overline{F}_{n1}^0(Z_{1i})+\overline{F}_{n2}^0(Z_{2i}))+\theta(\theta+1)\overline{F}_{n1}^0(Z_{1i})\overline{F}_{n2}^0(Z_{2i})\right) \\ & -(1+\delta_{1i}+\delta_{2i})\log\left(1-\theta(1-\overline{F}_{n1}^0(Z_{1i}))(1-\overline{F}_{n2}^0(Z_{2i}))\right) \\ & +(1-\delta_{1i})\log(\overline{F}_{n1}^0(Z_{1i}))+(1-\delta_{2i})\log(\overline{F}_{n2}^0(Z_{2i})) \\ & +(1-\delta_{1i})\,\delta_{2i}\log\left(1-\theta(1-\overline{F}_{n1}^0(Z_{1i}))\right)+\delta_{1i}(1-\delta_{2i})\log\left(1-\theta(1-\overline{F}_{n2}^0(Z_{2i}))\right)\}\end{aligned} \tag{5.57}$$

The reader may refer to Appendix D.4 for the calculation details.



(a) compact log-likelihood (5.54)

(b) full log-likelihood (5.55)

FIGURE 5.4: Log-likelihood functions for a model with an Ali-Mikhail-Haq ($\theta = -\frac{1}{2}$), 6% censoring ($a = 30$) and $n = 500$ observations

In each scenario, we simulate $M = 500$ samples of n observations each. The performances are assessed on the criterion of the mean squared error (MSE):

$$\mathbb{E}\left[(\hat{\theta} - \theta_0)^2\right] = \underbrace{\mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)^2\right]}_{\text{variance}} + \underbrace{\left(\mathbb{E}(\hat{\theta}) - \theta_0\right)^2}_{\text{bias}^2},$$

where $\hat{\theta}$ is the estimator for the true parameter value θ_0.

Choosing different parameter values of the censoring variable distributions (λ in the first scenario, a in the second one) enables us to assess the effect of censoring on the performance of the estimators. The performance of the maximum likelihood estimation procedure based on the mass-shifting estimator (*compact* likelihood approach) for the survival function is compared to that developed in Shih and Louis (1995) (*full* likelihood approach).

Using the programming language R (R Core Team), the functions *mvdc* and *rMvdc* (in the package *copula*) allow us to build a copula object with some marginal distributions and to simulate observations from it. We estimate the (univariate) Kaplan-Meier curves using the functions *survfit* and *Surv*

of the package *survival*. The *optimize* function (*stats* package) is used to determine the maximum of the log-likelihood functions. The results of the simulations are summarized in Tables 5.1 and 5.2.

In the first scenario (Table 5.1), the maximum likelihood estimator based on the mass-shifting estimator performs better than the *full* likelihood approach. The former estimator seems to benefit more from the increase in the sample size (from $n = 200$ to $n = 500$). The more correlated the variables X_1 and X_2 are, the more striking the advantage of the *compact* likelihood approach is. The advantage gained lies mainly in the bias component of the MSE. The variance aspects are more comparable between the two estimators.

TABLE 5.1: Simulation results for a Clayton copula

Model		Criterion	$n = 200$		$n = 500$	
			compact	full	compact	full
$\theta = \frac{1}{2}$	$\lambda = 0.05$	MSE	0.0199	0.0555	0.0073	0.0577
		bias^2	0.0011	0.0438	$6.9200 \cdot 10^{-05}$	0.0530
	(7% – 8% censoring)	variance	0.0188	0.0117	0.0072	0.0046
$(\tau = \frac{1}{5})$	$\lambda = 0.5$	MSE	0.0363	0.0415	0.0129	0.0242
		bias^2	$8.1300 \cdot 10^{-05}$	0.0098	$8.3000 \cdot 10^{-05}$	0.0130
	(52% – 55% censoring)	variance	0.0362	0.0318	0.0128	0.0112
$\theta = 1$	$\lambda = 0.05$	MSE	0.0364	0.2186	0.0138	0.2414
		bias^2	0.0013	0.2042	$7.5800 \cdot 10^{-05}$	0.2359
	(7% – 8% censoring)	variance	0.0351	0.0144	0.0137	0.0056
$(\tau = \frac{1}{3})$	$\lambda = 0.5$	MSE	0.0625	0.1038	0.0223	0.0830
		bias^2	$3.0800 \cdot 10^{-05}$	0.0506	0.0001	0.0660
	(52% – 55% censoring)	variance	0.0625	0.0532	0.0222	0.0170
$\theta = 2$	$\lambda = 0.05$	MSE	0.0864	1.1238	0.0324	1.2110
		bias^2	0.0007	1.1015	$2.0000 \cdot 10^{-06}$	1.2026
	(7% – 8% censoring)	variance	0.0857	0.0223	0.0324	0.0084
$(\tau = \frac{1}{2})$	$\lambda = 0.5$	MSE	0.1386	0.4498	0.0468	0.4577
		bias^2	0.0002	0.3380	0.0004	0.4196
	(52% – 55% censoring)	variance	0.1384	0.1118	0.0464	0.0381

Note: *compact* refers to the MLE based on the mass-shifting estimator. *full* refers to the MLE using the Shih and Louis (1995)'s estimator

For lightly linked variables (Kendall tau smaller than 0.2 in absolute value), the *full* MLE technique of Shih and Louis (1995) achieves better results (Table 5.2). The advantage of the *full* MLE is more pronounced when the weak correlation is combined with heavy censoring.

TABLE 5.2: Simulation results for an AMH copula

Model		Criterion	$n = 200$	$n = 500$
$\theta = -\frac{3}{4}$	$a = 30$ (6% censoring)	MSE of compact MLE	0.055	0.027
		ratio compact MSE/full MSE	0.864	1.007
$(\tau \approx -0.14)$	$a = 3$ (50% censoring)	MSE of compact MLE	0.302	0.286
		ratio compact MSE/full MSE	4.088	8.089
$\theta = -\frac{1}{2}$	$a = 30$ (6% censoring)	MSE of compact MLE	0.072	0.028
		ratio compact MSE/full MSE	1.195	1.398
$(\tau \approx -0.1)$	$a = 3$ (50% censoring)	MSE of compact MLE	0.278	0.233
		ratio compact MSE/full MSE	4.085	9.113
$\theta = \frac{1}{2}$	$a = 30$ (6% censoring)	MSE of compact MLE	0.027	0.009
		ratio compact MSE/full MSE	1.071	1.002
$(\tau \approx 0.13)$	$a = 3$ (50% censoring)	MSE of compact MLE	0.169	0.058
		ratio compact MSE/full MSE	5.707	5.583
$\theta = \frac{3}{4}$	$a = 30$ (6% censoring)	MSE of compact MLE	0.011	0.004
		ratio compact MSE/full MSE	0.484	0.247
$(\tau \approx 0.21)$	$a = 3$ (50% censoring)	MSE of compact MLE	0.067	0.012
		ratio compact MSE/full MSE	3.549	1.452

Note: *compact* refers to the MLE based on the mass-shifting estimator. *full* refers to the MLE using the Shih and Louis (1995)'s estimator

5.7 Illustration using Canadian life insurance data

These data were first used in Frees, Carriere, and Valdez (1995) and are now available in the package *CASdatasets* in the software R.

The data concerns 14,889 contracts in force with a large Canadian insurer during the period December 29, 1988 - December 31, 1993. The contracts were joint and contained last-survivor annuities in the payout status over the observation period. 5 columns are available:

- *EntryAgeM*: entry age (years) of the male
- *EntryAgeF*: entry age (years) of the female
- *DeathTimeM*: time of death (years from December 29, 1988) of the male (zero if not applicable)

- *DeathTimeF*: time of death (years from December 29, 1988) of the female (zero if not applicable)
- *AnnuityExpiredM*: date (years from December 29, 1988) that the annuity guarantee expired (if applicable).

3 observations have values of entry age that are abnormally low: observation 8446 has EntryAgeF = 0.2676, observation 8484 has EntryAgeM = 0.0657 and EntryAgeF = 0.4906, and observation 13815 has EntryAgeF = 1.1326. These observations are excluded from the analysis. Due to computational limits, we restrict our analysis to 3000 observations randomly sampled from the remaining 14886. Of those 3000 observations,

- 50 were uncensored (death has been observed for both individuals),
- $2,625$ were doubly censored,
- 262 were only censored on the female death time,
- 63 were only censored on the male death time.

We are interested in the lifetimes from birth. The median male lifetime in our sample is 68.12. The female median lifetime is 65.53. Figure (5.5) shows the dependence in lifetime in the couples.

Our data is $\{(Z_{1i}, Z_{2i}, \delta_{1i}, \delta_{2i})\}_{i=1}^{i=3000}$, with

- $Z_1 = EntryAgeM + DeathTimeM$,
- $Z_2 = EntryAgeF + DeathTimeF$,
- $\delta_1 = 1$ if $DeathTimeM > 0$,
- $\delta_2 = 1$ if $DeathTimeF > 0$.

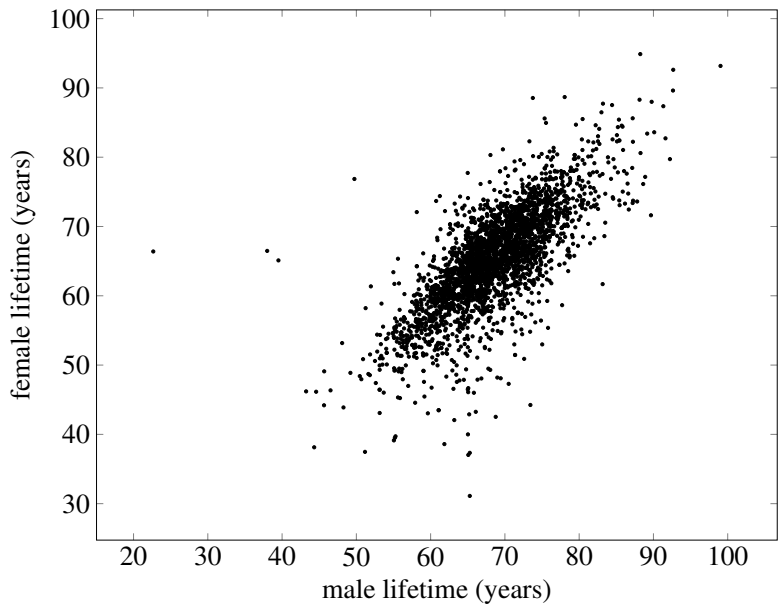


FIGURE 5.5: Lifetimes of a sample from the Canadian life insurance data

We assume a Clayton copula distribution for the pair of lifetimes. Although not as good a fit for the data as Nelsen (2006)'s copula number 4.2.20 (Luciano, Spreeuw, and Vigna, 2008), the Clayton copula allows relatively easier computations[1]. We estimate the (univariate) Kaplan-Meier curves using the functions *survfit* and *Surv* of the package *survival* of R (R Core Team). The *optimize* function (*stats* package) is used to determine the maximum of the log-likelihood functions.

The compact pseudo-likelihood maximum is obtained with 0.38 and the full pseudo-likelihood maximum estimator is 1.84 for a Clayton copula family structure.

5.8 Illustration using kidney patients' infection data

Let us also illustrate the two MLE procedures on the kidney patients's infection data, described in Section 4.3.4. Here too, like with the previous data (Canadian life insurance dataset), let us assume a Clayton copula dependence structure. We find that the *compact MLE* is 0.10 and the *full MLE* is 0.11. These values both yield a Kendall τ coefficient of approximately 0.05. This very weak dependence coefficient may indicate that the Clayton copula is not a good fit for the data.

[1] Cf. calculations in Appendix D.4 for the log-likelihood formulas with Nelsen (2006)'s copula number 4.2.20

5.9 Conclusion

The mass-shifting estimator presented in Section 4.2.2 stimulated the derivation of the semi-parametric likelihood estimator presented in this chapter. Under some mild assumptions about functionals of the mass-shifting estimator, the asymptotic distribution of this *compact MLE* was obtained. The maximization procedure was illustrated on simulated data and on the Canadian life insurance data. Our procedure was compared to the two-stage semi-parametric estimator of Shih and Louis (1995). The comparison on simulated data showed that our new maximum pseudo-likelihood estimator performs better as the correlation between the censored variables grows. Our estimator also seems to benefit more from the increase in the sample size. The advantage gained lies mainly in the bias component of the MSE. The variance aspects are more comparable between the two estimators.

Chapter 6

Conclusion and discussions

6.1 Concluding notes

At the outset of this thesis, our objective was to develop some inference procedures for copula-based models in the case of bivariate dependence. This has been done in both the uncensored and censored data cases.

The importance of Kendall functions for identifying Archimedean copula models led us to examine the distribution of the Kendall functions for survival functions from copulas in Chapter 2. We were able to derive a formula for the density. Moreover, the approach used allows for a representation that is useful in simulating the values of the survival function. But the difficulty in finding an analytical form of quantile functions in general limits the reach of the formula.

The well-known relation linking the Kendall function to the generator of Archimedean copulas allowed us to derive, in Chapter 3, two estimators for the generator: one based on a differential equation perspective and the other on an integral equation viewpoint. Asymptotic distributions of the estimators could be established under some conditions.

A novel estimator by Sen and Stute (2013) for the bivariate survival function in the case of right-censored random variables inspired the work in Chapters 4 and 5.

A plug-in estimator for Kendall's tau is derived in Chapter 4. Its asymptotic distribution was provided, using the influence function approach. The estimator was illustrated using both simulated and real-life data.

In Chapter 5, a maximum likelihood estimation for copula model parameters was developed using the estimator for the bivariate survival function. The maximization procedure was illustrated using simulated data and a Canadian life insurance dataset and compared to the estimator of Shih and Louis (1995). The comparison on simulated data showed that our new maximum pseudo-likelihood estimator shows performs better as the correlation between the censored lifetimes increases.

6.2 Future work

This thesis suggests some goodness-of-fit tests. First, a test for selecting among different Archimedean copula families may be derived by investigating the asymptotic behaviour of S_n (in 1.17) and T_n (in 1.18) in the case of right censored data. An estimator for the Kendall distribution (in replacement of 1.14) would be

$$K_n(t) = \sum_{i=1}^{n} p_i \mathbb{1}\{F^0_{\varepsilon,n}(Z_{1i}, Z_{2i}) \leq t\}, \tag{6.1}$$

with the mass-shifting estimator $F^0_{\varepsilon,n}$ and the weights p_i defined in section 4.2.2.

Secondly, another goodness-of-fit test may take advantage of either estimator of the generator provided in Chapter 3.

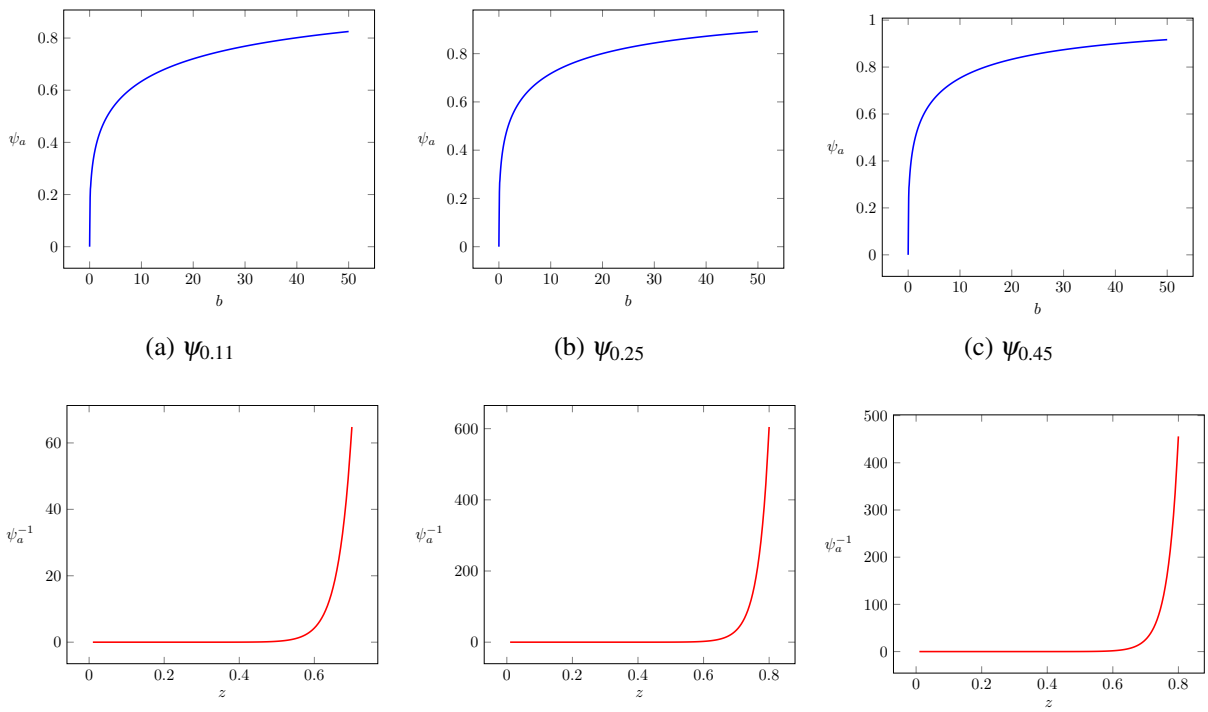
Some results of this thesis were achieved by making an assumption: Conjecture (4.2.1) was needed to derive the bivariate Kaplan-Meier estimator (*mass-shifting estimator*). Although reasonable, that assumption need to be formally established as a result.

Finally, with regards to the numerical analysis, the computation of the variance of the *compact* MLE presents some difficulty. More specifically, calculating the score function (formula (D.88) for a Clayton copula) with the software R sometimes yields divisions of the type $0/0$. A thorough examination of those specific situations and an appropriate treatment for them are necessary to allow for the calculation of the variance of the *compact* MLE.

Appendices

A Density of the survival function of a copula

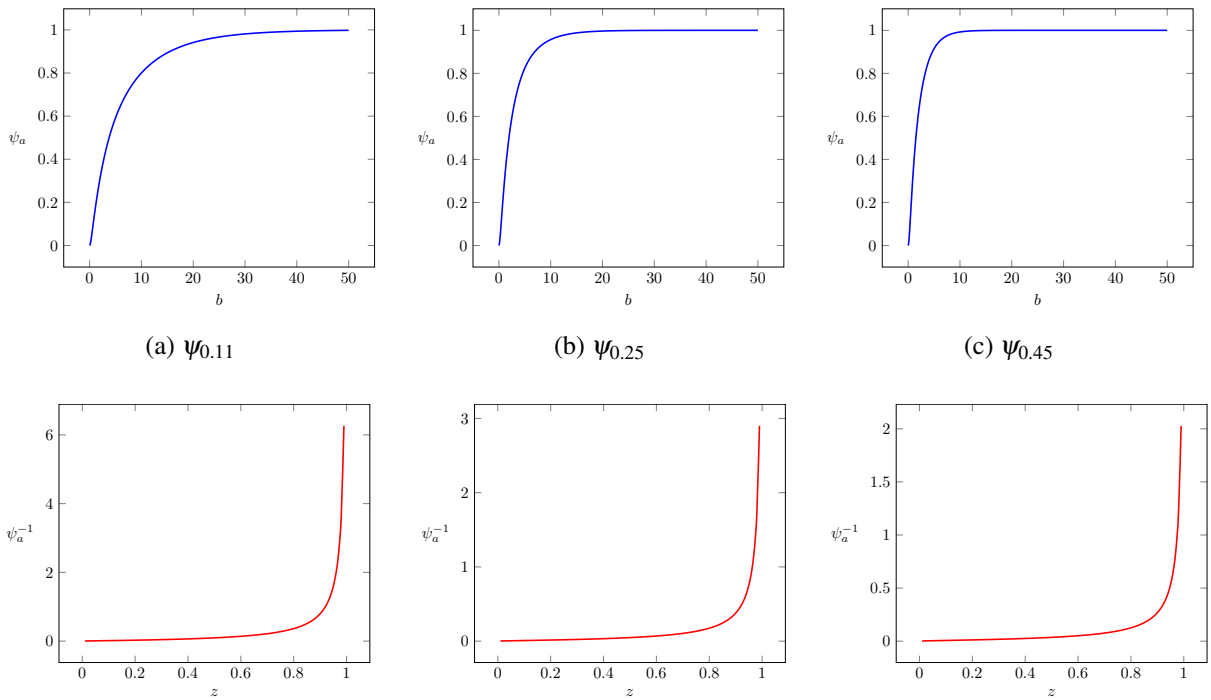
A.1 Graphs of the distribution function ψ_a and its inverse $\widehat{\psi}_a^{-1}$



(a) $\psi_{0.11}$ (b) $\psi_{0.25}$ (c) $\psi_{0.45}$

(d) $\widehat{\psi}_{0.11}^{-1}$ (e) $\widehat{\psi}_{0.25}^{-1}$ (f) $\widehat{\psi}_{0.45}^{-1}$

FIGURE A.1: Distribution function ψ_a and its estimated inverse $\widehat{\psi}_a^{-1}$ for $a = 0.11$, $a = 0.25$, and $a = 0.45$ in the case of the Gumbel copula with $\theta = 5$



(a) $\psi_{0.11}$ (b) $\psi_{0.25}$ (c) $\psi_{0.45}$

(d) $\widehat{\psi}_{0.11}^{-1}$ (e) $\widehat{\psi}_{0.25}^{-1}$ (f) $\widehat{\psi}_{0.45}^{-1}$

FIGURE A.2: Distribution function ψ_a and its estimated inverse $\widehat{\psi}_a^{-1}$ for $a = 0.11$, $a = 0.25$, and $a = 0.45$ in the case of the Ali-Mikhail-Haq copula with $\theta = 0.5$

A.2 Proof of Theorem 2.1

With an integrable function h defined on $[0,1]$:

$$\begin{aligned}
&\mathbb{E}\left(h\left(C^*(U_1,U_2)\right)\right)\\
&= \int_0^1\int_0^1 h\left(1-u_1-u_2+C(u_1,u_2)\right)\mathrm{d}C(u_1,u_2) \qquad \text{(A.2)}\\
&= \int_0^1\int_0^1 h\left(1-u_1-u_2+\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)\frac{-\varphi''\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)}{\left[\varphi'\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)\right]^3}\varphi'(u_1)\varphi'(u_2)\mathrm{d}u_1\mathrm{d}u_2 \qquad \text{(A.3)}
\end{aligned}$$

Writing $t_1 = \varphi(u_1), t_2 = \varphi(u_2)$, we have:

$$\mathbb{E}\left(h\left(C^*(U_1,U_2)\right)\right) = \int_0^\infty\int_0^\infty h\left(1-\varphi^{-1}(t_1)-\varphi^{-1}(t_2)+\varphi^{-1}(t_1+t_2)\right)\frac{-\varphi''\left(\varphi^{-1}(t_1+t_2)\right)}{\left[\varphi'\left(\varphi^{-1}(t_1+t_2)\right)\right]^3}\mathrm{d}t_1\mathrm{d}t_2. \qquad \text{(A.4)}$$

From the properties of the generator, φ^{-1} is decreasing with $\varphi^{-1} \geq 0, \varphi^{-1}(0) = 1, \varphi^{-1}(\infty) = 0$. Like φ, φ^{-1} is also continuous. Let us recall with Joe (2014) (page 32) that these properties suffice to make φ^{-1} a survival function. Let $\varphi^{-1}(t) = \mathbb{P}(T > t) = \overline{F}_0(t)$ for some random variable $T \geq 0$. It has density

$$f_0(t) = -\frac{1}{\varphi'\left(\varphi^{-1}(t)\right)}. \tag{A.5}$$

Further:

$$-f_0'(t) = -\frac{\varphi''\left(\varphi^{-1}(t)\right)}{\left(\varphi'\left(\varphi^{-1}(t)\right)\right)^3} \geq 0. \tag{A.6}$$

So that f_0 is decreasing and

$$\varphi^{-1}(t_1 + t_2) = \overline{F}_0(t_1 + t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} -f_0'(s_1 + s_2) \mathrm{d}s_1 \mathrm{d}s_2. \tag{A.7}$$

$-f_0'(s_1 + s_2)$ is the joint density of $(\varphi(U_1), \varphi(U_2))$ on $[0, \infty) \times [0, \infty)$. To see that, note first that the joint density of (U_1, U_2) is

$$\begin{aligned}
\frac{\partial^2}{\partial u_2 \partial u_1} C(u_1, u_2) &= \frac{\partial}{\partial u_2} \frac{\partial}{\partial u_1} \left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right) && \text{(A.8)} \\
&= \frac{\partial}{\partial u_2} \left\{ \frac{\frac{\mathrm{d}}{\mathrm{d}u_1} \varphi(u_1)}{\varphi'\left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right)} \right\} && \text{(A.9)} \\
&= \frac{\partial}{\partial u_2} \left\{ \frac{\varphi'(u_1)}{\varphi'\left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right)} \right\} && \text{(A.10)} \\
&= \varphi'(u_1) \frac{\partial}{\partial u_2} \frac{1}{\varphi'\left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right)} && \text{(A.11)} \\
&= \varphi'(u_1) \left\{ -\frac{\frac{\partial}{\partial u_2} \varphi'\left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right)}{\left(\varphi'\left(\varphi^{-1}\left(\varphi(u_1) + \varphi(u_2)\right)\right)\right)^2} \right\} && \text{(A.12)}
\end{aligned}$$

$$\frac{\partial^2}{\partial u_2 \partial u_1} C(u_1,u_2) = -\varphi'(u_1) \frac{\varphi''\left(\varphi(u_1)+\varphi(u_2)\right) \frac{\partial}{\partial u_2} \varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)}{\left(\varphi'\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)\right)^2} \tag{A.13}$$

$$= -\varphi'(u_1) \frac{\varphi'(u_2)\varphi''\left(\varphi(u_1)+\varphi(u_2)\right) \frac{1}{\varphi'\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)}}{\left(\varphi'\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)\right)^2} \tag{A.14}$$

$$= -\varphi'(u_1)\varphi'(u_2) \frac{\varphi''\left(\varphi(u_1)+\varphi(u_2)\right)}{\left(\varphi'\left(\varphi^{-1}\left(\varphi(u_1)+\varphi(u_2)\right)\right)\right)^3}. \tag{A.15}$$

Then derive the density of $(T_1,T_2) = (\varphi(U_1),\varphi(U_2))$ as

$$-\varphi'\left(\varphi^{-1}(t_1)\right)\varphi'\left(\varphi^{-1}(t_2)\right) \frac{\varphi''(t_1+t_2)}{\left(\varphi'\left(\varphi^{-1}(t_1+t_2)\right)\right)^3} \left| \det \begin{bmatrix} \frac{1}{\varphi'(\varphi^{-1}(t_1))} & 0 \\ 0 & \varphi'\left(\varphi^{-1}(t_2)\right) \end{bmatrix} \right|$$

$$= -\varphi'\left(\varphi^{-1}(t_1)\right)\varphi'\left(\varphi^{-1}(t_2)\right) \frac{\varphi''(t_1+t_2)}{\left(\varphi'\left(\varphi^{-1}(t_1+t_2)\right)\right)^3} \times \frac{1}{\varphi'\left(\varphi^{-1}(t_1)\right)\varphi'\left(\varphi^{-1}(t_2)\right)} \tag{A.16}$$

$$= -\frac{\varphi''(t_1+t_2)}{\left(\varphi'\left(\varphi^{-1}(t_1+t_2)\right)\right)^3} \tag{A.17}$$

$$= -f_0'(t_1+t_2) \tag{A.18}$$

Further:

$$1-\varphi^{-1}(t_1)-\varphi^{-1}(t_2)+\varphi^{-1}(t_1+t_2) = \int_0^{t_1}\int_0^{t_2} -f_0'(s_1+s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{A.19}$$

(Recall that $1-\mathbb{P}(X_1>x_1)-\mathbb{P}(X_2>x_2)+\mathbb{P}(X_1>x_1,X_2>x_2) = \mathbb{P}(X_1\le x_1,X_2\le x_2)$)

Thus:

$$\mathbb{E}\left(h\left(C^*(U_1,U_2)\right)\right) = \int_0^\infty\int_0^\infty h\left(1-\varphi^{-1}(t_1)-\varphi^{-1}(t_2)+\varphi^{-1}(t_1+t_2)\right)\left(-f_0'(t_1+t_2)\right)\mathrm{d}t_1\mathrm{d}t_2. \tag{A.20}$$

As any decreasing density on $[0,\infty)$ can be represented as a scale mixture of uniform densities (Williamson, 1956), $f_0(t)$ is of the form

$$f_0(t) = \int_t^\infty \frac{\mathrm{d}G(s)}{s} = \int_t^\infty \frac{g(s)}{s}\mathrm{d}s, \tag{A.21}$$

where G is a distribution function on $[0,\infty)$, assumed to be absolutely continuous with density g. It is clear from Equation (A.6) that

$$g(t) = -tf_0'(t) = -\frac{t\varphi''\left(\varphi^{-1}(t)\right)}{\left(\varphi'\left(\varphi^{-1}(t)\right)\right)^3}. \tag{A.22}$$

With

$$a = \frac{t_1}{t_1+t_2} \text{ and } b = t_1+t_2,$$

$$\begin{aligned}\mathbb{E}\left(h\left(C^*(U_1,U_2)\right)\right) &= \int_0^\infty \int_0^1 h\left(1-\varphi^{-1}(ab)-\varphi^{-1}\left((1-a)b\right)+\varphi^{-1}(b)\right) g(b)\mathrm{d}a\mathrm{d}b \\ &= 2\int_0^\infty \int_0^{\frac{1}{2}} h\left(1-\varphi^{-1}(ab)-\varphi^{-1}\left((1-a)b\right)+\varphi^{-1}(b)\right) g(b)\mathrm{d}a\mathrm{d}b. \end{aligned} \tag{A.23}$$

Let

$$\psi(a,b) = 1-\varphi^{-1}(ab)-\varphi^{-1}\left((1-a)b\right)+\varphi^{-1}(b). \tag{A.24}$$

Fix a and let $\psi_a(b) = \psi(a,b)$.

As

$$\psi_a(b) = \mathbb{P}\left(\varphi(U_1)\le ab, \varphi(U_2)\le (1-a)b\right) = \mathbb{P}\left(\frac{\varphi(U_1)}{a}\le b, \frac{\varphi(U_2)}{1-a}\le b\right), \tag{A.25}$$

we have that $\psi_a(b)$ is increasing in b.

Put

$$\psi_a(b) = z \iff b = \psi_a^{-1}(z).$$

We can then rewrite Equation (A.23) as

$$\mathbb{E}\left(h\left(C^*(U_1,U_2)\right)\right) = 2\int_0^{\frac{1}{2}}\int_0^1 h(z)g\left(\psi_a^{-1}(z)\right)\left(\frac{\partial}{\partial z}\psi_a^{-1}(z)\right)\mathrm{d}z\mathrm{d}a \quad \text{for} \quad z\in[0,1]$$

Hence, the density of $C^*(U_1,U_2)$ is

$$k^*(z) = 2\int_0^{\frac{1}{2}} g\left(\psi_a^{-1}(z)\right)\left(\frac{\partial}{\partial z}\psi_a^{-1}(z)\right)\mathrm{d}a, \quad \text{where} \quad z\in[0,1]. \tag{A.26}$$

B Limiting distribution of the generator estimators

B.1 Differential estimator φ_{n1}: proof of Theorem 3.1

We assumed that (Assumption 3.2.1):

$$\lim_{\varepsilon\downarrow 0}\frac{\varphi(1-\varepsilon)}{\varepsilon}=-\varphi'(1)=1.$$

Consider the estimator in (3.15):

$$\varphi_{n1}(w)=\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)\quad\text{for}\quad 0<w<1.$$

Further, using (3.12) with $\varepsilon=\varepsilon_n$,

$$\varphi(w)=\frac{\varphi(1-\varepsilon_n)}{\varepsilon_n}\cdot\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right). \tag{B.27}$$

Thus

$$\begin{aligned}
\varphi_{n1}(w)-\varphi(w)=&\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)-\frac{\varphi(1-\varepsilon_n)}{\varepsilon_n}\cdot\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right)\\
=&\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)-\frac{\varphi(1-\varepsilon_n)}{\varepsilon_n}\cdot\varepsilon_n\left[\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right)\right.\\
&\left.+\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)-\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)\right]\\
=&\left(1-\frac{\varphi(1-\varepsilon_n)}{\varepsilon_n}\right)\cdot\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right) &\text{(B.28)}\\
&+\frac{\varphi(1-\varepsilon_n)}{\varepsilon_n}\cdot\varepsilon_n\left[\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)-\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-1}\right)\right] &\text{(B.29)}
\end{aligned}$$

Consider the second term above (B.29). We have by Taylor's expansion:

$$\begin{aligned}
&\varepsilon_n\left[\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}\right)-\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right)\right]\\
=&\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right)\left(\left[\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K_n(t)-t}-\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right]+R_n\right)\ \text{with}\ |R_n|\le\frac{1}{2(1-\frac{1}{n})} &\text{(B.30)}\\
=&\varepsilon_n\exp\left(\int_w^{1-\varepsilon_n}\frac{\mathrm{d}t}{K(t)-t}\right)\left(-\int_w^{1-\varepsilon_n}\frac{(K_n(t)-K(t))\,\mathrm{d}t}{(K_n(t)-t)(K(t)-t)}+R_n\right) &\text{(B.31)}
\end{aligned}$$

$\sqrt{n}\,(K_n(t)-K(t))$ is weakly convergent (Barbe et al., 1996). Then,

$$\max_{0\le t\le 1}\frac{\sqrt{n}\,(K_n(t)-K(t))}{\sigma(t)}<\infty \quad \text{almost surely.} \tag{B.32}$$

Recalling also that $K_n(t)-t\to K(t)-t=-\frac{\varphi(t)}{\varphi'(t)}$, we have

$$-\int_w^{1-\varepsilon_n}\frac{(K_n(t)-K(t))\,\mathrm{d}t}{(K_n(t)-t)(K(t)-t)} \tag{B.33}$$

$$=-\frac{1}{\sqrt{n}}\log\varepsilon_n\int_w^{1-\varepsilon_n}\frac{\sigma(t)}{\log\varepsilon_n}\frac{\sqrt{n}\,(K_n(t)-K(t))\,\mathrm{d}t}{\sigma(t)(K_n(t)-t)(K(t)-t)} \tag{B.34}$$

$$\simeq O\left(\frac{\log\varepsilon_n}{\sqrt{n}}\int_w^{1-\varepsilon_n}\frac{\sigma(t)}{\log\varepsilon_n}\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\right)\mathrm{d}t \tag{B.35}$$

Equation (B.35) can be justified as follows: denote the integral in (B.33) by $Z_n(w)$. Then

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{\sqrt{n}}{-\log\varepsilon_n}Z_n(w)\right)$$

$$=\lim_{n\to\infty}\left(\frac{1}{(\log\varepsilon_n)^2}\int_w^{1-\varepsilon_n}\int_w^{1-\varepsilon_n}\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\Gamma(t,s)\left(\frac{\varphi'(s)}{\varphi(s)}\right)^2\mathrm{d}s\mathrm{d}t\right) \tag{B.36}$$

$$\le\lim_{n\to\infty}\left(\frac{1}{\log\varepsilon_n}\int_w^{1-\varepsilon_n}\sigma(t)\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\mathrm{d}t\right)^2, \tag{B.37}$$

since, by Cauchy-Schwartz inequality,

$$\Gamma(t,s)=\lim_{n\to\infty}\mathrm{Cov}\left(\sqrt{n}\,(K_n(t)-K(t)),\,\sqrt{n}\,(K_n(s)-K(s))\right)\le\sigma(t)\sigma(s) \tag{B.38}$$

From

$$\lim_{\varepsilon\downarrow 0}\frac{\varphi(1-\varepsilon)}{\varepsilon}=-1 \quad \text{(Assumption 3.2.1)} \qquad \text{and} \qquad \lim_{\varepsilon\downarrow 0}\frac{\sigma(1-\varepsilon)}{\varepsilon}=\sigma_0 \quad \text{(Assumption 3.2.2)},$$

it follows that

$$\max_{w\le t\le 1}\frac{\sigma(t)}{\varphi(t)}<\infty. \tag{B.39}$$

Then

$$\lim_{n\to\infty}\int_w^{1-\varepsilon_n}\sigma(t)\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\mathrm{d}t\Big/\log\varepsilon_n \tag{B.40}$$

$$=\lim_{n\to\infty}-\sigma(1-\varepsilon_n)\left(\frac{\varphi'(1-\varepsilon_n)}{\varphi(1-\varepsilon_n)}\right)^2\varepsilon_n \quad \text{by L'Hospital's rule} \tag{B.41}$$

$$=-\lim_{n\to\infty}(\varphi'(1-\varepsilon_n))^2\left(\frac{\varepsilon_n}{\varphi(1-\varepsilon_n)}\right)^2\frac{\sigma(1-\varepsilon_n)}{\varepsilon_n} \tag{B.42}$$

$$=-(-1)^2\times\left(\frac{1}{-1}\right)^2\times\sigma_0 \tag{B.43}$$

$$=-\sigma_0 \tag{B.44}$$

Thus,

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{\sqrt{n}}{-\log\varepsilon_n}Z_n(w)\right)\leq\sigma_0^2, \tag{B.45}$$

i.e, the sequence $\frac{\sqrt{n}Z_n(w)}{-\log\varepsilon_n}$, $n\to\infty$, is tight. Again, by L'Hospital's rule,

$$\lim_{n\to\infty}\mathbb{V}\left(\frac{\sqrt{n}}{-\log\varepsilon_n}Z_n(w)\right)$$

$$=\lim_{n\to\infty}\left(\frac{\varepsilon_n}{2\log\varepsilon_n}\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_w^x\int_w^x\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\Gamma(t,s)\left(\frac{\varphi'(s)}{\varphi(s)}\right)^2\mathrm{d}s\mathrm{d}t\right]_{x=1-\varepsilon_n}\right) \tag{B.46}$$

$$=\lim_{n\to\infty}\left(\frac{\varepsilon_n}{2\log\varepsilon_n}\left[\int_w^{1-\varepsilon_n}f(1-\varepsilon_n,s)\mathrm{d}s+\int_w^{1-\varepsilon_n}f(t,1-\varepsilon_n)\mathrm{d}t\right]\right) \tag{B.47}$$

$$=\lim_{n\to\infty}\left(\frac{\varepsilon_n}{\log\varepsilon_n}\int_w^{1-\varepsilon_n}f(1-\varepsilon_n,s)\mathrm{d}s\right), \tag{B.48}$$

where

$$f(t,s)=\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2\Gamma(t,s)\left(\frac{\varphi'(s)}{\varphi(s)}\right)^2 \tag{B.49}$$

is the integrand above; note that $f(t,s)=f(s,t)$.

Now, applying L'Hospital's rule one more time to

$$\frac{1}{\log \varepsilon_n} \int_w^{1-\varepsilon_n} f(1-\varepsilon_n, s) \mathrm{d}s,$$

we get

$$\lim_{n\to\infty} \mathbb{V}\left(\frac{\sqrt{n}}{-\log \varepsilon_n} Z_n(w)\right) = \lim_{n\to\infty} \varepsilon_n^2 f(1-\varepsilon_n, 1-\varepsilon_n) = \sigma_0^2, \tag{B.50}$$

since $\Gamma(1-\varepsilon_n, 1-\varepsilon_n) = \sigma^2(1-\varepsilon_n)$.

Remark B.1. *For continuous $f(x_1, x_2)$,*

$$\int_w^x \int_w^x f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 = F(x,x) - F(w,x) - F(x,w) + F(w,w), \tag{B.51}$$

where

$$\frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2) = f(x_1, x_2).$$

Thus, for $h > 0$:

$$\begin{aligned} &\int_w^{x+h} \int_w^{x+h} f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 - \int_w^x \int_w^x f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\ &= [F(x+h, x+h) - F(x,x)] - [F(w, x+h) - F(w,x)] - [F(x+h, w) - F(x,w)]. \end{aligned} \tag{B.52}$$

It follows that

$$\begin{aligned} &\lim_{h\downarrow 0} \frac{1}{h} \left[\int_w^{x+h} \int_w^{x+h} f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 - \int_w^x \int_w^x f(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2\right] \\ &= \int_w^1 f(x, x_2) \mathrm{d}x_2 + \int_w^1 f(x_1, x) \mathrm{d}x_1. \end{aligned} \tag{B.53}$$

So that

$$\int_w^{1-\varepsilon_n} \sigma(t) \frac{\sqrt{n}\left(K_n(t) - K(t)\right)/\sigma(t)}{\left(\varphi(t)/\varphi'(t)\right)^2} \mathrm{d}t = O_{\mathbb{P}}\left(\frac{\log \varepsilon_n}{\sqrt{n}}\right) \tag{B.54}$$

From the previous calculations and the fact that

$$\frac{1}{\sigma(t)} \sqrt{n}\left(K_n(t) - K(t)\right) \to Z \quad \text{for each} \quad w \le t < 1,$$

we derive the asymptotic behaviour:

If $\frac{\log \varepsilon_n}{\sqrt{n}} \to 0$ as $n \to \infty$, then, for $0 < w < 1$,

$$\frac{\sqrt{n}}{\log \varepsilon_n} \left(\varphi_{n1}(w) - \varphi(w)\right) \to \sigma_0 \varphi(w) Z \quad \text{in distribution, as} \quad n \to \infty,$$

where Z is a standard normal random variable.

B.2 Integral estimator φ_{n2}: proof of Theorem 3.2

Consider $h_n(t)$, estimator of $h(t) = -\varphi'(t)$, given by

$$h_n(t) = \frac{1}{K_n(t) - t} \int_t^1 h_n(v) \mathrm{d}v, \tag{B.55}$$

whereas

$$h(t) = \frac{1}{K(t) - t} \int_t^1 h(v) \mathrm{d}v. \tag{B.56}$$

Then $\varphi_{n2}(t) = \int_t^1 h_n(v) \mathrm{d}v$ is an estimator for $\varphi(t) = \int_t^1 h(v) \mathrm{d}v$, and we have the equations

$$\varphi_{n2}(t) = \int_t^1 \frac{\varphi_{n2}(w)}{K_n(w) - w} \mathrm{d}w \tag{B.57}$$

$$\varphi(t) = \int_t^1 \frac{\varphi(w)}{K(w) - w} \mathrm{d}w. \tag{B.58}$$

Hence

$$\begin{aligned}
&\sqrt{n}\left(\varphi_{n2}(w) - \varphi(w)\right) \\
&= \sqrt{n} \int_t^1 \left(\frac{\varphi_{n2}(w)}{K_n(w) - w} - \frac{\varphi(w)}{K(w) - w} \right) \mathrm{d}w && \text{(B.59)} \\
&= \sqrt{n} \int_t^1 \frac{\varphi_{n2}(w)\left(K(w) - w\right) - \varphi(w)\left(K_n(w) - w\right)}{\left(K_n(w) - w\right)\left(K(w) - w\right)} \mathrm{d}w && \text{(B.60)} \\
&= \sqrt{n} \int_t^1 \frac{\varphi_{n2}(w)\left(K(w) - w\right) + \varphi_{n2}(w) K_n(w) - \varphi_{n2}(w) K_n(w) - \varphi(w)\left(K_n(w) - w\right)}{\left(K_n(w) - w\right)\left(K(w) - w\right)} \mathrm{d}w && \text{(B.61)} \\
&= \sqrt{n} \int_t^1 \frac{\varphi_{n2}(w)\left(K(w) - K_n(w)\right) + \left(K_n(w) - w\right)\left(\varphi_{n2}(w) - \varphi(w)\right)}{\left(K_n(w) - w\right)\left(K(w) - w\right)} \mathrm{d}w && \text{(B.62)}
\end{aligned}$$

$$\sqrt{n}\,(\varphi_{n2}(w)-\varphi(w))$$
$$= \sqrt{n}\int_t^1 \left(\frac{\varphi_{n2}(w)\,(K(w)-K_n(w))}{(K_n(w)-w)\,(K(w)-w)} + \frac{\varphi_{n2}(w)-\varphi(w)}{K(w)-w}\right)\mathrm{d}w \tag{B.63}$$
$$= -\int_t^1 \frac{1}{(K_n(w)-w)\,(K(w)-w)}\varphi_{n2}(w)\,\sqrt{n}\,(K_n(w)-K(w))\,\mathrm{d}w + \int_t^1 \frac{\sqrt{n}\,(\varphi_{n2}(w)-\varphi(w))}{K(w)-w}\mathrm{d}w \tag{B.64}$$

C Kendall's tau estimation

C.1 Variance in a readily estimable form

Asymptotic variance of $\frac{1}{4}\sqrt{n}(\tau_n-\tau)$:

$$\frac{1}{16}\mathbb{V}_{as}\left(\sqrt{n}(\tau_n-\tau)\right) = \mathbb{E}\left\{\left(2\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t})\right)\left(2\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s}) + \int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right)\right\}$$
$$= 4\mathbb{E}\left(\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s})\right) \tag{C.65}$$
$$+4\mathbb{E}\left(\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right) \tag{C.66}$$
$$+\mathbb{E}\left(\int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right). \tag{C.67}$$

Square term 1 (C.65)

$$\mathbb{E}\left(\int L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{t})\int L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})F^0(\mathrm{d}\mathbf{s})\right) = \mathbb{E}\left(\int \overline{F}^0(\mathbf{t})L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{t})}{\overline{F}^0(\mathbf{t})}\int \overline{F}^0(\mathbf{s})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\right) \tag{C.68}$$
$$= \int\int \overline{F}^0(\mathbf{t})\overline{F}^0(\mathbf{s})v(\mathbf{t},\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}, \tag{C.69}$$

with

$$v(\mathbf{t},\mathbf{s}) = \mathbb{E}\left(L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\right) \tag{C.70}$$

Cross term (C.66)

From Sen and Stute (2013), the influence function $L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})$ of $\sqrt{n}\left(\overline{F}^0_{\varepsilon,n}(\mathbf{t}) - \overline{F}^0(\mathbf{t})\right)$ satisfies

$$L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} L_{\mathbf{u}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} = \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \alpha_n(\mathrm{d}\mathbf{u}) \quad \text{with initial condition} \quad L_{\mathbf{0}}(\overline{F}^0_{\varepsilon,n}) = 0, \tag{C.71}$$

where

$$\alpha_n(\mathrm{d}\mathbf{u}) = \frac{H_n^{11}(\mathrm{d}\mathbf{u})}{\overline{G}(\mathbf{u})} - \overline{H}_n(\mathbf{u}) \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{H}(\mathbf{u})} \tag{C.72}$$

Multiply Equation (C.71) by $\int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})$:

$$\begin{aligned} &L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s}) - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} L_{\mathbf{u}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s}) \\ &= \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \alpha_n(\mathrm{d}\mathbf{u}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s}) \end{aligned} \tag{C.73}$$

Take expectation:

$$\begin{aligned} &\mathbb{E}\left(L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right) - \mathbb{E}\left(\int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} L_{\mathbf{u}}(\overline{F}^0_{\varepsilon,n}) \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right) \\ &= \mathbb{E}\left(\int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \alpha_n(\mathrm{d}\mathbf{u}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right) \end{aligned} \tag{C.74}$$

Rearrange as:

$$\begin{aligned} &\underbrace{\mathbb{E}\left(L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right)}_{=:w_{\mathbf{t}}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \underbrace{\mathbb{E}\left(L_{\mathbf{u}}(\overline{F}^0_{\varepsilon,n}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right)}_{=w_{\mathbf{u}}} \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} \\ &= \mathbb{E}\left(\int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \alpha_n(\mathrm{d}\mathbf{u}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right) \end{aligned} \tag{C.75}$$

$w_{\mathbf{t}}$ is the solution to the inhomogeneous Volterra equation of the second kind:

$$w_{\mathbf{t}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} w_{\mathbf{u}} \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} = \mathbb{E}\left(\int \mathbb{1}\{\mathbf{u} \geq \mathbf{t}\} \alpha_n(\mathrm{d}\mathbf{u}) \int \overline{F}^0(\mathbf{s}) \alpha_n(\mathrm{d}\mathbf{s})\right) \quad \text{with condition} \quad w_{\mathbf{0}} = 0. \tag{C.76}$$

Square term 2 (C.67)

Just as with Equation (4.57) in Section 4.2.3,

$$\begin{aligned}
&\mathbb{E}\left(\int \overline{F}^0(\mathbf{t})\alpha_n(\mathrm{d}\mathbf{t})\int \overline{F}^0(\mathbf{s})\alpha_n(\mathrm{d}\mathbf{s})\right)\\
&= n^{-1}\left[\int \overline{F}^4(\mathbf{t})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})^2} + \int\int (1-\mathbb{1}\{\mathbf{t}\geq\mathbf{s}\})(1-\mathbb{1}\{\mathbf{t}<\mathbf{s}\})\overline{H}(\mathbf{t}\vee\mathbf{s})\overline{F}^2(\mathbf{t})\overline{F}^2(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{t})}{\overline{H}(\mathbf{t})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}\right]
\end{aligned} \tag{C.77}$$

D Maximum Pseudo-Likelihood Estimation

D.1 Multiplication (5.38) × (5.39)

$$
\begin{aligned}
&\Bigg\{\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\\
&+\int\phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}+\int\phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z})\Bigg\}\\
&\Bigg\{\int\left(L_{\mathbf{s_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{s_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}F^0(\mathrm{d}\mathbf{s})\\
&+\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}+\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\Bigg\}\\
&=\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\times\\
&\quad\int\left(L_{\mathbf{s_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{s_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}F^0(\mathrm{d}\mathbf{s})\\
&+2\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\times\\
&\quad\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\\
&+2\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\\
&\quad\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\\
&+\int\phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\\
&+2\int\phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)L_{\mathbf{z}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\\
&+\int\phi\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{z})\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})
\end{aligned}
$$

D.2 Variance of A_n in a readily estimable shape

Expectation 1

$$
\begin{aligned}
&\mathbb{E}\Bigg\{\int\int\Bigg(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Bigg)\Bigg|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\times\\
&\quad\Bigg(L_{\mathbf{s_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{s_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Bigg)\Bigg|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}F^0(\mathrm{d}\mathbf{z})F^0(\mathrm{d}\mathbf{s})\Bigg\}\\
&=\mathbb{E}\Bigg\{\int\int\Bigg(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}L_{\mathbf{s_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\\
&\quad+L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}L_{\mathbf{s_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\\
&\quad+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}L_{\mathbf{s_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\\
&\quad+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}L_{\mathbf{s_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\Bigg)\\
&\quad\overline{F}^0(\mathbf{z})\overline{F}^0(\mathbf{s})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\Bigg\}\\
&=\int\int\Big(v(\mathbf{z_1^0},\mathbf{s_1^0})+v(\mathbf{z_1^0},\mathbf{s_2^0})+v(\mathbf{z_2^0},\mathbf{s_1^0})+v(\mathbf{z_2^0},\mathbf{s_2^0})\Big)\\
&\quad\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\\
&\quad\overline{F}^0(\mathbf{z})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}
\end{aligned}
$$

From the linearity of the covariance, this last term becomes

$$
\begin{aligned}
&\int\int v(\mathbf{z},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2)\right)}\\
&\quad\overline{F}^0(\mathbf{z})\overline{F}^0(\mathbf{s})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})},
\end{aligned}
\tag{D.78}
$$

where

- $v(\mathbf{z},\mathbf{s})=\mathbb{E}\left(L_{\mathbf{z}}(\overline{F}^0_{\varepsilon,n})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\right)$ is the covariance of the influence function of $\overline{F}^0_{\varepsilon,n}$,
- $H^{11}(\mathbf{x})=H^{11}(x_1,x_2)=\mathbb{P}(Z_1\le x_1,Z_2\le x_2,\delta_1=1,\delta_2=1)$
- $\overline{H}(\mathbf{x})=\overline{H}(x_1,x_2)=\mathbb{P}(Z_1\ge x_1,Z_2\ge x_2)$.

Expectation 2

$$
\begin{aligned}
&\mathbb{E}\left\{\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\times\right.\\
&\qquad\left.\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\right\}\\
&=\int\int\mathbb{E}\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})L_{\mathbf{s}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\times\\
&\qquad\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\overline{F}^0(\mathbf{z})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\\
&=\int\int\left(v(\mathbf{z_1^0},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+v(\mathbf{z_2^0},\mathbf{s})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\times\\
&\qquad\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\overline{F}^0(\mathbf{z})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}
\end{aligned}
$$

Expectation 3

$$
\begin{aligned}
&\mathbb{E}\left\{\int\left(L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)+L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}F^0(\mathrm{d}\mathbf{z})\right.\\
&\qquad\left.\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}\\
&=\int\underbrace{\mathbb{E}\left\{L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{:=r_{\mathbf{z_1^0}}}F^0(\mathrm{d}\mathbf{z})\\
&+\int\underbrace{\mathbb{E}\left\{L_{\mathbf{z_2^0}}(\overline{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\overline{F}^0_1(z_1),\overline{F}^0_2(z_2)\right)}\int\phi\left(\overline{F}^0_1(s_1),\overline{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{:=r_{\mathbf{z_2^0}}}F^0(\mathrm{d}\mathbf{z})\\
&=\int(r_{\mathbf{z_1^0}}+r_{\mathbf{z_2^0}})\overline{F}^0(\mathbf{z})\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\\
&=\int(r_{\mathbf{z_1^0}}+r_{\mathbf{z_2^0}})\overline{F}^0(\mathbf{z})\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}
\end{aligned}
$$

From

$$
L_{\mathbf{z_1^0}}(\overline{F}^0_{\varepsilon,n})-\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}L_{\mathbf{u}}(\overline{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})}=\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}\alpha_n(\mathrm{d}\mathbf{u}),
$$

we have

$$
\begin{aligned}
&\mathbb{E}\left\{L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}\\
&-\mathbb{E}\left\{\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}L_{\mathbf{u}}(\bar{F}^0_{\varepsilon,n})\frac{F^0(\mathrm{d}\mathbf{u})}{\bar{F}^0(\mathbf{u})}\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}\\
&=\mathbb{E}\left\{\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}\alpha_n(\mathrm{d}\mathbf{u})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}.
\end{aligned}
$$

The equation can be rearranged as:

$$
\begin{aligned}
&\underbrace{\mathbb{E}\left\{L_{\mathbf{z_1^0}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{:=r_{\mathbf{z_1^0}}}\\
&-\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}\underbrace{\mathbb{E}\left\{L_{\mathbf{u}}(\bar{F}^0_{\varepsilon,n})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{:=r_{\mathbf{u}}}\frac{F^0(\mathrm{d}\mathbf{u})}{\bar{F}^0(\mathbf{u})}\\
&=\underbrace{\mathbb{E}\left\{\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}\alpha_n(\mathrm{d}\mathbf{u})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\theta_0)\Big|_{(u_1,u_2)=\left(\bar{F}^0_1(z_1),\bar{F}^0_2(z_2)\right)}\int\phi\left(\bar{F}^0_1(s_1),\bar{F}^0_2(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{o_{\mathbf{z_1^0}}}
\end{aligned}
$$

$r_{\mathbf{z_1^0}}$ is solution to the inhomogeneous Volterra equation of second kind:

$$
r_{\mathbf{z_1^0}}=o_{\mathbf{z_1^0}}+\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_1^0}\}r_{\mathbf{u}}\frac{F^0(\mathrm{d}\mathbf{u})}{\bar{F}^0(\mathbf{u})}\quad\text{with initial condition}\quad r_{\mathbf{0}}=0 \tag{D.79}
$$

Similarly, $r_{\mathbf{z_2^0}}$ is solution to:

$$
r_{\mathbf{z_2^0}}=o_{\mathbf{z_2^0}}+\int\mathbb{1}\{\mathbf{u}\geq\mathbf{z_2^0}\}r_{\mathbf{u}}\frac{F^0(\mathrm{d}\mathbf{u})}{\bar{F}^0(\mathbf{u})}\quad\text{with initial condition}\quad r_{\mathbf{0}}=0 \tag{D.80}
$$

Expectation 4

$$\mathbb{E}\left\{\int \phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right) L_{\mathbf{z}}(\overline{F}_{\varepsilon,n}^0)\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right) L_{\mathbf{s}}(\overline{F}_{\varepsilon,n}^0)\frac{F^0(\mathrm{d}\mathbf{s})}{\overline{F}^0(\mathbf{s})}\right\}$$
$$=\int\int v(\mathbf{z},\mathbf{s})\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}\frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}(\mathbf{s})}$$

Expectation 5

$$\mathbb{E}\left\{\int \phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right) L_{\mathbf{z}}(\overline{F}_{\varepsilon,n}^0)\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}$$
$$=\int \underbrace{\mathbb{E}\left\{\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right) L_{\mathbf{z}}(\overline{F}_{\varepsilon,n}^0)\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\right\}}_{:=e_{\mathbf{z}}}\frac{F^0(\mathrm{d}\mathbf{z})}{\overline{F}^0(\mathbf{z})}$$
$$=\int e_{\mathbf{z}}\frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}(\mathbf{z})}$$

From

$$L_{\mathbf{z}}(\overline{F}_{\varepsilon,n}^0)-\int \mathbb{1}\{\mathbf{u}\geq\mathbf{z}\}L_{\mathbf{u}}(\overline{F}_{\varepsilon,n}^0)\frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})}=\int \mathbb{1}\{\mathbf{u}\geq\mathbf{z}\}\alpha_n(\mathrm{d}\mathbf{u}),$$

we have

$$\mathbb{E}\left\{\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})L_{\mathbf{z}}(\overline{F}_{\varepsilon,n}^0)\right\}$$
$$-\mathbb{E}\left\{\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\int \mathbb{1}\{\mathbf{u}\geq\mathbf{z}\}L_{\mathbf{u}}(\overline{F}_{\varepsilon,n}^0)\frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})}\right\}$$
$$=\mathbb{E}\left\{\phi\left(\overline{F}_1^0(z_1),\overline{F}_2^0(z_2);\theta_0\right)\int \phi\left(\overline{F}_1^0(s_1),\overline{F}_2^0(s_2);\theta_0\right)\alpha_n(\mathrm{d}\mathbf{s})\int \mathbb{1}\{\mathbf{u}\geq\mathbf{z}\}\alpha_n(\mathrm{d}\mathbf{u})\right\}$$

The equation can be rearranged as:

$$
\begin{aligned}
& e_{\mathbf{z}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{z}\} e_{\mathbf{u}} \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} \\
& = \underbrace{\mathbb{E}\left\{ \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \int \phi\left(\overline{F}_1^0(s_1), \overline{F}_2^0(s_2); \theta_0\right) \alpha_n(\mathrm{d}\mathbf{s}) \int \mathbb{1}\{\mathbf{u} \geq \mathbf{z}\} \alpha_n(\mathrm{d}\mathbf{u}) \right\}}_{:= h_{\mathbf{z}}}.
\end{aligned}
$$

$e_{\mathbf{z}}$ is solution to:

$$
e_{\mathbf{z}} - \int \mathbb{1}\{\mathbf{u} \geq \mathbf{z}\} e_{\mathbf{u}} \frac{F^0(\mathrm{d}\mathbf{u})}{\overline{F}^0(\mathbf{u})} = h_{\mathbf{z}} \quad \text{with the initial condition} \quad e_{\mathbf{0}} = 0 \tag{D.81}
$$

Expectation 6

This term is already in readily estimable form, since, similarly to Equation (4.57) in Section 4.2.3,

$$
\begin{aligned}
& \mathbb{E}\left(\int \int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \phi\left(\overline{F}_1^0(s_1), \overline{F}_2^0(s_2); \theta_0\right) \alpha_n(\mathrm{d}\mathbf{z}) \alpha_n(\mathrm{d}\mathbf{s}) \right) \\
& = n^{-1} \Bigg[\int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \left(\overline{F}^0(\mathbf{z})\right)^2 \frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}^2(\mathbf{z})} \\
& \quad + \int \int \phi\left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2); \theta_0\right) \phi\left(\overline{F}_1^0(s_1), \overline{F}_2^0(s_2); \theta_0\right) \left(1 - \mathbb{1}\{\mathbf{z} \geq \mathbf{s}\}\right) \left(1 - \mathbb{1}\{\mathbf{z} < \mathbf{s}\}\right) \\
& \qquad \overline{H}(\mathbf{z} \vee \mathbf{s}) \overline{F}^0(\mathbf{z}) \overline{F}^0(\mathbf{s}) \frac{H^{11}(\mathrm{d}\mathbf{z})}{\overline{H}^2(\mathbf{z})} \frac{H^{11}(\mathrm{d}\mathbf{s})}{\overline{H}^2(\mathbf{s})} \Bigg]
\end{aligned}
$$

D.3 Variance estimate of A_n in matrix form

Summand 1

$$
\begin{aligned}
& \int \int v_n(\mathbf{z}, \mathbf{s}) \frac{\partial}{\partial u_1} \phi(u_1, u_2; \hat{\theta}) \Big|_{(u_1,u_2) = \left(\overline{F}_1^0(z_1), \overline{F}_2^0(z_2)\right)} \frac{\partial}{\partial u_2} \phi(u_1, u_2; \hat{\theta}) \Big|_{(u_1,u_2) = \left(\overline{F}_1^0(s_1), \overline{F}_2^0(s_2)\right)} \\
& \qquad \overline{F}_{\varepsilon,n}^0(\mathbf{z}) \overline{F}_{\varepsilon,n}^0(\mathbf{s}) \frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})} \frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})} \\
& = \sum_{i=1}^n \sum_{j=1}^n v_{ij} \frac{\partial}{\partial u_1} \phi(u_1, u_2; \hat{\theta}) \Big|_{(u_1,u_2) = \left(\overline{F}_1^0(z_{1i}), \overline{F}_2^0(z_{2i})\right)} \frac{\partial}{\partial u_2} \phi(u_1, u_2; \hat{\theta}) \Big|_{(u_1,u_2) = \left(\overline{F}_1^0(s_{1j}), \overline{F}_2^0(s_{2j})\right)} p_i p_j b_i b_j \\
& = (\Phi_1^*)^T \overline{\mathbb{F}}^{\mathbf{0}} \mathbf{BVB} \overline{\mathbb{F}}^{\mathbf{0}} \Phi_2^*
\end{aligned}
$$

Summand 2

$$
\begin{aligned}
&\int\int\left(v_n(\mathbf{z_1^0},\mathbf{s})\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})+v_n(\mathbf{z_2^0},\mathbf{s})\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2)\right)}\times\\
&\quad\phi\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2);\hat{\theta}\right)\overline{F}^0_{\varepsilon,n}(\mathbf{z})\frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}\frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})}\\
&=\sum_{i=1}^n\sum_{j=1}^n\left(v_n\left((Z_{1i},0),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})+\right.\\
&\qquad\left.v_n\left((0,Z_{2i}),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\right)\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)}\times\\
&\qquad\phi\left(\overline{F}^0_{n1}(s_{1j}),\overline{F}^0_{n2}(s_{2j});\hat{\theta}\right)p_ib_ib_j\\
&=\mathbf{b}^T\mathbf{V}^*\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b},
\end{aligned}
$$

where

$$
\begin{aligned}
(\mathbf{V}^*)_{ij}=&v_n\left((Z_{1i},0),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_1}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)}\\
&+v_n\left((0,Z_{2i}),(S_{1j},S_{2j})\right)\frac{\partial}{\partial u_2}\phi(u_1,u_2;\hat{\theta})\Big|_{(u_1,u_2)=\left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)}.
\end{aligned}
$$

Summand 3

$$
\int(\hat{r}_{\mathbf{z_1^0}}+\hat{r}_{\mathbf{z_2^0}})\overline{F}^0_{\varepsilon,n}(\mathbf{z})\frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}=(\widehat{\mathbf{r_1}}+\widehat{\mathbf{r_2}})^T\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b}
$$

Summand 4

$$
\begin{aligned}
&\int\int v_n(\mathbf{z},\mathbf{s})\phi\left(\overline{F}^0_{n1}(z_1),\overline{F}^0_{n2}(z_2);\hat{\theta}\right)\phi\left(\overline{F}^0_{n1}(s_1),\overline{F}^0_{n2}(s_2);\hat{\theta}\right)\frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})}\frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n(\mathbf{s})}\\
&=\sum_{i=1}^n\sum_{j=1}^n v_{ij}(\Phi)_i(\Phi)_jb_ib_j\\
&=\Phi^T\mathbf{BVB}\Phi,
\end{aligned}
$$

where $(\mathbf{V})_{ij}=v_{ij}=v_n(\mathbf{Z}_i,\mathbf{Z}_j)$ is the sample covariance of $L_{\mathbf{t}}(\overline{F}^0_{\varepsilon,n})$ at $(\mathbf{Z}_i,\mathbf{Z}_j)$.

Summand 5

$$\int \hat{e}_{\mathbf{z}} \frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n(\mathbf{z})} = \mathbf{b}^T \hat{\mathbf{e}}$$

Summand 6

$$\begin{aligned}
&n^{-1}\Bigg[\int \phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \hat{\theta}\right)\left(\overline{F}_{\varepsilon,n}^0(\mathbf{z})\right)^2 \frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n^2(\mathbf{z})} \\
&+ \int\int \phi\left(\overline{F}_{n1}^0(z_1), \overline{F}_{n2}^0(z_2); \hat{\theta}\right)\phi\left(\overline{F}_{n1}^0(s_1), \overline{F}_{n2}^0(s_2); \hat{\theta}\right)\left(1-\mathbb{1}\{\mathbf{z}\geq\mathbf{s}\}\right)\left(1-\mathbb{1}\{\mathbf{z}<\mathbf{s}\}\right) \\
&\qquad \times \overline{H}_n(\mathbf{z}\vee\mathbf{s})\overline{F}_{\varepsilon,n}^0(\mathbf{z})\overline{F}_{\varepsilon,n}^0(\mathbf{s})\frac{H_n^{11}(\mathrm{d}\mathbf{z})}{\overline{H}_n^2(\mathbf{z})}\frac{H_n^{11}(\mathrm{d}\mathbf{s})}{\overline{H}_n^2(\mathbf{s})}\Bigg] \\
&= \Phi^T\mathbf{B}(\overline{\mathbb{F}}^{\mathbf{0}})^2\mathbf{b} + \Phi^T\mathbf{B}\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{BDB}\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi \\
&= \Phi^T\mathbf{B}\overline{\mathbb{F}}^{\mathbf{0}}(\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{b} + \mathbf{BDB}\overline{\mathbb{F}}^{\mathbf{0}}\mathbf{B}\Phi)
\end{aligned}$$

D.4 Likelihood function derivation

Clayton copula

Clayton copula:

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \quad (u_1,u_2)\in[0,1]^2, \theta\in(0,\infty)$$

The density of a Clayton copula is, for $u_1,u_2\in[0,1], \theta\in(0,\infty)$,

$$c_\theta(u_1,u_2) = \frac{\partial^2}{\partial u_1 \partial u_2}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}} \tag{D.82}$$

$$= (\theta+1)(u_1u_2)^{-(\theta+1)}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{\left(-2-\frac{1}{\theta}\right)}. \tag{D.83}$$

The score function is

$$\phi(u_1,u_2) = \frac{\mathrm{d}}{\mathrm{d}\theta}\log\left(c_\theta(u_1,u_2)\right) \tag{D.84}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\log\left((\theta+1)(u_1u_2)^{-(\theta+1)}\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{\left(-2-\frac{1}{\theta}\right)}\right) \tag{D.85}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\log(\theta+1)-(\theta+1)\log(u_1u_2)-\left(2+\frac{1}{\theta}\right)\log\left(u_1^{-\theta}+u_2^{-\theta}-1\right)\right) \tag{D.86}$$

$$= \frac{1}{\theta+1}-\log(u_1u_2)-\left(\frac{-1}{\theta^2}\log\left(u_1^{-\theta}+u_2^{-\theta}-1\right)+\left(2+\frac{1}{\theta}\right)\frac{-u_1^{-\theta}\log(u_1)-u_2^{-\theta}\log(u_2)}{u_1^{-\theta}+u_2^{-\theta}-1}\right) \tag{D.87}$$

$$= \frac{1}{\theta+1}-\log(u_1u_2)+\frac{1}{\theta^2}\log\left(u_1^{-\theta}+u_2^{-\theta}-1\right)-\left(2+\frac{1}{\theta}\right)\frac{-u_1^{-\theta}\log(u_1)-u_2^{-\theta}\log(u_2)}{u_1^{-\theta}+u_2^{-\theta}-1}. \tag{D.88}$$

From a sample, we can evaluate the compact log-pseudo-likelihood as

$$\log\left(L(\theta)\right) = \sum_{i=1}^{n} p_i\left(\log(\theta+1)-(\theta+1)\log\left(F_{n1}^0(Z_{1i})F_{n2}^0(Z_{2i})\right)-\left(2+\frac{1}{\theta}\right)\log\left(F_{n1}^0(Z_{1i})^{-\theta}+F_{n2}^0(Z_{2i})^{-\theta}-1\right)\right). \tag{D.89}$$

And the full log-likelihood as:

$$\log\left(L^*(\theta)\right) = \sum_{i=1}^{n}\{\delta_{1i}\delta_{2i}\log\left(c_\theta(u_{1i},u_{2i})\right) \tag{D.90}$$

$$+\delta_{1i}\left(1-\delta_{2i}\right)\log\left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_1}\right)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})} \tag{D.91}$$

$$+\left(1-\delta_{1i}\right)\delta_{2i}\log\left(\frac{\partial C_\theta(u_1,u_2)}{\partial u_2}\right)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})} \tag{D.92}$$

$$+\left(1-\delta_{1i}\right)\left(1-\delta_{2i}\right)\log\left(C_\theta(u_{1i},u_{2i})\right)\} \tag{D.93}$$

$$= \sum_{i=1}^{n}\Big\{\delta_{1i}\delta_{2i}\left(\log(\theta+1)-(\theta+1)\log\left(u_{1i}u_{2i}\right)-\left(2+\frac{1}{\theta}\right)\log\left(u_{1i}^{-\theta}+u_{2i}^{-\theta}-1\right)\right) \tag{D.94}$$

$$-\delta_{1i}\left(1-\delta_{2i}\right)\left((\theta+1)\log\left(u_{1i}\right)+\left(\frac{1}{\theta}+1\right)\log\left(u_{1i}^{-\theta}+u_{2i}^{-\theta}-1\right)\right) \tag{D.95}$$

$$-\left(1-\delta_{1i}\right)\delta_{2i}\left((\theta+1)\log\left(u_{2i}\right)+\left(\frac{1}{\theta}+1\right)\log\left(u_{1i}^{-\theta}+u_{2i}^{-\theta}-1\right)\right) \tag{D.96}$$

$$-\left(1-\delta_{1i}\right)\left(1-\delta_{2i}\right)\frac{1}{\theta}\log\left(u_{1i}^{-\theta}+u_{2i}^{-\theta}-1\right)\Big\} \tag{D.97}$$

$$\log\left(L^*(\theta)\right) = \sum_{i=1}^{n} \{\delta_{1i}\delta_{2i}\log(\theta+1) - \delta_{1i}\delta_{2i}(\theta+1)\log\left(u_{1i}\right) - \delta_{1i}\delta_{2i}(\theta+1)\log\left(u_{2i}\right) \tag{D.98}$$

$$- \delta_{1i}\left(1-\delta_{2i}\right)(\theta+1)\log\left(u_{1i}\right) - \left(1-\delta_{1i}\right)\delta_{2i}(\theta+1)\log\left(u_{2i}\right) \tag{D.99}$$

$$- \left(\delta_{1i}\left(1-\delta_{2i}\right)\left(\frac{1}{\theta}+1\right) + \left(1-\delta_{1i}\right)\delta_{2i}\left(\frac{1}{\theta}+1\right) + \left(1-\delta_{1i}\right)\left(1-\delta_{2i}\right)\frac{1}{\theta} + \delta_{1i}\delta_{2i}\left(2+\frac{1}{\theta}\right)\right) \times \tag{D.100}$$

$$\log\left(u_{1i}^{-\theta} + u_{2i}^{-\theta} - 1\right)\Big\} \tag{D.101}$$

$$= \sum_{i=1}^{n} \{\delta_{1i}\delta_{2i}\log(\theta+1) \tag{D.102}$$

$$- (\theta+1)\left(\left(\delta_{1i}\left(1-\delta_{2i}\right) + \delta_{1i}\delta_{2i}\right)\log\left(u_{1i}\right) + \left(\left(1-\delta_{1i}\right)\delta_{2i} + \delta_{1i}\delta_{2i}\right)\log\left(u_{2i}\right)\right) \tag{D.103}$$

$$- \left(\frac{1-\delta_{1i}\delta_{2i}}{\theta} + \delta_{1i} + \delta_{2i} - 2\delta_{1i}\delta_{2i} + \delta_{1i}\delta_{2i}\left(2+\frac{1}{\theta}\right)\right)\log\left(u_{1i}^{-\theta} + u_{2i}^{-\theta} - 1\right)\Big\} \tag{D.104}$$

$$= \sum_{i=1}^{n} \{\delta_{1i}\delta_{2i}\log(\theta+1) - (\theta+1)\left(\delta_{1i}\log\left(u_{1i}\right) + \delta_{2i}\log\left(u_{2i}\right)\right) \tag{D.105}$$

$$- \left(\frac{1}{\theta} + \delta_{1i} + \delta_{2i}\right)\log\left(u_{1i}^{-\theta} + u_{2i}^{-\theta} - 1\right)\Big\} \tag{D.106}$$

$$= \sum_{i=1}^{n} \{\delta_{1i}\delta_{2i}\log(\theta+1) \tag{D.107}$$

$$- (\theta+1)\left(\left(\delta_{1i}\left(1-\delta_{2i}\right) + \delta_{1i}\delta_{2i}\right)\log\left(u_{1i}\right) + \left(\left(1-\delta_{1i}\right)\delta_{2i} + \delta_{1i}\delta_{2i}\right)\log\left(u_{2i}\right)\right) \tag{D.108}$$

$$- \left(\frac{1-\delta_{1i}\delta_{2i}}{\theta} + \delta_{1i} + \delta_{2i} - 2\delta_{1i}\delta_{2i} + \delta_{1i}\delta_{2i}\left(2+\frac{1}{\theta}\right)\right)\log\left(u_{1i}^{-\theta} + u_{2i}^{-\theta} - 1\right)\Big\} \tag{D.109}$$

$$= \sum_{i=1}^{n} \{\delta_{1i}\delta_{2i}\log(\theta+1) - (\theta+1)\left(\delta_{1i}\log\left(u_{1i}\right) + \delta_{2i}\log\left(u_{2i}\right)\right) \tag{D.110}$$

$$- \left(\frac{1}{\theta} + \delta_{1i} + \delta_{2i}\right)\log\left(u_{1i}^{-\theta} + u_{2i}^{-\theta} - 1\right)\Big\} \tag{D.111}$$

$$= \sum_{i=1}^{n} \left\{\delta_{1i}\delta_{2i}\log(\theta+1) - (\theta+1)\left(\delta_{1i}\log\left(\overline{F}_{n1}^{0}(Z_{1i})\right) + \delta_{2i}\log\left(\overline{F}_{n2}^{0}(Z_{2i})\right)\right)\right. \tag{D.112}$$

$$\left. - \left(\frac{1}{\theta} + \delta_{1i} + \delta_{2i}\right)\log\left(\overline{F}_{n1}^{0}(Z_{1i})^{-\theta} + \overline{F}_{n2}^{0}(Z_{2i})^{-\theta} - 1\right)\right\} \tag{D.113}$$

Ali-Mikhail-Haq copula

Ali-Mikhail-Haq copula:

$$C_\theta(u_1,u_2) = \frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)}, u_1,u_2 \in [0,1]^2, \theta \in [-1,1]. \tag{D.114}$$

$$\frac{\partial}{\partial u_1} C_\theta(u_1, u_2)$$

$$= \frac{\partial}{\partial u_1} \frac{u_1 u_2}{1-\theta(1-u_1)(1-u_2)} \tag{D.115}$$

$$= \frac{\frac{\partial}{\partial u_1}(u_1 u_2) \times (1-\theta(1-u_1)(1-u_2)) - u_1 u_2 \times \frac{\partial}{\partial u_1}(1-\theta(1-u_1)(1-u_2))}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.116}$$

$$= \frac{u_2(1-\theta(1-u_1)(1-u_2)) - u_1 u_2 (1-u_2)\theta}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.117}$$

$$= \frac{u_2((1-\theta(1-u_1)(1-u_2)) - u_1(1-u_2)\theta)}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.118}$$

$$= \frac{u_2(1+(-\theta+\theta u_1)(1-u_2) - u_1(1-u_2)\theta)}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.119}$$

$$= \frac{u_2\left(1-\theta(1-u_2)+\cancel{\theta u_1(1-u_2) - u_1(1-u_2)\theta}\right)}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.120}$$

$$= \frac{u_2(1-\theta(1-u_2))}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.121}$$

Similarly:

$$\frac{\partial}{\partial u_2} C_\theta(u_1, u_2) = \frac{u_1(1-\theta(1-u_1))}{(1-\theta(1-u_1)(1-u_2))^2} \tag{D.122}$$

The density of the copula is:

$$
\begin{aligned}
& c_\theta(u_1,u_2) \\
&= \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}C_\theta(u_1,u_2) && \text{(D.123)}\\
&= \frac{\partial}{\partial u_1}\frac{u_1\left(1-\theta(1-u_1)\right)}{\left(1-\theta(1-u_1)(1-u_2)\right)^2} && \text{(D.124)}\\
&= \frac{\frac{\partial}{\partial u_1}u_1\left(1-\theta(1-u_1)\right)\times\left(1-\theta(1-u_1)(1-u_2)\right)^2}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.125)}\\
&\quad -\frac{u_1\left(1-\theta(1-u_1)\right)\times\frac{\partial}{\partial u_1}\left(1-\theta(1-u_1)(1-u_2)\right)^2}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.126)}\\
&= \frac{\left(1-\theta(1-u_1)+\theta u_1\right)\left(1-\theta(1-u_1)(1-u_2)\right)^2}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.127)}\\
&\quad -\frac{u_1\left(1-\theta(1-u_1)\right)\times 2\frac{\partial}{\partial u_1}\left(1-\theta(1-u_1)(1-u_2)\right)\times\left(1-\theta(1-u_1)(1-u_2)\right)}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.128)}\\
&= \frac{\left(1-\theta+2\theta u_1\right)\left(1-\theta(1-u_1)(1-u_2)\right)^2}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.129)}\\
&\quad -\frac{2u_1\left(1-\theta(1-u_1)\right)\times\theta(1-u_2)\times\left(1-\theta(1-u_1)(1-u_2)\right)}{\left(1-\theta(1-u_1)(1-u_2)\right)^4} && \text{(D.130)}\\
&= \frac{\left(1-\theta(1-u_1)(1-u_2)\right)}{\left(1-\theta(1-u_1)(1-u_2)\right)^4}\left(\left(1-\theta+2\theta u_1\right)\left(1-\theta(1-u_1)(1-u_2)\right)-2u_1\left(1-\theta(1-u_1)\right)\theta(1-u_2)\right) && \text{(D.131)}\\
&= \frac{1}{\left(1-\theta(1-u_1)(1-u_2)\right)^3}\left(\left(1-\theta\right)\left(1-\theta(1-u_1)(1-u_2)\right)\right. && \text{(D.132)}\\
&\quad \left.+2\theta u_1\left(1-\theta(1-u_1)(1-u_2)\right)-2u_1\theta(1-u_2)+2\theta^2u_1(1-u_1)(1-u_2)\right) && \text{(D.133)}\\
&= \frac{1}{\left(1-\theta(1-u_1)(1-u_2)\right)^3}\left(\left(1-\theta\right)\left(1-\theta(1-u_1)(1-u_2)\right)\right. && \text{(D.134)}\\
&\quad \left.+2\theta u_1\cancel{-2\theta^2u_1(1-u_1)(1-u_2)}-2u_1\theta(1-u_2)+\cancel{2\theta^2u_1(1-u_1)(1-u_2)}\right) && \text{(D.135)}\\
&= \frac{1}{\left(1-\theta(1-u_1)(1-u_2)\right)^3}\left(\left(1-\theta\right)\left(1-\theta(1-u_1)(1-u_2)\right)+2\theta u_1-2\theta u_1(1-u_2)\right) && \text{(D.136)}\\
&= \frac{1}{\left(1-\theta(1-u_1)(1-u_2)\right)^3}\left(1-\theta(1-u_1)(1-u_2)-\theta+\theta^2(1-u_1)(1-u_2)+2\theta u_1-2\theta u_1(1-u_2)\right) && \text{(D.137)}
\end{aligned}
$$

$$c_\theta(u_1,u_2) = \frac{1}{(1-\theta(1-u_1)(1-u_2))^3}\big(1-\theta(1-u_1-u_2+u_1u_2)-\theta+\theta^2(1-u_1-u_2+u_1u_2) \tag{D.138}$$

$$+2\theta u_1 - 2\theta u_1(1-u_2)) \tag{D.139}$$

$$= \frac{1}{(1-\theta(1-u_1)(1-u_2))^3}\big(1-\theta+\theta u_1+\theta u_2-\theta u_1u_2-\theta+\theta^2-\theta^2u_1-\theta^2u_2+\theta^2u_1u_2+\cancel{2\theta u_1} \tag{D.140}$$

$$-\cancel{2\theta u_1}+2\theta u_1u_2\big) \tag{D.141}$$

$$= \frac{1}{(1-\theta(1-u_1)(1-u_2))^3}\big(1-\theta+\theta u_1+\theta u_2-\theta u_1u_2-\theta+\theta^2-\theta^2u_1-\theta^2u_2+\theta^2u_1u_2+2\theta u_1u_2\big) \tag{D.142}$$

$$= \frac{1}{(1-\theta(1-u_1)(1-u_2))^3}\big(1-2\theta+\theta^2+\theta(1-\theta)(u_1+u_2)+\theta(\theta+1)u_1u_2\big) \tag{D.143}$$

Compact log-likelihood (for $(u_{1i},u_{2i}) = \big(F^0_{n1}(Z_{1i}),F^0_{n2}(Z_{2i})\big)$):

$$\begin{aligned}
\log(L(\theta)) &= \log\left(\prod_{i=1}^n c_\theta(u_{1i},u_{2i})^{p_i}\right)\\
&= \sum_{i=1}^n p_i \log(c_\theta(u_{1i},u_{2i}))\\
&= \sum_{i=1}^n p_i\left\{-3\log(1-\theta(1-u_{1i})(1-u_{2i}))+\log\big(1-2\theta+\theta^2+\theta(1-\theta)(u_{1i}+u_{2i})+\theta(\theta+1)u_{1i}u_{2i}\big)\right\}\\
&= \sum_{i=1}^n p_i\left\{-3\log\big(1-\theta(1-F^0_{n1}(Z_{1i}))(1-F^0_{n2}(Z_{2i})\big)\right.\\
&\quad \left.+\log\big(1-2\theta+\theta^2+\theta(1-\theta)\big(F^0_{n1}(Z_{1i})+F^0_{n2}(Z_{2i})\big)+\theta(\theta+1)F^0_{n1}(Z_{1i})F^0_{n2}(Z_{2i})\big)\right\}
\end{aligned}$$

Full log-likelihood (for $(u_{1i},u_{2i}) = \left(\overline{F}^0_{n1}(Z_{1i}),\overline{F}^0_{n2}(Z_{2i})\right)$):

$$
\begin{aligned}
\log\left(L^*(\theta)\right) &= \log\left(\prod_{i=1}^n c_\theta(u_{1i},u_{2i})^{\delta_{1i}\delta_{2i}} \times \left(\frac{\partial}{\partial u_1}C_\theta(u_1,u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})}\right)^{\delta_{1i}(1-\delta_{2i})}\right.\\
&\qquad \left.\times\left(\frac{\partial}{\partial u_2}C_\theta(u_1,u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})}\right)^{(1-\delta_{1i})\delta_{2i}} \times C_\theta(u_{1i},u_{2i})^{(1-\delta_{1i})(1-\delta_{2i})}\right)\\
&= \sum_{i=1}^n \left\{\delta_{1i}\delta_{2i}\log\left(c_\theta(u_{1i},u_{2i})\right) + \delta_{1i}(1-\delta_{2i})\log\left(\frac{\partial}{\partial u_2}C_\theta(u_1,u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})}\right)\right.\\
&\qquad \left.+(1-\delta_{1i})\delta_{2i}\log\left(\frac{\partial}{\partial u_1}C_\theta(u_1,u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})}\right) + (1-\delta_{1i})(1-\delta_{2i})\log\left(C_\theta(u_{1i},u_{2i})\right)\right\}\\
&= \sum_{i=1}^n \delta_{1i}\delta_{2i}\left\{-3\log\left(1-\theta(1-u_{1i})(1-u_{2i})\right) + \log\left(1-2\theta+\theta^2+\theta(1-\theta)(u_{1i}+u_{2i})+\theta(\theta+1)u_{1i}u_{2i}\right)\right\}\\
&\qquad + \delta_{1i}\left(1-\delta_{2i}\right)\left\{\log(u_{2i}\left(1-\theta(1-u_{2i})\right)) - 2\log\left(1-\theta(1-u_{1i})(1-u_{2i})\right)\right\}\\
&\qquad + \left(1-\delta_{1i}\right)\delta_{2i}\left\{\log\left(u_{1i}\left(1-\theta(1-u_{1i})\right)\right) - 2\log\left(1-\theta(1-u_{1i})(1-u_{2i})\right)\right\}\\
&\qquad + \left(1-\delta_{1i}\right)\left(1-\delta_{2i}\right)\left\{\log(u_{1i}u_{2i}) - \log(1-\theta(1-u_{1i})(1-u_{2i}))\right\}\\
&= \sum_{i=1}^n \left\{\delta_{1i}\delta_{2i}\log\left(1-2\theta+\theta^2+\theta(1-\theta)(u_{1i}+u_{2i})+\theta(\theta+1)u_{1i}u_{2i}\right)\right.\\
&\qquad - \left(1+\delta_{1i}+\delta_{2i}\right)\log\left(1-\theta(1-u_{1i})(1-u_{2i})\right)\\
&\qquad + \left(1-\delta_{1i}\right)\log(u_{1i}) + \left(1-\delta_{2i}\right)\log(u_{2i})\\
&\qquad \left.+ \left(1-\delta_{1i}\right)\delta_{2i}\log\left(1-\theta(1-u_{1i})\right) + \delta_{1i}\left(1-\delta_{2i}\right)\log\left(1-\theta(1-u_{2i})\right)\right\}\\
&= \sum_{i=1}^n \left\{\delta_{1i}\delta_{2i}\log\left(1-2\theta+\theta^2+\theta(1-\theta)(\overline{F}^0_{n1}(Z_{1i})+\overline{F}^0_{n2}(Z_{2i}))+\theta(\theta+1)\overline{F}^0_{n1}(Z_{1i})\overline{F}^0_{n2}(Z_{2i})\right)\right.\\
&\qquad - \left(1+\delta_{1i}+\delta_{2i}\right)\log\left(1-\theta(1-\overline{F}^0_{n1}(Z_{1i}))(1-\overline{F}^0_{n2}(Z_{2i}))\right)\\
&\qquad + \left(1-\delta_{1i}\right)\log(\overline{F}^0_{n1}(Z_{1i})) + \left(1-\delta_{2i}\right)\log(\overline{F}^0_{n2}(Z_{2i}))\\
&\qquad \left.+ \left(1-\delta_{1i}\right)\delta_{2i}\log\left(1-\theta(1-\overline{F}^0_{n1}(Z_{1i}))\right) + \delta_{1i}\left(1-\delta_{2i}\right)\log\left(1-\theta(1-\overline{F}^0_{n2}(Z_{2i}))\right)\right\}
\end{aligned}
$$

Nelsen (2006)'s 4.2.20 copula

Nelsen (2006)'s copula number 4.2.20 is given by

$$
C_\theta(u_1,u_2) = \left[\log\left(\exp(u_1^{-\theta}) + \exp(u_2^{-\theta}) - e\right)\right]^{-\frac{1}{\theta}} \quad \text{for} \quad u_1,u_2 \in [0,1], \theta > 0. \tag{D.144}
$$

$$\frac{\partial}{\partial u_2}C_\theta(u_1,u_2) = \frac{\partial}{\partial u_2}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}} \tag{D.145}$$

$$= -\frac{1}{\theta}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{\partial}{\partial u_2}\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right) \tag{D.146}$$

$$= -\frac{1}{\theta}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{\frac{\partial}{\partial u_2}\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e} \tag{D.147}$$

$$= -\frac{1}{\theta}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{-\theta u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e} \tag{D.148}$$

$$= \left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e} \tag{D.149}$$

Similarly,

$$\frac{\partial}{\partial u_1}C_\theta(u_1,u_2) = \left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{u_1^{-\theta-1}\exp(u_1^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e} \tag{D.150}$$

Its density is

$$c_\theta(u_1,u_2) = \frac{\partial}{\partial u_1}\frac{\partial}{\partial u_2}C_\theta(u_1,u_2) \tag{D.151}$$

$$= \frac{\partial}{\partial u_1}\left\{\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\frac{u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\right\} \tag{D.152}$$

$$= \left\{\frac{\partial}{\partial u_1}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\right\}\frac{u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}+ \tag{D.153}$$

$$\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\times\frac{\partial}{\partial u_1}\left\{\frac{u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\right\} \tag{D.154}$$

$$= \left\{-\left(\frac{1}{\theta}+1\right)\frac{-\theta u_1^{-\theta-1}\exp(u_1^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-2}\right\}+ \tag{D.155}$$

$$\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1}\times u_2^{-\theta-1}\exp(u_2^{-\theta})\times\frac{\partial}{\partial u_1}\left\{\frac{1}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\right\} \tag{D.156}$$

$$c_\theta(u_1,u_2) = \left\{ (1+\theta)\,\frac{u_1^{-\theta-1}\exp(u_1^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-2}\right\} + \tag{D.157}$$

$$\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1} \times u_2^{-\theta-1}\exp(u_2^{-\theta}) \times \left\{\frac{(-1)(-\theta)u_1^{-\theta-1}\exp(u_1^{-\theta})}{\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)^2}\right\} \tag{D.158}$$

$$= \frac{(1+\theta)\,u_1^{-\theta-1}\exp(u_1^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-2} + \tag{D.159}$$

$$\frac{\theta u_1^{-\theta-1}\exp(u_1^{-\theta})u_2^{-\theta-1}\exp(u_2^{-\theta})}{\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)^2}\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-1} \tag{D.160}$$

$$= \frac{u_1^{-\theta-1}\exp(u_1^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\left\{(1+\theta)+\frac{\theta u_2^{-\theta-1}\exp(u_2^{-\theta})}{\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e}\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right\} \tag{D.161}$$

$$\left[\log\left(\exp(u_1^{-\theta})+\exp(u_2^{-\theta})-e\right)\right]^{-\frac{1}{\theta}-2} \tag{D.162}$$

Compact log-likelihood (at $(u_{1i},u_{2i}) = (F_{n1}^0(Z_{1i}),F_{n2}^0(Z_{2i}))$):

$$\log(L(\theta)) = \sum_{i=1}^{n} p_i\Big\{-(\theta+1)\log(u_{1i}) + u_{1i}^{-\theta} - \log\left(\exp(u_{1i}^{-\theta})+\exp(u_{2i}^{-\theta})-e\right) \tag{D.163}$$

$$+\log\left\{(1+\theta)+\frac{\theta u_{2i}^{-\theta-1}\exp(u_{2i}^{-\theta})}{\exp(u_{1i}^{-\theta})+\exp(u_{2i}^{-\theta})-e}\log\left(\exp(u_{1i}^{-\theta})+\exp(u_{2i}^{-\theta})-e\right)\right\} \tag{D.164}$$

$$-\left(\frac{1}{\theta}+2\right)\log\left[\log\left(\exp(u_{1i}^{-\theta})+\exp(u_{2i}^{-\theta})-e\right)\right]\Big\} \tag{D.165}$$

$$= \sum_{i=1}^{n} p_i\Big\{-(\theta+1)\log(F_{n1}^0(Z_{1i})) + F_{n1}^0(Z_{1i})^{-\theta} - \log\left(\exp(F_{n1}^0(Z_{1i})^{-\theta})+\exp(F_{n2}^0(Z_{2i})^{-\theta})-e\right) \tag{D.166}$$

$$+\log\left\{(1+\theta)+\frac{\theta F_{n2}^0(Z_{2i})^{-\theta-1}\exp(F_{n2}^0(Z_{2i})^{-\theta})}{\exp(F_{n1}^0(Z_{1i})^{-\theta})+\exp(F_{n2}^0(Z_{2i})^{-\theta})-e}\log\left(\exp(F_{n1}^0(Z_{1i})^{-\theta})+\exp(F_{n2}^0(Z_{2i})^{-\theta})-e\right)\right\} \tag{D.167}$$

$$-\left(\frac{1}{\theta}+2\right)\log\left[\log\left(\exp(F_{n1}^0(Z_{1i})^{-\theta})+\exp(F_{n2}^0(Z_{2i})^{-\theta})-e\right)\right]\Big\} \tag{D.168}$$

Full log-likelihood (with $(u_{1i}, u_{2i}) = (\overline{F}^0_{n1}(Z_{1i}), \overline{F}^0_{n2}(Z_{2i}))$):

$$\log(L^*(\theta)) = \sum_{i=1}^{n} \Bigg\{ \delta_{1i}\delta_{2i} \log(c_\theta(u_{1i}, u_{2i})) + \delta_{1i}(1-\delta_{2i}) \log\left(\frac{\partial}{\partial u_1} C_\theta(u_1, u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})} \right)$$

$$+ (1-\delta_{1i})\delta_{2i} \log\left(\frac{\partial}{\partial u_2} C_\theta(u_1, u_2)\Big|_{(u_1,u_2)=(u_{1i},u_{2i})} \right) + (1-\delta_{1i})(1-\delta_{2i}) \log(C_\theta(u_{1i}, u_{2i}))\Bigg\}$$

$$= \sum_{i=1}^{n} \Bigg\{ \delta_{1i}\delta_{2i} \Big\{ -(\theta+1)\log(u_{1i}) + u_{1i}^{-\theta} - \log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right) \tag{D.169}$$

$$+ \log\left\{ (1+\theta) + \frac{\theta u_{2i}^{-\theta-1} \exp(u_{2i}^{-\theta})}{\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e} \log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right) \right\} \tag{D.170}$$

$$- \left(\frac{1}{\theta} + 2\right) \log\left[\log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right)\right] \Big\} \tag{D.171}$$

$$+ \delta_{1i}(1-\delta_{2i}) \Big\{ -\left(\frac{1}{\theta} + 1\right) \log\left[\log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right)\right] - (\theta+1)\log(u_{1i}) + u_{1i}^{-\theta} \tag{D.172}$$

$$- \log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right) \Big\} \tag{D.173}$$

$$+ (1-\delta_{1i})\delta_{2i} \Big\{ -\left(\frac{1}{\theta} + 1\right) \log\left[\log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right)\right] - (\theta+1)\log(u_{2i}) + u_{2i}^{-\theta} \tag{D.174}$$

$$- \log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right) \Big\} \tag{D.175}$$

$$- \frac{1}{\theta}(1-\delta_{1i})(1-\delta_{2i}) \log\left[\log\left(\exp(u_{1i}^{-\theta}) + \exp(u_{2i}^{-\theta}) - e\right)\right] \Bigg\} \tag{D.176}$$

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