UNIFORMIZATION OF SHIMURA CURVES

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This is to certify that the thesis prepared

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Abstract

Uniformization of Shimura Curves

Jhan-Cyuan Syu

A Shimura curve is a moduli scheme parametrizing a certain family of abelian schemes. After taking the formal completion, one can see that it is related to the formal scheme $\widehat{\Omega}$ associated to the Drinfeld upper half plane. This is the well-known Cherednik-Drinfeld Theroem. To prove this theorem, we need to develop a modular description of $\widehat{\Omega}$ via Deligne's and Drinfeld's functor, and Cartier theory on formal modules is also used.

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Chapter 1

Introduction

What is Uniformization?

A natural and immediate question arising from the title of this thesis is that what a uniformization is. To illustrate this question and to grab a rough idea about it, let us take a look at two examples of uniformizations. The first one is the uniformization of Riemann surfaces, and the second one is the uniformization of elliptic curves.

Uniformization of Riemann Surfaces

Intuitively, a Riemann surface is a geometric object which locally looks like an open subset of the complex plane $\mathbf{C}$. To be more precise, a **Riemann surface** is a one-dimensional complex manifold. The most trivial example of a Riemann surface is the complex plane $\mathbf{C}$. Another example is the **Riemann sphere** $\mathbf{C}_\infty$, which is defined by $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$. In addition, we know that $\mathbf{C}_\infty$ is conformally equivalent to the projective line $\mathbf{P}^1(\mathbf{C})$, and $\mathbf{C}_\infty$ is a one-point compactification of $\mathbf{C}$. One more example is the **Poincaré upper half plane** $\mathcal{H} := \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Note that $\mathcal{H}$ is conformally equivalent to $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. One easily sees that these three examples are all simply connected. The first important result is that they are not conformally equivalent to each other, and they are the only three simply connected Riemann surfaces (up to conformal equivalence).

Theorem. (Uniformization of Riemann surfaces I) [FK92, Theorem IV.4.1]
If X is a simply connected Riemann surface, then X is conformally equivalent to one of the following: $\mathbf{C}$, $\mathbf{C}_\infty$ or $\mathcal{H}$.

Next question is: what can we say about general Riemann surfaces? Let X be an arbitrary Riemann surface. One can show that the universal covering $\widetilde{X}$ of X is also a Riemann surface. Since $\widetilde{X}$ is simply connected, we know that $\widetilde{X}$ is conformally equivalent to $\mathbf{C}$, $\mathbf{C}_\infty$ or $\mathcal{H}$ by the above theorem. Another property that should be used here is that the covering group G of the universal covering $\widetilde{X} \to X$ is a subgroup of $\operatorname{Aut} \widetilde{X}$, where $\operatorname{Aut} \widetilde{X}$ is the group of conformal automorphisms of $\widetilde{X}$. Note that the general linear group $\mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{C} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}.$$

We call such transformation a **Möbius transformation** (or a **fractional transformation**). One can prove that

$$\operatorname{Aut} \mathbf{C} \cong \mathrm{PU}_2(\mathbf{C}), \qquad \operatorname{Aut} \mathbf{C}_\infty \cong \mathrm{PGL}_2(\mathbf{C}), \qquad \operatorname{Aut} \mathcal{H} \cong \mathrm{PGL}_2(\mathbf{R}).$$

Here $\mathrm{PU}_2(\mathbf{C}) = \{M \in \mathrm{PGL}_2(\mathbf{C}) : M \text{ is represented by upper triangular matrices}\}$. Now we can pass $\widetilde{X}$ to X by modulo the G-action, and we conclude the result in the following:

Theorem. (Uniformization of Riemann surfaces II) [FK92, Theorem IV.5.6]
If X is a Riemann surface, then X is conformally equivalent to $\mathcal{X}/G$, where $\mathcal{X}$ is $\mathbf{C}$, $\mathbf{C}_\infty$ or $\mathcal{H}$, and G is a freely acting discontinuous group of Möbius transformations which preserves $\mathcal{X}$.

In this example, we see that uniformization of a Riemann surface means representing it as the quotient of an easier geometric object (namely, $\mathbf{C}$, $\mathbf{C}_\infty$ or $\mathcal{H}$) modulo a group action.

Uniformization of Elliptic Curves

An elliptic curve E over a field k is an algebraic curve over k of genus 1. By Riemann-Roch theorem for algebraic curves one can show that E is defined by the Weierstraß equation

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4 + a_6 \quad \text{for some } a_1, \cdots, a_6 \in k.$$

Moreover, one can simplify the Weierstraß equation as $Y^2 = X^3 + a_4'X + a_6'$ for some $a_4', a_6' \in k$ provided $\operatorname{char} k \neq 2, 3$.

Consider an elliptic curve E over $\mathbf{C}$. As what we saw above, $\mathbf{C}/\Lambda$ has a natural structure as a Riemann surface, where $\Lambda \subset \mathbf{C}$ is a lattice, and we also have addition on $\mathbf{C}/\Lambda$ inherited from the additive structure on $\mathbf{C}$. This gives $\mathbf{C}/\Lambda$ a structure as a complex Lie group. In fact, such Riemann surface has a relation with the set of $\mathbf{C}$-valued points of E via Weierstraß $\wp$-function. Note that the set $E(\mathbf{C})$ of $\mathbf{C}$-valued points of E also has a structure as a complex Lie group with group law inherited from the group law on E. To be more precise, there is a lattice $\Lambda \subset \mathbf{C}$ such that the map

$$\mathbf{C}/\Lambda \longrightarrow E(\mathbf{C}), \quad z \longmapsto [\wp(z) : \wp'(z) : 1]$$

is a complex analytic isomorphism of complex Lie groups. We state this result formally in the following theorem.

Theorem. (Uniformization of complex elliptic curves) [Si09, Corollary VI.5.1.1]
Let E be an elliptic curve over $\mathbf{C}$. Then the group of $\mathbf{C}$-valued points of E is isomorphic to a one-dimensional complex torus as complex Lie groups. That is, $E(\mathbf{C}) \cong \mathbf{C}/\Lambda$ for some lattice $\Lambda \subset \mathbf{C}$.

Next we turn our attention to the p-adic case, which is of course an interesting and natural problem in arithmetic. Naively, for an elliptic curve E over $\mathbf{Q}_p$ one may first conjecture that $E(\mathbf{Q}_p) \cong \mathbf{Q}_p/\Lambda$ for some discrete subgroup Λ. However, this is definitely impossible because there is no non-zero discrete subgroup of $\mathbf{Q}_p$. To conquer this problem, Tate came up with a genius idea that one shall consider the multiplicative structure instead of the additive structure. In the complex case, this can be achieved via the exponential map, and we get $\mathbf{C}^\times/q^{\mathbf{Z}}$ for some $q \in \mathbf{C}^\times$. This result turns out to have a p-adic analogue, and the statement is as follows.

Theorem. (Uniformization of p-adic elliptic curves) [Si09, Theorem V.5.3]
Let E be an elliptic curve over a p-adic filed k. If $|j(E)| > 1$, then there exists a unique $q \in \bar{k}^\times$ with $|q| < 1$ such that $E \cong E_q$ and $E(\bar{k}) \cong \bar{k}^\times/q^{\mathbf{Z}}$, where E_q is the elliptic curve defined by $Y^2 + XY = X^3 + a_4(q)X + a_6(q)$ for some $a_4(q), a_6(q) \in k$.

In the case of an elliptic curve E over either $\mathbf{C}$ or a p-adic field k we see that the uniformization is a group ($\mathbf{C}$ or $\bar{k}^\times$) modulo a subgroup (Λ or $q^{\mathbf{Z}}$).

Uniformization of Shimura Curves

The theorem on p-adic uniformization of Shimura curves was first proved by Drinfeld in [Dr76]. However, [Dr76] is a very condense article for only 9 pages, which makes it quite difficult to read. Therefore, Boutot and Carayol gave a more understandable exposition complementing [Dr76] with more background materials and illustrations in their article [BC91]. The treatment to this beautiful theorem on p-adic uniformization in this thesis mainly follows the scope of [BC91].

Let Δ be a division quaternion algebra over $\mathbf{Q}$ such that $\Delta \otimes_{\mathbf{Q}} \mathbf{R}$ is non-division, i.e., $\Delta \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})$. We write $\mathcal{O}_\Delta$ for a maximal order of Δ. Fix a prime $p \in \mathbf{N}$ such that We define a reductive group $\mathbf{G}_\Delta$ by

$$\mathbf{G}_\Delta : \mathbf{Alg}_{\mathbf{Q}} \longrightarrow \mathbf{Grp}, \quad R \longmapsto (\Delta \otimes_{\mathbf{Q}} R)^\times,$$

where $\mathbf{Alg}_{\mathbf{Q}}$ is the category of commutative $\mathbf{Q}$-algebras, and $\mathbf{Grp}$ is the category of groups. Take an open compact subgroup $\mathbf{U} \subseteq \mathbf{G}_\Delta(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$ which is small enough. Moreover, we assume $\mathbf{U} = \mathbf{U}_p\mathbf{U}^p$ with $\mathbf{U}_p \subset \mathbf{G}_\Delta(\mathbf{Q}_p)$ the unique maximal compact subgroup and $\mathbf{U}^p \subseteq \mathbf{G}_\Delta(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ a compact open subgroup ($\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}}$ is obtained from $\mathbf{A}_{\mathbf{Q},\mathrm{fin}}$ by excluding the p-part). The Shimura curve associated to $\mathbf{U}$ over $\mathbf{Z}_p$ is a projective $\mathbf{Z}_p$-scheme $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$ which parametrizes a certain family of abelian schemes with special $\mathcal{O}_\Delta$-actions and level $\mathbf{U}$-structures. The main theorem in this thesis is to uniformize $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$. To state this theorem, we need to introduce the notion of Drinfeld upper half plane.

The Drinfeld upper half plane over $\mathbf{Q}_p$ is $\Omega := \mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(\mathbf{Q}_p)$, and there is a natural $\mathrm{PGL}_2(\mathbf{Q}_p)$-action on Ω. We denote by $\mathrm{BT}_{\mathbf{Q}_p}$ the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbf{Q}_p)$. For each vertex s in $\mathrm{BT}_{\mathbf{Q}_p}$ we set $\mathbf{P}_s$ to be the canonical isomorphism class of the projective bundles of representatives of s, and for each edge $[s,s']$ in $\mathrm{BT}_{\mathbf{Q}_p}$ we set $\mathbf{P}_{[s,s']}$ the blow-up of $\mathbf{P}_s$ along s'. In addition, define Ω_s by removing all $\mathbf{F}_p$-rational points of $\mathbf{P}_s$, and define $\Omega_{[s,s']}$ by removing all $\mathbf{F}_p$-rational points of $\mathbf{P}_{[s,s']}$ except for the one defined by s'. Both Ω_s and $\Omega_{[s,s']}$ are schemes over $\mathbf{Z}_p$, and we denote by $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$ their formal completions with respect to special fibres. Since $\widehat{\Omega}_s$ and $\widehat{\Omega}_{s'}$ are open formal subscheme of $\widehat{\Omega}_{[s,s']}$, one has a gluing data $\{(\widehat{\Omega}_{[s,s']})_{[s,s']}, (\widehat{\Omega}_s)_s\}$. We then glue $(\widehat{\Omega}_{[s,s']})_{[s,s']}$ together along $(\widehat{\Omega}_s)_s\}$, and the resulting formal scheme id denoted by $\widehat{\Omega}$. Note that the $\mathrm{GL}_2(K)$-action on $\mathbf{Q}_p^2$ induces a natural $\mathrm{PGL}_2(\mathbf{Q}_p)$-action on $\mathrm{BT}_{\mathbf{Q}_p}$. Hence one has a $\mathrm{PGL}_2(\mathbf{Q}_p)$-action on $\widehat{\Omega}$.

Let $\bar{\Delta}$ be the division quaternion algebra over $\mathbf{Q}$ such that (1) $\bar{\Delta} \otimes_{\mathbf{Q}} \mathbf{R}$ is a division quaternion $\mathbf{R}$-algebra, (2) $\bar{\Delta} \otimes_{\mathbf{Q}} \mathbf{Q}_p \cong M_2(\mathbf{Q}_p)$, and (3) $\bar{\Delta} \otimes_{\mathbf{Q}} \mathbf{Q}_\ell \cong \Delta \otimes_{\mathbf{Q}} \mathbf{Q}_\ell$ for all rational primes $\ell \neq p$. We can similarly associate with $\bar{\Delta}$ a reductive group $\mathbf{G}_{\bar{\Delta}}$. Denote by $\breve{\mathbf{Z}}_p$ the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$. Now we are ready to state our main theorem.

THEOREM. (CEREDNIK-DRINFELD)
Set $Z_{\mathbf{U}} := \mathbf{U}^p \backslash \mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})/\mathbf{G}_{\bar{\Delta}}(\mathbf{Q})$. Then there is an isomorphism of formal $\mathbf{Z}_p$-schemes:

$$\widehat{\mathrm{Sh}}_{\mathbf{U},\mathbf{Z}_p} \cong \mathrm{GL}_2(\mathbf{Q}_p) \backslash (\widehat{\Omega} \mathbin{\widehat{\otimes}_{\mathbf{Z}_p}} \breve{\mathbf{Z}}_p \times Z_{\mathbf{U}}).$$

In order to prove CEREDNIK-DRINFELD THEOREM we need to develope the modular description of $\widehat{\Omega}$. In addition, the philosophy of transforming problems into formal geometry plays a crucial role. In formal geometry, Cartier theory establishes a bridge between formal group objects and linear algebraic objects, so we can study the geometric problems via tools from linear algebra.

ORGANIZATION

There are three chapters and two appendices in this thesis, and the main content of each of them is described below.

CHAPTER 1. The first chapter is about the construction of $\widehat{\Omega}$ and its modular description. Instead of $\mathbf{Q}_p$, we consider more generally a non-archimedean local field K. In this case, the Drinfeld upper half plane Ω is defined by $\mathbf{P}^1(\mathbf{C}_K) - \mathbf{P}^1(K)$, where $\mathbf{C}_K$ is the completion of the algebraic closure of K. We study the Bruhat-Tits tree BT_K of $\mathrm{PGL}_2(K)$ in SECTION 2.1, and we prove in SECTION 2.2 that Ω is a rigid analytic K-space by using the geometric realization of BT_K . SECTION 2.3 is devoted to the geometric construction of the formal scheme $\widehat{\Omega}$ over $\mathcal{O}_K$. As we said before, the modular description of $\widehat{\Omega}$ is essential for the proof of CEREDNIK-DRINFELD THEOREM. Such categorical constructions are studied in SECTION 2.4 and 2.5. As the final section, we discuss how $\mathrm{PGL}_2(K)$ acts on $\widehat{\Omega}$ in SECTION 2.6.

CHAPTER 2. As we mentioned previously, the group objects in formal geometry can be studied via linear algebra. Such theory is called CARTIER THEORY, and we give a survey on it in SECTION 3.1. For a more systematic treatment of Cartier theory, one can see Zink's book [Zi84]. SECTION 3.3 is about the modular description of the formal scheme $\widehat{\Omega} \mathbin{\widehat{\otimes}_{\mathcal{O}_K}} \mathcal{O}_{\breve{K}}$ appearing in the statement of CEREDNIK-DRINFELD THEOREM. To achieve the modular description, we need some preliminary works in rigidifications, which is SECTION 3.2.

CHAPTER 3. As the start of the chapter, we define Shimura curves in SECTION 4.1. Then we state our main theorem and do the first step analysis in SECTION 4.2. Before going into the proof of CEREDNIK-DRINFELD THEOREM, we introduce the notion of algebraizations in SECTION 4.3. Finally, SECTION 4.4 is about the proof of the main theorem.

APPENDIX A. This appendix collects some basic properties about quaternion algebras. The most important results of this appendix is the classification of quaternion algebras over local fields, which is given in SECTION A.3.

Appendix B. Abelian varieties and abelian schemes appear frequently in Chapter 4, so the aim of this appendix is to give a brief survey on them. We start with the definition of group schemes in Section B.1 and take a look at some important examples in Section B.2. Algebraic groups is treated in Section B.3, and an important class, reductive groups, is defined in Section B.4, where the most important example GL_n is given. The last two sections, Section B.5 and B.6, are about abelian varieties, abelian schemes and a special class of morphisms between them called isogenies.

Notations and Conventions

- As usual, $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ are the set of positive integers, of integers, of rational numbers and of real numbers, respectively. In addition, $\mathbf{N}_0$ is the set of all non-negative integers, i.e., $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. We also use the notation $\square_?$ for the subset of $\square$ possessing property $?$. For example, $\mathbf{R}_{\geq 0}$ is the set of all real numbers that are greater than or equal to 0.
- All rings are assumed to be commutative and unitary unless it is specified. On the other hand, every algebra is not necessarily commutative and unitary. However, once we say an algebra is commutative, it means this algebra is both commutative and unitary.
- We usually use T (or T_i, or T_{ij}) as variables. For example, $R[T]$ is the polynomial ring with coefficients in a ring R in variable T. In some cases, T will be a scheme or an algebra, but there is no risk for confusion because it will be clear to tell which case it is from the text.
- For a commutative ring R we denote by $M_2(R)$ the set of all 2-by-2 matrices with coefficients in R. In addition, we set $\mathrm{GL}_2(R) := \{A \in M_2(R) : \det A \in R^\times\}$. That is, $\mathrm{GL}_2(R)$ is the set of units of $M_2(R)$.
- Let X be a scheme with the structure sheaf $\mathscr{O}_X$. For any point $x \in X$, we denote by $\mathscr{O}_{X,x}$ the stalk of $\mathscr{O}_X$ at x. By definition, $\mathscr{O}_{X,x}$ is a local ring. We then write $\mathfrak{m}_{X,x}$ (or simply $\mathfrak{m}_x$) and $\mathbf{k}(x)$ for the maximal ideal and the residue field of $\mathscr{O}_{X,x}$, respectively. If $X = \operatorname{Spec} R$ is an affine scheme for some ring R, then we write $\mathscr{O}_R$ instead of $\mathscr{O}_X$.
- $\mathbf{Set}$ is the category of sets.
  $\mathbf{Grp}$ is the category of groups.
  $\mathbf{Mod}_R$ is the category of modules over a ring R.
  $\mathbf{Alg}_R$ is the category of commutative algebras over a ring R.
  $\mathbf{Sch}_S$ is the category of schemes over a scheme S. If $S = \operatorname{Spec} R$ is the affine scheme associated to a ring R, then we write $\mathbf{Sch}_R$ instead of $\mathbf{Sch}_{\operatorname{Spec} R}$. In the case $S = \operatorname{Spec} \mathbf{Z}$, we just simply write $\mathbf{Sch}$.

Chapter 2
Drinfeld Upper Half Plane

In this chapter, we are going to study the non-archimedean analogue of upper half plane, which is known as the Drinfeld upper half plane. Let us fix some notations.

Notation. Throughout this chapter, K is a non-archimedean local field with the ring of integers $\mathcal{O}_K$. The maximal ideal of $\mathcal{O}_K$ is denoted by $\mathfrak{m}$ with a given uniformizer π, and the residue field of $\mathcal{O}_K$ is denoted by k. We write p for the characteristic of k and q the cardinality of k. We denote by v the normalized valuation on K and by $|\cdot|$ the absolute value on K (we take the standard convention $|\cdot| = q^{-v(\cdot)}$). In addition, $\mathbf{C}_K$ is the completion of the algebraic closure of K, i.e., $\mathbf{C}_K = \widehat{\overline{K}}$. As the usual convention, $\mathcal{O}_{\mathbf{C}_K}$ is the ring of integers of $\mathbf{C}_K$. We again use the notation $|\cdot|$ for the unique extension of $|\cdot|$ on K to $\mathbf{C}_K$.

By the definition, $\mathbf{C}_K$ is complete. A natural question is that whether it is algebraically closed. This is a well-known result that $\mathbf{C}_K$ is algebraically closed and we briefly quote this result in the following:

Lemma. *Notations as the above. Then $\mathbf{C}_K$ is algebraically closed.*

2.1 The Bruhat-Tits Tree of $\mathrm{PGL}_2(K)$

Let n be a positive interger. An $\boldsymbol{\mathcal{O}_K}$**-lattice** in K^n is a free $\mathcal{O}_K$-submodule M of K^n with rank n. Two $\mathcal{O}_K$-lattices M and M' are **homothetic** if there exists $c \in K^\times$ such that $M = cM'$. We denote by $[M]$ the **homothety class** of M.

A **(simple) graph**[1] is a pair (V, E) consisting of a set V, whose elements are called **vertices**, and a set of unordered pairs of distinct vertices. We call an unordered pair (v, v') in E an **edge** joining the vertices v and v', and we usually write $[v, v']$ for the edge joining v and v'. A **path** of the graph (V, E) is a sequence $([v_i, v_{i+1}])_i$ of edges, where $v_i \in V$ are all distinct. We say the graph (V, E) is a **tree** if for any two vertices there is exactly one path joining them.

A graph is called **regular** if every vertex has the same number of adjacent vertices. We say a graph is **connected** if for any two vertices there is a path joining them. A **trial** of a graph is a sequence $([v_i, v_{i+1}])_i$ of distinct edges. A **cycle** of a graph is a trial where only the first and the last vertices are the same. We say a graph is **acyclic** if it has no cycles. It is clear that a graph is a tree if and only if it is connected and acyclic.

Now we are going to associate with the group $\mathrm{PGL}_2(K)$ a graph, and we will see how this combinatorial object helps us to study algebro-geometric problems.

Definition 2.1.1. The **Bruhat-Tits building** of $\mathrm{PGL}_2(K)$ is a graph BT_K with vertices and edges given by what follows:

(1) the set of vertices of BT_K is the set of homothety classes of $\mathcal{O}_K$-lattices of K^2 (for an $\mathcal{O}_K$-lattice $M \subset K^2$ the vertex in BT_K defined by M is denoted by $[M]$; this coincides with our notation for homothety classes given at the beginning of this section);

(2) two vertices s, s' are adjacent by an edge if there are $\mathcal{O}_K$-lattices M, M' such that $s = [M], s' = [M']$ and $\pi M \subsetneq M' \subsetneq M$. In this case, one can find an $\mathcal{O}_K$-basis e_1, e_2 for M such that $M' = \langle e_1, \pi e_2 \rangle_{\mathcal{O}_K}$, and we call (e_1, e_2) a **compatible basis** for the pair M, M' (see Lemma 2.1.3 for the proof of the existence of such compatible basis).

(Please see Figure 2.1 for a picture of BT_K when $q = 2$.)

Remark 2.1.2.

[1] There are some more general types of graphs such as multigraghs, directed graphs, graphs with loops. In this thesis, we only consider simple graphs and call them graphs for short.

(a) Note that the relation for s, s' being adjacent defined above is indeed a symmetric relation. If $s = [M]$ is adjacent to $s' = [M']$ with $\pi M \subsetneq M' \subsetneq M$, then $\pi M' \subsetneq \pi M \subsetneq M'$. Since $s = [M] = [\pi M]$, we know that s' is adjacent to s.

(b) It is an easy algebraic fact (some people call it the fourth isomorphism theorem) that we have the following one-one correspondence

$$\{\mathcal{O}_K\text{-submodules of } M \text{ containing } \pi M\} \longleftrightarrow \{\mathcal{O}_K\text{-submodules of } M/\pi M\}.$$

Note also that $\{\mathcal{O}_K\text{-submodules of } M/\pi M\} = \{k\text{-submodules of } M/\pi M\}$. Since M is a free $\mathcal{O}_K$-module of rank 2, one has $M/\pi M \cong k^2$. To count the number of vertices adjacent to $s = [M]$, we need to find the number of submodules M' of M such that $\pi M \subsetneq M' \subsetneq M$. Since $M' \neq M$, $M' \neq \pi M$ and $M/\pi M$ is a 2-dimensional k-vector spaces, each M' corresponds to a 1-dimensional subspace of $M/\pi M$. The isomorphism $M/\pi M \cong k^2$ tells us that the space of all 1-dimensional subspaces of $M/\pi M$ is isomorphic to $\mathbf{P}^1(k)$. Therefore, we obtain a bijection

$$\mathbf{P}^1(k) \longleftrightarrow \{s' \in \mathrm{BT}_K : s' \text{ is adjacent to } s = [M]\}.$$

If we fix an $\mathcal{O}_K$-basis e_1, e_2 for M and an isomorphism

$$M \xrightarrow{\sim} \mathcal{O}_K^2, \qquad \begin{array}{l} e_1 \longmapsto (1,0) \\ e_2 \longmapsto (0,1) \end{array}$$

then we can describe the vertices adjacent to $s = [M]$ more precisely. We first write down the following map

$$\mathbf{P}^1(k) \longrightarrow \{s' \in \mathrm{BT}_K : s' \text{ is adjacent to } s = [M]\}, \qquad [a:b] \longmapsto [\pi M + \langle \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K}]. \tag{2.1}$$

Here $\tilde{a}, \tilde{b} \in \mathcal{O}_K$ are lifts of $a, b \in k$, respectively. It is obvious that the map (2.1) is independent of the choices of lifts $\tilde{a}, \tilde{b}$. We can write $\pi M + \langle \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K}$ in a simplified form:

- Case I: $a \neq 0$. In this case, $\tilde{a}$ is a unit in $\mathcal{O}_K$. Then one has

$$\pi M + \langle \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K} = \langle \pi e_1, \pi e_2, \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K} = \langle \tilde{a}e_1 + \tilde{b}e_2, \pi e_2\rangle_{\mathcal{O}_K}. \tag{2.2}$$

- Case II: $b \neq 0$. In this case, $\tilde{b}$ is a unit in $\mathcal{O}_K$. Then we have

$$\pi M + \langle \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K} = \langle \pi e_1, \pi e_2, \tilde{a}e_1 + \tilde{b}e_2\rangle_{\mathcal{O}_K} = \langle \tilde{a}e_1 + \tilde{b}e_2, \pi e_1\rangle_{\mathcal{O}_K}. \tag{2.3}$$

The above two cases include all possibilities because a, b are not both zero by the definition of projective lines.

Let us close this part by summarizing what we have obtained so far:

(1) We have a bijection $\mathbf{P}^1(k) \longleftrightarrow \{s' \in \mathrm{BT}_K : s' \text{ is adjacent to } s = [M]\}$. For example, if $q = 5$, then for each vertex in BT_K there are 6 edges leaving it.

(2) If we fix a basis e_1, e_2 for M, then one can explicitly write down all vertices adjacent to $s = [M]$ by using (2.1), (2.2) and (2.3). For example, if $K = \mathbf{Q}_p$ and $M = \langle e_1, e_2\rangle_{\mathbf{Z}_p}$, then $\mathbf{P}^1(\mathbf{F}_p) = \{[1:0] = 0, [1:1], \cdots, [1:p-1], [0:1] = \infty\}$. Therefore, the set of vertices adjacent to $s = [M]$ is given by

$$\{[M_j] : M_j := \langle e_1 + je_2, pe_2\rangle_{\mathbf{Z}_\ell} \text{ for } j = 0, \cdots, p-1 \text{ and } M_p := \langle e_2, pe_1\rangle_{\mathbf{Z}_p}\}.$$

(c) Let us write down what happens generally like what we did in (2) of (b). Write $k = \{c_0 = 0, c_1 = 1, \cdots, c_{q-1}\}$, and let $\{\tilde{c}_i \in \mathcal{O}_K : \tilde{c}_i \equiv c_i \mod \mathfrak{m} \text{ for } i = 0, \cdots, q-1\}$ be a complete set of representatives of $\mathcal{O}_K/\mathfrak{m}$. Note that we have $\mathbf{P}^1(k) = \{[1:c_0], [1:c_1], \cdots, [1:c_{q-1}], [0:1] = \infty\}$. From (2.1), (2.2) and (2.3) one deduces that for every vertex $s = [M] \in \mathrm{BT}_K$ with $M = \langle e_1, e_2\rangle_{\mathcal{O}_K}$ the set of vertices adjacent to s is given by

$$\{[M_j] : M_j := \langle e_1 + \tilde{c}_j e_2, \pi e_2\rangle_{\mathcal{O}_K} \text{ for } j = 0, \cdots, q-1 \text{ and } M_q := \langle e_2, \pi e_1\rangle_{\mathcal{O}_K}\}.$$

(d) If $s = [M]$ and $s' = [M']$ are two adjacent vertices with a compatible basis e_1, e_2. Then s' corresponds to the point $0 = [1:0]$ in $\mathbf{P}^1(k)$ by (c). We will see in SECTION 2.3.3 that this realization of an adjacent vertex as a point in the projective line over k plays an important role in the construction of $\widehat{\Omega}_K$.

As we guaranteed before, we are going to show that if $M, M' \subset K^2$ are two $\mathcal{O}_K$-lattices such that $\pi M \subsetneq M' \subsetneq M$, then there exists a compatible basis for them. We first note that if $M, N \subset K^2$ are two $\mathcal{O}_K$-lattices, then for any basis e_1, e_2 for M and e_1', e_2' for N there exists $A \in \mathrm{GL}_2(K)$ such that

$$\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

This is because e_1, e_2 and e_1', e_2' are both K-basis for K^2. If we suppose further that $M = N$, then the matrix A turns out to be in $\mathrm{GL}_2(\mathcal{O}_K)$. Conversely, if A is in $\mathrm{GL}_2(\mathcal{O}_K)$, then $M = N$. We thus conclude

$$M = N \iff A \in \mathrm{GL}_2(\mathcal{O}_K).$$

Lemma 2.1.3. *If M, M' are two $\mathcal{O}_K$-lattices of K^2 such that $\pi M \subsetneq M' \subsetneq M$, then there exists an $\mathcal{O}_K$-basis e_1, e_2 for M such that $M' = \langle e_1, \pi e_2 \rangle_{\mathcal{O}_K}$.*

Proof. Let η_1, η_2 be an $\mathcal{O}_K$-basis for M. By Remark 2.1.2 (c) we have two cases: (notations are the same as there)

- ◇ Suppose $M' = \langle \eta_1 + \tilde{c}_j \eta_2, \pi \eta_2 \rangle_{\mathcal{O}_K}$ for some $j \in \{0, 1, \cdots, q-1\}$. Since

$$\begin{pmatrix} 1 & \tilde{c}_j \\ 0 & \pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \begin{pmatrix} 1 & \tilde{c}_j \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \tilde{c}_j \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K),$$

  we can take $e_1 := \eta_1 + \tilde{c}_j \eta_2, e_2 := \eta_2$ to satisfy the requirements.
- ◇ If $M' = \langle \eta_2, \pi \eta_1 \rangle_{\mathcal{O}_K}$, then $e_1 := \eta_2, e_2 := \pi \eta_1$ is the desired one.

■

Next we want to show that BT_K is a regular tree. First, we define a map $v_{\inf} : M_2(K) \to \mathbf{Z}$ by

$$v_{\inf} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \min\{v(a), v(b), v(c), v(d)\}.$$

There are some properties that will be used in the proof of next proposition.

(1) *For any two vertices s and s' one can find $\mathcal{O}_K$-lattices M with basis e_1, e_2 and M' with basis e_1', e_2' such that $s = [M]$, $s' = [M']$ and $\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ for some $A \in \mathrm{GL}_2(K) \cap M_2(\mathcal{O}_K)$ with $v_{\inf}(A) = 0$.*

Proof. Let $\overline{M}$ and $\overline{M}'$ be two $\mathcal{O}_K$-lattices in K^2 such that $s = [\overline{M}]$ and $s' = [\overline{M}']$, and take η_1, η_2 and η_1', η_2' any basis for $\overline{M}$ and $\overline{M}'$, respectively. Let A' be the matrix in $\mathrm{GL}_2(K)$ such that $\begin{pmatrix} \eta_1' \\ \eta_2' \end{pmatrix} = A' \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. If we set $\delta := v_{\inf}(A')$, then $\pi^{-\delta} A'$ is in $\mathrm{GL}_2(K) \cap M_2(\mathcal{O}_K)$ with $v_{\inf}(A') = 0$. Now we set $e_1' = \pi^{-\delta} \eta_1', e_2' = \pi^{-\delta} \eta_2'$ and $e_1 = \eta_1, e_2 = \eta_2$, and these satisfy the requirements.

(2) *Notations are as in* (1). *If $r := v(\det A) > 0$, then $\pi^r M \subsetneq M' \subsetneq M$, $M' \nsubseteq \pi^m M$ for all $m \in \mathbf{N}$ and $\pi^m M \nsubseteq M'$ for all $0 \leq m \leq r-1$.*

Proof. First note that $v(\det A) > 0$ indicates $M' \neq M$. Since $\pi^r A^{-1} \in \mathrm{GL}_2(K) \cap M_2(\mathcal{O}_K)$, the formula

$$\pi^r \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (\pi^r A^{-1}) A \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

indicates that $\pi^r M \subseteq M'$. In addition, $v(\det(\pi^r A^{-1})) = r$ implies that $\pi^r M \neq M'$. Since $v_{\inf}(A) = 0$ and $v(\det A) = r > 0$, the matrix $\pi^m A^{-1}$ can not be in $M_2(\mathcal{O}_K)$ for all $0 \leq m \leq r-1$

Proposition 2.1.4. *The graph BT_K is a regular tree.*

Proof. We have seen in Remark 2.1.2 (b) that BT_K is regular. To prove BT_K is a tree, it is equivalent to prove that it is connected and acyclic.

- $\diamond$ Let s and s' be two vertices. By the above discussion one can find two $\mathcal{O}_K$-lattices M and M' such that $s = [M]$, $s' = [M']$, $\pi^r M \subsetneq M' \subsetneq M$, $M' \not\subseteq \pi^m M$ for all $m \in \mathbf{N}$ and $\pi^m M \not\subseteq M'$ for all $0 \leq m \leq r-1$. The $\mathcal{O}_K$-module M/M' is a torsion module because it is killed by π^r. Since $\mathcal{O}_K$ is a DVR (in particular, a PID), one can apply the fundamental theorem for modules over PID (see, for example, [DF04, SECTION 12.1]) to conclude $M/M' \cong \mathcal{O}/(\pi^{r_1}) \oplus \mathcal{O}/(\pi^{r_2})$ for some $r_1, r_2 \in \mathbf{Z}_{\geq 0}$. We know that one of r_1, r_2 is exactly r: if they were both smaller than r, then M/M' was killed by $\pi^{\max\{r_1,r_2\}}$, i.e., $\pi^{\max\{r_1,r_2\}} M \subseteq M'$, which is a contradiction; if they were both greater than r, then M/M' could not be killed by π^r. We may assume $r_1 = r$ (then $0 \leq r_2 \leq r$). If $r_2 > 0$, then $M' \subseteq \pi^{r_2} M$, which is again a contradiction. Thus we know $M/M' \cong \mathcal{O}/(\pi^r)$. Therefore, we have a filtration $0 = N_r \subsetneq N_{r-1} \subsetneq \cdots \subsetneq N_0 = M/M'$ such that the successive quotients in the lifting $M' = M_r \subsetneq M_{r-1} \subsetneq \cdots \subsetneq M_0 = M$ are isomorphic to k. By LEMMA 2.1.3 this is a path from s to s'; that is, BT_K is connected.
- $\diamond$ Let $M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_0$ be a chain of $\mathcal{O}_K$-lattices defining a path in BT_K (i.e., $\pi M_i \subsetneq M_{i+1} \subsetneq M_i$ for all $0 \leq i \leq n-1$). We are going to show that M_n is not homothetic to M_0 (i.e., it is not a cycle), and it suffices to show that $M_n \not\subseteq \pi M_0$. We proceed by induction on n. There is nothing to prove when $n = 0, 1$. Suppose $n \geq 2$. The image of M_n and πM_{n-2} in $M_{n-1}/\pi M_{n-1} \cong k^2$ are two distinct line (if not, then $[M_n] - [M_{n-1}] - [M_{n-2}]$ is a cycle, which means there are two edges connecting $[M_n]$ and $[M_{n-2}]$; however, this is impossible by our definition of BT_K). Therefore, $M_{n-1} = M_n + \pi M_{n-2}$, and then we have

$$M_{n-1} \equiv M_n \pmod{\pi M_0}.$$

The induction hypothesis says that $M_{n-1} \not\subseteq \pi M_0$, so $0 \not\equiv M_{n-1} \equiv M_n \pmod{\pi M_0}$. Thus $M_n \not\subseteq \pi M_0$.

■

DEFINITION 2.1.5.

(a) Let $\mathrm{BT}_{K,\mathbf{R}}$ be the proportionality classes of K-vector space norms on K^2. To be more precise, $\mathrm{BT}_{K,\mathbf{R}}$ is the set of all K-vector space norms on K^2 modulo the following equivalence relation:

two norms $|\cdot|$ and $|\cdot|'$ are equivalent if and only if there exists $c \in K^\times$ such that $|\cdot| = c|\cdot|'$.

That is, two norms are equivalent if they are proportional to each other.

(b) The **geometric realization** of BT_K is the map

$$\mathrm{BT}_K \longrightarrow \mathrm{BT}_{K,\mathbf{R}}, \quad s \longmapsto |\cdot|_s,$$

which is defined as the following: choose any representative $M = \langle e_1, e_2 \rangle_{\mathcal{O}_K}$ for s, and we set the Gauß norm associated to M by

$$|\cdot|_M : K^2 \longrightarrow \mathbf{R}, \quad a_1 e_1 + a_2 e_2 \longmapsto \sup(|a_1|, |a_2|).$$

Then $|\cdot|_s$ is defined to be the class represented by $|\cdot|_M$. If $s, s' \in \mathrm{BT}_K$ are adjacent to each other with a compatible basis (e_1, e_2), for each $t \in (0,1)$ we define the point $(1-t)s + ts'$ on the edge between s and s' by

$$|\cdot|_t : K^2 \longrightarrow \mathbf{R}, \quad a_1 e_1 + a_2 e_2 \longmapsto \sup(|a_1|, q^t |a_2|).$$

The line segment joining s, s' is $[s, s'] := \{(1-t)s + ts' : 0 \leq t \leq 1\}$.

LEMMA 2.1.6. *The geometric realization* $\mathrm{BT}_K \to \mathrm{BT}_{K,\mathbf{R}}$ *is an injective map* (*i.e., distinct vertices are mapped to distinct points in* $\mathrm{BT}_{K,\mathbf{R}}$).

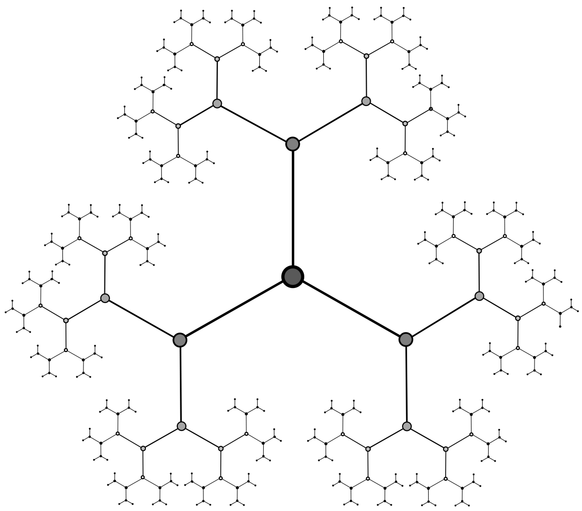

Figure 2.1: The Bruhat-Tits tree for the case $q = 2$.

2.2 Drinfeld Upper Half Plane as a Rigid Space

Recall that the Poincaré upper half plane is $\mathcal{H} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Consider at the same time the upper and lower half planes $\mathbf{C} - \mathbf{R} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\} \sqcup \{z \in \mathbf{C} : \operatorname{Im} z < 0\}$. If we define an equivalence relation $\sim$ on $\mathbf{C} - \mathbf{R}$ by $z_1 \sim z_2$ if and only if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = -\operatorname{Im} z_2$, then $\mathcal{H}$ is recovered by $\mathbf{C} - \mathbf{R}$ via $\mathcal{H} = (\mathbf{C} - \mathbf{R})/\sim$. In addition, $\mathbf{C} - \mathbf{R}$ can also be identified with $\mathbf{P}^1(\mathbf{C}) - \mathbf{P}^1(\mathbf{R})$.

Definition 2.2.1. The **Drinfeld upper half plane** attached to K is

$$\Omega := \mathbf{P}^1(\mathbf{C}_K) - \mathbf{P}^1(K).$$

We write $\operatorname{Hom}_K(K^2, \mathbf{C}_K)$ the set of K-bilinear maps from K^2 to $\mathbf{C}_K$. For any two $f_1, f_2 \in \operatorname{Hom}_K(K^2, \mathbf{C}_K)$ we say f_1 is homothetic to f_2 if and only if $f_1 = cf_2$ for some $c \in \mathbf{C}_K^\times$. In other words, f_1, f_2 are equivalent if and only if they are in the same orbit of the $\mathbf{C}_K^\times$-action on $\operatorname{Hom}_K(K^2, \mathbf{C}_K)$. As usual, we use the notation $\operatorname{Hom}_K(K^2, \mathbf{C}_K)/\mathbf{C}_K^\times$ for the orbit space. There is a bijection

$$(\operatorname{Hom}_K(K^2, \mathbf{C}_K) - \{0\})/\mathbf{C}_K^\times \xrightarrow{\sim} \mathbf{P}^1(\mathbf{C}_K), \quad f \longmapsto [f(1,0) : f(0,1)]. \tag{2.4}$$

To see this is indeed a bijection, note that we have an isomorphism

$$\operatorname{Hom}_K(K^2, \mathbf{C}_K) \xrightarrow{\sim} \mathbf{C}_K^2, \quad f \longmapsto (f(1,0), f(0,1))$$

because a map in $\operatorname{Hom}_K(K^2, \mathbf{C}_K)$ is determined by the image of $(1,0)$ and $(0,1)$. Thus we obtain (2.4) from

$$(\operatorname{Hom}_K(K^2, \mathbf{C}_K) - \{0\})/\mathbf{C}_K^\times \cong (\mathbf{C}_K^2 - \{0\})/\mathbf{C}_K^\times \cong \mathbf{P}^1(\mathbf{C}_K).$$

It is clear that the inverse map of (2.4) is given by

$$\mathbf{P}^1(\mathbf{C}_K) \xrightarrow{\sim} (\operatorname{Hom}_K(K^2, \mathbf{C}_K) - \{0\})/\mathbf{C}_K^\times, \quad [a : b] \longmapsto (f : (1,0) \mapsto a, (0,1) \mapsto b). \tag{2.5}$$

For each $z \in \Omega$ we associate an absolute value $|\cdot|_z$ on K^2 by setting $|\cdot|_z := |z(\cdot)|$. Here we view z as a map from K^2 to $\mathbf{C}$ via (2.4). Note that (2.5) associates with z a homothety class of K-bilinear maps from K^2 to $\mathbf{C}_K$, and $\mathrm{BT}_{K,\mathbf{R}}$ is the set of proportionality classes of norms on K^2. Therefore, we obtain a well-defined map

$$\lambda : \Omega \longrightarrow \mathrm{BT}_{K,\mathbf{R}}, \quad z \longmapsto |\cdot|_z.$$

Recall that we view every vertex in BT_K as a point in $\mathrm{BT}_{K,\mathbf{R}}$ via the geometric realization $\mathrm{BT}_K \hookrightarrow \mathrm{BT}_{K,\mathbf{R}}$ (see LEMMA 2.1.6). Next lemma is about the rigid analytic structure of Ω. Before stating that, we first recall some consequences about extension of non-archimedean local fields.

REMARK 2.2.2. Let F be a non-archimedean local field, and let $\mathbf{F}_{\ell^r}$ be the residue field of F for some prime $\ell \in \mathbf{N}$ and some $r \in \mathbf{N}$. We write v_F for the normalized valuation on F (i.e., $v(F^\times) = \mathbf{Z}$) and $|\cdot|_F := \ell^{-v_F(\cdot)r}$ the standard choice of absolute value on F. By the classification of local fields we know F is either a finite extension of $\mathbf{Q}_\ell$ or a finite extension of $\mathbf{F}_{\ell^r}((T))$. If L is a finite extension of F, then there exist unique extensions v_L of v_F and $|\cdot|_L$ of $|\cdot|_F$ Explicitly, these extensions are given by

$$v_L(\cdot) = \frac{1}{e(L/F)} v_F(\cdot) \quad \text{and} \quad |\cdot|_L = \left(|N_{L/F}(\cdot)|_F\right)^{1/[L:F]}, \tag{2.6}$$

where $e(L/F)$ is the ramification index of L/F and $N_{L/F} : L \to F$ is the norm map. A consequence is that

$$\textit{for each } n \in \mathbf{N} \textit{ there is a finite extension } L/F \textit{ such that } e(L/F) = n. \tag{2.7}$$

Let us see how to find such L. A monic polynomial $T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in \mathcal{O}_F[T]$ is called an **Eisenstein polynomial** if $a_i \in \mathfrak{m}_F$ and $a_0 \notin \mathfrak{m}_F^2$. Then one can show that

- ◇ [Alle, PROPOSITION 7.13] *every Eisenstein polynomial with coefficients in $\mathcal{O}_F$ is irreducible over F;*
- ◇ [Alle, PROPOSITION 7.14] *if α is a root of an Eisenstein polynomial of degree n over $\mathcal{O}_F$, then $F(\alpha)$ is totally ramified of degree n.*

Now for each $n \in \mathbf{N}$ we take $f_n(T) := T^n + \pi(T^{n-1} + \cdots + T_1 + 1)$ and take α_n an arbitrary root of f_n. Then the above two properties indicate that $F(\alpha_n)$ is totally ramified of degree n, i.e., $e(F(\alpha_n)/F) = n$.

Consequently, since the algebraic closure $\overline{F}$ is the union of all finite extension of F, (2.6) and (2.7) together imply that the valuation group (resp. value group) of $\overline{F}^\times$ is $\mathbf{Q}$ (resp. $\ell^{\mathbf{Q}}$). After taking completion, we then conclude that the valuation group (resp. value group) of $\mathbf{C}_F^\times$ is $\mathbf{R}$ (resp. $\mathbf{R}_{>0}$).

LEMMA 2.2.3. [BC91, I.2.3]
Let $s, s' \in \mathrm{BT}_K$ be two adjacent vertices. Take representatives M and M' for s and s' such that $\pi M \subsetneq M' \subsetneq M$ together with a compatible basis e_1, e_2. For each point $z \in \Omega$ we always take a representative (which is again denoted by z) such that $z(e_2) = 1$, and we set $\zeta := z(e_1)$[2]. Then we have the following formulae: (see FIGURE 2.2 *for a picture of an affinoid)*

$$\begin{aligned}
\lambda^{-1}(s) &= \{\zeta \in \mathbf{C}_K : |\zeta| \le 1\} - \bigcup_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\},\\
\lambda^{-1}(x_t) &= \{\zeta \in \mathbf{C}_K : |\zeta| = q^{-t}\} \quad \textit{for } x_t = (1-t)s + ts' \textit{ with } t \in (0,1),\\
\lambda^{-1}(s') &= \{\zeta \in \mathbf{C}_K : |\zeta| \le q^{-1}\} - \bigcup_{b \in \pi\mathcal{O}_K/\pi^2\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - b| < q^{-1}\},
\end{aligned}$$

$$\begin{aligned}
\lambda^{-1}([s,s']) = \{\zeta \in \mathbf{C}_K : |\zeta| \le 1\} &- \bigcup_{a \in (\mathcal{O}_K - \pi\mathcal{O}_K)/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\}\\
&- \bigcup_{b \in \pi\mathcal{O}_K/\pi^2\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - b| < q^{-1}\}.
\end{aligned}$$

PROOF. For any $t \in [0,1]$ we have the equivalent statements:

$$\begin{aligned}
z \in \lambda^{-1}((1-t)s + ts') &\iff |\cdot|_z = c \cdot |\cdot|_t \text{ for some } c \in K^\times\\
&\iff |z(c_1e_1 + c_2e_2)| = c \cdot |c_1e_1 + c_2e_2|_t \text{ for some } c \in K^\times \text{ and for all } c_1, c_2 \in K\\
&\iff |c_1\zeta + c_2| = c \cdot \sup(|c_1|, q^t|c_2|) \text{ for some } c \in K^\times \text{ and for all } c_1, c_2 \in K.
\end{aligned}$$

[2]We explain why we can always find a representative z such that $z(e_2) = 1$. This can be formulated as what follows:
For any $[a : b] \in \Omega$ and any $w = (w_1, w_2) \in K^2 - \{0\}$ one can always find a representative $z \in \mathbf{C}_K$ of $[a : b]$ such that $z(w) = 1$.
Let us prove this statement. If we view (a,b) as a map $K^2 \to \mathbf{C}_K$, then $(a,b)(w) = aw_1 + bw_2$. If $w_1a + w_2b \ne 0$, one can take the representative $z = (w_1a + w_2b)^{-1} \cdot (a,b)$ and then $z(w) = 1$. Now suppose $w_1a + w_2b = 0$. Notice that a, b are both non-zero; otherwise, say $b = 0$, $[a : b] = [1 : 0] \in \mathbf{P}^1(K)$. We then deduce w_1, w_2, a, b are all non-zero. Then $\frac{a}{b} = -\frac{w_2}{w_1} \in K^\times$, and this implies $[a : b] = [-\frac{w_2}{w_1}b : b] = [-\frac{w_2}{w_1} : 1] \in \mathbf{P}^1(K)$, which is a contradiction. This completes the proof. One more remark is that ζ lies in $\mathbf{C}_K - K$ because z is a point in $\Omega = \mathbf{P}^1(\mathbf{C}_K) - \mathbf{P}^1(K)$.

By taking $c_1 = 0$ we see that $c = q^{-t}$, so we conclude

$$z \in \lambda^{-1}((1-t)s + ts') \iff |c_1\zeta + c_2| = \sup(q^{-t}|a|, |b|) \text{ for all } c_1, c_2 \in K.$$

A simple observation is that

$$|c_1\zeta + c_2| = \sup(q^{-t}|c_1|, |c_2|) \text{ for all } c_1, c_2 \in K \iff |\zeta + c_2| = \sup(q^{-t}, |c_2|) \text{ for all } c_2 \in K.$$

The arrow from the left to the right is trivial. Suppose conversely that the right hand side holds. There is nothing to prove if $c_1 = 0$. For any $c_1 \neq 0$ we have $|c_1\zeta + c_2| = |c_1| \cdot |\zeta + \frac{c_2}{c_1}| = |c_1| \cdot \sup(q^{-t}, |\frac{c_2}{c_1}|) = \sup(q^{-t}|c_1|, |c_2|)$ for any $c_2 \in K$. Hence we obtain

$$z \in \lambda^{-1}((1-t)s + ts') \iff |\zeta + c| = \sup(q^{-t}, |c|) \text{ for all } c \in K. \tag{2.8}$$

(1) We first prove the formula for $\lambda^{-1}(s)$ (i.e., $t = 0$). Then (2.8) reads

$$z \in \lambda^{-1}(s) \iff |\zeta - c| = \sup(1, |c|) \text{ for all } c \in K. \tag{2.9}$$

If $|c| < 1$, then (2.9) implies $|\zeta - c| = 1$, which is equivalent to $|\zeta| = 1$ by the strong triangle inequality. If $|c| > 1$, then (2.9) and strong triangle inequality gives $|c| = |\zeta - c| = \sup(1, |c|) = |c|$ (because $|\zeta| = 1$), which gives no more information. If $|c| = 1$, (2.9) gives $|\zeta - 1| = 1$ for all $c \in K$ with $|c| = 1$. Altogether, we have

$$\lambda^{-1}(s) = \{\zeta \in \mathbf{C}_K : |\zeta| = 1\} - \bigcup_{c \in K, |c|=1} \{\zeta \in \mathbf{C}_K : |\zeta - c| < 1\}. \tag{2.10}$$

Note that there is an easy observation: for any $c \in K$, $\zeta \in \mathbf{C}_K$ and any $\varpi \in \mathfrak{m}$, $|\zeta - c - \varpi| < 1$ if and only if $|\zeta - c| < 1$. Indeed, if $|\zeta - c| < 1$, then $|\zeta - c - \varpi| \leq \sup(|\zeta - c|, |\varpi|) < 1$; conversely, if $|\zeta - c| \geq 1$, then the strong triangle inequality implies $|\zeta - c - \varpi| = |\zeta - c| \geq 1$. With this observation, we can rewrite (2.10) as

$$\lambda^{-1}(s) = \{\zeta \in \mathbf{C}_K : |\zeta| = 1\} - \bigcup_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\}.$$

Moreover, since $\{\zeta \in \mathbf{C}_K : |\zeta| < 1\} \subset \bigcup_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\}$ (by taking $a = 0$), we finally derive

$$\lambda^{-1}(s) = \{\zeta \in \mathbf{C}_K : |\zeta| \leq 1\} - \bigcup_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\}.$$

(2) Now look at the formula for $\lambda^{-1}(x_t)$ (i.e., $0 < t < 1$). Then (2.8) implies

$$z \in \lambda^{-1}(x_t) \iff |\zeta - c| = \sup(q^{-t}, |c|) \text{ for all } c \in K. \tag{2.11}$$

Remark 2.2.2 indicates that $|c| \neq q^{-t}$ for all $c \in K$. If $|c| < 1$, then (2.11) implies $|\zeta - c| = q^{-t}$. Thus $\zeta \in \mathbf{C}_K - K$ and $|\zeta| = q^{-t}$ (if ζ was in K, then $|\zeta - c|$ must be in the value set of K). If $|c| > q^{-t}$, then (2.11) and strong triangle inequality indicate $|c| = |\zeta - c| = \sup(q^{-t}, |c|) = |c|$ (because $|\zeta| = q^{-t}$). Hence

$$\lambda^{-1}(x_t) = \{\zeta \in \mathbf{C}_K : |\zeta| = q^{-t}\}.$$

(3) We turn to the formula for $\lambda^{-1}(s')$ (i.e., $t = 1$), and (2.8) indicates

$$z \in \lambda^{-1}(x) \iff |\zeta - c| = \sup(q^{-1}, |c|) \text{ for all } c \in K.$$

The argument is similar to (1): consider the cases $|c| < q^{-1}$, $|c| > q^{-1}$ and $|c| = q^{-1}$.

(4) By definition

$$\lambda^{-1}([s, s']) = \lambda^{-1}(s) \cup \lambda^{-1}(s') \cup \bigcup_{0<t<1} \lambda^{-1}(x_t). \tag{2.12}$$

Since one has

$$\bigcup_{0<t<1} \lambda^{-1}(x_t) = \{\zeta \in \mathbf{C}_K : q^{-1} < |\zeta| < 1\} \quad \text{and}$$

$$\lambda^{-1}(s) \cup \lambda^{-1}(s') = \{\zeta \in \mathbf{C}_K : |\zeta| \leq 1\} - \bigcup_{a \in \mathcal{O}_K/\pi\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - a| < 1\}$$
$$- \bigcup_{b \in \pi\mathcal{O}_K/\pi^2\mathcal{O}_K} \{\zeta \in \mathbf{C}_K : |\zeta - b| < q^{-1}\},$$

the final formula is proved.

■

We know that $\mathbf{P}^1(\mathbf{C}_K)$ is a rigid analytic K-space, whose rigid analytic structure is given by connected affinoid subsets (i.e., a closed disk with finitely many strongly pairwise disjoint open disks removed). The above lemma shows that we also have connected affinoid sybsets in Ω. (For the rigid analytic structure on projective lines one can see [GvdP80, EXAMPLE III.1.18].)

PROPOSITION 2.2.4. *The space Ω is a rigid analytic K-space.*

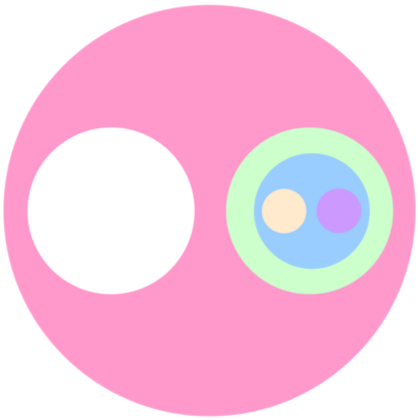

Figure 2.2: $\lambda^{-1}(s)$, $\lambda^{-1}(s')$ and $\lambda^{-1}([s, s'])$ for the case $q = 2$. The pink part is $\lambda^{-1}(s)$, and the blue part is $\lambda^{-1}(s)$, and the green part is $\lambda^{-1}([s, s'])$.

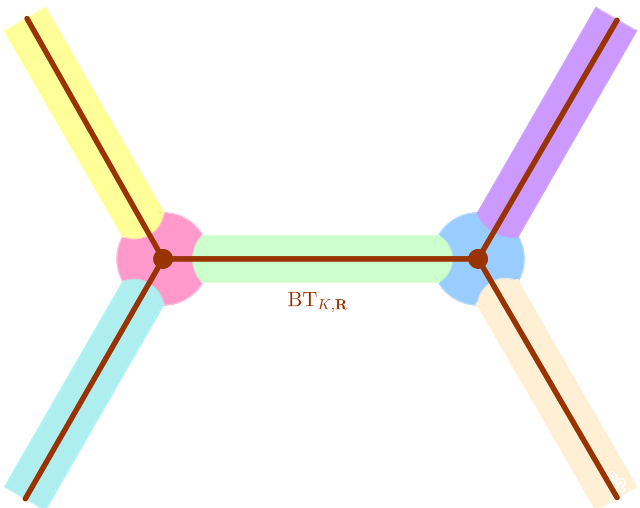

Figure 2.3: Ω as a tubular neoghborhood of $\mathrm{BT}_{K,\mathbf{R}}$ for the case $q = 2$. Pull the blue part and green part (in FIGURE 2.2) away from the paper, and push the white part into the paper. Then we obtain this picture, where $\mathrm{BT}_{K,\mathbf{R}}$ (the brown part) serve as a skeleton. Thus we may view Ω as a tubular neighborhood of $\mathrm{BT}_{K,\mathbf{R}}$.

2.3 The Formal Scheme Attached to Drinfeld Upper Half Plane

2.3.1 Projective Bundles

We first recall some fundamentals about the projective space associated to a quasi-coherent module. For a field F and $r \in \mathbf{N}$ the projective r-space $\mathbf{P}^r(F) \equiv \mathbf{P}(F^{r+1})$ over F is the space of all lines in F^{r+1}. In addition, we have the following isomorphism

$$\mathbf{P}^r(F) \xrightarrow{\sim} \{W \subseteq F^{r+1} : W \text{ is an } F\text{-vector subspace such that } \dim_F F^{r+1}/W = r\}. \tag{2.13}$$

Now let S be a scheme and $\mathscr{E}$ a quasi-coherent module over S. The above formulation motivates us to define a functor $\mathbf{P}(\mathscr{E}) : \mathbf{Sch}_S^{\mathrm{opp}} \longrightarrow \mathbf{Set}$ by

$$(T \xrightarrow{h} S) \longmapsto \{\mathscr{F} \subseteq h^*(\mathscr{E}) : \mathscr{F} \text{ is an } \mathscr{O}_T\text{-submodule such that } h^*(\mathscr{E})/\mathscr{F} \text{ is an invertible } \mathscr{O}_T\text{-module}\}. \tag{2.14}$$

One can show that this functor is representable, and it is represented by a relative projective spectrum.

Proposition 2.3.1. [GW20, Theorem 13.32]
*Let S be a scheme and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. The functor $\mathbf{P}(\mathscr{E})$ defined in (2.14) is represented by $\mathrm{Proj}_S(\mathrm{Sym}(\mathscr{E}))$. We again denote $\mathrm{Proj}_S(\mathrm{Sym}(\mathscr{E}))$ by $\mathbf{P}(\mathscr{E})$ and call it the **projective bundle** associated to $\mathscr{E}$.*

Now we take a look at a particular case, which is of our main interest in the later discussion. Let $S = \operatorname{Spec} R$ be an affine scheme associated to a ring R, and let $\mathscr{E} = \widetilde{M}$ with M an R-module. Note that in this case $\operatorname{Sym} \widetilde{M} = (\operatorname{Sym} M)^{\sim}$. Then

$$\mathbf{P}(\widetilde{M}) = \mathrm{Proj}_{\operatorname{Spec} R}(\operatorname{Sym} \widetilde{M}) = \operatorname{Proj}((\operatorname{Sym} M)^{\sim}(\operatorname{Spec} R)) = \operatorname{Proj}(\operatorname{Sym} M).$$

Specifically, if M is a free R-module of rank $r+1$ with a basis $\{x_0, \cdots, x_r\} \subset M$, then there is a unique isomorphism

$$\operatorname{Sym} M \xrightarrow{\sim} R[T_0, \cdots, T_n], \quad x_i \longmapsto T_i \ \ (i = 0, \cdots, r).$$

In this case, $\mathbf{P}(\widetilde{M}) \cong \operatorname{Proj} R[T_0, \cdots, T_r] = \mathbf{P}_R^r$. Thus we see that the notion of projective bundles recovers that of projective spaces.

2.3.2 Projective Bundles Associated to Lattices

Now we come back to our main issue, namely the construction of the formal scheme associated to Ω. Let M be a $\mathcal{O}_K$-lattice in K^2. The projective space $\mathbf{P}(M)$ associated to M is defined by $\mathbf{P}(M) := \mathbf{P}(\widetilde{M})$. The isomorphism $M \cong \mathcal{O}_K^2$ of $\mathcal{O}_K$-modules (which is not canonical) gives an isomorphism of $\mathcal{O}_K$-schemes

$$\mathbf{P}(M) \cong \operatorname{Proj} \mathcal{O}_K[T_0, T_1] = \mathbf{P}^1_{\mathcal{O}_K}. \tag{2.15}$$

Note that the above isomorphism is not canonical because it depends on the choice of the basis of M.

Viewing $\mathbf{P}(M)$ as a functor, the formula is given by (2.14). This functor can be described explicitly for the affine case. Recall that if $S = \operatorname{Spec} R$ is an affine scheme associated to a ring R, then we have an equivalence of categories:

$$\mathbf{Mod}_R \longrightarrow \mathbf{QCoh}_R, \qquad \begin{array}{ccc} M & \longmapsto & \widetilde{M} \\ \Gamma(\operatorname{Spec} R, \mathscr{F}) & \longleftarrow\!\shortmid & \mathscr{F} \end{array}$$

Here $\mathbf{QCoh}_R$ is the category of quasi-coherent $\mathscr{O}_R$-modules. Now take any commutative $\mathcal{O}_K$-algebra A. There is a natural $\mathcal{O}_K$-schemes $\operatorname{Spec} A \xrightarrow{h} \operatorname{Spec} \mathcal{O}_K$, and we have $h^*(\widetilde{M}) = (M \otimes_{\mathcal{O}_K} A)^{\sim}$. Therefore, the set of A-valued points of $\mathbf{P}(M)$ is given by

$$\begin{aligned} \mathbf{P}(M)(A) &= \{\mathscr{F} \subset (M \otimes_{\mathcal{O}_K} A)^{\sim} : \mathscr{F} \text{ is an } \mathscr{O}_A\text{-submodule such that } (M \otimes_{\mathcal{O}_K} A)^{\sim}/\mathscr{F} \text{ is invertible}\} \\ &= \{N \subset M \otimes_{\mathcal{O}_K} A : N \text{ is an } A\text{-submodule such that } M \otimes_{\mathcal{O}_K} A/N \text{ is free of rank } 1\} \\ &= \{M \otimes_{\mathcal{O}_K} A \twoheadrightarrow L : L \text{ is an } A\text{-module free of rank } 1\} \end{aligned} \tag{2.16}$$

Particularly, the set of $\mathcal{O}_K$-valued points, K-valued points and k-valued points of $\mathbf{P}(M)$ are given by:

$$\begin{aligned}
\mathbf{P}(M)(\mathcal{O}_K) &= \{N \subset M : N \text{ is an } \mathcal{O}_K\text{-submodule such that } M/N \text{ is free of rank } 1\},\\
\mathbf{P}(M)(K) &= \{N' \subset K^2 : N' \text{ is a } K\text{-vector subspace such that } \dim_K K^2/N' = 1\} = \mathbf{P}^1(K),\\
\mathbf{P}(M)(k) &= \{N'' \subset k^2 : N'' \text{ is a } k\text{-vector subspace such that } \dim_k k^2/N'' = 1\} = \mathbf{P}^1(k).
\end{aligned} \tag{2.17}$$

Notice that $\mathbf{P}(M)(K) = \mathbf{P}^1(K)$ is canonical, but $\mathbf{P}(M)(k) = \mathbf{P}^1(k)$ is not. Moreover, one can easily establish a bijection between $\mathbf{P}(M)(\mathcal{O}_K)$ and $\mathbf{P}(M)(K)$:

$$\mathbf{P}(M)(\mathcal{O}_K) \xrightarrow{\sim} \mathbf{P}(M)(K), \qquad \begin{array}{ccc} N & \longmapsto & N \otimes_{\mathcal{O}_K} K \\ N' \cap \mathcal{O}_K^2 & \longleftarrow\!\!\shortmid & N' \end{array}$$

Up to now, we see that $\mathbf{P}(M) \cong \mathbf{P}^1_{\mathcal{O}_K}$ as $\mathcal{O}_K$-schemes in (2.15). Note that $\mathcal{O}_K$ is a discrete valuation ring, so $\operatorname{Spec} \mathcal{O}_K$ consists of two points: the generic point, which is open and corresponds to the zero ideal of $\mathcal{O}_K$, and the special point (or the closed point), which is closed and corresponds to the maximal ideal $\mathfrak{m}$. The generic fibre and the special fibre of $\mathbf{P}(M)$ are

$$\begin{aligned}
\text{generic fibre of } \mathbf{P}(M) &\cong \mathbf{P}^1_{\mathcal{O}_K} \times_{\mathcal{O}_K} K \cong \operatorname{Proj} \mathcal{O}_K[T_0, T_1] \otimes_{\mathcal{O}_K} K = \operatorname{Proj} K[T_0, T_1] = \mathbf{P}^1_K,\\
\text{special fibre of } \mathbf{P}(M) &\cong \mathbf{P}^1_{\mathcal{O}_K} \times_{\mathcal{O}_K} k \cong \operatorname{Proj} \mathcal{O}_K[T_0, T_1] \otimes_{\mathcal{O}_K} k = \operatorname{Proj} k[T_0, T_1] = \mathbf{P}^1_k.
\end{aligned} \tag{2.18}$$

Since $\mathbf{P}^1_K(K) = \mathbf{P}^1(K)$ and $\mathbf{P}^1_k(k) = \mathbf{P}^1(k)$, one deduces from (2.17) and (2.18) that[3]

$$\begin{aligned}
\{K\text{-rational points of } \mathbf{P}(M)\} &= \{K\text{-rational points of its generic fibre}\},\\
\{k\text{-rational points of } \mathbf{P}(M)\} &= \{k\text{-rational points of its special fibre}\}.
\end{aligned} \tag{2.19}$$

Suppose M'' is homothetic to M, say $M = cM''$ for some $c \in K^\times$. The homothety induces a canonical $\mathcal{O}_K$-isomorphism

$$\operatorname{Hom}_{\mathcal{O}_K}(M, \mathcal{O}_K) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_K}(M'', \mathcal{O}_K), \quad f(\cdot) \longmapsto f(c\cdot).$$

This induces a canonical isomorphism $\operatorname{Sym}_{\mathcal{O}_K}(\operatorname{Hom}_{\mathcal{O}_K}(M, \mathcal{O}_K)) \cong \operatorname{Sym}_{\mathcal{O}_K}(\operatorname{Hom}_{\mathcal{O}_K}(M'', \mathcal{O}_K))$ and thus a canonical isomorphism of $\mathcal{O}_K$-schemes:

$$\mathbf{P}(M) \xrightarrow{\sim} \mathbf{P}(M''). \tag{2.20}$$

This leads us to define the projective space associated to a vertex in the Bruhat-Tits tree BT_K.

2.3.3 Construction of $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$

Definition 2.3.2. For a vertex $s \in \mathrm{BT}_K$ we take any representative M of s and define $\mathbf{P}_s = \mathbf{P}(M)$. The projective lines associated to different representatives are identified canonically via (2.20). As what we have seen, $\mathbf{P}_s$ is an $\mathcal{O}_K$-scheme, and we set $\widehat{\mathbf{P}}_s$ to be the formal completion of $\mathbf{P}_s$ along its special fibre.

From (2.18), we identify the special fibre of $\mathbf{P}_s$ with $\mathbf{P}^1_k$. Since the set of k-rational points is $\mathbf{P}^1(k)$, we then have the identification

$$\{\text{rational points of special fibre of } \mathbf{P}_s\} = \mathbf{P}^1(k).$$

In addition, the special fibre $\mathbf{P}^1_k$ is closed in $\mathbf{P}_s$ because it is the preimage of the closed point of $\operatorname{Spec} \mathcal{O}_K$. Similarly, the generic fibre of $\mathbf{P}^1_s$ can be identified with $\mathbf{P}^1_K$ via (2.18).

Definition 2.3.3. For a vertex $s \in \mathrm{BT}_K$ we define

$$\Omega_s = \mathbf{P}_s - \{k\text{-rational points of the special fibre of } \mathbf{P}_s\},$$

which is an open subscheme of $\mathbf{P}^1_s$, and we denote by $\widehat{\Omega}_s$ the formal completion of Ω_s along its special fibre. Note that $\Omega_s = \mathbf{P}_s - \mathbf{P}^1(k)$ by (2.19).

[3]If X is a scheme over $\mathcal{O}_K$, then one can show that $X(k) = \{k\text{-rational points of its special fibre}\}$. If we further assume that X is proper over $\mathcal{O}_K$, then one can prove $X(K) = \{K\text{-rational points of its special fibre}\}$

Remark 2.3.4. Let us argue that Ω_s is indeed an open subset of $\mathbf{P}_s$. It suffices to prove that $\mathbf{P}^1(k)$ is a closed subset of $\mathbf{P}^1_k$[4]. Since k is a finite field, $\mathbf{P}^1(k)$ is a finite set. If we can prove that every point in $\mathbf{P}^1(k)$ is a closed point in $\mathbf{P}^1_k$, then $\mathbf{P}^1(k)$ is a finite union of closed points and thus closed in $\mathbf{P}^1_k$. It remains to prove the following fact:

Let X be a scheme over a field k. Then every k-rational point of X is a closed point.

Note that a subset $V \subseteq X$ is closed if and only if for any open covering $(U_i)_{i\in I}$ of X the intersection $U_i \cap V$ is closed in U_i for all $i \in I$ (this is generally true for any topological space). Thus it suffices to consider $X = \operatorname{Spec} A$ is an affine scheme. To give a k-rational point of $\operatorname{Spec} A$ is equivalent to give the following commutative diagram:

$$\begin{tikzcd} & A \arrow[dr, "\varphi"] & \\ k \arrow[ur] \arrow[rr, equal] & & k \end{tikzcd}$$

The associated scheme morphism $\tilde{\varphi} : \operatorname{Spec} k \to \operatorname{Spec} A$ is given by $[(0)] \mapsto [\varphi^{-1}(0)]$, so we have to look at $\ker\varphi$. Passing to quotient gives (we again denote by φ the map $A/\ker\varphi \hookrightarrow k$)

$$\begin{tikzcd} & A/\ker\varphi \arrow[dr, hook, "\varphi"] & \\ k \arrow[ur] \arrow[rr, equal, "(*)"] & & k \end{tikzcd}$$

Since φ is injective and $(*)$ is the identity map (in particular, it is surjective), we then conclude φ is also surjective. This shows φ is an isomorphism and thus $\ker\varphi \subset A$ is a maximal ideal; that is, the k-rational point is closed.

Now for two vertices $s = [M]$ and $s' = [M']$ with a compatible basis e_1, e_2, i.e., $M = \langle e_1, e_2\rangle_{\mathcal{O}_K}$ and $M' = \langle e_1, \pi e_2\rangle_{\mathcal{O}_K}$. The projection $M \twoheadrightarrow M/M' \cong k$ induces the map of k-vector spaces:

$$M \otimes_{\mathcal{O}_K} k \xrightarrow{\varphi_{M,M'}} M/M' \otimes_{\mathcal{O}_K} k \cong k.$$

The map $\varphi_{M,M'}$ is surjective from the two-dimensional k-vector space $M \otimes_{\mathcal{O}_K} k$ to the one-dimensional k-vector space $M/M' \otimes_{\mathcal{O}_K} k$. Thus $\ker\varphi_{M,M'}$ is a one-dimensional subspace of $M \otimes_{\mathcal{O}_K} k$. In conclusion, if s' is adjacent to s, then s' defines a k-rational point in $\mathbf{P}_s$. Actually, we can describe this k-rational point more explicitly. Let $[1:0]$, $[1:1]$ and $[0:1]$ be the points in $\mathbf{P}^1(k)$ representing $0, 1$ and ∞, respectively. Since we have fixed a compatible basis e_1, e_2, one has the following characterizations

$$\langle e_1\rangle_k \longleftrightarrow [1:0] = 0, \quad \langle e_2\rangle_k \longleftrightarrow [0:1] = \infty, \quad \langle e_1 + e_2\rangle_k \longleftrightarrow [1:1] = 1.$$

The k-rational point in $\mathbf{P}_s$ defined by s' is the kernel of $\varphi_{M,M'}$, and it is clear that $\ker\varphi_{M,M'} = \langle e_1\rangle_k$. Therefore, s' defines the k-rational point $0 = [1:0]$ in $\mathbf{P}_s$, which coincides with what we have seen previously in Remark 2.1.2 (d). We have to emphasize again that this explicit characterization based on our choice of compatible basis.

Definition 2.3.5. Let $s, s' \in \mathrm{BT}_K$ be adjacent. Define $\mathbf{P}_{[s,s']}$ to be the blow-up of $\mathbf{P}_s$ along s' (i.e., along the point in $\mathbf{P}_s$ defined by s'), which is an $\mathcal{O}_K$-scheme. We write $\widehat{\mathbf{P}}_{[s,s']}$ for the formal completion of $\mathbf{P}_{s,s'}$ along its special fibre.

Definition 2.3.6. Let $s, s' \in \mathrm{BT}_K$ be adjacent. Set

$$\Omega_{[s,s']} := \mathbf{P}_{[s,s']} - \{k\text{-rational points on the special fibre of } \mathbf{P}_{[s,s']} \text{ except for } s'\}.$$

Then $\Omega_{[s,s']}$ is an $\mathcal{O}_K$-scheme, and we denote by $\widehat{\Omega}_{[s,s']}$ the formal completion of $\Omega_{[s,s']}$ along its special fibre. See Figure 2.4.

Definition 2.3.7. We define $\widehat{\Omega}$ to be the gluing of $(\widehat{\Omega}_{[s,s']})_{[s,s']\in\mathrm{BT}_K}$ along $(\widehat{\Omega}_s)_{s\in\mathrm{BT}_K}$. See Figure 2.5.

[4]Here we use an easy topological property: *Let X be a topological space, and let $Y \subseteq X$ be a closed subset endowed with subspace topology. If Z is a closed subset of Y, then Z is also a closed subset of X.*
By the definition of subspace topology there is an open subset U of X such that $U \cap Y = Y - Z$. Then $X - Z = U \cup (X - Y)$; that is, $X - Z$ is a union of two open subsets of X and thus open in X. Hence Z is closed in X.

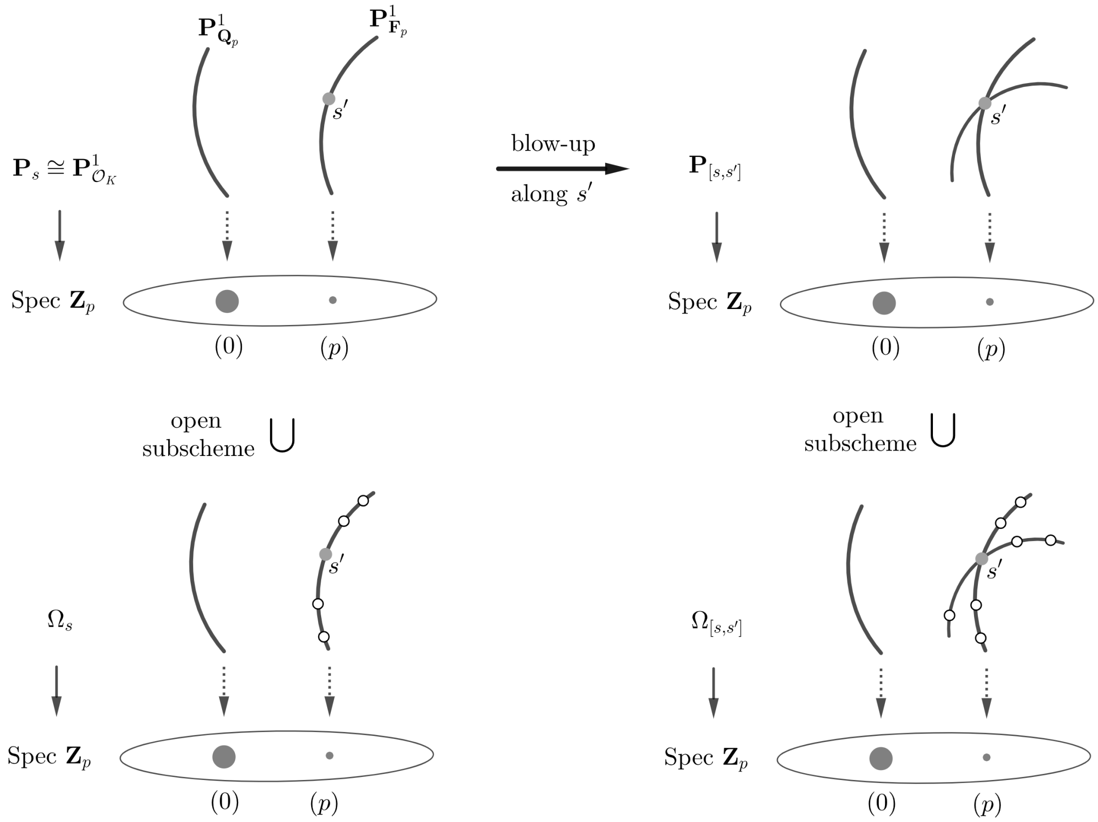

Figure 2.4: $\mathbf{P}_s$, $\mathbf{P}_{[s,s']}$, Ω_s and $\Omega_{[s,s']}$. The gray point is the k-rational point defined by s', and those blank points are k-rational points that we need to remove in the definitions of Ω_s and $\widehat{\Omega}_{[s,s']}$.

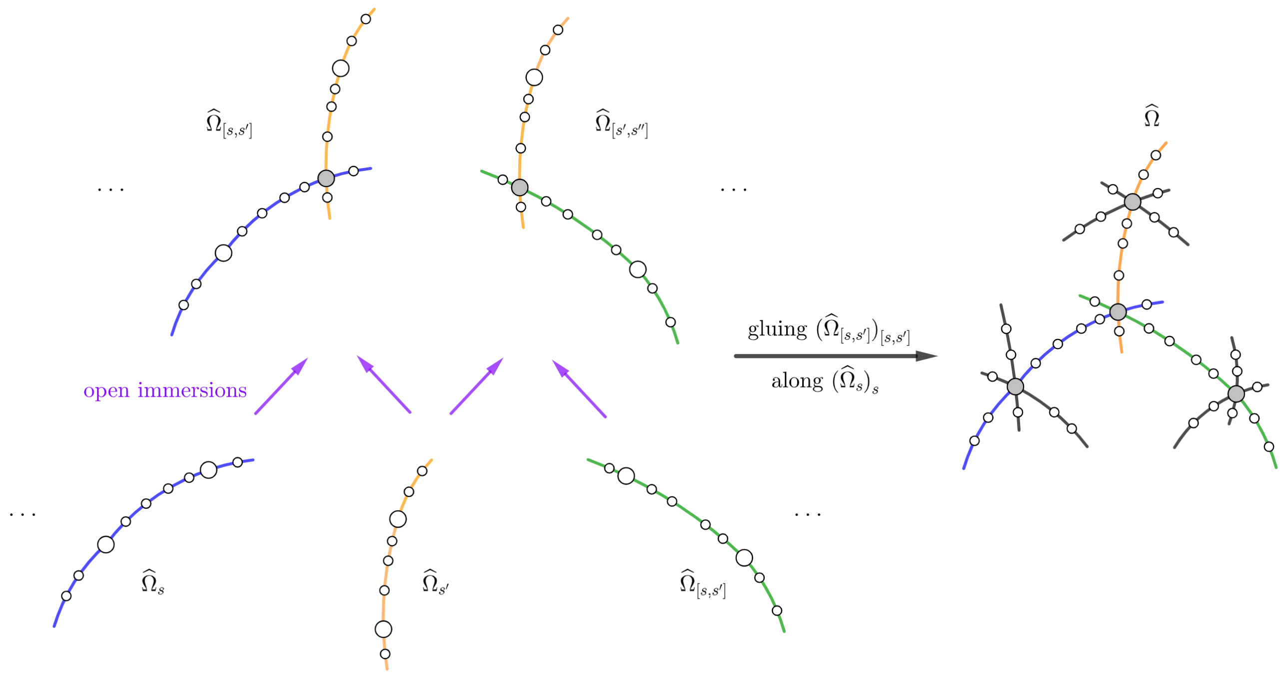

Figure 2.5: Construction of $\widehat{\Omega}$. One can find that $\widehat{\Omega}$ is a geometrization of BT_K by replacing vertices (resp. edges) of BT_K with some k-rational points (resp. with $\mathbf{P}^1_k - \{k\text{-rational points except for } 0, \infty\}$).

2.4 MODULAR DESCRIPTION OF $\widehat{\Omega}$: DELIGNE'S FUNCTOR

In this section, we are going to see the modular descriptions of $\widehat{\Omega}_s$ and $\widehat{\Omega}_{[s,s']}$.

NOTATION 2.4.1. Recall that $\mathbf{Alg}_{\mathcal{O}_K}$ is the category of commutative $\mathcal{O}_K$-algebra. Consider two subcategories of $\mathbf{Alg}_{\mathcal{O}_K}$:

(a) $\mathbf{Compl}_{\mathcal{O}_K}$ is the category of commutative $\mathcal{O}_K$-algebras which are complete with respect to $\mathfrak{m}$-adic topology.

(b) $\mathbf{Nilp}_{\mathcal{O}_K}$ the category of commutative unitary $\mathcal{O}_K$-algebras such that π is a nilpotent.

If R is an object in $\mathbf{Nilp}_{\mathcal{O}_K}$ and $\pi^n = 0$ in R for some $n \in \mathbf{Z}_{\geq 0}$, then R is an $\mathcal{O}_K/\mathfrak{m}^n$-algebra. Thus $\mathbf{Nilp}_{\mathcal{O}_K}$ is the "union" of categories of commutative $\mathcal{O}_K/\mathfrak{m}^n$-algebras for all $n \in \mathbf{Z}_{\geq 0}$. In addition, every R in $\mathbf{Nilp}_{\mathcal{O}_K}$ satisfies $R \cong \varprojlim R/\mathfrak{m}^n R$ because $\mathfrak{m}^n = 0$ for some $n \in \mathbf{Z}_{\geq 0}$. Therefore, $\mathbf{Nilp}_{\mathcal{O}_K}$ is a subcategory of $\mathbf{Compl}_{\mathcal{O}_K}$. Moreover, $\mathbf{Nilp}_{\mathcal{O}_K}$ is a full subcategory of $\mathbf{Compl}_{\mathcal{O}_K}$. In conclusion, there is a chain of categories with respect to inclusions:

$$\mathbf{Nilp}_{\mathcal{O}_K} \subset \mathbf{Compl}_{\mathcal{O}_K} \subset \mathbf{Alg}_{\mathcal{O}_K}.$$

Now let X be a scheme over $\mathcal{O}_K$, and let $\widehat{X}$ be the formal completion of X along its special fibre. Then the functors of points defined by X and $\widehat{X}$ on $\mathbf{Compl}_{\mathcal{O}_K}$ are as what follows:

$$h_X : \mathbf{Compl}_{\mathcal{O}_K} \longrightarrow \mathbf{Set}, \quad R \longmapsto \mathrm{Hom}_{\mathcal{O}_K}(\operatorname{Spec} R, X),$$
$$h_{\widehat{X}} : \mathbf{Compl}_{\mathcal{O}_K} \longrightarrow \mathbf{Set}, \quad R \longmapsto \mathrm{Hom}_{\mathcal{O}_K}(\operatorname{Spf} R, \widehat{X}).$$

A remark is that if $X = \operatorname{Spec} A$ and we endow A with the $\mathfrak{m}$-adic topology, then $\widehat{X} = \operatorname{Spf} \varprojlim A/\mathfrak{m}^n$.

LEMMA 2.4.2. *Let A be in $\mathbf{Alg}_{\mathcal{O}_K}$ endowed with the $\mathfrak{m}$-adic topology. Set $X := \operatorname{Spec} A$ and $\widehat{X}$ the formal completion of X along its special fibre. Then $h_X = h_{\widehat{X}}$.*

PROOF. There are two steps: first check that $h_X = h_{\widehat{X}}$ on $\mathbf{Nilp}_{\mathcal{O}_K}$, and then deduce $h_X = h_{\widehat{X}}$ on $\mathbf{Compl}_{\mathcal{O}_K}$.

◇ STEP 1. $h_X = h_{\widehat{X}}$ ON $\mathbf{Nilp}_{\mathcal{O}_K}$.

It suffices to prove that for each $R \in \mathbf{Nilp}_{\mathcal{O}_K}$

$$\mathrm{Hom}_{\mathcal{O}_K}(A, R) \cong \mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R),$$

where $\mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(-,-)$ denotes the set of continuous $\mathcal{O}_K$-algebra homomorphisms. If we write $\iota : A \to \varprojlim A/\mathfrak{m}^n$ for the canonical map, then we have a well-defined map

$$\mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R) \longrightarrow \mathrm{Hom}_{\mathcal{O}_K}(A, R), \quad f \longmapsto f \circ \iota.$$

To construct the inverse map, we note that $\mathfrak{m}^r R = 0$ for some $r \in \mathbf{N}$ because $R \in \mathbf{Nilp}_{\mathcal{O}_K}$. Given any map $g \in \mathrm{Hom}_{\mathcal{O}_K}(A, R)$, it induces $g^{(r)} : A/\mathfrak{m}^r A \to R/\mathfrak{m}^r R = R$. By compositing with the canonical projection $\mathrm{pr}_r : \varprojlim A/\mathfrak{m}^n A \to A/\mathfrak{m}^r A$, we then have

$$\mathrm{Hom}_{\mathcal{O}_K}(A, R) \longrightarrow \mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R), \quad g \longmapsto g^{(r)} \circ \mathrm{pr}_r.$$

One can easily check that this map is well-defined and is the inverse map.

◇ STEP 2. $h_X = h_{\widehat{X}}$ ON $\mathbf{Compl}_{\mathcal{O}_K}$.

Similarly, it suffices to prove that for each $R \in \mathbf{Compl}_{\mathcal{O}_K}$

$$\mathrm{Hom}_{\mathcal{O}_K}(A, R) \cong \mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R).$$

Note that $R \cong \varprojlim R/\mathfrak{m}^n R$, and $R/\mathfrak{m}^n R$ is in $\mathbf{Nilp}_{\mathcal{O}_K}$ for each $n \in \mathbf{N}$. Therefore,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R) &\cong \mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, \varprojlim R/\mathfrak{m}^n R) \\
&\cong \varprojlim \mathrm{Hom}_{\mathcal{O}_K,\mathrm{cont}}(\varprojlim A/\mathfrak{m}^n A, R/\mathfrak{m}^n R) \\
&\overset{(*)}{\cong} \varprojlim \mathrm{Hom}_{\mathcal{O}_K}(A, R/\mathfrak{m}^n R) \\
&\cong \mathrm{Hom}_{\mathcal{O}_K}(A, \varprojlim R/\mathfrak{m}^n R) \\
&\cong \mathrm{Hom}_{\mathcal{O}_K}(A, R),
\end{aligned}$$

where we use $h_X|_{\mathbf{Nilp}_{\mathcal{O}_K}} = h_{\widehat{X}}|_{\mathbf{Nilp}_{\mathcal{O}_K}}$ in $(*)$.

■

Definition 2.4.3. For each vertex $s = [M]$ in BT_K define the functor $\mathcal{F}_s$ on $\mathbf{Compl}_{\mathcal{O}_K}$ which associates each R in $\mathbf{Compl}_{\mathcal{O}_K}$ the set of isomorphism classes of pairs $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a free R-module of rank 1 and $\alpha : M \to \mathcal{L}$ is an $\mathcal{O}_K$-module homomorphism such that

$$\text{for each } x \in \operatorname{Spec} R/\mathfrak{m}R \text{ the map } \alpha_x : M/\mathfrak{m}M \to \mathcal{L} \otimes_R \mathbf{k}(x) \text{ is injective.} \tag{2.21}$$

If $R \in \mathbf{Nilp}_{\mathcal{O}_K}$, then $\operatorname{Spec} R = \operatorname{Spec} R/\mathfrak{m}R$ as topological spaces. Moreover, since the underlying topological space of $\operatorname{Spf} R$ consists of all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \supseteq (\mathfrak{m}R)^r$ for some $r \in \mathbf{N}$, one has $\operatorname{Spf} R = \operatorname{Spec} R$ as topological spaces.

Proposition 2.4.4. [BC91, Proposition I.4.2]
For every vertex $s \in \mathrm{BT}_K$ the functor $\mathcal{F}_s$ is represented by $\widehat{\Omega}_s$.

Proof. Take a representative M of the vertex s, i.e., $s = [M]$. We conduct the proof in three steps.

◇ Step 1. *$\mathcal{F}_s$ is a subfunctor of $\widehat{\mathbf{P}}_s$.*
Note that the projective line $\mathbf{P}(M)$ as a functor is given by

$$\begin{aligned} \mathbf{P}(M) : \mathbf{Sch}_{\mathcal{O}_K} &\longrightarrow \mathbf{Set} \\ S &\longmapsto \mathbf{P}(M)(S) = \{\widetilde{M} \otimes_{\mathscr{O}_{\mathcal{O}_K}} \mathscr{O}_S \twoheadrightarrow \mathscr{L} : \mathscr{L} \text{ is an invertible } \mathscr{O}_S\text{-module}\}/\cong \end{aligned}$$

We have the inclusion of functors $\mathcal{F}_s \hookrightarrow \mathbf{P}(M)$:

$$\mathcal{F}_s(R) \hookrightarrow \mathbf{P}(M)(R), \quad (\mathcal{L}, \alpha) \longmapsto \alpha \otimes \mathrm{id}_R : M \otimes_{\mathcal{O}_K} R \to \mathcal{L}.$$

Lemma 2.4.2 tells us that $\mathbf{P}(M)(R) = \widehat{\mathbf{P}}_s(R)$ for all $R \in \mathbf{Compl}_{\mathcal{O}_K}$, so we obtain the inclusion of functors $\mathcal{F}_s \hookrightarrow \widehat{\mathbf{P}}_s$. In other words, $\mathcal{F}_s$ is a subfunctor of $\widehat{\mathbf{P}}_s$.

◇ Step 2. Let $R \in \mathbf{Compl}_{\mathcal{O}_K}$ and $(\mathcal{L}, \alpha) \in \mathcal{F}_s(R)$. The pair $(\mathcal{L}, \alpha)$ is determined up to isomorphism by

$$\alpha : M \longrightarrow \mathcal{L}, \quad e_1 \longmapsto \zeta, \quad e_2 \longmapsto 1.$$

For any $x \in \operatorname{Spec} R/\mathfrak{m}R$ the condition (2.21) says that

$$M/\mathfrak{m}M = ke_1 \oplus ke_2 \longrightarrow \mathcal{L} \otimes_R \mathbf{k}(x), \quad e_1 \longmapsto \bar{\zeta}, \quad e_2 \longmapsto 1$$

is injective; that is, $\bar{\zeta}$ and 1 are k-linearly indepenent in $\mathcal{L} \otimes_R \mathbf{k}(x)$. In other words,

$$\bar{\zeta} - a \cdot 1 \neq 0 \text{ in } \mathcal{L} \otimes_R \mathbf{k}(x) \cong R \otimes_R \mathbf{k}(x) = \mathbf{k}(x) \text{ for all } a \in k. \tag{2.22}$$

By our definition $\widehat{\Omega}_s \subset \widehat{\mathbf{P}}_s - \{\infty\} = \widehat{\mathbf{A}}^1_{\mathcal{O}_K}$ and

$$\begin{aligned} \widehat{\Omega}_s(R) &= \operatorname{Hom}_{\mathcal{O}_K}(\operatorname{Spf} R, \widehat{\Omega}_s) \\ &= \left\{\varphi : \operatorname{Spf} R \to \widehat{A}^1_{\mathcal{O}_K} : \text{for each } x \in \operatorname{Spec} R/\mathfrak{m}R,\ \varphi \circ \iota_x \text{ does not factor through } \mathbf{P}^1(k) - \{\infty\}\right\}, \end{aligned} \tag{2.23}$$

where $\iota_x : \operatorname{Spec} \mathbf{k}(x) \to \operatorname{Spf} R$ is the canonical map. We can express the condition in (2.23) as the following diagram:

$$\begin{tikzcd} \operatorname{Spf} R \arrow[r, "\varphi"] & \widehat{A}^1_{\mathcal{O}_K} \\ \operatorname{Spec} \mathbf{k}(x) \arrow[u, "\iota_x"] \arrow[r, dashed, "\times"] & \widehat{\mathbf{P}}^1(k) - \{\infty\}. \arrow[u] \end{tikzcd} \tag{2.24}$$

We note that $\widehat{A}^1_{\mathcal{O}_K} = \operatorname{Spf} \mathcal{O}_K\langle T\rangle$[5], so every φ in (2.23) corresponds to a continuous homomorphism $\varphi^a : \mathcal{O}_K\langle T\rangle \to R$. In addition, points in $\mathbf{P}^1(k) - \{\infty\}$ are on 1-1 correspondence with $\operatorname{Spec} k[T]/(T-a)$ for all $a \in k$. Thus the

[5]The ring $\mathcal{O}_K\langle T\rangle$ is the Tate algebra in one variable over $\mathcal{O}_K$; see [Bo14, Section 2.2].

diagram (2.24) corresponds to the following diagram for all $a \in k$:

$$\begin{tikzcd} R \arrow[d] & \mathcal{O}_K\langle T\rangle \arrow[l, "\varphi^a"'] \arrow[d] \\ \mathbf{k}(x) & k[T]/(T-a). \arrow[l, dashed, "\times"] \end{tikzcd} \tag{2.25}$$

The map $\mathcal{O}_K\langle T\rangle \to R \to \mathbf{k}(x)$ is given by $T \mapsto \zeta \mapsto \bar{\zeta}$. Hence (2.25) is equivalent to $\bar{\zeta} - a \neq 0$ for all $a \in k$, which coincides with (2.22).

■

Definition 2.4.5. Let $s, s' \in \mathrm{BT}_K$ be adjacent vertices with representatives M, M' such that $\pi M \subsetneq M' \subsetneq M$. Define a functor $\mathcal{F}_{[s,s']} : \mathbf{Compl}_{\mathcal{O}_K} \to \mathbf{Set}$ by associating with each $R \in \mathbf{Compl}_{\mathcal{O}_K}$ an isomorphism class of tuples $(\mathcal{L}, \mathcal{L}', \alpha, \alpha', c, c')$, where $\mathcal{L}, \mathcal{L}'$ are free R-modules of rank 1, $\alpha : M \to \mathcal{L}$ and $\alpha' : M; \to \mathcal{L}'$ are $\mathcal{O}_K$-module homomorphisms, and $c' : \mathcal{L}' \to \mathcal{L}$ and $c : \mathcal{L} \to \mathcal{L}'$ are R-module homomorphisms, such that the following diagram commutes:

$$\begin{tikzcd} \pi M \arrow[r, hook] \arrow[d, "\alpha/\pi"] & M' \arrow[r, hook] \arrow[d, "\alpha'"] & M \arrow[d, "\alpha"] \\ \mathcal{L} \arrow[r, "c"] & \mathcal{L}' \arrow[r, "c'"] & \mathcal{L}. \end{tikzcd}$$

Moreover, $\mathcal{L}$ and $\mathcal{L}'$ satisfy the following condition:

$$\begin{aligned} &\text{for each } x \in \operatorname{Spec} R/\pi R \\ &\ker(\alpha_x : M/\pi M \to \mathcal{L} \otimes_R \mathbf{k}(x)) \subseteq M'/\pi M \ \text{ and } \ker(\alpha'_x : M'/\pi M' \to \mathcal{L}' \otimes_R \mathbf{k}(x)) \subseteq \pi M/\pi M'. \end{aligned} \tag{2.26}$$

Proposition 2.4.6. [BC91, Proposition I.4.4]
For any adjacent vertices $s, s' \in \mathrm{BT}_K$ the functor $\mathcal{F}_{[s,s']}$ is represented by $\widehat{\Omega}_{[s,s']}$.

Proof. Take M and M' be representatives of s and s' with a compatible basis e_1, e_2. One can see that $\mathcal{F}_{[s,s']}$ is a subfunctor of $\widehat{\mathbf{P}}_{[s,s']} - (\{\infty\} \sqcup \{\infty\})$[6] by an argument similar to Step 1 in the proof of Proposition 2.4.4. Since $M'/\pi M \cong ke_1$ and $\pi M/\pi M' \cong ke_2$, the condition (2.26) implies $\alpha(e_2)$ (resp. $\alpha'(e_1)$) generates $\mathcal{L}$ (resp. $\mathcal{L}'$). Thus the two pairs are uniquely determined, up to isomorphism, by

$$\begin{aligned} \alpha : M &\longrightarrow \mathcal{L}, \quad & e_1 &\longmapsto \zeta, \ & e_2 &\longmapsto 1, \\ \alpha' : M &\longrightarrow \mathcal{L}', \quad & e_1 &\longmapsto 1, \ & \pi e_2 &\longmapsto \eta. \end{aligned}$$

The commutative diagram in the definition of $\mathcal{F}_{[s,s']}$ implies $c = \eta$, $c' = \zeta$ and $\zeta\eta = \pi$. Therefore, if we write $\bar{\zeta}$ and $\bar{\eta}$ for the image of ζ and η in $\mathbf{k}(x)$, then

$$\bar{\zeta} - a \neq 0 \text{ and } \bar{\eta} - b \neq 0 \text{ for all } a, b \in k^\times. \tag{2.27}$$

Note that $\widehat{\Omega}_{[s,s']} \subset \widehat{\mathbf{P}}_{[s,s']} - (\{\infty\} \sqcup \{\infty\}) = \operatorname{Spf} \mathcal{O}_K\langle T_0, T_1\rangle$. For each $R \in \mathbf{Compl}_{\mathcal{O}_K}$ our construction of $\widehat{\Omega}_{[s,s']}$ gives

$$\widehat{\Omega}_{[s,s']}(R) = \{\varphi : \operatorname{Spf} R \to \operatorname{Spf} \mathcal{O}_K\langle T_0, T_1\rangle : \varphi \text{ satisfies the condition } (\star)\}, \tag{2.28}$$

where the condition $(\star)$ is:

$$\begin{gathered} \text{for each } x \in \operatorname{Spec} R/\mathfrak{m}R \text{ the map } \varphi \circ \iota_x \text{ factors through } (\mathbf{P}^1(k) - \{\infty\}) \sqcup (\mathbf{P}^1(k) - \{\infty\}) \\ \text{if and only if the image of } \operatorname{Spec} \mathbf{k}(x) \to (\mathbf{P}^1(k) - \{\infty\}) \sqcup (\mathbf{P}^1(k) - \{\infty\}) \text{ is } (0,0) \\ (\iota_x \text{ is the canonical map } \operatorname{Spec} \mathbf{k}(x) \to \operatorname{Spf} R). \end{gathered}$$

[6]Note that $\widehat{\mathbf{P}}_{[s,s']}$ has two branches: each branch is isomorphic to $\widehat{\mathbf{P}}_s$, and the two branches intersect at the point defined by s', i.e., the point 0 in $\mathbf{P}^1(k)$ (see Remark 2.1.2 (d) or Section 2.3).Therefore, there are two infinite points in $\widehat{\mathbf{P}}_{[s,s']}$, and $\widehat{\mathbf{P}}_{[s,s']} - (\{\infty\} \sqcup \{\infty\})$ means the space $\widehat{\mathbf{P}}_{[s,s']}$ with the two infinite points removed.

In other words, we have the commutative diagram

$$\begin{tikzcd} \operatorname{Spf} R \arrow[r, "\varphi"] & \operatorname{Spf} \mathcal{O}_K\langle T_0, T_1\rangle \\ \operatorname{Spec} \mathbf{k}(x) \arrow[u, "\iota_x"] \arrow[r, dashed, "(?)"] & (\mathbf{P}^1(k) - \{\infty\}) \sqcup (\mathbf{P}^1(k) - \{\infty\}) \arrow[u] \end{tikzcd} \tag{2.29}$$

and $(?)$ exists if and only if its image is $(0,0)$. Since points in $(\mathbf{P}^1(k) - \{\infty\}) \sqcup (\mathbf{P}^1(k) - \{\infty\})$ are in 1-1 correspondence with $\operatorname{Spec} k[T_0, T_1]/(T_0 - a, T_1 - b)$ for all $a, b \in k$. Thus the diagram (2.29) corresponds to the following diagram for all $a, b \in k^\times$:

$$\begin{tikzcd} R \arrow[d] & \mathcal{O}_K\langle T_0, T_1\rangle \arrow[l, "\varphi^a"'] \arrow[d] \\ \mathbf{k}(x) & k[T_0, T_1]/(T_0 - a, T_1 - b). \arrow[l, dashed, "\times"] \end{tikzcd} \tag{2.30}$$

The map $\mathcal{O}_K\langle T_0, T_1\rangle \to R \to \mathbf{k}(x)$ is given by $T_0 \mapsto \zeta \mapsto \bar{\zeta}$ and $T_1 \mapsto \eta \mapsto \bar{\eta}$. Hence (2.30) is equivalent to $\bar{\zeta} - a \neq 0$ and $\bar{\eta} - b \neq 0$ for all $a, b \in k^\times$, which coincides with (2.27). ■

2.5 MODULAR DESCRIPTION OF $\widehat{\Omega}$: DRINFELD'S FUNCTOR

2.5.1 THE STATEMENT

NOTATION 2.5.1. In this section, for each $R \in \mathbf{Alg}_{\mathcal{O}_K}$ we define

$$R[\Pi] := R[T]/(T^2 - \pi).$$

That is, $R[\Pi] = \{a_0 + a_1\Pi : a_0, a_1 \in R\}$ is a commutative ring with the relation $\Pi^2 = \pi$. It is clear that $R[\Pi]$ has a $(\mathbf{Z}/2\mathbf{Z})$-graded structure.

DEFINITION 2.5.2. Define a functor $\mathcal{F} : \mathbf{Nilp}_{\mathcal{O}_K} \to \mathbf{Set}$ as what follows. We associate with each $R \in \mathbf{Nilp}_{\mathcal{O}_K,\pi}$ a set $\mathcal{F}(R)$ of isomorphism classes of quadruples $(\mathscr{E}, \mathscr{T}, u, r)$ consisting of the following data: (set $S := \operatorname{Spec} R$)

(1) $\mathscr{E} = \mathscr{E}_0 \oplus \mathscr{E}_1$ is a constructible sheaf of flat $(\mathbf{Z}/2\mathbf{Z})$-graded $\mathcal{O}_K[\Pi]$-modules on $\operatorname{Spec} R$ with respect to Zariski topology.

(2) $\mathscr{T} = \mathscr{T}_0 \oplus \mathscr{T}_1$ is a sheaf of $(\mathbf{Z}/2\mathbf{Z})$-graded $\mathscr{O}_S[\Pi]$-modules such that the homogeneous components $\mathscr{T}_0$ and $\mathscr{T}_1$ are invertible sheaves on S.

(3) $u : \mathscr{E} \to \mathscr{T}$ is an $\mathcal{O}_K[\Pi]$-linear morphism of degree 0 such that $u \otimes_{\mathcal{O}_K} \mathscr{O}_S$ is injective.

(4) $r : \underline{K}^2 \to \mathscr{E}_0 \otimes_{\mathcal{O}_K} K$ is a K-linear isomorphism.

In addition, these data are required to satisfy the following conditions:

[C1] For $S_i \subseteq S$ the zero locus of $\Pi : \mathscr{T}_i \to \mathscr{T}_{i+1}$ $(i = 0, 1)$ the restriction of sheaves $\mathscr{E}_i|_{S_i}$ is a constant sheaf whose stalk is isomorphic to $\mathcal{O}_K^2$ for $i = 0, 1$.

[C2] For every geometric point x of S the map $\mathscr{E}_x/\Pi\mathscr{E}_x \to \mathscr{T} \otimes_R \mathbf{k}(x)/\Pi\mathscr{T} \otimes_R \mathbf{k}(x)$ induced by u is injective.

[C3] $(\bigwedge^2 \mathscr{E}_i)|_{S_i} = \pi^{-i}(\bigwedge^2(\Pi^i r\underline{\mathcal{O}_K}^2))|_{S_i}$ for $i = 0, 1$.

The definition of the functor $\mathcal{F}$ is a little bit complicated and contains some technical conditions, so let us make the first investigation before proving the representbility of $\mathcal{F}$ (THEOREM 2.5.4) in SECTION 2.5.2.

REMARK 2.5.3. Notations are the smae as the above definition.

(a) For a sheaf $\mathscr{F}$ on a locally ringed space X the **fibre** of $\mathscr{F}$ at a point $x \in X$ is defined to be

$$\mathscr{F}(x) \equiv \mathscr{F} \otimes \mathbf{k}(x) := \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathbf{k}(x).$$

If $\mathscr{G}$ is another sheaf on X together with a morphism $f : \mathscr{F} \to \mathscr{G}$, the **zero locus** of f is the set

$$\{x \in X : \mathscr{F}(x) \to \mathscr{G}(x) \text{ is a zero map}\}.$$

(b) Let us take a look at an example. Consider a commutative ring A and $X := \operatorname{Spec} A$. It is equivalent to give the following data:

(1) a morphism $\varphi : \mathscr{O}_X \to \mathscr{O}_X$ of $\mathscr{O}_X$-modules;
(2) a homomorphism $\varphi^a : A \to A$, $1 \mapsto f$ of A-modules.

Thus the zero locus of φ is given by

$$\{[\mathfrak{p}] \in \operatorname{Spec} A : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p},\ 1 \mapsto f \ \text{ is a zero map}\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : f \in \mathfrak{p}A_\mathfrak{p}\} = V(f),$$

where $V(f) := \{[\mathfrak{p}] \in \operatorname{Spec} A : f \in \mathfrak{p}\}$.

(c) Since $\mathscr{T}_0$ and $\mathscr{T}_1$ are invertible $\mathscr{O}_S$-modules on S, one can find an affine open covering $(U_j)_j$ of S such that $\mathscr{T}_0$ and $\mathscr{T}_1$ are isomorphic to $\mathscr{O}_{U_j}$ on U_j for each j. Now take $U = \operatorname{Spec} R'$ to be one of the U_j's (note that R' is an R-algebra; since π is a nilpotent in R, π is also a nilpotent in R'; thus $R' \in \mathbf{Nilp}_{\mathcal{O}_K}$). Then $\mathscr{T}_0|_U \cong \mathscr{O}_U$ and $\mathscr{T}_1|_U \cong \mathscr{O}_U$. From (b) let f_0 (resp. f_1) be the element in R' determining $\Pi : \mathscr{T}_0|_U \to \mathscr{T}_1|_U$ (resp. $\Pi : \mathscr{T}_1|_U \to \mathscr{T}_0|_U$). Since $\Pi^2 = \pi$, we have $f_0 f_1 = \pi$. From (b) we know that

$$\begin{aligned} V(f_0) &= \{[\mathfrak{p}] \in \operatorname{Spec} R' : f_0 \in \mathfrak{p}\} = \text{zero locus of } \Pi : \mathscr{T}_0|_U \to \mathscr{T}_1|_U = S_0 \cap U \\ V(f_1) &= \{[\mathfrak{p}] \in \operatorname{Spec} R' : f_1 \in \mathfrak{p}\} = \text{zero locus of } \Pi : \mathscr{T}_1|_U \to \mathscr{T}_0|_U = S_1 \cap U. \end{aligned}$$

Therefore,

$$(S_0 \cap U) \cup (S_1 \cap U) = V(f_0) \cup V(f_1) = V(f_1 f_2) = V(\pi) = \operatorname{Spec} R',$$

where the last equality holds because π is nilpotent in R'[7]. In conclusion, S_0 and S_1 are two closed subschemes of S such that $S = S_0 \cup S_1$. Moreover, if $[\mathfrak{p}]$ does not lie in $V(f_0)$, then the map $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p} \to R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ induced by $R' \to R'$, $1 \mapsto f_0$ is invertible. This indicates

$$\begin{aligned} U - V(f_0) &= \{\text{points in } U \text{ where } \Pi : \mathscr{F}_1|_U \to \mathscr{F}_0|_U \text{ is invertible}\}, \\ U - V(f_1) &= \{\text{points in } U \text{ where } \Pi : \mathscr{F}_0|_U \to \mathscr{F}_1|_U \text{ is invertible}\}. \end{aligned}$$

By passing the local to the global one then obtains

$$\begin{aligned} S - S_0 &= \{\text{points in } S \text{ where } \Pi : \mathscr{F}_1 \to \mathscr{F}_0 \text{ is invertible}\}, \\ S - S_1 &= \{\text{points in } S \text{ where } \Pi : \mathscr{F}_0 \to \mathscr{F}_1 \text{ is invertible}\}. \end{aligned}$$

(d) It follows directly from the definition that giving a triple $(\mathscr{E}, \mathscr{T}, u)$ is equivalent to give a commutative diagram of period 2

$$\begin{tikzcd} \cdots \arrow[r, "\Pi"] & \mathscr{E}_0 \arrow[r, "\Pi"] \arrow[d, "u_0"] & \mathscr{E}_1 \arrow[r, "\Pi"] \arrow[d, "u_1"] & \mathscr{E}_0 \arrow[r, "\Pi"] \arrow[d, "u_0"] & \cdots \\ \cdots \arrow[r, "\Pi"] & \mathscr{T}_0 \arrow[r, "\Pi"] & \mathscr{T}_1 \arrow[r, "\Pi"] & \mathscr{T}_0 \arrow[r, "\Pi"] & \cdots \end{tikzcd}$$

The main result in this section is that $\mathcal{F}$ is represented by the formal $\mathcal{O}_K$-scheme $\widehat{\Omega}$. We state this result in the following theorem, and we will sketch the proof of it in the next subsection.

Theorem 2.5.4. [BC91, Proposition 5.3]
The functor $\mathcal{F}$ is represented by the formal $\mathcal{O}_K$-scheme $\widehat{\Omega}$.

2.5.2 The Proof

This subsection is devoted to the proof of Theorem (2.5.4). Instead of giving the whole proof, we only construct the map form $\widehat{\Omega}$ to $\mathcal{F}$, and we refer to [BC91, I.5.5] for the verification that the map that we will construct is indeed an isomorphism.

[7] Note that for any ring A we have $\{$ nilpotent elements of $A\} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

DEFINITION 2.5.5. Let $s = [M]$ be a vertex in BT_K. Since $\bigwedge^2 M \subseteq \bigwedge^2 K^2 = K$ is an $\mathcal{O}_k$-submodule, there exists $n \in \mathbf{Z}$ such that $\bigwedge^2 M = \pi^n \mathcal{O}_K$. We say s is an **odd** (resp. **even**) vertex if n is odd (resp. even). If s is odd (resp. even), we always take a representative M for s such that $\bigwedge^2 M = \pi^{-1}\mathcal{O}_K$ (resp. $\bigwedge^2 M = \mathcal{O}_K$).

Note that the above definition is independent of the choice of the representative. If we take another representative cM with $c = \pi^m u \in K^\times$ ($m \in \mathbf{Z}$ and $u \in \mathcal{O}_K^\times$), then

$$\bigwedge^2 cM = c^2 \bigwedge^2 M = \pi^{2m} \cdot \pi^n \mathcal{O}_K.$$

Thus we get the same result after modulo 2.

CONSTRUCTION OF MAPS INTO $\mathcal{F}$

(1) *For $s = [M] \in \mathrm{BT}_K$ an odd vertex there is a natural transformation $\mathcal{F}_s \to \mathcal{F}$.*
Let $R \in \mathbf{Nilp}_{\mathcal{O}_K}$. We associate with each point $(\mathcal{L}, \alpha) \in \mathcal{F}_s(R)$ the diagram

$$\begin{tikzcd}
\mathscr{E}_0 = \underline{M} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_0=\alpha"] & \mathscr{E}_1 = \underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "u_1=\alpha"] & \mathscr{E}_0 = \underline{M} \arrow[d, "u_0=\alpha"] \\
\mathscr{T}_0 = \widetilde{\mathcal{L}} \arrow[r, "\Pi=\mathrm{id}"] & \mathscr{T}_1 = \widetilde{\mathcal{L}} \arrow[r, "\Pi=\pi"] & \mathscr{T}_0 = \widetilde{\mathcal{L}}
\end{tikzcd}$$

and the isomorphism $r : \underline{K}^2 \xrightarrow{\sim} \underline{M} \otimes K$ induced by the inclusion $M \hookrightarrow K^2$. Let us check that the resulting map $\mathcal{F}_s(R) \to \mathcal{F}(R)$ is well-defined.

[C1] Since $\Pi : \mathscr{T}_0 \to \mathscr{T}_1$ is the identity map, we have $S_0 = \varnothing$ and $S_1 = S$. By our definition both $\mathscr{E}_0$ and $\mathscr{E}_1$ are constant sheaves defined by M, so [C1] follows.

[C2] Let x be a geometric point of S. Notice that since $R \in \mathbf{Nilp}_{\mathcal{O}_K}$, one has $\operatorname{Spec} R = \operatorname{Spec} R/\pi R$. Then $\mathscr{E}_x = M \oplus M$ because both $\mathscr{E}_0$ and $\mathscr{E}_1$ are constant sheaves defined by M. The map Π on $\mathscr{E}_x$ is by definition as the following:

$$\Pi : \mathscr{E}_x = M \oplus M \longrightarrow \mathscr{E}_x = M \oplus M, \quad (t_0, t_1) \longmapsto (\pi t_1, t_0).$$

Hence we have $\Pi\mathscr{E}_x = \pi M \oplus M$ and $\mathscr{E}_x/\Pi\mathscr{E}_x = M/\pi M$. Similarly, we have $\mathscr{T} \otimes_R \mathbf{k}(x) = (\mathcal{L} \oplus \mathcal{L}) \otimes_R \mathbf{k}(x)$, and the map $\Pi : \mathscr{T} \to \mathscr{T}$ induces

$$\Pi : (\mathcal{L} \oplus \mathcal{L}) \otimes_R \mathbf{k}(x) \longrightarrow (\mathcal{L} \oplus \mathcal{L}) \otimes_R \mathbf{k}(x), \quad (l_0, l_1) \otimes f \longmapsto (\pi l_1, l_0) \otimes f.$$

Thus $\Pi\mathscr{T} \otimes_R \mathbf{k}(x) = (\pi\mathcal{L} \oplus \mathcal{L}) \otimes_R \mathbf{k}(x)$ and

$$\mathscr{T} \otimes_R \mathbf{k}(x)/\Pi\mathscr{T} \otimes_R \mathbf{k}(x) = (\mathcal{L}/\pi\mathcal{L}) \otimes_R \mathbf{k}(x) \cong R/\pi R \otimes_R \mathbf{k}(x) = \mathbf{k}(x)/\pi\mathbf{k}(x).$$

Since π is nilpotent in R and $\mathbf{k}(x)$ is an R-algebra, we then have $\pi^r = 0$ in $\mathbf{k}(x)$ for some $r \in \mathbf{N}$. Therefore, $\pi = 0$ in $\mathbf{k}(x)$ because $\mathbf{k}(x)$ is a field. this shows $\mathscr{T} \otimes_R \mathbf{k}(x)/\Pi\mathscr{T} \otimes_R \mathbf{k}(x) \cong \mathbf{k}(x)$. In addition, we have by definition that $\mathcal{L} \otimes_R \mathbf{k}(x) \cong R \otimes_R \mathbf{k}(x) = \mathbf{k}(x)$, so $\mathcal{L} \otimes_R \mathbf{k}(x) \cong \mathscr{T} \otimes_R \mathbf{k}(x)/\Pi\mathscr{T} \otimes_R \mathbf{k}(x)$. Combining altogether, we conclude that the composition

$$M/\pi M \longrightarrow \mathcal{L} \otimes_R \mathbf{k}(x) \xrightarrow{\sim} \mathscr{T} \otimes_R \mathbf{k}(x)/\Pi\mathscr{T} \otimes_R \mathbf{k}(x).$$

is injective (note that the first arrow is injective by (2.21)). Thus [C2] is verified.

[C3] Note that $\Pi \circ r(\mathcal{O}_K^2) = \mathcal{O}_K^2$ because of $\Pi = \mathrm{id}$ on $\mathscr{E}_0$. Thus we have

$$\bigwedge^2 M = \pi^{-1}\mathcal{O}_K = \pi^{-1} \bigwedge^2 \mathcal{O}_K^2 = \pi^{-1} \left(\bigwedge^2 \Pi \circ r(\mathcal{O}_K^2) \right).$$

(2) *For $s' = [M'] \in \mathrm{BT}_K$ an even vertex there is a natural transformation $\mathcal{F}_{s'} \to \mathcal{F}$.*
Let $R \in \mathbf{Nilp}_{\mathcal{O}_K}$. We associate to each point $(\mathcal{L}', \alpha') \in \mathcal{F}_{s'}(R)$ a diagram

$$\begin{tikzcd} \mathscr{E}_0 = \underline{M'} \arrow[r, "\Pi=\pi"] \arrow[d, "u_0=\alpha'"] & \mathscr{E}_1 = \underline{M'} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_1=\alpha'"] & \mathscr{E}_0 = \underline{M'} \arrow[d, "u_0=\alpha'"] \\ \mathscr{T}_0 = \widetilde{\mathcal{L}}' \arrow[r, "\Pi=\pi"] & \mathscr{T}_1 = \widetilde{\mathcal{L}}' \arrow[r, "\Pi=\mathrm{id}"] & \mathscr{T}_0 = \widetilde{\mathcal{L}}' \end{tikzcd}$$

and the isomorphism $r : \underline{K}^2 \xrightarrow{\sim} \underline{M'} \otimes K$ induced by the inclusion $M' \hookrightarrow K^2$. Let us check that the resulting map $\mathcal{F}_{s'}(R) \to \mathcal{F}(R)$ is well-defined.

[C1] Since $\Pi : \mathscr{T}_1 \to \mathscr{T}_0$ is the identity map, we have $S_1 = \varnothing$ and $S_0 = S$. By our definition both $\mathscr{E}_0$ and $\mathscr{E}_1$ are constant sheaves defined by M', so [C1] follows.
[C2] This is the same as what we did in (1).
[C3] Note that $\Pi \circ r(\mathcal{O}_K^2) = \pi \mathcal{O}_K^2$ be cause $\Pi = \pi$ on $\mathscr{E}_0$. Thus we have

$$\bigwedge^2 M = \mathcal{O}_K = \pi^{-1}\pi \bigwedge^2 \mathcal{O}_K^2 = \pi^{-1} \bigwedge^2 \pi\mathcal{O}_K^2 = \pi^{-1}\left(\bigwedge^2 \Pi \circ r(\mathcal{O}_K^2)\right).$$

(3) *For every edge $[s, s']$ in BT_K with s odd and s' even there is a natural transformation $\mathcal{F}_{[s,s']} \to \mathcal{F}$.*
Let $R \in \mathbf{Nilp}_{\mathcal{O}_K}$, and we are going to associate with each $(\mathcal{L}, \mathcal{L}', \alpha, \alpha', c, c') \in \mathcal{F}_{[s,s']}(R)$ a point in $\mathcal{F}(R)$ via the following constructions:

- ◇ Let S_0 be the zero locus of c', and let S_1 be the zero locus of c. Set $U_0 \subseteq S_0$ (resp. $U_1 \subseteq S_1$) to be the open subscheme such that c (resp. c') is invertible, and set $V := S_1 \cap S_2$. Since $c' \circ c = \pi$ by DEFINITION 2.4.5, REMARK 2.5.3 (c) implies

$$S = S_0 \cup S_1, \qquad S_0 = U_0 \sqcup V, \qquad S_1 = U_1 \sqcup V.$$

Inaddition, S_0, S_1 are both closed subschemes of S.

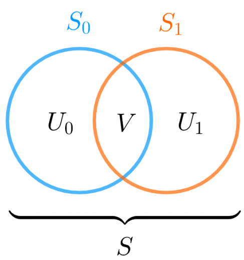

Figure 2.6: $S = S_0 \cup S_1 = (U_0 \sqcup V) \cup (U_1 \sqcup V)$.

- ◇ When we restrict to U_0, the map c is invertible and then the diagram

$$\begin{tikzcd} \mathscr{E}_0 = \underline{M'} \arrow[r, "\Pi=\pi"] \arrow[d, "u_0=\alpha'"] & \mathscr{E}_1 = \underline{M'} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_1=c^{-1}\alpha'"] & \mathscr{E}_0 = \underline{M'} \arrow[d, "u_0=\alpha'"] \\ \mathscr{T}_0 = \widetilde{\mathcal{L}}' \arrow[r, "\Pi=c'"] & \mathscr{T}_1 = \widetilde{\mathcal{L}} \arrow[r, "\Pi=c"] & \mathscr{T}_0 = \widetilde{\mathcal{L}}' \end{tikzcd}$$

together with $r : \underline{K}^2 \xrightarrow{\sim} \underline{M'} \otimes K$ induced by $M' \hookrightarrow K^2$ define a point in $\mathcal{F}(U_0)$ (one can see that this is well-defined like what we did in (2)). Thus we have $\mathcal{F}_{[s,s']}(U_0) \to \mathcal{F}(U_0)$.
- ◇ When we restrict to U_1, the map c' is invertible and then the diagram

$$\begin{tikzcd} \mathscr{E}_0 = \underline{M} \arrow[r, "\Pi=\mathrm{id}"] \arrow[d, "u_0=(c')^{-1}\alpha"] & \mathscr{E}_1 = \underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "u_1=\alpha"] & \mathscr{E}_0 = \underline{M} \arrow[d, "u_0=(c')^{-1}\alpha"] \\ \mathscr{T}_0 = \widetilde{\mathcal{L}}' \arrow[r, "\Pi=c'"] & \mathscr{T}_1 = \widetilde{\mathcal{L}} \arrow[r, "\Pi=c"] & \mathscr{T}_0 = \widetilde{\mathcal{L}}' \end{tikzcd}$$

together with $r : \underline{K}^2 \xrightarrow{\sim} \underline{M} \otimes K$ induced by $M \hookrightarrow K^2$ define a point in $\mathcal{F}(U_1)$ (one can see that this is well-defined like what we did in (1)). Thus we have $\mathcal{F}_{[s,s']}(U_1) \to \mathcal{F}(U_1)$.

◇ When we restrict to V, the diagram

$$\begin{tikzcd}
\mathscr{E}_0 = \underline{M'} \arrow[r, hook, "\Pi"] \arrow[d, "u_0=\alpha'"] & \mathscr{E}_1 = \underline{M} \arrow[r, "\Pi=\pi"] \arrow[d, "u_1=\alpha"] & \mathscr{E}_0 = \underline{M'} \arrow[d, "u_0=\alpha'"] \\
\mathscr{T}_0 = \widetilde{\mathcal{L}}' \arrow[r, "\Pi=c'"] & \mathscr{T}_1 = \widetilde{\mathcal{L}} \arrow[r, "\Pi=c"] & \mathscr{T}_0 = \widetilde{\mathcal{L}}'
\end{tikzcd}$$

together with $r : \underline{K}^2 \xrightarrow{\sim} \underline{M'} \otimes K$ induced by $M' \hookrightarrow K^2$ define a point in $\mathcal{F}(V)$ (this can be checked similarly as before).

◇ Now we are going to glue the data which we obtained previously into a point $(\mathscr{E}, \mathscr{T}, u, r)$ in $\mathcal{F}(R)$.

▷ Let $\mathscr{G}$ be the sheaf on $S_0 \sqcup S_1$ such that $\mathscr{G}|_{S_0} = \underline{M'}$ and $\mathscr{G}|_{S_1} = \underline{M}$. Write $\phi : S_0 \sqcup S_q$ for the canonical morphism. Note that $\phi_*\mathscr{G}|_V = \underline{M \oplus M'}$. In addition, we fix the following two inclusions

$$\begin{aligned} M &\hookrightarrow M \oplus M', \quad m \mapsto (m, \pi m); \\ M' &\hookrightarrow M \oplus M', \quad m' \mapsto (m', m'). \end{aligned}$$

We then view M and M' as submodules of $M \oplus M'$ via the above two inclusions. Now define $\mathscr{E}_0$ by

$$\begin{aligned} \mathscr{E}_0|_{U_0} &:= \phi_*\mathscr{G}|_{U_0}, \\ \mathscr{E}_0|_{U_1} &:= \phi_*\mathscr{G}|_{U_1}, \\ \mathscr{E}_0|_V &:= \underline{M'} \subset \underline{M \oplus M'}. \end{aligned}$$

Similarly, we define $\mathscr{E}_1$ as

$$\begin{aligned} \mathscr{E}_1|_{U_0} &:= \phi_*\mathscr{G}|_{U_0}, \\ \mathscr{E}_1|_{U_1} &:= \phi_*\mathscr{G}|_{U_1}, \\ \mathscr{E}_1|_V &:= \underline{M} \subset \underline{M \oplus M'}. \end{aligned}$$

It remains to define $\Pi : \mathscr{E}_0 \to \mathscr{E}_1$ and $\Pi : \mathscr{E}_1 \to \mathscr{E}_0$, and we will define separately on U_0, on U_1 and on V. These are given by

$$\Pi : \mathscr{E}_0 \longrightarrow \mathscr{E}_1, \quad \begin{cases} M' \xrightarrow{\pi} M' & \text{on } U_0, \\ M \xrightarrow{\mathrm{id}} M & \text{on } U_1, \\ M' \hookrightarrow M & \text{on } V, \end{cases} \qquad \Pi : \mathscr{E}_1 \longrightarrow \mathscr{E}_0, \quad \begin{cases} M' \xrightarrow{\mathrm{id}} M' & \text{on } U_0, \\ M \xrightarrow{\pi} M & \text{on } U_1, \\ M \xrightarrow{\pi} M' & \text{on } V. \end{cases}$$

▷ We set $\mathscr{T}_0 = \widetilde{\mathcal{L}}'$ and $\mathscr{T}_1 = \widetilde{\mathcal{L}}$ together with $\Pi = c' : \mathscr{T}_0 \to \mathscr{T}_1$ and $\Pi = c : \mathscr{T}_1 \to \mathscr{T}_0$.

▷ Set u_0 to be α' on S_0 and to be $(c')^{-1}\alpha$ on U_1, and set u_1 to be α on S_1 and to be $c^{-1}\alpha'$ on U_0.

▷ The isomorphism r is defined by the inclusions $M \hookrightarrow K^2$ and $M' \hookrightarrow K^2$.

2.6 $\mathrm{PGL}_2(K)$-ACTION ON $\widehat{\Omega}$

Up to now, there are two viewpoints on the formal $\mathcal{O}_K$-scheme $\widehat{\Omega}$: (1) the geometric construction given in SECTION 2.3; (2) the modular description given in SECTION 2.5. In this section, we will see how $\mathrm{PGL}_2(K)$ acts on $\widehat{\Omega}$ from these two viewpoints.

2.6.1 GEOMETRIC VIEWPOINT

The group $\mathrm{GL}_2(K)$ consisting of all invertible 2-by-2 matrices with entries in K. Equivalently, the group $\mathrm{GL}_2(K)$ contains all K-automorphisms on K^2. Therefore, $\mathrm{GL}_2(K)$ acts on the set of all $\mathcal{O}_K$-lattices of K^2. Since every element in the subgroup $K^\times \subset \mathrm{GL}_2(K)$ sends each $\mathcal{O}_K$-lattice to a homothetic $\mathcal{O}_K$-lattices, one has the $\mathrm{PGL}_2(K)$-action on the set of homothety classes of $\mathcal{O}_K$-lattices in K^2; in other words, $\mathrm{PGL}_2(K)$ acts on the set of vertices of BT_K. If M and M' are two $\mathcal{O}_K$-lattices of K^2 such that $\pi M \subsetneq M' \subsetneq M$ with a compatible basis e_1, e_2, then for any $g \in \mathrm{GL}_2(K)$ one has $gM = \langle ge_1, ge_2 \rangle_{\mathcal{O}_K}$ and $gM' = \langle ge_1, \pi(ge_2) \rangle_{\mathcal{O}_K}$. This shows $\pi(gM) \subsetneq gM' \subsetneq gM$ and ge_1, ge_2 is a compatible basis. Modulo the homothety relation, one knows $\mathrm{PGL}_2(K)$ also acts on the set of edges of BT_K. In conclusion, $\mathrm{PGL}_2(K)$ acts on the graph BT_K.

Recall from Section 2.3 that $\widehat{\Omega}$ is obtained by gluing all $(\widehat{\Omega}_{[s,s']})_{[s,s']}$ along $(\widehat{\Omega}_s)_s$. Then the $\mathrm{PGL}_2(K)$-action on BT_K induces for each $g \in \mathrm{PGL}_2(K)$:

$$g : \widehat{\Omega}_s \longmapsto \widehat{\Omega}_{gs} \quad \text{and} \quad g : \widehat{\Omega}_{[s,s']} \longmapsto \widehat{\Omega}_{[gs,gs']}.$$

One sees that the $\mathrm{PGL}_2(K)$-actions on $(\widehat{\Omega}_s)_s$ and $(\widehat{\Omega}_{[s,s']})_{[s,s']}$ are just permuting them. Therefore, we have the $\mathrm{PGL}_2(K)$-action on $\widehat{\Omega}$.

2.6.2 Categorical Viewpoint

Now we know how $\mathrm{PGL}_2(K)$ acts on $\widehat{\Omega}$, and we have seen in Section 2.5.2 the explicit map between $\mathcal{F}$ and $\widehat{\Omega}$. Combining all them together, one can see how $\mathrm{PGL}_2(K)$ acts on the functor $\mathcal{F}$. For the detailed proof please see [BC91, Proposition I.6.2].

Proposition 2.6.1. [BC91, Proposition I.6.2]
For any $g \in \mathrm{PGL}_2(K)$, any $R \in \mathbf{Nilp}_{\mathcal{O}_K}$ and for any $(\mathscr{E}, \mathscr{T}, u, r) \in \mathcal{F}(R)$ the formula

$$g \cdot (\mathscr{E}, \mathscr{T}, u, r) := ([n], \mathscr{T}[n], u[n], \Pi^n \circ r \circ g^{-1}) \quad \textit{with } n := v(\det g)$$

defines a $\mathrm{PGL}_2(K)$-action on $\mathcal{F}$ (and thus on $\widehat{\Omega}$). Here $[n]$ means to shift the $\mathbf{Z}/2\mathbf{Z}$-graded structure by $n \pmod 2$.

Chapter 3

Cartier Theory and Drinfeld's Theorem

Like in Chapter 2, we fix the following notations throughout this chapter. Notice that we further assume that the non-archimedean local field K is of characteristic 0.

Notation. K is a non-archimedean local field of characteristic 0 with the ring of integers $\mathcal{O}_K$. The maximal ideal of $\mathcal{O}_K$ is denoted by $\mathfrak{m}$ with a given uniformizer π, and the residue field of $\mathcal{O}_K$ is denoted by k. We write p for the characteristic of k and q the cardinality of k. Fix an algebraic closure $\bar{k}$ of k. We denote by v the normalized valuation on K and by $|\cdot|$ the absolute value on K (we take the standard convention $|\cdot| = q^{-v(\cdot)}$). In addition, $\mathbf{C}_K$ is the completion of the algebraic closure of K, i.e., $\mathbf{C}_K = \widehat{\overline{K}}$. As the usual convention, $\mathcal{O}_{\mathbf{C}_K}$ is the ring of integers of $\mathbf{C}_K$. We again use the notation $|\cdot|$ for the unique extension of $|\cdot|$ on K to $\mathbf{C}_K$. We write K^{nr} for the maximal unramified extension of K and write $\breve{K}$ for the completion of K^{nr} (note that π is also a uniformizer of K^{nr}; see Remark 2.2.2). Then $\mathcal{O}_{K^{\mathrm{nr}}}$ and $\mathcal{O}_{\breve{K}}$ are the ring of integers of K^{nr} and $\breve{K}$, respectively. Note that the residue field of $\mathcal{O}_{K^{\mathrm{nr}}}$ is $\bar{k}$.

Besides these, we let $\mathcal{D}$ be a quaternion algebra over K and let $\mathcal{O}_{\mathcal{D}}$ be the valuation ring of $\mathcal{D}$. Write L for the quadratic unramified extension of K containing in $\mathcal{D}$ (see Theorem A.3.2), and write $\mathcal{O}_L$ the ring of integers of L. The Galois group $\mathrm{Gal}(L/K)$ has cardinality 2, and we write σ for the non-trivial automorphism in $\mathrm{Gal}(L/K)$.

3.1 Cartier Theory for Formal Modules

Definition 3.1.1. We define the functor $\mathsf{W}_{\mathcal{O}_K} : \mathbf{Alg}_{\mathcal{O}_K} \to \mathbf{Alg}_{\mathcal{O}_K}$ by associating with each $R \in \mathbf{Alg}_{\mathcal{O}_K}$ the algebra $\mathsf{W}_{\mathcal{O}_K} := (R^{\mathbf{N}_0}, +, \cdot)$, where $R^{\mathbf{N}_0} := \prod_{i\in\mathbf{N}_0} R$ is a commutative $\mathcal{O}_K$-algebra with certain addition $+$ and multiplication $\cdot$ such that

$$\mathrm{Frob}_n : \mathsf{W}_{\mathcal{O}_K}(R) \longrightarrow R, \quad (x_0, x_1, x_2, \cdots) \longmapsto x_0^{q^n} + \pi x_1^{q^{n-1}} + \cdots + \pi^n a_n$$

is an $\mathcal{O}_K$-algebra homomorphism for every $n \in \mathbf{N}_0$. We call $\mathsf{W}_{\mathcal{O}_K}(R)$ the **ring of Witt vectors** (or the **Witt ring**) over R.

Definition 3.1.2. Let $R \in \mathbf{Alg}_{\mathcal{O}_K}$. The **Verschiebung map** is an endomorphism of $\mathsf{W}_{\mathcal{O}_K}(R)$ given by:

$$\mathsf{V} : \mathsf{W}_{\mathcal{O}_K}(R) \longrightarrow \mathsf{W}_{\mathcal{O}_K}(R), \quad x = (x_0, x_1, x_2, \cdots) \longmapsto {}^{\mathsf{V}}x := (0, x_0, x_1, \cdots).$$

In addition, there is an endomorphism $\mathsf{T} : \mathsf{W}_{\mathcal{O}_K}(R) \to \mathsf{W}_{\mathcal{O}_K}(R)$, $x \mapsto {}^{\mathsf{T}}x$ such that $\mathrm{Frob}_n \circ \mathsf{T} \equiv \mathrm{Frob}_{n+1} \pmod{\pi^{n+1}}$ for every $n \in \mathbf{N}_0$. We also define an $\mathcal{O}_K$-algebra homomorphism

$$[\cdot] : R \longrightarrow \mathsf{W}_{\mathcal{O}_K}(R), \quad a \longmapsto [a] := (a, 0, 0, \cdots).$$

Definition 3.1.3. For each $R \in \mathbf{Alg}_{\mathcal{O}_K}$ the **Dieudonné ring** associated to R is $\mathsf{W}_{\mathcal{O}_K}(R)[V, F]$, where F and V satisfy the relations

(1) $Fx = {}^{\mathsf{T}}xF$ for all $x \in \mathsf{W}_{\mathcal{O}_K}(R)$;
(2) $xV = V{}^{\mathsf{T}}x$ for all $x \in \mathsf{W}_{\mathcal{O}_K}(R)$;
(3) $VxF = {}^{\mathsf{V}}x$;
(4) $FV = \pi$.

Note that $\mathsf{W}_{\mathcal{O}_K}(R)[F, V]$ is non-commutative. A module over a Dieudonné ring is called a **Dieudonné module**.

Definition 3.1.4. Let $R \in \mathbf{Alg}_{\mathcal{O}_K}$. The ***V*-adic topology** on $\mathsf{W}_{\mathcal{O}_K}(R)[F, V]$ is the topology where the family of right ideals $(V^n)_{n\geq 0}$ forms a base of open neighborhoods of $0 \in \mathsf{W}_{\mathcal{O}_K}(R)[F, V]$. The **Cartier ring** $\mathsf{E}_{\mathcal{O}_K}(R)$ associated to R is defined to be the completion of the Dieudonné ring $\mathsf{W}_{\mathcal{O}_K}(R)[F, V]$ with respect to the V-adic topology.

Remark 3.1.5.

(a) By the explicit construction of completion every element in $\mathsf{E}_{\mathcal{O}_K}(R)$ can be written as the following form

$$\sum_{m\geq 0} V^m \tilde{a}_M \quad \text{with } \tilde{a}_m \in \mathsf{W}_{\mathcal{O}_K}(R)[F,V]/(V) \cong \mathsf{W}_{\mathcal{O}_K}(R)[F]. \tag{3.1}$$

Using the axioms in Definition 3.1.3 one can rewrite the element of the form (3.1) as

$$\sum_{m,n\geq 0} V^m[a_{m,n}]F^n \quad \text{with } a_{m,n} \in R \text{ such that } a_{m,n} = 0 \text{ for all } n \gg 0. \tag{3.2}$$

In addition, one can identify $\mathsf{W}_{\mathcal{O}_K}(R)$ as a subring of $\mathsf{E}_{\mathcal{O}_K}(R)$ via the injective $\mathcal{O}_K$-algebra homomorphism

$$\mathsf{W}_{\mathcal{O}_K}(R) \longrightarrow \mathsf{E}_{\mathcal{O}_K}(R), \quad (x_0, x_1, x_2, \cdots) \longmapsto \sum_{m\geq 0} V^m[x_m]F^m. \tag{3.3}$$

(b) There is a natural $(\mathbf{Z}/2\mathbf{Z})$-graded structure on $E_{\mathcal{O}_K}(R)$ given by

$$\deg F = \deg V = 1 \quad \text{and} \quad \deg[a] = 0 \text{ for all } a \in R.$$

From the explicit form in (3.2) the homogeneous components of $\mathsf{E}_{\mathcal{O}_K}(R)$ is given by

$$\begin{aligned}
\mathsf{E}_{\mathcal{O}_K}(R)_0 &:= \{\text{degree 0 elements}\} = \{\textstyle\sum V^m[a_{m,n}]F^n \in \mathsf{E}_{\mathcal{O}_K}(R) : m+n \text{ is even}\};\\
\mathsf{E}_{\mathcal{O}_K}(R)_1 &:= \{\text{degree 1 elements}\} = \{\textstyle\sum V^m[a_{m,n}]F^n \in \mathsf{E}_{\mathcal{O}_K}(R) : m+n \text{ is odd}\}.
\end{aligned}$$

It is clear that $\mathsf{E}_{\mathcal{O}_K}(R) = \mathsf{E}_{\mathcal{O}_K}(R)_0 \oplus \mathsf{E}_{\mathcal{O}_K}(R)_1$, and (3.3) implies $\mathsf{W}_{\mathcal{O}_K}(R) \subset \mathsf{E}_{\mathcal{O}_K}(R)_0$.

Definition 3.1.6. Let R be a commutative $\mathcal{O}_K$-algebra.

(a) A **Cartier $\mathcal{O}_K$-module over R** is a left $\mathsf{E}_{\mathcal{O}_K}(R)$-module M such that

  (1) M/VM is a free R-module of finite rank;
  (2) V is injective on M (i.e., the map $M \to M$ given by multiplication by V is injective);
  (3) M is complete with respect to the V-adic topology.

(b) A **formal $\mathcal{O}_K$-module over R** is a smooth formal group $\mathfrak{G}$ over R with an $\mathcal{O}_K$-action such that the induced $\mathcal{O}_K$-action on $\mathrm{Lie}(\mathfrak{G})$ is the same as the action of $\mathcal{O}_K$ on $\mathrm{Lie}(\mathfrak{G})$ via the R-module structure.

Theorem 3.1.7. (Cartier Theory I) [BC91, Theorem II.1.4]
Let R be an arbitrary commutative $\mathcal{O}_K$-algebra. Then there is an equivalence between the category of Cartier $\mathcal{O}_K$-modules over R and the category of formal $\mathcal{O}_K$-modules over R.

Definition 3.1.8. Let R be a commutative $\mathcal{O}_K$-algebra, and let $\Pi \in \mathcal{O}_{\mathcal{D}}$ be an element such that $\Pi^2 = \pi$ and $\Pi x = ({}^{\sigma}x)\Pi$ for ever $x \in L$.

(a) A **graded Cartier $\mathcal{O}_K[\Pi]$-module over R** is a $(\mathbf{Z}/2\mathbf{Z})$-graded Cartier $\mathcal{O}_K$-module M over R together with an $\mathsf{E}_{\mathcal{O}_K}(R)$-linear endomorphism $\Pi : M \to M$ such that $\Pi^2 = \pi$. If we write $M = M_0 \oplus M_1$ as the decomposition into homogeneous components, then M is **special** if M_0/VM_1 and M_1/VM_0 are both free R-modules of rank 1.

(b) A **formal $\mathcal{O}_{\mathcal{D}}$-module over R** is a formal $\mathcal{O}_K$-module $\mathfrak{G}$ over R together with an $\mathcal{O}_{\mathcal{D}}$-action extending the $\mathcal{O}_K$-action. Furthermore, $\mathfrak{G}$ is called **special** if $\mathrm{Lie}(\mathfrak{G})$ is a free $R \otimes_{\mathcal{O}_K} \mathcal{O}_L$-module of rank 1.

Theorem 3.1.9. (Cartier Theory II) [BC91, Theorem II.1.4]
Let R be an $\mathcal{O}_L$-algebra (in particular, R is an $\mathcal{O}_K$-algebra). Then there is an equivalence of categories between the category of graded Cartier $\mathcal{O}_K[\Pi]$-modules over R and the category of formal $\mathcal{O}_{\mathcal{D}}$-modules over R. Moreover, a graded Cartier $\mathcal{O}_K[\Pi]$-module over R is special if and only if the corresponding formal $\mathcal{O}_{\mathcal{D}}$-module over R is special.

3.2 Rigidifications

In this section, we are going to introduce some notions and summarize some properties about formal modules over $\bar{k}$. In fact, some statements in this section can be formulated in a more general setting; please see [BC91, Section II.5-II.7].

Definition 3.2.1. Let $\mathfrak{G}$ be a formal $\mathcal{O}_K$-module over $\bar{k}$. Then one can prove that its associated Cartier module (see Theorem 3.1.7) is free of finite rank over $\mathsf{W}_{\mathcal{O}_K}(\bar{k})$, and we call such rank the **height** of $\mathfrak{G}$.

Proposition 3.2.2. [BC91, Proposition II.5.1, II.5.2]

(a) *The height of every formal $\mathcal{O}_K$-module over $\bar{k}$ is divisible by 4.*

(b) *All special formal $\mathcal{O}_{\mathcal{D}}$-modules over $\bar{k}$ of height 4 are isogenous to each other.*

Notation 3.2.3. By Proposition 3.2.2 (b) we from now on fix a special formal $\mathcal{O}_{\mathcal{D}}$-module Φ over $\bar{k}$ of height 4.

Definition 3.2.4. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be two special formal $\mathcal{O}_{\mathcal{D}}$-modules over $\bar{k}$ of height 4 (note that they are isogenous by Proposition 3.2.2 (b)).

(a) A **quasi-isogeny** from $\mathfrak{G}_1$ to $\mathfrak{G}_2$ is an element in $\mathrm{Isog}_{\mathcal{O}_{\mathcal{D}}}(\mathfrak{G}_1, \mathfrak{G}_2) \otimes_{\mathcal{O}_K} K$. In other words, a map $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ is a quasi-isogeny if $\pi^n f$ is an $\mathcal{O}_{\mathcal{D}}$-isogeny for some $n \in \mathbf{Z}_{\geq 0}$.

(b) Let f be in $\mathrm{Isog}_{\mathcal{O}_{\mathcal{D}}}(\mathfrak{G}_1, \mathfrak{G}_2)$, and let N be the smallest non-negative integer such that $\pi^N f$ is an isogeny. We say f is of **height 0** if for every $n \geq N$ the rank of $\ker \pi^n f$ and $\ker[\pi^n]$ are the same.

Definition 3.2.5. Let $R \in \mathbf{Alg}_{\bar{k}}$ and $\mathfrak{G}$ a special formal $\mathcal{O}_{\mathcal{D}}$-module over R of height 4. A **rigidification** of $\mathfrak{G}$ is a quasi-isogeny $\rho : \Phi_R \to \mathfrak{G}$ of height 0, where Φ_R is the base change of Φ from $\bar{k}$ to R.

We will see in Section 3.3.1 how rigidified formal modules play important roles in the statement of Drinfeld's Theorem.

3.3 Drinfeld's Theorem

Notation 3.3.1. We introduce the following two categories of algebras (we have seen the first one in Section 2.4).

(a) Let $\mathbf{Nilp}_{\mathcal{O}_K}$ be the category of commutative $\mathcal{O}_K$-algebras such that π is a nilpotent.

(b) Let $\mathbf{Nilp}_{\mathcal{O}_{K^{\mathrm{nr}}}}$ be the category of commutative $\mathcal{O}_{K^{\mathrm{nr}}}$-algebras such that π is a nilpotent.

Definition 3.3.2.

(a) Define a functor

$$\mathcal{G} : \mathbf{Nilp}_{\mathcal{O}_K} \longrightarrow \mathbf{Set}, \quad R \longmapsto \{(\psi, \mathfrak{G}, \rho) : (\mathfrak{G}, \rho) \text{ modulo isomorphisms}\},$$

where the triple $(\psi, \mathfrak{G}, \rho)$ consists of

(1) a k-algebra homomorphism $\psi : \bar{k} \to R/\pi R$;

(2) a special formal $\mathcal{O}_{\mathcal{D}}$-module $\mathfrak{G}$ over R of height 4;

(3) a quasi-isogeny $\rho : \Phi_{R/\pi R} \to \mathfrak{G}_{R/\pi R}$ of height 0.

(b) Define a functor

$$\mathcal{G}^{\mathrm{nr}} : \mathbf{Nilp}_{\mathcal{O}_{K^{\mathrm{nr}}}} \to \mathbf{Set}, \quad R \longmapsto \{\text{isomorphism classes of } (\mathfrak{G}, \rho)\},$$

where the pair $(\mathfrak{G}, \rho)$ consists of

(1) a special formal $\mathcal{O}_{\mathcal{D}}$-module $\mathfrak{G}$ of height 4 over R;

(2) a quasi-isogeny $\rho : \Phi_{R/\pi R} \to \mathfrak{G}_{R/\pi R}$ of height 0.

The Drinfeld's Theorem (Theorem 3.3.13) states that there is some relation between $\mathcal{G}$ and the functor $\mathcal{F}$ that we studied in Section 2.5. In fact, we would like to work over $\mathcal{O}_{K^{\mathrm{nr}}}$ instead of over $\mathcal{O}_K$. Thus we should construct a quadruple $(\mathscr{E}, \mathscr{T}, u, r)$ from a pair $(\mathfrak{G}, \rho)$. This is our task in the following subsection.

3.3.1 Constructions

Definition 3.3.3. Let R be an algebra over $\mathcal{O}_L$. We define the non-commutative $\mathcal{O}_L$-algebra $\mathsf{W}_{\mathcal{O}_K}(R)[V,\Pi]$ such that V,Π satisfy the following relations:

(1) $\Pi V = V\Pi$;

(2) $\Pi x = x\Pi$ for all $x \in \mathsf{W}_{\mathcal{O}_K}(R)$;

(3) $xV = V^{\mathsf{V}}x$ for all $x \in \mathsf{W}_{\mathcal{O}_K}(R)$;

(4) $\Pi^2 = \pi$.

We set $\mathsf{E}_{\mathcal{O}_L}(R)$ to be the completion of $\mathsf{W}_{\mathcal{O}_K}[V,\Pi]$ with respect to the V-adic topology.

Similar to $\mathsf{E}_{\mathcal{O}_K}(R)$ there is a $\mathbf{Z}/2\mathbf{Z}$-graded structure on $\mathsf{E}_{\mathcal{O}_L}(R)$ by setting

$$\deg V = \deg \Pi = 1 \quad \text{and} \quad \deg x = 0 \text{ for all } x \in \mathsf{W}_{\mathcal{O}_K}(R).$$

Note also that for each $R \in \mathbf{Alg}_{\mathcal{O}_L}$ every graded $\mathcal{O}_K[\Pi]$-module R is naturally a graded $\mathsf{E}_{\mathcal{O}_L}(R)$-module by omitting the F-action. Recall from Definition 3.1.2 that we have an $\mathcal{O}_K$-algebra endomorphism $\mathcal{W}_{\mathcal{O}_K}(R) \to \mathsf{W}_{\mathcal{O}_K}(R)$ for every $R \in \mathbf{Alg}_{\mathcal{O}_K}$.

Definition 3.3.4. Let R be an algebra over $\mathcal{O}_L$.

(a) For any $\mathsf{W}_{\mathcal{O}_K}(R)$-module M we define M^{T} to be the $\mathsf{W}_{\mathcal{O}_K}(R)$-module M twisted by T; that is, the underlying set of M and M^{T} are the same, and for any $x \in \mathsf{W}_{\mathcal{O}_K}(R)$ and any $\alpha \in M^{\mathsf{T}}$ the action is given by

$$x \cdot \alpha := \mathsf{T}(x) \cdot \alpha.$$

The $\cdot$ on the left hand side is the $\mathsf{W}_{\mathcal{O}_K}(R)$-action on M^{T}, and the $\cdot$ on the right hand side is the $\mathsf{W}_{\mathcal{O}_K}(R)$-action on M. It is clear that M^{T} is again a $\mathsf{W}_{\mathcal{O}_K}(R)$-module. In addition, if M is an $\mathsf{E}_{\mathcal{O}_L}(R)$-module, then one can check that M^{T} is also an $\mathsf{E}_{\mathcal{O}_L}(R)$-module.

(b) Let M be an $\mathsf{E}_{\mathcal{O}_L}(R)$-module. Define the $\mathsf{E}_{\mathcal{O}_L}(R)$-module $\mathsf{N}(M)$ via the short exact sequence

$$0 \longrightarrow M^{\mathsf{T}} \xrightarrow{\iota_M} M \oplus M^{\mathsf{T}} \longrightarrow \mathsf{N}(M) \longrightarrow 0,$$

where $\iota_M : \alpha \mapsto (V\alpha, -\Pi\alpha)$ is an injection by (2) of Definition 3.1.6 (a). That is, $\mathsf{N}(M) = (M \oplus M^{\mathsf{T}})/\iota_M(M^{\mathsf{T}})$.

It follows from our construction that every $\mathsf{E}_{\mathcal{O}_L}(R)$-module homomorphism $M \to M'$ induces an $\mathsf{E}_{\mathcal{O}_L}(R)$-module homomorphism $\mathsf{N}(M) \to \mathsf{N}(M')$. If $M = M_0 \oplus M_1$ is a graded $\mathsf{E}_{\mathcal{O}_L}(R)$-module, then M^{T} is also a graded $\mathsf{E}_{\mathcal{O}_L}(R)$-module with the decomposition $M^{\mathsf{T}} = M_0^{\mathsf{T}} \oplus M_1^{\mathsf{T}}$. Moreover, $\mathsf{N}(M)$ is a graded $\mathsf{E}_{\mathcal{O}_L}(R)$-module with the decomposition

$$\mathsf{N}(M) = (M_0 \oplus M_0^{\mathsf{T}})/\iota_M(M^{\mathsf{T}}) \oplus (M_1 \oplus M_1^{\mathsf{T}})/\iota_M(M^{\mathsf{T}}).$$

In next lemma, we are going to construct a map $\mathsf{L}_M : M \to \mathsf{N}(M)$. To do this, we need to express each element in M in a concrete form.

Remark 3.3.5. Let $R \in \mathbf{Alg}_{\mathcal{O}_L}$, and let M be a special graded Cartier $\mathcal{O}_K[\Pi]$-module over R. From Definition 3.1.6 and 3.1.8 we know that M/VM is a free $\mathbf{Z}/2\mathbf{Z}$-graded R-module. Then we can take a homogeneous R-basis $\{\gamma_0,\gamma_1\} \subset M$. Namely, $\deg \gamma_i = i$ ($i = 0,1$) and every $\alpha \in M$ can be written uniquely as $\alpha \equiv [a_0]\gamma_0 + [a_1]\gamma_1 \mod VM$ with $a_0, a_1 \in R$. We call such basis $\{\gamma_0,\gamma_1\}$ a **homogeneous V-basis**. As a result, every $\alpha \in M$ can be uniquely written as

$$\alpha = [a_0]\gamma_0 + [a_1]\gamma_1 + V\alpha' \quad \text{with } a_0, a_1 \in R \text{ and } \alpha' \in M.$$

Lemma 3.3.6. [BC91, Proposition II.3.8]
Let $R \in \mathbf{Alg}_{\mathcal{O}_L}$, and let M be a special graded Cartier $\mathcal{O}_K[\Pi]$-module over R. Then there exists a unique map $\mathsf{L}_M : M \to \mathsf{N}(M)$ satisfying the following properties:

(1) *for any $\mathcal{O}_L$-algebra homomorphism $R \to R'$ the diagram*

$$\begin{tikzcd} M \arrow[r, "\mathsf{L}_M"] \arrow[d] & \mathsf{N}(M) \arrow[d] \\ M' \arrow[r, "\mathsf{L}_{M'}"] & \mathsf{N}(M') \end{tikzcd}$$

commutes, where $M' := M \widehat{\otimes}_{\mathsf{E}_{\mathcal{O}_L}(R)} \mathsf{E}_{\mathcal{O}_L}(R')$.

(2) *$F = \lambda_M \circ \mathsf{L}_M$, where $\lambda_M : \mathsf{N}(M) \to M$ is the $\mathsf{E}_{\mathcal{O}_L}(R)$-linear map induced by $M \oplus M^{\mathsf{T}} \to M$, $(\alpha, \beta) \mapsto \Pi\alpha + V\beta$* [1]*. Moreover, L_M can be described explicitly as what follows. Let $\{\gamma_0, \gamma_1\}$ be a homogeneous V-basis for M, and let $\alpha = [a_0]\gamma_0 + [a_1]\gamma_1 + V\alpha'$ be an element in M (same notations as in* Remark 3.3.5*). Then*

$$\mathsf{L}_M(\alpha) = [a_0^q]\mathsf{L}_M(\gamma_0) + [a_1^q]\mathsf{L}_M(\gamma_1) + [(\Pi\alpha', 0)], \tag{3.4}$$

where $[(\Pi\alpha', 0)]$ is the class in $\mathsf{N}(M)$ represented by $(\Pi\alpha', 0) \in M \oplus M^{\mathsf{T}}$.

Definition 3.3.7. Let R be an $\mathcal{O}_L$-algebra, and let M be a special graded Cartier $\mathcal{O}_K[\Pi]$-module over R.

(a) The map $\phi_M : \mathsf{N}(M) \to \mathsf{N}(M)$ is the graded $\mathsf{E}_{\mathcal{O}_L}(R)$-module endomorphism induced by the map

$$M \oplus M^{\mathsf{T}} \longrightarrow \mathsf{N}(M), \quad (\alpha, \beta) \longmapsto \mathsf{L}_M(\alpha) + [(\beta, 0)]\text{[2]},$$

where $[(\beta, 0)]$ is the class in $\mathsf{N}(M)$ represented by $(\beta, 0) \in M \oplus M^{\mathsf{T}}$.

(b) Define the graded $\mathcal{O}_K[\Pi]$-module $\mathscr{E}_M^{\square}$ by

$$\mathscr{E}_M^{\square} := \{z \in \mathsf{N}(M) : \phi_M(z) = z\}.$$

Up to now, to each $R \in \mathbf{Alg}_{\mathcal{O}_L}$ and each special graded Cartier $\mathcal{O}_K[\Pi]$-module M over R we associate a graded $\mathcal{O}_K[\Pi]$-module $\mathscr{E}_M^{\square}$. We want to use this to define a sheaf on $\operatorname{Spec} R$. To do this, we just need to define on the principal open subsets of $\operatorname{Spec} R$, i.e., on open subsets of the form $\operatorname{Spec} R_f$ with $f \in R$ (note that each R_f is an R-algebra). This leads to the following definition.

Definition 3.3.8. Let R be an $\mathcal{O}_L$-algebra, let $\mathfrak{G}$ be a special formal $\mathcal{O}_{\mathcal{D}}$-module over R of height 4, and let M be the corresponding special graded Cartier $\mathcal{O}_K[\Pi]$-module over R.

(a) The functor

$$\mathbf{Alg}_R \longrightarrow \{\text{graded } \mathcal{O}_K[\Pi]\text{-module}\}, \quad R' \longmapsto \mathscr{E}_{M \otimes_R R'}^{\square}$$

defines a sheaf of graded $\mathcal{O}_K[\Pi]$-module on $\operatorname{Spec} R$, and we denote such sheaf by $\mathscr{E}_{\mathfrak{G}}$.

(b) Define $\mathscr{T}_{\mathfrak{G}}$ to be the $\mathbf{Z}/2\mathbf{Z}$-graded $\mathscr{O}_R[\Pi]$-module $(M/VM)^{\sim}$.

(c) For each R-algebra R' we have a map $\mathscr{E}_{M \otimes_R R'}^{\square} \to (M/VM) \otimes_R R'$ by the following composition: (write $M' := M \otimes_R R'$)

$$\mathscr{E}_{M'}^{\square} \hookrightarrow \mathsf{N}(M') \longrightarrow M'/VM' = (M/VM) \otimes_R R',$$

where the first arrow follows from Definition 3.3.7, and the second arrow is the map induced by $M' \oplus (M')^{\mathsf{T}} \to M/VM$, $(\alpha, \beta) \mapsto \alpha \mod VM$. This defines an $\mathcal{O}_K$-linear morphism $u_{\mathfrak{G}} : \mathscr{E}_{\mathfrak{G}} \to \mathscr{T}_{\mathfrak{G}}$ of degree 0.

Proposition 3.3.9. [BC91, Proposition II.5.5, II.5.6]
Let R be an $\mathcal{O}_L$-algebra, let $\mathfrak{G}$ be a special formal $\mathcal{O}_{\mathcal{D}}$-module over R of height 4, and let M be the corresponding special graded Cartier $\mathcal{O}_K[\Pi]$-module over R.

(a) *We write $\mathscr{E}_{\mathfrak{G}}^{\square} = \mathscr{E}_{\mathfrak{G},0}^{\square} \oplus \mathscr{E}_{\mathfrak{G},1}^{\square}$ for the homogeneous decomposition. Then $\mathscr{E}_{\mathfrak{G},i}^{\square}$ is a free $\mathcal{O}_K$-module of rank 2 for every $i \in \{0, 1\}$.*

(b) *The map $\mathscr{E}_{\mathfrak{G}}^{\square}/\Pi\mathscr{E}_{\mathfrak{G}}^{\square} \to (M/VM)/\Pi(M/VM)$ induced by $u_{\mathfrak{G}}$ is injective.*

Definition 3.3.10. Let $R \in \mathbf{Alg}_{\mathcal{O}_L}$ such that $\pi R = 0$, and let $M = M_0 \oplus M_1$ be a graded $\mathcal{O}_K[\Pi]$-module over R. We say $i \in \{0, 1\}$ is **critical** if $\Pi M_i \subseteq VM_i$.

Notations are the same as the above definition. If $\mathfrak{G}$ is the corresponding formal $\mathcal{O}_{\mathcal{D}}$-module over R. We say i is critical for $\mathfrak{G}$ over R if and only if i is critical for M over R.

---

[1] Note that $\iota_M(M^{\mathsf{T}})$ lies in the kernel of $M \oplus M^{\mathsf{T}} \to M$, $(\alpha, \beta) \mapsto \Pi\alpha + V\beta$. Thus $\mathsf{N}(M) \to M$ is well-defined.

[2] By Lemma 3.3.6, particularly (3.4), $\mathsf{L}_M(V\alpha) = [(\Pi\alpha, 0)]$ for any $\alpha \in M$. Thus for any $(V\alpha, -\Pi\alpha) \in \iota_M(M^{\mathsf{T}})$ we have

$$(V\alpha, -\Pi\alpha) \longmapsto \mathsf{L}_M(V\alpha) + [(-\Pi\alpha)] = [(\Pi\alpha, 0)] + [(-\Pi\alpha)] = 0.$$

Hence ϕ_M is well-defined.

Proposition 3.3.11. [BC91, Proposition II.7.5]
Let $R \in \mathbf{Alg}_{\bar{k}}$ and $\mathfrak{G}$ a rigidified special formal $\mathcal{O}_{\mathcal{D}}$-module over R of height 4 with the rigidification $\rho : \Phi_B \to \mathfrak{G}$. Set $S := \operatorname{Spec} R$, and set S_i to be the closed subscheme of S such that i is critical for $\mathfrak{G}$ over S_i[3].

(a) *For each $i \in \{0,1\}$ the sheaf $\mathscr{E}_{\mathfrak{G},i}$ is constructible with respect to the Zariski topology on S, and the restriction $\mathscr{E}_{\mathfrak{G},i}|_{S_i}$ is a constant sheaf.*

(b) *We have an isomorphism $r_{(\mathfrak{G},\rho)} : \underline{K}^2 \xrightarrow{\sim} \mathscr{E}_{\mathfrak{G},0} \otimes_{\mathcal{O}_K} K$ such that the degree $[\mathscr{E}_{\mathfrak{G},i}|_{S_i} : \Pi^i r(\underline{\mathcal{O}_K}^2)]$ is i for each $i \in \{0,1\}$.*

3.3.2 The Statement

Theorem 3.3.12. (Drinfeld) [BC91, Theorem II.8.2, II.8.5]
Let $\mathcal{F}^{\mathrm{nr}}$ be the restriction of the functor $\mathcal{F}$ to $\mathbf{Nilp}_{\mathcal{O}_{K^{\mathrm{nr}}}}$. Since $\mathcal{F}$ is represented by the formal $\mathcal{O}_K$-scheme $\widehat{\Omega}$, one deduces that $\mathcal{F}^{\mathrm{nr}}$ is represented by the formal $\mathcal{O}_{\breve{K}}$-scheme $\widehat{\Omega}\, \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}$ (see Definition 2.5.2 *and* Theorem 2.5.4*). Then the natural transformation $\mathcal{G}^{\mathrm{nr}} \to \mathcal{F}^{\mathrm{nr}}$ given by*

$$\mathcal{G}^{\mathrm{nr}}(R) \longrightarrow \mathcal{F}^{\mathrm{nr}}(R), \quad (\mathfrak{G}, \rho) \longmapsto (\mathscr{E}_{\mathfrak{G}}, \mathscr{T}_{\mathfrak{G}}, u_{\mathfrak{G}}, r_{(\mathfrak{G},\rho)}) \quad \text{for each } R \in \mathbf{Nilp}_{\mathcal{O}_{K^{\mathrm{nr}}}}$$

is an isomorphism, and thus $\mathcal{G}^{\mathrm{nr}}$ is represented by the formal $\mathcal{O}_{\breve{K}}$-scheme $\widehat{\Omega}\, \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}$.

Recall that the definition of $(\mathscr{E}_{\mathfrak{G}}, \mathscr{T}_{\mathfrak{G}}, u_{\mathfrak{G}}, r_{(\mathfrak{G},\rho)})$ is given in Definition 3.3.8 and Proposition 3.3.11 (b), and the map $\mathcal{G}^{\mathrm{nr}}(R) \to \mathcal{F}^{\mathrm{nr}}(R)$ is well-defined by Proposition 3.3.9 and Proposition 3.3.11.

For $R \in \mathbf{Nilp}_{\mathcal{O}_K}$ a k-algebra homomorphism $\psi : \bar{k} \to R/\pi R$ can be lifted to a unique $\mathcal{O}_K$-algebra homomorphism $\tilde{\psi} : \mathcal{O}_{K^{\mathrm{nr}}} \to R$. Since $\mathcal{O}_{\breve{K}}$ is the completion of $\mathcal{O}_{K^{\mathrm{nr}}}$, the map $\tilde{\psi}$ can be extended uniquely to an $\mathcal{O}_K$-algebra homomorphism $\mathcal{O}_{\breve{K}} \to R$, which is again denoted by $\tilde{\psi}$. One can view R as an $\mathcal{O}_{\breve{K}}$-algebra, via $\tilde{\psi}$ and we denote it by $R_{\tilde{\psi}}$. Then for any $A \in \mathbf{Nilp}_{\mathcal{O}_K}$ we have

$$\operatorname{Hom}_{\mathcal{O}_K,\mathrm{cont}}(A \,\widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}, R) \longrightarrow \coprod_{\psi : \bar{k} \to R/\pi R} \operatorname{Hom}_{\mathcal{O}_{\breve{K}},\mathrm{cont}}(A \,\widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}, R_{\tilde{\psi}}). \tag{3.5}$$

It is obvious that (3.5) is injective. If we have a map $\vartheta \in \operatorname{Hom}_{\mathcal{O}_{\breve{K}},\mathrm{cont}}(A \,\widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}, R_{\tilde{\psi}})$, the following commutative diagram

$$\begin{tikzcd}
A \,\widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}} \arrow[r, "\vartheta"] & R_{\tilde{\psi}} \\
\mathcal{O}_{\breve{K}} \arrow[u] \arrow[r, "\mathrm{id}"] & \mathcal{O}_{\breve{K}} \arrow[u, "\tilde{\psi}"'] \\
\mathcal{O}_K \arrow[u] \arrow[r, "\mathrm{id}"] & \mathcal{O}_K \arrow[u]
\end{tikzcd}$$

indicates that ϑ is also a map in $\operatorname{Hom}_{\mathcal{O}_K,\mathrm{cont}}(A \,\widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}, R)$. Therefore, (3.5) is surjective and thus an isomorphism. By passing from local to global and using Theorem 3.3.12 we then obtain

$$\mathcal{G}(R) = \coprod_{\psi : \bar{k} \to R/\pi R} \mathcal{G}(R_{\tilde{\psi}}) = \coprod_{\psi : \bar{k} \to R/\pi R} \operatorname{Hom}_{\mathcal{O}_{\breve{K}}}(\operatorname{Spf} R_{\tilde{\psi}}, \widehat{\Omega}\, \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}) = \operatorname{Hom}_{\mathcal{O}_K}(\operatorname{Spf} R, \widehat{\Omega}\, \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}).$$

Therefore, we obtain the representability of the functor $\mathcal{G}$.

Theorem 3.3.13. (Drinfeld) [BC91, Theorem II.8.4]
The functor $\mathcal{G}$ is represented by the formal $\mathcal{O}_K$-scheme $\widehat{\Omega}\, \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\breve{K}}$.

[3] If we write $S_i = \operatorname{Spec} R/I_i$ for some ideal $I_i \subseteq R$, then i being critical for $\mathfrak{G}$ over S_i means that i is critical for $\mathfrak{G}$ over R/I_i.

CHAPTER 4

CHEREDNIK-DRINFELD THEOREM ON UNIFORMIZATION OF SHIMURA CURVES

In CHAPTER 2 and CHAPTER 3 our discussions are based on a given non-archimedean local field K. In this chapter the field K will be the field of p-adic numbers $\mathbf{Q}_p$, and the rational prime p will be fixed throughout this chapter in NOTATION 4.1.2. Thus $\Omega = \mathbf{P}^1(\mathbf{C}_p) - \mathbf{P}^1(\mathbf{Q}_p)$, and $\breve{\mathbf{Z}}_p$ is the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$.

4.1 SHIMURA CURVES

In this section, we are going to define a Shimura curve, the geometric object that we want to study. Roughly speaking, a Shimura curve is a geometric object represented a moduli functor of certain family of abelian schemes. That is, a Shimura curve parametrize a family of abelian schemes.

LEMMA 4.1.1. *Let Δ be a quaternion algebra over $\mathbf{Q}$. We associate with Δ a functor*

$$\mathbf{G}_\Delta : \mathbf{Alg}_\mathbf{Q} \longrightarrow \mathbf{Grp}, \quad R \longmapsto (\Delta \otimes_\mathbf{Q} R)^\times.$$

Then this defines a reductive group $\mathbf{G}_\Delta$ over $\mathbf{Q}$.

PROOF. Since $\overline{\mathbf{Q}}$ is algebraically closed, every number in $\mathbf{Q}$ has a square root in $\overline{\mathbf{Q}}$. Therefore, $\Delta \otimes_\mathbf{Q} \overline{\mathbf{Q}} \cong M_2(\overline{\mathbf{Q}})$ by EXAMPLE A.1.6 and REMARK A.1.5. This shows that $\mathbf{G}_{\Delta,\overline{\mathbf{Q}}} \cong \mathrm{GL}_{2,\overline{\mathbf{Q}}}$. By EXAMPLE B.4.5, we know that G_Δ is a reductive group. ■

Let $\mathcal{P}(\mathbf{Q})$ be the set of all places of $\mathbf{Q}$. We know

$$\mathcal{P}(\mathbf{Q}) = \{|\cdot|_p, |\cdot| = |\cdot|_\infty : p \text{ runs through all rational primes}\}.$$

We call $|\cdot|_p$'s finite places and $|\cdot| = |\cdot|_\infty$ the infinite place. For a place $v \in \mathcal{P}(\mathbf{Q})$ we write $v \nmid \infty$ if v is a finite place and write $v \mid \infty$ (or $v = \infty$) otherwise.

Let $\mathbf{A}_\mathbf{Q}$ be the **adele ring** of $\mathbf{Q}$. Recall by definition that

$$\mathbf{A}_\mathbf{Q} = \left\{ (x_v)_v \in \prod_{v \in \mathcal{P}(\mathbf{Q})} \mathbf{Q}_v \,\middle|\, x_v \in \mathcal{O}_{\mathbf{Q}_v} \text{ for all but finitely many } v \right\},$$

where $\mathbf{Q}_v$ is the completion of $\mathbf{Q}$ with respect to v. Note that for $v \nmid \infty$ the ring $\mathcal{O}_{\mathbf{Q}_v}$ is just the ring of integers of the local field $\mathbf{Q}_v$; for $v \mid \infty$ we take $\mathcal{O}_{\mathbf{Q}_v} = \mathbf{Q}_v$. In other words, $\mathbf{A}_\mathbf{Q}$ is the restricted product of $(\mathbf{Q}_v)_{v\in\mathcal{P}(\mathbf{Q})}$ with respect to $(\mathcal{O}_{\mathbf{Q}_v})_{v\in\mathcal{P}(\mathbf{Q})}$. Set $\mathcal{P}_{\mathrm{fin}}(\mathbf{Q})$ to be the set of all finite places of $\mathbf{Q}$. The **finite adele ring** is

$$\mathbf{A}_{\mathbf{Q},\mathrm{fin}} = \left\{ (x_v)_v \in \prod_{v \in \mathcal{P}_{\mathrm{fin}}(\mathbf{Q})} \mathbf{Q}_v \,\middle|\, x_v \in \mathcal{O}_{\mathbf{Q}_v} \text{ for all but finitely many } v \right\}.$$

In other words, $\mathbf{A}_{\mathbf{Q},\mathrm{fin}}$ is the restricted product of $(\mathbf{Q}_v)_{v\in\mathcal{P}_{\mathrm{fin}}(\mathbf{Q})}$ with respect to $(\mathcal{O}_{\mathbf{Q}_v})_{v\in\mathcal{P}_{\mathrm{fin}}(\mathbf{Q})}$. In addition, we define for each rational prime p the ring

$$\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}} = \left\{ (x_v)_v \in \prod_{v \in \mathcal{P}_{\mathrm{fin}}(\mathbf{Q}) - \{|\cdot|_p\}} \mathbf{Q}_v \,\middle|\, x_v \in \mathcal{O}_{\mathbf{Q}_v} \text{ for all but finitely many } v \right\}.$$

NOTATION 4.1.2. We fix some notations throughout this chapter. We fix an indefinite division quaternion algebra Δ over $\mathbf{Q}$, and we fix a maximal order $\mathcal{O}_\Delta$ of Δ (note that all maximal orders of Δ are conjugate). Write δ for the product of all rational primes where Δ is ramified[1], and fix a prime $p \in \mathbf{N}$ such that $p \mid \delta$. Let $\mathbf{U} \subseteq \mathbf{G}_\Delta(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$ be a compact open subgroup such that $\mathbf{U} = \mathbf{U}_p\mathbf{U}^p$ with $\mathbf{U}_p \subset \mathbf{G}_\Delta(\mathbf{Q}_p)$ the unique maximal compact subgroup[2] and $\mathbf{U}^p \subseteq \mathbf{G}_\Delta(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ a compact open subgroup. Moreover, we always assume $\mathbf{U}^p$ is small enough[3].

THEOREM 4.1.3. [BC91, THEOREM III.1.2]
For $\mathbf{U}$ as in NOTATION 4.1.2 we define a functor

$$\mathcal{M}_{\mathbf{U},\mathbf{C}} : \mathbf{Sch}_{\mathbf{C}} \longrightarrow \mathbf{Set}, \quad S \longmapsto \mathcal{M}_{\mathbf{U},\mathbf{C}}(S) = \{\textit{isomorphism classes of } (A, \iota, \bar{\nu})\},$$

where $(A, \iota, \bar{\nu})$ is a triple consisting of

(1) *an abelian scheme A of relative dimension 2 over S;*
(2) *a ring homomorphism $\mathcal{O}_\Delta \to \mathrm{End}_S(A)$;*
(3) *a level $\mathbf{U}$-structure $\bar{\nu}$ on A. Explicitly, if n is a positive integer such that $\mathbf{U}(n) := \{g \in (\mathcal{O}_\Delta \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times : g \cong 1 \pmod{n}\} \subseteq \mathbf{U}$, then there exists an étale covering $(U_i)_i$ of S together with an $\mathcal{O}_\Delta$-linear isomorphism $(A \times_S U_i)[n] \xrightarrow{\sim} \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z})$ as group schemes over U_i for each i and these isomorphisms are compatible; that is, for any i, j the diagram commutes:*

$$\begin{tikzcd}
(A \times_S U_i)[n] \arrow[r, "\sim"] & \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) \\
(A \times_S (U_i \times_S U_j))[n] \arrow[r, "\sim"] \arrow[u] \arrow[d] & \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) \arrow[u, "\mathrm{pr}_i"'] \arrow[d, "\mathrm{pr}_j"] \\
(A \times_S U_j)[n] \arrow[r, "\sim"] & \mathcal{O}_\Delta \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}),
\end{tikzcd}$$

where pr_i and pr_j are induced by the natural projections $U_i \times_S U_j \to U_i$ and $U_i \times_S U_j \to U_j$, respectively.

*Then the functor $\mathcal{M}_{\mathbf{U},\mathbf{C}}$ is represented by a $\mathbf{C}$-scheme, and we call such scheme the **Shimura curve associated to $\mathbf{U}$ over $\mathbf{C}$** and denote it by $\mathrm{Sh}_{\mathbf{U},\mathbf{C}}$.*

REMARK 4.1.4. Let us say some words about the above theorem.

(a) Let $f : X \to Y$ be a morphism of schemes, and let $d \in \mathbf{Z}_{\geq 0}$. The morphism f is **smooth of relative dimension d** at a point $x \in X$ if there are open neighborhoods U and $V := \mathrm{Spec}\, R$ of x and $f(x)$, respectively, such that $f(U) \subseteq V$ and there exists an open immersion $U \hookrightarrow \mathrm{Spec}\, R[T_1, \cdots, T_n]/(f_1, \cdots, f_{n-d})$ for some $n \geq d$ and some $f_1, \cdots, f_{n-d} \in R[T_1, \cdots, T_n]$ satisfying

$$\left(\frac{\partial f_i}{\partial T_j}(x)\right)_{i,j} \in M_{(n-d)\times n}(\mathbf{k}(x)) \text{ has rank } n - d.$$

We say f is smooth of relative dimension d if it is so at every point of X.

(b) A morphism $f : X \to Y$ is **unramified** at a point $x \in X$ if it satisfies two conditions:
 (1) f is locally of finite presentation (thus f is locally of finite type);
 (2) the ring $\mathscr{O}_{X,x}/\mathfrak{m}_{Y,f(x)}\mathscr{O}_{X,x}$ is a field and is finite separable over $\mathbf{k}(f(x))$.

We say f is an unramified morphism if it is unramified at every point of X.

(c) A morphism $f : X \to Y$ of schemes is **flat** at $x \in X$ if $\mathscr{O}_{X,x}$ is a flat $\mathscr{O}_{Y,f(x)}$-module. We say f is flat if it is flat at every point of X.

[1]We note that any division quaternion algebra over $\mathbf{Q}$ cannot be unramified at all finite places. This can be proved by a computation via Hilbert symbol; see [Vo21, SECTION 12.4].

[2]To be more concrete, the group $\mathbf{U}_p$ is the group of units of the unique maximal order of $\Delta \otimes_{\mathbf{Q}} \mathbf{Q}_p$.

[3]This is a technical condition which assures the moduli problems (see THEOREM 4.1.3 and 4.1.5) is a fine moduli problem. We will not discuss what "small enough" exactly means. In conclusion, if $\mathbf{U}^p$ is small enough, then $\mathbf{U}$ is also small enough. For each $n \in \mathbf{N}$ set $\mathbf{U}(n)$ to be the subgroup of $(\mathcal{O}_\Delta \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times$ consisting of elements congruent to 1 modulo n. For example, if $\mathbf{U} \subseteq \mathbf{U}(n)$ for some $n \geq 3$, then $\mathbf{U}$ is small enough.

(d) A morphism $f : X \to Y$ of schemes is called **étale** if it is smooth of relative dimension 0. Here we quote some equivalent statements for étale morphisms (see [EGAIV$_4$, COROLLAIRE 17.6.2]).
Let $f : X \to Y$ be a morphism of schemes, and let $x \in X$ be a point. Then the following statements are equivalent.

(1) *f is a étale at x;*
(2) *f is smooth and unramified at x;*
(3) *f is flat and unramified at x (note that generally a morphism smooth at x is also flat at x).*

There is another characterization of étale morphisms called INFINITESIMAL LIFTING PROPERTY (see THEOREM 4.4.6). This property will be used in the proof of CHEREDNIK-DRINFELD THEOREM (THEOREM 4.2.2).

(e) [EGAIV$_4$, PROPOSITION 17.6.3] *Let $f : X \to Y$ be a morphism locally of finite type, and suppose Y is locally noetherian. If $x \in X$ is a point such that $\mathbf{k}(x) \cong \mathbf{k}(f(x))$, then f is étale at x if and only if $\widehat{\mathscr{O}}_{X,x} \cong \widehat{\mathscr{O}}_{Y,f(x)}$.*

THEOREM 4.1.5. [BC91, DEFINITION III.3.2, THEOREM III.3.6]
For $\mathbf{U}$ as in NOTATION 4.1.2 we define a functor

$$\mathcal{M}_{\mathbf{U},\mathbf{Z}_p} : \mathbf{Sch}_{\mathbf{Z}_p} \longrightarrow \mathbf{Set}, \quad S \longmapsto \{\textit{isomorphism classes of } (A, \iota, \bar{\nu})\},$$

where $(A, \iota, \bar{\nu})$ is a triple consisting of

(1) *an abelian scheme A of relative dimension 2 over S;*
(2) *a ring homomorphism $\iota : \mathcal{O}_\Delta \to \mathrm{End}_S(A)$ such that the following statement holds.*
*Let $\mathbf{Z}_p^{(2)}$ be the ring of integers of the unique quadratic unramified extension of $\mathbf{Q}_p$, and we view $\mathbf{Z}_p^{(2)}$ as a subring of $\mathcal{O}_{\Delta_p}$ (up to conjugation). We ask that for each geometric point $x = \mathrm{Spec}\, F$ of S (F is an algebraically closed field) the $\mathbf{Z}_p^{(2)}$-action on $\mathrm{Lie}(A_x)$ decomposes into the sum of the two injections $\mathbf{Z}_p^{(2)} \otimes_{\mathbf{Z}_p} \mathbf{F}_p \xrightarrow{\sim} \mathbf{F}_{p^2} \hookrightarrow F$. In this case, we call the $\mathcal{O}_\Delta$-action on A defined by ι is **special**.*
(3) *a level $\mathbf{U}$-structure $\bar{\nu}$ on A.*

*Then the functor $\mathcal{M}_{\mathbf{U},\mathbf{Z}_p}$ is represented by a projective scheme $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$ over $\mathbf{Z}_p$, which is called the **Shimura curve associated to $\mathbf{U}$ over $\mathbf{Z}_p$**. Moreover, the generic fibre of $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$ is $\mathrm{Sh}_{\mathbf{U},\mathbf{C}} \otimes_{\mathbf{Q}} \mathbf{Q}_p$; that is, $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \cong \mathrm{Sh}_{\mathbf{U},\mathbf{C}} \otimes_{\mathbf{Q}} \mathbf{Q}_p$ as schemes over $\mathbf{Q}_p$.*

4.2 THE MAIN THEOREM

DEFINITION 4.2.1. Let notations be as in NOTATION 4.1.2. We define $\bar{\Delta}$ to be the quaternion algebra over $\mathbf{Q}$ such that[4]

$$\bar{\Delta}_v \cong \begin{cases} \text{a division quaternion } \mathbf{R}\text{-algebra} & \text{if } v = \infty; \\ M_2(\mathbf{Q}_p) & \text{if } v = p; \\ \Delta_v & \text{if } v \text{ is a rational prime coprime to } p. \end{cases}$$

Like in LEMMA 4.1.1, we can associate $\bar{\Delta}$ a functor

$$\mathbf{G}_{\bar{\Delta}} : \mathbf{Alg}_{\mathbf{Q}} \longrightarrow \mathbf{Grp}, \quad R \longmapsto (\bar{\Delta} \otimes_{\mathbf{Q}} R)^\times.$$

Again, this defines a reductive group $\mathbf{G}_{\bar{\Delta}}$ over $\mathbf{Q}$.

THEOREM 4.2.2. (CHEREDNIK-DRINFELD) [BC91, THEOREM III.5.3]
For $\mathbf{U}$ as in NOTATION 4.1.2 set $Z_{\mathbf{U}} := \mathbf{U}^p \backslash \mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})/\mathbf{G}_{\bar{\Delta}}(\mathbf{Q})$. Then there is an isomorphism of formal $\mathbf{Z}_p$-schemes

$$\widehat{\mathrm{Sh}}_{\mathbf{U},\mathbf{Z}_p} \cong \mathrm{GL}_2(\mathbf{Q}_p) \backslash (\widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p \times Z_{\mathbf{U}}).$$

Here $\widehat{\mathrm{Sh}}_{\mathbf{U},\mathbf{Z}_p}$ is the formal completion of $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$ along its special fibre, $\breve{\mathbf{Z}}_p$ is the ring of integers of the completion of the maximal unramified extension of $\mathbf{Q}_p$, and $\widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p \times Z_{\mathbf{U}}$ is defined by

$$\widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p \times Z_{\mathbf{U}} = \bigsqcup_{Z_{\mathbf{U}}} \widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p.$$

[4] Note that all division quaternion $\mathbf{R}$-algebras are isomorphic to each other by THEOREM A.3.1. Another remark is that $\bar{\Delta}$ is automatically a division algebra; see REMARK A.3.4.

The proof of Cherednik-Drinfeld Theorem will be demonstrated in Section 4.4, and the remaining of this section is devoted to the first step preparations toward the proof. There are two points: (1) a rough idea for the proof of Cherednik-Drinfeld Theorem; (2) understand the structure of $\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p \times Z_{\mathbf{U}})$.

To sum up our strategy of the proof in one sentence: we first show that the isomorphism in Cherednik-Drinfeld Theorem holds for their special fibres, and then one can conclude that the isomorphism holds for the whole spaces. The construction of the morphism between special fibres is in Section 4.4.1, and then we prove that such morphism is an isomorphism in Section 4.4.2. Finally, in Section 4.4.3 we refer to [BC91] to see how to deduce Cherednik-Drinfeld Theorem from the isomorphism on special fibres.

By taking special fibres it means that we need to take a base change to $\mathbf{F}_p$. In addition, we also need to take a base change to the algebraic closure $\overline{\mathbf{F}}_p$ because some properties behave better over an algebraically closed field. There will be various group actions appearing in our discussion, and we should be very careful about the objects and the groups acting on them. To avoid confusion, we list two notations that will occur in the sequel.

- $\widehat{\Omega}\otimes\overline{\mathbf{F}}_p$: This is the $\mathbf{F}_p$-scheme obtained by extending the scalar of $\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p$. Indeed,

$$(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p)\otimes_{\mathbf{Z}_p}\mathbf{F}_p = \widehat{\Omega}\otimes_{\mathbf{Z}_p}(\breve{\mathbf{Z}}_p\otimes_{\mathbf{Z}_p}\mathbf{F}_p) = \widehat{\Omega}\otimes\overline{\mathbf{F}}_p.$$

- $\widehat{\Omega}_{\overline{\mathbf{F}}_p}$: This is defined to be $\widehat{\Omega}\otimes\overline{\mathbf{F}}_p$, but distinct to the first point it is now considered as an $\overline{\mathbf{F}}_p$-scheme.

Next task is to understand the structure of $\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p \times Z_{\mathbf{U}})$. In the introduction of this thesis, we have seen some examples of uniformizations. In those examples, the uniformization theorems state that the geometric object concerning us can be characterized by the quotient a relatively simple geometric object. Thus it is natural to ask: is $\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p \times Z_{\mathbf{U}})$ a relatively simple geometric object? The answer is yes, but it is not so obvious and requires some efforts to see it.

The first thing is to describe the group actions appearing in Cherednik-Drinfeld Theorem. The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts naturally on $\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p \times Z_{\mathbf{U}}$ componentwisely:

- The $\mathrm{GL}_2(\mathbf{Q}_p)$-action on $\widehat{\Omega}$ is the way that we have seen in Section 2.6.
- The $\mathrm{GL}_2(\mathbf{Q}_p)$-action on $\breve{\mathbf{Z}}_p$ is induced by the $\mathrm{GL}_2(\mathbf{Q}_p)$-action on $\mathbf{Z}_p^{\mathrm{nr}}$: for any $g\in\mathrm{GL}_2(\mathbf{Q}_p)$ and any $x\in\mathbf{Z}_p^{\mathrm{nr}}$

$$g\cdot x := (\mathrm{Frob}^{\mathrm{nr}})^{-v(\det g)}x,$$

  where $\mathrm{Frob}^{\mathrm{nr}}:\mathbf{Z}_p^{\mathrm{nr}}\to\mathbf{Z}_p^{\mathrm{nr}}$ is the lift of the Frobenius map $\mathrm{Frob}:\overline{\mathbf{F}}_p\to\overline{\mathbf{F}}_p$, $t\mapsto t^p$.
- We identify $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ with $\mathbf{G}_{\Delta}(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ via the anti-isomorphism $g\mapsto g^{-1}$, so $\mathbf{U}^p$ is a subgroup of $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$. The group $\mathrm{GL}_2(\mathbf{Q}_p)$ acts on $Z_{\mathbf{U}}$ by acting on the p-component (note that $\mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p)\cong M_2(\mathbf{Q}_p)$).

Notation 4.2.3. We set $\mathrm{GL}_2^{\times}(\mathbf{Q}_p) := \{g\in\mathrm{GL}_2(\mathbf{Q}_p) : v(\det g)=0\}$.

The following is a sketch of how to understand the quotient

- One can prove that *the quotient* $\mathrm{GL}_2(\mathbf{Q}_p)\backslash Z_{\mathbf{U}}$ *is a finite set.* Write $\{x_1,\cdots,x_n\}$ for a complete set of representatives of $\mathrm{GL}_2(\mathbf{Q}_p)\backslash Z_{\mathbf{U}}$ with each x_i in $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$. If we set $\Gamma_i := \mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p)\cap x_i^{-1}\mathbf{U}^p x_i$ for each $i=1,\cdots,n$, then one can prove that

$$\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p \times Z_{\mathbf{U}}) = \bigsqcup_{i=1}^{n}\Gamma_i\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p).$$

- For each $i=1,\cdots,n$ there exists $r_i\in\mathbf{N}$ such that $p^{r_i}\equiv p^{r_i}\cdot\mathbf{1}=\begin{pmatrix}p^{r_i} & 0\\ 0 & p^{r_i}\end{pmatrix}$ is in Γ_i. To compute $\Gamma_i\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p)$, we split into two steps: first compute $\langle p^{r_i}\cdot\mathbf{1}\rangle\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p)$, and then take quotient by modulo Γ_i. Note that $p^{r_i}\cdot\mathbf{1}$ acts on $\widehat{\Omega}$ trivially because it is a scalar matrix. On the other hand, the fixed part of $\breve{\mathbf{Z}}_p$ by $p^{r_i}\cdot\mathbf{1}$-action is by definition given by

$$\breve{\mathbf{Z}}_p^{(\mathrm{Frob}^{\mathrm{nr}})^{-v(\det p^{r_i})}} = \breve{\mathbf{Z}}_p^{(\mathrm{Frob}^{\mathrm{nr}})^{-2r_i}} = \mathbf{Z}_p^{(2r_i)},$$

  where $\mathbf{Z}_p^{(2r_i)}$ is the ring of integers of the unramified extensions of $\mathbf{Q}_p$ of degree $2r_i$. Combining all these together, we then have

$$\langle p^{r_i}\cdot\mathbf{1}\rangle\backslash(\widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\breve{\mathbf{Z}}_p) = \widehat{\Omega}\,\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p^{(2r_i)}.$$

Now we take the the quotient of the above expression modulo Γ_i to obtain

$$\Gamma_i \backslash (\widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p) = \Gamma_i \backslash \widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{Z}_p^{(2r_i)}.$$

- Let Γ_i' be the image of Γ_i in $\mathrm{PGL}_2()\mathbf{Q}_p$. One can show that Γ_i' is a **Schottky group**. Moreover, it can be proved that $\Gamma_i \backslash \widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{Z}_p^{(2r_i)}$ can be written as

$$\Gamma_i \backslash \widehat{\Omega} \mathbin{\widehat{\otimes}}_{\mathbf{Z}_p} \mathbf{Z}_p^{(2r_i)} = \bigsqcup_1^{2r_i} \Gamma_i' \backslash \widehat{\Omega}.$$

The object $\bigsqcup_{\text{finite}} \Gamma_i' \backslash \widehat{\Omega}$ is called a **Mumford quotient**[5]. Under our assumption that $\mathbf{U}^p$ is small enough, one can show that Γ_i'-action on BT_K satisfies the following property: for any $\gamma \in \Gamma_i'$ and any vertex $s \in \mathrm{BT}_K$, $\mathrm{dist}(s, \gamma s) \geq 2$[6].
- If we take base change to $\overline{\mathbf{F}}_p$, then one has

$$(\bigsqcup_1^{2r_i} \Gamma_i' \backslash \widehat{\Omega}) \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p = \bigsqcup_1^{2r_i} \Gamma_i' \backslash \widehat{\Omega}_{\overline{\mathbf{F}}_p}.$$

4.3 ALGEBRAIZATIONS

PROPOSITION 4.3.1. [BC91, PROPOSITION III.2.1]
There is only one isogeny class of pairs (A, ι), where A is an abelian variety of dimension 2 over $\overline{\mathbf{F}}_p$ and ι is a map from $\mathcal{O}_{\mathfrak{D}}$ to $\mathrm{End}_{\overline{\mathbf{F}}_p}(A)$. Moreover, there is a $\mathbf{Q}$-algebra isomorphism

$$\mathrm{End}_{\mathcal{O}_\Delta}(A)_{\mathbf{Q}} := \mathrm{End}_{\mathcal{O}_\Delta}(A) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \bar{\Delta}.$$

SETUP 4.3.2. Let us fix an abelian variety A_0 over $\overline{\mathbf{F}}_p$ of dimension 2 with a special $\mathcal{O}_{\mathfrak{D}}$-action. Set Φ the corresponding special formal $\mathcal{O}_{\Delta_p}$-module (i.e., Φ is the formal completion of A_0 along its identity section; see REMARK B.5.4). By PROPOSITION 4.3.1 we fix the identification $\mathrm{End}_{\mathcal{O}_\Delta}(A_0)_{\mathbf{Q}} = \bar{\Delta}$ and thus $\mathbf{G}_{\bar{\Delta}}(\mathbf{Q}) = \mathrm{Aut}_{\mathcal{O}_\Delta}(A_0)_{\mathbf{Q}} := \mathrm{Aut}_{\mathcal{O}_\Delta}(A_0) \otimes_{\mathbf{Z}} \mathbf{Q}$. Passing to the formal completion, we then have

$$\bar{\Delta}_p = M_2(\mathbf{Q}_p) = \mathrm{End}_{\mathcal{O}_{\Delta_p}}(\Phi)_{\mathbf{Q}_p} \quad \text{and} \quad \mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p) = \mathrm{GL}_2(\mathbf{Q}_p) = \mathrm{Aut}_{\mathbf{O}_{\Delta_p}}(\Phi)_{\mathbf{Q}_p}.$$

Here $\mathrm{End}_{\mathcal{O}_{\Delta_p}}(\Phi)_{\mathbf{Q}_p} := \mathrm{End}_{\mathcal{O}_{\Delta_p}}(\Phi) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $\mathrm{Aut}_{\mathcal{O}_{\Delta_p}}(\Phi)_{\mathbf{Q}_p} := \mathrm{Aut}_{\mathcal{O}_{\Delta_p}}(\Phi) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. For each rational prime $\ell \neq p$ we define

$$T_\ell(\mathcal{O}_\Delta) := \mathcal{O}_\Delta \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \quad \text{and} \quad V_\ell(\mathcal{O}_\Delta) := T_\ell(\mathcal{O}_\Delta) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

Similarly, for an abelian variety A (over any field) we define $T_\ell(A)$ to be the ℓ-adic Tate module of A and $V_\ell(A) := T_\ell(A) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$. Then we fix the isomorphisms for every $\ell \neq p$

$$\nu_{0,\ell} : V_\ell(A_0) \xrightarrow{\sim} V_\ell(\mathcal{O}_\Delta)$$

Recall that we have introduced the category $\mathbf{Nilp}_{\mathcal{O}_K}$ in NOTATION 3.3.1, and now we are considering $K = \mathbf{Q}_p$.

DEFINITION 4.3.3. Let $R \in \mathbf{Nilp}_{\mathbf{Z}_p}$, let $S := \operatorname{Spec} R$, and let $\mathfrak{G}$ be a special formal $\mathcal{O}_{\Delta_p}$-module over R. An **algebraization** of $\mathfrak{G}$ is a pair (A, ε), where A is an abelian scheme over S, and $\varepsilon : \widehat{A} \xrightarrow{\sim} \mathfrak{G}$ is an $\mathcal{O}_\Delta$-equivariant isomorphism. If A is equipped with a level $\mathbf{U}$-structure, we then call (A, ε) an algebraization with level $\mathbf{U}$-structure. We write $\mathsf{Alg}_{\mathbf{U}}(\mathfrak{G})$ for the set of isomorphism classes of algebraization of $\mathfrak{G}$ with level $\mathbf{U}$-structure.

LEMMA 4.3.4. [BC91, III.6.2.2]
The set $\mathsf{Alg}_{\mathbf{U}}(\Phi)$ is bijective to the following two sets:
(1) *the set of isomorphism classes of triples $(A, \varepsilon, \bar{\nu})$ consisting of*

[5] For more details about Schottky groups and Mumford quotients please see Mumford's paper [Mu72] or the related books [GvdP80] and [Lü16].
[6] Note that we have proved in PROPOSITION 2.1.4 that BT_K is a tree. For any two vertices s and s' we denote by $\mathrm{dist}(s, s')$ the length of the path joining s and s', i.e., $\mathrm{dist}(s, s')$ is the number of edges of the path.

- *an abelian variety A over $\overline{\mathbf{F}}_p$ with an $\mathcal{O}_\Delta$-action;*
- *an $\mathcal{O}_\Delta$-equivariant isomorphism $\varepsilon : \widehat{A} \xrightarrow{\sim} \Phi$;*
- *a class $\bar{\nu}$ of $\mathcal{O}_\Delta$-linear isomorphisms ν's modulo the $\mathbf{U}^p$-action on the right hand side:*

$$\nu : \prod_{\ell \neq p} T_\ell(A) \xrightarrow{\sim} \prod_{\ell \neq p} T_\ell(\mathcal{O}_\Delta).$$

(2) *the set of isogeny classes of triples $(A, \varepsilon, \bar{\nu})$ consisting of*

- *an abelian variety A over $\overline{\mathbf{F}}_p$ with an Δ-action by isogenies;*
- *an $\mathcal{O}_\Delta$-equivariant quasi-isogeny $\varepsilon : \widehat{A} \xrightarrow{\sim} \Phi$;*
- *a class $\bar{\nu}$ of Δ-linear isomorphisms ν's modulo the $\mathbf{U}^p$-action on the right hand side:*

$$\nu : \prod_{\ell \neq p} V_\ell(A) \xrightarrow{\sim} \prod_{\ell \neq p} V_\ell(\mathcal{O}_\Delta).$$

DEFINITION 4.3.5. We define $\mathsf{Alg}(\Phi) := \varprojlim \mathsf{Alg}_{\mathbf{U}'}(\Phi)$, where the projective limit runs through all small enough compact open subgroup $\mathbf{U}' \subseteq \mathbf{G}_\Delta(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$. The set $\mathsf{Alg}(\Phi)$ has a natural $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$-action by using the characterization in LEMMA 4.3.4 (2). More precisely, write $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}}) = \mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p) \times \mathbf{G}_{\bar{\Delta}}(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$.

- The component $\mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p)$ acts on ε by composition. That is, for any $g \in \mathbf{G}_{\bar{\Delta}}(\mathbf{Q}_p) \cong \mathrm{GL}_2(\mathbf{Q}_p)$ we define

$$g \cdot \varepsilon : \widehat{A} \xrightarrow{\varepsilon} \Phi \xrightarrow{g} \Phi.$$

  Note that g acts on Φ by our identification (see SETUP 4.3.2).
- The component $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ acts on $\bar{\nu}$ by composition (note that $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}^p_{\mathbf{Q},\mathrm{fin}})$ acts on $\prod V_\ell(\mathcal{O}_\Delta)$ canonically).

From SETUP 4.3.2 we have a fixed element $(A_0, \varepsilon_0, \bar{\nu}_0)$ in $\mathsf{Alg}(\Phi)$ given by

$$\varepsilon_0 : \widehat{A}_0 \xrightarrow{=} \Phi \quad \text{and} \quad \bar{\nu}_0 = \text{class of} \prod_{\ell \neq p} \nu_{0,\ell}.$$

We then have a map

$$\varrho : \mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}}) \longrightarrow \mathsf{Alg}(\Phi), \quad g \longmapsto g \cdot (A_0, \varepsilon_0, \bar{\nu}_0).$$

In the statement of CHEREDNIK-DRINFELD THEOREM (THEOREM 4.2.2) there is a set $Z_{\mathbf{U}}$. Now we can see this set occurs naturally from our discussions on algebraization.

LEMMA 4.3.6. [BC91, III.6.2.3]
The action of $\mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})$ on $\mathsf{Alg}(\Phi)$ is transitive, and the stabilizer of $(A_0, \varepsilon_0, \bar{\nu}_0)$ is $\mathbf{G}_{\bar{\Delta}}(\mathbf{Q})$. Thus the map ϱ induces the bijections on sets:

$$\mathsf{Alg}(\Phi) \cong \mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})/\mathbf{G}_{\bar{\Delta}}(\mathbf{Q}) \quad \textit{and} \quad \mathsf{Alg}_{\mathbf{U}}(\Phi) \cong \mathbf{U}^p \backslash \mathbf{G}_{\bar{\Delta}}(\mathbf{A}_{\mathbf{Q},\mathrm{fin}})/\mathbf{G}_{\bar{\Delta}}(\mathbf{Q}) = Z_{\mathbf{U}}$$

4.4 THE PROOF

4.4.1 CONSTRUCTION OF $\overline{\Theta}$

Note that $(\widehat{\Omega} \otimes \mathbf{F}_p) \times Z_U$ is the disjoint union of $\#Z_{\mathbf{U}}$-copies of $\widehat{\Omega} \otimes \mathbf{F}_p$. We view $(\widehat{\Omega} \otimes \mathbf{F}_p) \times Z_U$ as a functor from $\mathbf{Sch}_{\mathbf{F}_p}$ to $\mathbf{Set}$. Certainly, we only need to consider affine $\mathbf{F}_p$-scheme $S = \operatorname{Spec} R$ such that S is connected (because then the image of a map from S to $(\widehat{\Omega} \otimes \mathbf{F}_p) \times Z_U$ just lies in one connected component, i.e., in one of $\widehat{\Omega} \otimes \mathbf{F}_p$'s). By applying THEOREM 3.3.13 with base change to $\mathbf{F}_p$ and LEMMA 4.3.6 we obtain that

$$(\widehat{\Omega} \otimes \mathbf{F}_p \times Z_U)(S) = \{(\psi, \mathfrak{G}, \rho, A, \varepsilon, \bar{\nu}) : (\mathfrak{G}, \rho) \text{ modulo isomorphisms}\},$$

where the tuple $(\psi, \mathfrak{G}, \rho, A, \varepsilon, \bar{\nu})$ consisting of

- an $\mathbf{F}_p$-algebra homomorphism $\psi : \overline{\mathbf{F}}_p \to R$;
- a special formal $\mathcal{O}_{\Delta_p}$-module $\mathfrak{G}$ over R;

- ◇ a quasi-isogeny $\rho : \Phi_B \to \mathfrak{G}$ of height 0;
- ◇ an algebrisation $(A, \varepsilon, \bar{\nu})$ with level $\mathbf{U}$-structure of Φ.

To give a connection between $\widehat{\Omega} \otimes \mathbf{F}_p \times Z_U$ and $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$, we need the following lemma:

LEMMA 4.4.1. [BC91, LEMMA III.6.3.1]
Let $R \in \mathbf{Alg}_{\mathbf{F}_p}$ and $S := \operatorname{Spec} R$. Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two special formal $\mathcal{O}_{\Delta_p}$-modules over R, let $f : \mathfrak{G}_1 \to \mathfrak{G}_2$ be a quasi-isogeny, and let (A_1, ε_1) be an algebrisation of $\mathfrak{G}_1$. Then there exists a unique algebraization (A_2, ε_2) of $\mathfrak{G}_2$ and a unique quasi-isogeny $g : A_1 \to A_2$ such that the diagram

$$\begin{array}{ccc} \widehat{A}_1 & \xrightarrow[\sim]{\varepsilon_1} & \mathfrak{G}_1 \\ {\scriptstyle \widehat{g}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \widehat{A}_2 & \xrightarrow[\sim]{\varepsilon_2} & \mathfrak{G}_2 \end{array}$$

is commutative. If A_1 has a level $\mathbf{U}$-structure, then A_2 also has a level $\mathbf{U}$-structure.

Keep the same notations before LEMMA 4.4.1. Applying LEMMA 4.4.1 with

$$\mathfrak{G}_1 = \Phi_B, \quad \mathfrak{G}_2 = \mathfrak{G}, \quad f = \rho, \quad A_1 = A \otimes_{\overline{\mathbf{F}}_p} B, \quad \varepsilon_1 = \varepsilon \otimes_{\overline{\mathbf{F}}_p} B,$$

we then obtain an algebraization (A_2, ε_2) with a level $\mathbf{U}$-structure of $\mathfrak{G}$. This pair (A_2, ε_2) with a level $\mathbf{U}$-structure is a point in $(\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)(R)$ by THEOREM 4.1.5 (with base change to $\mathbf{F}_p$).

To sum up, for any $R \in \mathbf{Alg}_{\mathbf{F}_p}$ such that $S := \operatorname{Spec} R$ is connected, we have a map

$$(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_U)(R) \longrightarrow (\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p)(R).$$

This induces a morphism $\Theta : (\widehat{\Omega} \otimes \overline{\mathbf{F}}_p) \times Z_U \to \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ of $\mathbf{F}_p$-scheme. Moreover, one can check that Θ is $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant, so we can conclude the following:

PROPOSITION 4.4.2. *There is a morphism $\overline{\Theta} : \mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_{\mathbf{U}}) \to \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ of $\mathbf{F}_p$-schemes.*

4.4.2 $\overline{\Theta}$ IS AN ISOMORPHISM

In last subsection, we constructed a morphism $\overline{\Theta} : \mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_{\mathbf{U}}) \to \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ of $\mathbf{F}_p$-schemes in PROPOSITION 4.4.2. In this subsection, we are going to prove that $\overline{\Theta}$ is indeed an isomorphism. Our strategy is to take a base change to $\overline{\mathbf{F}}_p$ and then apply FAITHFULLY FLAT DESCENT (LEMMA 4.4.12).

First of all, the map $\overline{\Theta}$ is a morphism between $\mathbf{F}_p$-schemes. By base change to $\overline{\mathbf{F}}_p$ we then obtain

$$\overline{\Theta}_{\overline{\mathbf{F}}_p} : \left[\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_{\mathbf{U}})\right] \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \longrightarrow \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p.$$

The following lemma tells us that the source of $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ can be written as a cleaner form.

LEMMA 4.4.3. *We have $\left[\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_U)\right] \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \cong \mathrm{GL}_2^{\times}(\mathbf{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})$.*

With the above lemma we will from now on consider $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ as the form $\mathrm{GL}_2^{\times}(\mathbf{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}}) \to \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$. It is not so clear how to prove $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is an isomorphism, but is is rather easy to check that whether the induced map on $\overline{\mathbf{F}}_p$-points is bijective, Indeed, $\overline{\mathbf{F}}_p$-points of $\mathrm{GL}_2^{\times}(\mathbf{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})$ and of $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$ can be explicitly described due to our categorical construction. Next lemma says that the induced map of $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ on $\overline{\mathbf{F}}_p$-points is bijective.

LEMMA 4.4.4. [BC91, III.6.4.2]
The morphism $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ induces bijection on the $\overline{\mathbf{F}}_p$-rational points.

We would like to show that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is an étale morphism. It is difficult to use those equivalent statements stated in REMARK 4.1.4 to check that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is étale. However, we have very explicit modular descriptions of the geometric objects.

Therefore, the INFINITESIMAL LIFTING PROPERTY is a suitable choice for us. Instead of proving that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is étale, we will prove that

$$\Theta_{\overline{\mathbf{F}}_p} : \widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}} \longrightarrow \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$$

is an étale morphism.

DEFINITION 4.4.5. Let B be an algebra over a ring R. A **thickening** of B is a surjective R-algebra homomorphism $\varphi : B' \twoheadrightarrow B$ such that $(\ker \varphi)^2 = 0$.

THEOREM 4.4.6. (INFINITESIMAL LIFTING PROPERTY) [BLR90, PROPOSITION 2.2.6]
Let $f : X \to Y$ be a morphism locally of finite presentation of schemes. Then the following statements are equivalent.

(a) *f is unramified (resp. smooth, resp. étale);*

(b) *for any $\Gamma(Y, \mathscr{O}_Y)$-algebra B and any thickening $B' \twoheadrightarrow B$ of B the canonical map*

$$\mathrm{Hom}_{\mathbf{Sch}_Y}(\mathrm{Spec}\, B', X) \longrightarrow \mathrm{Hom}_{\mathbf{Sch}_Y}(\mathrm{Spec}\, B, X)$$

is injective (resp. surjective, resp. bijective).

REMARK 4.4.7. Notations are as in THEOREM 4.4.6. We can reformulate the theorem as what follows:
The morphism $f : X \to Y$ is unramified (resp. smooth, resp. étale) if and only if for any commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, B \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\, B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists at most one (resp. at least one, resp. a unique) lifting $\mathrm{Spec}\, B' \to X$.

LEMMA 4.4.8. [BC91, II.6.4.3]
The morphism $\Theta_{\overline{\mathbf{F}}_p}$ is étale, and thus $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is also an étale morphism.

PROOF. (SKETCH)
Let R be an $\overline{\mathbf{F}}_p$-algebra, and let $R' \twoheadrightarrow R$ be a thickening of R. By the INFINITESIMAL LIFTING PROPERTY (THEOREM 4.4.6) one needs to show that the following two canonical maps

$$(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})(R') \longrightarrow (\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})(R) \quad \text{and} \quad (\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)(R') \longrightarrow (\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)(R)$$

are bijective. The situation could be understood more clearly from the diagram:

$$\begin{tikzcd} \mathrm{Spec}\, R \arrow[r, dashed] \arrow[d] \arrow[dr, dashed] & \widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}} \arrow[d, "\Theta_{\overline{\mathbf{F}}_p}"] \\ \mathrm{Spec}\, R' \arrow[r, dashed] \arrow[ur, dashed] & \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \end{tikzcd}$$

By the theory of commutative formal groups lifting a point $(\mathfrak{G}, \rho, A, \varepsilon\bar{\nu}) \in (\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})(R)$ to $(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}})(R')$ is equivalent to lifting the formal $\mathcal{O}_{\Delta_p}$-module $\mathfrak{G}$, and such lifting is unique (it is important that the base $\overline{\mathbf{F}}_p$ where we are working is algebraically closed). The argument for $(\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)(R') \longrightarrow (\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)(R)$ is quite similar (lifting an abelian variety over $\overline{\mathbf{F}}_p$ with an $\mathcal{O}_{\Delta}$-action and a level structure is equivalent to lifting its associated formal group; this is a theorem of Serre-Tate). ■

The next thing is to combine LEMMA 4.4.4 and LEMMA 4.4.8 to show that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is an isomorphism, and then we can conclude $\overline{\Theta}$ is an isomorphism by faithfully flat descent. What we will do is to extract the essential information and formulate our problem as a general statement in algebraic geometry. To begin with, let us take a look at some basic properties concerning about rational points of schemes over fields.

REMARK 4.4.9. In this remark, we are going to see some basic properties about schemes over fields.

(a) [GW20, PROPOSITION 3.33] *Let X be a scheme locally of finite type over a field k. Then the following statements are equivalent.*

  (1) *A point $x \in X$ is a closed point;*
  (2) *the field extension $\mathbf{k}(x)/k$ is finite;*
  (3) *the field extension $\mathbf{k}(x)/k$ is algebraic.*

(b) [GW20, PROPOSITION 3.36] *Let X be a scheme locally of finite type over an algebraically closed field k. Then $\{$closed points of $X\} = X(k)$.*
Since k is algebraically closed, there is no non-trivial algebraic extension over k. Then the assertion follows directly from (1) and (3) of part (a).

(c) Let X be a topological space and $Z \subseteq X$ a subset endowed with the subspace topology. We say Z is **very dense** in X if it satisfies the following equivalent statements:

  (1) The map is a bijection:

$$\{\text{open subsets of } X\} \longrightarrow \{\text{open subsets of } Z\}, \quad U \longmapsto U \cap Z.$$

  (2) The intersection of Z and every non-empty locally closed subset of X is non-empty.

It is obvious from the definition (statement (1)) that every very dense subset is a dense subset. Note that every dense subset (and thus every very dense subset) of a non-empty topological space is non-empty.

(d) [GW20, PROPOSITION 3.35] *Let X be a scheme locally of finite type over a field k. Then the set of closed points of X is very dense in X. In particular, if k is algebraically closed, then $X(k) \neq \varnothing$.*
By the characterization (2) in part (c) we should prove that every locally closed subset of X contains a closed point. It suffices to prove that for any affine open subsets $\operatorname{Spec} R \subseteq X$ every non-empty closed subset of $\operatorname{Spec} R$ contains a closed point. A non-empty closed subset of $\operatorname{Spec} R$ is of the form $V := V(\mathfrak{a})$ for some proper ideal $\mathfrak{a}$. Since $\mathfrak{a}$ is contained in some maximal ideal of R, one deduces that V contains a closed point x of $\operatorname{Spec} R$. Since $\operatorname{Spec} R$ is locally of finite type over k, (a) implies that $\mathbf{k}(x)/k$ is algebraic and thus x is also closed in X.

We know that injectivity (of morphisms of schemes) is not stable under base change, but surjectivity is (see [GW20, PROPOSITION 4.32]). We say a morphism is **universally injective** if every base change of it is still injective. The next lemma are two properties of universally injective morphisms.

LEMMA 4.4.10. [SP05, TAG 01S4, 02LC]

(a) *Let $f : X \to Y$ be a morphism of schemes. The following statements are equivalent.*

  (1) *The morphism f is universally injective.*
  (2) *The diagonal morphism $\Delta_{X/Y} : X \to X \times_Y X$ is surjective.*

(b) *Every universally injective étale morphism is an open immersion.*

PROPOSITION 4.4.11. [SE21]
Let k be an algebraically closed field, and let X and Y be two schemes locally of finite type over k. If a morphism $f : X \to Y$ over k is étale and bijective on k-rational points (i.e., the induced map $X(k) \to Y(k)$ is bijective), then f is an isomorphism.

PROOF. We first prove that f is universally injective by using LEMMA 4.4.10 (a); that is, we should show that $\Delta_{X/Y}$ is surjective. Note that $\Delta_{X/Y}(X)$ is a locally closed subset in $X \times_Y X$ (see [GW20, PROPOSITION 9.5]), so we write $\Delta_{X/Y}(X) = U \cap V$ for some open subset U and some closed subset V of $X \times_Y X$. Suppose $\Delta_{X/Y}(X) \subsetneq X \times_Y X$. Since

$$X \times_Y X - \Delta_{X/Y}(X) = (X \times_Y X - U) \cup (X \times_Y X - V),$$

we then have either $X \times_Y X - U \neq \varnothing$ or $X \times_Y X - V \neq \varnothing$.

(1) We check that this implies $X \times_Y X - \Delta_{X/Y}(X)$ contains a k-rational point.

  ◇ Assume $X \times_Y X - U \neq \varnothing$. Then $X \times_Y X - U$ is a non-empty closed subscheme of $X \times_Y X$ (with reduced structure). Since every closed immersion is locally of finite type and being locally of finite type is stable under composition, $X \times_Y X - U$ is a scheme of finite type over k. Then REMARK 4.4.9 (b) and (d) indicate that $X \times_Y X - U$ contains a k-rational point.

- ⋄ Assume $X \times_Y X - V \neq \varnothing$. Then $X \times_Y X - V$ is a non-empty open subscheme of $X \times_Y X$. Since every open immersion is locally of finite type and being locally of finite type is stable under composition, we see that $X \times_Y X - V$ is locally of finite type over k. Again, REMARK 4.4.9 (b) and (d) indicate that $X \times_Y X - V$ contains a k-rational point.

(2) Now we show that $\Delta_{X/Y}(X)(k) = X \times_Y X(k)$. The categorical property of fibre product gives

$$X \times_Y X(k) = \{(a, b) \in X(k) \times X(k) : f(a) = f(b)\},$$
$$\Delta_{X/Y}(X)(k) = \{(a, b) \in X(k) \times X(k) : a = b\}.$$

However, our assumption says that f is bijective on k-rational points, so $f(a) = f(b)$ implies $a = b$. Hence $\Delta_{X/Y}(X)(k) = X \times_Y X(k)$.

Combining (1) and (2), we get a contradiction, so $\Delta_{X/Y}(X) = X \times_Y X$. This proves that f is universally injective. Then LEMMA 4.4.10 (b) indicates that f is an open immersion. If f is not surjective, then $Y - f(X)$ is a non-empty closed subscheme of Y (with reduced structure). Apply the same argument as the first ⋄ in (1), we then conclude $Y - f(X)$ has a k-rational point, which is a contradiction because f is bijective on k-rational point. Therefore, f is a surjective open immersion; namely, f is an isomorphism. ■

Now we come back to our situation. We can apply PROPOSITION 4.4.11 to obtain that $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is an isomorphism. To furthermore obtain that $\overline{\Theta}$ is an isomorphism, we need to use a common technique called FATHFULLY FLAT DESCENT in algebraic geometry. A morphism $f : X \to Y$ of schemes is called **faithfully flat** if f is flat and surjective. The easiest example of faithfully flat morphisms is the morphism associated to a field extension. Indeed, for any field extension k'/k the morphism $f : \operatorname{Spec} k' \to \operatorname{Spec} k$ is surjective because $\operatorname{Spec} k$ contains only one point; moreover, k' is a flat k-module, so f is flat.

LEMMA 4.4.12. (FAITHFULLY FLAT DESCENT) [GW20, PROPOSITION 14.51]
Let $f : X \to Y$ be a morphism of schemes over a scheme S, and let $g : S' \to Y$ be a quasi-compact faithfully flat morphism of schemes. Set $X' := X \times_S S'$ and $Y' := Y \times_S S'$, and denote by $f' : X' \to Y'$ the morphism obtained by base change. If f' is an isomorphism, then f is also an isomorphism.

COROLLARY 4.4.13. *The morphism $\overline{\Theta}$ is an isomorphism.*

PROOF. By LEMMA 4.4.4 and LEMMA 4.4.8 one can apply PROPOSITION 4.4.11 to conclude $\overline{\Theta}_{\overline{\mathbf{F}}_p}$ is an isomorphism. From the cartesian diagram

$$\begin{array}{ccccc} \mathrm{GL}_2^\times(\mathbf{Q}_p)\backslash(\widehat{\Omega}_{\overline{\mathbf{F}}_p} \times Z_{\mathbf{U}}) & \xrightarrow{\overline{\Theta}_{\overline{\mathbf{F}}_p}} & \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes \overline{\mathbf{F}}_p & \longrightarrow & \overline{\mathbf{F}}_p \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_{\mathbf{U}}) & \xrightarrow{\overline{\Theta}} & \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes \mathbf{F}_p & \longrightarrow & \mathbf{F}_p \end{array}$$

we then conclude by FAITHFULLY FLAT DESCENT (LEMMA 4.4.12) that $\overline{\Theta}$ is an isomorphism. ■

4.4.3 THE FINAL STEP

Up to now, we have seen that the special fibres of $\mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \,\widehat{\otimes}_{\mathbf{Z}_p} \breve{\mathbf{Z}}_p \times Z_{\mathbf{U}})$ and of $\mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p}$ are isomorphic; that is,

$$\overline{\Theta} : \mathrm{GL}_2(\mathbf{Q}_p)\backslash(\widehat{\Omega} \otimes \overline{\mathbf{F}}_p \times Z_{\mathbf{U}}) \xrightarrow{\sim} \mathrm{Sh}_{\mathbf{U},\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p.$$

To derive the isomorphism in THEOREM 4.2.2 from the isomorphism $\overline{\Theta}$, we refer to [BC91, III.6.5].

Appendix A

Quaternion Algebras

A.1 Definitions

Definition A.1.1. Let A be an algebra over a field k. The **center** of A is defined to be $Z_k(A) = \{x \in A : xa = ax \text{ for all } a \in A\}$. It follows from definition that $k \subseteq Z_k(A)$.

(a) The algebra A is called **central** if $k = Z_k(A)$.

(b) The algebra A is called **simple** if 0 and A are the only two-sided ideals of A.

We quote the following theorem without proof.

Theorem A.1.2. [GS06, Theorem 2.2.1]
Let A be a finite dimensional algebra over a field k. The algebra A is central simple if and only if there exists $n \in \mathbf{N}$ and a finite field extension K/k such that $A \otimes_k K \cong M_n(K)$ as K-algebras. If it is the case, $\dim_k A$ is a square of some positive integer and $n = \sqrt{\dim_k A}$.

We call $\sqrt{\dim_k A}$ the **degree** of A and denote it by $\deg_k A$. If $\deg_k A = 1$, then $A = k$ and there is nothing interesting. Thus the first non-trivial but rather easy case is $\deg_k A = 2$. Among all central simple algebras of degree 2 there is a class of them called *quaternion algebras*. The toy model for the definition of general quaternion algebras is the Hamilton quaternion algebra. We will first recall the definition and some properties of Hamilton quaternion algebra and then give a generalization. Surprisingly, every central simple algebra of degree 2 is in fact isomorphic to some quaternion algebra, and this result will be proved in Corollary A.2.6.

We are going to define quaternion algebras and prove that they are central simple algebras in this subsection. Before we go to the definition, let us start with a survey of Hamilton quaternion algebra. This also plays a role as a motivational example for our general definition.

Example A.1.3. (Hamilton quaternion algebra)
Hamilton quaternion algebra $\mathbf{H}$ is an $\mathbf{R}$-algebra defined by

$$\mathbf{H} = \{x + yi + zj + wk \mid x, y, z, w \in \mathbf{R}\}$$

with $i^2 = j^2 = k^2 = -1$ and $ij = -(ji) = k$. (Caution: The symbol k here is not a field but just an abstract symbol with the stated operation laws.)

To see $\mathbf{H}$ is central, we note that $\mathbf{R} \subseteq Z(\mathbf{H})$ is obviously true and thus it remains to show the converse inclusion. If $x + yi + zj + wk$ is in the center of $\mathbf{H}$, then

$$(x + yi + zj + wk) \cdot i = i \cdot (x + yi + zj + wk) \quad \text{and} \quad (x + yi + zj + wk) \cdot j = j \cdot (x + yi + zj + wk).$$

By simple computation, the first formula gives $wj + (-z)k = -wj + zk$, and the second one gives $-wi + yj = wi - yk$. Since $\{1, i, j, k\}$ is a basis for $\mathbf{H}$ and 2 is invertible in $\mathbf{R}$, we then conclude $y = z = w = 0$.

Before proving that $\mathbf{H}$ is simple let us introduce the norm map on $\mathbf{H}$. For each $q = x + yi + zj + wk \in \mathbf{H}$, the *conjugate* of q is

$$\bar{q} := x - yi - zj - wk.$$

The conjugation gives a quasi-automorphism on $\mathbf{H}$; that is,

$$\overline{(\cdot)} : \mathbf{H} \xrightarrow{\sim} \mathbf{H}, \quad q \mapsto \bar{q}$$

and the multiplication reverses under conjugation, i.e., $\overline{q_1q_2} = \overline{q_2} \cdot \overline{q_1}$ for all $q_1, q_2 \in \mathbf{H}$. The the *norm map* is given by

$$N : \mathbf{H} \longrightarrow \mathbf{R}, \qquad q \longmapsto N(q) := q\bar{q}.$$

The norm map N is multiplicative: $N(q_1q_2) = N(q_1)N(q_2)$ for all $q_1, q_2 \in \mathbf{H}$. Indeed,

$$N(q_1q_2) = q_1q_2 \cdot \overline{q_1q_2} = q_1q_2 \cdot \overline{q_2} \cdot \overline{q_1} = N(q_2)q_1\overline{q_1} = N(q_1)N(q_2).$$

Note that we use the fact $\mathbf{R} = Z(\mathbf{H})$ in the above computation. On the other hand, we have the explicit expression of N by

$$N(q) = (x + yi + zj + wk) \cdot (x - yi - zj - wk) = x^2 + y^2 + z^2 + w^2.$$

Thus $N(q) = 0$ if and only if $q = 0$. Together with our definition $N(q) = q\bar{q}$ we see that every non-zero element is a unit in $\mathbf{H}$, and the inverse is given by

$$q^{-1} = \frac{\bar{q}}{N(q)} \qquad \text{provided } q \neq 0.$$

In conclusion, $\mathbf{H}$ is a division ring. Thus $\mathbf{H}$ is simple because every division ring is simple (a non-zero ideal of a division ring must contain a non-zero element which is also a unit, so every non-zero ideal is equal to the whole ring).

Although both $\mathbf{H}$ and $M_2(\mathbf{R})$ are 4-dimensional central simple $\mathbf{R}$-algebra, there are not isomorphic because $\mathbf{H}$ is a division ring but $M_2(\mathbf{R})$ not.

DEFINITION A.1.4. Let k be a field with $\operatorname{char} k \neq 2$. For any two elements $a, b \in k^\times$ the (generalized) **quaternion algebra** over k, denoted by $(a, b)_k$ (or simply by (a, b) if there is no confusion with the base field), is the k-algebra generated by the basis $\{1, i, j, ij\}$ with the following multiplication law

$$i^2 = a, \qquad j^2 = b, \qquad i \cdot j = ij, \qquad ij = -ji.$$

The basis $\{1, i, j, ij\}$ is called a **quaternion basis** for $(a, b)_k$. We always assume $\operatorname{char} k \neq 2$ whenever we are talking about quaternion algebras defined over k.

If $\operatorname{char} k = 2$, then $(a, b)_k$ is a commutative algebra because of $ij = -ji = ji$. Thus the assumption $\operatorname{char} k \neq 2$ is necessary in the definition of quaternion algebras. In fact, we can still define quaternion algebras over a field of characteristic 2, but the formulation is different from the one we give here (see [GS06, REMARK 1.1.8] or [Vo21, CHAPTER 6]).

REMARK A.1.5. *Let k be a field. If $a, b \in k^\times$ and $a \equiv a' \mod k^{\times 2}$, $b \equiv b' \mod k^{\times 2}$, then $(a, b)_k \cong (a', b')_k$ as k-algebras. In particular, $(a, b)_k \cong (b, a)_k$.*
Write $(a, b)_k = \operatorname{span}_k\{1, i, j, ij\}$, $(a', b')_k = \operatorname{span}_k\{1, i', j', i'j'\}$ and $a = a'u^2$, $b = b'v^2$ for some $u, v \in k^\times$. The first assertion follows from the following isomorphism

$$(a, b)_k \longrightarrow (a', b')_k, \quad i \longmapsto ui', \ j \longmapsto vj'.$$

For the second assertion, note first that we have the isomorphism

$$(a^3b^2, a^2b^3)_k \xrightarrow{\sim} (b, a)_k, \quad i \longmapsto abj', \ j \longmapsto abi'.$$

Now apply the first assertion to deduce

$$(b, a)_k \cong (a^3b^2, a^2b^3)_k \cong (a, b)_k.$$

EXAMPLE A.1.6. *Let k be a field and $b \in k^\times$ be any element. Then $(1, b)_k \cong M_2(k)$ as k-algebras.*
We consider the map

$$\varphi : (1, b)_k \longrightarrow M_n(k), \quad i \longmapsto I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \longmapsto J := \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

This map is well-defined because $I^2 = I_2$, $J^2 = bI_2$ and $IJ = -JI$ (here I_2 is the identity matrix). The injectivity of φ is obvious, and φ is surjective because

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{x + w}{2} I_2 + \frac{x - w}{2} I + \frac{\frac{y}{b} + z}{2} J + \frac{\frac{y}{b} - z}{2} IJ.$$

We will proceed similarly as what we do for Hamilton quaternion algebra to show that $(a,b)_k$ is central simple. We first write down analogue definitions for conjugates and norms.

DEFINITION A.1.7. Let $(a,b)_k$ be a quaternion algebra over k. For any $q = x + yi + zj + wij$ in $(a,b)_k$, the **conjugate** is

$$\bar{q} := x - yi - zj - wij.$$

The **norm map** on $(a,b)_k$ is given by

$$N : (a,b)_k \longrightarrow k, \quad q \longmapsto N(q) := q\bar{q}.$$

If $q = x + yi + zj + wij$, then we can write down the explicit expression of norm by simple computation:

$$N(x + yi + zj + wij) = x^2 - ay^2 - bz^2 + abw^2.$$

REMARK A.1.8. Here are some easy facts about conjugate and norm map: for any $q, q_1, q_2 \in (a,b)_k$

(a) $\overline{q_1 q_2} = \bar{q}_2 \cdot \bar{q}_1$.

(b) $N(q) = q\bar{q} = \bar{q}q$.

(c) $N(q_1 q_2) = N(q_1)N(q_2)$.

All these results follow from simple computations, so we omit their proof.

A.2 CLASSIFICATION OF CENTRAL SIMPLE ALGEBRAS OF DEGREE 2

LEMMA A.2.1. [GS06, PROPOSITION 1.1.7]
Consider the quaternion algebra $(a,b)_k$ over a field k. The following statements are equivalent:

(a) *$(a,b)_k$ is split.*

(b) *$(a,b)_k$ is not a division ring.*

(c) *There is a non-zero element whose norm is zero.*

(d) *b is in the image of the norm map $N_{k(\sqrt{a})/k} : k(\sqrt{a}) \to k$.*

PROOF.

(a)⇒(b) This is trivial.

(b)⇒(c) Suppose every non-zero element in $(a,b)_k$ also has non-zero norm. Then for any $q \in (a,b)_k$ we have $q^{-1} = \frac{\bar{q}}{N(q)}$.

(c)⇒(d) If $a \in k^{\times 2}$, then $k(\sqrt{a}) = k$ and there is nothing to prove. Suppose $a \notin k^{\times 2}$. By assumption, there is $q = x + yi + zj + wij \neq 0$ such that

$$N(q) = x^2 - ay^2 - bz^2 + abw^2 = 0 \implies (z^2 - aw^2)b = x^2 - ay^2.$$

Since a is not a square, $z^2 - aw^2 \neq 0$. Thus

$$b = \frac{x^2 - ay^2}{z^2 - aw^2} = \frac{(x + \sqrt{a}y)(x - \sqrt{a}y)}{(z + \sqrt{a}w)(z - \sqrt{a}w)} = \frac{N_{k(\sqrt{a})/k}(x + \sqrt{a}y)}{N_{k(\sqrt{a})/k}(z + \sqrt{a}w)} = N_{k(\sqrt{a})/k}\left(\frac{x + \sqrt{a}y}{z + \sqrt{a}w}\right).$$

(d)⇒(a) If a is a square, then $(a,b)_k \cong (1,b)_k$ by REMARK A.1.5. Then EXAMPLE A.1.6 implies $(a,b)_k \cong M_n(k)$. Now suppose a is not a square, i.e., $k(\sqrt{a})/k$ is a quadratic extension. Since b is in the image of $N_{k(\sqrt{a})/k}$, so is b^{-1}. Our assumption implies that there exists $r, s \in k$ (r, s not all zero) such that

$$b^{-1} = N_{k(\sqrt{a})/k}(r + s\sqrt{a}) = r^2 - as^2.$$

We define two elements in $(a,b)_k$:

$$u := rj + sij \quad \text{and} \quad v := (1+a)i + (1-a)ui.$$

Direct computation gives $u^2 = 1$, $v^2 = 4a^2$ and $uv = -vu$, and we can check that $\{1, u, v, uv\}$ is linearly independent over k. Hence $\{1, u, v, uv\}$ is a quaternion basis for $(a,b)_k$ and $(a,b)_k \cong (1, 4a^2)_k$. Again, EXAMPLE A.1.6 tells us that $(1, 4a^2) \cong M_2(k)$.

■

Proposition A.2.2. *Every quaternion algebra over a field k is central simple.*

Proof. If a quaternion algebra is split, then it is obviously a central simple algebra. Suppose $(a,b)_k$ is a non-split quaternion algebra over a field k. By Lemma A.2.1 (a) and (b), we know $(a,b)_k$ is a division ring; in particular, it is simple. Now let $q = x + yi + zj + wij \in Z((a,b)_k)$. Then

$$\begin{aligned} i \cdot q = q \cdot i &\implies xi + ay + zij + awj = xi + ay - zij - awj \implies z = w = 0 \\ j \cdot q = q \cdot j &\implies xj - yij = xj + yij \implies y = 0. \end{aligned}$$

Note that we use the condition $\operatorname{char} k \neq 2$ in the above computation. Since we know $k \subseteq Z((a,b)_k)$, one concludes $k = Z((a,b)_k)$. ■

Our main goal in this section is to classify all central simple algebras of degree 2. First of all, we will see in the following proposition that every central division algebra of degree 2 is isomorphic to a quaternion algebra. How about those that are non-division? To answer this question, we need to use Wedderburn's Theorem.

Proposition A.2.3. [GS06, Proposition 1.2.1]
Let k be a field with $\operatorname{char} k \neq 2$. Then every central division algebra over k of degree 2 is isomorphic to some quaternion k-algebra.

Proof. Let D be a central division algebra over k of degree 2. Take any element $q' \in D \setminus k$. Since $\dim_k D = 4$, the set $\{1, q', q'^2, q'^3, q'^4\}$ is linearly dependent and thus $f(q') = 0$ for some polynomial $f \in k[T]$. We may assume f is irreducible because D has no zero-divisor. Then we have a map

$$k[T]/(f) \longrightarrow D, \quad \bar{g} \longmapsto g(q').$$

This map is clearly injective, so we may view $k(q') \cong k[T]/(f)$ as a k-subalgebra of D (notice that $k(q')$ is a field). Since

$$4 = \dim_k D = \dim_{k(q')} D \cdot \dim_k k(q'),$$

we know that $[k(q') : k]$ divides 4. However, $[k(q') : k] \neq 1$ because $q' \notin k$, and $[k(q') : k] \neq 4$ because D is non-commutative. Hence $[k(q') : k] = 2$. We take $q \in k(q') \setminus k$ such that $a := q^2 \in k^\times$.

Consider the inner automorphism of D:

$$\operatorname{conj}_q : D \longrightarrow D, \quad s \longmapsto qsq^{-1}.$$

Since $\operatorname{conj}_q(\operatorname{conj}_q(s)) = q^2 s q^{-2} = asa^{-1} = s$ for all $s \in D$, one has $\operatorname{conj}_q^2 = \mathrm{id}_D$. This implies the eigenvalues of conj_q is ± 1. Note that q is an eigenvector of conj_q with respect to 1. Take $0 \neq r \in D$ be an eigenvector of conj_q with respect to -1, i.e.,

$$\operatorname{conj}_q(r) = -r \iff qrq^{-1} = -r \iff qr = -rq.$$

Claim: *The set $\{1, q, r, qr\}$ is linearly independent over k.*

First, $\{1, q\}$ is linearly independent because $q \notin k$. Note that r does not commute with q, and then $\{1, q, r\}$ is linearly independent; otherwise, r is a linear combination of $\{1, q\}$, which implies r commutes with q. Since $q \cdot qr = ar \neq -ar = qr \cdot q$, we know qr does not commute with q. Suppose $qr = x + yq + zr$ for some $x, y, z \in k$. Since qr does not commute with q, $z \neq 0$. Multiplying by q gives

$$q \cdot qr = q \cdot (x + yq + zr) \implies qr = -ayz^{-1} - xz^{-1}q + az^{-1}r.$$

So far we have two expressions of qr, so

$$x + yq + zr = -ayz^{-1} - xz^{-1}q + az^{-1}r \implies (x + ayz^{-1}) + (y + xz^{-1})q + (z - az^{-1})r = 0.$$

Since $\{1, q, r\}$ is linearly independent, one has $z - az^{-1}$, which implies $a = z^2$. However, $a = q^2$ is not a square in k, a contradiction. Hence we prove that $\{1, q, r, qr\}$ is linearly independent.

Now consider the automorphism $\mathrm{conj}_{r^2} : D \to D$, $s \mapsto r^2 s r^{-2}$. By using $qr = -rq$, it is easily check that conj_{r^2} fixes $\{1, q, r, qr\}$. This means r^2 commutes with all elements in D, so r^2 is in $Z(D) = k$.

To sum up, we have showed that $D = \mathrm{span}_k\{1, q, r, qr\}$, $a := q^2 \in k^\times$, $b := r^2 \in k^\times$ and $qr = -rq$. Thus $D = (a, b)_k$. ■

COROLLARY A.2.4. *Let k be a field with $\mathrm{char}\, k \neq 2$. Then every central division algebra over k of degree 2 contains a quadratic extension of k as a k-subalgebra.*

PROOF. This follows directly from the first paragraph of the proof of PROPOSITION A.2.3. ■

Now we are very close to prove that every central simple algebra over a field k with $\mathrm{char}\, k \neq 2$ is isomorphic to some quaternion algebra. To achieve this, we need to understand the general structure of a simple k-algebra.

THEOREM A.2.5. (WEDDERBURN) [FD93, THEOREM 1.11]
If A is a simple algebra over a field k, then $A \cong M_n(D)$ for some $n \in \mathbf{N}$ and some division k-algebra D.

COROLLARY A.2.6. *Let k be a field with $\mathrm{char}\, k \neq 2$. Every central simple k-algebra A of degree 2 is isomorphic a quaternion algebra. To be more precise,*

$$A \begin{cases} \cong M_2(k) \cong (1, b)_k \ \ \forall\, b \in k^\times & \text{if } A \text{ is not a division algebra;} \\ = \text{division quaternion algebra} & \text{if } A \text{ is a division algebra.} \end{cases}$$

PROOF. By WEDDERBURN'S THEOREM (THEOREM A.2.5), $A \cong M_n(D)$ for some $n \in \mathbf{N}$ and some division k-algebra D. By dimension reason, we know $n = 1$ or 2. If $n = 1$, then A is division. Thus PROPOSITION A.2.3 tells us that A is isomorphic to a quaternion algebra. If $n = 2$, then $\dim_k D$ must equal 1, i.e., $D = k$. EXAMPLE A.1.6 shows that $M_2(k)$ is also isomorphic to a quaternion algebra. ■

A.3 CLASSIFICATION OF QUATERNION ALGEBRAS

A.3.1 OVER LOCAL FIELDS

Recall that a local field is

- ◇ (archimedean) either $\mathbf{R}$ or $\mathbf{C}$, or
- ◇ (non-archimedean) a finite extension of $\mathbf{Q}_p$ for some prime $p \in \mathbf{N}$, or
- ◇ (non-archimedean) a finite extension of $\mathbf{F}_q(\!(T)\!)$, where q is a power of a rational prime.

Let k be a local field of characteristic not equal to 2, and let A be a quaternion algebra over k. From COROLLARY A.2.6 we know that $A \cong M_2(k)$ if A is non-division. We then want to know whether we can say anything as A is division. The following theorem tells us that the answer is surprisingly quite simple.

THEOREM A.3.1. [Vo21, MAIN THEOREM 13.3.2]
Let $k \neq \mathbf{C}$ be a local field of characteristic not equal to 2, and let A be a division quaternion algebra over k. Then A is unique up to k-algebra isomorphism.

Now we focus on a non-archimedean local field k. We know that for every $n \in \mathbf{N}$ there is a unique unramified extension of k of degree 2. From LEMMA A.2.1 and the proof of PROPOSITION A.2.3 we know that quaternion algebras are highly related to quadratic extensions. In the case of over non-archimedean local field, the structure of division quaternion algebra is explicit.

THEOREM A.3.2. [Vo21, THEOREM 13.3.11]
Let k be a non-archimedean local field not of characteristic 2 with a uniformizer π, and let k' be the quadratic unramified extension of k. If A is a quaternion algebra over k, then

$$A \text{ is division} \iff A \cong (k', \pi)_k.$$

A.3.2 Over Rationals

Definition A.3.3. Let A be a quaternion algebra over $\mathbf{Q}$.

(a) We say A is **ramified** at a place v of $\mathbf{Q}$ if $A \otimes_{\mathbf{Q}} \mathbf{Q}_v$ is a division ring ($\mathbf{Q}_v$ is the completion of $\mathbf{Q}$ with respect to v), and we call v **unramified** (or **split**) otherwise.

(b) The algebra A is called **definite** if A is ramified at ∞ and is called **indefinite** otherwise.

Remark A.3.4. We are going to consider the cases over $\mathbf{Q}$. Let $\mathcal{P}(\mathbf{Q})$ be the set of all places of $\mathbf{Q}$. We know

$$\mathcal{P}(\mathbf{Q}) = \{|\cdot|_p, |\cdot| = |\cdot|_\infty : p \text{ runs through all rational primes}\}.$$

We call $|\cdot|_p$'s finite places and $|\cdot| = |\cdot|_\infty$ the infinite place. In particular, there is only one infinite place of $\mathbf{Q}$, which is the usual absolute value, and $\mathbf{Q}_\infty = \mathbf{R}$. For a place $v \in \mathcal{P}(\mathbf{Q})$ we write $v \nmid \infty$ if v is a finite place and write $v \mid \infty$ (or $v = \infty$) otherwise.

Now let A be a quaternion algebra over $\mathbf{Q}$. Therefore, we have the following equivalent statements:

(1) A is unramified at all infinite places of $\mathbf{Q}$;

(2) A is indefinite (by definition, this means $A \otimes_{\mathbf{Q}} \mathbf{R}$ is non-division);

(3) $A \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})$.

Generally speaking, for a division algebra A over a field k the base change $A_K := A \otimes_k K$ to the field extension K/k is not necessarily division. However, if A is a division quaternion algebra over $\mathbf{Q}$ ramified at some place v, then $A \otimes_{\mathbf{Q}} \mathbf{Q}_v$ is again division.

Another important observation is that if A is a non-division quaternion algebra over a field k (with $\operatorname{char} k \neq 2$), then $A \otimes_k K$ is again non-division for every field extension K/k. To see this, first note that $A \cong M_2(k)$ by Corollary A.2.6. Then

$$A \otimes_k K \cong M_2(k) \otimes_k K \cong M_2(K),$$

which is again non-division.

Appendix B

Algebraic Groups and Abelian Varieties

B.1 Group Schemes

Definition B.1.1. Let $\mathcal{C}$ be a category with finite products[1], take $*$ to be a terminal object. A **group object** in $\mathcal{C}$ is an object G together with morphisms $e : * \to G$, $m : G \times G \to G$ and $i : G \to G$ such that the following diagrmas commute:

(1) (identity)

$$\begin{tikzcd} G \times * \arrow[r, "\mathrm{id} \times e"] \arrow[dr, "\mathrm{pr}_1"'] & G \times G \arrow[d, "m"] & * \times G \arrow[l, "e \times \mathrm{id}"'] \arrow[dl, "\mathrm{pr}_2"] \\ & G & \end{tikzcd}$$

where pr_1 and pr_2 are the canonical projection on the first and the second component, respectively.

(2) (inverse)

$$\begin{tikzcd} G \arrow[r, "(\mathrm{id}, i)"] \arrow[d] & G \times G \arrow[d, "m"] & G \arrow[l, "(i, \mathrm{id})"'] \arrow[d] \\ * \arrow[r, "e"] & G & * \arrow[l, "e"'] \end{tikzcd}$$

where the vertical on the left and the right are the unique maps given by the definition of the terminal object $*$.

(3) (associativity)

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G. \end{tikzcd}$$

Definition B.1.2. A **group scheme** over a scheme S is a group object G in the category $\mathbf{Sch}_S$ of S-schemes[2]. We also call the morphism $e : S \to G$ the **identity section** of G because $S \xrightarrow{e} G \to S$ is the identity map on S (this follows from the definition of G as an S-scheme). We say a group scheme G over S is **commutative** if $m \circ t = m$. Here $t : G \times_S G \to G \times_S G$ is the transposition map defined by the following commutative diagram:

$$\begin{tikzcd} G \times_S G \arrow[dr, "t"] \arrow[drr, bend left, "\mathrm{pr}_2"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \\ & G \times_S G \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & G \arrow[d] \\ & G \arrow[r] & S \end{tikzcd}$$

A group scheme G over S is called finite if the structural morphism $G \to S$ is a finite morphism (of schemes).

[1]Let $\mathcal{C}$ be a category, and let $(A_i)_{i \in I}$ be a family of objects in $\mathcal{C}$ indexed by a set I. The product of $(A_i)_{i \in I}$ is an object A in $\mathcal{C}$ together with morphisms $(\pi_i : A \to A_i)_{i \in I}$ satisfying the following universal property: if B is an object in $\mathcal{C}$ and $(\varphi_i : B \to A_i)_{i \in I}$ is a family of morphisms, then there exists a unique morphism $\varphi : B \to A$ such that $\varphi_i = \pi_i \circ \varphi$ for all $i \in I$. Such A can be shown to be unique up to isomorphism, and we denote X by $\prod_{i \in I} X_i$. We also define the product of an empty family a terminal object. We say that a category is with finite product if the product exists for every finite family of objects.

[2]In the category $\mathbf{Sch}_S$ the product is the fibre product $- \times_S -$.

PROPOSITION B.1.3. [Po17, PROPOSITION 5.1.7]
Let G be a scheme over a scheme S. Then G is a group scheme over S if the functor

$$G(\cdot) : \mathbf{Sch}_S \longrightarrow \mathbf{Set}, \quad T \longmapsto G(T) := \mathrm{Hom}_{\mathbf{Sch}_S}(T, G)$$

factors through the category $\mathbf{Grp}$. *That is, to give a group scheme structure on G, it is equivalent to giving a functor* $\mathscr{G} : \mathbf{Sch}_S^{\mathrm{opp}} \to \mathbf{Grp}$ *making the diagram commute*

$$\begin{tikzcd} \mathbf{Sch}_S^{\mathrm{opp}} \arrow[rr, "h_G"] \arrow[dr, "\mathscr{G}"'] & & \mathbf{Set} \\ & \mathbf{Grp} . \arrow[ur, "\textit{forgetful functor}"'] & \end{tikzcd}$$

Here h_G is the Yoneda functor associated to G; that is, $h_G : T \mapsto G(T)$.

PROOF. This follows directly from YONEDA'S LEMMA; for example, see [GW20, SECTION (4.2) and (4.15)]. ■

Let X and Y be two schemes over a scheme S. If $(U_i)_i$ is an affine open covering of X and $(f_i : U_i \to Y)_i$ is a compatible system of S-morphisms, then one can glue all f_i together to obtain an S-morphism $f : X \to Y$ such that $f|_{U_i} = f_i$ for all i (see [GW20, PROPOSITION 3.5]). For any scheme S we write $\mathbf{AffSch}_S$ for the category of affine S-schemes. We have the following equivalence of categories:

$$\mathbf{AffSch}_S \longrightarrow \mathbf{Alg}_{\Gamma(S,\mathscr{O}_S)}, \qquad \begin{array}{rcl} T & \longmapsto & \Gamma(T, \mathscr{O}_T) \\ \operatorname{Spec} A & \longleftarrow\!\shortmid & A \end{array}$$

The gluing property and the equivalence of categories rephrase PROPOSITION B.1.3 as the following corollary.

COROLLARY B.1.4. *For any scheme S it is equivalent to give the following data.*

(a) *A group scheme over S.*
(b) *A functor* $\mathbf{Sch}_S \to \mathbf{Grp}$.
(c) *A functor* $\mathbf{AffSch}_S \to \mathbf{Grp}$.
(d) *A functor* $\mathbf{Alg}_{\Gamma(S,\mathscr{O}_S)} \to \mathbf{Grp}$.

DEFINITION B.1.5. For two group schemes G, H over a scheme S a **group scheme homomorphism** is a morphism $\varphi : G \to H$ of S-schemes such that the following diagram commutes:

$$\begin{tikzcd} G \times_S G \arrow[r, "m_G"] \arrow[d, "(\varphi,\varphi)"'] & G \arrow[d, "\varphi"] \\ H \times_S H \arrow[r, "m_H"] & H. \end{tikzcd}$$

The **kernel** of φ, which is denoted by $\ker \varphi$, is defined by the cartesian diagram

$$\begin{tikzcd} \ker \varphi := G \times_H S \arrow[r] \arrow[d] & G \arrow[d, "\varphi"] \\ S \arrow[r, "e_H"] & H. \end{tikzcd}$$

REMARK B.1.6. Let G, H be two group schemes over a scheme S, and let $\varphi : G \to H$ be an S-group scheme homomorphism.

(a) Note that in abstract algebra the kernel of a group homomorphism is defined to be the fibre of the identity element. In scheme theory, the fibre of a morphism $f : X \to Y$ at $s \in S$ of schemes at $y \in Y$ is $\varphi^{-1}(y) \cong X_y := X \times_Y \mathbf{k}(y)$. Now come back to our case: H is a group scheme over S with the identity section $e_H : S \to H$. Thus the kernel of φ should be "the fibre of the identity", which exactly means the fibre product of φ and e_H: $\ker \varphi = G \times_H S$. This illustrates our definition of $\ker \varphi$.

(b) We can also describe the homomorphism φ with more categorical flavor.
φ is a natural transformation from the functors G to the functor H. To be more precise,

(c) With the categorical description of φ in (b), one obtains the following categorical characterization of $\ker\varphi$:
The kernel $\ker\varphi : \mathbf{Sch}_S \to \mathbf{Grp}$ *is a functor associating each S-scheme T the group* $\ker(G(T) \to H(T))$.

Definition B.1.7. Let G be a group scheme over a scheme S. A **subgroup scheme** of G is a group scheme G' over S which is also a closed subscheme of G' such that the closed immersion $G' \hookrightarrow G$ is a group scheme homomorphism. Moreover, we say G' is **normal** if $G'(T)$ is a normal subgroup of $G(T)$ for all S-scheme T.

Remark B.1.8. Let G, H be two group schemes over a scheme S, and let $\varphi : G \to H$ be an S-group scheme homomorphism.

(a) There is a categorical characterization of subgroup schemes as what follows.
G' is a subgroup scheme of G if and only if $G'(T)$ is a group with an injective group homomorphism $G'(T) \hookrightarrow G(T)$ for every $T \in \mathbf{Sch}_S$.

(b) By definition, $\ker\phi$ is a scheme. In fact, $\ker\phi$ is not just a scheme but moreover an S-subgroup scheme of G.

B.2 Examples

Example B.2.1. Let A be a commutative unitary ring.

(a) The **additive group scheme** of A is the scheme $\mathbf{G}_{a,A} := \operatorname{Spec} A[T]$ with $m : \mathbf{G}_{a,A} \times_A \mathbf{G}_{a,A} \to \mathbf{G}_{a,A}$ induced by

$$A[T] \longrightarrow A[T] \otimes_A A[T], \quad T \longmapsto T \otimes 1 + 1 \otimes T$$

Consider the functor

$$\mathbf{Alg}_A \longrightarrow \mathbf{Grp}, \quad B \longmapsto (B, +).$$

In short, this functor sends each A-algebra to its additive group. Since we have

$$(B, +) \cong \operatorname{Hom}_A(A[T], B) \cong \operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} B, \operatorname{Spec} A[T]) = \operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} B, \mathbf{G}_{a,A}),$$

one concludes that this functor is represented by $\mathbf{G}_{a,A}$. This illustrates why we call $\mathbf{G}_{a,A}$ an additive group.

(b) The **multiplicative group scheme** of A is the scheme $\mathbf{G}_{m,A} := \operatorname{Spec} A[T, T^{-1}]$ with the multiplicative map $m : \mathbf{G}_{m,A} \times_A \mathbf{G}_{m,A} \to \mathbf{G}_{m,A}$ induced by

$$A[T, T^{-1}] \longrightarrow A[T, T^{-1}] \otimes_A A[T, T^{-1}], \quad T \longmapsto T \otimes T$$

Consider the functor

$$\mathbf{Alg}_A \longrightarrow \mathbf{Grp}, \quad B \longrightarrow (B^\times, \times).$$

That is, this functor associate to each A-algebra its multiplicative group. From

$$(B^\times, \times) \cong \operatorname{Hom}_A(A[T, T^{-1}], B) \cong \operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} B, \operatorname{Spec} A[T, T^{-1}]) = \operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} B, \mathbf{G}_{m,A}),$$

we derive that this functor is represented by $\mathbf{G}_{m,A}$, and this tells us that $\mathbf{G}_{m,A}$ is indeed a "multiplicative group".

Example B.2.2. Let A be a commutative unitary ring.

(a) For any $m, n \in \mathbf{N}$ define the m-by-n matrix (algebraic) group $M_{m\times n,A} := \operatorname{Spec} A[T_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$. Look at the functor

$$\mathbf{Alg}_A \longrightarrow \mathbf{Grp}, \quad B \longmapsto (M_{m\times n}(B), +).$$

(b) For any $n \geq 1$ the **general linear (algebraic) group** over A is

$$\mathrm{GL}_{n,A} := \operatorname{Spec} A[T_{ij}, T : 1 \leq i \leq m, 1 \leq j \leq n]/(\det \cdot T - 1).$$

(c) For any $n \geq 1$ the **special linear (algebraic) group** is the scheme

$$\mathrm{SL}_{n,A} := \operatorname{Spec} A[T_{ij} : 1 \leq i, j \leq n]/(\det -1).$$

EXAMPLE B.2.3. Let G be an abstract group and S a scheme. The **constant group scheme** of G over S, which is denoted by S^G, is defined as follows. The underlying scheme of S^G is

$$\coprod_{g\in G} S_g, \quad \text{where } S_g = S \text{ for every } g \in G.$$

We define the identity map, inversion map and multiplication map by

$$\begin{aligned} &e : S \longrightarrow S^G, \quad s \longmapsto (s)_{g\in G};\\ &i : S_g \xrightarrow{\equiv} S_{g^{-1}} \quad \text{for each } g \in G;\\ &S_g \times_S S_h \xrightarrow{\sim} S_{gh} \quad \text{for any two } g, h \in G. \end{aligned}$$

EXAMPLE B.2.4. This example is extremely important in our main context toward the proof of Cerednik-Drinfeld Theorem. Let A be a group scheme over a scheme S. For each $n \in \mathbf{Z}$ we are going to define the **multiplication-by-n** map $[n]_A \equiv [n] : A \to A$ (sometimes we omit the subscript A if it is clear from the context). For each S-scheme T

$$[n](T) : A(T) \longrightarrow A(T), \quad x \longmapsto nx.$$

This defines the morphism by the categorical description.

EXAMPLE B.2.5. Let S be a scheme over a field k with $\operatorname{char} k = p > 0$. We define the **absolute Frobenius morphism** $\operatorname{Frob}_S : S \to S$ to be the k-morphism given by

- ◇ the map $\operatorname{Frob}_S : S \to S$ on the underlying topological space is just the identity;
- ◇ the morphism of sheaves $\operatorname{Frob}_S^\flat : \mathscr{O}_S \to \operatorname{Frob}_* \mathscr{O}_S = \mathscr{O}_S$ is the p-th power map; namely, for any open subset $U \subseteq S$ one has $\Gamma(U, \mathscr{O}_S) \to \Gamma(U, \mathscr{O}_S)$, $x \mapsto x^p$.

Note that $\operatorname{char} k = p > 0$ implies $p\Gamma(U, \mathscr{O}) = 0$ for all open subsets $I \subseteq S$, so $\Gamma(U, \mathscr{O}_S) \to \Gamma(U, \mathscr{O}_S)$, $x \mapsto x^p$ is indeed a k-algebra homomorphism.

Now consider an S-scheme $\pi : X \to S$. Since S is a k-scheme, one can view X as a k-scheme. Thus we also have the absolute Frobenius morphism $\operatorname{Frob}_X : X \to X$. Unfortunately, Frob_X is generally not an S-morphism. To make a modification, we first define $X^{(p)}$ to be the base change of X via $\operatorname{Frob}_S : S \to S$; that is, $X^{(p)}$ fits into the cartesian diagram

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ S \arrow[r, "\operatorname{Frob}_S"] & S \end{tikzcd}$$

Now we introduce the notion of **relative Frobenius morphism** $F_{X/S} : X \to X^{(p)}$ given by the following the universal property of $X^{(p)}$:

$$\begin{tikzcd} X \arrow[dr, "F_{X/S}", "\exists!"'] \arrow[drr, bend left, "\operatorname{Frob}_X"] \arrow[ddr, bend right, "\pi"'] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & S \arrow[r, "\operatorname{Frob}_S"] & S \end{tikzcd}$$

Notice that we have to check $\pi \circ \operatorname{Frob}_X = \operatorname{Frob}_S \circ \pi$ so that the above diagram makes sense. Since $\operatorname{Frob}_X = \operatorname{id}_X$ and $\operatorname{Frob}_S = \operatorname{id}_S$ for the underlying topological spaces X and S, respectively, $\pi \circ \operatorname{Frob}_X = \operatorname{Frob}_S \circ \pi = \pi$ for the underlying topological spaces. For any open subset $U \subseteq X$ we have

$$\begin{tikzcd} \Gamma(\pi(U), \mathscr{O}_S) \arrow[r, "(\cdot)^p"] \arrow[d, equal] & \Gamma(\pi(U), \mathscr{O}_S) \arrow[r, "\pi_U^\flat"] & \Gamma(U, \mathscr{O}_X) \arrow[d, equal] \\ \Gamma(\pi(U), \mathscr{O}_S) \arrow[r, "\pi_U^\flat"'] & \Gamma(U, \mathscr{O}_X) \arrow[r, "(\cdot)^p"'] & \Gamma(U, \mathscr{O}_X) \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, equal] & x^p \arrow[r, mapsto] & \pi_U^\flat(x^p) \arrow[d, equal] \\ x \arrow[r, mapsto] & \pi_U^\flat(x) \arrow[r, mapsto] & \left(\pi_U^\flat(x)\right)^p \end{tikzcd}$$

Then one has $\pi^\flat \circ \mathrm{Frob}_S^\flat = \mathrm{Frob}_X^\flat \circ \pi^\flat$. Therefore, we conclude the equality $\pi \circ \mathrm{Frob}_X = \mathrm{Frob}_S \circ \pi$ of morphisms of schemes.

Specifically, we can apply this construction to a the case of group schemes. If G is a group scheme over a field k with $\operatorname{char} k = p > 0$, then we let $X = G$ and $S = k$ to obtain the relative Frobenius morphism $F_{G/k} : G \to G$..

B.3 Algebraic Groups

Recall that fppf is the abbreviation of "faithfully flat and locally of finite presentation". Every morphsim $X \to \operatorname{Spec} k$ from a non-empty scheme X to the prime spectrum of a field k is faithfully flat (this follows immediately from the definition, see [GW20, Section 14.2]). Another fact (see [GW20, Remark 10.36]) is that if Y is a locally noetherian scheme, then a morphism $X \to Y$ of schemes is locally of finite presentation (resp. of finite presentation) if and only if it is locally of finite type (resp. of finite type). Therefore, we have the following easy consequence.

Lemma B.3.1. *A group scheme over a field k is fppf if and only if it is locally of finite type over k.*

Proposition B.3.2. [GW20, Proposition 16.50, Proposition 16.51, Corollary 16.52]
Let G be an fppf group scheme over a field k.

(a) *If k is perfect, then G_{red} is a smooth closed subgroup scheme of G over k.*

(b) *G is geometrically irreducible if and only if G is connected.*

Theorem B.3.3. [Po17, Theorem 5.2.16]
Let G be an fppf group scheme over a field k. The following statements are equivalent.

(a) *G is smooth over k.*

(b) *G is geometrically reduced over k.*

(c) *The local ring $\mathscr{O}_{G_{\bar{k}},e}$ is a reduced ring.*

(d) *Either $\operatorname{char} k = 0$, or $\operatorname{char} k = p > 0$ and the relative Frobenius morphism $F_{G/k} : G \to G^{(p)}$ is surjective.*

Definition B.3.4. An **algebraic group** is a group scheme of finite type over a field k.

Remark B.3.5.

(a) An **algebraic scheme** over a field k is a scheme of finite type over k. Thus an algebraic group over k is a group object in the category of algebraic k-scheme.

(b) Applying Corollary B.1.4, we then have the following equivalent definition of algebraic groups. *An algebraic group over a field k is a k-scheme*

Definition B.3.6. Let (G, m_G) and (H, m_H) be two algebraic groups over a field k. A **homomorphism** $\varphi : (G, m_G) \to (H, m_H)$ is a k-morphism which also a group scheme homomorphism.

Theorem B.3.7. [Po17, Theorem 5.2.20]
Every algebraic group is quasi-projective.

Definition B.3.8. Let G be an algebraic group over a field k.

(a) An **algebraic subgroup** (or briefly a **subgroup**) of G is a k-subscheme which is also a subgroup scheme of G.

(b) An algebraic subgroup H of G is called **normal** if H is normal subgroup scheme of G.

Definition B.3.9. A **linear algebraic group** (or an **affine algebraic group**) is an algebraic group whose underlying scheme is affine.

Theorem B.3.10. [Po17, Theorem 5.3.1]
Let G be an algebraic group over a field k. Then G is a linear algebraic group if and only if G is isomorphic to an algebraic subgroup of $\mathrm{GL}_{n,k}$.

B.4 Reductive Groups

Definition B.4.1. Let G be an algebraic group over a field k.

(a) A **composition series** of G is a chain

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

where G_i is a closed normal subgroup of G_{i+1} for each $0 \leq i \leq n-1$. We call the groups $(G_{i+1}/G_i)_{0 \leq i \leq n-1}$ the **successive quotient** of the composition series.

(b) We say G is **unipotent** if there is a composition series of $G_{\bar{k}}$ such that every successive quotient is a closed subgroup of $\mathbf{G}_{a,\bar{k}}$.

Theorem B.4.2. [Po17, Theorem 5.4.8]
Let G be an algebraic group over a field k. Then the following statements are equivalent.

(a) *G is unipotent.*

(b) *There is an embedding $G \hookrightarrow U_n$ of algebraic groups for some $n \in \mathbf{N}$.*

(c) *There is a composition series of G such that*

- *if $\operatorname{char} k = 0$, every successive quotient is isomorphic to $\mathbf{G}_{a,k}$;*
- *if $\operatorname{char} k = p > 0$, every successive quotient is isomorphic to one of the following: $\boldsymbol{\alpha}_p$, $\mathbf{G}_{a,k}$ or $k^{(\mathbf{Z}/p\mathbf{Z})^n}$ for some $n \geq 1$.*

All these statements imply the following:

(d) *Every element of $G(\bar{k})$ is unipotent.*

Moreover, if G is smooth, then (d) is equivalent to (a),(b) and (c).

Definition B.4.3. Let G be a smooth linear algebraic group over a field k.

(a) The **unipotent radical** of G, which is denoted by $R_u(G)$, is the unique maximal smooth connected unipotent normal subgroup of G.

(b) We say G is **reductive** if $R_u(G_{\bar{k}}) = \{1\}$.

Remark B.4.4. There are some differences of terminologies in various books. Let us take a look at them and investigate their relations a little bit.

(a) There are several versions of the definition of **varieties**. Let X be a scheme over a field k.

- In [GW20, Ha77, SP05], X is a variety if it is integral separated and of finite type. If X is only integral and of finite type but not necessarily separated, then it is called a **prevariety** in [GW20].
- In [Mi17], X is a variety if it is separated geometrically reduced and of finite type.
- In [Po17], X is a variety if it is separated and of finite type (namely, a prevariety in the sense of [GW20]).

(b) Now we are going to compare the definitions of reductive groups in [Mi17] and [Po17]. Let G be an affine algebraic group over a field k.

- In [Po17], G is a reductive group if it is smooth and $R_u(G_{\bar{k}}) = \{1\}$.
- In [Mi17], G is a reductive group if it is a connected variety and $R_u(G_{\bar{k}}) = \{1\}$.

In [Mi17], a variety is in particular geometrically reduced, so it is automatically smooth by Theorem B.3.3. Therefore, the definition of reductive groups in [Po17] is weaker than the one in [Mi17].

Example B.4.5. For any field k and any positive integer n the algebraic group $\mathrm{GL}_{n,k}$ is reductive. Note that $U_{n,k}$ is a smooth connected unipotent subgroup of $\mathrm{GL}_{n,k}$, but it is not a normal subgroup.

B.5 Abelian Varieties

Definition B.5.1. Let k be a field. An **abelian variety** over k is a connected, geometrically reduced and proper group scheme over k.

Remark B.5.2. We first quote two properties of morphisms. Let $f : X \to Y$ be a morphism of schemes.

(1) If f is proper, then f is of finite type (this is just the definition of proper morphisms).
(2) If f is proper and quasi-projective and Y is qcqs (i.e., quasi-compact and quasi-separated), then f is projective (see [GW20, Corollary 13.72]).

Therefore, an abelian varieties is particularly an algebraic group by (a). Moreover, Theorem B.3.3 and Proposition B.3.2 indicate that an abelian variety is automatically smooth and geometrically integral. On the mother hand, we have seen in Theorem B.3.7 that every algebraic group is quasi-projective, so (b) implies that an abelian variety is always projective (note that the affine scheme attached to a field is qcqs). To sum up, an abelian variety over a field k is an algebraic group G over k satisfying

(1) G is geometrically integral (this condition is equivalent to G is connected and smooth over k);
(2) G is projective (this condition is equivalent to G is proper over k).

Definition B.5.3. Let S be a scheme. An **abelian scheme** over S is a group scheme A over S which is proper smooth and has connected fibres.

Remark B.5.4.

(a) Recall a general property of schemes from [Ha77, Exercise 2.2.7].
Let X be a scheme and K a field. Then it is equivalent to give the following two datum:
   (1) *A morphism $f : \operatorname{Spec} K \to X$;*
   (2) *A point $x \in X$ and an inclusion $\mathbf{k}(x) \hookrightarrow K$ of fields.*

   More precisely, if we have a morphism $f : \operatorname{Spec} K \to X$, then we have an inclusion $\mathbf{k}(f(\operatorname{Spec} K)) \hookrightarrow K$ of fields; conversely, if we have a point $x \in X$ and an inclusion $\mathbf{k}(x) \hookrightarrow K$, then we have a morphism $f : \operatorname{Spec} K \to X$ such that $f(\operatorname{Spec} K) = x$.
   Now we apply the above property. Let X be a scheme. For each point $x \in X$, we always have a natural morphism $\operatorname{Spec} \mathbf{k}(x) \to X$ whose image is exactly x. Moreover, for each $x \in X$ we can associate naturally a geometric point $\bar{x}$ to it by $\operatorname{Spec} \overline{\mathbf{k}(x)} \to X$. Note that this morphism is given by the composition $\operatorname{Spec} \overline{\mathbf{k}(x)} \to \operatorname{Spec} \mathbf{k}(x) \to X$.

(b) We quote the following property from [GW20, Corollary 5.45].
Let k be a field and X, Y two schemes over k. Let $\mathbf{P}$ be any one of the properties: irreducible, connected, reduced, integral. Then X possesses $\mathbf{P}$ if and only if $X \times_k Y$ possesses $\mathbf{P}$.
Now we can give another equivalent definition of abelian schemes. An abelian scheme over a scheme S is a group scheme G over S which is proper smooth and has connected geometric fibres. Recall that a geometric fibre is the fibre of a geometric point. By (a) we have cartesian diagrams for any $s \in S$:

$$\begin{tikzcd} G_{\bar{s}} \arrow[r] \arrow[d] & G_s \arrow[r] \arrow[d] & G \arrow[d] \\ \operatorname{Spec} \overline{\mathbf{k}(s)} \arrow[r] & \operatorname{Spec} \mathbf{k}(s) \arrow[r] & S \end{tikzcd}$$

Then the quoting property implies that G_s is connected if and only if $G_{\bar{s}}$ is connected, so the two definitions are indeed equivalent.

(c) Let G be an abelian scheme over a scheme S. For each $s \in S$ consider the cartesian diagram

$$\begin{tikzcd} G_s \arrow[r] \arrow[d] & G \arrow[d] \\ \mathbf{k}(s) \arrow[r] & S \end{tikzcd}$$

From (b) we know that G_s is connected. In addition, smoothness and properness are stable under base change (see [GW20, Proposition 6.15, Proposition 12.58]). Thus we conclude that G_s is a connected proper smooth scheme over the field $\mathbf{k}(s)$, and this shows that G_s is an abelian variety over $\mathbf{k}(s)$.

(d) If G is an abelian scheme over a scheme S, then the identity section $e : S \to G$ is a closed immersion. To see this, we state and prove the following general property.
Let X be a scheme over a scheme Y. If the structural morphism $\varphi : X \to Y$ is separated, then any Y-morphism $\psi : Y \to X$ is a closed immersion (note that ψ is a section of φ, i.e., $\varphi \circ \psi = \mathrm{id}_Y$).

PROOF. Note that we have the following cartesian diagram:

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d, "\psi"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\Delta"] & X \times_Y X, \end{tikzcd}$$

where for any scheme T the map $\phi(T) : X(T) \to (X \times_Y X)(T)$ is given by $f \mapsto (f, \psi)$. Since $\varphi : X \to Y$ is separated, Δ is by definition a closed immersion. Therefore, ψ is also a closed immersion because being a closed immersion is stable under base change. ■

PROPOSITION B.5.5. [Ka14, THEOREM 1]
Every abelian scheme is commutative (as a group scheme).

B.6 ISOGENIES

DEFINITION B.6.1. A homomorphism $\varphi : A \to B$ of abelian schemes over a scheme S is called an **isogeny** if it is surjective and $\ker \varphi$ is finite, i.e., $\ker \varphi$ is a finite group scheme.

LEMMA B.6.2. [Ka14]

(a) *Every isogeny of abelian schemes is proper, locally free (thus flat) and finite.*

(b) *The composition of two isogenies of abelian schemes is again an isogeny of abelian schemes.*

DEFINITION B.6.3. Let $\varphi : A \to B$ be an isogeny of abelian schemes over a scheme S. The **degree** of φ, which is denoted by $\deg \varphi$, is the rank of $\ker \varphi$.

PROPOSITION B.6.4. *Let $\varphi : A \to B$ be an isogeny of abelian schemes over a scheme S, and let $n := \deg \varphi$.*

(a) *There exists an isogeny $\psi : B \to A$ such that $\varphi \circ \psi = [n]_B$ and $\psi \circ \varphi = [n]_A$.*

(b) *If S is a prime spectrum of a field (i.e., A, B are abelian varieties over k), then $n = [\mathbf{K}(A) : \mathbf{K}(B)]$, where $\mathbf{K}(A)$ and $\mathbf{K}(B)$ are the function fields of A and B, respectively.*

DEFINITION B.6.5. Let A, B be two abelian schemes over a scheme S. We say that A and B are **isogenous** if there exists an isogeny $\varphi : A \to B$. By PROPOSITION B.6.4 (a) it is equivalent to there exists an isogeny $\psi : B \to A$.

Obviously, being isogenous is reflexive because the identity map is of course an isogeny. As we have pointed out in the definition, being isogenous is symmetric by PROPOSITION B.6.4 (a). Finally, LEMMA B.6.2 (b) tells us that being isogenous is transitive. To sum up, we see that being isogenous is an equivalence relation in the category of abelian schemes over a scheme S.

PROPOSITION B.6.6. [Ka14, COROLLARY 1]
Let A be an abelian scheme over a scheme S, and let g be the relative dimension of A. For any $n \in \mathbf{Z}$ the map $[n] : A \to A$ is an isogeny of degree n^{2g}.

DEFINITION B.6.7. Let A, B be two abelian scheme over a scheme S. We denote by by $\mathrm{Isog}_S(A, B)$ the set of isogenies from A to B, and this set has a natural $\mathbf{Z}$-module structure with

$$n \cdot f := [n]_B \circ f \quad \text{for any } f \in \mathrm{Isog}(A, B) \text{ and } n \in \mathbf{Z}.$$

A **quasi-isogeny** from A to B is an element in $\mathrm{Isog}(A, B) \otimes_{\mathbf{Z}} \mathbf{Q}$.

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Bibliography

[Alle] P. Allen. *Nonarchimedean local fields*. Online notes, available at https://faculty.math.illinois.edu/~pballen/research/LocalFields.pdf.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Company, 1969.

[BC91] J.-F. Boutot and H. Carayol. *Uniformisation p-adique des courbes de Shimura: les théorèmes de Čerednik et de Drinfeld*. Courbes modulaires et courbes de Shimura, Astérisque, tome **196-197**, 1991.

[BGR84] S. Bosch, U. Gntzer and R. Remmert. *Non-archimedean analysis: a systematic approach to rigid analytic geometry*. Grundlehren der mathematischen Wissenschaften **261**, Springer-Verlag Berlin Heidelberg, 1984.

[BLR90] S. Bosch, L. Ltkebohmert and M. Raynaud. *Néron models*. Springer-Verlag Berlin Heidelberg, 1990.

[Bo14] S. Bosch. *Lectures on formal and rigid geometry*. Lecture Notes in Mathematics **2105**, Springer, 2014.

[DF04] D. S. Dummit and R. M. Foote. *Abstract algebra*. Third edition, John Siley and Sons, Inc., 2004.

[Dr76] V. G. Drinfeld. *Coverings of p-adic symmetric regions*. Functional Analysis and Its Applications volume **10**, 107–115, 1976.

[EGAIV$_4$] A. Grothendieck and J. A. Dieudonné. *Eléments de Géométrie Algébrique IV*. Publ. Math. IHÉS **32**, 1967.

[FD93] B. Farb and R. K. Dennis. *Non-commutative algebra*. Graduate Texts in Mathematics **144**, Springer, 1993.

[FK92] H. M. Farkas and I. Kra. *Riemann surfaces*. Second edition, Graduate Texts in Mathematics **71**, Springer, 1992.

[GS06] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics **101**, Cambridge University Press, 2006.

[GvdP80] L. Gerritzen and M. van der Put. *Schottky groups and Mumford curves*. Lecture Notes in Mathematics **817**, Springer, 1980.

[GW20] U. Görtz and T. Wedhorn. *Algebraic geometry I: schemes with examples and exercises*. Second edition, Springer Spektrum, 2020.

[Ha77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics **52**, Springer, 1977.

[Ka14] J. Kass. *Notes on abelian schemes*. Online notes, available at https://people.math.sc.edu/kassj/AbelianSchemes.pdf.

[La79] S. Lang. *Algebra*. Revised third edition, Graduate Texts in Mathematics **211**. Springer, 2002.

[Li02] Q. Liu. *Algebraic geometry and arithmetic curves*. Oxford Graduate Texts in Mathematics **6**, Oxford University Press, 2002.

[Lü16] W. Lütkebohmert. *Rigid geometry of curves and their Jacobians*. Ergebnisse der Mathematik und ihrer Grenzgebiete **61**, Springer, 2016.

[Ma86] H. Matsumura. *Commutative ring theory*. Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986.

[Mi15] P. Milione. *Shimura curves and their p-adic uniformization*. PhD thesis in Universitat de Barcelona, available at http://diposit.ub.edu/dspace/bitstream/2445/109649/1/MILIONE_PhD_THESIS.pdf, 2015.

[Mi17] J. S. Milne. *Algebraic groups: the theory of group schemes of finite type over a field*. Cambridge Studies in Advanced Mathematics **170**, Cambridge University Press, 2017.

[MO15] D. Mumford and T. Oda. *Algebraic geometry II*. Texts and Readings in Mathematics **73**, Hindustan Book Agency, 2015.

[Mu70] D. Mumford. *Abelian varieties: with appendices by C. P, Ramanujam and Yuri Manin*. Hindustan Book Agency, 1970.

[Mu72] D. Mumford. *An analytic construction of degenerating curves over complete local rings*. Compositio Math. 24.2, 129-174, 1972.

[Ne99] J. Neukirch. *Algebraic number theory*. Grundlehren der mathematischen Wissenschaften **323**, Springer-Verlag Berlin Heidelberg, 1999.

[Po17] B. Poonen. *Rational points on varieties*. Graduate Studies in Mathematics **186**, American Mathematical Society, 2017.

[SE21] StackExchange. https://math.stackexchange.com/questions/4157626/is-a-bijective-etale-morphism-from-a-quasi-projective-bbb-c-scheme-to-a-prope.

[Se79] J.-P. Serre. *Local Fields*. Graduate Texts in Mathematics **67**. Springer, 1979.

[Si09] J. H. Silverman. *The arithmetic of elliptic curves*. Second edition, Graduate Texts in Mathematics **106**.

[SP05] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu.

[Vo21] J. Voight. *Quaternion algebras*. Graduate Texts in Mathematics **288**. Springer, 2021.

[Zi84] T. Zink. *Cartiertheorie kommutativer formaler gruppen*. Teubner-Texts zur Mathematik, Leipzig, Teubner, 1984.