Symplectic aspects of Gaudin integrable systems and Szegö kernel variational method

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Abstract

Symplectic aspects of Gaudin integrable systems and Szegö kernel variational method

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In this thesis, we will study the symplectic aspects of classical Gaudin systems, an important type of integrable dynamical systems at both classical and quantum levels. After a review of integrability and the Lax representation of integrable dynamical systems, we will investigate the analytical properties of Gaudin model via its spectral curve. The main focus is to reconstruct the Lax matrix using the analytical information of the system and subsequently, provide a symplectic structure for the phase space. We will also calculate the symplectic potential in terms of action-angle coordinates using Szegö kernel variational method. A brief look into the spectral transform aspect as well as the study of variational properties of vector of Riemann constants will also follow.

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Contents

Chapter 1

Introduction

This work presents a study of rational Gaudin systems and their symplectic properties from an analytical perspective. Since the introduction of Guadin model in 1976, this has been a very attractive topic of study in the field of mathematical physics. The unique properties of Gaudin models make them quite significant in the study of both classical and quantum integrable dynamical systems.

Chapter 2 consists of a review of underlying concepts of integrability, Lax representation and some notions from symplectic geometry. We will then provide a general description of Gaudin model framework, followed by an overview of the analytical properties of Gaudin system and subsequently, the analytical approach to constructing the solutions of the system. Building on what we learn and the tools we develop in this analytical study of Gaudin models, we will then expand the work to the phase space and its symplectic structure. We are aiming to show that there exists a canonical way of constructing a symplectic structure via Darboux coordinates. This requires showing that the main components of symplectic structure, which are the canonical 1-form and 2-form, also known as symplectic potential and symplectic form, have a loosely speaking, natural representation in terms of the provided coordinates.

The classical Gaudin system has the rational matrix representation

$$\begin{cases} \Big(\sum_{j=1}^{m} \dfrac{A_j}{\lambda - \lambda_j}\Big)\Psi = \mu\Psi \\ \dfrac{d}{dt_j}\Psi = \dfrac{A_j}{\lambda - \lambda_j}\Psi \end{cases} \tag{1.0.1}$$

where $A_j \in \mathfrak{sl}(N, \mathbb{C})$ and Ψ a vector of dimension N. Lax representation paves the way to

the analytical study of integrable systems. Lax equation which is essentially an alternative representation of equations of motion, is centered around the Lax matrix $L(\lambda)$, λ being the spectral parameter. Setting the Lax pair $L(\lambda) = \sum_j \frac{A_j}{\lambda - \lambda_j}$ and $M_j(\lambda) = \frac{A_j}{\lambda - \lambda_j}$, with Ψ and μ being the eigenvector and eigenvalue of the Lax operator $L(\lambda)$, the Lax representation of Gaudin model becomes

$$\begin{cases} L(\lambda)\Psi = \mu\Psi \\ \dfrac{d}{dt_j}\Psi = M_j(\lambda)\Psi \end{cases} \tag{1.0.2}$$

Lax matrix contains all crucial information to describe the system so looking into its eigenvectors and eigenvalues will tell us a lot about the dynamical system in question. The key here is the characteristic polynomial of Lax matrix $\det(L(\lambda) - \mu\mathbf{1})$ which bridges the eigenvectors and eigenvalues. Since the Lax matrix is an element of $\mathfrak{sl}(N, \mathbb{C})$. The characteristic polynomial has N complex values μ (eigenvalues) associated to a given λ. This means that the characteristic polynomial defines an N-sheeted covering of extended complex plane (i.e. $\mathbb{CP}^1$ or Riemann sphere) hence it can be realized as a compact Riemann surface called the *spectral curve*. More specifically, the spectral curve Γ associated to the system is the compact Riemann surface described by the equation $\det(L(\lambda) - \mu\mathbf{1}) = 0$ and it has the genus

$$g = \frac{N(N-1)m}{2} - N + 1 \tag{1.0.3}$$

The eigenvectors, being meromorphic functions, now form a vector bundle on this surface and using the analytical data contained in the poles of Lax matrix (the data we will call *dynamical data*), and the fact that the field of meromorphic functions on a Riemann surface has a parametrization in terms of Jacobi theta functions, we will be able to reconstruct the components of the eigenvectors, hence the Lax matrix itself, via just the dynamical data of the system. More specifically, the goal is to reconstruct the matrix of eigenvectors

$$\hat{\Psi}(\lambda) = \begin{pmatrix} \psi_1(P_1) & \dots & \psi_1(P_N) \\ \vdots & \vdots & \vdots \\ \psi_N(P_1) & \dots & \psi_N(P_N) \end{pmatrix} \tag{1.0.4}$$

Then the Lax matrix can be reconstructed via

$$L(\lambda) = \hat{\Psi}(\lambda)\hat{\mu}\hat{\Psi}^{-1}(\lambda) \tag{1.0.5}$$

where $\hat{\mu} = \mathrm{diag}(\mu_1, ..., \mu_N)$, μ_j the eigenvalues corresponding to a given spectral parameter λ. With the assumption of $L(\lambda_0)$ being diagonal, the main result is the following formula for the components of eigenvectors:

$$\psi_k(P) = C_k \frac{\theta\big(\mathcal{A}(P) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(1)}) - \mathcal{A}(D) - \mathcal{K}\big)}{\theta\big(\mathcal{A}(P) - \mathcal{A}(D) - \mathcal{K}\big)} \frac{\theta(e + \int_{\lambda_0^{(1)}}^{P} \omega)}{\theta(e + \int_{\lambda_0^{(k)}}^{P} \omega)} \tag{1.0.6}$$

$\mathcal{A}$ being the Abel map, $\mathcal{K}$ the vector of Riemann constants, $\lambda_0^{(j)}$ the points above λ_0 on Γ and θ the Jacobi theta function on $\mathbb{C}^g$. This will be covered in chapter 3. We will also review some applications of Gaudin system, namely $\mathfrak{su}(2)$ Gaudin model, in describing dynamical systems in physics such as Lagrange top.

The work of chapter 3 will then be continued to the phase space of the system $\mathcal{M}$, characterized by the set of pairs $\{G_j, L_j\}$ modulo the right multiplication of G_j by diagonal matrices where $A_j = G_j L_j G_j^{-1}$, and also the quotient by the action of gauge group keeping the Lax equation invariant. The phase space turns out to be a $2g$-dimensional. This space can be equipped with a canonical symplectic structure, using either a set of Darboux coordinates satisfying standard Poisson bracket relations known as *dynamical variables*, or another set of Darboux coordinates , in this case action-angle variables. Considering the canonical 1-form

$$\Theta_{\mathcal{M}} = \sum_{j=1}^{m} \mathrm{tr}\, L_j G_j^{-1} dG_j \tag{1.0.7}$$

also known as *symplectic potential*, its exterior derivative $\omega_{\mathcal{M}} = d\Theta_{\mathcal{M}}$, gives the symplectic 2-form on the phase space. The symplectic structure of phase space is studied in chapters 4 and 5. We will provide two different parametrizations of $\mathcal{M}$: First, via the so-called *dynamical variables*. These are the coordinates of g points $\gamma_j = (\lambda_{\gamma_j}, \mu_{\gamma_j})$ on the spectral curve whose formal sum defines the *dynamical divisor*, i.e. the divisor of the poles of Lax matrix $L(\lambda)$. The second parametrization which is the one we are mostly interested in will be given by the so-called *action-angle* variables $\{q_j, I_j\}_{j=1}^{g}$.

The significance of action-angle variables comes from the fact that the angle variables

are coming from a torus and their canonical conjugates, action variables, are functions of constants of motion. Consequently, the Hamiltonian can be expressed in terms of action variables only. This provides a useful geometric description of the system and has important applications in quantum mechanical approach and perturbation theory.

The essence of chapter 4 and 5 may be summarized by the following:

$$\omega_{\mathcal{M}} = \sum_{j=1}^{g} d\mu_{\gamma_j} \wedge d\lambda_{\gamma_j} = \sum_{j=1}^{g} dI_j \wedge dq_j \tag{1.0.8}$$

The link between the symplectic structure of phase space and the analytical reconstruction of Lax matrix is that angle variables also appear in the theta functions expressing the eigenvectors as characteristics. The conjugate action variables will also be defined via the loop integrals of canonical meromorphic 1-form $\mu d\lambda$ over the a-cycles of homology group basis.

In chapter 6, we will turn our focus to the symplectic potential, that is a canonical 1-form generating the symplectic form $\omega_{\mathcal{M}}$ via its exterior derivative. Expressing the canonical 1-form in terms of action-angle variables requires the calculation of partial derivatives of G_j matrices with respect to angle and action variables. In other words, the following needs to be calculated:

$$\Theta_{\mathcal{M}} = \sum_{j=1}^{m} \mathrm{tr}(L_j G_j^{-1} \frac{\partial G_j}{\partial q_\gamma})\ dq_\gamma + \sum_{j=1}^{m} \mathrm{tr}(L_j G_j^{-1} \frac{\partial G_j}{\partial I_\gamma})\ dI_\gamma \tag{1.0.9}$$

While this is may not seem straightforward on the first sight, rewriting these matrices in terms of Szegö kernel, a $(\frac{1}{2}, \frac{1}{2})$-form on $\Gamma \times \Gamma$ given by

$$S_{pq}(P, Q) = \frac{\theta_{pq}(\mathcal{A}(P) - \mathcal{A}(Q))}{\theta_{pq}(0) E(P, Q)} \tag{1.0.10}$$

$E(P, Q)$ being the prime form, is the key to find required derivatives. The matrix of eigenvectors of $L(\lambda)$ can be expressed in terms of Szeg o kernel by:

$$\hat{\Psi}(\lambda)_{ab} = \frac{S_q(\lambda^{(b)}, \lambda_0^{(a)})}{S_q(\lambda^{(b)}, \lambda_0^{(1)})} \tag{1.0.11}$$

with q the vector of angle variables. Subsequently, the following identities relating the

matrices G_j and G_j^{-1} to Szegö kernel can be derived:

$$[G_j]_{ab} = \frac{S_q(\lambda^{(b)}, \lambda_0^{(a)})(\lambda - \lambda_0)}{\sqrt{d\lambda}\sqrt{d\lambda_0}}\Big|_{\lambda=\lambda_j}, \quad [G_j^{-1}]_{ab} = \frac{S_q(\lambda_0^{(b)}, \lambda^{(a)})(\lambda_0 - \lambda)}{\sqrt{d\lambda}\sqrt{d\lambda_0}}\Big|_{\lambda=\lambda_j} \tag{1.0.12}$$

Considering the fact that $A_j = G_j L_j G_j^{-1}$, the matrix G_j can be identified with $\hat{\Psi}(\lambda_j)$ in the vicinity of λ_j. We then apply the variational formulae below

$$\frac{\partial S(x,y)}{\partial q_\gamma} = -\oint_{a_\gamma} S_q(x,t) S_q(t,y) \tag{1.0.13}$$

and

$$\frac{\partial S_q(x,y)}{\partial I_\gamma} = -\frac{\pi i}{2} \sum_{x_k \text{ br. pts}} \operatorname*{res}_{t=x_k} v_\gamma(t) \frac{W_t[S_q(x,t), S_q(t,y)]}{d\lambda(t) d\mu(t)} \tag{1.0.14}$$

where v_γ are the basis of holomorphic differentials on the spectral curve Γ and W_t is the Wronskian with respect to t and the sum is taken over all branch points.

Once we have all of these, the required partial derivatives can be calculated. The end result conveniently turns out to be

$$\Theta_{\mathcal{M}} = \sum_{\gamma=1}^{g} I_\gamma dq_\gamma$$

Chapters 7 and 8 are a mainly a glance into future paths of research. The spectral transform viewpoint sheds light on this work from a different angle, via the link between the phase space of the system and the space of spectral data. The variations of vector of Riemann constant with respect to action variables is also an intriguing result that appears while studying the symplectic properties of Gaudin systems. Our observation is that

$$\frac{\partial \mathcal{K}_j}{\partial I_i} = \frac{\partial \mathcal{K}_i}{\partial I_j}$$

In chapter 8, we look into this directly, without relying on previous work, in the case of spectral curve being hyperelliptic. The idea is to prove the desired result for a specific choice of base point for the Abel map $\mathcal{A}$, then show that all the steps and calculations are essentially *invariant* under a change of basepoint. We will also provide a scalar potential function F whose gradient is the vector of Riemann constants, i.e. $\mathcal{K} = \nabla F$. Expanding the results to generic case could be a subject of future research.

Chapter 2

Integrable systems and Lax formalism

The roots of integrable systems can be traced back to the genesis of classical mechanics, beginning with the quest for exact solutions of Newtonian equations of motion. Other than the Kepler problem which was solved by Newton himself, only a handful of "integrable" systems are known to us even after almost two centuries of intense studies. Liouville finally provided a general framework to describe the cases where the equations of motions are solvable by quadrature, i.e. when the solution can be expressed in terms of some integrals and following the works of Poincaré in the late 19th century, the theory of integrable systems remained more or less dormant for several decades till the late 60s. Once Lax formulation was devised, the domain of integrable systems expanded significantly, leading to numerous results. More recently, these results have been extended to quantum mechanics which has led to a very active field of research.

Notion of Lax pairs involves presenting the equations of motion of the system in the form $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$ where the matrices $M(\lambda)$ and $L(\lambda)$ depend on dynamical variables as well as a parameter λ known as spectral parameter. The importance of Lax pair originates from the fact that evolution under Lax equation preserve the spectrum of Lax matrix L. This means that the curve defined by the characteristic polynomial of Lax matrix $\det(L(\lambda) - \mu I) = 0$ is time-independent. This so-called *spectral curve*, can be regarded as a Riemann surface and it will be used later on to reconstruct the solution of the system. This points at two aspects of the theory: group theoretic aspect which enters through the Lie algebra involved in the commutator $[M, L]$ and the complex analytical aspect that enters

through the spectral curve. Here we are interested in the latter.

Integrable systems are rather rare to come across so once we do, there are usually many constraints on the matrices $L(\lambda)$ and $M(\lambda)$. To obtain the solutions of the system in the analytical setting, we look for the eigenvector of $L(\lambda)$ corresponding to an eigenvalue of $L(\lambda)$ such as μ which is a point of the spectral curve. We will then utilize analytic tools to construct the solution in terms of theta functions.

2.1 Integrable dymanical systems and Lax representation

The integrable dynamical systems are defined through Liouville theorem, systems possessing n independent quantities F_j conserved under the time flow that are also in involution i.e. they commute with respect to the Poisson bracket. Such property, also known as *Liouville integrablity*, then leads to Liouville theorem which states *"The solutions of a Liouville integrable system are obtained by some integrals"*. The conserved quantities live on a $2n$ dimensional phase space and we require a set of coordinates on this space. Action-angle variables are a set of coordinates introduced for this purpose. The *angle variables* θ_j are angular variables describing the cycles on which the motion is taking place. The *action variables* I_j are defined as integrals of a differential 1-form known as the canonical form (which depends on F_j) over those cycles and they are canonically conjugated to angle variables.

Before getting to the idea of Lax representation, let us quickly refresh some of the basics of symplectic and Poisson geometry. Consider a Poisson manifold M, which is going to be the phase of the system later on, equipped with a Poisson bracket $\{\ ,\ \} : \mathcal{C}(M) \times \mathcal{C}(M) \to \mathcal{C}(M)$ defined on the algebra of all differentiable functions on M. The Hamiltonian of the system H will determine the time evolution of functions on M via $\dot{f} = \{H, f\}$. The important property of the Poisson bracket is that if f_1 and f_2 are conserved quantities, then so is $\{f_1, f_2\}$ since $\{H, f_1\} = \{H, f_2\} = 0$ implies that $\{H, \{f_1, f_2\}\} = 0$. A Poisson bracket is degenerate in general, which means that there is a function f on M such that $\{f, g\} = 0$ for all functions g. Such functions that lie in the center of the Poisson algebra are called *Casimir functions*. If the center is non-trivial, i.e. contains non-constant functions, one can reduce the dynamical system by setting all functions of the center to constant values. This defines a foliation of M into *symplectic leaves*. The Poisson bracket will be non-degenerate on the leaves. Now assuming that the manifold M is equipped with the

symplectic structure, i.e. a non-degenerate closed 2-form ω, locally expressed by

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$$

the Poisson bracket can be reconstructed from the symplectic structure by

$$\{f_1, f_2\} = -\sum_{ij} \omega^{ij} \partial_i f_1 \partial_j f_2$$

where ω^{ij} is the ij entry of the inverse of the matrix $[\omega_{ij}]$.

Lax pair is a new concept that emerged from the modern studies of integrable systems. A Lax pair L, M consists of two matrices as functions on the phase space of the system. The Hamiltonian equations of motion may be written as the following *Lax equation*

$$\dot{L} = \frac{dL}{dt} = [M, L] \tag{2.1.1}$$

The significance of Lax pair lies in the fact that it allows for an easy construction of conserved quantities. The solution of Lax equation is of the form

$$L(t) = G(t)L(0)G(t)^{-1} \tag{2.1.2}$$

where the invertible matrix G is given by $M = \dot{G}G^{-1}$. It follows that if $I(L)$ is a function of L invariant by the conjugation $L \to GLG^{-1}$, then $I(L(t))$ is a constant of motion. Such functions of functions of eigenvalues of L. Eigenvalues of $L(t)$ are preserved by time evolution hence Lax equation is called *isospectral*.

The following well-known and simple examples will further illuminate the above notions.

Example 1: For any Liouville integrable system, the Lax pair can be constructed in a tautological way. Assume we have a finite-dimensional Hamiltonian with n degrees of freedom, a Poisson bracket $\{,\}$ and Hamiltonian H. Liouville integrability means there are n independent integrals of motion $F_i,\ i = 1, ..., n$, in involution. By Liouville theorem, there exists a system of conjugate coordinates $I_i, \theta_i,\ i = 1, ..., n$, where I_i are only the functions of F_i. In these coordinates, the equations of motion take the form:

$$\dot{I}_j = 0, \quad \dot{\theta}_j = \frac{\partial H}{\partial I_j} \tag{2.1.3}$$

Now consider a Lie algebra generated by $\{H_i, E_i,\ i = 1, ..., n\}$ with the commutators:

$$[H_i, H_j] = [E_i, E_j] = 0, \qquad [H_i, E_j] = 2\delta_{ij} E_j \tag{2.1.4}$$

The Lax pair will be then given by:

$$L = \sum_{j=1}^{n} I_j H_j + 2 I_j \theta_j E_j, \qquad M = -\sum_{j=1}^{n} \frac{\partial H}{\partial I_j} E_j \tag{2.1.5}$$

It is easy to check that the equation $\dot{L} = [M, L]$ is equivalent to the equations of motion. This construction however, is useless since we are using the action-angle variables to construct the Lax pair hence we are assuming that we already know action-angle variables, in which case there is no need for Lax pair.

Example 2: Take the harmonic oscillator with the equations of motion:

$$\dot{q} = p, \quad \dot{p} = -\omega^2 q \tag{2.1.6}$$

Set the Lax pair:

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix} \tag{2.1.7}$$

Again, one can quickly check that the Lax equation $\dot{L} = [M, L]$ is equivalent to the equations of motion. The Hamiltonian can be written as $H = \frac{1}{2}(p^2 + \omega^2 q^2) = \frac{1}{4}\,\mathrm{tr}(L^2)$. This example may be generalized to n independent harmonic oscillators by writing the Lax matrices L and M in a block-diagonal form where each block is a 2×2 matrix as above. Now, the conserved quantities are $\mathrm{tr}(L^{2n}) = 2\sum (2F_i)^n$ where $2F_i = p_i^2 + \omega^2 q_i^2$ and $\mathrm{tr}(L^{2n+1}) = 0$, so that they are equivalent to the collection of F_i.

The Lax pairs that depend analytically on a parameter such as λ are Lax pairs with *spectral parameter* λ. The study of analytical properties of the Lax equation with spectral parameter $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$ leads to a significant insight into its structure and introduces many objects and concepts with which we will be working in the future.

In what follows we will review some examples of Lax pairs with spectral parameter:

Example 3: Euler top is a case where the Lax pair appears naturally. This is a case of a rotating solid body attached to a fixed point with no external force involved. The equations of motion are written in a moving frame with the origin at the fixed point of the top.

Consider the 3×3 matrices $J_{ij} = \epsilon_{ijk} J_k$ and $\Omega_{ij} = \epsilon_{ijk}\omega_k$ where $\vec{J}$ is the vector of angular momentum of the top in the moving frame given by $\vec{J} = I.\vec{\omega}$ where $I = \mathrm{diag}(I_1, I_2, I_3)$ is the inertia tensor and $\vec{\omega}$ is the rotation vector of the moving frame. The equation of motion is

$$\frac{d\vec{J}}{dt} = -\vec{\omega} \wedge \vec{J} \tag{2.1.8}$$

which can be rewritten in the matrix form $\dot{J} = [\Omega, J]$. One could think of setting the Lax pair as $L = J$ and $M = \Omega$ but this turns out to be problematic since the conserved quantities $\mathrm{tr}(L^n)$ either vanish or are the functions of $\vec{J}^2$. Hence some modifications are required. Define the diagonal matrix $\mathcal{I} = \mathrm{diag}(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3)$ where $\mathcal{I}_k = \frac{1}{2}(I_i + I_j - I_k)$ for I_i's the moments of inertia and (i, j, k) a cyclic permutation of $(1, 2, 3)$. The matrix J is now given by $J = \mathcal{I}\Omega + \Omega\mathcal{I}$. Now set the Lax pair as:

$$L(\lambda) = \mathcal{I}^2 + \frac{1}{\lambda}J, \qquad M(\lambda) = \lambda\mathcal{I} + \Omega \tag{2.1.9}$$

where λ is a free arbitrary parameter, the so-called spectral parameter. One can check that the Lax equation gives back the equation of motion and Now the Hamiltonian appears amongst the conserved quantities of the form $\mathrm{tr}\, L^n(\lambda)$.

Example 4: Another example of Lax pair with spectral parameter is the Lagrange top. This is when the top is in a gravitational field, so its weight should also be accounted for. In one case where two inertia moments are equal and the center of mass is located at a position like $(0, 0, h)$ with respect to the rotation axis containing a point to which the top is attached. The equations of motion are:

$$\frac{d\vec{J}}{dt} = -\vec{\omega} \wedge \vec{J} + \vec{h} \wedge \vec{P}, \qquad \frac{d\vec{P}}{dt} = -\vec{\omega} \wedge \vec{P} \tag{2.1.10}$$

where $\vec{P}$ is the weight and $\vec{h}$ is the vector from the fixed point to the center of mass, and $\vec{J}$ and $\vec{\omega}$ are as in previous example. The conserved quantities are $\vec{P}^2$, $\vec{J}.\vec{P}$ and the energy:

$$H = \frac{1}{2}(\vec{J}.I^{-1}.\vec{J}) - \vec{P}.\vec{h} \tag{2.1.11}$$

We now write the Lax pair as 4×4 matrices in block form:

$$L(\lambda) = \begin{pmatrix} 0 & Ih^t + \lambda^{-2}P^t \\ Ih + \lambda^{-2}P & \lambda^{-1}J \end{pmatrix}, \qquad M(\lambda) = \begin{pmatrix} 0 & \lambda h^t \\ \lambda h & \Omega \end{pmatrix} \tag{2.1.12}$$

where the 3×3 matrices J and Ω are as in the previous example, and h and P are 3×1 matrices corresponding to the vectors $\vec{h}$ and $\vec{P}$ of the Lagrange top. Calculating the left-hand side of $\dot{L} - [M, L] = 0$ and applying the Lagrange condition $I_1 = I_2$ confirms that the Lax equation is equivalent to the equations of motion.

2.2 Construction of Lax pair

Given an integrable system, there is no useful algorithm to construct a Lax pair. However, there is a general procedure, due to Zakharov and Shabat, to construct consistent Lax pairs paving the way to integrable systems. This is a general method of constructing matrices $L(\lambda)$ and $M(\lambda)$ depending on a spectral parameter λ such that the Lax equation is equivalent to the equations of motions of an integrable system. This method works around the analytical properties of $L(\lambda)$ and $M(\lambda)$ for $\lambda \in \mathbb{C}^*$. Considering the systems with a finite number of degrees of freedom, the main result expresses the possible forms of Lax matrices.

Suppose that the matrices $L(\lambda)$ and $M(\lambda)$ are of dimension $N \times N$. Assume furthermore that L and M are rational functions of the parameter λ. Let $\{\lambda_k\}$ be the set of their poles. Assuming no pole is at infinity, the Lax matrices can be generally written as:

$$L(\lambda) = L_0 + \sum_k L_k(\lambda) \quad \text{with} \quad L_k(\lambda) = \sum_{r=-n_k}^{-1} L_{kr}(\lambda - \lambda_k)^r \tag{2.2.1}$$

and

$$M(\lambda) = M_0 + \sum_k M_k(\lambda) \quad \text{with} \quad M_k(\lambda) = \sum_{r=-m_k}^{-1} M_{kr}(\lambda - \lambda_k)^r \tag{2.2.2}$$

where n_k and m_k are the order of poles at the point λ_k, and the coefficients L_{kr} and M_{kr} are matrices. We also assume that the positions of poles λ_k are constants independent of time. The following proposition clarifies the structure of the Lax pair:

Proposition: ([1], p. 36) *Assuming that $L(\lambda)$ has distinct eigenvalues in a neighbourhood of λ_k, one can perform a regular similarity transformation $G_k(\lambda)$ diagonalizing $L(\lambda)$ in a vicinity of λ_k:*

$$L(\lambda) = G_k(\lambda) A_k(\lambda) G_k^{-1}(\lambda) \tag{2.2.3}$$

where $A_k(\lambda)$ is diagonal and has a pole of order n_k at λ_k. As a result, the decomposition of $L(\lambda)$ and $M(\lambda)$ in singular parts reads as:

$$L(\lambda) = L_0 + \sum_k L_k(\lambda) \quad with \quad L_k(\lambda) = \Big(G_k(\lambda) A_k(\lambda) G_k^{-1}(\lambda)\Big)_- \tag{2.2.4}$$

$$M(\lambda) = M_0 + \sum_k M_k(\lambda) \quad with \quad M_k(\lambda) = \Big(G_k(\lambda) B_k(\lambda) G_k^{-1}(\lambda)\Big)_- \tag{2.2.5}$$

where $B_k(\lambda)$ is diagonal with a pole of order m_k at λ_k and $(\quad)_-$ denotes the singular part of the analytic expansion.

In the proximity of a singularity, $L(\lambda)$ and $M(\lambda)$ can be simultaneously diagonalized if the Lax equation holds true. In this diagonal "*gauge*", the Lax equation simply states that the matrix $A_k(\lambda)$ is conserved and $B_k(\lambda)$ is diagonal. Once we transform these results into the original gauge, we get the general solution of non-dynamical constraints on $M(\lambda)$:

Proposition: ([1], p. 38) *Let $L(\lambda)$ be a Lax matrix of the form (2.2.1). The general form of $M(\lambda)$ such that the orders of poles match on both sides of the Lax equation is $M = M_0 + \sum_k M_k$ with*

$$M_k = \Big(P_k(L, \lambda)\Big)_- \tag{2.2.6}$$

where $P_k(L, \lambda)$ is a polynomial in $L(\lambda)$ with coefficients rational in λ and the singular part is taken at $\lambda = \lambda_k$.

It is important to realize that the dynamical variables are the elements of the Lax matrix, or the elements of L_{kr}. Now the question of specifying a certain model of dynamical system essentially becomes the matter of choosing the number and the order of poles of the Lax matrix. Choosing the polynomials $P_k(L, \lambda)$ amounts to specifying the dynamical flows.

The above propositions give the general form of $M(\lambda)$ as far as the matrix structure and the λ-dependence are concerned. We should keep in mind however that the coefficients

of the polynomial P_k should be already known and given as functions of the elements of matrix L and they require further characterizations in order to obtain an integrable system. Let us now illustrate these constructions on some of the examples we saw before, the Euler and Lagrange tops.

Example (Euler top): From (2.1.9), we see that $L(\lambda)$ has a pole at 0 and $M(\lambda)$ has a pole at ∞. We now apply the above construction to remove this pole. There exists a polynomial $P(x) = \alpha x^2 + \beta x + \gamma$ such that $P(\mathcal{I}^2) = \mathcal{I}$ so need $\alpha = -\frac{1}{I_1 I_2 I_3}$. Redefining $M(\lambda)$ to $M(\lambda) - \lambda P(L(\lambda))$ gives $M = M_0 - (\alpha/\lambda)J^2$ with $M_0 = \Omega - \alpha(\mathcal{I}^2 J + J\mathcal{I}^2) - \beta J$. One can check that $M_0 = 0$ by computing $(\mathcal{I}_i - \mathcal{I}_j)(M_0)_{ij}$ using $P(\mathcal{I}_i^2) = \mathcal{I}_i$. Hence for the Euler top we can choose

$$M(\lambda) = -\frac{\alpha}{\lambda} J^2 \tag{2.2.7}$$

Notice that this new $M(\lambda)$ is such that $M(\lambda) = -\alpha(\lambda L^2)_-$. The Lax matrix of the Euler top $L(\lambda) = \mathcal{I}^2 + \lambda^{-1} J$ is of the form $L_0 + L_-$ with L_0 diagonal and non-dynamical. The eigenvalues of J are $0, \pm i\sqrt{\vec{J}^2}$ which are non-dynamical since $\vec{J}^2$ belongs to the center of Poisson bracket and has been fixed to a numerical value.

Example (Lagrange top): From (2.1.12), we get that $L(\lambda)$ has a pole at 0 and $M(\lambda)$ has a pole at ∞. We can remove this pole by redefining $M(\lambda)$ to $M(\lambda) - I^{-1}\lambda L(\lambda)$. We should however notice that since the eigenvalues of L_0 are degenerate in this case, M_0 cannot be expressed as a polynomial in L_0. The new $M(\lambda)$ can be written as $M = M_0 + M_-$ with $M_-(\lambda) = -I^{-1}(\lambda L(\lambda))_-$. For the Lagrange top the Lax matrix is again of the form $L_0 + L_-$ where L_0 is non-dynamical. For the singular part one gets, since J is antisymmetric, $A_- = \lambda^{-2}\,\mathrm{diag}(\sqrt{\vec{P}^2}, -\sqrt{\vec{P}^2}, 0, 0)$ which again belongs to the center of Poisson bracket and is non-dynamical.

2.3 Phase space as a symplectic manifold

Identifying the phase space of a dynamical system basically comes down to identifying the degrees of freedom. We start with an initial phase space that contains all dynamical variables and parameters of the system. This is usually an even-dimensional space with a symplectic structure. Then one needs to account for the invariants of the system to reduce the number of degrees of freedoms to the actual independent ones. This is done through a process known as Hamiltonian reduction.

Let us first clarify a few notions from symplectic geometry and Hamiltonian mechanics. If a Lie group G acts on a symplectic manifold M by symplectic diffeomorphism, the action of any one-parameter subgroup of G is locally Hamiltonian i.e. there exists a function H_X, locally defined on M such that $X.f = \{H_X, f\}$. Now, this action is said to be *Poissonian* if the Hamiltonians H_X of all one-parameter subgroups are globally defined, depend linearly on X and

$$H_{[X,Y]} = \{H_X, H_Y\} \tag{2.3.1}$$

If the action of a Lie group G on a symplectic manifold M is Poissonian, any $X \in \mathfrak{g}$, the Lie algebra of G, is associated with the function H_X such that $X.f = \{H_X, f\}$ and $X \to H_X$ is linear. Hence, there is a function $\mathcal{P} : M \to \mathfrak{g}^*$ such that $H_X(m) = \langle \mathcal{P}(m), X \rangle$ where $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{g}$ and its dual. The association $m \to \mathcal{P}(m) \in \mathfrak{g}^*$ is called the *moment map*.

We also need to know the idea of coadjoint orbits. Suppose G is a connected Lie group. The coadjoint action of G on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, is defined by:

$$(\mathrm{Ad}^*\, g.\eta)(X) = \eta(\mathrm{Ad}\, g^{-1}(X)), \quad g \in G, \eta \in \mathfrak{g}^*, X \in \mathfrak{g} \tag{2.3.2}$$

where Ad is the usual adjoint action. The important fact here is that coadjoint orbits in $\mathfrak{g}^*$ generated by this coadjoint action are equipped with a canonical symplectic form known as *Kostant-Kirillov* symplectic structure. At a point $\xi \in \mathfrak{g}^*$ of a coadjoint orbit, take any two tangent vectors $V_X = \mathrm{ad}^* X.\xi$ and $V_Y = \mathrm{ad}^* Y.\xi$. The Kostant-Kirillov symplectic form is defined by:

$$\omega_K(V_X, V_Y) = \xi([X, Y]) \tag{2.3.3}$$

and it is a closed and non-degenerate 2-form on any G-orbit.

Now let M be a symplectic manifold and G a group acting on M by the Poissonian action. Let $\mathcal{P}$ be the moment map, fix a particular value of the moment like μ and consider the set of points m in the phase space such that $\mathcal{P}(m) = \mu$. By the Noether theorem, the motion of the system takes place on the set:

$$M_\mu = \mathcal{P}^{-1}(\mu) \tag{2.3.4}$$

Assuming that μ is not a critical value of $\mathcal{P}$, there exists a tangent space at m, tangent to M_μ. Now the dimension of M_μ is given by

$$\dim M_\mu = \dim M - \dim G \tag{2.3.5}$$

The stabilizer of the moment μ, denoted by G_μ, is the group of all $g \in G$ such that $\mathrm{Ad}^*_g \mu = \mu$. The action of G_μ preserves M_μ so the *reduced phase space* $\mathcal{M}_\mu$ is defined as the quotient of M_μ by this action:

$$\mathcal{M}_\mu := M_\mu / G_\mu = \mathcal{P}^{-1}(\mu)/G_\mu \tag{2.3.6}$$

There are particular values of μ for which this quotient is not well-defined but for a generic set, $\mathcal{M}_\mu$ is well-defined and can be assumed to be a differentiable manifold.

The crucial property of reduced phase space is that it is naturally equipped with a symplectic structure and in particular, it is of even dimension.

Proposition: ([1], p. 529) *Let ξ and η be two vectors tangent to $\mathcal{M}_\mu$ at the point p. At a point $m \in M_\mu$ above p, consider two vectors ξ' and η' tangent to M_μ and projecting to ξ and η respectively. We then set:*

$$\omega_p(\xi, \eta) = \omega_m(\xi', \eta') \tag{2.3.7}$$

This is independent of the choice of representatives m, ξ', η' and defines a symplectic form on $\mathcal{M}_\mu$.

Thus, the phase space can be equipped with a well-defined symplectic form.

2.4 Symplectomorphism and symplectic potential

A *symplectomorphism* $\varphi : M \to N$ is a diffeomorphism from a symplectic manifold (M, ω_1) to another one (N, ω_2) that preserves the symplectic structure, i.e. the pullback of symplectic form ω_2 under φ is:

$$\varphi^* \omega_2 = \omega_1 \tag{2.4.1}$$

An example of symplectomorphism is the transformation between two canonical coordinates sets of a manifold, called *canonical transformation*. We will be dealing with a symplectomorphism of this type later as we will study two different sets of canonical coordinates on the phase space of Gaudin system as a symplectic manifold.

A *symplectic potential* which also sometimes known as *tautological one-form*, *canonical one-form* or *Liouville one-form* is a 1-form defined on the cotangent bundle T^*M of a manifold M. The exterior derivative of this form gives a symplectic form on T^*M which has a symplectic structure. The importance of symplectic potential is in relating the Hamiltonian and Lagrangian formalisms of a dynamical system. In local coordinates, say (p_i, q_i) is a point in cotangent bundle T^*M ($p_i \in T^*_q M,\ q_i \in M$), then the symplectic potential is given by:

$$\Theta_M = \sum_i p_i dq_i \tag{2.4.2}$$

and its exterior derivative is the canonical symplectic form $\omega = d\Theta_M = \sum_i dp_i \wedge dq_i$.

As we shall see later, the phase space of Gaudin system is an even-dimensional space and it can be equipped with a canonical symplectic structure. We will study this symplectic structure, the associated symplectic potential and we will find the canonical coordinates parameterizing the phase space.

2.5 Classical Gaudin model

In his 1970s papers [9], [10] and [11], M. Gaudin introduced a new class of quantum integrable models mostly for the purpose of studying quantum magnets. The simplicity and unique properties of Gaudin model immediately attracted wide attention amongst mathematical physicists, as the Gaudin model has many applications in the study of both classical and quantum integrable systems.

The Gaudin model can be understood and characterized in different ways, but the Lax approach is perhaps the most straightforward and suitable way for the purpose of this work. It will be done by setting the space of spectral parameter first, then introducing the Lax matrix and subsequently, identifying the phase space of the system. The dynamics of the system will then be determined by the Hamiltonian and Lax equation.

The classical Gaudin system is an integrable system defined on the punctured or marked

Riemann sphere. Assume $X \setminus \{\lambda_1, ..., \lambda_m\}$ is a punctured sphere with λ_i showing the positions of punctures or marked points. Let $A_k, k = 1, ..., m$ be elements in $\mathfrak{sl}(N, \mathbb{C})$.

We set the Lax matrix

$$L(\lambda) = \sum_k \frac{A_k}{\lambda - \lambda_k} \tag{2.5.1}$$

with the regularity at infinity condition which requires $\sum_k A_k = 0$ and $L(\lambda_0)$ being diagonal. Hence, the matrices A_k may be realized as the residues of the Lax matrix at the puncture points λ_k. The phase space of the system shall be realized as the number of degrees of freedom in all A_k. By considering the diagonalization

$$A_k = G_k L_k G_k^{-1} \tag{2.5.2}$$

with the choice of G_k up to right multiplication by a diagonal matrix and then taking quotient by the simultaneous left multiplication of an element of $\mathfrak{sl}(N, \mathbb{C})$, we obtain the phase space of the system $\mathcal{M}$. The canonical 1-form on this space is given by

$$\Theta_{\mathcal{M}} = \sum_k \mathrm{tr}\,(A_k G_k^{-1} dG_k) \tag{2.5.3}$$

The integrable system on this phase space is now defined by the following Hamiltonian

$$H = \sum_{k \neq l} \frac{\mathrm{tr}(A_k A_l)}{(\lambda - \lambda_k)(\lambda - \lambda_l)} = \sum_k \frac{H_k}{\lambda - \lambda_k} \tag{2.5.4}$$

where

$$H_k = \sum_{k \neq l} \frac{\mathrm{tr}(A_k A_l)}{\lambda_k - \lambda_l} \tag{2.5.5}$$

form a family of commuting Hamiltonians known as Gaudin Hamiltonians.

We have the following relation for the Hamiltonians H_k:

$$\{L(\lambda), H_k\} = -[A_k, L(\lambda)] \tag{2.5.6}$$

The *classical Gaudin system* is in fact the autonomous (time-independent) Hamiltonian system with the same Hamiltonians as above. Let $\lambda_1, ..., \lambda_m$ be the variables that define the position of the m punctured or marked points. The Gaudin model is often described

by the following set of equations

$$\frac{\partial A_j}{\partial t_i} = \frac{[A_j, A_i]}{\lambda_i - \lambda_j} \tag{2.5.7}$$

which are equivalent to the Lax formulation of the Gaudin model as we will see in section 3.1. The matrices A_j may be regarded as the residues of the Lax operator at $\lambda = \lambda_i$. Now consider m time variables $t_1, \ldots, t_m$; setting the other Lax matrix $M_j = \frac{A_j}{\lambda - \lambda_j}$, these equations for each time variable t_i can be rewritten in the Lax form:

$$\frac{dL(\lambda)}{dt_j} = [M_j(\lambda), L(\lambda)] \tag{2.5.8}$$

We will study various aspects of classical Gaudin systems such as spectral curve, explicit solutions and the symplectic structure of phase space in details in the following chapters.

2.6 Analytical approach to Lax equation

One of the main focus of the work presented here is on solving the Lax equation of Gaudin system through an analytical method which involves theta functions and subsequently constructing a set of Darboux coordinates using parameters appearing in the analytical solution of the system.

The general idea for solving the Lax equation with spectral parameter is as follows: The spectral curve, denoted by Γ is described by the characteristic equation for the eigenvalues of the Lax matrix $\det(L(\lambda) - \mu\mathbf{1}) = 0$. Since the Lax equation $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$ is isospectral, i.e. the eigenvalues of $L(\lambda)$ are time-independent, so is the spectral curve. At any point of the spectral curve, there exists an eigenvector of $L(\lambda)$ with eigenvalue μ. We will reconstruct the eigenvector from its analytical properties on Γ, namely information about poles, monodromy, asymptotic properties, etc. In particular, all the analytical information of the system is contained in the divisor of the poles of Lax matrices which we call *dynamical divisor*. The time evolution of this divisor is equivalent to a linear flow on the *Jacobian* of Γ. The Jacobian $J(\Gamma)$ of a Riemann surface (already equipped with a set of g a-cycles and g b-cycles as the basis of homology and a basis $\{\omega_j\}_{j=1}^g$ of holomorphic differentials on Γ) is defined as $J(\Gamma) = \mathbb{C}^g/(\mathbb{Z} + \mathcal{B}\mathbb{Z}^g)$ where $\mathcal{B}$ is the matrix of b-periods. Hence, if Γ is of genus g, the dimension of Jacobian is g as well. We will explicitly reconstruct the eigenvectors, which turn out to be comprised of meromorphic components on the Riemann

surface, using a special type of functions defined on the Jacobian of the Riemann surface, the theta functions. As a result, one can express dynamical variables in terms of theta functions. We then show that the symplectic structure can be "nicely" written in terms of coordinates of the points of the dynamical divisor. This exhibits the interplay between analytical data and separation of variables.

After fulfilling the aforementioned tasks, we will be in a position to clarify the link between integrable systems and Riemann surfaces. Let Γ be a Riemann surface of genus g and λ a meromorphic function on it. We assume that λ takes each value N times. Any other meromorphic function μ on Γ is an element in the field of rational functions of λ and μ. The choice of these functions allows us to present Γ as an N sheeted covering of the Riemann sphere through the covering $(\lambda, \mu) \to \lambda$.

We can interpret Γ as the spectral curve of a Lax matrix $L(\lambda)$ in the following way: Let $\lambda_0^{(1)}, ..., \lambda_0^{(N)}$ be the N points above $\lambda = \lambda_0$. Choose a divisor D of g points on Γ. From these data, we construct N linearly independent meromorphic functions $\psi_1 = 1$, ψ_k with a zero at $\lambda_0^{(1)}$ and poles at $D + \lambda_0^{(k)}$ for $k = 2, ..., N$. This uniquely determines ψ_k u to multiplication by a constant. Let $P_i = (\lambda, \mu_i)$ be the N points above λ. Define the $N \times N$ matrices $\hat{\Psi}_{ij} = \psi_i(P_j)$ and $\hat{\mu} = \mathrm{diag}(\mu_i)$, and

$$L = \hat{\Psi}\hat{\mu}\hat{\Psi}^{-1} \tag{2.6.1}$$

This matrix is a rational function of λ because it is a rational of $\lambda, \mu_1, ..., \mu_N$, invariant under the permutations of μ_j. It tends to zero at ∞ and Γ is the spectral curve of $L(\lambda)$ since it contains the eigenvalues of $L(\lambda)$. Note that L is defined only up to conjugation by diagonal matrices because we did not specify a normalization for the functions ψ_k.

We now introduce a time evolution such that Γ is time-independent but the dynamical divisor D depends on time. This is enough to establish the existence of a rational Lax equation:

$$\dot{L}(\lambda) = [M(\lambda), L(\lambda)], \qquad M(\lambda) = \dot{\hat{\Psi}}\hat{\Psi}^{-1} \tag{2.6.2}$$

To relate to Liouville integrability, we have to introduce a symplectic structure on the space of dynamical variables, i.e. phase space. Imposing a coadjoint orbit structure on the poles of $L(\lambda)$ automatically yields integrability once we have performed the Hamiltonian reduction by the diagonal group action of dimension $N - 1$. This produces a dynamical

system of dimension $2g$. The g angle variables are given by the Abel map of dynamical divisor appearing as the characteristic of the theta functions used in the construction of eigenvectors, which evolve linearly on the Jacobian of Γ, and the g action variables are contained in the moduli of the spectral curve, i.e. they belong to the set of deformation parameters characterizing the moduli space of the curve.

The conditions we impose on the moduli, coming from the coadjoint orbit structure and the Hamiltonian reduction, can be all written as the following:

$$\tilde{\mathrm{d}}(\mu d\lambda) \text{ is a holomorphic differential}$$

where $\tilde{\mathrm{d}}$ is the differential with respect to the dynamical variables, i.e. variables from the phase space while keeping λ constant. This means that the polar parts of $\mu d\lambda$ are non-dynamical. Since $\tilde{\mathrm{d}}(\mu d\lambda) = \sum_{k=1}^{g} \omega_k \tilde{\mathrm{d}} I_k$ (ω_j a basis of holomorphic differentials on Γ), we can see that there are exactly g independent dynamical variables, hence g action variables.

In this setting, the standard symplectic form on the phase space is given in terms of coordinates of the points of dynamical divisor D, denoted by $\gamma_j = (\lambda_{\gamma_j}, \mu_{\gamma_j})$. We will also see that the angle variables $\theta_j = \sum_{k=1}^{g} \int^{\gamma_k} \omega_j$ are canonically conjugated to the action variables $I_j = \oint_{a_j} \mu d\lambda$ and that:

$$\omega = \sum_{j=1}^{g} \tilde{\mathrm{d}}\mu_{\gamma_j} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_j} = \sum_{j=1}^{g} \tilde{\mathrm{d}} I_j \wedge \tilde{\mathrm{d}}\theta_j \tag{2.6.3}$$

To elaborate more on this result, starting from a Riemann surface $\Gamma(\lambda, \mu) = 0$, we specify g independent variables $F_1, ..., F_g$ by imposing the regularity condition for $\tilde{\mathrm{d}}(\mu d\lambda)$. We take g arbitrary points $\gamma_j = (\lambda_{\gamma_j}, \mu_{\gamma_j})$ and impose the symplectic structure $\omega = \sum_{j=1}^{g} \tilde{\mathrm{d}}\mu_{\gamma_j} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_j}$ on these data. We determine the g variables F_j by solving g equations demonstrating that the curve passes through g points γ_j:

$$\Gamma(\lambda_{\gamma_j}, \mu_{\gamma_j}; F_1, ..., F_g) = 0 \tag{2.6.4}$$

This determines F_j as symmetric functions of $\lambda_{\gamma_k}, \mu_{\gamma_k}$. The interesting result is that these function Poisson commute, i.e. $\{F_i, F_j\} = 0$, because the action variables Poisson commute too and they are independent functions of F_j.

It is would be beneficial to see a concrete application of this approach before developing it in great technical details. The following "Kowalevski top" is a good example to help us understand the analytical method:

The Kowalevski top: ([1], p. 169) Consider the motion of a top in a moving frame with the origin at a fixed point of the top. Assume the moments of inertia satisfy $I_1 = I_2 = 2I_3$ and that the center of mass is on the plane $x_3 = 0$ but away from the origin so the top has no rotational symmetry. We are free to choose the inertia axis up to a rotation around the third one, hence we can assume that the fixed point is on the first axis. Set the notation:

$$\vec{\omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \vec{P} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \quad \vec{h} = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix} \tag{2.6.5}$$

following the same equations of motion as in Lagrange top and writing them in components with $c = h/I_3$, we obtain:

$$\begin{cases} 2\dot{p} = qr \\ 2\dot{q} = -pr - c\gamma_3 \\ \dot{r} = c\gamma_2 \end{cases} \qquad \begin{cases} \dot{\gamma}_1 = r\gamma_2 - q\gamma_3 \\ \dot{\gamma}_2 = p\gamma_3 - r\gamma_1 \\ \dot{\gamma}_3 = q\gamma_1 - p\gamma_2 \end{cases} \tag{2.6.6}$$

The Hamiltonian

$$H = \frac{I_3}{2}(2p^2 + 2q^2 + r^2) - h\gamma_1 \tag{2.6.7}$$

and the quantities

$$\vec{P}^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \quad \vec{P}.\vec{J} = I_3(2p\gamma_1 + 2q\gamma_2 + r\gamma_3) \tag{2.6.8}$$

are conserved ($\vec{J}$ is the vector of angular momentum as previous examples). The symplectic leaves in this example are of dimension 4 so we need one more conserved quantity to show integrability. This quantity is known as Lowalevski conserved quantity and it is given by:

$$K = (p^2 - q^2 + c\gamma_1)^2 + (2pq + c\gamma_2)^2 \tag{2.6.9}$$

Set $\sigma_0 = 1$ and σ_j as the 2×2 so-called Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.6.10}$$

and the Kowalevski variables:

$$\begin{cases} z_1 = J_1 + iJ_2 \\ z_2 = J_1 - iJ_2 \end{cases} \qquad \begin{cases} \xi_1 = \gamma_1 + i\gamma_2 \\ \xi_2 = \gamma_1 - i\gamma_2 \end{cases} \tag{2.6.11}$$

Now define the Lax matrix as:

$$L(\lambda) = \begin{pmatrix} 0 & -\lambda\xi_2 & \frac{1}{2}z_2 & \lambda\gamma_3 \\ \lambda\gamma_1 & 0 & -\lambda\gamma_3 & -\frac{1}{2}z_1 \\ \frac{1}{2}z_1 & \lambda\gamma_3 & -J_3 & 2\frac{1}{\lambda} + \lambda\xi_1 \\ -\lambda\xi_3 & -\frac{1}{2}z_2 & -2\frac{1}{2} - \lambda\gamma_2 & J_3 \end{pmatrix} \tag{2.6.12}$$

The Lax matrix satisfies the symmetry properties:

$$L(-\lambda) = \Sigma_1^{-1} L(\lambda)^T \Sigma_1, \quad L(\lambda)^T = -\Sigma_2^{-1} L(\lambda) \Sigma_2, \quad L(-\lambda) = \Sigma_3^{-1} L(\lambda) \Sigma_3 \tag{2.6.13}$$

where the matrices Σ_j are given by:

$$\Sigma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{2.6.14}$$

In general, the Kowalevski top endures two external forces $\vec{\gamma}$ which is $\vec{P}$ here and $\vec{\gamma'}$. Here we restrict ourselves to the pure Kowalevski case which has no external forces other than wieght so $\vec{\gamma'} = 0$. The equation of the spectral curve Γ: $\det(L(\lambda) - \mu) = 0$ reads:

$$\mu^4 - \Big(\frac{\lambda^2}{2}\vec{\gamma}^2 - \frac{1}{4}H + \frac{1}{\lambda^2}\Big)\mu^2 + \frac{\lambda^4}{16}(\vec{\gamma}^2)^2 + \frac{\lambda^2}{16}((\vec{J}.\vec{\gamma})^2 - H\vec{\gamma}^2) + \frac{K}{256} = 0 \tag{2.6.15}$$

The Hamiltonians H and K are given by:

$$H = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) - 4\gamma_1 = \frac{1}{2}z_1 z_2 + J_3^2 - 2(\xi_1 + \xi_2) \tag{2.6.16}$$

$$K = (J_1^2 - J_2^2 + 8\gamma_1)^2 + (2J_1J_2 + 8\gamma_2)^2 = (z_1^2 + 8\xi_1)(z_2^2 + 8\xi_2) \tag{2.6.17}$$

Note that the coordinates λ and μ on the spectral curve appear only through λ^2 and μ^2 which is a consequence of the symmetries of $L(\lambda)$.

Let us clarify the solutions $\mu(\lambda)$ of the equation of the spectral curve around $\lambda = 0$ and

$\lambda = \infty$. Around $\lambda = \infty$, we have four branches:

$$\mu = \epsilon \frac{\sqrt{\vec{\gamma}^2}}{2}\lambda + i\epsilon' \frac{\vec{J}.\vec{\gamma}}{4\sqrt{\vec{\gamma}^2}} + O(\frac{1}{\lambda}) \tag{2.6.18}$$

where ϵ and ϵ' are independent $\pm$ signs. Around $\lambda = 0$, we get two branches with $\mu \to 0$ and two branches with $\mu \to \infty$:

$$\mu = \epsilon \frac{\sqrt{K}}{16}\lambda + O(\lambda^3), \quad \mu = \epsilon\Big(\frac{1}{\lambda} - \frac{H}{8}\lambda\Big) + O(\frac{1}{\lambda^3}) \tag{2.6.19}$$

Now recall that Γ is defined as the desingularization (resolving the singularity by adding the ∞ point) of the curve defined by (2.6.15). We are going to study Γ by considering it as consecutive coverings of simpler curves. Setting $\lambda^2 = z$ in (2.6.15) yields a curve C of the equation:

$$\mu^4 - \Big(\frac{z}{2}\vec{\gamma}^2 - \frac{1}{4}H + \frac{1}{z}\Big)\mu^2 + \frac{z^2}{16}(\vec{\gamma}^2)^2 + \frac{z}{16}((\vec{J}.\vec{\gamma})^2 - H\vec{\gamma}^2) + \frac{K}{256} = 0 \tag{2.6.20}$$

Γ is a two-sheeted covering of C. Setting $\mu^2 = y$, we get a curve E of the equation:

$$y^2 - \Big(\frac{z}{2}\vec{\gamma}^2 - \frac{1}{4}H + \frac{1}{z}\Big)y + \frac{z^2}{16}(\vec{\gamma}^2)^2 + \frac{z}{16}((\vec{J}.\vec{\gamma})^2 - H\vec{\gamma}^2) + \frac{K}{256} = 0 \tag{2.6.21}$$

and C is now a two-sheeted covering of E. E is an elliptic curve of genus 1. By setting $t = 1/z$ and $Y = ty - \frac{1}{4}\vec{\gamma}^2 + \frac{H}{8}t - \frac{1}{2}t^2$, the equation of E takes the form $Y^2 = tP_3(t)$, where $P_3(t)$ is the degree 3 polynomial:

$$P_3(t) = \frac{1}{4}t^3 - \frac{H}{8}t^2 + \Big(\frac{H^2}{64} + \frac{\vec{\gamma}^2}{4} - \frac{K}{256}\Big)t - \frac{(\vec{J}.\vec{\gamma})^2}{16} \tag{2.6.22}$$

The four branch points are obtained by solving $Y = 0$, so there is a branch point at $t = 0$ (or $z = \infty$) and three branch points at finite values of t (or z).

We now study the covering $C \to E$ coming from $\mu \to y = \mu^2$. This two-sheeted covering is branched only at $y = 0$ and $y = \infty$. The meromorphic function y on E takes each value three times because for a given y, z is determined by a degree 3 equation. Hence it has three zeros and three poles. Setting $y = 0$ in the equation of E gives

$$\frac{1}{16}(\vec{\gamma}^2)^2 + \frac{1}{16}t((\vec{J}.\vec{\gamma})^2 - H\vec{\gamma}^2) + \frac{K}{256}t^2 = 0 \tag{2.6.23}$$

yielding two points $(y = 0, t = t_1, t_2)$. The third point with $y = 0$ and the three points with $y = \infty$ occur when $t = 0$ and $t = \infty$. For $t \to \infty$ we have two points P_1, P_2 on E corresponding to the branches:

$$P_1 : y = t - \frac{H}{4} + O(\frac{1}{t}), \quad P_2 : y = \frac{K}{256}\frac{1}{t} + O(\frac{1}{t^2}) \tag{2.6.24}$$

Set t as the local parameter at ∞, now P_1 is a pole of y and P_2 is the third zero of y. For $t \to 0$, choose the local parameter $\sqrt{t}$ and we get two branches:

$$P_3 : y = \frac{\vec{\gamma}^2}{4}\frac{1}{t} \pm i\frac{\vec{J}.\vec{\gamma}}{4}\frac{1}{\sqrt{t}} + O(1) \tag{2.6.25}$$

showing that y has a double pole at $P_3 = (t = 0, y = \infty)$ on E. Of these six poles and zeros of y, four are branch points of the covering $C \to E$ because at P_3 the equation of C is singular. Since C is a desingularized curve, P_3 blows up to two points $\tilde{P}_3$ and $\tilde{P}_3'$ of C, and P_3 has two pre-images similar to other points in its neighbourhood. On the other hand, P_1 and P_2 are branch points and have just one pre-image each, $\tilde{P}_1$ and $\tilde{P}_2$ on C. Using the Riemann-Hurwitz formula $2g - 2 = N(g_0 - 2) + \nu$, where $g_0 = 1, N = 2$ and $\nu = 4$, we find that the genus of C is 3.

We finally study the covering $\Gamma \to C$ coming from $\lambda \to z = \lambda^2$ whose only ramification can be at $z = 0, \infty$. For a given z, there are 4 values of μ satisfying the equation of C so the meromorphic function z has 4 zeros and 4 poles. Four branches of Γ above $\lambda = \infty$ are already obtained in (2.6.18) which by definition, correspond to four points on the curve Γ. They project on the two branches of C given by (2.6.25), therefore the covering at points $\tilde{P}_3$ and $\tilde{P}_3'$ is unbranched. Similarly, above $\lambda = 0$ we have the four branches of Γ given by (2.6.19). Two points of Γ, $Q_{1,2} = (\mu \sim \pm\lambda\sqrt{K}/16)$ project on $\tilde{P}_2$ and the two points $Q_{3,4} = (\mu \sim \pm 1/\lambda)$ project on $\tilde{P}_1$, as seen from (2.6.24). So the covering is unbranched at these points. This covers all zeros and poles of z hence the covering $\Gamma \to C$ is unbranched. Applying the Riemann-Hurwitz formula again with $g_0 = 3, N = 2, \nu = 0$, gives $g = 5$ for the genus of the spectral curve Γ.

We are now able to proceed to the matrix of eigenvectors of $L(\lambda)$. Consider the eigenvector $\Psi(\lambda, \mu)$ satisfying $L(\lambda)\Psi = \mu\Psi$ at the point (λ, μ) of the spectral curve. We can easily find the eigenvectors of $L(\lambda)$ at the four points Q_1, Q_2, Q_3, Q_4 above $\lambda = 0$ with the normalization of the first component. Then putting them in columns gives the matrix $\hat{\Psi}(\lambda)$

as:

$$\hat{\Psi}(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i\zeta + O(\lambda) & -i\zeta + O(\lambda) & i\frac{z_1}{z_2} + O(\lambda) & -i\frac{z_1}{z_2} + O(\lambda) \\ \frac{-iz_2}{4}\zeta\lambda + O(\lambda^2) & \frac{iz_2}{4}\zeta\lambda + O(\lambda^2) & \frac{-4i}{z_2}\frac{1}{\lambda} + O(1) & \frac{4i}{z_2}\frac{1}{\lambda} + O(1) \\ \frac{-1}{4}z_1\lambda + O(\lambda^2) & \frac{-1}{4}z_1\lambda & -\frac{4}{z_2}\frac{1}{\lambda} + O(1) & \frac{4}{z_2}\frac{1}{\lambda} + O(1) \end{pmatrix} \tag{2.6.26}$$

The j-th column corresponds to the expansion at Q_j. Here $\zeta = \sqrt{(z_1^2 + 8\xi_1)/(z_2^2 + 8\xi_2)}$.

2.7 Szegö kernel method

We will see that for the Gaudin systems, the symplectic potential (i.e. the canonical 1-form) associated to the symplectic structure on the phase space is given by

$$\Theta_{\mathcal{M}} = \sum_{j=1} \operatorname{tr} A_j G_j^{-1} dG_j \tag{2.7.1}$$

where A_k is the diagonalization of $L(\lambda)$ in a neighbourhood of the pole λ_k by G_k. However, having the parameterization of phase space $\mathcal{M}$ by action-angle coordinates $\{\theta_j, I_j\}$, we need to express this 1-form in terms of action-angle variables. This requires calculating the partial derivatives of G_k with respect to action-angle variables, although establishing a connection between G_j and action-angle variables is indeed a challenge. We will bridge this gap using the Szegö kernel which is a $(\frac{1}{2}, \frac{1}{2})$ bidifferential form on $\Gamma \times \Gamma$, defined as:

$$S_{pq}(x, y) = \frac{\theta_{pq}(\mathcal{A}(x) - \mathcal{A}(y))}{\theta_{pq}(0)E(x, y)} \tag{2.7.2}$$

for a given characteristics (p, q) where $\mathcal{A}$ is the Abel map on the surface, E is the prime form and θ_{pq} is the theta function with characteristics (p, q) defined on the Jacobian of the surface. The useful fact is that G_k can be identified with the matrix of eigenvectors $\hat{\Psi}$ locally around λ_k and the components of eigenvectors will have already been expressed in terms of theta functions. Therefore with some work, the eigenvectors and subsequently $\hat{\Psi}$ can be written in terms of Szegö kernel as following:

Proposition: *The matrix of eigenvectors $\hat{\Psi}$ and its inverse in terms of Szegö kernel*

are:

$$\hat{\Psi}(\lambda)_{ab} = \frac{S_q(\lambda^{(b)}, \lambda_0^{(a)})}{S_q(\lambda^{(b)}, \lambda_0^{(1)})}, \quad \hat{\Psi}^{-1}(\lambda)_{ab} = \frac{S_q(\lambda_0^{(b)}, \lambda^{(a)})}{S_q(\lambda_0^{(b)}, \lambda^{(1)})} \tag{2.7.3}$$

where $\lambda^{(i)}$ *and* $\lambda_0^{(j)}$ *are points above* λ *and* λ_0 *on the spectral curve realized as an* N*-sheeted covering of* $\mathbb{CP}^1$.

The next step is to identify G_j with $\hat{\Psi}(\lambda_j)$ in the vicinity of λ_j given the fact that $A_j = G_j L_j G_j^{-1}$ and $L(\lambda) = \hat{\Psi}(\lambda)\hat{\mu}\hat{\Psi}(\lambda)^{-1}$. There are now previously established formulae for variations of Szegö kernel with respect to action-angle variables, namely

$$\frac{\partial S(x,y)}{\partial q_\gamma} = -\oint_{a_\gamma} S_q(x,t) S_q(t,y) \tag{2.7.4}$$

and

$$\frac{\partial S_q(x,y)}{\partial I_\gamma} = -\frac{\pi i}{2} \sum_{x_k \text{ br. pts}} \operatorname*{res}_{t=x_k} v_\gamma(t) \frac{W_t[S_q(x,t), S_q(t,y)]}{d\lambda(t) d\mu(t)} \tag{2.7.5}$$

where v_γ are the basis of holomorphic differentials on the spectral curve Γ and W_t is the Wronskian with respect to t and the sum is taken over all branch points. With these formulae applied, we will be able to calculate the required partial derivatives. The end result is that:

$$\Theta_{\mathcal{M}} = \sum_{j=1}^{g} I_j dq_j \tag{2.7.6}$$

Chapter 3

Classical Gaudin system

The Gaudin models, introduced by M. Gaudin in his 1976 paper [10], are a significant part of the theory of integrable systems and the subject of high interest amongst mathematical physicists. Their unique properties at both classical and quantum level, are in connection with a long-range type of interactions described by the commuting Hamiltonians of Gaudin system. We will later review some use cases of Gaudin models but here we are mostly interested in analytical construction of the solution. The classical Gaudin system is best expressed via its Lax representation. Once we have the Lax equation, the problem of finding the solution of the system becomes the problem of finding the eigenvectors of Lax matrix. The eigenvector equation is closely related to the characteristic equation of the matrix and in this case, the characteristic equation defines an algebraic curve, referred to as the *spectral curve*. The general idea is to find the eigenvectors at any given point of the spectral curve and then use them to reconstruct the Lax matrix. This boils down to finding the meromorphic functions in the components of eigenvectors which hold specific analytical properties. Therefore, we need to study the spectral curve first, then extract the analytical data of the system and try to construct suitable functions satisfying the required analytical properties.

3.1 Lax representation of Gaudin model

Suppose we have the following ordinary matrix differential equation

$$\partial_\lambda \Psi = L(\lambda)\Psi \tag{3.1.1}$$

where $L(\lambda)$ is a matrix-valued meromorphic function of $\lambda \in \mathbb{CP}^1$ and $\Psi(\lambda) \in \mathbb{C}^N$. λ is called the *spectral parameter*. Consider the following special case of equation (3.1.1):

Now set

$$L(\lambda) = \sum_{j=1}^{m} \frac{A_j}{\lambda - \lambda_j} \tag{3.1.2}$$

where the residues $A_j \in \mathfrak{sl}(N, \mathbb{C})$ are independent of λ. We also impose the following assumptions:

1) Regularity at infinity, which requires

$$\sum_{j=1}^{m} A_j = 0 \tag{3.1.3}$$

2) Ψ is holomorphic for all $\lambda \in \mathbb{CP}^1 \setminus \{\lambda_1, ..., \lambda_m\}$

3) Ψ has regular singularities at $\lambda = \lambda_j, j = 1, ..., m$

4) $L(\lambda)$ is diagonal at $\lambda = \lambda_0$

By choosing the normalization point $\lambda_0 = \infty$ and combining the equations (3.1.1) and (3.1.2) we get the equations:

$$\begin{cases} \partial_\lambda \Psi = \Big(\sum\limits_{j=1}^{m} \dfrac{A_j}{\lambda - \lambda_j} \Big) \Psi \\ \partial_{\lambda_j} \Psi = \dfrac{A_j}{\lambda - \lambda_j} \Psi \end{cases} \tag{3.1.4}$$

which leads to the following system of nonlinear ODE known as *Schlesinger equations*:

$$\begin{cases} \dfrac{\partial A_j}{\partial \lambda_i} = \dfrac{[A_i, A_j]}{\lambda_i - \lambda_j}, \qquad i \neq j \\ \dfrac{\partial A_i}{\partial \lambda_i} = \sum\limits_{j=1, j\neq i}^{m} \dfrac{[A_i, A_j]}{\lambda_i - \lambda_j} \end{cases} \tag{3.1.5}$$

Here we consider the autonomous analogue of this, it then becomes the so called *classical Gaudin system*:

$$\begin{cases} \Big(\sum\limits_{j=1}^{m} \dfrac{A_j}{\lambda - \lambda_j} \Big) \Psi = \mu \Psi \\ \dfrac{d}{dt_j} \Psi = \dfrac{A_j}{\lambda - \lambda_j} \Psi \end{cases} \tag{3.1.6}$$

Suppose $L(\lambda) = \sum_j \frac{A_j}{\lambda - \lambda_j}$ and $M_j(\lambda) = \frac{A_j}{\lambda - \lambda_j}$, then Ψ and μ can be regarded as the

eigenvector and eigenvalue of the operator $L(\lambda)$ and the t_j-time evolution equation of Ψ is determined by $L(\lambda)$ so

$$\begin{cases} L(\lambda)\Psi = \mu\Psi \\ \dfrac{d}{dt_j}\Psi = M_j(\lambda)\Psi \end{cases} \tag{3.1.7}$$

differentiating the first equation with respect to t_j gives

$$\begin{aligned} &\frac{d}{dt_j}L(\lambda)\Psi + L(\lambda)\frac{d}{dt_j}\Psi = \mu\frac{d}{dt_j}\Psi = \mu M_j(\lambda)\Psi = M_j(\lambda)L(\lambda)\Psi \\ &\implies \frac{d}{dt_j}L\Psi + LM_j\Psi = M_jL\Psi \implies \dot{L}\Psi = (M_jL - LM_j)\Psi \\ &\implies \dot{L} = [M_j, L] \end{aligned} \tag{3.1.8}$$

Equation (3.1.8) is called the *Lax equation* and it is an alternative form of representing the equations of motion of a system. The pair (L, M_j) is called the *Lax pair* and L the *Lax matrix* which depend on dynamical variables as well as the spectral parameter λ. The importance of Lax equation comes from the fact that it is an isospectral evolution equation for the Lax matrix $L(\lambda)$ i.e. the spectrum of the matrix is time-invariant.

Taking the Lax equation and substituting $L(\lambda)$ and $M_j(\lambda)$ gives

$$\frac{d}{dt}\Big(\sum_k \frac{A_k}{\lambda - \lambda_k}\Big) = \Big[\frac{A_j}{\lambda - \lambda_j}, \sum_k \frac{A_k}{\lambda - \lambda_k}\Big] \tag{3.1.9}$$

Subsequently, taking the residues of both sides at a λ_l yields the better known form of Gaudin system

$$\begin{cases} \dfrac{\partial A_j}{\partial t_k} = \dfrac{[A_j, A_k]}{\lambda_j - \lambda_k}, & j \neq k \\ \dfrac{\partial A_j}{\partial t_j} = \displaystyle\sum_{l \neq j} \frac{[A_j, A_l]}{\lambda_j - \lambda_l} \end{cases} \tag{3.1.10}$$

The analytical method of solving an integrable system is based on the study of eigenvector equation $L(\lambda)\Psi(\lambda, \mu) = \mu\Psi(\lambda, \mu)$ where $\Psi(\lambda, \mu)$ is the eigenvector with eigenvalue μ. Now consider Γ, the curve defined by the characteristic polynomial of $L(\lambda)$, i.e. $\det(L(\lambda) - \mu\mathbf{1}) = 0$ which is called the *spectral curve*. The immediate result of Lax equation being isospectral is that Γ is a time-independent curve. To avoid technical difficulties, we only

consider the maximal subset of the locus of $\det(L(\lambda) - \mu\mathbf{1}) = 0$ that makes Γ *non-singular*, i.e. all of its points have the property that the projection to either λ or μ component can be used to homeomorphically identify a neighbourhood of the point on the surface with a disc in the respective projection plane. We also require that all branch points of the spectral curve be simple. With these assumptions, Γ can be regarded as a compact Riemann surface and it is going to be central to the analytical solution method of integrable systems.

3.2 Analysis of the solution on the spectral curve

As noted, the analytical approach to solve the equations of motion revolves around the eigenvector equation of the Lax matrix. At any point of the spectral curve there exists an eigenvector of $L(\lambda)$ with eigenvalue μ. We can reconstruct the eigenvector Ψ from its analytical properties on Γ. Specifically, from all the so-called *dynamical* information of the system that are contained in the divisor of its poles. We will later use theta functions to explicitly reconstruct the eigenvectors from these analytical data.

The first step of the solution is to calculate the genus of the spectral curve defined by the Lax matrix L. Assume L depends on the spectral parameter λ as previously stated

$$L(\lambda) = \sum_{j=1}^{m} \frac{A_j}{\lambda - \lambda_j} \tag{3.2.1}$$

Let us denote the eigenvector of $L(\lambda)$ with eigenvalue μ by $\Psi(\lambda, \mu)$. A point on the spectral curve Γ is a pair (λ, μ) satisfying $\det(L(\lambda) - \mu\mathbf{1}) = 0$. As mentioned before, to avoid some technical complications, we only consider the non-singular points satisfying characteristic equation plus the desingularized infinity point. Branch points are also assumed to be simple. This then defines a non-singular, compact Riemann surface. Assuming that the Lax matrix L is $N \times N$, the equation of Γ is of the form

$$\Gamma(\lambda, \mu) = (-\mu)^N + \sum_{q=0}^{N-1} r_q(\lambda)\mu^q = 0 \tag{3.2.2}$$

The coefficients $r_q(\lambda)$ are polynomials of matrix elements of $L(\lambda)$ so they have poles at λ_j. We can see from equation (3.2.2) that the spectral curve appears as N-sheeted covering of the Riemann sphere. To a given point λ on the Riemann sphere, there corresponds N points on the curve whose coordinates are $(\lambda, \mu_1), ..., (\lambda, \mu_N)$, where μ_i are the solutions of

the algebraic equation $\Gamma(\lambda, \mu) = 0$. By definition, μ_i are the eigenvalues of $L(\lambda)$. We are going to determine the analytical properties of the eigenvector $\Psi(\lambda, \mu)$ and try to reconstruct $L(\lambda)$ from them.

Denote the N points above $\lambda = \infty$ on the spectral curve by $\infty_1, ..., \infty_N$. The regularity condition at infinity means that the points $\infty_1, ..., \infty_N$ are the regular points of the differential $\mu d\lambda$. Therefore $\text{res}_{\lambda=\infty} L(\lambda)d\lambda = 0$ which implies

$$\sum_{j=1}^{m} A_j = 0$$

We are now in the position to compute the genus of the spectral curve.

Theorem 1. ([1], p.126) *The genus of the spectral curve* Γ *is*

$$g_\Gamma = \frac{N(N-1)m}{2} - N + 1 \tag{3.2.3}$$

Proof. To compute the genus, we utilize the Riemann-Hurwitz formula which expresses the genus of an N-sheeted covering of a Riemann surface of genus g_0

$$2g - 2 = N(2g_0 - 2) + \nu \tag{3.2.4}$$

where ν is the branching index of the covering. Being a Riemann sphere, $g_0 = 0$ in this case. To find ν, we should calculate the number of branch points with multiplicity or in other words, the number of roots of $\Gamma(\lambda, \mu)$ in μ with multiplicity. This is also the number of zeros of $\partial_\mu \Gamma(\lambda, \mu)$. Since $\partial_\mu \Gamma(\lambda, \mu)$ is a meromorphic function, the number of its zeros equals the number of poles. These poles are located at the poles of $L(\lambda)$. Above λ_j, a pole of $L(\lambda)$, we have N branches of the form $\mu_j = \frac{l_j}{\lambda - \lambda_j} + ...$ where l_j is the eigenvalue of A_j (assumed to be distinct). On such branch we have

$$\partial_\mu \Gamma(\lambda, \mu)|_{(\lambda, \mu_j(\lambda))} = \prod_{i \neq j} (\mu_j(\lambda) - \mu_i(\lambda)) \tag{3.2.5}$$

which has a pole of order $N - 1$. Thus, summing over all branches, the total order of the pole over λ_j is $N(N-1)$. Now summing over all poles of $L(\lambda)$, we see that the total branching index is $\nu = N(N-1)m$ so the genus of the spectral curve is

$$g_\Gamma = \frac{N(N-1)m}{2} - N + 1$$

□

Going back to the problem of reconstructing the eigenvectors, we shall now see how the general procedure of solving an integrable system using its analytical properties on the spectral curve works. We say that the vector $\Psi(P)$ has a pole if one of its components has a pole. The divisor D of the poles of the eigenvector $\Psi(P)$ for P a finite point on the spectral curve contains all the dynamical information of the system, hence it is called *dynamical divisor*.

Let $P = (\lambda, \mu) \in \Gamma$ and assume that it is not a branch point of the surface so all eigenvalues of $L(\lambda)$ are distinct and the eigenspaces at P are one-dimensional. The eigenvector $\Psi(P)$ is of the form

$$\Psi(P) = \begin{pmatrix} \psi_1(P) \\ \vdots \\ \psi_N(P) \end{pmatrix} \tag{3.2.6}$$

We choose to normalize Ψ such that its first component $\psi_1(P)$ is equal to 1 at any point $P \in \Gamma$. We can see that each $\psi_j(P)$ depends analytically on P locally. These components are as a matter of fact, meromorphic functions.

Proposition: ([1], p.131) With the above normalization, the components of the eigenvectors $\Psi(P)$ at $P = (\lambda, \mu)$ are meromorphic functions on the spectral curve.

Proof. Take a point $P = (\lambda, \mu)$ on the spectral curve Γ. It satisfies the equation $\det(L(\lambda) - \mu\mathbf{1}) = 0$ and there is a unique eigenvector $\Phi(P)$ for the eigenvalue μ with the normalized first component $\psi_1(P) = 1$. Notice that the components of $\Psi(P)$ can be regarded as some minors (determinant of the square matrix obtained by removing a row and a column) of the matrix $L(\lambda) - \mu\mathbf{1}$ so written in terms of elements of $L(\lambda) - \mu\mathbf{1}$, they are meromorphic functions on Γ. By applying the normalization, we divide all components by the first component (another meromorphic function) hence they are still meromorphic functions on Γ. □

Thus, to each point (λ, μ) on Γ, we associate a meromorphic eigenvector $\Psi(P)$. At branch points however, several eigenvectors can be associated to the point. For a generic Lax matrix, the eigenspaces are one-dimensional even at branch points. Moreover, the eigenspaces around P admit a unique analytic continuation at P, regardless of the chosen branch. The associated eigenspace to each point of Γ defines an analytic line bundle (see

[1], p.131) known as the eigenvector bundle where the eigenvectors at each point lie.

Example: Let us briefly review a 2×2 case where $L(\lambda)$ is a 2×2 matrix to get a better understanding of the situation at branch points. Let

$$L(\lambda)=\begin{pmatrix} a(\lambda) & b(\lambda)\\ c(\lambda) & d(\lambda)\end{pmatrix}$$

which has eigenvalues

$$\mu_{\pm}(\lambda)=\frac{1}{2}(a(\lambda)+d(\lambda))\pm\frac{1}{2}\sqrt{\Delta(\lambda)}\,,\qquad \Delta(\lambda)=(a(\lambda)-d(\lambda))^2+4b(\lambda)c(\lambda)$$

The corresponding normalized eigenvectors are

$$\Psi_{\pm}=\begin{pmatrix}1\\ \psi_{\pm}\end{pmatrix}\,,\qquad \psi_{\pm}=\frac{d(\lambda)-a(\lambda)}{2b(\lambda)}\pm\frac{\sqrt{\Delta(\lambda)}}{2b(\lambda)}$$

Assume that λ_0 is a root of $\Delta(\lambda)=0$. As $\lambda\longrightarrow\lambda_0$, Ψ_+ and Ψ_- tend smoothly to the same limit except if one has also $b(\lambda_0)=0$. If $b(\lambda_0)\neq 0$, we can express $L(\lambda_0)$ in the basis given by $\Psi(\lambda_0)$ and $\binom{0}{1}$, hence

$$L(\lambda_0)\longrightarrow\begin{pmatrix}\frac{1}{2}(a(\lambda)+d(\lambda)) & b(\lambda_0)\\ 0 & \frac{1}{2}(a(\lambda)+d(\lambda))\end{pmatrix}$$

which means that $L(\lambda_0)$ is of the Jordan form and has only one eigenvector. However, if $b(\lambda_0)=0$, then we also have $a(\lambda_0)=d(\lambda_0)$. Assuming that $d(\lambda)-a(\lambda)$ and $b(\lambda)$ vanish to first order in $\lambda-\lambda_0$, then $\frac{d(\lambda)-a(\lambda)}{2b(\lambda)}$ tends to some limit ψ_0. We see that $\psi_{\pm}\sim\psi_0\pm\sqrt{\frac{c(\lambda)}{b(\lambda)}}$. Therefore if $c(\lambda_0)\neq 0$, we still have only one eigenvector of the form $\binom{0}{1}$. If $c(\lambda)$ also vanishes to first order in $\lambda-\lambda_0$, the matrix $L(\lambda_0)$ is diagonalizable and the eigenvectors $\Psi_{\pm}$ tend to different limits at λ_0. However, in this case we have $\Delta\sim(\lambda-\lambda_0)^2$ so the corresponding point (λ_0,μ_0) of the spectral curve is not a branch point, but a singular point which blows up to two points if we desingularize it. This analysis also applies to the branch points of order 2 in general case.

To understand the analytical properties of $\Psi(P)$, we first count the number of its poles in the following proposition:

Proposition: ([1], eq. 5.17) The number of poles of $\Psi(P)$ with normalized first component is $g+N-1$, where $g=g_\Gamma$ is the genus of spectral curve.

Proof. Let $\hat{\Psi}(\lambda)$ be the matrix of eigenvectors of $L(\lambda)$:

$$\hat{\Psi}(\lambda) = \begin{pmatrix} \psi_1(P_1) & \dots & \psi_1(P_N) \\ \vdots & \vdots & \vdots \\ \psi_N(P_1) & \dots & \psi_N(P_N) \end{pmatrix} \tag{3.2.7}$$

where P_j are the N points above λ. So by definition, $\hat{\Psi}(\lambda)$ is the matrix diagonalizing $L(\lambda)$. We also apply the normalization $\psi_1(P_j) = 1$. Now define:

$$W(\lambda) = [\det \hat{\Psi}(\lambda)]^2$$

$W(\lambda)$ is a well-defined rational function of λ on the Riemann sphere since the square of the determinant does not depend on the order of P_j. It has a double pole at the simple poles of $\Psi(\lambda)$. We just need to count its zeroes to get the number of poles.

We show that $W(\lambda)$ has a simple zero for values of λ corresponding to branch points of the covering. If so, the number of poles would be $\frac{\nu}{2}$. Recall that from (3.2.4), the number of branch points is $\nu = 2(N + g - 1)$ which concludes the result. Notice that $W(\lambda)$ only vanishes at branch points where two columns of $\hat{\Psi}(\lambda)$ are identical. Let $P_j = (\mu_j, \lambda)$, then $\Psi(P_j)$ are the eigenvectors of $L(\lambda)$ corresponding to the eigenvalues μ_j hence they are linearly independent when all μ_j are different. Therefore $W(\lambda)$ cannot vanish at such points. The other possibility of $W(\lambda)$ vanishing is when the vector $\Psi(P)$ vanishes itself which means all of its components must have a common zero but this is not possible since the first component is always 1. Now assume b_0 corresponds to a branch point which is of order 2. Let z be a local analytic parameter, the covering projection $P \to \lambda$ has the expression $\lambda = b_0 + \lambda_1 z^2 + O(z^3)$. The determinant vanishes to order z so W vanishes to order z^2 which is proportional to $\lambda - b_0$ thus $W(\lambda)$ has a simple zero at b_0. □

We also need to study the behaviour of the eigenvector around $\lambda = \lambda_0$. At the N points $\lambda_0^{(j)}$ above λ_0, the eigenvectors are proportional to canonical vectors e_i due to $L(\lambda)$ being diagonal at λ_0. We also assume that λ_0 is not a branch point. The compatibility with the normalization of $\psi_1(P) = 1$ gives us the following statement:

Proposition: ([1], 1st prop. of p.135) The k^{th} component of $\Psi(P)$, $\psi_k(P)$ must have a simple pole at $\lambda_0^{(k)}$ and vanish at $\lambda_0^{(1)}$ for $k = 2, ..., N$.

Proof. Around $\lambda_0^{(k)}$, $k = 1, ..., N$, the eigenspace of $L(\lambda)$ is spanned by a vector of the form

$V_k(\lambda) = e_k + O(\frac{1}{\lambda})$. The first component of V_k is $V_k^1 = \delta_{1k} + O(\frac{1}{\lambda})$. To gave Ψ normalized, we divide V_k by V_k^1. So:

$$\Psi(P)|_{P\to\lambda_0^{(1)}} = \begin{pmatrix} 1 \\ O(\frac{1}{\lambda}) \\ \vdots \\ O(\frac{1}{\lambda}) \end{pmatrix}, \quad \Psi(P)|_{P\to\lambda_0^{(k)}} = \begin{pmatrix} 1 \\ O(1) \\ \vdots \\ O(\lambda) \\ \vdots \\ O(1) \end{pmatrix}, \ k \geq 2$$

where $O(\lambda)$ at the k^{th} component has pole at $\lambda_0^{(k)}$. □

To recap, imposing the condition that $L(\lambda_0)$ is diagonal introduces $N-1$ poles at $\lambda_0^{(j)}, j = 2, ..., N$. The location of these poles is independent of time and they do not contain any dynamical information. Only the position of the other g poles has dynamical significance. Taking the formal sum of of these g dynamical poles gives us a degree g divisor D known as the *dynamical divisor*. Unlike the aforementioned non-dynamical poles, the dynamical divisor does depend on time so if necessary, it could also be denoted by $D(t)$. Since the vector $\Psi(P)$ possesses a pole if one of its components have a pole, we obtain by the last two prepositions that the divisor of the k^{th} component of the eigenvector $\Psi(P)$ is bigger than $(-D+\lambda_0^{(1)}-\lambda_0^{(k)})$. This information is sufficient to reconstruct the eigenvectors and the Lax matrix. The following proposition implies that having the analytical data of the eigenvectors, i.e. the dynamical divisor, ensures the existence of a meromorphic vector meeting the required analytical behaviour:

Proposition:([1], p. 135) Let D be a divisor of degree g on Γ. Up to normalization, there is a unique meromorphic function ψ_k with divisor $(\psi_k) \geq -D + \lambda_0^{(1)} - \lambda_0^{(k)}$.

Proof. This can deduced from Riemann-Roch theorem, which states that for a compact Riemann surface on genus g, we have

$$l(D) - l(K - D) = \deg(D) - g + 1$$

where $l(D)$ is the complex dimension of the vector space of meromorphic functions f on the surface, such that all the coefficients of the divisor $(f) + D$ are non-negative and K is the canonical divisor, i.e. the divisor associated to the canonical line bundle. An immediate

corollary of Riemann-Roch theorem is that

Corollary: ([1], p. 558) The dimension of the space meromorphic functions with at most k poles and at least h zeros on a Riemann surface of genus g is greater than or equal to $k-h+1-g$. The equality occurs when $k-h>2g-2$.

As a result, since ψ_k must have $g+1$ poles and one zero, the dimension of the space of functions such as ψ_k is 1. Therefore ψ_k is unique up to multiplication by a constant $\psi_k \to d_k\psi_k$. □

3.3 Reconstructing the solution from analytical data

Equipped with the functions $\psi_k(P)$ and the vector $\Psi(P) = \big(1, \psi_2(P), ..., \psi_N(P)\big)^T$, we can eventually construct the eigenvectors we have been looking for. The following theorem guarantees the existence of the matrix L which solves the Lax equation:

Theorem 2. ([1], p.136) *Given the spectral curve Γ, such that above the points λ_k, the N branches $\mu_1, ..., \mu_N$ are of the form $\mu_j = \frac{c_j}{\lambda-\lambda_k}$ + (regular terms) , there exists a unique matrix $L(\lambda)$, rational in λ, such that*

$$\big(L(\lambda) - \mu\mathbf{1}\big)\Psi(P) = 0$$

This matrix has poles at the points λ_k and satisfies the condition $L(\lambda_0) = \mathrm{diag}(a_1, ..., a_N)$. The matrix $L(\lambda)$ is given by

$$L(\lambda) = \hat{\Psi}(\lambda)\hat{\mu}\hat{\Psi}^{-1}(\lambda) \tag{3.3.1}$$

where $\hat{\mu} = \mathrm{diag}(\mu_1, ..., \mu_N)$.

This theorem states that once the spectral curve is given, all of the dynamical data are encoded into the divisor D. We should point out that the eigenvector $\Psi(P)$ here is defined up to left multiplication by diagonal matrices. On the Lax matrix, this amounts to conjugation by a constant diagonal matrix.

Before getting to the main theorem of this section, let us review three key objects from the theory of Riemann surfaces that play a significant role here. Suppose we have a compact Riemann surface Γ with a choice of canonical homology basis $H_1(\Gamma, \mathbb{Z}) = \mathbb{Z}\{a_1, b_1, ..., a_g, b_g\}$ and a basis of holomorphic differentials $\{\omega_j\}_{j=1}^g$ normalized with respect to a-cycles. The

Abel map $\mathcal{A}(P) : \Gamma \to \mathbb{C}^g$ is defined on the Riemann surface Γ by

$$\mathcal{A}(P) := \left(\int_{P_0}^{P} \omega_1, ..., \int_{P_0}^{P} \omega_g \right)^T \tag{3.3.2}$$

with a fixed point P_0. The integration paths are taken inside fundamental polygon of Γ, i.e. the simply connected domain obtained by cutting the surface along cycles a_j, b_j realized as loops on the surface. Therefore, the definition is independent of integration paths. On a formal sum of points (i.e. a divisor), it is defined by the natural linear combination of Abel maps on each individual point. Taking the values of $\mathcal{A}(P)$ inside the Jacobian of the surface $J(\Gamma) = \mathbb{C}^g/(\mathbb{Z}^g + \mathcal{B}\mathbb{Z}^g)$, $\mathcal{B}$ the matrix of b-periods, makes the Abel map well-defined.

The *vector of Riemann constants* $\mathcal{K}$ is defined as

$$\mathcal{K}_j := \frac{\tau_{jj}}{2} - \sum_{l=1}^{g} \int_{a_l} \mathcal{A}_j d\mathcal{A}_l \tag{3.3.3}$$

and it only depends on the choice of homology basis and the basepoint P_0.

We then need to fix a basis for Homology group and also holomorphic differentials. To avoid certain technical difficulties in future, more specifically, to avoid having non well-defined integrals of the meromorphic form $\mu d\lambda$ due to the integration path crossing the poles of the differential, we choose a basis of homology for the punctured Riemann surface, i.e. a canonical basis $\{a_i, b_i\}_{i=1}^g, \{l_i^{(k)}\}$ on $\Gamma \setminus \{\lambda_j^{(k)}\}$, $j = 1, ..., m,\ k = 1, ..., N$ where a- and b- cycles are as before and $\{l_i^{(k)}\}$ are small loops around $\{\lambda_j^{(k)}\}$, the points above λ_j.

We also fix a basis of holomophic differentials $\{w_j\}$ on Γ normalized with respect to a-cycles, i.e. $\oint_{a_j} \omega_k = \delta_{jk}$.

We also require the *Riemann theta functions*, an important class of complex analytic functions which will essentially be the building blocks of the eigenvectors. A Riemann theta function in g variables is defined by

$$\theta(z_1, ..., z_g) = \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left(2\pi i \langle \vec{n}, \vec{z} \rangle + \pi i \langle \mathcal{B}\vec{n}, \vec{n} \rangle\right) \tag{3.3.4}$$

It can be shown that the series on the right converges, hence it defines an analytic function on $\mathbb{C}^g$. The crucial application of theta functions for us is the fact that they can be used to express meromorphic functions on a Riemann surface of genus g. To show this, we recall the *Riemann's theorem*:

Theorem: ([1], eq.15.15) *Let* $z = (z_1, ..., z_g) \in \mathbb{Z}^g$, *arbitrary. Either the function*

$\theta(\mathcal{A}(p) - z)$ vanishes identically for $p \in \Gamma$ or it has exactly g zeros $p_1, ..., p_g$ such that

$$\mathcal{A}(p_1) + ... + \mathcal{A}(p_g) = z - \mathcal{K}$$

$\mathcal{K}$ being the vector of Riemann constants.

Using this, we wish to show the following lemma:

Lemma: The field of meromorphic functions on a Riemann surface can be parameterized by the ratio of theta functions.

proof. Assuming f is a meromorphic function with g poles at $\sigma_1, ..., \sigma_g$, an additional pole at q_+ and a specific zero at q_-. By the Riemann-Roch theorem, such function is unique up to a multiplication factor. Define the vectors

$$\begin{aligned} v &= \sum_{j=1}^{g} \mathcal{A}(\sigma_j) + \mathcal{K} \\ v_+ &= \mathcal{A}(q_+) + \sum_{j=2}^{g} \mathcal{A}(\sigma_j) + \mathcal{K} \\ v_- &= \mathcal{A}(q_-) + \sum_{j=2}^{g} \mathcal{A}(\sigma_j) + \mathcal{K} \\ v_0 &= v + v_+ + v_- \end{aligned}$$

and the function

$$f(p) = \frac{\theta(\mathcal{A}(p) - v_-)\theta(\mathcal{A}(p) - v_0)}{\theta(\mathcal{A}(p) - v)\theta(\mathcal{A}(p) - v_+)}$$

One can check using Riemann theorem that the factors in the denominator vanish at the points $\sigma_1, ..., \sigma_g$ and $q_+, \sigma_2, ..., \sigma_g$ respectively. Similarly, the two factors in the numerator vanish at $q_-, \sigma_2, ..., \sigma_g$ and g other points. The zeros at $\sigma_2, ..., \sigma_g$ are cancelled out between the numerator and denominator, therefore giving the correct poles and zeros. $f(p)$ can also be shown to be well-defined as we go round cycles on the surface using the properties of theta functions. □

We are now finally in a position to construct the solution of Gaudin system in terms of theta functions using the dynamical data of the system.

Theorem 3. ([1], eq. 5.40) *The components of eigenvector matrix $\hat{\Psi}(\lambda)$ subject to being diagonal at $L(\lambda_0)$ and the normalization $\psi_1(\lambda) = 1$ can be explicitly expressed in terms of*

theta functions as:

$$\psi_k(P) = C_k \frac{\theta(\mathcal{A}(P) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(1)}) - \mathcal{A}(D) - \mathcal{K})}{\theta(\mathcal{A}(P) - \mathcal{A}(D) - \mathcal{K})} \frac{\theta(e + \int_{\lambda_0^{(1)}}^{P} \omega)}{\theta(e + \int_{\lambda_0^{(k)}}^{P} \omega)} \tag{3.3.5}$$

where

- N points above λ_0 on the spectral curve are denoted by $\lambda_0^{(1)}, ..., \lambda_0^{(N)}$.

- D is the degree g dynamical divisor $D = (\gamma_1, ..., \gamma_g)$ composed of g dynamical poles of the eigenvectors of Lax matrix on the spectral curve.

- $\mathcal{A}$ is the Abel map and $\mathcal{K}$ is the vector of Riemann constants, both with the base point P_0.

- e is a vector in $\mathbb{C}^g$ such that $\theta(e) = 0$ and $\omega = (\omega_1, ..., \omega_g)$ a basis of holomorphic differentials on Γ.

Proof. We have already established that eigenvectors of L are meromorphic functions and their analytical information is given by the dynamical divisor D. As mentioned earlier, the Riemann's theorem can be used to construct a parametrization of meromorphic functions on the Riemann surface Γ in terms of theta functions. For a vector e in the divisor $D_\theta = \{z \mid \theta(z) = 0\}$, the function $\theta(e + \int_x^y \omega)$ does not vanish identically in y. By Riemann's theorem, this has g zeros $y_1, ..., y_g$ for given x. One of these zeros is x since $e \in D_\theta$, set $y_1 = x$. We now show that $y_2, ..., y_g$ are independent of x. By Riemann's theorem, we have

$$\mathcal{A}(y_1) + ... + \mathcal{A}(y_g) = \mathcal{A}(x) - e - \mathcal{K}$$

so $y_2, ..., y_g$ are determined by $\mathcal{A}(y_2) + ... + \mathcal{A}(y_g) = -e - \mathcal{K}$ which is independent of x. To use this to construct meromorphic functions, notice that $\frac{\theta(e + \int_{x_1}^{y} \omega)}{\theta(e + \int_{x_2}^{y} \omega)}$ has a zero at $y = x_1$ and a pole at $y = x_2$. The extra $g - 1$ zeros in the numerator and denominator cancel each other out.

The components of the eigenvector $\Psi(t, P)$ can now be explicitly written. Ψ_k has a pole at $\lambda_0^{(k)}$ and a zero at $\lambda_0^{(1)}$ so we need the block $\dfrac{\theta(e + \int_{\lambda_0^{(1)}}^{P} \omega)}{\theta(e + \int_{\lambda_0^{(k)}}^{P} \omega)}$. It also has g poles located at dynamical poles which are encoded into the dynamical divisor D. By Riemann's theorem and using the same analysis as above, if P is one of these g poles, then we have:

$$\mathcal{A}(P) - \mathcal{A}(D) - \mathcal{K} = -\sum \mathcal{A}(\text{other } g - 1 \text{ poles}) - \mathcal{K} = e + \mathcal{K} - \mathcal{K} = e \tag{3.3.6}$$

so putting $\theta(\mathcal{A}(P) - \mathcal{A}(D_t) - \mathcal{K})$ in the denominator gives us g dynamical poles. Finally, there should be no monodromy, we then need to assemble all these blocks so that the product remains unchanged when we loop around cycles in the basis of homology group of Γ so the sum of the arguments of the theta functions in the numerator should be equal to the sum of the arguments in the denominator for the whole expression to have no monodromy. We add the term $\theta(\mathcal{A}(P) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(1)}) - \mathcal{A}(D_t) - \mathcal{K})$ to the numerator to fulfil this requirement. Thus:

$$\psi_k(t, P) = C_k \frac{\theta(\mathcal{A}(P) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(1)}) - \mathcal{A}(D_t) - \mathcal{K})}{\theta(\mathcal{A}(P) - \mathcal{A}(D_t) - \mathcal{K})} \frac{\theta(e + \int_{\lambda_0^{(1)}}^{P} \omega)}{\theta(e + \int_{\lambda_0^{(k)}}^{P} \omega)} \tag{3.3.7}$$

where D_t denotes the dynamical divisor at time t. □

It is convenient to express this result in terms of theta functions with characteristic. Recall the following formula for a theta function with characteristic vector $[p, q]$

$$\theta[p, q](z, \tau) = e^{\pi i p^2 \tau + 2\pi i p(z+q)} \theta(z + p\tau + q, \tau) \tag{3.3.8}$$

setting $p = 0$, we can regard the theta function in (3.3.5) as a theta function with characteristic for $q = \mathcal{A}(D) + \mathcal{K}$ so we have

$$(q)_l = \mathcal{K}_l + \int_{P_0}^{D} \omega_l = \mathcal{K}_l + \sum_{i=1}^{g} \int_{P_0}^{\gamma_i} \omega_l \tag{3.3.9}$$

So with q as above and denoting the theta function with characteristic $[0, q]$ by θ_q, we can rewrite the components of $\hat{\Psi}(\lambda)$ as:

$$\psi_k(t, P) = C_k \frac{\theta_q(\mathcal{A}(P) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(1)}))}{\theta_q(\mathcal{A}(P))} \frac{\theta(e + \int_{\lambda_0^{(1)}}^{P} \omega)}{\theta(e + \int_{\lambda_0^{(k)}}^{P} \omega)} \tag{3.3.10}$$

We will later utilize the components of vector q to construct coordinates on the space of spectral covers, i.e. the phase space of our system of equations.

3.4 Time evolution of eigenvectors

One last thing that remains after reconstructing the eigenvectors from analytical and dynamical data is to show that the time evolution of the eigenvectors satisfying the required

analytical properties indeed matches the one given by the time evolution equation of the system $\frac{d}{dt_j}\Psi = M_j\Psi$. Recall that the time evoltion of the system is governed by the Lax equation

$$\frac{d}{dt_j}L(\lambda) = [M_j(\lambda), L(\lambda)] \tag{3.4.1}$$

Let t represent any of time variables t_j and for brevity, denote M_j by just M. Also assume that we have constructed the normalized eigenvector $\Psi(t, P)$ at time t and point $P = (\lambda, \mu)$ with the eigenvalue μ. By the Lax equation, we obtain:

$$\begin{aligned}
&\frac{d}{dt}L\Psi = \frac{d}{dt}(L\Psi) - L\frac{d}{dt}\Psi = ML\Psi - LM\Psi \\
&\implies \frac{d}{dt}(L\Psi) + L(M\Psi - \Psi) - M\mu\Psi = 0 \\
&\implies \frac{d}{dt}(\mu\Psi) - \mu(M\Psi) + L(M\Psi - \frac{d}{dt}\Psi) = 0 \\
&\implies L(M\Psi - \frac{d}{dt}\Psi) = \mu(M\Psi - \frac{d}{dt}\Psi)
\end{aligned} \tag{3.4.2}$$

This implies that $\dot{\Psi} - M\Psi$ must be a multiple of the eigenvector of L for the eigenvalue μ, which is $\Psi(t, P)$. Therefore

$$\frac{d}{dt}\Psi(t, P) - M(\lambda)\Psi(t, P) = C(t, P)\Psi(t, P) \tag{3.4.3}$$

where $C(t, P)$ is a scalar function. Normalizing the eigenvector $\Psi(t, P)$ so that its first component equals 1 gives

$$C(t, P) = \sum_j M_{1j}(\lambda)\psi_j(t, P) \tag{3.4.4}$$

By the analysis of the previous section, the normalized eigenvector $\Psi(t, P)$ has poles at the dynamical divisor $D(t)$ and the $N - 1$ points ∞_k. Now for infinitesimal dt, consider the function

$$\mathcal{N}(t, dt, P) \equiv 1 + C(t, P)dt = 1 + dt\sum_j M_{1j}(\lambda)\psi_j(t, P) \tag{3.4.5}$$

The equation (3.4.3) can be equivalently rewritten as

$$\mathcal{N}(t, dt, P)\Psi(t + dt, P) = (\mathbf{1} + dtM(\lambda))\Psi(t, P) + O(dt^2) \tag{3.4.6}$$

We can see that $\mathcal{N}(t, dt, P)$ as a meromorphic function of $P \in \Gamma$ normalizes the eigenvector with the time evolution given by $\dot{\Psi} = M\Psi$ derived from the Lax equation. The divisor of this meromorphic function is

$$(\mathcal{N}) = D(t + dt) + \sum_{k,i} P^{\circ}_{k,i} - D(t) - \sum_{k,i} P_{k,i} \tag{3.4.7}$$

where as before, $P_{k,i}$ are the points above λ_k and $P^{\circ}_{k,i}$ the zeros of $\mathcal{N}$. From (3.4.6), we see that $\mathcal{N}$ cancels the poles of $\Psi(t + dt, P)$ at $D(t + dt)$ and produces the poles of $\Psi(t, P)$ at $D(t)$. The poles at $\infty_2, ..., \infty_N$ are the same at both sides of (3.4.6) and do not appear in $\mathcal{N}$. Furthermore, since $M(\lambda)$ has a simple pole at λ_k, $\mathcal{N}$ also has a simple pole at N points $P_{k,i}$ above λ_k. $\mathcal{N}$ also needs to have extra zeros $P^{\circ}_{k,i}$ in order to match the number of poles. Since dt is small, and $\mathcal{N} = 1$ for $dt = 0$, the zeros are close to the poles, $D(t+dt)$ is close to $D(t)$ and on each sheet of the covering there is exactly one zero $P^{\circ}_{k,i}$ close to the simple pole $P_{k,i}$. Now by combining the theorem ([1], p. 143) and ([1], eq. 5.37), we obtain the following:

Theorem: *Let $\gamma_j(t)$ with $j = 1, ..., g$ be the points of the dynamical divisor $D(t)$. Let ω be any holomorphic differential on Γ. The time evolution of the points $\gamma_j(t)$ induced by the Lax equation $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$ is such that:*

$$\frac{d}{dt} \sum_{j=1}^{g} \int^{\gamma_j(t)} \omega = \sum_{k} \sum_{i=1}^{N} \mathrm{res}_{P_{k,i}} \left(\left[M(\lambda)\Psi(t, P) \right]_1 \omega \right) \tag{3.4.8}$$

where $[M(\lambda)\Psi(t, P)]_1 = \sum_j M_{1j}(\lambda)\psi_j(t, P)$ is the first component of the vector $M(\lambda)\Psi$.

which leads to the following result:

Corollary: ([1], eq. 5.38) The flow induced by the Lax equation on the eigenvector bundle of the spectral curve is a linear flow on the Jacobian of the curve, i.e.

$$\mathcal{A}(D(t)) - \mathcal{A}(D(0)) = tU^{(M)} \tag{3.4.9}$$

where

$$U^{(M)}_j = \sum_{k,i} \mathrm{res}_{P_{k,i}} \left(\left[M(\lambda)\Psi(t, P) \right]_1 \omega_j \right)$$

ω_j being the basis of holomorphic differentials. The equation (3.4.9) in conjunction with the explicit expression of eigenvectors (3.3.5) shows that the time evolution of the eigenvectors conincides with $\dot{\Psi} = M(\lambda)\Psi$.

3.5 $\mathfrak{su}(2)$ Gaudin models

In this section, we provide a brief survey of rational $\mathfrak{su}(2)$ Gaudin models and how the Lagrange top [see example 4, (2.1.10)] can be obtained from a two-body $\mathfrak{su}(2)$ rational Gaudin model.

First, we set the following basis for $\mathfrak{su}(2)$:

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

and recall the correspondence between $\mathbb{R}^3$ and $\mathfrak{su}(2)$

$$\mathbf{v} = (v^1, v^2, v^3) \longleftrightarrow \sigma = \frac{1}{2}\begin{pmatrix} -iv^3 & -iv^1 - v^2 \\ -iv^1 + v^2 & iv^3 \end{pmatrix}$$

which gives an isomorphism between $(\mathfrak{su}(2), [\cdot,\cdot])$ and the Lie algebra $(\mathbb{R}^3, \times)$ with $\times$ the vector product. This allows to identify the matrices of $\mathfrak{su}(2)$ with vectors in $\mathbb{R}^3$ and it also induces a scalar product on $\mathfrak{su}(2)$ from $\mathbb{R}^3$. Via this scalar product, the elements of dual space $\mathfrak{su}^*(2)$ may be identified with $\mathfrak{su}(2)$.

The Lie algebra of the N-body $\mathfrak{su}(2)$ Gaudin model is given by $\oplus^N\mathfrak{su}^*(2)$, the direct sum of N copies of $\mathfrak{su}^*(2)$, denoted by $\{v_i^\alpha\}_{\alpha=1}^3$, $1 \le i \le N$, the set of coordinate functions related to the i-th copy of $\mathfrak{su}(2)$. The Lie-Poisson bracket on $\oplus^N\mathfrak{su}^*(2)$ is

$$\{v_i^\alpha, v_j^\beta\} = -\delta_{ij}\,\epsilon_{\alpha\beta\gamma}\,v_i^\gamma \qquad 1 \le i, j \le N \tag{3.5.1}$$

where $\epsilon_{\alpha\beta\gamma}$ is the skew-symmetric tensor with $\epsilon_{123} = 1$. The dynamics of rational $\mathfrak{su}(2)$ Gaudin model is governed by the Lax matrix

$$L_G(\lambda) = \sigma_\alpha p^\alpha + \sum_{i=1}^N \frac{\sigma_\alpha v_i^\alpha}{\lambda - \lambda_i} = \mathbf{p} + \sum_{i=1}^N \frac{\mathbf{v}_i}{\lambda - \lambda_i} \tag{3.5.2}$$

where the distinct λ_i are complex parameters of the models. Here the constant vector

$\mathbf{p} \in \mathbb{R}^3$ needs to be added in order to get a sufficient number of functionally independent integrals of motion. To recall, an integral of motion is a function of the coordinates of the phase space which is constant along a certain trajectory. In other words, quantities conserved throughout the motion. The complete set of integrals of rational $\mathfrak{su}(2)$ model can then be constructed by computing the residues of Lax matrix at $\lambda = \lambda_i$ on the spectral curve $\det(L_G(\lambda) - \mu\mathbf{1}) = 0$ which then leads to the following $2N$ independent functions in involution. ([16], prop. 2):

$$H_i = \langle \mathbf{p}, \mathbf{v}_i \rangle + \sum_{i \neq j} \frac{\langle \mathbf{v}_i, \mathbf{v}_j \rangle}{\lambda_i - \lambda_j}, \qquad C_i = \frac{1}{2}\langle \mathbf{v}_i, \mathbf{v}_j \rangle, \qquad i = 1, ..., N \tag{3.5.3}$$

where H_i are the integrals of motion and C_i are Casimir functions i.e. functions that commute with any othe function with respect to the Poisson bracket. Essentially, H_i are Gaudin Hamiltonians of the system, as described by (2.5.5). The Hamiltonian of rational $\mathfrak{su}(2)$ Gaudin model is given by

$$H_G = \sum_{i=1}^{N} \eta_i H_i, \qquad \eta_i \neq \eta_j \in \mathbb{C} \tag{3.5.4}$$

An interesting specialization of this Hamiltonian is obtained by setting $\eta_i = \lambda_i$. In that case, it yields

$$H_G = \frac{1}{2}\sum_{i \neq j} \langle \mathbf{v}_i, \mathbf{v}_j \rangle + \sum_i \lambda_i \langle \mathbf{p}, \mathbf{v}_i \rangle \tag{3.5.5}$$

We may now express the equations of motion with respect to this Hamiltonian. ([16], prop. 5)

$$\dot{\mathbf{v}}_i = \Big[\lambda_i \mathbf{p} + \sum_{j=1}^{N} \mathbf{v}_j \ , \ \mathbf{v}_i \Big] \tag{3.5.6}$$

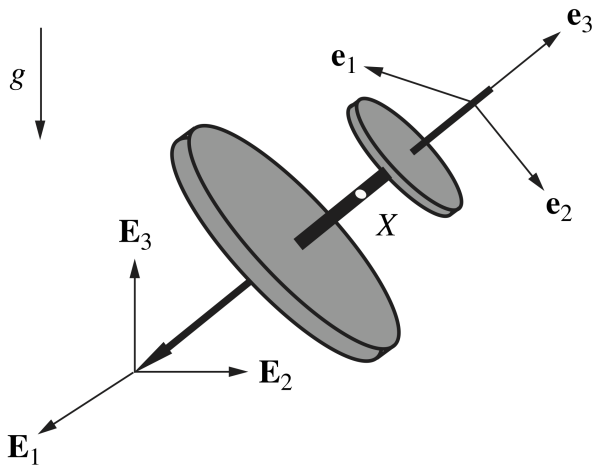
The equivalent Lax representation is

$$\dot{L}_G(\lambda) = [L_G(\lambda), M_G(\lambda)] \tag{3.5.7}$$

where

$$M_G(\lambda) = \sum_i \frac{\lambda_i \mathbf{v}_i}{\lambda - \lambda_i} \tag{3.5.8}$$

Lagrange top serves as a handy example of $\mathfrak{su}(2)$ Gaudin systems in two-body case; A 3-dimensional rigid body motion around a fixed point in a homogeneous field (gravity in this case) which is characterized by the inertia tensor $\mathrm{diag}(1, 1, \alpha)$, $\alpha \in \mathbb{R}$, meaning that the body is rotationally symmetric with respect to the third coordinate axis, and the fixed point lying on the symmetry axis as demonstrated below.



The standard form of equations of motion is given by the Euler-Poisson equations

$$\dot{J} = P \times X, \qquad \dot{X} = J \times X \tag{3.5.9}$$

where $J = (J_1, J_2, J_3)$ is the vector of angular momentum, $P = (0, 0, \alpha)$ is the weight vector along the gravity field and $X = (x_1, x_2, x_3)$ the vector pointing from the fixed point to the center of mass. This defines an integrable system with 2 degrees of freedom and the Hamiltonian is

$$H = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) + \alpha x_3 \tag{3.5.10}$$

The elements J_k, x_k are the generators of Lie-Poisson Euclidean algebra $\mathfrak{e}(3)$ defined by the Poisson brackets

$$\{J_k, J_l\} = J_m, \qquad \{J_k, x_l\} = x_m, \qquad \{x_k, x_l\} = 0 \tag{3.5.11}$$

where (klm) is the cyclic permutation of (123). This Lie-Poisson bracket has two Casimir

functions

$$C_1 = \sum_k x_k^2, \qquad C_2 = \sum_k x_k J_k \tag{3.5.12}$$

These Casimir functions are constant on symplectic leaves, which are four-dimensional symplectic manifolds in this case. The two commuting integrals of motion, i.e. Gaudin Hamiltonians, of the system are the Hamiltonian itself and J_3 since J_3 is a conserved quantity due to invariance of angular momentum under rotation about the direction of gravity field.

The Lagrange top can now be constructed from $\mathfrak{su}(2)$ Gaudin magnet model, originally introduced in [10] . This model has the Lax matrix

$$L_G(\lambda) = \sum_{j=1}^{m} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.5.13}$$

where $\lambda_j \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ are the parameters of the model and λ the spectral parameter as before. α indicates the intensity of magnetic field. Local variables $\{s_j^3, s_j^{\pm}\}_{j=1}^m$ are the generators of direct sum of m $\mathfrak{su}(2)$ spins.

As discussed in section 2.6, the nonlinear dynamics of the system defined by the equations (3.5.9) becomes linear on the Jacobian of the spectral curve

$$\Gamma_G : \qquad \det(L_G(\lambda) - \mu \mathbf{1}) = 0 \tag{3.5.14}$$

which by the formula (3.2.3), has the genus $g = m - 1$. The spectral curve Γ_G can be alternatively written as

$$v^2 = \alpha^2 + \sum_{j=1}^{m} \Big(\frac{H_j}{\lambda - \lambda_j} + \frac{s_j^2}{(\lambda - \lambda_j)^2} \Big) \tag{3.5.15}$$

Following (3.5.3), the commuting Gaudin Hamiltonians are

$$H_j = \sum_{k \neq j} \frac{2 s_j^3 s_j^k + s_j^+ s_j^- + s_j^- s_j^+}{\lambda_j - \lambda_k} + 2\alpha s_j^3 \tag{3.5.16}$$

Now considering the two-body case of this model where $m = 2$, the phase space is

$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Assume s_j^1 and s_j^2 are as $s_j^{\pm} = s_j^1 \pm s_j^2$. The Lax matrix is given by

$$L(\lambda) = \frac{L_1}{\lambda - \lambda_1} + \frac{L_2}{\lambda - \lambda_2} + \alpha \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.5.17}$$

where

$$L_k = \begin{pmatrix} is_k^3 & s_k^2 + is_k^1 \\ -s_k^2 + is_k^1 & -is_k^3 \end{pmatrix} \in \mathfrak{su}(2), \qquad k = 1, 2$$

By Inönü-Wigner contraction (see [16], section 3) from the rotation group $O(n+1)$ to the Euclidean group $E(n)$, the Lie-Poission algebra $\mathfrak{e}(3)$ for the Lagrange top can be obtained. The generators of this algebra $J_{\pm} = J_1 \pm iJ_2$, $x_{\pm} = x_1 \pm ix_2$, x_3 and J_3. The algebra $\mathfrak{e}(3)$ is generated by the natural extension of Poisson bracket to these generators. We skip the details of the contraction, but the end result is that the Lax matrix of Lagrange top becomes

$$L(\lambda) = \frac{1}{\lambda} \begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} x_3 & x_- \\ x_+ & -x_3 \end{pmatrix} + \alpha \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.5.18}$$

Here, α describes the intensity of gravitational field. The expanded form of spectral curve $\det(L(\lambda) - \mu \mathbf{1}) = 0$ is

$$\Gamma : \quad -\mu^2 = \frac{1}{\lambda^4} + \frac{2l}{\lambda^3} + \frac{2H}{\lambda^2} + \frac{2J_3}{\lambda} + \alpha^2 \tag{3.5.19}$$

Chapter 4

Symplectic structure of the system

Having studied the analytical aspects of Gaudin system and reconstructed the Lax matrix from analytical data, the next step is to study the symplectic properties of the system, specifically the symplectic structure of the phase space which is a crucial part of Hamiltonian formalism of integrable systems and the associated Poisson structure. As briefly discussed in section 2.3, the symplectic structure and properties are attributed to the space on which the motion and its orbits take place, that is, roughly speaking, the phase space of the system. Our aim in this section to define and understand the symplectic structure of this even-dimensional space and then express the canonical symplectic forms in terms of a suitable coordinate system, the one obtain by the dynamical data encoded in the dynamical divisor and also the action-angle variables which will follow in the next chapter.

4.1 The phase space of Gaudin system

In other to characterize the phase space of the system, we are going to diagonalize the coefficients in the rational Lax matrix of the form (3.1.2). Each A_j can be written as:

$$A_j = G_j L_j G_j^{-1} \tag{4.1.1}$$

where L_j is a diagonal matrix and $G_j, L_j \in \mathfrak{sl}(N, \mathbb{C})$. L_j is also chosen to be non-dynamical, that is it does not contain the dynamical data of the system.

Recall that the equation of the spectral curve $\det(L(\lambda) - \mu\mathbf{1}) = 0$, we expand it in its

polynomial form:

$$\Gamma(\lambda,\mu)=(-\mu)^N+\sum_{j=0}^{N-1}r_j(\lambda)\mu^j=0 \tag{4.1.2}$$

We have already provided a rigorous construction of the phase space in (2.3.6), denoted by $\mathcal{M}$. Here, we are going to define it directly for the case of Gaudin system (3.1.6). For the sake of simplicity and without the loss of precision, here we consider the phase space as the set of degrees of freedoms of system, which can be translated into all degrees of freedom in $A_j,\ j=1,...,m$ matrices, subject to $\sum A_j=0$ for regularity at ∞. Given the decomposition (4.1.1), A_j can be alternatively characterized by the set of all m pairs $(G_j,L_j),\ j=1,...,m$, modulo the right multiplication of G_j by diagonal matrices in addition to the action of gauge group, that is the action under which the Lax equation is invariant. Here, the said action is the left multiplication of all G_j by an element of $S\in\mathfrak{sl}(N,\mathbb{C})$. Under this action, the Lax pair transforms as:

$$L\to SLS^{-1},\qquad M\to SMS^{-1}+\partial_t SS^{-1}$$

and it is easy to verify the invariance of Lax equation under this action. To sum up, the phase space is given by

$$\mathcal{M}=\Big\{(G_j,L_j)_{j=1}^m,\ \sum_{j=1}^m G_jL_jG_j^{-1}=0\Big\}/\sim \tag{4.1.3}$$

where $\sim$ is the equivalence by the action of simultaneous left multiplication $G_j\to SG_j$, $S\in\mathfrak{sl}(N,\mathbb{C})$ and choice of G_j is up to right multiplication by a diagonal matrix. We now count the number of degrees of freedom in all (G_j,L_j) and then subtract those eliminated by the quotients. which leads to the following theorem concerning the dimension of the phase space:

Theorem 4. ([1], p.127) *The dimension of $\mathcal{M}$ is $2g$ and there are g independent action variables in the equation of the spectral curve* (4.1.2).

Proof. To count all the parameters, we start with the dynamical variables G_j which are $N\times N$ matrices so we have N^2 parameters for each. But notice that A_j is invariant under $G_j\to G_jD_j$ for a diagonal matrix D_j so the dimension of the orbit of A_j under this action is N^2-N. We have m of these orbits so the number of degrees of freedom so far is $(N^2-N)m$.

We also have the invariance under the right action of diagonal matrices $G_j \to G_j D$. The orbit of this action has the dimension $N-1$ (excluding the identity element as it does not act). The phase space $\mathcal{M}$ can now be obtained by the Hamiltonian reduction by this action. We need to reduce the total number of degrees of freedom of system by the number of constrained imposed by the action. This is done by first fixing a moment map, setting $A = GLG^{-1}$ as the moment map which yields $N-1$ constraints then we take the quotient by the specified action which has orbits of dimension $N-1$ so the total number of degrees of freedom $(N^2-N)m$ should be reduced by $2(N-1)$. Thus by using theorem (1) for the genus of the spectral curve we get:

$$\dim \mathcal{M} = (N^2-N)m - 2(N-1) = 2g$$

Now, to count the number of independent coefficients in (4.1.2), notice that $r_j(\lambda)$ is a rational function of λ and we also know its value at ∞ since $\mu_j \to a_j$ (Recall (**??**)). r_j is also a symmetrical function of $\mu_1, ..., \mu_N$ where μ_i are the eigenvalues of the Lax matrix $L(\lambda)$. Above $\lambda = \lambda_k$, they can be written as

$$\mu_j = \frac{c_j}{\lambda - \lambda_k} + \text{regular}$$

where c_j are fixed and non-dynamical since they are the elements of the diagonal matrix L_k which is non-dynamical, whereas the regular part of μ_j is dynamical. From (4.1.2), We can see that $r_j(\lambda)$ has a pole of order j at $\lambda = \lambda_k$ so it can be expressed using $j \times m$ parameters, which are actually the coefficients of all these poles. Summing over j we get $\frac{1}{2}N(N+1)m$ parameters overall. However, not all these parameters are independent because the coefficients c_j are non-dynamical. This implies that the highest order term in $r_j(\lambda)$ is fixed therefore it yields N constraints on the coefficients of $r_j(\lambda)$. We are now left with $\frac{1}{2}N(N-1)m = g + N - 1$ parameters.

It remains to take the symplectic quotient by the action of diagonal matrices. Consider the Hamiltonians

$$H_n = \frac{1}{n} \operatorname{res}_{\lambda=\infty} \operatorname{tr}(L^n(\lambda)) d\lambda \,, \qquad n = 1, ..., N-1 \tag{4.1.4}$$

i.e. the $\frac{1}{\lambda}$ term in $\operatorname{tr}(L^n(\lambda))$. These are functions of $r_j(\lambda)$ and they are the generators of the diagonal matrices action. Since they are the moment map for the diagonal action, they

would be fixed on the orbits so after taking the quotient, the number of parameters reduces by $N-1$ so we are left with g independent action variables. □

4.2 Symplectic structure of the phase space

Following theorem (4), Lax description of a dynamical system provides a natural coordinates on phase space with g independent action variables F_i which parameterize the spectral curve Γ and g points $\gamma_i = (\lambda_{\gamma_i}, \mu_{\gamma_i})$ which we called the *dynamical divisor* or *dynamical data*. We are essentially looking for expressing the symplectic form in these coordinates. The phase space of the system appears as a fibred space whose base is the space of moduli of the spectral curve, i.e. the set of parameters determining the deformations of spectral curve as a Riemann surface, which are explicitly described as the coefficients of the spectral curve equation $\Gamma(\lambda,\mu) = \det(L(\lambda) - \mu\mathbf{1})$ and the fibre at any given Γ is the Jacobian of Γ, which is a g-dimensional space over $\mathbb{C}$.

Before proceeding, we need to introduce a differential $\tilde{\mathrm{d}}$ on the phase space $\mathcal{M}$ which varies the variables of this space, action variables F_i and dynamical divisors $\lambda_{\gamma_i}, \mu_{\gamma_i}$, constrained to the spectral curve Γ. $\tilde{\mathrm{d}}$ differentiates any function of F_i, λ, μ by keeping λ constant. Consider a bundle consisting of the family of spectral curves Γ where the coefficients of Γ (in its polynomial form) depend on the moduli parameters, parameterizing the base space. By differentiating on this bundle of curves while keeping λ constant we obtain a horizontal differentiation, i.e. a connection on the bundle. Explicitly, for a function on this bundle, or in other words, a function f depending on F_i, λ, μ we take λ as a local parameter and define $\tilde{\mathrm{d}}f = \sum_i \partial_{F_i} f \tilde{\mathrm{d}} F_i$. At a branch point, we take μ as the local parameter and:

$$\tilde{\mathrm{d}}f = \partial_\mu f \tilde{\mathrm{d}}\mu + \sum_i \partial_{F_i} f \tilde{\mathrm{d}} F_i \tag{4.2.1}$$

where μ is obtained by differentiating $\Gamma(\lambda,\mu) = 0$ while keeping λ constant:

$$\tilde{\mathrm{d}}\mu = -\frac{1}{\partial_\mu \Gamma(\lambda,\mu)} \sum_i \partial_{F_i}\Gamma(\lambda,\mu)\tilde{\mathrm{d}}F_i \tag{4.2.2}$$

At a branch point of the covering map $P = (\lambda,\mu) \to \lambda$, we have $\partial_\mu\Gamma(\lambda,\mu) = 0$ so the differential $\tilde{\mathrm{d}}$ has a pole even if f is regular. If however, $f(P)$ only depends on $\lambda(P)$, then $\tilde{\mathrm{d}}f$ would be regular at the branch points.

Now, define the 1-form $\Theta_{\mathcal{M}}$, also known as *symplectic potential*, on $\mathcal{M}$ as

$$\Theta_{\mathcal{M}} = \sum_{j=1}^{m} \operatorname{tr} L_j G_j^{-1} \tilde{\mathrm{d}} G_j \tag{4.2.3}$$

The differential of the inverse matrix is given by $dM^{-1} = -M^{-1} dM M^{-1}$. Hence

$$\begin{aligned} d(L_j G_j dG_j^{-1}) &= d(L_j G_j^{-1}) \wedge dG_j = (dL_j G_j^{-1} + L_j dG_j^{-1}) \wedge dG_j \\ &= (dL_j G_j^{-1} - L_j G_j^{-1} dG_j G_j^{-1}) \wedge dG_j \end{aligned}$$

Taking the trace and summing over j, we obtain the 2-form $\omega_{\mathcal{M}}$

$$\begin{aligned} \omega_{\mathcal{M}} = \tilde{\mathrm{d}}\Theta_{\mathcal{M}} &= \sum_{j=1}^{m} \left(\operatorname{tr}(\tilde{\mathrm{d}} L_j \wedge G_j^{-1} \tilde{\mathrm{d}} G_j) - \operatorname{tr}(L_j G_j^{-1} \tilde{\mathrm{d}} G_j \wedge G_j^{-1} \tilde{\mathrm{d}} G_j) \right) \\ &= -\sum_{j=1}^{m} \operatorname{tr}(L_j G_j^{-1} \tilde{\mathrm{d}} G_j \wedge G_j^{-1} \tilde{\mathrm{d}} G_j) \end{aligned} \tag{4.2.4}$$

since L_j are non-dynamical, their differentiation with respect to dynamical variables vanishes. The goal is to show that $\omega_{\mathcal{M}}$ is a symplectic form on $\mathcal{M}$ and express it in terms of Darboux coordinates on $\mathcal{M}$.

Recall that we have an eigenvector $\Psi(\lambda)$ of the Lax matrix $L(\lambda)$ at each point $P(\lambda, \mu)$, up to normalization. Assume $\Psi^{(-1)}(P)$ is a dual eigenvector such that $\langle \Psi^{(-1)}(P), \Psi(P) \rangle = 1$ ($\langle \, , \rangle$ is the standard inner product).

We now define the 1-form ω on Γ which can be regarded as a 2-form on phase space $\mathcal{M}$.

$$\omega = \omega_1 + \omega_2 \tag{4.2.5}$$

where

$$\begin{aligned} \omega_1 &= \langle \Psi^{(-1)}(P), \tilde{\mathrm{d}} L(\lambda) \wedge \tilde{\mathrm{d}} \Psi(P) \rangle d\lambda \\ \omega_2 &= \langle \Psi^{(-1)}(P), \tilde{\mathrm{d}} \mu \wedge \tilde{\mathrm{d}} \Psi(P) \rangle d\lambda \end{aligned}$$

Knowing the dynamical divisor, $\Psi(P)$ is defined up to multiplication by a diagonal matrix independent of P. The eigenvectors at ∞ are normalized by

$$\psi_i(\infty_j) = \lambda \delta_{ij} + O(1), \qquad i, j = 2, ..., N \tag{4.2.6}$$

Define the following 2-form on $\mathcal{M}$:

$$\tilde{\omega} = \frac{1}{2}\sum_{k,i} \mathrm{res}_{P_{k,i}}(\omega) \tag{4.2.7}$$

where $P_{k,i}$ are the points above the poles λ_k of $L(\lambda)$. We then have the following theorem:

Theorem 5. ([1], eq. 5.61) *The form $\tilde{\omega}$ can be expressed in terms of the coordinates of the points of the dynamical divisor $(\lambda_{\gamma_i}, \mu_{\gamma_i})$ as:*

$$\tilde{\omega} = \sum_{i=1}^{g} \tilde{\mathrm{d}}\lambda_{\gamma_i} \wedge \tilde{\mathrm{d}}\mu_{\gamma_i} \tag{4.2.8}$$

Proof. The sum of residues of ω, as a form on Γ, obviously vanishes. We have four different types of the poles of ω. First are the dynamical poles of Ψ, then the poles at $P_{k,i}$ which come form Lax matrix L and μ, next the poles above $\lambda = \infty$ coming from Ψ and $d\lambda$ and finally the poles at the branch points of the covering coming from the poles of $\Psi^{(-1)}$ and $\tilde{\mathrm{d}}\mu$ in (4.2.2). We first compute the residues at the dynamical poles (λ_1, λ_g). The coordinates of these points are $\gamma_i = (\lambda_{\gamma_i}, \mu_{\gamma_i})$ for $i = 1, .., g$. Near such points, we can choose λ as a local parameter and we have $\Psi = \frac{1}{\lambda - \lambda_{\gamma_i}} \times$ (regular part), hence:

$$\tilde{\mathrm{d}}\Psi = \frac{\tilde{\mathrm{d}}\lambda_{\gamma_i}}{\lambda - \lambda_{\gamma_i}}(\Psi + O(1)) \;\Rightarrow\; \omega_1 \sim \langle \Psi^{(-1)}, \tilde{\mathrm{d}}A\Psi\rangle \wedge \tilde{\mathrm{d}}\lambda_{\gamma_i} \frac{d\lambda}{\lambda - \lambda_{\gamma_i}} \tag{4.2.9}$$

Since $(A-\mu)\Psi = 0$ and $\Psi^{(-1)}(A-\mu) = 0$, we have $(\tilde{\mathrm{d}}A - \tilde{\mathrm{d}}\mu)\Psi + (L-\mu)d\Psi = 0$. Multiplying this by $\Psi^{(-1)}$ we get $\langle \Psi^{(-1)}, \tilde{\mathrm{d}}A\Psi\rangle = \tilde{\mathrm{d}}\mu$, therefore:

$$\mathrm{res}_{\gamma_i}\, \omega_1 = \tilde{\mathrm{d}}\mu|_{\gamma_i} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_i} \tag{4.2.10}$$

Here $\tilde{\mathrm{d}}\mu$ is regarded as a meromorphic function on Γ. At γ_i, we have:

$$\tilde{\mathrm{d}}\mu|_{\gamma_i} = \tilde{\mathrm{d}}\mu_{\gamma_i} + \frac{\partial_\lambda \Gamma}{\partial_\mu \Gamma}\Big|_{\gamma_i} \tilde{\mathrm{d}}\lambda_{\gamma_i} \tag{4.2.11}$$

and the second term does not contribute to the wedge product in (4.2.10). The residues coming from ω_2 can be computed similarly, so we get:

$$\mathrm{res}_{\gamma_i}\, \omega = 2\tilde{\mathrm{d}}\mu_{\gamma_i} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_i} \tag{4.2.12}$$

We now show that there are no residues at the branch points. Looking at ω_1, At a branch point b, $\Psi^{(-1)}$ has a simple pole, $\tilde{\mathrm{d}}A$ is regular, $d\Psi$ has a simple pole due to (4.2.2) and the form $d\lambda$ has a simple zero. Thus, ω_1 has a simple pole at b. To compute its residue, it is enough to just keep the polar part of $\tilde{\mathrm{d}}\Psi$, i.e. to replace $\tilde{\mathrm{d}}\Psi$ by $\partial_\mu \Psi \tilde{\mathrm{d}}\mu$ (notice that μ is a local parameter around b). We then get:

$$\mathrm{res}_b\, \omega_1 = \mathrm{res}_b\, \langle \Psi^{(-1)}, \tilde{\mathrm{d}}A \partial_\mu \Psi \rangle \wedge \tilde{\mathrm{d}}\mu\, d\lambda = \mathrm{res}_b\, \langle \Psi^{(-1)}, (\tilde{\mathrm{d}}A - \tilde{\mathrm{d}}\mu)\partial_\mu \Psi \rangle \wedge \tilde{\mathrm{d}}\mu\, d\lambda \tag{4.2.13}$$

where the antisymmetry property of the wedge product is used in the last part of the equation to replace $\tilde{\mathrm{d}}A$ by $\tilde{\mathrm{d}}A - \tilde{\mathrm{d}}\mu$. By the eigenvector equation $(A-\mu)\Psi = 0$ and varying the point (λ, μ) on the spectral curve around b, we yield to

$$(A-\mu)\partial_\mu \Psi = \Psi - \frac{d\lambda}{d\mu}\frac{dA}{d\lambda}\Psi \tag{4.2.14}$$

where $\frac{d\lambda}{d\mu}$ vanishes at the branch points. We then differentiate and multiply on the left by $\Psi^{(-1)}$ and obtain:

$$\begin{aligned}\mathrm{res}_b\, \langle \Psi^{(-1)}, (\tilde{\mathrm{d}}A - \tilde{\mathrm{d}}\mu)\partial_\mu \Psi \rangle \wedge \tilde{\mathrm{d}}\mu\, d\lambda &= \mathrm{res}_b\, \langle \Psi^{(-1)}, \tilde{\mathrm{d}}\Psi \rangle \wedge \tilde{\mathrm{d}}\mu\, d\lambda \\ &\quad - \mathrm{res}_b\, \langle \Psi^{(-1)}, \tilde{\mathrm{d}}(\frac{d\lambda}{d\mu}\frac{dA}{d\lambda}\Psi) \rangle \wedge \tilde{\mathrm{d}}\mu\, d\lambda \end{aligned} \tag{4.2.15}$$

We can see that the first term on the right hand side is cancelled by the term $\mathrm{res}_b\, \omega_2$. The second term is equal to

$$\mathrm{res}_b \frac{\tilde{\mathrm{d}}\mu_b}{\mu - \mu_b} \wedge \tilde{\mathrm{d}}\mu\, d\lambda \tag{4.2.16}$$

To show this, note that $\zeta = \frac{d\lambda}{d\mu}\frac{dA}{d\lambda}\Psi$ vanishes at $b = (\lambda_b, \mu_b)$. Writing $\zeta = (\mu - \mu_b)\zeta_1$, we get $\tilde{\mathrm{d}}\zeta = -\frac{\tilde{\mathrm{d}}\mu_b}{\mu - \mu_b}\zeta + \tilde{\mathrm{d}}\mu\zeta_1 + \zeta_2$ with ζ_2 being regular. The ζ_1 term does not contribute to ω due to the antisymmetry of the wedge product and the ζ_2 term has no resdiue. Using (4.2.14), we get $\langle \Psi^{(-1)}, \frac{d\lambda}{d\mu}\frac{dA}{d\lambda}\Psi \rangle = 1$ yielding (4.2.16). This contribution is cancelled by the new form ω_3:

$$\omega_3 = \tilde{\mathrm{d}}(\log \partial_\mu \Gamma) \wedge \tilde{\mathrm{d}}\mu\, d\lambda \tag{4.2.17}$$

We will see that ω_3 has poles only at the branch points. At the branch point b, $\partial_\mu \Gamma$ has a

zero, so we write $\partial_\mu\Gamma = (\mu - \mu_b)C$ with C regular. The contribution of the point b to ω_3 is

$$\operatorname{res}_b \frac{\tilde{\mathrm{d}}\partial_\mu\Gamma}{\partial_\mu\Gamma} \wedge \tilde{\mathrm{d}}\mu \, d\lambda \tag{4.2.18}$$

The variation of $\partial_\mu\Gamma$ is

$$\tilde{\mathrm{d}}\partial_\mu\Gamma = \tilde{\mathrm{d}}(\mu - \mu_b)C + (\mu - \mu_b)\tilde{\mathrm{d}}C \tag{4.2.19}$$

The second term does not contribute to the residue since C is regular. The variation $\tilde{\mathrm{d}}\mu$ cancels again due to the antisymmetry of the wedge product and we are left with the contribution of $d\mu_b$ which cancels (4.2.16).

We now compute the third type of residues, the ones above $\lambda = \infty$. The normalization of the eigenvectors at ∞ is given by (4.2.6). Notice that $A = A_0 + O(\frac{1}{\lambda})$ and $\mu = \mu_i^0 + O(\frac{1}{\lambda})$ around ∞_i so $\tilde{\mathrm{d}}A$ and $\tilde{\mathrm{d}}\mu$ are $O(\frac{1}{\lambda})$. Furthermore, $\Psi^{(-1)}$ vanishes at ∞_i and $d\lambda$ has a double pole. Also ω_1 and ω_2 are regular at ∞_i since $\tilde{\mathrm{d}}\Psi(\infty_i) = O(1)$ due to the normalization condition. Finally, ω_3 is regular too because on the sheet $\mu = \mu_i(\lambda)$, we can write $\partial_\mu\Gamma = \prod_{j\neq i}(\mu_i - \mu_j)$ yielding $\tilde{\mathrm{d}}\log\partial_\mu\Gamma = O(\frac{1}{\lambda})$ hence $\tilde{\mathrm{d}}\log\partial_\mu\Gamma \wedge \tilde{\mathrm{d}}\mu = O(\frac{1}{\lambda^2})$ has a double zero which cancels out the pole of $d\lambda$ at ∞. All these show that ω has no residues at $\lambda = \infty$.

At the end, it remains to show that ω_3 has no other poles. ω_3 is clearly regular at the points of the dynamical divisor and does not contribute to the residues at these points. To compute the residue of ω at the points $P_{k,i}$ above λ_k, note that if $\partial_\mu\Gamma$ has a pole of order m at λ_k, then it can be written as $\partial_\mu\Gamma = \frac{c(\lambda)}{(\lambda-\lambda_k)^m}$ where $c(\lambda)$ is regular and non-vanishing. Since $\tilde{\mathrm{d}}\lambda = 0$ and $\tilde{\mathrm{d}}\lambda_k = 0$ we get $\tilde{\mathrm{d}}(\log\partial_\mu\Gamma) = \tilde{\mathrm{d}}\log c(\lambda)$ which is regular. At λ_k, note that $\tilde{\mathrm{d}}\mu$ is regular on all sheets above λ_k. This is because due to the form of $L(\lambda)$, we have $\hat{\mu} = L_k + (\text{regular})$. Since L_k describes the coadjoint orbit and is non-dynamical, we have $\tilde{\mathrm{d}}A_k = 0$ so ω_3 has no residue. □

We will now show that $\omega_{\mathcal{M}}$ and $-\tilde{\omega}$ coincide.

Theorem 6. ([1], eq. 5.66) *The form $\omega_{\mathcal{M}}$ is given by*

$$\omega_{\mathcal{M}} = -\sum_k \operatorname{tr}\left(L_k G_k^{-1}\tilde{\mathrm{d}}G_k \wedge G_k^{-1}\tilde{\mathrm{d}}G_k\right) = -\sum_{i=1}^{g} \tilde{\mathrm{d}}\lambda_{\gamma_i} \wedge \tilde{\mathrm{d}}\mu_{\gamma_i} = -\tilde{\omega} \tag{4.2.20}$$

This shows that $\omega_{\mathcal{M}}$ is the symplectic form on the orbit.

Proof. We first compute the residues of ω_1 at λ_k where only A_k contributes to the residue.

Using the diagonalization in (4.1.1), we can write the Lax matrix as $A = G_k L_k G_k^{-1}/(\lambda - \lambda_k)$ around $\lambda = \lambda_k$. So locally around λ_k we can identify the matrix $\hat{\Psi}(\lambda) = G_k$. Now consider the matrix of eigenvectors

$$\hat{\Psi}(\lambda) = \left[\psi_k(P_j)\right]_{jk} \tag{4.2.21}$$

where ψ_k is the k-th component of the vector $\Psi(\lambda)$ and $P_j = (\lambda, \mu_j)$ are the N points above λ. Also denote the diagonal matrix $\mathrm{diag}(\mu_1, ..., \mu_N)$ by $\hat{\mu}$. Hence

$$L(\lambda) = \hat{\Psi}(\lambda)\hat{\mu}\hat{\Psi}^{-1}(\lambda) \tag{4.2.22}$$

By (4.2.22), locally around λ_k, $\hat{\Psi}(\lambda)$ can be identified with G_k. More precisely, we have $\hat{\Psi}(\lambda) = G_k D_k$ and $\hat{\Psi}^{(-1)}(\lambda) = D_k^{-1} G_k^{-1}$ for a diagonal matrix D_k. The residues are obtained by integrating ω_1 along small circles $C_{k,i}$ around each of the N points $P_{k,i}$ above λ_k. We can choose these small circles so that they project on the base λ on a single small circle C_k around λ_k. We then get

$$\begin{aligned}\sum_{i=1}^{N} \mathrm{res}_{P_{i,k}}\, \omega_1 &= \sum_{i=1}^{N} \frac{1}{2\pi i} \oint_{C_{k,i}} \Psi^{(-1)}(P_i)\tilde{\mathrm{d}}L(\lambda) \wedge \tilde{\mathrm{d}}\Psi(P_i) \\ &= \frac{1}{2\pi i} \oint_{C_k} \mathrm{tr}\left(\hat{\Psi}^{(-1)}(\lambda)\tilde{\mathrm{d}}L(\lambda) \wedge \tilde{\mathrm{d}}\hat{\Psi}(\lambda)\right)\end{aligned} \tag{4.2.23}$$

using the fact that $\hat{\Psi}^{(-1)}(\lambda)$ is the matrix whose rows are the vectors $\Psi^{(-1)}(P_i)$. Since $\Psi(P_i), i = 1, ..., N$ forms a basis of eigenvectors, the integrand in (4.2.23) constructs the trace. Now using the identification of $\hat{\Psi}(\lambda)$ in terms of G_k gives

$$\begin{aligned}\mathrm{res}_{\lambda_k}\, \omega_1 &= \operatorname*{res}_{\lambda_k}\Big(\mathrm{tr}\left(D_k^{-1}G_k^{-1}(\tilde{\mathrm{d}}G_k L_k G_k^{-1} - G_k L_k G_k^{-1}\tilde{\mathrm{d}}G_k G_k^{-1}) \wedge (\tilde{\mathrm{d}}G_k D_k + G_k \tilde{\mathrm{d}}D_k)\right)/(\lambda - \lambda_k)\Big) d\lambda \\ &= -2\,\mathrm{tr}\left(L_k G_k^{-1}\tilde{\mathrm{d}}G_k \wedge G_k^{-1}\tilde{\mathrm{d}}G_k + G_k^{-1}\tilde{\mathrm{d}}G_k[L_k, \tilde{\mathrm{d}}D_k D_k^{-1}]\right)\end{aligned} \tag{4.2.24}$$

The last term vanishes because it is the commutator of two diagonal matrices. Finally, there is no contribution from since ω_2 is regular at λ_k because $\tilde{\mathrm{d}}\mu$ is regular on all the sheets above λ_k so it has no residue. Hence

$$\mathrm{res}_{\lambda_k}\, \omega_1 = -2\,\mathrm{tr}\left(L_k G_k^{-1}\tilde{\mathrm{d}}G_k \wedge G_k^{-1}\tilde{\mathrm{d}}G_k\right) \tag{4.2.25}$$

Therefore using (4.2.7) and theorem (5), the sum of (4.2.24) over all λ_k yields the

equation (4.2.20). □

Thus we have established that $\omega_{\mathcal{M}} = \sum_{j=1}^{g} \tilde{\mathrm{d}}\mu_{\gamma_j} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_j}$. This implies that the coordinates $(\lambda_{\gamma_i}, \mu_{\gamma_i})$ of the points of the dynamical divisor are indeed canonical coordinates on $\mathcal{M}$, the phase space of the system, and they satisfy the base identities of the non-degenerate Poisson bracket.

Chapter 5

Action-angle variables

We have already put a set of Darboux coordinates on the phase space of Gaudin system, $\mathcal{M}$. We are now going to introduce another set of coordinates on $\mathcal{M}$ using the characteristics q (3.3.9) of the theta function that appears in the solution of the Lax equation Ψ in (3.3.5). The components of the characteristics vector q minus the vector of Riemann's constants $\mathcal{K}$ happen to be what are known as *angle variables*. Angle variables along with their canonical conjugates known as *action variables* give a set of coordinates called *action-angle* which express a dynamical system in separable variables.

To understand the significance of action-angle variables, we need to go back to the very fundamental notions of integrable systems. Consider a dynamical Hamiltonian system with the phase space $\mathcal{M}$ of dimension $2n$ and a set of canonical coordinates p_j, q_j with the usual Poisson bracket relations. We then have a non-degenerate closed 2-form $\omega = \sum_j dp_j \wedge dq_j$ and let H be the Hamiltonian of the system.

Definition: The system is called *Liouville integrable* if it has n independent conserved quantities F_j such that $\{H, F_j\} = 0$ and they are in involution, i.e. $\{F_i, F_j\} = 0$.

The independence means that dF_j are linearly independent, or that the tangent space of the surface $F_j = f_j$ exists everywhere and has the dimension n. There cannot be more than n independent quantities in involution otherwise the Poisson bracket would be degenerate. The Hamiltonian H can then be written as a function of F_j. The crucial *Liouville theorem* states that:

The Liouville theorem: ([1], p.7) *The solution of the equations of motion of a Liouville integrable system is obtained by quadrature, i.e. the solution can be expressed in terms of integrals.*

Now let $\mathcal{M}_f$ be the level manifold described by $F_j(p,q) = f_j, j = 1, ..., n$. Under suitable compactness and connectivity conditions, $\mathcal{M}_f$ can be shown to be an n-dimensional torus T_n. This is where the angle variables become apparent as variables to describe the motion along the cycles, i.e. the tori. The torus T_n is isomorphic to a product of n circle C_j. Let θ_j be the angular variable parameterizing C_j and we call $\{\theta_j\}$ the set of *angle variables*.

The *action variables* I_j are defined as the integrals of the canonical 1-form $\alpha = \sum_j p_j dq_j$ over the cycles C_j:

$$I_j = \frac{1}{2\pi} \oint_{C_j} \alpha \tag{5.0.1}$$

I_j are functions of F_j which are conserved throughout the motion on the level manifold $\mathcal{M}_f$ and they are also independent so knowing the values of I_j determines $\mathcal{M}_f$.

To construct these coordinates in our setup, consider the homology basis $\{a_i, b_i\}_{i=1}^g, \{l_i^{(k)}\}$ on $\Gamma \setminus \{\lambda_j^{(k)}\}$, $j = 1, ..., m,\ k = 1, ..., N$, where a- and b- cycles are as before and $\{l_i^{(k)}\}$ are small loops around $\{\lambda_j^{(k)}\}$, the points above λ_j. Also set the basis of holomorphic differentials $\{\omega_k\}$ normalized by a-cycles: $\oint_{a_k} \omega_l = \delta_{kl}$. Now define the g angle variables by

$$\theta_k = \sum_{i=1}^{g} \int_{p_0}^{\gamma_i} \omega_k = \sum_{i=1}^{g} \int_{\lambda_0}^{\lambda_{\gamma_i}} \sigma_k(\lambda) d\lambda \tag{5.0.2}$$

where the integration paths are taken along the cycles a_k and the holomorphic differentials ω_k are written in terms of the local parameter λ as $\omega_k = \sigma_k(\lambda)d\lambda$ on integration paths. We notice that the expression in (5.0.2) is similar to the k-th component of Abel map of dynamical divisor $D = (\gamma_1, ..., \gamma_g)$ so the vector of angle variables $\vec{\theta} = (\theta_1, ..., \theta_g)$ can alternatively be expressed in terms of vector of Riemann's constants $\mathcal{K}$ and $\vec{q} = (q_1, ..., q_g)$, the characteristic of the theta functions appearing in the explicit solution of eigenvectors in (3.3.10) as:

$$\vec{\theta} = \vec{q} - \mathcal{K} = \mathcal{A}(D) \tag{5.0.3}$$

The components of q may be regarded as another set of angle variables. We are particularly interested in q angle variables due to the fact that some variational formulae for the Szegö kernel with respect to these variables are previously established. We will apply this framework in chapter 6 to directly calculate the symplectic potential generating the symplectic structure of the phase space in terms of action-angle variables. To define the

conjugate action variables, we first need a lemma.

Lemma 1: The differential form $\tilde{\mathrm{d}}(\mu d\lambda)$ is regular, i.e. it is of the form fdg, where as before $\tilde{\mathrm{d}}$ is the differential with respect to the moduli of spectral curve while keeping λ constant.

Proof: Taking the variation at constant λ produces poles at the branch points of the covering which are cancelled by corresponding zeros of $d\lambda$. The form $\mu d\lambda$ has poles at finite distance where $L(\lambda)$ has poles. Around a pole λ_k, we have

$$A_k(\lambda) = G_k(\lambda)L_k(\lambda)G_k^{-1}(\lambda) \tag{5.0.4}$$

and we assume that the polar part of the diagonal matrix $L_k(\lambda)$ is non-dynamical. Therefore, the singular part of μ is fixed under $\tilde{\mathrm{d}}$ and $\tilde{\mathrm{d}}(\mu d\lambda)$ is regular at λ_k. At $\lambda_k = \infty$, $d\lambda$ has a double pole. The dominant term of $\mu d\lambda$ is $c_i d\lambda$ when $\mu \to \infty_i = (\infty, c_i)$ and it is fixed under $\tilde{\mathrm{d}}$. The other terms are also kept fixed because of the conjugation action by diagonal matrices. The Hamiltonians generating this group action are given by (4.1.4). Assuming $\mu_i = c_i + \frac{d_i}{\lambda}$, we get

$$H_n = \sum_i c_i^{n-1} d_i$$

and after Hamiltonian reduction, these quantities will be kept fixed. So c_i and d_i are non-dynamical and $\tilde{\mathrm{d}}(\mu d\lambda)$ is also regular at infinity. □

We have already seen that the phase space has the dimension $2g$ so the g action variables I_k, canonically conjugated to θ_k, are constructed as:

$$I_k = \oint_{a_k} \mu d\lambda \tag{5.0.5}$$

Notice that the regular meromorphic differential $\mu d\lambda$ has poles at the points above λ_j and given the choice of homology basis for the punctured spectral curve, the path integrals above will be well-defined with respect to the homotopy class of the a-cycles.

The action-angle variables $\{q_k, I_k\}$ now provide another set of Darboux coordinates as we see in the following proposition:

Theorem 7. ([1], eq. 5.75) *Given that* $\tilde{\mathrm{d}}(\mu d\lambda)$ *is regular, we then have:*

$$\omega_{\mathcal{M}} = \sum_{i=1}^{g} \tilde{\mathrm{d}}\mu_{\gamma_i} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_i} = \sum_{i=1}^{g} \tilde{\mathrm{d}}I_i \wedge \tilde{\mathrm{d}}\theta_i = \sum_{i=1}^{g} \tilde{\mathrm{d}}I_i \wedge \tilde{\mathrm{d}}q_i \tag{5.0.6}$$

Proof. Since $\tilde{\mathrm{d}}(\mu d\lambda)$ is a regular form, it decomposes in the basis of holomorphic differentials as $\tilde{\mathrm{d}}(\mu d\lambda) = \sum_i \alpha_i \omega_i$. To find the coefficient α_i, we integrate both sides along the cycle a_l. So

$$\alpha_l = \oint_{a_l} \tilde{\mathrm{d}}(\mu d\lambda) = \tilde{\mathrm{d}} \oint_{a_l} \mu d\lambda = \tilde{\mathrm{d}} I_l \tag{5.0.7}$$

With $\omega_k = \sigma_k(\lambda) d\lambda$, we get

$$\tilde{\mathrm{d}}(\mu d\lambda) = \sum_k \tilde{\mathrm{d}} I_k \omega_k = \sum_k \tilde{\mathrm{d}} I_k \sigma_k(\lambda) d\lambda \tag{5.0.8}$$

Since the variations are taken at constant λ, then $\tilde{\mathrm{d}}(\mu d\lambda) = \tilde{\mathrm{d}}\mu d\lambda$ and $\tilde{\mathrm{d}}\mu$ decomposes on the $\tilde{\mathrm{d}} I_k$ by (4.2.2). Hence

$$\tilde{\mathrm{d}}\mu = \sum_k \frac{\partial \mu}{\partial I_k} \tilde{\mathrm{d}} I_k, \quad \frac{\partial \mu}{\partial I_k} = \sigma_k(\lambda) \tag{5.0.9}$$

By the definition of angle variables in (5.0.2) and using

$$\tilde{\mathrm{d}}\sigma_i(\lambda) = \sum_k \frac{\partial \sigma_i(\lambda)}{\partial I_k} \tilde{\mathrm{d}} I_k \tag{5.0.10}$$

we obtain

$$\tilde{\mathrm{d}}\theta_i = \sum_{j=1}^{g} \sigma_i(\lambda) \tilde{\mathrm{d}}\lambda_{\gamma_j} + \sum_{j=1}^{g} \int_{\lambda_0}^{\lambda_{\gamma_i}} \sum_k \frac{\partial^2 \mu}{\partial I_i \partial I_k} d\lambda \tilde{\mathrm{d}} I_k \tag{5.0.11}$$

Finally, we get

$$\begin{aligned} \omega_{\mathcal{M}} &= \sum_i \tilde{\mathrm{d}}\mu_{\gamma_i} \wedge \tilde{\mathrm{d}}\lambda_{\gamma_i} = \sum_{i,j} \tilde{\mathrm{d}} I_i \wedge \sigma_i(\lambda_{\gamma_i}) \tilde{\mathrm{d}}\lambda_{\gamma_i} \\ &= \sum_i \tilde{\mathrm{d}} I_i \wedge (\tilde{\mathrm{d}}\theta_i - \sum_j \int_{\lambda_0}^{\lambda_{\gamma_i}} \sum_k \frac{\partial^2 \mu}{\partial I_i \partial I_k} d\lambda \tilde{\mathrm{d}} I_k) = \sum_i \tilde{\mathrm{d}} I_i \wedge \tilde{\mathrm{d}}\theta_i \end{aligned} \tag{5.0.12}$$

where the second term in the parenthesis vanishes because $\partial_{I_i} \partial_{I_k} \mu$ is symmetrical with respect to indices i, k and $\tilde{\mathrm{d}} I_i \wedge \tilde{\mathrm{d}} I_k$ is antisymmetric. The last equality for q_i and I_i variables can be obtained from the variational identity for the vector of Riemann constants $\mathcal{K}$ that we will establish in chapter 8. It can also be directly derived from the explicit expression of the symplectic potential $\Theta_{\mathcal{M}}$ which will be calculated in the next chapter. □

Hence, I_k are canonically conjugated to q_k. The Poisson structure with these coordinates is:

$$\{I_i, I_j\} = 0, \quad \{I_i, q_j\} = \delta_{ij}, \quad \{q_i, q_j\} = 0$$

The set of action-angle variables, besides the aforementioned advantages, are also significant in the calculation of the canonical 1-form, also known as symplectic potential, since there are already well-known variational formulae with respect to action-angle variables. This matter will be studied in details in the next chapter.

Chapter 6

Symplectic potential

We previously saw in (4.2.3) that the symplectic potential of the Gaudin system is defined by the following 1-form:

$$\Theta_{\mathcal{M}} = \sum_{j=1}^{m} \mathrm{tr}(L_j G_j^{-1} dG_j) \tag{6.0.1}$$

The aim of this section is to express this potential in terms of the canonical action-angle coordinate of system

$$I_\gamma = \oint_{a_\gamma} \mu d\lambda \,, \quad q_\gamma = \Big(\mathcal{A}(D) + \mathcal{K}\Big)_\gamma \tag{6.0.2}$$

where the vector $\vec{q} = (q_\gamma)_{\gamma=1}^{g}$ is the characteristic of the theta function in the formula of the eigenvectors of Gaudin system in (3.3.5). In other words, we want to calculate the partial derivatives of G_j with respect to q_γ and I_γ in order to obtain

$$\Theta_{\mathcal{M}} = \sum_{\gamma=1}^{g}\sum_{j=1}^{m} \mathrm{tr}(L_j G_j^{-1} \frac{\partial G_j}{\partial q_\gamma})\, dq_\gamma + \sum_{\gamma=1}^{g}\sum_{j=1}^{m} \mathrm{tr}(L_j G_j^{-1} \frac{\partial G_j}{\partial I_\gamma})\, dI_\gamma \tag{6.0.3}$$

(in this section, we denote $\tilde{\mathrm{d}}$ by just d). The overall strategy of calculating (6.0.3) is that first, we write G_j as functions on the spectral curve. More specifically, we need to express G_j in terms of the components of the points coming from the spectral curve. This will be done via identifying G_j locally with $\hat{\Psi}$, the matrix of the eigenvectors of Lax matrix, and then expressing the elements of $\hat{\Psi}$ in terms of *Szegö kernel*. We will subsequently apply the previously established variational formulae for Szegö kernel to calculate the required partial derivatives.

6.1 Szegö kernel

A review of Szegö kernel and a few other lemmas is necessary before discussing the details of the calculation. We first need to introduce some holomorphic objects in order to define the Szegö kernel. As before, suppose we have a Riemann surface Γ of genus g with a choice of canonical homology basis and a normalized basis of holomorphic differentials $\{\omega_k\}$ on Γ. Let $\theta[\eta^*]$ be the genus g theta function of odd half-integer characteristic η^* corresponding the matrix of b-periods. Define the holomorphic differential $h(P)$ on Γ by:

$$h(x) = \sum_{k=1}^{g} \theta[\eta^*]_{z_k}(0)\omega_k(P) \tag{6.1.1}$$

where $\theta[\eta^*]_{z_k}$ denotes the derivative of theta function with respect to its k-th entry. All zeros of this differential are double and we can define the prime form on Γ by:

$$E(P,Q) = \frac{\theta[\eta^*](\mathcal{A}(P) - \mathcal{A}(Q))}{h(P)h(Q)} \tag{6.1.2}$$

and this definition is independent of the choice of any odd characteristic η^*.

Having the prime form, we can now define the *Szegö kernel*. A $(\frac{1}{2}, \frac{1}{2})$-form on $\Gamma \times \Gamma$ given by

$$S_{pq}(P,Q) = \frac{\theta_{pq}(\mathcal{A}(P) - \mathcal{A}(Q))}{\theta_{pq}(0)E(P,Q)} \tag{6.1.3}$$

Here we set the characteristic $p = 0$ and q to be the Abel map of the dynamical divisor D plus the verctor of Riemann constants $\mathcal{K}$ so $q = \mathcal{A}(D) + \mathcal{K}$. We will need the following property of Szegö kernel:

Proposition: The local expression of Szegö kernel when $x \to y$ is

$$S_{pq}(x,y) \simeq \frac{\sqrt{d\xi(x)}\sqrt{d\xi(y)}}{\xi(x) - \xi(y)} \tag{6.1.4}$$

where ξ is the local coordinate.

Proof. Assume $f(x,y) = \theta[\eta^*](\mathcal{A}(x) - \mathcal{A}(y))$. Since f has a zero at $x = y$, in a neighbourhood of $x = y$ with local coordinate ξ, it can be written as $f(x,y) = (\xi(x) - \xi(y))(1 + ...)$. By differentiating f with respect to x coordinate, we get $d_x f = \sum_j \theta_{z_j}(\mathcal{A}(x) - \mathcal{A}(y))v_j(x)$

therefore $h(x) = d_x f|_{x=y}$. So in the neighbourhood of $x = y$, we have

$$\begin{aligned} h(x) = d_{\xi(x)} f|_{\xi(x)=\xi(y)} &= \frac{\partial}{\partial \xi(x)}(\xi(x) - \xi(y))(1 + ...)d\xi(x) \\ &= d\xi(x)(1 + ...)|_{\xi(x)=\xi(y)} = d\xi(x) \end{aligned}$$

and similarly $h(y) = d\xi(y)$ since higher order terms vanish. So in this neighbourhood we have

$$E(x, y) = \frac{\theta[\eta^*](\mathcal{A}(x) - \mathcal{A}(y))}{\sqrt{h(x)}\sqrt{h(y)}} = \frac{\xi(x) - \xi(y)}{\sqrt{d\xi(x)}\sqrt{d\xi(y)}} + ...$$

and as $x \longrightarrow y$, we have

$$\frac{\theta_{pq}(\mathcal{A}(x) - \mathcal{A}(y))}{\theta_{pq}(0)} \longrightarrow 1$$

hence in the neighbourhood of $x = y$ as $x \longrightarrow y$, we get

$$S_{pq}(x, y) = \frac{\theta_{pq}(\mathcal{A}(x) - \mathcal{A}(y))}{\theta_{pq}(0)E(x, y)} = \frac{\sqrt{d\xi(x)}\sqrt{d\xi(y)}}{\xi(x) - \xi(y))} + ...$$

□

Recall $\hat{\Psi}(\lambda)$, the matrix of eigenvector of Lax matrix $L(\lambda)$ defined in (4.2.21). We now seek to express $\hat{\Psi}$ in terms of Szegö kernel. Then:

Theorem 8. *We have the following identities for $\hat{\Psi}$ and its inverse in terms of the Szegö kernel:*

$$\hat{\Psi}(\lambda)_{ab} = \frac{S_q(\lambda^{(b)}, \lambda_0^{(a)})}{S_q(\lambda^{(b)}, \lambda_0^{(1)})}, \quad \hat{\Psi}^{-1}(\lambda)_{ab} = \frac{S_q(\lambda_0^{(b)}, \lambda^{(a)})}{S_q(\lambda_0^{(b)}, \lambda^{(1)})} \tag{6.1.5}$$

Proof. The elements of the matrix defined by (6.1.5) are meromorphic functions. We need to show that the resulting matrix has similar analytical properties as the eigenvectors matrix $\hat{\Psi}$, that is, its rows have the same analytical properties as the eigenvectors ψ_k in (3.3.5). Thus, we must show that they have the same poles and zeros. The Szegö kernel with the characteristics $p = 0$ and the vector q as in (3.3.9) is given by

$$S_q(x, y) = \frac{\theta_q(\mathcal{A}(x) - \mathcal{A}(y))}{\theta_q(0)E(x, y)} \tag{6.1.6}$$

By (6.1.4), $S_q(\lambda^{(b)}, \lambda_0^{(a)})$ has a pole when $\lambda^{(b)} = \lambda_0^{(a)}$ or equally, when $\lambda = \lambda_0$ and we are the a-th point above it. This means the a-th row of $\hat{\Psi}$ i.e. the eigenvector ψ_a has a pole at

a-th point above λ_0. Notice that the base point of the Abel map as in (3.3.5) is set to be $\lambda_0^{(1)}$, the first point above λ_0 or equally, $\lambda_0^{(1)}$ in theorem (3). The first eigenvector ψ_1 has a pole at $\lambda_0^{(1)}$ as explained earlier, so once we apply the normalization ψ_1 by dividing the rows of $\hat{\Psi}$ by the fist row ψ_1, the denominator of other components tend to infinity at $\lambda_0^{(1)}$ (the numerators are finite) hence other rows have a zero at $\lambda_0^{(1)}$.

It remains to show that we also have g dynamical poles given by the dynamical divisor. Recall that the prime form in the denominator of Szegö kernel is defined as

$$E(x,y) = \frac{\theta[\eta^*](\mathcal{A}(x) - \mathcal{A}(y))}{h(x)h(y)} \tag{6.1.7}$$

for some odd half-integer characteristic $\eta^* = \eta_1\tau + \eta_2$. We also know that $\mathcal{K}$ is a half-period ([2], prop. 8.1.8) and so are $\mathcal{A}(\lambda_0^{(j)})$. Now consider odd half-integer the characteristic $\eta^* = \mathcal{A}(\lambda_0^{(k)}) - \mathcal{A}(D) - \mathcal{K}$. The components of k-th row of $\hat{\Psi}$ have the term $\theta[\eta^*](\mathcal{A}(\lambda^{(j)}) - \mathcal{A}(\lambda_0^{(k)}))$ in the denominator so when $\lambda^{(j)}$ is one of the dynamical poles i.e. one of the points in the divisor D, we get:

$$\begin{aligned}\theta[\eta^*](\mathcal{A}(\lambda^{(j)}) - \mathcal{A}(\lambda_0^{(k)})) &= \theta(\mathcal{A}(\lambda^{(j)}) - \mathcal{A}(\lambda_0^{(k)}) + \mathcal{A}(\lambda_0^{(k)}) - \mathcal{A}(D) - \mathcal{K}) \\ &= \theta(e) = 0\end{aligned} \tag{6.1.8}$$

where we applied (3.3.8) and the same analysis that was used in the proof of theorem (3) in (3.3.6). This implies that the k-th row of $\hat{\Psi}$ has g poles at the points of the dynamical divisor D therefore its poles and zeros are similar to those of ψ_k. As mentioned earlier, the prime form is independent of the characteristic as long as it is odd half-integer so the previous result also holds regardless of the odd half-integer characteristic used in the prime form.

To conclude, the analytical properties of the matrix defined in (6.1.5) coincide with the analytical properties of the matrix of eigenvectors (3.2.7). Hence (6.1.5) similarly represents the meromorphic functions described in (3.3.7) that solve the Lax equation. The boundary and normalization condition follows from the asymptotic expansion of the prime form.

As for the inverse, for $\Psi(\mu,\lambda) = \frac{S(\mu,\lambda)}{\sqrt{d\mu}\sqrt{d\lambda}}$, we have the well-known relation (see [14], eq. 4.13)

$$\Psi(\mu,\lambda)\Psi(\lambda,\nu) = \Psi(\mu,\nu)$$

for any arbitrary three points λ, μ and ν. Setting $\mu = \nu = \lambda_0$, we get

$$\Psi^{-1}(\lambda_0, \lambda) = \Psi(\lambda, \lambda_0)$$

which gives the formula for the elements of $\hat{\Psi}^{-1}$. □

The following identities relating the matrices G_j and G_j^{-1} to Szegö kernel can now be derived from (6.1.5) :

Lemma 2: The matrix G_j and its inverse are given by

$$[G_j]_{ab} = \frac{S_q(\lambda^{(b)}, \lambda_0^{(a)})(\lambda - \lambda_0)}{\sqrt{d\lambda}\sqrt{d\lambda_0}}\Big|_{\lambda=\lambda_j}, \quad [G_j^{-1}]_{ab} = \frac{S_q(\lambda_0^{(b)}, \lambda^{(a)})(\lambda_0 - \lambda)}{\sqrt{d\lambda}\sqrt{d\lambda_0}}\Big|_{\lambda=\lambda_j} \tag{6.1.9}$$

Proof. The matrix G_j differs from the matrix of eigenvectors by a multiplication by a diagonal matrix so it can be identified with $\hat{\Psi}(\lambda_j)$ without setting the normalization of the first component of eigenvectors, which is done by dividing by the first component of each column in $\hat{\Psi}$. This concludes the lemma. □

6.2 Differentiation of symplectic potential w.r.t. angle variables

Firstly, We have the following known variational formula for Szegö kernel:

Lemma 3:([12], eq. 2.18) The variation of Szegö kernel with respect to the angle variables is

$$\frac{\partial S_q(x,y)}{\partial q_\gamma} = -\oint_{a_\gamma} S_q(x,t)S_q(t,y) \tag{6.2.1}$$

We will also use the following lemma:

Lemma 4: Given two points $p, t \in \Gamma$ and the variable $\nu \in \Gamma$, and $\lambda(\ )$ representing the projection of the spectral curve points onto λ-argument, we have the following identity:

$$\sum_{a=1}^{N} S_q(t, \nu^{(a)})S_q(\nu^{(a)}, p) = S_q(t,p)\left(\frac{1}{\lambda(t) - \nu} - \frac{1}{\lambda(p) - \nu}\right)d\nu \tag{6.2.2}$$

Proof. For the variable $\nu \in \Gamma$, the expression $\sum_{a=1}^{N} S_q(t, \nu^{(a)})S_q(\nu^{(a)}, p)$ is a 1-form with respect to ν where ν is in fact the parameter on the base space ($\mathbb{CP}^1$) of the N-sheeted

cover and it has two simple poles at $\nu = \lambda(t)$ and $\nu = \lambda(p)$. Here λ is the map that projects from the cover to the base. Using the local expansion of Szegö kernel, we have to have the following form:

$$\sum_{a=1}^{N} S_q(t,\nu^{(a)})S_q(\nu^{(a)},p) = C(t,p)\left(\frac{1}{\lambda(t)-\nu} - \frac{1}{\lambda(p)-\nu}\right)d\nu$$
$$= S_q(t,p)\left(\frac{1}{\lambda(t)-\nu} - \frac{1}{\lambda(p)-\nu}\right)d\nu \tag{6.2.3}$$

The coefficient $C(t,p)$ must be the Szegö kernel $S_q(t,p)$ due to the singularity structure at $t = \nu^{(a)}$ and $p = \nu^{(a)}$. □

Now we can proceed to calculating the first part of (6.0.3). The calculation involves multiple stages so we have broken it down into each individual step and will elaborate on them.

Proposition: The following equality holds:

$$\sum_{j=1}^{m} \operatorname{tr}\left(L_j G_j^{-1}\frac{\partial G_j}{\partial q_\gamma}\right) = I_\gamma \tag{6.2.4}$$

Proof. Notice that from (4.2.22), $\hat{\Psi}(\lambda_j)$ can be identified with G_j in a neighbourhood of λ_j which results in lemma 2. We have

$$\sum_{j=1}^{m} \operatorname{tr}\left(L_j G_j^{-1}\frac{\partial G_j}{\partial q_\gamma}\right) = \sum_{j} \operatorname{tr}\left(L_j G_j^{-1}\frac{\partial \left[G_j\right]_{ab}}{\partial q_\gamma}\right) \tag{6.2.5}$$

We apply lemma 2 to write $\left[G_j\right]_{ab}$ in terms of Szegö kernel

$$(6.2.5) = \sum_{j} \operatorname{tr}\left(L_j G_j^{-1}\frac{\partial\left[\frac{S_q(\lambda^{(b)},\lambda_0^{(a)})(\lambda-\lambda_0)}{\sqrt{d\lambda}\sqrt{d\lambda_0}}\Big|_{\lambda=\lambda_j}\right]_{ab}}{\partial q_\gamma}\right) \tag{6.2.6}$$

We evaluate the Szegö kernel term at $\lambda = \lambda_j$ and write it as following for brevity:

$$(6.2.6) = \sum_{j} \operatorname{tr}\left(L_j G_j^{-1}\left[\frac{\partial}{\partial q_\gamma}\frac{S_q(\lambda_j^{(b)},\lambda_0^{(a)})(\lambda_j-\lambda_0)}{\sqrt{d\lambda_j}\sqrt{d\lambda_0}}\right]_{ab}\right) \tag{6.2.7}$$

The differential $\sqrt{d\lambda_j}$ is independent of q_γ so we move it out of the differentiation and then write G_j^{-1} in its matrix form:

$$(6.2.7) = \sum_j (\lambda_j - \lambda_0) \operatorname{tr}\left(L_j \left[G_j^{-1}\right]_{ba} \frac{1}{\sqrt{d\lambda_j}\sqrt{d\lambda_0}} \left[\frac{\partial}{\partial q_\gamma} S_q(\lambda_j^{(b)}, \lambda_0^{(a)})\right]_{ab}\right) \tag{6.2.8}$$

We apply the variational formula of Szegö kernel with respect to q_γ:

$$(6.2.8) = -\sum_j (\lambda_j - \lambda_0) \operatorname{tr}\left(L_j \left[G_j^{-1}\right]_{ba} \frac{1}{\sqrt{d\lambda_j}\sqrt{d\lambda_0}} \left[\oint_{t\in a_\gamma} S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})\right]_{ab}\right) \tag{6.2.9}$$

Now use the lemma 2 identity for G_j^{-1} in terms of Szegö kernel and merge the two $\sqrt{d\lambda_j}$ and $\sqrt{d\lambda_0}$'s

$$(6.2.9) = \sum_j (\lambda_j - \lambda_0)^2 \operatorname{tr}\left(L_j \left[\frac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)})}{d\lambda_j\, d\lambda_0}\right]_{ba} \left[\oint_{t\in a_\gamma} S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})\right]_{ab}\right) \tag{6.2.10}$$

and move the term $\dfrac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)})}{d\lambda_j}$ to the inside of the integral since it does not depend on the integral parameter t:

$$(6.2.10) = \sum_j (\lambda_j - \lambda_0)^2 \operatorname{tr}\left(L_j \oint_{t\in a_\gamma} \frac{\left[S_q(\lambda_0^{(a)}, \lambda_j^{(b)})\right]_{ba} \left[S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})\right]_{ab}}{d\lambda_j\, d\lambda_0}\right) \tag{6.2.11}$$

expanding the trace, we get:

$$(6.2.11) = \sum_j (\lambda_j - \lambda_0)^2 \sum_{a,b=1}^{N} (L_j)_{bb} \oint_{a_\gamma} \frac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)}) S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})}{d\lambda_j\, d\lambda_0} \tag{6.2.12}$$

We now use lemma 4 to continue (6.2.12). First we rearrange the summation by separating the terms only depending on b and writing the summation over a and b separately. Hence:

$$(6.2.12) = \sum_j (\lambda_j - \lambda_0)^2 \sum_{b=1}^{N} (L_j)_{bb} \oint_{a_\gamma} \frac{S_q(\lambda_j^{(b)}, t) \sum_a S_q(t, \lambda_0^{(a)}) S_q(\lambda_0^{(a)}, \lambda_j^{(b)})}{d\lambda_j\, d\lambda_0} \tag{6.2.13}$$

and swapping the place between integral and summations:

$$(6.2.13) = \oint_{a_\gamma} \sum_j (\lambda_j - \lambda_0)^2 \sum_{b=1}^{N} (L_j)_{bb} \frac{S_q(\lambda_j^{(b)}, t)}{d\lambda_j \, d\lambda_0} \sum_{a=1}^{N} S_q(t, \lambda_0^{(a)}) S_q(\lambda_0^{(a)}, \lambda_j^{(b)}) \tag{6.2.14}$$

Now substitute the last sum using lemma 4:

$$(6.2.14) = \oint_{a_\gamma} \sum_j (\lambda_j - \lambda_0)^2 \sum_{b=1}^{N} (L_j)_{bb} \frac{S_q(\lambda_j^{(b)}, t)}{d\lambda_j \, d\lambda_0} S_q(t, \lambda_j^{(b)}) \Big(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(\lambda_j^{(b)}) - \lambda_0} \Big) d\lambda_0 \tag{6.2.15}$$

The Lax matrix $L(\lambda)$ has the diagonalization $\mathrm{diag}(\mu_1, ..., \mu_N)$ for μ_i eigenvalues of $L(\lambda)$ at the point λ. So the diagonal matrix L_j, the diagonalization of $L(\lambda)$ near λ_j, can be identified with $\mathrm{diag}(\mu_j^{(1)}, ..., \mu_j^{(N)})$ given by the residues of $\mu d\lambda$ at the points $\lambda_j^{(b)}$ above λ_j.

$$(6.2.15) = \oint_{a_\gamma} \sum_j (\lambda_j - \lambda_0)^2 \sum_{b=1}^{N} (\mathrm{res}_{x=\lambda_j^{(b)}} \mu(x) d\lambda(x)) \Big\{ \frac{S_q(\lambda_j^{(b)}, t)}{d\lambda_j} S_q(t, \lambda_j^{(b)}) \Big(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(\lambda_j^{(b)}) - \lambda_0} \Big) \Big\} \tag{6.2.16}$$

The term in { } does not have a residue at $\lambda_j^{(b)}$ because its parameter t comes from a_γ so we move it into the $\mathrm{res}_{\lambda_j^{(b)}}$

$$\begin{aligned}(6.2.16) &= \oint_{a_\gamma} \sum_j (\lambda_j - \lambda_0)^2 \sum_{b=1}^{N} \mathrm{res}_{x=\lambda_j^{(b)}} \Big(\mu(x) d\lambda(x) \frac{S_q(x,t)}{d\lambda(x)} S_q(t,x) \Big(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(x) - \lambda_0} \Big) \Big) \\ &= \oint_{a_\gamma} \sum_j \sum_{b=1}^{N} \mathrm{res}_{x=\lambda_j^{(b)}} \Big((\lambda(x) - \lambda_0)^2 \mu(x) S_q(x,t) S_q(t,x) \Big(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(x) - \lambda_0} \Big) \Big)\end{aligned} \tag{6.2.17}$$

Given that $S_q(x,t)$ has a simple pole at $x = t$, and the fact that the sum of residues over the points λ_j plus the sum of residues over other poles is zero, (6.2.17) would be equal to the residue of the expression at $x = t$

$$\begin{aligned}(6.2.17) &= \oint_{a_\gamma} -\mathrm{res}_{x=t} \Big((\lambda(x) - \lambda_0)^2 \mu(x) S_q(x,t) S_q(t,x) \Big(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(x) - \lambda_0} \Big) \Big) \\ &= \oint_{a_\gamma} -\mathrm{res}_{x=t} \left((\lambda(x) - \lambda_0)^2 \mu(x) \frac{\sqrt{d\lambda(x)}\sqrt{d\lambda(t)}}{\lambda(x) - \lambda(t)} \frac{\sqrt{d\lambda(t)}\sqrt{d\lambda(x)}}{\lambda(t) - \lambda(x)} \Big(\frac{\lambda(x) - \lambda(t)}{(\lambda(t) - \lambda_0)(\lambda(x) - \lambda_0)} \Big) \right)\end{aligned}$$

$$= \oint_{a_\gamma} \mathrm{res}_{x=t} \left(\mu(x) \frac{d\lambda(x)\, d\lambda(t)}{\lambda(x) - \lambda(t)} \right) \tag{6.2.18}$$

where we used the local expansion of Szegö kernel at $x \to t$ for $S_q(x,t)$. Then move $d\lambda(t)$ out of residue to obtain

$$\begin{aligned} (6.2.18) &= \oint_{t \in a_\gamma} d\lambda(t)\, \mathrm{res}_{x=t} \frac{\mu(x)}{\lambda(x) - \lambda(t)} d\lambda(x) \\ &= \oint_{t \in a_\gamma} d\lambda(t).\mu(t) = \oint_{a_\gamma} \mu d\lambda = I_\gamma \end{aligned} \tag{6.2.19}$$

□

6.3 Differentiation of symplectic potential w.r.t. action variables

The next step is to calculate the variation with respect to action variables I_γ. This proves to be more challenging than the case of angle variables but the overall procedure is the same. First, we need the variational formula below:

Lemma 5:([12], eq. 2.42) The variational formula for Szegö kernel with respect to the action variables is:

$$\frac{\partial S_q(x,y)}{\partial I_\gamma} = -\frac{\pi i}{2} \sum_{x_k \text{ br. pts}} \underset{t=x_k}{\mathrm{res}}\, \omega_\gamma(t) \frac{W_t[S_q(x,t), S_q(t,y)]}{d\lambda(t) d\mu(t)} \tag{6.3.1}$$

where the sum is over all branch points and ω_γ are the basis of holomorphic differentials on the spectral curve. W_t is the *Wronskian*. For two functions f, g it is locally defined by $W_t[f,g] = f'g - fg'$ where f and g are functions of t.

We will apply this formula to the partial derivative $\dfrac{\partial G_j}{\partial I_\gamma}$. The result will be

Proposition: The following equality holds:

$$\sum_j \mathrm{tr} \left(L_j G_j^{-1} \frac{\partial G_j}{\partial I_\gamma} \right) = 0 \tag{6.3.2}$$

Proof. we start by again using the identification of G_j and Szegö kernel:

$$\sum_j \operatorname{tr}\left(L_j G_j^{-1} \frac{\partial G_j}{\partial I_\gamma}\right) = \sum_j \operatorname{tr}\left(L_j G_j^{-1} \frac{\partial \left[G_j\right]_{ab}}{\partial I_\gamma}\right) \tag{6.3.3}$$

and lemma 2 to write $\left[G_j\right]_{ab}$ in terms of Szegö kernel:

$$(6.3.3) = \sum_j \operatorname{tr}\left(L_j G_j^{-1}\left[\frac{\partial}{\partial I_\gamma} \frac{S_q(\lambda_j^{(b)}, \lambda_0^{(a)})(\lambda_j - \lambda_0)}{\sqrt{d\lambda_j}\sqrt{d\lambda_0}}\right]_{ab}\right) \tag{6.3.4}$$

then applying the variational formula of Szegö kernel w.r.t. I_γ results the following:

$$(6.3.4) = \frac{-\pi i}{2} \sum_j \operatorname{tr}\left(L_j \left[G_j^{-1}\right]_{ba} \frac{\lambda_j - \lambda_0}{\sqrt{d\lambda_j}\sqrt{d\lambda_0}}\left[\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{W\left[S_q(\lambda_j^{(b)}, t), S_q(t, \lambda_0^{(a)})\right]}{d\lambda(t) d\mu(t)}\right]_{ab}\right) \tag{6.3.5}$$

We apply lemma 2 once more to $\left[G_j^{-1}\right]_{ba}$ to get:

$$(6.3.5) = \frac{\pi i}{2} \sum_j \operatorname{tr}\left(L_j \left[\frac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)})(\lambda_j - \lambda_0)^2}{d\lambda_j\, d\lambda_0}\right]_{ba}\left[\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{W\left[S_q(\lambda_j^{(b)}, t), S_q(t, \lambda_0^{(a)})\right]}{d\lambda(t) d\mu(t)}\right]_{ab}\right) \tag{6.3.6}$$

we now rewrite and substitute the Wronskian by its definition

$$(6.3.6) = \frac{\pi i}{2} \sum_j$$
$$\operatorname{tr}\left(L_j \left[\frac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)})(\lambda_j - \lambda_0)^2}{d\lambda_j\, d\lambda_0}\right]_{ba}\left[\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(\lambda_j^{(b)}, t)' S_q(t, \lambda_0^{(a)}) - S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})'}{d\lambda(t) d\mu(t)}\right]_{ab}\right) \tag{6.3.7}$$

then we expand the trace to obtain

$$(6.3.7) = \frac{\pi i}{2} \sum_j$$
$$\sum_{a,b} \left[L_j\right]_{bb} \frac{S_q(\lambda_0^{(a)}, \lambda_j^{(b)})(\lambda_j - \lambda_0)^2}{d\lambda_j\, d\lambda_0} \sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(\lambda_j^{(b)}, t)' S_q(t, \lambda_0^{(a)}) - S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})'}{d\lambda(t) d\mu(t)} \tag{6.3.8}$$

The terms $S_q(\lambda_0^{(a)}, \lambda_j^{(b)})$ is independent of t so we can move it into the sum over branch points

$$(6.3.8) = \frac{\pi i}{2} \sum_j \sum_{a,b} [L_j]_{bb} \frac{(\lambda_j - \lambda_0)^2}{d\lambda_j \, d\lambda_0}$$
$$\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(\lambda_j^{(b)}, t)' S_q(t, \lambda_0^{(a)}) S_q(\lambda_0^{(a)}, \lambda_j^{(b)}) - S_q(\lambda_j^{(b)}, t) S_q(t, \lambda_0^{(a)})' S_q(\lambda_0^{(a)}, \lambda_j^{(b)})}{d\lambda(t) d\mu(t)} \tag{6.3.9}$$

and rearrange the summations to put the sums over a exactly before the terms where the a indices appear. Also the term $S_q(\lambda_0^{(a)}, \lambda_j^{(b)})$ does not depend on t so we can push it under the derivation $'$. Thus

$$(6.3.9) = \frac{\pi i}{2} \sum_j \sum_b [L_j]_{bb} \frac{(\lambda_j - \lambda_0)^2}{d\lambda_j \, d\lambda_0} \sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t)$$
$$\frac{S_q(\lambda_j^{(b)}, t)' (\sum_a S_q(t, \lambda_0^{(a)}) S_q(\lambda_0^{(a)}, \lambda_j^{(b)})) - S_q(\lambda_j^{(b)}, t) (\sum_a [S_q(t, \lambda_0^{(a)}) S_q(\lambda_0^{(a)}, \lambda_j^{(b)})]')}{d\lambda(t) d\mu(t)} \tag{6.3.10}$$

Now using lemma 4 to simplify the sums over a and keeping in mind that the derivative $'$ is with respect to t (more precisely, $\lambda(t)$), we can rewrite the last fraction:

$$(6.3.10) = \frac{\pi i}{2} \sum_j \sum_b [L_j]_{bb} \frac{(\lambda_j - \lambda_0)^2}{d\lambda_j \, d\lambda_0}$$
$$\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(\lambda_j^{(b)}, t)' S_q(t, \lambda_j^{(b)}) \left(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(\lambda_j^{(b)}) - \lambda_0} \right) d\lambda_0 - S_q(\lambda_j^{(b)}, t) \left[S_q(t, \lambda_j^{(b)}) \left(\frac{1}{\lambda(t) - \lambda_0} - \frac{1}{\lambda(\lambda_j^{(b)}) - \lambda_0} \right) d\lambda_0 \right) \Big]'}{d\lambda(t) d\mu(t)} \tag{6.3.11}$$

Expand the derivative $[S_q(t, \lambda_j^{(b)})(\lambda(t) - \lambda(\lambda_j^{(b)}))]'$:

$$[S_q(t, \lambda_j^{(b)})(\lambda(t) - \lambda(\lambda_j^{(b)}))]' = d\lambda(t) S_q(t, \lambda_j^{(b)}) + (\lambda(t) - \lambda_j) S_q(t, \lambda_j^{(b)})'$$

We also cancel out $d\lambda_0$'s, hence

$$(6.3.11) = \frac{-\pi i}{2} \sum_j \sum_b [L_j]_{bb} \frac{(\lambda_j - \lambda_0)^2}{d\lambda_j}$$

$$\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(\lambda_j^{(b)},t)' S_q(t,\lambda_j^{(b)})(\lambda(t)-\lambda_j) - S_q(\lambda_j^{(b)},t)\left[S_q(t,\lambda_j^{(b)})'(\lambda(t)-\lambda_j)+d\lambda(t) S_q(t,\lambda_j^{(b)})\right]}{(\lambda(t)-\lambda_0)(\lambda_j-\lambda_0)d\lambda(t)d\mu(t)} \tag{6.3.12}$$

Notice that here $'$ acts as the exterior derivative so $(\lambda(t)-\lambda_j)' = d\lambda(t)$. We then isolate $(\lambda(t)-\lambda_j)$ to get

$$(6.3.12) = \frac{-\pi i}{2} \sum_{j,b} [L_j]_{bb} \frac{(\lambda_j-\lambda_0)^2}{d\lambda_j} \sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{-S_q(\lambda_j^{(b)},t) S_q(t,\lambda_j^{(b)}) d\lambda(t) + \left(\lambda(t)-\lambda_j\right)\left[S_q(\lambda_j^{(b)},t)' S_q(t,\lambda_j^{(b)}) - S_q(\lambda_j^{(b)},t) S_q(t,\lambda_j^{(b)})'\right]}{(\lambda(t)-\lambda_0)(\lambda_j-\lambda_0)d\lambda(t)d\mu(t)} \tag{6.3.13}$$

Now substitute $\left[S_q(\lambda_j^{(b)},t)' S_q(t,\lambda_j^{(b)}) - S_q(\lambda_j^{(b)},t) S_q(t,\lambda_j^{(b)})'\right]$ with the Wronskian of $S_q(\lambda_j^{(b)},t)$ and $S_q(t,\lambda_j^{(b)})$. As before, we have $[L_j]_{bb} = \operatorname{res}_{\lambda_j^{(b)}} \mu d\lambda$, therefore

$$(6.3.13) = \frac{\pi i}{2} \sum_{j,b} \left(\operatorname*{res}_{x=\lambda_j^{(b)}} \mu d\lambda \right) \frac{(\lambda_j-\lambda_0)^2}{d\lambda_j} \sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \omega_\gamma(t) \frac{S_q(x,t)S_q(t,x)d\lambda(t) - (\lambda(t)-\lambda(x))W\left[S_q(x,t),S_q(t,x)\right]}{(\lambda(t)-\lambda_0)(\lambda_j-\lambda_0)d\lambda(t)d\mu(t)} \tag{6.3.14}$$

At branch points $t = x_k$, we have $d\lambda(t) = 0$. We are going to rearrange the summations, so

$$(6.3.14) = \frac{\pi i}{2} \sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \sum_{j,b} \operatorname*{res}_{x=\lambda_j^b} (\lambda(x)-\lambda_0)^2 \mu(x) \omega_\gamma(t) \frac{S_q(x,t)S_q(t,x)d\lambda(t) - (\lambda(t)-\lambda(x))W\left[S_q(x,t),S_q(t,x)\right]}{(\lambda(t)-\lambda_0)(\lambda(x)-\lambda_0)d\lambda(t)d\mu(t)} \tag{6.3.15}$$

The sum of the residues in the second summation can now be written as the negate of the sum of the residues over the rest of the poles $\{p\}$, hence:

$$(6.3.15) = \frac{\pi i}{2}$$

$$\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k} \sum_{p} -\operatorname*{res}_{x=p} (\lambda(x)-\lambda_0)^2 \mu(x)\omega_\gamma(t) \frac{S_q(x,t)S_q(t,x)d\lambda(t) - (\lambda(t)-\lambda(x))W\left[S_q(x,t), S_q(t,x)\right]}{(\lambda(t)-\lambda_0)(\lambda(x)-\lambda_0)d\lambda(t)d\mu(t)} \tag{6.3.16}$$

The other poles of the last expression are when $x = t$ so we write $S_q(x,t)$ in local coordinates as $x \to t$. Doing so, the Wronskian becomes

$$W\Big[f(t)\sqrt{dt}, g(t)\sqrt{dt}\Big] = (f'g - fg')(dt)^2$$

so

$$W\left[\frac{\sqrt{dx}}{x-t}\sqrt{dt}, \frac{\sqrt{dx}}{t-x}\sqrt{dt}\right] = \left(\frac{\sqrt{dx}}{(x-t)^2}\frac{\sqrt{dx}}{t-x} - \frac{-\sqrt{dx}}{(t-x)^2}\frac{\sqrt{dx}}{x-t}\right)(dt)^2 = 0$$

thus

$$\begin{aligned}(6.3.16) &= \frac{\pi i}{2}\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k}\left(-\operatorname*{res}_{x=t} (\lambda(x)-\lambda_0)^2 \mu(x)\omega_\gamma(t) \frac{\frac{\sqrt{dx}\sqrt{dt}}{x-t}\frac{\sqrt{dt}\sqrt{dx}}{t-x}d\lambda(t)}{(\lambda(t)-\lambda_0)(\lambda(x)-\lambda_0)d\lambda(t)d\mu(t)}\right) \\ &= \frac{\pi i}{2}\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k}\left(-\operatorname*{res}_{x=t} \mu(x)\omega_\gamma(t) \frac{dxdt}{(x-t)^2 d\mu(t)}\right) \end{aligned} \tag{6.3.17}$$

The residue of $\dfrac{\mu(x)}{(x-t)^2}dx$ part at $x = t$ is $\mu'(t)$, so

$$(6.3.17) = -\frac{\pi i}{2}\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k}\left(\mu'(t)\omega_\gamma(t)\frac{dt}{d\mu(t)}\right) = -\frac{\pi i}{2}\sum_{x_k \text{ BP}} \operatorname*{res}_{t=x_k}\omega_\gamma(t) \tag{6.3.18}$$

Since ω_γ are holomorphic differentials forming a basis of holomorphic differentials on the surface Γ, they have no poles on Γ therefore the last sum in (6.3.18) is zero. Thus the claim of the theorem is proved. □

Combining the last two propositions, we have:

Theorem 9. *The canonical symplectic potential $\Theta_{\mathcal{M}}$ in action-angle coordinates is*

$$\Theta_{\mathcal{M}} = \sum_{\gamma=1}^{g} I_\gamma dq_\gamma \tag{6.3.19}$$

The main take-away of (6.3.19) is that the set of action-angle variables introduced here indeed provide a set of Darboux coordinates that canonically parameterize the phase space, and the symplectic form $\omega_{\mathcal{M}}$ has the following canoncial epression in terms of action-angle variables

$$\omega_{\mathcal{M}} = \sum_{j=1}^{g} dI_j \wedge dq_j$$

The implications of this result from the spectral transform perspective will be reviewed in the following chapter.

Chapter 7

Spectral transform

We will now provide a different viewpoint to the solution of the Gaudin system revolving around the *inverse spectral problem*, which reconstructs the Lax matrix from a set of *spectral data*. This is essentially what we did in section (3.4) where we reconstructed the Lax matrix from the dynmaical data of the system. Here, we will step further back and revisit the problem via the *space of spectral data* standpoint. Starting from the phase space $\mathcal{M}$ of the system, which is defined as:

$$\mathcal{M} = \Big\{(G_j, L_j)_{j=1}^m,\ \sum_{j=1}^m G_j L_j G_j^{-1} = 0\Big\}/\sim \tag{7.0.1}$$

where $G_j \in \mathfrak{sl}(N,\mathbb{C})$ and L_j are diagonal $N \times N$ matrices such that the eigenvalues of each L_j are all different. $\sim$ is the equivalence relation that identifies the sets whose G_j components differ by a simultaneous multiplication $G_j \to G_j S$ for $S \in \mathfrak{sl}(N,\mathbb{C})$ and L_j are kept unchanged. To an arbitrary point of $\mathcal{M}$ such as $\{G_j, L_j\}_{j=1}^m$, we associate the Lax matrix

$$L(\lambda) = \sum_{j=1}^m \frac{A_j}{\lambda - \lambda_j}, \quad A_j = G_j L_j G_j^{-1}$$

The condition $\sum_{j=1}^m G_j L_j G_j^{-1} = 0$ ensures the regularity of $L(\lambda)$ at ∞. The phase space $\mathcal{M}$ has the symplectic form $\omega_{\mathcal{M}}$ with the symplectic potential $\Theta_{\mathcal{M}}$ as discussed before. The form $\omega_{\mathcal{M}}$ is invariant under the simultaneous transformation $G_j \to S G_j$ and it is also non-degenerate.

We now introduce the *space of spectral data* $\mathcal{S}$, defined as

$$\mathcal{S} = \left\{ \{Q_j\}_{j=2}^{N}, q \in \mathbb{C}^g, \{R_j\}_{j=1}^{m} \right\} \tag{7.0.2}$$

where $Q_j(z)$ are arbitrary rational functions with poles of order j at λ_j such that the j-differential $Q_j(z)(dz)^j$ is holomorphic at $z = \infty$ and R_j are diagonal matrices.

To avoid certain technical complications, we require the spectral curve $\det(L(\lambda) - \mu\mathbf{1}) = 0$ to have simple branch points and also be non-nodal. A *nodal* curve is an algebraic curve such that all of its points are either smooth or have a neighbourhood which is analytically isomorphic to a neighbourhood of the origin in the locus of the equation $xy = 0$ in $\mathbb{C}^2$, i.e. nodal points. The subspace of $\mathcal{M}$ defined by the condition of spectral curve being nodal is a subspace of comdimension 1 and we denote it by $\mathcal{D}$. Now the direct *spectral transform*

$$F : \mathcal{M} \setminus \mathcal{D} \to \mathcal{S}$$

depending on the m points λ_j is now defined as follows. Take a point in $\mathcal{M}$ and its associated Lax matrix $L(\lambda)$. Define Q_j as the j-th invariant polynomial of $L(\lambda)$, i.e. $Q_j = \operatorname{tr}\left(L(\lambda)^j\right)$. The curve Γ defined by the polynomial $\det(L(\lambda) - \mu\mathbf{1})$ has invariant polynomials of $L(\lambda)$ as the coefficients of terms in μ which we set as Q_j. The genus of Γ is calculated in (3.2.3). The vector $q \in \mathbb{C}^g$ will be defined similarly to the angle variables q before as in (5.0.3), hence $q = \mathcal{A}(D) + \mathcal{K}$. As for the diagonal matrices R_j, the choice is not unique. Notice that A_j are invariant under the multiplication of G_j by a diagonal matrix $G_j \to G_j D$. We can fix a reference matrix G_j^0, then define R_j as $G_j = G_j^0 R_j$.

The inverse spectral transform can also be constructed explicitly. Given a point in $\mathcal{S}$, i.e. a set of rational functions Q_j, $N \times N$ diagonal matrices R_j and a vector $q \in \mathbb{C}^g$, choose a basepoint $z_0 \in \mathbb{C}$ (it can be shown later that the inverse spectral transform does not depend on the choice of z_0). Define the curve Γ by

$$\Gamma(\lambda, \mu) = \mu^N + Q_2(\lambda)\mu^{N-2} + ... + Q_n(\lambda) = 0 \tag{7.0.3}$$

The key object on this curve as we saw in previous sections is the meromorphic differential $\mu d\lambda$. Denote the theta function on Γ by θ and the locus $\theta(q) = 0$ by (θ). We shall define

the inverse spectral transform

$$F^{-1} : \mathcal{S} \setminus (\theta) \to \mathcal{M}$$

using the Szegö kernel (see (6.1.3)) with characteristic q, which is

$$S_q(x,y) = \frac{\theta(\mathcal{A}(x) - \mathcal{A}(y) + q)}{\theta(q)E(x,y)}$$

By following the steps in section 6.2, we will be able to write G_j and L_j in terms of Szegö kernel. Define $\hat{\Psi}$ (basically, the matrix of eigenvectors) via

$$\hat{\Psi}_{jk}(\lambda) = \psi(\lambda^{(k)}, \lambda_0^{(j)}), \qquad \psi(x, x_0) = S_q(x, x_0)\frac{x - x_0}{\sqrt{dx}\sqrt{dx_0}}$$

Now the point $\{G_j, L_j\} \in \mathcal{M}$ associated to the point $\{Q_j, q, R_j\} \in \mathcal{S} \setminus (\theta)$ by the inverse spectral transform F^{-1} is given by

$$(L_j)_{kk} = \operatorname{res}|_{\lambda_j^{(k)}} \mu d\lambda, \qquad G_j = \hat{\Psi}(\lambda_j) R_j \tag{7.0.4}$$

where $\lambda_j^{(k)}$ are the points above λ_j on the curve Γ.

The moduli space of the spectral curves described by (7.0.3) is also a matter of interest since it is central to understanding the deformations of spectral curve. Fixing the points λ_j, we will show that the moduli space is of the dimension $g + m(N-1)$. To construct the local coordinates on this space, first we have $m(N-1)$ variables in the form of the independent residues of the differential $\mu d\lambda$ at the points $\lambda_j^{(k)}$, denoted by

$$z_j^{(k)} = \operatorname{res}|_{\lambda_j^{(k)}} \mu d\lambda, \qquad j = 1, ..., m, \quad k = 1, ..., N-1$$

We now define the coordinates $\mu_j^{(k)}$ for $j = 1, ..., m,\ k = 1, ..., N-1$ as

$$\begin{aligned} \mu_j^{(1)} &= z_j^{(1)} \\ \mu_j^{(k)} &= z_j^{(k)} - z_j^{(k-1)} \end{aligned}$$

Next, consider a homology basis $\{a_i, b_i\}_{i=1}^g, \{l_i^{(k)}\}$ on $\Gamma \setminus \{\lambda_j^{(k)}\}$, $j = 1, ..., m,\ k = 1, ..., N$ with $\{l_i^{(k)}\}$ being small loops around $\{\lambda_j^{(k)}\}$, the points above λ_j. The a-periods of the

differential $\mu d\lambda$, i.e. the action variables as discussed in (5.0.5),

$$I_r = \oint_{a_r} \mu d\lambda, \qquad r = 1, \dots, g$$

alongside the variables $\mu_j^{(k)}$, the set $\{I_r, \mu_j^{(k)}\}$ gives us local coordinates on the moduli space of curves described by (7.0.3).

By adding the vector q and the diagonal matrices R_j, we obtain the local coordinates on the space of spectral data $\mathcal{S}$. Le q_j be the components of the vector $q \in \mathbb{C}^g$, and the variables $\rho_j^{(k)}, j = 1, \dots, m,\ k = 1, \dots, N-1$ such that

$$\log r_j^{(k)} = \rho_j^{(k)} - \rho_j^{(k-1)}$$

where $R_j = \mathrm{diag}(r_j^{(1)}, \dots, r_j^{(N)})$. Then the following gives us the full set of coordinates on $\mathcal{S}$

$$\left\{ \{q_r, I_r\}_{r=1}^g, \{\mu_j^{(k)}, \rho_j^{(k)}\}, j = 1, \dots, m,\ k = 1, \dots, N-1 \right\}$$

and the symplectic form on $\mathcal{S}$ is now given by

$$\omega_{\mathcal{S}} = \sum_{r=1}^{g} dI_r \wedge dq_r + \sum_{j=1}^{m} \sum_{k=1}^{N-1} d\rho_j^{(k)} \wedge d\mu_j^{(k)} \tag{7.0.5}$$

hence the pairs (I_i, q_i) and $(\rho_j^{(k)}, \mu_j^{(k)})$ form a canonical set of Darboux coordinates.

7.1 Spectral transform as symplectomorphism

The consequence of the main result of chapter 6 (6.3.19) is that the spectral transform between the spaces $\mathcal{M}$ and $\mathcal{S}$ (both equipped with a symplectic structure) can be regarded as a symplectomorphism, i.e. a diffeomorphism preserving the symplectic structure (see section 2.4). All we need to do is to show that the pullback of $\Theta_{\mathcal{S}}$, the symplectic potential on $\mathcal{S}$ under the map : $\mathcal{M} \to \mathcal{S}$ (which sends the coordinates $\{G_j, L_j\}$ to $\{q_r, I_r.\rho_j^{(k)}, \mu_j^{(k)}\}$) is equal to $\Theta_{\mathcal{M}}$, the symplectic potential on $\mathcal{M}$. In other words,

Theorem 10. *The symplectic potential $\Theta_{\mathcal{M}}$ is given by*

$$\Theta_{\mathcal{M}} = F^*\Theta_{\mathcal{S}} = F^*\left(\sum_{r=1}^{g} I_r dq_r + \sum_{j=1}^{m}\sum_{k=1}^{N-1} \mu_j^{(k)} d\rho_j^{(k)}\right) \tag{7.1.1}$$

Proof. We provide a quick sketch of the proof. We have already shown in chapter 6 that the contribution of dI_r and dq_r terms are given by the first sum. The overall procedure to calculate the contribution of $\mu_j^{(k)}$ and $\rho_j^{(k)}$ terms will follow the same steps, only that this time the basis of holomorphic differentials ω_j should be replaced by the differentials $\omega_{\lambda_j^{(k)},\lambda_j^{(k-1)}}$ and instead of the variational formula (6.3.1) for the variation of Szegö kernel with respect to I_j, we apply the following one

$$\frac{1}{2\pi i}\frac{\partial}{\partial \mu_j^{(k)}} S_q(x,y)\Big|_{z(x),z(y)} = -\frac{\pi i}{2}\sum_{\text{br. pts.}} \text{res}|_{t=x_j} \frac{w_{\lambda_j^{(k)},\lambda_j^{(k-1)}}(t)\, W_t[S_q(x,t),\, S_q(t,y)]}{dz(t)dy(t)}$$

where $\omega_{x,y}(t)$ is the Abelian differential of the third kind with residues ± 1 at x and y. $\omega_{x,y}(t)$ is also normalized by $\oint_{a_k}\omega_{x,y} = 0$. Using this variational formula, the contribution of $d\mu_j^{(k)}$ term is given by

$$\Theta_{\mathcal{S}}(d\mu_j^{(k)}) = \frac{\pi}{2}\sum_{\text{br. pts.}} \text{res}_{t=x_i}\, \omega_{\lambda_j^{(k)},\lambda_j^{(k-1)}}$$

which is zero due to the assumption of genericity for the spectral curve that none of the branch points project to the poles λ_j. □

The following corollary provides a set of canonical Darboux coordinates on the space $\mathcal{M}$:

Corollary: The symplectic form $\omega_{\mathcal{M}} = d\Theta_{\mathcal{M}}$ on the space $\mathcal{M}$ coincides with the pullback of the symplectic form $\omega_{\mathcal{S}}$ (7.0.5) on the space of spectral data $\mathcal{S}$.

$$\omega_{\mathcal{M}} = F^*\omega_{\mathcal{S}}$$

Therefore $\omega_{\mathcal{M}}$ can be written as below in terms of spectral variables:

$$\omega_{\mathcal{M}} = F^*\left(\sum_{r=1}^{g} dI_r \wedge dq_r + \sum_{j=1}^{m}\sum_{k=1}^{N-1} d\mu_j^{(k)} \wedge d\rho_j^{(k)}\right) \tag{7.1.2}$$

Chapter 8

Variations of vector of Riemann constants

In this section, we will be discussing the variation of vector of Riemann constants with respect to the action variables. This is based on an observation made earlier while studying the symplectic properties of the system but here, we will look into it directly without relying on any previous result. First, we consider the hyperelliptic case and show that the vector of Riemann constants in this case is the gradient of some function on the moduli space of the spectral curve. The general case of results obtained here is to be pursued in the future.

8.1 Hyperelliptic case

The natural expectation and requirement rising while studying the symplectic structure of the phase space is that both $\{I_j, q_j\}_{j=1}^g$ and $\{I_j, q_j - \mathcal{K}_j\}_{j=1}^g$ be sets of Darboux coordinates. In other words, the symplectic form $\omega_{\mathcal{M}}$ have the same canonical expression in terms of both, so:

$$\sum_j dI_j \wedge dq_j = \sum_j dI_j \wedge d(q_j - \mathcal{K}_j)$$

which implies

$$\sum_j dI_j \wedge d\mathcal{K}_j = \sum_j dI_j \wedge (\sum_i \frac{\partial \mathcal{K}_j}{\partial I_i} dI_i) = \sum_{j,i} \frac{\partial \mathcal{K}_j}{\partial I_i} dI_j \wedge dI_i = 0 \tag{8.1.1}$$

The terms $j = i$ vanish in the last sum, and having $dI_j \wedge dI_i = -dI_i \wedge dI_j$ yields:

$$\sum_{j,l} \frac{\partial \mathcal{K}_j}{\partial I_i} dI_j \wedge dI_i = \sum_{j<i} (\frac{\partial \mathcal{K}_j}{\partial I_i} - \frac{\partial \mathcal{K}_i}{\partial I_j}) dI_j \wedge dI_i = 0 \tag{8.1.2}$$

For this to hold, the following variational identity needs to be true:

$$\frac{\partial \mathcal{K}_j}{\partial I_i} = \frac{\partial \mathcal{K}_i}{\partial I_j} \tag{8.1.3}$$

We wish to prove (8.1.3) via direct calculation. In order to simplify the matters, we will first compute an alternative expression for the vector of Riemann constants in the case of Γ being a hyperelliptic surface. We then apply known variational formulas on the space of holomorphic differentials to the explicit expression of $\mathcal{K}$. Let us recall a few ingredients first, the matrix of b-periods:

$$\tau_{ij} := \oint_{b_i} \omega_j \tag{8.1.4}$$

and the vector of Riemann constants:

$$\mathcal{K}_j = \frac{\tau_{jj}}{2} - \sum_{l=1}^{g} \int_{a_l} \mathcal{A}_j d\mathcal{A}_l \tag{8.1.5}$$

where $\mathcal{A}$ is the Abel map. Given $\mathcal{A}_j(P) = \int_{P_0}^{P} \omega_j$, (8.1.5) can be alternatively written as

$$\mathcal{K}_{jj} = \frac{\tau_{jj}}{2} - \sum_{l=1}^{g} \oint_{a_l} (\omega_l \int_{P_0}^{x} \omega_j) \tag{8.1.6}$$

where x is the parameter from the integral over a_l.

Theorem 11. *The variation of the components of vector of Riemann constants with respect to the action coordinates is index-symmetric, i.e.*

$$\frac{\partial \mathcal{K}_j}{\partial I_i} = \frac{\partial \mathcal{K}_i}{\partial I_j}$$

Proof. We first need to find an explicit expressions for the Abel map. Given the genus g of Γ, the number of branch points would be $2g + 2$ by Riemann-Hurwitz formula, let $P_j, j = 1, ..., 2g + 2$ be these branch points. The hyperelliptic surface has an involution J which can be viewed as a rotation by π about the axis that passes though $2g + 2$ branch

points. Let β_k be an oriented curve from P_{2k-1} to P_{2k} for $k = 1, ..., g+1$. The cycle b_k can now be defined as β_k followed by $J\beta_k$ but in the opposite direction or $-J\beta_k$. We also set a_k to be the curve connecting a point on b_k to b_{g+1} and then back to the initial point on b_k. The set $\{a_k, b_k\}_{k=1}^{g}$ forms a canonical basis of homology on Γ.

To simplify the calculation of Abel map as much as possible, We set P_1 as its basepoint to. At the end, we will show that all the calculations would remain valid if we change the basepoint of the Abel map. First we need to establish a few identities to do so (see [7], VII.1.1). Let $\tau^{(k)}$ denote the k-th column of the period matrix τ and $\vec{\omega} = (\omega_1, ..., \omega_g)$; by the fact that J essentially acts as a multiplication by -1 on the space of holomorphic differentials, for $k = 1, ..., g$ we have:

$$\tau^{(k)} = \int_{b_k} \vec{\omega} = \int_{\beta_k} \vec{\omega} - \int_{J\beta_k} \vec{\omega} = \int_{\beta_k} \vec{\omega} - \int_{J\beta_k} J\vec{\omega} = 2\int_{\beta_k} \vec{\omega} = 2\int_{P_{2k-1}}^{P_{2k}} \vec{\omega} \tag{8.1.7}$$

To calculate the $k = g+1$ case, we need the intersection numbers of the cycles. For $j = 1, ..., g$ we have

$$\hat{i}(a_j, b_{g+1}) = -1, \qquad \hat{i}(b_j, b_{g+1}) = 0$$

so up to homology we have $-b_{g+1} = b_1 + ... + b_g$ which in conjunction with (8.1.7) implies

$$\int_{P_{2g+1}}^{P_{2g+2}} \vec{\omega} = \frac{1}{2}(\tau^{(1)} + ... + \tau^{(g)}) \tag{8.1.8}$$

We also introduce the curves $\hat{\alpha}_j$ joining P_{2j} to P_{2j+1} and set $\alpha_j = \hat{\alpha}_j - J\hat{\alpha}_j$. Having the intersection numbers

$$\hat{i}(\alpha_j, a_k) = 0, \qquad \hat{i}(\alpha_j, b_k) = 0, \qquad \hat{i}(\alpha_j, b_j) = 1, \qquad \hat{i}(\alpha_j, b_{j+1}) = -1$$

for $j \neq k, k+1$, we get $\alpha_j = a_j - a_{j+1}$ for $j = 1, ..., g$ ($a_{g+1} = 0$). Thus:

$$\begin{aligned} 2\int_{P_{2k}}^{P_{2k+1}} \vec{\omega} = 2\int_{\hat{\alpha}_k} \vec{\omega} &= \int_{\hat{\alpha}_k} \vec{\omega} - \int_{J\hat{\alpha}_k} J\vec{\omega} = \int_{\hat{\alpha}_k} \vec{\omega} - \int_{J\hat{\alpha}_k} \vec{\omega} = \int_{\alpha_k} \vec{\omega} \\ &= \int_{a_k} \vec{\omega} - \int_{a_{k+1}} \vec{\omega} = e^{(k)} - e^{(k+1)} \qquad k = 1, ..., g-1 \end{aligned} \tag{8.1.9}$$

and $2\int_{P_{2g}}^{P_{2g+1}} \vec{\omega} = e^{(g)}$, using the normalization $\oint_{a_k} \omega_j = \delta_{kj}$. We can now compute the Abel

map at branch points:

$$
\begin{aligned}
&\mathcal{A}(P_1) = 0 \\
&\mathcal{A}(P_2) = \int_{P_1}^{P_2} \vec{\omega} = \frac{1}{2}\tau^{(1)} \\
&\mathcal{A}(P_3) = \int_{P_1}^{P_2} \vec{\omega} + \int_{P_2}^{P_3} \vec{\omega} = \frac{1}{2}(\tau^{(1)} + e^{(1)} - e^{(2)}) \\
&\mathcal{A}(P_4) = \mathcal{A}(P_3) + \int_{P_3}^{P_4} \vec{\omega} = \frac{1}{2}(\tau^{(1)} + \tau^{(2)} + e^{(1)} - e^{(2)}) \\
&\mathcal{A}(P_5) = \mathcal{A}(P_4) + \int_{P_4}^{P_5} \vec{\omega} = \frac{1}{2}(\tau^{(1)} + \tau^{(2)} + e^{(1)} - e^{(3)}) \\
&\quad \vdots \\
&\mathcal{A}(P_{2k+1}) = \frac{1}{2}(\tau^{(1)} + ... + \tau^{(k)} + e^{(1)} - e^{(k+1)}), \quad k = 1, ..., g-1 \\
&\mathcal{A}(P_{2k+2}) = \frac{1}{2}(\tau^{(1)} + ... + \tau^{(k+1)} + e^{(1)} - e^{(k+1)}), \quad k = 1, ..., g-1 \\
&\quad \vdots \\
&\mathcal{A}(P_{2g+1}) = \frac{1}{2}(\tau^{(1)} + ... + \tau^{(g)} + e^{(1)}) \\
&\mathcal{A}(P_{2g+2}) = \frac{1}{2}e^{(1)}
\end{aligned}
\tag{8.1.10}
$$

subsequently:

$$
\sum_{j=1}^{g} \mathcal{A}(P_{2j+1}) = \frac{1}{2}\Big(g\tau^{(1)} + (g-1)\tau^{(2)} ... + \tau^{(g)} + ge^{(1)} - e^{(2)} - ... - e^{(g)}\Big) \tag{8.1.11}
$$

We will now show that:

$$
\mathcal{K} = -\sum_{j=1}^{g} \mathcal{A}(P_{2j+1}) \tag{8.1.12}
$$

Recall that we have $\theta(0) \neq 0$ for the theta-function. So $\theta \circ \mathcal{A}$ does not vanish identically on Γ. As discussed previously, we know that $\theta \circ \mathcal{A}$ has g zeros on Γ, say $Q_1, ..., Q_g$. By Riemann's theorem, these zeros satisfy:

$$
\sum_{j=1}^{g} \mathcal{A}(Q_j) + \mathcal{K} = 0 \tag{8.1.13}
$$

$\mathcal{A}(P_j)$ are half-periods so they can be written as $\frac{1}{2}(\epsilon + \tau\epsilon')$ for the characteristic $\left[\begin{smallmatrix}\epsilon \\ \epsilon'\end{smallmatrix}\right]$. The

half-periods can be classified as odd or even depending on the parity of the characteristic. We have

$$\theta\Big(\frac{\epsilon+\tau\epsilon'}{2}\Big)=\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0)$$

hence θ vanishes at odd half-periods. Notice that $\mathcal{A}(P_{2j+1}), j=1,...,g$ are odd half-periods so these g points are the zeros of θ on Γ therefore (8.1.13) implies that:

$$\sum_{j=1}^{g}\mathcal{A}(P_{2j+1})=-\mathcal{K} \tag{8.1.14}$$

Combining with (8.1.11), we can write the j-th component of vector $\mathcal{K}$ as

$$-\mathcal{K}_j=\frac{1}{2}(g\tau_{j1}+(g-1)\tau_{j2}+...+\tau_{jg})+c_j \tag{8.1.15}$$

where $c_j=-1/2$ if $j=2,...,g$ and $c_1=g/2$.

We now have the following variational formula ([3], eq. 4.2) for the period matrix with respect to the a-periods of the differential $v=\mu d\lambda$:

$$\frac{\partial\tau_{\alpha\beta}}{\partial A_\gamma}=-2\pi i\sum_{x_i\ \text{br. pts.}}\frac{\omega_\gamma}{d\ln(v/\xi)}(x_i)\,\text{res}_{x_i}\frac{\omega_\alpha\omega_\beta}{v} \tag{8.1.16}$$

where the sum is taken over the branch points and ξ is the local coordinate at x_i. (8.1.16) can be rewritten in the more convenient form:

$$\frac{\partial\tau_{\alpha\beta}}{\partial A_\gamma}=-2\pi i\sum_{x_i\ \text{br. pts.}}\operatorname*{res}_{x_i}\frac{\omega_\alpha\omega_\beta\omega_\gamma}{d\xi d(v/d\xi)} \tag{8.1.17}$$

In our context, A_γ is the same as I_γ so by (8.1.15), the partial derivaties of $\mathcal{K}$ with respect to the action variables will be:

$$-\frac{\partial\mathcal{K}_j}{\partial I_\gamma}=\frac{1}{2}(g\frac{\partial\tau_{j1}}{\partial I_\gamma}+(g-1)\frac{\partial\tau_{j2}}{\partial I_\gamma}+...+\frac{\partial\tau_{jg}}{\partial I_\gamma}) \tag{8.1.18}$$

and by (8.1.17) we can see that the partials $\dfrac{\partial\tau_{jk}}{\partial I_\gamma}$ are symmetric with respect to permutating

indices, specifically $j \leftrightarrow \gamma$ so we conclude that:

$$\frac{\partial \mathcal{K}_j}{\partial I_\gamma} = \frac{\partial \mathcal{K}_\gamma}{\partial I_j} \tag{8.1.19}$$

□

Our next objective is to find a scalar 'potential' function in terms of which the vector of Riemann constants can be expressed.

Theorem 12. *There exists a function F of the moduli of the spectral curve, i.e. the action variables, whose gradient is the vector of Riemann constants, that is, $\mathcal{K} = \nabla F$.*

Proof. Let us denote the b-periods of v by B_j so $B_j = \oint_{b_j} \mu d\lambda$. Define the function $\tilde{F}$ as

$$\tilde{F} = \frac{1}{2} \sum_{j=1}^{g} I_j B_j \tag{8.1.20}$$

First, we show that the period matrix τ can be obtained by the second derivatives of $\tilde{F}$:

$$\tau_{\alpha\beta} = \frac{\partial^2 \tilde{F}}{\partial I_\alpha \partial I_\beta} \tag{8.1.21}$$

Notice that by the variational formula $\partial v / \partial A_\alpha = \omega_\alpha$ we get:

$$\frac{\partial B_\beta}{\partial I_\alpha} = \frac{\partial \int_{b_\beta} v}{\partial I_\alpha} = \int_{b_\beta} \frac{\partial v}{\partial I_\alpha} = \int_{b_\beta} \omega_\alpha = \tau_{\beta\alpha} = \tau_{\alpha\beta} \tag{8.1.22}$$

Thus,

$$\frac{\partial \tilde{F}}{\partial I_\alpha} = \frac{1}{2}\Big(B_\alpha + \sum_j I_j \frac{\partial B_j}{\partial I_\alpha}\Big) = \frac{1}{2}\Big(B_\alpha + \sum_j I_j \tau_{\alpha j}\Big) \tag{8.1.23}$$

and taking the second derivatives leads to

$$\frac{\partial^2 \tilde{F}}{\partial I_\alpha \partial I_\beta} = \frac{1}{2}\Big(\frac{\partial B_\alpha}{\partial I_\beta} + \tau_{\alpha\beta} + \sum_j I_j \frac{\partial \tau_{\alpha j}}{\partial I_\beta}\Big) = \tau_{\alpha\beta} + \frac{1}{2} \sum_j I_j \frac{\partial \tau_{\alpha j}}{\partial I_\beta} \tag{8.1.24}$$

The last sum turns out to be zero; Notice that using the symmetry of derivatives of τ with respect to the indices, it can be written as

$$\sum_j I_j \frac{\partial \tau_{\alpha j}}{\partial I_\beta} = \sum_j I_j \frac{\partial \tau_{\alpha\beta}}{\partial I_j} = \Big(\sum_j I_j \frac{\partial}{\partial I_j}\Big) \tau_{\alpha\beta} \tag{8.1.25}$$

and $\sum_j I_j \frac{\partial}{\partial I_\beta}$ is in fact the scaling operator which generates the map $\lambda \mapsto c\lambda$. The period matrix is invariant under such scaling hence (8.1.25) equals zero. We then get

$$\tau_{\alpha\beta} = \frac{\partial^2 \tilde{F}}{\partial I_\alpha \partial I_\beta} \tag{8.1.26}$$

Therefore by (8.1.15), $\mathcal{K}$ can be expressed as

$$-\mathcal{K}_j = \frac{1}{2}\Big(g\frac{\partial^2 \tilde{F}}{\partial I_j \partial I_1} + (g-1)\frac{\partial^2 \tilde{F}}{\partial I_j \partial I_2} + ... + \frac{\partial^2 \tilde{F}}{\partial I_j \partial I_g}\Big) + c_j \tag{8.1.27}$$

So by defining F as

$$F = -\frac{1}{2}\Big(g\frac{\partial \tilde{F}}{\partial I_1} + (g-1)\frac{\partial \tilde{F}}{\partial I_2} + ... + \frac{\partial \tilde{F}}{\partial I_g}\Big) - \sum_{k=1}^{g} c_k I_k \tag{8.1.28}$$

and having $\mathcal{K}_j = \frac{\partial F}{\partial I_j}$, we can express $\mathcal{K}$ in gradient form:

$$\mathcal{K} = \nabla F \tag{8.1.29}$$

□

It finally remains to show that if we change the basepoint of the Abel map to any other arbitrary point, the index symmetry of partial derivatives and the scalar potential function can still be established in a similar way. We know that the change of basepoint results in a translation of Abel map in the Jacobian of the curve so essentially for each of the equalities in (8.1.10), we get an extra constant term. Suppose we change the basepoint from P_1 to $\tilde{P}$, then:

$$\tilde{\mathcal{A}}(P_j) = \int_{\tilde{P}}^{P_j} \vec{\omega} = \int_{P_1}^{P_j} \vec{\omega} - \int_{P_1}^{\tilde{P}} \vec{\omega} = \mathcal{A}(P_j) - \vec{R} \tag{8.1.30}$$

the extra constant term being $\vec{R} = \int_{P_1}^{\tilde{P}} \vec{\omega}$. Going back to the definition of Riemann vector:

$$\mathcal{K}_j = \frac{\tau_{jj}}{2} - \sum_{l=1}^{g} \int_{a_l} \mathcal{A}_j d\mathcal{A}_l \tag{8.1.31}$$

substituting $\mathcal{A}$ by $\tilde{\mathcal{A}}$ leads to an extra $-R_j\omega_l$ term in the integrand so $\tilde{\mathcal{K}}_j$ gets an extra $\sum_l \int_{a_l} R_j\omega_l = gR_j$ constant term. Therefore the index symmetry property still holds and we

can similarly use the scalar function F to express $\tilde{\mathcal{K}}$ in terms of its gradient as in (8.1.29), this time the extra terms gR_jI_j should be included in (8.1.28) too.

8.2 Generic case

Proving the variational identity (8.1.3) in the case of the spectral curve being a generic Riemann surface requires establishing a variational formula by combining the results from [3] and [13]. First, we need to define the following (multi-valued) $\frac{g(1-g)}{2}$-differential

$$c(x) = \frac{1}{W[\omega_1, \dots, \omega_g](x)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \theta(\mathcal{K}^x)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} \omega_{\alpha_1}(x) \dots \omega_{\alpha_g}(x) \tag{8.2.1}$$

where the sum is taken over the permutations of $\alpha_1, ..., \alpha_g$ and W is the Wronskian determinant of g holomorphic differentials at at the point x, defined by

$$W[\omega_1, ..., \omega_g](x) = \det_{1\le\alpha,\beta\le g} \parallel \omega_\beta^{(\alpha-1)}(x) \parallel$$

where ω_β here represents $\omega_\beta = \omega_\beta d\lambda$ in slight abuse of notation. Now define the multi-valued differential of two variables $\sigma(x, y)$ by

$$\sigma(x, y) = \Big(\frac{c(x)}{c(y)}\Big)^{1/(1-g)} \tag{8.2.2}$$

The following variational formula can be established:

Proposition: Assume $x \in \Gamma$ such that its projection on the moduli space of the curve is independent of I_α. Then the variational formula

$$\frac{\partial \mathcal{K}_\beta}{\partial I_\alpha}\Big|_{\lambda(x)} = \sum_{j=1}^{p} \mathrm{res}_{t=x_j} \frac{\omega_\alpha(t)\, \omega_\beta(t) d_t \log \frac{\sigma(t,x_0)E(t,x)^{g-1}}{\sqrt{\mu(t)d\lambda(t)}}}{d\lambda(t)\, d\mu(t)} \tag{8.2.3}$$

Proof. By the equation (2.55) of [13], we have the following formula for the variation of vector of Riemann constants with the basepoint P with respect to the coordinates of the space of holomorphic Abelian differentials on Γ, rewritten in the notation of this work:

$$\frac{\partial \mathcal{K}_\alpha^P}{\partial I_k}\Big|_{\lambda(P)} = \frac{1}{2\pi i} \oint_{t\in a_k} \frac{\omega_\alpha(t)}{\mu(t)d\lambda(t)} d_t \log \frac{\sigma(t, x_0)E(t, P)^{g-1}}{\sqrt{\omega_\alpha(t)}} \tag{8.2.4}$$

This can be extended to the space of meromorphic differentials with simple poles as described in [12]. Using the same framework as [4], the variational formula (8.2.3) can be obtained from (8.2.4). □

The formula (8.2.3) immediately shows the symmetry of variation with respect to α and β, hence proving the general case of variational identity (8.1.3).

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