Boundedness of Operators on Local Hardy Spaces and Periodic Solutions of Stochastic Partial Differential Equations with Regime-Switching

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Abstract

Boundedness of Operators on Local Hardy Spaces and Periodic Solutions of Stochastic Partial Differential Equations with Regime-Switching

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In the first part of the thesis, we discuss the boundedness of inhomogeneous singular integral operators suitable for local Hardy spaces as well as their commutators. First, we consider the equivalence of different localizations of a given convolution operator by giving minimal conditions on the localizing functions; in the case of the Riesz transforms this results in equivalent characterizations of h^1. Then, we provide weaker integral conditions on the kernel of the operator and sufficient and necessary cancellation conditions to ensure the boundedness on local Hardy spaces for all values of p. Finally, we introduce a new class of atoms and use them to establish the boundedness of the commutators of inhomogeneous singular integral operators with bmo function.

In the second part of the thesis, we investigate periodic solutions of a class of stochastic partial differential equations driven by degenerate noises with regime-switching. First, we consider the existence and uniqueness of solutions to the equations. Then, we discuss the existence and uniqueness of periodic measures for the equations. In particular, we establish the uniqueness of periodic measures by proving the strong Feller property and irreducibility of semigroups associated with the equations. Finally, we use the stochastic fractional porous medium equation as an example to illustrate the main results.

Contribution of Authors

The first part of the thesis is based on [30–32].

The work [31] and [32]:

Galia Dafni: Discussion on the question, providing ideas, proofreading the papers, writing proofs.

Chun Ho Lau: Writing part of the preliminary, some of proofs with Claudio Vasconcelos, proofreading of the proofs.

Tiago Picon: Discussion on the question, providing ideas, proofreading the papers, writing proofs.

Claudio Vasconcelos: Discussion on the question and providing ideas, writing introduction and some of the proofs.

The work [30]:

Galia Dafni: Introducing the problems, providing ideas, writing some proofs, reviewing and editing.

Chun Ho Lau: Discussing on the problems, providing ideas, writing most of the proofs, proofreading.

The second part of the thesis is based on [73].

Wei Sun: Introducing the problems, providing ideas, writing some proofs, reviewing and editing.

Chun Ho Lau: Discussing on the problems, providing ideas, writing most of the proofs, proofreading.

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Contents

Chapter 1

Introduction

1.1 Introduction to Part I

Singular integral operators and pseudo-differential operators play crucial roles in harmonic analysis. They can be applied to solving partial differential equations and studying the regularity of the solutions. For instance, the Riesz transforms, an important example of singular integral operators, can be used to express the second derivatives of the solution to the Poisson equation on $\mathbb{R}^n$, $\Delta u = f$, in terms of f. It is well known that singular integral operators with the Hörmander condition (8) are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$. However, it is not possible that such operators are bounded on L^1. Therefore, the boundedness of the endpoint case becomes an important problem. In many cases, one can prove the boundedness of singular integral operators from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, where $H^1(\mathbb{R}^n)$ is the real Hardy space. Determining the optimal conditions on the kernels of these operators for this boundedness to hold is still a topic of current research – see for example [48, 109, 110] for recent boundedness results with weakened hypotheses on the kernel.

However, the real Hardy spaces $H^p(\mathbb{R}^n)$ are not closed under multiplication by smooth functions, which makes them unsuitable for working with partial differential equations. To include this property, Goldberg [46] introduced the local Hardy spaces $h^p(\mathbb{R}^n)$, which contain both the real Hardy space $H^p(\mathbb{R}^n)$ and the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. He provided an atomic theory for $h^p(\mathbb{R}^n)$, assuming the atoms have exact cancellation on a small scale. This raises a question: what is the optimal "cancellation" condition for all scales? For $\frac{n}{n+1} < p < 1$, Komori [69] imposed a cancellation condition through studying molecular theory. Dafni and Yue [34] studied a cancellation condition for $h^1(\mathbb{R}^n)$ atoms. In the joint work with Dafni, Picon, and Vasconcelos [31], we studied the approximate cancellation condition for atoms and molecules for all $0 < p \leqslant 1$. See Section 4.1 for a discussion of the atoms and molecules for $h^p(\mathbb{R}^n)$.

Goldberg [46] introduced local Riesz transform by localizing the multiplier away from the origin on the Fourier side, and used them to characterize $h^1(\mathbb{R}^n)$, while in [45] he introduced the local Hilbert transform through localizing near the origin to characterize $h^1(\mathbb{R})$. In [30], we provided a minimal condition on the localizing functions so that both types of transforms indeed characterize $h^1(\mathbb{R}^n)$ – see Theorem 3.0.3.

Because of the localization of local Hardy spaces, it is common to consider pseudo-differential operators instead of singular integral operators. Some partial results on the boundedness of pseudo-differential operators are listed in Theorem 2.5.29. Recently, Ding, Han and Zhu [35] generalized pseudo-differential operators to operators which they called inhomogeneous singular integral operators, and established separate sufficient and necessary cancellation conditions for boundedness on $h^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p < 1$. Can one have the same sufficient and necessary cancellation conditions for the boundedness of such operators? In [31], we provided a sufficient condition to ensure the boundedness on $h^p(\mathbb{R}^n)$ of an inhomogeneous singular integral operator, and in [32], we showed that the same condition is indeed sufficient – see Theorem 5.1.4 and 5.2.5.

Another story begins with the work Coifman, Rochberg and Weiss [21]. They showed that the commutator, $[b, T]$, of a singular integral operator T with a BMO function b is bounded on $L^p(\mathbb{R}^n)$ and use this to establish a weak factorization theorem. Another well-known result is due to Uchiyama [116], who showed that if $b \in \mathrm{CMO}(\mathbb{R}^n)$, then $[b, T]$ is compact. Recently, Hytönen [63] characterized all $L^p \to L^q$ boundedness of the commutators of singular integral operators $[b, T]$ with suitable b.

What happens in the endpoint case $p = 1$? Harboure, Segovia and Torrea [55] showed that it is impossible for the commutator of the Hilbert transform with a non-constant BMO function to be bounded on $H^1(\mathbb{R})$. Peréz [93] gave an example to show that $[b, T]$ is not bounded from L^1 to weak L^1, and provided an atomic space that is suitable for the boundedness of $[b, T]$. Ky [71] then found the largest space that leads to boundedness of $[b, T] = bT(\cdot) - T(b\cdot)$ from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Ky's space contains the atomic space introduced by Peréz, but it is not known whether they are the same – see Section 2.4 for the background on commutators. Yang, Wang and Chen [124], Hung and Ky [62] proved boundedness of $[b, T]$ on the Hardy space by restricting b to lie in a subclass of BMO of functions of logarithmic mean oscillation, and T in a class of pseudo-differential operators. In [30], we prove the boundedness of commutators of inhomogeneous singular integral operators with bmo functions on a new atomic space (Theorem 6.3.4). Since that space coincides with h^1 if $b \in \mathrm{lmo}(\mathbb{R}^n)$, we obtain the boundedness of such commutators on h^1, extending the results of Hung and Ky – see Corollary 6.3.5.

1.2 Introduction to Part II

Stochastic partial differential equations (SPDEs) have emerged as a fundamental tool for modeling random phenomena in diverse fields of science, including physics, biology, and ecosystems. There are many disciplines of SPDEs that are interesting, such as the existence and uniqueness of solutions, regularity of solutions, large-time behaviours, and asymptotic properties with small diffusion coefficients, among others. Notably, the works by experts, such as [28, 77, 94], have provided an extensive overview of SPDEs, their applications, and historical development. In this thesis, we concentrate on the existence and uniqueness of periodic measures for SPDEs.

The investigation of ergodicity for time-homogeneous SPDEs can be approached systematically through several sources, including Da Prato and Zabczyk's book [27], the survey by Maslowski and Seidler [83], and relevant references provided therein. In the past decades, many new results have been obtained for the existence and uniqueness of invariant measures. Here we list some of them which motivated our study. For instance, Hairer and Mattingly's work [54] established the ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. Romito and Xu [98] discussed the invariant measures of the 3D stochastic Navier-Stokes equations driven by mildly degenerate noise. Xie [121] obtained the uniqueness of invariant measures for general SPDEs driven by non-degenerate Lévy noise. Wang [118] utilized Harnack inequalities to investigate the ergodicity of SPDEs, while Gess and Röckner [44] explored the regularity and characterization of quasilinear SPDEs driven by degenerate Wiener noise. Zhang [127] considered invariant measures of 3D stochastic MHD-α models driven by degenerate noise, and Neuß [89] studied the ergodicity for singular-degenerate stochastic porous medium equations.

If stochastic equations have time-inhomogeneous coefficients, we generally do not expect the existence of invariant measures; instead, we consider periodic measures. Many existing results study the periodic behaviour of stochastic differential equations (SDEs) and SPDEs. For example, in [68], Khasminskii systematically studied periodically varying properties of SDEs driven by Wiener noise. In [129], Zhang et al. investigated the existence and uniqueness of periodic solutions of SDEs driven by Lévy noise. In [52], Guo and Sun generalized Doob's celebrated theorem on the uniqueness of invariant measures for time-homogeneous Markov processes to obtain the ergodicity and uniqueness of periodic solutions for non-autonomous SDEs driven by Lévy noises. For some other results related to periodic measures of SDEs, we refer the reader to Da Prato and Tudor [26], Xu et al. [122, 123], Chen et al. [17], Hu and Xu [61], and Ji et al. [64]. In [40], Feng and Zhao showed that there exist pathwise random periodic solutions to some SPDEs. In [25], Da Prato and Debussche investigated the long time behaviour of solutions to the 2D Stochastic Navier-Stokes equations with a time-periodic forcing term. In [18], Cheng and Liu used the variational approach to study recurrent properties of solutions to SPDEs driven by Wiener noise. Under suitable conditions, in particular, assuming strict monotonicity, they showed that the recurrent solutions are globally asymptotically stable in the mean-square sense. In [126], Yuan and Bao used the semigroup method to establish the exponential stability for a class of finite regime-switching SPDEs driven by Lévy noise. The method of our work is completely different from the methods of [18] and [126]. We shall investigate the ergodicity of SPDEs with countable regime-switching by considering the strong Feller property and irreducibility of the corresponding time-inhomogeneous semigroups.

The study of SDEs with hybrid switching has become increasingly important in different research areas such as biology, wireless communications, and engineering, as well as mathematical finance. We refer the reader to the monographs [82, 125] and references therein for detailed discussions.

In the second part of this thesis, our goal is to investigate the existence and uniqueness of periodic measures for a class of SPDEs with regime-switching. The model consists of two component processes, $X(t)$ and $\Lambda(t)$, whose state spaces are continuous and discrete. The evolution of $X(t)$ is described by an SPDE driven by degenerate Lévy noise. More flexibility can be added to applications by introducing regime switching $\Lambda(t)$ to the random dynamical system. Furthermore, our work extends previous studies on hybrid systems in finite dimensions to include some interesting hybrid systems in infinite dimensions. We shall give a detailed explanation of the model in Chapter 7.

1.3 Intermezzo: Connection between Harmonic Analysis and Probability Theory

This thesis is separated into two parts, harmonic analysis and stochastic processes. These two research areas have been developed independently but are closely related. Here, we give a brief description of the connection between these areas. Readers may refer to [36, 95] for a detailed discussion of the relation between harmonic analysis and probability theory.

Harmonic analysis and stochastic processes are linked by some of their fundamental objects, the Laplace operator and Brownian motion. The Laplacian Δ plays a crucial role in harmonic analysis and has been widely studied, while Brownian motion is a cornerstone in stochastic processes. The infinitesimal generator of a d-dimensional Brownian motion is precisely $\frac{1}{2}\Delta$. Additionally, the solution to the Dirichlet problem on an open set with continuous boundary conditions, including non-differentiable boundaries, can be written explicitly in terms of the expectation of the Brownian motion. We refer the readers to the book by Karatzas and Shreve [66, Chapters 4 and 5] for the explanation of the relationship between SDEs and PDEs.

Furthermore, martingale theory and harmonic analysis are closely connected. A breakthrough paper is by Burkholder, Gundy and Silverstein [14]. They established a Poisson integral characterization of $H^p(\mathbb{R})$ for any $0 < p < \infty$ by considering Brownian motion on $\mathbb{R}^2$, which led to the fundamental results of Fefferman and Stein [39]. Another example is the dyadic martingale and expansion in the Haar system. The conditional expectation with respect to the k-th filtration can be realized as the orthogonal projection onto the subspace generated by the Haar functions with scale $\leqslant k$. Stein [108, Chapter IV Section 6 Part D] elucidated the relation between wavelets and martingales and referenced further reading. The martingale analogues are also of great interest. Herz did much pioneering work in this area, in particular on martingale Hardy spaces and BMO [57, 58]. The book [67] is about BMO martingales, while [19, Chapter 10 and Appendix A.8] discuss the martingale Hardy spaces. There are active research areas in the martingale analogue of various spaces nowadays, such as [78, 91, 111, 120]. Meanwhile, studying stochastic versions of singular integral operators has allowed for investigating stochastic partial differential equations as shown in [79, 80].

Additionally, the paraproduct method is another harmonic analysis tool that has applications in probability theory, particularly in dealing with singular SPDEs where the noise term is not well-defined. In [50], Gubinelli and Imkeller provided a new method to discuss the existence and uniqueness of solutions to such SPDEs because of the ill-defined term in the classical sense. Later, Bailleul and Bernicot [49] generalized it to the higher-order paraproducts and applied it to singular SPDEs.

It is well-known that Sobolev spaces play an important role in harmonic analysis and the theory of PDEs. In Chapter 10, we use the stochastic fractional porous medium equation as a key example to illustrate the main theory. This example relies heavily on the theory of Sobolev spaces.

1.4 Structure of Thesis

This thesis is chapter-based and most of the material is taken from [30–32, 73].

In Chapter 2, we provide a comprehensive background for studying real and local Hardy spaces, and some known results on the boundedness of certain kinds of operators. Chapter 3 discusses the equivalence of different localizations of a convolution operator. In Chapter 4, we discuss the molecular theory for $h^p(\mathbb{R}^n)$ and prove the Hardy's inequality as an illustration of it. In Chapter 5, we focus on the sufficient and necessary conditions for the boundedness of an inhomogeneous singular integral operators on h^p. We end the first part in Chapter 6, which discusses lmo, the atomic commutator space, and the boundedness of commutators $[b, T]$.

In Chapter 7, we provide the necessary background and the framework of the SPDE models. We give the assumptions and main theorems in Chapter 8. All proofs of the theorems stated in Chapter 8 are given in Chapter 9. Finally, in Chapter 10, we use the stochastic fractional porous medium equation to illustrate the main theorem, Theorem 8.0.5.

Part I

Boundedness of Operators on Local Hardy Spaces

Chapter 2

Background and Preliminary Results

In this chapter, we will provide the background knowledge of Hardy spaces, the space of bounded mean oscillations (BMO), Calderón–Zygmund singular integral operators and their commutators with BMO functions, and non-homogeneous analogues.

We shall first fix some notations throughout Part I. We shall write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to denote the set of all non-negative integers. For any real number a, the expression $\lfloor a \rfloor$ denotes the greatest integer of a, namely $\sup\{n \in \mathbb{Z} : n \leqslant a\}$. We also write $A \lesssim B$ if there is a constant C independent of B such that $A \leqslant CB$, and we write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. The implicit constants may vary from line to line. If we want to emphasize the dependence on the implicit constants, we will write $\lesssim_X$ to represent the constants depending on X.

We denote the ball in $\mathbb{R}^n$ centered at x with radius r by $B(x, r)$. If a ball $B \subset \mathbb{R}^n$ is given, we will use $r(B)$ to denote its radius. For a ball $B = B(x, r)$ and $c > 0$, we write $cB := B(x, cr)$. For a set $A \subset \mathbb{R}^n$, we denote the complement of A to be $A^c := \mathbb{R}^n \backslash A$ and its indicator function to be χ_A. We will denote the Lebesgue measure of a measurable set B as $|B|$. It will be clear from the context whether $|\cdot|$ means the absolute value, the $\mathbb{R}^n$ norm, or the Lebesgue measure of a measurable set.

We write $L^p(\mathbb{R}^n)$, $0 < p < \infty$, to be the space of functions such that $|f|^p$ is an integrable function with respect to Lebesgue measure on $\mathbb{R}^n$, and

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} |f|^p \right)^{\frac{1}{p}}.$$

This is a norm when $p \geqslant 1$ and a quasi-norm when $0 < p < 1$. If the underlying space is clear, we omit the expression $\mathbb{R}^n$. If $|f|^p$ is integrable on every compact subset of $\mathbb{R}^n$, we write $f \in L^p_{loc}(\mathbb{R}^n)$. For $p = \infty$, $L^\infty(\mathbb{R}^n)$ is defined to be the space of all essentially bounded measurable functions defined on $\mathbb{R}^n$, and

$$\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

We also define $L^\infty_{loc}(\mathbb{R}^n)$ to be the space of the space of all locally bounded measurable functions defined on $\mathbb{R}^n$. The sequence space ℓ^p with $p > 0$ is defined to be all complex sequences $\{\lambda_j\}_{j=1}^{\infty}$ that satisfy

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}} < \infty.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of smooth with rapid decay functions, is the space of functions such that $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta(\phi)(x)| < \infty$ for all multiindices $\alpha, \beta \in (\mathbb{N}_0)^n$, where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$ and ∂_x^β means $\partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$ given $\beta = (\beta_1, \cdots, \beta_n)$. We denote $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ if $\alpha = (\alpha_1, \cdots, \alpha_n)$. We denote the dual of $\mathcal{S}(\mathbb{R}^n)$ to be $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. We write $f * g$ for the convolution of f and g; this is a well-defined function when one is in $\mathcal{S}(\mathbb{R}^n)$ and the other is in $\mathcal{S}'(\mathbb{R}^n)$. The space of all smooth functions is $C^\infty(\mathbb{R}^n)$, and the space of all smooth functions with compact support is $C_c^\infty(\mathbb{R}^n)$.

Moreover, in this thesis the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$, denoted by $\widehat{f}$ or $\mathcal{F}(f)$, is defined to be

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi} dy,$$

where $y \cdot \xi$ is the inner product of $\mathbb{R}^n$, while the inverse Fourier transform of g is denoted to be $\mathcal{F}^{-1}(g)$, which can be computed by

$$\mathcal{F}^{-1}(g)(x) := \int_{\mathbb{R}^n} g(\xi) e^{2\pi i \xi\cdot x} d\xi$$

given $g \in \mathcal{S}(\mathbb{R}^n)$. By duality, we use the same notation for the Fourier transform and inverse Fourier transform if the distribution is in $\mathcal{S}'(\mathbb{R}^n)$.

2.1 Real Hardy Spaces

In this section, we will provide the equivalent definitions of Hardy spaces and some properties of Hardy spaces.

The development of real Hardy spaces, which we will denote by $H^p(\mathbb{R}^n)$, came after the rich theory of complex (holomorphic) Hardy spaces, such as on the disk $\mathbb{D}$ or the upper half plane $\mathbb{R}_+^2$. In the critical paper by Fefferman and Stein [39], they proved Theorem 2.1.1 (in different wording).

In order to state the theorem and define $H^p(\mathbb{R}^n)$, we begin with the definition of bounded distributions. We say $f \in \mathcal{S}'(\mathbb{R}^n)$ is a *bounded distribution* if $\phi * f \in L^\infty(\mathbb{R}^n)$ for any $\phi \in \mathcal{S}(\mathbb{R}^n)$. The Poisson kernel is given by

$$P(x) := \frac{c_n}{(1+|x|^2)^{\frac{n+1}{2}}},$$

where $c_n = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$. For $t > 0$, we write $P_t(x) = t^{-n}P(xt^{-1})$. Then $P_t * f \in C^\infty(\mathbb{R}^n)$ for a bounded distribution f. Now, we can define the *nontangential maximal function of the Poisson integral of* f to be

$$M_P(f)(x) := \sup_{|x-y|\leqslant t<\infty} |P_t * f(y)|.$$

We also consider other maximal functions. Fix $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi \neq 0$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We write again $\phi_t(x) = t^{-n}\phi(xt^{-1})$. We define *the smooth maximal function of f with respect to ϕ*

$$M_\phi(f)(x) := \sup_{t>0} |\phi_t * f(y)|,$$

and define *the grand maximal function of f* to be

$$M_{\mathcal{F}_N}(f)(x) := \sup\{M_\phi(f)(x) : \phi \in \mathcal{S}_{\mathcal{F}_N}\},$$

where $\mathcal{S}_{\mathcal{F}_N} := \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta(\phi)(x)| \leqslant 1, |\alpha|, |\beta| \leqslant N\}$ and N is a positive integer.

Theorem 2.1.1 (See [47, Theorem 2.1.4] or [108, Chapter III, Theorem 1])**.**
Let $0 < p < \infty$. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then the following are equivalent.

1. *There exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $M_\phi(f) \in L^p(\mathbb{R}^n)$.*
2. *There exists $N \in \mathbb{N}$ such that $M_{\mathcal{F}_N}(f) \in L^p(\mathbb{R}^n)$.*
3. *The distribution f is a bounded distribution and $M_P(f) \in L^p(\mathbb{R}^n)$.*

In this case, we have

$$\|M_\phi f\|_{L^p} \approx \|M_{\mathcal{F}_N}(f)\|_{L^p} \approx \|M_P(f)\|_{L^p}.$$

Now we can define real Hardy spaces.

Definition 2.1.2. *Let $0 < p \leqslant \infty$. Then, we say $f \in H^p(\mathbb{R}^n)$ if one of the conditions in Theorem 2.1.1 holds and the H^p norm of f is defined to be the corresponding L^p norm.*

That is, if $f \in H^p(\mathbb{R}^n)$, we have

$$\|f\|_{H^p} \approx \|M_\phi f\|_{L^p} \approx \|M_{\mathcal{F}_N}(f)\|_{L^p} \approx \|M_P(f)\|_{L^p},$$

where $N = N_p + 1$.

We shall remark that although for $p < 1$, $\|\cdot\|_{L^p}$ is not a norm as remarked before, we still call the "norm" of H^p, $\|\cdot\|_{H^p}$, a norm. The following theorem says they coincide with $L^p(\mathbb{R}^n)$ for $p > 1$.

Theorem 2.1.3 (See [47, Theorem 2.1.2])**.** *For $1 < p \leqslant \infty$. The space $L^p(\mathbb{R}^n) = H^p(\mathbb{R}^n)$ with equivalent norms, that is, $\|M_\phi(f)\|_{L^p(\mathbb{R}^n)} \approx \|f\|_{L^p(\mathbb{R}^n)}$. Moreover, $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.*

We will see that the inclusion $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is proper – see Corollary 2.1.11.

Proposition 2.1.4 (See [47, Proposition 2.1.10 (c)])**.**
The Hardy spaces $H^p(\mathbb{R}^n)$ $(0 < p \leqslant 1)$ are complete quasi-metrizable spaces. For $p = 1$, $H^1(\mathbb{R}^n)$ is a complete normed space.

Indeed, we have two more characterizations of Hardy spaces. However, before that, we introduce two more types of functions. Because H^p consists of distributions and checking the norm of a distribution in H^p using the maximal function is difficult, it is natural to ask if there are any "simple" elements in this space that can generate all elements in H^p.

Definition 2.1.5 (See [108, Chapter III Section 2])**.** *Let $0 < p \leqslant 1 \leqslant q \leqslant \infty$ with $p < q$. We say a is a (p, q) atom if there exists a ball $B = B(x_0, r)$ such that*

1. $\operatorname{supp}(a) \subset B$;
2. $\|a\|_{L^q(\mathbb{R}^n)} \leqslant |B|^{\frac{1}{q}-1}$;
3. $\displaystyle\int_B (x - x_0)^\alpha a(x)dx = 0$ *for all* $|\alpha| \leqslant \lfloor n(p^{-1} - 1)\rfloor$ *and* $\alpha \in \mathbb{N}_0^n$.

From now on, we denote $N_p := \lfloor n(p^{-1} - 1)\rfloor$ and $\gamma_p := n(p^{-1} - 1)$.

Definition 2.1.6 (See [3, Definition 1.1])**.** *Let $0 < p \leqslant 1 \leqslant q \leqslant \infty$ with $p < q$ and $\lambda > n(\frac{q}{p} - 1)$. We say M is a (p, q, λ) molecule with respect to a ball $B = B(x_0, r)$ if there exists $C > 0$ independent of B and M such that*

1. $\displaystyle\int_B |M(x)|^q dx \leqslant Cr^{n(1-\frac{q}{p})}$;
2. $\displaystyle\int_{B^c} |x - x_0|^\lambda |M(x)|^q dx \leqslant Cr^{\lambda+n(1-\frac{q}{p})}$;
3. $\displaystyle\int_{\mathbb{R}^n} (x - x_0)^\alpha M(x)dx = 0$ *for all* $|\alpha| \leqslant N_p$ *and* $\alpha \in \mathbb{N}_0^n$.

Slightly different definitions of molecules can be found on [42, 85].

We shall remark that some authors may prefer to use $\displaystyle\int_{\mathbb{R}^n}$ in both Item 1 and 2 in Definition 2.1.6. Indeed, using Item 2 in Definition 2.1.6,

$$\int_{B^c} |M(x)|^q dx \leqslant \int_{B^c} |M(x)|^q |x - x_0|^\lambda r^{-\lambda} dx \leqslant Cr^{n(1-\frac{q}{p})+\lambda} r^{-\lambda}$$

and this implies

$$\int_{\mathbb{R}^n} |M(x)|^q dx \leqslant Cr^{n(1-\frac{q}{p})}.$$

Using Item 1 in Definition 2.1.6,

$$\int_B |x - x_0|^\lambda |M(x)|^q dx \leqslant r^\lambda \int_B |M(x)|^q dx \leqslant Cr^{\lambda+n(1-\frac{q}{p})}.$$

Thus,

$$\int_{\mathbb{R}^n} |x - x_0|^\lambda |M(x)|^q dx \leqslant C' r^{\lambda+n(1-\frac{q}{p})}.$$

From Item 1 and 2 in Definition 2.1.6, we can estimate

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^n}(x-x_0)^\alpha M(x)dx\right| \\
&\leqslant \left|\int_B (x-x_0)^\alpha M(x)dx\right| + \left|\int_{B^c}(x-x_0)^\alpha M(x)dx\right| \\
&\leqslant r^{|\alpha|}|B|^{1-q^{-1}}\|M\|_{L^q(B)} + \sum_{j=1}^{\infty}(2^{j+1}r)^{|\alpha|-\frac{\lambda}{q}+n(1-q^{-1})}\left(\int_{2^j r\leqslant|x-x_0|<2^{j+1}r}|x-x_0|^\lambda|M(s)|^q ds\right)^{\frac{1}{q}} \\
&\lesssim r^{|\alpha|+n(1-q^{-1})+n(q^{-1}-p^{-1})} + \sum_{j=0}^{\infty}(2^j r)^{|\alpha|-\frac{\lambda}{q}}(2^{j+1}r)^{n(1-q^{-1})}r^{\lambda q^{-1}+n(q^{-1}-p^{-1})} \\
&\approx r^{|\alpha|-n(p^{-1}-1)}. \qquad (1)
\end{aligned}
$$

We will use this later. This estimate does not consider Item 3 in Definition 2.1.6.

The following proposition tells us that a, M are the elements in $H^p(\mathbb{R}^n)$.

Proposition 2.1.7 (See [108, Chapter III Section 2.1])**.**
Let $0 < p \leqslant 1 \leqslant q$ with $p < q$ and $\lambda > n(\frac{q}{p}-1)$. Then, for any (p,q) atoms a and (p,q,λ) molecules M, we have $\|a\|_{H^p(\mathbb{R}^n)} \lesssim 1$ and $\|M\|_{H^p(\mathbb{R}^n)} \lesssim 1$.

We refer the proof of this proposition to the proof of Proposition 4.1.4 with $\omega = 0$. The key is the fact that the remainder term a_ω has the support and size condition of an H^p atom with the same cancellation condition as the molecule M.

The proof uses Hardy-Littlewood maximal operator, which is defined by

$$
Mf(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|dy, \qquad (2)
$$

where the supremum is taken over all balls B that containing x.

Theorem 2.1.8. *Let M be the Hardy-Littlewood maximal operator. Then,*

1. *for any $f \in L^p(\mathbb{R}^n)$ with $1 < p \leqslant \infty$, we have $\|Mf\|_{L^p} \leqslant A_p\|f\|_{L^p}$.*
2. *for any $\alpha > 0$,*
$$
|\{x \in \mathbb{R}^n : |Mf(x)| > \alpha\}| \leqslant \frac{C}{\alpha}\|f\|_{L^1}.
$$

This theorem is a classical result in harmonic analysis. We refer to [108, Chapter I Section 3] for a proof of the theorem.

Theorem 2.1.9. *Let $0 < p \leqslant 1 \leqslant q$ with $p \neq q$ and $\lambda > n(\frac{q}{p}-1)$. Let $f \in H^p(\mathbb{R}^n)$. Then,*

1. *there exists a sequence $\{\gamma_j\}_j \in \ell^p$ and a sequence of (p,q) atoms $\{a_j\}_j$ such that $f = \sum_{j=1}^{\infty}\gamma_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$; moreover, $\|f\|_{H^p} \approx \inf(\sum_{j\in\mathbb{N}}|\gamma_j|^p)^{\frac{1}{p}}$, where the infimum is taken over all such possible decomposition of f;*
2. *there exists a sequence $\{\gamma'_j\}_j \in \ell^p$ and a sequence of (p,q,λ) molecules $\{M_j\}_j$ such that $f = \sum_{j=1}^{\infty}\gamma'_j M_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$; moreover, $\|f\|_{H^p} \approx \inf(\sum_{j\in\mathbb{N}}|\gamma'_j|^p)^{\frac{1}{p}}$, where the infimum is taken over all such possible decomposition of f.*

We shall call such a decomposition of f as in Theorem 2.1.9 Item 1 as *atomic decomposition* and in Item 2 a *molecular decomposition*. Indeed, the first proof of atomic decomposition on $\mathbb{R}$ is by Coifman [20], and on $\mathbb{R}^n$ is by Latter [72]. The first discussion of the molecular decomposition on $\mathbb{R}^n$ is by Taibleson and Weiss [112].

From Theorem 2.1.9, we have the following properties.

Corollary 2.1.10. *For $0 < p \leqslant 1 \leqslant q \leqslant \infty$ and $p < q$, the space $H^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$.*

Corollary 2.1.11. *If $f \in H^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p \leqslant 1$, then*

$$\int_{\mathbb{R}^n} f(x)dx = 0; \tag{3}$$

more generally, if $f \in H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$, we have

$$\int_{\mathbb{R}^n} x^\alpha f(x)dx = 0 \quad \forall |\alpha| \leqslant N_p. \tag{4}$$

A proof of Corollary 2.1.11 without using atoms can be found in [103, Proposition 1.38].

Although every distribution in $H^p(\mathbb{R}^n)$ has global cancellation, it can still be positive locally. Indeed, we have the following theorem.

Theorem 2.1.12 (Stein's $L \log L$ theorem, [106])**.**
Suppose $f \in H^1(\mathbb{R}^n)$ is positive on $B \subset \mathbb{R}^n$; then for any compact subset $K \subset B$, f can be identify as a function in K with

$$\int_K |f(x)| \log(2 + |f(x)|)dx < \infty.$$

Moreover, the Fourier transform of an H^p distribution behaves well.

Proposition 2.1.13 (See [108, Chapter III Section 5.4])**.** *Let $f \in H^p(\mathbb{R}^n)$ for $0 < p \leqslant 1$.*

1. *The Fourier transform $\widehat{f}(\xi)$ is continuous on $\mathbb{R}^n$ and $|\widehat{f}(\xi)| \lesssim |\xi|^{n(p^{-1}-1)}\|f\|_{H^p}$ for all $\xi \in \mathbb{R}^n$.*

2. *Moreover,*

$$\lim_{|\xi| \to 0} \frac{\widehat{f}(\xi)}{|\xi|^{n(p^{-1}-1)}} = 0.$$

3. *We have Hardy's inequality:*

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^p}{|\xi|^{n(2-p)}} d\xi \lesssim \|f\|_{H^p}^p. \tag{5}$$

Using atomic decomposition, one can prove Item 3. We remark that the proof of Hardy's inequality in [31] (Theorem 4.2.1) works in $H^p(\mathbb{R}^n)$ when $\omega = 0$.

Although atomic decomposition is robust, we have to be careful when we consider the boundedness of an operator on $H^p(\mathbb{R}^n)$ because Bownik [10] gave a counter-example of a linear functional which when acting on $(1, \infty)$ atoms is bounded uniformly, but does not admit a bounded extension from H^1 to L^1. Fortunately, for $H^1(\mathbb{R}^n)$ the "finite atomic norm" is equivalent to the infinite one in the following cases:

Theorem 2.1.14 ([84, Theorem 3.1]).

1. *If $q < \infty$, then*
$$\inf\{\sum_{j=1}^{N} |\lambda_j| : \exists N \in \mathbb{N},\ f = \sum_{j=1}^{N} \lambda_j a_j,\ a_j \text{ is } (1,q)\text{-atom}\} \approx \|f\|_{H^1}.$$

2. *If $q = \infty$, then*
$$\inf\{\sum_{j=1}^{N} |\lambda_j| : \exists N \in \mathbb{N},\ f = \sum_{j=1}^{N} \lambda_j a_j,\ a_j \text{ is } (1,q)\text{-atom and continuous}\} \approx \|f\|_{H^1}.$$

Before we end this section, we should mention that there are additional essential characterizations of real Hardy spaces using the Lusin area integral, harmonic functions, wavelets, and other important tools. We refer the reader to [108, Chapter III, Section 4] for the characterization in terms of the area integral and harmonic functions, which is first established by Fefferman and Stein [39] and connected to the work of Burkholder, Gundy, and Silverstein [14], and [85] for the characterization using wavelets. We also refer to the book [42] for a detailed development of the theory and history of real Hardy spaces, and in particular the dual of the Hardy spaces, which we will use below.

2.2 Bounded Mean Oscillation

We start with the definition of bounded mean oscillation.

Definition 2.2.1 ([65]). *Let $b \in L^1_{loc}(\mathbb{R}^n)$. We say $b \in \mathrm{BMO}(\mathbb{R}^n)$ if*
$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx < \infty,$$
where $b_B = \dfrac{1}{|B|}\displaystyle\int_B b(y)dy =: \fint_B b(y)dy$ and the supremum is taken over all possible balls $B \subset \mathbb{R}^n$.

From the definition, $\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \leqslant 2\|f\|_{L^\infty(\mathbb{R}^n)}$ whenever $f \in L^\infty(\mathbb{R}^n)$; hence, $L^\infty(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ and this inclusion is proper. One example is $\log(|x|) \in \mathrm{BMO}(\mathbb{R}^n)\backslash L^\infty(\mathbb{R}^n)$.

It is true that if b is a constant function, $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 0$. Therefore, to be more rigorous, the space $(\mathrm{BMO}(\mathbb{R}^n)/\mathbb{R}, \|\cdot\|_{\mathrm{BMO}})$ is a normed space. Moreover, we have the following.

Theorem 2.2.2. *The space $(\mathrm{BMO}(\mathbb{R}^n)/\mathbb{R}, \|\cdot\|_{\mathrm{BMO}})$ is a Banach space.*

From now on, we will continue to use $\mathrm{BMO}(\mathbb{R}^n)$ to denote $\mathrm{BMO}(\mathbb{R}^n)/\mathbb{R}$.

Here we shall list some basic properties of $\mathrm{BMO}(\mathbb{R}^n)$ without proof. The reader may refer to [47, Chapter 3] or [108, Chapter IV].

Proposition 2.2.3.

1. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then for any ball $B_1 \subset B_2$,*

$$|b_{B_1} - b_{B_2}| \leqslant \frac{|B_2|}{|B_1|}\|b\|_{\mathrm{BMO}}; \tag{6}$$

indeed, one has a better estimate

$$|b_{B_1} - b_{B_2}| \leqslant \log\left(\frac{|B_2|}{|B_1|} + 1\right)\|b\|_{\mathrm{BMO}}.$$

2. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. For any $x_0 \in \mathbb{R}^n$, $r > 0$, and $\delta > 0$,*

$$r^\delta \int_{\mathbb{R}^n} \frac{|b(x) - b_{B(x_0,r)}|}{(r + |x - x_0|)^{n+\delta}} dx \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

3. *Suppose there exists a family of constants $\{k_B\}_B$ such that $\sup_B \frac{1}{|B|}\int_B |b(x) - k_B| dx < \infty$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$. We shall remark that the optimal k_B in this case is a median value of b over B.*

John and Nirenberg proved the following inequality.

Theorem 2.2.4 ([65]).
Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, there exist $c_1, c_2 > 0$ such that for any $\alpha > 0$ and ball B,

$$|\{x \in B : |b(x) - b_B| > \alpha\}| \leqslant c_1 \exp\left(- \frac{c_2\alpha}{\|finiteb\|_{\mathrm{BMO}}}\right)|B|.$$

By the John-Nirenberg inequality,

Corollary 2.2.5.
For any $b \in \mathrm{BMO}(\mathbb{R}^n)$, then for all $c < c_2$, where c_2 is given by Theorem 2.2.4, we have

$$\frac{1}{|B|}\int_B e^{c|b(x) - b_B|} dx < \infty,$$

where c_2 is from Theorem 2.2.4. Moreover, for any $0 < p < \infty$ and ball B,

$$\|b\|^p_{\mathrm{BMO}} \leqslant \frac{1}{|B|}\int_B |b(x) - b_B|^p dx \lesssim_p \|b\|^p_{\mathrm{BMO}}.$$

In particular, $b \in L^p_{loc}(\mathbb{R}^n)$.

This corollary tells us that if we define

$$\|b\|_{\mathrm{BMO}^p} := \sup_B \left(\frac{1}{|B|}\int_B |b(x) - b_B|^p dx\right)^{\frac{1}{p}},$$

and set $b \in \mathrm{BMO}^p(\mathbb{R}^n)$ to be the space of all functions that satisfy $\|b\|_{\mathrm{BMO}^p} < \infty$, then we have $\mathrm{BMO}^p(\mathbb{R}^n) \cong \mathrm{BMO}(\mathbb{R}^n)$ and $\|b\|_{\mathrm{BMO}^p} \approx \|b\|_{\mathrm{BMO}}$.

With the atomic decomposition Theorem 2.1.9, for given $b \in \mathrm{BMO}$, we first define the linear functional on **finite linear combinations of** $(1, q)$ **atoms** by

$$\langle b, f\rangle := \sum_{j=1}^{N} \lambda_j \int_{\mathbb{R}^n} b(y) a_j(y) dy,$$

where $f = \sum_{j=1}^{N} \lambda_j a_j$. Then, if a is an $(1, q)$ atom with $\operatorname{supp}(a) \subset B$, we have

$$|\langle b, a\rangle| \leqslant \left| \int_{\mathbb{R}^n} [b(y) - b_B] a(y) dy \right| \leqslant \left(\frac{1}{|B|} \int_B |b - b_B|^q \right)^{\frac{1}{q}} \leqslant \|b\|_{\mathrm{BMO}^q} \approx \|b\|_{\mathrm{BMO}}.$$

We have the Fefferman duality theorem.

Theorem 2.2.6 ([38, 39])**.** *1. Given $b \in \mathrm{BMO}(\mathbb{R}^n)$, we can extend the operator $\langle b, \cdot\rangle$ to $H^1(\mathbb{R}^n)$ continuously with*

$$|\langle b, f\rangle| \lesssim \|b\|_{\mathrm{BMO}} \|f\|_{H^1}.$$

2. Conversely, every linear functional on $H^1(\mathbb{R}^n)$ can be written as $\langle b, \cdot\rangle$ for some $b \in \mathrm{BMO}(\mathbb{R}^n)$ with

$$\|b\|_{\mathrm{BMO}} \lesssim \|\langle b, \cdot\rangle\|_{(H^1)^*}.$$

Therefore, we can conclude that $(H^1)^* \cong \mathrm{BMO}$.

The predual of $H^1(\mathbb{R}^n)$ is a smaller subspace of $\mathrm{BMO}(\mathbb{R}^n)$ called $\mathrm{CMO}(\mathbb{R}^n)$. This also gives us a crucial difference between $H^1(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ because $L^1(\mathbb{R}^n)$ has no predual.

Definition 2.2.7. *The space $\mathrm{CMO}(\mathbb{R}^n)$ is defined to be the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\mathrm{BMO}}$.*

Uchiyama proved an equivalent definition of CMO.

Theorem 2.2.8 ([116, Lemma in Section 3])**.**
Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then, a function b is in $\mathrm{CMO}(\mathbb{R}^n)$ if and only if the following conditions hold:

1. $\displaystyle \lim_{a \to 0} \sup_{|B| = a} \frac{1}{|B|} \int_B |b - b_B| = 0;$

2. $\displaystyle \lim_{a \to \infty} \sup_{|B| = a} \frac{1}{|B|} \int_B |b - b_B| = 0;$

3. $\displaystyle \lim_{|x| \to \infty} \frac{1}{|x + B|} \int_{x+B} |b - b_{x+B}| = 0$ *for each ball B.*

Moreover,

Theorem 2.2.9 ([22])**.** *We have $(\mathrm{CMO}(\mathbb{R}^n))^* \cong H^1(\mathbb{R}^n)$.*

We shall remark that the space CMO in Coifman and Weiss [22] is called VMO. Here we use CMO to distinguish it from the space VMO introduced by Sarason [101].

Definition 2.2.10. *The space* $\mathrm{VMO}(\mathbb{R}^n)$ *consists of* $b \in L^1_{loc}(\mathbb{R}^n)$ *such that*

$$\lim_{R\to 0} \sup_{r(B)<R} \frac{1}{|B|} \int_B |b(x) - b_B| dx = 0, \tag{7}$$

We remark that (7) gives the same condition in the Theorem 2.2.8 Item 1.
Therefore, by Theorem 2.2.8, we have the following inclusion

$$\mathrm{CMO}(\mathbb{R}^n) \subset \mathrm{VMO}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n).$$

Sarason [101] showed that the space $\mathrm{VMO}(\mathbb{R}^n)$ is the same as the $\mathrm{BMO}(\mathbb{R}^n)$-closure of uniformly continuous functions. These spaces do not allow jumps and we have $L^\infty \not\subset \mathrm{CMO}(\mathbb{R}^n)$ and $L^\infty \not\subset \mathrm{VMO}(\mathbb{R}^n)$ because L^∞ contains function that have jump discontinuity. For example, consider $b(x) = \chi_{[0,\infty)}(x)$, which is in $L^\infty(\mathbb{R}) \subset \mathrm{BMO}(\mathbb{R})$; however, when we consider the ball $B = (-r, r)$, the average is over B is $\frac{1}{2}$ and the integral

$$\begin{aligned}\frac{1}{|B|} \int_B |f(x) - f_B| dx &= \frac{1}{2r}\left(\int_0^r (1 - \frac{1}{2}) dx + \int_{-r}^0 (\frac{1}{2} - 0) dx \right)\\ &= \frac{r}{2r} = \frac{1}{2} \nrightarrow 0\end{aligned}$$

as $r \to 0$. This can be generalized to any bounded functions with jumps.

Unlike in $L^\infty(\mathbb{R}^n)$, if $f, g \in \mathrm{BMO}(\mathbb{R}^n)$, the product fg may not be in $\mathrm{BMO}(\mathbb{R}^n)$, even if one of them is in $L^\infty(\mathbb{R}^n)$. This can be seen by considering $f(x) = \chi_{[0,\infty)}(x) \in L^\infty(\mathbb{R})$ and $g(x) = \log(|x|) \in \mathrm{BMO}(\mathbb{R})$ on $\mathbb{R}$. The product $f(x)g(x) = \chi_{[0,\infty)}(x) \log(|x|)$ is not in $\mathrm{BMO}(\mathbb{R})$ because the oscillation on intervals centred at zero is unbounded due to the differences between the averages on the right and left halves - see [47, Example 3.1.4].

In order to have a pointwise multiplier, one needs one of the functions to be in a smaller space.

Definition 2.2.11. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *We say* $b \in \mathrm{LMO}(\mathbb{R}^n)$ *if*

$$\sup_B \frac{\log(1 + [r(B)]^{-1})}{|B|} \int_B |b(x) - b_B| dx < \infty.$$

Bounded functions of logarithmic mean oscillation (LMO) have been identified as the pointwise multipliers of H^1 and BMO on the circle [105], on the sphere [74], and on spaces of homogeneous type [16] by considering locally H^1 instead.

A generalization of LMO is called a ψ-generalized Campanato space.

Definition 2.2.12. *Let* $k \in \mathbb{N}_0$, $1 \leqslant q < \infty$, *and* $\psi : (0, \infty) \to (0, \infty)$. *We define*

$$L_k^{q,\psi}(\mathbb{R}^n) := \Big\{ f \in L^q_{loc}(\mathbb{R}^n) : \exists\ C > 0 \ s.t.\ \forall\ B \subset \mathbb{R}^n, \\ \left(\fint_B |f(y) - (P_B^k f)(y)|^q dy \right)^{\frac{1}{q}} \leqslant C\, \psi(r(B)) \Big\},$$

where $P_B^k f(y)$ *is the unique polynomial of degree less than or equal to* k *that has the same moments as* f *over* B *up to order* k. *For the case* $q = \infty$, *with the same notation as above, we define*

$$L_k^{\infty,\psi}(\mathbb{R}^n) := \Big\{ f \in L^\infty_{loc}(\mathbb{R}^n) :\ \exists\ C > 0\ s.t.\ \forall\ B,\ \|f - (P_B^k f)\|_{L^\infty(B)} \leqslant C\psi(r(B)) \Big\}.$$

The space $L^{q,\psi}_k(\mathbb{R}^n)$ is considered as a quotient space of the above classes of functions modulo all polynomials of degree less than or equal to k.

Given $f \in L^{q,\psi}_k(\mathbb{R}^n)$, we define

$$\|f\|_{L^{q,\psi}_k} := \sup_B \frac{1}{\psi(r(B))}\left(\fint_B |f(y)-(P^k_B f)(y)|^q dy\right)^{\frac{1}{q}}$$

if $1 \leqslant q < \infty$, and

$$\|f\|_{L^{\infty,\psi}_k} := \sup_B \frac{1}{\psi(r(B))}\|f(y)-(P^k_B f)(y)\|_{L^\infty(B)},$$

where the supremum in both cases is taken over all balls in $\mathbb{R}^n$.

With this terminology, we can see that $L^{q,1}_0(\mathbb{R}^n) \cong \mathrm{BMO}(\mathbb{R}^n)$, here 1 in the superscript is the constant function 1, and $L^{1,\psi}_k(\mathbb{R}^n) \cong \mathrm{LMO}(\mathbb{R}^n)$ if we take $\psi(r) = \frac{1}{\log(1+r^{-1})}$. Note that if ψ is increasing and has doubling property, i.e. $\psi(2r) \lesssim \psi(r)$, then by following the proof of the John-Nirenberg inequality [104] or the proof in [75], $L^{1,\psi}_k(\mathbb{R}^n) \cong L^{p,\psi}_k(\mathbb{R}^n)$ for all $1 \leqslant p < \infty$.

Definition 2.2.13. *We define the space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ to be all f such that*

1. *if $0 < \gamma < 1$, $|f(x+h)-f(x)| \leqslant C|h|$ for all $x, h \in \mathbb{R}^n$;*
2. *if $\gamma = 1$, $|f(x+h)-2f(x)+f(x-h)| \leqslant C|h|$ for all $x, h \in \mathbb{R}^n$;*
3. *if $\gamma = N+\theta$ and $\theta \in (0,1)$, f is continuously differentiable up to order N, and the N-th order derivatives are in $\dot{\Lambda}_\theta(\mathbb{R}^n)$;*
4. *if $\gamma = N$, f is continuously differentiable up to order $N-1$, and the $(N-1)$-th order derivatives are in $\dot{\Lambda}_1(\mathbb{R}^n)$.*

If $\gamma = N+\theta$, we need to consider the space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ modulo all polynomials up to degree $\lfloor\gamma\rfloor$; if $\gamma = N$, we need to consider the space $\dot{\Lambda}_\gamma(\mathbb{R}^n)$ modulo all polynomials up to degree $\lfloor\gamma\rfloor - 1$.

The seminorm of $\dot{\Lambda}_\gamma$ is: if $\gamma \notin \mathbb{N}$,

$$\|f\|_{\dot{\Lambda}_\gamma} := \sup_{|\beta|=\lfloor\gamma\rfloor} \sup_{h\neq 0,x} |h|^{\lfloor\gamma\rfloor-\gamma}|f(x+h)-f(x)|;$$

and if $\gamma \in \mathbb{N}$,

$$\|f\|_{\dot{\Lambda}_\gamma} := \sup_{|\beta|=\lfloor\gamma\rfloor-1} \sup_{h\neq 0,x} |h|^{-1}|f(x+h)-2f(x)+f(x-h)|.$$

Theorem 2.2.14 (See [42, Chapter III Theorem 5.30])**.** *We have $\dot{\Lambda}_\gamma(\mathbb{R}^n) \cong L^{1,\psi}_{\lfloor\gamma\rfloor}(\mathbb{R}^n)$ with $\psi(r) = r^\gamma$, with equivalent norms.*

With this terminology, we have

Theorem 2.2.15 (See [42, Chapter III Section 5])**.** *Let $0 < p < 1$. The dual space of $H^p(\mathbb{R}^n)$ is $\dot{\Lambda}_{n(p^{-1}-1)}(\mathbb{R}^n)$.*

For the relation between Campanato spaces, Lipschitz and Zygmund spaces, we refer to [42, Chapter III Section 5]; for more on Campanato spaces on different domains and their generalizations, we refer to the exposition [97].

2.3 Singular Integral Operators

In this section, we will provide the definitions of Calderón–Zygmund singular integral operators and also the definition of the Hilbert transform and Riesz transforms.

We shall start with the classical singular integral operators of convolution type.

Definition 2.3.1. *Let $0 < \delta \leqslant 1$. A function K defined on $\mathbb{R}^n \backslash \{0\}$ is called δ-kernel of convolution type if there exists $C > 0$ such that*

1. $|K(x)| \leqslant C|x|^{-n}$ *for all x;*
2. $|K(x-y) - K(x)| \leqslant C|y|^{\delta}|x|^{-n-\delta}$ *if $2|y| \leqslant |x|$;*
3. $\left| \int_{\varepsilon<|x|<M} K(x)dx \right| \leqslant C$ *for all $0 < \varepsilon < M < \infty$.*

We define the Calderón–Zygmund singular integral operator T associated with a δ-kernel of convolution type K to be

$$Tf(x) := P.V. \int_{\mathbb{R}^n} K(x-y)f(y)dy := \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} K(x-y)f(y)dy \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

The following are typical results on the boundedness of the singular integral operators. We refer the reader to [108, Chapter 1 and 3] or [47, Chapter 2] for the proof.

Theorem 2.3.2.
Let T be a Calderón–Zygmund singular integral operator T associated with a δ-kernel of convolution type. Then,

1. *it is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$;*
2. *it is of weak type $(1,1)$, i.e., for all $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$,*
$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \lesssim_{T,n} \frac{\|f\|_{L^1}}{\alpha};$$
3. *it is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and also from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$;*
4. *if we further assume that $\lim_{\varepsilon \to 0, M \to \infty} \int_{\varepsilon<|x|<M} K(x)dx = 0$, then T is bounded on $H^1(\mathbb{R}^n)$.*

The critical examples of Calderón–Zygmund singular integral operator T associated with a δ-kernel of convolution type are the Hilbert transform for $n = 1$ and Riesz transform for $n \geqslant 2$.

Definition 2.3.3. *The Hilbert transform on $\mathbb{R}$ is defined to be*

$$Hf(x) := P.V. \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy;$$

and j-th Riesz transform on $\mathbb{R}^n$ for $j = 1, \cdots, n$ is defined to be

$$R_j f(x) := P.V. \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy.$$

We remark that the Hilbert transform is the one-dimensional Riesz transform. The following theorem tells us that these operators characterize the Hardy space H^1.

Theorem 2.3.4 (see [107, Chapter VII, Section 3.2 Corollary 1])**.** *Let $f \in L^1(\mathbb{R}^n)$. Then $f \in H^1(\mathbb{R}^n)$ if and only if $f \in L^1(\mathbb{R}^n)$ and $R_j(f) \in L^1(\mathbb{R}^n)$ for $j = 1, \dots, n$. Moreover, we have*

$$\|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j f\|_{L^1(\mathbb{R}^n)} \approx \|f\|_{H^1}.$$

We can generalize the condition of the pointwise smoothness of K to an integral condition, called the Hörmander condition, namely, replace Condition 2 in Definition 2.3.1 by

$$\sup_{y \neq 0} \int_{2|y| \leqslant |x|} |K(x-y) - K(x)| dx < \infty. \tag{8}$$

Theorem 2.3.2 still holds for these K – see [107, Chapter II Section 3].

We now move on to a more general form of singular integral operators. The following definition is taken from [107, Chapter VII Section 3.2] or [47, Section 4.1.1].

Definition 2.3.5. *Let $0 < \delta \leqslant 1$. We say T is a singular integral operator associated by a δ-kernel K if K, defined on $\mathbb{R}^{2n} \backslash \{(x,x) : x \in \mathbb{R}^n\}$, is locally integrable, and there is a constant $C > 0$ such that*

1. $|K(x,y)| \leqslant C|x-y|^{-n}$ *for all x;*
2. $|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C|y-z|^{\delta}|x-y|^{-n-\delta}$ *if* $2|y-z| \leqslant |x-y|$*;*
3. *the operator T is given (formally) by*

$$\langle Tf, g \rangle = \int \int K(x,y) f(y) g(x) dy dx, \qquad \forall\ f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with disjoint supports}$$

and extends to a bounded operator on $L^2(\mathbb{R}^n)$.

Theorem 2.3.6 ([47, Section 4.2])**.** *The conclusions of Theorem 2.3.2 also hold for a Calderón–Zygmund singular integral operator T associated with a δ-kernel.*

Here is also a good place to introduce strongly singular integral operators. Strongly singular integral operators of convolution type were introduced by Fefferman in [37].

Definition 2.3.7 ([3])**.** *Let $0 < \delta \leqslant 1$ and $0 < \alpha \leqslant 1$. We say T is a strongly singular integral operator associated by K if K, defined on $\mathbb{R}^{2n} \backslash \{(x,x) : x \in \mathbb{R}^n\}$, is locally integrable, and there is a constant $C > 0$ such that*

1. $|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C|y-z|^{\delta}|x-y|^{-n-\frac{\delta}{\alpha}}$ *if* $2|y-z|^{\alpha} \leqslant |x-y|$*;*
2. *the operator T is given by*

$$\langle Tf, g \rangle = \int \int K(x,y) f(y) g(x) dy dx, \qquad \forall\ f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with disjoint supports}$$

and extends to a bounded operator on $L^2(\mathbb{R}^n)$;

3. *there exists* $\frac{n(1-\alpha)}{2} \leqslant \beta < \frac{n}{2}$ *such that both* T *and its adjoint* T^* *can be extended from* $L^q(\mathbb{R}^n)$ *to* $L^2(\mathbb{R}^n)$ *with* $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}$.

Álvarez and Milman proved the following boundedness for a strongly singular integral operator.

Theorem 2.3.8 ([3, Theorem 2.1, 2.2])**.**
Let T *be a strongly singular integral operator. Then,*

1. *it can be extended to a bounded operator from* $L^\infty(\mathbb{R}^n)$ *to* $\mathrm{BMO}(\mathbb{R}^n)$*;*
2. *if we further assume that* $T^*(1) = 0$, T *can be extended to a bounded operator on* H^p *for* $p_0 < p \leqslant 1$, *where*
$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta(\frac{\delta}{\alpha} + \frac{n}{2})}{n(\frac{\delta}{\alpha} - \delta + \beta)}.$$

An example of a strongly singular integral operator given by Álvarez and Milman is a class of *pseudo-differential operators*. Pseudo-differential operators are closely related to the non-homogeneous Hardy space, which will be discussed in Section 2.5.3.

2.4 Commutators

This section provides definitions of commutators as well as some boundedness results.

Definition 2.4.1. *Let* T *be an operator. Let* b *be a function. The commutator of* T *with* b *is formally defined to be*
$$[b, T](f) := b(Tf) - T(bf).$$

The story of commutators of Calderón–Zygmund singular integral operator with BMO starts from the outstanding paper by Coifman, Rochberg, and Weiss.

Theorem 2.4.2 ([21])**.**
Let T *be a singular integral operator and* $b \in \mathrm{BMO}(\mathbb{R}^n)$*. Then the operator* $[b, T]$*, originally defined on* $\mathcal{S}(\mathbb{R}^n)$*, can be extended to a continuous operator on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$*, and* $\|[b, T]\|_{L^p \to L^p} \lesssim \|b\|_{\mathrm{BMO}}$*. Conversely, if* $[b, R_j]$ *for* $j = 1, \cdots, n$ *are bounded on some* $L^p(\mathbb{R}^n)$ *with* $1 < p < \infty$*, then* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $\|b\|_{\mathrm{BMO}} \lesssim \sum_{j=1}^n \|[b, R_j]\|_{L^p \to L^p}$*.*

Another significant result is the compactness of the commutator.

Theorem 2.4.3 ([116])**.**
Let $b \in \mathrm{BMO}(\mathbb{R}^n)$*. Then* $b \in \mathrm{CMO}(\mathbb{R}^n)$ *if and only if* $[b, T]$ *is a compact operator on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty$ *and for all Calderón–Zygmund singular integral operators* T*.*

However, Pérez [93] gave an example that $[b, T]$ is not of weak type $(1, 1)$. Moreover, there is no hope of having $H^1 - L^1$ type boundedness for general b and any singular integral operator T by the following theorem.

Theorem 2.4.4. [55, Theorem 3.1] *Let* b *be a locally integrable function. Then the following are equivalent:*

1. $[b, H]$ *is bounded from* $L^\infty_c(\mathbb{R})$ *to* $\mathrm{BMO}(\mathbb{R})$.

2. $[b, H]$ *is bounded from* $H^1(\mathbb{R})$ *to* $L^1(\mathbb{R})$.

3. The function b *is constant almost everywhere.*

Here, L^∞_c *is the space of all bounded functions with compact support.*

Earlier, Pérez introduced a new type of atoms instead of $H^1(\mathbb{R}^n)$ atoms.

Definition 2.4.5 ([93], Definition 1.3)**.** *For* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *a* b*-atom is a function* a *such that for some cube* Q

(i) $\mathrm{supp}(a) \subset Q$;

(ii) $\|a\|_{L^\infty} \leqslant |Q|^{-1}$;

(iii) $\int a = 0$ *and* $\int ab = 0$.

Pérez then proceeded to define an atomic space, which he called H^1_b (Ky [71] uses $\mathcal{H}^1_b$ for this space) by taking all $f \in L^1$ which can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j are b-atoms and $\{\lambda_j\} \in \ell^1$. Note that since b-atoms are also H^1 atoms, such a decomposition will converge in the H^1 norm with $\|f\|_{H^1} \leqslant \sum |\lambda_j|$, and hence Pérez's space is contained in H^1.

Perez [93] proved that $[b, T](a) \in L^1$ for all b-atoms a. However, this does not imply automatically that $[b, T](f) \in L^1$ where $f \in H^1_b$ because we do not know that the norm of a finite sum of b-atoms is equivalent to the norm of an infinite sum of b-atoms (i.e. that a version of Theorem 2.1.14 holds in this case).

Afterwards, Ky [71] introduced another approach, which is using an appropriate maximal function.

Definition 2.4.6 ([71], Definition 2.2)**.** *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *be nontrivial. The space* $H^1_b(R^n)$ *consists of* $f \in H^1(\mathbb{R}^n)$ *such* $[b, \mathfrak{M}_{nt}]f \in L^1(\mathbb{R}^n)$, *where*

$$[b, \mathfrak{M}_{nt}]f(x) := \mathfrak{M}_{nt}[b(x)f(\cdot) - b(\cdot)f(\cdot)](x)$$

and $\mathfrak{M}_{nt}$ *is the non-tangential grand maximal function defined by*

$$\mathfrak{M}_{nt}f(x) := \sup\{|f * \phi_t(y)| : |y - x| < t, \varphi \in \mathcal{A}\}$$

with

$$\mathcal{A} = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \|\phi\|_\infty + \|\nabla\phi\|_\infty \leqslant (1 + |x|^2)^{-n-1}\}.$$

The norm on H^1_b *is given by* $\|f\|_{H^1_b} := \|f\|_{H^1}\|b\|_{\mathrm{BMO}} + \|[b, \mathfrak{M}_{nt}]f\|_{L^1}$.

Ky showed that $H^1_b(R^n)$ is the biggest space on which the commutator is bounded, and that it contains the Pérez space.

We will end this section by introducing several more operators that are related to the discussion of commutators. The following two maximal functions were introduced in [21, Section 6] in the one-dimensional case. Setting

$$M(b, f)(x) := \sup_{I \ni x} \big|(b(x) - b_I)f_I\big|,$$

it was shown in [21, Theorem IX] that $b \in \mathrm{BMO}(\mathbb{R})$ if and only if $f \mapsto M(b,f)$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. The proof uses [21, Theorem X] which states that for a different maximal function,

$$N(b,f)(x) := \sup_{I \ni x} |(bf)_I - b(x) f_I|,$$

$f \mapsto N(b,f)$ is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ when $b \in \mathrm{BMO}(\mathbb{R})$.

The commutator of the Hardy-Littlewood maximal operator M (see (2)) with multiplication by b, that is

$$[M,b](f)(x) := M(bf)(x) - b(x)Mf(x),$$

was studied in [1,6,88] and shown to be bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$ and b is bounded below.

Finally, one has the following maximal function,

$$M_b f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)| dy, \tag{9}$$

which was studied in [1,41,102] and is sometimes denoted by C_b. In this case, as in [21], one has that M_b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Moreover, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, [1, Corollary 1.11] gives the pointwise domination $M_b f(x) \lesssim \|b\|_{\mathrm{BMO}} M^2 f(x)$ for $f \in L^1_{loc}(\mathbb{R}^n)$, and therefore

$$\|M_b\|_{L^p \to L^p} \lesssim \|b\|_{\mathrm{BMO}}. \tag{10}$$

2.5 Non-homogeneous Analogues

In this section, we will discuss the non-homogeneous analogues of Hardy spaces, bounded mean oscillations, singular integral operators (including pseudo-differential operators) and their commutators. We will separate them into subsections.

2.5.1 Local Hardy spaces

The local Hardy spaces $h^p(\mathbb{R}^n)$ $(p > 0)$ are introduced by Goldberg [45,46]. As in the case of the usual Hardy spaces, local Hardy spaces have different characterizations.

Theorem 2.5.1. [46, Theorem 1] *For $0 < p < \infty$, the following are equivalent:*

1. *For a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \psi \neq 0$, the maximal function*

$$\mathcal{M}_\psi(f)(x) := \sup_{0<t<1} |\psi_t * f(x)|$$

is in $L^p(\mathbb{R}^n)$.

2. *There exists $N \in \mathbb{N}$ such that*

$$\mathcal{M}_{\mathcal{F}_N}(f)(x) := \sup\{\mathcal{M}_\phi(f)(x) : \phi \in \mathcal{S}_{\mathcal{F}}\} \in L^p(\mathbb{R}^n),$$

where $\mathcal{S}_{\mathcal{F}_N} := \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta(\phi)(x)| \leqslant 1, \forall\ |\alpha|, |\beta| \leqslant N\}$.

In this case, $\|\mathcal{M}_\psi(f)\|_{L^p} \approx \|\mathcal{M}_{\mathcal{F}_N}(f)\|_{L^p}$.

Definition 2.5.2. *We say $f \in h^p(\mathbb{R}^n)$ if $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfies one of the conclusion in Theorem 2.5.1, and the h^p norm of f is defined to be the corresponding L^p norm.*

That is, if $f \in h^p(\mathbb{R}^n)$, we have

$$\|f\|_{h^p} \approx \|\mathcal{M}_\psi f\|_{L^p} \approx \|\mathcal{M}_{\mathcal{F}_N}(f)\|_{L^p},$$

where $N = N_p + 1$.

We shall also remark that Goldberg provided equivalent definitions using the Poisson kernel of the strip $\mathbb{R}^n \times (0,1)$.

Comparing Theorem 2.1.1 with 2.5.1, we can see that we localize the variable t from $(0,\infty)$ to $(0,1)$ in the definition of the maximal function and therefore we can conclude that $H^p(\mathbb{R}^n) \subset h^p(\mathbb{R}^n)$. Still, $H^p(\mathbb{R}^n) = h^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$. We will see that the definition of h^p allows us to contain more elements than those with exact cancellations.

Example: Write $\phi(x) = e^{-x^2}$ and $\phi_r(x) = \frac{1}{r}\phi(\frac{x}{r})$. Consider $\mathcal{M}_\phi(\phi_r)$ and $M_\phi(\phi_r)$. Using the Fourier transform, we can see that

$$\phi_t * \phi_r(x) = \frac{1}{\sqrt{t^2+r^2}} e^{-\frac{x^2}{t^2+r^2}}.$$

Furthermore,

$$\sup_{0<t<1} \phi_t * \phi_r(x) = \begin{cases} \frac{1}{\sqrt{1+r^2}} e^{-\frac{x^2}{r^2+1}}, & \text{if } x^2 > \frac{1+r^2}{2}, \\ \frac{1}{\sqrt{2e}|x|}, & \text{if } \frac{r^2}{2} \leqslant x^2 \leqslant \frac{1+r^2}{2}, \\ \frac{1}{r} e^{-\frac{x^2}{r^2}}, & \text{if } x^2 < \frac{r^2}{2} \end{cases}$$

while

$$\sup_{t>0} \phi_t * \phi_r(x) = \begin{cases} \frac{1}{\sqrt{2e}|x|}, & \text{if } x^2 \geqslant \frac{r^2}{2}, \\ \frac{1}{r} e^{-\frac{x^2}{r^2}}, & \text{if } x^2 < \frac{r^2}{2} \end{cases}.$$

One can see that the first one is in $L^1(\mathbb{R})$, and the latter one is not in $L^1(\mathbb{R})$. Moreover, we can compute that

$$\|\mathcal{M}_\phi(\phi_r)\|_{L^1(\mathbb{R})} = \frac{\sqrt{\pi}}{2} + \frac{1}{\sqrt{2e}} \log(1 + r^{-2}).$$

From this calculation, we can see that the exact cancellation property no longer holds for h^1, and, as we will see later, the "log" term plays an important role here.

In fact, the range of $0 < t < 1$ in Defintion 2.5.2 is not mandatory; one can also consider $0 < t < T$ and get the same set of functions – see [34] for $h^1(\mathbb{R}^n)$. For $h^p(\mathbb{R}^n)$, we introduced the following maximal function.

Definition 2.5.3 ([32])**.** *Given $0 < T < \infty$ and $x \in \mathbb{R}^n$, consider the family*

$$\mathcal{F}_k^{T,x} = \left\{ \phi \in C^\infty(\mathbb{R}^n) : \ \operatorname{supp}(\phi) \subset B(x,t), \ 0 < t < T \text{ and } \|\partial^\alpha \phi\|_{L^\infty} \leqslant t^{-n-|\alpha|} \text{ for all } |\alpha| \leqslant k \right\}.$$

We define the local grand maximal function associated to the family $\mathcal{F}_k^{T,x}$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\mathcal{M}_{\mathcal{F}_k}(f)(x) = \sup_{\phi \in \mathcal{F}_k^{T,x}} |\langle f, \phi\rangle| ,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

Lemma 2.5.4 ([32, Lemma 1])**.** *Let $f \in h^p(\mathbb{R}^n)$. If $k \in \mathbb{N}$ is such that $\frac{n}{n+k} < p \leqslant \frac{n}{n+k-1}$ (i.e. $k = N_p + 1$) then*

$$\|\mathcal{M}_{\mathcal{F}_k}(f)\|_{L^p} \leqslant C_{n,p,T} \|f\|_{h^p},$$

where $C_{n,1,T} \lesssim 1 + \max\{0, \log T\}$ for $p = 1$ and $C_{n,p,T} \lesssim \max\{1, T^{n(1/p-1)}\}$ for $p < 1$.

Before we provide more characterizations of h^p, we give some of the properties shown by Goldberg.

Theorem 2.5.5 ([46])**.** *Suppose $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in h^p(\mathbb{R}^n)$. Then, $\phi f \in h^1(\mathbb{R}^n)$. In particular, if $\phi \in C_c^\infty(\mathbb{R}^n)$, $\phi f \in h^1(\mathbb{R}^n)$.*

From this, some authors may suggest the name "localizable Hardy spaces" for h^p because we can localize the distribution without leaving the space. In contrast, the usual Hardy spaces H^p are not localizable because multiplying by a C_c^∞ function may destroy the cancellation property (Corollary 2.1.11). Moreover, Theorem 2.5.5 allows us to consider the local Hardy spaces on manifolds and domains.

The local Hardy space $h^p(\mathbb{R}^n)$ is not only closed under multiplication by $\mathcal{S}(\mathbb{R}^n)$ functions, but also contains $\mathcal{S}(\mathbb{R}^n)$ as a dense subset.

Theorem 2.5.6 ([46])**.** *We have $\mathcal{S}(\mathbb{R}^n) \subset h^p(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$ for all $0 < p < \infty$.*

We also have atomic decomposition and molecular decomposition for h^1. Before stating the theorems, we shall introduce the atoms and molecules.

Definition 2.5.7. *Let $0 < p \leqslant 1 \leqslant q \leqslant \infty$ with $p < q$. We say a is an (p, q) atom for $h^p(\mathbb{R}^n)$ if there exists a ball $B = B(x_0, r)$ such that*

1. $\operatorname{supp}(a) \subset B$;
2. $\|a\|_{L^q(B)} \leqslant |B|^{-\frac{1}{p}+\frac{1}{q}}$;
3. *if* $r < 1$, $\displaystyle\int_{\mathbb{R}^n} (x - x_0)^\alpha a(x)dx = 0$ *for all* $|\alpha| \leqslant N_p$.

Theorem 2.5.8 ([46, Lemma 5])**.** *Let $0 < p \leqslant 1$ and $f \in h^p(\mathbb{R}^n)$. Then, there exist a sequence $\{\gamma_j\}_j \in \ell^p$ and a sequence of (p, ∞) atoms for h^p, $\{a_j\}_j$, such that $f = \sum_{j=1}^\infty \gamma_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and in $h^p(\mathbb{R}^n)$; moreover, $\|f\|_{h^p} \approx \inf(\sum_{j\in\mathbb{N}} |\gamma_j|^p)^{\frac{1}{p}}$, where the infimum is taken over all such possible decomposition of f.*

Comparing Theorem 2.1.9 and 2.5.8, we can see that the definition of atom for h^p has the additional restriction *if* $r < 1$ for Item (3).

From the atomic decomposition, we have the following property.

Proposition 2.5.9. *For $0 < p \leqslant 1 \leqslant q \leqslant \infty$ and $p < q$, the space $h^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ is dense in $h^p(\mathbb{R}^n)$.*

Unlike for the spaces H^p, Corollary 2.1.11 is no longer true for $h^p(\mathbb{R}^n)$ by observing that we do not require the cancellation condition when $r(B) \geqslant 1$. For example, the function $\frac{1}{2023}\chi_{[0,2023]}$ will be an $h^1(\mathbb{R})$ atom.

We are now ready to prove Lemma 2.5.4.

Proof of Lemma 2.5.4.
The proof is taken from the proof of [32, Lemma 1]. Since the atomic decomposition converges in the sense of distributions, and $\mathcal{M}_{\mathcal{F}_k}$ is sub-linear, it suffices to prove that

$$\|\mathcal{M}_{\mathcal{F}_k}(a)\|_{L^p} \leqslant C$$

for a (p, ∞) atom a supported on some ball $B = B(x_0, r) \subset \mathbb{R}^n$. Indeed, writing $f = \sum_{j\in\mathbb{N}} \lambda_j\, a_j$, this will give

$$\|\mathcal{M}_{\mathcal{F}_k}(f)\|_{L^p} \leqslant \left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\, \|\mathcal{M}_{\mathcal{F}_k}(a)\|_{L^p}^p\right)^{1/p} \leqslant C\left(\sum_{j\in\mathbb{N}} |\lambda_j|^p\right)^{1/p}$$

and we can take the decomposition so that the right-hand-side is bounded by a constant multiple of $\|f\|_{h^p}$.

So fix a and split

$$\|\mathcal{M}_{\mathcal{F}_k}(a)\|_{L^p}^p = \int_{B(x_0,2r)} [\mathcal{M}_{\mathcal{F}_k}(a)(x)]^p dx + \int_{\mathbb{R}^n\setminus B(x_0,2r)} [\mathcal{M}_{\mathcal{F}_k}(a)(x)]^p dx.$$

To deal with the first integral, note that for any $\phi \in \mathcal{F}_k^{T,x}$ one has

$$\left|\int a\phi\right| \leqslant \|a\|_{L^\infty}\, \|\phi\|_{L^\infty}\, |B(x_0, r) \cap B(x, t)| \leqslant C_n\, r^{-\frac{n}{p}}.$$

Then

$$\int_{B(x_0,2r)} [\mathcal{M}_{\mathcal{F}_k}(a)(x)]^p dx \leqslant C_{n,p}\, r^{-n}\, |B(x_0, 2r)| \simeq C_{n,p}.$$

When $x \notin B(x_0, 2r)$, note that $\int a\phi$ vanishes unless $B(x, t) \cap B(x_0, r) \neq \varnothing$, hence $r \leqslant \frac{|x-x_0|}{2} < t < T$. Thus, if $r \geqslant 1$ we have

$$\left|\int a\phi\right| \leqslant \|a\|_{L^1} \|\phi\|_{L^\infty} \leqslant C_n\, r^{n\left(1-\frac{1}{p}\right)} t^{-n} \leqslant C_n|x - x_0|^{-n},$$

and therefore

$$\int_{\mathbb{R}^n\setminus B(x_0,2r)} [\mathcal{M}_{\mathcal{F}_k}(a)(x)]^p dx \lesssim \int_{2r<|x-x_0|<2T} |x-x_0|^{-np} dx \lesssim \int_{2<|x-x_0|<2T} |x-x_0|^{-np} dx < \infty.$$

Note that the integral on the right is of the order of $\log T$ when $p = 1$ and $T^{n(1-p)}$ when $p < 1$.

For $0 < r < 1$, we have the standard $H^p(\mathbb{R}^n)$ argument, using the moment conditions of a up to the order $N_p = k-1$ and the Taylor expansion of $\phi \in \mathcal{F}_k^{T,x}$ to write

$$\begin{aligned}\left|\int a(y)\phi(y)dy\right| &= \left|\int \left[\phi(y) - \sum_{|\alpha|\leqslant k-1} C_\alpha\, \partial^\alpha \phi(x-x_0)(y-x_0)^\alpha\right] a(y)dy\right| \\ &\leqslant \sum_{|\alpha|=k} C_\alpha\, \|\partial^\alpha \phi\|_{L^\infty}\, r^{|\alpha|+n}\, \|a\|_{L^\infty} \\ &\leqslant C_n\, t^{-n-k}\, r^{k+n\left(1-\frac{1}{p}\right)}.\end{aligned}$$

Then

$$\int_{\mathbb{R}^n\setminus B(x_0,2r)} [\mathcal{M}_{\mathcal{F}_k}(a)(x)]^p dx \leqslant C_{n,p}\, r^{kp+np-n} \int_{|x-x_0|>2r} |x-x_0|^{p(-k-n)} dx < \infty,$$

since $p > n/(n+k)$. □

There are different definitions of atoms for $h^p(\mathbb{R}^n)$. Komori defined an $h^p(\mathbb{R}^n)$ atom, for $\frac{n}{n+1} < p < 1$, as follows.

Definition 2.5.10 ([69]).
We say a is an $(h^p, 1)$ atom for $\frac{n}{n+1} < p < 1$ if

- $\operatorname{supp}(a) \subset B(x_0, r)$ *for some* $x_0 \in \mathbb{R}^n$ *and* $r > 0$;
- $\|a\|_{L^\infty} \leqslant r^{-\frac{n}{p}}$;
- $|\int a| \leqslant 1$.

In other words, there is room for non-vanishing moment conditions for an $h^p(\mathbb{R}^n)$ atom. However, one cannot include $p = 1$ in the definition. We will explain this in Section 4.1. For $p = 1$, Dafni and Yue gave the following approximate cancellation conditions on atoms for $h^1(\mathbb{R}^n)$.

Definition 2.5.11 ([34]). *Fix $R > 0$, $1 < q \leqslant \infty$. We say a is an R-approximate $(1, q)$ atom if*

- $\operatorname{supp}(a) \subset B(x_0, r)$ *for some* $x_0 \in \mathbb{R}^n$ *and* $r > 0$;
- $\|a\|_{L^q} \leqslant |B(x_0, r)|^{\frac{1}{q}-1}$;
- $|\int a| \leqslant [\log(1 + \frac{R}{r})]^{-1}$.

Next, we discuss molecular theory. Unlike for the H^p spaces, the molecular theory for h^p was not fully understood before the work [31]. The first discussion of molecular decomposition for $h^p(\mathbb{R}^n)$ is by Komori [69].

Definition 2.5.12 ([69]).
Let $\alpha > \gamma_p$ and $\frac{n}{n+1} < p < 1$. We say M is an $(h^p, 1, \alpha)$ molecule centered at x_0 if there is $r > 0$ such that

- $\displaystyle\int_{|x-x_0|\leqslant 2r} |M(x)|dx \leqslant r^{n(1-p^{-1})}$;

- $\displaystyle\int_{|x-x_0|\geqslant 2r} |M(x)||x-x_0|^{\alpha}dx \leqslant r^{\alpha+n(1-p^{-1})}$;
- $\displaystyle\left|\int_{\mathbb{R}^n} M(x)dx\right| \leqslant 1.$

We will introduce atoms and molecules for general p in Chapter 4.

Finally, we end this subsection by introducing Hardy's inequality for $h^1(\mathbb{R})$. Goldberg claimed this inequality in [45] without proof, and it was proved by Dafni and Liflyand [33]. The n-dimensional version is stated in Theorem 4.2.1.

Theorem 2.5.13 ([33, Theorem 1])**.** *For any $f \in h^1(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \frac{|\widehat{f}(\xi)|^2}{1+|\xi|} \lesssim \|f\|_{h^1(\mathbb{R})}.$$

2.5.2 Non-homogeneous BMO

We start with the definition of non-homogeneous BMO.

Definition 2.5.14. *Let $b \in L^1_{loc}(\mathbb{R}^n)$. We say $b \in \mathrm{bmo}(\mathbb{R}^n)$, the non-homogeneous* BMO *if*

$$\|b\|_{\mathrm{bmo}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|}\int_B |b(x)-c_B|dx < \infty, \tag{11}$$

where the constant c_B is given by

$$c_B := \begin{cases} b_B = \displaystyle\fint_B b, & \text{if } r(B) < 1, \\ 0, & \text{if } r(B) \geqslant 1. \end{cases} \tag{12}$$

for each ball B, and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

We will continue to use this notation when b is a fixed bmo function.

We remark that

$$\sup_B \fint_B |b(x)-c_B|dx \approx \sup_{r(B)<1} \fint_B |b(x)-b_B|dx + \sup_{r(B)\geqslant 1} \fint_B |b(x)|dx.$$

Therefore, this explains why we call the space non-homogeneous. In some contexts, the space $\mathrm{bmo}(\mathbb{R}^n)$ may be called as "local BMO"; however, it is more appropriate to "local BMO" to be the space of $b \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\|b\|_{\mathrm{BMO_{loc}}(\mathbb{R}^n)} := \sup_{r(B)<R} \fint_B |b(x)-b_B|dx < \infty$$

for some fixed R – see [114]. This space is strictly larger than BMO and $\dot{\Lambda}_\gamma \subset \mathrm{BMO}_{loc}$. Also, there is a space called "small bmo", which is related to the product setting, see [24].

We observe that the constant function c has bmo norm c. As a result, unlike BMO, we do not need to take bmo modulo all the constant functions to obtain a Banach space. In other words,

Theorem 2.5.15. *The space* $(\mathrm{bmo}(\mathbb{R}^n), \|\cdot\|_{\mathrm{bmo}(\mathbb{R}^n)})$ *is a complete normed space.*

Let us state some basic properties of $\mathrm{bmo}(\mathbb{R}^n)$.

Proposition 2.5.16.

1. *(*[34, Lemma 6.1]*) Let* $b \in \mathrm{bmo}(\mathbb{R}^n)$. *The averages* $|b_B| \lesssim \dfrac{\|b\|_{\mathrm{bmo}}}{\log(1+[r(B)]^{-1})}$.
2. *Let* $b \in \mathrm{bmo}(\mathbb{R}^n)$. *For any* $x_0 \in \mathbb{R}^n$, $r > 0$, $\delta > 0$, *and* $1 \leqslant p < \infty$,
$$r^\delta \int_{\mathbb{R}^n} \frac{|b(x) - c_{B(x_0,r)}|^p}{(r + |x - x_0|)^{n+\delta}} dx \lesssim_{n,p,\delta} \|b\|^p_{\mathrm{bmo}(\mathbb{R}^n)}.$$
3. *We have the relations* $L^\infty(\mathbb{R}^n) \subset \mathrm{bmo}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ *and these inclusions are proper.*

Proof of Proposition 2.5.16.

1. Since we will use this argument later, it is good to provide proof. Let $b \in \mathrm{bmo}(\mathbb{R}^n)$ and B be a ball in $\mathbb{R}^n$. If $r(B) \geqslant 1$, then $|b_B| \leqslant \frac{1}{|B|}\int_B |b(x)|dx \leqslant \|b\|_{\mathrm{bmo}(\mathbb{R}^n)}$ and we use the fact that $\frac{1}{\log(1+r^{-1})}$ is an increasing function on $\mathbb{R}$. When $r(B) < 1$, there exists $N \in \mathbb{N}$ such that $2^{N-1}r(B) < 1 \leqslant 2^N r(B)$. With this N, we obtain
$$\begin{aligned}
|b_B| &\leqslant \sum_{j=1}^N |b_{2^j B} - b_{2^{j-1}B}| + |b_{2^N B}| \\
&\leqslant \sum_{j=1}^N |b_{2^j B} - b_{2^{j-1}B}| + \|b\|_{\mathrm{bmo}(\mathbb{R}^n)} \\
&\leqslant \sum_{j=1}^N 2\|b\|_{\mathrm{bmo}(\mathbb{R}^n)} + \|b\|_{\mathrm{bmo}(\mathbb{R}^n)} \\
&\leqslant (2N+1)\|b\|_{\mathrm{bmo}(\mathbb{R}^n)} \lesssim \frac{\|b\|_{\mathrm{bmo}(\mathbb{R}^n)}}{\log(1+[r(B)]^{-1})}.
\end{aligned}$$
We have used the estimate (6) for small balls.
2. This proof is from the work [30]. By translation invariance of $\mathrm{bmo}(\mathbb{R}^n)$, we may assume $x_0 = 0$. Denote $B(0, 2^k r)$ by B_k, $k = 0, 1, 2, \ldots$. Then using (14) and the fact that for $2^j r < 1$
$$|c_{B_j} - c_{B_{j-1}}| = |b_{B_j} - b_{B_{j-1}}| \leqslant 2^n \|b\|_{\mathrm{bmo}} \tag{13}$$
and Item 1, we have
$$\begin{aligned}
&r^\delta \int_{\mathbb{R}^n \setminus B_0} \frac{|b(x) - c_{B_0}|^p}{|x|^{n+\delta}} dx \\
&\lesssim \sum_{k=0}^\infty 2^{-k\delta} |B_k|^{-1} \int_{B_{k+1}\setminus B_k} \left(|b(x) - c_{B_{k+1}}|^p + |c_{B_{k+1}} - c_{B_0}|^p\right) dx \\
&\lesssim \sum_{k=0}^\infty 2^{-k\delta} \|b\|^p_{\mathrm{bmo}} (1 + (\min\{k+1, \log_+ r^{-1}\})^p) dx \lesssim \|b\|^p_{\mathrm{bmo}}.
\end{aligned}$$

3. The first inclusion is proper by considering $\log(\frac{1}{|x|})$ on $[-1,1]$. We can see that $\log|x|$ is not in bmo since as $R \to \infty$,

$$\frac{1}{R}\int_1^{R+1} |\log(x)|dx = (1+R^{-1})\log(R+1) - 1 \to \infty.$$

$\square$

We also have a John-Nirenberg inequality for $\mathrm{bmo}(\mathbb{R}^n)$.

Theorem 2.5.17 ([34, Theorem 3.1])**.**
For $b \in \mathrm{bmo}(\mathbb{R}^n)$, there exist $C, c > 0$ such that for any ball B,

$$|\{x \in B : |b(x) - c_B| > \lambda\}| \leqslant C|B|e^{-\lambda c/\|b\|_{\mathrm{bmo}}}$$

for all $\lambda > 0$. As a result, one gets that

$$\|b\|_{\mathrm{bmo}} \approx \|b\|_{\mathrm{bmo}^p} := \left(\sup_B \fint_B |b(x) - c_B|^p dx\right)^{1/p} \tag{14}$$

for $1 \leqslant p < \infty$, with constants depending on p.

Goldberg proved that the dual of $h^1(\mathbb{R}^n)$ is the space $\mathrm{bmo}(\mathbb{R}^n)$.

Theorem 2.5.18 ([46, Corollary 1])**.** *We have $(h^1(\mathbb{R}^n))^* \cong \mathrm{bmo}(\mathbb{R}^n)$ in the sense of Theorem 2.2.6.*

Similarly to the usual BMO case, we have the following definition.

Definition 2.5.19.

1. *The space $\mathrm{vmo}(\mathbb{R}^n)$ is the $\mathrm{bmo}(\mathbb{R}^n)$-closure of all bounded uniformly continuous functions.*

2. *The space $\mathrm{cmo}(\mathbb{R}^n)$ is the $\mathrm{bmo}(\mathbb{R}^n)$-closure of $C_c^\infty(\mathbb{R}^n)$.*

Theorem 2.5.20. [29] *Let $b \in \mathrm{bmo}(\mathbb{R}^n)$.*

1. *A function $b \in \mathrm{vmo}(\mathbb{R}^n)$ iff b satisfies (7).*

2. *A function $b \in \mathrm{cmo}(\mathbb{R}^n)$ iff b satisfies (7) and*

$$\lim_{R\to\infty} \sup_{\substack{|B|\geqslant 1 \\ B\subset(B(0,R))^c}} \fint_B |b| = 0. \tag{15}$$

One should notice that vmo in [29] is cmo in this thesis. Also,

Theorem 2.5.21 ([29])**.** *We have $(\mathrm{cmo}(\mathbb{R}^n))^* \cong h^1(\mathbb{R}^n)$.*

Therefore, we have $\mathrm{cmo}(\mathbb{R}^n) \subset \mathrm{vmo}(\mathbb{R}^n) \subset \mathrm{bmo}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ (in the sense of sets). As for BMO, if $f, g \in \mathrm{bmo}(\mathbb{R}^n)$, we do not expect that $fg \in \mathrm{bmo}(\mathbb{R}^n)$. One needs to consider a smaller space.

Definition 2.5.22. *Let $b \in L^1_{loc}(\mathbb{R}^n)$.*

1. *We say $b \in \mathrm{LMO}_{loc}(\mathbb{R}^n)$ if*

$$\|b\|_{\mathrm{LMO}_{loc}(\mathbb{R}^n)} := \sup_{r(B)<1} \frac{[\log(1 + r(B)^{-1})]}{|B|} \int_B |b(x) - b_B| dx < \infty.$$

2. *We define $\mathrm{lmo}(\mathbb{R}^n) := \mathrm{LMO}_{loc}(\mathbb{R}^n) \cap \mathrm{bmo}(\mathbb{R}^n)$ and*

$$\|b\|_{\mathrm{lmo}(\mathbb{R}^n)} := \|b\|_{\mathrm{LMO}_{loc}(\mathbb{R}^n)} + \sup_{r(B)\geqslant 1} |b|_B < \infty.$$

Bonami and Fetou [9] showed that the pointwise multipliers of $\mathrm{bmo}(\mathbb{R}^n)$ are the elements of $L^\infty(\mathbb{R}^n) \cap \mathrm{lmo}(\mathbb{R}^n)$. Furthermore, the space $\mathrm{lmo}(\mathbb{R}^n)$ is useful in PDE. In [113], coefficients with logarithmic mean oscillation condition of a parabolic equation are considered, while in [7, 8], a range of such conditions are imposed on the initial vorticity in the Euler and Navier-Stokes equations. We note that the latter articles use a little different notation (for example, bmo there refers to "local BMO" rather than Goldberg's non-homogeneous space) and an L^2 oscillation, which is equivalent to the L^1 one by the John-Nirenberg inequality for these spaces (see [104]).

From the definition of $\mathrm{lmo}(\mathbb{R}^n)$, because $[\log(1 + r(B)^{-1})]^{-1} \to 0$ as $r \to 0$, functions in $\mathrm{LMO}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfy (7) and therefore $\mathrm{lmo}(\mathbb{R}^n) \subset \mathrm{vmo}(\mathbb{R}^n)$. If $b \in \mathrm{LMO}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$, then b will also satisfy condition (15). In fact, if $|B| \geqslant 1$ and $B \subset \mathbb{R}^n \backslash B(0, R) \subset \mathbb{R}^n$, then

$$\fint_B |b| \leqslant \|b\|_{L^p(B(0,R)^c)} \to 0$$

as $R \to \infty$. Thus, $\mathrm{LMO}_{\mathrm{loc}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \subset \mathrm{cmo}(\mathbb{R}^n)$.

We give some examples for $\mathrm{lmo}(\mathbb{R}^n)$.

Example 1. The first example is $L(x) = \log\left(\log\left(\frac{2e}{|x|}\right)\right)\chi_{[-2,2]}(x)$. Brezis and Nirenberg [11] showed that $L \in \mathrm{VMO}(\mathbb{R})$. We shall show $L(x) \in \mathrm{LMO}_{\mathrm{loc}}(\mathbb{R})$. We will use the fact that for any interval I,

$$\fint_I |L(x) - L_I| dx \approx \fint_I \fint_I |L(x) - L(y)| dy dx.$$

For simplicity, we write $\theta(r) = [\log(1 + r^{-1})]^{-1}$. Let $0 < r < 1$, $x_0 \in [-2, 2]$, and $0 < \varepsilon \leqslant 1$.

<u>Case 1: if $|x_0| \geqslant (1 + \varepsilon)r$</u>

Without loss of generality, we assume $x_0 \geqslant (1 + \varepsilon)r$. Then,

$$\begin{aligned}\int_{x_0-r}^{x_0+r} \int_{x_0-r}^{x_0+r} |L(x) - L(y)| dy dx &\leqslant \int_{x_0-r}^{(x_0+r)} \int_{x_0-r}^{(x_0+r)} |x - y||L'(x_0 - r)| dy dx \\ &\leqslant C\varepsilon^{-1} r^2 [\log(\frac{2e}{\varepsilon r})] \leqslant C'\varepsilon^{-1} r^2 [\theta(\varepsilon r)].\end{aligned}$$

We have used that $|L'(x)| = [x\log(\frac{2e}{x})]^{-1}$ and it is decreasing on $(0,2]$; also the fact that there are $0 < c < C$ such that for all $x \leqslant 2$,

$$c \leqslant \frac{\log(\frac{2e}{x})}{\theta(x)} \leqslant C.$$

In particular, if $|x_0| \geqslant 2r$, we have

$$\frac{1}{(2r)^2}\int_{x_0-r}^{x_0+r}\int_{x_0-r}^{x_0+r} |L(x)-L(y)|dydx \leqslant C'\theta(r),$$

where the constant C' is independent of x_0.

Case 2: $|x_0| \leqslant 2r$

First, we can reduce to the case $[0,3r]$. If we let $I = [x_0-r, x_0+r]$ and $0 \in I$, then by evenness of L, we have

$$\begin{aligned}\int_{x_0-r}^{x_0+r}\int_{x_0-r}^{x_0+r} |L(x)-L(y)|dydx &\leqslant \int_{-3r}^{3r}\int_{-3r}^{3r} |L(x)-L(y)|dydx \\ &= 4\int_{0}^{3r}\int_{0}^{3r} |L(x)-L(y)|dydx\end{aligned}$$

and this shows that we can focus on the case $[0,r]$ (as r is arbitrarily fixed).

Now, we write $I = (0,r]$ and $I_j = (\frac{r}{2^j}, \frac{r}{2^{j-1}}]$ for $j \in \mathbb{N}$, and we have

$$\fint_I |L(x)-L_I|dx \leqslant 2\int_I |L(x)-L_{I_1}|dx.$$

Now by observing $I = \bigcup_{j=1}^{\infty} I_j$, we can write as

$$\frac{1}{|I|}\int_I |L(x)-L_{I_1}|dx \leqslant \sum_{j=1}^{\infty} 2^{-j}\left(\frac{1}{|I_j|}\int_{I_j} |L(x)-L_{I_j}|dx + |L_{I_j}-L_{I_1}|\right) =: (A).$$

For the first term, the center of I_j is $\frac{3r}{2^{j+1}}$ and the radius of I_j is $\frac{r}{2^{j+1}}$, which satisfies Case 1. Hence, we can apply Case 1 with $\varepsilon = 1$.

For the second term, write $J_k = I_{k-1} \cup I_k$ and observe that $[\log(1+r^{-1})]^{-1}$ is an increasing concave function, then for $j \geqslant 2$

$$\begin{aligned}|L_{I_j}-L_{I_1}| &\leqslant \sum_{k=2}^{j} |L_{I_k}-L_{I_{k-1}}| \\ &\leqslant C\sum_{k=2}^{j}\left[\frac{1}{|J_k|}\int_{J_k} |L(x)-L_{J_k}|dx\right] \\ &\leqslant C\sum_{k=2}^{j}\theta\big(3\cdot 2^{-k}r\big) \\ &\leqslant C(j-1)\theta\left(\sum_{k=2}^{j}\frac{3\cdot 2^{-k}r}{j-1}\right) \leqslant C(j-1)\cdot\theta(2r).\end{aligned}$$

We have used Case 1 with $\varepsilon = \frac{2}{3}$. Now, we have

$$(A) \leqslant \sum_{j=1}^{\infty} 2^{-j}\Big[[\theta(r)] + (j-1)[\theta(2r)]\Big] \leqslant C'[\theta(2r)] \leqslant C''[\theta(r)].$$

In the last line, we have used the fact that

$$1 \leqslant \frac{\theta(2r)}{\theta(r)} \leqslant 2$$

for all $r > 0$.

Therefore, if $|x_0| \leqslant 2r$, we have

$$\fint_I |L(x) - L_I| dydx \leqslant C'''\theta(r)$$

and the constant C''' is independent of x_0.

Combining both cases, there exists $C > 0$ such that

$$\fint_{[x_0-r,x_0+r]} |L(x) - L_{[x_0-r,x_0+r]}|dx \leqslant C\theta(r) = \frac{C}{\log(1+\frac{1}{r})},$$

which implies that

$$\sup_{0<r<1} \log(1+\frac{1}{r}) \fint_{[x_0-r,x_0+r]} |L(x) - L_{[x_0-r,x_0+r]}|dx \leqslant C < \infty.$$

Example 2. Following the idea in Section 3.1 in [13], let $\Lambda(x) = \sum_{j\in\mathbb{Z}} a_j L(x-n_j)$, where a_j is bounded and n_j satisfy $n_{j+1} - n_j \geqslant 2023$. Using the same argument as in the first example, we can show that $\Lambda \in \mathrm{LMO}_{\mathrm{loc}}(\mathbb{R})$. Moreover, if $\{a_j\} \in \ell^1(\mathbb{Z})$, then we can also see that for $p \geqslant 1$,

$$\|\Lambda\|_{L^p}^p = C \int_0^\infty \exp(-e^{y^{\frac{1}{p}}})dy < \infty.$$

Example 3. The function $\ell_\alpha(x) = (\log(|x|^{-1}))^\alpha \chi_{[-1,1]}$ for $\alpha \in (0,1]$ is not in $\mathrm{LMO}_{\mathrm{loc}}(\mathbb{R})$. For $\alpha \in (0,1)$, $\ell_\alpha(x) \in \mathrm{vmo}(\mathbb{R})\backslash\mathrm{LMO}_{\mathrm{loc}}(\mathbb{R})$. By considering the interval $B = (r, 2r)$ where $0 < r < \frac{1}{2}$ and using the Mean Value Theorem, we have

$$\begin{aligned}\frac{1}{r^2}\int_r^{2r}\int_r^{2r} |\ell_\alpha(x) - \ell_\alpha(y)|dydx &\geqslant \frac{1}{r^2}\int_r^{2r}\int_r^{2r} \frac{\alpha|x-y|}{(2r)[\log((2r)^{-1})]^{1-\alpha}}dxdy \\ &= \frac{\alpha}{24}\frac{1}{[\log((2r)^{-1})]^{1-\alpha}},\end{aligned}$$

which gives us

$$\lim_{r\to 0} \frac{\log(1+\frac{1}{r})}{[\log((2r)^{-1})]^{1-\alpha}} = \infty.$$

For $\alpha = 1$, it is not even in $\mathrm{vmo}(\mathbb{R})$, so it cannot be in $\mathrm{LMO}_{\mathrm{loc}}(\mathbb{R})$.

Finally, we introduce the non-homogeneous Zygmund space.

Definition 2.5.23. *The non-homogeneous Zygmund space is defined to be* $\Lambda_\gamma(\mathbb{R}^n) := \dot{\Lambda}_\gamma(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *The norm is given by*

$$\|f\|_{\Lambda_\gamma} := \|f\|_{\dot{\Lambda}_\gamma} + \|f\|_{L^\infty}.$$

Example 4. In fact, $\Lambda_\gamma(\mathbb{R}^n) \subset \mathrm{lmo}(\mathbb{R}^n)$ for all $\gamma > 0$. Let $b \in \Lambda_\gamma(\mathbb{R}^n)$. From the boundedness of b, we have

$$\sup_{r(B)\geqslant 1} \frac{1}{|B|} \int_B |b(x)|dx \leqslant \|b\|_{\Lambda_\gamma}.$$

Therefore, we need to show that $\|b\|_{\mathrm{LMO}_{\mathrm{loc}}} < \infty$. Let $B = B(x_0, r)$.

$$\begin{aligned}\log(1+r^{-1}) ⨍_B |b(y) - b_B|dy &\leqslant \log(1+r^{-1}) \frac{r^\gamma}{r^\gamma} ⨍_B |b(y) - b_B|dy \\ &\leqslant \log(1+r^{-1}) r^\gamma \|b\|_{\Lambda_\gamma}.\end{aligned}$$

Since $\log(1+r^{-1})r^\gamma \leqslant C_\gamma$ for $0 < r < 1$, we can conclude that $\|b\|_{\mathrm{LMO}_{\mathrm{loc}}} < \infty$.

As expected, Goldberg also identified the dual of h^p spaces:

Theorem 2.5.24 ([46, Theorem 5])**.**
Let $0 < p < 1$. *The dual space of* $h^p(\mathbb{R}^n)$ *is* $\Lambda_{n(p^{-1}-1)}(\mathbb{R}^n)$.

2.5.3 Pseudo-differential operators and commutators

We begin with the definition of a symbol.

Definition 2.5.25. *Let* $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, *and* $\rho, \delta \in [0,1]$. *We say* $a \in S^m_{\rho,\delta}$ *if for any* $\alpha, \beta \in \mathbb{N}_0$ *there exists* $A_{\alpha,\beta}$ *such that*

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant A_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|} \tag{16}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. *We also define the pseudo-differential operator associated with symbol* a, T_a, *to be*

$$T_a f(x) := \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

given $f \in \mathcal{S}(\mathbb{R}^n)$.

Some authors may write $OpS^m_{\rho,\delta}$ to denote the space of all pseudo-differential operators associated with a symbol $a \in S^m_{\rho,\delta}$.

We have that $T_a : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ continuously for all $\rho, \delta \in [0,1]$ – see [108, Chapter VI Section 1.3 and Chapter VII Section 1.1.1].

Two notable examples of pseudo-differential operators are multipliers and multiplication operators. More precisely, if $a(x,\xi) = m(\xi) \in S^0_{1,0}$, then $\widehat{T_a f}(\xi) = m(\xi)\widehat{f}(\xi)$, which is also known as a Fourier multiplier operator; if $a(x,\xi) = M(x) \in S^m_{1,0}$, then $T_a f(x) = M(x)f(x)$ by using the Fourier inversion formula, and this operator is known as a multiplication operator. However, the multiplication operators by functions in $\mathrm{BMO}(\mathbb{R}^n)$ or $\mathrm{bmo}(\mathbb{R}^n)$ are not pseudo-differential operators because these functions do not satisfy (16).

Pseudo-differential operators are an example of generalized singular integral operators. The proof of the following proposition can be found in [108] or [2].

Proposition 2.5.26. *Let $a \in S^m_{\rho,\delta}$ with $0 < \rho \leqslant 1$ and $0 \leqslant \delta \leqslant 1$. Let*

$$K(x,y) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} a(x,\xi)\psi(\varepsilon\xi)d\xi$$

for a $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ on $|\xi| \leqslant 1$, where the limit is taken in $\mathcal{S}'(\mathbb{R}^n)$. Then, we can write $T_a f(x) := \int_{\mathbb{R}^n} K(x,y)f(y)dy$, K is smooth away from the diagonal, and K satisfies

$$\sup_{|\alpha|+|\beta|=M} |\partial_x^\alpha \partial_y^\beta K(x,y)| \lesssim_{\alpha,\beta} |x-y|^{-\frac{M+m+n}{\rho}}$$

given that $x \neq y$ and $M \in \mathbb{N}$ such that $M+m+n > 0$. Furthermore, for any $N \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0$

$$\sup_{|x-y|\geqslant 1} |x-y|^N |\partial_x^\alpha \partial_y^\beta K(x,y)| \lesssim_{\alpha,\beta,N} 1.$$

The pseudo-differential operators associated to an $S^m_{\rho,\delta}$ symbol is a strongly singular integral operator if $0 < \delta \leqslant \rho < 1$ and $-\frac{n}{2} < m \leqslant -\frac{n(1-\rho)}{2}$ – see [3, Section 3].

We now introduce an important example of pseudo-differential operators.

Definition 2.5.27. [46] *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi \equiv 1$ on $B(0,1)$. For $j = 1, \cdots, n$, we define the j-th local Riesz transform to be*

$$r_j f := \mathcal{F}^{-1}\left(i(1-\varphi)\frac{\xi_j}{|\xi|}\widehat{f}\right), \tag{17}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

These are pseudo-differential operators with symbols in $S^0_{1,0}$ because the symbol $(1 - \varphi(\xi))\frac{\xi_j}{|\xi|}$ is smooth away from 0 and it is identically 0 in $B(0,1)$. Moreover, the symbol is bounded above by a constant for all ξ, so $m = 0$; when we differentiate the symbols, we reduce the power of ξ by 1. Thus, the symbol is in $S^0_{1,0}$.

Goldberg used these operators to characterize $h^1(\mathbb{R}^n)$.

Theorem 2.5.28 ([46, Theorem 2])**.** *A distribution $f \in h^1(\mathbb{R}^n)$ if and only if $f \in L^1(\mathbb{R}^n)$ and $r_j(f) \in L^1(\mathbb{R}^n)$ for all $j = 1, \cdots, n$.*

Later, Peloso and Secco [92] provided a characterization using local Riesz transform for some range of $p \leqslant 1$.

Next, we discuss the boundedness of pseudo-differential operators, which means there exists an extension of T such that it is a bounded operator. For $a \in S^0_{1,0}$ symbols, T_a is bounded on $L^2(\mathbb{R}^n)$, see [108, Chapter VI Section 2]. For $a \in S^0_{1,1}$, it may not be bounded on $L^2(\mathbb{R}^n)$, but it is bounded on $\Lambda_\gamma(\mathbb{R}^n)$ for $\gamma > 0$, see [108, Chapter VII Section 1.2 and 1.3]. If $a \in S^m_{1,0}$, then T_a maps $W^{s,2}(\mathbb{R}^n)$ to $W^{s-m,2}(\mathbb{R}^n)$ $(s \in \mathbb{R})$, where $W^{s,2}(\mathbb{R}^n)$ is the generalized Sobolev space and its norm is $\|f\|_{W^{s,p}} := \|(I-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^n)}$, see [100, Theorem 2.6.11]; T_a also maps $\Lambda_\gamma(\mathbb{R}^n)$ to $\Lambda_{\gamma-m}(\mathbb{R}^n)$ where $\gamma > m$.

Now we focus on the boundedness on local Hardy spaces. Let us first explain why the classical Calderón–Zygmund singular integral operators are unsuitable for h^1.

Example. Consider $f = \chi_{[-\frac{1}{2},\frac{1}{2}]}$, which is an h^1 atom on $\mathbb{R}$, and the Hilbert transform. Then we have

$$H(f)(t) = \frac{1}{\pi} \log \left| \frac{t+\frac{1}{2}}{t-\frac{1}{2}} \right|$$

and it is not in $L^1(\mathbb{R})$. Indeed, for $t \geqslant 1$, one has $Hf(t) \geqslant \frac{c}{(t-\frac{1}{2})}$ for some $c > 0$.

Next, we will record some results about the boundedness of pseudo-differential operators acting on h^p, which we will not use in later chapters.

Theorem 2.5.29.

1. [46, Theorem 4] *If $a \in S^0_{1,0}$, then T_a can be extended to a bounded operator on $h^p(\mathbb{R}^n)$ for any $0 < p \leqslant 1$.*

2. [115, Chapter 1 Section 2] *If $a \in S^0_{1,\delta}$ for $0 < \delta < 1$, then T_a can be extended to a bounded operator on $h^1(\mathbb{R}^n)$.*

3. [60, Theorem 4.1] *If $a \in S^{-n(1-\rho)/2}_{\rho,\delta}$ for $\delta \leqslant \rho \leqslant 1$ and $0 \leqslant \delta < 1$, then T_a can be extended to a bounded operator on $h^1(\mathbb{R}^n)$.*

4. [59] *If $a \in S^m_{1,\delta}$ for $-n < m \leqslant 0$, $0 \leqslant \delta < 1$, then T_a can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to $h^q(\mathbb{R}^n)$ for any $p \leqslant q \leqslant \frac{pn}{n-\alpha p}$ and $0 < p \leqslant 1$. Moreover, if $m < 0$, T_a maps continuously $h^p(\mathbb{R}^n)$ to $h^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)$ if $p < \frac{n}{-m}$; to $\mathrm{bmo}(\mathbb{R}^n)$ if $p = \frac{n}{-m}$; and to $\Lambda_{\alpha-n/p}$ if $p > \frac{n}{-m}$.*

5. [90, Theorem 4.1] *Let $0 < p \leqslant 1$. If $a \in S^m_{\rho,\rho}$ with $0 \leqslant \rho < 1$ and $m = -n(1-\rho)(p^{-1} - \frac{1}{2})$, then T_a can be extended to a bounded operator on $h^p(\mathbb{R}^n)$.*

6. [59] *Let $0 < p \leqslant 1$ and $-n < m \leqslant 0$. If $a \in S^m_{\rho,\delta}$ with $0 < \delta \leqslant \rho < 1$, then T_a can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to $h^q(\mathbb{R}^n)$ where q satisfies $-m = n[p^{-1} - q^{-1} + (1-\rho)(q^{-1} - \frac{1}{2})]$.*

7. [59] *Let $0 < p \leqslant 1$. If $a \in S^{-n(p^{-1}-\frac{1}{2})}_{0,0}$, then T_a can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.*

Finally, we state some results related to commutators of pseudo-differential operators with some functions.

Theorem 2.5.30 ([70, Theorem 1])**.** *Let b be a function such that $\nabla b \in \Lambda_\alpha$ for $0 < \alpha < 1$. Suppose $a \in S^1_{1,0}$. Then, $[b, T_a]$ acts continuously on $h^p(\mathbb{R}^n)$ for all $\frac{n}{n+\alpha} \leqslant p \leqslant 1$.*

Theorem 2.5.31. *Let $b \in L^{1,\psi}_0(\mathbb{R}^n)$ with $\psi(r) = \dfrac{(1+r^n)^\theta}{\log(e+r^{-n})}$ and $\theta \geqslant 0$.*

1. [124] *Suppose $a \in S^0_{1,\delta}$ with $0 \leqslant \delta < 1$, then $[b, T]$ is bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

2. [62] *Suppose $a \in S^m_{\rho,\delta}$ with $0 \leqslant \delta < 1$, $0 < \rho \leqslant 1$, $\delta \leqslant \rho$ and $-n-1 < m \leqslant -(n+1)(1-\rho)$, then $[b, T]$ is bounded on $h^1(\mathbb{R}^n)$.*

2.5.4 Inhomogeneous singular integral operators

Another generalization of singular integral operators that fits local Hardy spaces is called inhomogeneous Calderón–Zygmund singular integral operators, which [35] introduced.

Definition 2.5.32. *A locally integrable function K defined on $\mathbb{R}^n \times \mathbb{R}^n$ away from the diagonal is called a (μ, δ)-inhomogeneous standard kernel if there exist $\mu > 0$ and $0 < \delta \leqslant 1$ such that*

$$|K(x,y)| \leqslant C \ \min\left\{\frac{1}{|x-y|^n}, \ \frac{1}{|x-y|^{n+\mu}}\right\}, \qquad x \neq y, \tag{18}$$

and

$$|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)| \leqslant C\frac{|y-z|^\delta}{|x-z|^{n+\delta}} \tag{19}$$

for all $|x-z| \geqslant 2|y-z|$. A linear bounded operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is called inhomogeneous Calderón-Zygmund operator if

1. *T extends to a continuous operator from $L^2(\mathbb{R}^n)$ to itself;*
2. *T is associated to an $(\mu, \delta)-$inhomogeneous standard kernel given by*

$$\langle Tf, g\rangle = \int\int K(x,y)f(y)g(x)dydx, \qquad \forall \ f, g \in \mathcal{S}(\mathbb{R}^n) \textit{ with disjoint supports.}$$

Theorem 2.5.33 ([35])**.** *Let T be an inhomogeneous Calderón-Zygmund operator satisfying the pointwise controls (18) and (19). Suppose $n/(n + \min\{\delta, \mu\}) < p < 1$. If $T^*(1) \in \dot{\Lambda}_{\gamma_p}(\mathbb{R}^n)$, then T is a bounded operator from $h^p(\mathbb{R}^n)$ to itself. Moreover, if T is a bounded operator from $h^p(\mathbb{R}^n)$ to itself, then $T^*(1) \in \Lambda_{\gamma_p}(\mathbb{R}^n)$.*

This thesis will also generalize this result to wider class of kernels and for all $0 < p \leqslant 1$.

One example of a $(\mu, 1)$-inhomogeneous standard kernel is $K(x,y) = \frac{\phi(x-y)}{x-y}$, where $\phi(t) \equiv 1$ on $|t| \leqslant 1$, $\phi(t) \in (0,1)$ for $|t| \in (1,2)$, $\phi(t) \equiv 0$ on $|t| \geqslant 2$ and $\phi \in C_c^\infty(\mathbb{R})$. It is true that $K(x,y) \equiv 0$ if $|x-y| \geqslant 2$ and

$$|K(x,y)| \leqslant \frac{1}{|x-y|} \qquad \text{if } |x-y| \leqslant 2.$$

Moreover, if $2|y-z| \leqslant |x-y|$,

$$|K(x,y) - K(x,z)| \leqslant \frac{|\phi(x-y) - \phi(x-z)|}{|x-y|} + \frac{|y-z|}{|x-y|^2} \leqslant C\frac{|y-z|}{|x-y|^2}.$$

Indeed, this example is the kernel of the local (or localized) Hilbert transform introduced by Goldberg in [45]:

$$\mathcal{H}f(x) := P.V. \int K(x,y)f(y)dy. \tag{20}$$

He claimed the operator $\mathcal{H}$ characterizes $h^1(\mathbb{R})$, in the same way that the usual Hilbert transform characterizes $H^1(\mathbb{R})$. This claim was proved by Dafni and Liflyand [33]. Comparing with Theorem 2.5.28, we see that there are two different characterizations of h^1. Some natural questions arise: how are these operators related? What are sufficient conditions on the localizing functions so that those operators can characterize h^1? We will answer these questions in the next chapter.

Chapter 3

Equivalent Localization of Singular Integral Operators

This chapter will establish a relation between localization in physical spaces near 0 and frequency space away from 0. It is taken from [30] with minor changes.

Let K be a δ-kernel of convolution type (Definition 2.3.1) that satisfies (8) instead. We define two types of operators via localizations of the kernel K. On the one hand, we look at operators of the form

$$T^{\psi}(f) := T(f - \psi * f)$$

for suitable functions ψ. These are modelled on Goldberg's localized Riesz transforms in (17), with $\psi = \mathcal{F}^{-1}(\varphi)$. On the other hand, based on the definition of the local Hilbert transform (20), we look at the operators T_η, associated with the kernel $K\eta$, for a class of functions η.

We now show that under certain weak smoothness and decay conditions on η, the kernel $K\eta$ satisfies the same conditions as K and so we can associate to it an operator T_η in the same way that we associate T to K, and this operator enjoys the same boundedness properties as T.

Lemma 3.0.1. *Suppose that η is a bounded function and*

$$\sup_{y \neq 0} \int_{|x| \geqslant 2|y|} \frac{|\eta(x-y) - \eta(x)|}{|x|^n} dx < \infty.$$

Then $K\eta$ satisfies $|K(x)\eta(x)| \leqslant \|\eta\|_{L^\infty} |x|^{-n}$ and (8).

Proof of Lemma 3.0.1.
We have that $|K(x)\eta(x)| \leqslant |K(x)| \|\eta\|_{L^\infty} \leqslant \|\eta\|_{L^\infty} |x|^{-n}$, and for any $y \neq 0$, we get

$$\begin{aligned}
&\int_{|x| \geqslant 2|y|} |K(x-y)\eta(x-y) - K(x)\eta(x)| dx \\
&\leqslant \int_{|x| \geqslant 2|y|} |[K(x-y) - K(x)]\eta(x-y) + K(x)[\eta(x-y) - \eta(x)]| dx \\
&\leqslant \|\eta\|_{L^\infty} \int_{|x| \geqslant 2|y|} |K(x-y) - K(x)| dx + \int_{|x| \geqslant 2|y|} \frac{|\eta(x-y) - \eta(x)|}{|x|^n} dx \\
&\leqslant C_{K,\eta}.
\end{aligned}$$

$\square$

Note that the condition on η in Lemma 3.0.1 is satisfied whenever η is δ-Lipschitz for some $\delta > 0$, and the Lipschitz constant in $B(x, |x|/2)$ decays like $|x|^{-\delta}$. In particular, if η has compact support, then it is enough to require $\eta \in \mathrm{Lip}_\delta$.

Next we show that $K\eta$ satisfies Definition 2.3.1 Item 3. For this we need further assumptions on η guaranteeing that η has a certain decay at infinity and $\eta - 1$ vanishes sufficiently fast at the origin. Note that we do not require that $\eta \equiv 1$ in a neighborhood of the origin. For example, if $\eta \in \mathrm{Lip}_\delta$ then it suffices that $\eta(0) = 1$.

Lemma 3.0.2. *If η satisfies,*

$$\int_{0<|x|<1} \frac{|\eta(x) - 1|}{|x|^n} dx + \int_{|x|\geqslant 1} \frac{|\eta(x)|}{|x|^n} dx < \infty, \tag{21}$$

then

$$\sup_{0<r<R<\infty} \left| \int_{r<|x|<R} K(x)\eta(x)dx \right| \leqslant C_{\eta,K}.$$

Proof of Lemma 3.0.2.
Without loss of generality, we assume that $0 < r < 1 < R$. For $0 < r < R$, by the assumptions on K and η,

$$\begin{aligned}
&\left| \int_{r<|y|<R} K(y)\eta(y)dy \right| \\
&\leqslant \left| \int_{r<|y|<1} K(y)[\eta(y) - 1]dy \right| + C_K + \left| \int_{1\leqslant|y|<R} K(y)\eta(y)dy \right| \\
&\lesssim_K \int_{0<|y|<1} \frac{|\eta(y) - 1|}{|y|^n} dy + \int_{|y|\geqslant 1} \frac{|\eta(y)|}{|y|^n} dy + 1 \leqslant C_{\eta,K},
\end{aligned}$$

where C_K is the constant from Definition 2.3.1 Item 3 of K and the constant $C_{\eta,K}$ is independent of r and R. $\square$

If η satisfies the hypotheses of both lemmas, we get the boundedness of T_η on $L^2(\mathbb{R}^n)$; and hence the boundedness on $L^p(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ to weak-L^1.

Theorem 3.0.3. *Suppose η satisfies the hypotheses of Lemmas 3.0.1 and 3.0.2, and ψ satisfies*

$$\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \qquad \int \psi = 1,$$

$$\int_{|x|\geqslant 1} \left[\frac{1}{|x|^n} \int_{|y|\geqslant|x|/2} |\psi(y)|dy \right] dx < \infty \tag{22}$$

and

$$\int_{|x|\geqslant 1} \left[\int_{|y|\leqslant|x|/2} \frac{|\psi(x - y) - \psi(x)|}{|y|^n} dy \right] dx < \infty. \tag{23}$$

Then the operator T_η is bounded on $h^1(\mathbb{R}^n)$ if and only if T^ψ is bounded on $h^1(\mathbb{R}^n)$.

Note that condition (22) will hold, for example, if the function $|x|^\eta\psi(x)$ is integrable for some $\eta > 0$, while (23) will hold if $\psi \in \mathrm{Lip}_\alpha$ for $0 < \alpha \leqslant 1$ and $|x|^\alpha\|\psi\|_{\mathrm{Lip}_\alpha(B(x,|x|/2))}$ is integrable.

Proof of Theorem 3.0.3.
We will show that the operator $T_E := T_\eta - T^\psi$ is bounded on $L^1(\mathbb{R}^n)$, and from that deduce that T_E is bounded on $h^1(\mathbb{R}^n)$.

For $0 < \varepsilon < 1/2$, let $K_\varepsilon(x) = \chi_{|x|>\varepsilon}K(x)$ be the usual truncation of the kernel. Applying the same to $K\eta$, we consider $(K\eta)_\varepsilon - K_\varepsilon + K_\varepsilon * \psi$. Set

$$K_*(x) = \sup_{\varepsilon>0} |(K\eta)_\varepsilon(x) - K_\varepsilon(x) + K_\varepsilon * \psi(x)|.$$

We claim that $K_* \in L^1(\mathbb{R}^n)$.

For the local estimate, we write

$$K_*(x) \leqslant |K(x)(\eta(x) - 1)| + T_*(\psi)(x),$$

where $T_*(\psi)(x) = \sup_{\varepsilon>0}|K_\varepsilon * \psi(x)|$. Recall that the maximal operator T_* has the same boundedness as T (see [108, Section I.7],[47, Section 4.2.2]). Thus we have

$$\int_{|x|<1} K_*(x)dx \lesssim \int_{|x|<1} \frac{|\eta(x) - 1|}{|x|^n}dx + \|\psi\|_{L^2} < \infty$$

For $|x| \geqslant 1$, we can write

$$K_*(x) \leqslant |K\eta(x)| + \sup_{\varepsilon>0}|K(x) - K_\varepsilon * \psi(x)|.$$

The first term is integrable for $|x| \geqslant 1$ by condition (21) on η, so it remains to bound the integral of the second term. We do this by fixing ε and obtaining a pointwise estimate on $|K(x) - K_\varepsilon * \psi(x)|$ in terms of ψ.

Using the hypotheses on ψ, we can write, for $|x| \geqslant 1$ and $\varepsilon < 1/2$,

$$\begin{aligned}\left|K(x) - \int_{|x-y|>\varepsilon} K(x)\psi(y)dy\right| &\leqslant |K(x)|\int_{|x-y|\leqslant\varepsilon}|\psi(y)|dy \\ &\lesssim |x|^{-n}\int_{|y|\geqslant|x|/2}|\psi(y)|dy. \end{aligned} \tag{24}$$

Thus it remains to consider

$$\begin{aligned}&\int_{|x-y|>\varepsilon}[K(x) - K(x-y)]\psi(y)dy \\ &= \int_{\left\{\substack{|y|\leqslant|x|/2 \\ |x-y|\geqslant|x|/2}\right\}} + \int_{\left\{\substack{|y|\geqslant|x|/2 \\ |x-y|\geqslant|x|/2}\right\}} + \int_{\varepsilon<|x-y|\leqslant|x|/2}[K(x) - K(x-y)]\psi(y)dy \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

To estimate I_1, note that $B(0,|x|/2) \subset B(x,|x|/2)^c$ so we can use the smoothness condition (8) on K to write

$$\begin{aligned}
\int_{|x|\geqslant 1} \sup_{\epsilon>0} |I_1| &\leqslant \int_{|x|\geqslant 1} \int_{|y|\leqslant|x|/2} |K(x)-K(x-y)||\psi(y)|dydx \\
&\lesssim \int_{\mathbb{R}^n} \int_{|x|\geqslant 2|y|} |K(x)-K(x-y)|dx|\psi(y)|dy \\
&\leqslant C\|\psi\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

For I_2, we use the decay bound on K (Definition 2.3.1 Item 1) to get, as in (24),

$$\begin{aligned}
\int_{|x|\geqslant 1} \sup_{\epsilon>0} |I_2| &\leqslant \int_{|x|\geqslant 1} \int_{\left\{\substack{|y|\geqslant|x|/2 \\ |x-y|\geqslant|x|/2}\right\}} (|K(x)||\psi(y)| + |K(x-y)||\psi(y)|)dy \\
&\lesssim \int_{|x|\geqslant 1} \frac{1}{|x|^n} \int_{|y|\geqslant|x|/2} |\psi(y)|dy.
\end{aligned}$$

Finally, for I_3, we again use the decay bound (Definition 2.3.1 Item 1) as well as the cancellation condition on K (Definition 2.3.1 Item 3) to obtain

$$\begin{aligned}
|I_3| &\leqslant \left| \int_{\varepsilon<|x-y|\leqslant|x|/2} \left[K(x)\psi(y) - K(x-y)(\psi(y)-\psi(x))\right] dy \right| \\
&\qquad + \left| \int_{\varepsilon<|x-y|\leqslant|x|/2} K(x-y)\psi(x)dy \right| \\
&\lesssim \int_{|y|\geqslant|x|/2} \frac{|\psi(y)|}{|x|^n}dy + \int_{|x-y|\leqslant|x|/2} \frac{|\psi(y)-\psi(x)|}{|x-y|^n}dy + |\psi(x)| \\
&\lesssim \frac{1}{|x|^n} \int_{|y|\geqslant|x|/2} |\psi(y)|dy + \int_{|y'|\leqslant|x|/2} \frac{|\psi(x-y')-\psi(x)|}{|y'|^n}dy' + |\psi(x)|.
\end{aligned}$$

Since the last estimate is independent of ε, we can bound $\int_{|x|\geqslant 1} \sup_{\varepsilon>0} |I_3|$ by

$$\int_{|x|\geqslant 1} \frac{1}{|x|^n} \int_{|y|\geqslant|x|/2} |\psi(y)|dy + \int_{|x|\geqslant 1} \int_{|y'|\leqslant|x|/2} \frac{|\psi(x-y')-\psi(x)|}{|y'|^n}dy' + \|\psi(x)\|_{L^1}.$$

Combining (24), Conditions (22) and (23) on ψ and the estimates above, we get

$$\int_{|x|\geqslant 1} \sup_{\varepsilon>0} |K(x) - K_\varepsilon * \psi(x)| < \infty$$

and as a result, $K_* \in L^1(\mathbb{R}^n)$. Let

$$K_E(x) = \lim_{\varepsilon\to 0}((K\eta)_\varepsilon - K_\varepsilon + K_\varepsilon * \psi)(x) = K\eta(x) - K(x) + \lim_{\varepsilon\to 0} K_\varepsilon * \psi(x).$$

This limit exists for almost every $x \in \mathbb{R}^n$ by the properties of T_* (see [107, P. 45]). By the Dominated Convergence Theorem, $K_E \in L^1(\mathbb{R}^n)$ and

$$\lim_{\varepsilon\to 0}((K\eta)_\varepsilon - K_\varepsilon + K_\varepsilon * \psi) * f = K_E * f \quad \text{for } f \in L^1(\mathbb{R}^n).$$

Let $f \in h^1(\mathbb{R}^n)$. Then $T_E f \in L^1(\mathbb{R}^n)$ with

$$\|T_E f\|_{L^1(\mathbb{R}^n)} \leqslant \|K_E\|_{L^1(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover, the localized Riesz transforms r_j, $j = 1, \ldots, n$ of Goldberg (see Theorem 2.5.28) commute with convolution with the L^1 function K_E, so

$$r_j(T_E(f)) = T_E(r_j(f)).$$

Since $r_j(f) \in L^1(\mathbb{R}^n)$, hence $T_E(r_j(f)) \in L^1(\mathbb{R}^n)$, we have

$$\begin{aligned}
\|T_E f\|_{h^1(\mathbb{R}^n)} &\lesssim \|T_E f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|r_j(T_E(f))\|_{L^1(\mathbb{R}^n)} \\
&\leqslant \|K_E\|_{L^1(\mathbb{R}^n)} (\|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|r_j(f)\|_{L^1(\mathbb{R}^n)}) \\
&\lesssim \|f\|_{h^1(\mathbb{R}^n)}.
\end{aligned}$$

□

Corollary 3.0.4. *Suppose that η and ψ satisfy assumptions in Theorem 3.0.3. For $j = 1, \ldots, n$, define, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\mathcal{R}_{j,\eta} f(x) := c_n \lim_{\varepsilon \to 0} \int_{|x-y| \geqslant \varepsilon} \frac{(x_j - y_j)\eta(x-y)}{|x-y|^{n+1}} f(y) dy$$

and

$$\mathcal{R}_{j,\psi} f(x) := R_j(f - \psi * f),$$

where $c_n = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$. Then

1. *$\mathcal{R}_{j,\eta}$ and $\mathcal{R}_{j,\psi}$ both map $h^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$;*
2. *for $f \in L^1(\mathbb{R}^n)$, $f \in h^1(\mathbb{R}^n) \iff \mathcal{R}_{j,\eta}(f) \in L^1(\mathbb{R}^n), j = 1, ..., n \iff \mathcal{R}_{j,\psi}(f) \in L^1(\mathbb{R}^n), j = 1, ..., n$.*

As seen above, because convolution operators commute with the r_j, the boundedness from $h^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ implies the boundedness on $h^1(\mathbb{R}^n)$, and extra cancellation conditions on the kernel are not needed (see also the remark following the proof of [45, Theorem 4]).

Chapter 4

A New Molecular Theory of $h^p(\mathbb{R}^n)$

In this chapter, we will discuss a generalization of Definition 2.5.10 and 2.5.11, as well as the Definition 2.5.12. Moreover, we will consider the Hardy inequality in higher dimensions on local Hardy space.

This chapter is the joint work [31,32].

4.1 New Atoms and Molecules

In this section, we consider the cancellation conditions in the definition of $h^p(\mathbb{R}^n)$ atoms. They are different in Definition 2.5.10 and 2.5.11. Moreover, we cannot simply take $p = 1$ in Definition 2.5.10 as shown in the example below.

Example: In the paper [31, Example 3.4], we have shown that for each $r > 0$ there exists a function a, which is supported in an interval of length r, such that a has vanishing moments up to order $n(p^{-1}-1)-1$, and if a has bounded norm in h^p, then the highest order moment must decay logarithmically in r. We shall consider two particular cases, namely, on $h^1(\mathbb{R})$ and $h^{\frac{1}{2}}(\mathbb{R})$.

Let us start with

$$a_0(x) := \frac{\phi(r)}{r}\chi_{[-\frac{r}{2},\frac{r}{2}]}(x).$$

Here $\phi : [0,\infty) \to [0,1]$. We can see that this function satisfies $\mathrm{supp}(a) \subset [-\frac{r}{2},\frac{r}{2}]$ and $\|a\|_{L^\infty} \leqslant \frac{\phi(r)}{r}$.

We take $f(x) := \log(|x|^{-1})\chi_{[-1,1]}(x) \in \mathrm{bmo}(\mathbb{R})$. Since $f(x) \geqslant \log(2r^{-1})$ for all $x \in [-\frac{r}{2},\frac{r}{2}]$, by using Theorem 2.5.18,

$$\phi(r)\log(2r^{-1}) \leqslant \int a_0(x)f(x)dx = \left|\int a_0(x)f(x)dx\right| \leqslant \|f\|_{\mathrm{bmo}}\|a\|_{h^1}$$

If we want a_0 satisfies $\|a\|_{h^1} \leqslant 1$, then we have $\phi(r) \lesssim [\log(r^{-1})]^{-1}$. In other words, one cannot replace ϕ by a constant.

To establish the case for $h^{\frac{1}{2}}(\mathbb{R})$, we shall now translate a_0 to the right by $\frac{r}{2}$ units and extending it to $[-r,0]$ in such a way that the resulting function is odd. More precisely,

$$\widetilde{a_0}(x) := \frac{\phi(r)}{r}\chi_{[0,r]}(x) - \frac{\phi(r)}{r}\chi_{[-r,0]}(x).$$

Note that $\int \widetilde{a_0} = 0$ and $\int x\widetilde{a_0}(x)dx = \phi(r)r$; so we normalize $\widetilde{a_0}$ so that $\int xa_1(x)dx = \phi(r)$, *i.e.*

$$a_1(x) := \frac{\phi(r)}{r^2}\chi_{[0,r]}(x) - \frac{\phi(r)}{r^2}\chi_{[-r,0]}(x).$$

Similar to above, take $g(x) := x\log(|x|)\eta(x) \in \Lambda_1(\mathbb{R})$, where $\eta \in C_c^\infty(\mathbb{R})$ with $\eta \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\eta \equiv 0$ outside $[-2, 2]$; and by restricting ourselves to $r < \frac{1}{2}$, we have

$$\begin{aligned}
\left|\int a_1(x)g(x)dx - \int xa_1(x)\log(r)dx\right| &\leqslant \left|\int_{-r}^{r} a_1(x)x\log\left(\frac{|x|}{r}\right)dx\right| \\
&\leqslant \left|2\int_0^1 a_1(ry)ry\log(y)rdy\right| \\
&\leqslant 2r^2\frac{1}{r}\int_0^1 y\log(y^{-1})dy = \frac{1}{2}r < \frac{1}{4}.
\end{aligned}$$

This implies

$$\phi(r)\log(r) \leqslant \frac{1}{4} + \left|\int a_1(x)g(x)dx\right| \lesssim 1 + \|a_1\|_{h^{\frac{1}{2}}}.$$

A similar explanation shows that $\phi(r) \lesssim [\log(r^{-1})]^{-1}$.

We now give the definition of an approximate atoms for all $0 < p \leqslant 1$. Recall that $\gamma_p = n(p^{-1} - 1)$ and $N_p = \lfloor n(p^{-1} - 1)\rfloor$.

Definition 4.1.1 ([31, Definition 3.2])**.** *Let $0 < p \leqslant 1 \leqslant s \leqslant \infty$ with $p < s$, $\omega \geqslant 0$, and define $\varphi_p : (0,\infty) \to (0,\infty)$ by*

$$\varphi_p(t) := \left[\log\left(1 + \frac{1}{\omega t}\right)\right]^{-\frac{1}{p}},$$

where $\varphi_p(t) = 0$ in the limiting case $\omega = 0$. We say that a measurable function a is a (p, s, ω) atom (for $h^p(\mathbb{R}^n)$) if there exists a ball $B \subset \mathbb{R}^n$ such that

1. *$supp(a) \subset B$;*
2. *$\|a\|_{L^s(\mathbb{R}^n)} \leqslant r(B)^{n(\frac{1}{s}-\frac{1}{p})}$;*
3. $$\begin{cases} \left|\int_B a(x)(x - x_B)^\alpha dx\right| \leqslant \omega, & \text{if } |\alpha| < \gamma_p, \\ \left|\int_B a(x)(x - x_B)^\alpha dx\right| \leqslant \varphi_p(r(B)) & \text{if } |\alpha| = N_p = \gamma_p. \end{cases}$$

Proposition 4.1.2 ([31, Proposition 3.3])**.** *If a is a (p, s, ω) atom, then $\|a\|_{h^p} \lesssim_{p,s,\omega} 1$ and the constant is independent of a.*

Proof of Proposition 4.1.2.
This proof is taken from the proof of [31, Proposition 3.3]. Let a be a (p, s, ω) atom supported in $B = B(x_B, r)$. The idea is to split according to

Split

$$\|\mathcal{M}_\phi a\|_{L^p}^p = \int_{2B} \left(\sup_{0<t<1} |\phi_t * a(x)| \right)^p dx + \int_{(2B)^c} \left(\sup_{0<t<1} |\phi_t * a(x)| \right)^p dx.$$

Using the fact that $\sup_{0<t<1} |\phi_t * a(x)| \leqslant C_\phi \, Ma(x)$, where M denotes the Hardy-Littlewood maximal function, by Theorem 2.1.8 it follows that

$$\int_{2B} \left(\sup_{0<t<1} |\phi_t * a(x)| \right)^p dx \leqslant C_{\phi,s} |2B|^{1-\frac{p}{s}} \|a\|_{L^s}^p \leqslant C_{\phi,s,p,n} \; r^{n\left(1-\frac{p}{s}\right)} r^{n\left(\frac{p}{s}-1\right)} = C_{\phi,s,p,n}.$$

Note the last estimate holds for all $0 < p \leqslant 1$. For $s = 1$ and $p < 1$, using that M is of weak type $(1,1)$ and following the argument in [42, Lemma 3.1 p. 248], we have

$$\begin{aligned}
\int_{2B} \left(\sup_{0<t<1} |\phi_t * a(x)| \right)^p dx &\leqslant \int_{2B} |Ma(x)|^p dx \\
&= \int_0^\infty p\alpha^{p-1} |\{x \in 2B : |Ma(x)| > \alpha\}| d\alpha \\
&\leqslant |2B| \int_0^{r^{-np^{-1}}} p\alpha^{p-1} d\alpha + \int_{r^{-np^{-1}}}^\infty p\alpha^{p-1} \frac{\|a\|_{L^1}}{\alpha} d\alpha \\
&\lesssim_p r^n r^{-n} + r^{n(1-p^{-1})} r^{-np^{-1}(p-1)} \\
&\lesssim 1.
\end{aligned}$$

Now we deal with the estimate outside $2B$. From the Taylor expansion of the function $y \mapsto \phi_t(x-y)$ up to order N_p, we may write

$$\begin{aligned}
\sup_{0<t<1} |\phi_t * a(x)| = \sup_{0<t<1} \Bigg| &\int_{\mathbb{R}^n} \sum_{|\alpha| \leqslant N_p - 1} C_\alpha \, \partial^\alpha \phi_t(x - x_B) \; (x_B - y)^\alpha a(y) dy \\
&+ \int_{\mathbb{R}^n} \sum_{|\alpha| = N_p} C_\alpha \, \partial^\alpha \phi_t(x - x_B + c(x_B - y)) \; (x_B - y)^\alpha a(y) dy \Bigg|
\end{aligned}$$

for some $c \in (0,1)$. As $|x - x_B| \geqslant 2r$ and $|y - x_B| \leqslant r$, we have $|x - x_B + c(x_B - y)| \geqslant |x - x_B|/2$.

For $\phi \in \mathcal{S}(\mathbb{R}^n)$, we will use the bound $|\partial^\alpha \phi(x)| \leqslant C_\alpha |x|^{-N}$, where $N > 0$, depending on $|\alpha|$, will be chosen conveniently. Breaking the integral into the integrals over the regions $2r < |x - x_B| \leqslant 2$ (empty if $r \geqslant 1$) and $|x - x_B| \geqslant 2$, we take $N = n + |\alpha|$ for the first region and $N = n + N_p + 1$ for the second one. Since the supremum in t is taken over $(0,1)$, we have $t^{-n-|\alpha|+n+N_p+1} \leqslant 1$ for all $|\alpha| \leqslant N_p$. Thus

$$\begin{aligned}
&\int_{(2B)^c} (\sup_{0<t<1} |\phi_t * a(x)|)^p dx \\
&\leqslant \int_{2r<|x-x_B|\leqslant 2} \left(\sup_{0<t<1} \sum_{|\alpha|\leqslant N_p} C_\alpha\, t^{-n-|\alpha|} \left| \frac{x-x_B}{t} \right|^{-n-|\alpha|} \left| \int_{\mathbb{R}^n} a(y)(y-x_B)^\alpha dy \right| \right)^p dx \\
&\qquad + \int_{|x-x_B|\geqslant 2} \left(\sup_{0<t<1} \sum_{|\alpha|\leqslant N_p} C_\alpha\, t^{-n-|\alpha|} \left| \frac{x-x_B}{t} \right|^{-n-N_p-1} \left| \int_{\mathbb{R}^n} a(y)(y-x_B)^\alpha dy \right| \right)^p dx \\
&\lesssim \sum_{|\alpha|\leqslant N_p} \left| \int_{\mathbb{R}^n} a(y)(y-x_B)^\alpha dy \right|^p \left(\int_{2r<|x-x_B|\leqslant 2} |x-x_B|^{-np-|\alpha|p} dx \right. \\
&\qquad\qquad \left. + \int_{|x-x_B|\geqslant 2} |x-x_B|^{-p(n+N_p+1)} dx \right). \\
&\lesssim \sum_{|\alpha|\leqslant N_p} \left| \int_{\mathbb{R}^n} a(y)(y-x_B)^\alpha dy \right|^p \int_{2r<|x-x_B|\leqslant 2} |x-x_B|^{-np-|\alpha|p} dx + C_{n,p,\omega}.
\end{aligned}$$

Here we have used the fact that $p(n+N_p+1) > n$ to bound the terms involving the integral over $|x-x_B| \geqslant 2$ and the estimate

$$\left| \int a(x)(x-x_B)^\alpha \right| \lesssim_{\omega,p} 1$$

for $r(B) \geqslant 1$.

The other terms are nonzero only when $r < 1$. In the case $p \neq n/(n+k)$ for any $k \in \mathbb{N}$, meaning $N_p < \gamma_p$, we have $-np-|\alpha|p > -n$ for all $|\alpha| \leqslant N_p$ so the integral over $|x-x_B| \leqslant 2$ is convergent, and together with condition 3 in Definition 4.1.1, this gives a bound which is a constant multiple of ω.

The same bound also works when $p = n/(n+k)$, $k \in \mathbb{N}$, but $|\alpha| < N_p$. When $|\alpha| = N_p = \gamma_p$, we have $-np-|\alpha|p = -n$ and therefore $\int_{2r<|x-x_B|\leqslant 2} |x-x_B|^{-np-|\alpha|p} dx \approx \log r^{-1}$. Using condition 3 again, this time with the log bound on the moments, gives a multiple of $\log r^{-1}\, \varphi_p(r)^p$, which is bounded for $r \leqslant 1$. $\square$

Next, we shall establish the approximate molecular theory.

Definition 4.1.3. [31, Definition 3.5] *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$, $\lambda > n\left(\frac{s}{p} - 1\right)$, and ω, φ_p be as in Definition 4.1.1. We say that a measurable function M is a (p, s, λ, ω) molecule (for $h^p(\mathbb{R}^n)$) if there exists a ball $B \subset \mathbb{R}^n$ and a constant $C > 0$ such that*

(M1) $\|M\|_{L^s(B)} \leqslant C\, r(B)^{n\left(\frac{1}{s}-\frac{1}{p}\right)}$;

(M2) $\left(\int_{B^c} |M(x)|^s\, |x-x_B|^\lambda dx \right)^{1/s} \leqslant C\, r(B)^{\frac{\lambda}{s}+n\left(\frac{1}{s}-\frac{1}{p}\right)}$;

(M3) $$\begin{cases} \left| \int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx \right| \leqslant \omega, & \textit{if } |\alpha| < \gamma_p, \\ \left| \int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx \right| \leqslant \varphi_p(r(B)) & \textit{if } |\alpha| = N_p = \gamma_p. \end{cases}$$

We call the molecule "normalized" if $C = 1$.

Proposition 4.1.4. [31, Proposition 3.7] *If M is a normalized (p, s, λ, ω)-molecule, then $\|M\|_{h^p} \lesssim_{p,n,s,\lambda,\omega} 1$ and the constant is independent of M.*

Proof of Proposition 4.1.4.
This proof is taken from the proof of [31, Proposition 3.7]. The proof is inspired by the classical molecular decomposition for real Hardy spaces. We will outline only the main ideas, highlighting the parts that diverge from the classical proof, which can be consulted in [42, Theorem III.7.16], preserving some of the notation from that proof below. Let M be a normalized (p, s, λ, ω) molecule associated to a ball $B = B(x_B, r) \subset \mathbb{R}^n$. We show that

$$M = \sum_{k=0}^{\infty} t_k\, a_k + \sum_{k=0}^{\infty} s_k\, b_k + a_\omega, \tag{25}$$

where a_k, b_k are (p, s) and (p, ∞) atoms with full cancellation, respectively, and a_ω is a (p, s, ω) atom. Moreover,

$$\sum_{k=0}^{\infty} |t_k|^p < \infty \text{ and } \sum_{k=0}^{\infty} |s_k|^p < \infty$$

independently of B. Let $B_0 = E_0 = B$, $B_k = B(x_B, 2^k r)$, $E_k = B_k \backslash B_{k-1}$ and $M_k(x) = M(x)\chi_{E_k}(x)$ for $k \in \mathbb{N}$. Let $P_k = \sum_{|\alpha| \leqslant N_p} m_\alpha^k \phi_\alpha^k$ to be the polynomial of degree at most N_p, restricted to the set E_k, for which, for every $|\alpha| \leqslant N_p$,

$$⨍_{E_k} P_k(x)\ (x - x_B)^\alpha dx = m_\alpha^k ⨍_{E_k} \phi_\alpha^k(x)\ (x - x_B)^\alpha dx = m_\alpha^k := ⨍_{E_k} M(x)(x - x_B)^\alpha dx \tag{26}$$

and

$$|P_k(x)| \leqslant C_{n,p} ⨍_{E_k} |M(x)| dx \tag{27}$$

for some constant independent of k. This is done by choosing the polynomials ϕ_α^k to have β-th moment equal to $|E_k|$ when $\beta = \alpha$, and zero otherwise, and noting that $(2^k r)^{|\alpha|} |\phi_\alpha^k(x)| \leqslant C$ uniformly in k.

Setting, for $|\alpha| \leqslant N_p$, for $k \in \mathbb{N}_0$

$$\nu_\alpha := \sum_{j=0}^{\infty} |E_j| m_\alpha^j = \int_{\mathbb{R}^n} M(x)(x - x_B)^\alpha dx,$$

$$N_\alpha^k := \sum_{j=k+1}^{\infty} |E_j| m_\alpha^j = \int_{E_k^c} M(x)(x - x_B)^\alpha dx$$

we can represent the sum of P_k as

$$\begin{aligned}
\sum_{k=0}^{\infty} P_k(x) &= \sum_{|\alpha|\leqslant N_p} \left(\sum_{k=1}^{\infty} (N_\alpha^{k-1} - N_\alpha^k)|E_k|^{-1}\phi_\alpha^k(x) + (\nu_\alpha - N_\alpha^0)|E_0|^{-1}\phi_\alpha^0(x) \right) \\
&= \sum_{|\alpha|\leqslant N_p} \left(\sum_{k=0}^{\infty} N_\alpha^k |E_{k+1}|^{-1}\phi_\alpha^{k+1}(x) - \sum_{k=0}^{\infty} N_\alpha^k |E_k|^{-1}\phi_\alpha^k(x) + \nu_\alpha |E_0|^{-1}\phi_\alpha^0(x) \right) \\
&= \sum_{k=0}^{\infty} \sum_{|\alpha|\leqslant N_p} \Phi_\alpha^k(x) + \sum_{|\alpha|\leqslant N_p} \nu_\alpha \, |E_0|^{-1} \, \phi_\alpha^0(x),
\end{aligned}$$

where

$$\Phi_\alpha^k(x) = N_\alpha^{k+1} \left[|E_{k+1}|^{-1}\phi_\alpha^{k+1}(x) - |E_k|^{-1}\phi_\alpha^k(x) \right], \quad k \in \mathbb{N}_0.$$

Note that last sum appears since we do not have the vanishing moment conditions on the molecule compared with the case H^p.

This allows us to decompose M as follows:

$$M = \sum_{k=0}^{\infty} (M_k - P_k) + \sum_{k=0}^{\infty} \sum_{|\alpha|\leqslant N_p} \Phi_\alpha^k(x) + \sum_{|\alpha|\leqslant N_p} \nu_\alpha \, |E_0|^{-1} \, \phi_\alpha^0(x) = S_1 + S_2 + S_3. \quad (28)$$

We deal first with the terms S_1. From conditions (M1) and (M2) one gets

$$\|M_k\|_{L^s} \leqslant 2^{\frac{\lambda}{s}} \, C_{n,s} \, |B_k|^{\frac{1}{s}-\frac{1}{p}} \, (2^k)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}, \quad (29)$$

and it follows from (27) that $\|P_k\|_{L^s} \leqslant C_{n,p} \, \|M_k\|_{L^s}$. Moreover, from (26) we get that $M_k - P_k$ has vanishing moments up the order N_p. Thus $M_k - P_k$ is a multiple of a (p, s) atom. Writing $(M_k - P_k)(x) = t_k \, a_k(x)$ where $t_k = \|M_k - P_k\|_{L^s} \, |B_k|^{\frac{1}{p}-\frac{1}{s}}$, $a_k(x) = \dfrac{M_k(x) - P_k(x)}{\|M_k - P_k\|_{L^s}} \, |B_k|^{\frac{1}{s}-\frac{1}{p}}$ and note that from (29) one gets

$$\sum_{k=0}^{\infty} |t_k|^p \leqslant 2^{\frac{\lambda p}{s}} \, C_{n,p,s} \sum_{k=0}^{\infty} (2^k)^{-\frac{\lambda p}{s}+n\left(1-\frac{p}{s}\right)} = C_{n,p,s,\lambda} < \infty \quad (30)$$

provided $\lambda > n\,(s/p - 1)$. We point out that the closer λ gets to $n(s/p - 1)$, the bigger the constant appearing in (30).

For S_2, we claim that $\Phi_\gamma^j(x)$ is a multiple of a (p, ∞) atom with full cancellation conditions. The cancellation follows from the moment conditions on ϕ_α^k. For the size condition, from Hölder inequality and (29), for every $|\alpha| \leqslant N_p$ and $k \in \mathbb{N}$, one has

$$|N_\alpha^k| \leqslant C_{n,p,s,\lambda} \, |B_k|^{1-\frac{1}{p}} (2^k r)^{|\alpha|} (2^k)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}.$$

Hence, since $(2^k r)^{|\alpha|} |\phi_\alpha^k(x)| \leqslant C$ uniformly, it follows

$$\left| N_\alpha^k |E_k|^{-1} \phi_\alpha^k(x) \right| \leqslant C_{n,p,s,\lambda} \, |B_k|^{-\frac{1}{p}} (2^k)^{-\frac{\lambda}{s}+n\left(\frac{1}{p}-\frac{1}{s}\right)}.$$

Therefore, writing $\sum_{|\alpha|\leqslant N_p} \Phi_\alpha^j(x) = s_k\, b_k(x)$, where $s_k = C_{n,p,s,\lambda}\,(2^k)^{-\frac{\lambda}{n}+n\left(\frac{1}{p}-\frac{1}{s}\right)}$ for some appropriate constants $C_{n,p,s,\lambda}$, we get that b_k are (p,∞) atoms and

$$\sum_{k=0}^{\infty} |s_k|^p = C_{n,p,s,\lambda} \sum_{k=0}^{\infty} (2^k)^{-\frac{\lambda p}{s}+n\left(1-\frac{p}{s}\right)} < \infty, \tag{31}$$

where again we used that $\lambda > n\,(s/p-1)$.

Finally, for S_3 let

$$a_\omega = \sum_{|\alpha|\leqslant N_p} \nu_\alpha\, |E_0|^{-1}\, \phi_\alpha^0(x).$$

This function is supported on $E_0 = B_0$ and, proceeding as in (1), (M1) and (M2) give

$$|\nu_\alpha| = \left| \int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx \right| \lesssim r^{|\alpha|+n\left(1-\frac{1}{p}\right)}.$$

The L^s-estimate then follows immediately from the fact that $r^{|\alpha|}|\phi_\alpha^0(x)| \leqslant C$:

$$\begin{aligned}
\left\| \sum_{|\alpha|\leqslant N_p} \nu_\alpha\, |E_0|^{-1}\phi_\alpha^0 \right\|_{L^s} &\leqslant \sum_{|\alpha|\leqslant N_p} |N_\alpha^0|\, |E_0|^{-1} \left(\int_{E_0} |\phi_\alpha^0(x)|^s dx \right)^{\frac{1}{s}} \\
&\leqslant \sum_{|\alpha|\leqslant N_p} |\nu_\alpha|\, |E_0|^{\frac{1}{s}-1}\, r^{-|\alpha|} \lesssim r^{n\left(\frac{1}{s}-\frac{1}{p}\right)}.
\end{aligned}$$

It remains to show the moment conditions on a_ω, which follow immediately from (M3), since by the choice of ν_α and ϕ_α^0, the moments of a_ω are the same as those of M. Indeed, for $|\beta| \leqslant N_p$,

$$\begin{aligned}
\int a_\omega(x)\,(x-x_B)^\beta dx &= \sum_{|\alpha|\leqslant N_p} \nu_\alpha \left(|E_0|^{-1} \int_{E_0} \phi_\alpha^0(x)(x-x_B)^\beta dx \right) \\
&= \nu_\alpha = \int_{\mathbb{R}^n} M(x)(x-x_B)^\beta dx.
\end{aligned}$$

Thus a_ω is a multiple of a (p,s,ω) atom. □

From Proposition 4.1.2 and 4.1.4 and Theorem 2.5.8, we have

Theorem 4.1.5 ([31, Corollary 3.8])**.**
Let $0 < p \leqslant 1 \leqslant q$ with $p < q$ and $\lambda > n(\frac{q}{p}-1)$.

1. *Given $f \in h^p(\mathbb{R}^n)$, there exists a sequence $\{\gamma_j\}_j \in \ell^p$ and a sequence of (p,q,ω) atoms for h^p, $\{a_j\}_j$, such that $f = \sum_{j=1}^{\infty} \gamma_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$; moreover, $\|f\|_{h^p} \approx \inf(\sum_{j\in\mathbb{N}} |\gamma_j|^p)^{\frac{1}{p}}$, where the infimum is taken over all such possible decomposition of f.*

2. *Given $f \in h^p(\mathbb{R}^n)$, there exists a sequence $\{\gamma_j\}_j \in \ell^p$ and a sequence of (p,q,λ,ω) molecules for h^p, $\{M_j\}_j$, such that $f = \sum_{j=1}^{\infty} \gamma_j' M_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$; moreover, $\|f\|_{h^p} \approx \inf(\sum_{j\in\mathbb{N}} |\gamma_j'|^p)^{\frac{1}{p}}$, where the infimum is taken over all such possible decomposition of f.*

4.2 Hardy's Inequality

As an example of our new molecular theory, we will use it to prove the following.

Theorem 4.2.1 ([31, Theorem 4.1])**.** *Let $0 < p \leqslant 1$. Then, there exists $C > 0$ such that*

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^p}{(1+|\xi|)^{n(2-p)}} d\xi \leqslant C\|f\|_{h^p}^p \quad \forall\ f \in h^p(\mathbb{R}^n).$$

Lemma 4.2.2 ([31, Lemma 4.2])**.** *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$ and $\lambda > n\left(\frac{s}{p} - 1\right)$. Suppose M satisfies conditions (M1) and (M2) with respect to the ball $B = B(x_B, r) \subset \mathbb{R}^n$. Then the Fourier transform of M satisfies*

$$|\widehat{M}(\xi)| \lesssim |\xi|^{\gamma}\, r^{\gamma-\gamma_p} + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left|\int_{\mathbb{R}^n} M(x)(x-x_B)^{\alpha} dx\right| \tag{32}$$

for any $\gamma \in (\gamma_p, \frac{\lambda}{s} - \frac{n}{s'})$ and integer N with $N < \gamma \leqslant N+1$.

Proof of Lemma 4.2.2.
The proof is taken from the paper [31]. Since the absolute value of the Fourier transform is preserved under translation of the function, we may assume $x_B = 0$. For $\xi = 0$, we see that equality holds by in (32) by considering the $\alpha = 0$ term in the sum on the right-hand-side, so we need only consider $\xi \neq 0$.

Suppose first that $\gamma = N+1 < \frac{\lambda}{s} - \frac{n}{s'}$. Denoting $e^{-2\pi i x\cdot\xi}$ by $\varphi(x)$, we write $P_{N,\varphi,0}(x) = \sum_{|\alpha|\leqslant N} C_\alpha\,(\partial^\alpha\varphi)(0)x^\alpha$ for its Taylor polynomial of order N at 0, and use the formula for the remainder to get, for $t \in (0,1)$,

$$\begin{aligned}
|\widehat{M}(\xi)| &= \left|\int_{\mathbb{R}^n} M(x)\,[\varphi(x) - P_{N,\varphi,0}(x)]\,dx + \sum_{|\alpha|\leqslant N} C_\alpha\,(\partial^\alpha\varphi)(0)\int_{\mathbb{R}^n} M(x)x^\alpha dx\right| \\
&\leqslant \left|\int_{\mathbb{R}^n} M(x) \sum_{|\alpha|=N+1} C_\alpha\,(\partial^\alpha\varphi)(tx)x^\alpha dx\right| + \sum_{|\alpha|\leqslant N} C_\alpha\,|2\pi\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^\alpha dx\right| \\
&\lesssim |\xi|^{N+1}\int_{\mathbb{R}^n} |M(x)|\,|x|^{N+1}dx + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|}\left|\int_{\mathbb{R}^n} M(x)x^\alpha dx\right|.
\end{aligned} \tag{33}$$

Similarly to (1), from conditions (M1) and (M2) of the molecule and Hölder's inequality, one has

$$\begin{aligned}
\int_{\mathbb{R}^n} |M(x)|\,|x|^{N+1}dx &\leqslant r^{\frac{n}{s'}+N+1}\,\|M\|_{L^s(B)} + \|M\,|\cdot|^{\frac{\lambda}{s}}\,\|_{L^s(B^c)}\,\|\,|\cdot|^{-\frac{\lambda}{s}+N+1}\|_{L^{s'}(B^c)} \\
&= r^{\frac{n}{s'}+N+1}\,\|M\|_{L^s(B)} + r^{-\frac{\lambda}{s}+N+1+\frac{n}{s'}}\,\|M\,|\cdot|^{\frac{\lambda}{s}}\,\|_{L^s(B^c)} \\
&\lesssim r^{N+1-\gamma_p},
\end{aligned}$$

where the convergence of the integral follows from the assumption that $N+1 < \frac{\lambda}{s} - \frac{n}{s'}$. This gives the result in the case $\gamma = N+1$.

Now suppose $\gamma < N+1$. Recalling that $\xi \neq 0$, we write

$$\widehat{M}(\xi) = \int_{|x|\geqslant|\xi|^{-1}} e^{-2\pi i x\cdot\xi} M(x)dx + \int_{|x|\leqslant|\xi|^{-1}} e^{-2\pi i x\cdot\xi} M(x)dx =: I_1 + I_2.$$

We estimate the first integral using Hölder's inequality, together with the bound the $L^s(\mathbb{R}^n)$ norm of $M(x)|x|^{\frac{\lambda'}{s}}$ with $\lambda' = s(\gamma + \frac{n}{s'}) < \lambda$, as follows:

$$|I_1| \leqslant \int_{|x|\geqslant|\xi|^{-1}} |M(x)|dx \leqslant \|M\,|\cdot|^{\frac{\lambda'}{s}}\,\|_{L^s(\mathbb{R}^n)}\,\|\,|\cdot|^{-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\geqslant|\xi|^{-1})} \leqslant r^{\gamma-\gamma_p}|\xi|^{\gamma}.$$

For the second integral, we again proceed via the Taylor expansion of $\varphi(x) = e^{-2\pi i x\cdot\xi}$, to get, as in (33)

$$\begin{aligned}
|I_2| &\lesssim |\xi|^{N+1} \int_{|x|\leqslant|\xi|^{-1}} |M(x)|\,|x|^{N+1}dx + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{|x|\leqslant|\xi|^{-1}} M(x)x^\alpha dx \right| \\
&\lesssim |\xi|^{N+1} \int_{|x|\leqslant|\xi|^{-1}} |M(x)|\,|x|^{\frac{\lambda'}{s}}|x|^{N+1-\frac{\lambda'}{s}}dx \\
&\quad + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x)x^\alpha dx - \int_{|x|\geqslant|\xi|^{-1}} M(x)x^\alpha dx \right| \\
&\lesssim |\xi|^{N+1}\|M\,|\cdot|^{\frac{\lambda'}{s}}\,\|_{L^s(\mathbb{R}^n)}\,\|\,|\cdot|^{N+1-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\leqslant|\xi|^{-1})} \\
&\quad + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \int_{|x|\geqslant|\xi|^{-1}} |M(x)||x|^\alpha dx + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x)x^\alpha dx \right| \\
&\lesssim |\xi|^{N+1} r^{\frac{\lambda'}{s}-\frac{n}{s'}-\gamma_p} |\xi|^{-(N+1-\frac{\lambda'}{s}+\frac{n}{s'})} + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|}\|M\,|\cdot|^{\frac{\lambda'}{s}}\,\|_{L^s(\mathbb{R}^n)}\,\|\,|\cdot|^{|\alpha|-\frac{\lambda'}{s}}\|_{L^{s'}(|x|\geqslant|\xi|^{-1})} \\
&\quad + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x)x^\alpha dx \right| \\
&\lesssim r^{\gamma-\gamma_p}|\xi|^\gamma + \sum_{|\alpha|\leqslant N} |\xi|^{|\alpha|} \left| \int_{\mathbb{R}^n} M(x)x^\alpha dx \right|.
\end{aligned}$$

Here we have used that $\gamma = \frac{\lambda'}{s} - \frac{n}{s'} < N+1$ for the local integrability and that $|\alpha| \leqslant N < \gamma = \frac{\lambda'}{s} - \frac{n}{s'}$ implies $s'(|\alpha| - \frac{\lambda'}{s}) < -n$. This concludes the case $\gamma < N+1$. $\square$

For a molecule, the above estimate on the Fourier transform and the control of the moments allows us to prove the following more refined version of Hardy's inequality:

Lemma 4.2.3 ([31, Lemma 4.3])**.** *If $1 \leqslant s \leqslant 2$ with $p < s$ and M is a (p, s, λ, ω) molecule in $h^p(\mathbb{R}^n)$ associated to the ball $B = B(x_B, r)$, then for $a > 0$,*

$$\int_{\mathbb{R}^n} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi \leqslant C_{a,\omega}. \tag{34}$$

In the homogeneous case, $\omega = 0$, we recover Hardy's inequality for $H^p(\mathbb{R}^n)$ (Proposition 2.1.13 Item 3). For $\omega > 0$, taking $a = \omega^{-1}$ shows that Goldberg's Hardy inequality holds uniformly for molecules with a constant depending on ω. Applying the Fourier transform to the molecular decomposition of f in Theorem 4.1.5 gives the result of the Theorem 4.2.1.

Proof of Lemma 4.2.3.
The proof is taken from the paper [31]. To show (34) we split integral in the following way:

$$\int_{\mathbb{R}^n} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi = \int_{|\xi|<r^{-1}} + \int_{|\xi|>r^{-1}} := I_1 + I_2.$$

Control of I_2: Applying the Hölder and Hausdorff-Young inequalities, one gets

$$\begin{aligned}\int_{|\xi|>r^{-1}} \frac{|\widehat{M}(\xi)|^p}{(a\omega + |\xi|)^{n(2-p)}} d\xi &\leqslant \|\widehat{M}\|_{L^{s'}}^p \left(\int_{|\xi|>r^{-1}} |\xi|^{-\frac{n(2-p)}{1-p/s'}} d\xi\right)^{1-\frac{p}{s'}} \\ &\lesssim \|M\|_{L^s}^p \, r^{n(2-p)-n(1-\frac{p}{s'})} \left(\int_{|\xi|>1} |\xi|^{-\frac{n(2-p)}{1-p/s'}} d\xi\right)^{1-\frac{p}{s'}} \\ &\leqslant C.\end{aligned}$$

Here we've used condition (M1), and the integrability of the second term follows since

$$1 > p\left(1 - \frac{1}{s'}\right) \quad \Leftrightarrow \quad -\frac{n(2-p)}{1-p/s'} < -n.$$

Control of I_1: Taking $N = N_p$ and $\gamma \in \left(\gamma_p, \frac{\lambda}{s} - \frac{n}{s'}\right) \cap (N_p, N_p + 1\,]$ in Lemma 4.2.2, one has

$$\begin{aligned} I_1 &\lesssim r^{p(\gamma-\gamma_p)} \int_{|\xi|<r^{-1}} |\xi|^{p\gamma}(a\omega + |\xi|)^{n(p-2)} d\xi \\ &\quad + \sum_{|\alpha|\leqslant N_p} \left|\int_{\mathbb{R}^n} M(x)(x - x_B)^\alpha dx\right|^p \int_{|\xi|<r^{-1}} |\xi|^{|\alpha|p}(a\omega + |\xi|)^{n(p-2)} d\xi. \\ &:= I_3 + I_4 \end{aligned}$$

For I_3, using that $(1 + |\xi|)^{n(p-2)} \leqslant |\xi|^{n(p-2)}$ we get

$$\begin{aligned} |I_3| &\leqslant r^{p(\gamma-\gamma_p)} \int_{|\xi|<r^{-1}} |\xi|^{n(p-2)+p\gamma} d\xi \\ &\simeq r^{\,p\gamma+n(p-1)}\, r^{-p\gamma-n(p-2)-n} = 1, \end{aligned}$$

where the integrability follows from $p\gamma > p\gamma_p = n(1-p)$.

For I_4, using the approximate moment conditions (M3) of the molecule when $\omega > 0$, we get

$$\begin{aligned}
&\sum_{|\alpha|\leqslant N_p}\left|\int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx\right|^p \int_{|\xi|<r^{-1}} |\xi|^{|\alpha|p}(a\omega+|\xi|)^{n(p-2)}d\xi \\
&=\sum_{|\alpha|\leqslant N_p}\left|\int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx\right|^p (a\omega)^{np-n+|\alpha|p}\int_{|\xi|<(a\omega r)^{-1}} |\xi|^{|\alpha|p}(1+|\xi|)^{n(p-2)}d\xi \\
&\leqslant \sum_{|\alpha|\leqslant N_p}\left|\int_{\mathbb{R}^n} M(x)(x-x_B)^\alpha dx\right|^p (a\omega)^{np-n+|\alpha|p}\int_1^{1+(a\omega r)^{-1}} t^{p|\alpha|+np-n-1}dt \\
&\leqslant \sum_{|\alpha|<\gamma_p}\omega^p\,(a\omega)^{(|\alpha|-\gamma_p)p}\int_1^\infty t^{(|\alpha|-\gamma_p)p-1}dt+\sum_{|\alpha|=\gamma_p=N_p}\left[\log\left(1+\frac{1}{\omega r}\right)\right]^{-1}\int_1^{1+(a\omega r)^{-1}} t^{-1}dt \\
&\leqslant C_{a,\omega,p}+\sum_{\substack{|\alpha|=N_p\\ N_p\in\mathbb{N}_0}}\left[\log\left(1+\frac{1}{\omega r}\right)\right]^{-1}\log\left(1+\frac{1}{a\omega r}\right) \\
&\leqslant C_{a,\omega,p}.
\end{aligned}$$

$\square$

Chapter 5

Boundedness of Inhomogeneous Singular Integral Operators on $h^p(\mathbb{R}^n)$

In this chapter, we discuss sufficient conditions and necessary conditions for the boundedness of inhomogeneous singular integral operators on $h^p(\mathbb{R}^n)$. It is based on [31, 32].

5.1 Sufficient Conditions

We shall define a more general class of singular integral operators than Definition 2.5.32.

Definition 5.1.1. *We say T is an inhomogeneous Calderón–Zygmund operator with L^s_δ integral-type condition associated with K, where $1 \leqslant s < \infty$, $\mu > 0$ and $\delta > 0$ if*

1. *the kernel K satisfies (18), i.e. $|K(x,y)| \leqslant C \min\{|x-y|^{-n},\ |x-y|^{-n-\mu}\}$, $x \neq y$;*

2. *for each $x, z \in \mathbb{R}^n$ there exist two polynomials $P^1_{x,z}(y)$ and $P^2_{x,z}(y)$ with degree at most $\lfloor \delta \rfloor$ such that*

$$\left(\int_{A_j(z,r)} |K(x,y) - P^1_{x,z}(y)|^s + |K(y,x) - P^2_{x,z}(y)|^s dx \right)^{\frac{1}{s}} \lesssim |A_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta} \tag{35}$$

   for $|y-z| < r$, where $0 < r < 1$, $j \in \mathbb{N}$,

$$A_j(z,r) := \left\{ x \in \mathbb{R}^n :\ 2^j r \leqslant |x-z| < 2^{j+1} r \right\}.$$

3. *the operator T is given by*

$$\langle Tf, g \rangle = \int \int K(x,y) f(y) g(x) dy dx, \qquad \forall\ f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with disjoint supports}$$

   and extends to a bounded operator on $L^2(\mathbb{R}^n)$.

If K satisfies (35) for some $s_0 > 1$, then K also satisfies (35) for all $1 \leqslant s \leqslant s_0$ using Hölder's inequality.

Remark.

1. We compare this definition with the one in [31]. In the paper [31], the following condition is imposed. For all for $|y-z| < r$, where $0 < r < 1$, $j \in \mathbb{N}$

$$\left(\int_{A_j(z,r)} |K(x,y) - K(x,z)|^s + |K(y,x) - K(z,x)|^s dx \right)^{\frac{1}{s}} \lesssim |A_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta}.$$

The condition (35) is a generalization because we can take $P^1_{x,z}(y) = K(x,z)$ and $P^2_{x,z}(y) = K(z,x)$, which are constant polynomials in the y-variable.

2. In [96, Section 4.2], they imposed the condition that $K \in C^{\lfloor \delta \rfloor}$ and

$$\left(\int_{A_j(z,r)} |\partial_2^\beta K(x,y) - \partial_2^\beta K(x,z)|^s + |\partial_1^\beta K(y,x) - \partial_1^\beta K(z,x)|^s dx \right)^{\frac{1}{s}}$$
$$\lesssim r^{-\lfloor \delta \rfloor} |A_j(z,r)|^{\frac{1}{s}-1} 2^{-j\delta}, \tag{36}$$

for all for $|y-z| < r$, where $0 < r < 1$, $j \in \mathbb{N}$, and ∂_1 and ∂_2 denote the partial derivative with respect to the first variable and the second variable respectively. We show that the condition (36) implies the condition (35). We take the polynomials

$$P^1_{x,z}(y) := \sum_{|\beta| \leqslant \lfloor \delta \rfloor} c_\beta (y-z)^\beta \partial_2^\beta K(z,y); \quad P^2_{x,z}(y) := \sum_{|\beta| \leqslant \lfloor \delta \rfloor} c_\beta (y-z)^\beta \partial_1^\beta K(z,x),$$

namely the Taylor polynomials of K expanding in the y-variable centered at z. Then, if we expand K using Taylor's theorem with derivative remainder, we have

$$K(x,y) = \sum_{|\beta| \leqslant \lfloor \delta \rfloor - 1} c_\beta (y-z)^\beta \partial_2^\beta K(x,z) + \sum_{|\beta| = \lfloor \delta \rfloor} c_\beta (y-z)^\beta \partial_2^\beta K(x, \xi_{y,z})$$

and

$$K(y,x) = \sum_{|\beta| \leqslant \lfloor \delta \rfloor - 1} c_\beta (y-z)^\beta \partial_2^\beta K(z,x) + \sum_{|\beta| = \lfloor \delta \rfloor} c_\beta (y-z)^\beta \partial_2^\beta K(\xi'_{y,z}, x)$$

for some $\xi_{y,z}, \xi'_{y,z}$ lying on the line segment joining y and z, and for some constant c_β depending on β only.

Therefore,

$$K(x,y) - P^1_{x,z}(y) = \sum_{|\beta| = \lfloor \delta \rfloor} c_\beta (y-z)^\beta [\partial_2^\beta K(x, \xi_{y,z}) - \partial_2^\beta K(x,z)]$$

and $|\xi_{y,z} - y| \leqslant |y - z| < r$. We now can apply condition (36) and get

$$\begin{aligned}
&\left(\int_{A_j(z,r)} |K(x,y) - P^1_{x,z}(y)|^s dx\right)^{\frac{1}{s}} \\
&\leqslant \sum_{|\beta| = \lfloor\delta\rfloor} |c_\beta| r^{\lfloor\delta\rfloor} \left(\int_{A_j(z,r)} |\partial_2^\beta K(x, \xi_{y,z}) - \partial_2^\beta K(x,z)|^s dx\right)^{\frac{1}{s}} \\
&\lesssim \sum_{|\beta| = \lfloor\delta\rfloor} |c_\beta| r^{\lfloor\delta\rfloor} r^{-\lfloor\delta\rfloor} |A_j(z,r)|^{\frac{1}{s}-1}\, 2^{-j\delta} \\
&\lesssim_\delta |A_j(z,r)|^{\frac{1}{s}-1}\, 2^{-j\delta}.
\end{aligned}$$

The same argument shows that

$$\begin{aligned}
&\left(\int_{A_j(z,r)} |K(y,x) - P^2_{x,z}(y)|^s dx\right)^{\frac{1}{s}} \\
&\leqslant \sum_{|\beta| = \lfloor\delta\rfloor} |c_\beta| r^{\lfloor\delta\rfloor} \left(\int_{A_j(z,r)} |\partial_2^\beta K(\xi'_{y,z}, x) - \partial_2^\beta K(z,x)|^s dx\right)^{\frac{1}{s}} \\
&\lesssim_\delta |A_j(z,r)|^{\frac{1}{s}-1}\, 2^{-j\delta}.
\end{aligned}$$

Therefore, the condition (35) is weaker than (36).

3. Condition (35) should be compared with Campanato-type condition given in Definition 2.2.12. The idea of subtracting some polynomial instead of subtracting the unique polynomial as in Definition 2.2.12 can be traced back to the work of Calderón [15], where he defined the maximal function

$$N(F,x) := \sup_{\rho > 0} \rho^{-m} \left(\fint_{B(x,\rho)} |F(y) - P(x,y)|^s dy\right)^{\frac{1}{s}}$$

assuming there exists some polynomial $P(x,\cdot)$ of degree at most $m-1$ ($m \in \mathbb{N}$) such that $N(F,x) < \infty$.

Before we provide the sufficient conditions, we need to define the meaning of $T^*(f)$. We will define this for f being a monomial.

Definition 5.1.2 ([31, Definition 5.1])**.** *Let $N \in \mathbb{N}_0$. We denote by $L^2_{c,N}(\mathbb{R}^n)$ the space of all $g \in L^2(\mathbb{R}^n)$ compactly supported functions such that $\int g(x)x^\alpha = 0$ for all $|\alpha| \leqslant N$. For such an α, define $T^*(x^\alpha)$ in the distributional sense by*

$$\langle T^*(x^\alpha), g\rangle := \langle x^\alpha, T(g)\rangle = \int_{\mathbb{R}^n} x^\alpha T g(x) dx \tag{37}$$

for all $g \in L^2_{c,N}(\mathbb{R}^n)$.

The space $L^2_{c,N_p}(\mathbb{R}^n)$ corresponds to multiples of $(p,2)$ atoms in $H^p(\mathbb{R}^n)$. That $T^*(x^\alpha)$ is well-defined by (37) has been stated for standard Calderón-Zygmund operators in [86, p. 23].

We need to verify that the integral on the right-hand-side of (5.1.2) converges absolutely. From the definition of the space $L^2_{c,N}(\mathbb{R}^n)$, for any $N \in \mathbb{N}_0$ we have

$$L^2_{c,N+1}(\mathbb{R}^n) \subset L^2_{c,N}(\mathbb{R}^n) \subset L^2_{c,0}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n).$$

Proposition 5.1.3 ([31, Proposition 5.2]). *Let T be an operator given by Definition 5.1.1 with $s = 1$ and some $\mu, \delta > 0$. Then $x^\alpha T g \in L^1(\mathbb{R}^n)$ for all $g \in L^2_{c,\lfloor\delta\rfloor}(\mathbb{R}^n)$ given that $|\alpha| < \min\{\mu, \delta\}$.*

Proof of Proposition 5.1.3.
The proof is taken from the proof in [31, Proposition 5.2] with adaptation. Let $g \in L^2_{c,0}(\mathbb{R}^n)$, fix a ball $B = B(x_B, r)$ containing the support of f, and write

$$\int_{\mathbb{R}^n} |x^\alpha T g(x)| dx = \int_{2B} |x^\alpha T g(x)| dx + \int_{(2B)^c} |x^\alpha T g(x)| dx.$$

From the boundedness of T on $L^2(\mathbb{R}^n)$ we get

$$\int_{2B} |x^\alpha T g(x)| dx \leqslant \|x^\alpha\|_{L^\infty(2B)} \, |2B|^{\frac{1}{2}} \, \|Tg\|_{L^2} \lesssim r^{|\alpha| + \frac{n}{2}} \|g\|_{L^2} < \infty.$$

Suppose now that $r \geqslant 1$. The estimation of the second integral follows by (18):

$$\begin{aligned}
\int_{(2B)^c} |x|^{|\alpha|} |Tg(x)| dx &\leqslant \sum_{j \in \mathbb{N}} \int_{A_j(x_B, r)} \int_B |K(x,y)| \, |g(y)| \, (|x_B| + 2^j r)^{|\alpha|} dy dx \\
&\lesssim \|g\|_{L^2} \, r^{\frac{n}{2}} \sum_{|\alpha'| \leqslant |\alpha|} |x_B|^{|\alpha| - |\alpha'|} \sum_{j \in \mathbb{N}} (2^j r)^{|\alpha'|} \int_{A_j(x_B, r)} |x - x_B|^{-n-\mu} dx \\
&\lesssim \|g\|_{L^2} \sum_{|\alpha'| \leqslant |\alpha|} |x_B|^{|\alpha| - |\alpha'|} \, r^{|\alpha'| + \frac{n}{2} - \mu} \sum_{j \in \mathbb{N}} (2^j)^{|\alpha'| - \mu} < \infty
\end{aligned}$$

since $|\alpha'| \leqslant |\alpha| < \mu$.

For $r < 1$, since g has vanishing moments up to order $\lfloor\delta\rfloor$, the bound follows by applying (35):

$$\begin{aligned}
\int_{(2B)^c} |Tg(x) x^\alpha| dx &\leqslant \sum_{j \in \mathbb{N}} \int_{A_j(x_B, r)} \int_B |K(x,y) - P^1_{x, x_B}(y)| \, |g(y)| \, |x|^{|\alpha|} dy dx \\
&\lesssim \|g\|_{L^2} \sum_{|\alpha'| \leqslant |\alpha|} |x_B|^{|\alpha| - |\alpha'|} \, r^{|\alpha'| + \frac{n}{2} - \delta} \sum_{j \in \mathbb{N}} (2^j)^{|\alpha'| - \delta} < \infty
\end{aligned}$$

since $|\alpha'| < \delta$. This completes the proof. □

Theorem 5.1.4 (cf. [31, Theorem 5.3]). *Let $0 < p \leqslant 1$ and T be an operator given by Definition 5.1.1 for some $1 \leqslant s \leqslant 2$ with $p < s$ and $\lfloor\kappa\rfloor = N_p$. Then T can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to itself provided that $\min\{\mu, \delta\} > \gamma_p$ and there exists $C > 0$ such that for any ball $B \subset \mathbb{R}^n$ with $r(B) < 1$ and $\alpha \in (\mathbb{N}_0)^n$ with $|\alpha| \leqslant N_p$,*

$$f = T^*[(\cdot - x_B)^\alpha] \quad \textit{satisfies} \quad \left(\fint_B |f(y) - P_B^{N_p}(f)(y)|^2 dy \right)^{1/2} \leqslant C \, \Psi_{p,\alpha}(r(B)), \tag{38}$$

where $P_B^{N_p}(f)$ is the polynomial of degree $\leqslant N_p$ that has the same moments as f over B up to order N_p, and for φ_p is as in Definition 4.1.1,

$$\Psi_{p,\alpha}(t) := \begin{cases} t^{\gamma_p} & \text{if } |\alpha| < \gamma_p, \\ t^{\gamma_p}\varphi_p(t) & \text{if } |\alpha| = \gamma_p = N_p \in \mathbb{N}_0, \end{cases}$$

with $\omega > 0$.

<u>*Proof of Theorem 5.1.4*</u>.
The proof is taken from the work [31] with adaptation. Let a be a $(p, 2)$ atom in the sense of Definition 2.5.7, supported in $B := B(x_B, r)$. We will show that Ta is a (p, s, λ, ω) molecule for λ satisfying $\gamma_p < \frac{\lambda}{s} - \frac{n}{s'} < \min\{\mu, \delta\}$. By Theorem 4.1.5 and the h^p analogue of the results of [84], this suffices to show the boundedness of T on h^p.

As $1 \leqslant s \leqslant 2$, condition (M1) follows from L^2-continuity of T:

$$\int_{2B} |Ta(x)|^s dx \lesssim |B|^{1-\frac{s}{2}} \|Ta\|_{L^2}^s \lesssim |B|^{1-\frac{s}{2}} \|a\|_{L^2}^s \lesssim r(B)^{n(1-\frac{s}{p})}.$$

For (M2), suppose first $r \geqslant 1$. From condition (18) it follows that for $|x - x_B| > 2r$ we have $|K(x, y)| \lesssim |x - x_B|^{-n-\mu}$ for all $y \in B$ and therefore

$$|Ta(x)| \leqslant \int_B |K(x, y)|\, |a(y)| dy \lesssim \|a\|_{L^2} |B|^{1/2} \int_B |x - x_B|^{-n-\mu} \lesssim r^{-\gamma_p} |x - x_B|^{-n-\mu}.$$

Then, for λ satisfying $\frac{\lambda}{s} - \frac{n}{s'} < \mu$, which means $\lambda - s(n + \mu) < -n$, we have

$$\int_{(2B)^c} |Ta(x)|^s \, |x - x_B|^\lambda dx \lesssim r^{-s\gamma_p} \int_{(2B)^c} |x - x_B|^{\lambda - s(n+\mu)} dx \lesssim r^{\lambda + n\left(1 - \frac{s}{p}\right)}.$$

Condition (M3) follows from (1).

Suppose now $r < 1$. Using the vanishing moment condition of the atom and (35) we can apply Minkowski inequality for integrals to get

$$\begin{aligned}
&\int_{(2B)^c} |Ta(x)|^s |x - x_B|^\lambda dx \\
&= \sum_{j=0}^{\infty} \int_{A_j(x_B, r)} \left| \int_B [K(x, y) - P^1_{x, x_B}(y)] a(y) dy \right|^s |x - x_B|^\lambda dx \\
&\leqslant \sum_{j=0}^{\infty} \left\{ \left[\int_{A_j(x_B, r)} \left(\int_B |K(x, y) - P^1_{x, x_B}(y)|\; |a(y)|\; |x - x_B|^{\frac{\lambda}{s}} dy \right)^s dx \right]^{\frac{1}{s}} \right\}^s \\
&\leqslant \sum_{j=0}^{\infty} (2^{j+1} r)^\lambda \left\{ \int_B |a(y)| \left[\int_{A_j(x_B, r)} |K(x, y) - P^1_{x, x_B}(y)|^s \; dx \right]^{\frac{1}{s}} dy \right\}^s \\
&\lesssim \sum_{j=0}^{\infty} (2^j r)^\lambda \; (2^j r)^{-n(s-1)} \; 2^{-js\delta} \; \left(\int_B |a(y)| dy \right)^s \\
&\leqslant \sum_{j=0}^{\infty} (2^j r)^\lambda \; (2^j r)^{-n(s-1)} \; 2^{-js\delta} \; r^{-s\gamma_p} \\
&= C \; r^{\lambda + n\left(1 - \frac{s}{p}\right)} \sum_{j=0}^{\infty} 2^{j[\lambda - n(s-1) - s\delta]} = C' \; r^{\lambda + n\left(1 - \frac{s}{p}\right)}
\end{aligned}$$

assuming $\lambda < n(s-1) + s\delta$, which is the same as $\frac{\lambda}{s} - \frac{n}{s'} < \delta$. Finally, in order to verify that (M3) holds, note that for $r < 1$ the function a is in particular a $(p, 2)$ atom in $H^p(\mathbb{R}^n)$ with full cancellation condition. From condition (38), setting $f = T^*[(\cdot - x_B)^\alpha]$, we have, by (37),

$$\begin{aligned} \left| \int Ta(x)(x - x_B)^\alpha dx \right| &= |\langle f, a\rangle| \leqslant \left(\int_B |f(y) - P_B^{N_p}(f)(y)|^2 dy \right)^{1/2} \|a\|_{L^2(B)} \\ &\lesssim \Psi_p(r)|B|^{1/2}\|a\|_{L^2(B)} \\ &\lesssim \Psi_p(r) r^{-\gamma_p} \end{aligned}$$

which is bounded by $C_{n,p}$ if $|\alpha| < \gamma_p$ and by $\varphi_p(r(B))$ if $|\alpha| = \gamma_p$ from (38). □

We now define inhomogeneous strongly singular integral operators.

Definition 5.1.5 ([31, Section 5.2])**.** *We say T is an inhomogeneous strongly singular integral operator with L^s_δ integral-type condition associated with K, where $1 \leqslant s < \infty$, $\mu > 0$ and $\delta > 0$ if*

1. *the kernel K satisfies (18);*
2. *for each $x, z \in \mathbb{R}^n$ there exist two polynomials $P^1_{x,z}(y)$ and $P^2_{x,z}(y)$ with degree at most $\lfloor \delta \rfloor$ such that*

$$\begin{aligned} &\left(\int_{A_j(z, r^\rho)} |K(x,y) - P^1_{x,z}(y)|^s + |K(y,x) - P^2_{x,z}(y)|^s dx \right)^{\frac{1}{s}} \\ &\lesssim |A_j(z, r^\rho)|^{\frac{1}{s} - 1 + \frac{\delta}{n}\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}} \end{aligned} \tag{39}$$

   for $|y - z| < r$, where $0 < r < 1$, $j \in \mathbb{N}$, $\delta > 0$, $0 < \rho \leqslant \sigma \leqslant 1$, and

$$A_j(z, r) := \left\{ x \in \mathbb{R}^n : \ 2^j r \leqslant |x - z| < 2^{j+1} r \right\}$$

3. *the operator T is given by*

$$\langle Tf, g\rangle = \int \int K(x,y) f(y) g(x) dy dx, \qquad \forall \ f, g \in \mathcal{S}(\mathbb{R}^n) \text{ with disjoint supports}$$

   and extends to a bounded operator on $L^2(\mathbb{R}^n)$ as well as $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for some $\beta \in [(1-\rho)\frac{n}{2}, \frac{n}{2})$ and $\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}$.

Theorem 5.1.6 ([31, Theorem 5.8])**.** *Let $0 < p \leqslant 1$ and T an operator in Definition 5.1.5 for some $\delta > 0$ and $1 \leqslant s \leqslant 2$ with $p < s$. Then T can be extended to a bounded operator from $h^p(\mathbb{R}^n)$ to itself provided that*

$$\max\left\{ \frac{n}{n+\mu}, p_0 \right\} < p \leqslant 1 \quad \textit{where} \quad \frac{1}{p_0} := \frac{1}{2} + \frac{\beta\left(\frac{\delta}{\sigma} + \frac{n}{2}\right)}{n\left(\frac{\delta}{\sigma} - \delta + \beta\right)}$$

and the cancellation condition (38) holds.

Proof of Theorem 5.1.6.
The proof is taken from the paper [31] with adaptation. Let a be a $(p,2)$ atom in the sense of Definition 2.5.7, supported in $B := B(x_B, r)$. We will show that Ta is a (p,s,λ,ω) molecule for λ satisfying

$$\gamma_p < \frac{\lambda}{s} - \frac{n}{s'} < \min\left\{\mu, \gamma_{p_0}\right\}, \quad \gamma_{p_0} := n\left(\frac{1}{p_0} - 1\right) = -\frac{n}{2} + \frac{\beta\left(\frac{n}{2} + \frac{\delta}{\sigma}\right)}{\beta + \frac{\delta}{\sigma} + \delta}.$$

If $r \geqslant 1$, conditions (M1) and (M2) will follow by the same arguments presented in the proof of Theorem 5.1.4, provided $\frac{\lambda}{s} - \frac{n}{s'} < \mu$.

Suppose now that $r < 1$. Analogously to [3, Lemma 2.1], we will actually show some better estimates on Ta. In fact, since $1 \leqslant s \leqslant 2$, from the stronger continuity $L^q - L^2$ assumption it follows

$$\int_B |Ta(x)|^s dx \leqslant |B|^{1-\frac{s}{2}} \|Ta\|_{L^2}^s \lesssim |B|^{1-\frac{s}{2}} \|a\|_{L^q}^s \lesssim |B|^{1+\frac{s}{q}-s} \|a\|_{L^2}^s \lesssim |B|^{1-\frac{s}{p}+s\left(\frac{1}{q}-\frac{1}{2}\right)}$$

and so (M1) holds since $1/q - 1/2 \geqslant 0$.

To show (M2), consider $0 < \rho \leqslant \sigma \leqslant 1$, where ρ is a parameter that will be chosen conveniently later. Denote by $2B^\rho := B(x_B, 2r^\rho)$ and split

$$\begin{aligned}\int_{\mathbb{R}^n} |Ta(x)|^s |x - x_B|^\lambda dx &= \int_{2B^\rho} |Ta(x)|^s |x - x_B|^\lambda dx + \int_{(2B^\rho)^c} |Ta(x)|^s |x - x_B|^\lambda dx \\ &:= I_1 + I_2.\end{aligned}$$

To estimate I_1, we use the $L^q - L^2$ continuity again and get

$$\begin{aligned}\int_{2B^\rho} |Ta(x)|^s |x - x_B|^\lambda dx &\lesssim r^{\lambda\rho} |B^\rho|^{1-\frac{s}{2}} \|Ta\|_{L^2}^s \lesssim r^{\rho\lambda + n\left(\rho - \frac{s\rho}{2}\right)} \|a\|_{L^q}^s \\ &\lesssim r^{\rho\lambda + n\left[\rho - \frac{s\rho}{2} + s\left(\frac{1}{q} - \frac{1}{p}\right)\right]} \lesssim r^{\lambda + n\left(1 - \frac{s}{p}\right)}\end{aligned}$$

assuming

$$\lambda \leqslant -n\left(1 - \frac{s}{2}\right) + \frac{ns}{1-\rho}\left(\frac{1}{q} - \frac{1}{2}\right). \tag{40}$$

Note that this control would not be possible using only the L^2-boundedness. For I_2, we use

(39) and then

$$
\begin{aligned}
&\int_{(2B^\rho)^c} |Ta(x)|^s |x-x_B|^\lambda dx \\
&\leqslant \sum_{j=0}^{\infty} (2^j r^\rho)^\lambda \left\{ \int_B |a(y)| \left[\int_{A_j(x_B, r^\rho)} |K(x,y) - P^1_{x,x_B}(y)|^s dx \right]^{\frac{1}{s}} dy \right\}^s \\
&\lesssim \sum_{j=0}^{\infty} (2^j r^\rho)^\lambda \left(|A_j(x_B, r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}} \right)^s \|a\|^s_{L^2} |B|^{\frac{s}{2}} \\
&\lesssim \sum_{j=0}^{\infty} (2^j r^\rho)^\lambda \left(|A_j(x_B, r^\rho)|^{\frac{1}{s}-1+\frac{\delta}{n}\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)} 2^{-\frac{j\delta}{\rho}} \right)^s r^{sn\left(1-\frac{1}{p}\right)} \\
&= C\, r^{\rho\lambda + n\left[s+\frac{s\delta}{n} - s\rho\left(1-\frac{1}{s}+\frac{\delta}{n\sigma}\right) - \frac{s}{p}\right]} \sum_{j=0}^{\infty} 2^{j\left[\lambda - n(s-1) - \frac{s\delta}{\sigma}\right]} \qquad (41) \\
&\lesssim r^{\rho\lambda + n\left[\rho\left(1-\frac{s}{2}\right) + s\left(\frac{1}{q}-\frac{1}{p}\right)\right]} \lesssim r^{\lambda + n\left(1-\frac{s}{p}\right)}
\end{aligned}
$$

in which we choose ρ to be such that

$$
s + \frac{s\delta}{n} - \rho\left(s - 1 + \frac{s\delta}{n\sigma}\right) = \rho\left(1 - \frac{s}{2}\right) + \frac{s}{q}, \quad \text{i.e.} \quad \rho := \frac{n\left(1-\frac{1}{q}\right)+\delta}{\frac{n}{2}+\frac{\delta}{\sigma}}.
$$

The convergence of the series in (41) follows from (40), since by the choice of ρ we have

$$
-n\left(1-\frac{s}{2}\right) + \frac{ns}{1-\rho}\left(\frac{1}{q}-\frac{1}{2}\right) < n(s-1) + s\delta < n(s-1) + \frac{s\delta}{\sigma}.
$$

In particular, the restriction on λ for this particular choice of ρ is

$$
\lambda \leqslant -n\left(1-\frac{s}{2}\right) + \frac{s\beta\left(\frac{n}{2}+\frac{\delta}{\sigma}\right)}{\beta + \frac{\delta}{\sigma} + \delta}.
$$

For the validity of (M3) we proceed in the same way as in the proof of Theorem 5.1.4. Therefore, Ta is a (p, s, λ, ω) molecule provided that $\max\left\{\dfrac{n}{n+\mu}, p_0\right\} < p \leqslant 1$. $\square$

5.2 Necessary Conditions

In this subsection, we will prove that the cancellation conditions (38) are also necessary for the singular (or strongly singular) integral operator to be bounded on $h^p(\mathbb{R}^n)$.

As we have seen in the example in Section 2.5.1, the function $\frac{1}{r}e^{-\frac{x^2}{r^2}}$ has h^1 norm $\frac{\sqrt{\pi}}{2} + \frac{1}{\sqrt{2e}}\log(1+r^{-2})$ while its integral is $\frac{\sqrt{\pi}}{2}$. Therefore, the ratio of its integral to its h^1 norm is approximately $[\log(1+r^{-1})]^{-1}$. Let us now observe the general case for compactly supported distributions in $h^p(\mathbb{R}^n)$.

Proposition 5.2.1 ([32, Proposition 1]).
Let $g \in h^p(\mathbb{R}^n)$ be supported in $B(x_0, r)$ for some $x_0 \in \mathbb{R}^n$ and $0 < r < 1$. Then for $\alpha \in \mathbb{N}_0^n$, the moments $\langle g, (\cdot - x_0)^\alpha \rangle$ are well-defined and satisfy

$$|\langle g, (\cdot - x_0)^\alpha \rangle| \leqslant \begin{cases} C_{\alpha,p} \, \|g\|_{h^p} & \text{if } |\alpha| < \gamma_p; \\ C_{\alpha,p} \, \|g\|_{h^p} \left[\log\left(1 + \frac{1}{r}\right) \right]^{-1/p} & \text{if } |\alpha| = \gamma_p = N_p. \end{cases} \tag{42}$$

Proof of Proposition 5.2.1.
The proof is taken from [32].

Since g is a distribution of compact support and hence acts on $C^\infty(\mathbb{R}^n)$, we can define $\langle g, (\cdot - x_0)^\alpha \rangle$ unambiguously for any multi-index $\alpha \in \mathbb{N}_0^n$, and $\langle g, (\cdot - x_0)^\alpha \rangle = \langle g, \phi \rangle$ for all $\phi \in C^\infty(\mathbb{R}^n)$ such that $\phi(y) = (y - x_0)^\alpha$ on the support of g.

By a translation argument we may assume that $x_0 = 0$. For each unit vector on $v \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leqslant N_p$, we choose $\phi_0^{v,\alpha}$ satisfying the following conditions:

(i) $\phi_0^{v,\alpha} \in C_c^\infty(\mathbb{R}^n)$ with support in $B\left(\frac{v}{2}, 2\right)$ and $\|\partial^\beta \phi_0^{v,\alpha}\|_{L^\infty} \leqslant 2^{|\beta|-2n}$ for all $|\beta| \leqslant N_p + 1$;

(ii) $\phi_0^{v,\alpha}(y) = C_\alpha \, y^\alpha$ for all $|y| < 1$ for some constant C_α depending only on n and α;

(iii) $\int \phi_0^{v,\alpha}(y) dy \neq 0$.

For each $x \in \mathbb{R}^n$ with $|x| > \frac{r}{2}$ we define

$$\phi^{x,\alpha}(y) = \frac{1}{|x|^n} \, \phi_0^{\frac{x}{|x|},\alpha}\left(\frac{y}{2|x|}\right).$$

We claim $\phi^{x,\alpha} \in \mathcal{F}_k^{T,x}$ for $T = 2$ and $k \leqslant N_p + 1$. Indeed, note first that $\operatorname{supp}(\phi^{x,\alpha}) \subset B(x,t)$ for $t = 4|x|$ since if $|y - x| > t$ we have

$$\left| \frac{y}{2|x|} - \frac{x}{2|x|} \right| = \frac{|y-x|}{2|x|} \geqslant \frac{t}{2|x|} = 2$$

and then $\phi_0^{\frac{x}{|x|},\alpha}(y/2|x|) = 0$. Moreover, for $|\beta| \leqslant N_p + 1$,by assumption (i),

$$\left\| \partial^\beta \, \phi^{x,\alpha} \right\|_{L^\infty} = 2^{-|\beta|} |x|^{-n-|\beta|} \left\| \partial^\beta \, \phi_0^{\frac{x}{|x|},\alpha} \right\|_{L^\infty} \leqslant t^{-n-|\beta|}.$$

On the support of g, $|y| < r$ and $|x| > \frac{r}{2}$ so $\frac{|y|}{2|x|} < 1$ and by assumption (ii), $\phi^{x,\alpha}(y) = \frac{C_\alpha \, y^\alpha}{|x|^{n+|\alpha|}}$. Hence

$$\mathcal{M}_{\mathcal{F}_k}(g)(x) = \sup_{\phi \in \mathcal{F}_k^{T,x}} |\langle g, \phi \rangle| \geqslant |\langle g, \phi^{x,\alpha} \rangle| = C_\alpha |x|^{-n-|\alpha|} \, |\langle g, (\cdot - x_0)^\alpha \rangle| \, .$$

When $|\alpha| = \gamma_p = N_p$, this gives

$$\begin{aligned} \|g\|_{h^p}^p &\geqslant \int_{\frac{r}{2} < |x| < \frac{r+1}{2}} [\mathcal{M}_{\mathcal{F}_k}(f)(x)]^p dx \\ &\geqslant C_\alpha \, |\langle g, (\cdot - x_0)^\alpha \rangle|^p \int_{\frac{r}{2} < |x| \leqslant \frac{r+1}{2}} |x|^{-p(n+|\alpha|)} dx \\ &\geqslant C_\alpha \, |\langle g, (\cdot - x_0)^\alpha \rangle|^p \, \log\left(1 + \frac{1}{r}\right). \end{aligned}$$

When $|\alpha| < \gamma_p$, we consider $1 < |x| < \frac{3}{2}$. Since in particular $|x| > \frac{r}{2}$, the same calculations as above give

$$\begin{aligned}\|g\|_{h^p}^p &\geqslant \int_{1<|x|<\frac{3}{2}} [\mathcal{M}_{\mathcal{F}_k}(g)(x)]^p dx \geqslant C_\alpha \left|\langle g, (\cdot - x_0)^\alpha \rangle\right|^p \int_{1<|x|\leqslant\frac{3}{2}} |x|^{-p(n+|\alpha|)} dx \\ &= C_{n,\alpha,p} \left|\langle g, (\cdot - x_0)^\alpha \rangle\right|^p .\end{aligned}$$

□

Before we move on to a larger class of $h^p(\mathbb{R}^n)$ distributions, we shall observe that in the decomposition of a molecule (25) in the proof of Proposition 4.1.4, a molecule can be written as a sum of three components, the first two terms are infinite sums of H^p atoms, which are distributions in $H^p(\mathbb{R}^n)$, while the last term is an h^p atom. This observation leads to the following definition.

Definition 5.2.2 ([32, Definition 2])**.**
Let $\mathcal{C} > 0$ be a constant. We say $\mathscr{M} \in \mathcal{S}'(\mathbb{R}^n)$ is a $\mathcal{C}$-pseudo-molecule in $h^p(\mathbb{R}^n)$ associated to the ball B if $\mathscr{M} = g + h$ in $\mathcal{S}'(\mathbb{R}^n)$, where $g \in h^p(\mathbb{R}^n)$ supported in B, $h \in H^p(\mathbb{R}^n)$, and

$$\|g\|_{h^p} + \|h\|_{H^p} \leqslant \mathcal{C}.$$

Proposition 5.2.3 ([32, Proposition 2])**.** *Let $0 < p \leqslant 1$ and $\mathscr{M}$ a $\mathcal{C}$-pseudo-molecule in $h^p(\mathbb{R}^n)$ associated to the ball $B = B(x_0, r)$ with $0 < r < 1$. Then for $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leqslant N_p$, the moments $\langle \mathscr{M}, (\cdot - x_0)^\alpha \rangle$ are well-defined and satisfy*

$$\left|\langle \mathscr{M}, (\cdot - x_0)^\alpha \rangle\right| \lesssim \begin{cases} C_{\alpha,p}\,\mathcal{C} & \text{if } |\alpha| < \gamma_p; \\ \\ C_{\alpha,p}\,\mathcal{C}\left[\log\left(1+\dfrac{1}{r}\right)\right]^{-1/p} & \text{if } |\alpha| = \gamma_p = N_p. \end{cases} \tag{43}$$

<u>*Proof of Proposition 5.2.3.*</u>
The following proof is taken directly from [32]. Writing $\mathscr{M} = g + h$ as in Definition 5.2.2, since $h \in H^p(\mathbb{R}^n)$ satisfies vanishing moment conditions up the order N_p, we have $\langle h, (\cdot - x_0)^\alpha \rangle = 0$ (the pairing here is the one between H^p and its dual space, the homogeneous Lipschitz space $\dot{\Lambda}_{n(1/p-1)}$).

For $g \in h^p(\mathbb{R}^n)$ supported in B, the moments $\langle g, (\cdot - x_0)^\alpha \rangle$ can be defined as in Proposition 5.2.1. Thus we can set

$$\langle \mathscr{M}, (\cdot - x_0)^\alpha \rangle := \langle g, (\cdot - x_0)^\alpha \rangle + \langle h, (\cdot - x_0)^\alpha \rangle = \langle g, (\cdot - x_0)^\alpha \rangle.$$

If $\mathscr{M}$ has an alternative decomposition $g' + h'$ satisfying the conditions of Definition 5.2.2, then we must have that $g - g' = h' - h \in H^p$ and therefore the moments of g' are the same as those of g.

The estimates (43) now follow immediately from (42). □

Now we give an example of pseudo-molecules, which are (p, s, λ, ω) molecules (Definition 4.1.3) without (M3).

Definition 5.2.4 ([32, Definition 3])**.** *Let $0 < p \leqslant 1 \leqslant s < \infty$ with $p < s$, $\lambda > n\,(s/p - 1)$, and $C > 0$. We say that a measurable function $M \in h^p(\mathbb{R}^n)$ is a (p, s, λ, C) pre-molecule in $h^p(\mathbb{R}^n)$ if there exist a ball $B(x_0, r) \subset \mathbb{R}^n$ and a constant $C > 0$ such that*

(M1) $\|M\|_{L^s(B)} \leqslant C\, r^{n\left(\frac{1}{s}-\frac{1}{p}\right)}$;

(M2) $\|M\,|\cdot - x_0|^{\frac{\lambda}{s}}\,\|_{L^s(B^c)} \leqslant C\, r^{\frac{\lambda}{s}+n\left(\frac{1}{s}-\frac{1}{p}\right)}$.

To see it is a $\mathcal{C}$-pseudo-molecule for some $\mathcal{C}$, we will follow the proof of [32, Lemma 2]. We observe that from the decomposition (25), the condition (M3) is only used when we estimate the term a_ω. If we take $h := \sum_{k=0}^{\infty} t_k a_k + \sum_{k=0}^{\infty} s_k b_k$ and $g := a_\omega$, then from (30) and (31),

$$\|h\|_{H^p(\mathbb{R}^n)} \leqslant C_{n,p,s,\lambda}$$

and

$$\|g\|_{h^p(\mathbb{R}^n)} \leqslant \|M\|_{h^p(\mathbb{R}^n)} + \|h\|_{h^p(\mathbb{R}^n)} \leqslant \|M\|_{h^p(\mathbb{R}^n)} + \|h\|_{H^p(\mathbb{R}^n)} \leqslant \|M\|_{h^p(\mathbb{R}^n)} + C_{n,p,s,\lambda}.$$

We can now state the main theorem in this subsection.

Theorem 5.2.5 ([32, Theorem 1]).
Let $0 < p \leqslant 1$, $\mathcal{C} > 0$, and T be a linear and bounded operator on $h^p(\mathbb{R}^n)$ that maps each $(p,2)$ atom in $h^p(\mathbb{R}^n)$ into a $\mathcal{C}$-pseudo-molecule centered in the same ball as the support of the atom. Then the following cancellation conditions must hold:

For any ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $r < 1$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leqslant N_p := \lfloor \gamma_p \rfloor$, $\gamma_p := n(\frac{1}{p} - 1)$,

$$f = T^*[(\cdot - x_0)^\alpha] \quad \textit{satisfies} \quad \left(\fint_B |f(y) - P_B^{N_p}(f)(y)|^2 dy \right)^{1/2} \leqslant C\, \Psi_{p,\alpha}(r), \tag{44}$$

where $P_B^{N_p}(f)$ is the polynomial of degree less then or equal to N_p that has the same moments as f over B up to order N_p, and

$$\Psi_{p,\alpha}(t) := \begin{cases} t^{\gamma_p} & \textit{if } |\alpha| < \gamma_p, \\ t^{\gamma_p}\left[\log\left(1 + \frac{1}{t}\right)\right]^{-1/p} & \textit{if } |\alpha| = \gamma_p = N_p. \end{cases}$$

Proof of Theorem 5.2.5.
The proof is taken from [32]. Let $0 < p \leqslant 1$ and T be a linear and bounded operator on $h^p(\mathbb{R}^n)$ that maps each $(p,2)$ atom in $h^p(\mathbb{R}^n)$ into a pseudo-molecule centered in the same ball as the support of the atom.

As Definition 5.1.2 and Proposition 5.1.3 rely on the specific form of the operators considered there, namely those with a nice kernel, we first need to make sense of the cancellation conditions (44) in this more general context. Fix $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant N_p$. We want to show $T^*\left[(\cdot - x_0)^\alpha\right]$ is well defined locally in the following sense.

Fix a ball $B = B(x_0, r) \subset \mathbb{R}^n$ with $r < 1$. We will show that $T^*\left[(\cdot - x_0)^\alpha\right]$ can be identified with f in $(L^2_{c,N_p}(B))^*$. Here $L^2_{c,N_p}(B)$ denotes the space of functions in $L^2(B)$ with vanishing moments up to order N_p, and its dual space can be identified with the quotient of $L^2(B)$ by the subspace $\mathcal{P}_{N_p}$ of polynomials of order up to N_p. We then have

$$\|f\|_{(L^2_{c,N_p}(B))^*} := \sup_{\substack{\psi \in L^2_{c,N_p}(B) \\ \|\psi\|_{L^2(B)} \leqslant 1}} |\langle f, \psi \rangle| = \inf_{P \in \mathcal{P}_{N_p}} \|f - P\|_{L^2(B)} = \|f - P_B^{N_p}(f)\|_{L^2(B)}, \tag{45}$$

where $P_B^{N_p}(f)$ is the element of $\mathcal{P}_{N_p}$ with the same moments as f over B up to order N_p.

Given a $\psi \in L^2_{N_p}(B)$ with $\|\psi\|_{L^2(B)} \leqslant 1$, let

$$a(x) = \psi(x)\,|B|^{\frac{1}{2}-\frac{1}{p}}.$$

Note that a is a $(p,2)$ atom supported on B (strictly speaking we have $\mathrm{supp}(a) \subset \overline{B}$ but in the calculation of the norm we may always take ψ of compact support in B). By the boundedness assumptions on T, $\|Ta\|_{h^p(\mathbb{R}^n)} \lesssim \|a\|_{h^p(\mathbb{R}^n)} \leqslant C$ independent of a and $\mathscr{M} = Ta$ is a pseudo-molecule, where the choice of the constant $\mathcal{C}$ in Definition 5.2.2 should be consistent with the norm of T. Therefore, by (43),

$$\begin{aligned} |\langle T^*[(\cdot - x_0)^\alpha],\, a\rangle| &:= |\langle(\cdot - x_0)^\alpha,\, Ta\rangle| \\ &\leqslant \begin{cases} C_{\alpha,p}\,\mathcal{C} & \text{if } |\alpha| < \gamma_p, \\ C_{\alpha,p}\,\mathcal{C}\Big[\log\Big(1+\frac{1}{r}\Big)\Big]^{-1/p} & \text{if } |\alpha| = \gamma_p = N_p. \end{cases} \end{aligned}$$

Replacing a by ψ, we see that the left-hand-side defines a bounded linear functional $f \in (L^2_{c,N_p}(B))^*$ with

$$\begin{aligned} |\langle f,\, \psi\rangle| &= |B|^{\frac{1}{p}-\frac{1}{2}}\,|\langle T^*[(\cdot - x_0)^\alpha],\, a\rangle| \\ &\leqslant \begin{cases} C_{\alpha,p}|B|^{\frac{1}{p}-\frac{1}{2}}\,\mathcal{C} & \text{if } |\alpha| < \gamma_p, \\ C_{\alpha,p}|B|^{\frac{1}{p}-\frac{1}{2}}\,\mathcal{C}\Big[\log\Big(1+\frac{1}{r}\Big)\Big]^{-1/p} & \text{if } |\alpha| = \gamma_p = N_p. \end{cases} \end{aligned}$$

Thus by (45), we have

$$\begin{aligned} \left(\fint_B |f - P_{N_p}(f)|^2\right)^{1/2} &= |B|^{-\frac{1}{2}}\left(\int_B |f - P_{N_p}(f)|^2\right)^{1/2} \\ &= |B|^{-\frac{1}{2}} \sup_{\substack{\psi\in L^2_{N_p}(B) \\ \|\psi\|_{L^2(B)}\leqslant 1}} |\langle f,\, \psi\rangle| \\ &\leqslant \begin{cases} C_{n,p}\, r^{\gamma_p} & \text{if } |\alpha| < \gamma_p, \\ C_{n,p}\, r^{\gamma_p}\Big[\log\Big(1+\frac{1}{r}\Big)\Big]^{-1/p} & \text{if } |\alpha| = \gamma_p = N_p, \end{cases} \\ &= C_{n,p}\ \Psi_{p,\alpha}(r). \end{aligned}$$

□

Since a special case of pseudo-molecules are pre-molecules (see Definition 5.2.4), we obtain the following T^* characterization result, in the spirit of [86, Proposition 4].

Corollary 5.2.6 ([32, Corollary 1])**.**
Let $0 < p \leqslant 1$ and $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a linear and bounded operator that maps each $(p,2)$ atom in $h^p(\mathbb{R}^n)$ into a pre-molecule centered in the same ball as the support of the atom. Then the cancellation conditions (44) hold if and only if T is bounded on $h^p(\mathbb{R}^n)$.

Proof of Corollary 5.2.6.
This proof is directly taken from [32].

One direction follows from the fact that pre-molecules are pseudo-molecules and Theorem 5.2.5: if $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is bounded on $h^p(\mathbb{R}^n)$ and takes each $(p, 2)$ atom to a pre-molecule centered in the same ball as the support of the atom, then it satisfies the hypotheses of the Theorem and the cancellation conditions (44) hold.

We now show the converse. Suppose $T : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. If, for some appropriate fixed constants s, λ and C, T takes each $(p, 2)$ atom in $h^p(\mathbb{R}^n)$ to a (p, s, λ, C) pre-molecule centered in the same ball as the support of the atom, and in addition it satisfies the cancellation conditions (44), then we want to show that it maps each $(p, 2)$ atoms to a bona fide molecule M as in Definition 4.1.3. By Proposition 4.1.4 such a molecule will have h^p norm bounded by a constant (depending on s, λ and C), so the atomic decomposition and the continuity of T on $\mathcal{S}'(\mathbb{R}^n)$ will give us the boundedness of T on $h^p(\mathbb{R}^n)$.

Since the size conditions (M1) and (M2) in Definition 5.2.4 are identical to the ones in Definition 4.1.3, it just remains to verify that (M3) in Definition 4.1.3 holds for some ω. This follows from the cancellation conditions (44) on T in the same way as at the end of the proof of Theorem 5.1.4. That argument does not use the specific properties of T besides the cancellation conditions and, of course, the definition of $T^* [(\cdot - x_0)^\alpha]$, which, as shown in the proof Theorem 5.2.5 above, is well defined precisely because T takes atoms to pre-molecules, which are pseudo-molecules. The constant ω in (M3) will depend on the constant C in (44). $\square$

In the case of inhomogeneous (strongly) Calderón–Zygmund operators, we can conclude that (44) are both sufficient and necessary conditions for the boundedness of such operators on $h^p(\mathbb{R}^n)$.

Theorem 5.2.7 ([32, Theorem 2])**.**
Let $0 < p \leqslant 1$ and T a (strongly) inhomogeneous Calderón–Zygmund operator given by Definition 5.1.1 (Definition 5.1.5, respectively). Then, T is bounded on $h^p(\mathbb{R}^n)$ iff the condition (44) holds.

Proof of Theorem 5.2.7.
The proof is taken from [32].

Assume $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a strongly singular inhomogeneous Calderón–Zygmund operator. This means it extends continuously from $L^2(\mathbb{R}^n)$ to itself, from $L^q(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, where

$$\frac{1}{q} = \frac{1}{2} + \frac{\beta}{n}, \quad \text{for some } \frac{n}{2}(1-\sigma) \leqslant \beta < \frac{n}{2}, \quad 0 < \sigma \leqslant 1.$$

The size conditions (18) and (35) (or (39) if it is a strongly singular integral operator) on the kernel, together with the boundedness assumptions on T, with no further cancellation assumption, imply that if a is a $(p, 2)$ atom in $h^p(\mathbb{R}^n)$, then Ta satisfies the size conditions of a molecule in $h^p(\mathbb{R}^n)$, namely (M1) and (M2) in Definition 5.2.4, as shown in the proofs of Theorem 5.1.4 and 5.1.6. The desired result is then a consequence of Corollary 5.2.6. $\square$

Chapter 6

Commutator of Inhomogeneous Singular Integral Operator

This chapter will discuss a new type of atom that generalizes Pérez atoms to the non-homogeneous case. Moreover, we will establish some properties of this space. We will apply these atoms to establish the boundedness of the commutator of an inhomogeneous singular integral operators with a bmo function.

This chapter is based on the work [30, Sections 4 and 5].

6.1 Commutator Hardy Spaces

We shall start with a local version of Ky's maximal operators.

Definition 6.1.1 ([30, Definition 4.2]). *Given $b \in L^2_{loc}(\mathbb{R}^n)$, we define the commutator maximal function of $f \in L^2_{loc}(\mathbb{R}^n)$ to be*

$$\mathcal{M}_b f(x) := [b, \mathcal{M}](x) := \mathcal{M}[b(x)f(\cdot) - b(\cdot)f(\cdot)](x),$$

where $\mathcal{M}$ here is the local grand maximal function in Definition 2.5.3 with $T = 1$ and $k = 1$.

We therefore have that for $b, f \in L^2_{loc}(\mathbb{R}^n)$,

$$\mathcal{M}_b f(x) = \sup_{\phi} \left|\langle b(x)f(\cdot) - b(\cdot)f(\cdot), \phi\rangle\right| = \sup_{\phi}\left|\int_{\mathbb{R}^n}[b(x) - b(y)]f(y)\phi(y)dy\right|.$$

Since $(b(x) - b)f \in L^1_{loc}(\mathbb{R}^n)$ for almost every $x \in \mathbb{R}^n$, the integral is well-defined almost everywhere, and by the size and support conditions on ϕ,

$$\mathcal{M}_b f(x) \lesssim M_b f(x). \tag{46}$$

For fixed $\phi \in C^1(\mathbb{R}^n)$ such that $\text{supp}(\phi) \subset B(x,r)$ and $\|\phi\|_{L^\infty} \leqslant |B(x,r)|^{-1}$. If $g \in L^2_{loc}(\mathbb{R}^n)$, then the function $g\phi$ is a constant multiple of an h^1-atom with norm $\|g\phi\|_{h^1}$ depending on r. Therefore, using Theorem 2.5.18, we have

$$\int_{\mathbb{R}^n}[b - b(x)]a\phi = {}_{\text{bmo}}\langle b - b(x), a\phi\rangle_{h^1}, \tag{47}$$

where ${}_{\mathrm{bmo}}\langle\cdot,\cdot\rangle_{h^1}$ denotes the dual pairing between h^1 and bmo.

From this observation, we can also define the following extension of the commutator maximal function directly for $f \in h^1(\mathbb{R}^n)$ with $b \in \mathrm{bmo}(\mathbb{R}^n)$.

Definition 6.1.2. *Given $b \in \mathrm{bmo}(\mathbb{R}^n)$, we define the commutator maximal function $\mathcal{M}'_b$ of $f \in h^1(\mathbb{R}^n)$ to be*

$$\mathcal{M}'_b f(x) := \sup_{\phi} |{}_{\mathrm{bmo}}\langle b - b(x), f\phi\rangle_{h^1}|,$$

where ${}_{\mathrm{bmo}}\langle\cdot,\cdot\rangle_{h^1}$ denotes the dual pairing between h^1 and bmo, and the supremum is taken over all $\phi \in C^1(\mathbb{R}^n)$ with $\mathrm{supp}(\phi) \subset B(x,t)$ for some $0 < t < 1$, $\|\phi\|_{L^\infty} \leqslant |B(x,t)|^{-1}$ and $\|\nabla\phi\|_{L^\infty} \leqslant [t|B(x,t)|]^{-1}$.

This maximal function is well-defined because $f\phi \in h^1(\mathbb{R}^n)$ and $b - b(x) \in \mathrm{bmo}(\mathbb{R}^n)$ for almost every x. From (47), we have $\mathcal{M}'_b(f)(x) = \mathcal{M}_b(f)(x)$ for all $f \in h^1(\mathbb{R}^n) \cap L^2_{loc}(\mathbb{R}^n)$. Thus, $\mathcal{M}'_b(f)(x) \lesssim M_b(f)(x)$ for all $f \in h^1(\mathbb{R}^n) \cap L^2_{loc}(\mathbb{R}^n)$.

The following is the definition of the local version of Pérez atoms. It also appears in [62].

Definition 6.1.3. *Fix $b \in \mathrm{bmo}(\mathbb{R}^n)$. We say a is a Pérez h^1_b atom if for some ball $B(x_0,r)$ we have*

1. $\mathrm{supp}(a) \subset B(x_0,r)$
2. $\|a\|_{L^2(\mathbb{R}^n)} \leqslant |B(x_0,r)|^{-\frac{1}{2}}$
3. $\displaystyle\int a = 0$ *and* $\displaystyle\int ab\ = 0$ *whenever* $r < 1$.

For $f \in L^1(\mathbb{R}^n)$, we say $f \in h^1_{\mathrm{Perez},b}(\mathbb{R}^n)$ if f can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where a_j are Pérez h^1_b atoms and $\{\lambda_j\} \in \ell^1$. For such functions, we define

$$\|f\|_{h^1_{\mathrm{Perez},b}} := \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

We note that a Pérez h^1_b atom is a $(1,2)$ atom in h^1 and therefore the decomposition converges in h^1 and we have $\|f\|_{h^1} \lesssim \sum |\lambda_j|$. Moreover, each Pérez h^1_b atom supported in a ball $B = B(x_0,r)$ also satisfies that $a(b - c_B) \in h^1$, where c_B is as in (12). In fact, we have that $a(b - c_B)$ is supported in B, has integral zero whenever $r < 1$, and by (14)

$$\begin{aligned}\|a(b - c_B)\|_{L^s(\mathbb{R}^n)} &\leqslant \|a\|_{L^2(\mathbb{R}^n)}\|b - c_B\|_{L^p(\mathbb{R}^n)} \\ &\leqslant |B(x_0,r)|^{-\frac{1}{2}}\|b\|_{\mathrm{bmo},p}|B(x_0,r)|^{\frac{1}{p}} = \|b\|_{\mathrm{bmo},p}|B(x_0,r)|^{\frac{1}{s}-1} \end{aligned} \tag{48}$$

for $1 \leqslant s < 2$ and $\frac{1}{2} + \frac{1}{p} = \frac{1}{s}$; taking $s > 1$ makes $a(b - c_B)$ a multiple of a $(1,s)$ atom and we have

$$\|a(b - c_B)\|_{h^1} \lesssim \|b\|_{\mathrm{bmo},p}.$$

These two observations allow us to have the following generalization of Pérez h^1_b atoms.

Definition 6.1.4 ([30, Definition 4.5]). *Fix $b \in \mathrm{bmo}(\mathbb{R}^n)$ and let $C_b = \|b\|_{\mathrm{bmo},2}$ as defined in (14). We say a is an approximate h^1_b atom if for some ball $B(x_0, r)$ we have*

1. $\mathrm{supp}(a) \subset B(x_0, r)$;
2. $\|a\|_{L^2(\mathbb{R}^n)} \leqslant |B(x_0, r)|^{-\frac{1}{2}}$;
3. $\left|\int a\right| \leqslant \dfrac{1}{[\log(1 + r^{-1})]^2}$ *and* $\left|\int a(b - c_B)\right| \leqslant \dfrac{C_b}{\log(1 + r^{-1})}$.

For $f \in L^1(\mathbb{R}^n)$, we say $f \in h^1_{\mathrm{atom},b}(\mathbb{R}^n)$ if f can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where a_j are approximate h^1_b atoms and $\{\lambda_j\} \in \ell^1$. For such functions, we define

$$\|f\|_{h^1_{\mathrm{atom},b}} := \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all such decompositions.

The choice of $C_b = \|b\|_{\mathrm{bmo},2}$ guarantees that, by Cauchy-Schwarz, Condition 2 in Definition 6.1.4 implies Condition 3 in Definition 6.1.4 when $r \geqslant 1$. Thus every Pérez h^1_b atom is an approximate h^1_b atom, and

$$h^1_{\mathrm{Perez},b}(\mathbb{R}^n) \subset h^1_{\mathrm{atom},b}(\mathbb{R}^n).$$

Moreover, compared with Definition 2.5.11 for $R = 1$ and $q = 2$, we see, as in (48), that not only is every approximate h^1_b atom a an approximate $(1, 2)$ atom (up to a factor of $\log 2$), but also $a(b - c_B)$ is a multiple of an approximate $(1, s)$ atom for $1 < s < 2$ and $\frac{1}{2} + \frac{1}{p} = \frac{1}{s}$, with

$$\|a(b - c_B)\|_{h^1} \lesssim \|b\|_{\mathrm{bmo},p}. \tag{49}$$

Recall that in h^1 (or h^p), every atom (or molecule) with approximate cancellation conditions can be written as an infinite linear combination of atoms with exact cancellation, see Theorem 4.1.5. Our question is whether it is possible to decompose an approximate h^1_b atom into Pérez h^1_b atoms with exact cancellation. Although we do not achieve the exact cancellation against b, we give a partial result in showing that every approximate h^1_b atom can be written as a finite sum of atoms of integral zero. It will be interesting to see whether it is possible to express approximate h^1_b atoms in terms of Pérez h^1_b atoms or if there is a counterexample to show approximate $h^1_{\mathrm{atom},b}(\mathbb{R}^n)$ atoms is indeed a larger space.

Proposition 6.1.5 ([30, Proposition 4.1]).
Suppose A is an approximate h^1_b atom with respect to the ball $B(x_0, r)$, and $r < 1$. Then $A = \sum \lambda_j a_j$ where the sum is **finite**, *a_j are approximate h^1_b atoms supported in balls B_j, respectively, and in addition, when $r(B_j) < 1$,*

$$\int a_j = 0.$$

Moreover, for some constant depending only on n, $\sum |\lambda_j| \leqslant C_n$.

Proof of Proposition 6.1.5.
The proof is taken from the work [30].

Set $\alpha = \int A \neq 0$ (otherwise we are done). Denote $B(0, 2^j r)$ by B_j, $j = 0, 1, 2, \dots, k$, where $2^{k-1} r < 1 \leqslant 2^k r$, and let $\eta_j = \frac{\chi_{B_j}}{|B_j|}$ be the normalized characteristic function of B_j. Then

$$A = A - \alpha\eta_0 + \sum_{j=1}^{k} \alpha(\eta_{j-1} - \eta_j) + \alpha\eta_k = \sum_{i=0}^{k+1} A_j$$

where

$$\begin{aligned}
&A_0 := A - \alpha\eta_0,\\
&A_j := \alpha(\eta_{j-1} - \eta_j), \quad j = 1, \dots, k, \quad \text{and}\\
&A_{k+1} := \alpha\eta_k.
\end{aligned}$$

First observe that $\operatorname{supp}(A_j) \subset B_j$ for $j = 1, ..., k$, and $\operatorname{supp}(A_{k+1}) \subset B_k$. Moreover, since $\int \eta_j = 1$ for all j, we have that $\int A_j = 0$ for $j = 0, ..., k$.

Write $A_j = \lambda_j a_j$ where

$$a_0 := \frac{1}{2} A_0, \quad \lambda_0 := 2, \quad a_{k+1} := \alpha^{-1} A_{k+1}, \quad \lambda_{k+1} := \alpha,$$

and

$$a_j := \frac{A_j}{\alpha 2^n \log(1 + [2^j r]^{-1})}, \quad \lambda_j := \alpha 2^n \log(1 + [2^j r]^{-1}), \quad j = 1, \dots, k.$$

Since $|\alpha| \leqslant \|A\|_{L^1} \leqslant 1$, $\|\eta_j\|_{L^\infty} \leqslant |B_j|^{-1}$, and $2^n \log(1 + [2^j r]^{-1}) > 1$ for $j = 1, \dots, k$, we have that $\|a_0\|_{L^2} \leqslant |B_0|^{-\frac{1}{2}}$, $\|a_j\|_{L^\infty} \leqslant |B_j|^{-1}$ for $j = 1, ..., k$, and $\|a_{k+1}\|_{L^\infty} \leqslant |B_k|^{-1}$. Thus, all the a_j satisfy Conditions 1 and conditions in Definition 6.1.4, and in addition $\int a_j = 0$ for $j = 0, ..., k$. Since $r(B_k) = 2^k r \geqslant 1$, the latter condition is not required for a_{k+1}.

To verify Condition (3) in Definition 6.1.4 , we need to check that the a_j satisfy the approximate cancellation condition against b. For $j = 0$, we get the approximate cancellation for a_0 from that of A:

$$\left| \int a_0 b \right| = \frac{1}{2} \left| \int Ab - \alpha \int b\eta_0 \right| = \frac{1}{2} \left| \int A[b - b_B] \right| \leqslant \frac{C_b}{\log(1 + r^{-1})}.$$

For $j = 1, ..., k$, we have, using (13),

$$\begin{aligned}
\left| \int a_j b \right| &= \frac{1}{2^n \log(1 + [2^j r]^{-1})} \left| \int b(\eta_{j-1} - \eta_j) \right|\\
&= \frac{1}{2^n \log(1 + [2^j r]^{-1})} |b_{2^{j-1}B} - b_{2^j B}|\\
&\leqslant \frac{\|b\|_{\mathrm{bmo}}}{\log(1 + [2^j r]^{-1})},
\end{aligned}$$

and for $j = k + 1$, since $r(B_k) = 2^k r \geqslant 1$,

$$\left| \int a_{k+1} b \right| = |b_{B_k}| \leqslant \frac{\|b\|_{\mathrm{bmo}}}{\log(1 + (2^k r)^{-1})}.$$

If, as noted after Definition 6.1.4, we take $C_b = \|b\|_{\mathrm{bmo},2} \geqslant \|b\|_{\mathrm{bmo}}$, then we can conclude that a_j for $j = 0, ..., k+1$ are all approximate h^1_b atoms.

Finally, we have

$$\begin{aligned}
\sum |\lambda_j| &= 2 + \alpha 2^n \sum_{j=1}^{k} \log(1 + [2^j r]^{-1}) + \alpha \\
&\leqslant 3 + \frac{2^n}{[\log(1 + r^{-1})]^2} \sum_{j=1}^{k} \log(1 + [2^j r]^{-1}) \\
&\leqslant 3 + \frac{2^n k \log(1 + r^{-1})}{[\log(1 + r^{-1})]^2} \\
&\leqslant 3 + \frac{2^n (\log_2 r^{-1} + 1)}{[\log(1 + r^{-1})]} \leqslant C_n.
\end{aligned}$$

□

6.2 Relations between the Spaces

The discussion following the definitions of the atomic spaces in the previous section gave us the inclusions

$$h^1_{\mathrm{Perez},b}(\mathbb{R}^n) \subset h^1_{\mathrm{atom},b}(\mathbb{R}^n) \subset h^1(\mathbb{R}^n).$$

In the trivial case, *i.e.* when b is a constant function, the cancellation conditions against b reduce to the conditions on the integral of the atoms. Thus every h^1 atom with exact cancellation is a Pérez h^1_b atom, so

$$h^1_{\mathrm{Perez},b}(\mathbb{R}^n) = h^1_{\mathrm{atom},b}(\mathbb{R}^n) = h^1(\mathbb{R}^n). \tag{50}$$

In the next section, we will see later that the second equality can hold for b in a nontrivial subspace of bmo. The next proposition is an analogue of $\mathcal{M}_b$ acting to the atoms.

Proposition 6.2.1 ([30, Proposition 4.2])**.** *Let $b \in \mathrm{bmo}(\mathbb{R}^n)$. Then for every h^1_b atom a,*

$$\|\mathcal{M}_b a\|_{L^1} \lesssim \|b\|_{\mathrm{bmo}}.$$

Proof of Proposition 6.2.1.
The proof is taken from [30].

Let a be an approximate h^1_b atom with support in $B = B(x_0, r)$. We want to bound $\mathcal{M}_b(a)$.

For the local estimate, using (46), we can apply the L^p boundedness of the maximal commutator operator $M_b f$ defined in (9), with the bound (10). Thus, using the L^2 size condition on a we have

$$\int_{2B} \mathcal{M}_b(a) \leqslant |2B|^{1/2} \|M_b a\|_{L^2(2B)} \lesssim |B|^{\frac{1}{2}} \|a\|_{L^2} \|M_b\|_{L^2 \mapsto L^2} \lesssim \|b\|_{\mathrm{bmo}}.$$

Next we handle the integral on $(2B)^c$. Note that

$$\left| \int_{B(x,t)} \phi(y)[b(x) - b(y)] a(y) dy \right| \neq 0$$

implies that there exists $y \in B(x,t) \cap B(x_0,r)$, which in turn implies

$$r \leqslant \frac{|x-x_0|}{2} \leqslant |x-x_0| - |x_0-y| \leqslant |x-y| \leqslant t < 1. \tag{51}$$

This cannot happen when $r \geqslant 1$, so the integral vanishes for all test functions ϕ and $\mathcal{M}_b(a) = 0$ on $(2B)^c$.

For $r < 1$, we may assume, by Proposition 6.1.5, that $\int a = 0$. Thus we have, for almost all $x \in (2B)^c$,

$$\begin{aligned}
&\left|\int [b(x)-b(y)]a(y)\phi(y)dy\right| \\
&\leqslant \left|\int [b(x)-b(y)]a(y)[\phi(y)-\phi(x_0)]dy\right| + \left|\phi(x_0)\int b(y)a(y)dy\right| \\
&\leqslant |b(x)-b_B|\|\nabla\phi\|_{L^\infty} r \int |a(y)|dy + \|\nabla\phi\|_{L^\infty} r \int |b(y)-b_B||a(y)|dy \\
&\qquad + \|\phi\|_{L^\infty}\left|\int b(y)a(y)dy\right| \\
&\lesssim \frac{|b(x)-b_B|r}{|x-x_0|^{n+1}} + \frac{r\|b\|_{\mathrm{bmo},2}}{|x-x_0|^{n+1}} + \frac{C_b}{|x-x_0|^n \log(1+r^{-1})},
\end{aligned} \tag{52}$$

where in the last step we used the conditions on ϕ, (51), the L^1 estimate on $a(b-b_B)$ (see (48) with $s=1$, $p=2$), and condition (3) in Definition 6.1.4. Since the estimate above is independent of ϕ, it holds for $\mathcal{M}_b(a)(x)$ and therefore, again using (51),

$$\begin{aligned}
\int_{(2B)^c} \mathcal{M}_b(a)(x)dx &= \int_{2r<|x-x_0|<2} \mathcal{M}_b(a)(x)dx \\
&\lesssim \int_{(2B)^c} \frac{|b(x)-b_B|r}{|x-x_0|^{n+1}} + \int_{(2B)^c} \frac{r\|b\|_{\mathrm{bmo},2}}{|x-x_0|^{n+1}} \\
&\qquad + \int_{2r<|x-x_0|<2} \frac{C_b}{|x-x_0|^n \log(1+r^{-1})} \\
&\lesssim \|b\|_{\mathrm{bmo}}.
\end{aligned} \tag{53}$$

Here we have used Proposition 2.5.16 Item 2, (14), and the fact that $C_b = \|b\|_{\mathrm{bmo},2}$. ☐

If we take an h^1 atom a without assuming any cancellation against b, the boundedness of the maximal function $\mathcal{M}_b$ turns out to be equivalent to whether $a(b-c_B)$ belongs to h^1, where B is the ball containing the support of a. Such boundedness automatically gives us the approximate cancellation against b.

Proposition 6.2.2 ([30, Proposition 4.3]). *Let $b \in \mathrm{bmo}(\mathbb{R}^n)$, b nontrivial. Suppose a is an h^1 atom supported in a ball $B = B(x_0,r)$, and has vanishing integral when $r<1$. Then*

$$\|\mathcal{M}_b a\|_{L^1} \lesssim \|a(b-c_B)\|_{h^1} + \|b\|_{\mathrm{bmo}} \lesssim \|\mathcal{M}_b a\|_{L^1} + \|b\|_{\mathrm{bmo}}.$$

Moreover,

$$\left|\int ab\right| \lesssim \min\left\{\frac{\|\mathcal{M}_b a\|_{L^1} + \|b\|_{\mathrm{bmo}}}{\log(1+r^{-1})}, \|b\|_{\mathrm{bmo}}\right\}.$$

Proof of Proposition 6.2.2.
The proof is taken from [30].

We assume that a satisfies an L^2 size condition. First note that by (48), $a(b-c_B) \in L^s(\mathbb{R}^n)$ for $1 < s < 2$, and it has compact support, $a(b-c_B)$ is in $h^1(\mathbb{R}^n)$. By Definition 6.1.1, we have, for almost every $x \in \mathbb{R}^n$,

$$\mathcal{M}_b(a)(x) = \mathcal{M}([b(x)-b]a)(x) \leqslant |b(x)-c_B|\mathcal{M}(a)(x) + \mathcal{M}([b-c_B]a)(x),$$

hence

$$\begin{aligned}\|\mathcal{M}_b(a)\|_{L^1} &\leqslant \int |b(x)-c_B|\mathcal{M}(a)(x)dx + \|\mathcal{M}([b-c_B]a)\|_{L^1}\\ &\lesssim \int |b(x)-c_B|\mathcal{M}(a)(x)dx + \|a(b-c_B)\|_{h^1}.\end{aligned}$$

Conversely,

$$\mathcal{M}([b-c_B]a)(x) \leqslant \mathcal{M}_b(a)(x) + |b(x)-c_B|\mathcal{M}(a)(x),$$

so

$$\|a(b-c_B)\|_{h^1} \leqslant \|\mathcal{M}_b(a)\|_{L^1} + \int |b(x)-c_B|\mathcal{M}(a)(x)dx.$$

It thus remains to show that the integral of $|b-c_B|\mathcal{M}(a)$ is controlled by $\|b\|_{\mathrm{bmo}}$. The arguments are similar to those in the proof of Proposition 6.2.1. First, we have

$$\int_{2B} |b-c_B|\mathcal{M}(a) \leqslant \|b-c_B\|_{L^2(2B)}\|Ma\|_{L^2(2B)} \lesssim \|b\|_{\mathrm{bmo}}|B|^{\frac{1}{2}}\|a\|_{L^2} \lesssim \|b\|_{\mathrm{bmo}}.$$

Moreover, (51) implies that if $r \geqslant 1$, $\mathcal{M}(a)$ vanishes outside $2B$. For $r < 1$, we can use the cancellation condition on a to write, as in (52), for every test function ϕ, and for almost every $x \in (2B)^c$,

$$\begin{aligned}|b(x)-c_B|\left|\int a(y)\phi(y)dy\right| &\leqslant |b(x)-b_B|\|\nabla\phi\|_{L^\infty} r \int |a(y)|dy\\ &\lesssim \frac{|b(x)-b_B|r}{|x-x_0|^{n+1}}.\end{aligned}$$

The integral on $(2B)^c$ is then estimated as in the first term of (53).

By applying Proposition 5.2.1 to $a(b-c_B) \in h^1(\mathbb{R}^n)$, followed by the estimates above, we have

$$\left|\int ab\right| = \left|\int a(b-c_B)\right| \lesssim \frac{\|a(b-c_B)\|_{h^1}}{\log(1+r^{-1})} \lesssim \frac{(\|\mathcal{M}_b a\|_{L^1} + \|b\|_{\mathrm{bmo}})}{\log(1+r^{-1})}.$$

The other upper bound follows from the duality of bmo and h^1, or by Cauchy-Schwartz. $\square$

It is interesting to note that if we had not assumed $\int a = 0$ in the hypotheses of Proposition 6.2.2 when $r < 1$, we would have ended up having to estimate an extra term of the form

$$\int_{2r<|x-x_0|<2} \frac{|b(x)-b_B|}{|x-x_0|^n}dx\left|\int a\right|.$$

Looking at the proof of Proposition 2.5.16 Item 2, if $\delta = 0$ and $p = 1$, the integral in x can be estimated by $\|b\|_{\mathrm{bmo}}(\log \frac{1}{r})^2$. Therefore, we would need a cancellation condition of the form $\left|\int a\right| \leqslant [\log(1 + r^{-1})]^{-2}$, as in Condition 3 of Definition 6.1.4.

Combining Propositions 6.1.5, 6.2.1 and 6.2.2, we get the following.

Corollary 6.2.3. *Let $f \in h^1(\mathbb{R}^n)$. Then the following are equivalent.*

1. *The function $f \in h^1_{\mathrm{atom},b}(\mathbb{R}^n)$.*

2. *There exist a sequence $\{\lambda_j\} \in \ell^1$ and a collection of h^1 atoms $\{a_j\}$, where B_j denotes the ball containing the support of a_j and a_j has vanishing integral when $r(B_j) < 1$, so that $f = \sum_j \lambda_j a_j$ and $\mathcal{M}_b a_j$ are uniformly bounded in $L^1(\mathbb{R}^n)$.*

3. *There exist a sequence $\{\lambda_j\} \in \ell^1$ and a collection of h^1 atoms $\{a_j\}$, where B_j denotes the ball containing the support of a_j and a_j has vanishing integral when $r(B_j) < 1$, so that $f = \sum_j \lambda_j a_j$ and*

$$\sum_j \lambda_j a_j (b - c_{B_j}) \text{ converges absolutely in } h^1(\mathbb{R}^n).$$

One may also consider what $\mathcal{M}'_b(f)$ tells us.

Proposition 6.2.4. *Let $f \in h^1(\mathbb{R}^n)$ and $b \in \mathrm{bmo}(\mathbb{R}^n)$. Then the following are equivalent.*

1. *The function f satisfies $\mathcal{M}'_b(f) \in L^1(\mathbb{R}^n)$.*

2. *There exist a sequence $\{\lambda_j\} \in \ell^1$ and a collection of h^1 atoms $\{a_j\}$, where B_j denotes the ball containing the support of a_j and a_j has vanishing integral when $r(B_j) < 1$, so that $f = \sum_j \lambda_j a_j$ and*

$$\sum_j \lambda_j a_j (b - c_{B_j}) \text{ converges in } h^1(\mathbb{R}^n).$$

We remark that Item 1 implies Item 2 for an arbitrary atomic decomposition of f, while Item 2 implies Item 1 when there is an atomic decomposition for which the series $\sum_j \lambda_j a_j (b - c_{B_j})$ is in $h^1(\mathbb{R}^n)$, and that of course the norm inequalities depend on which decomposition is used.

Proof of Proposition 6.2.4.
Consider a function $f \in h^1$ and take an atomic decomposition $f = \sum_j \lambda_j a_j$ into h^1 atoms with exact cancellation for $r < 1$.

Since for $j \in \mathbb{N}$ one has $|b(x) - c_{B_j}|\left|\int a_j \phi\right| \leqslant |b - c_{B_j}|\mathcal{M}(a_j)(x)$, where ϕ is in Definition 6.1.2, and the proof of Proposition 6.2.2 gives

$$\||b - c_{B_j}|\mathcal{M}(a_j)\|_{L^1} \lesssim \|b\|_{\mathrm{bmo}},$$

the series $\displaystyle\sum_j \lambda_j [b(x) - c_{B_j}] \int a_j \phi$ converges absolutely in $L^1(\mathbb{R}^n)$ and hence converges absolutely for almost every x.

Furthermore, for fixed x, we have

$$\sum_{j=1}^{\infty}\int_{\mathbb{R}^n}|\lambda_j||b(y)-c_{B_j}||a_j(y)||\phi(y)|dy \leqslant \|\phi\|_{L^\infty}\|b\|_{\mathrm{bmo}}\sum_{j=1}^{\infty}|\lambda_j|. \tag{54}$$

Therefore, we can write the dual pairing as a difference of two absolutely convergent series for almost every $x \in \mathbb{R}^n$:

$$\begin{aligned}
{}_{\mathrm{bmo}}\langle b-b(x), f\phi\rangle_{h^1} &= \sum_{j=1}^{\infty}\lambda_j\, {}_{\mathrm{bmo}}\langle b-b(x), a_j\phi\rangle_{h^1}\\
&= \sum_{j=1}^{\infty}\lambda_j\int_{\mathbb{R}^n}[b(y)-b(x)]a_j(y)\phi(y)dy\\
&= \sum_{j=1}^{\infty}\lambda_j\int_{\mathbb{R}^n}[b(y)-c_{B_j}]a_j(y)\phi(y)dy - \sum_{j=1}^{\infty}\lambda_j[b(x)-c_{B_j}]\int_{\mathbb{R}^n}a_j(y)\phi(y)dy\\
&= \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\lambda_j[b(y)-c_{B_j}]a_j(y)\phi(y)dy - \sum_{j=1}^{\infty}\lambda_j[b(x)-c_{B_j}]\int_{\mathbb{R}^n}a_j(y)\phi(y)dy.
\end{aligned}$$

Because of (54), we can could Fubini's theorem to the first term to interchange the sum and the integral.

Taking supremum over all ϕ, we have

$$\mathcal{M}_b'(f)(x) \leqslant \mathcal{M}_\phi\bigg(\sum_{j=1}^{\infty}\lambda_j[b-c_{B_j}]a_j\bigg)(x) + \sum_j|\lambda_j||b(x)-c_{B_j}|\mathcal{M}(a_j)(x)$$

and

$$\mathcal{M}\bigg(\sum_{j=1}^{\infty}\lambda_j[b-c_{B_j}]a_j\bigg)(x) \leqslant \mathcal{M}_b'(f)(x) + \sum_j|\lambda_j||b(x)-c_{B_j}|\mathcal{M}(a_j)(x),$$

from which we can conclude the equivalence of the statements.

□

In the remainder of this section, we consider the case $b \in \mathrm{lmo}(\mathbb{R}^n)$. We now come to the result which gives us the second equality in (50) even when b is not constant. Recall that the pointwise multipliers of $\mathrm{bmo}(\mathbb{R}^n)$, and hence $h^1(\mathbb{R}^n)$, were identified as the elements of $L^\infty \cap \mathrm{lmo}(\mathbb{R}^n)$ in [9]. It is therefore not surprising that we have the following results.

Theorem 6.2.5 ([30, Theorem 4.6])**.** *Let $b \in \mathrm{lmo}(\mathbb{R}^n)$. Then $h^1_{\mathrm{atom},b}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$. Moreover,*

$$\|f\|_{h^1} \leqslant \|f\|_{h^1_{\mathrm{atom},b}} \lesssim \frac{\|b\|_{\mathrm{LMO}_{loc}}}{\|b\|_{\mathrm{BMO}_{loc}}}\|f\|_{h^1}.$$

Proof of Theorem 6.2.5.
This is taken from [30]. It was already observed following Definition 6.1.4 that for $b \in \mathrm{bmo}$, h^1_b atoms are approximate h^1 atoms (up to a factor of $\log 2$) and therefore $h^1_b(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$ with $\|f\|_{h^1} \lesssim \|f\|_{h^1_{\mathrm{atom},b}}$. We only need the lmo condition for the reverse inclusion.

Let a be an h^1 atom with support in the ball $B = B(x_0, r)$, with L^2 size condition and such that $\int a = 0$. This means that conditions (1), (2), and the first part of (3) in Definition 6.1.4 are satisfied, and we only need to check the approximate cancellation against b. Since this holds automatically for $r \geqslant 1$, we assume $r < 1$. By the L^2 size condition on a,

$$\left| \int_{\mathbb{R}^n} a(x)[b(x) - b_B]dx \right| \leqslant \frac{1}{\log(1+r^{-1})} \frac{\log(1+r^{-1})}{|B|^{\frac{1}{2}}} \left(\int_B |b(x) - b_B|^2 dx \right)^{\frac{1}{2}}$$
$$\leqslant \frac{\|b\|_{\mathrm{LMO}_{\mathrm{loc},2}}}{\log(1+r^{-1})},$$

where

$$\|b\|_{\mathrm{LMO}_{\mathrm{loc},2}} := \sup_{r(B)<1} \log(1 + r(B)^{-1}) \left(\fint_B |b(x) - b_B|^2 dx \right)^{\frac{1}{2}}.$$

Since

$$\gamma := \frac{\log 2 \|b\|_{\mathrm{BMO}_{\mathrm{loc},2}}}{\|b\|_{\mathrm{LMO}_{\mathrm{loc},2}}} \leqslant 1$$

and $\log 2\|b\|_{\mathrm{BMO}_{\mathrm{loc},2}} \leqslant \|b\|_{\mathrm{bmo},2} = C_b$, we have that γa is an h^1_b atom.

Finally, for $f \in h^1$ with an atomic decomposition $f = \sum_{j=1}^{\infty} \lambda_j a_j$, we can regard it as a decomposition using h^1_b atoms multiplied by γ^{-1}, so $f \in h^1_{\mathrm{atom},b}(\mathbb{R}^n)$ with

$$\|f\|_{h^1_{\mathrm{atom},b}} \leqslant \gamma^{-1} \sum_{j=1}^{\infty} |\lambda_j|.$$

Taking the infimum over all such decompositions, we get $\|f\|_{h^1_{\mathrm{atom},b}} \leqslant \gamma^{-1}\|f\|_{h^1}$. The inequality in the statement of the theorem is obtained by applying the John-Nirenberg inequality for LMO - see [104]. □

We give a partial converse to the theorem.

Proposition 6.2.6 ([30, Proposition 4.4])**.** *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose there exists a constant C_b such that every h^1 atom a satisfies*

$$\left| \int ab \right| \leqslant \min \left\{ C_b, \frac{C_b \log(2)}{\log(1+r^{-1})} \right\},$$

where r is the radius of the ball B containing the support of a. Then $b \in \mathrm{lmo}(\mathbb{R}^n)$.

Proof of Proposition 6.2.6.
The proof is taken from [30].

We first handle a ball B with radius $r < 1$. We define $s(x) := \mathrm{sgn}[b(x) - b_B]$ and

$a(x) := \frac{s(x)-s_B}{|B|}\chi_B(x)$. Then a is an h^1 atom. Moreover, we can write

$$\begin{aligned}\log(1+r^{-1}) \fint_B |b(x)-b_B|dx &= \log(1+r^{-1}) \fint_B s(x)(b(x)-b_B)dx \\ &= \frac{\log(1+r^{-1})}{|B|}\int_B (s(x)-s_B)(b(x)-b_B)dx \\ &= \log(1+r^{-1})\int_B a(x)b(x)dx \\ &= \log(1+r^{-1})\left|\int_B ab\right| \\ &\leqslant C_b.\end{aligned}$$

If $r \geqslant 1$, we define $a(x) := \frac{\operatorname{sgn}[b(x)]}{|B|}\chi_B(x)$. A similar argument shows that

$$\frac{1}{|B|}\int_B |b(x)|dx = \left|\int_B a(x)b(x)dx\right| \leqslant C_b.$$

Therefore, we can conclude that $b \in \operatorname{lmo}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{lmo}} \leqslant C_b$. □

In a similar vein, we give a partial converse to Proposition 6.2.1.

Proposition 6.2.7 ([30, Proposition 4.5]). *Let $b \in \operatorname{bmo}(\mathbb{R}^n)$. Suppose there exists a constant κ_b such that every h^1 atom a satisfies*

$$\|\mathcal{M}_b a\|_{L^1} \leqslant \kappa_b.$$

Then b satisfies

$$\sup_{r(B)<1}\frac{1}{|B|}\||b-b_B|\chi_B\|_{h^1} + \sup_{r(B)\geqslant 1}\frac{1}{|B|}\||b|\chi_B\|_{h^1} < \infty.$$

Proof of Proposition 6.2.7.
The proof is taken from [30].

Fix a ball B with radius $r < 1$. As in the previous proof, set $s(x) := \operatorname{sgn}[b(x)-b_B]$ and $a(x) := \frac{s(x)-s_B}{|B|}\chi_B(x)$. Then a is an h^1 atom, and from Proposition 6.2.2,

$$\begin{aligned}\||b-b_B|\frac{\chi_B}{|B|}\|_{h^1} &= \|(b-b_B)s(x)\frac{\chi_B}{|B|}\|_{h^1} \\ &\leqslant \|a(b-b_B)\|_{h^1} + |s_B|\left\|\frac{(b-b_B)\chi_B}{|B|}\right\|_{h^1} \\ &\lesssim \|\mathcal{M}_b(a)\|_{L^1} + \|b\|_{\mathrm{bmo}} + |s_B|\left\|\frac{(b-b_B)\chi_B}{|B|}\right\|_{h^1} \\ &\lesssim \|\mathcal{M}_b(a)\|_{L^1} + \|b\|_{\mathrm{bmo}}.\end{aligned}$$

In the last step we used the fact that $|s_B| \leqslant 1$ and $\frac{(b-b_B)\chi_B}{|B|\|b\|_{\mathrm{bmo},2}}$ is a $(1,2)$ atom.

For B with radius $r \geqslant 1$, we again put $a(x) = \frac{\operatorname{sgn}(b(x))\chi_B(x)}{|B|}$ and use Proposition 6.2.2:

$$\||b|\frac{\chi_B}{|B|}\|_{h^1} = \|a(b-c_B)\|_{h^1} \lesssim \|\mathcal{M}_b(a)\|_{L^1} + \|b\|_{\mathrm{bmo}}.$$

□

One interesting corollary from Propositions 6.2.1 and 6.2.7 is the following.

Corollary 6.2.8 ([30, Corollary 4.2]). *Let $b \in \mathrm{bmo}(\mathbb{R}^n)$. Then the following are equivalent.*

1. *The function b is in $\mathrm{lmo}(\mathbb{R}^n)$.*

2. *The function b satisfies*

$$A_b := \sup_{r(B)<1} \frac{1}{|B|} \||b - b_B|\chi_B\|_{h^1} + \sup_{r(B)\geqslant 1} \frac{1}{|B|} \||b|\chi_B\|_{h^1} < \infty.$$

Proof of Corollary 6.2.8.
The proof is taken from [30].

Suppose $b \in \mathrm{lmo}(\mathbb{R}^n)$. Then from the proof of Theorem 6.2.5, for an h^1 atom a, γa is an h^1_b atom, hence by Proposition 6.2.1

$$\|\mathcal{M}_b a\|_{L^1} \lesssim \gamma^{-1}\|b\|_{\mathrm{bmo}} \approx \kappa_b := \frac{\|b\|_{\mathrm{LMO_{loc}}}}{\|b\|_{\mathrm{BMO_{loc}}}}\|b\|_{\mathrm{bmo}},$$

which is exactly the hypothesis of Proposition 6.2.7.

Conversely, suppose $A_b < \infty$. Observe that from Proposition 5.2.1, we can estimate

$$\sup_{r(B)<1} \frac{\log(1 + [r(B)]^{-1})}{|B|} \int_B |b(x) - b_B| dx \lesssim \sup_{r(B)<1} \frac{1}{|B|} \||b - b_B|\chi_B\|_{h^1}.$$

Meanwhile, using the fact that $\|f\|_{L^1(\mathbb{R}^n)} \leqslant \|f\|_{h^1(\mathbb{R}^n)}$, we have

$$\sup_{r(B)\geqslant 1} \frac{1}{|B|} \int_B |b(x)| dx \leqslant \sup_{r(B)\geqslant 1} \frac{1}{|B|} \||b|\chi_B\|_{h^1}.$$

Thus $\|b\|_{\mathrm{lmo}} \lesssim A_b < \infty$. ☐

6.3 Boundedness of Commutators on Atomic Spaces

Theorem 6.3.1 (cf. [30, Theorem 5.1]). *Suppose $b \in \mathrm{bmo}(\mathbb{R}^n)$ and T is a inhomogeneous singular integral operator defined in Definition 5.1.1 with $s > 2$, $0 < \delta \leqslant 1$, $\eta > 0$ and $P^1_{x,z}(y) = K(x, x_B) = P^2_{x,z}(y)$. Then $[b, T](a) \in L^1(\mathbb{R}^n)$ and*

$$\|[b, T]a\|_{L^1} \lesssim \|b\|_{\mathrm{bmo}}$$

for all h^1_b atoms a. Thus, $[b, T] : h^1_{\mathrm{finite},b}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ continuously, where $h^1_{\mathrm{finite},b}(\mathbb{R}^n)$ denotes the space of finite linear combinations of h^1_b atoms.

Proof of Theorem 6.3.1.
The proof is adapted from [30].

Let a be an h^1_b atom with $\mathrm{supp}(a) \subset B = B(x_0, r)$. Note that we can write the commutator acting on a as $[b, T](a) = (b - c_B)T(a) - T(a(b - c_B))$.

We first show that $T(a(b-c_B)) \in L^1(\mathbb{R}^n)$. By the L^p boundedness of T for $p=\frac{3}{2}$, we have

$$\begin{aligned}
\|T(a(b-c_B))\|_{L^1(2B)} &\leqslant |2B|^{\frac{1}{3}} \|T(a(b-c_B))\|_{L^{\frac{3}{2}}(2B)} \\
&\lesssim |B|^{\frac{1}{3}} \|T\|_{L^{\frac{3}{2}} \to L^{\frac{3}{2}}} \|a(b-c_B)\|_{L^{\frac{3}{2}}(\mathbb{R}^n)} \\
&\lesssim \|T\|_{L^{\frac{3}{2}} \to L^{\frac{3}{2}}} |B|^{\frac{1}{3}} \|b-c_B\|_{L^6(B)} \|a\|_{L^2(B)} \\
&\lesssim \|T\|_{L^{\frac{3}{2}} \to L^{\frac{3}{2}}} |B|^{-\frac{1}{6}} \|b-c_B\|_{L^6(B)} \lesssim \|T\|_{L^{\frac{3}{2}} \to L^{\frac{3}{2}}} \|b\|_{\mathrm{bmo},6}.
\end{aligned}$$

Looking at $x \in (2B)^c$, first consider the case $r<1$. From Proposition 6.1.5, we may assume $\int a = 0$. Since $x \notin \operatorname{supp}(a(b-b_B))$, we can use the kernel representation to write

$$\begin{aligned}
&\|T(a(b-b_B))\|_{L^1((2B)^c)} \\
&\leqslant \int_{(2B)^c} \left| \int_B K(x,y)a(y)(b(y)-b_B)dy \right| dx \\
&\leqslant \int_{(2B)^c} \left| \int_B [K(x,y)-K(x,x_0)]a(y)(b(y)-b_B)dy \right| dx + \int_{(2B)^c} |K(x,x_0)| \left| \int ab \right| dx \\
&=: I + II.
\end{aligned}$$

We first estimate I.

$$\begin{aligned}
I &\lesssim \int_{(2B)^c} \left(\int_B |K(x,y)-K(x,x_0)|^s dy \right)^{\frac{1}{s}} \frac{1}{|B|^{\frac{1}{2}}} \left(\int_B |b(y)-b_B|^{\frac{2s}{s-2}} dy \right)^{\frac{1}{2}-\frac{1}{s}} dx \\
&\lesssim \frac{\|b\|_{\mathrm{bmo},\frac{2s}{s-2}}}{|B|^{\frac{1}{s}}} \int_{(2B)^c} \left(\int_B |K(x,y)-K(x,x_0)|^s dy \right)^{\frac{1}{s}} dx \\
&=: \widetilde{I}.
\end{aligned}$$

As $2|y-x_0| < 2r \leqslant |x-x_0|$, we can use the smoothness of K (Condition (2) in Definition 5.1.1).

Therefore, we have

$$\begin{aligned}
\widetilde{I} &\lesssim \frac{\|b\|_{\mathrm{bmo},\frac{2s}{s-2}}}{|B|^{\frac{1}{s}}} \sum_{j=1}^{\infty} |A_j(x_0,r)|^{1-\frac{1}{s}} \left(\int_{A_j(x_0,r)} \int_B |K(x,y)-K(x,x_0)|^s dy dx \right)^{\frac{1}{s}} \\
&\lesssim \frac{\|b\|_{\mathrm{bmo},\frac{2s}{s-2}}}{|B|^{\frac{1}{s}}} \sum_{j=1}^{\infty} |A_j(x_0,r)|^{1-\frac{1}{s}} \left(\int_B \int_{A_j(z,r)} |K(x,y)-K(x,x_0)|^s dy dx \right)^{\frac{1}{s}} \\
&\lesssim \frac{\|b\|_{\mathrm{bmo},\frac{2s}{s-2}}}{|B|^{\frac{1}{s}}} \sum_{j=1}^{\infty} |A_j(z,r)|^{1-\frac{1}{s}} \left(\int_B |A_j(x_0,r)|^{1-s} 2^{-js\delta} dx \right)^{\frac{1}{s}} \\
&\lesssim \|b\|_{\mathrm{bmo}} \sum_{j=1}^{\infty} 2^{-j\delta} \lesssim \|b\|_{\mathrm{bmo}}.
\end{aligned}$$

Using the decay property of K and the cancellation of a against b, recalling that we chose

$C_b = \|b\|_{\mathrm{bmo},2}$, we have

$$
\begin{aligned}
II &\leqslant \int_{(2B)^c} |K(x,x_0)| \left| \int ab \right| dx \\
&\lesssim \left(\int_{2r \leqslant |x-x_0| \leqslant 1} \frac{1}{|x-x_0|^n} dx + \int_{|x-x_0|\geqslant 1} \frac{1}{|x-x_0|^{n+\mu}} dx \right) \frac{C_b}{\log(1+r^{-1})} \\
&\lesssim \|b\|_{\mathrm{bmo},2} \frac{\log(1+r^{-1})+1}{\log(1+r^{-1})} \lesssim \|b\|_{\mathrm{bmo}}.
\end{aligned}
$$

When $r \geqslant 1$, using the fact that $2|y-x_0| \leqslant |x-x_0|$ implies $|x-y| \geqslant \frac{1}{2}|x-x_0|$, the decay of K again gives

$$
\begin{aligned}
\|T(a(b-c_B))\|_{L^1((2B)^c)} &\leqslant \int_{(2B)^c} \left| \int_B K(x,y)a(y)b(y)dy \right| dx \\
&\lesssim \int_{|x-x_0|\geqslant 2r} \frac{1}{|x-x_0|^{n+\mu}} \|b\|_{\mathrm{bmo},2} \\
&\lesssim \int_{|x-x_0|\geqslant 2} \frac{1}{|x-x_0|^{n+\mu}} \|b\|_{\mathrm{bmo},2} \lesssim \|b\|_{\mathrm{bmo}}.
\end{aligned}
$$

Now we can handle the term $(b-c_B)T(a)$. Using the boundedness of T on L^2, we have

$$
\begin{aligned}
\|(b-c_B)T(a)\|_{L^1(2B)} &\leqslant \|b-c_B\|_{L^2(2B)} \|T(a)\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|b\|_{\mathrm{bmo},2} |B|^{\frac{1}{2}} \|T\|_{L^2\to L^2} \|a\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|b\|_{\mathrm{bmo}} \|T\|_{L^2\to L^2}.
\end{aligned}
$$

When $r < 1$, we apply the cancellation of a to get

$$
\begin{aligned}
&\|(b-c_B)T(a)\|_{L^1((2B)^c)} \\
&\leqslant \int_{(2B)^c} |b(x)-c_B| \left| \int_B [K(x,y)-K(x,x_0)]a(y)dy \right| dx \\
&\leqslant \frac{1}{|B|^{\frac{1}{2}}} \int_{(2B)^c} |b(x)-c_B| \left(\int_B |K(x,y)-K(x,x_0)|^2 dy \right)^{\frac{1}{2}} dx \qquad (55)
\end{aligned}
$$

Then, using Fubini's theorem and (35), we have

$$
\begin{aligned}
(55) &\leqslant \frac{1}{|B|^{\frac{1}{2}}} \sum_{j=1}^{\infty} \int_{A_j(x_0,r)} |b(x)-c_B| \bigg(\int_B |K(x,y)-K(x,x_0)|^2 dy \bigg)^{\frac{1}{2}} dx \\
&\leqslant \frac{1}{|B|^{\frac{1}{2}}} \sum_{j=1}^{\infty} \bigg(\int_{A_j(x_0,r)} |b(x)-c_B|^2 dx \bigg)^{\frac{1}{2}} \bigg(\int_{A_j(x_0,r)} \int_B |K(x,y)-K(x,x_0)|^2 dy dx \bigg)^{\frac{1}{2}} \\
&\lesssim \frac{1}{|B|^{\frac{1}{2}}} \sum_{j=1}^{\infty} \bigg(\int_{A_j(x_0,r)} |b(x)-c_B|^2 dx \bigg)^{\frac{1}{2}} \bigg(\int_B |A_j(x_0,r)|^{-1} 2^{-2j\delta} dy \bigg)^{\frac{1}{2}} \\
&\lesssim \sum_{j=1}^{\infty} \bigg[\bigg(\int_{2^{j+1}B} |b(x)-b_{2^{j+1}B}|^2 dx \bigg)^{\frac{1}{2}} + |2^{j+1}B|^{\frac{1}{2}} (j+1) \|b\|_{\mathrm{bmo}} \bigg] |A_j(x_0,r)|^{-\frac{1}{2}} 2^{-j\delta} \\
&\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^j B|^{\frac{1}{2}}} \bigg[\bigg(\int_{2^{j+1}B} |b(x)-b_{2^{j+1}B}|^2 dx \bigg)^{\frac{1}{2}} + |2^{j+1}B|^{\frac{1}{2}} (j+1) \|b\|_{\mathrm{bmo}} \bigg] 2^{-j\delta} \\
&\lesssim \sum_{j=1}^{\infty} (j+2) \|b\|_{\mathrm{bmo}} 2^{-j\delta} \lesssim \|b\|_{\mathrm{bmo}}.
\end{aligned}
$$

When $r \geqslant 1$, similar to the estimate of $\|T(a(b-c_B))\|_{L^1((2B)^c)}$, we have, again by Theorem 2.5.16,

$$
\begin{aligned}
\|(b-c_B)T(a)\|_{L^1((2B)^c)} &\leqslant \int_{(2B)^c} |b(x)| \bigg| \int_B K(x,y)a(y)dy \bigg| dx \\
&\leqslant \int_{(2B)^c} \frac{|b(x)|}{|x-x_0|^{n+\mu}} dx \\
&\leqslant \int_{(2B)^c} r^{\varepsilon} \frac{|b(x)-c_B|}{|x-x_0|^{n+\mu}} dx \lesssim \|b\|_{\mathrm{bmo}}. \qquad (56)
\end{aligned}
$$

We have thus shown $\|[b,T]a\|_{L^1} \lesssim \|b\|_{\mathrm{bmo}}$ for all h^1_b atoms a. Finally, $f \in h^1_{\mathrm{finite},b}(\mathbb{R}^n)$ means it can be written as a finite linear combination $\sum_k \gamma_k a_k$ of h^1_b atoms, so by linearity and taking the infimum over all such decompositions, we have

$$
\|[b,T]f\|_{L^1(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{bmo}} \inf \sum_k |\gamma_k|.
$$

$\square$

It is not possible to extend the conclusion to all $h^1_{\mathrm{atom},b}(\mathbb{R}^n)$ because it is unknown whether the infimum $\inf \sum_k |\gamma_k|$ taken over all finite linear combinations of atoms is comparable to $\|f\|_{h^1_{\mathrm{atom},b}}$, the infimum taken over all possible atomic decomposition. However, if $h^1_{\mathrm{atom},b}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$, the infimum over the finite linear combinations of atoms is comparable to the h^1 norm. In this case, verifying the boundedness on one h^1 atom is enough to obtain a bounded extension to the entire space (see [84, Theorem 3.1, Corollary 3.4] for the case of H^1 and [56, Proposition 7.1] for the case of h^1 on a space of homogeneous type). This gives us the following corollary.

Corollary 6.3.2 ([30, Corollary 5.1]). *Let $b \in \mathrm{lmo}(\mathbb{R}^n)$, and T be an inhomogeneous singular integral operator given by Theorem 6.3.1. Then there exists a unique extension L_b of $[b, T]$ that is bounded from $h^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.*

In particular, this corollary implies the boundedness of the commutators $[b, \mathcal{R}_{j,\eta}]$ and $[b, \mathcal{R}_{j,\psi}]$ defined in Corollary 3.0.4, given that the resulting localized convolution kernels have extra decay away from the origin, which requires some extra assumptions on η and ψ in addition to those in Theorem 3.0.3, and $b \in \mathrm{lmo}(\mathbb{R}^n)$.

We can also compare this corollary with the results of [62]. While our assumption $b \in \mathrm{lmo}$ is stronger than their assumption on b in Theorem 2.5.31, which allows the oscillation to grow on large balls, and they obtain h^1 to h^1 boundedness, their results only apply to a class of pseudo-differential operators, which have kernels with much better decay and smoothness (see [62, Proposition 2.1]).

We next discuss the boundedness from $h^1_{\mathrm{finite},b}(\mathbb{R}^n)$ to h^1. We first impose some approximate cancellation conditions, the same idea as in Definition 5.1.2.

Definition 6.3.3 ([30, Definition 5.2]).
Suppose T is an inhomogeneous singular integral operator. For $b \in \mathrm{bmo}$, define $T^*_B(b)$, relative to this ball B, in the distributional sense, by

$$\langle T^*_B(b), g\rangle := \int_{\mathbb{R}^n} (b - b_B)Tg(x)dx, \qquad \forall\, g \in L^2_0(B). \tag{57}$$

As in (55), the conditions on the kernel and the vanishing integral of g guarantee that the integral on the right-hand-side of (57) converges absolutely and $T^*_B(b)$ is a bounded linear functional on $L^2_0(B)$ with norm bounded by a constant multiple of $\|b\|_{\mathrm{bmo}}|B|^{1/2}$. Therefore, we can identify $T^*_B(b)$ with an equivalence class of functions in $L^2(B)$ modulo constants. In particular, we can impose the following condition, denoting $f := T^*_B(b)$, without ambiguity:

$$\left(\fint_B |f - f_B|^2\right)^{1/2} \leqslant \frac{1}{\log(1 + r(B)^{-1})}. \tag{58}$$

Theorem 6.3.4 ([30, Theorem 5.3]). *Let $b \in \mathrm{bmo}(\mathbb{R}^n)$. Suppose T is a inhomogeneous singular integral operator defined in Definition 5.1.1 with $s > \frac{3}{2}$, $0 < \delta \leqslant 1$, $\eta > 0$ and $P^1_{x,z}(y) = K(x, x_B) = P^2_{x,z}(y)$. If $T^*(1) \in \mathrm{LMO}_{loc}(\mathbb{R}^n)$, and, for any ball B with $r(B) < 1$, $f := T^*_B(b)$ satisfies (58), then*

$$\|[b, T](a)\|_{h^1} \leqslant C\|b\|_{\mathrm{bmo}}$$

for all h^1_b atoms a.

Proof of Theorem 6.3.4.
The proof is adapted from [30].

Let a be an h^1_b atom supported in a ball $B = B(x_0, r)$; as above, we may assume that when $r < 1$, $\int a = 0$. Again write $[b, T](a) = (b - c_B)T(a) - T(a(b - c_B))$.

The assumption $T^*(1) \in \mathrm{LMO}_{\mathrm{loc}}(\mathbb{R}^n)$ allows us to apply Theorem 5.1.4 get the boundedness of T on h^1; thus from (49) we get

$$\|T(a(b - c_B))\|_{h^1} \lesssim \|a(b - c_B)\|_{h^1} \lesssim \|b\|_{\mathrm{bmo}}.$$

It remains to estimate the term $\|(b-c_B)T(a)\|_{h^1}$. We will show that $M=(b-c_B)T(a)$ is a multiple of an $(1,\frac{3}{2},\lambda,1)$ molecule (see Definition 4.1.3) associated to the ball $2B$, where we choose

$$\lambda=\frac{n}{2}+\varepsilon,\quad 0<\varepsilon<\frac{3}{2}\min(\delta,\mu).$$

We first verify condition (M1). Proceeding as in (48) and using the boundedness of T on L^2, we have

$$\begin{aligned}\|(b-b_B)T(a)\|_{L^{\frac{3}{2}}(2B)} &\lesssim \|(b-b_B)\|_{L^6(2B)}\|T(a)\|_{L^2}\\ &\lesssim \|b-b_B\|_{L^6(2B)}\|a\|_{L^2}\\ &\lesssim \|b\|_{\mathrm{bmo},6}r^{-\frac{n}{3}}.\end{aligned}$$

Next, we need to show condition M2: $\int_{(2B)^c}|M(x)|^{\frac{3}{2}}|x-x_0|^{\lambda}dx\lesssim r^{\varepsilon}$. Suppose that $r<1$. Note that

$$\begin{aligned}&\int_{(2B)^c}|(b(x)-b_B)T(a)(x)|^{\frac{3}{2}}|x-x_0|^{\lambda}dx\\ &=\int_{(2B)^c}|b(x)-b_B|^{\frac{3}{2}}\left|\int_B[K(x,y)-K(x,x_0)]a(y)dy\right|^{\frac{3}{2}}|x-x_0|^{\lambda}dx\\ &\leqslant |B|^{\frac{1}{2}}\int_{(2B)^c}\int_B|b(x)-b_B|^{\frac{3}{2}}|K(x,y)-K(x,x_0)|^{\frac{3}{2}}|a(y)|^{\frac{3}{2}}|x-x_0|^{\lambda}dydx\\ &\leqslant |B|^{\frac{1}{2}}\sum_{j=1}^{\infty}\int_B\int_{A_j(x_0,r)}|b(x)-b_B|^{\frac{3}{2}}|K(x,y)-K(x,x_0)|^{\frac{3}{2}}|a(y)|^{\frac{3}{2}}|x-x_0|^{\lambda}dxdy\\ &\lesssim |B|^{\frac{1}{2}}\sum_{j=1}^{\infty}2^{j\lambda}r^{\lambda}\int_B\int_{A_j(x_0,r)}|b(x)-b_B|^{\frac{3}{2}}|K(x,y)-K(x,x_0)|^{\frac{3}{2}}|a(y)|^{\frac{3}{2}}dxdy\\ &\lesssim |B|^{\frac{1}{2}}\sum_{j=1}^{\infty}2^{j\lambda}r^{\lambda}\int_B\left(\int_{A_j(x_0,r)}|b-b_B|^{\widetilde{s}}\right)^{\frac{3}{2\widetilde{s}}}\left(\int_{A_j(x_0,r)}|K(\cdot,y)-K(\cdot,x_0)|^{s}\right)^{\frac{3}{2s}}|a(y)|^{\frac{3}{2}}dy\end{aligned}\tag{59}$$

where $\widetilde{s}=\frac{3s}{2s-3}$.

Since

$$\begin{aligned}&\left(\int_{A_j(x_0,r)}|b-b_B|^{\widetilde{s}}\right)^{\frac{3}{2\widetilde{s}}}\\ &\lesssim |B|^{\frac{3}{2\widetilde{s}}}(2^j)^{\frac{3}{2\widetilde{s}}}\left[\left(\int_{B(x_0,2^{j+1}r)}|b-b_{B(x_0,2^{j+1}r)}|^{\widetilde{s}}\right)^{\frac{3}{2\widetilde{s}}}+\left(\min\{j+1,\log_+(r^{-1})\}\|b\|_{\mathrm{bmo}}\right)^{\frac{3}{2}}\right]\\ &\lesssim |B|^{\frac{3}{2\widetilde{s}}}(2^j)^{\frac{3n}{2\widetilde{s}}}j^{\frac{3}{2}}\|b\|_{\mathrm{bmo}}^{\frac{3}{2}}\end{aligned}$$

and using (35) of K,

$$\left(\int_{A_j(x_0,r)}|K(\cdot,y)-K(\cdot,x_0)|^{s}\right)^{\frac{3}{2s}}\leqslant |A_j(x_0,r)|^{\frac{3}{2s}-\frac{3}{2}}2^{-\frac{3}{2}j\delta}\lesssim |B|^{\frac{3}{2s}-\frac{3}{2}}2^{nj(\frac{3}{2s}-\frac{3}{2})-\frac{3}{2}j\delta},$$

substituting into (59) we have

$$
\begin{aligned}
(59) &\lesssim |B|^{\frac{1}{2}} \sum_{j=1}^{\infty} 2^{j\lambda} r^{\lambda} j^{\frac{3}{2}} |B|^{\frac{3}{2s}} (2^j)^{\frac{3n}{2s}} \|b\|_{\mathrm{bmo}}^{\frac{3}{2}} |B|^{\frac{3}{2s}-\frac{3}{2}} 2^{nj(\frac{3}{2s}-\frac{3}{2})-\frac{3}{2}j\delta} |B|^{-\frac{1}{2}} \\
&\lesssim \|b\|_{\mathrm{bmo}}^{\frac{3}{2}} \sum_{j=1}^{\infty} r^{-\frac{1}{2}n+\lambda} j^{\frac{3}{2}} 2^{j(\lambda-\frac{n}{2}-\frac{3}{2}\delta)} \lesssim \|b\|_{\mathrm{bmo}}^{\frac{3}{2}} r^{-\frac{1}{2}n+\lambda} = r^{\varepsilon} \|b\|_{\mathrm{bmo}}^{\frac{3}{2}}.
\end{aligned}
$$

Here we have used the fact that $\lambda - \frac{n}{2} - \frac{3}{2}\delta < 0$.

Finally, condition (58) on $f = T_B^*(b)$ gives us

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} [b(x) - b_B] T(a)(x) dx \right| = \left| \langle T_B^*(b), a \rangle \right| &\leqslant \left(\int_B |f - f_B|^2 \right)^{1/2} \|a\|_{L^2(B)} \\
&\leqslant [\log(1 + r(B)^{-1})]^{-1}.
\end{aligned}
$$

Now looking at the case $r \geqslant 1$, we can proceed as in (56) to write

$$
\begin{aligned}
&\int_{(2B)^c} |(b(x) - c_B) T(a)(x)|^{\frac{3}{2}} |x - x_0|^{\lambda} dx \\
&\lesssim |B|^{\frac{1}{2}} \int_{(2B)^c} \int_B |b(x) - c_B|^{\frac{3}{2}} |K(x,y)|^{\frac{3}{2}} |a(y)|^{\frac{3}{2}} |x - x_0|^{\lambda} dy dx \\
&\lesssim \int_{(2B)^c} \frac{|b(x) - c_B|^{\frac{3}{2}}}{|x - x_0|^{\frac{3}{2}n + \frac{3}{2}\mu - \lambda}} dx \\
&\lesssim \|b\|_{\mathrm{bmo}} r^{\lambda - \frac{n}{2} - \frac{3}{2}\mu} \lesssim \|b\|_{\mathrm{bmo}} r^{\varepsilon}.
\end{aligned}
$$

Here we have used Theorem 2.5.16 and the fact that $r \geqslant 1$, $\varepsilon < \frac{3}{2}\mu$.

To proceed to the one with $|\int a| \leqslant [\log(1 + r^{-1})]^{-2}$, note that if we write $a = \sum_{j=1}^N \lambda_j a_j$ according to Proposition 6.1.5, we have

$$
\|[b, T](a)\|_{h^1} \lesssim \|b\|_{\mathrm{bmo}} \sum |\lambda_j| \lesssim \|b\|_{\mathrm{bmo}}.
$$

$\square$

Similarly to Corollary 6.3.2, when $h^1_{b,atom} = h^1$, the boundedness extends from one atom to all functions in the space.

Corollary 6.3.5 ([30, Corollary 5.2])**.** *Assuming the hypotheses of Theorem 6.3.4, if in addition $b \in \mathrm{lmo}(\mathbb{R}^n)$, then $[b, T]$ is bounded from $h^1(\mathbb{R}^n)$ to $h^1(\mathbb{R}^n)$.*

Part II

Periodic Solutions of Stochastic Partial Differential Equations with Regime-Switching

Chapter 7

Preliminary of Part II and Framework

This chapter provides the background and definitions that will be used in later chapters as well as the framework for the SPDE model. We fix some notations for Part II of this thesis. We use the set $\mathbb{N} := \{1, 2, 3, \ldots\}$ and denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_{0,\infty} = \mathbb{N} \cup \{0, \infty\}$. We also use the notation $B_b(H)$ ($C_b(H)$) to denote the space of all bounded measurable functions (bounded continuous functions, respectively) defined on H.

In addition, we work with a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and denote the expectation of a random variable X with respect to $\mathbb{P}$ by $\mathbb{E}(X)$. Then, we have $\mathbb{P}(A) = \mathbb{E}(1_A)$, where 1_A is the indicator function of an event A. We also fix a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ such that $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ is the complete filtered probability space throughout Part II.

The remaining part of this chapter is organized as follows. In Section 7.1, we introduce definitions related to functional analysis. In Section 7.2, we give the basic definitions of stochastic integrals and their properties. In Section 7.3, we recall some concepts related to ergodicity. In Section 7.4, we describe the framework of our SPDE model.

7.1 Analysis Preliminary

7.1.1 Gelfand triple

We consider a *Gelfand triple* (V, H, V^*) such that $V \subseteq H \subseteq V^*$. Here, $(H, \langle\cdot,\cdot\rangle_H)$ is a real separable Hilbert space, $(V, |\cdot|_V)$ is a real reflexive Banach space that is continuously and densely embedded into H, and V^* is the dual of V. The dual pairing of V and V^* is denoted by $\langle\cdot,\cdot\rangle$, and if $h \in H, v \in V$ then $\langle h, v\rangle = \langle h, v\rangle_H$.

7.1.2 Operators

Let H_1, H_2 be two separable Hilbert spaces. The set $L(H_1; H_2)$ is defined to be the space of all bounded linear operators from H_1 to H_2 with the norm denoted by $\|\cdot\|_{L(H_1;H_2)}$. In this thesis, the following two subspaces of $L(H_1; H_2)$ are considered.

Definition 7.1.1. *Let $T \in L(H_1; H_2)$, $\{e_k\}_{k=1}^\infty$ and $\{e'_k\}_{k=1}^\infty$ be a orthonormal basis of H_1 and H_2, respectively.*

1. *We say T is a trace class operator and we write $T \in L_1(H_1; H_2)$ if*

$$\sum_{k=1}^{\infty} |\langle T(e_k), e'_k \rangle_{H_2}| < \infty.$$

2. *We say T is a Hilbert-Schmidt operator and we write $T \in L_2(H_1; H_2)$ if*

$$\sum_{k=1}^{\infty} |T(e_k)|_{H_2}^2 < \infty.$$

The definitions are independent of the choices of basis. This can be seen by writing one basis in terms of the second basis and interchanging the sums. For shortening the notations, if $H_2 = H_1 = H$, we write $L(H)$, $L_1(H)$ and $L_2(H)$ as $L(H; H)$, $L_1(H; H)$ and $L_2(H; H)$ respectively. For a Hilbert space H, it is also known that $T \in L_1(H)$ iff there exists $Q \in L_2(H)$ such that $T = Q^*Q$ (see [99, Theorem 12.33]).

In our applications, we are interested in the case that $H_1 = H_2 = H$. We define the Hilbert-Schmidt norm for $L_2(H)$ to be

$$\|T\|_{L_2(H)} := \left(\sum_{j=1}^{\infty} |T(e_j)|_H^2 \right)^{\frac{1}{2}},$$

where the $\{e_j\}$ is an orthonormal basis of H and the trace norm to be

$$\|T\|_{L_1(H)} := \sum_{j=1}^{\infty} |\langle T(e_j), e_j \rangle_H|.$$

We refer to [23, Chapter 3] for a detailed discussion. We also denote the identity operator on H as I, which is not in $L_1(H)$ nor $L_2(H)$ if H is infinite-dimensional since these are classes of compact operators.

7.1.3 Gronwall's inequalities

In this thesis, we use the following two forms of Gronwall's lemma.

Proposition 7.1.2. *If $\beta(t)$ is non-negative, $\alpha(x) \geqslant 0$, and $u(t)$ satisfies*

$$u(t) \leqslant \alpha(t) + \int_{t_0}^{t} \beta(s) u(s) ds,$$

then

$$u(t) \leqslant \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s) e^{\int_s^t \beta(r) dr} ds.$$

Proposition 7.1.3. *Suppose u is differentiable on I and satisfies*

$$u'(t) \leqslant a(t)u(t) + b(t) \quad \forall\ t \in (t_0, T),$$

for some integrable $a(t)$ and $b(t)$ on $[t_0, T]$. Then,

$$u(t) \leqslant e^{\int_0^t a(s) ds} u(0) + \int_0^t b(s) e^{\int_s^t a(r) dr} ds$$

for all $t \in [t_0, T]$.

7.2 Stochastic Integrals

7.2.1 Wiener integrals

We now introduce H-valued Wiener processes.

Definition 7.2.1. *Let $Q \in L_1(H)$, non-negative and symmetric. An H-valued adapted stochastic process $W = \{W(t)\}_{t \geqslant 0}$ is called Q-Wiener process if*

1. *for any $0 \leqslant t_0 < t_1 < \cdots < t_n$, the set $\{W(t_i) - W(t_{i-1})\}_{i=1}^n$ is an independent collection of random variables;*
2. *$W(t) - W(s)$ has normal distribution with mean 0 and variance $(t-s)Q$.*
3. *$W(0) = 0$; and*
4. *$t \mapsto W(t)$ is continuous a.s.*

We can express $W(t)$ as $\sum_{j \in \mathbb{N}} \sqrt{\lambda_j} W_j(t) e_j$, where $\{\lambda_j\}$ are the eigenvalues (all positive) of Q with eigenvectors $\{e_j\}$, $W_j(t) := (\lambda_j)^{-\frac{1}{2}} \langle W(t), e_j \rangle_H$ and each W_j are independent one dimensional Brownian motion, see [81, Theorem 2.119].

If $Q \in L_1(H)$, then one can directly define (see [77, Page 53] and [81, Section 2.10]) the stochastic integral $\int_0^T f(t) dW(t)$ with respect to Q-Wiener process for progressively measurable (and adapted) f that satisfies

$$\int_0^T \|f(t) Q^{\frac{1}{2}}\|_{L_2(H)}^2 dt < \infty \quad a.s. \tag{60}$$

For $Q \notin L_1(H)$, we need another way to define Wiener processes.

Definition 7.2.2 (See [81, Definition 2.120])**.** *Let Q be a non-negative and symmetric operator on H. An H-valued adapted stochastic process $W = \{W(t)\}_{t \geqslant 0}$ is called Wiener process with covariance Q if we have the following formal series*

$$W(t) = \sum_{j=1}^{\infty} w_j(t) Q^{\frac{1}{2}}(e_j)$$

where $\{e_j\}_{j \in \mathbb{N}}$ are the eigenvectors of Q and $\{w_j(t)\}_{j \in \mathbb{N}}$ are independent real-valued standard Brownian motions.

For such a Wiener process, the series does not converges in H but in a larger Hilbert space, see [81, Theorem 2.121], so we can define $\langle W(t), h \rangle_H$ for all $h \in H$. We refer the reader to [81, Section 2.9] or [28, Sections 4.1.2] for the construction of Wiener processes with generalized covariance Q. We can also define the stochastic integral for progressively measurable functions, but the integrand still has to satisfy (60), which requires a stronger assumption on f. One particular choice of $Q \notin L_1(H)$ is the identity operator I and the Wiener process with this covariance operator is called the cylindrical Wiener process. For a detailed discussion of the construction of the stochastic integrals with respect to Wiener processes, we refer the reader to [28, Chapter 4] and [77, Chapter 2].

We give some useful results on stochastic integrals with respect to Wiener processes.

Theorem 7.2.3 (See [81, Sections 2.10 and 2.11] or [28, Chapter 4])**.** *Let $T > 0$ and W be a Q-Wiener process. Suppose f satisfies (60). Then, for $t \in [0, T]$,*

1. $\mathbb{E}\left(\int_0^t f(s) dW(s) \right) = 0$;

2. $\int_0^t f(s) dW(s)$ *is a H-valued and continuous local martingale;*

3. $\mathbb{E}\left(\left| \int_0^t f(s) dW(s) \right|_H^2 \right) = \mathbb{E}\left(\int_0^t \| f(s) Q^{\frac{1}{2}} \|_{L_2(H)}^2 ds \right)$.

7.2.2 Poisson integrals

We introduce the Poisson measure on a measurable and separable Banach space $(E, \mathcal{E})$. The space $\mathcal{M}(E; \mathbb{N}_{0,\infty})$ is defined to be the space of all $\mathbb{N}_{0,\infty}$-valued measures defined on $(E, \mathcal{E})$. We equip $\mathcal{M}(E; \mathbb{N}_{0,\infty})$ with a σ-field (also known as σ-algebra) $\mathscr{A}$ generated by the map $\mathcal{M}(E; \mathbb{N}_{0,\infty}) \ni \rho \mapsto \rho(A)$ for all $A \in \mathcal{E}$.

Definition 7.2.4. *A map $N : \Omega \times \mathcal{E} \to \mathbb{N}_{0,\infty}$ is called Poisson random measure if*

1. *$N(\omega, \cdot)$ is a measure on E for any $\omega \in \Omega$ and $N(\cdot, A)$ is a random variable on $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t \geqslant 0}, \mathbb{P})$ for any $A \in \mathcal{E}$;*

2. *for any $A \in \mathcal{E}$ that $\lambda(A) := \mathbb{E}(N(\cdot, A)) < \infty$, the random variable $N(\omega, A)$ is a Poisson random variable with parameter η, that is*
$$\mathbb{P}(\{\omega \in \Omega : N(\omega, A) = n\}) = \frac{[\lambda(A)]^n}{n!} e^{-\lambda(A)}.$$

3. *for any pairwise disjoint sets $A_1, \cdots, A_n \in \mathcal{E}$, $\{N(\cdot, A_i)\}_{i=1}^n$ are independent random variables.*

We write $N(A)$ instead of $N(\cdot, A)$. Moreover, the measure λ defined on $\mathcal{E}$ is called the intensity measure of N.

Definition 7.2.5. *Let N be a Poisson random measure with intensity λ. The compensated Poisson measure is defined*
$$\widetilde{N}(A) := N(A) - \lambda(A) \qquad \forall A \in \mathcal{E}.$$

We are particularly interested in the Poisson random measure defined on $[0, \infty) \times Z$, where $(Z, \mathcal{Z})$ is a measurable Banach space. We say N is stationary in t if for any $t \geqslant 0$ and $\mathcal{T} \times A \in \mathscr{B}([0, \infty)) \times \mathcal{Z}$, the law of the random variables $N(\mathcal{T} \times A)$ and $N((\mathcal{T} + t) \times A)$ are the same. It is equivalent to saying the intensity measure is of the form $dt\nu(d\sigma)$ for some non-negative σ-additive measure on Z. We assume that ν is finite.

Definition 7.2.6. *We say p is a Poisson point process corresponding to $N(dt, dz)$ if p is Z-valued adapted process such that*
$$N(\omega, (0, t] \times A) = \#\{s : p(s, \omega) \in A\} \quad \forall A \in \mathcal{Z}, \ t \in (0, T].$$

Definition 7.2.7. *We say a point process p is stationary if $p(\cdot,\cdot)$ and $p(\cdot+t,\cdot)$ have the same probability distribution for all $t>0$.*

We denote the collection

$$\mathscr{B}_{\mathscr{P}} := \{A \subset [0,\infty)\times\Omega : A \cap ([0,t]\times\Omega) \in \mathscr{B}([0,t]) \otimes \mathscr{F}_t \quad \forall t \in [0,T]\}.$$

By [130, Corollary 3.2.26], we can define the stochastic integral for H-valued function f that is measurable with respect to $\mathscr{B}_{\mathscr{P}} \otimes \mathcal{Z}$ (also known as progressively measurable) that is

$$\int_0^T \int_Z \mathbb{E}(|f(s,\omega,z)|_H^r)\nu(dz)ds < \infty,$$

where $1 \leqslant r \leqslant 2$. We remark that Zhu [130] introduced Poisson integrals for the Banach spaces E of martingale type $p \in [1,2]$, of which the Hilbert space H is (see [130, Remark 3.2.10]).

Now we state some properties of stochastic integrals with respect to $\widetilde{N}$.

Theorem 7.2.8 ([130, Theorem 3.3.2])**.** *Suppose f that is measurable with respect to $\mathscr{B}_{\mathscr{P}} \otimes \mathcal{Z}$ that is*

$$\int_0^T \int_Z \mathbb{E}(|f(s,\cdot,z)|_H^r)\nu(dz)ds < +\infty,$$

for $1 \leqslant r \leqslant 2$. Then, for all $t \in [0,T]$,

1. $\mathbb{E}\left(\int_0^t \int_Z f(s,\cdot,z)\widetilde{N}(ds,dz)\right) = 0$;
2. $\mathbb{E}\left(\left|\int_0^t \int_Z f(s,\cdot,z)\widetilde{N}(ds,dz)\right|_H^r\right) \leqslant C_p \int_0^t \int_Z \mathbb{E}(|f(s,\cdot,z)|_H^r)\nu(dz)ds$; *in particular, if* $p=2$, $\mathbb{E}\left(\left|\int_0^t \int_Z f(s,\cdot,z)\widetilde{N}(ds,dz)\right|_H^2\right) = \int_0^t \int_Z \mathbb{E}(|f(s,\cdot,z)|_H^2)\nu(dz)ds.$
3. *Moreover, the stochastic integral with respect to $\widetilde{N}$ has a càdlàg modification.*

Next, we define the stochastic integral with respect to Poisson random measure N. For our purpose, we only consider the case where $|z|_Z \geqslant 1$. In this case, there are only finitely number of jumps by time T, and we denote them by $0 < t_1 < t_2 < \cdots < t_n < T$. Then we can define

$$\int_0^T \int_{|z|_Z \geqslant 1} g(s,\omega,z)N(dt,dz) := \sum_{j=1}^n g(s,\omega,p(t_j)(\omega)).$$

7.2.3 Itô's formula

In this thesis, we apply the following infinite-dimensional Itô's formula.

Theorem 7.2.9 (Itô's formula)**.**
Suppose (V,H,V^*) is a Gelfand triple. Suppose W is a I-Wiener process; N is a Poisson random measure defined on $[0,\infty)\times Z$, where Z is a Banach space, independent of W; and

$$X(t) = \int_0^t A(s)ds + \int_0^t B(s)dW(s) + \int_0^t \int_{|z|_Z<1} G(s,z)\widetilde{N}(ds,dz),$$

where $A(s)$ is V^*-valued, $B(s) \in L_2(H)$ for all s, and $G(s,z)$ is H-valued. Then we have (a.s.)

$$
\begin{aligned}
|X(t)|_H^2 = |X(0)|_H^2 &+ \int_0^t 2\langle A(s), \overline{X}(s)\rangle ds + \int_0^t \|B(s)\|_{L_2(H)}^2 ds \\
&+ 2\int_0^t \langle X(s), B(s)dW(s)\rangle_H + \\
&+ \int_0^t \int_{|z|_Z<1} \big(|X(s)+G(s,z)|_H^2 - |X(s)|_H^2 - |G(s,z)|_H^2\big)\widetilde{N}(ds,dz) \\
&+ \int_0^t \int_{|z|_Z<1} \big(|X(s)+G(s,z)|_H^2 - |X(s)|_H^2\big)\nu(dz)ds,
\end{aligned}
$$

where $\overline{X}(t)$ is a V-valued process that is $\overline{X}(t) = X(t)$ for $dt \times d\mathbb{P}$-a.e.

We will also apply the following argument in later chapter.

$$
|X(t)|_H^\gamma - |X(0)|_H^\gamma = \int_0^t \frac{d}{ds}\left(\big[|X(s)|_H^2\big]^{\frac{\gamma}{2}}\right)ds = \int_0^t \frac{\gamma}{2}|X(s)|_H^{\gamma-2}\frac{d}{ds}\left(|X(s)|_H^2\right)ds \quad (61)
$$

7.3 Ergodicity

In this section, we recall the concepts related to ergodicity. We refer to [27] for the comprehensive discussion of ergodicity of SPDEs.

Definition 7.3.1.

- *We say $P(s,x;t,A)$ for $t \geqslant s \geqslant 0$, $x \in H$ and $A \in \mathscr{B}(H)$ is a Markovian transition function if*
  1. *for each $x \in H$ and $t \geqslant s \geqslant 0$, $P(s,x;t,\cdot)$ is a probability measure on $(H, \mathscr{B}(H))$;*
  2. *for each $A \in \mathscr{B}(H)$ and $t \geqslant s \geqslant 0$, the function $x \mapsto P(s,x;t,A)$ is a $\mathscr{B}(H)$ measurable function defined on H; and*
  3. *for each $x \in H$ and $A \in \mathscr{B}(H)$, we have $P(s,x;s,A) = \chi_A(x)$.*
- *For this $P(s,x;t,A)$, define $P_{s,t}$ as a linear operator on $B_b(H)$ by*

  $$P_{s,t}f(x) := \int_H f(y)P(s,x;t,dy) \qquad \forall f \in B_b(H).$$

  Such $\{P_{s,t}\}_{t\geqslant s}$ is called Markovian transition semigroup.

It is clear that if f is a characteristic function of a measurable set A, then $P_{s,t}f(x) = P_{s,t}(1_A)(x) = P(s,x;t,A)$.

Definition 7.3.2. *Let $\{P_{s,t}\}_{t\geqslant s}$ be a Markovian transition semigroup.*

- *We say $P_{s,t}$ is strong Feller at $t_0 > 0$ if $P_{t_0,s}f \in C_b(H)$ whenever $f \in B_b(H)$.*
- *We say $P_{s,t}$ is strong Feller if $P_{s,t}$ is strong Feller at all $t > 0$.*

Definition 7.3.3. *Let $\{P_{s,t}\}_{t\geqslant s}$ be a Markovian transition semigroup.*

- *We say $P_{s,t}$ is irreducible at $t_0 > s$ if for any non-empty set $A \in \mathscr{B}(H)$ and $x \in H$, the transition semigroup satisfies*

$$P_{t_0,s}(x, A) = P_{t_0,s}(\chi_A)(x) > 0.$$

- *We say $P_{s,t}$ is irreducible if $P_{s,t}$ is irreducible at all $t > s$.*

7.4 Framework

This section provides the framework of our SPDE model. Let (V, H, V^*) be a Gelfand triple. Let $\{W(t)\}_{t\geqslant 0}$ be an H-valued cylindrical Wiener process (*i.e.* $Q = I$) on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ which we fixed at the beginning. To ease the notation, we let Z be a real Banach space with norm $|\cdot|$ instead of $|\cdot|_Z$. Let N be a Poisson random measure on $(Z, \mathscr{B}(Z))$ with intensity measure ν. *We assume that W and N are independent.*

We consider the SPDE

$$\begin{aligned} dX(t) &= A(t, X(t), \Lambda(t))dt + B(t, X(t), \Lambda(t))dW(t) \\ &\quad + \int_{\{|z|<1\}} G(t, X(t), \Lambda(t), z)\widetilde{N}(dt, dz) \\ &\quad + \int_{\{|z|\geqslant 1\}} J(t, X(t), \Lambda(t), z)N(dt, dz) \end{aligned} \tag{62}$$

with $X(0) = x \in H$. From now on, we assume that the functions $A : [0, \infty) \times V \times \mathbb{N} \to V^*$, $B : [0, \infty) \times V \times \mathbb{N} \to L_2(H)$ and $G, J : [0, \infty) \times V \times \mathbb{N} \times Z \to H$ are all measurable. We also assume that the process $\Lambda(t)$ has the state space $\mathbb{N}$ such that when $\Delta \to 0$,

$$\mathbb{P}(\Lambda(t + \Delta) = j|\Lambda(t) = i, X(t) = x) = \begin{cases} q_{ij}(x)\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + q_{ij}(x)\Delta + o(\Delta), & \text{if } i = j. \end{cases} \tag{63}$$

Hereafter, we assume that $\{q_{ij}\}$ are Borel measurable functions on H; $q_{ij}(x) \geqslant 0$ for any $x \in H$ and $i, j \in \mathbb{N}$ with $i \neq j$; and $\sum_{j\in\mathbb{N}} q_{ij}(x) = 0$ for any $x \in H$ and $i \in \mathbb{N}$. In the remaining discussion, we assume

(Q0)

$$L := \sup_{x\in H, i\in\mathbb{N}} \sum_{j\neq i} q_{ij}(x) < \infty.$$

According to Guo and Sun [51], $\Lambda(t)$ can be written as a stochastic integral with respect to a Poisson random measure. More precise, for each $x \in H$ and $i, j \in \mathbb{N}$ that $i \neq j$, define $q_{i0}(x) = 0$ and

$$\Delta_{ij}(x) := \left[\sum_{m=0}^{j-1} q_{im}(x), \sum_{m=0}^{j} q_{im}(x) \right).$$

Set

$$\Gamma(x, i, r) = \sum_{j\in\mathbb{N}} (j - i)1_{\Delta_{ij}(x)}(r), \qquad (x, i, r) \in H \times \mathbb{N} \times [0, L].$$

Then, $\Lambda(t)$ can be written as

$$d\Lambda(t) = \int_{[0,L]} \Gamma(X(t-), \Lambda(t-), r) N_1(dt, dr), \tag{64}$$

where N_1 is a Poisson random measure with the Lebesgue measure on $[0, L]$ as its characteristic measure. *We assume that N_1 is independent of W and N henceforth.*

Chapter 8

Main Results

In this chapter, we will state the main results. All the statements are from work [73]. We start with the existence and uniqueness of solutions. Therefore, we first impose the following assumption.

Assumption 1. *Suppose that there exist $\alpha > 1$, $\beta \geqslant 0$, $\theta > 0$, $K \in \mathbb{R}$, $\gamma < \frac{\theta}{2\beta}$, $c > 0$, $\rho \in L^{\infty}_{loc}(V; [0,\infty))$ and $C \in L^{\frac{\beta+2}{2}}_{loc}([0,\infty); [0,\infty))$ such that for $v_1, v_2, v \in V$, $i \in \mathbb{N}$ and $t \in [0,\infty)$,*

(HC) *(Hemicontinuity) $s \mapsto \langle A(t, v_1 + sv_2, i), v\rangle$ is continuous on $\mathbb{R}$.*

(LM) *(Local monotonicity)*

$$\begin{aligned}
& 2\langle A(t,v_1,i) - A(t,v_2,i), v_1 - v_2\rangle + \|B(t,v_1,i) - B(t,v_2,i)\|^2_{L_2(H)} \\
& + \int_{\{|z|<1\}} |G(t,v_1,i,z) - G(t,v_2,i,z)|^2_H \nu(dz) \\
\leqslant\ & [K + \rho(v_2)]|v_1 - v_2|^2_H.
\end{aligned}$$

(C) *(Coercivity)*

$$\begin{aligned}
& 2\langle A(t,v,i), v\rangle + \|B(t,v,i)\|^2_{L_2(H)} + \int_{\{|z|<1\}} |G(t,v,i,z)|^2_H \nu(dz) \\
\leqslant\ & C(t) - \theta|v|^{\alpha}_V + c|v|^2_H.
\end{aligned}$$

(G1) *(Growth of A)*

$$|A(t,v,i)|^{\frac{\alpha}{\alpha-1}}_{V^*} \leqslant [C(t) + c|v|^{\alpha}_V](1 + |v|^{\beta}_H).$$

(G2) *(Growth of B and H)*

$$\|B(t,v,i)\|^2_{L_2(H)} + \int_{\{|z|<1\}} |G(t,v,i,z)|^2_H \nu(dz) \leqslant C(t) + \gamma|v|^{\alpha}_V + c|v|^2_H.$$

(Gβ) *(Growth of H in $L^{\beta+2}$)*

$$\int_{\{|z|<1\}} |G(t,v,i,z)|^{\beta+2}_H \nu(dz) \leqslant [C(t)]^{\frac{\beta+2}{2}} + c|v|^{\beta+2}_H.$$

(Gρ) *(Growth of ρ)*

$$\rho(v) \leqslant c(1+|v|_V^\alpha)(1+|v|_H^\beta).$$

Now we can state the theorem of the existence and uniqueness of solutions to the equations (62) and (64).

Theorem 8.0.1. *Suppose that Assumption 1 and condition (Q0) hold. Let $T > 0$, $x \in H$ and $i \in \mathbb{N}$. Then, there exists a unique $H \times \mathbb{N}$-valued adapted càdlàg process $\{(X(t), \Lambda(t))\}_{t\in[0,T]}$ such that*

1. *any $dt \times \mathbb{P}$-equivalent class $\widehat{X}$ of X is in $L^\alpha([0,T];V)\bigcap L^2([0,T];H)$, $\mathbb{P}$-a.s.;*
2. *for any V-valued progressively measurable $dt \times \mathbb{P}$-version $\overline{X}$ of $\widehat{X}$, the following holds for all $t \in [0,T]$ and $\mathbb{P}$-a.s.:*

$$\begin{aligned} X(t) = x &+ \int_0^t A(s,\overline{X}(s),\Lambda(s))ds + \int_0^t B(s,\overline{X}(s),\Lambda(s))dW(s) \\ &+ \int_0^t \int_{\{|z|<1\}} G(s,\overline{X}(s),\Lambda(s),z)\widetilde{N}(ds,dz) \\ &+ \int_0^t \int_{\{|z|\geqslant 1\}} J(s,\overline{X}(s),\Lambda(s),z)N(ds,dz); \end{aligned} \tag{65}$$

3. *$\Lambda(0) = i$ and Equation (64) holds.*

By the standard argument, we know that $\{(X(t), \Lambda(t))\}_{t\geqslant 0}$ is a Markov process (cf. [43, Theorem 4.8] and [5, Theorem 6.4.5]). We now shall establish the strong Feller property and irreducibility under the assumption that the noise is degenerate.

Assumption 2. *Suppose that $\alpha \geqslant 2$ and the following conditions hold:*

1. ***(LipB)*** *For any $n \in \mathbb{N}$, there exists $C_n > 0$ such that*

$$\|B(t,v_1,i) - B(t,v_2,i)\|_{L_2(H)} \leqslant C_n|v_1 - v_2|_H$$

for all $v_1, v_2 \in V$ with $|v_1|_H, |v_2|_H \leqslant n$, $t \geqslant 0$ and $i \in \mathbb{N}$.

2. *There exist $\lambda \in [2,\infty)\cap(\alpha-2,\infty)$, $\{B_n\} \subset L_2^{+,s}(H)$ and $n_0 \in \mathbb{N}$ such that the following conditions hold:*

(N) *For any $n \in \mathbb{N}$, $t \geqslant 0$, $v \in V$ with $|v|_H \leqslant n$ and $i \in \mathbb{N}$,*

$$B(t,v,i)[B(t,v,i)]^* \geqslant B_n^2.$$

(M) *Each B_n has a Moore-Penrose pseudo-inverse $B_n^{-1} : V \to H$ and for any $n \geqslant n_0$, there exist $\widetilde{K_n} \geqslant 0$ and $\delta_n > 0$ such that*

$$\begin{aligned} &2\langle A(t,v_1,i) - A(t,v_2,i), v_1 - v_2\rangle + \|B(t,v_1,i) - B(t,v_2,i)\|_{L_2(H)}^2 \\ &+ \int_{\{|z|<1\}} |G(t,v_1,i,z) - G(t,v_2,i,z)|_H^2 \nu(du) \\ \leqslant\ & -\delta_n|B_n^{-1}(v_1 - v_2)|_H^\lambda |v_1 - v_2|_H^{\alpha-\lambda} + \widetilde{K_n}|v_1 - v_2|_H^2 \end{aligned}$$

for all $v_1, v_2 \in V$, $t \geqslant 0$ and $i \in \mathbb{N}$.

Here we denote $L_2^{+,s}(H)$ to be all Hilbert-Schmidt operators that are positive and self-adjoint.

Theorem 8.0.2. *Suppose that Assumption 1 holds with $C \in L^\infty_{loc}([0,\infty);(0,\infty))$, Assumption 2 and condition (Q0) hold. Then, the transition semigroup $\{P_{s,t}\}$ of $(X(t),\Lambda(t))$ is strong Feller.*

To obtain irreducibility, we need some assumptions on the domain of the operator A. For $\varpi > 0$, define

$$D(A,\varpi) := \left\{(v,i) \in V \times \mathbb{N} : A(t,v,i) \in H \text{ and } \int_0^t |A(s,v,i)|_H^\varpi ds < \infty,\ \forall t \in [0,\infty)\right\}.$$

We impose the following assumptions.

(D) There exists $\varpi > 2$ such that $\overline{D(A,\varpi)} = H \times \mathbb{N}$.

(Q1) For any distinct $i,j \in \mathbb{N}$, there exist an open set $U \subset H$ and $j_1,\dots,j_r \in \mathbb{N}$ with $j_p \neq j_{p+1}$, $j_1 = i$ and $j_r = j$ such that $q_{j_p j_{p+1}}(x) > 0$ for $p = 1,\dots,r-1$ and $x \in U$.

Assumption 3. *Assumption 1 holds with $C \in L^\infty_{loc}([0,\infty);(0,\infty))$, $\gamma = 0$ and the exponent α in condition (Gρ) replaced by some $\alpha' \in (1,\alpha)$.*

Theorem 8.0.3. *Suppose that Assumption 3 and conditions (D), (Q1) hold. Then, the transition semigroup $\{P_{s,t}\}$ of $(X(t),\Lambda(t))$ is irreducible.*

From the definition, Condition (M) implies Condition (LM). We remark that to obtain the existence and uniqueness of the solutions as well as the irreducibility, and we may relax to assuming the coefficients are locally monotone, *i.e.* (LM). However, to establish the strong Feller property, one has to assume a stronger assumption that the coefficients are only monotone, *i.e.* (M). Monotonicity plays an important role in establishing strong Feller property and studying Harnack inequalities, *e.g.* , [76, 118, 128].

Finally, we give the definition of periodic measures and state the main theorem of Part II of the thesis.

Definition 8.0.4. *A probability measure μ_0 on $\mathscr{B}(H)$ is said to be an ℓ-periodic measure for $\{Y(t)\}_{t\geqslant 0}$ if the following condition holds:*

- *$Y(0)$ has distribution μ_0 implies that the joint distribution of $Y(t_1+k\ell),\dots,Y(t_n+k\ell)$ is independent of k for all $0 \leqslant t_1 < \cdots < t_n$ and $n \in \mathbb{N}$.*

We remark that this definition can be applied to any Polish space E with its Borel σ-algebra. Stating in terms of Hilbert space is just to reduce complexity.

We need to impose one more assumption concerning about the function $Q := (q_{ij}(x))$.

(Q2) There exists a positive increasing function f on $\mathbb{N}$ satisfying

$$\lim_{j\to\infty} f(j) = \infty, \quad \sup_{x\in H,\, i\in\mathbb{N}} \sum_{j\neq i}[f(j)-f(i)]q_{ij}(x) < \infty, \quad \lim_{i\to\infty} \sup_{x\in H} \sum_{j\neq i}[f(j)-f(i)]q_{ij}(x) = -\infty.$$

Theorem 8.0.5. *Let $\ell > 0$. Suppose that functions A, B, G, J are all ℓ-periodic with respect to t, the embedding of V into H is compact, Assumption 3 holds with $C \in L^\infty([0,\infty);(0,\infty))$, Assumption 2 and conditions (D), (Q0), (Q1), (Q2) hold, and*

$$\lim_{n\to\infty} \sup_{|v|_V > n, t \geqslant 0, i \in \mathbb{N}} \Big\{ -\theta |v|_V^\alpha + c|v|_H^2 + \int_{\{|z|\geqslant 1\}} \left[|J(t,v,i,z)|_H^2 + 2\langle v, J(t,v,i,z)\rangle_H \right] \nu(dz) \Big\} = -\infty. \tag{66}$$

Then,

(i) Equations (62) and (64) have a unique solution $\{(X(t),\Lambda(t))\}_{t\geqslant 0}$;

(ii) The transition semigroup $\{P_{s,t}\}$ of $\{(X(t),\Lambda(t))\}_{t\geqslant 0}$ is strong Feller and irreducible;

(iii) The hybrid system $\{(X(t),\Lambda(t))\}_{t\geqslant 0}$ has a unique ℓ-periodic measure μ_0;

(iv) Let $\mu_s(A) = \mathbb{P}_{\mu_0}((X(s),\Lambda(s)) \in A)$ for $A \in \mathscr{B}(H\times\mathbb{N})$ and $s \geqslant 0$. Then, for any $s \geqslant 0$ and $\varphi \in L^2(H\times\mathbb{N};\mu_s)$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} P_{s,s+i\ell}\varphi = \int_{H\times\mathbb{N}} \varphi d\mu_s \quad \text{in} \quad L^2(H\times\mathbb{N};\mu_s). \tag{67}$$

Chapter 9

Proofs

All the proofs of this chapter are taken from the paper [73] with more details.

9.1 Proof of Theorem 8.0.1

We first consider the case that $\Lambda(t) \equiv i$ for fixed $i \in \mathbb{N}$. The existence and uniqueness of solutions to SPDEs without regime-switching has been considered by [12].

Theorem 9.1.1. *([12, Theorem 1.2]) Under the assumptions of Theorem 8.0.1, there exists a unique H-valued adapted càdlàg process $\{X(t)\}_{t\in[0,T]}$ such that*

1. *any $dt \times \mathbb{P}$-equivalent class $\widehat{X}$ of X is in $L^{\alpha}([0,T];V)\bigcap L^2([0,T];H)$, $\mathbb{P}$-a.s.;*
2. *for any V-valued progressively measurable $dt \times \mathbb{P}$-version $\overline{X}$ of $\widehat{X}$, the following holds for all $t \in [0,T]$ and $\mathbb{P}$-a.s.:*

$$
\begin{aligned}
X(t) = x &+ \int_0^t A(s,\overline{X}(s))ds + \int_0^t B(s,\overline{X}(s))dW(s) \\
&+ \int_0^t \int_{\{|z|<1\}} G(s,\overline{X}(s),z)\widetilde{N}(ds,dz) \\
&+ \int_0^t \int_{\{|z|\geqslant 1\}} J(s,\overline{X}(s),z)N(ds,dz).
\end{aligned}
$$

We remark that if the coefficients are random, *i.e.* a mapping from $[0,\infty) \times V \times \Omega$ or $[0,\infty) \times V \times Z \times \Omega$ that are progressively measurable, Theorem 9.1.1 still holds.

Using Theorem 9.1.1, we can construct the solution with regime-switching.

Proof of Theorem 8.0.1.
Using Theorem 9.1.1, we know that for any $(x,i) \in H \times \mathbb{N}$, there exists a unique H-valued

adapted process $X^{(i)}(t)$ such that

$$\begin{aligned}X^{(i)}(t) &= x + \int_0^t A(s, \overline{X^{(i)}}(s), i)ds + \int_0^t B(s, \overline{X^{(i)}}(s), i)dW(s) \\ &\quad + \int_0^t \int_{\{|z|<1\}} G(s, \overline{X^{(i)}}(s), i, z)\widetilde{N}(ds, dz) \\ &\quad + \int_0^t \int_{\{|z|\geqslant 1\}} J(s, \overline{X^{(i)}}(s), i, z)N(ds, dz), \end{aligned} \tag{68}$$

where $\overline{X^{(i)}}$ is a V-valued progressively measurable $dt \times \mathbb{P}$-version. Let $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_n < \cdots$ be the set of all jump points of the stationary point process $p_1(t)$ corresponding to the Poisson random measure $N_1(dt, dr)$. From condition (Q0), we have $\lim_{n\to\infty} \sigma_n = \infty$ almost surely.

Now we construct the processes (X, Λ) and its progressively measurable version. For $t \in [0, \sigma_1)$, define

$$(X(t), \Lambda(t)) = (X^{(i)}(t), i), \qquad \overline{X}(t) = \overline{X^{(i)}}(t). \tag{69}$$

Set

$$\Lambda(\sigma_1) = i + \sum_{j\in\mathbb{N}} (j - i)1_{\Delta_{ij}(X^{(i)}(\sigma_1 -))}(p_1(\sigma_1)).$$

Then, (65) holds for $t \in [0, \sigma_1)$.

Let

$$\widetilde{W}(t) = W(t + \sigma_1) - W(\sigma_1), \quad \widetilde{p}(t) = p(t + \sigma_1), \quad \widetilde{p_1}(t) = p_1(t + \sigma_1).$$

Set

$$\begin{aligned}&X^{(\Lambda(\sigma_1))}(0) = X^{(i)}(\sigma_1), \\ &(\widetilde{X}(t), \widetilde{\Lambda}(t)) = (X^{(\Lambda(\sigma_1))}(t), \Lambda(\sigma_1)), \qquad \forall t \in [0, \sigma_2 - \sigma_1), \\ &\widetilde{\Lambda}(\sigma_2 - \sigma_1) = \Lambda(\sigma_1) + \sum_{j\in\mathbb{N}} (j - \Lambda(\sigma_1))1_{\widetilde{A}(j)}(\widetilde{p}_1(\sigma_2 - \sigma_1)),\end{aligned}$$

where

$$\widetilde{A}(j) = \Delta_{\Lambda(\sigma_1),j}(X^{(\Lambda(\sigma_1))}((\sigma_2 - \sigma_1)-)).$$

Then, for $t \in [\sigma_1, \sigma_2)$, we define

$$(X(t), \Lambda(t)) = (\widetilde{X}(t - \sigma_1), \widetilde{\Lambda}(t - \sigma_1)), \qquad \overline{X}(t) = \overline{\widetilde{X}}(t).$$

which together with (69) gives the unique solution on the time interval $[0, \sigma_2)$. Continuing this procedure inductively, we define $(X(t), \Lambda(t))$ on the time interval $[0, \sigma_n)$ for each n. Therefore, $(X(t), \Lambda(t))$ is the unique (càdlàg) solution to the hybrid system (62) and (64) since $\lim_{n\to\infty} \sigma_n = \infty$ almost surely. Finally, since $\widehat{X}$ is $L^\alpha([\sigma_i, \sigma_{i+1}); V) \bigcap L^2([\sigma_i, \sigma_{i+1}); H)$ $\mathbb{P}$-a.s., we conclude that $\widehat{X}$ is also in $L^\alpha([0, T]; V) \bigcap L^2([0, T]; H)$, $\mathbb{P}$-a.s. □

We remark that Theorem 8.0.1 still holds if the coefficients are random and progressively measurable. This observation may be useful for the future work.

9.2 Proof of Theorem 8.0.2

We again first consider the strong Feller property without regime-switching and then the case with regime-switching.

9.2.1 Without regime-switching

Zhang [128] already considered the strong Feller property for SPDEs driven by Wiener processes without regime-switching. We will follow his idea and generalize it to SPDEs driven by Lévy noises. We first consider the case that there is no large jump (*i.e.* $J \equiv 0$).

Theorem 9.2.1. *Under the assumptions of Theorem 8.0.2, the transition semigroup* $\{P_{s,t}\}$ *of* $X(t)$ *is strong Feller.*

To simplify notation, we drop the dependence on i. Fix $T > 0$. We need a lemma.

Lemma 9.2.2. *Suppose that Assumption 1 holds with* $\alpha \geqslant 2$*,* $J \equiv 0$ *and the following conditions hold:*

(i) There exists $K_2 > 0$ *such that*

$$|[B(t,v_1) - B(t,v_2)]^*(v_1 - v_2)|_H \leqslant K_2(|v_1 - v_2|_H^2 \wedge |v_1 - v_2|_H)$$

for all $t \in [0,T]$ *and* $v_1, v_2 \in V$.

(ii) There exist $\lambda \in [2,\infty) \cap (\alpha - 2, \infty)$*,* $\overline{B} \in L_2^{+,s}(H)$*,* $\delta > 0$*,* $\widetilde{K} \geqslant 0$ *such that*

$$B(t,v)[B(t,v)]^* \geqslant \overline{B}^2,$$

and

$$\begin{aligned}
&2\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle + \|B(t,v_1) - B(t,v_2)\|_{L_2(H)}^2 \\
&\quad + \int_{\{|z|<1\}} |G(t,v_1,z) - G(t,v_2,z)|_H^2 \nu(du) \\
&\leqslant -\delta|\overline{B}^{-1}(v_1 - v_2)|_H^\lambda |v_1 - v_2|_H^{\alpha-\lambda} + \widetilde{K}|v_1 - v_2|_H^2
\end{aligned} \tag{70}$$

for all $t \in [0,T]$ *and* $v, v_1, v_2 \in V$.

Then, $P_{s,t}f$ *is* $\frac{\lambda+2-\alpha}{2\lambda}$*-Hölder continuous for any* $f \in B_b(H)$.

<u>*Proof of Lemma 9.2.2*</u>.
We follow the method of [128, Lemma 3.1]. Let $\varepsilon \in (0,1)$ satisfying $0 \vee (\alpha - 2) < \lambda(1-\varepsilon) < (2\alpha - 2) \wedge \alpha$. Take $\alpha' \in (0,\varepsilon)$, whose value will be determined at the end of the proof. For $x, y \in H$, consider

$$\begin{aligned}
dX(t) =& A(t,X(t))dt + B(t,X(t))dW(t) + \int_{\{|z|<1\}} G(t,X(t),z)\widetilde{N}(dt,dz), \\
dY(t) =& A(t,Y(t))dt + B(t,Y(t))dW(t) + \int_{\{|z|<1\}} G(t,Y(t),z)\widetilde{N}(dt,dz) \\
&\quad + |x-y|_H^{\alpha'} \frac{X(t) - Y(t)}{|X(t) - Y(t)|_H^\varepsilon} dt,
\end{aligned}$$

with $X(0) = x$ and $Y(0) = y$, respectively. Since $X(t)$ is an adapted cádlág process, it is progressively measurable; thus, we have the unique solution $Y(t)$ using Theorem 9.1.1, the random coefficients case. To verify the additional drift term satisfies Assumption 1, but this is done in [117, Theorem A.2] and in fact we have

$$\left\langle \frac{X(t) - v_1}{|X(t) - v_1|_H^{\varepsilon}} - \frac{X(t) - v_2}{|X(t) - v_2|_H^{\varepsilon}}, v_1 - v_2 \right\rangle_H \leqslant 0.$$

Define

$$\tau_n := \inf\left\{ t > 0 : |X(t) - Y(t)|_H \leqslant \frac{1}{n} \right\}, \quad n \in \mathbb{N},$$

and

$$\tau := \lim_{n\to\infty} \tau_n.$$

From now on, to simplify notation, we write $A(t, X(t))$ instead of $A(t, \overline{X}(t))$ *etc.* unless it is necessary to distinguish them.

Using Itô's formula (Theorem 7.2.9), we find that for $t < \tau$,

$$\begin{aligned}
&|X(t) - Y(t)|_H^2 - |x - y|_H^2 \\
&= \int_0^t \left(2\langle A(u, X(u)) - A(u, Y(u)), X(u) - Y(u)\rangle + \|B(u, X(u)) - B(u, Y(u))\|_{L_2(H)}^2 \right) du \\
&\quad + \int_0^t 2\langle X(u) - Y(u), [B(u, X(u)) - B(u, Y(u))] dW(u)\rangle_H \\
&\quad + \int_0^t \int_{\{|z|<1\}} \left[|X(u) - Y(u) + G(u, X(u), z) - G(u, Y(u), z)|_H^2 - |X(u) - Y(u)|_H^2 \right] \widetilde{N}(du, dz) \\
&\quad + \int_0^t \int_{\{|z|<1\}} |G(u, X(u), z) - G(u, Y(u), z)|_H^2 \nu(dz) du \\
&\quad - 2|x - y|_H^{\alpha'} \int_0^t |X(u) - Y(u)|_H^{2-\varepsilon} du,
\end{aligned} \tag{71}$$

here we have used the fact that $\overline{X} = X$ for almost every point in $dt \times \mathbb{P}$ (recall that we define the additional drift term is defined using V-valued progressive measurable version of X, $\overline{X}$.)

Then equation (71) and hypothesis *(ii)* implies that

$$\begin{aligned}
&|X(t) - Y(t)|_H^2 - |x - y|_H^2 \\
&\leqslant \int_0^t \Big[-\delta |\overline{B}^{-1}(X(u) - Y(u))|_H^{\lambda} |X(u) - Y(u)|_H^{\alpha-\lambda} \\
&\qquad + \widetilde{K} |X(u) - Y(u)|_H^2 - 2|x - y|_H^{\alpha'} |X(u) - Y(u)|_H^{2-\varepsilon} \Big] du \\
&\quad + \int_0^t 2\langle X(u) - Y(u), [B(u, X(u)) - B(u, Y(u))] dW(u)\rangle_H \\
&\quad + \int_0^t \int_{\{|z|<1\}} \left[|X(u) - Y(u) + G(u, X(u), z) - G(u, Y(u), z)|_H^2 - |X(u) - Y(u)|_H^2 \right] \widetilde{N}(du, dz).
\end{aligned}$$

Set

$$r := \frac{\alpha - \lambda(1 - \varepsilon)}{2}.$$

We have $r \in (0,1)$.

By (70) and (61), and the assumption on B, we obtain that for $t < \tau_n$,

$$
\begin{aligned}
&|X(t)-Y(t)|_H^{2-2r} - |x-y|_H^{2-2r} \\
&\leqslant \int_0^t -\delta(1-r)|\overline{B}^{-1}(X(u)-Y(u))|_H^{\lambda}|X(u)-Y(u)|^{-\lambda\varepsilon}du + \int_0^t \widetilde{K}(1-r)|X(u)-Y(u)|_H^{2-2r}du \\
&\quad - 2(1-r)|x-y|_H^{\alpha'}\int_0^t |X(u)-Y(u)|_H^{2-\varepsilon-2r}du \\
&\quad - r(1-r)K_2^2\int_0^t \big(|X(u)-Y(u)|_H^{2-2r} \wedge |X(u)-Y(u)|_H^{-2r}\big)du \\
&\quad + 2(1-r)\int_0^t |X(u)-Y(u)|_H^{-2r}\langle X(u)-Y(u), [B(u,X(u))-B(u,Y(u))]dW(u)\rangle_H \\
&\quad + \int_0^t\int_{\{|z|<1\}}\Big[|X(u)-Y(u)+G(u,X(u),z)-G(u,Y(u),z)|_H^{-2r} \\
&\qquad\quad - |X(u)-Y(u)|_H^{-2r}\Big]\widetilde{N}(du,dz).
\end{aligned}
\tag{72}
$$

Taking expectation from both sides, we have

$$
\mathbb{E}(|X(t)-Y(t)|_H^{2-2r}) \leqslant |x-y|_H^{2-2r} + \int_0^t \widetilde{K}(1-r)\mathbb{E}(|X(u)-Y(u)|_H^{2-2r})du;
$$

we then use Gronwall's lemma and obtain

$$
\int_0^{t\wedge\tau_n}\mathbb{E}(|X(s)-Y(s)|_H^{2-2r})ds \leqslant e^{\widetilde{K}(1-r)t}|x-y|_H^{2-2r}.
$$

Hence,

$$
\begin{aligned}
\int_0^{T\wedge\tau_n}\delta(1-r)\mathbb{E}\bigg(\frac{|\overline{B}^{-1}(X(t)-Y(t))|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}\bigg)du &\leqslant \int_0^{T\wedge\tau_n}\widetilde{K}(1-r)e^{\widetilde{K}(1-r)t}|x-y|_H^{2-2r}dt \\
&\leqslant e^{\widetilde{K}(1-r)t}|x-y|_H^{2-2r},
\end{aligned}
$$

which leads

$$
\mathbb{E}\bigg[\int_0^{T\wedge\tau_n}\frac{|\overline{B}^{-1}(X(t)-Y(t))|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}dt\bigg] \leqslant \frac{e^{(1-r)\tilde{K}T}}{\delta(1-r)}|x-y|_H^{2-2r}.
$$

Further, by Fatou's lemma, we get

$$
\mathbb{E}\bigg[\int_0^{T\wedge\tau}\frac{|\overline{B}^{-1}(X(t)-Y(t))|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}dt\bigg] \leqslant \frac{e^{(1-r)\tilde{K}T}}{\delta(1-r)}|x-y|_H^{2-2r}.
\tag{73}
$$

Define

$$
\eta_n := \inf\bigg\{t>0 : \int_0^t \frac{|\overline{B}^{-1}(X(s)-Y(s))|_H^{\lambda}}{|X(s)-Y(s)|_H^{\lambda\varepsilon}}ds \geqslant n\bigg\}.
$$

By (73), we get $\lim\limits_{n\to\infty}\eta_n \geqslant T\wedge\tau$. Set

$$
\xi_n := \eta_n \wedge \tau_n \wedge T.
$$

Then, we have that $\lim_{n\to\infty} \xi_n = T \wedge \tau$.

Define

$$\widetilde{W}(t) := W(t) + \int_0^{t\wedge\tau} |x-y|_H^{\alpha'} \frac{[B(s,Y(s))]^*(B(s,Y(s))[B(s,Y(s))]^*)^{-1}(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}} ds.$$

Thus, $\{\widetilde{W}(s)\}_{s\in[0,\xi_n\wedge t]}$ is a cylindrical Wiener process on H under the probability measure $R_{t\wedge\xi_n}\mathbb{P}$. Define

$$E^{-1}(t,v) := (B(t,v)[B(t,v)]^*)^{-1}B(t,v),$$

and

$$\begin{aligned} R_t : = \exp\bigg\{ &- |x-y|_H^{\alpha'} \int_0^{t\wedge\tau} \left\langle \frac{(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}}, E^{-1}(s,Y(s))dW(s) \right\rangle_H \\ &- \frac{|x-y|_H^{2\alpha'}}{2} \int_0^{t\wedge\tau} \frac{|E^{-1}(s,Y(s))(X(s)-Y(s))|_H^2}{|X(s)-Y(s)|_H^{2\varepsilon}} ds \bigg\}. \end{aligned}$$

We will show that $\{\widetilde{W}(s)\}_{s\in[0,T]}$ is a cylindrical Wiener process on H under $R_T\mathbb{P}$.

By (72) and the definition of $\widetilde{W}$, we obtain that for all $t<\tau_n$,

$$\begin{aligned} &|X(t)-Y(t)|_H^{2-2r} - |x-y|_H^{2-2r} \\ &\leqslant \int_0^t (1-r)\bigg[-\delta\cdot \frac{|\overline{B}^{-1}(X(u)-Y(u))|_H^{\lambda}}{|X(u)-Y(u)|_H^{\lambda\varepsilon}} + \widetilde{K}|X(u)-Y(u)|_H^{2-2r} \\ &\qquad - 2|x-y|_H^{\alpha'}|X(u)-Y(u)|_H^{2-2r-\varepsilon}\bigg] du \\ &\quad + 2(1-r)\int_0^t \left\langle \frac{X(u)-Y(u)}{|X(u)-Y(u)|_H^{2r}}, [B(u,X(u))-B(u,Y(u))]d\widetilde{W}(u) \right\rangle_H \\ &\quad + 2(1-r)|x-y|_H^{\alpha'} \int_0^t \left\langle \frac{X(u)-Y(u)}{|X(u)-Y(u)|_H^{2r}}, \frac{[B(u,X(u))-B(u,Y(u))]E^{-1}(u,Y(u))(X(u)-Y(u))}{|X(u)-Y(u)|_H^{\varepsilon}} \right\rangle_H du \\ &\quad + \int_0^t \int_{\{|z|<1\}} \Big| X(u)-Y(u)+G(u,X(u),z)-G(u,Y(u),z)|_H^{-2r} - |X(u)-Y(u)|_H^{-2r}\Big] \widetilde{N}(du,dz). \end{aligned} \tag{74}$$

By Young's inequality and condition (i), we obtain that for $t<\tau$,

$$\begin{aligned} &\left| 2(1-r)|x-y|_H^{\alpha'} \left\langle \frac{X(t)-Y(t)}{|X(t)-Y(t)|_H^{2r}}, \frac{[B(t,X(t))-B(t,Y(t))]E^{-1}(s,Y(s))(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}} \right\rangle_H \right| \\ &\leqslant 2(1-r)|x-y|_H^{\alpha'}\cdot K_2\big(|X(t)-Y(t)|_H^2 \wedge |X(t)-Y(t)|_H\big)|X(t)-Y(t)|_H^{-2r-\varepsilon}|\overline{B}^{-1}[X(t)-Y(t)]|_H \\ &\leqslant \frac{\lambda-1}{\lambda}\Big(\delta^{-\frac{1}{\lambda}}2^{1+\frac{1}{\lambda}}(1-r)^{1-\frac{1}{\lambda}}|x-y|_H^{\alpha'}\cdot K_2\big(|X(t)-Y(t)|_H^{2-2r}\wedge|X(t)-Y(t)|_H^{1-2r}\big)\Big)^{\frac{\lambda}{\lambda-1}} \\ &\quad + \frac{1}{\lambda}\left(\left[\frac{(1-r)\delta}{2}\right]^{\frac{1}{\lambda}} \frac{|\overline{B}^{-1}[X(t)-Y(t)]|_H}{|X(t)-Y(t)|_H^{\varepsilon}}\right)^{\lambda} \\ &\leqslant \frac{\lambda-1}{\lambda}\cdot 2^{\frac{\lambda+1}{\lambda-1}}\delta^{-\frac{1}{\lambda-1}}(1-r)|x-y|_H^{\frac{\alpha'\lambda}{\lambda-1}}K_2^{\frac{\lambda}{\lambda-1}}\left(|X(t)-Y(t)|_H^{\frac{\lambda(2-2r)}{\lambda-1}}\wedge|X(t)-Y(t)|_H^{\frac{\lambda(1-2r)}{\lambda-1}}\right) \\ &\quad + \frac{(1-r)\delta}{2\lambda}\cdot\frac{|\overline{B}^{-1}[X(t)-Y(t)]|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}. \end{aligned}$$

Since $|X(t)-Y(t)|_H^{\frac{\lambda(1-2r)}{\lambda-1}} \leqslant |X(t)-Y(t)|_H^{2-2r}$ if $|X(t)-Y(t)|_H \geqslant 1$ and $|X(t)-Y(t)|_H^{\frac{\lambda(2-2r)}{\lambda-1}} \leqslant |X(t)-Y(t)|_H^{2-2r}$ if $|X(t)-Y(t)|_H \leqslant 1$, we get

$$\begin{aligned}
&\left|2(1-r)|x-y|_H^{\alpha'}\left\langle \frac{X(t)-Y(t)}{|X(t)-Y(t)|_H^{2r}}, \frac{[B(t,X(t))-B(t,Y(t))]E^{-1}(s,Y(s))(X(s)-Y(s))}{|X(s)-Y(s)|_H^{\varepsilon}}\right\rangle_H\right| \\
&\leqslant C_{\lambda,\delta,K_2}(1-r)|x-y|_H^{\frac{\alpha'\lambda}{\lambda-1}}|X(t)-Y(t)|_H^{2-2r} + \frac{(1-r)\delta}{2}\cdot\frac{|\overline{B}^{-1}[X(t)-Y(t)]|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}},
\end{aligned}$$

where C_{λ,δ,K_2} depends only on δ, λ and K_2. Thus, for $t<\tau_n$, we have that

$$\begin{aligned}
&|X(t)-Y(t)|_H^{2-2r} - |x-y|_H^{2-2r} \\
&\leqslant \int_0^t \Bigg[-\frac{(1-r)\delta}{2}\frac{|\overline{B}^{-1}(X(u)-Y(u))|_H^{\lambda}}{|X(u)-Y(u)|_H^{\lambda\varepsilon}} + (1-r)\widetilde{K}|X(u)-Y(u)|_H^{2-2r} \\
&\quad + C_{\lambda,\delta,K_2}(1-r)|x-y|_H^{\frac{\alpha'\lambda}{\lambda-1}}|X(u)-Y(u)|^{2-2r}\Bigg]du \\
&\quad + 2(1-r)\int_0^t \left\langle \frac{X(u)-Y(u)}{|X(u)-Y(u)|_H^{2r}}, [B(u,X(u))-B(u,Y(u))]d\widetilde{W}(u)\right\rangle_H \\
&\quad + \int_0^t\int_{\{|z|<1\}}\Big[|X(u)-Y(u)+G(u,X(u),z)-G(u,Y(u),z)|_H^{-2r} \\
&\qquad - |X(u)-Y(u)|_H^{-2r}\Big]\widetilde{N}(du,dz).
\end{aligned}$$

By Gronwall's lemma, we get

$$\mathbb{E}_{R_{s\wedge\xi_n}\mathbb{P}}\left(\int_0^{s\wedge\eta_n}\frac{|\overline{B}^{-1}(X(t)-Y(t))|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}dt\right) \leqslant \frac{2\exp\left\{\left[C_{\lambda,\delta,K_2}|x-y|_H^{\frac{\alpha'\lambda}{\lambda-1}}+\widetilde{K}\right](1-r)s\right\}}{(1-r)\delta}|x-y|_H^{2-2r}. \tag{75}$$

By the definition of $\widetilde{W}$ and (75), we get

$$\begin{aligned}
&\sup_{s\in[0,T],n\in\mathbb{N}}\mathbb{E}\big[R_{s\wedge\xi_n}\log(R_{s\wedge\xi_n})\big] \\
&= \sup_{s\in[0,T],n\in\mathbb{N}}\mathbb{E}_{R_{s\wedge\xi_n}\mathbb{P}}\Bigg[|x-y|_H^{\alpha'}\int_0^{s\wedge\xi_n}\left\langle\frac{E^{-1}(t,Y(t))(X(t)-Y(t))}{|X(t)-Y(t)|_H^{\varepsilon}}, dW(t)\right\rangle_H \\
&\qquad - \frac{|x-y|_H^{2\alpha'}}{2}\int_0^{s\wedge\xi_n}\frac{|E^{-1}(t,Y(t))(X(t)-Y(t))|_H^2}{|X(t)-Y(t)|_H^{2\varepsilon}}dt\Bigg] \\
&= \sup_{s\in[0,T],n\in\mathbb{N}}\frac{|x-y|_H^{2\alpha'}}{2}\mathbb{E}_{R_{s\wedge\xi_n}\mathbb{P}}\left[\int_0^{s\wedge\xi_n}\frac{|E^{-1}(t,Y(t))(X(t)-Y(t))|_H^2}{|X(t)-Y(t)|_H^{2\varepsilon}}dt\right] \\
&\leqslant \sup_{s\in[0,T],n\in\mathbb{N}}\frac{|x-y|_H^{2\alpha'}}{2}\mathbb{E}_{R_{s\wedge\xi_n}\mathbb{P}}\left[\left(\int_0^{s\wedge\xi_n}\frac{|E^{-1}(t,Y(t))(X(t)-Y(t))|_H^{\lambda}}{|X(t)-Y(t)|_H^{\lambda\varepsilon}}dt\right)^{\frac{2}{\lambda}}T^{\frac{\lambda-2}{\lambda}}\right] \\
&\leqslant \left(\frac{T}{2}\right)^{\frac{\lambda-2}{\lambda}}\cdot\frac{\exp\left\{\left[C_{\lambda,\delta,K_2}|x-y|_H^{\frac{\alpha'\lambda}{\lambda-1}}+\widetilde{K}\right](1-r)\frac{2T}{\lambda}\right\}}{[(1-r)\delta]^{\frac{2}{\lambda}}}\cdot|x-y|_H^{\frac{4(1-r)}{\lambda}+2\alpha'} \qquad (76) \\
&< \infty.
\end{aligned}$$

Then $\{R_{s\wedge\xi_n}\}_{s\in[0,T],n\in\mathbb{N}}$ and thus $\{R_s\}_{s\in[0,T]}$ is uniform integrable. Hence, $\{\widetilde{W}(t)\}_{t\in[0,T]}$ is a cylindrical Wiener process under $R_T\mathbb{P}$. Moreover, $\{Y(t)\}_{t\geqslant 0}$ satisfies

$$dY(t) = A(t,Y(t))dt + B(t,Y(t))d\widetilde{W}(t) + \int_{\{|z|<1\}} G(t,Y(t),z)\widetilde{N}(dt,dz)$$

with $Y(0)=y$. This implies that $\{Y(t)\}_{t\geqslant 0}$ is also a solution to (62) with $J\equiv 0$.

We now show that $P_{s,t}f$ is $\frac{\lambda+2-\alpha}{2\lambda}$-Hölder continuous for any $f\in B_b(H)$. To simplify notation, we only give the proof for the case that $s=0$. The proof for the case that $s>0$ is completely similar.

Let $f\in B_b(H)$, $0\leqslant t\leqslant T$ and $x,y\in H$. We have

$$\begin{aligned}
|P_{0,t}f(x)-P_{0,t}f(y)| &= |\mathbb{E}[f(X(t))-R_tf(Y(t))]| \\
&\leqslant \big|\mathbb{E}[f(Y(t))-R_tf(Y(t))]\big| + \big|\mathbb{E}\big\{[f(X(t))-f(Y(t))]1_{\{\tau\geqslant t\}}\big\}\big| \\
&\leqslant |f|_{L^\infty}\big\{\mathbb{E}[|1-R_t|]+2\mathbb{P}(\tau\geqslant t)\big\}. \qquad (77)
\end{aligned}$$

By (73), (76) and the inequality

$$|1-e^x|\leqslant xe^x+2|x|,\quad x\in\mathbb{R},$$

there exists $C'_{r,\widetilde{K},T,\delta,\lambda,K_2}>0$ such that if $|x-y|_H\leqslant 1$,

$$\begin{aligned}
\mathbb{E}[|1-R_t|] &\leqslant \mathbb{E}[R_t\log R_t]+2\mathbb{E}[|\log R_t|] \\
&\leqslant C'_{r,\widetilde{K},T,\delta,\lambda,K_2}\Big[|x-y|_H^{\frac{2(1-r)}{\lambda}+\alpha'}+|x-y|_H^{\frac{4(1-r)}{\lambda}+2\alpha'}\Big]. \qquad (78)
\end{aligned}$$

Next, we estimate $\mathbb{P}(\tau\geqslant t)$. Similar to (74), we can show that for $s<\tau_n$,

$$\begin{aligned}
&|X(s)-Y(s)|_H^\varepsilon - |x-y|_H^\varepsilon \\
&\leqslant \frac{\widetilde{K}\varepsilon}{2}\int_0^s |X(u)-Y(u)|_H^\varepsilon du - \varepsilon s|x-y|_H^{\alpha'} \\
&\quad + \int_0^s 2|X(u)-Y(u)|_H^{\varepsilon-2}\langle X(u)-Y(u),[B(u,X(u))-B(u,Y(u))]dW(u)\rangle_H \\
&\quad + \int_0^s\int_{\{|z|<1\}}\Big[|X(u)-Y(u)+G(u,X(u),z)-G(u,Y(u),z)|_H^\varepsilon \\
&\qquad\quad - |X(u)-Y(u)|_H^\varepsilon\Big]\widetilde{N}(du,dz),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{E}\big[|X(s\wedge\tau_n)-Y(s\wedge\tau_n)|_H^\varepsilon\big] \leqslant& |x-y|_H^\varepsilon + \int_0^s \frac{\widetilde{K}\varepsilon}{2}\mathbb{E}\big[|X(u)-Y(u)|_H^\varepsilon\big]du \\
&- \varepsilon|x-y|_H^{\alpha'}\mathbb{E}[s\wedge\tau_n] \\
=& |x-y|_H^\varepsilon + \int_0^s \frac{\widetilde{K}\varepsilon}{2}\mathbb{E}\big[|X(u\wedge\tau_n)-Y(u\wedge\tau_n)|_H^\varepsilon\big]du \\
&- \varepsilon|x-y|_H^{\alpha'}\mathbb{E}[s\wedge\tau_n]. \qquad (79)
\end{aligned}$$

By Gronwall's lemma, we get

$$\mathbb{E}\big[|X(s\wedge\tau_n)-Y(s\wedge\tau_n)|_H^{\varepsilon}\big]\leqslant e^{s\widetilde{K}\varepsilon/2}|x-y|_H^{\varepsilon}.$$

Then,

$$\int_0^t \mathbb{E}\big[|X(s\wedge\tau_n)-Y(s\wedge\tau_n)|_H^{\varepsilon}\big]ds\leqslant \frac{2}{\widetilde{K}\varepsilon}(e^{t\widetilde{K}\varepsilon/2}-1)|x-y|_H^{\varepsilon},$$

which together with (79) implies that

$$\mathbb{E}[t\wedge\tau_n]\leqslant \varepsilon^{-1}(e^{t\widetilde{K}\varepsilon/2})|x-y|_H^{\varepsilon-\alpha'}.$$

Thus, we have that

$$\mathbb{P}(\tau>t)\leqslant \liminf_{n\to\infty}\mathbb{P}(\tau_n\geqslant t)\leqslant \liminf_{n\to\infty}\frac{\mathbb{E}[t\wedge\tau_n]}{t}\leqslant \frac{e^{t\widetilde{K}\varepsilon/2}}{t\varepsilon}|x-y|_H^{\varepsilon-\alpha'}. \tag{80}$$

Now we can choose

$$\alpha':=\frac{\lambda+2-\alpha}{2\lambda}.$$

Therefore, by (77), (78) and (80), we obtain that for any $x,y\in H$ with $|x-y|_H<1$,

$$\begin{aligned}
&|P_{0,t}f(x)-P_{0,t}f(y)|\\
\leqslant\ & |f|_{L^\infty}\left\{C'_{r,\widetilde{K},T,\delta,\lambda,K_2}\left[|x-y|_H^{\frac{2(1-r)}{\lambda}+\alpha'}+|x-y|_H^{\frac{4(1-r)}{\lambda}+2\alpha'}\right]+\frac{2e^{t\widetilde{K}\varepsilon/2}}{t\varepsilon}|x-y|_H^{\varepsilon-\alpha'}\right\}\\
\leqslant\ & 2\left[C'_{r,\widetilde{K},T,\delta,\lambda,K_2,\varepsilon}+\frac{e^{t\widetilde{K}\varepsilon/2}}{t\varepsilon}\right]|f|_{L^\infty}|x-y|_H^{\frac{\lambda+2-\alpha}{2\lambda}}.
\end{aligned}$$

□

Proof of Theorem 9.2.1.
We first consider the case that $J\equiv 0$. For $R>0$, define

$$B_R(t,v):=\begin{cases}B(t,v) & \text{if } |v|_H\leqslant R,\\ B(t,\frac{Rv}{|v|_H}) & \text{if } |v|_H>R.\end{cases}$$

Denote by $X(s,w;t)$ the solution to (62) with $X(s)=w$, $w\in H$ for fixed $i\in\mathbb{N}$. Suppose $|w|_H<R$. Define

$$\tau_R^w:=\inf\{t>s:|X(s,w;t)|_H\geqslant R\}.$$

Let $\{X_R(s,w;t)\}$ be the unique solution to the SPDE:

$$dX_R(t)=A(t,X_R(t))dt+B_R(t,X_R(t))dW(t)+\int_{\{|z|<1\}}G(t,X_R(t),z)\widetilde{N}(dt,dz)$$

with $X(s)=w$. Denote by $\{P_{s,t}^R\}$ the transition semigroup of $\{X_R(s,w;t)\}$. Suppose $x,y\in H$ with $|x|_H,|y|_H<R$. By the uniqueness of solutions, we find that $X(s,x;t)=X_R(s,x;t)$ and $X(s,y;t)=X_R(s,y;t)$ for all $t<\tau_R^x\wedge\tau_R^y$. Let $R>n_0$, where n_0 is given in condition (M). By conditions (N), (LipB), (M) and replacing δ_n, $\widetilde{K}_n$, $\overline{B}_n$ with δ_{n_0}, $K_{\widetilde{n_0}}$, B_{n_0}, respectively, we can apply Lemma 9.2.2 to show that $\{P^R_{s,t}\}$ is strong Feller.

Let $f \in B_b(H)$. We have

$$\begin{aligned}
&|P_{s,t}f(x) - P_{s,t}f(y)| \\
&\leqslant |\mathbb{E}[\{f(X(s,x;t)) - f(X(s,y;t))\}1_{\{\tau^x_R \wedge \tau^y_R > t\}}]| + 2|f|_{L^\infty}\big[\mathbb{P}(\tau^x_R \leqslant t) + \mathbb{P}(\tau^y_R \leqslant t)\big] \\
&= |\mathbb{E}[\{f(X_R(s,x;t)) - f(X_R(s,y;t))\}1_{\{\tau^x_R \wedge \tau^y_R > t\}}]| + 2|f|_{L^\infty}\big[\mathbb{P}(\tau^x_R \leqslant t) + \mathbb{P}(\tau^y_R \leqslant t)\big] \\
&\leqslant |P^R_{s,t}f(x) - P^R_{s,t}f(y)| + 2|f|_{L^\infty}\big[\mathbb{P}(\tau^x_R \leqslant t) + \mathbb{P}(\tau^y_R \leqslant t)\big]. \qquad (81)
\end{aligned}$$

By Theorem 7.2.9, we get

$$\begin{aligned}
&|X(s,x;t)|^2_H - |x|^2_H \\
&= \int_s^t \bigg[2\langle A(u, X(s,x;u)), X(s,x;u)\rangle + \|B(u, X(s,x;u))\|^2_{L_2(H)} \\
&\quad + \int_{\{|z|<1\}} |G(u, X(s,x;u), z)|^2_H \nu(dz)\bigg] du + \int_s^t 2\langle B(u, X(s,x;u)), dW(u)\rangle_H \\
&\quad + \int_s^t \int_{\{|z|<1\}} \big[|X(s,x;u) + G(u, X(s,x;u), z)|^2_H - |X(s,x;u)|^2_H\big] \widetilde{N}(du, dz) \\
&\leqslant t \sup_{u\in[0,t]} |C(u)| + \int_s^t \Big(- \theta |X(s,x;u)|^\alpha_V + c|X(s,x;u)|^2_H \Big) du \\
&\quad + \int_s^t 2\langle B(u, X(s,x;u)), dW(u)\rangle_H \\
&\quad + \int_s^t \int_{\{|z|<1\}} \big[|X(s,x;u) + G(u, X(s,x;u), z)|^2_H - |X(s,x;u)|^2_H\big] \widetilde{N}(du, dz).
\end{aligned}$$

Then, by Gronwall's lemma, we obtain that

$$\mathbb{E}\big[|X(s,x;t)|^2_H\big] \leqslant \left(\sup_{u\in[0,t]} |C(u)| + |x|^2_H\right) c e^{c(t-s)},$$

and also

$$\int_s^t \mathbb{E}[|X(s,x;u)|^\alpha_V] du \leqslant \frac{\sup_{u\in[0,t]} |C(u)| + |x|^2_H}{\theta} \cdot e^{c(t-s)} + \frac{\sup_{u\in[0,t]} |C(u)|}{\theta} \cdot (t-s).$$

By the Burkholder-Davis-Gundy inequality, there exists $C' > 0$ such that

$$\begin{aligned}
&\mathbb{E}[\sup_{s\leqslant u\leqslant t} |X(s,x;u)|^2_H] \\
&\leqslant C'\mathbb{E}\bigg[\int_s^t \Big\{\|B(t, X(s,x;u))\|^2_{L_2(H)} + \int_{\{|z|<1\}} |G(t, X(s,x;u), z)|^2_H \nu(dz)\Big\} du\bigg] \\
&\leqslant C'\mathbb{E}\bigg[\int_s^t \Big\{\sup_{u\in[0,t]} |C(u)| + \gamma |X(s,x;u)|^\alpha_V + c|X(s,x;u)|^2_H\Big\} du\bigg] \\
&\leqslant \frac{C'\left(\sup_{u\in[0,t]} |C(u)| + |x|^2_H\right)}{\theta} \cdot \Big[(\gamma+\theta)(t-s) + (\gamma + c\theta)e^{c(t-s)}\Big],
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{P}(\tau_R^x \leqslant t) &\leqslant \mathbb{P}(\sup_{u\in[s,t]} |X(s,x;u)|_H \geqslant R) \\
&\leqslant \frac{\mathbb{E}[\sup_{u\in[s,t]} |X(s,x;u)|_H^2]}{R^2} \\
&\leqslant \frac{C'\left(\sup_{u\in[0,t]} |C(u)| + |x|_H^2\right)}{\theta R^2} \cdot \Big[(\gamma+\theta)(t-s) + (\gamma + c\theta)e^{c(t-s)}\Big].
\end{aligned}$$

Hence for $l \in (0, R - |x|_H)$, we have that

$$\begin{aligned}
&\sup_{\{w\in H:|w-x|_H\leqslant l\}} \mathbb{P}(\tau_R^w \leqslant t) \\
&\leqslant \frac{C'\left(\sup_{u\in[0,t]} |C(u)| + [|x|_H + l]^2\right)}{\theta R^2} \cdot \Big[(\gamma+\theta)(t-s) + (\gamma + c\theta)e^{c(t-s)}\Big]. \qquad (82)
\end{aligned}$$

Therefore, $P_{s,t}f$ is continuous at x by (81) and (82). Since $x \in H$ is arbitrary, the proof for the case that $J \equiv 0$ is complete.

We now consider the case that $J \not\equiv 0$. Let $\{X(t)\}_{t\geqslant 0}$ be the unique solution to the SPDE (62) with arbitrary J. Let $\{Z(t)\}_{t\geqslant 0}$ be the unique solution to the SPDE (62) with $J \equiv 0$.

Denote by $P^Z(s,x;t,\mathcal{X})$ the transition semigroup of $\{Z(t)\}_{t\geqslant 0}$, where $x \in H, \mathcal{X} \in \mathscr{B}(H)$ and $0 \leqslant s < t < \infty$. Define $\zeta_1 := \inf\{u > s : N([s,u], \{|z| \geqslant 1\}) = 1\}$, which is the first jump time of $u \mapsto N([s,u], \{|z| \geqslant 1\})$ after time s. Then, by conditioning on ζ_1, we get

$$\begin{aligned}
&P^X(s,x;t,\mathcal{X}) \\
&\quad = e^{-\nu(\{|z|\geqslant 1\})(t-s)} P^Z(s,x;t,\mathcal{X}) \\
&\qquad + \int_s^t \int_{\{|x_2|\geqslant 1\}} \int_H e^{-\nu(\{|z|\geqslant 1\})(t_1-s)} P^X(t_1, x_1 + J(t_1,x_1,x_2); t, \mathcal{X}) P^Z(s,x;t_1,dx_1)\nu(dx_2)dt_1.
\end{aligned}$$

Repeating this procedure, we get

$$P^X(s,x;t,\mathcal{X}) = e^{-\nu(\{|z|\geqslant 1\})(t-s)} \left[P^Z(s,x;t,\mathcal{X}) + \sum_{k=1}^{\infty} \Psi_k \right], \qquad (83)$$

where

$$\begin{aligned}
\Psi_k = &\int\cdots\int_{s<t_1<\cdots<t_k<t} \int_{\{|x_2|\geqslant 1\}} \cdots \int_{\{|x_{2k}|\geqslant 1\}} \int_{H^n} P^Z(s,x;t_1,dx_1) P^Z(t_1, x_1 + J(t_1,x_1,x_2); t_2, dx_3) \\
&\times \cdots P^Z(t_k, x_{2k-1} + J(t_k, x_{2k-1}, x_{2k}); t, \mathcal{X}) \nu(dx_2)\nu(dx_4)\cdots\nu(dx_{2k}) dt_1 dt_2 \cdots dt_k.
\end{aligned}$$

Since we have shown that the transition semigroup of $\{Z(t)\}_{t\geqslant 0}$ is strong Feller, $P^Z(s,x;t,\mathcal{X})$ and Ψ_k, $k \in \mathbb{N}$, are all continuous with respect to x. Then, by (83), we conclude that $P^X(s,x;t,\mathcal{X})$ is lower semi-continuous with respect to x. Therefore, the transition semigroup $\{P_{s,t}\}$ is strong Feller by [87, Proposition 6.1.1]. □

9.2.2 With regime-switching

Proof of Theorem 8.0.2.

Denote the transition probability function of $(X(t), \Lambda(t))$ by $\{P(s,(x,i),t,B \times \{j\}) : 0 \leqslant$

$s < t, (x,i) \in H \times \mathbb{N}, B \in \mathscr{B}(H), j \in \mathbb{N}\}$. For $i \in \mathbb{N}$ and $g \in C^{1,2}([0,\infty) \times H; \mathbb{R})$, define

$$
\begin{aligned}
\mathscr{L}_i g(t,x) \quad := \quad & g_t(t,x) + \langle A(t,x,i), g_x(t,x)\rangle + \frac{1}{2}\mathrm{trace}(B^T(t,x,i)g_{xx}(t,x)B(t,x,i)) \\
& + \int_{\{|z|<1\}} [g(t,x+G(t,x,i,z)) - g(t,x) - \langle g_x(t,x), G(t,x,i,z)\rangle]\nu(\mathrm{d}z) \\
& + \int_{\{|z|\geqslant 1\}} [g(t,x+J(t,x,i,z)) - g(t,x)]\nu(\mathrm{d}z). \qquad (84)
\end{aligned}
$$

For $(x,i) \in H \times \mathbb{N}$, let $X^{(i)}(t)$ be defined by (68). We also define $\widetilde{X^{(i)}}(t)$ to be the killing process with generator $\mathscr{L}_i + q_{ii}$. Then, for $f \in B_b(H)$,

$$
\mathbb{E}[f(\widetilde{X^{(i)}}(t))] = \mathbb{E}\bigg[f(X^{(i)}(t))\exp\bigg\{\int_0^t q_{ii}(X^{(i)}(u))du\bigg\}\bigg].
$$

Let $\widetilde{P^{(i)}}(s,x;\cdot)$ be the transition probability function of $\widetilde{X^{(i)}}(t)$. Then, for $0 \leqslant s < t$, $B \in \mathscr{B}(H)$ and $j \in \mathbb{N}$, we have

$$
\begin{aligned}
& P(s,(x,i);t,B\times\{j\}) \\
= \quad & \delta_{ij}\widetilde{P^{(i)}}(s,x;t,B) \\
& + \int_s^t \int_H P(t',(x',j'),t,B\times\{j\})\bigg(\sum_{j'\in\mathbb{N}\setminus\{i\}} q_{ij'}(x')\bigg)\widetilde{P^{(i)}}(s,x;t',dx')dt'.
\end{aligned}
$$

Repeating this procedure, we get

$$
P(s,(x,i);t,B\times\{j\}) = \delta_{ij}\widetilde{P^{(i)}}(s,x;t,B) + \sum_{k=1}^n \Psi_k + U_n,
$$

where

$$
\begin{aligned}
\Psi_k = \int\cdots\int_{s<t_1<\cdots<t_k<t} & \sum_{j_0,\ldots,j_k}\int_{H^k} q_{j_{k-1},j_k}(x_k)\widetilde{P^{(j_k)}}(t_k,x_k;t,B) \\
& \times q_{j_{k-2},j_{k-1}}(x_{k-1})\widetilde{P^{(j_{k-1})}}(t_{k-1},x_{k-1};t_k,dx_k)\cdots q_{i,j_1}(x_1)\widetilde{P^{(j_1)}}(t_1,x_1;t_2,dx_2) \\
& \times \widetilde{P^{(i)}}(s,x;t_1,dx_1)dt_1\cdots dt_k
\end{aligned}
$$

and the sum is over

$$
j_0 = i,\ j_\ell \in \mathbb{N}\setminus\{j_{\ell-1}\} \text{ for } \ell \in \{1,\ldots,k-1\},\ j_k = j;
$$

$$
\begin{aligned}
U_n = \int\cdots\int_{s<t_1<\cdots<t_{n+1}<t} & \sum_{j_0,\ldots,j_{n+1}}\int_{H^{n+1}} q_{j_n,j_{n+1}}(x_{n+1})P(t_{n+1},x_{n+1};t,B\times\{j\}) \\
& \times q_{j_{n-1},j_n}(x_n)\widetilde{P^{(j_n)}}(t_n,x_n;t,B)\cdots q_{i,j_1}(x_1)\widetilde{P^{(j_1)}}(t_1,x_1;t_2,dx_2) \\
& \times \widetilde{P^{(i)}}(s,x;t_1,dx_1)dt_1\cdots dt_{n+1}
\end{aligned}
$$

and the sum is over

$$j_0 = i,\ j_\ell \in \mathbb{N}\backslash\{j_{\ell-1}\} \text{ for } \ell \in \{1, \ldots, n+1\}.$$

By condition (Q0), we find that $U_n \leqslant \frac{[(t-s)L]^{n+1}}{(n+1)!}$. Letting $n \to \infty$, we get

$$P(s,(x,i);t,B \times \{j\}) = \delta_{ij}\widetilde{P^{(i)}}(s,x;t,B) + \sum_{k=1}^{\infty} \Psi_k. \tag{85}$$

By Theorem 9.2.1, we know that the transition semigroup of $X^{(i)}(t)$ is strong Feller. Then, following the argument of [119, Lemma 4.5], we can show that the semigroup of $\widetilde{X^{(i)}}(t)$ is also strong Feller. Thus, we conclude that $\widetilde{P^{(i)}}(s,x;t,B)$ and Ψ_k for $k \in \mathbb{N}$ are all continuous with respect to x. Using the fact that $\mathbb{N}$ is equipped with a discrete metric, we conclude that $P(s,(x,i);t,B\times\{j\})$ is lower semi-continuous with respect to (x,i). Therefore, the transition semigroup $\{P_{s,t}\}$ of $(X(t),\Lambda(t))$ is strong Feller by [87, Proposition 6.1.1]. □

9.3 Proof of Theorem 8.0.3

First, we consider the case that $i \in \mathbb{N}$ is fixed. To simplify notation, we drop the dependence on i. Define the first jump time of $\{X(t)\}_{t\geqslant 0}$ by

$$\zeta_1 := \inf\{t > 0 : N([0,t],\{|z| \geqslant 1\}) = 1\},$$

which is exponentially distributed with rate $\nu(\{|z| \geqslant 1\})$. Let $\{X(t)\}_{t\geqslant 0}$ be the unique solution to the SPDE (62) with arbitrary J. Let $\{Z(t)\}_{t\geqslant 0}$ be the unique solution to the SPDE (62) with $J \equiv 0$. We have that $Z(t) = X(t)$ for $t < \zeta_1$. Hence, to obtain the irreducibility of $\{P_{s,t}\}$, we may assume without loss of generality that $J \equiv 0$.

Denote by $\{X^x(t)\}_{t\geqslant 0}$ the solution to Equation (62) with $X(0) = x$, $x \in H$. Let $T, M, R > 0$, $t_1 \in (0,T)$, $y(0) \in D(A_H)$ and $\{y(t)\}_{t\in[0,T]}$ be the solution to the following equation:

$$\begin{aligned} dy(t) &= A(t,y(t))dt - \frac{M}{T-t_1}(y(t)-y(0))dt, \quad t > t_1, \\ y(t) &= X^x(t_1)1_{\{|X^x(t_1)|_H \leqslant R\}}, \quad t \leqslant t_1. \end{aligned} \tag{86}$$

Lemma 9.3.1. *Let $m := 2M - (K + \rho(y(0)) + 1)T > 0$. Then,*

$$|y(t)-y(0)|_H^2 \leqslant e^{-\frac{m(t-t_1)}{T-t_1}}(R+|y(0)|_H)^2 + \int_{t_1}^t e^{-\frac{m(t-s)}{T-t_1}}|A(s,y(0))|_H^2 ds, \quad t \in [t_1,T], \tag{87}$$

$$\int_{t_1}^T |y(t)-y(0)|_H^2 dt \leqslant \frac{T-t_1}{m}\left[(R+|y(0)|_H)^2 + \int_{t_1}^T |A(s,y(0))|_H^2 ds\right], \tag{88}$$

and there exists $\vartheta > 0$, which is independent of t_1, such that

$$\int_{t_1}^T \rho(y(s))ds \leqslant \vartheta(T-t_1)^{\frac{\alpha-\alpha'}{\alpha}}. \tag{89}$$

Proof of Lemma 9.3.1.
By (LM) and Theorem 7.2.9, we get

$$\begin{aligned}
&|y(t)-y(0)|_H^2 \\
&= \int_{t_1}^t 2\langle A(u,y(u))-A(u,y(0)), y(u)-y(0)\rangle du - \frac{2M}{T-t_1}\int_{t_1}^t |y(u)-y(0)|_H^2 du \\
&\quad + \int_{t_1}^t 2\langle A(u,y(0)), y(t)-y(0)\rangle du \\
&\leqslant \left(K+\rho(y(0))+1-\frac{2M}{T-t_1}\right)\int_{t_1}^t |y(u)-y(0)|_H^2 du + \int_{t_1}^t |A(u,y(0))|_H^2 du \\
&\leqslant -\frac{m}{T-t_1}\int_{t_1}^t |y(u)-y(0)|_H^2 du + \int_{t_1}^t |A(u,y(0))|_H^2 du.
\end{aligned}$$

Then, by Gronwall's lemma, we get

$$|y(t)-y(0)|_H^2 \leqslant e^{-\frac{m(t-t_1)}{T-t_1}}(R+|y(0)|_H)^2 + \int_{t_1}^t e^{-\frac{m(t-s)}{T-t_1}}|A(s,y_0)|_H^2 ds.$$

Hence (87) and (88) hold.

By (C) and Theorem 7.2.9, we get

$$\begin{aligned}
|y(t)|_H^2 - |y(t_1)|_H^2 &= \int_{t_1}^t \left[2\langle A(u,y(u)), y(u)\rangle - \frac{2M}{T-t_1}\langle y(u)-y(0), y(u)\rangle_H\right] du \\
&\leqslant \int_{t_1}^t \left[C(u) - \theta|y(u)|_V^\alpha + c|y(u)|_H^2 - \frac{2M}{T-t_1}\langle y(u)-y(0), y(u)\rangle_H\right] du.
\end{aligned}$$

Let $p \geqslant 2$. We have

$$\begin{aligned}
&|y(t)|_H^p - |y(t_1)|_H^p \\
&\leqslant \int_{t_1}^t \Bigg[\frac{p\sup_{v\in[0,u]}|C(v)|}{2}|y(u)|_H^{p-2} - \frac{\theta p}{2}|y(u)|_H^{p-2}|y(u)|_V^\alpha + \frac{cp}{2}|y(u)|_H^p \\
&\quad - \frac{pM}{T-t_1}|y(u)|_H^{p-2}\langle y(u)-y(0), y(u)\rangle_H\Bigg] du \\
&\leqslant \int_{t_1}^t \Bigg[\frac{(p-2)\sup_{v\in[0,u]}|C(v)|+cp}{2}|y(u)|_H^p + \sup_{v\in[0,u]}|C(v)| - \frac{\theta p}{2}|y(u)|_H^{p-2}|y(u)|_V^\alpha \\
&\quad + \frac{pM}{T-t_1}|y(0)|_H|y(u)|_H^{p-1} - \frac{pM}{T-t_1}|y(u)|_H^p\Bigg] du.
\end{aligned}$$

Set

$$\widehat{c} = \frac{(p-2)\sup_{u\in[0,t]}|C(u)|+cp}{2}.$$

Then, for $t \in [t_1, T]$, we have

$$\begin{aligned}
&e^{-\widehat{c}t}|y(t)|_H^p + \frac{\theta p}{2}\int_{t_1}^t e^{-\widehat{c}s}|y(s)|_H^{p-2}|y(s)|_V^\alpha ds + \frac{pM}{T-t_1}\int_{t_1}^t e^{-\widehat{c}s}|y(s)|_H^p ds \\
&\leqslant e^{-\widehat{c}t_1}|y(t_1)|_H^p + \frac{pM}{T-t_1}|y(0)|_H\int_{t_1}^t e^{-\widehat{c}s}|y(s)|_H^{p-1} ds + \frac{(1-e^{-\widehat{c}(t-t_1)})\sup_{u\in[0,t]}|C(u)|}{\widehat{c}e^{\widehat{c}t_1}},
\end{aligned} \tag{90}$$

which implies that

$$\begin{aligned}
&\frac{pM}{T-t_1}\int_{t_1}^{t} e^{-\widehat{c}s}|y(s)|_H^p ds\\
\leqslant& e^{-\widehat{c}t_1}|y(t_1)|_H^p+\frac{pM}{T-t_1}|y(0)|_H\int_{t_1}^{t} e^{-\widehat{c}s}|y(s)|_H^{p-1}ds+\frac{(1-e^{-\widehat{c}(t-t_1)})\sup_{u\in[0,t]}|C(u)|}{\widehat{c}e^{\widehat{c}t_1}}.
\end{aligned}$$

Further, by Young's inequality, we get

$$\begin{aligned}
\int_{t_1}^{t}|y(s)|_H^p ds \leqslant& \frac{T-t_1}{pM}e^{\widehat{c}(t-t_1)}|y(t_1)|_H^p+|y(0)|_H\int_{t_1}^{t}e^{\widehat{c}(t-s)}|y(s)|_H^{p-1}ds\\
&+\frac{T-t_1}{pM}\cdot\frac{(e^{\widehat{c}t}-e^{\widehat{c}t_1})\sup_{u\in[0,t]}|C(u)|}{\widehat{c}e^{\widehat{c}t_1}}\\
\leqslant& \frac{T-t_1}{pM}e^{\widehat{c}(t-t_1)}R^p+|y(0)|_H e^{\widehat{c}(t-t_1)}\int_{t_1}^{t}|y(s)|_H^{p-1}ds\\
&+\frac{T-t_1}{pM}\cdot\frac{e^{\widehat{c}(t-t_1)}\sup_{u\in[0,t]}|C(u)|}{\widehat{c}}\\
\leqslant& \frac{T-t_1}{pM}e^{\widehat{c}(t-t_1)}\left(R^p+\frac{\sup_{u\in[0,t]}|C(u)|}{\widehat{c}}\right)\\
&+\frac{2^{p-1}|y(0)|_H^p e^{p\widehat{c}(t-t_1)}}{p}\cdot(T-t_1)+\frac{1}{2}\int_{t_1}^{t}|y(s)|_H^p ds.
\end{aligned}$$

Therefore, there exists $\vartheta_1(p)>0$, which is independent of t_1, such that

$$\int_{t_1}^{T}|y(s)|_H^p ds\leqslant\vartheta_1(p)(T-t_1). \tag{91}$$

Further, by (90) and (91), we conclude that there exists $\vartheta_2(p)>0$, which is independent of t_1, such that

$$\int_{t_1}^{T}|y(s)|_V^{\alpha}|y(s)|_H^{p-2}ds\leqslant\vartheta_2(p). \tag{92}$$

By Assumption 3, we get

$$\rho(y(t))\leqslant c(1+|y(t)|_V^{\alpha'}+|y(t)|_H^{\beta}+|y(t)|_V^{\alpha'}|y(t)|_H^{\beta}). \tag{93}$$

(91) implies that

$$\int_{t_1}^{T}|y(t)|_H^{\beta}dt\leqslant\vartheta_1(\beta)(T-t_1). \tag{94}$$

By (91), (92) and Hölder 's inequality, we get

$$\int_{t_1}^{T}|y(t)|_V^{\alpha'}dt\leqslant\left(\int_{t_1}^{T}|y(t)|_V^{\alpha}dt\right)^{\frac{\alpha'}{\alpha}}(T-t_1)^{\frac{\alpha-\alpha'}{\alpha}}\leqslant[\vartheta_2(2)]^{\frac{\alpha'}{\alpha}}(T-t_1)^{\frac{\alpha-\alpha'}{\alpha}}, \tag{95}$$

and

$$\int_{t_1}^{T}|y(t)|_V^{\alpha'}|y(t)|_H^{\beta}dt \leqslant \left(\int_{t_1}^{T}|y(t)|_V^{\alpha}dt\right)^{\frac{\alpha'}{\alpha}}\left(\int_{t_1}^{T}|y(t)|_H^{\frac{\alpha\beta}{\alpha-\alpha'}}dt\right)^{\frac{\alpha-\alpha'}{\alpha}}$$

$$\leqslant [\vartheta_2(2)]^{\frac{\alpha'}{\alpha}}\left[\vartheta_1\left(\frac{\alpha\beta}{\alpha-\alpha'}\right)(T-t_1)\right]^{\frac{\alpha-\alpha'}{\alpha}}. \tag{96}$$

Therefore, by (93)–(96), we conclude that there exists $\vartheta > 0$, which is independent of t_1, such that (89) holds. This completes the proof of Lemma 9.3.1. □

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. We consider the following equation:

$$\begin{aligned} d\widetilde{X}^n(t) &= A(t,\widetilde{X}^n(t))dt + B(t,\widetilde{X}^n(t))dW(t) + \int_{\{|z|<1\}} G(t,\widetilde{X}^n(t),z)\widetilde{N}(dt,dz) \\ &\quad - \frac{M}{T-t_1}(\varepsilon B_n^{-1}+I)^{-1}(y(t)-y(0))\chi_{[t_1,T]}(t)dt, \\ \widetilde{X}^n(0) &= x, \end{aligned} \tag{97}$$

where $\{y(t)\}_{t\in[0,T]}$ is the solution to (86). Note that $\widetilde{X}^n(t_1) = X(t_1)$. By Theorem 9.1.1 and (87), we know that (97) has a unique solution $\{\widetilde{X}^n(t)\}_{t\in[0,T]}$.

Lemma 9.3.2. *There exists $\vartheta > 0$, which is independent of $\varepsilon, n, T, M, R, t_1$, such that*

$$\begin{aligned} &\sup_{n\in\mathbb{N}}\left\{\mathbb{E}\left[\sup_{s\in[0,T]}|\widetilde{X}^n(s)|_H^2\right]\right\} \\ \leqslant & e^{\vartheta T}\left(\vartheta(T+|x|_H^2) + \frac{M^2}{m(T-t_1)}\left[(R+|y(0)|_H)^2 + \int_{t_1}^{T}|A(s,y(0))|_H^2 ds\right]\right). \end{aligned}$$

Proof of Lemma 9.3.2.
We will obtain the estimation for the solution $\widetilde{X}^n$ by following the argument of [128] and carefully handling the dependence of constants. By Theorem 7.2.9, we get

$$\begin{aligned} &|\widetilde{X}^n(t)|_H^2 - |x|_H^2 \\ &= \int_0^t\Big[2\langle A(u,\widetilde{X}^n(u)),\widetilde{X}^n(u)\rangle - \frac{2M}{T-t_1}1_{[t_1,T]}(u)\langle(\varepsilon B_n^{-1}+I)^{-1}(y(u)-y(0)),\widetilde{X}^n(u)\rangle_H \\ &\quad + \|B(u,\widetilde{X}^n(u))\|_{L_2(H)}^2\Big]du + \int_0^t 2\langle\widetilde{X}^n(u),B(u,\widetilde{X}^n(u))dW(u)\rangle_H \\ &\quad + \int_0^t\int_{\{|z|<1\}}[2\langle\widetilde{X}^n(u),G(u,\widetilde{X}^n(u),z)\rangle_H]\widetilde{N}(du,dz) \\ &\quad + \int_0^t\int_{\{|z|<1\}}[|\widetilde{X}^n(u)+G(u,\widetilde{X}^n(u),z)|_H^2 - |\widetilde{X}^n(u)|_H^2 - 2\langle\widetilde{X}^n(u),G(u,\widetilde{X}^n(u),z)\rangle_H]N(du,dz). \end{aligned} \tag{98}$$

Define

$$I_1(t) := 2 \sup_{s\in[0,t]} \left| \int_0^s \langle \widetilde{X}^n(u), B(u, \widetilde{X}^n(u)) dW(u) \rangle_H \right|,$$

$$I_2(t) := 2 \sup_{s\in[0,t]} \left| \int_0^s \int_{\{|z|<1\}} \langle \widetilde{X}^n(u), G(u, \widetilde{X}^n(u), z) \rangle_H \widetilde{N}(du, dz) \right|,$$

$$I_3(t) := \sup_{s\in[0,t]} \left| \int_0^s \int_{\{|z|<1\}} [|\widetilde{X}^n(u) + G(u, \widetilde{X}^n(u), z)|_H^2 \right.$$
$$\left. - |\widetilde{X}^n(u)|_H^2 - 2\langle \widetilde{X}^n(u), G(u, \widetilde{X}^n(u), z) \rangle_H] N(du, dz) \right|,$$

$$I_4(t) := \sup_{s\in[0,t]} \left| \int_0^s \frac{2M}{T - t_1} 1_{[t_1,T]}(u) \langle (\varepsilon B_n^{-1} + I)^{-1}(y(u) - y(0)), \widetilde{X}^n(u) \rangle_H du \right|.$$

Then, by (C) and (98), we get

$$\sup_{u\in[0,t]} |\widetilde{X}^n(u)|_H^2 + \theta \int_0^t |\widetilde{X}^n(u)|_V^\alpha du$$
$$\leqslant |x|_H^2 + \int_0^t (C(u) + c|\widetilde{X}^n(u)|_H^2) du + I_1(t) + I_2(t) + I_3(t) + I_4(t). \tag{99}$$

By the Burkholder-Davis-Gundy inequality and Assumption 3, we get

$$\begin{aligned}\mathbb{E}[I_1(t)] &\leqslant C'\mathbb{E}\left[\left\{ \int_0^t |[B(u, \widetilde{X}^n(u))]^* \widetilde{X}^n(u)|_H^2 du \right\}^{\frac{1}{2}}\right] \\ &\leqslant C'\mathbb{E}\left[\left\{ \int_0^t \|B(u, \widetilde{X}^n(u))\|_{L_2(H)}^2 |\widetilde{X}^n(u)|_H^2 du \right\}^{\frac{1}{2}}\right] \\ &\leqslant C'\left\{ \kappa \mathbb{E}\left[\sup_{u\in[0,t]} |\widetilde{X}^n(u)|_H^2 \right] \right\}^{\frac{1}{2}} \left\{ \frac{1}{\kappa} \mathbb{E}\left[\int_0^t [C(u) + c|\widetilde{X}^n(u)|_H^2] du \right] \right\}^{\frac{1}{2}}\end{aligned}$$

for some constants $C', \kappa > 0$. Then,

$$\mathbb{E}[I_1(t)] \leqslant \kappa \mathbb{E}\left[\sup_{u\in[0,t]} |\widetilde{X}^n(u)|_H^2 \right] + \frac{(C')^2}{4\kappa} \mathbb{E}\left[\int_0^t [C(u) + c|\widetilde{X}^n(u)|_H^2] du \right]. \tag{100}$$

Similarly, we get

$$\mathbb{E}[I_2(t)] \leqslant \kappa \mathbb{E}\left[\sup_{u\in[0,t]} |\widetilde{X}^n(u)|_H^2 \right] + \frac{(C')^2}{4\kappa} \mathbb{E}\left[\int_0^t [C(u) + c|\widetilde{X}^n(u)|_H^2] du \right]. \tag{101}$$

By Assumption 3, we get

$$\begin{aligned}\mathbb{E}[I_3(t)] &\leqslant \mathbb{E}\left[\int_0^t \int_{\{|z|<1\}} |G(u, \widetilde{X}^n(u), z)|_H^2 \nu(dz) du \right] \\ &\leqslant \mathbb{E}\left[\int_0^t [C(u) + c|\widetilde{X}^n(u)|_H^2] du \right].\end{aligned} \tag{102}$$

By (88), we get

$$
\begin{aligned}
I_4(t) &\leqslant \int_{t_1}^{t} \frac{M^2}{(T-t_1)^2} \|(\varepsilon B_n^{-1}+I)^{-1}\|_{L(H)}^2 |y(u)-y(0)|_H^2 du + \int_{t_1}^{T} |\widetilde{X}^n(u)|_H^2 du \\
&\leqslant \frac{M^2}{m(T-t_1)} \Big[(R+|y(0)|_H)^2 + \int_{t_1}^{T} |A(u,y(0))|_H^2 du \Big] + \int_{t_1}^{T} |\widetilde{X}^n(u)|_H^2 du. \qquad (103)
\end{aligned}
$$

Take $\kappa = \frac{1}{4}$. Thus, by (99)–(103), we obtain that for $t \geqslant t_1$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{u\in[0,t]} |\widetilde{X}^n(u)|_H^2 \Big] \\
&\leqslant 2|x|_H^2 + \frac{2M^2}{m(T-t_1)} \Big[(R+|y(0)|_H)^2 + \int_{t_1}^{T} |A(u,y(0))|_H^2 du \Big] \\
&\quad + [2(C')^2+1] T \sup_{t\in[0,T]} |C(t)| + 2(2(C')^2 c + c + 1) \Big(\int_0^t \mathbb{E}\Big[\sup_{r\in[0,u]} |\widetilde{X}^n(r)|_H^2 \Big] du \Big).
\end{aligned}
$$

Therefore, the proof is complete with the aid of Gronwall's lemma. □

Lemma 9.3.3. *Let $T>0$, $y(0)\in D(A,\varpi)$ for some $\varpi>2$ and $\eta,\delta\in(0,1)$. Then, there exist $M,R>0$ and $t_1\in(0,T)$ such that for any $n\in\mathbb{N}$ we can find an $\varepsilon\in(0,1)$ satisfying*

$$
\mathbb{P}(|\widetilde{X}^n(T)-y(0)|_H > \delta) < \eta.
$$

Proof of Lemma 9.3.3.
First, note that

$$
\begin{aligned}
\{|\widetilde{X}^n(T)-y(0)|_H > \delta\} &\subset \Big\{ |\widetilde{X}^n(T)-y(T)|_H + |y(T)-y(0)|_H > \delta \Big\} \\
&\subset \Big\{ |\widetilde{X}^n(T)-y(T)|_H > \frac{\delta}{2} \Big\} \bigcup \Big\{ |y(T)-y(0)|_H > \frac{\delta}{2} \Big\}. \qquad (104)
\end{aligned}
$$

For $t \in [t_1, T]$, by Theorem 7.2.9, (LM) and Assumption 3, we get

$$\begin{aligned}
&|\widetilde{X}^n(t) - y(t)|_H^2 - |\widetilde{X}^n(t_1) - y(t_1)|_H^2 \\
&= \int_{t_1}^t \big(2\langle A(u, \widetilde{X}^n(u)) - A(u, y(u)), \widetilde{X}^n(u) - y(u)\rangle + \|B(u, \widetilde{X}^n(u))\|_{L_2(H)}^2\big) du \\
&\quad + 2\int_{t_1}^t \langle \widetilde{X}^n(u) - y(u), B(u, \widetilde{X}^n(u)) dW(u)\rangle_H + \int_{t_1}^t \int_{\{|z|<1\}} |G(u, \widetilde{X}^n(u), z)|_H^2 \nu(dz) du \\
&\quad + \int_{t_1}^t \int_{\{|z|<1\}} [|\widetilde{X}^n(u) - y(u) + G(u, \widetilde{X}^n(u), z)|_H^2 - |\widetilde{X}^n(u) - y(u)|^2] \widetilde{N}(du, dz) \\
&\quad - \frac{2M}{T - t_1} \int_{t_1}^t \langle ([(\varepsilon B_n^{-1} + I)^{-1} - I](y(u) - y(0)), \widetilde{X}^n(u) - y(u)\rangle_H du \\
&\leqslant \int_{t_1}^t \big[(K + \rho(y(u)) + 2c + 1)|\widetilde{X}^n(u) - y(u)|_H^2 + (C(u) + 2c|y(u)|_H^2)\big] du \\
&\quad + \frac{M^2}{(T - t_1)^2} \int_{t_1}^t |[(\varepsilon B_n^{-1} + I)^{-1} - I](y(u) - y(0))|_H^2 du \\
&\quad + 2\int_{t_1}^t \langle \widetilde{X}^n(u) - y(u), B(u, \widetilde{X}^n(u)) dW(u)\rangle_H \\
&\quad + \int_{t_1}^t \int_{\{|z|<1\}} [|\widetilde{X}^n(u) - y(u) + G(u, \widetilde{X}^n(u), z)|_H^2 - |\widetilde{X}^n(u) - y(u)|^2] \widetilde{N}(du, dz). \qquad (105)
\end{aligned}$$

Set

$$\vartheta' = (K + 2c + 1) T^{\frac{\alpha'}{\alpha}} + \vartheta.$$

Then, by (89), (105) and Gronwall's lemma, we get

$$\begin{aligned}
&\mathbb{E}[|\widetilde{X}^n(T) - y(T)|_H^2] \\
&\leqslant e^{\vartheta'(T - t_1)^{\frac{\alpha - \alpha'}{\alpha}}} \Big\{ \mathbb{E}[|\widetilde{X}^n(t_1)|_H^2 1_{\{|\widetilde{X}^n(t_1)|_H > R\}}] + (T - t_1) \sup_{t \in [0,T]} |C(t)| + 2c\mathbb{E}\Big[\int_{t_1}^T |y(u)|_H^2 du\Big] \\
&\qquad + \frac{M^2}{(T - t_1)^2} \mathbb{E}\Big[\int_{t_1}^T |[(\varepsilon B_n^{-1} + I)^{-1} - I](y(u) - y(0))|_H^2 du\Big]\Big\}. \qquad (106)
\end{aligned}$$

Choose a sufficiently large R such that

$$\mathbb{E}[|X(t_1)|_H^2 1_{\{|X(t_1)|_H > R\}}] < \frac{\delta^2 \eta}{32}. \qquad (107)$$

Then, by (87), we can choose a sufficiently large M and hence sufficiently large m such that

$$\begin{aligned}
|y(T) - y(0)|_H^2 &\leqslant e^{-m}(R + |y(0)|_H)^2 + \left(\frac{T(\varpi - 2)}{m\varpi}\right)^{1 - \frac{2}{\varpi}} \left(\int_0^T |A(s, u)|_H^{\frac{\varpi}{\varpi}} ds\right)^{\frac{2}{\varpi}} \\
&< \frac{\delta}{2}. \qquad (108)
\end{aligned}$$

Next, we choose a t_1 which is sufficiently close to T such that

$$e^{\vartheta'(T - t_1)^{\frac{\alpha - \alpha'}{\alpha}}} < 2, \qquad (T - t_1) \sup_{t \in [0,T]} |C(t)| < \frac{\delta^2 \eta}{32}, \qquad (109)$$

and by (91),

$$\mathbb{E}\bigg[\int_{t_1}^{T}|y(u)|_H^2du\bigg]<\frac{\delta^2\eta}{64c}. \tag{110}$$

With R, M, t_1 chosen as above, note that for any $n\in\mathbb{N}$ and $t\in[t_1,T]$,

$$\begin{aligned}|[(\varepsilon B_n^{-1}+I)^{-1}-I](y(t)-y(0))|_H &\leqslant 2|y(t)-y_0|_H,\\ \lim_{\varepsilon\to 0^+}|[(\varepsilon B_n^{-1}+I)^{-1}-I](y(t)-y(0))|_H &= \lim_{\varepsilon\to 0^+}|\varepsilon B_n^{-1}(\varepsilon B_n^{-1}+I)^{-1}(y(t)-y(0))|_H=0.\end{aligned}$$

By (88), we find that $|y(t)-y(0)|_H^2\in L^1([t_1,T]\times\Omega)$. Then, by the dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0^+}\mathbb{E}\bigg[\int_{t_1}^{T}|[(\varepsilon B_n^{-1}+I)^{-1}-I](y(t)-y(0))|_H^2dt\bigg]=0.$$

Thus, for any $n\in\mathbb{N}$, there exists $\varepsilon>0$ such that

$$\frac{M^2}{(T-t_1)^2}\mathbb{E}\bigg[\int_{t_1}^{T}|[(\varepsilon B_n^{-1}+I)^{-1}-I](y(s)-y(0))|_H^2ds\bigg]<\frac{\delta^2\eta}{32}. \tag{111}$$

Therefore, the proof is complete by (104) and (106)–(111). □

The remaining argument is the same as [128, Theorem 1.1], but for completeness, we include the proof here.

Let $0\leqslant s<T$, $y\in H$ and $\delta\in(0,1)$. We will show that

$$\mathbb{P}(X(T)\notin B_H(y,\delta)|X(s)=x)<1,\quad \forall x\in H,$$

where $B_H(y,\delta):=\{z\in H:|z-y|<\delta\}$. To simplify notation, we only give the proof for the case that $s=0$. The proof for the case that $s>0$ is completely similar.

Let $\{X(t)\}_{t\geqslant 0}$ be the unique solution of (62) with $X(0)=x$ and $\Lambda(t)\equiv i$. By (D), there exists $y(0)\in D(A_H)$ such that $|y-y(0)|_H<\frac{\delta}{2}$. Let $\eta\in(0,1)$. By Lemmas 9.3.2 and 9.3.3, there exists $n_0>|x|$ such that for any $n\geqslant n_0$ we can find an $\varepsilon\in(0,1)$ satisfying

$$\frac{1}{n^2}\sup_{m\in\mathbb{N}}\mathbb{E}\bigg[\sup_{t\in[0,T]}|\widetilde{X}^m(t)|_H^2\bigg]<\frac{\eta}{2},\qquad \mathbb{P}\bigg(|\widetilde{X}^n(T)-y(0)|_H>\frac{\delta}{2}\bigg)<\frac{\eta}{2}.$$

Then, we have that

$$\mathbb{P}\bigg(|\widetilde{X}^n(T)-y|_H>\delta\bigg)<\frac{\eta}{2}. \tag{112}$$

Define $\overline{\tau}_n:=\inf\{t:|\widetilde{X}^n(t)|_H\geqslant n\}$. Then,

$$\mathbb{P}(\overline{\tau}_n\leqslant T)\leqslant\mathbb{P}\bigg(\sup_{t\in[0,T]}|\widetilde{X}^n(t)|_H\geqslant n\bigg)\leqslant\frac{\mathbb{E}\big[\sup_{t\in[0,T]}|\widetilde{X}^n(t)|_H^2\big]}{n^2}<\frac{\eta}{2},$$

which together with (112) implies that

$$\mathbb{P}\left(\{\overline{\tau}_n\leqslant T\}\bigcup\{|\widetilde{X}^n(T)-y|_H>\delta\}\right)<\eta. \tag{113}$$

Define (abuse of notation)

$$E^{-1}(t,v) := (B(t,v))[B(t,v)]^*)^{-1}B(t,v), \quad v \in V.$$

For $v \in V$ with $|u|_H \leqslant n$, we have $B(t,u)[B(t,u)]^* \geqslant B_n^2$. Hence,

$$|E^{-1}(t,v)(\varepsilon B_n^{-1}+I)^{-1}h|_H^2 \leqslant |B_n^{-1}(\varepsilon B_n^{-1}+I)^{-1}h|_H^2 \leqslant \frac{|h|_H^2}{\varepsilon^2}$$

for all $h \in H$ and $u \in V$ with $|u|_H \leqslant n$. Then, (87), we get

$$\begin{aligned}&\int_{t_1}^{T\wedge\overline{\tau}_n} |E^{-1}(s,\widetilde{X}^n(s))(\varepsilon B_n^{-1}+I)^{-1}[y(s)-y(0)]|_H^2 ds\\ \leqslant&\frac{1}{\varepsilon^2}\int_{t_1}^{T\wedge\overline{\tau}_n} |y(s)-y(0)|_H^2 ds < \infty.\end{aligned}$$

Define

$$\widetilde{W}(t) := W(t) - \frac{M}{T-t_1}\int_{t_1}^{t\wedge\overline{\tau}_n}[B(t,v)]^*(B(t,v))[B(t,v)]^*)^{-1}(\varepsilon B_n^{-1}+I)^{-1}(y(s)-y(0))ds$$

and

$$\begin{aligned}\widetilde{R}(t) \quad := \quad \exp\Bigg\{&\frac{M}{T-t_1}\int_{t_1}^{t\wedge\overline{\tau}_n}\langle(\varepsilon B_n^{-1}+I)^{-1}(y(s)-y(0)), E^{-1}(s,\widetilde{X}^n(s))dW(s)\rangle_H\\ &-\frac{M^2}{2(T-t_1)^2}\int_{t_1}^{t\wedge\overline{\tau}_n}|E^{-1}(s,\widetilde{X}^n(s))(\varepsilon B_n^{-1}+I)^{-1}(y(s)-y(0))|_H^2 ds\Bigg\}.\end{aligned}$$

Then, $\{\widetilde{W}(t)\}_{t\in[0,T]}$ is a cylindrical Wiener process under the probability measure $\mathbb{Q} := \widetilde{R}(T)\mathbb{P}$, which is equivalent to $\mathbb{P}$. By (113), we get

$$\mathbb{Q}\left(\{\overline{\tau}_n \leqslant T\}\bigcup\{|\widetilde{X}^n(T)-y|_H > \delta\}\right) < 1. \tag{114}$$

Define

$$\tau_n' := \inf\{t : |X(t)|_H \geqslant n\}.$$

By Girsanov's theorem, we know that the joint distribution of $(\{X(t)1_{\tau_n'>T}\}_{t\in[0,T]}, \tau_n')$ under $\mathbb{P}$ is the same as that of $(\{\widetilde{X}^n(t)1_{\overline{\tau}_n>T}\}_{t\in[0,T]}, \overline{\tau}_n)$ under $\mathbb{Q}$. Therefore, by (114), we get

$$\begin{aligned}\mathbb{P}(|X(T)-y|_H > \delta) &\leqslant \mathbb{P}\left(\{\tau_n' \leqslant T\}\bigcup\{\tau_n' > T, |X(T)-y|_H > \delta\}\right)\\ &= \mathbb{Q}\left(\{\overline{\tau}_n \leqslant T\}\bigcup\{\overline{\tau}_n > T, |\widetilde{X}^n(T)-y|_H > \delta\}\right)\\ &\leqslant \mathbb{Q}\left(\{\overline{\tau}_n \leqslant T\}\bigcup\{|\widetilde{X}^n(T)-y|_H > \delta\}\right)\\ &< 1.\end{aligned}$$

The proof is complete when there is no regime-switching.

For the general case, we follow the argument in [51, Theorem 3]. For $(x,i) \in H \times \mathbb{N}$, let $X^{(i)}(t)$ be defined by Equation (68). We also define $\widetilde{X^{(i)}}(t)$ to be the killing process with generator $\mathscr{L}_i + q_{ii}$. The idea is that using (85), the process $\widetilde{X^{(i)}}(t)$, and assumption (Q1), the right-hand-side of (85) is positive if B is a non-empty open subset in H, which implies the irreducibility. This completes the proof of Theorem 8.0.3.

9.4 Proof of Theorem 8.0.5

Claims *(i)* and *(ii)* follow from Theorems 8.0.1, 8.0.2 and 8.0.3. To prove claims *(iii)* and *(iv)*, we are going to apply [52, Theorem 3.13], which also holds with the state space $\mathbb{R}^m$ replaced by $H \times \mathbb{N}$.

Let $(X(t), \Lambda(t))$ be the unique solution to the hybrid system (62) and (64) with initial value $(x, i) \in H \times \mathbb{N}$. For $g \in C^{1,2}([0, \infty) \times H \times \mathbb{N}; \mathbb{R})$, define

$$\mathscr{A} g(t, x, i) := \mathscr{L}_i g(\cdot, \cdot, i)(t, x) + \sum_{j \in \mathbb{N}} [g(t, x, j) - g(t, x, i)] q_{ij}(x),$$

where $\mathscr{L}_i$ is defined by (84). Define $\mathcal{V}(t, x, i) = |x|_H^2 + f(i)$, where f is given by condition (Q2). Then,

$$\begin{aligned}\mathscr{A}\mathcal{V}(t, x, i) &= 2\langle A(t, x, i), x\rangle + \|B(t, x, i)\|_{L_2(H)}^2 + \int_{\{|z|<1\}} |G(t, x, i, z)|_H^2 \nu(dz) \\ &\quad + \int_{\{|z| \geqslant 1\}} \left[|J(t, x, i, z)|_H^2 + 2\langle x, J(t, x, i, z)\rangle_H\right] \nu(dz) + \sum_{j \in \mathbb{N}} [f(j) - f(i)] q_{ij}(x).\end{aligned}$$

Thus, by conditions (C), (66) and (Q2), we obtain that

$$\begin{aligned}&\lim_{|y|_H + i \to \infty} \inf_{t \geqslant 0} \mathcal{V}(t, y, i) = \infty, \\ &\lim_{n \to \infty} \sup_{|y|_V + i > n, i \in \mathbb{N}, t \geqslant 0} \mathscr{A}\mathcal{V}(t, y, i) = -\infty, \qquad (115) \\ &\sup_{y \in V, i \in \mathbb{N}, t \geqslant 0} \mathscr{A}\mathcal{V}(t, y, i) < \infty.\end{aligned}$$

For $n \in \mathbb{N}$, define the stopping time T_n by

$$T_n := \inf\{t \geqslant 0 : |X(t)|_V \vee \Lambda(t) \geqslant n\}.$$

For $t \geqslant 0$, by [53, Theorem 2], we get

$$\begin{aligned}&\mathbb{E}[\mathcal{V}(t \wedge T_n, X(t \wedge T_n), \Lambda(t \wedge T_n))] \\ &= \mathbb{E}[\mathcal{V}(0, X(0), \Lambda(0))] + \mathbb{E}\left[\int_0^{t \wedge T_n} \mathscr{A}\mathcal{V}(u, X(u), \Lambda(u)) du\right]. \qquad (116)\end{aligned}$$

Define

$$A_n := -\sup_{|y|_V + k > n, t \geqslant 0} \mathscr{A}\mathcal{V}(t, y, k).$$

By (115), we get $\lim_{n \to \infty} A_n = \infty$.

We have

$$\mathscr{A}\mathcal{V}(u, X(u), \Lambda(u)) \leqslant -1_{\{|X(u)|_V + k \geqslant n\}} A_n + \sup_{|y|_V + k < n, u \geqslant 0} \mathscr{A}\mathcal{V}(u, y, k).$$

Then, there exist positive constants ϵ_1 and ϵ_2 such that for sufficiently large n,

$$\mathbb{E}\left[\int_0^{t \wedge T_n} 1_{\{|X(u)|_V + k \geqslant n\}} du\right] \leqslant \frac{\epsilon_1 t + \epsilon_2}{A_n}. \qquad (117)$$

Letting $n \to \infty$ in (117), we get

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T P(0,(x,i);u,B_n^c)du = 0, \tag{118}$$

where $B_n^c = \{(y,k) \in H \times \mathbb{N} : |y|_V + k \geqslant n\}$.

By (116), we find that there exists $\lambda > 0$ such that

$$\mathbb{E}[\mathcal{V}(s,X(s),\Lambda(s))] \leqslant \lambda s + \mathcal{V}(0,x,i), \quad s > 0,$$

which, together with Chebyshev's inequality, implies that

$$P(0,(x,i);s,B_n^c) \leqslant \frac{\lambda s + \mathcal{V}(0,x,i)}{\inf_{|y|_V > n,\, i\in\mathbb{N}, t>0} \mathcal{V}(t,y,i)}.$$

Hence, there exists a sequence of positive integers $\gamma_n \uparrow \infty$ such that

$$\lim_{n\to\infty} \left\{ \sup_{(x,i)\in B_{H\times\mathbb{N}}(\gamma_n),\, s\in(0,\ell)} P(0,(x,i);s,B_n^c) \right\} = 0, \tag{119}$$

where $B_{H\times\mathbb{N}}(\iota) := \{(y,k) \in H \times \mathbb{N} : |y|_H + k < \iota\}$ for $\iota > 0$.

By the assumption that functions A, B, H, J are all ℓ-periodic with respect to t, we find that the transition semigroup $\{P_{s,t}\}$ is ℓ-periodic, i.e.,

$$P(s,(x,i);t,A) = P(s+\ell,(x,i);t+\ell,A), \quad \forall 0 \leqslant s < t, x \in H, i \in \mathbb{N}, A \in \mathscr{B}(H\times\mathbb{N}). \tag{120}$$

Since the embedding of V into H is compact, combining the periodicity and the strong Feller property of $\{P_{s,t}\}$ with (118), (119) and following the argument of [68, Theorem 3.2 and Remark 3.1], we conclude that $\{(X(t),\Lambda(t))\}_{t\geqslant 0}$ has an ℓ-periodic measure μ_0.

By Theorems 8.0.2 and 8.0.3, we know that $\{P_{s,t}\}$ is strong Feller and irreducible. Then, following the same argument of [52, Lemma 3.12], we can show that $\{P_{s,t}\}$ is regular. That is, for any $0 \leqslant s < t$, the transition probability measures $P(s,(x,i);t,\cdot)$ are mutually equivalent for all $(x,i) \in H \times \mathbb{N}$. Further, by virtue of (120), we conclude that there exists a unique family of probability measures $\{\eta_s\}$ on $\mathscr{B}(H\times\mathbb{N})$ that is ℓ-periodic with respect to $\{P_{s,t}\}$, i.e.,

$$\eta_s(A) = \int_H P(s,x;s+\ell,A)\eta_s(dx), \qquad \forall s \geqslant 0, A \in \mathscr{B}(H\times\mathbb{N}).$$

Hence, we obtain the uniqueness of periodic measures, namely, $\mu_0 := \eta_0$. Finally, following the same argument of the proof of [52, Theorem 3.13], we obtain (67). This finishes the proof.

Chapter 10

Example

This chapter provides an example explaining how Theorem 8.0.5 can be applied. We consider stochastic fractional porous medium equations. We first explain how to choose the coefficients A, B, G, J and the Gelfand triple, and then we verify all conditions of Theorem 8.0.5. This section is also based on [73].

Equations:

The equations read

$$\begin{aligned} dX(t) &= A(t, X(t), \Lambda(t))dt + B(t, X(t), \Lambda(t))dW(t) \\ &\quad + \int_{\{|z|<1\}} G(t, X(t), \Lambda(t), z)\widetilde{N}(dt, dz) \\ &\quad + \int_{\{|z|\geqslant 1\}} J(t, X(t), \Lambda(t), z)N(dt, dz) \end{aligned}$$

and

$$\mathbb{P}(\Lambda(t+\Delta) = j | \Lambda(t) = i, X(t) = x) = \begin{cases} q_{ij}(x)\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + q_{ij}(x)\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where W is a (cylindrical) I-Wiener process on H and N is a Poisson random measure on a real Banach space $(Z, \mathscr{B}(Z))$ with intensity measure ν. We assume that W and N are independent.

Choices of H and V:

Let $d \geqslant 1$ and $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open set with smooth boundary. For $\gamma > 0$, let Δ be the Dirichlet Laplacian on $\mathcal{O}$ and set $\mathfrak{L} = -(-\Delta)^\gamma$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots$ be the eigenvalues of $-\Delta$ and $\{e_j\}$ the corresponding unit eigenvectors. The operator $\mathfrak{L}$ is given by $\mathfrak{L}(e_j) := -\lambda_j^\gamma e_j$ for $j \in \mathbb{N}$. Let $r > 1$. Denote by $d\mu$ the normalized Lebesgue measure on $\mathcal{O}$ and define

$$V = L^{r+1}(\mathcal{O}; d\mu), \qquad H = H^{-\gamma}(\mathcal{O}; d\mu),$$

where $H^{-\gamma}(\mathcal{O}; d\mu)$ is the completion of $L^2(\mathcal{O}; d\mu)$ with respect to the norm for $f \in L^2(\mathcal{O}; \mu)$

$$|f|_{H^{-\gamma}} = \left(\int_{\mathcal{O}} |(-\Delta)^{-\frac{\gamma}{2}} f|^2 d\mu\right)^{\frac{1}{2}}. \tag{121}$$

Note that the embedding of V into H is compact. Here $H^{-\gamma}(\mathcal{O}; d\mu)$ is called the generalized Sobolev space (do not confuse with Hardy spaces introduced in Part I).

Choice of q_{ij}:

Let $\{q_{ij}\}$ be measurable functions defined on H such that one of the following conditions is satisfied:

(a) There exist $m \in \mathbb{N}$ and $M > 0$ such that for all $x \in H$,

$$
\begin{aligned}
&q_{ij}(x) = 0 \quad \text{if } |i-j| > m, \\
&q_{ij}(x) \in (0, M] \quad \text{if } 0 < |i-j| \leqslant m,
\end{aligned}
$$

and

$$
\inf_{x \in H, i > m, j \in [i-m, i)} q_{ij}(x) > \sup_{x \in H, i > m, j \in (i, i+m]} q_{ij}(x).
$$

(b)

$$
0 < \inf_{x \in H,\, j \neq i} \{j^{1+\delta} q_{ij}(x)\} < \sup_{x \in H,\, j \neq i} \{j^{1+\delta} q_{ij}(x)\} < \infty \quad \text{for some } \delta > 0.
$$

Then, conditions (Q0), (Q1) and (Q2) (setting $f(i) = i^2$ under condition (a), and $f(i) = i^{\delta/2}$ under condition (b)) hold (cf. [51, Examples 1 and 2]).

Choice of A:

Let $\ell > 0$ and g be a measurable function on $[0, \infty) \times \mathbb{N}$ such that $g(\cdot, i) \in L^\infty[0, \infty)$ is ℓ-periodic for each $i \in \mathbb{N}$ and $\sup_{t \in [0, \ell), i \in \mathbb{N}} |g(t, i)| < \infty$. For $t \in [0, \infty)$, $s \in \mathbb{R}$ and $x \in V$, define

$$
\Psi(s) = |s|^{r-1} s, \qquad \Phi(t, s, i) = g(t, i) s,
$$

and

$$
A(t, x, i) = \kappa(t, i) \mathfrak{L}(\Psi(x)) + \Phi(t, x, i),
$$

where $\kappa(\cdot, i)$ is ℓ-periodic for each $i \in \mathbb{N}$ and $k_1 \leqslant \kappa(t, i) \leqslant k_2$ for all $t \in [0, \infty)$, $i \in \mathbb{N}$ and some positive constants $k_1 < k_2$. Then, there exists $C_r > 0$ such that for all $t \in [0, \infty)$, $x_1, x_2 \in V$ and $i \in \mathbb{N}$,

$$
\langle A(t, x_1, i) - A(t, x_2, i), x_1 - x_2 \rangle \leqslant -C_r \kappa(t, i) |x_1 - x_2|_V^{r+1} + g(t, i) |x_1 - x_2|_H^2. \tag{122}
$$

Note that, for $\varpi > 2$, there exists $\iota_\varpi > 0$ such that

$$
|A(s, x, i)|_{V^*}^{\varpi} \leqslant \iota_\varpi \big(|\kappa(s, i)|^{\varpi} |\mathfrak{L}(\Psi(x))|_{V^*}^{\varpi} + |\Phi(s, x, i)|_{V^*}^{\varpi} \big), \qquad \forall s \geqslant 0, x \in V, i \in \mathbb{N}.
$$

Choice of B:

Suppose $\frac{1}{2} < s \leqslant \frac{\gamma}{d}$. Define $B_0 \in L_2(H)$ by

$$
B_0 e_j = j^{-s} e_j, \quad j \in \mathbb{N}.
$$

Let $b > 0$ and $\{b_j\}_{j\in\mathbb{N}}$ be measurable functions defined on $[0,\infty) \times H \times \mathbb{N}$ satisfying the following conditions:

$$
\begin{aligned}
&b_j(t,x,i) = b_j(t+\ell,x,i), \quad t \in [0,\infty), x \in H, i \in \mathbb{N}, j \in \mathbb{N},\\
&|b_j(t,x,i) - b_j(t,y,i)| \leqslant b|x-y|_H, \quad t \in [0,\ell), x, y \in H, i \in \mathbb{N}, j \in \mathbb{N},\\
&\sup_{t\in[0,\ell),x\in H,i\in\mathbb{N},j\in\mathbb{N}} |b_j(t,x,i)| \leqslant b,\\
&\inf_{t\in[0,\ell),|x|_H\leqslant n,i\in\mathbb{N},j\in\mathbb{N}} b_j(t,x,i) > 0, \qquad \forall n \in \mathbb{N}.
\end{aligned}
$$

Define

$$B(t,x,i)e_j = b_j(t,x,i)j^{-s}e_j, \quad t \in [0,\infty), x \in H, i \in \mathbb{N}, j \in \mathbb{N}.$$

As an explicit example, similar to [128, Example 4.6], we may let

$$b_j(t,x,i) := \frac{b'(t,i)}{1 + j^{-\frac{2\gamma}{d}}|\langle x, e_j\rangle_{L^2(\mathcal{O})}|},$$

where b' is a measurable function on $[0,\infty) \times \mathbb{N}$ such that $b'(\cdot,i)$ is ℓ-periodic for each $i \in \mathbb{N}$ and $0 < \inf_{t\in[0,\ell),i\in\mathbb{N}} b'(t,i) \leqslant \sup_{t\in[0,\ell),i\in\mathbb{N}} b'(t,i) < \infty$.

Choices of G and J:

Suppose that $c > 0$, $K > 0$, and $G, J : [0,\infty) \times V \times \mathbb{N} \times Z \to H$ are measurable functions satisfying the following conditions:

- for any $t \geqslant 0$, $x \in V$, $i \in \mathbb{N}$ and $z \in Z$, we have $G(t,x,i,z) = G(t+\ell,x,i,z)$ and $J(t,x,i,z) = J(t+\ell,x,i,z)$;
- for any $t \geqslant 0$, $x_1, x_2 \in V$, $i \in \mathbb{N}$ and $z \in Z$,

$$\int_{\{|z|<1\}} |G(t,x_1,i,z) - G(t,x_2,i,z)|_H^2 \nu(dz) \leqslant K|x_1 - x_2|_H^2; \tag{123}$$

- for any $t \geqslant 0$, $x \in V$, $i \in \mathbb{N}$ and $z \in Z$,

$$\int_{\{|z|<1\}} |G(t,x,i,z)|_H^2 \nu(dz) \leqslant c(1 + |x|_H^2) \tag{124}$$

  and

$$\int_{\{|z|\geqslant 1\}} |J(t,x,i,z)|_H^2 \nu(dz) \leqslant c. \tag{125}$$

We now verify all the conditions of Theorem 8.0.5.

(V, H, V^*) is a Gelfand triple with compact embedding:

Here we define $H^\gamma(\mathcal{O})$ to be the set of all $f \in L^2(\mathcal{O};\mu)$ such that $|f|^2_{H^\gamma(\mathcal{O})} := \sum_{k=1}^\infty \lambda_k^\gamma |\langle f, e_k\rangle_{L^2(\mathcal{O})}|^2 = \int_{\mathcal{O}} |(-\Delta)^{\frac{\gamma}{2}} f|^2 d\mu < \infty$ for $\gamma > 0$. Without further mentioning, all integrals are with respect to μ. From the definition, $H^0(\mathcal{O}) = L^2(\mathcal{O})$. Moreover, the operator $(-\Delta)^\gamma$, which is originally defined on $C_0^\infty(\mathcal{O})$, can be extended to a bounded operator from $H^\gamma(\mathcal{O})$ to $H^{-\gamma}(\mathcal{O})$.

Since $(-\Delta)^\gamma : H^\gamma(\mathcal{O}) \to (H^\gamma(\mathcal{O}))^*$, see [4, Section 2.1], and satisfies

$$\langle(-\Delta)^\gamma u, v\rangle_{L^2} = \int_{\mathcal{O}} (-\Delta)^\gamma u(x) v(x) \mu(dx) \leqslant |u|_{H^\gamma} |v|_{H^\gamma}.$$

Hence, we have

$$\langle(-\Delta)^\gamma u, (-\Delta)^\gamma v\rangle_{(H^\gamma(\mathcal{O}))^*} = \langle u, v\rangle_{H^\gamma}.$$

Thus, following [77, Lemma 4.1.12], we have $(-\Delta)^{-\gamma} : (H^\gamma)^* \to H^\gamma$ and

$$\langle u, v\rangle_{(H^\gamma)^*} = \langle(-\Delta)^{-\gamma} u, (-\Delta)^{-\gamma} v\rangle_{H^\gamma}.$$

Therefore, we can identify $H^{-\gamma}$ and H^γ by $(-\Delta)^\gamma$, see that $|\cdot|_{H^{-\gamma}}$ is the same as (121) and the space $H^{-\gamma}$ is the $|\cdot|_{H^{-\gamma}}$-closure of L^2.

Next, we find that $C_0^\infty(\mathcal{O}) \subset H^\gamma$ since for arbitrary $g \in L^2(\mathcal{O})$,

$$|\langle(-\Delta)^{\frac{\gamma}{2}} f, g\rangle_{L^2}| = |\langle(-\Delta)^{\lfloor\frac{\gamma}{2}\rfloor+1} f, (-\Delta)^{\frac{\gamma}{2}-\lfloor\frac{\gamma}{2}\rfloor-1} g\rangle| \leqslant |f|_{C_c^\infty} \lambda_1^{\frac{\gamma}{2}-\lfloor\frac{\gamma}{2}\rfloor-1} |g|_{L^2} < \infty.$$

Therefore, we have the inclusions

$$C_0^\infty(\mathcal{O}) \subset H^\gamma(\mathcal{O}) \subset L^2(\mathcal{O}) \subset L^{1+\varepsilon}(\mathcal{O})$$

for any $0 < \varepsilon < 1$. Thus, we have $\overline{H^\gamma}^{|\cdot|_{L^{1+\varepsilon}}} = L^{1+\varepsilon}$ because $\overline{C_0^\infty}^{|\cdot|_{L^{1+\varepsilon}}} = L^{1+\varepsilon}$; and this implies for any $r > 1$

$$L^{1+r}(\mathcal{O}) \subset H^{-\gamma}(\mathcal{O}); \qquad \overline{L^{1+r}}^{|\cdot|_{H^{-\gamma}(\mathcal{O})}}(\mathcal{O}) = H^{-\gamma}(\mathcal{O})$$

and $|v|_{H^{-\gamma}(\mathcal{O})} \leqslant \lambda_1^{-\frac{\gamma}{2}} |v|_{L^{1+r}(\mathcal{O})}$. Hence, if $V = L^{1+r}(\mathcal{O})$ and $H = H^{-\gamma}(\mathcal{O}) \cong H^\gamma(\mathcal{O})$, then (V, H, V^*) is a Gelfand triple.

Finally, the operator $(-\Delta)^\gamma : L^{1+r^{-1}}(\mathcal{O}) \to V^*$. Using the argument in [77, Lemma 4.1.13], we have $|(-\Delta)^\gamma v|_{V^*} = |v|_{L^{1+r^{-1}}}$. Thus, we have

$$|v|_{V^*} = |(-\Delta)^{-\gamma} v|_{L^{1+r^{-1}}} \leqslant |(-\Delta)^{-\gamma} v|_{L^2} \leqslant \lambda_1^{-\frac{\gamma}{2}} |v|_{L^{1+r}(\mathcal{O})}.$$

In terms of V and H, we have $|v|_{V^*} \leqslant |v|_H \leqslant \lambda_1^{-\frac{\gamma}{2}} |v|_V$ for any $v \in V$.

By Rellich-Kondrashov theorem, the embedding $H^\gamma(\mathcal{O}) \subset L^{1+\varepsilon}(\mathcal{O})$ is compact. Therefore, the embedding $L^{1+r}(\mathcal{O}) \subset H^{-\gamma}(\mathcal{O})$ is compact.

(Q0), (Q1) and (Q2) hold:
We refer to [51, Examples 1 and 2].

(HC) holds:
Note that $A(t, v_1 + sv_2, i) = \kappa(t, i)\mathfrak{L}(\Psi(v_1 + sv_2)) + \Phi(t, v_1 + sv_2, i)$. The second term is continuous in s as it is linear in the second variable. The first term is also continuous as $\mathfrak{L}$ is a linear operator and the function $\Psi(s)$ is continuous. Therefore, $\langle A(t, v_1 + sv_2, i), v\rangle$ is continuous in s.

(C) holds:

First, for $x \in V$

$$\begin{aligned} 2\langle A(t,x,i),x\rangle &= 2\langle \kappa(t,i)\mathfrak{L}(\Psi(x)),x\rangle + 2g(t,i)|x|_V^2 \\ &= -2\kappa(t,i)|x|_V^{r+1} + 2g(t,i)|x|_V^2 \\ &\leqslant -2k_1|x|_V^{r+1} + 2\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)||x|_H^2 \end{aligned}$$

For the coefficient B,

$$\|B(t,x,i)\|^2_{L_2(H)} = \sum_{j=1}^{\infty}|B(t,x,i)e_j|_H^2 \leqslant \sum_{j=1}^{\infty}|b_j(t,x,i)|^2 j^{-2s} \leqslant bC_s,$$

where we have used the fact that $2s > 1$. Finally, from (124), we can conclude that

$$\begin{aligned} &2\langle A(t,x,i),x\rangle + \|B(t,x,i)\|^2_{L_2(H)} + \int_{\{|z|<1\}}|G(t,x,i,z)|_H^2\nu(dz) \\ &\leqslant c + C_sb - 2k_1C_r|x|_V^{r+1} + \big(2\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)| + c\big)|x|_H^2. \end{aligned}$$

(G1) holds:

$$\begin{aligned} |A(t,x,i)|_{V^*}^{1+r^{-1}} &\leqslant 2^{1+r^{-1}}\left(|\kappa(t,i)\mathfrak{L}(\Psi(x))|_{V^*}^{1+r^{-1}} + |g(t,i)|^{1+r^{-1}}|x|_{V^*}^{1+r^{-1}}\right) \\ &\leqslant 2^{1+r^{-1}}\left(k_2^{1+r^{-1}}|\Psi(x)|^{1+r^{-1}}_{L^{1+r^{-1}}(\mathcal{O})} + C\big(\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)|\big)^{1+r^{-1}}|x|_{V^*}^{1+r^{-1}}\right) \\ &\leqslant 2^{1+r^{-1}}\left(k_2^{1+r^{-1}}|x|_V^{1+r} + C\big(\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)|\big)^{1+r^{-1}}|x|_V^{1+r^{-1}}\right) \\ &\leqslant A_{r,k_2,g}|x|_V^{1+r} + B_{r,g}, \end{aligned}$$

where the constants $A_{r,k_2,g} := 2^{1+r^{-1}}\Big[k_2^{1+r^{-1}} + C\big(\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)|\big)^{1+r^{-1}}\Big]$ and $B_{r,g} = 2^{1+r^{-1}}C\big(\sup_{t\in[0,\ell),i\in\mathbb{N}}|g(t,i)|\big)^{1+r^{-1}}$. We have used the fact that $|x|_{V^*} \leqslant C|x|_V$, and $|x|_V^{1+r^{-1}} \leqslant (|x|_V^{1+r} + 1)$.

(G2) and (Gβ) hold:

This follows from the computation above. Moreover, when $\beta = 0$, Condition (Gβ) reduces to Condition (G2).

(LipB) holds:

Note that for B, we have

$$\begin{aligned}
&\|B(t,x_1,i)-B(t,x_2,i)\|_{L_2(H)}^2 \\
&= \sum_{j=1}^{\infty} \|[B(t,x_1,i)-B(t,x_2,i)]e_j\|_H^2 \\
&\leqslant \sum_{j=1}^{\infty} j^{-2s}|b_j(t,x_1,i)-b_j(t,x_2,i)|^2 \\
&\leqslant \sum_{j=1}^{\infty} b|x_1-x_2|_H^2 j^{-2s} \leqslant C_{s,b}|x_1-x_2|_H^2,
\end{aligned}$$

where we have used the fact that $2s > 1$.

(N) holds:
Take

$$B_n := \left(\inf_{t\in[0,\ell),|x|_H\leqslant n, i\in\mathbb{N}, j\in\mathbb{N}} b_j(t,x,i)\right)B_0, \quad n \in \mathbb{N}.$$

Then, for all $j,k \in \mathbb{N}$,

$$\langle B(t,x,i)[B(t,x,i)]^*(e_j), e_k\rangle_H = b_j(t,x,i)b_k(t,x,i)\langle B_0e_j, B_0e_k\rangle_H \geqslant \langle B_n^2 e_j, e_k\rangle_H.$$

Hence, $B(t,x,i)[B(t,x,i)]^* \geqslant B_n^2$.

(M) holds:
By (122) and [118, Theorem 2.4.1], we get

$$\begin{aligned}
&\langle A(t,x_1,i)-A(t,x_2,i), x_1-x_2\rangle \\
&\leqslant -\kappa(t,i)C_r|x_1-x_2|_V^{r+1} + g(t,i)|x_1-x_2|_H^2 \\
&\leqslant -\kappa(t,i)C_r|B_n^{-1}(x_1-x_2)|_H^{\lambda}|x_1-x_2|_H^{r+1-\lambda} + g(t,i)|x_1-x_2|_H^2.
\end{aligned}$$

Finally, Condition (M) holds for G from (123).

(D) holds:
Define $D(A,\varpi) := C_0^\infty \times \mathbb{N}$. Then, we have $\overline{C_0^\infty}^{|\cdot|_H} = H$, and for $(x,i) \in D(A,\varpi)$ and $t \geqslant 0$, we have

$$\int_0^t |A(s,x,i)|_H^{\varpi} ds < \infty.$$

Thus, Condition (D) holds.

(66) **holds:**
Since $\alpha = r+1 > 2$, $|x|_H^2 \leqslant C|x|_V^2$ and the fact that $\lim_{d\to\infty} -\theta d^{r+1} + CKd^2 = -\infty$, we can deduce (66) holds with aid of (125).

Therefore, all assertions of Theorem 8.0.5 hold.

Bibliography

[1] Mujdat Agcayazi, Amiran Gogatishvili, Kerim Koca, and Rza Mustafayev, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan **67** (2015), no. 2, 581–593. MR3340187

[2] Josefina Álvarez and Jorge Hounie, *Estimates for the kernel and continuity properties of pseudo-differential operators*, Ark. Mat. **28** (1990), no. 1, 1–22. MR1049640

[3] Josefina Álvarez and Mario Milman, *H^p continuity properties of Calderón-Zygmund-type operators*, J. Math. Anal. Appl. **118** (1986), no. 1, 63–79. MR849442

[4] Harbir Antil, Johannes Pfefferer, and Sergejs Rogovs, *Fractional operators with inhomogeneous boundary conditions: analysis, control, and discretization*, Commun. Math. Sci. **16** (2018), no. 5, 1395–1426. MR3900347

[5] David Applebaum, *Lévy processes and stochastic calculus*, Second, Cambridge Studies in Advanced Mathematics, vol. 116, Cambridge University Press, Cambridge, 2009. MR2512800

[6] Jesús Bastero, Mario Milman, and Francisco J. Ruiz, *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3329–3334. MR1777580

[7] Frédéric Bernicot, Tarek Elgindi, and Sahbi Keraani, *On the inviscid limit of the 2D Navier-Stokes equations with vorticity belonging to BMO-type spaces*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **33** (2016), no. 2, 597–619. MR3465387

[8] Frédéric Bernicot and Taoufik Hmidi, *On the global well-posedness for Euler equations with unbounded vorticity*, Dyn. Partial Differ. Equ. **12** (2015), no. 2, 127–155. MR3361245

[9] Aline Bonami and Justin Feuto, *Products of functions in hardy and lipschitz or bmo spaces* (2010), 57–71. MR2603175

[10] Marcin Bownik, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proc. Amer. Math. Soc. **133** (2005), no. 12, 3535–3542. MR2163588

[11] Haim Brezis and Louis Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), no. 2, 197–263. MR1354598

[12] Zdzisław Brzeźniak, Wei Liu, and Jiahui Zhu, *Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise*, Nonlinear Anal. Real World Appl. **17** (2014), 283–310. MR3158475

[13] Almut Burchard, Galia Dafni, and Ryan Gibara, *Vanishing mean oscillation and continuity of rearrangements*, 2022.

[14] Donald L. Burkholder, Richard F. Gundy, and Martin L. Silverstein, *A maximal function characterization of the class H^p*, Trans. Amer. Math. Soc. **157** (1971), 137–153. MR274767

[15] Alberto P. Calderón, *Estimates for singular integral operators in terms of maximal functions*, Studia Math. **44** (1972), 563–582. MR348555

[16] Der-Chen Chang and Song-Ying Li, *On the boundedness of multipliers, commutators and the second derivatives of Green's operators on H^1 and BMO*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 2, 341–356. MR1736232

[17] Feng Chen, Yuecai Han, Yong Li, and Xue Yang, *Periodic solutions of Fokker-Planck equations*, J. Differential Equations **263** (2017), no. 1, 285–298. MR3631307

[18] Mengyu Cheng and Zhenxin Liu, *Periodic, almost periodic and almost automorphic solutions for SPDEs with monotone coefficients*, Discrete Contin. Dyn. Syst. Ser. B **26** (2021), no. 12, 6425–6462. MR4342169

[19] Samuel N. Cohen and Robert J. Elliott, *Stochastic calculus and applications*, Second, Probability and its Applications, Springer, Cham, 2015. MR3443368

[20] Ronald R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274. MR358318

[21] Ronald R. Coifman, Richard Rochberg, and Guido Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. MR412721

[22] Ronald R. Coifman and Guido Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645. MR447954

[23] John B. Conway, *A course in operator theory*, Graduate Studies in Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2000. MR1721402

[24] Mischa Cotlar and Cora Sadosky, *Two distinguished subspaces of product BMO and Nehari-AAK theory for Hankel operators on the torus*, Integral Equations Operator Theory **26** (1996), no. 3, 273–304. MR1415032

[25] Giuseppe Da Prato and Arnaud Debussche, *2D stochastic Navier-Stokes equations with a time-periodic forcing term*, J. Dynam. Differential Equations **20** (2008), no. 2, 301–335. MR2385713

[26] Giuseppe Da Prato and Constantin Tudor, *Periodic and almost periodic solutions for semilinear stochastic equations*, Stochastic Anal. Appl. **13** (1995), no. 1, 13–33. MR1313204

[27] Giuseppe Da Prato and Jerzy Zabczyk, *Ergodicity for infinite-dimensional systems*, London Mathematical Society Lecture Note Series, vol. 229, Cambridge University Press, Cambridge, 1996. MR1417491

[28] ———, *Stochastic equations in infinite dimensions*, Second, Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014. MR3236753

[29] Galia Dafni, *Local VMO and weak convergence in h^1*, Canad. Math. Bull. **45** (2002), no. 1, 46–59. MR1884133

[30] Galia Dafni and Chun Ho Lau, *h^1 boundedness of localized operators and commutators with bmo and lmo*, 2023. To appear in “Recent progress in function theory and operator theory”, Contemp. Math.

[31] Galia Dafni, Chun Ho Lau, Tiago Picon, and Claudio Vasconcelos, *Inhomogeneous cancellation conditions and Calderón-Zygmund type operators on h^p*, Nonlinear Anal. **225** (2022), Paper No. 113110, 22. MR4475438

[32] ———, *Necessary cancellation conditions for the boundedness of operators on local hardy spaces* (2022), available at 2210.05786.

[33] Galia Dafni and Elijah Liflyand, *A local Hilbert transform, Hardy's inequality and molecular characterization of Goldberg's local Hardy space*, Complex Anal. Synerg. **5** (2019), no. 1, Paper No. 10, 9. MR3936310

[34] Galia Dafni and Hong Yue, *Some characterizations of local* bmo*and h^1 on metric measure spaces*, Anal. Math. Phys. **2** (2012), no. 3, 285–318. MR2958361

[35] Wei Ding, YongSheng Han, and YuePing Zhu, *Boundedness of singular integral operators on local Hardy spaces and dual spaces*, Potential Anal. **55** (2021), no. 3, 419–441. MR4313085

[36] Richard Durrett, *Brownian motion and martingales in analysis*, Wadsworth Mathematics Series, Wadsworth International Group, Belmont, CA, 1984. MR750829

[37] Charles Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36. MR257819

[38] ———, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588. MR280994

[39] Charles Fefferman and Elias M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. MR447953

[40] Chunrong Feng and Huaizhong Zhao, *Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding*, J. Funct. Anal. **262** (2012), no. 10, 4377–4422. MR2900470

[41] José Garcia-Cuerva, Eleonor Harboure, Carlos Segovia, and José L. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40** (1991), no. 4, 1397–1420. MR1142721

[42] José Garcia-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Mathematics Studies, vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104. MR807149

[43] Leszek Gawarecki and Vidyadhar Mandrekar, *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*, Probability and its Applications (New York), Springer, Heidelberg, 2011. MR2560625

[44] Benjamin Gess and Michael Röckner, *Stochastic variational inequalities and regularity for degenerate stochastic partial differential equations*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3017–3045. MR3605963

[45] David Goldberg, *Local Hardy spaces*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, 1979, pp. 245–248. MR545262

[46] ______, *A local version of real Hardy spaces*, Duke Math. J. **46** (1979), no. 1, 27–42. MR523600

[47] Loukas Grafakos, *Modern Fourier analysis*, Second, Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009. MR2463316

[48] Loukas Grafakos and Cody B. Stockdale, *A limited-range Calderón-Zygmund theorem*, Bull. Hellenic Math. Soc. **63** (2019), 54–63. MR3999602

[49] Massimiliano Gubinelli, *A panorama of singular SPDEs*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures, 2018, pp. 2311–2338. MR3966852

[50] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75. MR3406823

[51] Xiaoxia Guo and Wei Sun, *Periodic solutions of hybrid jump diffusion processes*, Front. Math. China **16** (2021), no. 3, 705–725. MR4277379

[52] ______, *Periodic solutions of stochastic differential equations driven by Lévy noises*, J. Nonlinear Sci. **31** (2021), no. 2, Paper No. 32, 39. MR4227839

[53] István Gyöngy and Nicolai V. Krylov, *On stochastics equations with respect to semimartingales. II. Itô formula in Banach spaces*, Stochastics **6** (1981/82), no. 3-4, 153–173. MR665398

[54] Martin Hairer and Jonathan C. Mattingly, *Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing*, Ann. of Math. (2) **164** (2006), no. 3, 993–1032. MR2259251

[55] Eleonor Harboure, Carlos Segovia, and José L. Torrea, *Boundedness of commutators of fractional and singular integrals for the extreme values of* p, Illinois J. Math. **41** (1997), no. 4, 676–700. MR1468874

[56] Ziyi He, Dachun Yang, and Wen Yuan, *Real-variable characterizations of local Hardy spaces on spaces of homogeneous type*, Math. Nachr. **294** (2021), no. 5, 900–955. MR4288480

[57] Carl Herz, H_p*-spaces of martingales,* $0 < p \leqslant 1$, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **28** (1973/74), 189–205. MR372987

[58] ______, *Bounded mean oscillation and regulated martingales*, Trans. Amer. Math. Soc. **193** (1974), 199–215. MR353447

[59] Gustavo Hoepfner, Rafael Kapp, and Tiago Picon, *On the continuity and compactness of pseudodifferential operators on localizable Hardy spaces*, Potential Anal. **55** (2021), no. 3, 491–512. MR4313088

[60] Jorge G. Hounie and Rafael Kapp, *Pseudodifferential operators on local Hardy spaces*, J. Fourier Anal. Appl. **15** (2009), no. 2, 153–178. MR2500920

[61] Hongxiao Hu and Liguang Xu, *Existence and uniqueness theorems for periodic Markov process and applications to stochastic functional differential equations*, J. Math. Anal. Appl. **466** (2018), no. 1, 896–926. MR3818151

[62] Ha Duy Hung and Luong Dang Ky, *An Hardy estimate for commutators of pseudo-differential operators*, Taiwanese J. Math. **19** (2015), no. 4, 1097–1109. MR3384681

[63] Tuomas P. Hytönen, *The* L^p*-to-*L^q *boundedness of commutators with applications to the jacobian operator*, J. Math. Pures Appl. (9) **156** (2021), 351–391 (English, with English and French summaries). MR4338459

[64] Min Ji, Weiwei Qi, Zhongwei Shen, and Yingfei Yi, *Existence of periodic probability solutions to Fokker-Planck equations with applications*, J. Funct. Anal. **277** (2019), no. 11, 108281, 41. MR4013829

[65] Fritz John and Louis Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426. MR131498

[66] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, Second, Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940

[67] Norihiko Kazamaki, *Continuous exponential martingales and BMO*, Lecture Notes in Mathematics, vol. 1579, Springer-Verlag, Berlin, 1994. MR1299529

[68] Rafail Khasminskii, *Stochastic stability of differential equations*, second, Stochastic Modelling and Applied Probability, vol. 66, Springer, Heidelberg, 2012. With contributions by G. N. Milstein and M. B. Nevelson. MR2894052

[69] Yasuo Komori-Furuya, *Calderón-Zygmund operators on $H^p(\mathbf{R}^n)$*, Sci. Math. Jpn. **53** (2001), no. 1, 65–73. MR1821600

[70] ———, *Commutators of pseudo-differential operators on local Hardy spaces*, Acta Sci. Math. (Szeged) **77** (2011), no. 3-4, 489–501. MR2906517

[71] Luong Dang Ky, *Bilinear decompositions and commutators of singular integral operators*, Trans. Amer. Math. Soc. **365** (2013), no. 6, 2931–2958. MR3034454

[72] Robert H. Latter, *A characterization of $H^p(\mathbf{R}^n)$ in terms of atoms*, Studia Math. **62** (1978), no. 1, 93–101. MR482111

[73] Chun Ho Lau and Wei Sun, *Periodic measures for a class of spdes with regime-switching*, 2023. To appear in Stochastics and Dynamics.

[74] Song-Ying Li, *Toeplitz operators on Hardy space $H^p(S)$ with $0 < p \leqslant 1$*, Integral Equations Operator Theory **15** (1992), no. 5, 807–824. MR1177324

[75] Wenming Li, *John-Nirenberg type inequalities for the Morrey-Campanato spaces*, J. Inequal. Appl. (2008), Art. ID 239414, 5. MR2379517

[76] Wei Liu, *Harnack inequality and applications for stochastic evolution equations with monotone drifts*, J. Evol. Equ. **9** (2009), no. 4, 747–770. MR2563674

[77] Wei Liu and Michael Röckner, *Stochastic partial differential equations: an introduction*, Universitext, Springer, Cham, 2015. MR3410409

[78] Long Long, Ferenc Weisz, and Guangheng Xie, *Real interpolation of martingale Orlicz Hardy spaces and BMO spaces*, J. Math. Anal. Appl. **505** (2022), no. 2, Paper No. 125565, 23. MR4300991

[79] Emiel Lorist, *Vector-valued harmonic analysis with applications to SPDE*, Ph.D. Thesis, 2021 (English).

[80] Emiel Lorist and Mark Veraar, *Singular stochastic integral operators*, Anal. PDE **14** (2021), no. 5, 1443–1507. MR4307214

[81] Qi Lü and Xu Zhang, *Mathematical control theory for stochastic partial differential equations*, Springer International Publishing, Cham, 2021 (eng).

[82] Xuerong Mao and Chenggui Yuan, *Stochastic differential equations with Markovian switching*, Imperial College Press, London, 2006. MR2256095

[83] Bohdan Maslowski and Jan Seidler, *Invariant measures for nonlinear SPDE's: uniqueness and stability*, 1998, pp. 153–172. Equadiff 9 (Brno, 1997). MR1629692

[84] Stefano Meda, Peter Sjögren, and Maria Vallarino, *On the H^1-L^1 boundedness of operators*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2921–2931. MR2399059

[85] Yves Meyer, *Wavelets and operators*, Different perspectives on wavelets (San Antonio, TX, 1993), 1993, pp. 35–58. MR1267996

[86] Yves Meyer and Ronald Coifman, *Wavelets*, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997. Calderón-Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger. MR1456993

[87] Sean P. Meyn and Richard L. Tweedie, *Markov chains and stochastic stability*, Communications and Control Engineering Series, Springer-Verlag London, Ltd., London, 1993. MR1287609

[88] Mario Milman and Tomas Schonbek, *Second order estimates in interpolation theory and applications*, Proc. Amer. Math. Soc. **110** (1990), no. 4, 961–969. MR1075187

[89] Marius Neuß, *Ergodicity for Singular-Degenerate Stochastic Porous Media Equations*, Journal of Dynamics and Differential Equations (2021Feb).

[90] Lassi Päivärinta and Erkki Somersalo, *A generalization of the Calderón-Vaillancourt theorem to L^p and h^p*, Math. Nachr. **138** (1988), 145–156. MR975206

[91] Maciej Paluszynski and Jacek Zienkiewicz, *A remark on atomic decompositions of martingale Hardy's spaces*, J. Geom. Anal. **31** (2021), no. 9, 8866–8878. MR4302201

[92] Marco M. Peloso and Silvia Secco, *Local Riesz transforms characterization of local Hardy spaces*, Collect. Math. **59** (2008), no. 3, 299–320. MR2452309

[93] Carlos Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), no. 1, 163–185. MR1317714

[94] Szymon Peszat and Jerzy Zabczyk, *Stochastic partial differential equations with Lévy noise*, Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007. An evolution equation approach. MR2356959

[95] Karl E. Petersen, *Brownian motion, Hardy spaces and bounded mean oscillation*, London Mathematical Society Lecture Note Series, No. 28, Cambridge University Press, Cambridge-New York-Melbourne, 1977. MR0651556

[96] Tiago Picon and Claudio Vasconcelos, *On the Continuity of Strongly Singular Calderón–Zygmund-Type Operators on Hardy Spaces*, Integral Equations Operator Theory **95** (2023), no. 2, Paper No. 9. MR4565858

[97] Humberto Rafeiro, Natasha Samko, and Stefan Samko, *Morrey-Campanato spaces: an overview*, Operator theory, pseudo-differential equations, and mathematical physics, 2013, pp. 293–323. MR3025501

[98] Marco Romito and Lihu Xu, *Ergodicity of the 3D stochastic Navier-Stokes equations driven by mildly degenerate noise*, Stochastic Process. Appl. **121** (2011), no. 4, 673–700. MR2770903

[99] Walter Rudin, *Functional analysis*, Second, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR1157815

[100] Michael Ruzhansky and Ville Turunen, *Pseudo-differential operators and symmetries*, Pseudo-Differential Operators. Theory and Applications, vol. 2, Birkhäuser Verlag, Basel, 2010. Background analysis and advanced topics. MR2567604

[101] Donald Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. MR377518

[102] Carlos Segovia and José L. Torrea, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat. **35** (1991), no. 1, 209–235. Conference on Mathematical Analysis (El Escorial, 1989). MR1103616

[103] Stephen Semmes, *A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller*, Comm. Partial Differential Equations **19** (1994), no. 1-2, 277–319. MR1257006

[104] Xian Liang Shi and Alberto Torchinsky, *Functions of vanishing mean oscillation*, Math. Nachr. **133** (1987), 289–296. MR912434

[105] David A. Stegenga, *Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean oscillation*, Amer. J. Math. **98** (1976), no. 3, 573–589. MR420326

[106] Elias M. Stein, *Note on the class L log L*, Studia Math. **32** (1969), 305–310. MR247534

[107] ———, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095

[108] ———, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR1232192

[109] Soichiro Suzuki, *The Calderón-Zygmund theorem with an L^1 mean Hörmander condition*, J. Fourier Anal. Appl. **27** (2021), no. 2, Paper No. 10, 11. MR4226507

[110] ———, *On a generalization of the Hörmander condition*, Proc. Amer. Math. Soc. Ser. B **9** (2022), 286–296. MR4436582

[111] Kristóf Szarvas and Ferenc Weisz, *Mixed martingale Hardy spaces*, J. Geom. Anal. **31** (2021), no. 4, 3863–3888. MR4236545

[112] Mitchell H. Taibleson and Guido Weiss, *The molecular characterization of Hardy spaces*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, 1979, pp. 281–287. MR545267

[113] Lin Tang, *Interior h^1 estimates for parabolic equations with LMO coefficients*, Canad. J. Math. **62** (2010), no. 1, 202–217. MR2597030

[114] ———, *Weighted local Hardy spaces and their applications*, Illinois J. Math. **56** (2012), no. 2, 453–495. MR3161335

[115] Michael E. Taylor, *Tools for PDE*, Mathematical Surveys and Monographs, vol. 81, American Mathematical Society, Providence, RI, 2000. Pseudodifferential operators, paradifferential operators, and layer potentials. MR1766415

[116] Akihito Uchiyama, *On the compactness of operators of Hankel type*, Tohoku Math. J. (2) **30** (1978), no. 1, 163–171. MR467384

[117] Feng-Yu Wang, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. **35** (2007), no. 4, 1333–1350. MR2330974

[118] ———, *Harnack inequalities for stochastic partial differential equations*, SpringerBriefs in Mathematics, Springer, New York, 2013. MR3099948

[119] Fubao Xi, George Yin, and Chao Zhu, *Regime-switching jump diffusions with non-Lipschitz coefficients and countably many switching states: existence and uniqueness, Feller, and strong Feller properties*, Modeling, stochastic control, optimization, and applications, 2019, pp. 571–599. MR3970185

[120] Qi Xia and Chuanzhou Zhang, *The quasi-martingale Hardy spaces with variable exponents*, Adv. Math. (China) **50** (2021), no. 2, 277–289. MR4259565

[121] Bin Xie, *Uniqueness of invariant measures of infinite dimensional stochastic differential equations driven by Lévy noises*, Potential Anal. **36** (2012), no. 1, 35–66. MR2886453

[122] Daoyi Xu, Yumei Huang, and Zhiguo Yang, *Existence theorems for periodic Markov process and stochastic functional differential equations*, Discrete Contin. Dyn. Syst. **24** (2009), no. 3, 1005–1023. MR2505691

[123] Daoyi Xu, Bing Li, Shujun Long, and Lingying Teng, *Moment estimate and existence for solutions of stochastic functional differential equations*, Nonlinear Anal. **108** (2014), 128–143. MR3238298

[124] Jie Yang, Yuzhao Wang, and Wenyi Chen, *Endpoint estimates for the commutator of pseudo-differential operators*, Acta Math. Sci. Ser. B (Engl. Ed.) **34** (2014), no. 2, 387–393. MR3174086

[125] George Yin and Chao Zhu, *Hybrid switching diffusions*, Stochastic Modelling and Applied Probability, vol. 63, Springer, New York, 2010. Properties and applications. MR2559912

[126] Chenggui Yuan and Jianhai Bao, *On the exponential stability of switching-diffusion processes with jumps*, Quart. Appl. Math. **71** (2013), no. 2, 311–329. MR3087425

[127] Rangrang Zhang, *Existence and uniqueness of invariant measures of 3D stochastic MHD-α model driven by degenerate noise*, Appl. Anal. **101** (2022), no. 2, 629–654. MR4392132

[128] Shao-Qin Zhang, *Irreducibility and strong Feller property for non-linear SPDEs*, Stochastics **91** (2019), no. 3, 352–382. MR3925662

[129] Xinhong Zhang, Ke Wang, and Dingshi Li, *Stochastic periodic solutions of stochastic differential equations driven by Lévy process*, J. Math. Anal. Appl. **430** (2015), no. 1, 231–242. MR3347211

[130] Jiahui Zhu, *A study of spdes w.r.t. compensated poisson random measures and related topics*, PhD. Thesis, 2010.