Absolutely continuous invariant measures for piecewise convex maps of an interval with countable (infinite) number of branches

A H M Mahbubur Rahman

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By: **A H M Mahbubur Rahman**

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Dr. Jason Bramburger

____________________ Co-Supervisor
Dr. Pawel Góra

____________________ Co-Supervisor
Dr. Md Shafiqul Islam

Approved by ____________________
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Department of Mathematics and Statistics

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Dr. Pascale Sicotte, Dean
Faculty of Arts and Science

Abstract

Absolutely continuous invariant measures for piecewise convex maps of an interval with countable (infinite) number of branches

A H M Mahbubur Rahman, Ph.D.

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This thesis delves into three areas of research on dynamical systems. First, it explores the existence and exactness of Absolutely Continuous Invariant Measures (ACIM) for piecewise convex maps with countable (infinite) number of branches. Second, it employs Ulam's method to approximate the density function of these ACIMs. Third, it investigates the existence of Absolutely Continuous Invariant Measures for piecewise concave maps using the technique of conjugation.

For the first topic, we examine the existence and uniqueness of ACIMs within two distinct classes, denoted as $\mathcal{T}_{pc}^{\infty}(I)$ and $\mathcal{T}_{pc}^{\infty,0}(I)$, which together encompass piecewise convex maps $\tau : I = [0,1] \to [0,1]$ with countable number of branches. We establish the necessary conditions under which these maps possess a unique ACIM, presenting multiple illustrative examples of ACIM existence. Our findings are based on the analysis of the Frobenius-Perron operator associated with these maps, utilizing analytical techniques to gain insights into the Frobenius-Perron operator's properties.

The main purpose of the second part of this thesis is to approximate τ by the map τ_n, where we construct a sequence τ_n with a finite number of branches. Then, approximate τ_n by Ulam's method. Since piecewise convex maps have countable (infinite) number of branches, the convergence of Ulam's method becomes more challenging, and complexity makes it harder to find a suitable sequence of approximating functions that can accurately analyze the behavior of this system across all branches.

The primary contribution of this Ph.D. thesis lies in the generalization of the existence of absolutely continuous invariant measures for piecewise convex maps defined on an interval with an infinite number of branches. In the case of $\mathcal{T}_{pc}^{\infty}(I)$, we examine piecewise convex maps with an infinite

number of branches and arbitrary countable number of limit points for partition points separated from 0. For $\mathcal{T}_{pc}^{\infty,0}(I)$, we consider piecewise convex maps with countable number of branches and partition points that converge to 0. Throughout the thesis, we investigate Absolutely Continuous Invariant Measures (ACIM) for $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ and $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$, along with exploring non-autonomous dynamical systems of maps within these classes and scrutinize the existence of ACIMs for their limit maps.

Furthermore, we investigate the approximation for ACIMs associated with piecewise convex maps with an infinite number of branches by employing Ulam's method. This computational approach is a practical way to estimate the density functions of ACIMs and thereby facilitate their numerical analysis. We then extended our research area on ACIM for piecewise concave maps with countable number of branches. We examine the existence and uniqueness of ACIMs for two distinct classes, $\mathcal{T}_{pcv}^{\infty}(I)$ and $\mathcal{T}_{pcv}^{\infty,1}(I)$, which encompass piecewise concave mappings denoted as σ. We utilize the concept of conjugation with piecewise convex maps τ to demonstrate that σ conserves a normalized absolutely continuous invariant measure with a density that exhibits increasing behavior.

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Chapter 1

Introduction

In the realm of discrete dynamical systems, the focus lies on understanding the long-term patterns exhibited by trajectories as they evolve through the iteration of a map. The presence of chaos within deterministic dynamical systems introduces an inherent limitation in our ability to forecast the future behavior of these systems, given a particular set of initial conditions. Hence, it is natural to adopt a statistical perspective when describing the entire system's behavior. This approach seeks to characterize the dynamics by establishing the presence of an invariant measure and studying its ergodic properties. By considering the system from a statistical viewpoint, we focus on the long-term average behavior rather than attempting to predict individual trajectories. We aim to identify a measure that remains unchanged under the system's evolution, capturing its essential characteristics. This invariant measure provides a statistical description of the system's dynamics, enabling us to analyze its properties and make probabilistic predictions. Observing absolutely continuous invariant measures (ACIM) in computer simulations carries significant implications, such as in molecular dynamics simulations of a gas, simulations of a chaotic system like the double pendulum, quantum mechanical simulations, climate simulations, aerospace engineering, traffic simulations, and so on [5, 41]. It indicates that absolutely continuous invariant measures (ACIM) can adequately describe the system's behavior using probability densities and continuous distributions. In general, let X be a metric space, $\mathcal{B}$ be a $\sigma-$algebra of subsets of X, and μ be a measure on $\mathcal{B}$. The Birkhoff Ergodic theorem states that if $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic with respect to an invariant measure μ and E is a measurable subset of X then the orbit of almost every point of X occurs in the set

E with the asymptotic frequency $\mu(E)$. If the measure μ is absolutely continuous, this occurs for points forming a set of positive Lebesgue measure, so for a physically meaningful set of points. The Birkhoff Theorem does not say anything about the existence of invariant measures. The existence of an absolutely continuous invariant measure is one of the most important problems in the ergodic theory and dynamical systems. If a dynamical system possesses an absolutely continuous invariant measure, it often indicates that the system is indeed ergodic. The Frobenius-Perron operator plays an essential role in understanding the existence and properties of ACIM. This operator is a fundamental tool in ergodic theory, enabling the study of invariant measures and the long-term statistical behavior of dynamical systems.

In [32], Lasota and Yorke established the existence of absolutely continuous invariant measures for piecewise expanding maps. In [33], the authors investigated the exactness and the existence of ACIM for piecewise convex transformations with a finite number of branches with a strong repeller. In this context, the authors in [33] considered the following properties as fundamental properties for the proof of the existence of ACIM: (i) the F-P operator P_τ maps non-increasing functions to non-increasing functions; (ii) If $f : [0,1] \to \mathbb{R}^+$ is non-increasing, then $\| P_\tau f \|_\infty$ is bounded by $A \| f \|_\infty + B \| f \|_1$, where $A < 1$ and B are some constants. Similar results were also demonstrated in [5] for convex transformations with a finite number of branches. In [38], the author studied the ACIM for a piecewise convex map on $[0,1]$ with countable number of branches where 1 is the limit point of partition points. We generalize these results to more general classes of piecewise convex maps.

Recently, there has been a burgeoning interest in non-autonomous dynamical systems. In such systems, each map τ_n from the family $\{\tau_n\}_{n=1}^{\infty}$ applies at the n-th step within the system. Carvalho et al. [8], Cheban [9, 10, 11], Chepyzhov and Vishik [12], Haraux [24], Kloeden and Rasmussen [30] studied non-autonomous dynamical systems and their global attractors. The author [45] introduced the generalization of the Sinai-Ruelle-Bowen (SRB) (originally conceived in the 1970s) measure to non-autonomous systems. In [22, 17], P. Góra et al. studied the generalization of Krylov-Bogoliubov Theorem and Straube's Theorem for non-autonomous dynamical systems of continuous maps on a compact space. Furthermore, they investigated ACIMs of the limit map for non-autonomous dynamical systems of piecewise expanding maps.

S. M. Ulam suggested numerical computations of stationary densities of invariant measures for dynamical systems [42]. Ulam's method is one of the most used and the best-understood numerical methods for the approximation of stationary densities of absolutely continuous invariant measures for deterministic maps and random maps. For piecewise expanding deterministic transformations, T-Y Li [34] first proved the convergence of Ulam's approximation. Subsequently, researchers extended Ulam's method to encompass one-dimensional and higher-dimensional expanding deterministic transformations. For piecewise expanding interval maps, Bose and Murray presented the convergence rate of Ulam's method in [3]. In the context of higher-dimensional Jablonski transformations, Boyarsky and Lou proved the convergence of Ulam's method in [6]. For piecewise expanding and C^2 transformations, Ding and Zhou proved the convergence of Ulam's method in [16]. On random maps with constant probabilities, Froyland confirmed the convergence of Ulam's method and presented the rate of convergence in [19]. Góra and Boyasrsky proved the convergence of Ulam's method for position-dependent random maps in [5]. In [35], Miller proved the convergence of Ulam's method for piecewise convex transformations with a finite number of branches with a strong repeller. J. Ding [14] developed and presented piecewise linear and piecewise quadratic Markov finite approximation methods for piecewise convex maps with a finite number of branches. If piecewise convex maps have countable (infinite) number of branches, the convergence of Ulam's method becomes more challenging and complex. This complexity makes it harder to find a suitable sequence of approximating functions that can accurately capture the behavior of this system across all branches. In [20], the author presented a class of maps with countable (infinite) number of branches without any absolutely continuous invariant measure. In [21], Góra and Boyarsky presented an approximation method for invariant measures for piecewise continuous maps with countable number of branches. Here, we consider countable number of branches to mean an infinite number of branches. A set is countably infinite if its elements can be put into one-to-one correspondence with the natural numbers. For instance, the set of all natural numbers $\{1, 2, 3, 4, ...\}$ is countably infinite because we can list its elements one after the other, and each natural number corresponds to a unique element in the set.

The main objective of this thesis is the study of the existence and exactness of absolutely continuous invariant measures for piecewise convex maps with countable number of branches. We also investigate the existence of ACIMs of limit maps for a non-autonomous dynamical system of piecewise convex maps with countable (infinite) number of branches. We explore numerical methods for approximating ACIMs for piecewise convex maps with countable (infinite) number of branches by applying Ulam's method. While there are several results on piecewise convex maps with a finite number of branches, there is only one work about such maps with an infinite number of branches [38]. The existence and approximation of ACIMs of piecewise concave maps with an infinite number of branches are also studied. In most applications, the Lebesgue measure is the predominant choice. When we opt for a singular measure, it often renders actual points imperceptible. While such measures have theoretical existence, they typically lack practical relevance. In practice, we commonly rely on the Lebesgue measure and frequently work with measures that are absolutely continuous with respect to the Lebesgue measure.

In Chapter 3, we scrutinize the ACIMs for two classes, $\mathcal{T}_{pc}^{\infty}(I)$ and $\mathcal{T}_{pc}^{\infty,0}(I)$, of piecewise convex maps with countable (infinite) number of branches. We study absolutely continuous invariant measures of maps τ in the first class $\mathcal{T}_{pc}^{\infty}(I)$, where $\tau : I = [0,1] \to [0,1]$ has a countable number of branches with an arbitrary countable number of limit points of partition points separated from 0. For the second class $\mathcal{T}_{pc}^{\infty,0}(I)$, we assume: there exists a countable (infinite) partition $\{0 = a_0 < \cdots < a_{0,-n} < a_{0,-(n-1)} < \cdots < a_{0,-2} < a_{0,-1} = a_1, a_2, a_3, \ldots, a_n, \ldots\}$ of $I = [0,1]$ with $\lim_{n\to\infty} a_{0,-n} = 0$. Here, we also consider non-autonomous dynamical systems of maps in $\mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$ and study the existence of acim of the limit map. We give several examples of piecewise convex maps with a countable number of branches.

In the next Chapter, i.e., Chapter 4, we use Ulam's method for approximation of f^* where f^* is the actual stationary density of absolutely continuous invariant measure μ for the piecewise convex map τ with countable number of branches. Ulam's method does not guarantee uniqueness in the approximation of f^*, but when we deal with exactness, the density is unique (almost everywhere) with respect to the Lebesgue measure. We construct a sequence $\{\tau_n\}_{n=1}^{\infty}$ of piecewise convex maps with a finite number of branches such that $\tau_n \to \tau$ almost uniformly. We apply Ulam's method to τ_n and compute an approximation $f_{n,k}$ of the actual density f_n of τ_n and prove that $f_{n,k} \to f_n$ as $k \to \infty$. Finally, we prove that $f_{n,k} \to f^*$ as $n \to \infty, k \to \infty$. It's important to note that our notion of "approximation" does not rely on any specific norm. In this sense, approximation means $\tau_n \to \tau$

almost uniformly. We also illustrate by numerical examples.

In Chapter 5, we extend our research area on absolutely continuous invariant measures for piecewise concave maps with countable number of branches. We investigate the existence and approximation of ACIMs for two classes, $\mathcal{T}_{pcv}^{\infty}(I)$ and $\mathcal{T}_{pcv}^{\infty,1}(I)$, of piecewise concave maps σ with a strong repellor. We give some examples of piecewise concave maps with countable (infinite) number of branches and exhibit that if any convex maps have an ACIM, then piecewise concave maps has also an ACIM. One fascinating aspect of piecewise concave maps is that they can be conjugated to piecewise convex maps on [0, 1], which allows us to use results from the theory of piecewise convex maps to study their properties.

Chapter 2

Preliminaries

2.1 Review of Necessary Facts for Dynamical System

2.1.1 Review of Measure Theory

In this Section, we recap some important definitions and theorems of measure theory. The interested reader may consult the books " Chapter 2, [40] or Chapter 1, [19]". Most of the material can be found in [18] and [5].

Let $(X, \mathcal{B}, \mu)$ be a measure space where X is a non-empty set, $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{B}$. We call it a **probability space** or *normalized measure space* if $\mu(X) = 1$. If X is a countable union of sets of finite measure, then we say that μ is a ***σ-finite*** measure.

Definition 2.1.1. Let ν and μ be two measure on same measure space $(X, \mathcal{B})$. We say that ν is ***absolutely continuous*** with respect to μ, denoted by $\nu \ll \mu$, if for any $E \in \mathcal{B}$

$$\mu(E) = 0 \Longrightarrow \nu(E) = 0.$$

For absolutely continuous measures, the following theorem is useful:

Theorem 2.1.2. *[18] $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta \implies \nu(E) < \epsilon$.*

If $\nu \ll \mu$, then it is possible to represent ν in terms of μ. Now, we want to state the Radon-Nikodym theorem, which is related to absolutely continuous measures.

Theorem 2.1.3. *[18] Let $(X, \mathcal{B})$ be a measure space and let ν and μ be two σ-finite positive normalized measures on $(X, \mathcal{B})$. If $\nu \ll \mu$, then there exists a unique $f \in L^1(X, \mathcal{B}, \mu)$ such that for every $A \in \mathcal{B}$,*

$$\nu(A) = \int_A f d\mu.$$

The function f is called the Radon-Nikodym derivative, $d\nu/d\mu$, or a density of ν with respect to μ. So, the Radon-Nikodym theorem states that if ν is absolutely continuous with respect to μ, and both measures are σ-finite, then ν has a density, or "Radon-Nikodym derivative," with respect to μ. The Frobenius-Perron operator (see page 12, Section 2.3) serves as a bridge between the dynamics of the transformation and the associated Radon-Nikodym derivative or density function [43], which encapsulates how the transformation affects the probability distribution over its state space.

Definition 2.1.4. Let $(X, \mathcal{B}, \mu)$ be a normalized measure space.
Let $\mathcal{D}(I, \mathcal{B}, \mu) = \{f \in L^1(\mu) : f \in\geq 0 \text{ and } ||f||_1 = 1\}$ denote the space of probability density functions. A function $f \in \mathcal{D}(I, \mathcal{B}, \mu)$, then $\mu_f(A) = \int f d\mu \ll \mu$ is a measure and f is called the density of μ_f and is written as $\frac{d\mu_f}{d\mu}$.

2.2 Overview of Ergodic Theory with Measure-Preserving Transformations

Ergodic theory deals with studying the long-term statistical behavior of dynamical systems, particularly those that exhibit chaotic or random-like properties. However, the presence of chaos renders it impossible for deterministic dynamical systems to accurately predict their long-term behavior from any specific set of initial conditions. Despite this, it is still possible to derive statistical conclusions regarding chaotic systems using ergodic theory. For a more detailed understanding of ergodicity and its applications in chaos theory, interested individuals may consult works such as "Chapter 3, [5] or Chapter 1, [13] (1980 in English translation)".

Let $(X, \mathcal{B}, \mu)$ as a normalized measure space.

Definition 2.2.1. A transformation $\tau : X \to X$ is said to be measurable on a measure space $(X, \mathcal{B}, \mu)$ if $\tau^{-1}(\mathcal{B}) \subset \mathcal{B}$, i.e., $A \in \mathcal{B} \implies \tau^{-1}(A) \in \mathcal{B}$, where $\tau^{-1}(A) = \{x \in X : \tau(x) \in A\}$. Additionally, μ is said to be ***τ-invariant*** or τ preserves measure μ if for every $A \in \mathcal{B}$,

$$\mu(\tau^{-1}(A)) = \mu(A). \tag{2.2.2}$$

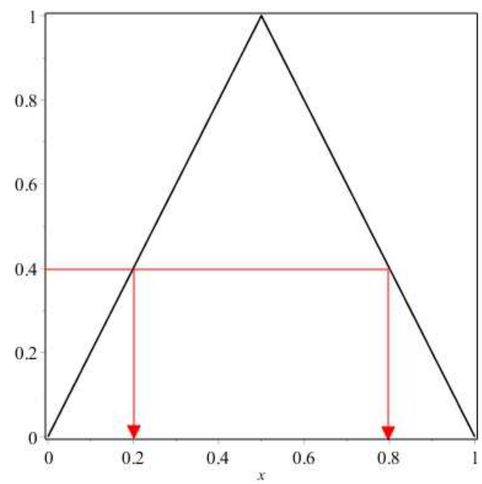

Figure 2.1: Tent map τ for equation 2.2.3.

For example, consider a transformation $\tau : [0,1] \to [0,1]$ defined as

$$\tau(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}) \\ 2 - 2x, & x \in [\frac{1}{2}, 1] \end{cases}. \tag{2.2.3}$$

Suppose $A = [0, 0.4]$, so $\tau^{-1}(A) = [0, 0.2] \cup [0.8, 1]$.

$\mu(A) = \mu([0, 0.4]) = 0.4$ and $\mu([0, 0.2] \cup [0.8, 1]) = 0.2 + 0.2 = 0.4$.

Therefore, $\mu(\tau^{-1}(A)) = \mu(A)$.

If the precise information regarding all the members of Borel set $\mathcal{B}$ is unavailable, it can be challenging to verify whether τ preserves a measure. Employing a $\pi-$system can provide a valuable approach to determining whether τ preserves a measure.

Definition 2.2.4. A family $\mathcal{P}$ of subsets of X is called a ***π-system*** if and only if for any $A, B \in \mathcal{P}$, $A \cap B$ is also in $\mathcal{P}$.

Theorem 2.2.5. *[5] Let $\tau : X \to X$ be a measurable transformation on a normalized measure space $(X, \mathcal{B}, \mu)$. Let $\mathcal{P}$ be a π-**system** that generates $\mathcal{B}$. If $\mu(\tau^{-1}(A)) = \mu(A)$ for any $A \in \mathcal{P}$, then τ preserves measure μ.*

Definition 2.2.6. Let $(X, \mathcal{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ preserve μ. The quadruple $(X, \mathcal{B}, \mu, \tau)$ is called a ***dynamical system***.

In the study of dynamical systems, the primary concern is the investigation of the properties exhibited by the sequence of points $\{\tau^n(x)\}_{n \geq 0}$ called the orbit or the trajectory of the point x. The nth iterate of τ is denoted by τ^n i.e.

$$\tau^n(x) = \underbrace{\tau \circ \tau \circ ... \circ \tau(x)}_{n\text{-times}}.$$

If τ has an invariant measure, then the orbit starting in a specified set, returns to that initial set (state) infinitely many times. The Poincaré Recurrence Theorem exactly tells us this.

Theorem 2.2.7. (Poincaré Recurrence Theorem)*[5] Let $(X, \mathcal{B}, \mu)$ be a normalized measure space and $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving transformation. Let $E \in \mathcal{B}$ be such that $\mu(E) > 0$. Then, almost all points of E return infinitely often to E under iteration of τ.*

Definition 2.2.8. A measure-preserving transformation $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic if for any invariant set $A \in \mathcal{B}$, such that $\tau^{-1}(A) = A$, $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

So, we can say that a dynamical system is ergodic if it is indecomposable, that is if every invariant measurable set has a measure 0 or 1 [1].

Definition 2.2.9. Let $(X, \mathcal{B}, \mu, \tau)$ be a dynamical system. A set $B \in \mathcal{B}$ is called

(i) τ-invariant if $\tau^{-1}(B) = B$,

(ii) almost τ-invariant if $\mu(\tau^{-1}B \triangle B) = 0$.

Here $\tau^{-1}B \triangle B = (\tau^{-1}B \setminus B) \cup (B \setminus \tau^{-1}B)$ and $\triangle$ - the symmetric difference of sets.

Similarly, a measurable function is called τ-invariant if $f \circ \tau = f$ and almost τ-invariant if $f \circ \tau = f$ μ-a.e.

The ***Birkhoff Ergodic Theorem*** is one of the cornerstones of ergodic theory. This theorem says the time averages along the trajectories are equal to the space averages [13].

Theorem 2.2.10. *[5] Let transformation* $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ *be a measure-preserving, where* $(X, \mathcal{B}, \mu)$ *is* σ*-finite, and* $f \in L^1(\mu)$*. Then for almost every* $x \in X$*, there exists a function* $f^* \in L^1(\mu) \left(= \int_X |f| d\mu < \infty\right)$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) \ = f^*, \ \ \mu - a.e.$$

Furthermore, $f^* \circ \tau = f^*$ μ*- a.e. and if* $\mu(X) < \infty$*, then*

$$\int_X f^* d\mu = \int_X f d\mu.$$

The function f^* is invariant i.e.

$$f^*(\tau^n(x)) = f^*(x), \qquad n \geqslant 0$$

Corollary 2.2.11. *[5] If* τ *is ergodic, then* f^* *is constant* μ*-a.e. and if* $\mu(X) < \infty$*, then*

$$f^* = \frac{1}{\mu(X)} \int_X f d\mu \qquad a.e.$$

Thus, if $\mu(X) = 1$ *and* τ *is ergodic and* $f = \chi_E$ *where* $E \in \mathcal{B}$*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_E(\tau^k(x)) = \mu(E), \qquad \mu - a.e.,$$

and thus the orbit of almost every point of X *occurs in the set* E *with the asymptotic relative frequency* $\mu(E)$*.*

If τ *is ergodic, then the above Corollary 2.2.11 states that the time average equals the space average and convergence is also true, i.e.,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^k(x)) = \frac{1}{\mu(X)} \int_X f(x) d\mu.$$

Definition 2.2.12. Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a transformation preserving the measure μ.

(i) τ is ***ergodic*** if and only if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\tau^{-k} A \cap B) = \mu(A)\mu(B)$$

(ii) τ is ***weakly mixing*** if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(\tau^{-k} A \cap B) - \mu(A)\mu(B)| = 0$$

(iii) τ is ***strongly mixing*** if for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(\tau^{-n} A \cap B) = \mu(A)\mu(B)$$

(iv) τ is ***exact*** if for every $A \in \mathcal{B}$, $\mu(A) > 0$, and $\tau(A) \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(\tau^{n}(A)) = 1.$$

Moreover, the exactness of τ implies that τ is strongly mixing, but the converse is not generally true.

Now, we introduce the functions of bounded variation. Let $[a, b] \subset \mathbb{R}$ be a bounded interval and let λ denote Lebesgue measure on $[a, b]$. We define $\mathcal{P} = \{I_i = [x_{i-1}, x_i) : i = 1, 2, \ldots, n\}$ a partition of $[a, b]$. The points $\{x_0, x_1, \ldots, x_n\}$ are called end-point of the partition $\mathcal{P}$.

Definition 2.2.13. Let $f : I = [a, b] \longrightarrow \mathbb{R}$ and let $\mathcal{P} = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of $I = [a, b]$.

(i) f is called of bounded variation on $[a, b]$ if there is $M \in \mathbb{R}^+$ such that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M$$

for all partitions $\mathcal{P}$.

(ii) $V_I f$ is called total variation or, the variation of f on I

$$V_I f = \sup_{\mathcal{P}} \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\}.$$

Here $V_I(\cdot)$ denotes the variation of a function on $[0, 1]$ and BV(I) is the space of function of bounded variation on I equipped with the norm

$$\| \cdot \|_{BV} = V_I(\cdot) + \| \cdot \|_1,$$

where $\| \cdot \|_1$ denotes the norm on $L^1(I, \mathcal{B}, \mu)$.

Recall the definitions of τ-invariance of ν and $\nu \ll \mu$. If ν satisfies these properties, then we say that ν is absolutely continuous invariant measure for τ on $(X, \mathcal{B}, \mu)$.

2.3 The Frobenius-Perron Operator

The Frobenius-Perron Operator is a linear operator that determines the transformation of density [37]. This operator was first introduced by Kuzmin R. O. [5]. The Frobenius-Perron operator plays an essential role in the existence of acim. We define non-singular transformation and then define the Frobenius-Perron operator.

Definition 2.3.1. Let $(X, \mathcal{B}, \mu)$ be a measure space and $\tau : X \to X$ be a measurable transformation on $(X, \mathcal{B}, \mu)$. Then τ is called non-singular if $\mu(\tau^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ such that $\mu(A) = 0$.

Definition 2.3.2. Let $(I, \mathcal{B}, \lambda)$ be a measure space and $\tau : I \to I$ be a non-singular transformation on $(I, \mathcal{B}, \lambda)$. Let λ be the normalized Lebesgue measure on I and let $\mu \ll \lambda$, where f is the density for μ. The operator $P_\tau : L^1 \to L^1$ called the ***Frobenius-Perron operator*** associated with τ is defined by

$$\int_A P_\tau f \; d\lambda = \int_{\tau^{-1}(A)} f \; d\lambda, \; \forall A \in \mathcal{B}, \forall f \in L^1(\mu).$$

Here, the Lebesgue measure, λ, is normalized if and only if it is on an interval of length 1. In any other case, we can speak about a measure equivalent to the Lebesgue measure. In particular, the Lebesgue measure of $\mathbf{R}$ is infinity, and we cannot normalize it.

Let $A = [a, x] \subset I$. We obtain

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}(A=[a,x])} f \; d\lambda \; a.e.$$

and if τ has countable number of monotonic branches then P_τ has the explicit representation [5]:

$$P_\tau f(x) = \sum_{w \in \{\tau^{-1}(x)\}} \frac{f(w)}{|\tau'(w)|}. \tag{2.3.3}$$

Note that: $(\tau^{-1}(x))' = \frac{1}{\tau'(\tau^{-1}(x))}$. For any value of x, the set $\{\tau^{-1}(x)\}$ consists of at most countably many points. Here is the short proof of the equation (2.3.3).

Since τ is monotonic on each $(a_{i-1}, a_i), i = 1, 2, \dots,$ we define an inverse function for each $\tau|_{(a_{i-1}, a_i)}$.

Let $\phi_i = \tau^{-1}|_{B_i}$, where $B_i = \tau([a_{i-1}, a_i])$.

Then $\phi_i : B_i \to [a_{i-1}, a_i]$ and $\tau^{-1}(A) = \cup_{i=1}^{\infty} \phi_i(B_i \cap A)$, where the sets $\{\phi_i(B_i \cap A)\}_{i=1}$ are mutually disjoint.

Now,

$$\int_A P_\tau f d\lambda = \sum_{i=1}^{\infty} \int_{\phi_i(B_i \cap A)} f d\lambda \quad = \quad \sum_{i=1}^{\infty} \int_{(B_i \cap A)} f(\phi_i(x)) |\phi_i'(x)| d\lambda$$

where we have used the change of variable formula for each i. We obtain,

$$\int_A P_\tau f d\lambda = \sum_{i=1}^{\infty} \int_A f(\phi_i(x)) |\phi_i'(x)| \chi_{B_i}(x) d\lambda \quad = \quad \int_A \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1}, a_i)}(x) d\lambda$$

Since A is arbitrary,

$$P_\tau f(x) = \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1},a_i)}(x)$$

for any $f \in L^1$.

The existence and uniqueness of $P_\tau f$ are established through the use of the Radon-Nikodym Theorem. The Radon-Nikodym Theorem ensures the existence and uniqueness of the Radon-Nikodym derivative under certain conditions. These conditions typically involve the absolute continuity of the probability density function (PDF) with respect to the measure μ and the measure-preserving property of the transformation τ. When these conditions are met, the Frobenius-Perron operator $P_\tau f$ exists and is unique. One of the main properties of $P_\tau f$ is that its fixed points are the densities of invariant measures under τ [5].

Some important ***properties***[5] of the Frobenius-Perron operator:

Let $P_\tau : L^1 \to L^1$ be the Frobenius-Perron operator associated with τ. Then

(i) Linearity: P_τ is a linear operator. Let $f, g \in L^1$, and α, β be constants. Then,

$$P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g.$$

(ii) Positivity: If $f \in L^1$ and $f \geq 0$, then $P_\tau f \geq 0$.

(iii) Preservation of Integral: $\int_I P_\tau f d\lambda = \int_I f d\lambda$.

(iv) Contraction: P_τ is contraction, i.e., $\|P_\tau f\|_1 \leq \|f\|_1 \qquad$ for any $f \in L^1$.

(v) Composition: Let $\tau_1, \tau_2 : I \to I$ be non-singular transformations. Then

$$P_{\tau_1 \circ \tau_2} f = P_{\tau_1} \circ P_{\tau_2} f.$$

In particular, $P_{\tau^n} = P_\tau^n$ for any integer $n \geq 1$.

The Frobenius-Perron operator is an adjoint operator of the **Koopman operator**. The basic idea is to use a suitable change of variables that transforms the Koopman operator into a form that is equivalent to the Frobenius-Perron operator. Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measurable transformation. The operator $U_\tau : L^\infty \to L^\infty$ defined by

$$U_\tau f = f \circ \tau$$

is called the Koopman operator and $\|U_\tau f\|_\infty \leq \|f\|_\infty$, for any $f \in L^\infty$.
Here, we show that $||U_\tau f||_\infty \leq ||f||_\infty$.
Since the Koopman operator $U_\tau : L^\infty \to L^\infty$ defined by

$$U_\tau f = f \circ \tau.$$

Let for any $x \in I = [0, 1]$, then we have $||f||_\infty = \sup_{x \in I} |f| = 1$.
Now $||U_\tau f||_\infty = \sup_{x \in I} |(U_\tau f)(x)| = \sup_{x \in I} |f(\tau(x))|$.
Let for any $x \in I, |f(\tau(x))| \leq ||f||_\infty = 1$. Since this inequality holds for all $x \in I$, we have

$$||U_\tau f||_\infty = \sup_{x \in I} |f(\tau(x))| \leq 1.$$

Thus for any $f \in L^\infty$ with $||f||_\infty = 1$, we have shown that $||U_\tau f||_\infty \leq 1$.

Proposition 2.3.4. *[5] Let $(I, \mathcal{B}, \mu)$ be a normalized measure space and let $\tau : I \to I$ be a non-singular transformation. If μ is a τ - invariant measure, then*

(i) $\|P_\tau f\|_p \leq \|f\|_p$, when $1 \leq p \leq +\infty$.

(ii) τ is exact $\Longleftrightarrow$ for any $f \in \mathcal{D}(I, \mathcal{B}, \mu)$,

$$\lim_{n \to \infty} P^n_{\tau,\mu} f = \mathbf{1}$$

where $\mathcal{D}(I, \mathcal{B}, \mu)$ denotes the probability density functions on the measure space $(I, \mathcal{B}, \mu)$ and $\mathbf{1}$ is *a constant function equal to 1 everywhere.*

The following proposition tells us why we need the Frobenius-Perron operator for the existence of invariant measures.

Proposition 2.3.5. *[5] Let $\tau : I \longrightarrow I$ be non-singular and $\mu \ll \lambda$. Then*

$$P_\tau f^* = f^* \ \ a.e. \Longleftrightarrow \mu(A) = \int_A f^* d\lambda,$$

is a τ-invariant measure.

This proposition tells us a density function f^* is a fixed point of Frobenius-Perron operator P_τ if and only if it is the density of a τ-invariant measure μ, absolutely continuous with respect to a measure λ.

Now, we consider $I = [0, 1]$ with normalized Lebesgue measure λ on I. Let $\mathcal{T}(I)$ denotes the class of transformation $\tau : I = [0, 1] \to I$ that satisfy the following conditions:

(1) τ is piecewise monotonic and expanding, i.e. there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i), i = 1, 2, \ldots, N\}$ of I such that $\tau|_{[a_{i-1}, a_i)}$ is C^1 and $|\tau_i'(x)| \geq \alpha > 1$ for any i and for all $x \in (a_{i-1}, a_i)$;

(2) $\frac{1}{\tau'(x)}$ is a function of bounded variation, where $\tau'(x)$ is the appropriate one-sided derivative at the end points of $\mathcal{P}$.

Comment: Since τ is piecewise monotonic and expanding, so for any i, $\tau_i'(x) \geq \alpha > 1$. Thus $\frac{1}{\tau'(x)} < 1$. If it is not then we don't have an ACIM. If $\tau_i'(x) \leq 1$ then we have no ACIM and it will have a attracting fixed point and attracting fixed point has no ACIM.

Lasota - Yorke theorem [32] provides the existence of absolutely continuous invariant measures for a class of point transformation of the unit interval $[0, 1]$ to itself. Originally, Lasota and Yorke assumed piecewise C^2 instead of assumption (2).

Theorem 2.3.6. (Lasota - Yorke) *Let $\tau \in \mathcal{T}(I)$. Then for any $f \in L^1[0, 1]$ the sequence*

$$\frac{1}{n} \sum_{i=1}^{n} P_\tau^i f$$

is convergent in norm to $f^* \in L^1[0,1]$. The limit function has the following properties:

(1) $f \geq 0 \implies f^* \geq 0$.

(2) $\int_0^1 f^* d\lambda = \int_0^1 f d\lambda$.

(3) $P_\tau f^* = f^*$ and consequently the measure $d\mu^* = f^* d\lambda$ is invariant under τ.

(4) The function $f^* \in BV[0,1]$. Moreover, there exists a constant β independent to the choice of initial f such that

$$V_{[0,1]} f^* \leq \beta \|f\|_1.$$

Lemma 2.3.7. *Let $\tau \in \mathcal{T}(I)$. Then there exist constants $0 < \alpha < 1, C > 0$, and $R > 0$ such that for any $f \in BV(I)$ and any $n \geq 1$,*

$$\|P_\tau^n f\|_{BV} \leq C\alpha^n \|f\|_{BV} + R\|f\|_1.$$

The inequality above is called the Lasota-Yorke inequality, and different versions of this inequality play essential roles in the theory of absolutely continuous invariant measures. Below, we will use a similar inequality but with different norms, since our transformations are not necessarily piecewise expanding.

2.4 Piecewise convex transformations

Definition 2.4.1. Let $I = [0,1]$. A transformation $\tau : I \longrightarrow I$ is convex if $\forall x, y \in I$, and $\eta \in [0,1]$, we have

$$\tau(\eta x + (1-\eta)y) \leq \eta\tau(x) + (1-\eta)\tau(y).$$

A function is strictly convex if $\tau(\eta x + (1-\eta)y) < \eta\tau(x) + (1-\eta)\tau(y)$ whenever $x \neq y$. A convex transformation is continuous on $(0,1)$ and thus measurable with respect to the Lebesgue measure.

Let $I = [0,1]$ and $\mathcal{T}_{pc}(I)$ be the class of transformations $\tau : I \longrightarrow I$ that satisfy the following conditions:

(i) there exists a partition $\mathcal{P} = \{0 = a_0 < \cdots < a_N = 1\}$ of I such that $\tau_{[a_{i-1},a_i)}$ is continuous, and convex, $i = 1, \ldots, N$;

(ii) $\tau(a_{i-1}) = 0, \tau'(a_{i-1}) > 0, \;\; i = 1, \ldots, N$;

(iii) $\tau'(0) = \alpha > 1$.

Transformations in $\mathcal{T}_{pc}(I)$ are called piecewise convex maps with strong repellers.

A convex function is differentiable except at a countable set of points, and its derivative τ' is non-decreasing. In particular, this means that (ii) implies

$$\tau'(x) > \tau'(a_{i-1}) > 0, \;\; x \in [a_{i-1}, a_i)$$

and $\tau|_{[a_{i-1},a_i)}$ is increasing for $i = 1, \ldots, N$.

Lasota and Yorke [33] proved the existence of an ACIM with respect to the Lebesgue measure for one-dimensional piecewise convex maps with a strong repellor. The following propositions, Lemma and Theorem, are proved in [5].

Definition 2.4.2. A set S is countably infinite if S has a one-to-one correspondence with $\mathbb{N}$ i.e., the elements of S can be arranged in an infinite sequence $a_0, a_1, a_2, \ldots,$ where a_i is distinct from a_j for $i \neq j$ and every element of S is listed.

Proposition 2.4.3. *[5] Let $\tau \in \mathcal{T}_{pc}(I)$ and f be a non-increasing function. Then, $P_\tau(f)$ is also non-increasing.*

Lemma 2.4.4. *If $f \geq 0$ and f is non-increasing, then $f(x) \leq \frac{1}{x}\lambda(f)$, for $x \in [0,1]$, where*

$$\lambda(f) = \int_{I=[0,1]} f d\lambda.$$

Proposition 2.4.5. *Let $\tau \in \mathcal{T}_{pc}(I)$. If $f : [0,1] \to \mathbb{R}^+$ is non-increasing, then*

$$||P_\tau f||_\infty \leq \frac{1}{\alpha}||f||_\infty + C||f||_1,$$

where $C = \sum_{i=2}^{r}(a_{i-1} \cdot \tau'(a_{i-1}))^{-1}$.

Theorem 2.4.6. *[5] Let $\tau \in \mathcal{T}_{pc}(I)$. Then τ admits an absolutely continuous invariant measure, $\mu = f^*\lambda$, and the density f^* is non-increasing.*

Theorem 2.4.7. *[33] Let $\tau : I = [0,1] \to [0,1]$ and $\tau \in \mathcal{T}_{pc}$. Then, there exists the unique normalized absolutely continuous measure $\mu = g\lambda$ that is invariant under τ. The system $(I, \mathcal{B}, \mu; \tau)$ is exact and the density g is bounded and decreasing. Moreover, $\lim_n P_\tau^n f = g$ in $L^1(I, \lambda)$, for any $f \in L^1(I, \lambda)$ where λ is the Lebesgue measure on $[0,1]$ and P_τ is the Frobenius-Perron operator corresponding to τ.*

2.5 Non-autonomous Dynamical Systems

Definition 2.5.1. [22] Let (X, μ) be a compact metric space and $\tau_n : X \to X$ be a sequence of maps such that τ_n converges uniformly to a limit map τ, where τ is a continuous map. Then, the non-autonomous dynamical system is defined by

$$x_{m+1} = \tau_m(x_m), \quad m = 0, 1, 2, ...$$

where τ_0 is the identity and $x_0 \in X$. When the system starts at x_0, the first iteration, x_1, is given by $\tau_0(x_0)$, and since τ_0 is the identity map, $x_1 = \tau_0(x_0) = x_0$. This ensures that the initial condition is preserved after the first iteration.

For $n > m$, we write

$$\tau_{(m,n)} = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_{m+1} \circ \tau_m.$$

In particular,

$$\tau_{(0,n)} = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_1 \circ \tau_0.$$

A trajectory of a point x in the phase space is $x, \tau_1(x), \tau_2(\tau_1(x)), \ldots$. Let $T_n = \tau_n \circ \tau_{n-1} \circ$

$\dots \tau_2 \circ \tau_1$. Then the above trajectory of the non-autonomous dynamical system can be written as $x, T_1(x), T_2(x), \dots, T_n(x), \dots$.

The Krylov–Bogoliubov Theorem is a powerful tool in the domain of non-autonomous dynamical systems because it addresses the challenges posed by explicit time dependence and provides conditions under which solutions exist and are unique.

Theorem 2.5.2. (Krylov-Bogoliubov Theorem) *[5] Let X be a compact metric space and let the transformation $\tau : X \to X$ be continuous. Then there exists a $\tau-$ invariant normalized measure on X.*

Note that: Krylov-Bogoliubov Theorem does not tell about absolute continuity. But this result establishes that every continuous transformation on a compact metric space is guaranteed to have an invariant measure.

Theorem 2.5.3. (Extension of the Krylov-Bogoliubov Theorem) *[22] Let $\{\tau_n\}$ be a sequence of transformations defining a non-autonomous dynamical system on the compact metric space X with a continuous limit τ. We assume that the τ_n's converge uniformly to τ. Let η be a fixed probability measure on X. Define the measures $\mu_n = \frac{1}{n}\sum_{i=1}^{n} \nu_i$, where $\nu_i = (\tau_{(0,i)})_* \eta$. Let μ be a $*$- weak limit point of the sequence $\{\mu_n\}_{n\geq 1}$. Then μ is a τ- invariant measure, i.e., $\tau_* \mu = \mu$.*

Straube Theorem provides a sufficient condition for μ to be absolutely continuous.

Theorem 2.5.4. (Straube Theorem) *[22] Let $(X, \mathcal{B}, \nu)$ be a normalized measure space and let $\{\tau_n\}$ be a sequence of non-singular transformations defining a non-autonomous dynamical system on X. We do not assume that the limit τ is continuous. Assume there exists $\delta > 0$ and $0 < \alpha < 1$ such that*

$$\nu(E) < \delta \implies \sup_{k\geq 1} \nu\left(\tau_{(0,k)}^{-1}(E)\right) < \alpha \quad ,$$

for all $E \in \mathcal{B}$. Then there exists a $\tau-$ invariant normalized measure μ which is absolutely continuous with respect to ν.

2.6 Markov transformations and approximation of invariant measure of dynamical systems by Ulam's method

Markov transformation is a piecewise monotonic transformation such that each interval of the partition is mapped onto a union of intervals of the partition. The Frobenius-Perron operator can be defined in terms of the Markov transformation matrix. To approximate the fixed point of the Frobenius-Perron operator P_τ, we use the fixed point of a matrix operator known as the Markov operator. S. Ulam introduced Ulam's conjecture [2] as part of a comprehensive set of intriguing open problems in applied mathematics, one of which pertained to the approximation of Frobenius-Perron operators.

Conjecture 2.6.1. Ulam's Conjecture: *[2]*

- *A finite rank approximation of the Frobenius-Perron operator by Eq. (2.6.4); and*
- *the conjecture that the dominant eigenvector (corresponding to eigenvalue equal to 1) weakly approximates the invariant distribution of the Frobenius-Perron operator.*

Definition 2.6.2. Let $\tau : I = [0,1] \to [0,1]$ and let $\mathcal{P} = \{0 = a_0 < a_1 < \cdots < a_n = 1\}$ be a partition of I. Let $I_i = (a_{i-1}, a_i), i = 1, 2, \ldots, n$ and $\tau_i = \tau|_{I_i}$. If τ_i is homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, i.e., some interval $\left(a_{j(i)}, a_{k(i)}\right)$, then τ is said to be Markov transformation. The partition $\mathcal{P}$ is said to be a Markov partition with respect to the function τ. If each τ_i is linear on I_i, then τ is a piecewise linear Markov transformation.

Theorem 2.6.3. *[5] Let $\tau : I = [0,1] \to [0,1]$ be a piecewise linear Markov transformation on the partition $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$. Then there exists an $n \times n$ matrix $\mathbf{M}_\tau$ such that $P_\tau f = \mathbf{M}_\tau f$ for every piecewise constant function $f = (f_1, f_2, \ldots, f_n)$ on the partition $\mathcal{P}$. The matrix $\mathbf{M}_\tau = (m_{ij})_{1 \le i,j \le n}$ is defined*

$$m_{ij} = \frac{a_{ij}}{|\tau_i'|} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}, 1 \le i, j \le n, \tag{2.6.4}$$

where $(a_{ij})_{1 \le i,j \le n}$ is the incidence matrix induced by τ and $\mathcal{P}$. Here λ denotes the normalized Lebesgue measure on I and $\{I_i\}_{i=1}^{k}$ is a finite family of connected sets with nonempty and adjoint *interiors that cover I i.e., $I = \cup_{i=1}^{k} I_i$, and indexed in terms of nested refinements.*

Example 2.6.5. Let $\tau : [0,1] \to [0,1]$ be a piecewise linear Markov transformation on the partition $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ defined by

$$\tau(x) = \begin{cases} 2x + \frac{1}{2}, & x \in 0 \leq x \leq \frac{1}{4} \\ -x + \frac{5}{4}, & x \in \frac{1}{4} < x \leq \frac{1}{2} \\ -3x + \frac{9}{4}, & x \in \frac{1}{2} < x \leq \frac{3}{4} \\ 4x - 3, & x \in \frac{3}{4} < x \leq 1 \end{cases}.$$

Here we use equation (2.6.4) for finding the all elements of the matrix $\mathbf{M}_\tau$.

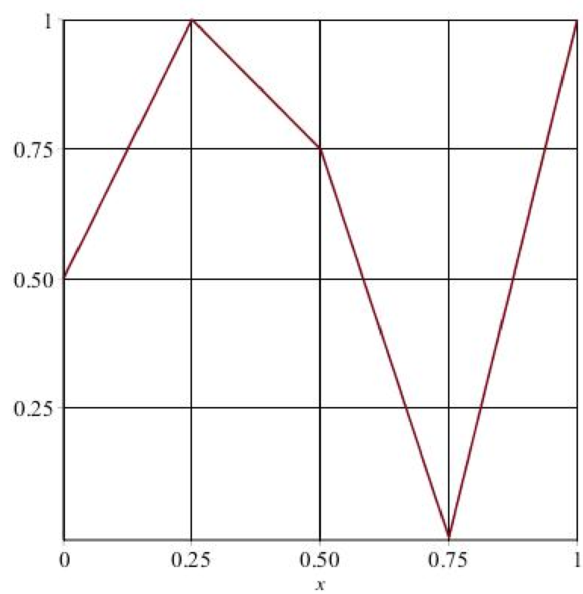

Figure 2.2: The map τ for Example 2.6.5.

Now using $i, j = 1, 2, 3, 4$ successively, then

$$m_{11} = \frac{\lambda\left(I_1 \cap \tau^{-1}(I_1)\right)}{\lambda(I_1)} = \frac{\lambda([0, \frac{1}{4}] \cap \tau^{-1}([0, \frac{1}{4}]))}{\lambda([0, \frac{1}{4}]} = \frac{\lambda([0, \frac{1}{4}] \cap ([-\frac{1}{8}, -\frac{1}{4}]))}{\lambda([0, \frac{1}{4}]} = 0$$

$$m_{13} = \frac{\lambda\left(I_1 \cap \tau^{-1}(I_3)\right)}{\lambda(I_1)} = \frac{\lambda([0, \frac{1}{4}] \cap ([0, \frac{1}{8}]))}{\lambda([0, \frac{1}{4}]} = \frac{1}{2}$$

$$m_{14} = \frac{\lambda\left(I_1 \cap \tau^{-1}(I_4)\right)}{\lambda(I_1)} = \frac{\lambda([0, \frac{1}{4}] \cap ([\frac{1}{8}, \frac{1}{4}]))}{\lambda([0, \frac{1}{4}]} = \frac{1}{2}$$

$$m_{31} = \frac{\lambda\left(I_3 \cap \tau^{-1}(I_1)\right)}{\lambda(I_3)} = \frac{\lambda([\frac{1}{2}, \frac{3}{4}] \cap ([\frac{2}{3}, \frac{3}{4}]))}{\lambda([\frac{1}{2}, \frac{3}{4}]} = \frac{1}{3}$$

$$m_{41} = \frac{\lambda\left(I_4 \cap \tau^{-1}(I_1)\right)}{\lambda(I_4)} = \frac{\lambda([\frac{3}{4}, 1] \cap ([\frac{3}{4}, \frac{13}{16}]))}{\lambda([\frac{1}{2}, \frac{3}{4}]} = \frac{1}{4}$$

Similarly, we can find the rest of the elements $m_{12} = 0, m_{32} = m_{33} = \frac{1}{3}, m_{34} = 0, m_{42} = m_{43} = m_{44} = \frac{1}{4}$. Thus, the matrix approximation of the F-P operator has the form for map τ :

$$\mathbf{M}_\tau = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

The resulting $\mathbf{M}_\tau$ may be interpreted as a transfer matrix, for which it is easy to check that all row sums are 1, i.e., $\sum_j m_{ij} = 1\, \forall j$.

Let $f = (f_1, f_2, f_3, f_4)$, where $f_i = f|_{I_i}, I_i = [\frac{i-1}{4}, \frac{i}{4}], i = 1, 2, 3, 4$. The normalized density of the map τ (Figure 2.2) is the left eigenvector of $\mathbf{M}_\tau$ with eigenvalue 1. $P_\tau f = f$ reduces to $f\mathbf{M}_\tau = f$ which is a system of linear equation. It can be shown that

$$f = (2, 2, 3, 4).$$

Since determining the fixed point of the Frobenius-Perron operator P_τ of τ or invariant density of τ is generally challenging, especially considering the increased complexity associated with the Frobenius-Perron equation $P_\tau f = f$. It is required to approximate the F-P operator P_τ using Ulam's method. S. M. Ulam [42] first introduced the approximation of the Frobenius-Perron operator. If the map τ is piecewise linear and Markov, we can find the Frobenius-Perron operator in a matrix form. Therefore, it is possible to find the density or invariant measure because the Frobenius-Perron equation $P_\tau f = f$ is a system of linear equations. Ulam's method stands as a widely employed and thoroughly comprehended approach for numerically computing the stationary density of invariant measures in dynamical system. Li [34] first proved the convergence of Ulam's approximation for one-dimensional piecewise expanding transformations. Miller [35] proved convergence for piecewise convex maps with a finite number of pieces.

Let $\tau : I = [0,1] \to [0,1]$ be a piecewise C^2- map with $\inf_{x\in[0,1]} |\tau'(x)| > 2$. Let $\mathcal{P}^{(n)} = \{I_1, I_2, \ldots, I_n\}$ be a partition of $[0,1]$ into subintervals of equal length and let $\mathbf{M}_\tau$ be the matrix transition probabilities, defined in (2.6.4), between the elements of $\mathcal{P}^{(n)}$ for map $\tau : I \to I$. Let $L^{(n)}$ be the $n-$dimensional linear subspace of L^1 which is the finite element space generated by $\{\chi_i\}_{i=1,\ldots,n}$ where χ_i denotes the characteristic function for the interval I_i. We introduce the operator $Q_n : L^1 \to L^{(n)}$, defined by

$$Q_n(f) = \sum_{i=1}^{n} \frac{1}{\lambda(I_i)} \left(\int_{I_i} f d\lambda \right) \chi_i.$$

Since $\{I_i\}_{i=1}^n$ is an equipartition of I i.e. $I = \cup_{i=1}^n I_i$ and $\lambda(I_i) = \frac{1}{n}$.

$$Q_n(f) = \sum_{i=1}^{n} n \left(\int_{I_i} f d\lambda \right) \chi_i = \left(n \int_{I_1} f d\lambda, n \int_{I_2} f d\lambda, \ldots, \int_{I_n} f d\lambda \right).$$

Let $f = (f_1, f_2, \ldots, f_n) \in L^{(n)}$.

Let P_τ be the Frobenius-Perron operator of τ and $P_n : L^{(n)} \to L^{(n)}$ be a finite approximation of P_τ, defined by

$$P_n f = \mathbf{M}_\tau f = \sum_{j=1}^{n} m_{ij} f_j \chi_j.$$

Then, we have

$$P_n f = Q_n P_\tau f.$$

More generally, for $f \in L^1$, we have

$$P_n Q_n f = Q_n P_\tau Q_n f.$$

The following Lemmas and Theorem are proved in [34].

Lemma 2.6.6. *For $f \in L^1$, the sequence $Q_n f$ converges in L^1 to f as $n \to \infty$.*

Lemma 2.6.7. *For $f \in L^{(n)}$ we have $P_n f = Q_n P_\tau f$.*

Lemma 2.6.8. *For f in $L^{(n)}$, the sequence $P_n f$ converges to $P_\tau f$ in L^1 as $n \to \infty$.*

Lemma 2.6.9. *The sequence $\left\{V_{[0,1]} f_n\right\}$ is bounded, where $P_n f_n = f_n$.*

If P_τ has a unique fixed point, then the sequence of fixed points f_n of P_n are expected to converge to that fixed point as n approaches infinity, according to the Ulam's method. The following theorem proves this.

Theorem 2.6.10. *[34] Let $\tau : [0,1] \to [0,1]$ be a piecewise C^2- function with $M = \inf |\tau'| > 2$. Suppose P_τ has a unique fixed point. Then, for any positive integer n, P_n has a fixed point f_n in $L^{(n)}$ with $\| f \| = 1$ and $\{f_n\}$ converges in L^1 to the fixed point of P_τ.*

Note: Lasota-Yorke inequality has a constant $\alpha = \frac{2}{\inf_{x \in [0,1]} |\tau'(x)|} < 1$.

2.7 Some Facts from Functional Analysis

Here we defined $\mathcal{D}$ or $\mathcal{D}(X, \mathcal{B}, \mu)$ as a set of densities in L^1.

Definition 2.7.1. Let $\mathcal{F}$ be a linear space. A function $|| \cdot || : \mathcal{F} \to R^+$ is called a norm if it has the following properties:

- $||f|| = 0 \iff f = 0$
- $||\alpha f|| = |\alpha| ||f||$
- $||f + g|| \leq ||f|| + ||g||$,

for $f, g \in \mathcal{F}$ and $\alpha \in R^+$. The space $\mathcal{F}$ is called a normed linear space.

Definition 2.7.2. A linear operator $P : L^1 \to L^1$ is called Markov if it has the following properties:

- P is positive, i.e., $f \geq 0 \implies Pf \geq 0$, for any $f \in L^1$;
- $||P||_1 \leq 1$ and $||Pf||_1 = ||f||_1$ for $f \geq 0,\ f \in L^1$.

Definition 2.7.3. Let $(X, \mathcal{B}, \mu)$ be a measure space and $P : L^1 \to L^1$ a Markov operator. Then $\{P^n\}$ is said to be **asymptotically stable** if there exists a unique $f^* \in \mathcal{D}$ such that $Pf^* = f^*$ and

$$\lim_{n\to\infty} \| P^n f - f^* \|_1 = 0 \text{ for every } f \in \mathcal{D}. \tag{2.7.4}$$

Definition 2.7.5. A function $h \in L^1$ is a **lower bound function** for a Markov operator $P : L^1 \to L^1$ if

$$\lim_{n\to\infty} \| (P^n f - h)^- \|_1 = 0 \text{ for every } f \in \mathcal{D}. \tag{2.7.6}$$

Equation (2.7.4) may be rewritten as

$$(P^n f - h)^- = \epsilon_n,$$

where $\| \epsilon_n \|_1 \to 0$ as $n \to \infty$ or, explicitly, as

$$P^n f \geq h - \epsilon_n.$$

Theorem 2.7.7. *Let $P : L^1 \to L^1$ be a Markov operator. $\{P^n\}$ is asymptotically stable if and only if there is a nontrivial lower bound function for P.*

Definition 2.7.8. Let K be a set of functions defined on a measure space $(X, \mathcal{B}, \mu)$. Then K is uniformly bounded in L^∞ if there exists a constant M such that $\sup_{f\in K} \| f \|_\infty \leq M$.

Definition 2.7.9. A set K of functions in L^1 is said to be **weakly compact** if every sequence $\{f_n\}$ in K has a subsequence $\{f_{n_k}\}$ that converges weakly to a function $f \in L^1(K)$.

Definition 2.7.10. Let a sequence of functions $\{f_n\}$ defined on a measure space $(X, \mathcal{B}, \mu)$, where for each f_n is Lebesgue integrable in $L^1(X)$. The sequence $\{f_n\}$ is said to be **weakly convergent** in $L^1(X)$ if it converges to a limit function f in $L^1(X)$.

Theorem 2.7.11. (The Dunford-Pettis Theorem) *[5] Let a sequence* $\{f_n\}_{n=1}^{\infty}, f_n \in L^1, n = 1, 2, \ldots$ *satisfy the following conditions:*

(i) $\| f_n \|_1 \leq M$ *for some* M;

(ii) $\forall \epsilon > 0 \ \exists \ \delta > 0$ *such that for any* $A \in \mathcal{B}$, *if* $\mu(A) < \delta$ *then for all* n,

$$|\int_A f_n d\mu| < \epsilon.$$

Then $\{f_n\}$ *contains a weakly convergent subsequence, i.e.,* $\{f_n\}$ *is weakly compact in* L^1.

Theorem 2.7.12. *Let a sequence* $\{f_n\}$ *be uniformly bounded in* L^{∞}. *Then* $\{f_n\}$ *is weakly compact in* L^1.

Proof. Let $(X, \mathcal{B}, \mu)$ be a normalized measure space.

Since $\{f_n\}$ is uniformly bounded in L^{∞}, i.e., $\| f_n \|_{\infty} \leq M$ for some M.

Now, we can write

$$f_n(x) \leq \sup_{x \in X} f_n(x) \leq M, \ \forall n.$$

Which implies,

$$\int_X f_n(x) d\mu \leq \int_X M d\mu = M\mu(X) = M.$$

Therefore, $\| f_n \|_1 \leq M$.

Now suppose that $\mu(A) < \frac{\epsilon}{M}$ for any $A \in \mathcal{B}$. So, $\int_A |f_n| d\mu < \int_A M d\mu = \epsilon$.

Using the Dunford-Pettis Theorem, $\{f_n\}$ is weakly compact in L^1.

□

Theorem 2.7.13. (Yoshida– Kakutani Theorem) *[5] Let* $\mathcal{F}$ *be a Banach space and let* $T : \mathcal{F} \to \mathcal{F}$ *be a bounded linear operator. Assume there exists* $M > 0$ *such that* $\| T^n \| \leq M, n = 1, 2, \ldots$.

Furthermore, if for any $f \in E \subset \mathcal{F}$, the sequence $\{f_n\}$, where

$$f_n = \frac{1}{n}\sum_{k=1}^{n} T^k f,$$

contains a subsequence $\{f_{n_k}\}$ which converges weakly in $\mathcal{F}$, then for any $f \in \bar{E}$,

$$\frac{1}{n}\sum_{k=1}^{n} T^k f \to f^* \in \mathcal{F}$$

(norm convergence) and $T(f^*) = f^*$.

Theorem 2.7.14. (Helly's Theorem) *[5] Let $F = \{f\}$ be an infinite family of functions on an interval $[0, 1]$. If $|f(x)| \leq K,\ V_{[a,b]}f \leq K\ \forall f \in F$, then there exists a sequence $\{f_n\} \subset F$ such that $f_n \to f^*\ \forall x \in [a, b]$, and $V_{[a,b]}f^* \leq K$.*

Chapter 3

ACIM for piecewise convex maps with countable (infinite) number of branches

3.1 Introduction

The existence and properties of absolutely continuous invariant measures of deterministic dynamical systems reflect their long-time behaviour and play an important role in understanding most of their chaotic nature [5, 31, 32]. Let $\mathcal{B}$ a Borel σ-algebra of subsets of $I = [0,1]$ and λ be the normalized Lebesgue measure on I. Let $\tau : I \to I$ be a non-singular measurable transformation. A measure μ on $\mathcal{B}$ is τ-invariant or τ preserve μ if $\mu(\tau^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$. The Frobenius-Perron operator $P_\tau : L^1(I, \mathcal{B}, \lambda) \to L^1(I, \mathcal{B}, \lambda)$ of τ plays an important role for the existence, approximations, and properties of ACIMs. The Frobenius-Perron operator P_τ is defined by

$$\int_A P_\tau f \; d\lambda = \int_{\tau^{-1}(A)} f \; d\lambda, \; \forall A \in \mathcal{B}, \forall f \in L^1. \tag{3.1.1}$$

If τ has countable number of monotonic on each $[b_{i-1}, b_i]$, then it can be shown that the Frobenius-Perron operator P_τ has the following representation [5] :

$$P_\tau f(x) = \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau[b_{i-1}, b_i)}(x) = \sum_{z \in \{\tau^{-1}(x)\}} \frac{f(z)}{|\tau'(z)|}, \tag{3.1.2}$$

where $\tau_i^{-1}, i = 1, 2, \ldots, n, \ldots$ are inverse branches of τ on I. In [32], Lasota and Yorke proved the existence of absolutely continuous invariant measures for piecewise expanding maps. In this Chapter, we consider transformations that are not necessarily expanding, i.e., their derivatives may be smaller than 1, but they possess another property that makes them very special: piecewise convexity. In [33], the authors studied the exactness and the existence of absolutely continuous invariant measures (ACIM) for piecewise convex transformations with a finite number of branches and with a strong repeller. The authors in [33] considered the following properties as primary factors for the proof of the existence of ACIM (i) the F-P operator P_τ maps non-increasing functions to non-increasing function; (ii) If $f : [0, 1] \to \mathbb{R}^+$ is non-increasing, then $||P_\tau f||_\infty$ is bounded by $A||f||_\infty + B||f||_1$, where $A < 1$ and B are some constants. In [5], similar results proved for convex transformations with a finite number of branches. In this chapter, we consider two classes $\mathcal{T}_{pc}^{\infty}(I), \mathcal{T}_{pc}^{\infty,0}(I)$ of piecewise convex maps $\tau : I = [0, 1] \to [0, 1]$ with countable number of branches. In Section 3.2, we study absolutely continuous invariant measures of maps τ in the first class $\mathcal{T}_{pc}^{\infty}(I)$, where $\tau : I = [0, 1] \to [0, 1]$ has a countable number of branches with an arbitrary countable number of limit points of partition points separated from 0. In Section 3.3, we study absolutely continuous invariant measures of maps τ in the second class $\mathcal{T}_{pc}^{\infty,0}(I)$, where $\tau : I = [0, 1] \to [0, 1]$ has a countable number of branches. We assume there exists a countable partition $\{0 = a_0 < \cdots < a_{0,-n} < a_{0,-(n-1)} < \cdots < a_{0,-2} < a_{0,-1} = a_1, a_2, a_3, \ldots, a_n, \ldots\}$ of $I = [0, 1]$ with $\lim_{n\to\infty} a_{0,-n} = 0$. In Section 3.4, we study ACIMs of non-autonomous dynamical systems of piecewise convex maps with a countable number of branches. The exactness of maps in $\mathcal{T}_{pc}^{\infty}(I), \mathcal{T}_{pc}^{\infty,0}(I)$ is presented in Section 3.6.

Recently, there has been an increasing interest in non-autonomous dynamical systems [36, 17]. A non-autonomous dynamical system of a family of maps $\{\tau_n\}_{n=1}^{\infty}$ is a system that acts on the space by application of τ_n in the n-th step. In [22], P. Góra et al. studied the generalization of Krylov-Bogoliubov Theorem and Straube's Theorem for non-autonomous dynamical systems of continuous maps on a compact space. Moreover, they learned the ACIMs of the limit map for non-autonomous dynamical systems of piecewise expanding maps. In section 3.4 of this Chapter, we consider non-autonomous dynamical systems of maps in $\mathcal{T}_{pc}^{\infty}(I), \mathcal{T}_{pc}^{\infty,0}(I)$ and study the existence of ACIM of the limit map.

3.2 ACIMs for piecewise convex maps on $[0, 1]$ with countable number of limit points of partition points separated from 0

Consider $(I = [0, 1], \mathcal{B}, \lambda)$ be a measure space, where λ is the Lebesgue measure on I and $\mathcal{B}$ is the Borel σ-algebra on I. Let $\{0 = a_0, a_1, a_2, \dots, a_n, \dots\}$ be a countable partition of I such that $a_0 < a_1$ and all $a_2, a_3, \dots \in [a_1, 1]$. We do not assume the sequence $\{a_2, \dots, a_n, \dots\}$ to be increasing or decreasing. For any $i \in \{0, 1, 2, \dots\}$, let $n(i)$ be the index such that the interval $[a_i, a_{n(i)}]$ does not contain any other points of the partition. If a_k is the limit point of decreasing subsequence of a_n's, the $n(k)$ is not defined. This notation allows us to consider maps with more than one, even an infinite number of limit points of the partition points. We assume that the set of such limit points has Lebesgue measure 0. We say that $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ if

(1) $\tau_0 = \tau|_{[0,a_1)}$ is continuous and convex;

$\tau_i = \tau|_{[a_i,a_{n(i)})}$ is continuous and convex, $i = 1, 2, \cdots$;

(2) $\tau(a_i) = 0, \tau'(a_i) > 0, i = 1, 2, \dots;$

(3) $\tau(0) = 0, \tau'(0) = \alpha_1 > 1;$

(4) $\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} < \infty.$

Remarks: If the Condition (2) does not satisfy for some i, then there will be a problem with showing that if f is non-increasing, then $P_\tau f$ is non-increasing, Lemma 3.2.2. Also, if $\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} > \infty$, then we will show one example in this chapter that τ has no ACIM.

For $\tau \in \mathcal{T}_{pc}^{\infty}(I)$, $f \in L^1(I)$, $f \geq 0$ the Frobenius-Perron operator P_τ is defined as

$$P_\tau f(x) = \frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i,a_{n(i)})}(x) \tag{3.2.1}$$

The proof of equation (3.2.1) is analogous to the proof of equation (2.3.3) in Chapter 2.

Lemma 3.2.2. *Let $\tau \in \mathcal{T}_{pc}^{\infty}(I)$, $f \in L^1(I)$, $f \geq 0$, f non-increasing. Then*

(1) $P_\tau(f) \in L^1(I)$.

(2) $P_\tau(f) \geq 0$.

(3) $P_\tau(f)$ *is non-increasing.*

(4) $\parallel P_\tau(f) \parallel_\infty \leq C \parallel f \parallel_\infty$, *where* $C = \left(\frac{1}{\alpha_1} + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)}\right)$.

Proof. (1)

$$\int_I P_\tau(f) d\lambda = \int_{\tau^{-1}(I)} f d\lambda = \int_I f d\lambda.$$

Therefore, $P_\tau(f) \in L^1(I)$

(2) Note that

$$P_\tau f(x) = \frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i,a_{n(i)})}(x).$$

We want to show that $\frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x)$ is non-negative. Since $\tau : [0,1] \to [0,1]$, $\tau_0 = \tau_{[0,a_1)}$ and $\tau_i = \tau_{[a_i,a_{n(i)})}$ are continuous and convex, τ_0 is positive and hence $\tau_0^{-1}(x) \geq 0$. Since $f \geq 0$, $f(\tau_0^{-1}(x)) \geq 0$. The derivative of a convex function is non-decreasing. Thus, by property 2, $\tau'(\tau_0^{-1}(x)) \geq 0$ and hence, $\frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x)$ is non-negative. Similarly, $\frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i,a_{n(i)})}(x)$ is also non-negative.

(3) Assume that $f \in L^1(I)$, $f \geq 0$, f non-increasing. Let $0 \leq x < y \leq 1$. We will show that for any $i = 0, 1, 2, \ldots,$

$$\frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i,a_{n(i)})}(x) \geq \frac{f(\tau_i^{-1}(y))}{\tau'(\tau_i^{-1}(y))}\chi_{\tau[a_i,a_{n(i)})}(y). \tag{3.2.3}$$

Let us fix $i \in \{0, 1, 2, \dots\}$. Now, $\tau_i = \tau|_{[a_i, a_{n(i)})}$ is increasing and $\tau(a_i) = 0$. If $x \notin \tau[a_i, a_{n(i)})$ then $y \notin \tau[a_i, a_{n(i)})$. In other words, if $\chi_{\tau[a_i, a_{n(i)})}(x) = 0$ then $\chi_{\tau[a_i, a_{n(i)})}(y) = 0$. Thus,

$$\chi_{\tau[a_i, a_{n(i)})}(x) \geq \chi_{\tau[a_i, a_{n(i)})}(y).$$

If both $x, y \in \tau|_{[a_i, a_{n(i)})}$ then $\tau_i^{-1}(x) < \tau_i^{-1}(y)$ and thus,

$$f(\tau_i^{-1}(x)) > f(\tau_i^{-1}(y)).$$

Moreover, since τ' is non-decreasing and $\tau_i^{-1}(x) < \tau_i^{-1}(y)$, we have $\tau'\left(\tau_i^{-1}(x)\right) < \tau'\left(\tau_i^{-1}(y)\right)$ and thus,

$$\frac{1}{\tau'(\tau_i^{-1}(x))} \geq \frac{1}{\tau'(\tau_i^{-1}(y))}.$$

Combining above inequalities, we have proved (3.2.3).

(4) We have

$$\begin{aligned} P_\tau f(x) &= \frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i, a_{n(i)})}(x) \\ &\leq \frac{f(0)}{\tau'(0)} + \sum_{i=1}^{\infty} \frac{\parallel f \parallel_\infty}{\tau'(a_i)} = \frac{f(0)}{\alpha_1} + \sum_{i=1}^{\infty} \frac{\parallel f \parallel_\infty}{\tau'(a_i)} \\ &\leq \left(\frac{1}{\alpha_1} + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} \right) \parallel f \parallel_\infty . \end{aligned}$$

□

Proposition 3.2.4. *If $f \geq 0$ and f is non-increasing, then $f(x) \leq \frac{1}{x}\lambda(f),\ for\ x \in [0,1]$, where*

$$\lambda(f) = \int_0^1 f(x) d\lambda(x).$$

Proof. For any $0 < x \leq 1$, from the figure 3.1 we have

$$\lambda(f) = \int_0^1 f(x) d\lambda(x) \geq \int_0^x f(x) d\lambda(x) \geq x \cdot f(x).$$

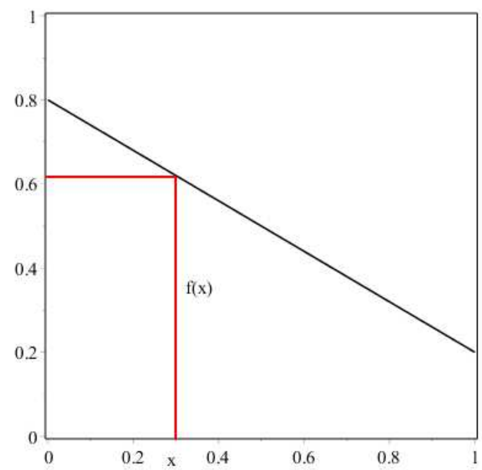

Figure 3.1: Graph for proposition 3.2.4.

For more detailed, since $f \geq 0$ and f is non-increasing. Let $t \in [0, x]$ and $t \leq x$. So

$$\begin{aligned} f(t) \geq f(x) &\implies f(t) - f(x) \geq 0 \\ &\implies \int [f(t) - f(x)]\, dt \geq 0 \end{aligned}$$

Now,

$$\begin{aligned} \int_0^x f(t)dt &= \int_0^x f(x)dt + \int_0^x [f(t) - f(x)]\, dt \\ &\geq \int_0^x f(x)dt = x \cdot f(x) \end{aligned}$$

□

Lemma 3.2.5. *If* $f : [0, 1] \to \mathbb{R}^+$ *is non-increasing and* $\tau \in \mathcal{T}_{pc}^{\infty}(I)$*, then*

$$\| P_\tau(f) \|_\infty \leq \frac{1}{\alpha_1} \| f \|_\infty + D \| f \|_1, \tag{3.2.6}$$

where $D = \left(\sum_{i=1}^{\infty} \frac{1}{a_i} \frac{1}{\tau'(a_i)} \right)$.

Proof. Since f is non-increasing, $f(0) \geq ||f||_\infty$, and by part 3 of Lemma 3.2.2,

$$P_\tau f(0) \geq \| P_\tau f \|_\infty .$$

$$
\begin{aligned}
P_\tau f(0) &= \frac{1}{\tau'(0)} f(0) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(0))}{\tau'(\tau_i^{-1}(0))} \\
&\leq \frac{1}{\alpha_1} f(0) + \sum_{i=1}^{\infty} \frac{f(a_i)}{\tau'(a_i)} \\
&\leq \frac{1}{\alpha_1} f(0) + \sum_{i=1}^{\infty} \frac{\lambda(f)}{a_i} \frac{1}{\tau'(a_i)} \\
&\leq \frac{1}{\alpha_1} \parallel f \parallel_\infty + \left(\sum_{i=1}^{\infty} \frac{1}{a_i} \frac{1}{\tau'(a_i)} \right) \parallel f \parallel_1 .
\end{aligned}
$$

□

Theorem 3.2.7. *Let $\tau \in \mathcal{T}_{pc}^{\infty}(I)$. Then τ admits an absolutely continuous invariant measure $\mu = f^* \cdot \lambda$ with non-increasing density function f^*.*

Proof. Let $f = \mathbf{1}$ and consider the sequence $\{P_\tau^k f\}_{k=0}^{\infty}$. Since f is non-increasing, then by part 4 of Lemma 3.2.2 we can apply Lemma 3.2.5 iteratively and obtain

$$
\begin{aligned}
\parallel P_\tau^k f \parallel_\infty &= \parallel P_\tau \left(P_\tau^{k-1} f \right) \parallel_\infty \leq \frac{1}{\alpha_1} \parallel \left(P_\tau^{k-1} f \right) \parallel_\infty + D \parallel \left(P_\tau^{k-1} f \right) \parallel_1 \\
&\leq \frac{1}{\alpha_1} \left(\frac{1}{\alpha_1} \parallel \left(P_\tau^{k-2} f \right) \parallel_\infty + D \parallel \left(P_\tau^{k-2} f \right) \parallel_1 \right) + D \parallel \left(P_\tau^{k-1} f \right) \parallel_1 \\
&\cdots \\
&\leq \frac{1}{\alpha_1^k} \parallel f \parallel_\infty + D \left(\parallel P_\tau^{k-1} f \parallel_1 + \frac{1}{\alpha_1} \parallel P_\tau^{k-2} f \parallel_1 + \cdots + \frac{1}{\alpha_1^{k-1}} \parallel P_\tau^2 f \parallel_1 \right) \\
&\leq 1 + D \left(1 + \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_1^{i-1}} \right) = 1 + \frac{D}{1 - \frac{1}{\alpha_1}}.
\end{aligned}
$$

So the sequence $\{P_\tau^k\}_{k=0}^{\infty}$ is uniformly bounded in L^∞ and thus weakly compact in L^1. By Yosida-Kakutani theorem, $\frac{1}{k} \sum_{j=1}^{k} P_\tau^j f$ converges in L^1 to a P_τ invariant function f^*. It is non-increasing since it is the limit of non-increasing functions. □

3.3 ACIMs for piecewise convex maps on $[0,1]$ with countable number of branches where 0 is a limit point of partition points

Consider $(I=[0,1],\mathcal{B},\lambda)$ be a measure space, where λ is the Lebesgue measure on I and $\mathcal{B}$ is the Lebesgue σ-algebra on I. Let $\{0=a_0,a_1,a_2,\dots,a_n,\dots\}$ be a countable (infinite) partition of I such that $a_0<a_1$ and all $a_2,a_3,\dots\in[a_1,1]$. We do not assume the sequence $\{a_2,\dots,a_n,\dots\}$ to be increasing or decreasing. For any $i\in\{1,2,\dots\}$, let $n(i)$ be the index such that the interval $[a_i,a_{n(i)}]$ does not contain any other points of the partition. If a_k is the limit point of decreasing subsequence of a_n's, the $n(k)$ is not defined. Moreover, we consider a decreasing infinite sequence $\{a_{0,-n}\}$ of partition points in $(a_0,a_1]$ such that $\{0=a_0<\dots<a_{0,-n}<a_{0,-(n-1)}<\dots<a_{0,-2}<a_{0,-1}=a_1\}$ and $\lim_{n\to\infty}a_{0,-n}=0=a_0$. We say that $\tau\in\mathcal{T}_{pc}^{\infty,0}(I)$ if

(1) $\tau_{0,-(j+1)}=\tau|_{(a_{0,-(j+1)},a_{0,-j}]}$ is continuous and convex, j=1, 2, ...;

$\tau_i=\tau|_{[a_i,a_{n(i)})}$ is continuous and convex, $i=1,2,\cdots$;

(2) $\tau(a_{0,-j})=0,\tau'(a_{0,-j})>0,j=1,2,\dots;$

$\tau(a_i)=0,\tau'(a_i)>0,i=1,2,\dots;$

(3) $\sum_{i=1}^{\infty}\frac{1}{\tau'(a_i)}<\infty;$

(4) $D_1=\sum_{j=1}^{\infty}\frac{1}{\tau'(a_{0,-j})}<1.$

Remark 3.3.1. The Condition (3) and Condition (4) can be replaced by the following Condition 3^+, where

(3^+) $\sum_{j=1}^{\infty}\frac{1}{\tau'(a_{0,-j})}+\sum_{i=1}^{\infty}\frac{1}{\tau'(a_i)}<\infty.$

If condition (3^{+}) is satisfied, then we can find a $J \geq 1$ such that

$$\sum_{j=J}^{\infty} \frac{1}{\tau'(a_{0,-j})} < 1,$$

and after proper renaming of the partition points, the Condition (3) and Condition (4) are satisfied.

Lemma 3.3.2. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$, $f \in L^1(I)$, $f \geq 0$, f non-increasing. Then*

(1) $P_\tau(f) \in L^1(I)$.

(2) $P_\tau(f) \geq 0$.

(3) $P_\tau(f)$ *is non-increasing.*

(4) $\| P_\tau(f) \|_\infty \leq C \| f \|_\infty$, *where* $C = \left(D_1 + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} \right)$.

Proof. The proof is similar to the proof of Lemma 3.2.2. □

Lemma 3.3.3. *If $f : [0,1] \to \mathbb{R}^+$ is non-increasing and $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$. Then*

$$\| P_\tau(f) \|_\infty \leq D_1 \| f \|_\infty + D \| f \|_1, \tag{3.3.4}$$

where $D = \sum_{i=1}^{\infty} \frac{1}{a_i \tau'(a_i)}$.

Proof. The proof is similar to the proof of Lemma 3.2.5. □

Theorem 3.3.5. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$. Then τ admits an absolutely continuous invariant measure $\mu = f^* \cdot \lambda$ with non-increasing density function f^*.*

Proof. The proof is similar to the proof of Theorem 3.2.7. □

3.4 ACIMs of the limit map for non-autonomous dynamical systems of piecewise convex maps

We consider non-autonomous piecewise convex dynamical system $x_{m+1} = \tau_m(x_m), m = 0, 1, \ldots,$ where we assume that τ_0 is the identity map, $x_0 \in [0,1]$, it means that the mapping function τ_0 for the initial state x_0 is such that $\tau_0(x_0) = x_0$ for all values of x_0 within the interval $[0, 1]$, and $\tau_m \in \mathcal{T}_{pc}^{\infty}(I)$ or $\tau_m \in \mathcal{T}_{pc}^{\infty,0}(I), m = 1, 2, \ldots$. Also, we assume that all τ_m satisfy the Lasota-Yorke inequality (3.2.5) with common constants α_1 and D or inequality (3.3.4) with common constants D_1 and D. We write,

$$\tau_{(m,n)} = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_m.$$

In particular,

$$\tau_{(0,n)} = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_0.$$

Let $T_n = \tau_n \circ \tau_{n-1} \circ \cdots \circ \tau_2 \circ \tau_1$, where $\tau_k \in \mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I), k = 1, 2, \ldots, n$.

Proposition 3.4.1. *Let f be a non-increasing function. Then, $P_{T_n} f$ is also non-increasing.*

Proof. We use mathematical induction. For $n = 1, P_{T_n} f = P_{T_1} f = P_{\tau_1} f$ is non-increasing by Lemma 3.2.2 or 3.3.4. Assume that $P_{T_n} f$ non-increasing. Then, $P_{T_{n+1}} f = P_{\tau_{n+1}} (P_{T_n} f)$ is non-increasing by Lemma 3.2.2 or 3.3.4. □

Proposition 3.4.2. *Let $\{\tau_n\}_{n=1}^{\infty}$ be a sequence of transformations such that $\tau_n \in \mathcal{T}_{pc}^{\infty}(I)$, or $\tau_n \in \mathcal{T}_{pc}^{\infty,0}(I), n = 1, 2, \ldots$, τ_n satisfy the Lasota-Yorke inequality (3.2.5) or (3.3.4) with common constants α_1 and D or D_1 and D, and τ_n converges uniformly to a map τ. Then, for any non-increasing density f, the sequence $f_n = P_{T_n} f$ forms a pre-compact set in L^1 and any convergent subsequence converges to a density of an ACIM of the limit map τ.*

Proof. Let $r = \max(\frac{1}{\alpha_1}, D_1)$ and let $f \equiv 1$. Since for class 1 and class 2 transformation $\alpha_1 > 1$ and $D_1 < 1$ respectively. Then f is non-increasing. Note that $P_{T_n} = P_{\tau_n} \circ P_{\tau_{n-1}} \circ \ldots P_{\tau_2} \circ P_{\tau_1}$.

We can apply inequality (3.2.5) or (3.3.4) consecutively and obtain

$$\begin{aligned}
\| P_{T_n} f \|_\infty &\le r \| P_{T_{n-1}} f \|_\infty + D \| P_{T_{n-1}} f \|_1 \\
&\le r \left(r \| P_{T_{n-2}} f \|_\infty + D \| P_{T_{n-2}} f \|_1 \right) + D \| P_{T_{n-1}} f \|_1 \\
&\dots \\
&\le r^n \| f \|_\infty + D \left(\| P_{T_{n-1}} f \|_1 + r \| P_{T_{n-2}} f \|_1 + \dots + r^{n-1} \| P_{T_1} f \|_1 \right) \\
&\le 1 + D \left(1 + r + \dots + r^{n-1} + \dots \right) \\
&= 1 + \frac{D}{1-r}.
\end{aligned}$$

Thus, the functions $P_{T_n} f$ are uniformly bounded and thus weakly compact in L^1.

Let $f_n = P_{T_n} f$. Let $\{f_{n_k}\}$ be a weakly convergent subsequence with limit f^*. Since the functions f_{n_k} are all decreasing and uniformly bounded, they are also of uniformly bounded variation. By Helly's Theorem, every bounded sequence has a subsequence that converges pointwise. Now let we have another subsequence $f_{n_{k_j}}$ convergent to some g a.e.. Since $f_{n_{k_j}}$ converges weakly to f^*, we get $f^* = g$. Thus, f_{n_k} converges to f^* pointwise. Now, by the Lebesgue Dominated Convergence Theorem, we get $f_{n_k} \to f^*$ in L^1. We will prove that the measures $f d\lambda$ and $(P_\tau f) d\lambda$ are equal. It is enough to show that for any $h \in C(I)$, $\int h(f - P_\tau f) d\lambda = 0$.

First, we estimate $\int h(P_{\tau_n} F - P_\tau F) d\lambda$, for any density F. By conjugacy to Koopman operator, we have

$$\left| \int h(P_{\tau_n} F - P_\tau F) d\lambda \right| \le \int F |h \circ \tau_n - h \circ \tau| d\lambda \le \omega_h(\sup |\tau_n - \tau|), \tag{3.4.3}$$

where ω_h is the modulus of continuity of the function h. For more detailed calculations, see A.1 in Appendix.

To simplify the notation, we will skip the subindex k. Let us define

$$g_n = \frac{1}{n}(f_1 + f_2 + \dots + f_n).$$

Since $f_n \to f^*$ in L^1 we also have $g_n \to f^*$ in L^1. We have

$$P_\tau f^* = P_\tau(\lim_{n\to\infty} g_n) = \lim_{n\to\infty} P_\tau g_n.$$

We will show that $\int(P_\tau g_n - g_n)hd\lambda$ converges to 0, for any $h \in C(I)$. Let us fix an $\epsilon > 0$. Since $\|\tau_n - \tau\|_\infty \to 0$, as $n \to \infty$ we can find $N \geq 1$ such that for all $n \geq N$ we have $\omega_h(\|\tau_n - \tau\|_\infty) < \epsilon$. Let $M_h = \sup |h|$.

We can write

$$P_\tau g_n - g_n = \frac{1}{n}\left(P_\tau f_1 + P_\tau f_2 + \cdots + P_\tau f_{n-1} + P_\tau f_n\right) - \frac{1}{n}\left(f_1 + f_2 + \cdots + f_{n-1} + f_n\right)$$
$$= \frac{1}{n}\left(P_\tau f_n - f_1\right) + \frac{1}{n}\sum_{i=1}^{n-1}\left(P_\tau f_i - f_{i+1}\right) = \frac{1}{n}\left(P_\tau f_n - f_1\right) + \frac{1}{n}\sum_{i=1}^{n-1}\left(P_\tau f_i - P_{\tau_{i+1}} f_i\right).$$

Using the estimate (A.1.2) we obtain

$$\left|\int(P_\tau g_n - g_n)hd\lambda\right| \leq \int \frac{1}{n}\left|P_\tau f_n - f_1\right||h|d\lambda$$
$$+ \left|\frac{1}{n}\sum_{i=1}^{N}\left(P_\tau f_i - P_{\tau_{n+1}} f_i\right)\right||h|d\lambda + \left|\frac{1}{n}\sum_{i=N+1}^{n-1}\left(P_\tau f_i - P_{\tau_{i+1}} f_i\right)\right||h|d\lambda$$
$$\leq \frac{1}{n}2M_h + \frac{1}{n}N2M_h + \frac{1}{n}(n-1-N-1)M_h,$$

which converges to 0 as $n \to +\infty$. This completes the proof of the Theorem. □

3.5 Examples

Example 3.5.1. Consider the piecewise convex map $\tau : [0,1] \to [0,1]$ with countable number of branches on the countable partition $\left\{\left[\frac{n}{2+n}, \frac{n+1}{3+n}\right)\right\}_{n=0}^{\infty}$ of $[0,1]$ defined as

$$\tau(x) = \frac{2x}{1-x} \quad (mod\ 1). \tag{3.5.2}$$

See Figure 3.1 for the graph of τ.

$\tau(x)$ is piecewise continuous on the countable partition $\left\{\left[\frac{n}{2+n}, \frac{n+1}{3+n}\right)\right\}_{n=0}^{\infty}$ of $[0,1]$. We use the notation from Section 3.2. Here, $a_0 = 0, a_1 = \frac{1}{3}, a_2 = \frac{1}{2}, a_3 = \frac{3}{5}, a_4 = \frac{2}{3}$ as so on. Now,

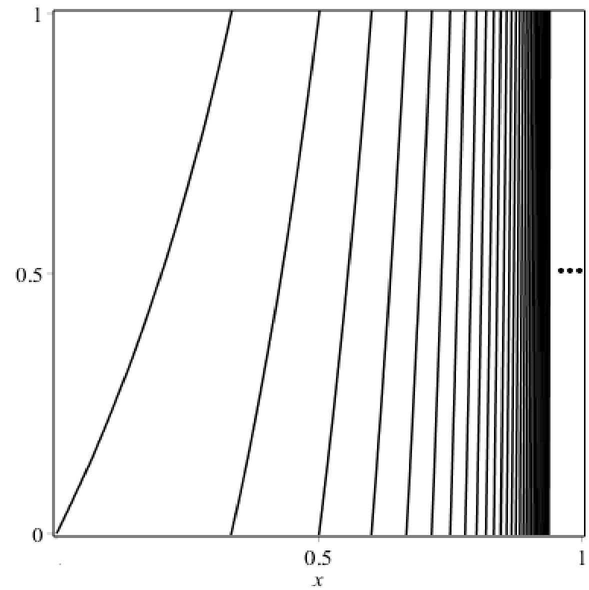

Figure 3.2: Piecewise convex map with infinitely many branches for Example 3.5.1

$\tau'(x) = \frac{2}{(1-x)^2}$ which is increasing on $[a_i, a_{n(i)}], i \geq 1$. Thus τ is piecewise convex. Now, $\forall a_i \in [0,1],\ \ i = 1,2,3,\ldots,\ \tau'(a_i) = \frac{2}{(1-a_i)^2} > 0 \implies \frac{1}{\tau'(a_i)} = \frac{(1-a_i)^2}{2}$, i.e., $\frac{1}{\tau'(a_0)} = \frac{(1-a_0)^2}{2} = \frac{1}{2}, \frac{1}{\tau'(a_1)} = \frac{(1-\frac{1}{3})^2}{2} = \frac{2}{9}, \frac{1}{\tau'(a_2)} = \frac{(1-\frac{1}{2})^2}{2} = \frac{2}{16}, \ldots$. Therefore,

$$\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} = 2 \cdot \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \right) \approx 1.2898 < \infty.$$

From the above calculations, τ satisfies all conditions in Section 3.2. Thus, $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ and hence by Theorem 3.2.7, τ has an ACIM.

Example 3.5.3. Consider the piecewise convex map $\tau : [0,1] \to [0,1]$ with countable number of branches defined as

$$\tau(x) = \frac{1}{\frac{2n+1}{n(n+1)} - x} - n \quad \text{on} \quad \left[\frac{1}{n+1}, \frac{1}{n} \right] \tag{3.5.4}$$

See Figure 3.2. We show that τ satisfies conditions of Section 3.3.

Condition 1: $\tau(x)$ is piecewise continuous and convex on its domain.

Condition 2: Here, $\tau'(x) = \frac{1}{(\frac{2n+1}{n(n+1)} - x)^2}$ which is increasing on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. Thus τ is piecewise convex. Now, $\forall n \in \mathbb{N}, \tau'(\frac{1}{n}) = (n+1)^2 > 0$.

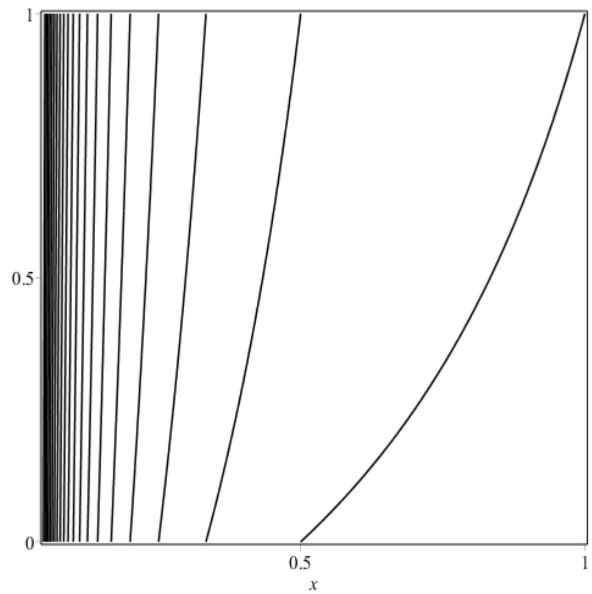

Figure 3.3: Piecewise convex map with countable number of branches, for Example 3.5.3

Condition (3): Condition (3) is obviously satisfied since in this example we have only one interval in between $\frac{1}{2}$ and 1, and $\tau'(\frac{1}{2}) = 1$, which means $\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} = 1 < \infty$.

Condition (4):

$D_1 = \sum_{j=1}^{\infty} \frac{1}{\tau'(a_{0,-j})} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \approx 0.6449 < 1$. Thus, $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$ and hence by Theorem 3.3.5, τ has an ACIM.

Example 3.5.5. [20] Let $I = [0,1], A = \{0\} \cup \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}$ and $J_n = \left[\frac{1}{n+1}, \frac{1}{n}\right]$ for $n = 1, 2, \ldots$. We define $\tau : \tau|_{J_1}(x) = 2x - 1$; for any $n = 2, 3, \ldots, \tau|_{J_n}$ is an increasing linear function such that $\tau(J_n) = \left(0, \frac{1}{n-1}\right]$; $\tau(0) = 0$. We define

$$\frac{1}{\tau'(x)} = \begin{cases} \frac{1}{2}\ , & x \in \text{Int}(J_1) \\ \frac{n-1}{n(n+1)}\ , & x \in \text{Int}(J_n), n = 2, 3, \ldots; \\ 0 \ , & x \in A. \end{cases}$$

$\tau'(x) = 2$, when $x \in [1/2, 1]$ which implies $\frac{1}{\tau'(x)} = \frac{1}{2}$, and when $x \in (0, 1/2)$, $\frac{1}{\tau'(x)} = \frac{n-1}{n(n+1)}$

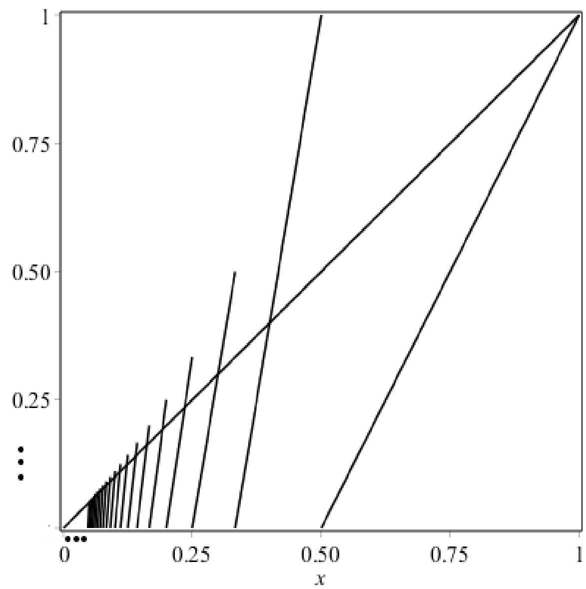

Figure 3.4: For Example 3.5.5.

where $n = 2, 3, 4, \ldots$. Thus, $D_1 = \sum_{j=1}^{\infty} \frac{1}{\tau'(a_{0,-j})} = \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{2}{3\cdot 4} + \frac{3}{4\cdot 5} + \cdots = +\infty > 1$, and condition (4) is not satisfied. It is proven in [20] that τ has no finite ACIM and any interval is mapped after a finite number of iterations onto the whole interval $I = [0, 1]$.

3.6 Exactness of piecewise convex maps with countable number of branches

Theorem 3.6.1. *Let* $\tau : I = [0, 1] \to [0, 1]$ *satisfies Conditions* (1)– (4) *in Section 3.2. Then there exists the unique normalized absolutely continuous measure* λ_g *that is invariant under* τ*. The system* $(I, \mathcal{B}, \lambda_g; \tau)$ *is exact and the density* g *is bounded and decreasing. Moreover,* $\lim_n P_\tau^n f = g$ *in* $L^1(I, \lambda)$*, for any* $f \in L^1(I, \lambda)$ *where* λ *is the Lebesgue measure on* $[0, 1]$ *and* P_τ *is the Frobenius-Perron operator corresponding to* τ*.*

Proof. We follow [33] closely. We will prove that operator P_τ admits a lower bound function. We are going to construct a nontrivial lower function for P_τ. First, we will prove that the set

$$S = \cup_{n=0}^{\infty} \tau^{-n} (\{a_0, a_1, a_2, \ldots\})$$

is dense in $[0,1]$. Suppose it is not true. Then there exists an interval $[x_0, y_0] \subset [0,1]$ such that

$$\tau^n([x_0, y_0]) \cap \{a_0, a_1, a_2, \dots\}$$

is empty for all $n = 0, 1, 2, \cdots$. Therefore, for each n, the points $x_n = \tau^n(x_0)$ and $y_n = \tau^n(y_0)$ belong to the same interval $(a_i, a_{n(i)})$. Recall from the 1st paragraph of Section 3.2, $n(i)$ is the index such that the interval $[a_i, a_{n(i)}]$ does not contain any other points of the partition $\{0 = a_0, a_1, a_2, \dots\}$ such that $a_0 < a_1$ and all $a_2, a_3, \dots \in [a_1, 1]$. If $x_n, y_n \in (a_0, a_1)$, we have by the convexity of $\tau_1 = \tau|_{[a_0,a_1)}$, see Figure 3.4,

$$\tan\theta_1 = \frac{\tau_1(x_n)}{\epsilon_1} \geq \frac{\tau_1(x_n)}{x_n} \quad \text{and} \quad \tan\theta_2 = \frac{\tau_1(y_n)}{\epsilon_2} \geq \frac{\tau_1(y_n)}{y_n}; \;\; \epsilon_1, \epsilon_2 > 0.$$

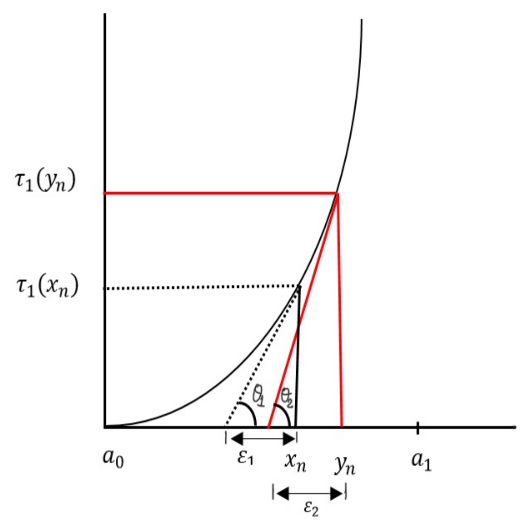

Figure 3.5: A graph for Equation 3.6.2.

Now,

$$\tan\theta_2 \geq \tan\theta_1$$

$$\implies \frac{\tau_1(y_n)}{y_n} \geq \frac{\tau_1(x_n)}{x_n}$$

$$\implies \frac{y_{n+1}}{x_{n+1}} = \frac{\tau_1(y_n)}{\tau_1(x_n)} \geq \frac{y_n}{x_n}. \tag{3.6.2}$$

If $x_n, y_n \in (a_i, a_{n(i)})$ with $i \geq 1$, similarly we have

$$\frac{y_{n+1}}{x_{n+1}} = \frac{\tau_i(y_n)}{\tau_i(x_n)} \geq \frac{y_n - a_1}{x_n - a_1} \geq \frac{y_n}{x_n}\left(\frac{1 - a_1 x_n / y_n}{1 - a_1}\right).$$

Since $\frac{y_{n+1}}{x_{n+1}} \geq \frac{y_n}{x_n}$, we have $\frac{x_{n+1}}{y_{n+1}} \leq \frac{x_n}{y_n} \leq \cdots \leq \frac{x_0}{y_0}$ and thus, $\implies 1 - a_1 \frac{x_n}{y_n} \geq 1 - a_1 \frac{x_0}{y_0}$. Therefore, $\frac{1 - a_1 x_n / y_n}{1 - a_1} \geq \frac{1 - a_1 x_0 / y_0}{1 - a_1}$ and consequently

$$\frac{y_{n+1}}{x_{n+1}} \geq q \frac{y_n}{x_n} \quad \text{where} \quad q = \frac{1 - a_1 x_0 / y_0}{1 - a_1} > 1. \tag{3.6.3}$$

Since $\tau_1'(x) \geq \tau_1'(0) > 1$, the points x_n, y_n cannot belong to (a_0, a_1) for almost all n. For infinitely many n's we have $x_n, y_n > a_1$ and, according to Equation (3.6.2) and (3.6.3), $\lim_n (\frac{y_n}{x_n}) = \infty$. Since $\limsup_n x_n \geq a_1$, this in turn implies $\limsup_n y_n = \infty$ which is impossible. Thus, S is dense in $[0, 1]$.

Second, we claim that for n sufficiently large $P_\tau^n 1_\triangle$ is a decreasing function, where $1_\triangle$ be the characteristic function of an interval $\triangle = [d_0, d_1]$ with the endpoints belonging to the set S. Here P_τ^n is the Frobenius-Perron operator corresponding to τ^n. The function τ^n satisfies condition analogous to Conditions 1– 4 in Section 3.2. Let

$$\{0 = a_0^{(n)}, a_1^{(n)} \ldots, a_n^{(n)}, \ldots\}$$

be the partition corresponding to τ^n i.e. τ^n is convex on each interval $[a_i^{(n)}, a_{n(i)}^{(n)})$ and $\tau^n(a_i^{(n)}) = 0$. We see that

$$\{a_0^{(n)}, \ldots, a_n^{(n)}, \ldots\} = \tau^{-n+1}\{a_0, \ldots, a_n, \ldots\}$$

if we assume for simplicity that $\tau(1) = 0$. Moreover, $\forall i \geq 0$, $\tau(a_i) = 0$, and hence $a_i \in \tau^{-1}(0) \subset \tau^{-1}(\{a_0 = 0, a_1 \ldots, a_n, \ldots\})$. Thus,

$$\{a_0, \ldots, a_n, \ldots\} \subset \tau^{-1}(\{a_0, \ldots, a_n, \ldots\})$$

it follows by induction that

$$\tau^{-n+1}(\{a_0, \dots, a_n, \dots\}) \subset \tau^{-n}(\{a_0, \dots, a_n, \dots\})$$

which shows that the system of partition is decreasing. Since $d_0, d_1 \in S$ there is an integer n_0 sufficiently large such that d_i belongs to the partition $\{a_0^{(n)}, \dots, a_{n_i}^{(n)}, \dots\}$ for $n \geq n_0$. The operator P_τ^n is the Frobenius-Perron operator for τ^n and so it may be written as

$$P_\tau^n f(x) = \frac{f(\tau_0^{-n}(x))}{(\tau^n)'(\tau_0^{-n}(x))} \chi_{\tau_n[0^{(n)}, a_1^{(n)})}(x) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-n}(x))}{(\tau^n)'(\tau_i^{-n}(x))} \chi_{\tau_n[a_i^{(n)}, a_{i+1}^{(n)})}(x)$$

In particular for $f = 1_\triangle$ and $i \geq n_0$ we have

$$P_\tau^n 1_\triangle = \frac{1}{(\tau^n)'(\tau_0^{-n}(x))} \chi_{\tau_n[0,a_1)}(x) + \sum_{i=1}^{\infty} \frac{1}{(\tau^n)'(\tau_i^{-n}(x))} \chi_{\tau^n[a_i^{(n)}, a_{i+1}^{(n)})}(x)$$

Since the right side of the equation is decreasing so, $P_\tau^n 1_\triangle$ has the same property.

Finally, let D_0 be a subset of $L^1(I, \lambda)$ consisting of all functions of the form

$$f(x) = \sum_{i=1}^{\infty} c_k 1_{\triangle_k}(x), \quad c_k \geq 0$$

where the endpoints of the intervals $\triangle_k$ belong to S. Since S is dense in $[0, 1]$, the set D_0 is dense in $L^1(D_0, \lambda)$. Now, we construct a lower function for P_τ. Let $f \in D_0$ be an arbitrary function. There exists $n_0 = n_0(f)$ such that $P_\tau^n f$ is decreasing for $n \geq n_0$. No decreasing density on $(0, 1]$ exceeds $1/x$. In fact for any decreasing $\bar{f}$ we have

$$1 \geq \int_0^x \bar{f}(s) ds \geq x \cdot \bar{f}(x).$$

In particular we have $P_\tau^n f(x) \leq 1/x$ for $n \geq n_0$. Applying this estimate, using lemma 3.3.2, to the equality

$$P_\tau^{n+1} f(0) = \frac{1}{\tau'(0)} P_\tau^n f(0) + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} P_\tau^n f(a_i)$$

we obtain

$$P_\tau^{n+1} f(0) \leq \frac{1}{\alpha_1} P_\tau^n f(0) + D$$

where $D = \sum_{i=1}^{\infty} \frac{1}{a_i \tau'(a_i)}$. From Condition 3 of Section 3.2, we have $\alpha_1 > 1$ and by an induction argument (and using theorem 3.2.7) we obtain

$$P_\tau^{n+n_0} f(0) \leq (\frac{1}{\alpha_1})^n P_\tau^{n_0} f(0) + \frac{D}{1-\frac{1}{\alpha_1}} \leq 1 + \frac{D}{1-\frac{1}{\alpha_1}}$$

Now let $K = D/(1 - 1/\alpha_1) + 1$. For n sufficiently large, say $n \geq n_1$, we have $P_\tau^n f(0) \leq K$. Define $h = \frac{1}{2} 1_{[0,1/2K]}$. We will prove that

$$P_\tau^n f(x) \geq h(x) \;\; for \;\; n \geq n_1. \tag{3.6.4}$$

Suppose not. Then there is $x_0 \in [0, 1/(2K)]$ such that $P_\tau^n f(x_0) < h(x_0) = \frac{1}{2}$ and

$$1 = \int_0^{x_0} P_\tau^n f dx + \int_{x_0}^1 P_\tau^n f dx$$

For some $x \in [0, x_0]$ by the MVT, $\int_0^{x_0} P_\tau^n f dx = x_0 P_\tau^n f(x)$. Again $x \geq 0$, so $P_\tau^n f(x) \leq P_\tau^n f(0) \leq K$. Similarly for second part of the integral and consequently

$$1 = \int_0^{x_0} P_\tau^n f dx + \int_{x_0}^1 P_\tau^n f dx < x_0 K + \frac{1}{2}(1 - x_0) \leq \frac{1}{2K} K + \frac{1}{2} = 1$$

which is impossible.

□

Theorem 3.6.5. *Let $\tau : I = [0,1] \to [0,1]$ satisfies Conditions (1)– (4) in Section 3.3. Then there exists the unique normalized absolutely continuous measure λ_g that is invariant under τ. The system $(I, \mathcal{B}, \lambda_g; \tau)$ is exact and density g is bounded and decreasing. Moreover, $\lim_n P_\tau^n f = g$ in $L^1(I, \lambda)$, for any $f \in L^1(I, \lambda)$ where λ is the Lebesgue measure on $[0,1]$ and P_τ is the Frobenius-Perron operator corresponding to τ.*

Proof. The proof is similar to the proof of Theorem 3.6.1. Recall from the beginning of section

3.3, we consider a decreasing countable sequence $\{a_{0,-n}\}$ of partition points in $(a_0, a_1]$ such that $\{0 = a_0 < \cdots < a_{0,-n} < a_{0,-(n-1)} < \cdots < a_{0,-2} < a_{0,-1} = a_1\}$ and $\lim_{n\to\infty} a_{0,-n} = 0 = a_0$, and $\{0 = a_0, a_1, a_2, \ldots, a_n, \ldots\}$ be a countable partition of I such that $a_0 < a_1$ and all $a_2, a_3, \cdots \in [a_1, 1]$. We do not assume the sequence $\{a_2, \ldots, a_n, \ldots\}$ to be increasing or decreasing. Here we will prove that the set $S = \cup_{j=J}^{\infty} \tau^{-n}(\{a_{0,-j}\}) \cup \ \cup_{i=1}^{\infty} \tau^{-n}(\{a_1, a_2, \ldots, a_n, \ldots\})$ is dense in $[0, 1]$ where $J \geq 1$. Suppose it is not true. Then there exists an interval $[x_0, y_0] \subset [0, 1]$ such that

$$\tau^n([x_0, y_0]) \cap \{0 = a_0 < \cdots < a_{0,-n} < a_{0,-(n-1)} < \cdots < a_{0,-2} < a_{0,-1} = a_1, \ldots, a_n, \ldots\}$$

is empty for all $n = 0, 1, 2, \ldots$. This means that for each n the points $x_n = \tau^n(x_0)$ and $y_n = \tau^n(y_0)$ belong to the same interval $(a_{0,-(j+1)}, a_{0,-j}] \cup (a_i, a_{n(i)}), i = 1, 2, \cdots, j = 1, 2, \cdots$. If $x_n, y_n \in (a_{0,-(j+1)}, a_{0,-j}] \cup (a_i, a_{n(i)})$, we have by applying the similar techniques of the proof of Theorem 3.6.1,

$$\frac{y_{n+1}}{x_{n+1}} = \frac{\tau_1(y_n)}{\tau_1(x_n)} \geq \frac{y_n}{x_n} \tag{3.6.6}$$

$$\frac{y_{n+1}}{x_{n+1}} \geq q\frac{y_n}{x_n} \ \ where \ \ q = \frac{1 - a_1 x_0/y_0}{1 - a_1} > 1 \tag{3.6.7}$$

Since the derivation of τ on (a_0, a_1) is positive, the points x_n, y_n cannot belong to (a_0, a_1) for almost all n. For infinitely many $n's$ we have $x_n, y_n > a_1$ and, according to (3.6.6) and (3.6.7), $\lim_n(\frac{y_n}{x_n}) = \infty$. Since $\limsup_n x_n \geq a_1$, this in turn implies $\limsup_n y_n = \infty$ which is impossible. Then S is dense in $[0, 1]$. The remaining part of the proof is very similar to the corresponding part in the proof of Theorem 3.6.1.

□

Chapter 4

Ulam's method for computing stationary densities of invariant measures for piecewise convex maps

4.1 Introduction

In this chapter, we use Ulam's method for the approximation of f^*, an invariant density of a map τ or equivalently a fixed point of the Frobenius-Perron operator P_τ. Fixed points of the Frobenius-Perron operator P_τ of $\tau \in \mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$ are the stationary densities of τ. The F- P operator P_τ is an infinite dimensional operator. Except for some simple cases (where τ is piecewise linear Markov and the Frobenius-Perron operator has a matrix representation), it is not easy to obtain an analytical solution of the F-P equation $P_\tau f^* = f^*$.

Asymptotic stability of stationary densities and weakly attracting repellors for piecewise convex maps are studied by Inoue in [25] and [26]. Recently, Inoue [27] contemplated invariant measures for random piecewise convex maps with a finite number of branches. However, the literature on stationary densities of ACIMs for piecewise convex maps with countable number of branches is not affluent.

In Chapter 3, we studied ACIMs of maps in two classes $\mathcal{T}_{pc}^{\infty}(I), \mathcal{T}_{pc}^{\infty,0}(I)$ of piecewise convex maps $\tau : I = [0,1] \to [0,1]$ with countable (infinite) number of branches. It is proved in [23] that any $\tau \in \mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$ has a stationary density f^* of absolutely continuous invariant measure μ.

Numerical computations of stationary densities of invariant measures for dynamical systems were suggested by Ulam [42]. For piecewise expanding deterministic transformations, T-Y Li [34] first proved the convergence of Ulam's approximation. Since then, Ulam's method has been applied to one and higher-dimensional expanding deterministic transformations. For piecewise expanding interval maps, Bose and Murray presented the convergence rate of Ulam's method in [3]. In the context of higher-dimensional Jablonski transformations, Boyarsky and Lou proved the convergence of Ulam's method in [6]. For piecewise expanding and C^2 transformations, the convergence of Ulam's is proved by Ding and Zhou in [16]. In the case of random maps with constant probabilities, Froyland proved the convergence of Ulam's method and presented the rate of convergence in [19]. Góra and Boyasrsky proved the convergence of Ulam's method for position-dependent random maps in [5]. However, there are few results on the approximation for stationary densities of invariant measures for piecewise convex maps. In [35], Miller proved the convergence of Ulam's method for piecewise convex transformations with a finite number of branches with a strong repeller. J. Ding [14] developed and presented piecewise linear and piecewise quadratic Markov finite approximation methods for piecewise convex maps with a finite number of branches. If piecewise convex maps have countable number of branches, then the convergence of Ulam's method becomes more challenging and complex. This complexity makes it harder to find a suitable sequence of approximating functions that can accurately capture the behavior of this system across all branches. In [21], Góra and Boyarsky presented an approximation method for invariant measures for piecewise continuous maps with countable number of branches. As far as our knowledge goes, Ulam's method for piecewise convex maps with infinitely many branches has not been investigated so far. In Section 4.2, we introduce notations and review the existence of stationary densities of absolutely continuous invariant measures for $\tau \in \mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$. In Section 4.3.1, we construct a sequence $\{\tau_n\}_{n=1}^{\infty}$ of maps $\tau_n : [0,1] \to [0,1]$ s.t. τ_n has finite number of branches and τ_n converges to τ almost uniformly. Using supremum norms and Lasota-Yorke type inequalities, we prove the existence of densities f_n of ACIMs μ_n for τ_n. In Section 4.3.1, we apply Ulam's method to τ_n and compute an approximation $f_{n,k}$ of f_n and prove that $f_{n,k} \to f_n$ as $k \to \infty$. Finally, in Section 4.3.2, we prove that $f_{n,k} \to f^*$ where f^* is the actual stationary density of absolutely continuous invariant measures for the piecewise convex map τ with countable number of branches. In Section 4.4, we present numerical examples.

4.2 Stationary densities of ACIMs for piecewise convex maps in class $\mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$.

In this section, we review results on the existence of stationary densities of absolutely continuous invariant measures (ACIMs) of piecewise convex maps with countable (infinite) number of branches. We closely follow Chapter 3.

4.2.1 Stationary densities for piecewise convex maps with countable (infinite) branches and limit points of partition points separated from 0

Consider $(I = [0,1], \mathcal{B}, \lambda)$ be a measure space, where λ is the Lebesgue measure on I and $\mathcal{B}$ is the Borel σ-algebra on I. Let $\{0 = a_0, a_1, a_2, \dots, a_n, \dots\}$ be a countable partition of I such that $a_0 < a_1$ and all $a_2, a_3, \dots \in [a_1, 1]$. We do not assume the sequence $\{a_2, \dots, a_n, \dots\}$ to be increasing or decreasing. For any $i \in \{0, 1, 2, \dots\}$, let $n(i)$ be the index such that the interval $[a_i, a_{n(i)}]$ does not contain any other points of the partition. If a_k is the limit point of decreasing subsequence of a_n's, the $n(k)$ is not defined. We say that $\tau \in \mathcal{T}_{pc}^{\infty}(I)$ if

(1) $\tau_0 = \tau|_{[0,a_1)}$ is continuous and convex;

$\tau_i = \tau|_{[a_i, a_{n(i)})}$ is continuous and convex, $i = 1, 2, \cdots$;

(2) $\tau(a_i) = 0, \tau'(a_i) > 0, i = 1, 2, \dots$;

(3) $\tau(0) = 0, \tau'(0) = \alpha_1 > 1$;

(4) $\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} < \infty$.

For $\tau \in \mathcal{T}_{pc}^{\infty}(I), f \in L^1(I), f \geq 0$ the Frobenius-Perron operator P_τ is defined as

$$P_\tau f(x) = \frac{f(\tau_0^{-1}(x))}{\tau'(\tau_0^{-1}(x))}\chi_{\tau[0,a_1)}(x) + \sum_{i=1}^{\infty} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))}\chi_{\tau[a_i,a_{n(i)})}(x) \tag{4.2.1}$$

The following results are proved in Chapter 3.

Lemma 4.2.2. *Let $\tau \in \mathcal{T}_{pc}^{\infty}(I), f \in L^1(I), f \geq 0, f$ non-increasing. Then*

(1) $P_\tau(f) \in L^1(I)$.

(2) $P_\tau(f) \geq 0$.

(3) $P_\tau(f)$ *is non-increasing.*

(4) $\| P_\tau(f) \|_\infty \leq C \| f \|_\infty$, *where* $C = \left(\frac{1}{\alpha_1} + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)}\right)$.

Proposition 4.2.3. *If $f \geq 0$ and f is non-increasing, then $f(x) \leq \frac{1}{x}\lambda(f),\ for\ x \in [0,1]$, where*

$$\lambda(f) = \int_0^1 f(x) d\lambda(x).$$

Proof. For any $0 < x \leq 1$, we have

$$\lambda(f) = \int_0^1 f(x) d\lambda(x) \geq \int_0^x f(x) d\lambda(x) \geq x \cdot f(x).$$

□

Lemma 4.2.4. *If $f : [0,1] \to \mathbb{R}^+$ is non-increasing and $\tau \in \mathcal{T}_{pc}^{\infty}(I)$. Then*

$$\| P_\tau(f) \|_\infty \leq \frac{1}{\alpha_1} \| f \|_\infty + D \| f \|_1, \tag{4.2.5}$$

where $D = \left(\sum_{i=1}^{\infty} \frac{1}{a_i} \frac{1}{\tau'(a_i)}\right)$.

Theorem 4.2.6. *Let $\tau \in \mathcal{T}_{pc}^{\infty}(I)$. Then τ admits an absolutely continuous invariant measure $\mu = f^* \cdot \lambda$ with non-increasing density function f^*.*

4.2.2 Stationary densities of ACIMs of maps with countable (infinite) number of branches where 0 is a limit point of partition points

Let $\{0 = a_0, a_1, a_2, \dots, a_n, \dots\}$ be a countable (infinite) partition of I such that $a_0 < a_1$ and all $a_2, a_3, \dots \in [a_1, 1]$. We do not assume the sequence $\{a_2, \dots, a_n, \dots\}$ to be increasing or decreasing. For any $i \in \{1, 2, \dots\}$, let $n(i)$ be the index such that the interval $[a_i, a_{n(i)}]$ does not contain any other points of the partition. If a_k is the limit point of decreasing subsequence of a_n's, the $n(k)$ is not defined. Moreover, we consider a decreasing infinite sequence $\{a_{0,-n}\}$ of partition points in $(a_0, a_1]$ such that $\{0 = a_0 < \cdots < a_{0,-n} < a_{0,-(n-1)} < \cdots < a_{0,-2} < a_{0,-1} = a_1\}$ and $\lim_{n\to\infty} a_{0,-n} = 0 = a_0$. We say that $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$ if

(1) $\tau_{0,-(j+1)} = \tau|_{(a_{0,-(j+1)}, a_{0,-j}]}$ is continuous and convex, j=1, 2, …;
$\tau_i = \tau|_{[a_i, a_{n(i)})}$ is continuous and convex, $i = 1, 2, \cdots$;

(2) $\tau(a_{0,-j}) = 0, \tau'(a_{0,-j}) > 0, j = 1, 2, \dots;$

$\tau(a_i) = 0, \tau'(a_i) > 0, i = 1, 2, \dots;$

(3) $\sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} < \infty;$

(4) $D_1 = \sum_{j=1}^{\infty} \frac{1}{\tau'(a_{0,-j})} < 1.$

Remark 4.2.7. Condition 3 and Condition 4 can be replaced by the following Condition 3^+, where

3^+. $\sum_{j=1}^{\infty} \frac{1}{\tau'(a_{0,-j})} + \sum_{i=1}^{\infty} \frac{1}{\tau'(a_i)} < \infty.$

If 3^+ is satisfied, then we can find a $J \geq 1$ such that

$$\sum_{j=J}^{\infty} \frac{1}{\tau'(a_{0,-j})} < 1,$$

and after proper renaming of the partition points, Conditions 3 and Condition 4 are satisfied.

The following results are proved in [23].

Lemma 4.2.8. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$, $f \in L^1(I)$, $f \geq 0$, f non-increasing. Then*

(1) $P_\tau(f) \in L^1(I)$.

(2) $P_\tau(f) \geq 0$.

(3) $P_\tau(f)$ *is non-increasing.*

(4) $\| P_\tau(f) \|_\infty \leq C \| f \|_\infty$, *where* $C = \left(D_1 + \sum_{i=1}^\infty \frac{1}{\tau'(a_i)} \right)$.

Lemma 4.2.9. *If $f : [0,1] \to \mathbb{R}^+$ is non-increasing and $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$. Then*

$$\| P_\tau(f) \|_\infty \leq D_1 \| f \|_\infty + D \| f \|_1, \tag{4.2.10}$$

where $D = \sum_{i=1}^\infty \frac{1}{a_i \tau'(a_i)}$.

Theorem 4.2.11. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$. Then τ admits an absolutely continuous invariant measure $\mu = f^* \cdot \lambda$ with non-increasing density function f^*.*

4.3 Ulam's method for piecewise convex maps with countable (infinite) number of branches

4.3.1 Approximation of piecewise convex maps with an infinite number of branches by piecewise convex maps with finite number of branches

Let $\tau : [0,1] \to [0,1]$ be a piecewise convex map in $\mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$ with countable (infinite) number of branches. In this section, we will describe the most difficult case when 0 is the limit of the partition points for the map $\tau \in \mathcal{T}_{pc}^{\infty,0}$. We assume that there are no other limit points of the partition points. The other cases, i.e., where there are such limit points or $\tau \in \mathcal{T}_{pc}^{\infty}$, are done similarly. Thus, the map τ is like in Section 4.2.2. For simplicity, we change the notation by renaming the partition points. Let $a_n = a_{0,-n}$, $n = 1, 2, \ldots$. Then, the assumptions (3) and (4) of Section 4.2.2 are restated as:

(3') There exists an $N \geq 1$ such that

$$D_1 = \sum_{n=N+1}^{\infty} \frac{1}{\tau'(a_n)} < 1. \tag{4.3.1}$$

Then, the Lemma 4.2.8 and Lemma 4.2.9 of Section 4.2.2 hold with changed constants

$$\begin{aligned} C &= D_1 + \sum_{n=1}^{N} \frac{1}{\tau'(a_n)} = \sum_{n=1}^{\infty} \frac{1}{\tau'(a_n)}, \\ D &= \sum_{n=1}^{N} \frac{1}{a_n \tau'(a_n)}. \end{aligned} \tag{4.3.2}$$

For $n \geq N$, we construct a sequence $\{\tau_n\}_{n=N}^{\infty}$ of maps $\tau_n : [0,1] \to [0,1]$ s.t. τ_n has finite number of branches and τ_n converges to τ almost uniformly (see definition 4.3.8 and proof of lemma 4.3.9). Using supremum norms and Lasota-Yorke type inequalities, we prove that the existence of stationary densities f_n of ACIMs μ_n for τ_n. We approximate $\tau : [0,1] \to [0,1]$ with the following sequence of maps $\tau_n : [0,1] \to [0,1]$, $n \geq N$, with finite number of branches:

$$\tau_n(x) = \begin{cases} x/a_n \ , \ 0 \leq x < a_n; \\ \tau(x) \ , \ a_n \leq x \leq 1. \end{cases}$$

In the following, we show that for each $n \geq 0$, the map τ_n has an absolutely continuous invariant measure. Each τ_n is a piecewise convex map with finite partition $\{1 = a_0, a_1, a_2, \cdots, a_n, a_{n+1} = 0\}$ and τ_n satisfies following conditions:

(1) $\tau_{n_j} = \tau_n|(a_j, a_{j-1}]$ is continuous and convex, $j = 1, 2, \cdots, n+1$;

(2) $\tau_n(a_j) = 0, \tau_n'(a_j) > 0, j = 1, \cdots, n+1$;

(3) $\tau_n(0) = 0, \tau_n'(0) = \frac{1}{a_n} > 1$.

(4) $\sum_{j=1}^{n+1} \frac{1}{\tau'(a_j)} < \infty$.

Let $f \in L^1(I), f \geq 0$. Then, the Frobenius-Perron operator P_{τ_n} is defined as

$$P_{\tau_n} f(x) = \sum_{j=1}^{n+1} \frac{f(\tau_{n_j}^{-1}(x))}{\tau_n'(\tau_{n_j}^{-1}(x))} \chi_{\tau_n(a_j, a_{j-1}]}(x) \tag{4.3.3}$$

Lemma 4.3.4. *Let $f \in L^1(I), f \geq 0, f$ non-increasing. Then*

(1) $P_{\tau_n}(f) \in L^1(I)$ for each $n = 1, 2, \ldots$.

(2) $P_{\tau_n}(f) \geq 0$ for each $n = 1, 2, \ldots$.

(3) $P_{\tau_n}(f)$ is non-increasing for each $n = 1, 2, \ldots$.

(4) $\| P_{\tau_n}(f) \|_\infty \leq C_n \| f \|_\infty \leq (1 + C) \| f \|_\infty$ for each $n = 1, 2, \ldots$ where $C_n = \left(a_n + \sum_{j=1}^{n} \frac{1}{\tau_n'(a_j)} \right)$.

Proof. (1)

$$P_{\tau_n}(f) = \int_{\tau_n^{-1}(I)} f d\lambda = \int_I f.$$

Therefore, $P_{\tau_n}(f) \in L^1(I)$

(2) Note that

$$P_{\tau_n} f(x) = \sum_{j=1}^{n+1} \frac{f(\tau_{n_j}^{-1}(x))}{\tau_n'(\tau_{n_j}^{-1}(x))} \chi_{\tau_n(a_j, a_{j-1}]}(x).$$

Each of the branches $\frac{f(\tau_{n_j}^{-1}(x))}{\tau_n'(\tau_{n_j}^{-1}(x))} \chi_{\tau_n(a_j, a_{j-1}]}(x)$ is non-negative.

(3) Each of the branches $\frac{f(\tau_{n_j}^{-1}(x))}{\tau_n'(\tau_{n_j}^{-1}(x))} \chi_{\tau_n(a_j, a_{j-1}]}(x)$ is non-increasing since $\tau_{n_j}^{-1}$ is increasing, f is non-increasing and τ_n' is increasing.

(4) We have

$$
\begin{aligned}
P_{\tau_n} f(x) &= \sum_{j=1}^{n+1} \frac{f(\tau_{n_j}^{-1}(x))}{\tau_n'(\tau_{n_j}^{-1}(x))} \chi_{\tau_n(a_j, a_{j-1}]}(x) \\
&\leq \sum_{j=1}^{n+1} \frac{\| f \|_\infty}{\tau_n'(\tau_{n_j}^{-1}(x))} \leq \sum_{j=1}^{n+1} \frac{\| f \|_\infty}{\tau_n'(a_j)} \\
&= \left(a_n + \sum_{j=1}^{n} \frac{1}{\tau_n'(a_j)} \right) \| f \|_\infty \\
&\leq (1 + C) \| f \|_\infty
\end{aligned}
$$

□

Lemma 4.3.5. *If $f : [0,1] \to \mathbb{R}^+$ is non-increasing, then for each $n \geq N$,*

$$
\| P_{\tau_n}(f) \|_\infty \leq (a_n + D_1) \| f \|_\infty + D \| f \|_1, \tag{4.3.6}
$$

Proof. Since f is non-increasing, $f(0) \geq \| f \|_\infty$, and by Lemma 4.3.4, $P_{\tau_n} f(0) \geq \| P_{\tau_n} f \|_\infty$. Now,

$$
\begin{aligned}
P_{\tau_n} f(0) &= \sum_{j=1}^{n+1} \frac{f(\tau_{n_j}^{-1}(0))}{\tau_n'(\tau_{n_j}^{-1}(0))} \chi_{\tau_n(a_j, a_{j-1}]}(0) \\
&= \frac{1}{\tau_n'(0)} f(0) + \sum_{j=N}^{n} \frac{f(\tau_{n_j}^{-1}(0))}{\tau_n'(\tau_{n_j}^{-1}(0))} + \sum_{j=1}^{N} \frac{f(\tau_{n_j}^{-1}(0))}{\tau_n'(\tau_{n_j}^{-1}(0))} \\
&= a_n f(0) + \sum_{j=N}^{n} \frac{f(a_{n_j})}{\tau_n'(a_{n_j})} + \sum_{j=1}^{N} \frac{f(a_{n_j})}{\tau_n'(a_{n_j})} \\
&\leq (a_n + D_1) f(0) + \sum_{j=1}^{N} \frac{\lambda(f)}{a_j} \frac{1}{\tau_n'(a_j)} \\
&\leq (a_n + D_1) \| f \|_\infty + D \| f \|_1 .
\end{aligned}
$$

□

Theorem 4.3.7. *For each $n \in \mathbb{N}$, τ_n admits an absolutely continuous invariant measure $\mu_n = f_n^* \cdot \lambda$ with non-increasing density function f_n^*.*

Proof. Our proof works for all $n \geq n_0$, where we have $a_{n_0} + D_1 < 1$ and gives a uniform estimate. For the $n < n_0$ the claim follows from Subsection 4.2.1.

Let $f = \mathbf{1}$ and consider the sequence $\{P^k_{\tau_n} f\}_{k=0}^{\infty}$. Clearly f is non-increasing. Then by part 3 of Lemma 4.3.4 we can apply Lemma 4.3.5 iteratively and obtain

$$
\begin{aligned}
& \| P^k_{\tau_n} f \|_\infty = \| P_{\tau_n}\left(P^{k-1}_{\tau_n} f\right) \|_\infty \leq (a_n + D_1) \| \left(P^{k-1}_{\tau_n} f\right) \|_\infty + D \| \left(P^{k-1}_{\tau_n} f\right) \|_1 \\
\leq\ & (a_n + D_1)\left((a_n + D_1) \| \left(P^{k-2}_{\tau_n} f\right) \|_\infty + D \| \left(P^{k-2}_{\tau_n} f\right) \|_1\right) + D \| \left(P^{k-1}_{\tau_n} f\right) \|_1 \\
\cdots & \\
\leq\ & (a_n + D_1)^k \| f \|_\infty + D\Big(\| P^{k-1}_{\tau_n} f \|_1 + (a_n + D_1) \| P^{k-2}_{\tau_n} f \|_1 \\
& \qquad\qquad + \cdots + (a_n + D_1)^{k-1} \| P^2_{\tau_n} f \|_1\Big) \\
\leq\ & (a_n + D_1)^k \| f \|_\infty + D\left(1 + (a_n + D_1) + \cdots + (a_n + D_1)^{k-1}\right) \\
\leq\ & (a_{n_0} + D_1)^k \| f \|_\infty + \frac{D}{1 - (a_{n_0} + D_1)}.
\end{aligned}
$$

So the sequence $\{P^k_{\tau_n} f\}_{k=0}^{\infty}$ is uniformly bounded and weakly compact. By Yosida-Kakutani theorem, $\frac{1}{k}\sum_{j=1}^{k} P^j_{\tau_n} f$ converges in L^1 to a P_{τ_n} invariant function f^*_n. It is non-increasing since it is the limit of non-increasing functions. □

Definition 4.3.8. Let τ_n, τ are maps on $[0,1]$ into itself and τ_n, τ are defined as above. We say that τ_n converges to τ almost uniformly if, given $\epsilon > 0$, there exists a measurable set $A_\epsilon \subset [0,1]$, $\lambda(A_\epsilon) > 1 - \epsilon$, such that $\tau_n \to \tau$ uniformly on A_ϵ.

Lemma 4.3.9. *τ_n converges to τ almost uniformly.*

Proof. Let $\epsilon > 0$. Choose the decreasing partition $\{1 = a_0, a_1, a_2, \cdots, a_n, a_{n+1} = 0\}$ of $[0,1]$ for τ_n such that $a_n < \epsilon$. Let $A_\epsilon = (a_n, 1)$. Then, $\lambda(A_\epsilon) = 1 - a_n > 1 - \epsilon$. Since $\tau_n = \tau$ on A_ϵ this completes the proof.

□

4.3.2 Ulam's method for approximation

In general, most dynamical systems do not possess Markov properties, implying that they lack a Markov partition, and most partitions designed for these systems will not qualify as Markov partitions (section (4.2.2), example 4.1(c), [2]). However, when we encounter such systems, the corresponding the Frobenius-Perron operator can be accurately represented by an operator of finite rank. Markov transformation is a piecewise monotonic transformation such that each interval of the partition is mapped onto a union of intervals of the partition. The Frobenius-Perron operator can be defined in terms of the Markov transformation matrix. We can approximate the fixed point of the Frobenius-Perron operator P_τ by the fixed point of a matrix operator, which we call the Markov operator. If the map τ is piecewise linear and Markov, we can find the Frobenius-Perron operator in a matrix form. Therefore, it is easy to find the density or invariant measure because the Frobenius-Perron equation $P_\tau f = f$ is a system of linear equations. In the deterministic case, the matrix approximation of the F-P operator has the form

$$\mathbf{M}_\tau = \left(\frac{\lambda\left(\tau^{-1}(J_j) \cap J_i\right)}{\lambda(J_i)} \right)_{1 \le i,j \le k}$$

where λ denotes the normalized Lebesgue measure on J and $\{J_i\}_{i=1}^k$ is a finite family of connected sets with nonempty and adjoint interiors that cover J i.e., $J = \cup_{i=1}^k J_i$, and indexed in terms of nested refinements.

Example 4.3.10. Let $\tau : [0,1] \to [0,1]$ be a piecewise linear Markov transformation on the partition $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ defined by

$$\tau(x) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ -3x + \frac{7}{4}, & \frac{1}{4} < x \le \frac{1}{2} \\ 3x - \frac{5}{4}, & \frac{1}{2} < x \le \frac{3}{4} \\ -2x + \frac{5}{2}, & \frac{3}{4} < x \le 1 \end{cases}.$$

Here we use the above matrix form of the F-P operator for finding all elements of the matrix $\mathbf{M}_\tau$.

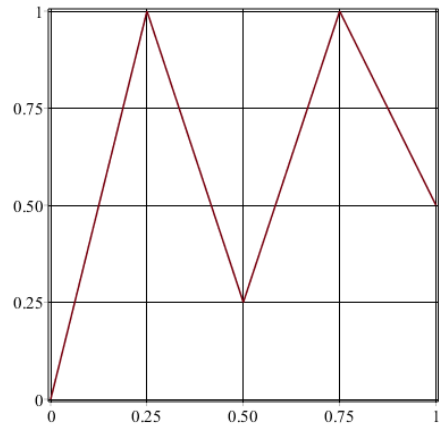

Figure 4.1: Markov map , τ, with partition shown for Example 4.3.10.

Now using $i, j = 1, 2, 3, 4$ successively, then

$$m_{11} = \frac{\lambda\left(J_1 \cap \tau^{-1}(J_1)\right)}{\lambda(J_1)} = \frac{\lambda([0, \frac{1}{4}] \cap \tau^{-1}([0, \frac{1}{4}]))}{\lambda([0, \frac{1}{4}])} = \frac{\lambda([0, \frac{1}{4}] \cap ([0, \frac{1}{16}]))}{\lambda([0, \frac{1}{4}])} = \frac{1}{4};$$

$$m_{12} = \frac{\lambda\left(J_1 \cap \tau^{-1}(J_2)\right)}{\lambda(J_1)} = \frac{\lambda([0, \frac{1}{4}] \cap \tau^{-1}([\frac{1}{4}, \frac{1}{2}]))}{\lambda([0, \frac{1}{4}])} = \frac{\lambda([0, \frac{1}{4}] \cap ([\frac{1}{16}, \frac{1}{8}]))}{\lambda([0, \frac{1}{4}])} = \frac{1}{4};$$

$$m_{21} = \frac{\lambda\left(J_2 \cap \tau^{-1}(J_1)\right)}{\lambda(J_2)} = \frac{\lambda([\frac{1}{4}, \frac{1}{2}] \cap \tau^{-1}([0, \frac{1}{4}]))}{\lambda([\frac{1}{4}, \frac{1}{2}])} = \frac{\lambda([\frac{1}{4}, \frac{1}{2}] \cap ([\frac{1}{2}, \frac{7}{12}]))}{\lambda([0, \frac{1}{4}])} = 0;$$

$$m_{22} = \frac{\lambda\left(J_2 \cap \tau^{-1}(J_2)\right)}{\lambda(J_2)} = \frac{\lambda([\frac{1}{4}, \frac{1}{2}] \cap ([\frac{1}{2}, \frac{5}{12}]))}{\lambda([\frac{1}{4}, \frac{1}{2}])} = \frac{1}{3};$$

$$m_{33} = \frac{\lambda\left(J_3 \cap \tau^{-1}(J_3)\right)}{\lambda(J_3)} = \frac{\lambda([\frac{1}{2}, \frac{3}{4}] \cap ([\frac{7}{12}, \frac{2}{3}]))}{\lambda([\frac{1}{2}, \frac{3}{4}])} = \frac{1}{3};$$

$$m_{44} = \frac{\lambda\left(J_4 \cap \tau^{-1}(J_4)\right)}{\lambda(J_4)} = \frac{\lambda([\frac{3}{4}, 1] \cap ([\frac{3}{4}, \frac{7}{8}]))}{\lambda([\frac{3}{4}, 1])} = \frac{1}{2}.$$

Similarly, we can find the rest of the elements $m_{13} = m_{14} = \frac{1}{4}, m_{21} = 0, m_{23} = m_{24} = \frac{1}{3}, m_{31} = 0, m_{32} = m_{34} = \frac{1}{3}, m_{34} = 0, m_{41} = m_{42} = 0, m_{43} = \frac{1}{4}$.

Thus, the matrix approximation of the F-P operator has the form for map τ :

$$\mathbf{M}_\tau = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

The resulting $\mathbf{M}_\tau$ may be interpreted as a transfer matrix, for which it is easy to check that all row sums are 1, i.e., $\sum_j m_{ij} = 1 \; \forall j$.

Let $f = (f_1, f_2, f_3, f_4)$, where $f_i = f|_{I_i}, I_i = [\frac{i-1}{4}, \frac{i}{4}], i = 1, 2, 3, 4$. The normalized density of the map τ (Figure 4.1) is the left eigenvector of $\mathbf{M}_\tau$ with eigenvalue 1. $P_\tau f = f$ reduces to $f\mathbf{M}_\tau = f$ which is a system of linear equation. It shows that $f = (0, 1, 2, 2)$.

Ulam's method is often used to describe the process of using Ulam's conjecture (see Chapter 2, Conjecture 2.6.1). In this subsection, first, we describe Ulam's method for finite-dimensional approximation $P_{n,k}$, n and k denoted by the number of branches and number of partitions, respectively, of the Perron-Frobenius operator P_{τ_n} of τ_n. Ulam's method computes $f_{n,k}$ on a partition of k subintervals of the state space as an approximation of the actual stationary density function f_n^* of $\tau_n,\ n \geq 1$. Moreover, we show that $f_{n,k}$ converges to f_n^* as $k \to \infty$. We closely follow [34], [35] and [14]. Let τ_n be an approximation of $\tau \in \mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$. Then, by Theorem 4.3.7, τ_n has an absolutely continuous invariant measure μ_n with stationary density function f_n^*. The approximation f_n^* is carried out using a two-step process. Initially, we approximate τ by the map τ_n with a finite number of branches. Then, we further approximate τ_n using Ulam's method. In our case, we don't need approximation in the L^1 norm. So, it's not an approximation of any norm. In this sense, approximation means τ_n converges τ almost uniformly. Now, we describe Ulam's method for approximating f_n^*. Let k be a positive integer. Let $\mathcal{P}^{(k)} = \{J_1, J_2, \ldots, J_k\}$ be a partition of the interval $[0, 1]$ into k equal subintervals. Now, construct the matrix

$$\mathbf{M}_{\tau_n}^{(k)} = \left(\frac{\lambda\left(\tau_n^{-1}(J_j) \cap J_i\right)}{\lambda(J_i)} \right)_{1 \le i,j \le k}.$$

Let $L^{(k)} \subset L^1([0,1], \lambda)$ be a subspace of L^1 consisting of functions which are constant on elements of the partition $\mathcal{P}^{(k)}$. We will represent functions in $L^{(k)}$ as vectors: vector $f = [f_1, f_2, \dots, f_k]$ correspond to the function $f = \sum_{i=1}^{k} f_i \chi_{J_i}$. We introduce the operator $Q^{(k)} : L^1 \to L^{(k)}$, defined by

$$Q^{(k)}(f) = \sum_{i=1}^{k} \left(\frac{1}{\lambda(J_i)} \int_{J_i} f d\lambda \right) \chi_{J_i} = \left[\frac{1}{\lambda(J_1)} \int_{J_1} f d\lambda, \dots, \frac{1}{\lambda(J_k)} \int_{J_k} f d\lambda \right] \tag{4.3.11}$$

Let $f = [f_1, f_2, \dots, f_k] \in L^{(k)}$. We define the operator $P_{\tau_n}^{(k)} : L^{(k)} \to L^{(k)}$ by

$$P_{\tau_n}^{(k)} f \quad = \quad \left(\mathbf{M}_{\tau_n}^{(k)} \right)^{Trans} \cdot ([f_1, f_2, \dots, f_k]) \tag{4.3.12}$$

which is a finite-dimensional approximation to the operator P_{τ_n}. A^{Trans} denotes the transpose of the matrix A.

Then, we have

$$P_{\tau_n}^{(k)} f = Q^{(k)} P_{\tau_n} f.$$

More generally, for $f \in L^1$, we have

$$P_{\tau_n}^{(k)} Q^{(k)} f = Q^{(k)} P_{\tau_n} Q^{(k)} f.$$

The following Lemma will be used several times in the sequel.

Lemma 4.3.13. *Let $\{g_n\}_{n=1,2,\dots}$ be a sequence of non-increasing functions uniformly bounded in L^∞. If $g_n \to h$, as $n \to \infty$, weakly in L^1, then the convergence is also in L^1 and a.e.*

Proof. Since g_n's are non-increasing and uniformly bounded in L^∞, they are also of uniformly bounded variation. By Helly's Theorem [Rudin, 1976], there is a subsequence g_{n_k} convergent a.e. to some function h_1. Since $g_{n_k} \to h_1$ weakly in L^1 we have $h_1 = h$. Considering all possible subsequences we prove that $g_n \to h$ a.e. Since g_n converge to h a.e. and they are uniformly bounded in L^∞, the convergence is also in L^1 (e.g., by Lebesgue Dominated Convergence theorem). □

Since each map τ_n is exact [Lasota and Yorke, 1982], by Proposition 1.2 of [Hunt and Miller, 1992], we obtain that the invariant densities $f_{n,k}$ of $P_{\tau_n}^{(k)}$ are unique.

Lemma 4.3.14. *The invariant density $f_{n,k}$ of $P_{\tau_n}^{(k)}$ τ_n is non-increasing for any $n, k > 1$.*

Proof. Let $f \in L^(k)$. Since $P_{\tau_n}^{(k)} f = Q^{(k)} P_{\tau_n} f$, and both operators P_{τ_n} and $Q^{(k)}$ transform non-increasing functions into non-increasing functions, the operator $P_{\tau_n}^{(k)}$ also have this property. Let $f = 1$ be a constant function understood as $[1, 1, \dots 1] \in L^(k)$. It is non-increasing. Thus, all the functions $(P_{\tau_n})^m f, m = 1, 2, , , ,$ are non-increasing. Similarly, as the estimate (15) was obtained, they can be shown to be uniformly bounded in L^∞ and thus weakly compact in L^1. Then, Yosida-Kakutani theorem [Yosida and Kakutani , 1941] shows that the sequence $\frac{1}{s} \sum \left(P_{\tau_n}^{(k)} \right)^m f$ converges in L^1 to the invariant density $f_{n,k}$. By Lemma 4.3.13 the convergence is also a.e. and $f_{n,k}$ is non-increasing. □

Using Ulam's method and corresponding convergence analysis described in [34, 35, 14], we prove the following theorem.

Theorem 4.3.15. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$ be a piecewise convex map with countably many branches. Let $\{\tau_n\}_{n=1}^\infty$ be the approximating sequence of piecewise convex maps with finite numbers of branches where τ_n are defined in the previous Sub-Section 4.3.1. If $f_{n,k}$ is a normalized fixed point of $P_{\tau_n}^{(k)}, k = 1, 2, \dots,$ defined in (4.3.12), then the sequence $\{f_{n,k}\}_{k=1}^\infty$ is weakly pre-compact in L^1. Any limit point f_n^* of the sequence $\{f_{n,k}\}_{k=1}^\infty$ is a fixed point of P_{τ_n}.*

Proof. Let $P_{\tau_n}^{(k)}$ be the Ulam's approximation of the Frobenius-Perron operator P_{τ_n} of τ_n. Let $Q^{(k)}$ be the isometric projection defined in (4.3.11). It can be shown that (see: (4), page 3 [35]; def. (2.1), page 5 [34]):

$$P_{\tau_n}^{(k)} Q^{(k)} f = Q^{(k)} P_{\tau_n} f. \tag{4.3.16}$$

From lemma 2.5 of [35], we get

$$\| Q^{(k)} f \|_\infty \leq \| f \|_\infty . \tag{4.3.17}$$

Equation (4.3.17) implies that $Q^{(k)}$ does not increase in L^∞ norm.

From equation (4.3.6), P_{τ_n} satisfies the following Lasota-Yorke type inequality

$$\| P_{\tau_n} f \|_\infty \leq (a_n + D_1) \| f \|_\infty + D \| f \|_1 . \tag{4.3.18}$$

Using equation (4.3.16) and (4.3.17) we obtain

$$\| P_{\tau_n}^{(k)} f \|_\infty = \| Q^{(k)} P_{\tau_n} f \|_\infty \leq \| P_{\tau_n} f \|_\infty \leq (a_n + D_1) \| f \|_\infty + D \| f \|_1 . \tag{4.3.19}$$

Now, $f_{n,k}$ is a normalized fixed point of the approximate of the F-P operator $P_{\tau_n}^{(k)}$, then we can write

$$\| f_{n,k} \|_\infty = \| P_{\tau_n}^{(k)} f_{n,k} \|_\infty .$$

From equation (4.3.17) and equation (4.3.18) we obtain

$$\| P_{\tau_n}^{(k)} f_{n,k} \|_\infty = \| Q^{(k)} P_{\tau_n} f_{n,k} \|_\infty \leq \| P_{\tau_n} f_{n,k} \|_\infty \leq (a_n + D_1) \| f_{n,k} \|_\infty + D \| f_{n,k} \|_1 . \tag{4.3.20}$$

Thus, we have

$$\| f_{n,k} \|_\infty \leq (a_n + D_1) \| f_{n,k} \|_\infty + D \| f_{n,k} \|_1$$

$$\| f_{n,k} \|_\infty \leq \frac{D}{1 - (a_n + D_1)} \| f_{n,k} \|_1 \tag{4.3.21}$$

This shows that the densities $f_{n,k}$ are uniformly bounded in L^∞ and thus form a precompact set in the weak topology of L^1. There exists a subsequence f_{n,k_j} that converges weakly in L^1 to some limit function $\tilde{f}$ for fixed n. Since all f_{n,k_j}'s are decreasing, they are also of uniformly bounded variation. By Helly's Theorem, there is a further subsequence $f_{n_{j_s}}$ convergent a.e. to some function $\tilde{\tilde{f}}$. Since the functions $f_{n_{j_s}}$ are uniformly bounded, by Lebesgue dominated convergence theorem , we obtain that

$$f_{n_{j_s}} \to \tilde{\tilde{f}} \text{ , in } L^1.$$

This implies that $\tilde{\tilde{f}} = \tilde{f}$. Since the same reasoning applies to any subsequence of f_{n,k_j} we obtain that

$$f_{n_j} \to \tilde{f}\ , \text{ in } L^1.$$

Now, we want to show that if $f_{n,k_j} \to \tilde{f}$, $n-$fixed, $k_j \to \infty$ then for any continuous function g we have

$$\int g\left(\tilde{f} - P_{\tau_n}\tilde{f}\right) d\lambda = 0$$

Now,

$$|\int g\left(\tilde{f} - P_{\tau_n}\tilde{f}\right) d\lambda| \leq |\int g\left(\tilde{f} - f_{n,k_j}\right) d\lambda| + |\int g\left(f_{n,k_j} - P_{\tau_n}\tilde{f}\right) d\lambda|$$

$$= |\int g\left(\tilde{f} - f_{n,k_j}\right) d\lambda| + |\int g\left(P_{\tau_n}^{(k_j)} f_{n,k_j} - P_{\tau_n}\tilde{f}\right) d\lambda|$$

$$\leq |\int g\left(\tilde{f} - f_{n,k_j}\right) d\lambda| + |\int g\left(P_{\tau_n}^{(k_j)} f_{n,k_j} - P_{\tau_n}^{(k_j)}\tilde{f}\right) d\lambda| + |\int g\left(P_{\tau_n}^{(k_j)}\tilde{f} - P_{\tau_n}\tilde{f}\right) d\lambda|$$

Since g, being continuous, is bounded and $f_{n,k_j} \to \tilde{f}$ weakly in L^1, so first term goes to 0.

Since

$$f_{n_j} \to \tilde{f}\ , \text{ in } L^1$$

and $P_{\tau_n}^{(k_j)}$ is a norm 1 operator the second integral goes to 0. From the lemma 2.2 of T-Y Li, for $\tilde{f} \in L^1$, the sequence $Q^{(k_j)}\tilde{f}$ converges in L^1 to $\tilde{f}$ as $k_j \to \infty$.

Which implies, $P_{\tau_n}^{(k_j)}\tilde{f} = Q^{(k_j)} P_{\tau_n}\tilde{f} \to P_{\tau_n}\tilde{f}$ in L^1 as $k_j \to \infty$. Therefore, the third term also goes to zero.

□

Theorem 4.3.22. *Let $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$ be a piecewise convex map with countably many branches. As described at the beginning of subsection 4.3.1, let $\{\tau_n\}_{n=1}^{\infty}$ be the approximating sequence of piecewise convex maps with finite numbers of branches. Let $P_{\tau_n}^{(k)}$, $k = 1, 2, \ldots$ be the sequence of Ulam's operators approximating operators P_{τ_n}. Let $f_{n,k}$ be the normalized (in L^1) fixed point of $P_{\tau_n}^{(k)}$. Then, the family $\{f_{n,k}\}_{n=1,2,\ldots,k=1,2,\ldots}$ is uniformly bounded in L^∞ and weakly compact in L^1. If f_{n_j,k_j}, $j = 1, 2, \ldots$ is a weakly convergent subsequence, then it converges in L^1 (and almost everywhere) to a function f which is a fixed point of P_τ, $P_\tau f = f$.*

Proof. The stationary densities $\{f_n\}_{n\geq 1}$ of $\{\tau_n\}_{n\geq 1}$ are uniformly bounded in L^∞ by Theorem 4.3.7. Moreover, the densities $f_{n,k}$, the piecewise constant approximations of f_n's are also uniformly bounded in L^∞ by formula (4.3.21). Thus, the set $\{f_{n,k}\}_{k=1}^\infty$ is weakly compact in L^1. Assume that $\{f_{n,k}\}_{k=1}^\infty$ has weakly convergent subsequence $\{f_{n_j,k_j}\}$ with limit f. Since the functions $\{f_{n_j,k_j}\}$ are decreasing and uniformly bounded, they also have uniformly bounded variations. By Helly's Theorem there is a further subsequence $\{f_{n_{j_s},k_{j_s}}\}$ that converges pointwise to some function h. Since subsequence of $\{f_{n,k}\}_{k=1}^\infty$ converges weakly to f, we have $f = h$. Thus, $\{f_{n,k}\}_{k=1}^\infty$ converges to f pointwise. By the Lebesgue Dominated Convergence Theorem, $f_{n,k} \to f$ in L^1.

It remains to show that f is a fixed point of P_τ, $P_\tau f = f$. We will show that the measures $fd\lambda$ and $(P_\tau f)d\lambda$ are equal. It is enough to show that for any $g \in C(I)$, we have $\int g(f - P_\tau f)d\lambda = 0$. To simplify the notation we assume that the whole sequence $f_{n,k}$ converges to f. We have,

$$
\begin{aligned}
&|\int g(f - P_\tau f)d\lambda| \leq |\int g(f - f_{n,k})d\lambda| + |\int g(f_{n,k} - P_{\tau_n}^{(k)} f_{n,k})d\lambda| \\
&+|\int g(P_{\tau_n}^{(k)} f_{n,k} - P_{\tau_n} f_{n,k})d\lambda| + |\int g(P_{\tau_n} f_{n,k} - P_{\tau_n} f)d\lambda| + |\int g(P_{\tau_n} f - P_\tau f)d\lambda|.
\end{aligned}
$$

Since $f_{n,k} \to f$ in L^1, the first term goes to 0 and $k \to \infty$. Since $f_{n,k}$ are fixed points of $P_{\tau_n}^{(k)}$, the second term is 0.

Third integral:

$$
\int g(P_{\tau_n}^{(k)} f_{n,k} - P_{\tau_n} f_{n,k})d\lambda = \int g(Q^{(k)}(P_{\tau_n} f_{n,k}) - P_{\tau_n} f_{n,k})d\lambda
$$

We have $Q^{(k)}h \to h$ in L^1 for any $h \in L^1$. Since the densities $\{f_{n,k}\}$ form a pre-compact set in L^1, the convergence is uniform on this set. The third integral converges to 0 as $k \to +\infty$.

Since $f_{n,k} \to f$ in L^1 and P_{τ_n}'s are norm 1 operators, the fourth integral converges to 0 (as $n, k \to +\infty$).

The last integral. By properties of the Frobenius-Perron operator, we have

$$
|\int g(P_{\tau_n} f - P_\tau f)d\lambda = |\int (g \circ \tau_n - g \circ \tau) f d\lambda
$$

To show that this converges to 0, we fix an $\varepsilon > 0$. Let $M = \sup |g|$. For integrable function f we can find a $\delta_1 > 0$ such that $\int_A |fd\lambda| < \varepsilon/2M$ for any set A with $\lambda(A) < \delta_1$. Since $\tau_n \to \tau$ almost uniformly, we can find a set A with $\lambda(A) < \delta_1$ such that $\tau_n \to \tau$ uniformly on A^c. Function g is continuous, so for its modulus of continuity ω_g we can find a δ_2 such that $\omega_g(\delta_2) < \varepsilon$. Now, we can find an $N \geq 1$ such that for $n \geq N$ we have $|\tau_n - \tau| < \delta_2$ on A^c. Then, for $n \geq N$ we write

$$
\begin{aligned}
&|\int (g \circ \tau_n - g \circ \tau) f d\lambda \leq \int |g \circ \tau_n - g \circ \tau)| f d\lambda \\
&= \int_A |g \circ \tau_n - g \circ \tau| f d\lambda + \int_{A^c} |g \circ \tau_n - g \circ \tau| f d\lambda \leq 2M \cdot \varepsilon/2M + \omega_g(\delta_2) \cdot 1 \leq 2\varepsilon.
\end{aligned}
$$

This shows that the last integral converges to 0 as $n \to +\infty$. □

4.4 Examples

Example 4.4.1. Consider the piecewise expanding and piecewise linear map $T : [0,1] \to [0,1]$ with countable number of branches defined as

$$
T(x) = i(i+1)\left(x - \frac{1}{i+1}\right) \quad \text{on} \quad \left[\frac{1}{\mathrm{i}+1}, \frac{1}{\mathrm{i}}\right], \mathrm{i} = 1, 2, \cdots . \tag{4.4.2}
$$

See Figure 4.1 for a graph of T. It shows that the Lebesgue measure is invariant under T. Derivative

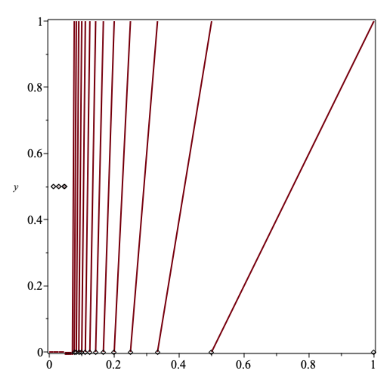

Figure 4.2: The graph of the piecewise expanding and piecewise linear map T, for Example, 4.4.1.

of T is $i(i+1)$ and its reciprocal, i.e., $\frac{1}{T'(x)} = \frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$. Therefore, $\sum_{i=1}^{\infty} \frac{1}{T'(x)} = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1$, which means that the Lebesgue measure is invariant under T. The slope of T on $\left[\frac{1}{i+1}, \frac{1}{i}\right]$ is $\frac{1}{\frac{1}{i}-\frac{1}{i+1}}$ and therefore, the inverse of T has slope $\frac{1}{i} - \frac{1}{i+1}$. Then, with the density $f = \mathbf{1}$, $P_T f = \sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1$. Thus, $P_T f = f$ and $f = \mathbf{1}$ is the invariant density of T. Now, consider the conjugation $h : [0,1] \to [0,1]$ defined by $h(x) = 1 - (1-x)^2$. We construct the piecewise convex map $\tau : [0,1] \to [0,1]$ with countable number of branches defined by $\tau = h^{-1} \circ T \circ h$. See Figure 4.2 for a graph of τ. The piecewise convex map τ is topologically

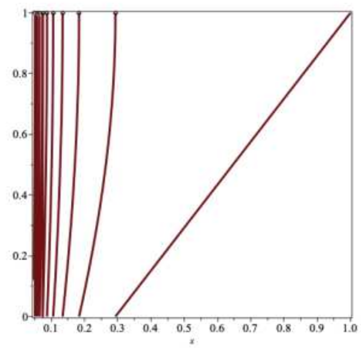

Figure 4.3: The graph of the piecewise convex map τ with countable number of branches for Example 4.4.1.

conjugated to the piecewise linear and piecewise expanding map T via the conjugation h. Therefore, the stationary density g of τ is given by $g = f \circ h \times |h'|$ (see proof of Theorem 5.2.2 in Chapter 5). Now, $h(x) = 1 - (1-x)^2$. Hence, $g(x) = f(h(x)) \times |h'(x)| = |2(1-x)|$. See Figure 4.3 for a graph of the stationary density g of τ.

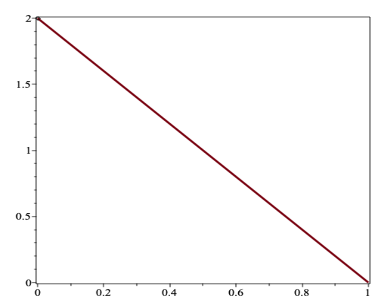

Figure 4.4: The graph of the stationary density g of the piecewise convex map τ with countable number of branches for Example 4.4.1.

Now, we find the first few branches (from right) of τ on $[0,1]$. Note that $\tau(x) = (h^{-1} \circ T \circ h)(x)$, where $h(x) = 1 - (1-x)^2$, $h^{-1}(x) = 1 - \sqrt{(1-x)}$. See Figure 4.4 for graphs of h and h^{-1}.

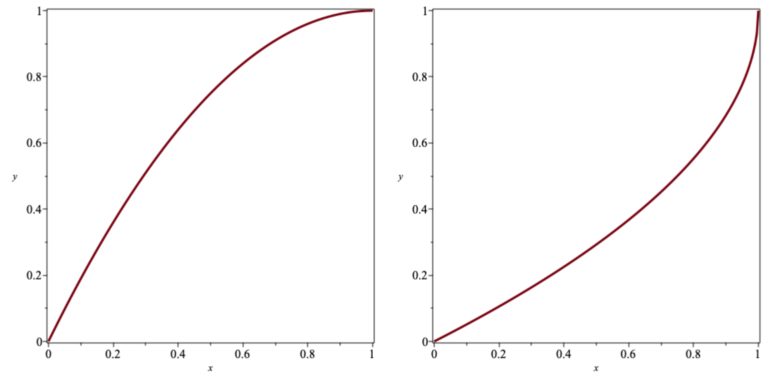

Figure 4.5: Graphs of t H (left) and h^{-1} (right).

The map T is piecewise onto the partition

$$\left\{\left[\frac{1}{i+1},\frac{1}{i}\right]\right\}_{i=1}^{\infty}=\left\{\dots,\frac{1}{22},\frac{1}{21},\frac{1}{20},\frac{1}{19},\dots,\frac{1}{7},\frac{1}{6},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{1}{2},1\right\}.$$

Moreover, $h\left(1-\sqrt{\frac{i}{i+1}}\right)=\frac{1}{i+1}, i=1,2,3,\dots$. Thus, the map τ is defined on the partition

$$\left\{\left[1-\sqrt{\frac{i}{i+1}},1-\sqrt{\frac{i-1}{i}}\right]\right\}_{i=1}^{\infty}=\left\{\dots,1-\sqrt{\frac{4}{5}},1-\sqrt{\frac{3}{4}},1-\sqrt{\frac{2}{3}},1-\sqrt{\frac{1}{2}},1\right\}.$$

If $x\in[1-\sqrt{\frac{1}{2}},1]$, then $h(x)\in[\frac{1}{2},1]$. If $x\in[\frac{1}{2},1]$, then $T(x)\in[0,1]$. If $x\in[0,1]$, then $h^{-1}(x)\in[0,1]$. Thus, if $x\in[1-\sqrt{\frac{1}{2}},1]$, then $\tau(x)=\tau=(h^{-1}\circ T\circ h)(x)\in[0,1]$. Moreover, on $[1-\sqrt{\frac{1}{2}},1]$,

$$\begin{aligned}\tau(x) &= (h^{-1}\circ T\circ h)(x)\\ &= h^{-1}(T(1-(1-x)^2))\\ &= h^{-1}(2(1-(1-x)^2)-1)\\ &= 1-\sqrt{1-(1-2(1-x)^2)}\\ &= 1-\sqrt{2}(1-x).\end{aligned}$$

In a similar way, we can find other branches of τ on the partition $\left\{\left[1-\sqrt{\frac{i}{i+1}},1-\sqrt{\frac{i-1}{i}}\right]\right\}_{i=1}^{\infty}=$

$\left\{\ldots, 1-\sqrt{\frac{3}{4}}, 1-\sqrt{\frac{2}{3}}, 1-\sqrt{\frac{1}{2}}, 1\right\}$.

The first few, starting from 1 to the left, the branches of τ on

$$\left[1-\sqrt{\frac{1}{2}}, 1\right], \left[1-\sqrt{\frac{2}{3}}, 1-\sqrt{\frac{1}{2}}\right], \left[1-\sqrt{\frac{3}{4}}, 1-\sqrt{\frac{2}{3}}\right], \left[1-\sqrt{\frac{4}{5}}, 1-\sqrt{\frac{3}{4}}\right], \left[1-\sqrt{\frac{5}{6}}, 1-\sqrt{\frac{4}{5}}\right]$$

are $1-\sqrt{2}(1-x), 1-\sqrt{6(1-x)^2-3}, 1-\sqrt{12(1-x)^2-8}, 1-\sqrt{20(1-x)^2-15}, 1-\sqrt{30(1-x)^2-24}$ respectively.

Now, consider the following sequence $\{\tau_n\}_{n\geq 0}$ of piecewise convex map $\tau_n : [0,1] \to [0,1]$ with finite number of branches:

$$\tau_n(x) = \begin{cases} \frac{1}{1-\sqrt{\frac{n}{n+1}}} x \ , \ 0 \leq x < 1 - \sqrt{\frac{n}{n+1}} \\ \tau(x) \ , \ 1 - \sqrt{\frac{n}{n+1}} \leq x \leq 1. \end{cases}$$

See Figure 4.6 for a graph of τ_n with $n = 5$. The sequence $\{\tau_n\}_{n\geq 0}$ of piecewise convex map

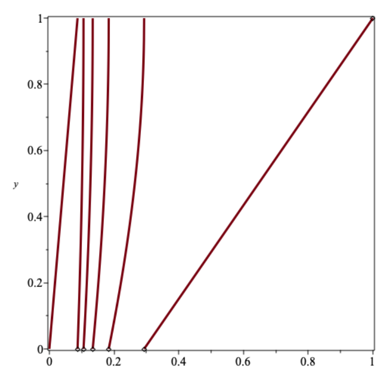

Figure 4.6: Piecewise convex map τ_n with finite number of branches ($n = 5$).

$\tau_n : [0,1] \to [0,1]$ with finite number of branches converges almost uniformly to τ with countable number of branches. In Figure 4.7, we present a graph of approximate stationary density $f_{n,k}, n = 5, k = 100$ via Ulam's method of the actual stationary density of $f_n, n = 5$ of the piecewise convex map $\tau_n, n = 5$ with a finite number of branches. Note that $\tau_n, n = 5$ is an approximation of the piecewise convex map τ. In Figure 4.7, we present the density of the piecewise convex map τ with a finite number of branches and the graph of the approximate stationary density $f_{n,k}, n = 5, k = 100$ via Ulam's method. In Figure 4.8, we present the graph of the actual density g (in red) of

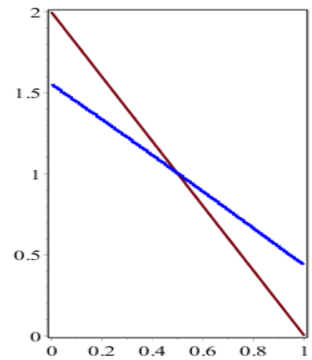

Figure 4.7: The approximate stationary density $f_{n,k}, n = 5, k = 100$ via Ulam's method of the piecewise convex map τ_n with finite number of branches ($n = 5$).

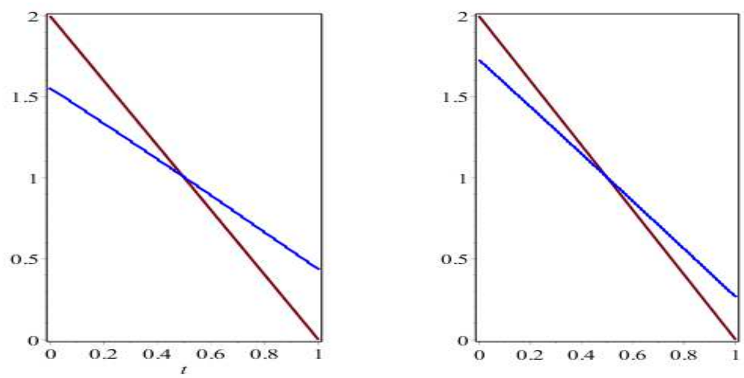

Figure 4.8: The graph of the actual invariant density g of the piecewise convex map τ with infinite number of branches(in red) and the graph of of the approximating density $f_{n,k}$ (in blue): $n = 5, k = 1000$ on the left and $n = 10, k = 1000$ on the right hand side.

the piecewise convex map τ with countable number of branches and graphs of the approximate stationary densities $f_{n,k}$ (in blue) via Ulam's method for maps τ_n with a finite number of branches. Numerical computations are performed for a number of cases. In the following table, we present the L^1 norm error $||g - f_{n,k}||_1$. Note that for each n, τ_n is a map with a finite number of branches, approximating the piecewise convex map τ in Figure 4.3 with countable number of branches.

n	k	$\|\|g - f_{n,k}\|\|_1$
5	100	0.2195470623
5	1000	0.2195243505
6	100	0.1943541673
6	1000	0.1943541673
7	1000	0.1742407352
10	1000	0.133493040

The above table shows that as we increase n, the L^1 norm error $||g - f_{n,k}||_1$ gets smaller. For the fixed n, the increasing of k is ineffective. For example, the errors for $k = 100$ and $k = 1000$ are almost identical. The main method to lower the error is to increase the number of branches of τ_n. Theorem 4.3.22 confirms that for large n and large k the L^1 norm error is close to 0.

Example 4.4.3. Consider the piecewise convex map $\tau : [0, 1] \to [0, 1]$ with countable number of branches defined as

$$\tau(x) = \frac{1}{\frac{2i+1}{i(i+1)} - x} - i \quad \text{on} \quad \left[\frac{1}{i+1}, \frac{1}{i}\right], i = 1, 2, \cdots . \tag{4.4.4}$$

See Figure 4.9 for a graph of τ which is defined on the countable number of partition $b_0 = 1, b_1 = \frac{1}{2}, b_2 = \frac{1}{3}, \cdots, b_n = \frac{1}{n+1}, \cdots$ of $[0, 1]$. It is shown in [23] that $\tau \in \mathcal{T}_{pc}^{\infty,0}(I)$ and hence by Theorem

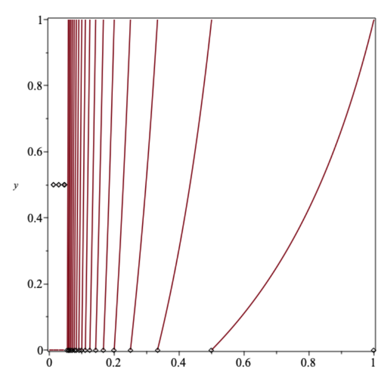

Figure 4.9: Piecewise convex map with countable number of branches, for example, 4.4.3.

4.2.6, τ has an acim. Now, consider the following sequence $\{\tau_n\}_{n\geq 0}$ of piecewise convex map $\tau_n : [0, 1] \to [0, 1]$ with finite number of branches:

$$\tau_n(x) = \begin{cases} (n+1)x \ , \ 0 \leq x < \frac{1}{n+1}; \\ \\ \tau(x) \ , \ \frac{1}{n+1} \leq x \leq 1. \end{cases}$$

See Figure 4.10 for a graph of τ_n with $n = 4, 8$. The sequence $\{\tau_n\}_{n\geq 0}$ of piecewise convex map

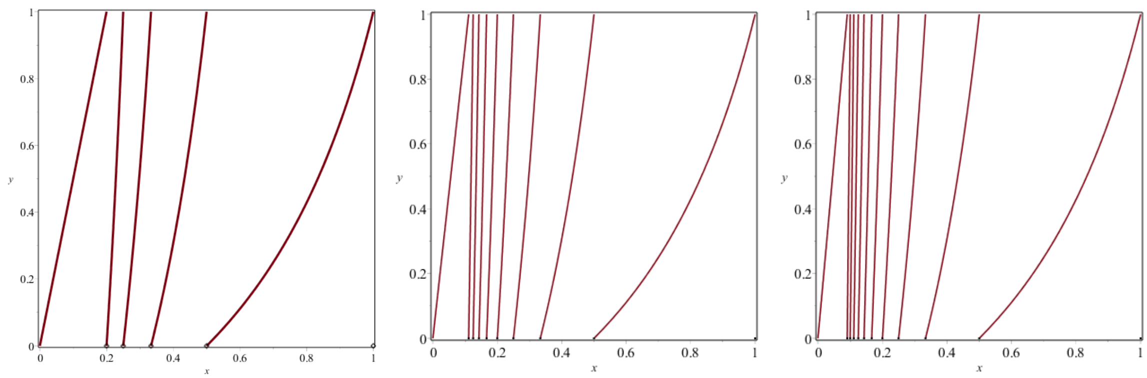

Figure 4.10: Piecewise convex map τ_n with finite number of branches ($n = 4$, $n = 8$ and $n = 10$).

$\tau_n : [0, 1] \to [0, 1]$ with finite number of branches converges almost uniformly to τ. In Figure 4.11, we compare three graphs of approximate stationary density $f_{n,k}, n = 4, k = 60, k = 120$ and $k = 240$ respectively via Ulam's method of the actual stationary density of $f_n, n = 4$ of the piecewise convex map $\tau_n, n = 4$ with finite number of branches. In Figure 4.12, we present a graph of Ulam's approximation $f_{n,k}, n = 10, k = 1000$ of the actual invariant density of $f_n, n = 10$ of the piecewise convex map $\tau_n, n = 10$ with a finite number of branches. By Theorem 4.3.22, it is also an approximation of the invariant density f^* of the map τ. The same Figure 4.12 shows also the approximation $f_{n,k}, n = 10, k = 1000$, as the densities $f_{10,1000}$ and $f_{10,500}$ are indistinguishable at this scale. We have $||f_{10,1000} - f_{10,500}||_1 \sim 0.00055$. In Figure 4.13, we show the enlargement of both graphs on the small subinterval.

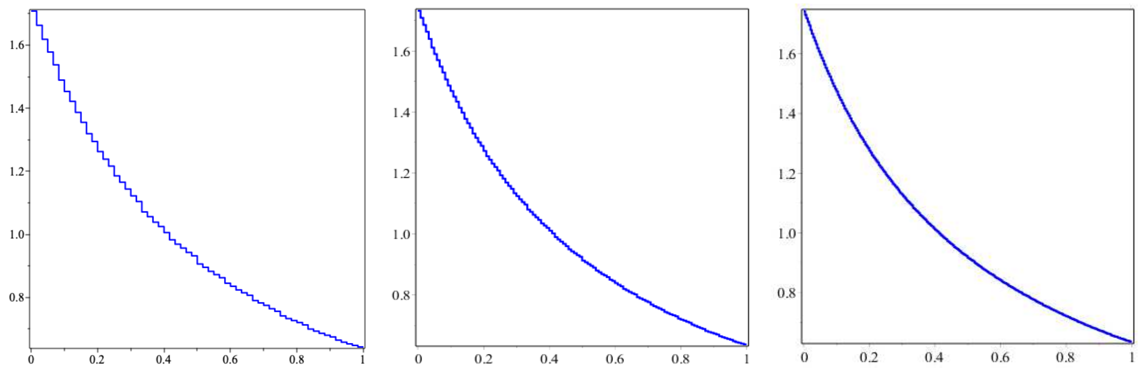

Figure 4.11: The graphs of approximate stationary density $f_{n,k}, n = 4, k = 60, k = 120$ and $k = 240$ respectively via Ulam's method of the actual stationary density of $f_n, n = 4$ of the piecewise convex map $\tau_n, n = 4$ with finite number of branches. The map $\tau_n, n = 4$ is an approximation of the piecewise convex map τ in Figure 4.9

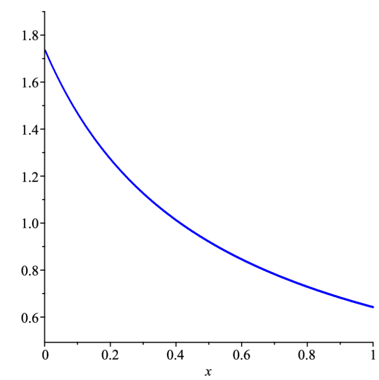

Figure 4.12: A graph of Graph of the Ulam's approximation $f_{n,k}, n = 10, k = 1000$.

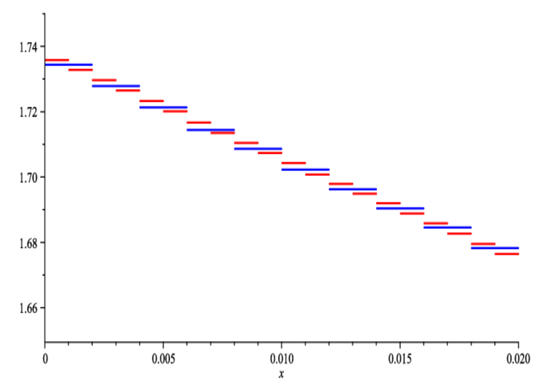

Figure 4.13: Enlargement of the approximating densities $f_{10,1000}$ (in red) and $f_{10,500}$ (in blue) on $[0, 0.02]$.

Chapter 5

ACIMs for Piecewise concave maps with countable (infinite) number of limit points of partition points

5.1 Introduction

In the previous two Chapters, we focused on analyzing piecewise convex maps characterized by a countable (infinite) number of branches. Now, let's shift our attention to another significant category of maps: concave maps. Piecewise concave maps on the interval [0, 1] with infinite number of branches are another important class in dynamical systems, with applications in areas such as optimization, economics, and physics. In [15], the existence of ACIMs is proved for a class of piecewise concave interval maps with a finite number of branches. One interesting aspect of piecewise concave maps is that they can be conjugated to piecewise convex maps on [0, 1], which allows us to use results from the theory of piecewise convex maps to study their properties. Conjugation is a powerful tool in dynamical systems that allows us to transform one system into another while preserving certain properties, such as the existence of invariant measures.

In the last two Chapters, the existence of a unique normalized absolutely continuous invariant measure is proved for two classes, $\mathcal{T}_{pc}^{\infty}(I)$ and $\mathcal{T}_{pc}^{\infty,0}(I)$, of piecewise convex mapping and we also proved the convergence of Ulam's approximation method for computing the invariant measure of piecewise convex mapping. In this Chapter, we want to show similar results for two classes, $\mathcal{T}_{pcv}^{\infty}(I)$ and $\mathcal{T}_{pcv}^{\infty,1}(I)$, of piecewise concave maps σ. We use conjugation of τ that is defined in Chapter 3 (page: 31), which implies that σ preserves a normalized absolutely continuous invariant measure whose density is an increasing function.

5.2 ACIMs for piecewise concave maps on [0, 1] with countable (infinite) number of branches

5.2.1 Piecewise concave maps with countable (infinite) number of limit points separated from 1

Consider $(I, \mathcal{B}, \lambda)$ be a measure space, where λ is the Lebesgue measure on $I = [0, 1]$ and $\mathcal{B}$ is the Borel σ-algebra on I. Let $\{1 = a_0, a_1, a_2, \dots, a_n, \dots\}$ be a countable partition of I such that $a_0 > a_1$ and all $a_2, a_3, \dots \in [0, a_1]$. We do not assume the sequence $\{a_2, \dots, a_n, \dots\}$ to be increasing or decreasing. For any $i \in \{0, 1, 2, \dots\}$, let $n(i)$ be the index such that the interval $[a_{n(i)}, a_i]$ does not contain any other points of the partition. If a_k is the limit point of increasing subsequence of a_n's, the $n(k)$ is not defined. We say that a non-singular transformation $\sigma \in \mathcal{T}_{pcv}^{\infty}(I)$ if

(1) $\sigma_1 = \sigma|_{(a_1,1]}$ is continuous and concave;
 $\sigma_i = \sigma|_{(a_{n(i)},a_i]}$ is continuous and concave, $i = 1, 2, \cdots$;

(2) $\sigma(a_i) = 1, \sigma'(a_i) > 0, i = 1, 2, \dots$;

(3) $\sigma(a_0) = 1, \sigma'(a_0) = \alpha_1 > 1$;

(4) $\sum_{i=1}^{\infty} \frac{1}{\sigma'(a_i)} < \infty$.

For $\sigma \in \mathcal{T}_{pcv}^{\infty}(I), g \in L^1(I), g \geq 0$ the Frobenius-Perron operator P_σ is defined as

$$P_\sigma g(x) = \frac{g(\sigma_1^{-1}(x))}{\sigma'(\sigma_1^{-1}(x))}\chi_{\sigma(a_1,1]}(x) + \sum_{i=1}^{\infty} \frac{g(\sigma_i^{-1}(x))}{\sigma'(\sigma_i^{-1}(x))}\chi_{\sigma(a_{n(i)},a_i]}(x) \tag{5.2.1}$$

We construct a piecewise convex map $\tau : [0,1] \to [0,1]$ with countable number of branches such that it is conjugated to σ. Consider the diffeomorphism $h : [0,1] \to [0,1]$ defined by $h(x) = 1 - x$. It can be shown (see A.2 in Appendix for proof) that the map $\tau : [0,1] \to [0,1]$ defined by $\tau = h^{-1} \circ \sigma \circ h$ is a piecewise convex map with countable number of branches and τ belong to the class $\mathcal{T}_{pc}^{\infty}(I)$ of piecewise convex maps with countable number of branches (see Chapter 3).

If f is a τ-invariant density then $g = \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'|$ is a σ-invariant density according to the following theorem.

Theorem 5.2.2. *[5] Let $(I, \mathcal{B}, \mu, \tau)$ and $(I, \mathcal{B}, \nu, \sigma)$ be the dynamical system and let $\tau : [0,1] \to [0,1]$ be nonsingular. Let $h : [0,1] \to [0,1]$ be a diffeomorphism. Then*

$$P_\tau f = f \text{ implies } P_\sigma g = g, \text{ where } \sigma = h \circ \tau \circ h^{-1} \text{ and } g = \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'|,$$

i.e. if f is a $\tau-$ invariant density, then g is a $\sigma-$invariant density.

Proof. Let $P_\tau f = f$. Using the properties of Frobenius-Perron operator we have,

$$\begin{aligned} P_\sigma\left(P_h f\right) &= P_{h\circ\tau\circ h^{-1}}\left(P_h f\right) \\ &= P_h \circ P_\tau \circ P_{h^{-1}\circ h} f \\ &= P_h f. \end{aligned}$$

To prove $P_\sigma g = g$, we need to show that $P_h f = g$. We have

$$\begin{aligned} P_h f(x) &= \sum_{i=1}^{n} \left(f \circ h_i^{-1}\right) \cdot |\left(h_i^{-1}\right)'| \chi_{[a_{i-1},a_i)]} \\ &= \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'| \\ &= g \end{aligned}$$

since h is monotonic and also a diffeomorphism on $[0, 1]$. Thus, we have

$$P_\sigma g = P_\sigma \left(P_h f\right) = P_h f = g.$$

□

We can also prove the existence of the ACIM of σ directly without using the conjugation.

Proposition 5.2.3. *If* $g \geq 0$ *and* g *is non-decreasing, then* $(1 - x)\, g(x) \leq \lambda(g),\ for\ x \in [0, 1]$, *where*

$$\lambda(g) = \int_0^1 g(x) d\lambda(x).$$

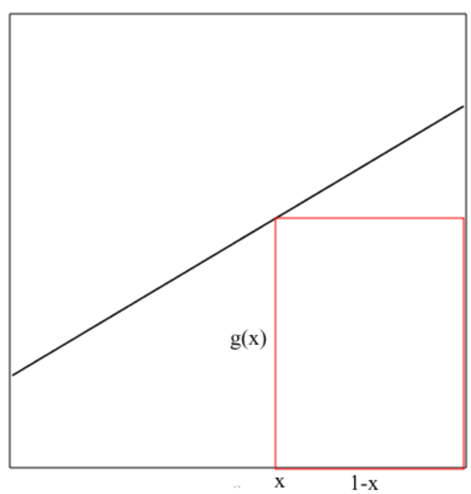

Figure 5.1: Graph for Prop. 5.2.3.

Proof. For any $0 < x \leq 1$, from the Figure 5.1

$$(1-x)\, g(x) \leq \int_x^1 g(t)dt \leq \int g = \lambda(g).$$

Since $g \geq 0$ and g is non-decreasing. Let $\forall t \in [x, 1]$ and $t \geq x, \implies g(t) \geq g(x)$. Therefore,

$$\int_x^1 g(t)dt \geq \int_x^1 g(x)dt = (1-x)g(x).$$

□

Lemma 5.2.4. *If* $g : [0,1] \to \mathbb{R}^+$ *is non-decreasing and* $\sigma \in \mathcal{T}_{pcv}^{\infty}(I)$. *Then*

$$\| P_\sigma(g) \|_\infty \leq \frac{1}{\alpha_1} \| g \|_\infty + D \| g \|_1, \tag{5.2.5}$$

where $D = \left(\sum_{i=1}^{\infty} \frac{1}{1-a_i} \frac{1}{\sigma'(a_i)} \right)$.

Proof. Since g is non-decreasing, $g(1) \geq \| g \|_\infty$, and so, $P_\sigma g(1) \geq \| P_\sigma g \|_\infty$.

$$P_\sigma g(1) \quad = \quad \frac{1}{\sigma'(1)} g(1) + \sum_{i=1}^{\infty} \frac{g(\sigma_i^{-1}(1))}{\sigma'(\sigma_i^{-1}(1))} \leq \frac{1}{\alpha_1} g(1) + \sum_{i=1}^{\infty} \frac{g(a_i)}{\sigma'(a_i)}$$

Since g is non-decreasing, from Proposition 5.2.3,

$$\lambda(g) \geq (1-a_i) \cdot g(a_i)$$

Therefore,

$$P_\sigma g(1) \leq \frac{1}{\alpha_1} g(1) + \sum_{i=1}^{\infty} \frac{\lambda(g)}{1-a_i} \frac{1}{\sigma'(a_i)} \leq \frac{1}{\alpha_1} \| g \|_\infty + \left(\sum_{i=1}^{\infty} \frac{1}{1-a_i} \frac{1}{\sigma'(a_i)} \right) \| g \|_1 .$$

□

Theorem 5.2.6. *Let* $\sigma \in \mathcal{T}_{pcv}^{\infty}(I)$. *Then* σ *admits an absolutely continuous invariant measure* $\nu = g^* \cdot \lambda$ *with non-decreasing density function* g^*.

Proof. The proof is analogous to the proof of Theorem 3.2.7 in Chapter 3. □

5.2.2 Piecewise concave maps with countable (infinite) number of branches where 1 is a limit point of partition points

Consider $(I, \mathcal{B}, \lambda)$ be a measure space, where λ is the Lebesgue measure on $I = [0, 1]$ and $\mathcal{B}$ is the Borel σ-algebra on I. Let $\{1 = a_0, a_1, a_2, \ldots, a_n, \ldots\}$ be a countable (infinite) partition of I such that $a_0 > a_1$ and all $a_2, a_3, \cdots \in [0, a_1]$. We do not assume the sequence $\{a_2, \ldots, a_n, \ldots\}$ to be increasing or decreasing. For any $i \in \{0, 1, 2, \ldots\}$, let $n(i)$ be the index such that the interval $[a_{n(i)}, a_i]$ does not contain any other points of the partition. If a_k is the limit point of increasing subsequence of a_n's, the $n(k)$ is not defined. Moreover, we consider an increasing sequence $\{a_{1,n}\}$ of partition points in $(a_1, a_0]$ such that $\{1 = a_0 > \cdots > a_{1,(n+1)} > a_{1,n} > a_{1,(n-1)} > \cdots > a_{1,2} > a_{1,1} = a_1\}$ and $\lim_{n\to\infty} a_{1,n} = a_0 = 1$. We say that a non-singular transformation $\sigma \in \mathcal{T}_{pcv}^{\infty,1}(I)$, if

(1) $\sigma_{1,j} = \sigma|_{[a_{1,j+1}, a_{1,j})}$ is continuous and concave, j=1, 2, . . . ;

$\sigma_i = \sigma|_{[a_i, a_{n(i)})}$ is continuous and concave, $i = 1, 2, \cdots$;

(2) $\sigma(a_{1,j}) = 1, \sigma'(a_{1,j}) > 0, j = 1, 2, \ldots;$

$\sigma(a_i) = 1, \sigma'(a_i) > 0, i = 1, 2, \cdots$;

(3) $\sum_{i=1}^{\infty} \frac{1}{\sigma'(a_i)} < \infty.$

(4) $D_1 = \sum_{j=1}^{\infty} \frac{1}{\sigma'(a_{1,j})} < 1.$

Remark 5.2.7. Condition (3) and Condition (4) can be replaced by the following Condition $(4)^+$, where

(3)$^+$. $\sum_{j=1}^{\infty} \frac{1}{\sigma'(a_{1,j})} + \sum_{i=1}^{\infty} \frac{1}{\sigma'(a_i)} < \infty$.

We construct a piecewise convex map $\tau : [0,1] \to [0,1]$ with countable number of branches such that it is conjugating to σ. Consider the diffeomorphism $h : [0,1] \to [0,1]$ defined by $h(x) = 1-x$. It can be easily shown (see A.2 in Appendix for proof) that the map $\tau : [0,1] \to [0,1]$ defined by $\tau = h^{-1} \circ \sigma \circ h$ is a piecewise convex map with countable number of branches and τ belong to the class $\mathcal{T}_{pc}^{\infty}(I) \cup \mathcal{T}_{pc}^{\infty,0}(I)$ of piecewise convex maps with countable number of branches (see Chapter 3). If f is a τ-invariant density then $g = \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'|$ is a σ-invariant density according to Theorem 5.2.2.

Again, we can prove the existence of the ACIM of σ directly.

Lemma 5.2.8. *If $g : [0,1] \to \mathbb{R}^+$ is non-decreasing and $\sigma \in \mathcal{T}_{pcv}^{\infty,1}(I)$. Then*

$$\| P_\sigma(g) \|_\infty \leq D_1 \| g \|_\infty + D \| g \|_1, \tag{5.2.9}$$

where $D = \sum_{i=1}^{\infty} \frac{1}{1-a_i} \frac{1}{\sigma'(a_i)}$.

Proof. The proof is analogous to the proof of Lemma 5.2.4. □

Theorem 5.2.10. *Let $\sigma \in \mathcal{T}_{pcv}^{\infty,1}(I)$. Then σ admits an absolutely continuous invariant measure $\nu = g^* \cdot \lambda$ with non-decreasing density function g^*.*

Proof. The proof is analogous to the proof of Theorem 3.2.7 in Chapter 3. □

5.3 Ulam's method for piecewise concave maps with countable (infinite) number of branches

5.3.1 Approximation of piecewise concave maps with countable (infinite) number of branches by piecewise concave maps with finite number of branches

Let $\sigma : [0,1] \to [0,1]$ be a piecewise concave map in $\mathcal{T}_{pcv}^{\infty}(I) \cup \mathcal{T}_{pcv}^{\infty,1}(I)$ with countable number of branches. In this section, we will describe the most difficult case when 1 is the limit of the partition points for map $\sigma \in \mathcal{T}_{pcv}^{\infty,1}$. We assume that there are no other limit points of the partition points. The other cases, i.e., where there are such limit points or $\sigma \in \mathcal{T}_{pcv}^{\infty}$, are done similarly. Thus, the map σ is like in Section 5.2.2. For simplicity, we change the notation by renaming the partition points. Let $a_n = a_{1,n}$, $n = 1, 2, \ldots$. Then, the assumptions (3) and (4) of Section 5.2.2 are restated as:

(3′) There exists an $N \geq 1$ such that

$$D_1 = \sum_{n=N+1}^{\infty} \frac{1}{\sigma'(a_n)} < 1. \tag{5.3.1}$$

Then, the Lemma 5.3.4 of Section 5.2.2 hold with changed constants

$$D = \sum_{n=1}^{N} \frac{1}{1-a_n} \frac{1}{\sigma'(a_n)} \text{ and } D_1 = \sum_{n=N+1}^{\infty} \frac{1}{\sigma'(a_n)} < 1 \tag{5.3.2}$$

For $n \geq N$, we construct a sequence $\{\sigma_n\}_{n=N}^{\infty}$ of maps $\sigma_n : [0,1] \to [0,1]$ s.t. σ_n has finite number of branches and σ_n converges to σ almost uniformly. Using supremum norms and Lasota-Yorke type inequalities, we prove the existence of stationary densities g_n of ACIMs ν_n for σ_n. We approximate $\sigma : [0,1] \to [0,1]$ with the following sequence of maps $\sigma_n : [0,1] \to [0,1]$, $n \geq N$, with finite number of branches:

$$\sigma_n(x) = \begin{cases} x/a_n \ , \ a_n \leq x \leq 1; \\ \sigma(x) \ , \ 0 \leq x \leq a_n. \end{cases}$$

In the following, we show that for each $n \geq 0$, the map σ_n has an absolutely continuous invariant measure. It shows that each σ_n is a piecewise convex maps with finite partition $\{0 = a_0, a_1, a_2, \cdots, a_n, a_{n+1} = 1\}$ and σ_n satisfies following conditions:

(1) $\sigma_{n_j} = \sigma_n|(a_j, a_{j-1}]$ is continuous and concave, $j = 1, 2, \cdots, n+1$;

(2) $\sigma_n(a_j) = 1, \sigma_n'(a_j) > 0, j = 1, \cdots, n+1$;

(3) $\sigma_n(1) = 1, \sigma_n'(1) = \frac{1}{1-a_n} > 1$.

(4) $\sum_{j=1}^{n+1} \frac{1}{\sigma'(a_j)} < \infty$.

Let $g \in L^1(I), g \geq 0$. Then, the Frobenius-Perron operator P_{σ_n} is defined as

$$P_{\sigma_n} g(x) = \sum_{j=1}^{n+1} \frac{g(\sigma_{n_j}^{-1}(x))}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \chi_{\sigma_n[a_j, a_{j-1})}(x) \tag{5.3.3}$$

Lemma 5.3.4. *Let $g \in L^1(I), g \geq 0, g$ non-decreasing. Then*

(1) $P_{\sigma_n}(g) \in L^1(I)$ for each $n = 1, 2, \ldots$.

(2) $P_{\sigma_n}(g) \geq 0$ for each $n = 1, 2, \ldots$.

(3) $P_{\sigma_n}(g)$ is non-decreasing for each $n = 1, 2, \ldots$.

(4) $\| P_{\sigma_n}(g) \|_\infty \leq C_n \| g \|_\infty \leq (1 + C) \| g \|_\infty$ for each $n = 1, 2, \ldots$ where $C_n = \left(1 - a_n + \sum_{j=1}^{n} \frac{1}{\sigma_n'(a_j)}\right)$.

Proof. (1)

$$P_{\sigma_n}(g) = \int_{\sigma_n^{-1}(I)} g d\lambda = \int_I g.$$

Therefore, $P_{\sigma_n}(g) \in L^1(I)$

(2) Note that

$$P_{\sigma_n} g(x) = \sum_{j=1}^{n+1} \frac{g(\sigma_{n_j}^{-1}(x))}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \chi_{\sigma_n[a_j, a_{j-1})}(x).$$

Each of the branches $\frac{g(\sigma_{n_j}^{-1}(x))}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \chi_{\sigma_n[a_j, a_{j-1})}(x)$ is non-negative.

(3) Each of the branches $\frac{g(\sigma_{n_j}^{-1}(x))}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \chi_{\sigma_n[a_j, a_{j-1})}(x)$ is non-decreasing since $\sigma_{n_j}^{-1}$ is decreasing, g is non-decreasing and σ_n' is decreasing.

(4) We have

$$
\begin{aligned}
P_{\sigma_n} g(x) &= \sum_{j=1}^{n+1} \frac{g(\sigma_{n_j}^{-1}(x))}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \chi_{\sigma_n[a_j,a_{j-1})}(x) \\
&\leq \sum_{j=1}^{n+1} \frac{\| g \|_\infty}{\sigma_n'(\sigma_{n_j}^{-1}(x))} \leq \sum_{j=1}^{n+1} \frac{\| g \|_\infty}{\sigma_n'(a_j)} \\
&= \left(1 - a_n + \sum_{j=1}^{n} \frac{1}{\sigma_n'(a_j)} \right) \| g \|_\infty \\
&\leq (1+C) \| g \|_\infty
\end{aligned}
$$

□

Lemma 5.3.5. *If $g : [0,1] \to \mathbb{R}^+$ is non-decreasing, then for each $n \geq N$,*

$$
\| P_{\sigma_n}(g) \|_\infty \leq (a_n + D_1) \| g \|_\infty + D \| g \|_1, \tag{5.3.6}
$$

Proof. Since g is non-decreasing, $g(1) \geq \| g \|_\infty$, and by Lemma 5.3.4, $P_{\sigma_n} g(1) \geq \| P_{\sigma_n} g \|_\infty$. Now,

$$
\begin{aligned}
P_{\sigma_n} g(1) &= \sum_{j=1}^{n+1} \frac{g(\sigma_{n_j}^{-1}(1))}{\sigma_n'(\sigma_{n_j}^{-1}(1))} \chi_{\sigma_n(a_j,a_{j-1}]}(1) \\
&= \frac{1}{\sigma_n'(1)} g(1) + \sum_{j=N}^{n} \frac{g(\sigma_{n_j}^{-1}(1))}{\sigma_n'(\sigma_{n_j}^{-1}(1))} + \sum_{j=1}^{N} \frac{g(\sigma_{n_j}^{-1}(1))}{\sigma_n'(\sigma_{n_j}^{-1}(1))} \\
&= (1-a_n) g(1) + \sum_{j=N}^{n} \frac{g(a_{n_j})}{\sigma_n'(a_{n_j})} + \sum_{j=1}^{N} \frac{g(a_{n_j})}{\sigma_n'(a_{n_j})} \\
&\leq (1 - a_n + D_1) g(1) + \sum_{j=1}^{N} \frac{\lambda(g)}{a_j} \frac{1}{\sigma_n'(a_j)} \\
&\leq (1 - a_n + D_1) \| g \|_\infty + D \| g \|_1 .
\end{aligned}
$$

□

Theorem 5.3.7. *For each $n \in \mathbb{N}$, σ_n admits an absolutely continuous invariant measure $\nu_n = g_n^* \cdot \lambda$ with non-decreasing density function g_n^*.*

Proof. The proof is analogous to the proof Theorem 4.3.7 in Chapter 4. □

Definition 5.3.8. Let σ_n, σ are maps on $[0, 1]$ into itself and σ_n, σ are defined as above. We say that σ_n converges to σ almost uniformly if, given $\epsilon > 0$, there exists a measurable set $A_\epsilon \subset [0, 1], \lambda(A_\epsilon) > 1 - \epsilon$, such that $\sigma_n \to \sigma$ uniformly on A_ϵ.

Lemma 5.3.9. *σ_n converges to σ almost uniformly.*

Proof. Let $\epsilon > 0$. Choose the increasing partition $\{0 = a_0, a_1, a_2, \cdots, a_n, a_{n+1} = 1\}$ of $[0, 1]$ for σ_n such that $1 - a_n < \epsilon$. Let $A_\epsilon = (0, a_n)$. Then, $\lambda(A_\epsilon) = a_n > 1 - \epsilon$. Since $\sigma_n = \sigma$ on A_ϵ the proof is complete.

□

5.3.2 Ulam's method

In this subsection, first, we describe Ulam's method for finite-dimensional approximation $P_{n,k}$, n and k denoted by the number of branches and number of partitions, respectively, of the Frobenius-Perron operator P_{σ_n} of σ_n. Ulam's method computes $g_{n,k}$ on a partition of k subintervals of the state space as an approximation of the actual stationary density function g_n of $\sigma_n,\ n \geq 1$. Moreover, we show that $g_{n,k}$ converges to g_n as $k \to \infty$. Let σ_n is an approximation of $\sigma \in \mathcal{T}^{\infty}_{pcv}(I) \cup \mathcal{T}^{\infty,1}_{pcv}(I)$. Then, by the Theorem 5.3.7, σ_n has an absolutely continuous invariant measure ν_n with stationary density function g_n. The approximation g_n is carried out using a two-step process. In the beginning, we approximate σ by the map σ_n with a finite number of branches. Then, we further approximate σ_n by using Ulam's method. In our case, we don't need approximation in the L^1 norm. So, it's not an approximation of any norm. In this sense, approximation means τ_n converges τ almost uniformly. Now, we describe Ulam's method for approximating g_n. Let k be a positive integer. Let $\mathcal{P}^{(k)} = \{J_1, J_2, \ldots, J_k\}$ be a partition of the interval $[0, 1]$ into k equal subintervals. In the deterministic case, we construct the matrix approximation of the F-P operator as the form

$$\mathbf{M}_{\sigma_n} = \left(\frac{\lambda\left(\sigma_n^{-1}(J_j) \cap J_i\right)}{\lambda(J_i)} \right)_{1 \leq i,j \leq k}$$

where λ denotes the normalized Lebesgue measure on J and $\{J_i\}_{i=1}^k$ is a finite family of connected sets with nonempty and adjoint interiors that cover J i.e., $J = \cup_{i=1}^k J_i$, and indexed in terms of nested refinements. Let $L^{(k)} \subset L^1([0,1],\lambda)$ be a subspace of L^1 consisting of functions which are constant on elements of the partition $\mathcal{P}^{(k)}$. We will represent functions in $L^{(k)}$ as vectors: vector $g = [g_1, g_2, \ldots, g_k]$ corresponds to the function $g = \sum_{i=1}^k g_i \chi_{J_i}$. Let $Q^{(k)}$ be the isometric projection of L^1 onto $L^{(k)}$:

$$Q^{(k)}(g) = \sum_{i=1}^{k} \left(\frac{1}{\lambda(J_i)} \int_{J_i} g d\lambda \right) \chi_{J_i} = \left[\frac{1}{\lambda(J_1)} \int_{J_1} g d\lambda, \ldots, \frac{1}{\lambda(J_k)} \int_{J_k} g d\lambda \right] \qquad (5.3.10)$$

Let $g = [g_1, g_2, \ldots, g_k] \in L^{(k)}$. We define the operator $P_{\sigma_n}^{(k)} : L^{(k)} \to L^{(k)}$ by

$$P_{\sigma_n}^{(k)} g \quad = \quad (\mathbf{M}_{\sigma_n})^{Trans} \cdot ([g_1, g_2, \ldots, g_k]) \qquad (5.3.11)$$

which is a finite-dimensional approximation to the operator P_{σ_n}. A^{Trans} denotes the transpose of the matrix A.

Then, we have

$$P_{\sigma_n}^{(k)} g = Q^{(k)} P_{\sigma_n} g.$$

More generally, for $g \in L^1$, we have

$$P_{\sigma_n}^{(k)} Q^{(k)} g = Q^{(k)} P_{\sigma_n} Q^{(k)} g.$$

The following Lemma will be used several times in the sequel.

Lemma 5.3.12. *Let $\{g_n\}_{n=1,2,\ldots}$ be a sequence of non-increasing functions uniformly bounded in L^∞. If $g_n \to h$, as $n \to \infty$, weakly in L^1, then the convergence is also in L^1 and a.e.*

Proof. The proof is analogous to the proof of the Lemma 4.3.13. in Chapter 4. □

Lemma 5.3.13. *The invariant density $g_{n,k}$ of $P_{\tau_n}^{(k)}$ τ_n is non-increasing for any $n, k > 1$.*

Proof. The proof is analogous to the proof of the Lemma 4.3.14 in Chapter 4. □

Using Ulam's method and corresponding convergence analysis described in [34, 35, 14], the following theorem can be proved.

Theorem 5.3.14. *Let $\sigma \in \mathcal{T}_{pcv}^{\infty,1}(I)$ be a piecewise concave map with countably many branches. Let $\{\sigma_n\}_{n=1}^{\infty}$ be the approximating sequence of piecewise convex maps with finite numbers of branches where σ_n are defined in the previous Sub-Section 5.3.1. If $g_{n,k}$ is a normalized fixed point of $P_{\sigma_n}^{(k)}, k = 1, 2, \ldots,$ defined in (5.3.11), then the sequence $\{g_{n,k}\}_{k=1}^{\infty}$ is weakly pre-compact in L^1. Any limit point g_n^* of the sequence $\{g_{n,k}\}_{k=1}^{\infty}$ is a fixed point of P_{σ_n}.*

Proof. The proof is analogous to the proof of Theorem 4.3.15 in Chapter 4. □

Theorem 5.3.15. *Let $\sigma \in \mathcal{T}_{pcv}^{\infty,1}(I)$ be a piecewise concave map with countably many branches. As described at the beginning of subsection 3.1, let $\{\sigma_n\}_{n=1}^{\infty}$ be the approximating sequence of piecewise convex maps with finite numbers of branches. Let $P_{\sigma_n}^{(k)}$, $k = 1, 2, \ldots$ be the sequence of Ulam's operators approximating operators P_{σ_n}. Let $g_{n,k}$ be the normalized (in L^1) fixed point of $P_{\sigma_n}^{(k)}$. Then, the family $\{g_{n,k}\}_{n=1,2,\ldots,k=1,2,\ldots}$ is weakly compact in L^1 and uniformly bounded in L^∞. If g_{n_j,k_j}, $j = 1, 2, \ldots$ is a weakly convergent subsequence, then it converges in L^1 (and almost everywhere) to a function f which is a fixed point of P_σ, $P_\sigma g = g$.*

Proof. The proof is analogous to the proof of Theorem 4.3.15 in Chapter 4.

□

5.4 Examples

Example 5.4.1. Consider the piecewise concave map $\sigma : [0, 1] \to [0, 1]$ with countable (infinite) number of branches defined by $\sigma = h_1^{-1} \circ \tau \circ h_1$, where $h_1 : [0, 1] \to [0, 1]$ is the diffeomorphism defined by $h_1(x) = 1 - x$ and $\tau : [0, 1] \to [0, 1]$ is a piecewise convex map with countable number of branches defined by $\tau = h^{-1} \circ T \circ h$, $h : [0, 1] \to [0, 1]$ is the conjugation defined by $h(x) = 1 - (1 - x)^2$, $T : [0, 1] \to [0, 1]$ is the piecewise expanding and piecewise linear map with countable number of branches defined as

$$T(x) = i(i+1)\left(x - \frac{1}{i+1}\right) \quad \text{on} \quad \left[\frac{1}{\mathrm{i}+1}, \frac{1}{\mathrm{i}}\right], \mathrm{i} = 1, 2, \cdots . \qquad (5.4.2)$$

See Figure 5.2 for a few branches of σ and a few branches of τ. Moreover, see Figure 5.3 for a graph of the piecewise expanding and piecewise linear map T, for Example, 5.4.1.

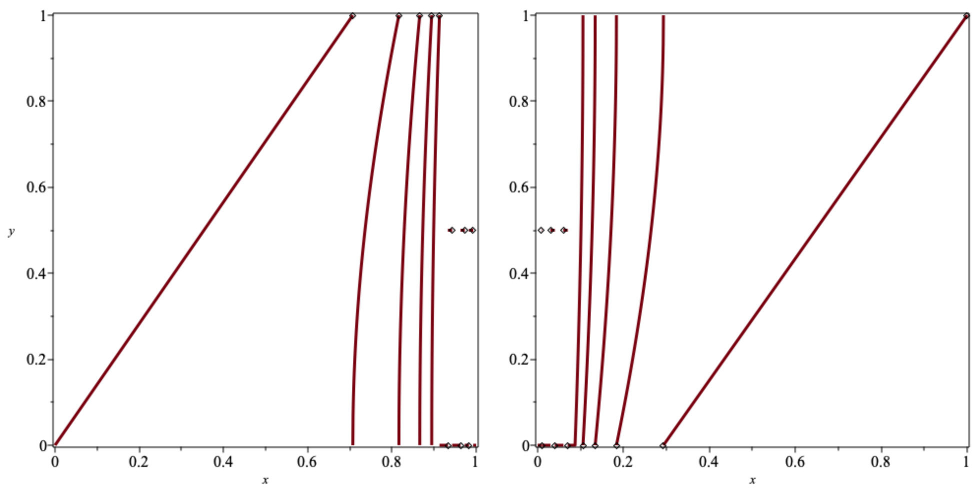

Figure 5.2: Graphs of σ (left) and τ (right) for Example 5.4.1.

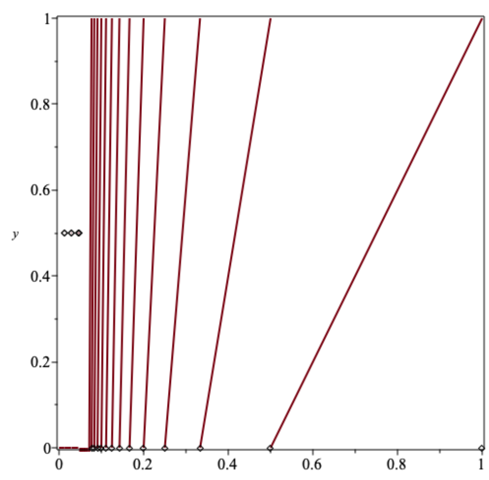

Figure 5.3: The graph of the piecewise expanding and piecewise linear map T, for Example 5.4.1.

It is shown in Chapter 4 that $f(x) = |2(1-x)|$ is the stationary density of the piecewise convex map τ with countable number of branches. Therefore, $g(x) = f(h_1(x)) \times |h_1'(x)| = 2|x|$ is the stationary density of σ. See Figure 5.4 for a graph of σ.

Now, we find the first few branches (from right) of σ on $[0,1]$. Note that $\sigma(x) = (h_1^{-1} \circ \tau \circ h_1)(x)$, where $h_1(x) = 1-x, h_1^{-1}(x) = 1-x$. The piecewise convex map τ is piecewise onto on the partition $\left\{\ldots, 1-\sqrt{\frac{5}{6}}, 1-\sqrt{\frac{4}{5}}, 1-\sqrt{\frac{3}{4}}, 1-\sqrt{\frac{2}{3}}, 1-\sqrt{\frac{1}{2}}, 1\right\}$ of $[0,1]$. Moreover,

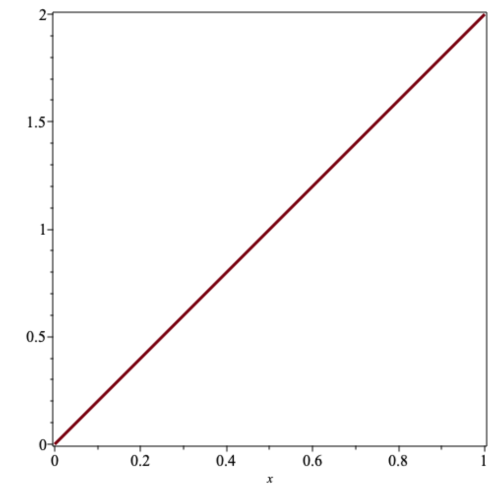

Figure 5.4: The graph of the stationary density g of the piecewise concave map σ with countable (infinite) number of branches for Example 5.4.1.

$h_1(1) = 0, h_1\left(1-\sqrt{\frac{1}{2}}\right) = \sqrt{\frac{1}{2}}, h_1\left(1-\sqrt{\frac{2}{3}}\right) = \sqrt{\frac{2}{3}}, h_1\left(1-\sqrt{\frac{3}{4}}\right) = \sqrt{\frac{3}{4}}, h_1\left(1-\sqrt{\frac{4}{5}}\right) = \sqrt{\frac{4}{5}}, h_1\left(1-\sqrt{\frac{5}{6}}\right) = \sqrt{\frac{5}{6}}, \ldots$. Therefore, the piecewise concave map σ is defined on the partion $\mathcal{P} = \left\{\sqrt{\frac{i}{i+1}}\right\} = \left\{0, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots, 1\right\}$ of $[0,1]$. If $x \in [0, \sqrt{\frac{1}{2}}]$, then $h_1(x) \in [1-\sqrt{\frac{1}{2}}, 1]$. If $x \in [1-\sqrt{\frac{1}{2}}, 1]$ then $\tau(x) \in [0,1]$. If $x \in [0,1]$, then $h^{-1}(x) \in [0,1]$. Thus, if $x \in [0, \sqrt{\frac{1}{2}}]$, then $\sigma(x) = (h_1^{-1} \circ \tau \circ h)(x) \in [0,1]$. Moreover, on $[0, \sqrt{\frac{1}{2}}]$,

$$\begin{aligned}
\sigma(x) &= (h_1^{-1} \circ \tau \circ h)(x) \\
&= h_1^{-1}(\tau(1-x)) \\
&= h^{-1}(1-\sqrt{2}(1-(1-x)) \\
&= h^{-1}(1-\sqrt{2}x) \\
&= \sqrt{2}x.
\end{aligned}$$

Similarly, we can find other branches of σ on the partition

$$\mathcal{P} = \left\{\sqrt{\frac{i}{i+1}}\right\} = \left\{0, \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}, \ldots, 1\right\} \text{ of } [0,1].$$

The branches $(n = 4)$ on $\left[\sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}\right]$, $\left[\sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}\right]$, $\left[\sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}\right]$, $\left[\sqrt{\frac{4}{5}}, \sqrt{\frac{5}{6}}\right]$ are $\sqrt{6x^2 - 3}$, $\sqrt{12x^2 - 8}$, $\sqrt{20x^2 - 15}$, $\sqrt{30x^2 - 24}$ respectively. Now, consider the following sequence $\{\sigma_n\}_{n \geq 0}$ of piecewise concave map $\sigma_n : [0, 1] \to [0, 1]$ with finite number of branches:

$$\sigma_n(x) = \begin{cases} \frac{1}{1-\sqrt{\frac{n}{n+1}}}\left(x - \sqrt{\frac{n}{n+1}}\right) , \sqrt{\frac{n}{n+1}} \leq x \leq 1; \\ \sigma(x) , \ 0 \leq x < \sqrt{\frac{n}{n+1}}. \end{cases}$$

See Figure 5.5 for a graph of σ_n with $n = 5$. The sequence $\{\sigma_n\}_{n \geq 0}$ of piecewise concave map

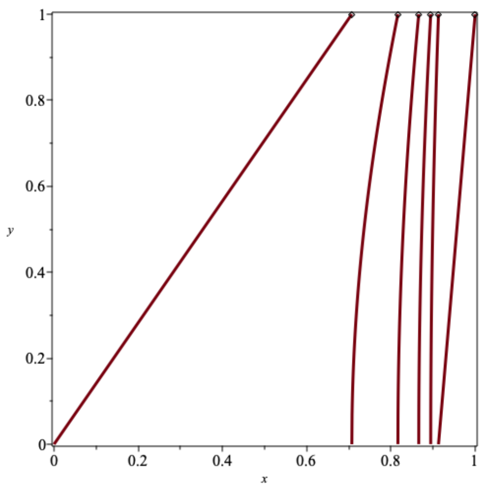

Figure 5.5: Piecewise concave map σ_n with finite number of branches ($n = 5$).

$\sigma_n : [0, 1] \to [0, 1]$ with finite number of branches converges almost uniformly to σ with countable (infinite) number of branches. In Figure 5.6, we present a graph of approximate stationary density $g_{n,k}, n = 5, k = 100$ via Ulam's method of the actual stationary density of $g_n, n = 5$ of the piecewise concave map $\sigma_n, n = 5$ with a finite number of branches. Note that $\sigma_n, n = 5$ is an approximation of the piecewise concave map σ.

In Figure 5.7, we present a graph of the actual density g of the piecewise concave map σ with countable number of branches and graphs of the approximate stationary density $g_{n,k}, n = 5, k = 100$ and $n = 10, k = 1000$ via Ulam's method of the actual stationary density of $g_n, n = 5$ and 10 of the piecewise concave map $\sigma_n, n = 5$ and 10 respectively with finite number of branches. Note that $\sigma_n, n = 5$ is an approximation of the piecewise concave map σ in Figure 5.2 and therefore, the L^1 norm error $\| \ g - g_{n,k} \ \|_1 = 0.2205242549$ with $n = 5, k = 100$ is not very small but for

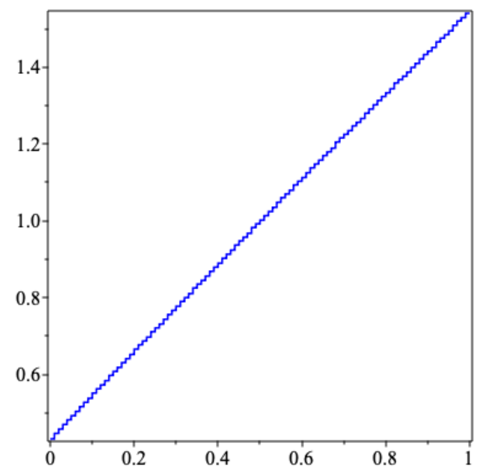

Figure 5.6: The approximate stationary density $g_{n,k}, n = 5, k = 100$ via Ulam's method of the piecewise concave map σ_n with finite number of branches ($n = 5$).

$n = 10, k = 1000$ the L^1 norm error $\| g - g_{n,k} \|_1 = 0.1334973586689$ which is smaller than $n = 5, k = 100$. Theorem 5.3.14 confirms that for large n and large k, the L^1 norm error is close to 0.

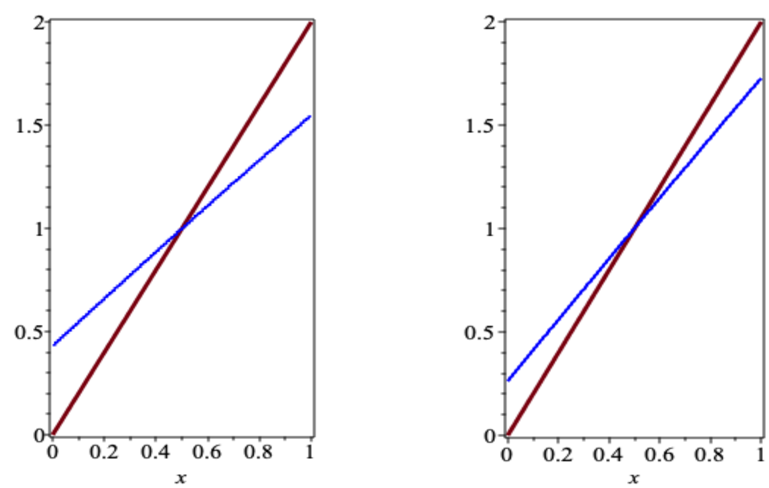

Figure 5.7: The graph of the actual invariant density g of the piecewise concave map σ with infinite number of branches (in red) and the graph of the approximating density $g_{n,k}$ (in blue): $n = 5, k = 100$ on the left and $n = 10, k = 1000$ on the right hand side.

Example 5.4.3. Consider the piecewise concave map $\sigma : [0, 1] \to [0, 1]$ with countable number of branches on the partition $\{\left[\frac{2}{3+n}, \frac{2}{2+n}\right]\}_{n=0}^{\infty}$ of $I = [0, 1]$ defined as

$$\sigma(x) = 1 - (\frac{2}{x} - 2)(mod\ 1). \tag{5.4.4}$$

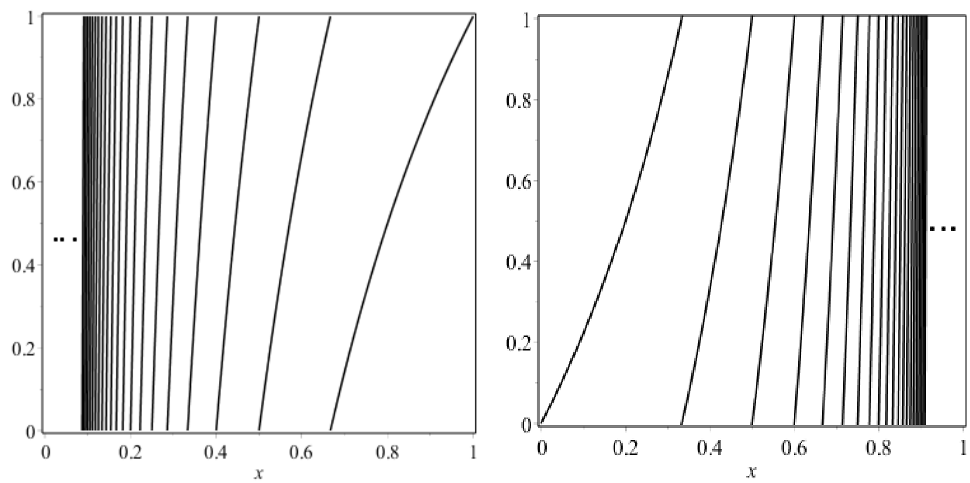

Figure 5.8: Graphs of σ (left) and τ (right) for Example 5.4.3.

In the following, we show that σ satisfies conditions of Theorem 5.2.6, i.e., $\sigma \in \mathcal{T}_{pcv}^{\infty}(I)$:

Condition (1): $\sigma(x)$ is piecewise continuous and concave on $\left[\frac{2}{3+n}, \frac{2}{2+n}\right]$. Since $\sigma(x) = 1 - \left(\frac{2}{x} - 2\right)(mod 1)$ and $\sigma'(x) = \frac{2}{x^2}$ is decreasing on $\left[\frac{2}{3+n}, \frac{2}{2+n}\right]$. Thus σ is piecewise concave.

Condition (2): $\sigma(\frac{2}{2+n}) = 1 - \left(\frac{2}{\frac{2}{2+n}} - 2\right) + n = 1$, $\sigma'(\frac{2}{2+n}) = \left(\frac{1}{2} + \frac{n}{4}\right)^2 > 0\ \forall n \in \mathbb{N}$.

Condition (3): Clearly, $\sigma(1) = 1, \sigma'(1) = 2(= \alpha) > 1$

Condition (4): we have only one interval in between $\frac{2}{3}$ and 1, and $\sigma'\left(\frac{2}{3}\right) = \frac{9}{2}$. Thus $\sum_{i=1}^{\infty} \frac{1}{\sigma'(a_i)} = \frac{2}{9} < \infty$.

Thus $\sigma \in \mathcal{T}_{pcv}^{\infty}(I)$ and hence by Theorem 5.2.6, σ has an acim.

Now, consider the piecewise convex map $\tau : [0, 1] \to [0, 1]$ with countable number of branches on the partition $\{[\frac{n}{2+n}, \frac{n+1}{3+n})\}_{n=0}^{\infty}$ of $[0, 1]$ defined as

$$\tau(x) = \frac{2x}{1 - x} \quad (mod\ 1). \tag{5.4.5}$$

See Figure 5.9 for the graph of σ (right) and τ (left). Consider the diffeomorphism $h : [0, 1] \to [0, 1]$ defined by $h(x) = 1-x$. We have $h \circ \tau = \sigma \circ h$. Hence τ is topologically conjugate to σ. By Theorem

5.2.2, if τ has a unique acim μ with density f, then it is easy to find the stationary density g of σ. In fact, $g = \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'|$.

Example 5.4.6. Consider a piecewise concave map $\sigma : [0,1] \to [0,1]$ with countable (infinite) number of partitions $\{1 = a_0, a_1, a_2, \dots\}$ of $I = [0,1]$ defined as

$$\sigma(x) = 1 - \frac{1}{\frac{1+n-n^2}{n(n+1)} + x} + n \quad on \quad \left[\frac{n-1}{n}, \frac{n}{n+1}\right]. \tag{5.4.7}$$

We want to show that ν is an acim for σ.

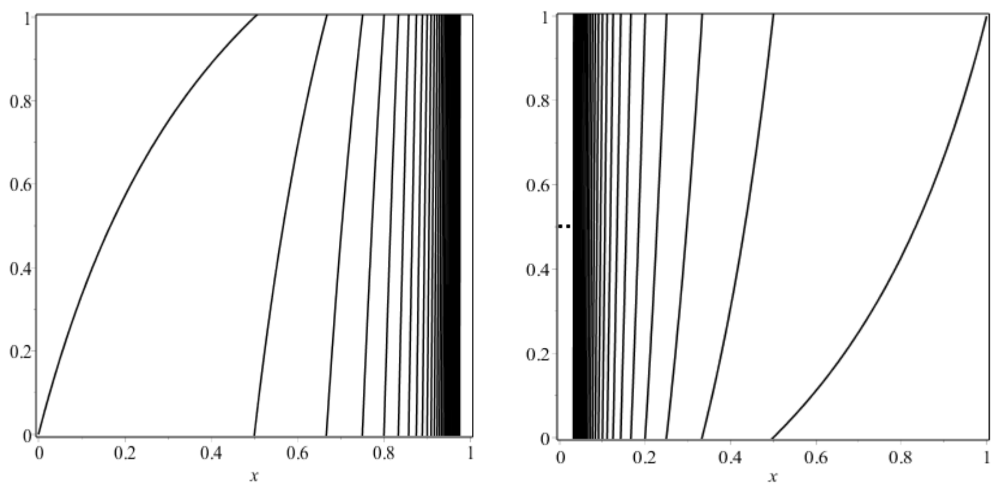

Figure 5.9: Graphs of σ and τ for example 5.4.6.

From Chapter 3, Consider the piecewise convex map $\tau : [0,1] \to [0,1]$ with infinite number of branches defined as

$$\tau(x) = \frac{1}{\frac{2n+1}{n(n+1)} - x} - n \quad on \quad \left[\frac{1}{n+1}, \frac{1}{n}\right]. \tag{5.4.8}$$

See Figure 5.10 for the graph of σ and τ.

Condition 1:

Here $\sigma(x)$ is piecewise continuous and concave on its domain. Since $\sigma'(x) = \frac{1}{\left(\frac{1+n-n^2}{n(n+1)}+x\right)^2}$ is decreasing on $\left[\frac{n-1}{n}, \frac{n}{n+1}\right]$. Thus σ is piecewise concave.

Conditions $2, 3$:

Since in this example we have only one interval in between 0 and $\frac{1}{2}$;

$\sigma(\frac{1}{2}) = 1 > 0$, and $\sigma'\left(\frac{1}{2}\right) = 1 > 0$. Thus $\sum_{i=1}^{\infty} \frac{1}{\sigma'(a_i)} = 1 < \infty$.

Condition 4:

$\sum_{j=1}^{\infty} \frac{1}{\sigma'(a_{0,j})} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} - 1 \equiv 0.6449 < 1$. We know that $\tau(x)$ is piecewise continuous on the countable partition $[\frac{1}{n+1}, \frac{1}{n}]$ of $[0, 1]$. Since μ is a unique acim for τ.

Let $h : [0, 1] \to [0, 1]$ be a diffeomorphism defined as

$$h(x) = 1 - x.$$

Here, $h(x)$ is linear with slopes -1, and we have $h \circ \tau = \sigma \circ h$. By Theorem 5.2.2, if τ has a unique ACIM μ with density f, then it is easy to find the stationary density g of σ. In fact, $g = \left(f \circ h^{-1}\right) \cdot |\left(h^{-1}\right)'|$.

Chapter 6

Conclusion

In this thesis, we dealt with the problems of the existence and exactness of ACIMs of some chaotic dynamical systems in one dimension.

First, we defined two new classes of transformation, $\mathcal{T}_{pc}^{\infty}(I), \mathcal{T}_{pc}^{\infty,0}(I)$ of piecewise convex maps with countable (infinite) number of branches. We investigated the properties of these classes that enable us to derive a unique ACIM for the transformations in these classes. We determined the density function using Ulam's method for these new classes.

Our study extended to non-autonomous dynamical systems within these defined classes, focusing on the existence of ACIMs for the limit map, τ. We established a significant result, demonstrating that the τ-invariant density could be obtained as the limit of a sequence of densities, $P_{T_n}f$, where T_n represents the composition of the first n-maps in the non-autonomous system, for a non-increasing density f.

We discussed the invariant density using the Frobenius-Perron operator in Chapter 3. But generally, the fixed point or invariant density is not found easily. In Chapter 4, our main purpose was to approximate the F-P operator by a sequence of finitely dimensional operators. Determining the fixed point of the Frobenius-Perron operator P_τ of τ is generally challenging. It was required to approximate the F-P operator P_τ using any of the approximation methods. We used Ulam's approximation. We introduced an operator $Q^{(k)}$ that projected $L^1 \rightarrow L^{(k)}$ and used the finite dimensional approximation $P_{\tau_n}^{(k)}$ of the F-P operator P_{τ_n} of τ_n.

In Chapter 5, we explored the dynamics of new families of transformations, $\mathcal{T}_{pcv}^{\infty}(I), \mathcal{T}_{pcv}^{\infty,1}(I)$, focusing on the conjugation of a piecewise concave map to a convex map previously introduced in Chapter 3. This conjugation provided insights into the dynamics of these maps and enabled us to prove the existence of an invariant density.

This research has endeavored to elucidate certain obscure aspects within the field of dynamical systems, particularly with respect to the existence and stability of ACIMs.

Moving forward, for future research, our research directions will focus on investigating ACIMs for random maps within the class of piecewise convex maps with a countable number of branches. This exploration will include both constant and position-dependent probabilities, as we can use Ulam's method to approximate ACIM in terms of random maps with countable partitions. Additionally, we aim to explore the concept of sustainability within this framework, providing new perspectives and solutions to this complex problem. Overall, the insights and methods developed in this thesis will pave the way for future research and lead to a deeper understanding of ACIM in chaotic dynamical systems.

Appendix A

A.1 Inequality (3.4.3):

Recall that the Koopman operator $U_\tau : L^\infty \to L^\infty$ is defined by

$$U_\tau g = g \circ \tau.$$

Proposition A.1.1. *[5] If $f \in L^1$ and $g \in L^\infty$, then $\langle P_\tau f, g\rangle = \langle f, U_\tau g\rangle$, i.e.,*

$$\int_I (P_\tau f) \cdot g d\lambda = \int_I f \cdot U_\tau g d\lambda.$$

Now from inequality 3.4.3:

$$\begin{aligned}
\left|\int h(P_{\tau_n}F - P_\tau F)d\lambda\right| &\leq \int |(P_{\tau_n}F - P_\tau F)h|\, d\lambda \\
&= \int |(P_{\tau_n}Fh - P_\tau Fh)|\, d\lambda \\
&= \int |F(U_{\tau_n}h - U_\tau h)|\, d\lambda \\
&= \int F|h \circ \tau_n - h \circ \tau| d\lambda \\
&\leq \omega_h(\sup |\tau_n - \tau|).
\end{aligned}$$

A.2 τ is conjugated to σ

Proposition A.2.1. *Let σ be a piecewise concave map with countable number of branches. Consider the diffeomorphism $h : [0,1] \to [0,1]$ defined by $h(x) = 1-x$. Show that the map $\tau : [0,1] \to [0,1]$ defined by $\tau = h^{-1} \circ \sigma \circ h$ is a piecewise convex map with countable number of branches.*

Proof. Here the diffeomorphism $h : [0,1] \to [0,1]$ defined by $h(x) = 1 - x$ is linear and non-increasing. so, $h^{-1}(x) = 1 - x$ is also linear and non-increasing on its domain.

Consider σ is a piecewise concave map with countable number of branches.

Since the composition of the concave map with a non-increasing function is convex, i.e., $\sigma \circ h$ is convex.

Again $h^{-1}(x) = 1 - x$ is also linear and non-increasing which conclude that

$$\tau = h^{-1} \circ \sigma \circ h$$

is a piecewise convex map with countable number of branches. Since the composition of functions does not change the number of branches. □

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