Prismatic Dieudonné Theory for Truncated Barsotti-Tate Groups

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Abstract

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The aim of this thesis is to classify truncated Barsotti-Tate groups over p-torsion free quasi-syntomic rings via a semilinear category which is a prismatic analogue of the category truncated displays introduced by Lau-Zink. This rests crucially on the classification of p-divisible groups over quasi-syntomic rings due to Anschütz-Le Bras and an argument of Beilinson which was used by Kisin to deduce a similar classification of truncated Barsotti-Tate groups over rings of integers of p-adic fields in terms of certain Breuil-Kisin modules.

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Contents

Notations and conventions

All the rings appearing in this article are commutative with unit.

Throughout this article we fix a prime number p.

The complexes used in this article are always cochain complexes. A contravariant functor applied to cochain complexes are understood to also flip the sign of the degrees, so that it maps cochain complexes to cochain complexes. If M is an object of some additive category, then we denote by $M[0]$ the cochain complex with M sitting in degree 0 and 0 elsewhere.

We use $\simeq$ for canonical isomorphisms and $\approx$ for non-canonical ones. However $\simeq$ is used for all quasi-isomorphisms in derived categories. The symbol $=$ is used in more restrictive settings, namely equality as elements of some set.

We stick to the notation in [Sta18] for the definitions in algebraic geometry and category theory.

If $f : A \to B$ is a ring homomorphism and M an A-module, then denote by f^*M the B-module $M \otimes_{A,f} B$, the base change of M along $f : A \to B$. In particular, if $\sigma : A \to A$ is an endomorphism of a ring A, then σ^*M is $M \otimes_{A,\sigma} A$, seen as an A-module via the second factor.

In order to avoid set theoretic issues in defining the sites, we choose once and for all a strong limit cardinal κ, and all the rings in the article are assumed to have cardinality less than κ.

1 Introduction

The aim of this thesis is to classify the truncated Barsotti-Tate groups over p-torsion free quasi-syntomic rings using prismatic cohomology, as envisioned in Remark 1.4.11 of [AL23].

Definition 1. A *quasi-syntomic* (short for p-completely quasi-syntomic) ring is a derived p-complete ring R with bounded p^∞-torsion such that the absolute cotangent complex $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$. A *quasiregular semiperfectoid* ring is a quasi-syntomic ring R which admits a surjection from an integral perfectoid ring.

The precise definition of the terminologies used in the definition will appear in later parts. A lot of the interesting rings occuring in p-adic geometry are quasi-syntomic. For example integral perfectoid rings and p-complete local complete intersection Noetherian rings are quasi-syntomic.

The choice of the class of quasi-syntomic ring as base rings, which was also made in [AL23], naturally occurs in calculting prismatic cohomology. Namely, a quasi-syntomic ring can be covered by quasiregular semiperfectoid rings, while the absolute prismatic cohomology of quasiregular semiperfectoid rings are concentrated in degree 0. Thus we may use the absolute prismatic cohomology of a quasi-syntomic ring without working with derived objects. We will do the classification for p-torsion free quasiregular semiperfectoid rings first and then extend it to general p-torsion free quasi-syntomic rings using descent. The requirement that the base ring is p-torsion free is a technical condition that allows applying the truncation argument described below.

The strategy is to use the following observations:

(1) The category of standard truncated Barsotti-Tate groups of level n over R, i.e. the truncated Barsotti-Tate groups over R that are the p^n-torsion subgroups of some p-divisible group over R, can be embedded fully faithfully into the bounded derived category of the p-divisible groups.

(2) Every truncated Barsotti-Tate group can be made standard if we pass to a pro-étale cover of the base scheme. (Note that we will mostly use the terminology ind-étale rather than pro-étale in the technical arguments since we work exclusively in the affine situation.)

(1) was first used in [Kis06], while (2) was noticed by Lau in [Lau13].

We may obtain a classification of standard truncated Barsotti-Tate groups using (1) and the results in [AL23], and then extend the classification to all truncated Barsotti-Tate groups by pro-étale descent.

2 Preliminaries

In this section we collect several preliminary results for later use.

2.1 I-completed commutative algebra

In this subsection we introduce the technical tool of derived completeness. Throughout this subsection we fix a ring R and a finitely generated ideal $I \subset R$.

Definition 2. An object M in $D(R)$ is call *derived I-complete* if for all $f \in I$ we have that

$$\mathrm{RHom}_R(R_f, M) \simeq 0$$

. An R-module M is called derived I-complete if the complex $M[0]$ is derived I-complete.

It is clear that if two of the objects in a distinguished triangle are derived I-complete, then so is the third. In particular derived I-complete modules are closed under extensions. There is an extensive theory of derived I-completeness as in [Sta18, Tag 091N]. We need the following basic results.

Remark 3. By [Sta18, Tag 091Q], the subset of elements $f \in R$ such that

$$\mathrm{RHom}_R(R_f, M) \simeq 0$$

is a radical ideal. Thus we may detect derived I-completeness by only checking the condition $\mathrm{RHom}_R(R_f, M) \simeq 0$ for f ranging in a set of generators of I. In particular, if $R \to S$ is a ring map and M is an object of $D(S)$, then the image of I in S generate the ideal IS in S, so M is derived IS-complete if and only if it is derived I-complete as an object of $D(R)$.

Lemma 4. *A complex is derived I-complete if and only if all the cohomology groups are derived I-complete.*

Proof. [Sta18, Tag 091P]. □

Lemma 5. *Derived I-complete modules are closed under kernels and cokernels.*

Proof. Let $f : M \to N$ be a map of derived I-complete R-modules. Applying Lemma 4 to the complex $M \xrightarrow{f} N$, where M sits in degree 0 yields that $\ker f \simeq H^0(M \to N)$ and $\operatorname{coker} f \simeq H^1(M \to N)$ are derived I-complete. □

Derived completeness of a module is a suitable relaxation of classical completeness.

Lemma 6. *Suppose M is an R-module.*

(1) If M is classically I-complete, i.e. $M \simeq \lim_n M/I^n M$, then M is derived I-complete.

(2) Conversely, if M is derived I-complete and $M \to \lim_n M/I^n M$ is injective, then M is classically I-complete.

Proof. [Sta18, Tag 091R]. □

As with classical completeness, there is a derived completion functor.

Proposition 7. *For K ranging in objects in $D(R)$, there is a functorially associated morphism $K \to K^\wedge$ such that for all $K \in D(R)$ the object $K^\wedge$ is derived I-complete, and for all $K, L \in D(R)$ such that L is derived I-complete, the morphism*

$$\mathrm{RHom}_R(K^\wedge, L) \to \mathrm{RHom}_R(K, L)$$

induced by $K \to K^\wedge$ is a quasi-isomorphism.

Proof. [Sta18, Tag 091V]. □

The functor $(-)^\wedge$ is called the *derived completion*. Occasionally we will write $(-)^\wedge_I$ to emphasize the dependence on I.

The universal property of the derived completion implies the following useful lemma.

Lemma 8. *Let K be an object in $D(R)$. We have that $K^\wedge \otimes^{\mathbb{L}}_R R/I \simeq K \otimes^{\mathbb{L}}_R R/I$.*

Proof. We will use the universal property of the derived tensor product as in [Sta18, Tag 0GMT]. Every element $f \in I$ acts by zero on the complex $K \otimes^{\mathbb{L}}_R R/I$, so $K \otimes^{\mathbb{L}}_R R/I$ is derived I-complete. Then the natural map $K \to K \otimes^{\mathbb{L}}_R R/I$ factors through $K^\wedge$ by the universal property. The universal property of the derived tensor product then gives a map $K^\wedge \otimes^{\mathbb{L}}_R R/I \to K \otimes^{\mathbb{L}}_R R/I$. The composition $K \otimes^{\mathbb{L}}_R R/I \to K^\wedge \otimes^{\mathbb{L}}_R R/I \to K \otimes^{\mathbb{L}}_R R/I$ is the identity since the map $K \to K^\wedge \to K \otimes^{\mathbb{L}}_R R/I$ is just the natural $K \to K \otimes^{\mathbb{L}}_R R/I$ by definition of $K^\wedge \to K \otimes^{\mathbb{L}}_R R/I$. The other composition $K^\wedge \otimes^{\mathbb{L}}_R R/I \to K \otimes^{\mathbb{L}}_R R/I \to K^\wedge \otimes^{\mathbb{L}}_R R/I$ is also the identity since by the universal property of $K^\wedge$, we only need to check that precomposing this map with $K \otimes^{\mathbb{L}}_R R/I$ yields the natural $K \otimes^{\mathbb{L}}_R R/I \to K^\wedge \otimes^{\mathbb{L}}_R R/I$, and this is true as $K \otimes^{\mathbb{L}}_R R/I \to K^\wedge \otimes^{\mathbb{L}}_R R/I \to K \otimes^{\mathbb{L}}_R R/I$ is the identity. □

The following lemma is very useful.

Lemma 9. *(derived Nakayama) If K is a derived I-complete object in $D(R)$, then $K \simeq 0$ if and only if $K \otimes^{\mathbb{L}}_R R/I \simeq 0$.*

Proof. This is [Sta18, Tag 0G1U]. □

Now we turn to the notion of I-complete flatness.

Definition 10. Let $a \leq b$ be indices in $\mathbb{Z} \bigcup \{\pm\infty\}$, such that at least one of a, b is finite. An object K in $D(R)$ is said to have I-complete Tor-amplitude in $[a, b]$ if for any R/I-module M, the complex $M \otimes^{\mathbb{L}}_{R/I} (K \otimes^{\mathbb{L}}_R R/I)$ has vanishing cohomology outside $[a, b]$. K is called *I-completely flat* if it has I-complete Tor-amplitude in $[0, 0]$. An R-module M is called I-completely flat if the complex $M[0]$ is I-completely flat. M is called I-completely faithfully flat if M is I-completely flat and M/IM is faithfully flat over R/I.

The notion of I-complete flatness is stable under base change.

Lemma 11. *Let $R \to S$ be a ring homomorphism. If $K \in D(R)$ has I-complete* Tor*-amplitude in $[a,b]$, then $K \otimes_R^{\mathbb{L}} S$ has IS-complete* Tor*-amplitude in $[a,b]$.*

Proof. Using [Sta18, Tag 08YU] we can calculate that

$$K \otimes_R^{\mathbb{L}} S \otimes_S^{\mathbb{L}} S/IS \simeq K \otimes_R^{\mathbb{L}} R/I \otimes_{R/I}^{\mathbb{L}} S/IS$$

So for any S/IS-module N, we have

$$(K \otimes_R^{\mathbb{L}} S \otimes_S^{\mathbb{L}} S/IS) \otimes_{S/IS}^{\mathbb{L}} N \simeq K \otimes_R^{\mathbb{L}} R/I \otimes_{R/I}^{\mathbb{L}} N$$

has vanishing cohomology outside $[a,b]$. □

Lemma 12. *Let $R \to S$ be an I-completely flat ring map and $M \in D(S)$ an object having IS-complete* Tor*-amplitude in $[a,b]$, then M has I-complete* Tor*-amplitude in $[a,b]$ as an object in $D(R)$.*

Proof. Suppose that N is an R/I-module.

$$\begin{aligned} M \otimes_R^{\mathbb{L}} R/I \otimes_{R/I}^{\mathbb{L}} N &= M \otimes_S^{\mathbb{L}} S \otimes_R^{\mathbb{L}} R/I \otimes_{R/I}^{\mathbb{L}} N \\ &= M \otimes_S^{\mathbb{L}} S/IS \otimes_{R/I}^{\mathbb{L}} N \\ &= (M \otimes_S^{\mathbb{L}} S/IS) \otimes_{S/IS}^{\mathbb{L}} (N \otimes_{R/I}^{\mathbb{L}} S/IS) \end{aligned}$$

as $R \to S$ is I-completely flat, $N \otimes_{R/I}^{\mathbb{L}} S/IS$ is still concentrated in degree 0. Thus $M \otimes_R^{\mathbb{L}} R/I \otimes_{R/I}^{\mathbb{L}} N$ has vanishing cohomology outside $[a,b]$ since $M \otimes_S^{\mathbb{L}} S/IS$ has Tor-amplitude in $[a,b]$. □

In particular, a composition of I-completely flat R-algebra maps is still I-completely flat.

Fix $\{f_1, \dots, f_n\}$ a finite set of elements of I that generates I.

Definition 13. An R-algebra S is said to *have tame $\{f_1, \dots, f_n\}^\infty$-torsion* if the pro-systems $\{S/I^kS\}_k$ and $\{\mathrm{Kos}(S; f_1^k, \dots, f_n^k\}_k$ are pro-isomorphic, i.e. isomorphic as pro-objects of R-modules.

By [Sta18, Tag 0625], the definition is independant of multiplying $f_1, \dots, f_n$ by an invertible $n \times n$ matrix with coefficients in R. In particular, it is independant of the order of f_i.

Remark 14. We do not know if the notion of having tame $\{f_1, \dots, f_n\}^\infty$-torsion only depends on I.

Lemma 15. *Let $a \leq b$ be indices in $\mathbb{Z} \bigcup \{\pm\infty\}$. If $M \in D(R)$ has I-complete* Tor*-amplitude in $[a,b]$, then $M \otimes_R^{\mathbb{L}} R/I^n$ has* Tor*-amplitude in $[a,b]$.*

Proof. We use induction on n. The case $n = 1$ is true by assumption. Suppose $M \otimes^{\mathbb{L}}_R R/I^n$ has Tor-amplitude in $[a, b]$. For any R/I^{n+1}-module N, we have an exact sequence

$$0 \to I^n N \to N \to N/I^n N \to 0$$

And $I^n.I^n N = 0$ since $I^{2n} \subset I^{n+1}$. So we only need to show that $M \otimes^{\mathbb{L}}_R R/I^{n+1} \otimes^{\mathbb{L}}_{R/I^{n+1}} N$ has vanishing cohomology outside $[a, b]$ for R/I^{n+1}-modules N such that $I^n N = 0$. In this case N is also an R/I^n-module, and

$$M \otimes^{\mathbb{L}}_{R/I^{n+1}} N \simeq M \otimes^{\mathbb{L}}_{R/I^n} N$$

which has vanishing cohomology outside $[a, b]$ by induction hypothesis. □

Lemma 16. *Let $a, b \in \mathbb{Z} \bigcup \{+\infty\}, a \leq b$ be indices, where a is finite. Suppose that S has tame $\{f_1, \ldots, f_n\}^\infty$-torsion and $M \in D(S)$ a derived IS-complete object with IS-complete* Tor-*amplitude in $[a, b]$. Then M has vanishing cohomology outside $[a, b]$.*

Proof. As M is derived IS-complete, it is the Rlim of $M \otimes_S \mathrm{Kos}(S; f_1^k, \ldots, f_n^k)$ by [Sta18, Tag 091Z].(Note that the Koszul complex is K-flat, so the derived tensor product is just the usual tensor product.) By tameness assumption M is also Rlim of $M \otimes^{\mathbb{L}}_S S/I^m S$. Lemma 15 shows that each $M \otimes^{\mathbb{L}}_S S/I^m S$ has Tor-amplitude in $[a, b]$. By [Sta18, Tag 0654], each $M \otimes^{\mathbb{L}}_S S/I^m S$ can be represented by a K-flat complex $K^\bullet_m$ with each term a flat $S/I^m S$-module and $K^i_m \simeq 0$ if $i \notin [a, b]$. Since Rlim is a right derived functor, the object $\mathrm{Rlim}_m K^\bullet_m$ has vanishing cohomology outside $[a, +\infty]$. So the statement is proved if $b = +\infty$. If b is finite, then we may use a resolution of $S/I^m S$ to calculate $M \otimes^{\mathbb{L}}_S S/I^m S$, so that the term at degree b of $M \otimes^{\mathbb{L}}_S S/I^m S$ is $K^b_m/I^m K^b_m$. Thus the transition maps between different m induces surjections on the cohomology at degree b. Using the exact sequence [Sta18, Tag 0CQE] and Mittag-Leffler, we see that $\mathrm{Rlim}_m M \otimes^{\mathbb{L}}_S S/I^m S$ has vanishing cohomology outside $[a, b]$. □

Lemma 17. *If S is an R-algebra with tame $\{f_1, \ldots, f_n\}^\infty$-torsion and M is a derived IS-complete, IS-completely flat object in $D(S)$. Then $M \simeq N[0]$, where N is a classically IS-complete S-module.*

Proof. Lemma 16 shows that M is concentrated in degree 0. Moreover, M is the Rlim of $M \otimes^{\mathbb{L}}_S S/I^m S$, while Lemma 15 shows that $M \otimes^{\mathbb{L}}_S S/I^m S$ is the flat module $M/I^m M$ sitting in degree 0. The transition maps are clearly surjective, so the Rlim is the usual limit sitting in degree 0 by Mittag-Leffler. Then $M \simeq \lim_m M/I^m M$. □

For the lemmas above to be useful, we need a way to produce rings of tame $\{f_1, \ldots, f_n\}^\infty$-torsion.

Lemma 18. *If S is a ring with tame $\{f_1, \ldots, f_n\}^\infty$-torsion and S' an IS-completely flat S-algebra, then S' has tame $\{f_1, \ldots, f_n\}^\infty$-torsion.*

Proof. The Koszul complex $\mathrm{Kos}(S'; f_1^m, \ldots, f_n^m)$ is the derived tensor product

$$\mathrm{Kos}(S; f_1^m, \ldots, f_n^m) \otimes_S^{\mathbb{L}} S'$$

while by Lemma 15 we have $S'/I^m S' \simeq M \otimes_S^{\mathbb{L}} S/I^m S$. Thus the two pro-systems are the derived base change from those for S, and are pro-isomorphic. □

Lemma 19. *Suppose that $I = (f)$ is principal. If S is an R-algebra with bounded f^∞-torsion, then S has tame $\{f\}^\infty$-torsion. Here a module M is said to have bounded f^∞-torsion if $M[f^\infty] = M[f^c]$ for some c.*

Proof. The complex $\mathrm{Kos}(S; f^m)$ has H^{-1} the module $S[f^m]$, H^0 the module $S/f^m S$ and all other cohomology groups 0. The transition map $\mathrm{Kos}(S; f^{m+1}) \to \mathrm{Kos}(S; f^m)$ induces the multiplication by f map on H^{-1} and the quotient map on H^0. If c is such that $S[f^\infty] = S[f^c]$, then $\mathrm{Kos}(S; f^{m+c+1}) \to \mathrm{Kos}(S; f^m)$ induces 0 on H^{-1} for every m. Thus the cones of the natural maps $\mathrm{Kos}(S; f^m) \to S/f^m S$ form a pro-zero system. □

Let τ denote a Grothendieck topology on the category of schemes coarser than (or equal to) the fpqc topology and finer than (or equal to) the Zariski topology, and such that any fpqc subcover of a cover in τ is still a cover in τ. Denote by (Aff/R) the category of affine schemes over R, and $(\mathrm{Aff}/R)_\tau$ the category (Aff/R) equipped with the topology τ.

Suppose the ideal $I \subset R$ is generated by $\{f_1, \ldots, f_n\}$. If S is a derived IS-complete R-algebra with tame $\{f_1, \ldots, f_n\}^\infty$-torsion, then let $(I\mathrm{Aff}/S)$ be the opposite of the category of derived IS-complete S-algebras with tame $\{f_1, \ldots, f_n\}^\infty$-torsion. Note that the notation $(I\mathrm{Aff}/S)$ does not explicitly indicate the dependance on the chosen $\{f_1, \ldots, f_n\}$ and on the ring S. This will not be a problem in the sequel since there will be no confusion of these choices. In fact most of the time we work with $I = (p) \subset \mathbb{Z}$.

Lemma 20. *If $T \to T', T \to T''$ are maps in $(I\mathrm{Aff}/S)^{\mathrm{op}}$ with $T \to T''$ being IT-completely flat, then $(T' \otimes_T^{\mathbb{L}} T'')^\wedge_{IT'}$ is IT'-completely flat over T', is concentrated in degree 0, and gives a pullback of T'' and T' in $(I\mathrm{Aff}/S)$.*

Proof. We calculate that

$$\begin{aligned}(T'' \otimes_T^{\mathbb{L}} T')^\wedge_{IT'} \otimes_{T'}^{\mathbb{L}} T'/IT' &\simeq T'' \otimes_T^{\mathbb{L}} T' \otimes_{T'}^{\mathbb{L}} T'/IT' \\ &\simeq T'' \otimes_T^{\mathbb{L}} T/IT \otimes_{T/IT}^{\mathbb{L}} T'/IT'\end{aligned}$$

is concentrated in degree 0 and there given by a flat T'/IT'-module. This shows that $(T' \otimes_T^{\mathbb{L}} T'')^\wedge_{IT}$ is IT'-completely flat over T', which has tame $\{f_1, \ldots, f_n\}^\infty$-torsion. By Lemma 17 and Lemma 18, $(T' \otimes_T^{\mathbb{L}} T'')^\wedge_{IT}$ is concentrated in degree 0, and has tame $\{f_1, \ldots, f_n\}^\infty$-torsion. Then it lies in the category $(I\mathrm{Aff}/S)$. Its being the pull back follows from the universal property of the derived completion and the derived tensor product. □

Since pullbacks in $(I\mathrm{Aff}/S)$ along I-completely flat maps exist and remain I-completely flat, we may define a Grothendieck topology on $(I\mathrm{Aff}/T)$ by declaring the covers to be $\{T_i \to T\}$ where the ring maps $T \to T_i$ are IT-completely flat and $\{T_i/IT_i \to T/IT\}$ lies in τ. We call this topology the I-completely τ topology. Let $(I\mathrm{Aff}/S)_\tau$ denote $(I\mathrm{Aff}/S)$ with the I-completely τ topology,

Lemma 21. *An arbitrary cover $\{T_i \to T\}_{i\in J}$ in $(I\mathrm{Aff}/S)_\tau$ can be refined into a cover of the form $T' \to T$.*

Proof. Since the family $\{T_i/IT_i \to T/IT\}_{i\in J}$ is an fpqc cover, there is a finite subset $J' \subset J$ such that $\{T_i/IT_i \to T/IT\}_{i\in J'}$ is still an fpqc cover. By assumption on τ, $\{T_i/IT_i \to T/IT\}_{i\in J'}$ is a cover in τ. Since τ is finer than the Zariski topology, $\prod_{i\in J'} T_i/IT_i \to T/IT$ is also a cover in τ. Then $\prod_{i\in J'} T_i$ also lies in $(I\mathrm{Aff}/S)$, and $\prod_{i\in J'} T_i \to T$ refines the cover $\{T_i \to T\}_{i\in J}$. □

Lemma 22. *In this lemma $I = (p)$. Suppose R has bounded p^∞-torsion and $M \in D(R)$. If M is derived p-complete and p-completely flat, then $M \simeq N[0]$ where N is a classically p-complete R-module. Moreover, M/p^nM is flat over R/p^n and we have*

$$M \otimes_R R[p^n] \simeq M[p^n]$$

Proof. This is [BMS19], Lemma 4.7. The fact that M is concentrated in degree 0 and M/p^nM is flat over R/p^n follows from Lemma 17 and Lemma 15, in view of Lemma 19. □

2.2 I-completely fpqc descent

In this subsection we collect a few results about I-completely fpqc descent.

We first prove the exactness of the I-completed Amitsur complex.

Suppose R_0 is a ring and $I = (f_1, \dots, f_n) \subset R_0$ an ideal. Let R be a derived I-complete R_0-algebra with tame $\{f_1, \dots, f_n\}^\infty$-torsion and S a derived IR-complete, IR-completely flat R-algebra.

Definition 23. For a derived IR-complete, IR-completely flat R-module M, the IR-completed Amitsur complex, $\mathrm{Ami}(M; R \to S)$ is defined to be the augmented complex $A^\bullet$ where

$$A^n = (M \otimes^{\mathbb{L}}_R C(R,S)^n)^\wedge_{IR}$$

and $C(R,S)^\bullet$ is the augmented Čech nerve

$$0 \to R \to S \to (S \otimes^{\mathbb{L}}_R S)^\wedge_{IR} \to (S \otimes^{\mathbb{L}}_R S \otimes^{\mathbb{L}}_R S)^\wedge_{IR} \to \cdots$$

with R lying in degree -1.

Note that each term $(M \otimes^{\mathbb{L}}_R C(R,S)^n)^\wedge_{IR}$ is derived IR-complete and IR-completely flat over R. Thus each term is indeed concentrated in degree 0 by Lemma 17.

Proposition 24. *If $R \to S$ is IR-completely faithfully flat, then the Amitsur complex* $\mathrm{Ami}(M; R \to S)$ *is acyclic.*

We first need a lemma.

Lemma 25. *Suppose $n \geq -1, m \geq 1$ are integers. Under the assumption of Proposition 24, A^n is classically I-complete, and*

$$A^n \otimes_R R/I^m R \simeq M/I^m M \otimes_{R/I^m R} S/I^m S \otimes_{R/I^m R} \cdots \otimes_{R/I^m R} S/I^m S$$

Proof. Note that A^n is I-completely flat over $C(R,S)^n$, which is in turn I-completely flat over R. Thus A^n is a classically IR-complete R-module sitting in degree 0 by Lemma 17. Moreover, the module $A^n \otimes_R R/I^m R$ can be calculated by $A^n \otimes^{\mathbb{L}}_R R/I^m R$ since the latter is concentrated in degree 0 by Lemma 15. The same Lemma implies that $S \otimes^{\mathbb{L}}_R R/I^m R \simeq S/I^m S[0]$. We calculate that

$$\begin{aligned} A^n \otimes^{\mathbb{L}}_R R/I^m R &\simeq (M \otimes^{\mathbb{L}}_R S \otimes^{\mathbb{L}}_R \cdots \otimes^{\mathbb{L}}_R S)^{\wedge}_{IR} \otimes^{\mathbb{L}}_R R/I^m R \\ &\simeq (M \otimes^{\mathbb{L}}_R S \otimes^{\mathbb{L}}_R \cdots \otimes^{\mathbb{L}}_R S) \otimes^{\mathbb{L}}_R R/I^m R \\ &\simeq M/I^m M \otimes^{\mathbb{L}}_{R/I^m R} S/I^m S \otimes^{\mathbb{L}}_{R/I^m R} \cdots \otimes^{\mathbb{L}}_{R/I^m R} S/I^m S \\ &\simeq M/I^m M \otimes_{R/I^m R} S/I^m S \otimes_{R/I^m R} \cdots \otimes_{R/I^m R} S/I^m S \end{aligned}$$

using the fact that $M/I^m M$ and $S/I^m S$ are flat over $R/I^m R$ by Lemma 15. The third quasi-isomorphism follows by repeatedly replacing $\cdots \otimes^{\mathbb{L}}_R S \otimes^{\mathbb{L}}_R R/I^m R$ with $\cdots \otimes^{\mathbb{L}}_R R/I^m R \otimes^{\mathbb{L}}_{R/I^m R} S/I^m S$. □

Poof of Proposition 24. Note that $M/I^m M$ and $S/I^m S$ are flat over $R/I^m R$. Moreover, nilpotent extentions of rings do not change the underlying topological space of the Spec, so $R/I^m R \to S/I^m S$ is still faithfully flat. Then the sequence $A^\bullet \otimes_R R/I^m R$ is the classical Amitsur complex for the module $M/I^m M$ and the ring map $R/I^m R \to S/I^m S$. Thus it is acyclic by [Sta18, Tag 023M]. We have a bounded below acyclic complex of inverse systems

$$0 \to \{A^{-1} \otimes_R R/I^m R\}_m \to \{A^0 \otimes_R R/I^m R\}_m \to \cdots$$

The transition maps in each of the inverse systems are surjective. We may break the acyclic complex into short exact sequences

$$\begin{gathered} 0 \to \{A^{-1} \otimes_R R/I^m R\}_m \to \{A^0 \otimes_R R/I^m R\}_m \to \operatorname{coker} d^{-1} \to 0 \\ 0 \to \operatorname{coker} d^{-1} \to \{A^1 \otimes_R R/I^m R\}_m \to \operatorname{coker} d^0 \to 0 \\ \cdots \end{gathered}$$

Note that the transition maps in $\operatorname{coker} d^i$ are also surjective. Thus we can apply Rlim to each of them to obtain short exact sequences of the limits, and get an acyclic complex

$$0 \to \lim_m A^{-1} \otimes_R R/I^m R \to \lim_m A^0 \otimes_R R/I^m R \to \cdots$$

But for each index $n \geq -1$, Lemma 25 shows that A^n is the limit of $A^n \otimes_R R/I^m R$. So we conclude that the complex $\mathrm{Ami}(M; R \to S)$ is acyclic. □

We now collect a few lemmas about finite locally free modules. In these lemmas R is allowed to be an arbitrary ring.

Lemma 26. *If M is a finite locally free R-module and N is an arbitrary R-module, then* $\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_R(M, R) \otimes_R N$.

Proof. The isomorphism $\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_R(M, R) \otimes_R N$ is true for N a finite free R-module since $\mathrm{Hom}_R(M, -)$ preserves direct sums. But $\mathrm{Hom}_R(M, -)$ also commutes with filtered colimits and cokernels since M is finite locally free. The tensor product also commutes with filtered colimits and cokernels. In writing the module N as a cokernel of free modules (of possibly infinite rank), we have that $\mathrm{Hom}_R(M, N) \simeq \mathrm{Hom}_R(M, R) \otimes_R N$ is true for all N. □

Lemma 27. *Let M be a finite locally free R-module, then the module M^*, defined as* $\mathrm{Hom}_R(M, R)$ *is also finite locally free. We have a canonical categorical equivalence $M \mapsto M^{**}$ from finite locally free R-modules to itself. Moreover, for any R-algebra S, we have that $(M \otimes_R S)^* \simeq M^* \otimes_R S$, where $(M \otimes_R S)^*$ is the dual of $M \otimes_R S$ as an S-module.*

Proof. Let $R_{f_i}, i = 1, \ldots, n$ be a Zariski cover of R such that $M \otimes_R R_{f_i}$ is free of rank n_i. We calculate with the Lemma 26

$$\begin{aligned}\mathrm{Hom}_R(M, R) \otimes_R R_{f_i} &\simeq \mathrm{Hom}_R(M, R_{f_i})\\ &\simeq \mathrm{Hom}_{R_{f_i}}(M \otimes_R R_{f_i}, R_{f_i})\\ &\simeq (R_{f_i})^{n_i}\end{aligned}$$

So $\mathrm{Hom}_R(M, R)$ is also finite locally free.

The map $M \to M^{**}$ given by $m \mapsto (f \mapsto f(m))$ is an isomorphism on each $\operatorname{Spec} R_{f_i}$, so an isomorphism of R-modules. Suppose $f : M \to N$ is a homomorphism of finite locally free R-modules. We denote by $f^* : N^* \to M^*$ its dual, and $f^{**} : M^{**} \to N^{**}$ its double dual. We need to show commutativity fo the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^{**} & \xrightarrow{f^{**}} & N^{**} \end{array}$$

An element $m \in M$ is mapped to $\zeta \mapsto \zeta(m), \zeta \in M^*$ in M^{**}, and f^{**} maps the latter to $\zeta \mapsto f^*(\zeta)(m)$ for $\zeta \in N^*$. On the other hand, $f(m) \in N$ is mapped to $\zeta \mapsto \zeta(f(m)) \in N^{**}$. But this is exactly $\zeta \mapsto f^*(\zeta)(m)$ by definition of f^*.

Finally,

$$\begin{aligned}\mathrm{Hom}_S(M \otimes_R S, S) &\simeq \mathrm{Hom}_R(M, S)\\ &\simeq \mathrm{Hom}_R(M, R) \otimes_R S\end{aligned}$$

again by the Lemma 26. □

Lemma 28. *If M, N are finite locally free modules over R, then there is a canonical isomorphism $(M \otimes_R N)^* \simeq M^* \otimes_R N^*$.*

Proof. Using Lemma 26 we calculate that

$$\begin{aligned}(M \otimes_R N)^* &\simeq \mathrm{Hom}(M \otimes_R N, R) \\ &\simeq \mathrm{Hom}(M, \mathrm{Hom}(N, R)) \\ &\simeq \mathrm{Hom}(M, R) \otimes_R \mathrm{Hom}(N, R)\end{aligned}$$

□

Lemma 29. *If R is derived IR-complete and M is a finite locally free module over R, then M is derived IR-complete and IR-completely flat.*

Proof. The functor $\mathrm{RHom}_R(R_f, -)$ commutes with direct sums for all $f \in IR$, so a finite direct sum of R-modules is derived IR-complete if and only if each summand is derived IR-complete. But M is a direct summand of a finite free R-module, so M is derived IR-complete. Since M is a projective module, the derived tensor product $M \otimes_R^{\mathbb{L}} R/IR$ is the same as $M \otimes_R R/IR$, which is concentrated in degree 0. M/IM is flat over R/IR by base change. □

Proposition 30. *Let* $\mathsf{FinProj}$ *be the opposite of the category of pairs (A, M) where A is a ring in $(I\mathrm{Aff}/R)$ and M a finite locally free A-module. We equip* $\mathsf{FinProj}$ *with the forgetful functor to $(I\mathrm{Aff}/R)$. Then* $\mathsf{FinProj} \to (I\mathrm{Aff}/R)$ *is a stack for the I-completed fpqc topology whose pullbacks are $(M \otimes_R S, S) \to (M, R)$ corresponding to the map $M \to M \otimes_R S, m \mapsto m \otimes 1$.*

Proof. The universal property of the usual tensor product implies that the morphisms $(M \otimes_R S, S) \to (M, R)$ in the statement are strongly Cartesian, as in [Sta18, Tag 02XK]. We choose these maps as pullbacks in order to make $\mathsf{FinProj} \to (I\mathrm{Aff}/R)$ a fibred category.

Suppose that S is some object of $(I\mathrm{Aff}/R)$. If M, N are two finite locally free S-modules, then $\mathrm{Hom}_{S'}(M \otimes_S S', N \otimes_S S') \simeq (M^* \otimes_S N) \otimes_S S'$ for any $S \to S'$ by Lemma 26. The module $(M^* \otimes_S N) \otimes_S S'$ is further isomorphic to $((M^* \otimes_S N) \otimes_S^{\mathbb{L}} S')^\wedge_I$ by Lemma 29. Thus the complex obtained by applying $S' \mapsto \mathrm{Hom}_{S'}(M \otimes_S S', N \otimes_S S')$ to the Čech nerve of a cover $T' \to T$, which corresponds to a ring map $T \to T'$, is exactly the Amitsur complex $A^\bullet((M^* \otimes_S N) \otimes_S T; T \mapsto T')$. So $S' \mapsto \mathrm{Hom}_{S'}(M \otimes_S S', N \otimes_S S')$ is a sheaf by the exactness of the I-completed Amitsur complex.

Suppose that we have a descent datum of finite locally free module for a cover $S \to S'$. We denote by S^i the derived IS-completion of the $(i+1)$-fold derived tensor product of S' over S. For notational conveniance, let $\iota_{a_0, \ldots, a_n} : S^n \to S^{n+m}$, where $0 \le a_0 \le \cdots \le a_n \le n+m$ to be the inclusion to the $a_0, \ldots, a_n$-th factors. The notation $\otimes_{S^n, a_0, \ldots, a_n} S^{n+m}$ means the tensor product along $\iota_{a_0, \ldots, a_n}$. Then the descent datum is a finite locally free S'-module M and an isomorphism $\varphi : M \otimes_{S^0, 0} S^1 \to M \otimes_{S^0, 1} S^1$ satisfying the cocycle condition, i.e. a commutative diagram

$$\begin{array}{ccc} M \otimes_{S^0,0} S^2 & \xrightarrow{\varphi\otimes\iota_{0,2}} & M \otimes_{S^0,2} S^2 \\ & \searrow^{\varphi\otimes\iota_{0,1}} \quad \swarrow_{\varphi\otimes\iota_{1,2}} & \\ & M_{\otimes_{S^0,1} S^2} & \end{array}$$

By Lemma 25, base changing the descent datum along $S \to S/IS$ gives a descent datum of finite locally free modules for $S/IS \to S'/IS'$ as in [Sta18, Tag 023G]. Then classical fpqc descent [Sta18, Tag 023N] gives a finite locally free S/IS-module N such that $N \otimes_{S/IS} S'/IS' \simeq M \otimes_{S'} S'/IS'$. Since the ring S is classically IS-complete, the pair (S, IS) is henselian and the S/IS module N lifts to a finite locally free S-module P by [Sta18, Tag 0D4A]. The map $P \otimes_S S' \to P \otimes_S S' \otimes_{S'} S'/IS' \simeq M \otimes_{S'} S'/IS'$ lifts along the surjections $M \to M \otimes_{S'} S'/IS'$, so we have a map $P \otimes_S S' \to M$. This map is an isomorphism after modulo IS', and $P \otimes_S S', M$ are finite locally free S'-modules, so derived Nakayama implies that $P \otimes_S S' \to M$ is an isomorphism. Thus we have proved that the descent datum is effective. □

Proposition 31. *Suppose that M is a finite locally free R-algebra. The functor*

$$h_M : (I\mathrm{Aff}/R)^{\mathrm{op}} \to \mathsf{Sets}$$

given by $S \mapsto \mathrm{Hom}_{R-\mathrm{alg}}(M, S)$ is a sheaf with respect to the IR-completely fpqc topology.

Proof. Let $S \to S'$ be a IS-completely faithfully flat ring map with S, S' lying in $(I\mathrm{Aff}/R)$. By Lemma 15, the ring maps $S/I^nS \to S'/I^nS'$ are flat. Then the map is faithfully flat since the spectrum does not change if we enlarge n. In particular the map is injective.

Lemma 26 shows that $\mathrm{Hom}_R(M, S) \simeq M^* \otimes_R S$ for all R-algebra S. As M^* is flat, we get an exact sequence by tensoring M^* with the I-completed Amitsur complex.

$$0 \to \mathrm{Hom}_R(M, S) \to \mathrm{Hom}_R(M, S') \to \mathrm{Hom}_R(M, (S' \otimes^{\mathbb{L}}_S S')^{\wedge}_{IS})$$

In other words, $\mathrm{Hom}_R(M, S)$ is the equalizer of

$$\mathrm{Hom}_R(M, S') \rightrightarrows \mathrm{Hom}_R(M, (S' \otimes^{\mathbb{L}}_S S')^{\wedge}_{IS})$$

where the two maps are induced by the two natural $S' \to (S' \otimes^{\mathbb{L}}_S S')^{\wedge}_{IS}$.

Let $a_i, i = 1, 2, \ldots, m$ be a set of generators of M. For any R-algebra T, the R-algebra homomorphisms $M \to T$ form the equalizer of

$$\mathrm{Hom}_R(M, S) \rightrightarrows \left(\bigoplus_{i=1}^{m^2} T \right) \oplus T$$

where one map sends f to $((f(a_i)f(b_i))_{i,j}, f(1))$ and the other sends f to $((f(a_ib_i))_{i,j}, 1)$. Since equalizers commute with equalizers, $\mathrm{Hom}_{R-\mathrm{alg}}(M, S)$ is the equalizer of

$$\mathrm{Hom}_{R-\mathrm{alg}}(M, S') \rightrightarrows \mathrm{Hom}_{R-\mathrm{alg}}(M, (S' \otimes_S^{\mathbb{L}} S')_I^{\wedge})$$

□

2.3 Quillen exact categories

In this section we introduce the construction of the derived category of a Quillen exact category. We follow the notes [Büh10].

Definition 32. A *Quillen exact category* is an additive category $\mathcal{C}$ with an exact structure, i.e. a specified set $\mathcal{E}$ of pairs (i, p) where i is the kernel of p and p is the cokernel of i, satisfying the following conditions:

(1) For any object A in $\mathcal{C}$, the pairs $(\mathrm{id}_A, 0)$ and $(0, \mathrm{id}_A)$ belong to $\mathcal{E}$.

(2) Suppose (i_1, p_1) and (i_2, p_2) are in $\mathcal{E}$. If i_1 and i_2 are composable, then there is some p_3 such that $(i_1 \circ i_2, p_3)$ lies in $\mathcal{E}$. If p_1 and p_2 are composable, then there is some i_3 such that $(i_3, p_1 \circ p_2)$ lies in $\mathcal{E}$.

(3) Suppose that (i, p) lies in $\mathcal{E}$ where $i : A_1 \to A_2$ and $p : A_2 \to A_3$. If $f : A_1 \to B$ is any morphism, then the pushout j of i along f exists, and (j, q) lies in $\mathcal{E}$ for some q. If $g : C \to A_2$ is any morphism, then the pullback π of p along g exists, and (ι, π) lies in $\mathcal{E}$ for some ι.

The elements of $\mathcal{E}$ are called the short exact sequences. Any i (resp. p) appearing in some pair (i, p) in $\mathcal{E}$ is called an admissible monic (resp. admissible epic).

Note that the class of admissible monics and admissible epics are closed under compositions with isomorphisms.

Clearly an abelian category is a Quillen exact category with the usual notion of short exact sequences. More generally, we have the following lemma, stated in [Büh10] as Lemma 10.20.

Lemma 33. *Suppose $\mathcal{A}$ is an abelian category and $\mathcal{C}$ a full subcategory of $\mathcal{A}$. If $\mathcal{C}$ is closed under extensions, then $\mathcal{C}$ is a Quillen exact category with the exact structure given by the exact sequences in $\mathcal{A}$ whose terms are in $\mathcal{C}$.*

Let $\mathcal{C}$ be a Quillen exact category. We may construct its derived category following the usual recipe. Namely, let $\mathrm{Ch}(\mathcal{C})$ be the category of cochain complexes in $\mathcal{C}$. The notion of chain homotopy works without change in $\mathrm{Ch}(\mathcal{C})$, and the exact structure allows us to identify the acyclic complexes in $\mathrm{Ch}(\mathcal{C})$ with respect to which we may form a Verdier quotient. More precisely,

Definition 34. A complex $K^\bullet \in \mathrm{Ch}(\mathcal{C})$ is called acyclic if for every n, the differential $K^n \to K^{n+1}$ factors as $K^n \to Z^{n+1} \to K^{n+1}$, where $K^n \to Z^{n+1}$ is an admissible epic and $Z^{n+1} \to K^{n+1}$ is an admissible monic, such that for every n, the sequence $Z^n \to K^n \to Z^{n+1}$ is a short exact sequence.

Clearly these notions agree with the usual ones when $\mathcal{C}$ is abelian.

Definition 35. A Quillen exact category $\mathcal{C}$ is called *idempotent complete* if any morphism of the shape $p : A \to A$ in $\mathcal{C}$ satisfying $p^2 = p$ has a kernel.

The definition of quasi-isomorphisms is significantly easier when $\mathcal{C}$ is idempotent complete. So we only give the definition under this assumption.

Definition 36. Suppose $\mathcal{C}$ is an idempotent complete Quillen exact category. A map of complexes $C^\bullet \to D^\bullet$ in $\mathrm{Ch}(\mathcal{C})$ is called a quasi-isomorphism if its cone is acyclic.

The full category of the homotopy category of $\mathrm{Ch}(\mathcal{C})$ spanned by the acyclic complexes is thick, so we can take the Verdier quotient $D(\mathcal{C})$. The Verdier quotient may also be viewed as formally inverting quasi-isomorphisms. There are also versions of the derived category with boundedness condition $D^+(\mathcal{C})$, $D^-(\mathcal{C})$ and $D^b(\mathcal{C})$, as usual. More details of this construction are included in Section 10.4 of [Büh10].

Finally we make some definition about functors between exact categories.

Definition 37. Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between Quillen exact categories. F is called exact if it takes exact sequences to exact sequences. F is said to reflects exactness if $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is a short exact sequence in $\mathcal{D}$ implies that $A \xrightarrow{f} B \xrightarrow{g} C$ is a short exact sequence in $\mathcal{C}$. F is said to exhibits $\mathcal{C}$ as a fully exact subcategory if F is fully faithful, exact and reflects exactness.

2.4 Embedding into the derived category

It is well-known that an abelian category embeds fully faithfully into its derived category as complexes concentrated in degree 0. We now investigate the embedding problem in slightly more generality. The idea of using the following discussion to embed finite locally free group schemes (Zariski-locally) into the derived category of p-divisible groups was proposed by Beilinson. See [Kis06], Section 2.3.

Suppose $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is a full additive subcatgory closed under extensions. (In particular, closed under isomorphisms.) Then $\mathcal{C}$ can be viewed as a Quillen exact category by declaring short exact sequences in $\mathcal{C}$ to be short exact sequences in $\mathcal{A}$ whose terms are in $\mathcal{C}$. Let $\mathcal{E}$ be a full additive subcategory of $\mathcal{A}$, closed under isomorphisms. Furthermore, we assume the following conditions on $\mathcal{C}$ and $\mathcal{E}$.

- $\mathcal{C}$ is idempotent complete as a Quillen exact category
- If f is a monomorphism of $\mathcal{A}$ with domain and target both in $\mathcal{C}$, then $\operatorname{coker} f$ is an object of $\mathcal{C}$
- Every object of $\mathcal{E}$ is the kernel of a morphism in $\mathcal{C}$ which is an epimorphism in $\mathcal{A}$

- If f is a monomorphism of $\mathcal{A}$ whose domain is in $\mathcal{E}$ and target is in $\mathcal{C}$, then $\operatorname{coker} f$ is an object of $\mathcal{C}$

Lemma 38. *If $0 \to A_0 \to A_1 \to \cdots \to A_n$ is an exact sequence in $\mathcal{A}$ such that $A_i \in \mathcal{C}$ for all i, then the cokernel of $A_{n-1} \to A_n$ lies in $\mathcal{C}$.*

Proof. We prove the statement by induction on n. The case $n = 0$ is trivial. Suppose we have an exact sequence $0 \to A_0 \to \cdots \to A_n \to A_{n+1}$ in $\mathcal{A}$ with all the terms objects of $\mathcal{C}$. By induction hypothesis the cokernel of $A_{n-1} \to A_n$ lies in $\mathcal{C}$. Then $\operatorname{coker}(A_{n-1} \to A_n) \to A_{n+1}$ is a monomorphism in $\mathcal{A}$ by exactness. Thus it has a cokernel in $\mathcal{C}$, which coincides with the cokernel of $A_n \to A_{n+1}$. □

Lemma 39. *A bounded below complex in $\mathrm{Ch}(\mathcal{C})$ is acyclic if and only if it is acyclic as an object of $\mathrm{Ch}(\mathcal{A})$.*

Proof. If $C^\bullet$ is acyclic as a complex in $\mathcal{C}$, then the factorizations $C^n \to Z^{n+1} \to C^{n+1}$ provide a factorization in $\mathcal{A}$.

Conversely, suppose $C^\bullet$ is an acyclic bounded below complex. By Lemma 38, the cokernels of $C^n \to C^{n+1}$ all lie in $\mathcal{C}$. We may take $Z^n = \operatorname{coker}(C^{n-1} \to C^n)$. By construction $Z^n \to C^n \to Z^{n+1}$ is an exact sequence in $\mathcal{A}$, and thus also an exact sequence in $\mathcal{C}$. So $C^n \to Z^{n+1} \to C^{n+1}$ give the required factorization for $C^\bullet$ to be acyclic. □

Lemma 40. *Suppose $C^\bullet$ is a bounded below complex in $\mathcal{C}$ which has vanishing cohomology in degrees < 0 when regarded as a complex in $\mathcal{A}$. Let $K^\bullet$ be the complex with $K^i \simeq 0$ when $i < 0$, $K^0 \simeq \operatorname{coker}(C^{-1} \to C^0)$ and $K^i \simeq C^i$ for $i > 0$. Then $K^\bullet$ is a complex in $\mathcal{C}$, and the natural map $C^\bullet \to K^\bullet$ is a quasi-isomorphism in $\mathrm{Ch}(\mathcal{C})$.*

Proof. That K^0 lies in $\mathcal{C}$ follows from Lemma 38. The cone of $C^\bullet \to K^\bullet$ is a bounded below complex in $\mathcal{C}$ which is acyclic regarded as a complex in $\mathcal{A}$. Thus the cone is acyclic and $C^\bullet \to K^\bullet$ is a quasi-isomorphism. □

Now we are ready to prove the embedding.

Proposition 41. *The category $\mathcal{E}$ embeds fully faithfully into $D^b(\mathcal{C})$ by sending an object X to the complex $K^0 \to K^1$, where $K^0 \to K^1$ is an epimorphism in $\mathcal{A}$ with K^i in $\mathcal{C}$ and $\ker(K^0 \to K^1) \simeq X$.*

Proof. Let $\mathcal{H}$ be the full subcategory of $D^b(\mathcal{C})$ spanned by the complexes $K^0 \to K^1$ where K^i lie in $\mathcal{C}$, $K^0 \to K^1$ is an epimorphism in $\mathcal{A}$ and $\ker(K^0 \to K^1)$ lies in $\mathcal{E}$. We show that $H^0 : \mathcal{H} \to \mathcal{E}$ is fully faithful, which will prove the embedding.

Suppose a morphism $f : K^\bullet \to L^\bullet$ in $\mathcal{H}$ is mapped to 0 by H^0. f factorizes as the composition $K^\bullet \xrightarrow{g} M^\bullet \xleftarrow{s} L^\bullet$, where g, s are maps of chain complexes and s is a quasi-isomorphism. By Lemma 40, we may assume that $M^i \simeq 0$ for $i < 0$. Then the map g, viewed as a map of complexes in $\mathcal{A}$, induces 0 on the H^0. So $g^0(H^0(K^\bullet)) = 0$ in M^0. But $K^0 \to K^1$ is surjective. So the map $K^0 \to M^0$ factors through K^1. Let $h^i : K^i \to M^{i-1}$ be the map $K^1 \to M^0$

induced by $K^0 \to M^0$ for $i = 1$, and 0 otherwise. h is a map of chain complexes in $\mathcal{C}$ since the latter is a full subcategory of $\mathcal{A}$. Then h is a null homotopy for g. This proves faithfulness.

Now suppose there's a map $f : H^0(K^\bullet) \to H^0(L^\bullet)$. Denote $P = H^0(K^\bullet)$, $Q = H^0(L^\bullet)$. Let M^0 be $K^0 \oplus L^0$. We have a map (in $\mathcal{A}$) $(\mathrm{id}, f) : P \to P \oplus Q$, whose composition with the natural inclusions give $g : P \to M^0$. g is a monomorphism since $\mathrm{id} : P \to P$ is. By assumption g has a cokernel $M^0 \to M^1$ which also lies in $\mathcal{C}$.

Note that $H^0(M^\bullet) = \ker(M^0 \to M^1) = P \xrightarrow{g} M^0$. So the compositions $H^0(M^\bullet) \to M^0 \xrightarrow{\mathrm{pr}_2} L^0 \to L^1$ and $H^0(M^\bullet) \to M^0 \xrightarrow{\mathrm{pr}_1} K^0 \to K^1$ are 0, and the projections give maps of chain complexes $M^\bullet \to K^\bullet$, $M^\bullet \to L^\bullet$. We have a commutative diagram

$$\begin{array}{ccccc} P & \xleftarrow[\mathrm{id}]{} & P & \xrightarrow{f} & Q \\ \downarrow & & \downarrow & & \downarrow \\ K^0 & \longleftarrow & M^0 & \longrightarrow & L^0 \end{array}$$

Now we check that $M^\bullet \to K^\bullet$ is a quasi-isomorphism. The mapping cone consists of only three non-zero terms $M^0 \to M^1 \oplus K^0 \to K^1$. We need to check that this sequence is a short exact sequence. But this can be checked in $\mathcal{A}$, and the map induced on H^0 is $H^0(M^\bullet) \simeq P$, an isomorphism. Also, it follows immediately from the diagram that $K^\bullet \leftarrow M^\bullet \to L^\bullet$ induces f on H^0. This proves fullness. $\square$

2.5 Finite locally free group schemes

In this subsection we collect some basic results about finite locally free group schemes. Let R be a ring.

Definition 42. A *finite locally free group scheme* over R is an abelian sheaf on $(\mathrm{Aff}/R)_{\mathrm{fppf}}$ representable by an R-algebra that is finite locally free as an R-module.

Remark 43. Since we require representability of the sheaf, the topology we use on (Aff/R) is not important. We could use fpqc, syntomic, étale, or Zariski topology instead.

Remark 44. Suppose $f : G \to H$ is a map of finite locally free group schemes over R which is an injective map of sheaves (on $(\mathrm{Aff}/R)_{\mathrm{fppf}}$). Then [Sta18, Tag 035D] implies f is finite because G is finite over R and H is separeted over R. And thus f is proper by [Sta18, Tag 01WN]. But morphisms to affine schemes are the same as ring maps of global sections in the other direction, so f is a monomorphism in the category of R-schemes. We conclude that f is in fact a closed immersion by [Sta18, Tag 04XV].

Conversely, if $f : G \to H$ is a closed immersion, then f is also a monomorphism in the category of R-schemes, and in particular injective as a map of sheaves on $(\mathrm{Aff}/R)_{\mathrm{fppf}}$.

Next we discuss the Cartier duality.

Proposition 45. *If G is a finite locally free group scheme over R, then the sheaf $\mathscr{H}\mathrm{om}_{(\mathrm{Aff}/R)_{\mathrm{fppf}}}(G, \mathbb{G}_m)$ is also a finite locally free group scheme.*

Proof. Let A be the R-algebra representing G. The commutative group scheme structure on G makes A into a commutative cocommutative Hopf algebra. Since A is finite locally free over R, we can apply Lemma 27 to see that the dual R-module A^* is also a finite locally free R-module. Thanks to Lemma 28, we can define the multiplication, comultiplication, unit, counit and antipodal on A^* by the dual of comultiplication, multiplication, counit, unit and antipodal on A respectively. The axioms of Hopf algebras are preserved by changing the direction of arrows, so A^* is also a commutative cocommutative Hopf algebra.

We write $\mathscr{H}\mathrm{om}(-,-)$ for $\mathscr{H}\mathrm{om}_{(\mathrm{Aff}/R)}(-,-)$ in the rest of the proof. We claim that A^* represents the sheaf $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$. For all R-algebra S,

$$\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)(S) \simeq \mathrm{Hom}_{\mathsf{Hopf}}(S[T, T^{-1}], A \otimes S)$$

can be identified with the set of group-like elements in $A \otimes_R S$, i.e. the elements $x \in A \otimes_R S$ such that $\Delta(x) = x \otimes x$ where Δ is the comultiplication. On the other hand, Lemma 27 shows that

$$\mathrm{Hom}_R(A^*, S) \simeq \mathrm{Hom}_S(A^* \otimes_R S, S) \simeq \mathrm{Hom}_S((A \otimes_R S)^*, S) \simeq A \otimes_R S$$

Then, by definition of the multiplicative structure on A^*, the R-algebra homomorphisms $A^* \to S$ are the elements $x \in A \otimes_R S$ such that $\xi \otimes \zeta(\Delta x) = \xi(x)\zeta(x)$ for all $\xi, \zeta \in A^* \otimes_R S \simeq (A \otimes_R S)^*$, and that $\eta(x) = 1$, where $\eta = \epsilon^*(1)$ and $\epsilon^* : S \to A^* \otimes_R S$ is the dual of the counit $\epsilon : A \otimes S \to S$. The first requirement forces x to be a group-like element. We check the second requirement for group-like elements. We need to verify that the composition $S \xrightarrow{\epsilon^*} (A \otimes_R S)^* \xrightarrow{x} S$ is the identity. Taking duals of the S-modules, we reduce to showing that $S \xrightarrow{x} A \otimes_R S \xrightarrow{\epsilon} S$ is the identity. That is, the counit ϵ take the group-like element x to 1. The compatibility of the comultiplication and counit implies that $\epsilon \otimes \mathrm{id}(\Delta(x))$ is the identity, but $\Delta(x) = x \otimes x$, so $\epsilon(x)x = x$. Moreover the compatibility of the antipodal and the comultiplication implies that x is a unit in $S \otimes_R A$. Thus $\epsilon(x) = 1$. We conclude that the R-algebra morphisms $A^* \to S$ are in bijection with the group-like elements of $A \otimes S$, which is also the S-valued points of $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$.

Finally we show that the group structure on $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$ agrees with that defined by the comultiplication and counit of A^*. Suppose x, y are two S-valued points of $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$, which we identify with two group-like elements in $A \otimes_R S$. The multiplication on $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$ is defined by the multiplication on $\mathbb{G}_m$, which translated to

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Hopf}}(S[T, T^{-1}], A \otimes_R S)^2 &\simeq \mathrm{Hom}_{\mathsf{Hopf}}(S[T, T^{-1}] \otimes_S S[T, T^{-1}], A \otimes_R S) \\ &\to \mathrm{Hom}_{\mathsf{Hopf}}(S[T, T^{-1}], A \otimes_R S)\end{aligned}$$

where the latter map is the comultiplication of $\mathbb{G}_m$. The product of x, y in $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$ is thus xy. We now show that the product of x, y as S-valued points of $\operatorname{Spec} A^*$ is also xy. x, y associate to the ring homomorphisms $(A \otimes_R S)^* \to S$ dual to the maps $S \xrightarrow{x,y} A \otimes_R S$. And we need to show that the composition

$$(A \otimes_R S)^* \to (A \otimes_R S)^* \otimes_S (A \otimes_R S)^* \to S$$

is the ring homomorphism associated to xy. Taking the dual, this is equivalent to

$$S \to (A \otimes_R S) \otimes_S (A \otimes_R S) \to A \otimes_R S$$

mapping $1 \mapsto x \otimes y \mapsto xy$, which is true since the comultiplication on $(A \otimes_R S)^*$ is the dual of the multiplication on $A \otimes_R S$. Lastly the counit $(A \otimes_R S)^* \to S$ is the dual of the unit $S \to A \otimes_R S$, which corresponds to the group-like element 1. It correspond to the identity element of $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$. □

Definition 46. If G is a finite locally free group scheme over R, then the group scheme representing the sheaf $\mathscr{H}\mathrm{om}(G, \mathbb{G}_m)$ is called the Cartier dual of G. We denote the Cartier dual of G by $G^\vee$.

Remark 47. Since base change of schemes along a morphism $S \to T$ of schemes correspond to restriction of sheaves from (Sch/S) to (Sch/T), and the internal hom commutes with restriction, the Cartier dual commutes with base change.

Proposition 48. *The Cartier duality functor is an order* 2 *autoequivalence of the category of finite locally free group schemes.*

Proof. We need to prove that $G \mapsto (G^\vee)^\vee$ is equivalent to the identity functor.

Suppose G is a finite locally free group scheme represented by the Hopf algebra A. Then $(G^\vee)^\vee$ is represented by A^{**}, which is canonically isomorphic to A as an R-module by Lemma 27. Again by Lemma 27, the double duality maps a homomorphism of R-modules to itself under the isomorphism $A \simeq A^{**}$. Thus the Hopf algebra structure on A^{**} is identified with the Hopf algebra structure on A by construction, and also a morphism $f : A \to B$ of Hopf algebras is identified with itself under the isomorphisms $A \simeq A^{**}$ and $B \simeq B^{**}$. □

Proposition 49. *The Cartier duality functor takes an injection (as sheaves) $f : G \to H$ of finite locally free group schemes into a faithfully flat morphism of schemes $f^\vee : H^\vee \to G^\vee$.*

Proof. This proof is adapted from Lemma 2.4 of [Rub]. Suppose $f : G \to H$ is an injection of finite locally free group schemes. Remark 44 shows that f is a closed immersion. Then the map $A(H) \to A(G)$ of coordinate rings is surjective. Now we prove that $A(G)^* \to A(H)^*$ is faithfully flat.

The main theorem of Chapter 14 of [Wat79] asserts that an injective map of Hopf algebras over a field is faithfully flat. Let $R \to k$ be a residue field of R. Then $A(H) \otimes_R k \to A(G) \otimes_R k$ is surjective and so $(A(G) \otimes_R k)^* \to (A(H) \otimes_R k)^*$ is injective. By Lemma 27, $(A(H) \otimes_R k)^* \simeq A(H)^* \otimes_R k$ and similarly for $A(G)$. So $A(G)^* \otimes_R k \to A(H)^* \otimes_R k$ is injective. These two base changes are still Hopf algebras, and thus the map $A(G)^* \otimes_R k \to A(H)^* \otimes_R k$ is faithfully flat. Now $G^\vee, H^\vee$ are finite and flat over R, and all maps on fibres $f_s^\vee : H_s^\vee \to G_s^\vee, s \in \operatorname{Spec} R$ are faithfully flat, so $f^\vee : H^\vee \to G^\vee$ is faithfully flat by the fibre criterion of flatness [Sta18, Tag 039E]. □

Remark 50. Suppose $f : G \to H$ is a map of finite locally free groups schemes which is a surjective map of sheaves. Then $0 \to \ker f \to G \to H \to 0$ is an exact sequence of sheaves, and we have an injection $f^\vee : H^\vee \to G^\vee$ by applying $\mathscr{H}\mathrm{om}(-, \mathbb{G}_m)$. As double Cartier duality takes a morphism back to itself, Proposition 49 shows that f is faithfully flat.

Conversely, let $G \to H$ be a faithfully flat morphism of finite locally free group schemes. If $A(G), A(H)$ are the coordinate rings of G, H respectively, then $A(H) \to A(G)$ is of finite presentation as a ring map since both are of finite presentation over R. Therefore we can apply [Sta18, Tag 05VM] to obtain that $G \to H$ is surjective as a map of sheaves.

In fact in this case $A(G)$ is even of finite presentation as a module over $A(H)$ since $A(H) \to A(G)$ is finite. Then $A(G)$ is finite locally free over $A(H)$ by flatness. Moreover, the kernel of $G \to H$ is flat over R, and is thus a finite locally free group scheme.

Proposition 51. *The Cartier duality functor is exact.*

Proof. The Cartier dual is left exact since the internal hom is left exact. We only need to prove that the Cartier duality takes injective maps of finite locally free group schemes into surjective maps. Proposition 49 shows that an injection $G \to H$ of finite locally free group schemes is taken to a faithfully flat morphism $H^\vee \to G^\vee$. Then Remark 50 implies that $H^\vee \to G^\vee$ is surjective. □

Remark 52. Combining Remark 44 and Remark 50, we can give a scheme theoretic description of exactness of finite locally free group schemes. Namely, a sequence

$$0 \to G \to H \to K \to 0$$

of finite locally free group schemes is exact if and only if $H \to K$ is faithfully flat and G fits in the pullback square of schemes

$$\begin{array}{ccc} G & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow{\scriptstyle 1} \\ H & \longrightarrow & K \end{array}$$

Finally we give definitions of the Frobenius and the Verschiebung that will be useful in the sequel.

Definition 53. Suppose $pR = 0$ and G a finite locally free group scheme over R. We denote by F the relative Frobenius $G \to G^{(p)}$, where $G^{(p)}$ denotes $G \times_{\operatorname{Spec} R, x \mapsto x^p} \operatorname{Spec} R$. Since the Cartier duality commutes with base change, $(G^\vee)^{(p)} \simeq (G^{(p)})^\vee$. The morphism $V : G^p \to G$ corresponding to the relative Frobenius $F : G^\vee \to (G^\vee)^{(p)}$ under Cartier duality is called the *Verschiebung*.

3 Prismatic cohomology

3.1 Prisms

Definition 54. A *δ-ring* is a pair (R, δ) where R is an $\mathbb{Z}_{(p)}$-algebra and $\delta : R \to R$ is a map of sets, such that

$$\begin{aligned} \delta(0) = \delta(1) &= 0 \\ \delta(x+y) &= \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} x^i y^{p-i} \\ \delta(xy) &= x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y) \end{aligned}$$

Note that $p \mid \binom{p}{i}$ is true for all $0 < i < p$, so the definition is indeed valid without p being invertible. If there is no risk of confusion, we may also refer to the δ-ring (R, δ) with only R.

Definition 55. If (R, δ) is a δ-ring, then we call the map $\varphi : R \to R, x \mapsto x^p + p\delta(x)$ the Frobenius lift of (R, δ). If no confusion is possible, then we will also simply say that φ is the Frobenius on R.

Lemma 56. *The Frobenius on a δ-ring is a ring homomorphism whose reduction modulo p is the usual Frobenius of characteristic p rings.*

Proof. We need to show that $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(xy) = \varphi(x)\varphi(y)$. We calculated that

$$\begin{aligned} \varphi(x+y) &= (x+y)^p + p\delta(x+y) \\ &= x^p + y^p + \sum_{i=1}^{p-1} \binom{p}{i} x^i y^{p-i} + p\delta(x+y) \\ &= x^p + y^p + p\delta(x) + p\delta(y) + \sum_{i=1}^{p-1} \binom{p}{i} x^i y^{p-i} - \sum_{i=1}^{p-1} \binom{p}{i} x^i y^{p-i} \\ &= \varphi(x) + \varphi(y) \end{aligned}$$

and also

$$\begin{aligned} \varphi(xy) &= x^p y^p + p\delta(xy) \\ &= x^p y^p + p(x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y)) \\ &= \varphi(x)\varphi(y) \end{aligned}$$

The reduction modulo p part follows immediately from the expression $x \mapsto x^p + p\delta(x)$. □

Definition 57. A *prism* is a pair (A, I) where A is a δ-ring and I an ideal in A, satisfying the following conditions:
(1) I is finite locally free of rank 1 as an A-module.
(2) A is derived (p, I)-complete.
(3) $p \in I + \varphi(I)A$.

In practice the ideal I is usually principal, i.e. $I = (d)$ where d is not a zero-divisor in A. A particularily important example is that $I = (d)$ where d is not a zero divisor in A such that $\delta(d)$ is a unit. In this case the condition (3) is verified as $p \in (p\delta(d)) \subset (d, d^p + p\delta(d))$.

There is some rigidity involving the ideal I.

Lemma 58. *If $(A, I) \to (B, J)$ is a morphism of prisms, i.e. a δ-ring map $A \to B$ such that I is mapped into J, then $J = IB$. Conversely, if (A, I) is a prism, $(A, I) \to (B, IB)$ a map of δ-rings such that B is derived (p, I)-complete and IB is finite locally free of rank 1 over B, then (B, IB) is also a prism.*

Proof. [BS22], Lemma 3.5.

The proof of the converse implication is easy so we recall it. The only condition that needs to be checked is that $p \in IB + \varphi_B(IB)B$. But the map $A \to B$ is compatible with the Frobenii on A and B, so $\varphi_B(IB)$ contains the image of $\varphi_A(I)A$. □

In particular, if A is a δ-ring and $I \subset J$ are two ideals such that (A, I) and (A, J) are both prisms, then $I = J$.

Definition 59. A prism (A, I) is called *bounded* if the ring A/I has bounded p^∞-torsion, and *transversal* if A/I is p-torsion free.

Lemma 60. *If $(A, (d))$ is a transversal prism, then A is p-torsion free. In this case (p^m, d^n) and (d^m, p^n) are regular sequences on A for arbitrary $m, n \geq 1$.*

Proof. Suppose $x \in A$ is such that $px = 0$. Since A/d is p-torsion free, $x \in (d)$. Then $x = dy$ for some $y \in A$. But then $dpy = 0$, and thus $py = 0$. Continuing this process, we can find a sequence $\{x_n\}$ such that $x_n = dx_{n+1}$ and $x_0 = x$. But A is derived d-complete, and [Sta18, Tag 091P] implies that $\lim_{\times d} A \simeq 0$. So we have that $x = 0$.

For a transversal prism $(A, (d))$, both p and d are not zero divisors. Then p^m, d^n are not zero divisors. We show by induction that A/d^nA is p^m-torsion free for all $m, n \geq 1$. When $n = 1$ this is the assumption. Suppose we have shown the claim whenever $n < n_0$. As d is not a zero divisor, we have an exact sequence

$$0 \to A/d^{n_0-1}A \xrightarrow{d} A/d^{n_0}A \to A/dA \to 0$$

Applying $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}/p^m\mathbb{Z}, -)$ to the exact sequence yields an exact sequence

$$(A/d^{n_0-1}A)[p^m] \to (A/d^{n_0}A)[p^m] \to (A/dA)[p^m]$$

where the first and the third terms are 0 by induction hypothesis. Thus $(A/d^{n_0}A)[p^m] \simeq 0$. So (d^n, p^m) is a regular sequence.

Finally, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{d^n} & A & \longrightarrow & A/d^nA & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle p^m} & & \\ 0 & \longrightarrow & A & \xrightarrow{d^n} & A & \longrightarrow & A/d^nA & \longrightarrow & 0 \end{array}$$

The snake lemma implies that $A/p^mA \xrightarrow{d^n} A/p^mA$ is injective. So (p^m, d^n) is also a regular sequence. ☐

Lemma 61. *A bounded prism $(A, (d))$ has tame $\{d, p\}^\infty$-torsion.*

Proof. By assumption A is d-torsion free. Then we have exact sequences

$$0 \to A/d \xrightarrow{d^n} A/d^{n+1} \to A/d^n \to 0$$

If $(A/d)[p^\infty] = (A/d)[p^c]$, then we show by induction that $(A/d^n)[p^\infty] = (A/d^n)[p^{nc}]$. The case $n = 1$ is the assumption. Suppose we have case n. Let $x \in (A/d^{n+1})[p^\infty]$. Then the image of x in A/d^n is killed by p^{nc} by induction hypothesis. So $p^{nc}x$ lies in the image of A/d, and is a p^∞-torsion. Then $p^cp^{nc}x = 0$ since $(A/d)[p^\infty] = (A/d)[p^c]$. This completes the induction step.

Using again that A is d-torsion free, we have that

$$\mathrm{Kos}(A; d^m, p^n) \simeq \mathrm{Kos}(A/d^m; p^n)$$

while the latter has H^{-1} equal to $(A/d^m)[p^n]$, H^0 equal to $A/(d^m, p^n)$ and $H^i = 0$ if $i \neq -1, 0$. Thus the transition maps

$$\mathrm{Kos}(A; d^m, p^{n+mc+1}) \to \mathrm{Kos}(A; d^m, p^n)$$

induces 0 on H^{-1}. We conclude that the pro-systems

$$\{\mathrm{Kos}(A; d^m, p^n)\}_{m,n}, \{A/(d^m, p^n)\}_{m,n}$$

are pro-isomorphic. Note that $(d^m, p^n) \subset I^{\min(m,n)}$ and $I^{2(m+n)} \subset (d^m, p^n)$. Thus the ideals $(d^m, p^n)_{m,n}$ and I^m are cofinal, so $\{A/(d^m, p^n)\}$ and $\{A/I^n\}_n$ are pro-isomorphic. ☐

Remark 62. In this subsection we will often state results about prisms while only giving the argument when the ideal in the prism is pricipal. The general statements are proved in [BS22]. The idea of the generalization is to replace the Koszul complexes in Definition 13 with a more general form $\mathrm{Kos}(J_1, \dots, J_n)$ where J_n are generalized Cartier divisors.

There is an analogue of Lemma 22 for bounded prisms.

Lemma 63. *Let (A, I) be a bounded prism and $M \in D(A)$ a derived (p, I)-complete, (p, I)-completely flat complex. Then $M \simeq M_0[0]$, where M_0 is a classically (p, I)-complete A-module. Moreover, for any $n \geq 0$, $M[I^n] = 0$ and the module M/I^nM has bounded p^∞-torsion.*

Proof. This is [BS22], Lemma 3.7. If $I = (d)$ is principal, then we can apply Lemma 61 to see that A has tame $\{d, p\}^\infty$-torsion. Thus the part that M is concentrated in degree 0 and there given by a classically (p, d)-complete module follows from Lemma 17. ☐

Definition 64. Let (A, I) be a bounded prism and R an A/I-algebra. The *relative prismatic site* $(R/A)_{\mathbb{\Delta}}$ is defined to be the category of pairs $((B, IB), f)$, where (B, IB) is a bounded prism over (A, I) and $f : R \to B/IB$ a map of A/I-algebras. Morphisms from $((B', IB'), f')$ to $((B, IB), f)$ are defined to be maps of δ-rings $B \to B'$ such that the diagram commutes

$$\begin{array}{ccc} & R & \\ {}^{f}\swarrow & & \searrow^{f'} \\ B/IB & \longrightarrow & B'/IB' \end{array}$$

We equip $(R/A)_{\mathbb{\Delta}}$ with the following topology: covers are given by morphisms

$$((B', IB'), f') \to ((B, IB), f)$$

where $B \to B'$ is (p, I)-completely faithfully flat.

Definition 65. Let R be a ring. The *absolute prismatic site* $(R)_{\mathbb{\Delta}}$ is defined to be the category of pairs $((B, IB), f)$, where (B, IB) is a bounded prism and $f : R \to B/IB$ a map of rings. Morphisms from $((B', IB'), f')$ to $((B, IB), f)$ are defined to be maps of δ-rings $B \to B'$ such that the diagram commutes

$$\begin{array}{ccc} & R & \\ {}^{f}\swarrow & & \searrow^{f'} \\ B/IB & \longrightarrow & B'/IB' \end{array}$$

We equip $(R/A)_{\mathbb{\Delta}}$ with the following topology: covers are given by morphisms

$$((B', IB'), f') \to ((B, IB), f)$$

where $B \to B'$ is (p, I)-completely faithfully flat.

We will omit the map $f : R \to B/IB$ in the notation $((B, I), f)$ if no confusion is possible. We need to verify the site axioms for the definition. Namely, we need to show the following.

Lemma 66. *Let $(B', JB') \to (B, J) \leftarrow (B'', JB'')$ be two morphisms in either $(R/A)_{\mathbb{\Delta}}$ or $(R)_{\mathbb{\Delta}}$ where the map $B \to B'$ is (p, J)-completely flat. Then*

$$((B' \otimes^{\mathbb{L}}_B B'')^{\wedge}_{(p,J)}, J(B' \otimes^{\mathbb{L}}_B B'')^{\wedge}_{(p,J)})$$

is a prism and serves as the pullback of the diagram $(B', JB') \to (B, J) \leftarrow (B'', JB'')$.

Proof. This is the first part of [BS22], Corollary 3.12. We give here the argument for the case $J = (d)$ is principal.

Let C be $(B' \otimes^{\mathbb{L}}_B B'')^{\wedge}_{(p,d)}$. Lemma 11 implies that C is (p,d)-completely flat over B''. Then we can apply Lemma 17 to see that C is concentrated in degree 0, and is classically (p,d)-complete. Moreover, $C \otimes^{\mathbb{L}}_{B''} B''/(p,d)^n B''$ is concentrated in degree 0 by Lemma 15, and thus coincides with $C/(p,d)^n C$. But we also have that

$$\begin{aligned} C \otimes^{\mathbb{L}}_{B''} B''/(p,d)^n B'' &\simeq B' \otimes^{\mathbb{L}}_B B'' \otimes^{\mathbb{L}}_{B''} B''/(p,d)^n B'' \\ &\simeq B'/(p,d)^n B' \otimes^{\mathbb{L}}_{B/(p,d)^n} B''/(p,d)^n B'' \\ &\simeq B'/(p,d)^n B' \otimes_{B/(p,d)^n} B''/(p,d)^n B'' \end{aligned}$$

So in fact C is the classical (p,d)-completion of $B' \otimes_B B''$, which inherits the structure of a δ-ring by Remark 2.7 and Lemma 2.17 of [BS22].

In view of Lemma 58, it remains to show that C is d-torsion free. The ring B'' is d-torsion free since $(B'', (d))$ is a prism, so

$$C[d] \simeq H^{-1}(C \otimes^{\mathbb{L}}_{B''} B''/dB'')$$

By Lemma 11, $C \otimes^{\mathbb{L}}_{B''} B''/dB''$ is $(p,d) = (p)$-completely flat over B''/dB'', which has bounded p^∞-torsion since $(B'', (d))$ is a bounded prism. Lemma 17 then implies that $C \otimes^{\mathbb{L}}_{B''} B''/dB''$ is concentrated in degree 0. Thus $C[d] \simeq 0$. □

Definition 67. The *prismatic structural sheaf* on $(R/A)_{\mathbb{\Delta}}$ or $(R)_{\mathbb{\Delta}}$ is the functor $\mathcal{O}_{\mathbb{\Delta}} : (B, J) \mapsto B$. We also define the functor $\overline{\mathcal{O}}_{\mathbb{\Delta}} : (B, J) \mapsto B/J$.

Lemma 68. *The functors $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ are indeed sheaves on $(R/A)_{\mathbb{\Delta}}$ and $(R)_{\mathbb{\Delta}}$, and the sheaves have vanishing higher cohomology on each object (B, J).*

Proof. This statement is shown in the second part of [BS22], Corollary 3.12. Here we give the argument assuming that $J = (d)$ is principal.

Lemma 66 implies that the Čech nerve of $(B', (d)) \to (B, (d))$ in $(R/A)_{\mathbb{\Delta}}$ or $(R)_{\mathbb{\Delta}}$ is calculated in the category $((p,d)\mathrm{Aff}/B)$. Applying the functor $\mathcal{O}_{\mathbb{\Delta}}$ to the Čech nerve gives the (p,d)-completed Amitsur complex $\mathrm{Ami}(B; B \to B')$, which is acyclic by Lemma 61 and Lemma 24. The sheaf property and vanishing of higher Čech cohomology follow.

For the functor $\overline{\mathcal{O}}_{\mathbb{\Delta}}$, we note that each term of the Čech nerve is d-torsion free, so the derived tensor product $\mathrm{Ami}(B; B \to B') \otimes^{\mathbb{L}}_B B/dB$ can be calculated as applying $\otimes_B B/dB$ to each term of the complex, thanks to the acyclic assembly lemma. But this is exactly the complex obtained by applying $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ to the Čech nerve of $(B', (d)) \to (B, (d))$, which is then acyclic. Again this shows the sheaf property and the vanishing of higher Čech cohomology.

Equipped with the knowledge that all higher Čech cohomology of $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ for all covers $(B', (d)) \to (B, (d))$ vanish, [Sta18, Tag 03F9] implies that all higher cohomology on the object $(B, (d))$ vanish. □

Definition 69. Let (A, I) be a bounded prism and R be a p-completely smooth A/I-algebra, then the prismatic cohomology $\mathbb{\Delta}_{R/A}$ of R is defined to be the complex

$$R\Gamma((R/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$$

The Hodge-Tate cohomology $\overline{\Delta}_{R/A}$ is defined to be the complex

$$R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}_{\Delta}})$$

For general A/I-algebra R, the derived prismatic cohomology $\Delta_{R/A}$ of R is defined to be the left Kan extension of $\Delta_{-/A}$ from p-completely smooth A/I-algebras. And the derived Hodge-Tate cohomology $\overline{\Delta}_{R/A}$ is defined to be the left Kan extension of $\overline{\Delta}_{-/A}$ from p-completely smooth A/I-algebras.

To be more precise, the left Kan extension is done in the following steps: for a general A/I-algebra R, we fix a simplicial resolution $C^\bullet \to R$ of R by p-completely A/I-algebras C^i, calculate the prismatic cohomology $\Delta_{C^i/A}$ or the Hodge-Tate cohomology $\overline{\Delta}_{C^i/A}$, and then take the homotopy colimits of the diagram $\Delta_{C^\bullet/A}$ or $\overline{\Delta}_{C^\bullet/A}$. It is claimed in Construction 7.6 of [BS22] that this construction does not depend on the choice of the simplicial resolution. In particular, we may resolve a p-completely smooth A/I-algebra R by itself to see that its derived prismatic cohomology (resp. its derived Hodge-Tate cohomology) agrees with its prismatic cohomology (resp. its Hodge-Tate cohomology). Thus we are justified to use the same symbol to denote either the derived or the non-derived version of the prismatic cohomology and the Hodge-Tate cohomology.

Proposition 70 (formally étale localization for derived prismatic cohomology)**.** *Let (A, I) be a bounded prism and $R \to R'$ map of p-complete $\overline{A}$-algebra such that $(L_{R'/R})_p^\wedge \simeq 0$, then $\overline{\Delta}_{R'/A} \simeq (\overline{\Delta}_{R/A} \otimes_R^{\mathbb{L}} R')^\wedge$.*

Proof. [BL22], Proposition 4.1.13. ☐

Finally we introduce the Nygaard filtration.

Definition 71. Let (A, I) be a prism. The *Nygaard filtration*, denoted by $\mathcal{N}^{\geq n}A \subset A$ for $n \geq 0$, is defined to be

$$\mathcal{N}^{\geq n}A = \{x \in \Delta_R \mid \varphi(x) \in I^n\}$$

The Nygaard filtration is a decreasing filtration on A. However it is not separated in general. From the definition it is clear that the Nygaard filtration is separated if and only if φ is injective, which is not always the case.

3.2 Quasi-syntomic rings

We will use the cotangent complex of a ring map, defined in [Sta18, Tag 08PN]. The fundamental distinguished triangle for the cotangent complex, [Sta18, Tag 08QS], will be used in the sequel without citing the reference everytime.

Definition 72. A ring A is called *quasi-syntomic* if it is derived p-complete with bounded p^∞-torsion, and the cotangent complex $L_{A/\mathbb{Z}_p} \in D(A)$ has p-complete Tor-amplitude in $[-1, 0]$.

A map $A \to B$ of derived p-complete rings with bounded p^∞-torsion is called *quasi-syntomic* if B is p-completely flat over A, and the relative cotangent complex $L_{B/A} \in D(B)$ has p-complete Tor amplitude in $[-1, 0]$.

Remark 73. As we require quasi-syntomic rings to have bounded p^∞-torsion, it is equivalent to require that they are p-adically complete in either the classical sense or the derived sense by Lemma 17.

Although we define quasi-syntomic rings to be derived p-complete, it is sometimes useful to consider non-complete rings.

Lemma 74. *Let $R \to S$ be a p-completely flat map of rings where R lies in $(p\mathrm{Aff}/\mathbb{Z}_p)$. Then $(L_{S^\wedge_p/R})^\wedge_p \simeq (L_{S/R})^\wedge_p$*

Proof. By assumption $S^\wedge_p$ is concentrated in degree 0 and classically p-complete. Consider the chain of ring maps $S \to S^\wedge_p \to S/p$. By the fundamental triangle of the cotangent complex we have a distinguished triangle

$$L_{S^\wedge_p/S} \otimes^{\mathbb{L}}_{S^\wedge_p} S/p \to L_{(S/p)/S} \to L_{(S/p)/S^\wedge_p} \xrightarrow{+1}$$

But $S/p \simeq S^\wedge_p \otimes^{\mathbb{L}}_S S/p$, so [Sta18, Tag 08QZ] implies that $L_{(S/p)/S} \to L_{(S/p)/S^\wedge_p}$ is a quasi-isomorphism. Then $L_{S^\wedge_p/S} \otimes^{\mathbb{L}}_{S^\wedge_p} S/p \simeq 0$, and derived Nakayama implies that $(L_{S^\wedge_p/S})^\wedge_p \simeq 0$.

Now consider the chain of ring maps $R \to S \to S^\wedge_p$. We have a distinguished triangle

$$(L_{S/R} \otimes^{\mathbb{L}}_S S^\wedge_p)^\wedge_p \to (L_{S^\wedge_p/R})^\wedge_p \to (L_{S^\wedge_p/S})^\wedge_p \xrightarrow{+1}$$

Thus we have a quasi-isomorphism

$$(L_{S/R} \otimes^{\mathbb{L}}_S S^\wedge_p)^\wedge_p \to (L_{S^\wedge_p/R})^\wedge_p$$

But applying derived Nakayama to the natural

$$(L_{S/R} \otimes^{\mathbb{L}}_S S)^\wedge_p \to (L_{S/R} \otimes^{\mathbb{L}}_S S^\wedge_p)^\wedge_p$$

shows that $(L_{S/R} \otimes^{\mathbb{L}}_S S^\wedge_p)^\wedge_p \simeq (L_{S/R})^\wedge_p$. Thus we conclude that $(L_{S/R})^\wedge_p \simeq (L_{S^\wedge_p/R})^\wedge_p$. □

Definition 75. A quasi-syntomic map $A \to B$ is called a *quasi-syntomic cover* if B is p-completely faithfully flat over A. For a ring R, the quasi-syntomic topology on $(p\mathrm{Aff}/R)$ is defined to be the Grothendieck topology with covers being $\{T_i \to T\}$ where each ring map $T \to T_i$ is quasi-syntomic, and $\{T_i/p \to T/p\}$ is an fpqc cover.

Let R be a quasi-syntomic ring. The big quasi-syntomic site over R, denoted by $(R)_{\mathrm{QSYN}}$, is defined to be the opposite of the category of R-algebras with bounded p^∞-torsion, equipped with the quasi-syntomic topology. And the small quasi-syntomic site, $(R)_{\mathrm{qsyn}}$, is defined to be the category opposite of that of all quasi-syntomic map $R \to R'$, with the quasi-syntomic topology.

Lemma 76. *If R is quasi-syntomic, $R \to R'$ is quasi-syntomic, then R' is also quasi-syntomic.*

Proof. The fundamental triangle for $\mathbb{Z}_p \to R \to R'$ is

$$L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R' \to L_{R'/\mathbb{Z}_p} \to L_{R'/R} \xrightarrow{+1}$$

We calculate that

$$\begin{aligned} L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R' \otimes^{\mathbb{L}}_{R'} R'/p &\simeq L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R/p \otimes^{\mathbb{L}}_{R/p} R'/p \\ &\simeq (L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R/p) \otimes_{R/p} R'/p \end{aligned}$$

And the last term has Tor amplitude in $[-1, 0]$ in $D(R'/p)$. So $L_{R/\mathbb{Z}_p} \otimes^{\mathbb{L}}_R R'$ has p-complete Tor amplitude in $[-1, 0]$. But also $L_{R'/R}$ has p-complete Tor amplitude in $[-1, 0]$. We conclude that $L_{R'/\mathbb{Z}_p}$ has p-complete Tor amplitude in $[-1, 0]$. □

In the quasi-syntomic site, Bhatt, Morrow and Scholze singled out a particularly well behaved class of rings. The following is the Definition 4.20 of [BMS19].

Definition 77. Let R be a quasi-syntomic ring. R is called *quasiregular semiperfectoid* if there is a perfectoid ring S mapping to R, and the ring R/p is semiperfect. Moreover, R is called *quasiregular semiperfect* if it is quasiregular semiperfectoid and is an $\mathbb{F}_p$-algebra.

Remark 78. Equivalently, we may define R to be quasiregular semiperfectoid if it is quasi-syntomic and there is a perfectoid ring S mapping surjectively to R. For the equivalence of the two, see Remark 4.22 in [BMS19].

A convenient feature of a quasiregular semiperfectoid ring is the existence of an initial prism $(\Delta_R, (d))$.

Proposition 79. *Let R be quasiregular semiperfectoid. The category of prisms (A, I) equipped with a map $R \to A/I$ has an initial object $(\Delta_R, (d))$.*

Proof. [BS22], Proposition 7.2. □

Remark 80. As shown in Proposition 7.10 of [BS22], The prism $(\Delta_R, (d))$ is bounded, and $R \to \overline{\Delta}_R \simeq \Delta_R/d$ is p-completely faithfully flat.

We need several results about the Nygaard filtration in the initial prism of a quasiregular semiperfectoid ring.

Proposition 81. *If R is quasiregular semiperfectoid, then $\Delta_R/\mathcal{N}^{\geq 1}\Delta_R \simeq R$, where the isomorphism is induced by φ.*

Proof. [BS22], Theorem 12.2. □

Lemma 82. *Suppose R is quasiregular semiperfectoid and $(\Delta_R, (d))$ the initial prism. For every $n \geq 1$, the image of $\mathcal{N}^{\geq 1}\Delta_R$ in Δ_R/p^n coincides with the ideal $\{x \in \Delta_R/p^n \mid \varphi(x) \in d\Delta_R/p^n\}$.*

Proof. The Nygaard filtration $\mathcal{N}^{\geq 1}\Delta_R$ is the kernel of the composition

$$\Delta_R \xrightarrow{\varphi} \Delta_R \to \Delta_R/d$$

and the image is identified with R. Since the map $R \to \Delta_R/d$ is p-completely faithfully flat by Remark 80, Lemma 22 then implies that $R/p^n \to \Delta_R/(d, p^n)$ is faithfully flat, and thus injective. Then the image of

$$\Delta_R/p^n \xrightarrow{\varphi} \Delta_R/p^n \to \Delta_R/(p^n, d)$$

is isomorphic to R/p^n.

It is clear that the image of $\mathcal{N}^{\geq 1}\Delta_R$ in Δ_R/p^n lies in the ideal

$$\{x \in \Delta_R/p^n \mid \varphi(x) \in d\Delta_R/p^n\}$$

Moreover, the quotient of Δ_R/p^n by the image of $\mathcal{N}^{\geq 1}\Delta_R$ is also R/p^n. Thus the two ideals coincide. □

3.3 Ind-étale base changes of quasi-syntomic rings

Definition 83. A ring homomorphism $R \to S$ is called *ind-étale* if S is a filtered colimit of étale R-algebras.

Let R be a ring and $I \subset R$ a finitely generated ideal. A ring map $R \to S$ is called *I-ind-étale* (short for I-completely ind-étale) if $S \otimes_R^{\mathbb{L}} R/I$ is concentrated in degree 0 and there given by an ind-étale R/I-algebra.

Lemma 84. *If $R \to S$ is ind-étale, where R is p-complete with bounded p^∞-torsion, then $R \to S_p^\wedge$ is p-ind-étale.*

Proof. $S_p^\wedge \otimes_R^{\mathbb{L}} R/p \simeq S \otimes_R^{\mathbb{L}} R/p \simeq S/p$ is concentrated in degree 0 and ind-étale over R/p. □

Lemma 85. *If $R \to S_i$ are p-ind-étale for i objects of some filtered category, then $R \to (\operatorname{colim}_i S_i)_p^\wedge$ is p-ind-étale.*

Proof. We calculate that

$$\begin{aligned}(\operatorname{colim}_i S_i)_p^\wedge \otimes_R^{\mathbb{L}} R/p &\simeq (\operatorname{colim}_i S_i) \otimes_R^{\mathbb{L}} R/p \\ &\simeq \operatorname{colim}_i(S_i \otimes_R^{\mathbb{L}} R/p)\end{aligned}$$

Each $S_i \otimes_R^{\mathbb{L}} R/p$ is concentrated in degree 0 and given by an ind-étale R/p-algebra. Thus $(\operatorname{colim}_i S_i)_p^\wedge \otimes_R^{\mathbb{L}} R/p$ is concentrated in degree 0 and given by an ind-étale R/p-algebra. □

Proposition 86 (Algebraization)**.** *Let R be a p-complete ring with bounded p^∞-torsion. If $R \to S$ is a p-ind-étale map where S is also derived p-complete, then S is the derived p-completion of an ind-étale R-algebra.*

Proof. Lemma 17 implies that S is classically p-complete. Since $R/p \to S/p$ is ind-étale, [Sta18, Tag 097P] gives an ind-étale map $R \to S'$ such that $S'/p \simeq S/p$. Let $S' \simeq \operatorname{colim}_{i\in I} S_i$ where each S_i is étale over R. The maps $S_i \to S'/p \to S/p$ lift uniquely to some $S_i \to S/p^n$ for each n as S_i is étale. Then these maps assemble into a map $S_i \to \lim_n S/p^n \simeq S$. Uniqueness of the liftings ensures that these maps are compatible with transition maps between S_i's, so they give a map $S' \to S$. Then $(S')^\wedge_p \to S$ is a map between derived p-complete, p-completely flat R-algebras that reduces to identity modulo p, and derived Nakayama (Lemma 9) shows that it is an isomorphism. $\square$

Corollary 87. *If $R \to R'$ is a p-ind-étale map with R having bounded p^∞-torsion, then $(L_{R'/R})^\wedge_p \simeq 0$. In particular, $R \to R'$ is quasi-syntomic.*

Proof. Using [Sta18, Tag 08S9], i.e. taking the cotangent complex commutes with filtered limits, the cotangent complex of an ind-étale ring map is 0. Then we conclude by algebrazaition and Lemma 74. $\square$

Proposition 88. *If R is quasiregular semiperfectoid and $R \to S$ is p-complete and p-ind-étale, then S is also quasiregular semiperfectoid.*

Proof. By Lemma 76 and Corollary 87, S is also quasi-syntomic. If R' is a perfectoid ring with a map $R' \to R$, then the compostion $R' \to R \to S$ is a map from a perfectoid ring to S. Now we check that S/p is semiperfect. Write $S/p \simeq \operatorname{colim}_{i\in I} L_i$, where I is a filtered category and each $R/p \to L_i$ is étale. By [Sta18, Tag 0EBS], the relative Frobenius $L_i \otimes_{R/p,\varphi} R/p \to L_i$ is an isomorphism. But R/p is semiperfect, so the map $L_i \to L_i \otimes_{R/p,\varphi} R/p$ is surjective. So each L_i is semiperfect. The colimit $\operatorname{colim}_{i\in I} L_i$ commutes with the forgetful to $\mathbb{F}_p$-modules, and filtered colimits in $\mathbb{F}_p$-modules are exact, so $S/p \simeq \operatorname{colim}_{i\in I} L_i$ is semiperfect. $\square$

Proposition 89. *If R is quasiregular semiperfectoid and $R \to S$ is p-complete and p-ind-étale, then there is a unique map of prisms $(\mathbbl{\Delta}_R, (d)) \to (\mathbbl{\Delta}_S, (d))$ inducing the dashed arrow in the following commutative diagram*

$$\begin{array}{ccc} R & \longrightarrow & \overline{\mathbbl{\Delta}}_R \\ \downarrow & & \downarrow \\ S & \longrightarrow & \overline{\mathbbl{\Delta}}_S \end{array}$$

The map $\overline{\mathbbl{\Delta}}_R \to \overline{\mathbbl{\Delta}}_S$ is p-ind-étale, is p-completely faithfully flat if $R \to S$ is p-completely faithfully flat. The map of prisms $\mathbbl{\Delta}_R \to \mathbbl{\Delta}_S$ is (p,d)-ind-étale, and is (p,d)-completely faithfully flat if $R \to S$ is p-completely faithfully flat.

Proof. By the Proposition 88, S is also quasiregular semiperfectoid. The existence and uniqueness of the map $\mathbbl{\Delta}_R \to \mathbbl{\Delta}_S$ follow from the universal property of $\mathbbl{\Delta}_R$.

The Proposition 7.10 of [BS22] allows us to compute $\mathbb{\Delta}_R$ and $\mathbb{\Delta}_S$ as the prismatic cohomology $\mathbb{\Delta}_{R/A}$ and $\mathbb{\Delta}_{S/A}$, where (A, d) is a perfect prism such that A/d maps to R. The formally étale localization of the derived prismatic cohomology implies that $\overline{\mathbb{\Delta}}_{S/A} \simeq (\overline{\mathbb{\Delta}}_{R/A} \otimes^{\mathbb{L}}_R S)^{\wedge}_p$. Using that $R \to S$ is p-completely flat, we calculated that

$$\begin{aligned}\overline{\mathbb{\Delta}}_{S/A} \otimes^{\mathbb{L}}_{\overline{\mathbb{\Delta}}_{R/A}} \overline{\mathbb{\Delta}}_{R/A}/p &\simeq (S \otimes^{\mathbb{L}}_R \overline{\mathbb{\Delta}}_{R/A})^{\wedge}_p \otimes^{\mathbb{L}}_{\overline{\mathbb{\Delta}}_{R/A}} \overline{\mathbb{\Delta}}_{R/A}/p \\ &\simeq S \otimes^{\mathbb{L}}_R \overline{\mathbb{\Delta}}_{R/A}/p \\ &\simeq (S \otimes^{\mathbb{L}}_R R/p) \otimes^{\mathbb{L}}_{R/p} \overline{\mathbb{\Delta}}_{R/A}/p \\ &\simeq (S/p \otimes_{R/p} \overline{\mathbb{\Delta}}_{R/A}/p)[0]\end{aligned}$$

Then $\overline{\mathbb{\Delta}}_{R/A} \to \overline{\mathbb{\Delta}}_{S/A}$ is p-ind-étale, and is p-completely faithfully flat if $R \to S$ is p-completely faithfully flat.

For the last part, we calculate that

$$\begin{aligned}\mathbb{\Delta}_S \otimes^{\mathbb{L}}_{\mathbb{\Delta}_R} \mathbb{\Delta}_R/(p, d) &\simeq \mathbb{\Delta}_S \otimes^{\mathbb{L}}_{\mathbb{\Delta}_R} \overline{\mathbb{\Delta}}_R \otimes^{\mathbb{L}}_{\overline{\mathbb{\Delta}}_R} \overline{\mathbb{\Delta}}_R/p \\ &\simeq \overline{\mathbb{\Delta}}_S \otimes^{\mathbb{L}}_{\overline{\mathbb{\Delta}}_R} \overline{\mathbb{\Delta}}_R/p \\ &\simeq (S/p \otimes_{R/p} \overline{\mathbb{\Delta}}_{R/A}/p)[0]\end{aligned}$$

The second quasi-isomorphism uses the fact that both $\mathbb{\Delta}_R$ and $\mathbb{\Delta}_S$ are d-torsion free. We conclude that $\mathbb{\Delta}_R \to \mathbb{\Delta}_S$ is (p, d)-ind-étale. If $R \to S$ is p-completely faithfully flat, then $\overline{\mathbb{\Delta}}_R \to \overline{\mathbb{\Delta}}_S$ is p-completely faithfully flat, so $\mathbb{\Delta}_R \to \mathbb{\Delta}_S$ is (p, d)-completely faithfully flat. □

3.4 Prismatic structural sheaf

Let R be a quasi-syntomic ring.

We first introduce, as in [AL23], Section 4.1, several functors between the topoi

$$\mathrm{Sh}((R)_{\mathbb{\Delta}}), \mathrm{Sh}((R)_{\mathrm{QSYN}}), \mathrm{Sh}((R)_{\mathrm{qsyn}})$$

Let u be the functor $(R)_{\mathbb{\Delta}} \to (R)_{\mathrm{QSYN}}$, defined by $(B, I) \mapsto B/I$. u is cocontinuous by [AL23], Proposition 3.3.8. Then u induces a morphism of topoi

$$\begin{aligned}u_* &: \mathrm{Sh}((R)_{\mathbb{\Delta}}) \to \mathrm{Sh}((R)_{\mathrm{QSYN}}) \\ u^{-1} &: \mathrm{Sh}((R)_{\mathrm{QSYN}}) \to \mathrm{Sh}((R)_{\mathbb{\Delta}})\end{aligned}$$

Let ρ be the inclusion functor $(R)_{\mathrm{qsyn}} \to (R)_{\mathrm{QSYN}}$. ρ is continuous cocontinuous, so it induces the following three functors:

$$\rho^{-1} : \mathrm{Sh}((R)_{\mathrm{QSYN}}) \to \mathrm{Sh}((R)_{\mathrm{qsyn}}), \rho_*, \rho_! : \mathrm{Sh}((R)_{\mathrm{qsyn}}) \to \mathrm{Sh}((R)_{\mathrm{QSYN}})$$

where ρ^{-1}, ρ_* define a morphism of topoi and $\rho_!$ is the left adjoint to ρ^{-1}. For more details on the construction of the functors, see [Sta18, Tag 00XR]. In

[AL23], ρ^{-1} is denoted by ϵ_* and $\rho_!$ by $\epsilon^\natural$. However $\rho_!$ doesn't commute with finite limits, so $\rho_!(=\epsilon^\natural), \rho^{-1} = (\epsilon_*)$ do not define a morphism of topoi in the other direction.

For consistency with [AL23], we denote $v_* = \rho^{-1}u_*$ despite v_* is not the push forward functor in a morphism of topoi.

Then we define a few sheaves on the prismatic site and the small quasi-syntomic site. Let $\mathcal{O}_{\mathbb{\Delta}}$ be the prismatic structural sheaf, $\mathcal{I}_{\mathbb{\Delta}}$ the sheaf $(B, I) \mapsto I$ on the prismatic site, and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ the quotient $\mathcal{O}_{\mathbb{\Delta}}/\mathcal{I}_{\mathbb{\Delta}}$. Let $\mathcal{N}^{\geq 1}\mathcal{O}_{\mathbb{\Delta}}$ be the kernel of the composition

$$\mathcal{O}_{\mathbb{\Delta}} \xrightarrow{\varphi} \mathcal{O}_{\mathbb{\Delta}} \to \overline{\mathcal{O}_{\mathbb{\Delta}}}$$

By [BS22], Corollary 3.12, the section of $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ on (B, I) is in fact just B/I. Thus the section of $\mathcal{N}^{\geq 1}\mathcal{O}_{\mathbb{\Delta}}$ on (B, I) is the Nygaard filtration $\mathcal{N}^{\geq 1}B$.

Furthermore, define the following sheaves on $(R)_{\text{qsyn}}$:

$$\mathcal{O}^{\text{pris}} = v_*\mathcal{O}_{\mathbb{\Delta}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\text{pris}} = v_*\mathcal{O}_{\mathbb{\Delta}}, \mathcal{I}^{\text{pris}} = v_*\mathcal{I}_{\mathbb{\Delta}}$$

Let $\mathcal{O}$ denote the structural sheaf on $(R)_{\text{qsyn}}$.

Suppose that $n \geq 1$. We denote by $\mathcal{O}_n^{\text{pris}}$ the sheaf $\mathcal{O}^{\text{pris}} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ on the site $(R)_{\text{qsyn}}$, and by $\mathcal{N}^{\geq 1}\mathcal{O}_n^{\text{pris}}$ the image of $\mathcal{N}^{\geq 1}\mathcal{O}^{\text{pris}}$ in $\mathcal{O}_n^{\text{pris}}$. If $n = \infty$, then $\mathcal{O}_n^{\text{pris}}$ is defined to be $\mathcal{O}^{\text{pris}}$. In addition, we denote by $\mathcal{O}_n$ the sheaf $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ where $\mathcal{O}$ is the quasi-syntomic structural sheaf. Again $\mathcal{O}_\infty$ is simply defined to be $\mathcal{O}$. By applying Lemma 82 to all quasiregular semiperfectoid rings in $(R)_{\text{qsyn}}$, we see that $\mathcal{N}^{\geq 1}\mathcal{O}_n^{\text{pris}}$ is also the kernel of the composition

$$\mathcal{O}_n^{\text{pris}} \xrightarrow{\varphi} \mathcal{O}_n^{\text{pris}} \to \mathcal{O}_n^{\text{pris}}/\mathcal{I}^{\text{pris}}$$

We write formally R/p^∞ to mean R, and $\mathbb{\Delta}_R/p^\infty$ to mean $\mathbb{\Delta}_R$.

Proposition 90. *Suppose R a p-torsion free quasi-syntomic ring and $n \geq 1$ or $n = \infty$. Then*

$$R\Gamma((R)_{\text{qsyn}}, \mathcal{O}_n) \simeq R/p^n[0]$$

If R is further supposed to be quasiregular semiperfectoid, then

$$R\Gamma((R)_{\text{qsyn}}, \mathcal{O}_n^{\text{pris}}) \simeq \mathbb{\Delta}_R/p^n[0]$$
$$R\Gamma((R)_{\text{qsyn}}, \mathcal{N}^{\geq 1}\mathcal{O}_n^{\text{pris}}) \simeq \ker(\mathbb{\Delta}_R/p^n \to R/p^n)[0]$$

Proof. Lemma 21 shows that all covers in $(R)_{\text{qsyn}}$ can be refined into a cover of the form $S' \to S$. In view of [Sta18, Tag 03F9], that $R\Gamma((R)_{\text{qsyn}}, \mathcal{O}) = R$ follows from Lemma 24 Since all the objects S in $(R)_{\text{qsyn}}$ are p-completely flat over S, Lemma 22 implies that the sheaf $\mathcal{O}$ is p-torsion free, so we have an exact sequence

$$0 \to \mathcal{O} \xrightarrow{p^n} \mathcal{O} \to \mathcal{O}_n \to 0$$

This implies that $R\Gamma((R)_{\text{qsyn}}, \mathcal{O}_n) \simeq R/p^n[0]$.

From now on we suppose that R is quasiregular semiperfectoid. The site $(R)_{\Delta}$ has an initial object Δ_R by Proposition 79. Then Lemma 68 implies that $R\Gamma((R)_{\Delta}, \mathcal{O}_{\Delta}) \simeq \Delta_R[0]$. Using the Leray spectral sequence associated to the morphism of topoi

$$u_*\mathrm{Sh}((R)_{\Delta}) \to \mathrm{Sh}((R)_{\mathrm{QSYN}})$$

we can compute that $R\Gamma((R)_{\mathrm{QSYN}}, u_*\mathcal{O}_{\Delta}) = \Delta_R$. Let $\mathcal{C}$ be the covers in $(R)_{\mathrm{qsyn}}$ of the form $R' \to R''$ where R'' is quasiregular semiperfectoid. By [BMS19], Lemma 4.30, all covers in $(R)_{\mathrm{qsyn}}$ can be refined by the covers in $\mathcal{C}$, and all terms of the Čech nerve of a such cover are quasiregular semiperfectoid. The higher Čech cohomology all vanish for covers in $\mathcal{C}$ on the site $(R)_{\mathrm{QSYN}}$ by $R\Gamma((R)_{\mathrm{QSYN}}, u_*\mathcal{O}_{\Delta}) = \Delta_R$. But for a cover in $(R)_{\mathrm{qsyn}}$, the Čech cohomology of $\rho^{-1}u_*\mathcal{O}_{\Delta}$ can also be calculated using the site $(R)_{\mathrm{QSYN}}$. So the higher Čech coholomogy for $\mathcal{O}^{\mathrm{pris}}$ on covers in $\mathcal{C}$ all vanish. Then [Sta18, Tag 03F9] implies that all higher cohomology of $\mathcal{O}^{\mathrm{pris}}$ on R vanish.

As R is p-torsion free, each quasiregular semiperfectoid ring S in $(R)_{\mathrm{qsyn}}$ is p-torsion free. By [BS22], Proposition 7.10, we know that $\overline{\Delta_S}$ is p-completely flat over S, and thus p-torsion free by Lemma 22. Thus Δ_S is a transversal prism, and is p-torsion free by Lemma 60. As all objects in $(R)_{\mathrm{qsyn}}$ are covered by some quasiregular semiperfectoid ring, the sheaf $\mathcal{O}^{\mathrm{pris}}$ is p-torsion free, and we have an exact sequence

$$0 \to \mathcal{O}^{\mathrm{pris}} \xrightarrow{p^n} \mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}}_n \to 0$$

Applying $R\Gamma((R)_{\mathrm{qsyn}}, -)$ to the exact sequence gives $R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{O}^{\mathrm{pris}}_n) \simeq \Delta_R/p^n[0]$

Finally, apply $R\Gamma((R)_{\mathrm{qsyn}}, -)$ to the exact sequence

$$0 \to \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}_n \to \mathcal{O}^{\mathrm{pris}}_n \to \mathcal{O}_n \to 0$$

and use the fact that the induced $\Delta_R/p^n \to R/p^n$ is surjective, we can conclude that $R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}_n) \simeq \ker(\Delta_R/p^n \to R/p^n)[0]$. □

We collect a few results about finite locally free modules.

Lemma 91. *Let A be a ring with tame $\{f_1, \ldots, f_n\}^\infty$-torsion. Write $I = (f_1, \ldots, f_n)$. Let $\mathcal{C}$ be a site and $\mathcal{O}$ a sheaf of A-algebras on $\mathcal{C}$ (here the notation $\mathcal{O}$ is in conflict with the structural sheaf on $(R)_{\mathrm{qsyn}}$ but it is too standard to change) such that*

(a) Any cover $\{U_i \to U\}_{i\in J}$ in $\mathcal{C}$ can be refined into a cover $\{V_i \to U\}_{i\in J'}$ where J' is finite.

(b) Each $\mathcal{O}(U)$ is derived I-complete with tame $\{f_1, \ldots, f_n\}^\infty$-torsion.

(c) For every cover $\{U_i \to U\}_{i\in J}$ in $\mathcal{C}$, the (opposite of the) family $\{\mathcal{O}(U) \to \mathcal{O}(U_i)\}_{i\in J}$ is an I-completely fpqc cover.

(d) For a cover $\{U_i \to U\}$ and a morphism $V \to U$, we have that $\mathcal{O}(U_i \times_U V) \simeq (\mathcal{O}(U_i) \otimes^{\mathbb{L}}_{\mathcal{O}(U)} \mathcal{O}(V))^{\wedge}_I$.

If $\mathcal{M}$ is a finite locally free $\mathcal{O}$-module, then:

(1) For every object $U \in \mathcal{C}$, $\mathcal{M}(U)$ is a finite locally free $\mathcal{O}(U)$-module.

(2) For every morphism $V \to U$ in $\mathcal{C}$, we have $\mathcal{M}(V) \simeq \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V)$.

(3) For every object $U \in \mathcal{C}$, $R\Gamma(U, \mathcal{M}) \simeq \mathcal{M}(U)[0]$.

Proof. Suppose that $\{U_i \to U\}_{i\in J}$ is a cover in $\mathcal{C}$ such that

$$\mathcal{M}|_{U_i} \simeq \mathcal{O}|_{U_i}^{n_i}$$

for some integers n_i. We may suppose that J is finite.

We have free $\mathcal{O}(U_i)$-modules $\mathcal{O}(U_i)^{n_i}$, together with isomorphisms $t_{i,j} : \mathcal{O}_{i,j}^{n_i} \simeq \mathcal{M}(R_{i,j}) \simeq \mathcal{O}_{i,j}^{n_j}$, where $\mathcal{O}_{i,j} \simeq \mathcal{O}(U_i \times_U U_j)$. The isomorphisms $t_{i,j}$ satisfy the cocycle condition by construction. Thus we have a descent datum of finite locally free modules for the I-completely fpqc cover (opposite to) $\{\mathcal{O}(U) \to \mathcal{O}U_i\}_{i\in J}$, which is effective by Proposition 30. Suppose that the descent datum is isomorphic to the canonical descent datum associated to the finite locally free $\mathcal{O}(U)$-module M. Then M is classically I-complete, and M/I^nM is the kernel of

$$\prod_{i\in J}(\mathcal{O}(U_i)/I^n)^{n_i} \to \prod_{i,j\in J}(\mathcal{O}(U_i \times_U U_j)/I^n)^{n_i}$$
$$(x_i)_i \mapsto (x_i - t_{i,j}(x_j))_{i,j}$$

Note that inverse limit is left exact, so the sheaf property of $\mathcal{M}$ forces $\mathcal{M}(U)$ to be isomorphic to M.

If $V \to U$ is a morphism in $\mathcal{C}$, then $\{(V \times_U U_i\}_{i\in J}$ is a cover of V. Since $\mathcal{M}|_{U_i}$ is free over $\mathcal{O}|_{U_i}$ for every i, we have that

$$\mathcal{M}((V \times_U U_i) \simeq \mathcal{M}(U_i) \otimes_{\mathcal{O}(U_i)} \mathcal{O}(V \times_U U_i)$$

For any $n \geq 1$, $\mathcal{O}(U_i) \otimes^{\mathbb{L}}_{\mathcal{O}(U)} \mathcal{O}(U)/I^n$ is concentrated in degree 0 and flat over $\mathcal{O}(U)/I^n$ by Lemma 15 and Lemma 17. Then we have that

$$\mathcal{O}(U_i \times_U V)/I^n \simeq \mathcal{O}(U_i)/I^n \otimes_{\mathcal{O}(U)/I^n} \mathcal{O}(V)/I^n$$

Then the base change of the descent datum in the last paragraph to $\mathcal{O}(V)/I^n$ is isomorphic to the canonical descent datum associated to the finite locally free $\mathcal{O}(V)/I^n$-module $\mathcal{M}(V)/I^n\mathcal{M}(V)$. By flatness, the base change to $\mathcal{O}(V)/I^n$ commutes with taking cohomology. So we have that

$$\mathcal{M}(U)/I^n\mathcal{M}(U) \otimes_{\mathcal{O}(U)/I^n} \mathcal{O}(V)/I^n \simeq \mathcal{M}(V)/I^n\mathcal{M}(V)$$

Taking the limit over all n and using the fact that finite locally free modules over $\mathcal{O}(U), \mathcal{O}(V)$ are classically p-complete gives that

$$\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \simeq \mathcal{M}(V)$$

Now we calculate $R\Gamma(U, \mathcal{M})$. We first claim that for every object V, $R\Gamma(V, \mathcal{O})$ is concentrated in degree 0. Let $\{V_i \to V\}_{i\in K}$ be a cover. We may assume that K is finite. The condition (d) implies that the Čech complex $C^\bullet(\{V_i \to V\}_{i\in K}, \mathcal{O})$ is the stupid truncation to degrees ≥ 0 of the I-completed Amitsur complex

$$\mathrm{Ami}(\mathcal{O}(V); \mathcal{O}(V) \to \prod_{i\in K} \mathcal{O}(V_i))$$

which is concentrated in degree 0 by Proposition 24. As this is true for any cover, [Sta18, Tag 03F9] implies that $R\Gamma(V, \mathcal{O})$ is concentrated in degree 0.

As $R\Gamma(U_i, \mathcal{O}) \simeq \mathcal{O}(U_i)[0]$, we have that $R\Gamma(U_i, \mathcal{M}) \simeq (\mathcal{O}(U_i))^{n_i}[0]$. Using the Čech-to-derived spectral sequence we see that the cohomologies of $R\Gamma(U, \mathcal{M})$ can be calculated as those of the Čech complex $C^\bullet(\{U_i \to U\}_{i\in J}, \mathcal{M})$. But $\mathcal{M}(V) \simeq M \otimes_{\mathcal{O}(U)} \mathcal{O}(V)$ where $M \simeq \mathcal{M}(U)$, which implies that the Čech complex $C^\bullet(\{U_i \to U\}_{i\in J}, \mathcal{M})$ is the stupid truncation to degrees ≥ 0 of the I-completed Amitsur complex $\mathrm{Ami}(M, \mathcal{O}(U) \to \prod_{i\in J} \mathcal{O}(U_i))$. We conculde by Proposition 24 that $R\Gamma(U, \mathcal{M}) \simeq M[0]$. □

Lemma 92. *Suppose that R is a p-torsion free quasi-syntomic ring and $n \geq 1$ or $n = \infty$. If $\mathcal{M}$ is a finite locally free $\mathcal{O}_n$-module (on $(R)_{\mathrm{qsyn}}$), then $\mathcal{M}(R)$ is a finite localy free R/p^n-module, and for every $R \to R'$ in $(R)_{\mathrm{qsyn}}$ we have that $\mathcal{M}(R') \simeq \mathcal{M}(R)\otimes_R R'$. The cohomology can be calculated as $R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{M}) \simeq \mathcal{M}(R)[0]$.*

Proof. Here we take $\mathcal{C}$ to be $(R)_{\mathrm{qsyn}}$ and $\mathcal{O}$ to be $\mathcal{O}_n$ in Lemma 91. Lemma 90 implies that $\mathcal{O}_n(R') \simeq R'/p^n$ for every R'. Then the conditions (a) - (d) are trivial to verify. □

Lemma 93. *Suppose that R is p-torsion free quasi-syntomic. If $\mathcal{M} \to \mathcal{N}$ is a surjection of finite locally free $\mathcal{O}_n$-modules, then $\mathcal{M}(R) \to \mathcal{N}(R)$ is a surjection of finite locally free R/p^n-modules.*

Proof. Let $f : \mathcal{M} \to \mathcal{N}$ be a surjection of finite locally free $\mathcal{O}_n$-modules. Passing to a cover that trivializes both $\mathcal{M}$ and $\mathcal{N}$, we may check that $\ker f$ is also finite locally free over $\mathcal{O}_n$. Taking global section of the exact sequence

$$0 \to \ker f \to \mathcal{M} \to \mathcal{N} \to 0$$

and using that $R\Gamma((R)_{\mathrm{qsyn}}, \ker f)$ is concentrated in degree 0, we obtain that the map $\mathcal{M}(R) \to \mathcal{N}(R)$ is also surjective. But $\mathcal{M}(R), \mathcal{N}(R)$ are finite locally free R/p^n-modules, so the surjection splits. Note that $\mathcal{M}(R') \simeq \mathcal{M}(R) \otimes_R R'$ and $\mathcal{N}(R') \simeq \mathcal{M}(R) \otimes_R R'$, so the splitting $\mathcal{N}(R) \to \mathcal{M}(R)$ induces a splitting $\mathcal{N} \to \mathcal{M}$. □

Lemma 94. *Suppose that R is a quasiregular semiperfectoid ring. If $\mathcal{M}$ is a finite locally free $\mathcal{O}_{\Delta}$-module (on $(R)_{\Delta}$), then $\mathcal{M}(\Delta_R)$ is a finite locally free Δ_R-module. For every $(A, (d)) \in (R)_{\Delta}$ we have that $\mathcal{M}((A, (d))) \simeq \mathcal{M}(\Delta_R) \otimes_{\Delta_R} A$. Moreover,*

$$R\Gamma((R)_{\Delta}, \mathcal{M}) \simeq \mathcal{M}(\Delta_R)[0]$$

Proof. Here we take $\mathcal{C}$ to be $(R)_{\Delta}$ and $\mathcal{O}$ to be $\mathcal{O}_{\Delta}$ in Lemma 91. The conditions (a) - (d) are trivial to verify. □

Lemma 95. *Let R be quasiregular semiperfectoid. There are equivalences of categories between the followings that preserves direct sums:*

(1) finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules.

(2) finite locally free $\mathcal{O}_{\mathbb{\Delta}}$-modules.

(3) finite locally free $\mathbb{\Delta}_R$-modules.

In particular, finite locally free $\mathcal{O}^{\mathrm{pris}}$-modules are direct summands of finite free $\mathcal{O}^{\mathrm{pris}}$-modules.

Moreover, the equivalences are exact.

Proof. The equivalence of (1) and (2) is [AL23], Proposition 4.1.4. The equivalence between (2) and (3) is constructed by the global section functor. It is indeed an equivalence by Lemma 94.

The vanishing of cohomology in Lemma 94 also implies that the global section functor from (2) to (3) is exact. The global section functor from (1) to (3) is exact by Lemma 90. While if we have an exact sequence of $\mathbb{\Delta}_R$-modules, then the exact sequence splits. The split exact sequence will give split exact sequences in the categories (1) and (2). So the functors from (3) to (1) and (2) are also exact. □

Finally we show the repleteness of the topoi in this subsection. We follow [BS13] for the definition.

Definition 96. A topos X is called *replete* if the surjections in X are closed under sequential limits.

Lemma 97. *Suppose that C is a site where all covers $\{U_i \to U\}$ can be refines by a cover of the form $V \to U$, and limits of countable towers of covers of the form $V \to U$ exist and are still covers, i.e. if we have a diagram $\cdots \to U_2 \to U_1 \to U$ such that $U_1 \to U$ and $U_{i+1} \to U_i$ are covers for all i, then the limit of the diagram $\cdots \to U_2 \to U_1$ exists and is still a cover of U. Then the topos $\mathrm{Sh}(C)$ is replete.*

Proof. Suppose $\{F_i\}$ is an inverse system in $\mathrm{Sh}(C)$ indexed by $\mathbb{N}$ such that $F_{i+1} \to F_i$ is surjective for every i. Let n be some index and $a_0 \in F_i(S_0)$, where S_0 is some object of C. We obtain by induction a tower of covers $\cdots \to S_2 \to S_1 \to S_0$ and $a_k \in F_{n+k}(S_k)$ such that the map of sheaves $F_{n+k} \to F_{n+k-1}$ takes a_k to the restriction of a_{k-1} to S_k. By assumption the limit S_∞ of $\cdots S_2 \to S_1 \to S_0$ exists and is a cover of S_0. Then we have elements $b_i \in F_{n+i}(S_\infty)$ whose image under $F_{n+i} \to F_{n+i-1}$ is b_{i-1}, and b_0 is the restriction of a_0 to S_∞. So we have constructed a cover $S_\infty \to S_0$ and an element $(b_i) \in \lim F_i(S_\infty)$ whose projection in $F_n(S_\infty)$ is the restriction of a_0. This shows that $\lim F_i \to F_n$ is surjective. □

Lemma 98. *The topoi $\mathrm{Sh}((R)_{\mathrm{qsyn}})$, $\mathrm{Sh}((R)_{\mathrm{QSYN}})$ and $\mathrm{Sh}((R)_{\mathbb{\Delta}})$ are replete.*

Proof. We need to verify the conditions of Lemma 97 for the sites $(R)_{\mathrm{qsyn}}, (R)_{\mathrm{QSYN}}$ and $(R)_{\mathbb{\Delta}}$. The part about refining covers onto the form $V \to U$ follows from Lemma 21 for $(R)_{\mathrm{QSYN}}$ and $(R)_{\mathrm{qsyn}}$, since a finite direct product of rings with bounded p^∞-torsion also has bouned p^∞-torsion, and taking the cotangent complex commutes with finite direct products. It is also true for $(R)_{\Delta}$ by definition of the topology on $(R)_{\Delta}$.

Now we check the condition concerning limits of covers. Let $\cdots \to S_2 \to S_1 \to S$ be a tower of covers in $(R)_{\mathrm{QSYN}}$, and S_∞ be the complex $(\mathrm{colim}_i\, S_i)^\wedge_p$. Then $S_\infty \otimes^{\mathbb{L}}_S S/p \simeq (\mathrm{colim}_i\, S_i \otimes^{\mathbb{L}}_S S/p)[0]$, so S_∞ is p-completely faithfully flat over S, and is thus concentrated in degree 0 and classically p-complete by Lemma 22. Moreover, $S_\infty[p^n] \simeq S_\infty \otimes_S S[p^n]$ by the same Lemma. Thus S_∞ is an object of $(R)_{\mathrm{QSYN}}$. By the universal property of the derived completion, S_∞ is the limit of $\cdots \to S_2 \to S_1$ in the category $(R)_{\mathrm{QSYN}}$.

To check the condition for $(R)_{\mathrm{qsyn}}$, we assume further that the ring maps $S_i \to S_{i+1}$ and $S \to S_1$ are quasi-syntomic and p-completely faithfully flat. Then $S_i \to S$ is quasi-syntomic for all i. Using Lemma 74 and the fact that the cotangent complex commutes with filtered colimits, we calculate that

$$\begin{aligned}(L_{S_\infty/S})^\wedge_p &\simeq (L_{\mathrm{colim}_i\, S_i/S})^\wedge_p \\ &\simeq (\mathrm{colim}_i\, L_{S_i/S})^\wedge_p\end{aligned}$$

has p-complete Tor-amplitude in $[-1, 0]$. So S_∞ is also an object of $(R)_{\mathrm{qsyn}}$. The rest follows from the arguments in the last paragraph.

Finally, we check the condition for $(R)_{\Delta}$. Suppose that $\cdots \to (B_2, IB_2) \to (B_1, IB_1) \to (B, I)$ is a tower of covers in $(R)_{\Delta}$. We will only prove the claim when $I = (d)$ is principal. The general case is [BS23], Remark 2.4. Let B_∞ be the complex $(\mathrm{colim}_i\, B_i)^\wedge_{(p,d)}$. Again B_∞ is (p, d)-completely faithfully flat over B, so Lemma 63 implies that B_∞ is concentrated in degree 0. The ring $\mathrm{colim}_i\, B_i$ has a natural structure of a δ-ring by [BS22], Remark 2.7. Also, $B_\infty \otimes^{\mathbb{L}}_B B/(p, d)^n$ is concentrated in degree 0 by Lemma 15, and thus coincides with $B_\infty/(p, d)^n B_\infty$. While $B_\infty \otimes^{\mathbb{L}}_B B/(p, d)^n \simeq (\mathrm{colim}_i\, B_i) \otimes^{\mathbb{L}}_B B/(p, d)^n B$, the latter also coincides with $\mathrm{colim}_i\, B_i \otimes_B B/(p, d)^n B$ since it is concentrated in degree 0. Thus B_∞ is the classical (p, d)-completion of $\mathrm{colim}_i\, B_i$, which inherits a structure of a δ-ring by [BS22], Lemma 2.17. Note that B is d-torsion free, so $B[d] \simeq H^{-1}(B_\infty \otimes^{\mathbb{L}}_B B/dB)$. But $B_\infty \otimes^{\mathbb{L}}_B B/dB$ is p-completely flat over B/dB, which has bounded p^∞-torsion. So Lemma 22 implies that $B_\infty \otimes^{\mathbb{L}}_B B/dB$ is concentrated in dergee 0. Therefore B_∞ is d-torsion free. In view of Lemma 58, $(B_\infty, (d))$ is a prism. Moreover, $(B_\infty/dB_\infty)[p^n] = (B_\infty/dB_\infty) \otimes_{B/dB} B/dB[p^n]$ by the same Lemma, so $(B_\infty, (d))$ is indeed an object of $(R)_{\Delta}$. The universal property of the derived completion implies that B_∞ is the limit of the tower $\cdots (B_2, (d)) \to (B_1, (d)) \to (B, (d))$. □

Lemma 99. *Suppose that X is a replete topos and*

$$0 \to \{F_i\} \to \{G_i\} \to \{H_i\} \to 0$$

is an exact sequence of inverse systems in X indexed by $\mathbb{N}$ and the maps $F_{i+1} \to F_i$ are surjective for all i. Then the map $\lim G_i \to \lim H_i$ is surjective.

Proof. This follows directly from [BS13], Lemma 3.1.8. □

4 Truncated Barsotti-Tate groups

In this section we recall the definition of and several useful results about truncated Barsotti-Tate groups. For the moment let R be any ring.

4.1 Finite locally free group schemes as quasi-syntomic sheaves

In this subsection we show that finite locally free group schemes can be seen as sheaves on $(R)_{\mathrm{QSYN}}$.

The notion of a syntomic morphism is useful.

Definition 100. An algebra A over a field k is called a local complete intersection if there are elements $f_1, \ldots, f_n \in A$ generating the unit ideal such that each A_{f_i} is of the form $k[x_1, \ldots, x_{m_i}]/(g_1^{(i)}, \ldots, g_{n_i}^{(i)})$ and $\dim(A_{f_i}) = m_i - n_i$.

A ring map $R \to S$ is called syntomic if it is flat, of finite presentation and for each $s \in \operatorname{Spec} R$ the base change to $k(s)$ is a local complete intersection map $k(s) \to S_s$.

Lemma 101. *If $R \to S$ is a syntomic ring map, then $L_{S/R}$ has* Tor*-amplitude in* $[-1, 0]$.

Proof. Combine [Sta18, Tag 07D3] and [Sta18, Tag 08SL]. □

Lemma 102. *If G is a finite locally free group scheme over R, then G is syntomic over R. A morphism of finite locally free group schemes that is surjective as a map of fppf sheaves is syntomic.*

Proof. Finite locally free group schemes are syntomic by [Mes72], Lemma II.3.2.6.

Suppose $f : G \to H$ is a surjective map of finite locally free group schemes. By Remark 50, f is faithfully flat and of finite presentation (as algebra or as module). And the structural map $G \to \operatorname{Spec} R$ is syntomic. Then [Sta18, Tag 05B7] implies that f is syntomic. □

Remark 103. Finite locally free group schemes are also sheaves on the syntomic site $(\mathrm{Aff}/R)_{\mathrm{syn}}$. Injectivity of a morphism of finite locally free group schemes are clearly the same whether we use the fppf or syntomic topology on (Aff/R).

As the fppf topology is finer than the syntomic topology, a morphism $f : G \to H$ of finite locally free group schemes is surjective as a map of fppf sheaves if it is surjective as a map of syntomic sheaves. Lemma 102 implies the converse is true. Indeed, f being surjetive as a map of fppf sheaves implies that f is faithfully flat. If $T \to H$ is a T-valued point of H, then $T \times_H G \to G$ is a $T \times_H G$-valued point of G that maps to $T \to H$, and $T \times_H G$ is a syntomic cover since $G \to H$ is a syntomic cover.

Suppose R is derived p-complete with bounded p^∞-torsion. Let $\Xi : (R)_{\mathrm{QSYN}} \to (\mathrm{Aff}/R)$ be the inclusion functor. If $S \in (R)_{\mathrm{QSYN}}$ and $S' \to S$ is a syntomic cover corresponding to the ring map $S \to S'$, then S' is in particular flat over S. So $(S')^\wedge_p$ is p-completely flat over S, and thus concentrated in degree 0. The cotangent complex $L_{S'/S}$ has Tor-amplitude in $[-1,0]$ since $S \to S'$ is syntomic, by Lemma 101. Then $(L_{(S')^\wedge_p/S})^\wedge_p$ has p-complete Tor-amplitude in $[-1,0]$. We conclude that $(S')^\wedge_p$ is quasi-syntomic over S, so also quasi-syntomic over R by Lemma 76. So the functor Ξ is cocontinuous if we equip (Aff/R) with the syntomic topology. We then have an exact functor $\Xi^{-1} : \mathrm{Sh}((\mathrm{Aff}/R)_{\mathrm{syn}}) \to \mathrm{Sh}((R)_{\mathrm{QSYN}})$. Recall that we have an exact functor $\rho^{-1} : \mathrm{Sh}((R)_{\mathrm{QSYN}}) \to \mathrm{Sh}((R)_{\mathrm{qsyn}})$.

Proposition 104. *Let G a finite locally free group scheme over R. Sheafification is not required for calculating $\Xi^{-1}G$. Moreover, $\rho^{-1}\Xi^{-1}$ induces a fully faithful embedding from the category of finite locally free group schemes to $\mathrm{Ab}((R)_{\mathrm{qsyn}})$.*

Proof. The composition $G \circ \Xi$ sends an object $T \in (R)_{\mathrm{qsyn}}$ to the set of ring homomorphisms $A(G) \to T$, where $A(G)$ is the ring of functions of G. Then $G \circ \Xi$ is even a p-completely fpqc sheaf as shown in Proposition 31.

Lemma 102 shows that G is syntomic over R, but G is also derived p-complete, so it is quasi-syntomic over R by Lemma 101. Then the category of finite locally free group schemes is a full subcategory of $(R)_{\mathrm{qsyn}}$, so it embeds fully faithfully via the Yoneda embedding from $(R)_{\mathrm{qsyn}}$ to $\mathrm{Sh}((R)_{\mathrm{qsyn}})$, which takes G to its functor of points on $(R)_{\mathrm{qsyn}}$. This agrees with $G \mapsto \rho^{-1}\Xi^{-1}G$. So the restriction is fully faithful. □

Lemma 105. *Let R be a ring such that p lies in the Jacobson radical of R. A morphism $G \to H$ of finite locally free group schemes over R is injective (resp. surjective) if and only if its base change to all residue field of R of characteristic p are injective (resp. surjective).*

A sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of finite locally free group schemes over R is exact if and only if the composition $G_1 \to G_3$ is 0, and it is exact after base change to all residue fields of R of characteristic p.

Proof. Denote by $G_i^{(k)}$ the base change of G_i along $R \to k$.

By Remark 44 and Remark 50, injectivity (resp. surjectivity) of a morphism is equivalent to being a closed immersion (resp. faithfully flat). Both properties are preserved by arbitrary base change. If $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact, then $G_2 \to G_3$ is faithfully flat and G_1 is the fibre product $G_2 \times_{G_3} \operatorname{Spec} R$ where the map $\operatorname{Spec} R \to G_3$ is the unit section. In this case $G_2^{(k)} \to G_3^{(k)}$ is faithfully flat by base change, and $G_1^{(k)}$ is the corresponding fibre product since fibre product commutes with base change. Thus $0 \to G_1^{(k)} \to G_2^{(k)} \to G_3^{(k)} \to 0$ is exact.

Conversely, suppose that $G \to H$ is a morphism such that all $G^{(k)} \to H^{(k)}$ are injective for k a residue field of R of characteristic p. Let $A(G), A(H)$ be the coordinate rings of G, H respectively. Then $A(H) \otimes_R k \to A(G) \otimes_R k$ is surjective for all such k. We have by Nakayama that the maps induced by $A(H) \to A(G)$ on the stalks of R at the maximal ideal $\mathfrak{m}$ is surjective whenever $R/\mathfrak{m}$ is of characteristic p. But p lies in the Jacobson radical of R, so all the maximal ideals of R contain p. Thus $A(H) \to A(G)$ is surjective, which implies that $G \to H$ is a closed immersion. Since the Cartier duality is compatible with base change and interchanges injectivity and surjectivity of a morphism of finite locally free group schemes, we get the claim for surjectivity by duality.

Suppose that $0 \to G_1^{(k)} \to G_2^{(k)} \to G_3^{(k)} \to 0$ is exact for every residue field of characteristic p $R \to k$. We have by the above that $G_1 \to G_2$ is injective and $G_2 \to G_3$ is surjective. Then $G_2 \to G_3$ is faithfully flat, and thus the kernel is a finite locally free group scheme. (see Remark 50) We have a factorization $G_1 \to \ker(G_2 \to G_3) \to G_2$, and $G_1 \to \ker(G_2 \to G_3)$ is an isomorphism after base change to k for all k. Let $B = A(G_1)$ and $C = A(\ker(G_2 \to G_3))$ be the respective coordinate rings. Then we have a map of R-algebras $C \to B$ whose base changes to all characteristic p residue fields of R are isomorphisms. Again it is surjective by Nakayama. Moreover, B and C are both finite locally free as R-modules, and they have the same rank at all closed points. Thus $C \to B$ is a surjection between finite locally free R-modules of the same rank, and is thus bijective. Then we conclude that $G_1 \simeq \ker(G_2 \to G_3)$, and the sequence is exact. □

Proposition 106. *A morphism $f : G \to H$ of finite locally free group schemes is injective (resp. surjective) if and only if $\Xi^{-1} f : \Xi^{-1} G \to \Xi^{-1} H$ is injective (resp. surjective).*

A sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of finite locally free group schemes is exact as fppf sheaves if and only if it is exact as sheaves on $(R)_{\mathrm{QSYN}}$.

Proof. The forward direction follows from Remark 103 and the exactness of Ξ^{-1}.

If $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact as sheaves on $(R)_{\mathrm{QSYN}}$, then full faithfulness implies that the composition $G_1 \to G_3$ is 0. Note that all the residue fields of R of characteristic p lie in $(R)_{\mathrm{QSYN}}$. Thus the reverse direction follows from Lemma 105. □

4.2 Truncated Barsotti-Tate groups

All the materials in the subsection are well-known. We follow [Ill85].

Definition 107. A *truncated Barsotti-Tate group of level n*, or a BT_n, is a sheaf G of abelian groups on $(\mathrm{Aff}/R)_{\mathrm{fppf}}$ satisfying

(a) G is a finite locally free group scheme;

(b) G is killed by p^n, and flat over $\mathbb{Z}/p^n\mathbb{Z}$;

(c) if $n = 1$ and $R_0 = R/p$, G_0 the base change of G to R_0, then the sequence

$$G \xrightarrow{F} G^{(p)} \xrightarrow{V} G$$

is exact (as fppf sheaves).

Remark 108. If G is a $\mathbb{Z}/p^n\mathbb{Z}$-module on the fppf site, then G is flat over $\mathbb{Z}/p^n\mathbb{Z}$ if and only if for all $i = 0, 1, \dots, n$, the sequence

$$0 \to G[p^{n-i}] \to G \xrightarrow{p^{n-i}} G[p^i] \to 0$$

is exact. Indeed, the exactness of the sequence is equivalent to the image of p^{n-i} being the kernel of p^i for all i, which is shown to be equivalent to G being flat over $\mathbb{Z}/p^n\mathbb{Z}$ in [Mes72], Lemma I.1.1.

Proposition 109. *If G is a BT_n and $m \leq n$, then the subsheaf $G[p^m]$-torsions is a BT_m.*

Proof. For $m \geq 2$ this is shown in [Mes72], Remark I.2.3, while the extra condition in $m = 1$ is verified in [Mes72], Proposition II.3.3.11. □

Definition 110. A *p-divisible group* over R is a sheaf of abelian groups on $(\mathrm{Aff}/R)_{\mathrm{fppf}}$ satisfying:

(a) G is p^∞-torsion, i.e. for any $U \in (\mathrm{Aff}/R)$ and $s \in G(U)$, there is some $n \geq 1$ such that $p^n s = 0$;

(b) the multiplication by p map $\cdot p : G \to G$ is a surjection of sheaves;

(c) $G[p]$ is a finite locally free group scheme.

Let $p - \mathrm{div}(R)$ be the full subcategory of $\mathrm{Ab}((R)_{\mathrm{fppf}})$ spanned by all the p-divisible groups.

Remark 111. For a p-divisible group G, there are exact sequences

$$0 \to G[p] \to G[p^{n+1}] \xrightarrow{\cdot p} G[p^n] \to 0$$

for all $n \geq 1$. Since finite locally free group schemes are closed under extensions (See [DG70], III 4, 1.9), we can show by induction that all the $G[p^n]$ are finite locally free group schemes.

Proposition 112. *If G is a p-divisible group, then $G[p^n]$ is a BT_n for all n.*

Proof. As multiplication by p is surjective on G, we have exact sequences

$$0 \to G[p^i] \to G[p^n] \xrightarrow{\cdot p^i} G[p^{n-i}] \to 0$$

for all positive integers $i \leq n$. Thus the proposition follows from [Mes72], Remark I.2.3 and Proposition II.3.3.11. □

Lemma 113. *If G is a BT_1, then the rank of the coordinate ring of G is of the form p^h, where h is a locally constant $\mathbb{N}$-valued function on $\mathrm{Spec}\, R$.*

Proof. We call the rank of the coordinate ring of a finite locally free group scheme its group order. Since the coordinate ring is finite locally free over R, the order is a locally constant function on $\mathrm{Spec}\, R$. It remains to show that for every residue field $R \to k$, the order of $G \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k$ is a power of p. The order is also preserved after base changing to the algebraic closure of k. Then we only need to show that a finite group scheme H over an algebraically closed field k, such that multiplication by p on H is zero, has order a power of p.

We have the connected-étale exact sequence

$$0 \to H^0 \to H \to H^{\text{ét}} \to 0$$

by [Sti09], Proposition 37. If k has characteristic 0, then $H^0 \simeq 0$ and H is a constant group scheme. The order of H is the cardinality of $H(k)$. But $H(k)$ is an $\mathbb{F}_p$-vector space by assumption, and thus has cardinality a power of p. If k has characteristic $\ell \neq p$, then by [Sti09], Corollary 50 (1), the order of H^0 is a power of ℓ. Then the order, some power of ℓ, kills H^0 by [Sti09], Theorem 6. But p also kills H^0, so 1 kills H^0 and $H^0 \simeq 0$. Again H is a constant group scheme, so it has order a power of p. Finally, assume that k has characteristic p. Then [Sti09], Corollary 50 (1) implies that the order of H^0 is a power of p, while the order of $H^{\text{ét}}$ is also a power of p since it is constant. As the order is multiplicative in exact sequences, the order of H is also a power of p. □

Definition 114. If G is a BT_n for a p-divisible group, we define the *height* of G to be the locally constant function h on $\operatorname{Spec} R$ such that the rank of the coordinate ring of $G[p]$ is p^h.

Remark 115. If $R \to R'$ is a ring map and H is a BT_n or a p-divisible group over R with height h, then the base change of G to R' has height the composition $\operatorname{Spec} R' \to \operatorname{Spec} R \xrightarrow{h} \mathbb{N}$.

Proposition 116. *(a) If $0 \to G_1 \to G_2 \to G_3 \to 0$ is a short exact sequence of fppf sheaves and either $G_1, G_2 \in p-\operatorname{div}(R)$ or $G_1, G_3 \in p-\operatorname{div}(R)$, then the third is also in $p-\operatorname{div}(R)$;*

(b) If $H \to G$ is an injection of fppf sheaves, where H is finite locally free and G a p-divisible group, then G/H is a p-divisible group;

Proof. Part (a) is [Mes72], I.(2.4.3). Part (b) is [BBM82], Lemme 3.3.12. □

Remark 117. Since G is a p-divisible group, we have an exact sequence

$$0 \to G[p] \to G[p^{n+1}] \to G[p^n] \to 0$$

Thus Proposition 51 shows that the Cartier dual $p^\vee : G[p^n]^\vee \to G[p^{n+1}]^\vee$ is injective. Moreover the map $p : G[p^{n+1}]^\vee \to G[p^n]^\vee$ corresponds to the natural injection under the Cartier duality, so it is surjective. Then the colimit of $\{G[p^n]^\vee\}$ along $p^\vee$ is a p-divisible group. We call the colimit the Cartier dual of G, and denote it by $G^\vee$.

Lemma 118. *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be a sequence of abelian sheaves on any site such that each G_i is of p^∞-torsion and $\cdot p : G_i \to G_i$ are surjective. Then the follows are equivalent:*

(1) $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact.

(2) For each n, the sequence $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$ is exact.

(3) The composition $G_1 \to G_3$ is 0, and $0 \to G_1[p] \to G_2[p] \to G_3[p] \to 0$ is exact.

Proof. Suppose $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact. Note that $G_i[p^n] \simeq \mathscr{H}\mathrm{om}(\underline{\mathbb{Z}/p^n\mathbb{Z}}, G_i)$. The complex $\underline{\mathbb{Z}} \xrightarrow{\cdot p^n} \underline{\mathbb{Z}}$ is a projective resolution of $\underline{\mathbb{Z}/p^n\mathbb{Z}}$, so $\mathscr{E}\mathrm{xt}^1(\underline{\mathbb{Z}/p^n\mathbb{Z}}, G_1) \simeq G_1/p^nG_1$. But p is surjective on G_1, so $G_1/p^nG_1 \simeq 0$.

Then the long exact sequence gives an exact sequence $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$. This proves that (1) implies (2). That (2) implies (1) follows from the exactness of filtered colimits, [Sta18, Tag 03CO], (5).

(2) trivially implies (3). Assume (3) is true. Using the exact sequences $0 \to G_i[p^n] \to G_i[p^{n+1}] \to G_i[p] \to 0$ and the 3×3 lemma we can prove inductively that $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$ is exact for each n. □

For defining the Tate module, a site where covers are closed under cofiltered limit is convenient. Thus we assume from now on that R is derived p-complete with bounded p^∞-torsion.

Proposition 119. *The functor Ξ^{-1} induces a fully faithful, exact and exactness-reflecting embedding of the category of p-divisible groups into $\mathrm{Ab}((R)_{\mathrm{QSYN}})$.*

Proof. A morphism $G \to H$ between p-divisible groups can be identified with a compatible system of maps $G[p^n] \to H[p^n]$. So the embedding is fully faithful by Proposition 104.

If G is a p-divisible group, then the maps $p : G[p^{n+1}] \to G[p^n]$ are surjective, and thus $p : (\Xi^{-1}G)[p^{n+1}] \to (\Xi^{-1}G)[p^n]$ is surjective. Moreover Ξ^{-1} commutes with taking p^n-torsions by the explicit formula for Ξ^{-1}. Then we can apply Lemma 118 to reduce exactness of p-divisible groups and their images in $\mathrm{Ab}((R)_{\mathrm{QSYN}})$ to the exactness of finite locally free group schemes. Thus we conclude by Proposition 106. □

Definition 120. Let G be a p-divisible group over R. The *Tate module* of G, denoted by T_pG, is defined to be the following sheaf on $(R)_{\mathrm{QSYN}}$

$$\lim_n G[p^n]$$

where the transition maps $G[p^{n+1}] \to G[p^n]$ is given by multiplication by p.

Proposition 121. *Let $G_1 \to G_2 \to G_3$ be maps of p-divisible groups, the followings are equivalent,*

(1) $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact;

(2) $0 \to T_pG_1 \to T_pG_2 \to T_pG_3 \to 0$ is exact;

(3) $0 \to G_3^\vee \to G_2^\vee \to G_1^\vee \to 0$ is exact.

Proof. (1) $\Rightarrow$ (2): By Lemma 118 we have exact sequences $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$. Then the required left exactness follows from the left exactness of the section fucntor and of limits, and the surjectivity of $T_pG_2 \to T_pG_3$ follows from Lemma 99 as $\mathrm{Sh}((R)_{\mathrm{QSYN}})$ is replete.

(2) $\Rightarrow$ (1): For any p-divisible group G we have $G[p^n] \simeq T_pG \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$. If $(a_1, a_2, \ldots)$ is a p-torsion element in $T_pG(S)$, then $pa_i = 0$. But $a_i = pa_{i+1}$, so the only possibility is that $a_i = 0$ for all i. Then T_pG is p-torsion free. Thus $\mathrm{Tor}_1^{\mathbb{Z}}(T_pG_3, \mathbb{Z}/p^n\mathbb{Z}) \simeq 0$. Taking tensor product of $0 \to T_pG_1 \to T_pG_2 \to T_pG_3 \to 0$ with $\mathbb{Z}/p^n\mathbb{Z}$, we obtain the required exact sequences.

For (1) $\Rightarrow$ (3), we again reduce to showing the exactness of p^n-torsions for all n. But then it follows from Proposition 51. and (3) $\Rightarrow$ (1) follows by applying the duality functor twice. □

4.3 The stack BT_n

Let $\mathcal{C}$ be the category of morphisms of schemes $G \to S$, where S is an affine scheme, and G is a BT_n over S, with morphisms being commutative squares (which aren not neccessarily pull back squares)

$$\begin{array}{ccc} G' & \longrightarrow & G \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Since arbitrary base changes for BT_n's are BT_n's, the functor $\mathcal{C} \to (\mathrm{Aff}/\mathbb{Z})$ is a fibered category.

Definition 122. The fibered category BT_n is defined to be the $\mathcal{C} \to (\mathrm{Aff}/\mathbb{Z})$ constructed in the paragraph above. There is also a restriction BT_n to $(p\mathrm{Aff}/\mathbb{Z}_p)$, which is again a fibered category.

There is also a groupoid version of the stack.

Definition 123. Let $\mathscr{BT}_n$ be the subcategory of BT_n with the same objects, but only pull back squares as morphisms. Again it's a fibered category over $(\mathrm{Aff}/\mathbb{Z})$, which has a restriction to $(p\mathrm{Aff}/\mathbb{Z}_p)$.

Proposition 124. BT_n *is a stack on* $(p\mathrm{Aff}/\mathbb{Z}_p)_{\mathrm{fpqc}}$.

Proof. BT_n is a full subcategory of the fibered category of finite locally free groups schemes. By [AL23], Proposition A.2 in the appendix, the fibered category of finite locally free group schemes is a stack over $(p\mathrm{Aff}/\mathbb{Z}_p)_{\mathrm{fpqc}}$. It remains to show that the conditions distinguishing a BT_n from a finite locally free group scheme is local in the p-completely fpqc topology.

Note that a morphism of finite locally free group schemes is injective if and only if the map on the coordinate rings is surjective. We claim that surjectivity of a map between finite locally free modules can be checked after a p-completely fpqc cover. Let $f : M \to N$ be such a map and $R \to R'$ a p-completely fpqc cover. Suppose that

$$f \otimes 1 : M \otimes_R R' \to N \otimes_R R'$$

is surjective. Then the induced map $M \otimes_R R' \otimes_{R'} R'/p \to N \otimes_R R' \otimes_{R'} R'/p$ is also surjective. But $M \otimes_R R' \otimes_{R'} R'/p \simeq M/pM \otimes_{R/p} R'/p$ and similarly for N, and $R/p \to R'/p$ is faithfully flat, so $M/pM \to N/pN$ is surjective. Since R is classically p-complete by Lemma 22, p lies in the Jacobson radical of R. Nakayama then implies that $M \to N$ is surjective. Thus injectivity of maps between finite locally free group schemes can be check after a p-completely fpqc cover. As base change commutes with Cartier duality and with taking kernel, surjectivity and exactness can also be checked after a p-completely fpqc cover. But the extra conditions for a finite locally free group scheme to be a BT_n can all be rephrased as injectivity, surjectivity or exactness statements. So BT_n form a stack on $(p\mathrm{Aff}/\mathbb{Z}_p)_{\mathrm{fpqc}}$. □

The groupoid version is only needed to prove the results in the next subsection. We denote by $\mathscr{BT}_n^h$ the sub-(fibred category) of $\mathscr{BT}_n$ consisting of BT_n's of height constantly h where h is some natural number. The definition of the height of a BT_n only depends on its p-torsion subgroup, so the truncation $\mathscr{BT}_{n+1} \to \mathscr{BT}_n$, $G \mapsto G[p^n]$ preserves the height. The following classical theorem is enough for our purpose.

Theorem 125 (Grothendieck-Illusie)**.** *$\mathscr{BT}_n^h$ is an algebraic stack of finite type on $(\mathrm{Aff}/\mathbb{Z})_{\mathrm{fpqc}}$ with affine diagonal, and there is an affine scheme X and a morphism $X \to \mathscr{BT}_n^h$ which is a smooth presentation. The truncations $\mathscr{BT}_{n+1}^h \to \mathscr{BT}_n^h$ are smooth and surjective.*

Proof. The argument that $\mathscr{BT}_n^h$ is an algebraic stack of finite type over $\operatorname{Spec} \mathbb{Z}$ with affine diagonal, and that there is a amooth presentation of $\mathscr{BT}_n^h$ by an affine scheme, is in the beginning of Section 2 of [Lau08]. The part about the truncation morphism being smooth and sujective is [Lau08], Theorem 2.1, which depends on [Ill85] Théorèm 4.4. □

4.4 Standard BT_n

Definition 126. A BT_n G is called *standard* if there is a p-divisible group H such that $G = H[p^n]$.

Lemma 127. *Let $R \to S$ be a smooth cover, then there is an étale cover $R \to R'$ such that the map $R' \to S \otimes_R R'$ has a retract.*

Proof. [Gro67], Corollaire 17.16.3 (ii). □

Lemma 128. *Suppose R is a ring and G a BT_n over R, then there is a smooth cover $R \to S$ and a BT_{n+1} H over S such that $G \otimes_R S$ is isomorphic to $H[p^n]$.*

Proof. Since the height is locally constant on $\operatorname{Spec} R$, it is constant on connected components. By arguing on each connected component of $\operatorname{Spec} R$ and glue together the results, we may assume that G has constant height h.

We will use that the truncation morphism from $\mathscr{BT}_{n+1}^h$ to $\mathscr{BT}_n^h$ is smooth and surjective.

Let $\operatorname{Spec} R \to \mathscr{BT}_n$ be the group scheme G under the 2-Yoneda embedding. Then we have a pullback square

$$\begin{array}{ccc} U & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \mathscr{BT}_{n+1} & \longrightarrow & \mathscr{BT}_n \end{array}$$

where U is an algebraic stack.

Since $\mathscr{BT}_n^h$ has affine diagonal, [Sta18, Tag 0GQE] implies that the morphism $\operatorname{Spec} R \to \mathscr{BT}_n^h$ is affine. Then $U \to \mathscr{BT}_{n+1}^h$ is also affine by base change. Let $T \to \mathscr{BT}^h_{n+1}$ be a smooth presentation, and V the pullback as in the diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathscr{BT}^h_{n+1} \end{array}$$

The morphism $V \to T$ is affine, so V is also an affine scheme. Moreover, $V \to U$ is smooth surjective. Thus $V \to \operatorname{Spec} R$ is amooth surjective, and we have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \operatorname{Spec} R \\ \downarrow & & \downarrow \\ \mathscr{BT}^h_{n+1} & \longrightarrow & \mathscr{BT}^h_{n} \end{array}$$

which gives us a BT_{n+1} over V whose truncation is the base change of G to V. □

Proposition 129. *Let G be a BT_n over R. There is an ind-étale cover $R \to R'$ such that the base change of G to R' is standard.*

Proof. We use the Lemma 128 inductively.

Let $G_0 = G$ and $R_0 = R$. Let $m \in \mathbb{N}$ and assume that we have constructed R_m an étale cover of R, G_m a BT_{n+m} over R_m such that $G \otimes_R R_m \simeq G_m[p^n]$. Apply the Lemma 128 to G_m, we have a smooth cover $R_m \to S$ and a BT_{n+m+1} H over S such that $G_m \otimes_{R_m} S \simeq H[p^{n+m}]$. Now use the Lemma 127, we have an étale cover $R_m \to R_{m+1}$ such that $R_{m+1} \to S \otimes_{R_m} R_{m+1}$ has a retract. Base changing H along the map $S \to S \otimes_{R_m} R_{m+1} \to R_{m+1}$ gives a BT_{n+m+1} over R_{m+1} which we denote by G_{m+1}. Since truncation commutes with base change, we have that $G \otimes_{R_m} R_{m+1} \simeq G_{m+1}[p^n]$. This completes the inductive step.

With all the R_m, G_m constructed, we can take $R' = \operatorname{colim} R_m$ and $G_\infty = \operatorname{colim} G_m \otimes_{R_m} R'$. R' is ind-étale over R while G_∞ is a p-divisible group over R' such that $G \otimes_R R' \simeq G_\infty[p^n]$. □

As base change of standard BT_n's are standard BT_n's, we get the p-completed version for free.

Corollary 130. *Let G be a BT_n over R, where R is assumed to be p-complete with bounded p^∞-torsion. There is an p-ind-étale cover $R \to R'$ such that the base change of G to R' is standard.*

4.5 Truncation from p-divisible groups

We would like to invoke the Proposition 41 to embed the standard BT_n's into the derived category of p-divisible groups. Let $\mathcal{A}$ be the abelian category $\mathrm{Ab}((R)_{\mathrm{fppf}})$.

Lemma 131. *The exact category $p-\mathrm{div}(R)$ is idempotent complete, i.e. for any morphism $f : X \to X$ in $p-\mathrm{div}(R)$ such that $f^2 = f$, X has a direct sum decomposition $Y \oplus Z$ such that f is id on Y and 0 on Z.*

Proof. The category $\mathcal{A}$ is abelian, so in particular idempotent complete. Let $X \simeq Y \oplus Z$, where $Y, Z \in \mathcal{A}$ and f restricts to id on Y and 0 on Z. Fix some $\operatorname{Spec} S \in (R)_{\mathrm{fppf}}$. Every element x in $Y(S)$ (resp. $Z(S)$) can be embedded into $X(S)$ as $(x, 0)$ (resp. $(0, x)$), and thus is killed by some power of p since X (resp. Z) is a p-divisible group. Moreover, there is some cover $S \to S'$ and $r \in X(S')$ such that $pr = (x, 0)$ (resp. $pr = (0, x)$) in $X(S')$. Let $r = (x', y')$ with $x' \in Y(S')$ and $y' \in Z(S')$, then $px' = x$ (resp. $py' = x$) in $Y(S')$ (resp. $Z(S')$). So the conditions (a), (b) in the definition of p-divisible groups are verified by Y and Z.

Note that $X[p] \simeq Y[p] \oplus Z[p]$. Also, f restricts to a map $f[p] : X[p] \to X[p]$ with $Z[p]$ being the kernel. Then $Z[p]$ is a kernel of finite locally free group schemes, and is thus representable. $Y[p]$, being the kernel of $1 - f[p]$, is also representable. Let $\mathcal{O}(X[p]), \mathcal{O}(Y[p])$ and $\mathcal{O}(Z[p])$ be the R-algebras that represent $X[p]$, $Y[p]$ and $Z[p]$. The inclusion and projection give a pair of map between $Z[p]$ and $X[p]$, whose composition is $\mathrm{id}_{Z[p]}$. Going to the opposite category, we have a pair of maps between $\mathcal{O}(Z[p])$ and $\mathcal{O}(X[p])$, whose composition is $\mathrm{id}_{\mathcal{O}(Z[p])}$. Thus $\mathcal{O}(Z[p])$ is a direct summand of $\mathcal{O}(X[p])$, and is finite locally free. So $Z \simeq \ker f$ is a p-divisible group, and the same reasoning goes for Y. Then $X \simeq Y \oplus Z$ is a direct sum decomposition in $p - \mathrm{div}(R)$. □

Proposition 132. *The category of standard BT_n's embeds fully faithfully into the derived category $D^b(p - \mathrm{div}(R))$ as complexes of the shape $H \xrightarrow{p^n} H$ sitting in degree $0, 1$, where H is a p-divisible group.*

Proof. We need to verify the conditions for the Proposition 41. That $p-\mathrm{div}(R)$ is a fully exact subcategory of $\mathcal{A}$ is true by definition. The category $p - \mathrm{div}(R)$ is idempotent complete by Lemma 131. Also Lemma 116 implies that if $G \to H$ is an injective map of fppf sheaves where either of the two cases are true:

- G, H are both p-divisible groups
- G is a BT_n and H is a p-divisible group

Then the cokernel is also a p-divisible group. Finally each standard BT_n is the kernel of some $H \xrightarrow{p^n} H$ by definition. □

5 Dieudonné theory

5.1 Prismatic Dieudonné theory

In this subsection we briefly introduce the prismatic Dieudonné theory of [AL23].

Suppose R is a quasi-syntomic ring.

Definition 133. A *prismatic Dieudonné crystal* over R is a finite locally free $\mathcal{O}^{\mathrm{pris}}$-module $\mathcal{M}$ equipped with a φ-linear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, such that the cokernel of the linearization $\varphi_{\mathcal{M}} : \varphi^*\mathcal{M} \to \mathcal{M}$ has cokernel killed by $\mathcal{I}^{\mathrm{pris}}$. A prismatic Dieudonné crystal is said to be admissible if the image of

$$\mathcal{M} \xrightarrow{\varphi_{\mathcal{M}}} \mathcal{M} \to \mathcal{M}/\mathcal{I}^{\mathrm{pris}}\mathcal{M}$$

is a finite locally free $\mathcal{O}$-module $\mathcal{F}$ such that the map $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}^{\mathrm{pris}}/\mathcal{I}^{\mathrm{pris}} \to \mathcal{M}/\mathcal{I}^{\mathrm{pris}}\mathcal{M}$ is injective.

For a p-divisible group G, we just write $\Xi^{-1}G$ for the sheaf $\rho^{-1}\Xi^{-1}G$ on $(R)_{\mathrm{qsyn}}$.

Theorem 134. *For a p-divisible group G over R, the $\mathcal{O}^{\mathrm{pris}}$-module*

$$\mathcal{M} = \mathscr{E}\mathrm{xt}^1_{(R)_{\mathrm{qsyn}}}(\Xi^{-1}G, \mathcal{O}^{\mathrm{pris}})$$

equipped with the natural Frobenius $\varphi_{\mathcal{M}}$ induced by the Frobenius on $\mathcal{O}^{\mathrm{pris}}$ is an admissible prismatic Dieudonné crystal. The functor $G \mapsto (\mathcal{M}, \varphi_{\mathcal{M}})$ is a contravariant equivalence between the category of p-divisible groups over R to that of admissible prismatic Dieudonné crystals over R.

Proof. [AL23], Theorem 4.6.10. □

For a quasiregular semiperfectoid ring R, there is also a version of the equivalence using $\mathbb{\Delta}_R$-modules.

Definition 135. Let R be a quasiregular semiperfectoid ring and $(\mathbb{\Delta}_R, d)$ the initial prism. A *prismatic Dieudonné module* is a finite locally free $\mathbb{\Delta}_R$-module equipped with a φ-linear map $\varphi_M : M \to M$ such that the cokernel of the linearization $\varphi_M : \varphi^*M \to M$ is killed by d. A prismatic Dieudonné module is said to be *admissible* if the image of the composition

$$M \xrightarrow{\varphi_M} M \to M/dM$$

is a finite locally free R-module F such that $F \otimes_R \overline{\mathbb{\Delta}}_R \to M/dM$ is injective.

Theorem 136. *Let R be a quasiregular semiperfectoid ring. For a p-divisible group G over R, the $\mathbb{\Delta}_R$-module $M = \mathrm{Ext}^1_{(R)_{\mathrm{qsyn}}}(\Xi^{-1}G, \mathcal{O}^{\mathrm{pris}})$ equipped with the natural Frobenius φ_M induced by the Frobenius on $\mathbb{\Delta}_R$ is an admissible prismatic Dieudonné module. The functor $G \mapsto (M, \varphi_M)$ is a contravariant equivalence between the category of p-divisible groups over R to that of admissible prismatic Dieudonné modules over R.*

Proof. This is a combination of Theorem 134 and [AL23], Proposition 4.1.13. □

5.2 Exactness of the prismatic Dieudonné functor

In this subsection we investigate how the prismatic Dieudonné functor interacts with exactness.

For R a quasi-syntomic ring and G a p-divisible group or a finite locally free group scheme over R, we denote $\mathcal{M}_{\Delta}(G)$ the sheaf $\mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta})$. Our notation is slightly inconsistent with that of [AL23] as we use $\mathcal{M}_{\Delta}(G)$ for a sheaf on $(R)_{\Delta}$. The sheaf $v_*\mathcal{M}_{\Delta}(G)$ on $(R)_{\mathrm{qsyn}}$ is the prismatic Dieudonné crystal of [AL23]. We denote also by $\mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G)$ the $\mathcal{O}^{\mathrm{pris}}$-module $\mathscr{E}\mathrm{xt}^1_{(R)_{\mathrm{qsyn}}}(\Xi^{-1}G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}})$. If R is quasiregular semiperfectoid, then we denote $M_{\Delta}(G)$ the Δ_R-module $\mathrm{Ext}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta})$ and $\mathcal{N}^{\geq 1}M_{\Delta}(G)$ the Δ_R-module $\mathrm{Ext}^1_{(R)_{\mathrm{qsyn}}}(\Xi^{-1}G, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}})$.

Lemma 137. *Let R be quasi-syntomic and $G \to H$ an injection of finite locally free group schemes over R. Then the map of sheaves*

$$\mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}H, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta})$$

is surjective.

Proof. By a theorem of Raynaud, [BBM82], Théorème 3.1.1, there is a Zariski cover $R \to R'$ and an abelian scheme A over R' such that there is a closed immersion of group schemes $H \to A$. Let R_1 be the classical p-completion of R' and A_1 the base change of A to R_1, which is an abelian scheme over R_1. Since $G \to H$ is also a closed immersion by Lemma 44, G is also a closed sub-(group scheme) of A_1 after base changing to R_1. Moreover, the closed immersions $G \to A_1, H \to A_1$ have cokernel A_2, A_3 respectively, which are also abelian schemes as G, H are finite locally free group schemes. Let B_1, B_2, B_3 be the formal completion of A_1, A_2, A_3 along the closed subscheme defined by the sheaf of ideal (p) respectively.

Now we can use the results in [AL23], Section 4.5 to calculate Ext groups for B_1, B_2, B_3. The sheaf $\mathscr{E}\mathrm{xt}^2_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}B_2, \mathcal{O}_{\Delta})$ vanishes by [AL23], Theorem 4.5.6. We can then apply $\mathrm{R}\mathscr{H}\mathrm{om}_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}(-), \mathcal{O}_{\Delta})$ to the exact sequence

$$0 \to H \to B_1 \to B_3 \to 0$$

to obtain a surjection

$$\mathscr{E}\mathrm{xt}^1_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}B_1, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}H, \mathcal{O}_{\Delta})$$

and similarly with H replaced by G. Note that the injection $G \to A$ factors through H, and thus the surjections has a corresponding factorization. So we have a surjection

$$\mathscr{E}\mathrm{xt}^1_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}H, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(R_1)_{\Delta}}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta})$$

Since $R \to R'$ is a Zariski cover, $R \to R_1$ is a quasi-syntomic cover of R. By the localizing property of $\mathscr{E}\mathrm{xt}^1$, we conclude that

$$\mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}H, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta})$$

is surjective. ∎

Proposition 138. *Suppose R is quasi-syntomic and $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of p-divisible groups over R. Then the sequence*

$$0 \to \mathcal{M}_{\Delta}(G_3) \to \mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1) \to 0$$

is exact.

Proof. The proof is essentially taken from [AL23], Proposition 4.6.8 but here we work on the prismatic site instead.

Since the functor $u^{-1}\Xi^{-1}$ is exact, we have an exact sequence

$$0 \to u^{-1}\Xi^{-1}G_1 \to u^{-1}\Xi^{-1}G_2 \to u^{-1}\Xi^{-1}G_3 \to 0$$

of sheaves on $(R)_{\Delta}$. Moreover, exactness of $u^{-1}\Xi^{-1}$ also implies that the map $p : u^{-1}\Xi^{-1}G \to u^{-1}\Xi^{-1}G$ is still surjective for any p-divisible group G. Then the argument of [AL23], Remark 4.2.2 applies to show that $\mathscr{H}\mathrm{om}(u^{-1}\Xi^{-1}G, \mathcal{O}_{\Delta}) \simeq 0$ for any p-divisible group G. Applying $\mathrm{R}\mathscr{H}\mathrm{om}_{(R)_{\Delta}}(-, \mathcal{O}_{\Delta})$ and use the vanishing of $\mathscr{H}\mathrm{om}(u^{-1}\Xi^{-1}G_1, \mathcal{O}_{\Delta})$, we have an exact sequence

$$0 \to \mathcal{M}_{\Delta}(G_3) \to \mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1)$$

Now we prove surjectivity of $\mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1)$. For each integer n, Lemma 137 gives an exact sequence

$$\mathcal{M}_{\Delta}(G_3[p^n]) \to \mathcal{M}_{\Delta}(G_2[p^n]) \to \mathcal{M}_{\Delta}(G_1[p^n]) \to 0$$

and also

$$\mathcal{M}_{\Delta}(G_3[p^{n+1}]) \to \mathcal{M}_{\Delta}(G_3[p^n])$$

is surjective. Since the topos $\mathrm{Sh}((R)_{\Delta})$ is replete, Lemma 99 implies that

$$\lim_n \mathcal{M}_{\Delta}(G_2[p^n]) \to \lim_n \mathcal{M}_{\Delta}(G_1[p^n])$$

is surjective. But we have by the proof of [AL23], Proposition 4.6.5 that

$$\mathcal{M}_{\Delta}(G_i) \simeq \lim_n \mathcal{M}_{\Delta}(G_i[p^n])$$

And surjectivity follows. □

Corollary 139. *If R is quasiregular semiperfectoid, then the functor $M_{\Delta}(-)$ is exact.*

Proof. The sheaf $\mathcal{M}_{\Delta}(G)$ is finite locally free for every p-divisible group G by [AL23], Proposition 4.6.5. Then we can apply Lemma 95 to Proposition 138. □

Lemma 140. *Suppose that $R = k$ is a perfect field of characteristic p and $0 \to G_1 \to G_2 \to G_3 \to 0$ is a sequence of p-divisible groups over k. If the sequence*

$$0 \to \mathcal{M}_{\Delta}(G_3) \to \mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1) \to 0$$

is exact, then the sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is also exact.

Proof. Since $\mathcal{M}_{\Delta}(G_i)$ is finite locally free over $\mathcal{O}_{\Delta}$ which is p-torsion free, each $\mathcal{M}_{\Delta}(G_i)$ is p-torsion free. Taking $\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z}$ with the exact sequence given in the statement, we have an exact sequence

$$0 \to \mathcal{M}_{\Delta}(G_3)/p^n \to \mathcal{M}_{\Delta}(G_2)/p^n \to \mathcal{M}_{\Delta}(G_1)/p^n \to 0$$

while on the other hand we have $\mathcal{M}_{\Delta}(G_i)/p^n \simeq \mathcal{M}_{\Delta}(G_i[p^n])$ as is proved in the proof of [AL23], Proposition 4.6.5. But the equivalence of exact categories in [AL23], Theorem 5.1.4 implies that $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$ is exact. We conclude by Lemma 118 that $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact. ☐

Lemma 141. *Suppose that R is quasiregular semiperfectoid and*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is a sequence of p-divisible groups over R such that the composition $G_1 \to G_3$ is 0 and the base change of the sequence to every characteristic p residue field of R is exact. Then

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is exact.

Proof. The exactness of $0 \to G_1 \to G_2 \to G_3 \to 0$ is equivalent to that of $0 \to G_1[p^n] \to G_2[p^n] \to G_3[p^n] \to 0$ for all n, while the latter can be checked after base change to all residue field of R of characteristic p by Lemma 105. ☐

Proposition 142. *Suppose R is quasiregular semiperfectoid and $G_1 \to G_2 \to G_3$ two morphisms of p-divisible groups over R. If the sequence*

$$0 \to \mathcal{M}_{\Delta}(G_3) \to \mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1) \to 0$$

is exact, then the sequence

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is also exact.

Proof. Let $R \to k$ be a residue field of characteristic p. By assumption R/p is semiperfect, so k is perfect. The inclusion functor $(k)_{\Delta} \to (R)_{\Delta}$ induced by $R \to k$ is continuous cocontinuous, so we have an exact functor $w^{-1} : \mathrm{Sh}((R)_{\Delta}) \to \mathrm{Sh}((k)_{\Delta})$.

Denote $\Xi_R^{-1} : \mathrm{Sh}((\mathrm{Aff}/R)_{\mathrm{syn}}) \to \mathrm{Sh}((R)_{\mathrm{QSYN}})$ and $\Xi_k^{-1} : \mathrm{Sh}((\mathrm{Aff}/k)_{\mathrm{syn}}) \to \mathrm{Sh}((k)_{\mathrm{QSYN}})$ the functors as in Proposition 119. Let u_R^{-1}, u_k^{-1} be the functor from the big quasi-syntomic site to the prismatic site over R, k respectively. Finally let $G_i^{(k)}$ be the base change of G_i to k.

By the explicit expression of the functors, we have $u_k^{-1}\Xi_k^{-1}G_i^{(k)} \simeq w^{-1}u_R^{-1}\Xi_R^{-1}G_i$. Since the $\mathscr{E}\mathrm{xt}^1$ sheaf localizes, we have isomorphisms

$$\mathscr{E}\mathrm{xt}^1_{(k)_{\Delta}}(u_k^{-1}\Xi_k^{-1}G_i^{(k)}, \mathcal{O}_{\Delta}) \simeq w^{-1}\,\mathscr{E}\mathrm{xt}^1_{(R)_{\Delta}}(u_R^{-1}\Xi_R^{-1}G_i, \mathcal{O}_{\Delta})$$

The exactness of w^{-1} implies that

$$0 \to \mathscr{E}\mathrm{xt}^1_{(k)_{\Delta}}(u_k^{-1}\Xi_k^{-1}G_3^{(k)}, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(k)_{\Delta}}(u_k^{-1}\Xi_k^{-1}G_2^{(k)}, \mathcal{O}_{\Delta}) \to \mathscr{E}\mathrm{xt}^1_{(k)_{\Delta}}(u_k^{-1}\Xi_k^{-1}G_1^{(k)}, \mathcal{O}_{\Delta}) \to 0$$

is exact. Then Lemma 140 implies that

$$0 \to G_1^{(k)} \to G_2^{(k)} \to G_3^k \to 0$$

is exact. We conclude by Lemma 141 that

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

is exact. □

Corollary 143. *If R is quasiregular semiperfectoid, then a sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of p-divisible groups is exact if and only if the sequence*

$$0 \to M_{\Delta}(G_3) \to M_{\Delta}(G_2) \to M_{\Delta}(G_1) \to 0$$

is exact.

Proof. This is the combination of Corollary 139 and Proposition 142, noting that the exact categories of finite locally free $\mathcal{O}_{\Delta}$-modules and of Δ_R-modules are equivalent by Lemma 95. □

Now we investigate the exactness of the Hodge filtration.

Remark 144. The natural map induced by the inclusion $\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}}$ maps $\mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G)$ (resp. $\mathcal{N}^{\geq 1}M_{\Delta}(G)$) injectively into $v_*\mathcal{M}_{\Delta}(G)$ (resp. $M_{\Delta}(G)$). This follows from the exact sequence

$$0 \to \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}} \to \mathcal{O}^{\mathrm{pris}} \to \mathcal{O} \to 0$$

and the fact that $\mathscr{H}\mathrm{om}_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}) = 0$ (resp. $\mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(G, \mathcal{O}) = 0$) which is shown in [AL23], Remark 4.2.2.

Proposition 145. *Suppose R quasiregular semiperfectoid with initial prism $(\Delta_R, (d))$, and G a p-divisible group over R. Then there is an exact sequence*

$$0 \to T_pG^{\vee} \to \mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G) \xrightarrow{\frac{\varphi}{d}-1} v_*\mathcal{M}_{\Delta}(G) \to 0$$

on $(R)_{\mathrm{qsyn}}$.

Proof. [AL23], Remark 4.8.4. (Note that our $\mathcal{M}_{\Delta}(G)$ is a sheaf on $(R)_{\Delta}$, and $v_*\mathcal{M}_{\Delta}(G)$ is the $\mathcal{M}_{\Delta}(G)$ in [AL23] by [AL23], Lemma 4.2.4.) □

Proposition 146. *If R is quasiregular semiperfectoid and $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of p-divisible groups over R, then the sequence*

$$0 \to \mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G_3) \to \mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G_2) \to \mathcal{N}^{\geq 1}\mathcal{M}_{\Delta}(G_1) \to 0$$

is exact.

Proof. Proposition 121 shows that

$$0 \to T_p G_3^\vee \to T_p G_2^\vee \to T_p G_1^\vee \to 0$$

is exact on $(R)_{\mathrm{QSYN}}$, and thus also exact on $(R)_{\mathrm{qsyn}}$. Proposition 138 and the exact equivalence in Lemma 95 shows that

$$0 \to v_* \mathcal{M}_{\Delta}(G_3) \to v_* \mathcal{M}_{\Delta}(G_2) \to v_* \mathcal{M}_{\Delta}(G_1) \to 0$$

is exact. By functoriality the map

$$\mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G_3) \to \mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G_1)$$

is 0. We conclude that

$$0 \to \mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G_3) \to \mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G_2) \to \mathcal{M}_{\Delta}(G_1) \to 0$$

is exact by the exact sequence in Proposition 145 and the 3×3 lemma. □

The remaining part of this subsection is devoted to proving that the sequence of global sections of Proposition 146 is exact.

Lemma 147. *Suppose R is quasi-syntomic. Then $\mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G)$ coincides with the submodule $\varphi_M^{-1}(\mathcal{I}^{\mathrm{pris}} M)$, where $M = v_* \mathcal{M}_{\Delta}(G)$ and φ_M is induced by the Frobenius on $\mathcal{O}^{\mathrm{pris}}$.*

Proof. We only need to check the statement for the sections of the sheaves on quasiregular semiperfectoid rings since any object in $(R)_{\mathrm{qsyn}}$ can be covered by some quasiregular semiperfectoid ring. So assume R is quasiregular semiperfectoid and $(\Delta_R, (d))$ the initial prism. Note that $\mathcal{O}^{\mathrm{pris}}$ and $\mathcal{N}^{\geq 1} \mathcal{O}^{\mathrm{pris}}$ are derived p-complete, so [AL23], Remark 4.2.2 shows that $\mathscr{H}om_{(R)_{\mathrm{qsyn}}}(S, \mathcal{O}^{\mathrm{pris}}) \simeq 0$ and $\mathscr{H}om_{(R)_{\mathrm{qsyn}}}(S, \mathcal{O}^{\mathrm{pris}}) \simeq 0$ for any sheaf S such that multiplication by p on S is a surjection. Using the exact sequence

$$0 \to T_p G \to \lim_{\times p} \Xi^{-1} G \to \Xi^{-1} G \to 0$$

in [AL23], Lemma 4.2.5 and the vanishing of the $\mathscr{H}om$ sheaves, we have that

$$v_* \mathcal{M}_{\Delta}(G)(R) \simeq \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_p G, \mathcal{O}^{\mathrm{pris}})$$

and

$$\mathcal{N}^{\geq 1} \mathcal{M}_{\Delta}(G)(R) \simeq \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_p G, \mathcal{N}^{\geq 1} \mathcal{O}^{\mathrm{pris}})$$

Suppose $f \in \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_p G, \mathcal{O}^{\mathrm{pris}})$. As $\mathcal{O}^{\mathrm{pris}}$ is d-torsion free, $\varphi_M(f)$ lies in $dM(R)$ if and only if for any $R' \in (R)_{\mathrm{qsyn}}$, R' quasiregular semiperfectoid and $s \in T_p G(R')$, the evaluation $\varphi_M(f)(R')(s)$ lies in $d\mathcal{O}^{\mathrm{pris}}(R')$. But $\varphi_M(f)(R')(s) = \varphi(f(R')(s))$ by the definition of φ_M. Therefore $\varphi_M(f) \in dM(R)$ if and only if for all R' and s as above, $f(R')(s)$ lies in $\mathcal{N}^{\geq 1} \mathcal{O}^{\mathrm{pris}}(R')$. And we conclude that f lies in $\varphi_M^{-1}(dM)(R)$ if and only if f lies in the submodule $\mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_p G, \mathcal{N}^{\geq 1} \mathcal{O}^{\mathrm{pris}})$. □

Denote temporarily by $\mathcal{L}(G)$ the quotient $(v_*\mathcal{M}_{\mathbb{A}}(G))/\mathcal{N}^{\geq 1}\mathcal{M}_{\mathbb{A}}(G)$.

Lemma 148. *For R quasi-syntomic and G a p-divisible group over R, the sheaf $\mathcal{L}(G)$ is finite locally free over $\mathcal{O}$.*

Proof. By Lemma 147, the quotient $(v_*\mathcal{M}_{\mathbb{A}}(G))/\mathcal{N}^{\geq 1}\mathcal{M}_{\mathbb{A}}(G)$ identifies with the image of the composition $v_*\mathcal{M}_{\mathbb{A}}(G) \xrightarrow{\varphi_{\mathcal{M}}} v_*\mathcal{M}_{\mathbb{A}}(G) \to v_*\mathcal{M}_{\mathbb{A}}(G)/\mathcal{I}^{\mathrm{pris}}\mathcal{M}_{\mathbb{A}}(G)$. The latter is finite locally free over $\mathcal{O}$ since $v_*\mathcal{M}_{\mathbb{A}}(G)$ is an admissible prismatic Dieudonné crystal. □

Lemma 149. *If R is quasiregular semiperfectoid with initial prism $(\mathbb{A}_R, (d))$ and G a p-divisible group over R, then the following sequence is exact*

$$0 \to \mathcal{N}^{\geq 1}M_{\mathbb{A}}(G) \to M_{\mathbb{A}}(G) \to \mathcal{L}(G)(R) \to 0$$

Proof. Recall that $\mathcal{N}^{\geq 1}\mathcal{M}_{\mathbb{A}}(G) \simeq \mathscr{H}\mathrm{om}_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}})$. Using again the exact sequence

$$0 \to T_pG \to \lim_{\times p} \Xi^{-1}G \to \Xi^{-1}G \to 0$$

We know that $\mathcal{N}^{\geq 1}M_{\mathbb{A}} \simeq \mathrm{Hom}_{(R)_{\mathrm{qsyn}}}(T_pG, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}})$. Therefore we have that

$$\mathcal{N}^{\geq 1}M_{\mathbb{A}}(G) \simeq \Gamma((R)_{\mathrm{qsyn}}, \mathcal{N}^{\geq 1}\mathcal{M}_{\mathbb{A}}(G))$$

Then everything except surjectivity of $M_{\mathbb{A}}(G) \to \mathcal{L}(G)(R)$ follows from applying $R\Gamma((R)_{\mathrm{qsyn}}, -)$ to the exact sequence

$$0 \to \mathcal{N}^{\geq 1}\mathcal{M}_{\mathbb{A}}(G) \to v_*\mathcal{M}_{\mathbb{A}}(G) \to \mathcal{L}(G) \to 0$$

Let $\mathcal{T}$ denote the sheaf

$$v_*\mathcal{M}_{\mathbb{A}}(G)/\mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.v_*\mathcal{M}_{\mathbb{A}}(G)$$

which is also finite locally free over $\mathcal{O}$, and maps surjectively to $\mathcal{L}(G)$. Lemma 93 then implies that $\mathcal{T}(R) \to \mathcal{L}(G)(R)$ is surjective.

Now we show that $M_{\mathbb{A}}(G) \to \mathcal{T}(R)$ is surjective. By Proposition 90,

$$R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}) = \mathcal{N}^{\geq 1}\mathbb{A}_R$$

Lemma 95 allows us to write $v_*\mathcal{M}_{\mathbb{A}}(G)$ as a direct summand of a finite free $\mathcal{O}^{\mathrm{pris}}$-module. Then $R\Gamma((R)_{\mathrm{qsyn}}, \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.v_*\mathcal{M}_{\mathbb{A}}(G))$ is concentrated in degree 0. Applying $R\Gamma((R)_{\mathrm{qsyn}}, -)$ to the exact sequence

$$0 \to \mathcal{N}^{\geq 1}\mathcal{O}^{\mathrm{pris}}.v_*\mathcal{M}_{\mathbb{A}}(G) \to v_*\mathcal{M}_{\mathbb{A}}(G) \to \mathcal{T} \to 0$$

yields that $M_{\mathbb{A}}(G) \to \mathcal{T}(R)$ is surjective.

Then we conclude that $M_{\mathbb{A}}(G) \to \mathcal{L}(G)(R)$ is surjective. □

Proposition 150. *If R is quasiregular semiperfectoid and $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of p-divisible groups over R, then*

$$0 \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_3) \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_2) \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_1) \to 0$$

is exact.

Proof. Use the exact sequences in Proposition 138, Proposition 146 and the 3×3 lemma, we have an exact sequence

$$0 \to \mathcal{L}(G_3) \to \mathcal{L}(G_2) \to \mathcal{L}(G_1) \to 0$$

But $\mathcal{L}(G_i)$ is finite locally free over $\mathcal{O}$, so we have

$$0 \to \mathcal{L}(G_3)(R) \to \mathcal{L}(G_2)(R) \to \mathcal{L}(G_1)(R) \to 0$$

Then use the exact sequences in Lemma 149, Proposition 139 and the 3×3 lemma again, we have that

$$0 \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_3) \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_2) \to \mathcal{N}^{\geq 1} M_{\mathbb{\Delta}}(G_1) \to 0$$

is exact. □

5.3 Pixels

We introduce in this subsection the objects we use to classify BT_n over quasiregular semiperfectoid rings, the pixels. Pixels are modeled after the truncated displays in [LZ18], Definition 1.1.

For the moment fix a ring A and a map of A-modules $\kappa : J \to A$. Assume that A has an endomorphism φ, and there is a φ-linear map $\varphi_1 : J \to A$ whose image generates A as an A-module.

Definition 151. A *prepixel* over (A, J) is a tuple $(P, Q, \iota, \epsilon, \dot{F}, F)$ where P, Q are A-modules, $\iota : Q \to P$, $\epsilon : P \otimes_A J \to Q$ are two A-linear maps and $F : P \to P$, $\dot{F} : Q \to P$ are two φ-linear maps, satisfying the following conditions: compositions $\iota \circ \epsilon$ and $\epsilon \circ (\mathrm{id} \otimes \iota)$ are the respective action maps via κ; the equality $\dot{F} \circ \epsilon(x \otimes v) = \varphi_1(v) F(x)$ holds for all $x \in P$ and $v \in J$.

Denote by $\mathsf{PrePxl}(A, J)$ the category of prepixels over (A, J), with morphisms given by morphisms of pairs that commute with the structural maps.

Proposition 152. *The category* $\mathsf{PrePxl}(A, J)$ *is an abelian category.*

Proof. Let PM be the category of pairs of A-modules. There is a natural forgetful functor $G : \mathsf{PrePxl}(A, J) \to \mathsf{PM}$, which is faithful and conservative since the additional requirement for a morphism in PM to be a morphism in $\mathsf{PrePxl}(A, J)$ is the commutation with the structural maps. Thus, in order to show that $\mathsf{PrePxl}(A, J)$ is abelian, it suffices to show that $\mathsf{PrePxl}(A, J)$ has kernels and cokernels, and formations of kernels and cokernels commute with G.

Let $(f, g) : (P_1, Q_1, \iota_1, \epsilon_1, \dot{F}_1, F_1) \to (P_2, Q_2, \iota_2, \epsilon_2, \dot{F}_2, F_2)$ be a morphism in $\mathsf{PrePxl}(A, J)$. ι_1 and $\dot{F}_1$ maps $\ker g$ to $\ker f$, and ϵ_1 maps the image of

$\ker f \otimes_A J \to P_1 \otimes_A J$ to $\ker g$. So these maps restrict to a prepixel structure on $(\ker f, \ker g)$. Similarly $\iota_2, \epsilon_2, \dot{F}_2$ induce a prepixel structure on $(\operatorname{coker} f, \operatorname{coker} g)$. Note that if a morphism $(P_3, Q_3) \to (P_1, Q_1)$ factors through $(\ker f, \ker g)$ as a morphism in PM, then the factorization automatically lies in $\mathsf{PrePxl}(A, J)$, and similarly for cokernels. Thus kernels and cokernels exist in $\mathsf{PrePxl}(A, J)$, and are calculated by kernels and cokernels in PM. $\square$

From now on assume that R is quasiregular semiperfectoid and is p-torsion free. In this case $\overline{\Delta}_R$ is also p-torsion free since it is p-completely flat over R by Remark 80. Thus Δ_R is a transversal prism, and Lemma 60 shows that Δ_R is p-torsion free. By [BS22], Theorem 12.2, the image of the divided Frobenius $\varphi_1 : \mathcal{N}^{\geq 1}\Delta_R \to \Delta_R$ generates Δ_R. Then we may take (A, J) to be $(\Delta_R, \mathcal{N}^{\geq 1}\Delta_R)$. Denote by $\mathsf{PrePxl}(R)$ the category $\mathsf{PrePxl}(\Delta_R, \mathcal{N}^{\geq 1}\Delta_R)$.

Remark 153. Let $(P, Q, \iota, \epsilon, \dot{F}, F)$ be a prepixel over R. By assumption there is some $\alpha_i \in A$ and $v_i \in J$ such that $\sum_i \alpha_i \varphi_1(v_i) = 1$. Then for all $x \in P$, we have that

$$\begin{aligned} F(x) &= \sum_i \alpha_i \varphi_1(v_i) F(x) \\ &= \sum_i \alpha_i \dot{F} \circ \epsilon(x \otimes v_i) \end{aligned}$$

So F is determined by $\dot{F}$.

Moreover, for $x \in Q$, we have

$$F(\iota(x)) = \sum_i \alpha_i \dot{F}(\epsilon(u_i \otimes \iota(x))) = d\dot{F}(x)$$

Thus $\dot{F}$ is determined by F if P is d-torsion free.

Lemma 154. *If M is a finite locally free module over Δ_R/p^n, then the action map $M \otimes_{\Delta_R} \mathcal{N}^{\geq 1}\Delta_R \to M$ is injective.*

Proof. By Lemma 60, the ring Δ_R is p-torsion free. Thus $\Delta_R \xrightarrow{p^n} \Delta_R$ is a projective resolution of Δ_R/p^n. Since R is also p-torsion free, we can calculate that $\operatorname{Tor}_1^{\Delta_R}(\Delta_R/p^n, R) = 0$. As M is a direct summand of a finite free Δ_R/p^n-module and Tor commutes with direct sums, we have $\operatorname{Tor}_1^{\Delta_R}(M, R) = 0$. Tensoring the exact sequence

$$0 \to \mathcal{N}^{\geq 1}\Delta_R \to \Delta_R \to R \to 0$$

with M gives the desired injectivity. $\square$

In particular, the map $\mathcal{N}^{\geq 1}\Delta_R \otimes_{\Delta_R} \Delta_R/p^n \to \Delta_R/p^n$ is injective. We identify the Δ_R-module $\mathcal{N}^{\geq 1}\Delta_R \otimes_{\Delta_R} \Delta_R/p^n$ with its image which we denote by $\mathcal{N}_{R,n}$. We will also write $\Delta_{R,n}$ for Δ_R/p^n in the sequel. By Lemma 60, the ring $\Delta_{R,n}$ is d-torsion free. Then there is a φ-linear map $\varphi_1 : \mathcal{N}_{R,n} \to \Delta_{R,n}$ defined by $x \mapsto \frac{\varphi(x)}{d}$. Theorem 12.2 of [BS22] implies that φ_1 is surjective after linearization. If $n = \infty$, then we define $\Delta_{R,n}$ to be Δ_R and $\mathcal{N}_{R,n}$ to be $\mathcal{N}^{\geq 1}\Delta_R$.

Definition 155. Suppose $n \in \mathbb{N}_{\geq 1}$ or $n = \infty$. A *pixel of level* n over R is a prepixel $(P, Q, \iota, \epsilon, \dot{F}, F)$ over R satisfying the following conditions:

(1) P is finite locally free over $\mathbb{\Delta}_{R,n}$;

(2) $\operatorname{coker} \iota$ and $\operatorname{coker} \epsilon$ are finite locally free over R/p^n;

(3) $\dot{F}(Q)$ generates P;

(4) the map $\operatorname{coker} \epsilon \to P/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P$ induced by ι is injective.

Denote by $\mathsf{Pxl}_n(R)$ the full subcategory of $\mathsf{PrePxl}(R)$ spanned by the pixels of level n.

Remark 156. Suppose $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a prepixel over R where P is finite locally free over $\mathbb{\Delta}_R/p^n$ (possibly $n = \infty$), such that the condition (4) in the definition of a pixel is also verified. Lemma 154 shows that the map $\mathcal{N}^{\geq 1} \otimes_{\mathbb{\Delta}_R} P \to P$ is injective. Then ϵ has to be injective. This, together with the injectivity of the map $\operatorname{coker} \epsilon \to P/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P$, implies that $\iota : Q \to P$ is injective. Indeed, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}^{\geq 1}\mathbb{\Delta}_R P & \xrightarrow{\epsilon} & Q & \longrightarrow & Q/\mathrm{im}(\epsilon) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{N}^{\geq 1}\mathbb{\Delta}_R P & \longrightarrow & P & \longrightarrow & P/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P & \longrightarrow & 0 \end{array}$$

We conclude from the snake lemma that $\iota : Q \to P$ is injective.

We identify Q with a submodule of P via ι in the following.

Remark 157. This remark is adapted from [AL23], Remark 4.1.7.

Suppose that $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a prepixel over R. Suppose further that P is finite locally free over either $\mathbb{\Delta}_R$ or $\mathbb{\Delta}_R/p^n$, and the cokernel of $\varphi^*(P) \xrightarrow{F} P$ is killed by d.

Note that p is in the Jacobson radical of $\mathbb{\Delta}_R$ and φ is a Frobenius lift, so all the maximal ideals $\mathfrak{m} \subset \mathbb{\Delta}_R$ are fixed by φ. Then $\varphi^*(P)$ is also finite locally free, and it has the same rank as P at all closed points As $F : \varphi^*(P) \to P$ is surjective after inverting d, it is actually an isomorphism after inverting d since P is finite locally free. Then $F : \varphi^*(P) \to P$ is injective as P is d-torsion free. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \varphi^*(Q) & \xrightarrow{\dot{F}} & P & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d} & & \downarrow & & \\ 0 & \longrightarrow & \varphi^*(P) & \xrightarrow{F} & P & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

where K, L are defined to be the respective cokernels. The right vertical map is 0 since by assumption $dL = 0$. Then the snake lemma gives an exact sequence

$$0 \to K \to \varphi^*(P/Q) \to P/dP$$

So the condition that $\dot{F}(Q)$ generates P is equivalent to that $\varphi^*(P/Q) \xrightarrow{F} P/dP$ is injective. But P/Q is an $R \simeq \Delta_R/\mathcal{N}^{\geq 1}\Delta_R$-module, so

$$\varphi^*(P/Q) \simeq P/Q \otimes_R R \otimes_{\Delta_R,\varphi} \Delta_R \simeq P/Q \otimes_R \Delta_R/d$$

To summerize, $\dot{F}(Q)$ generates P if and only if the map

$$P/Q \otimes_R \Delta_R/d \to P/dP$$

induced by F is injective.

Pixels have a normal decomposition that is similar to the normal decomposition of truncated displays in [LZ18]. We need a few lemmas.

Lemma 158. *The pair $(\Delta_{R,n}, \mathcal{N}_{R,n})$ is henselian.*

Proof. Note that by [AL23], Lemma 4.1.28, the pair $(\Delta_R, \mathcal{N}^{\geq 1}\Delta_R)$ is henselian. Also we have that $R \simeq \Delta_R/\mathcal{N}^{\geq 1}\Delta_R$ is classically p-complete. Thus [Sta18, Tag 0DYD] implies that

$$(\Delta_R/p^n, (p^n, \mathcal{N}^{\geq 1}\Delta_R)/p^n\Delta_R)$$

is henselian. The latter ideal agrees with $\mathcal{N}_{R,n}$. □

The lemma includes the fact that the ideal $\mathcal{N}_{R,n}$ lies in the Jacobson radical of $\Delta_{R,n}$.

Proposition 159. *Suppose $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n over R. There are two finite locally free $\Delta_{R,n}$-modules L, T and isomorphisms $P \simeq L \oplus T$, $Q \simeq (L \otimes_{\Delta_{R,n}} \mathcal{N}_{R,n}) \oplus T$ such that the map ι identifies with $\kappa_L \oplus \mathrm{id}$ and ϵ identifies with $\mathrm{id} \oplus \kappa_T$, where κ_L and κ_T are the action map of $\mathcal{N}_{R,n}$ on L, T respectively. Such a decomposition is called a* normal decomposition.

Once a decomposition $P \simeq L \oplus T, Q \simeq (\mathcal{N}_{R,n} \otimes_{\Delta_{R,n}} L) \oplus T$ is fixed, choices of $\dot{F}, F$ on (P, Q, ι, ϵ) as in the structure of a pixel are in bijection with φ-linear isomorphisms (i.e. φ-linear maps which are isomorphisms after linearization) $\Phi : L \oplus T \to L \oplus T$.

Proof. This proof is adapted from [AL23], Proposition 4.1.22 and Proposition 1.3 of [LZ18].

By assumption $Q/\mathcal{N}_{R,n}P$ lies in the exact sequence

$$0 \to Q/\mathcal{N}_{R,n}P \to P/\mathcal{N}_{R,n}P \to P/Q \to 0$$

As P/Q is finite locally free over R/p^n, $P/\mathcal{N}_{R,n}P \simeq P/Q \oplus T'$, where T' is the image of $Q/\mathcal{N}_{R,n}P$ in $P/\mathcal{N}_{R,n}P$. (In fact T' is isomorphic to $Q/\mathcal{N}_{R,n}P$.) Since $\Delta_{R,n} \to R/p^n$ is henselian, there is some finite locally free $\Delta_{R,n}$-module T whose reduction modulo $\mathcal{N}_{R,n}$ is T'. Let L be a finite locally free $\Delta_{R,n}$-module lifting P/Q. Then the map $P \to P/\mathcal{N}_{R,n}P \to P/\iota(Q) \oplus T'$ lifts along the surjection $L \oplus T \to P/\iota(Q) \oplus T'$ since P is projective. The map $P \to L \oplus T$ is surjective since it is surjective modulo $\mathcal{N}^{\geq 1}\Delta_R$, which lies in the Jacobson radical of Δ_R.

Moreover, the $\mathbb{\Delta}_{R,n}$-modules $P, L \oplus T$ have the same rank at all closed points. Thus $P \to L \oplus T$ is an isomorphism.

The composition $Q \xrightarrow{\iota} P \simeq L \oplus T \to T$ equals $Q/\mathcal{N}_{R,n}P \to T'$ after modulo $\mathcal{N}_{R,n}$, and is thus surjective. then there is a map $T \to Q$ such that $T \to Q \xrightarrow{\iota} P \simeq L \oplus T \to T$ is the identity. Consider the map $(\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R,n}} L) \oplus T \to Q \to P \simeq L \oplus T$. The composition equals the map $\kappa_L \oplus \mathrm{id} : (\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R,n}} L) \oplus T \to L \oplus T$. We have a commutative diagram with exact rows (unadorned tensor products are over $\mathbb{\Delta}_{R,n}$)

$$\begin{array}{ccccccccc}
 & & \mathcal{N}_{R,n} \otimes (L \oplus T) & \longrightarrow & \mathcal{N}_{R,n} \otimes L \oplus T & \longrightarrow & T/\mathcal{N}_{R,n}T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{im}(\epsilon) & \longrightarrow & Q & \longrightarrow & Q/\mathcal{N}_{R,n}P & \longrightarrow & 0
\end{array}$$

where the left and right vertical maps are surjective. Thus the map $\mathcal{N}_{R,n} \otimes L \oplus T \to Q$ is surjective.

It remains to check that the map $(\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R,n}} L) \oplus T \to Q$ is injective. As R is p-torsion free, $\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R,n}} L \to L$ is injective by Remark 156, and consequently $(\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R/p^n}} L) \oplus T \to P$ is injective. This implies that $(\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_{R,n}} L) \oplus T \to Q$ is injective.

Finally we show the bijective correspondence of the Frobenius structures. Fix (P, Q, ι, ϵ) part of a pixel and L, T a normal decomposition.

Suppose we have $\dot{F}, F$ on (P, Q, ι, ϵ), then we may define Φ to be the direct sum of the φ-linear maps $L \to P \xrightarrow{F} P$ and $T \to Q \xrightarrow{\dot{F}} P$. Φ is surjective after linearization since the composition $(\mathcal{N}_{R,n} \otimes_{\mathbb{\Delta}_R} L) \oplus T \to L \oplus T \xrightarrow{\Phi} P$ agrees with $\dot{F}'$ which is surjective after linearization. Since $\varphi^*(L \oplus T), L \oplus T$ are finite locally free modules of the same rank, Φ is an isomophism.

Conversely, suppose we have a φ-linear isomorphism $\Phi : L \oplus T \to L \oplus T$. We denote its restriction to L, T by Φ_L, Φ_T respectively. Then we may define $\dot{F}$ to be $(\varphi_1 \otimes \Phi_L) \oplus \Phi_T$ and F to be $\Phi_L \oplus d\Phi_T$. The image of $\dot{F}$ generates $L \oplus T$ since Φ is a φ-linear isomorphism and $\varphi^*(\mathcal{N}_{R,n} \otimes L) \to \varphi^*(L)$ induced by φ_1 is surjective. Therefore $\dot{F}, F$ defines a structure of pixel on (P, Q, ι, ϵ). □

Remark 160. If $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n, then the module Q and the map $\iota : Q \to P$ are determined by P and F. In fact, ι identifies Q with the submodule $F^{-1}(dP)$. This is a combination of Lemma 82 and [AL23], Lemma 4.1.23.

Thus to specify a pixel of level n, it is enough to give a φ-module over $\mathbb{\Delta}_R/p^n$ that satisfies certain conditions. To be precise, let $\mathcal{C}(R)$ denote the category of pairs (M, φ_M) where M is a finite locally free $\mathbb{\Delta}_R/p^n$-module and $\varphi_M : M \to M$ is a φ-linear endomorphism satisfying that $M/\varphi_M^{-1}(dM)$ is finite locally free over R/p^n and the map $(M/\varphi_M^{-1}(dM)) \otimes_{R/p^n} \mathbb{\Delta}_R/(d, p^n) \to M/dM$ is injective.

We show that $\mathcal{C}(R)$ is equivalent to $\mathsf{Pxl}_n(R)$. Suppose that $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n. Then $\iota : Q \to P$ is the inclusion $F^{-1}(dP) \to P$. And Remark 157 implies that the map $(P/F^{-1}(dP)) \otimes_{R/p^n} \mathbb{A}_R/(d, p^n) \to P/dP$ is injective. Thus the forgetful functor $(P, Q, \iota, \epsilon, \dot{F}, F) \mapsto (P, F)$ sends a pixel of level n to an object of $\mathcal{C}(R)$. Conversely, if (M, φ_M) is an object of $\mathcal{C}(R)$, then the same Remarks implie that $(M, \varphi_M^{-1}(dM), \iota_M, \epsilon_M, \varphi_M/d, \varphi_M)$ is a pixel of level n, where ι_M is the inclusion map $\varphi_M^{-1}(dM) \to M$ and ϵ_M is the action map. The construction is inverse to the forgetful functor. Therefore the forgetful functor induces an equivalence of categories.

We use the category of prepixels, instead of just φ-modules, because the category of prepixels naturally induces the exact structure of all pixels (of all different levels) which we will use in Lemma 169.

Remark 161. In view of [Hen20], Remark 3.10 and Lemma 3.12 and the normal decomposition of pixels, the category of pixels of level n over a p-torsion free perfectoid ring R is equivalent to $\mathrm{BK}_n(R)$, which was shown in [Hen20] to be (contravariant) equivalent to the category of BT_n's.

We can now discuss the base change of pixels. Let $R \to R'$ be a quasi-syntomic map between quasiregular semiperfectoid ring, and $\mathbb{A}_R \to \mathbb{A}_{R'}$ be the map of prisms obtained from the universal property of $\mathbb{A}_R$. Lemma 22 implies that R' is also p-torsion free. Note, however, that the map $\mathbb{A}_R \to \mathbb{A}_{R'}$ is not neccessarily (p, d)-completely faithfully flat.

Definition 162. Suppose $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n over R. Let $P \simeq L \oplus T$ and $Q \simeq (\mathcal{N}_{R,n} \otimes_{\mathbb{A}_R} L) \oplus T$ be a normal decomposition and Φ the φ-linear map on $L \oplus T$ as in Proposition 159. Let L', T' be the tensor product of L, T with $\mathbb{A}_{R'}$ respectively, and Φ' the base change of Φ. Then we have a pixel $(P', Q', \iota', \epsilon', \dot{F}', F')$ of level n over R' of which L', T' being a normal decomposition and Φ' corresponds to $\dot{F}', F'$ under the bijection in Proposition 159. We call $(P', Q', \iota', \epsilon', \dot{F}', F')$ the base change of $(P, Q, \iota, \epsilon, \dot{F}, F)$ to R'.

Remark 163. We need to verify that the base change defined using a normal decomposition is independant of the choice of a normal decomposition.

Under the equivalence in Remark 160, we verify that the base change of pixels agrees with the functor $(M, \varphi_M) \mapsto (M \otimes_{\mathbb{A}_R} \mathbb{A}_{R'}, \varphi_M \otimes_{\mathbb{A}_R} \mathbb{A}_{R'})$ from $\mathcal{C}(R)$ to $\mathcal{C}(R')$. Note that this statement implies that if (M, φ_M) is an object of $\mathcal{C}(R)$, then

$$(M \otimes_{\mathbb{A}_{R,n}} \mathbb{A}_{R',n}, \varphi_M \otimes_{\mathbb{A}_{R,n}} \mathbb{A}_{R',n})$$

is an object of $\mathcal{C}(R')$. Suppose $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n and $P \simeq L \oplus T, Q \simeq (\mathcal{N}_{R,n} \otimes_{\mathbb{A}_{R,n}} L) \oplus T, \Phi : P \to P$ is a normal decomposition. By definition the base change to R' is $(P', Q', \iota', \epsilon', \dot{F}', F')$, and $P' \simeq P \otimes_{\mathbb{A}_{R,n}} \mathbb{A}_{R',n}$. Moreover, F' is constructed as $\Phi'|_L \oplus d\, \Phi'|_T$, where Φ' is $\Phi \otimes_{\mathbb{A}_{R,n}} \mathbb{A}_{R',n}$. After taking the forgetful functor, we get the desired formula for the base change.

Proposition 164. *Let* $\mathsf{QRSP}(R)$ *be the opposite of the category of quasiregular semiperfectoid* R*-algebras that are quasi-syntomic over* R*, equipped with the* p*-ind-étale topology. Then the pixels of level* n *form a stack on* $\mathsf{QRSP}(R)$*.*

Proof. If S lies in $\mathsf{QRSP}(R)$, then $R \to S$ is p-completely flat while S is derived p-complete. Thus S is also p-torsion free by Lemma 22. If $S \to S'$ is a p-ind-étale cover in $\mathsf{QRSP}(R)$, then $\Delta_S \to \Delta_{S'}$ is (p,d)-completely faithfully flat by Proposition 89. Note that each Δ_S/p^i is d-torsion free, thus Δ_S/p^n has tame $\{p,d\}^\infty$-torsion. Then Proposition 30 shows that the category of finite locally free modules over Δ_S/p^n is equivalent to that of descent datum of finite locally free modules for the cover $\Delta_S/p^n \to \Delta_{S'}/p^n$.

But a pixel of level n can be represented in a base change compatible way by two finite locally free $\Delta_{R,n}$-modules L, T, 4 maps $L \to L, T \to T, L \to T, T \to L$ and an isomorphism $\varphi^*(L \oplus T) \to L \oplus T$. The module $\varphi^*(L \oplus T)$ is also finite locally free. Then it is clear that pixels form a stack on $\mathsf{QRSP}(R)$. □

Proposition 165. *There is a base change compatible equivalence*

$$\Upsilon : \mathrm{DM}^{\mathrm{adm}}(R) \to \mathsf{Pxl}_\infty(R)$$

Proof. This is the level ∞ case of the equivalence in Remark 163. □

Remark 166. From the proof of the comparison between pixels of level ∞ and windows we can see that the condition $\operatorname{coker} \epsilon$ being finite locally free over R in the definition of a pixel of level ∞ follows from the others.

Proposition 167. *Let $n \geq 1$ be an integer and $(P, Q, \iota, \epsilon, \dot{F}, F)$ a pixel of level ∞, then $(P', Q', \iota', \epsilon', \dot{F}', F')$ is a pixel of level n, where $P' = P/p^nP$, $Q = Q/p^nQ$ and the structural maps are induced from those on P, Q. This construction defines a truncation functor $\tau_n : \mathsf{Pxl}_\infty(R) \to \mathsf{Pxl}_n(R)$.*

Proof. It is clear that P/p^nP is finite locally free over $\Delta_{R,n}$, and

$$\operatorname{coker}(\iota \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \simeq (\operatorname{coker} \iota) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}, \operatorname{coker}(\epsilon \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \simeq (\operatorname{coker} \epsilon) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$$

are finite locally free over R/p^n. That $\dot{F}(Q/p^nQ)$ generates P/p^nP follows from the same statement for P and Q. Finally we show that $Q/(p^nQ + \mathrm{im}(\epsilon)) \to P/(p^nP + \mathcal{N}^{\geq 1}\Delta_R P)$ is injective. Apply $- \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ to the exact sequence

$$0 \to Q/\mathrm{im}(\epsilon) \to P/\mathcal{N}^{\geq 1}\Delta_R P \to P/Q \to 0$$

Note that P/Q is finite locally free over R, and thus p-torsion free. So

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z}, P/Q) = 0$$

, and $Q/(p^nQ + \mathrm{im}(\epsilon)) \to P/(p^nP + \mathcal{N}^{\geq 1}\Delta_R P)$ is injective. □

Proposition 168. *The truncation functor from $\mathsf{Pxl}_\infty(R)$ to $\mathsf{Pxl}_n(R)$ is essentially surjective.*

Proof. If $(P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level n, then we have a normal decomposition $P \simeq L \oplus T, Q \simeq (\mathcal{N}_{R,n} \otimes_{\mathbb{A}_{R,n}} L) \oplus T$ and $\Phi : L \oplus T \to L \oplus T$ determined by $\dot{F}, F$ as in Proposition 159. Then L, T can be lifted to finite locally free $\mathbb{A}_R$-modules $\tilde{L}, \tilde{T}$, and we can lift Φ to a φ-linear $\tilde{\Phi}$. Then we can construct a structure of a pixel of level ∞ on $\tilde{P} = \tilde{L} \oplus \tilde{T}$ and $\tilde{Q} = (\mathcal{N}^{\geq 1}\mathbb{A}_R \otimes_{\mathbb{A}_R} \tilde{L}) \oplus \tilde{T}$ using $\tilde{\Phi}$. □

Lemma 169. *If $\underline{M} = (P, Q, \iota, \epsilon, \dot{F}, F)$ is a pixel of level ∞, $\underline{M''} = (P'', Q'', \iota'', \epsilon'', \dot{F}'', F'')$ a pixels of level m for some $m \geq 1$ (possibly also $m = \infty$) and $f : \underline{M} \to \underline{M'}$ a surjective map of prepixels. Then the kernel of f is also a pixel of level ∞.*

Proof. Suppose we have an exact sequence $0 \to \underline{M'} \to \underline{M} \to \underline{M''} \to 0$ of prepixels and $\underline{M}$ is a pixel of level ∞, $\underline{M''}$ a pixel of level m. Let $\underline{M} = (P, Q, \iota, \epsilon, \dot{F}, F)$ and similarly for $\underline{M'}$ and $\underline{M''}$.

Note that $F(Q)$ (resp. $F''(Q'')$) generates dP (resp. dP''), so the cokernels of $\varphi^*(P) \xrightarrow{F} P$ and $\varphi^*(P'') \xrightarrow{F''} P''$ are killed by d. And by Remark 157, $\varphi^*(P'') \xrightarrow{F''} P''$ is injective. We have a commutative diagram of exact rows

$$\begin{array}{ccccccccc} & & \varphi^*(P') & \longrightarrow & \varphi^*(P) & \longrightarrow & \varphi^*(P'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F''} & & \\ 0 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & P'' & \longrightarrow & 0 \end{array}$$

The snake lemma then shows that the cokernel of $\varphi^*(P') \xrightarrow{F'} P'$ is also killed by d.

Since P is finite locally free over $\mathbb{A}_R$, $P/p^m P$ is finite locally free over $\mathbb{A}_R/p^m$. The map $P/p^m P \to P''$ is then a surjection between finite locally free $\mathbb{A}_R/p^m$-modules. Thus P'' is a direct summand of $P/p^m P$. Suppose $P/p^m P \simeq P'' \oplus K$. Since $(\mathbb{A}_R, (p^m))$ is a henselian pair, there are finite locally free $\mathbb{A}_R$-modules L, M lifting P'', K. The map $P/p^m P \to P'' \oplus K$ lifts to a map $P \to L \oplus M$, which is surjective by Nakayama and thus an isomorphism. Under this isomorphism the kernel of $P \to P''$ identified with $p^m L \oplus M$. But $\mathbb{A}_R$ is p-torsion free, so $P' \simeq p^m L \oplus M$ is a finite locally free $\mathbb{A}_R$-module. The same reasoning applies to $P/Q \to P''/Q''$, implying that P'/Q' is finite locally free over R.

To check that $\dot{F}(Q')$ generates P', we may apply Remark 157 to reduce to showing that $P'/Q' \otimes_R \mathbb{A}_R/d \to P'/dP'$ is injective. But $P'/Q' \otimes_R \mathbb{A}_R/d \to P/Q \otimes_R \mathbb{A}_R/d$ is injective since P''/Q'' is finite locally free over R. Thus injectivity of $P'/Q' \otimes_R \mathbb{A}_R/d \to P'/dP'$ follows from that of $P/Q \otimes_R \mathbb{A}_R/d \to P/dP$.

Finally the injectivity of $\operatorname{coker} \epsilon' \to P'/\mathcal{N}^{\geq 1}\mathbb{A}_R P'$ follows from that of $\operatorname{coker} \epsilon \to P/\mathcal{N}^{\geq 1}\mathbb{A}_R P$ since $\operatorname{coker} \epsilon' \to \operatorname{coker} \epsilon$ is injective. □

Lemma 170. $\mathsf{Pxl}_\infty(R)$ *is closed under extensions in* $\mathsf{PrePxl}(R)$.

Proof. Suppose we have an exact sequence $0 \to \underline{M'} \to \underline{M} \to \underline{M''} \to 0$ of prepixels and $\underline{M'}, \underline{M''}$ are pixels of level ∞. Let $\underline{M} = (P, Q, \iota, \epsilon, \dot{F}, F)$ and similarly for $\underline{M}'$ and $\underline{M}''$. Then clearly $P \simeq P' \oplus P''$ (resp. $P/Q \simeq P'/Q' \oplus P''/Q''$) is finite locally free over $\mathbb{\Delta}_R$ (resp. R). Suppose $t \in P$ arbitrary element. Then the image of t in P'' equals $\sum_i \alpha_i \dot{F}''(u_i)$ for some $\alpha_i \in \mathbb{\Delta}_R$ and $u_i \in Q''$. The map $Q \to Q''$ is surjective by assumption, so each u_i can be lifted to some $v_i \in Q$. Now $\sum_i \alpha_i \dot{F}(v_i) - t$ is mapped to 0 in P'', so it lies in the image of P'. But then $\sum_i \alpha_i \dot{F}(v_i) - t = \sum_j \beta_j \dot{F}'(w_j)$ for some $\beta_j \in \mathbb{\Delta}_R$ and w_j in Q'. So $t = \sum_i \alpha_i \dot{F}(v_i) - \sum_j \beta_j \dot{F}(w_j)$. This shows that P is generated by $\dot{F}(Q)$. Finally we have exact sequences

$$
\begin{aligned}
0 \to Q'/\mathrm{im}(\epsilon') \to &Q/\mathrm{im}(\epsilon) \to Q''/\mathrm{im}(\epsilon'') \to 0\\
0 \to P'/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P' \to &P/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P \to P''/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R P'' \to 0
\end{aligned}
$$

Then the 3×3 lemma shows that

$$0 \to Q/\mathrm{im}(\epsilon) \to P/\mathcal{N}^{\geq 1}\mathbb{\Delta}_R \to P/Q \to 0$$

is exact. □

Lemma 170 shows that $\mathsf{Pxl}_\infty(R)$ is a Quillen exact category. Thus it makes sense to talk about $D^b(\mathsf{Pxl}_\infty(R))$.

Proposition 171. *For $n \geq 1$ an integer, the category $\mathsf{Pxl}_n(R)$, embeds fully faithfully into $D^b(\mathsf{Pxl}_\infty(R))$ via $\tau_n\underline{M} \mapsto (\underline{M} \xrightarrow{p^n} \underline{M})$.*

Proof. We verify the conditions of Proposition 41 for our situation, taking the opposite category everywhere. $\mathsf{Pxl}_\infty(R)$ is idempotent complete since it is anti-equivalent to $p-\mathrm{div}(R)$. If $\underline{M} \to \underline{M}'$ is a surjection of prepixels such that either of the following two conditions are true:

- $\underline{M}, \underline{M}'$ lie in $\mathsf{Pxl}_\infty(R)$
- $\underline{M}$ lies in $\mathsf{Pxl}_\infty(R)$ and $\underline{M}'$ lies in $\mathsf{Pxl}_n(R)$ for some $n \geq 1$

Then Proposition 169 implies that the kernel is a pixel of level ∞. Finally pixels in $\mathsf{Pxl}_n(R)$ are cokernels of pixels of level ∞ by Proposition 168. □

5.4 Truncated Dieudonné theory

In this subsection we state and prove the main theorem of this article.

Proposition 172. *Suppose that R is a p-torsion free quasiregular semiperfectoid ring. The prismatic Dieudonné functor induces a contravariant fully faithful embedding from the category of standard BT_n's to $\mathsf{Pxl}_n(R)$ for every n. The equivalence is compatible with quasi-syntomic base change.*

Proof. By the Propositions 139, 150 and 142, the contravariant functor

$$p - \mathrm{div}(R) \xrightarrow{\mathcal{M}_{\mathbb{\Delta}}(-)} \mathrm{DM}^{\mathrm{adm}}(R) \to \mathsf{Pxl}_\infty(R)$$

is exact and reflects exactness, where the exact structure on $\mathsf{Pxl}_\infty(R)$ is induced from the abelian category $\mathsf{PrePxl}(R)$. Thus we have a contravariant equivalence of bounded derived categories

$$D^b(p-\mathrm{div}(R)) \to D^b(\mathsf{Pxl}_\infty(R))$$

Moreover the equivalence is compatible with the termwise quasi-syntomic base change of the complexes by Proposition 165.

Let $G = H[p^n]$ be a standard BT_n over R, where H is a p-divisible group. Then G is embedded into $D^b(p-\mathrm{div}(R))$ as the complex $H \xrightarrow{p^n} H$. This complex is taken by the prismatic Dieudonné functor to $M \xrightarrow{p^n} M$, where M is the pixel of level ∞ corresponding to $M_{\mathbb{\Delta}}(H)$ under the equivalence in Proposition 165. But this is exactly the embedding of $\tau_n M$ into $D^b(\mathsf{Pxl}_\infty(R))$ in Proposition 171. Thus the functor $G \mapsto \tau_n M$ is fully faithful. The embedding of a pixel of level n into $D^b(\mathsf{Pxl}_\infty(R))$ is compatible with base change, so the overall functor $G \mapsto \tau_n M$ is compatible with quasi-syntomic base change. □

Theorem 173. *Suppose that R is a p-torsion free quasiregular semiperfectoid ring. The prismatic Dieudonné functor induces a contravariant equivalence from the category of BT_n's over R to pixels of level n over R. The equivalence is compatible with quasi-syntomic base changes between p-torsion free quasiregular semiperfectoid rings.*

Proof. We use the contravariant equivalence in Proposition 172.

Proposition 164 shows that pixels of level n satisfy p-ind-étale descent, and Proposition 124 shows that BT_n's satisfy p-completely fpqc descent.

Let G be a BT_n over R. By Corollary 130, there is a p-ind-étale cover $R \to R'$ such that the base change of G to R' is standard. We denote by D the canonical descent datum obtained from G for the cover $R \to R'$. Then D is mapped to a descent datum D' of pixels of level n associated to $R \to R'$. Since pixels satisfy p-ind-étale descent, the descent datum D' is effective, giving a pixel M of level n over R. Then $G \mapsto M$ is a functor from BT_n to $\mathsf{Pxl}_n(R)$. Since both BT_n's and pixels of level n satisfy p-ind-étale descent, and the functor mapping standard BT_n to pixels is fully faithful, we conclude that $G \mapsto M$ is fully faithful.

The functor is essentially surjective since by Proposition 168, all the pixels of level n can be obatined as a truncation of a pixel of level ∞, and pixels of level ∞ are in equivalence with p-divisible groups.

That the equivalence is compatible with quasi-syntomic base changes follows from the same fact for the equivalence in Proposition 172. □

Corollary 174. *All BT_n over a p-torsion free quasiregular semiperfectoid ring R are standard.*

Proof. This is a combination of the classification and Proposition 168. □

The passing from the quasiregular perfectoid case to the quasi-syntomic case is achieved by descent.

Definition 175. Suppose that R is a p-torsion free quasi-syntomic ring. Let $\mathrm{QRSP}(R)$ be the category of objects $R \to R'$ in $(R)_{\mathrm{qsyn}}$ such that R' is a quasiregular semiperfectoid ring, with morphisms $R' \to R''$ being quasi-syntomic ring maps between R and R'. Note that all objects in $\mathrm{QRSP}(R)$ are p-torsion free.

A *crystal of pixels* of level n over R is an association $M : S \mapsto M_S$ where S is an object of $\mathrm{QRSP}(R)$ and M_S is an object of $\mathsf{Pxl}_n(S)$, such that if $S' \to S$ is a morphism in $\mathrm{QRSP}(R)$ corresponding to the ring map $R \to R'$, then $M_{S'}$ is the base change of M_S along $S \to S'$.

A morphism $f : M \to N$ between two crystals of pixels of level n is defined to be a system of morphisms $f_S : M_S \to N_S$ of pixels for S ranging in $\mathrm{QRSP}(R)$ such that for all morphisms $S \to S'$ in $\mathrm{QRSP}(R)$, $f_{S'}$ is the base change of f_S to S'.

Remark 176. If R is a p-torsion free quasiregular semiperfectoid ring, and $n \geq 1$, then evaluating at the ring R itself gives an equivalence between crystals of pixels of level n over R to pixels of level n over R.

Theorem 177. *Suppose that R is a p-torsion free quasi-syntomic ring. The prismatic Dieudonné functor induces a contravariant equivalence between the category of truncated BT_n over R to that of the crystals of pixels of level n over R.*

Proof. We send G a BT_n over R to the crystal $S \mapsto M_S$, where M_S is the pixel over S associated to the BT_n over S the base change of G to S. If $f : G \to H$ is a morphism of BT_n, then f induces morphisms $f_S : H_S \to G_S$, where G_S, H_S are the base changes of G, H to S respectively. We send f to the morphisms of pixels corresponding to f_S for all S. As the equivalence in Theorem 173 is compatible with quasi-syntomic base changes, $f_{S'}$ is the base change of f_S for all morphisms $S \to S'$ in $\mathrm{QRSP}(R)$.

Conversely, given a crystal of pixels over R, we can find a quasi-syntomic cover $S \to R$ corresponding to the ring map $R \to S$, where S is quasiregular semiperfectoid. Then $(S \otimes^{\mathbb{L}}_R S)^\wedge_p$ is also quasiregular semiperfectoid by [BMS19], Lemma 4.30. As S and $(S \otimes^{\mathbb{L}}_R S)^\wedge_p$ are derived p-complete and p-completely flat over R, they are also p-torsion free by 22. Theorem 173 then gives a descent datum of BT_n with respect to the cover $S \to R$. The descent datum is effective since BT_n satisfy p-completely fpqc descent. Thus we obtain a BT_n over R. Again by quasi-syntomic descent we see that the BT_n constructed is independant of the choice of S.

Finally we show that the functor is fully faithful. Let G, H be two BT_n. If two morphisms $f, g : G \to H$ induces the same morphism of crystals of pixels, then $f_S = g_S$ for all quasiregular semiperfectoid R-algebra lying in $(R)_{\mathrm{qsyn}}$. But then $f = g$ since BT_n form a stack. Now suppose that we have a morphism F from the crystal of pixels assoticted to G to that of H. Then we have a system of morphisms $f_S : G_S \to H_S$ for all objects in $\mathrm{QRSP}(R)$. Choose some quasiregular semiperfectoid R-algebra S such that $R \to S$ is a quasi-syntomic cover. Again by the sheaf property of morphisms between BT_n, we have a morphism $F : G \to H$ inducing f_S. If S' is another object in $\mathrm{QRSP}(R)$, then $(S \otimes^{\mathbb{L}}_R S')^\wedge_p$ is an object of $\mathrm{QRSP}(R)$, and $(S \otimes^{\mathbb{L}}_R S')^\wedge_p$ covers S'. The morphism $f_{(S\otimes^{\mathbb{L}}_R S')^\wedge_p}$ is the base change of f_S to $(S \otimes^{\mathbb{L}}_R S')^\wedge_p$, so the same as $F_{(S\otimes^{\mathbb{L}}_R S')^\wedge_p}$. We know that the base change of $F_{S'}$ to $(S \otimes^{\mathbb{L}}_R S')^\wedge_p$ is $F_{(S\otimes^{\mathbb{L}}_R S')^\wedge_p}$, and the map $\mathrm{Mor}(G_{S'}, H_{S'}) \to \mathrm{Mor}(G_{(S\otimes^{\mathbb{L}}_R S')^\wedge_p}, H_{(S\otimes^{\mathbb{L}}_R S')^\wedge_p})$ since S' is covered by $(S \otimes^{\mathbb{L}}_R S')^\wedge_p$. We conclude that the morphism $f_{S'}$ has to be F_S. So the morphism f between the crystals of pixels is induced by F. $\square$

Note that by Remark 160, a crystal of pixels of level n can be determined by the data of pairs (M_S, φ_S) for each $S \in \mathrm{QRSP}(R)$, where M_S is a finite locally free Δ_S/p^n-module, $\varphi_S : M_S \to M_S$ a φ-linear map, subject to certain conditions. Thus the following conjecture seems reasonable. However we are not able to prove it.

Definition 178. A *truncated prismatic Dieudonné crystal* of level n over R is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ where $\mathcal{M}$ is a $\mathcal{O}_n^{\mathrm{pris}}$-module and $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ a φ-linear map, satisfying the following conditions:

(1) The module $\mathcal{M}$ is finite locally free over $\mathcal{O}_n^{\mathrm{pris}}$.
(2) The cokernel of the linearization $\varphi^*\mathcal{M} \to \mathcal{M}$ of $\varphi_{\mathcal{M}}$ is killed by $\mathcal{I}^{\mathrm{pris}}$.
(3) The image of the composition

$$\mathcal{M} \xrightarrow{\varphi_{\mathcal{M}}} \mathcal{M} \to \mathcal{M}/\mathcal{I}^{\mathrm{pris}}\mathcal{M}$$

is a finite locally free $\mathcal{O}_n$-module $\mathcal{F}$ such that the map

$$\mathcal{F} \otimes_{\mathcal{O}_n} \mathcal{O}_n^{\mathrm{pris}}/\mathcal{I}^{\mathrm{pris}}\mathcal{O}_n^{\mathrm{pris}} \to \mathcal{M}/\mathcal{I}^{\mathrm{pris}}\mathcal{M}$$

induced by $\varphi_{\mathcal{M}}$ is injective.

The category of truncated prismatic Dieudonné crystal of level n over R is denoted by $\mathrm{DM}_n(R)$.

Conjecture 179. *Suppose that R is a p-torsion free quasi-syntomic ring. The prismatic Dieudonné functor induces a contravariant equivalence between the category of BT_n's over R and the category $\mathrm{DM}_n(R)$.*

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