Frames for measures on Cantor sets

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Abstract

Frames for measures on Cantor sets

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In this study of frames for measures on Cantor sets, we consider four measures with support contained in a Cantor set. These are the mass distribution measure, the Hausdorff measure of the appropriate dimension restricted to the Cantor set, the unique measure from Hutchinson's theorem for self-similar sets and the unique Lebesgue-Stieltjes measure with respect to the Cantor-Lebesgue function. For the ternary and quaternary Cantor sets, respectively, we show that these four measures give the same measure μ. This allows us to study frames in $L^2(\mu)$.

While the theory of frames is well developed, the literature on frames on Cantor sets is recent and limited. Central in defining frames of exponentials on Cantor sets is the set of integers (hereafter called spectrum) obtained from the Fourier transform of each measure supported on the corresponding Cantor set. After giving some background on frames, we follow the work of Jorgensen and Pedersen (1998) to find the spectrum of the mass distribution measure on the quaternary Cantor set from its Fourier transform and show that we do have an orthonormal basis, which is a special case of a frame. We also present the result of Jorgensen and Pedersen (1998) for the mass distribution measure on the ternary Cantor set, that it is not possible to have a spectrum that yields an orthonormal set of exponentials. However, this leads to the question: can we show the existence of a frame from the spectrum of the mass distribution measure on the ternary Cantor set? Recent work (for example, Lev (2018), Picioroaga and Weber (2017)) study this question but it remains an open problem.

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In closing, I would like to thank my family for their support during my studies at Concordia.

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Chapter 1

Introduction

In this study of frames for measures on Cantor sets, we consider four measures with support contained in a Cantor set. For any Borel set $A \subset \mathbb{R}$, these are:

- mass distribution measure $\mu_m(A)$
- Hausdorff measure restricted to a Cantor set C, $\mathscr{H}^s(A \cap C)$ with dimension s
- unique measure $\mu_H(A)$ from Hutchinson's theorem for self-similar sets.
- unique Lebesgue-Stieltjes measure $\mu_F(A)$ where F is the Cantor-Lebesgue function extended to $\mathbb{R}$

We prove a theorem that these four measures are equivalent. The necessary elements employed in the proof include:

- The construction process of the ternary and quaternary Cantor sets that can be either by removal of specific open intervals or by successive application of Iterated Functions System (IFS).
- The characteristics of the ternary Cantor set including its cardinality and its corresponding Cantor-Lebesgue function. Also, the construction of the quaternary Cantor set comes with its corresponding Cantor-Lebesgue function.

- The definitions and characteristics of the aforementioned four measures on these Cantor sets including for some of these, their respective relation with the corresponding Cantor-Lebesgue functions. In particular, we give
  - a more comprehensive proof than the original one [14, pp. 14-15], that Hausdorff measure of the ternary Cantor set is equal to 1 that includes some original additions and details.
  - our own proof that mass distribution measure satisfies the recursive relation that defines the measure on self-similar sets provided by Hutchinson's theorem

In studying vector spaces, basis arises to be of a notion of paramount importance. Having a basis represents an ideal tool to represent every vector in a given vector space by a linear expansion in terms of basis elements. However, that ideal tool often imposes requirements on the basis elements such as to be linearly independent and orthogonal with respect to an inner product. If additional requirements need to be satisfied, then that ideal tool becomes difficult or sometimes impossible to sustain. So, to obtain a more flexible tool, we have to relax these requirements. That is, new elements are added to the original basis to satisfy additional requirements but these new elements need not be independent and perhaps be orthogonal with respect to the inner product. This gives an "extended basis" that is called a frame and it has the same property as a basis which is that every vector in a given vector space can be represented by a linear expansion in terms of frame elements.

While the theory of frames is well developed (see [4],[5]), the literature on frames on Cantor sets is recent and limited, see [10],[11],[12],[23],[24]. Central in defining frames of exponentials on Cantor sets is the set of integers obtained from the Fourier transform of each measure supported on the corresponding Cantor set. That set of integers is hereafter called a spectrum. Jorgensen and Pedersen [20] and [21] were among the first authors to discuss the spectrum of a measure supported on the corresponding Cantor set. In this expository work, we follow some parts of their work to arrive at these spectra.

After giving some background and elements on frames, we find the spectrum of the mass distribution measure on the quaternary Cantor set from its Fourier transform and show that we do have an orthonormal basis which is a special case of a frame. The mass distribution measure on the binary Cantor set is in fact the Lebesgue measure and the spectrum of that measure leads to Fourier series which is also a frame. Next, we show that for mass distribution measure on the ternary Cantor set it is not possible to have a spectrum that yields an orthonormal set of exponentials to conclude that we cannot have a frame for the ternary Cantor set. However, this leads to the question that can we show the existence of a frame from the spectrum of the mass distribution measure on the ternary Cantor set? Recent work by Lev [23], Picioroaga and Weber [24] study this question but it remains an open problem.

1.1 Statement of originality

The original contributions of this thesis can be summarized as follows:

(a) For some Cantor set in $\mathbb{R}$, we considered four measures with support contained in Cantor set. We show in Theorem 3.0.1 that these measures give the same measure. In particular, we give our own proof that the mass distribution measure for the ternary and quaternary Cantor sets satisfies the recursive relation for Hutchinson's measure for self-similar sets. While this result may be known, we have not found it proved in the literature.

(b) The proof of Theorem 3.1.7 is based on a sketch of the proof by Falconer [14, pp. 14-15], but we provide more details to show that the Hausdorff measure of the ternary Cantor set is 1 and its dimension is $\log 2/\log 3$.

(c) We provide results from calculations that complete the characterization of the mathematical objects encountered in this work as follows.

   (i) The construction process of the ternary Cantor set easily brings the conclusion that it spreads over $[0,1/3]\cup[2/3,1]$ but the construction of the quaternary Cantor set does not lead to such straightforward conclusion. We provide calculations and a limiting process in Appendix A, showing that it spreads over $[0,1/6] \cup [1/2,2/3]$.

(ii) We provide calculations in the form of graphs that illustrate the convergence of sequence of functions giving the Cantor-Lebesgue function for the ternary (Figure 2.4) and quaternary (Figure 2.6) Cantor sets.

(iii) We provide calculations in Appendix I showing that the Fourier transform of the ternary and quaternary measures, $\hat{\mu}_3(t)$ and $\hat{\mu}_4(t)$ respectively, can be obtained from their corresponding Cantor-Lebesgue functions, F and W respectively, considered as distribution function of their corresponding measure. This represents an alternate approach to the one used by Jorgensen and Pedersen [20].

(d) In Appendix J, we extend in general form, the derivations in one dimension of the Fourier transform of the ternary and quaternary measure, μ_3 and μ_4, respectively of scale 3 and 4 respectively, to odd and even scales higher than 3 and 4 . The key element in doing so is to establish general Iterative Function Systems (IFSs) that each leads to a Cantor set $C \subset [0,1]$ of Lebesgue measure $m(C) = 0$.

Chapter 2

Construction of Cantor Sets

From the closed interval $[0,1]$, the construction of Cantor sets stems from a sequence of removals of open intervals. It results in two sequences:

1. a first sequence of $(C_n)_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$, C_n is a finite union of disjoint closed and bounded intervals.

2. a second sequence of deleted open intervals used in calculating the total length of the removed intervals.

In this chapter, we first describe the application of this construction process for creating the ternary Cantor set. While that process is in general classic, we introduce Iterated Function Systems (IFS) for constructing Cantor sets as IFS offer a formulation necessary for the topics presented later in this work.

2.1 Classic Construction

Constructing the ternary Cantor set (also called the middle-third Cantor set) calls for the removal of a proportional "middle-third" open interval in each of the closed and bounded intervals from a given construction level to obtain the next.

2.1.1 Construction of ternary Cantor set in $\mathbb{R}$

The construction process start with the closed interval $[0,1]$, labelled $C_0^{(3)}$, by removing the middle-third open interval leaving two closed intervals of equal length 1/3 with their union labelled $C_1^{(3)}$:

$$C_1^{(3)} = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \tag{2.1}$$

Removing the middle-third open interval means the removal of an open interval of length 1/3 from the middle of the closed interval $[0,1]$. It should be noted that 1/3 represents a proportion of the closed interval $[0,1]$ of length equal to 1. Also, observe the pattern where from construction level 0, $C_0^{(3)}$ is the "parent" interval and the removal of the middle-third open interval creates two "child" closed intervals at construction Level 1 in a union as given in eq. (2.1).

The construction continues by the removal of the middle-third open interval from each of the closed intervals in the union $C_1^{(3)}$. These removals amount to removing a proportion of $1/3$ of each closed interval of length 1/3 which amounts to the removal of an open interval of length 1/9. So at Level 1, in $C_1^{(3)}$ we have two "parent" intervals and the removals create at Level 2, four identical "child" closed intervals, two for each "parent" interval. These four identical "child" intervals form the union $C_2^{(3)}$:

$$C_2^{(3)} = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \tag{2.2}$$

Next, we reach Level 3 from Level 2 by the removal of the middle-third open interval of each of the "parent" intervals in $C_2^{(3)}$ that creates two "child" intervals. These removals create at Level 3, eight identical "child" closed intervals that form the union $C_3^{(3)}$:

$$C_3^{(3)} = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right] \tag{2.3}$$

Observe from the construction of these three levels that in the creation of two "child" intervals from a "parent" interval, the length of the removal represents the same proportion for each level.

This gives us an important property of the Cantor set: its self-similarity across the scales [8].

Continuing this process recursively we obtain for each $n \in \mathbb{N}$ a set C_n that is the finite union of 2^n closed intervals of length $1/3^n$. The following Figure 2.1 illustrates this proportional construction for the first three steps:

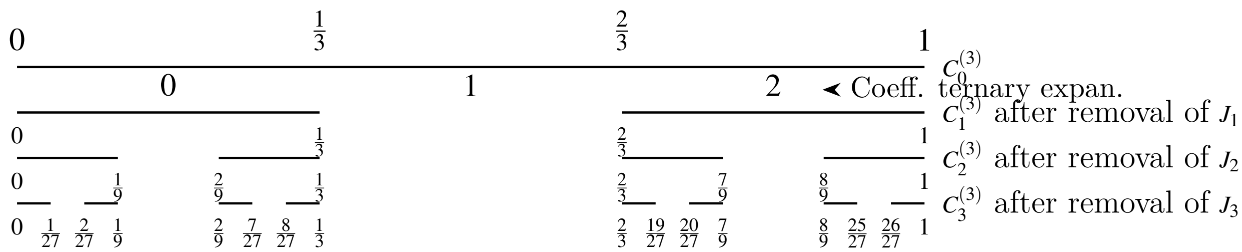

Figure 2.1: Few construction levels of the ternary Cantor set

where:

$$
\begin{aligned}
J_1 &= \left(\tfrac{1}{3}, \tfrac{2}{3}\right) \\
J_2 &= \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right) \\
J_3 &= \left(\tfrac{1}{27}, \tfrac{2}{27}\right) \cup \left(\tfrac{7}{27}, \tfrac{8}{27}\right) \cup \left(\tfrac{19}{27}, \tfrac{20}{27}\right) \cup \left(\tfrac{25}{27}, \tfrac{26}{27}\right)
\end{aligned} \tag{2.4}
$$

Figure 2.1 shows the unions $C_0^{(3)}$ to $C_3^{(3)}$ to be nested downward so $C_0^{(3)} \supset C_1^{(3)} \supset C_2^{(3)} \supset C_3^{(3)} \supset \ldots \supset C_n^{(3)} \ldots$. Moreover, each of these $C_n^{(3)}$ is closed being a finite union of disjoint closed sets (intervals). This leads to:

Definition 2.1.1 (Cantor set; $C^{(3)}$).

$$C^{(3)} = \bigcap_{n=1}^{\infty} C_n^{(3)} \tag{2.5}$$

Observe that by construction if y is the endpoint of some close subinterval of a given $C_n^{(3)}$ then it is also the endpoint of some of the subintervals of $C_{n+1}^{(3)}$. At each step of the construction, endpoints are never removed. It follows that $y \in C_n^{(3)}$ $\forall n \in \mathbb{N}$. Thus, by definition, $C^{(3)}$ contains all the endpoints generated in the construction of that Cantor set. Since arbitrary intersection of closed sets (intervals) in $\mathbb{R}$ is closed, then $C^{(3)}$ is closed in $\mathbb{R}$.

So, the question arises as what, besides the endpoints, is in the Cantor set? An answer to this question can be obtain by referring back to Figure 2.1 where the removal of:

- J_1 from $C_0^{(3)}$ gives $C_1^{(3)}$ and the Lebesgue measure of J_1 is $m(J_1) = m((1/3, 2/3)) = 1/3$
- $C_2^{(3)}$ obtained by the removal of J_2 from C_1 where $m(J_2) = 2/9 = 2^1/3^2$
- $C_3^{(3)}$ obtained by the removal of J_3 from C_2 where $m(J_3) = 4/27 = 2^2/3^3$.

Inductively $C_n^{(3)}$ is obtained by the removal of an open interval of length $1/3^n$ from each closed interval in C_{n-1} that contains 2^{n-1} closed intervals. So the removed disjoint intervals when constructing C_n from C_{n-1}, have total measure equal to $2^{n-1}/3^n$. Since the sets $\{J_n\}_{n=1}^{\infty}$ are pairwise disjoint, additivity gives:

$$m([0,1] \setminus C^{(3)}) = m(\bigcup_{n=1}^{\infty} J_n) = \sum_{n=1}^{\infty} m(J_n) = \sum_{i=n}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2}\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 1 \tag{2.6}$$

Thus, starting with the closed interval $[0,1]$ with measure $m([0,1]) = 1$, eq. (2.6) tells us the measure of all the removed intervals is 1. Then there should be nothing left of $[0,1]$. We established above that this is not the case since at each step of the construction, endpoints are never removed which gives that $C^{(3)} \neq \emptyset$. This seems at first like a paradox. However, we showed that $m([0,1] \setminus C^{(3)}) = 1$ and with $m([0,1]) = 1$ and $[0,1] \setminus C^{(3)} \cap C^{(3)} = \emptyset$ then $m(C^{(3)}) = 0$. So, continuing this construction process to the limit we are left with what, a priori, might be a “small” countable set of numbers, since the endpoints are rational numbers, which can be viewed as the “dust” left from the initial closed interval $[0,1]$ at the limit of the construction process [3, p. 26]. The next section aims at showing the Cantor set may not be such a small set after all. In fact, it concludes that both $C^{(3)}$ and $[0,1]$ have the same cardinality i.e. the Cantor set is uncountable!

2.1.2 Characteristics of the ternary Cantor Set in $\mathbb{R}$

This section presents three lemmas on bijections between the half-open set $[0,1)$ and binary and ternary expansions. We use these in a theorem on the cardinality of the ternary Cantor set: $\text{card}(C^{(3)})$. We start by defining two sets B and B^0, used throughout this section.

Definition 2.1.2. B: The set of all binary expansions $0.b_1b_2b_3\ldots$ corresponding to sequences of 0's and 1's.

Definition 2.1.3. B^0: The set of all binary expansions $0.b_1b_2b_3\ldots$ corresponding to sequences of 0's and 1's which do not have a tail of all 1's

Definition 2.1.4. $B^1 = B^0 \cup \{0.111111\ldots \text{ (base 2) } \} \subset B$.

Definition 2.1.5. T: the set of ternary expansions $0.a_1a_2a_3\ldots$ corresponding to sequences of 0's and 2's.

We then use the following bijections to establish the cardinality of the ternary Cantor set.

2.1.2.1 Bijection between $[0,1)$ and B^0

Lemma 2.1.6. There exists a map

$$\begin{aligned} f:[0,1) &\longrightarrow B^0 \\ x &\longmapsto 0.b_1b_2b_3\ldots \text{ with } b_i = 0 \text{ or } 1 \end{aligned} \tag{2.7}$$

which is a bijection. In addition, that mapping combined with the subdivison process in Figure 2.2 gives binary expansions which do not have a tail of all 1's.

Proof.

Starting with half-open interval $I^{(0)} = [0,1)$ we employ the subdivision process illustrated in Figure 2.2:

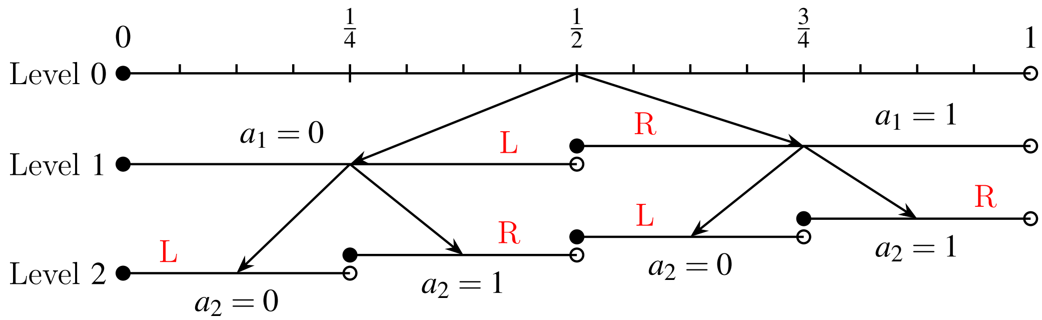

Figure 2.2: Subdivision process employed in showing bijection between $[0,1]$ and binary expansions.

We observe the subdivision process creates at each level a set of disjoint half-open intervals whose union is $[0,1)$. Let

$$s_k = 0.b_1 b_2 \ldots b_k(\text{base } 2) = \sum_{i=1}^{k} \frac{b_i}{2^i}. \tag{2.8}$$

We will use I_k below for labelling the closed interval that has s_k as its left endpoint.

Step 1: First, we use induction to show that $\forall x \in [0,1)$ we have:

$$0 \leq x - s_n < \frac{1}{2^n} \; \forall n \in \mathbb{N} \tag{2.9}$$

(a): Level $n=1$: Cut $[0,1)$ in $2^1 = 2$ pieces of equal length $1/2 = 1/2^1$. There is a unique j either equal to 0 or 1 such that $x \in I_j^{(1)} = [\frac{j}{2}, \frac{j+1}{2})$. Let $a_1 = j$ then:

$$x \in I_{s_1}^{(1)} = \left[\frac{a_1}{2}, \frac{a_1+1}{2}\right), \; \frac{a_1}{2} = s_1 \leq x \text{ and } 0 \leq x - \frac{a_1}{2} < \frac{1}{2} \tag{2.10}$$

This ensures the choice of the subinterval $I_{s_1}^{(1)}$ is unique and we have $I_{s_1}^{(1)} \subset I^{(0)} = [0,1)$. Note on symbol $I_{s_1}^{(1)}$: superscript "(1)": interval at Level 1, subscript "s_1" (see eq. (2.8)): interval at Level 1 corresponding to the value of a_1 i.e. if $a_1 = 0$ then first interval else if $a_1 = 1$ then second interval.

(b): Level $n=2$: Cut $[0,1)$ in $2^2 = 4$ pieces of equal length $1/4 = 1/2^2$. This implies that $I_{s_1}^{(1)}$ is cut in two pieces of equal length $1/4$. There is a unique j either equal to 0 or 1 such that $x \in I_j^{(1)} = [\frac{a_1}{2} + \frac{j}{2^2}, \frac{a_1}{2} + \frac{j+1}{2^2})$. Let $a_2 = j$ then:

$$\begin{aligned} s_2 &= \frac{a_1}{2} + \frac{a_2}{2^2} \\ x \in I_{s_2}^{(2)} &= \left[\frac{a_1}{2} + \frac{a_2}{2^2}, \frac{a_1}{2} + \frac{a_2+1}{2^2}\right), \ 0 \leq x - s_2 < \frac{1}{2^2} \end{aligned} \tag{2.11}$$

(c): Induction Hypothesis:

(i) Assume for $n = k$ that:

$$\begin{aligned} x \in I_{s_k}^{(k)} &= \left[\sum_{i=1}^{k-1} \frac{a_i}{2^i} + \frac{a_k}{2^k}, \sum_{i=1}^{k-1} \frac{a_i}{2^i} + \frac{a_k+1}{2^k}\right) \\ &= \left[\sum_{i=1}^{k} \frac{a_i}{2^i}, \sum_{i=1}^{k} \frac{a_i}{2^i} + \frac{1}{2^k}\right) \end{aligned} \tag{2.12}$$

(ii) with $I_{s_k}^{(k)} \subset I_{s_{k-1}}^{(k-1)}$

(iii) Also, we formulate the condition in eq. (2.12) as:

$$0 \leq x - s_k < \frac{1}{2^k} \tag{2.13}$$

with $s_k = \sum_{i=1}^{k} \frac{a_i}{2^i}$

(d): Induction Step: Let $n = k+1$. Since $x \in I_{s_k}^{(k)}$, we cut $I_{s_k}^{(k)}$ in two pieces of length $1/2^{k+1}$. There is a unique j either equal to 0 or 1 such that $x \in I_j^{(k+1)}$. Let $a_{k+1} = j$ then we have

$$x \in I_{s_{k+1}}^{(k+1)} = \left[\sum_{i=1}^{k+1} \frac{a_i}{2^i}, \sum_{i=1}^{k+1} \frac{a_i}{2^i} + \frac{1}{2^{k+1}}\right) \tag{2.14}$$

to obtain

$$0 \leq x - s_{k+1} < \frac{1}{2^{k+1}} \tag{2.15}$$

and $I_{s_{k+1}}^{(k+1)} \subset I_{s_k}^{(k)}$. Again this ensure the choice of $I_{s_{k+1}}^{(k+1)}$ is unique.

(e): Therefore

$$0 \leq x - \sum_{i=1}^{n} \frac{a_i}{2^i} < \frac{1}{2^n} \quad \forall n \in \mathbb{N} \tag{2.16}$$

Step 2: Now, let $\varepsilon > 0$, by the Archimedean Principle $\exists N \in \mathbb{N}$ such that $0 < \frac{1}{2^N} < \varepsilon$, then $\forall n \geq N$:

$$0 \leq |x - s_n| < \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon \quad \text{so} \quad |x - s_n| < \varepsilon. \tag{2.17}$$

Step 3: Since inequality (2.17) is $\forall n \geq N$ and $\varepsilon > 0$ is arbitrary, we get [6, p. 3]:

$$x = \lim_{n\to\infty} s_n = \sum_{i=1}^{\infty} \frac{a_i}{2^i} \tag{2.18}$$

Step 4: In the above, the selection of the a_n's shows that for each $x \in [0,1)$ there exists a unique sequence $\{a_1, a_2, \ldots\}$ corresponding to a binary expansion in B^0. It also shows that x is in each of these nested downward half-open intervals $I_{s_n}^{(n)}$ $\forall n \in \mathbb{N}$ giving the intersection of these is $\{x\}$. To see this:

(a) Let $x \in \bigcap_{n=1}^{\infty} I_{s_n}^{(n)}$.

(b) If $y \in \bigcap_{n=1}^{\infty} I_{s_n}^{(n)}$ then $\forall n \in \mathbb{N}$,

$$|x - y| < \ell(I_{s_n}^{(n)}) = 1/2^n \underset{n}{\rightarrow} 0. \tag{2.19}$$

(c) This implies that $x = y$ and we get that

$$\bigcap_{n=1}^{\infty} I_{s_n}^{(n)} = \{x\}. \tag{2.20}$$

(d) Therefore the construction gives a well-defined mapping from $[0,1)$ to B^0. That is for any $x \in [0,1)$ we get a unique sequence of 0's and 1's.

Step 5: Since that unique sequence corresponds to a binary expansion that gives x, the mapping $f : [0,1) \longrightarrow B^0$ is one-to-one.

Step 6: From inequality (2.16) we have that $s_n \leq x \; \forall n \in \mathbb{N}$. Therefore s_n is the left-hand side endpoint of one of the 2^n subintervals on Level n.

Step 7: Now, to show that f is onto, let $(a_n)_{n\in\mathbb{N}} \in B^0$. Consider the partial sum $s_N = \sum_{i=1}^{N} \frac{a_i}{2^i}$ then each term $\frac{a_i}{2^i} \leq \frac{1}{2^n}$ and $\{s_N\}_{N\in\mathbb{N}}$ is a monotone increasing sequence. Then:

$$s_N = \sum_{i=1}^{N} \frac{a_i}{2^i} \leq \sum_{i=1}^{N} \frac{1}{2^i} = 1 - \frac{1}{2^N} \underset{N}{\rightarrow} 1 \tag{2.21}$$

So $0 \leq s_N \leq 1 \; \forall N \in \mathbb{N}$ and $\{s_N\}_{N\in\mathbb{N}}$ is bounded and monotone increasing sequence that converges to a real number $x \in [0,1)$. Since $(a_n)_{n\in\mathbb{N}}$ is arbitrary, then for each $(a_n)_{n\in\mathbb{N}} \in B^0$ there is $x \in [0,1)$ such that $f(x) = 0.a_1a_2\ldots = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$.

Step 8: Therefore $f : [0,1) \longrightarrow B^0$ is onto and with f one-to-one, f is a bijection between $[0,1)$ and B^0.

Step 9: Let $x \in [0,1)$ and suppose for contradiction that $x = 0.b_1b_2\ldots b_k11111\ldots$ with:

(a) $b_1, b_2, b_3, \ldots, b_{k-1}$ can be either 0's or 1's

(b) $b_k = 0$ where k is the maximum value of i for which $b_i = 0$

(c) k could be 0 then the sequence would be $0.111111\ldots)_{\text{base 2}}$ which is equal to 1 but $1 \notin [0,1)$. This leads to an additional assumption that $k \geq 1$.

(d) Example: consider $x = 0.01111\ldots)_{\text{base 2}}$ where $k = 1$, $b_1 = 0$ and $b_i = 1 \; \forall i \geq 2$. So

$$x = 0.01111\ldots)_{\text{base 2}} = \sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{4} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{4}\frac{2}{1} = \frac{1}{2} \Longleftrightarrow 0.1000\ldots)_{\text{base 2}} = \frac{1}{2} \tag{2.22}$$

Step 10: By assumption $b_i = 1 \; \forall i \geq k+1$ and this implies for subsequent levels $n \geq k+1$ that the right interval will always be chosen for where x lands giving a sequence $LRRR\ldots$ giving that x does not land across the middle point p of $I_{s_{k-1}}^{(k-1)}$. In fact, x "stays" within the left $I_{s_k}^{(k)}$ but we get that

$$x = s_k + \sum_{i=k+1}^{\infty} \frac{b_i}{2^i} = s_k + \sum_{i=k+1}^{\infty} \frac{1}{2^i} = s_k + \frac{1}{2^k} \tag{2.23}$$

which contradicts the fact that $|x - s_k| < 1/2^k$. This completes the proof.

□

Remark 2.1.1. From Figure 2.2, we observe the subdivision process creates a binary tree. As we step from one level $k-1$ to the next level k, the choice $I_{s_k}^{(k)} \subset I_{s_{k-1}}^{(k-1)}$ being unique is either on the left "L" or on the right "R" half-open interval subset of $I_{s_{k-1}}^{(k-1)}$. This gives a unique path across the binary tree, "$LRLRLLRLR\ldots$" for example.

2.1.2.2 Bijection between Cantor set and ternary expansions

Lemma 2.1.7. There exists a map

$$\begin{aligned} g : C^{(3)} &\longrightarrow T \\ x &\longmapsto 0.a_1a_2a_3\ldots \text{ with } a_i = 0 \text{ or } 2 \end{aligned} \tag{2.24}$$

which is a bijection.

Proof. Let $C^{(3)}$ be the ternary Cantor set. We will identify $C^{(3)}$ with the subset T of $[0,1]$ consisting of all numbers having a ternary expansion $\sum_{i=1}^{\infty} \frac{a_i}{3^i}$ with a_i equal to either 0 or 2. To see this, consider again the proportional construction process of middle-third open interval removal of each subinterval as illustrated below for 2 steps:

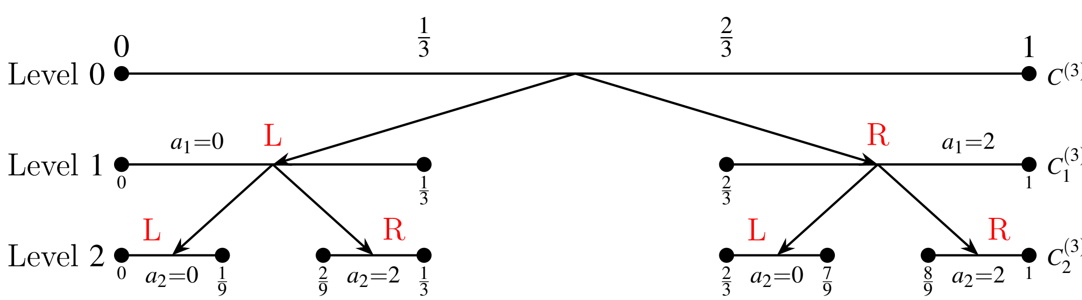

Figure 2.3: Proportional construction of ternary Cantor set by middle-third open interval removal.

Remark 2.1.2. When comparing Figure 2.3 to Figure 2.2, we observe they have in common the characteristic of being a binary tree. For $C^{(3)}$, each level k is a union of disjoint sets with a distance between them of at least $1/3^k$ and these disjoint sets are closed intervals. In particular, including the right endpoint of these closed intervals implies that some ternary expansion in 0's and 2's will have a tail of 2's. For example, $1/3$ is one of these right endpoints. It can be expressed as 0.1 (base 3) but this is not acceptable as it corresponds to the removed middle-third open interval $(1/3, 2/3)$. Instead, we have $1/3 = 0.1$ (base 3) $= 0.0\overline{2}$. For a given $x \in C^{(3)}$, stepping from one level to the next gives a unique path across the binary tree.

Let

$$t_k = 0.a_1a_2\dots a_k = \sum_{i=1}^{k} \frac{a_i}{3^i}. \tag{2.25}$$

We will use t_k below for labelling closed interval that has t_k as its left endpoint.

Step 1: First, for $x \in C^{(3)}$, $C^{(3)} \subset [0,1]$, we use induction to show that

$$0 \le x - \sum_{i=1}^{n} \frac{a_i}{3^i} \le \frac{1}{3^n} \ \forall n \in \mathbb{N} \tag{2.26}$$

(a) Level $n = 1$: If x does not belong to the middle-third open interval of the initial interval $I^{(0)} = [0,1]$, then there is a unique j equal to either 0 or 2 such that $x \in I^{(1)}_{t_1} = [\frac{j}{3}, \frac{j+1}{3}]$. Let $a_1 = j$ then:

$$x \in I^{(1)}_{t_1} = \left[\frac{a_1}{3}, \frac{a_1+1}{3}\right], \ \frac{a_1}{3} \le x \text{ and } 0 \le x - \frac{a_1}{3} \le \frac{1}{3}. \tag{2.27}$$

This ensures the choice of the subinterval $I^{(1)}_{t_1}$ is unique and we have $I^{(1)}_{t_1} \subset I^{(0)} = [0,1)$. When comparing eq. (2.27) to eq. (2.10), the last inequality is no longer strict because each level k is a union of disjoint closed sets with a distance between them of at least $1/3^k$, so, $x \in C^{(3)}$ could be equal to the right endpoint of one of these disjoint closed intervals.

(b) Induction Hypothesis:

(i) Assume for $n = k$ that:

$$x \in I_{t_k}^{(k)} = \left[\sum_{i=1}^{k-1} \frac{a_i}{3^t} + \frac{a_i}{3^i}, \sum_{i=1}^{k-1} \frac{a_i}{3^i} + \frac{a_i + 1}{3^i} \right] \\ = \left[\sum_{i=1}^{k} \frac{a_i}{3^i}, \sum_{i=1}^{k} \frac{a_i}{3^i} + \frac{1}{3^i} \right] \tag{2.28}$$

with $I_{t_k}^{(k)} \subset I_{t_{k-1}}^{(k-1)} \subset \ldots \subset I_{t_2}^{(2)} \subset I_{t_1}^{(1)} \subset I^{(0)} = [0,1]$.

(ii) Also, we formulate the condition in eq. (2.28) as:

$$0 \leq x - t_k \leq \frac{1}{3^k} \tag{2.29}$$

(c) Induction Step: Let $n = k+1$. Deletion of the middle-third open interval of $I_{t_k}^{(k)}$ implies there is a unique j equal to either 0 or 2 such that $x \in I_j^{(k+1)}$. Let $a_{k+1} = j$ then we have

$$x \in I_{t_{k+1}}^{(k+1)} = \left[t_{k+1}, t_{k+1} + \frac{1}{3^{k+1}} \right] \tag{2.30}$$

with

$$0 \leq x - t_{k+1} \leq \frac{1}{3^{k+1}} \tag{2.31}$$

and $I_{t_{k+1}}^{(k+1)} \subset I_{t_k}^{(k)}$. Again this ensure the choice of $I_{t_{k+1}}^{(k+1)}$ is unique.

(d) Therefore

$$0 \leq x - \sum_{i=1}^{n} \frac{a_i}{3^i} \leq \frac{1}{3^n} \; \forall n \in \mathbb{N} \tag{2.32}$$

Step 2: Now, let $\varepsilon > 0$, by the Archimedean Principle $\exists N \in \mathbb{N}$ such that $0 < \frac{1}{3^N} < \varepsilon$ then $\forall n \geq N$:

$$\left| x - \sum_{i=1}^{n} \frac{a_i}{3^i} \right| \leq \frac{1}{3^n} \leq \frac{1}{3^N} < \varepsilon \text{ so } \left| x - \sum_{i=1}^{n} \frac{a_i}{3^i} \right| < \varepsilon. \tag{2.33}$$

Step 3: Since inequality (2.33) is $\forall n \geq N$ and $\varepsilon > 0$ is arbitrary, we get [6, p. 3]:

$$x = \lim_{n \to \infty} t_n = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \tag{2.34}$$

Step 4: In the above, the selection of the a_n's shows that for each $x \in C^{(3)}$ there exists a unique sequence $\{a_1, a_2, \ldots\}$ corresponding to a ternary expansion in T. It also shows that x is in each of these nested downward closed remaining intervals $I_{t_n}^{(n)}$ $\forall n \in \mathbb{N}$, giving the intersection of these, is $\{x\}$. To see this:

(a) Let $x \in \bigcap_{n=1}^{\infty} I_{t_n}^{(n)}$.

(b) If $y \in \bigcap_{n=1}^{\infty} I_{t_n}^{(n)}$ then $\forall n \in \mathbb{N}$,

$$|x - y| < m(I_{s_n}^{(n)}) = 1/3^n \underset{n}{\rightarrow} 0 \tag{2.35}$$

(c) This implies that $x = y$ and we get

$$\bigcap_{n=1}^{\infty} I_{s_n}^{(n)} = \{x\}. \tag{2.36}$$

(d) Therefore the construction gives a well-defined mapping from $C^{(3)}$ to T. That is for any $x \in C^{(3)}$ we get a sequence of 0's and 2's in a unique way.

Step 5: Since that unique sequence corresponds to a ternary expansion that gives x, the mapping $g : C^{(3)} \longrightarrow T$ is one-to-one.

Step 6: Now, to show that g is onto, let $(a_n)_{n \in \mathbb{N}} \in T$ be a sequence of 0's and 2's that does not have a tail with all 2's. Consider the partial sum $t_N = \sum_{i=1}^{N} \frac{a_i}{3^i}$ then each term $\frac{a_i}{3^i} \leq \frac{2}{3^i}$ and $\{t_N\}_{N \in \mathbb{N}}$ is a monotone increasing sequence. Then:

$$t_N = \sum_{i=1}^{N} \frac{a_i}{3^i} \leq \sum_{i=1}^{N} \frac{2}{3^i} = 1 - \frac{1}{3^N} \underset{N}{\rightarrow} 1 \tag{2.37}$$

So $0 \leq t_N \leq 1 \; \forall N \in \mathbb{N}$ and $\{t_N\}_{N \in \mathbb{N}}$ is bounded and monotone increasing sequence that converges to a real number x. Since $(a_n)_{n\in\mathbb{N}}$ is arbitrary, then for each $(a_n)_{n\in\mathbb{N}} \in T$ there is $x \in C^{(3)}$ such that $g(x) = 0.a_1a_2\ldots a_k = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$.

Step 7: Therefore $g : C^{(3)} \longrightarrow T$ is onto and with g one-to-one, g is a bijection between $C^{(3)}$ and T.

□

From inequality [2.32] and eq. (2.25) we have that $t_n \leq x \leq t_n + \frac{1}{3^n} \; \forall n \in \mathbb{N}$. Therefore t_n and $t_n + \frac{1}{3^n}$ are respectively the left-hand and right-hand endpoint of one of the 2^n closed intervals on Level n.

Remark 2.1.3. The proof of Lemma 2.1.7 shows the endpoints are dense in $C^{(3)}$. That is, we showed that $x = lim_{n\to\infty} t_n$ and $x \in C^{(3)}$.

2.1.2.3 Cardinality of ternary Cantor set

Lemma 2.1.8. The cardinality of the ternary Cantor set $C^{(3)}$ is equal to the cardinality of $[0,1]$.

$$\text{card}(C^{(3)}) = \text{card}([0,1]) \tag{2.38}$$

Thus, $C^{(3)}$ is uncountable.

Proof. We establish the cardinality of ternary Cantor set as follows:

Step 1: Lemma 2.1.6 proves that $f : [0,1) \longrightarrow B^0$ is a bijection.

Step 2: Identifying the number 1 with the expansion $0.11111\ldots$ base 2, we have from Lemma 2.1.6, the bijection $[0,1] \longrightarrow B^1 = B^0 \cup \{0.11111\ldots)$ base $2 \subset B$. Given that $B^1 \subset B$, we have that

$$\text{card}(B^1) \leq \text{card}(B) \tag{2.39}$$

Step 3: Consider $h : T \longrightarrow B$, h is division by 2 which is a bijection.

Step 4: Lemma 2.1.7 proves that $g : C^{(3)} \longrightarrow T$ is a bijection. Since the composition of bijections is a bijection we can write that $h \circ g : C^{(3)} \longrightarrow B$ is a bijection. From this result and from ineq. (2.39) we obtain:

$$\begin{aligned} \mathrm{card}(C^{(3)}) = \mathrm{card}(B) &\geq \mathrm{card}(B^1) = \mathrm{card}([0,1]) \\ \mathrm{card}(C^{(3)}) &\geq \mathrm{card}[0,1] \end{aligned} \tag{2.40}$$

but $C^{(3)} \subset [0,1]$ giving that

$$\mathrm{card}(C^{(3)}) \leq \mathrm{card}([0,1]). \tag{2.41}$$

Step 5: From ineq.(2.40), ineq.(2.41) and Bernstein's Theorem [3, p. 24] we have:

$$\mathrm{card}(C^{(3)}) = \mathrm{card}([0,1]) \tag{2.42}$$

Therefore $C^{(3)}$ is uncountable.

□

2.1.2.4 Other characteristics of ternary Cantor set

Lemma 2.1.9. The ternary Cantor set is perfect.

Proof. We use the property that $C^{(3)}$ is a closed set.

Step 1: Let $x \in C^{(3)}$, $\varepsilon > 0$ and let a neighbourhood G of x be $G = (x - \varepsilon, x + \varepsilon)$.

Step 2: By the construction process of the ternary Cantor set, we have at construction level n that x is in some closed interval in $C_n^{(3)}$ for each $n \in \mathbb{N}$.

Step 3: We take $n \in \mathbb{N}$ such that $\varepsilon > \frac{1}{3^n} > 0$. This implies that $(x-\varepsilon, x+\varepsilon)$ contains the two endpoints of this closed interval in C_n.

Step 4: We get that for each $n \in \mathbb{N}$, $(x-\varepsilon, x+\varepsilon) \setminus \{x\} \cap C_n^{(3)} \neq \emptyset$ and x is an accumulation point of $C^{(3)}$.

Step 5: Since x is arbitrary then every point of $C^{(3)}$ is an accumulation point. In addition, $C^{(3)}$ is closed, so it contains all its accumulation points. Therefore $C^{(3)} = C^{(3)'}$ and $C^{(3)}$ is perfect.

□

Lemma 2.1.10. Ternary Cantor set is nowhere dense.

Proof. We use a proof by contradiction.

Step 1: Assume that $C^{(3)}$ does contain an open interval.

Step 2: Let $a, b \in [0,1]$ $a < b$ $a, b \in \mathbb{R}$ with a and b fixed. In addition let $(a,b) \subset C^{(3)}$ with length $\ell(b-a) = b-a > 0$.

Step 3: We have that $C^{(3)} = \bigcap_{n=1}^{\infty} C_n^{(3)}$ and $(a,b) \subset C^{(3)}$ gives that $\forall n$, (a,b) is contained in one of the interval of $C_n^{(3)}$. That is, an interval cannot be contained in the union of two of more separated intervals.

Step 4: This implies that $(b-a) < \frac{1}{3^n}$ and as $n \longrightarrow \infty$ we have that $\frac{1}{3^n} \longrightarrow 0$ then $(b-a) \longrightarrow 0$. Contradiction since $(b-a) > 0$. Therefore $C^{(3)}$ contains no non-empty open intervals and the interior of $C^{(3)}$, $C^{(3)^\circ} = \emptyset$ giving that $C^{(3)}$ is nowhere dense.

□

2.1.3 Cantor-Lebesgue function for ternary Cantor set

This section gives the definition of the Cantor-Lebesgue function for the ternary Cantor set using Definition 2.1.2 of B and the notation in Section 2.1.2.3. From Section 2.1.2.3, we have that the map $h \circ g : C^{(3)} \longrightarrow B$ is a bijection. We apply the map

$$\begin{aligned} g' : B &\longrightarrow [0,1] \\ 0.b_1 b_2 b_3 \ldots &\longmapsto \textstyle\sum_{i=1}^{\infty} \frac{b_i}{2^i}. \end{aligned} \tag{2.43}$$

to the map $h \circ g$ to give the definition of the Cantor-Lebesgue (C-L) function:

Definition 2.1.11. The Cantor-Lebesgue function $f : C^{(3)} \longrightarrow [0,1]$ is defined by :

$$f = g' \circ h \circ g \left(\sum_{n=1}^{\infty} \frac{2b_n}{3^n} \right) = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \ \ (b_n = \{0,1\}) \tag{2.44}$$

$$\sum_{n=1}^{\infty} \frac{2b_n}{3^n} \longmapsto \sum_{n=1}^{\infty} \frac{b_n}{2^n} \ \ (b_n = \{0,1\}) \tag{2.45}$$

Lemma 2.1.12. The domain of the Cantor-Lebesgue function $f : C^{(3)} \longrightarrow [0,1]$ can be extended to the whole interval $[0,1]$ with the extended function denoted by F.

Proof.

Let u and v be respectively the left and right endpoints of the same open interval (u,v) removed in step n in the construction of the ternary Cantor set. These endpoints have the following form:

$(u,v) = (0.a_1a_2a_3\ldots a_n1, 0.a_1a_2a_3\ldots a_n2)$ both base 3 with $a_i = 0$ or 2 for $1 \le i \le n$. In particular, u can be written as $u = 0.a_1a_2a_3\ldots a_n0\overline{2}$. Then

$$\begin{aligned} f(u) &= \textstyle\sum_{k=1}^{n} \frac{a_k}{2} \frac{1}{2^k} + \frac{0}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{1}{2^k} \\ &= \textstyle\sum_{k=1}^{n} \frac{a_k}{2} \frac{1}{2^k} + \frac{1}{2^{n+1}} \quad \text{since} \quad \sum_{k=n+2}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}} \end{aligned} \tag{2.46}$$

on the other hand

$$f(v) = \sum_{k=1}^{n} \frac{a_k}{2}\frac{1}{2^k} + \frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty}\frac{0}{2^k} = f(u) \tag{2.47}$$

We conclude that f takes on the same value on the endpoints of the removed intervals. Therefore, since the union of $C^{(3)}$ with all the removed middle third intervals is $[0,1]$, we can define the extension of $f(x)$, $F : [0,1] \longrightarrow [0,1]$ as $F(x) = f(x)$ for $x \in C^{(3)}$ and for any $y \in (u,v)$, a removed interval, $F(y) = f(u) = f(v)$. □

Lemma 2.1.13. The extended Cantor-Lebesgue function $F : [0,1] \longrightarrow [0,1]$ is continuous on $[0,1]$.

Proof. This proof requires analysing three cases:

Case (i) Let $x \notin C^{(3)}$, that is, $x \in I \subset [0,1] \setminus C^{(3)}$ where I is an open interval removed in the construction of $C^{(3)}$. By Lemma 2.1.12, F is constant on I giving that F is continuous.

Case (ii) Let $x \in C^{(3)}$ and let $\varepsilon > 0$, then by the Archimedean Principle [6, p. 2], there exist $N \in \mathbb{N}$ such that $0 < 1/2^N < \varepsilon$.

- For n sufficiently large and $n > N$ let $\delta = 1/3^n > 0$. Consider the open interval $U = (x - 1/3^n, x + 1/3^n)$.
- Let $y \in U$, if $y \in C^{(3)}$ and $|x - y| < 1/3^n$, then x and y lie in the same closed interval in the union $C_n^{(3)}$ (see Figure 2.1). So, the ternary expansions of x and y must agree for the first n terms. We can then write

$$|F(x) - F(y)| < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} < \frac{1}{2^N} < \varepsilon \tag{2.48}$$

Hence, F is continuous at $x \in C^{(3)}$.

Case (iii) Similarly, let $x \in C^{(3)}$ and let $\varepsilon > 0$ with $N \in \mathbb{N}$ such that $0 < 1/2^N < \varepsilon$ and $U = (x - 1/3^n, x + 1/3^n)$. Let $y \in U$, if $y \notin C^{(3)}$, then there exists an endpoint $z \in C^{(3)}$ close to x such that $F(y) = F(z)$ with $|x - z| < 1/3^n$. Again x and z must agree for the first

n terms. Then we can write:

$$|F(x)-F(y)| = |F(x)-F(z)| < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} < \frac{1}{2^N} < \varepsilon \tag{2.49}$$

Hence, F is continuous at $x \in C^{(3)}$.

Therefore, from Cases (i),(ii) and (iii), $F(x):[0,1] \longrightarrow [0,1]$ is continuous on $[0,1]$. □

Lemma 2.1.14. The extended Cantor-Lebesgue function $F:[0,1] \longrightarrow [0,1]$ is increasing on $[0,1]$.

Proof. First we show that F is increasing on $C^{(3)}$ and then extend that result to $[0,1]$

(i) Let $u,v \in C^{(3)}$ with $u<v$ where they are expressed by a ternary expansion $u=\sum_{k=1}^{\infty} u_k/3^k$ and $v=\sum_{k=1}^{\infty} v_k/3^k$.

(ii) u and v are in the same closed interval for each of the construction level down to level n. The closed interval may change when stepping from one level to the next but still it contains u and v until reaching some construction level $n+1$.

(iii) For construction level $n+1$, the coefficients $u_{n+1} < v_{n+1}$ and they are both in $\{0,2\}$ giving that $u_{n+1}=0$ and $v_{n+1}=2$ giving that v is in a later closed interval.

(iv) So, we can write

$$\begin{aligned} F(u) &= \textstyle\sum_{k=1}^{n} \frac{u_k}{2}\frac{1}{2^k} + \frac{0}{2}\frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{u_k}{2}\frac{1}{2^k} \\ &\leq \textstyle\sum_{k=1}^{n} \frac{v_k}{2}\frac{1}{2^k} + \frac{1}{2^{n+1}} \quad \text{since} \quad \sum_{k=n+2}^{\infty} \frac{u_k}{2}\frac{1}{2^k} < \sum_{k=n+2}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n+1}} \\ &\leq \textstyle\sum_{k=1}^{n} \frac{v_k}{2}\frac{1}{2^k} + \frac{1}{2^{n+1}} + \sum_{k=n+2}^{\infty} \frac{v_k}{2}\frac{1}{2^k} = F(y) \end{aligned} \tag{2.50}$$

To obtain that $F(x)$ is increasing on $C^{(3)}$.

(v) Let $u,v \in [0,1]$ with $u<v$ and assume for contradiction that $F(u)>F(v)$. By Lemma 2.1.12, there exist $u',v' \in C^{(3)}$ such that $F(u)=F(u')$ and $F(v)=F(v')$. Choosing u' to be the left endpoint of the removed interval where u lies and similarly for v', we have by assumption that $F(u) \neq F(v)$, so u' and v' cannot be the endpoints of the same interval. Then, $u' < v'$ since $u < v$. But $u', v' \in C^{(3)}$ so we obtain by assumption that $F(u) = F(u') > F(v') = F(v)$. This is a contradiction since we showed that $F(u') \leq F(v')$ on $C^{(3)}$.

(vi) Therefore, $\forall u, v \in [0,1]$ with $u < v$, $F(u) \leq F(v)$ and $F : [0,1] \longrightarrow [0,1]$ is increasing on $[0,1]$.

□

2.1.3.1 Characterization of $F(x)$ as a fixed point

The extended Cantor-Lebesgue function $F(x)$ is well-defined, continuous, monotone increasing on $[0,1]$. The left endpoint of each removed middle third open interval is in the form of a finite sequence $0.a_1a_2a_3\ldots a_n1$ (base 3) ($a_{n+1} = 1$) which is equal to $0.a_1a_2a_3\ldots a_na_{n+1}\overline{2}$ (base 3), a ternary expansion of 0's and 2's with $a_{n+1} = 0$ and a tail of 2's. For such a number x, the extended Cantor-Lebesgue is given by:

$$\begin{aligned} F(x) &= \textstyle\sum_{i=1}^{n} \frac{b_i}{2^i} + \frac{0}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \frac{1}{2^{n+4}} + \cdots \quad (b_i = \frac{a_i}{2} \in \{0,1\}) \\ &= \textstyle\sum_{i=1}^{n} \frac{b_i}{2^i} + \frac{1}{2^{n+1}} \end{aligned} \tag{2.51}$$

Lemma 2.1.15. By the definition of the Cantor-Lebesgue function, the following identity holds:

$$F(x) = \begin{cases} \frac{1}{2}F(3x) & \text{for} \quad 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2}F(3x-2) + \frac{1}{2} & \text{for} \quad \frac{2}{3} \leq x \leq 1. \end{cases} \tag{2.52}$$

Proof. We start by few facts:

- Let $x = 0.a_1a_2a_3\ldots$ (base 3) with $a_k = \{0,2\}$, then the map $x \longmapsto 3x$ corresponds to the left shift of the sequence $\{a_1, a_2, a_3, \ldots\}$ to become $\{a_2, a_3, a_4, \ldots\}$.
- The ternary representation for $0 \leq x \leq 1/3$ is of the form $0.0a_2a_3\ldots$ (base 3) and for $2/3 \leq x \leq 1$ is of the form $0.2a_2a_3\ldots$ (base 3). Also, 2.0 (base 10) = 2.0 (base 3).

- Let $x = 0.b_1b_2b_3\ldots$ (base 2) with $b_k = \{0,1\}$, then x divided by 2 corresponds to the right shift of the sequence $\{b_1, b_2, b_3, \ldots\}$ to become $\{0, b_1, b_2, b_3, \ldots\}$.

Step 1 Let $0 \le x \le 1/3$, then x is of the form $0.0a_2a_3a_3a_4\ldots$ (base 3) with $a_k = \{0,2\}$, so $F(x) = 0.0b_2b_3b_3b_4\ldots$ (base 2) with $b_i = a_i/2$ and $3x = 0.a_2a_3a_3a_4\ldots$.

Step 2 By definition of F, $F(3x) = 0.b_2b_3b_3b_4\ldots$ (base 2) but $F(3x)/2 = 0.0b_2b_3b_3b_4\ldots$ (base 2) $= F(x)$.

Step 3 Let $2/3 \le x \le 1$, then x is of the form $0.2a_2a_3a_3a_4\ldots$ (base 3) with $a_k = \{0,2\}$, so $F(x) = 0.1b_2b_3b_3b_4\ldots$ (base 2) with $b_i = a_i/2$ and $3x - 2 = 2.a_2a_3a_3a_4\ldots - 2.0 = 0.a_2a_3a_3a_4\ldots$.

Step 4 By definition of F, $F(3x-2) = 0.b_2b_3b_3b_4\ldots$ (base 2) but $1/2 + F(3x-2)/2 = (0.1 + 0.0b_2b_3b_3b_4\ldots) = (0.1b_2b_3b_3b_4\ldots)$ (base 2) $= F(x)$.

Step 5 For $1/3 < x < 2/3$, $F(x) = 1/2$ as defined in extending the Cantor-Lebesgue function over $[0,1]$.

Therefore, the identity in eq. (2.52) holds. □

Definition 2.1.16. Let $(\mathcal{B}([0,1], \|\cdot\|_\infty = \sup_{x\in[0,1]}|\cdot|)$ be the Banach space of all uniformly bounded real-valued functions on $[0,1]$ with the supremum norm. Define a sequence of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ as follows:

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{for} \quad 0 \le x \le \frac{1}{3}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2) & \text{for} \quad \frac{2}{3} \le x \le 1. \end{cases} \tag{2.53}$$

where the functions $f_n \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$ with $n = \{1,2,3,\ldots\}$ and $f_0 : [0,1] \longrightarrow \mathbb{R}$ is arbitrary.

Appendix E presents a proof that $(\mathcal{B}([0,1], \|\cdot\|_\infty)$ is complete.

As a preliminary to Proposition 1, we give the definition of a contraction, following Elaydi[13, p. 318] and adapted to the present context:

Definition 2.1.17. A map $H : \mathcal{B}([0,1], \|\cdot\|_\infty) \to \mathcal{B}([0,1], \|\cdot\|_\infty)$ is said to be a contraction if for some $0 < \alpha < 1$, we have:

$$\|H(g_1) - H(g_2)\|_\infty \leq \alpha \|g_1 - g_2\|_\infty \quad \forall g_1, g_2 \in \mathcal{B}([0,1], \|\cdot\|_\infty) \tag{2.54}$$

Proposition 1. The Cantor-Lebesgue function F for the ternary Cantor set, is the unique element of$(\mathcal{B}([0,1], \|\cdot\|_\infty)$ for which the identity in eq. (2.52) holds. If $f_0 \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$, then the sequence $\{f_n\}_{n=0}^{\infty}$ converges uniformly to F.

Proof.

Step 1 Define a map $H : (\mathcal{B}([0,1], \|\cdot\|_\infty) \to (\mathcal{B}([0,1], \|\cdot\|_\infty)$ by

$$H(g)(x) = \begin{cases} \frac{1}{2}g(3x) & \text{for} \quad 0 \leq x \leq \frac{1}{3}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}g(3x-2) & \text{for} \quad \frac{2}{3} \leq x \leq 1. \end{cases} \tag{2.55}$$

We need to show that $H(g)(x)$ is bounded on $[0,1]$, that is $\|H(g)\|_\infty < \infty$. This is done as follows:

(i) let $g(x) \in \mathcal{B}([0,1], \|\cdot\|_\infty)$ so $\|g\|_\infty < \infty$ on $[0,1]$.

(ii) if $x \in [0, 1/3]$ we have

$$\begin{aligned} H(g)(x) &= \tfrac{1}{2}g(3x) \\ |H(g)(x)| &= \tfrac{1}{2}|g(3x)| \leq \tfrac{1}{2}\|g\|_\infty \end{aligned} \tag{2.56}$$

(iii) if $x \in (1/3, 2/3)$ we have

$$H(g)(x) = \frac{1}{2} \tag{2.57}$$

(iv) if $x \in [2/3,1]$ we have

$$\begin{aligned} H(g)(x) &= \tfrac{1}{2}+\tfrac{1}{2}g(3x-2) \\ |H(g)(x)| &\leq \tfrac{1}{2}+\tfrac{1}{2}|g(3x-2)| \leq \tfrac{1}{2}+\tfrac{1}{2}\|g\|_\infty \end{aligned} \tag{2.58}$$

From eqs. (2.56), (2.57) and (2.58) we take

$$\begin{aligned} \|H(g)\|_\infty &\leq \max\{\tfrac{1}{2}\|g\|_\infty, \tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}\|g\|_\infty\} \\ \|H(g)\|_\infty &\leq \tfrac{1}{2}+\tfrac{1}{2}\|g\|_\infty < \infty \end{aligned} \tag{2.59}$$

and we conclude that $H(g)(x)$ is bounded on $[0,1]$ that is $H(g) \in \mathcal{B}([0,1], \|\cdot\|_\infty)$

Step 2 We need to show that H is a contraction and this done by applying the mapping H to any two $g_1, g_2 \in \mathcal{B}([0,1], \|\cdot\|_\infty)$ as follows:

(i) if $x \in [0,1/3]$ we have

$$\begin{aligned} H(g_1)(x)-H(g_2)(x) &= \tfrac{1}{2}(g_1(3x)-g_2(3x)) \\ |H(g_1)(x)-H(g_2)(x)| &= \tfrac{1}{2}|(g_1(3x)-g_2(3x))| \leq \tfrac{1}{2}\|g_1-g_2\|_\infty \end{aligned} \tag{2.60}$$

(ii) if $x \in (1/3,2/3)$ we have

$$H(g_1)(x)-H(g_2)(x) = \frac{1}{2}-\frac{1}{2} = 0 \tag{2.61}$$

(iii) if $x \in [2/3,1]$ we have

$$\begin{aligned} H(g_1)(x)-H(g_2)(x) &= \tfrac{1}{2}(g_1(3x-2)-g_2(3x-2)) \\ |H(g_1)(x)-H(g_2)(x)| &= \tfrac{1}{2}|(g_1(3x-2)-g_2(3x-2))| \leq \tfrac{1}{2}\|g_1-g_2\|_\infty \end{aligned} \tag{2.62}$$

From eqs. (2.60), (2.61) and (2.62) we take

$$\begin{aligned} \|H(g_1) - H(g_2)\|_\infty &\leq \max\{0, \tfrac{1}{2}\|g_1 - g_2\|_\infty\} \\ \|H(g_1) - H(g_2)\|_\infty &\leq \tfrac{1}{2}\|g_1 - g_2\|_\infty \end{aligned} \tag{2.63}$$

showing that the mapping H is a contraction. Applying the Banach contraction principle (see Appendix F) to the mapping H, we have that there is a unique fixed point $u_0 \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$ such that $H(u_0) = u_0$ and by Lemma 2.1.15, $u_0 = F$.

□

Consider the sequence $\{f_n\} \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$ in Definition 2.1.16. In the proof of the Banach contraction principle (see Appendix F), the choice of the initial function $f_0(x)$ is arbitrary as long as $f_0 \in \mathcal{B}([0,1], \|\cdot\|_\infty)$. Usually, $f_0(x) = x$ and Figure 2.1.3.1 presents graphs of $f_n(x)$ for increasing n. We observe that $f_n(x)$ converges relatively quickly to a function $f_n(x)$ for $n = 100000$ graphically close to the Cantor-Lebesgue function F. Since f_0 is arbitrary, Appendix F presents results when f_0 is a bounded step function on $[0,1]$ and we can observe the same convergence as in Figure 2.4.

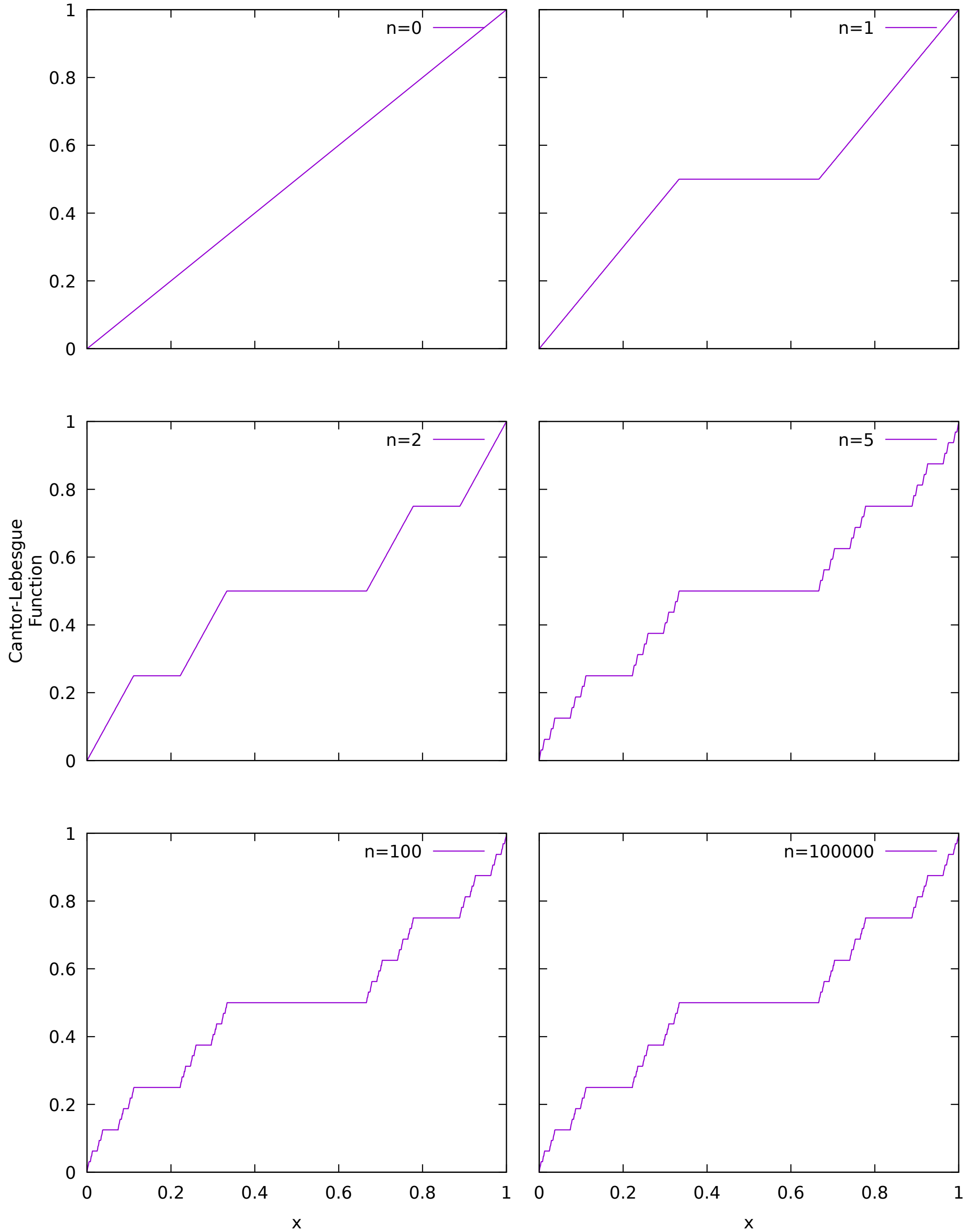

Figure 2.4: Graph of the functions $f_n(x)$ for $n = 0, 1, 2, 5, 100, 100000$ for the ternary Cantor set

Remark 2.1.18. Adding the condition $F(y) = 0$ for $y < 0$ or $y > 1$ to the identity in Lemma 2.1.15 for the Cantor-Lebesgue function, allows $F(x)$ to be written as a single expression:

$$F(x) = \frac{1}{2}F(3x) + \frac{1}{2} + \frac{1}{2}F(3x-2) \quad \forall x \in [0,1] \tag{2.64}$$

which will be revisited in the next section.

2.2 Construction of Cantor sets by IFS

Iterated Function Systems (IFS) can be used to construction Cantor sets. We present two IFS's: one for the ternary Cantor set and the other one for a quaternary Cantor. These IFS's will be used later in this work.

2.2.1 Construction of ternary Cantor set by IFS

Starting with the closed interval $C_0^{(3)} = [0,1]$ being the construction level 0, we apply the following IFS

$$\begin{aligned} T_0(x) &= \frac{x}{3} \\ T_1(x) &= \frac{x+2}{3} \end{aligned} \tag{2.65}$$

to the endpoints of $[0,1]$ to obtain the endpoints of the two closed intervals at construction level 1: $T_0(0) = 0$, $T_0(1) = 1/3$, $T_1(0) = 2/3$ and $T_1(1) = 1$ or $C_1^{(3)} = T_0(C_0^{(3)}) \cup T_1(C_0^{(3)})$. To obtain the subsequent construction levels we continue to apply the IFS in eq.(2.65) to give that $C_{n+1}^{(3)} = T_0(C_n^{(3)}) \cup T_1(C_n^{(3)})$. Eqs. (2.1) and (2.2) give the results for construction level 2 and 3 respectively, illustrated in Figure 2.5(a).

Since T_0 and T_1 are strictly increasing and continuous linear maps on $\mathbb{R}$, their inverses

$$\begin{aligned} T_0^{-1}(x) &= 3x \\ T_1^{-1}(x) &= 3x-2 \end{aligned} \tag{2.66}$$

have the same properties. Then the single expression for Cantor-Lebesgue function, eq. (2.64), can be written:

$$F(x)=\frac{1}{2}F(T_0^{-1}(x))+\frac{1}{2}+\frac{1}{2}F(T_1^{-1}(x)) \quad \forall x\in[0,1] \tag{2.67}$$

recalling the condition $F(y)=0$ for $y<0$ and $y>1$.

2.2.2 Construction of a quaternary Cantor set by IFS

We present the construction of a particular quaternary Cantor set for the reason that it used later in this work. The construction levels come from the repeated application of the following Iterated Function System (IFS):

$$\begin{aligned} \tau_0(x) &= \frac{x}{4} \\ \tau_1(x) &= \frac{x+2}{4} \end{aligned} \tag{2.68}$$

Since $\tau_0(x)$ and $\tau_1(x)$ are strictly increasing and continuous linear maps on $\mathbb{R}$, their inverses

$$\begin{aligned} \tau_0^{-1}(x) &= 4x \\ \tau_1^{-1}(x) &= 4x-2 \end{aligned} \tag{2.69}$$

have the same properties.

This particular quaternary Cantor set could be constructed by removal of open intervals. Starting from $[0,1]$ and dividing in four equal parts we remove the open intervals $(1/4,1/2)$

and $(3/4,1]$ where $[0,1/4]$ and $[1/2,3/4]$ are the remaining closed interval at construction level 1. $(3/4,1]$ is an open interval by the following consideration: let $[a,b] \subset \mathbb{R}$ with $a < c < b$, let $[a,c] \subset [a,b]$, with $[a,c]$ closed, then $[a,b] \setminus [a,c] = (c,b]$ is an open set being the complement of $[a,c]$ in $[a,b]$. The process continue by dividing $[0,1/4]$ in four equal parts removing the open intervals $(1/16,1/8)$ and $(3/16,1/4]$. That process is applied to $[1/2,3/4]$ to give at construction level 2 the following remaining closed intervals: $[0,1/16]$, $[1/8,3/16]$, $[1/2,9/16]$ and $[5/8,11/16]$ as illustrated in Figure 2.5(b).

Starting with the closed interval $C_0^{(4)} = [0,1]$ being the construction level 0, we apply the IFS in eq, (2.68 to the endpoints of $[0,1]$ to obtain the endpoints of the two closed intervals at construction level 1: $\tau_0(0) = 0$, $\tau_0(1) = 1/4$, $\tau_1(0) = 1/2$ and $\tau_1(1) = 3/4$ or $C_1^{(4)} = T_0(C_0^{(4)}) \cup T_1(C_0^{(4)})$. To obtain the subsequent construction levels we continue to apply the IFS in eq.(2.68) to give that $C_{n+1}^{(4)} = \tau_0(C_n^{(4)}) \cup \tau_1(C_n^{(4)})$. Figure 2.5(b) illustrates the results for construction level 2 and 3 respectively.

The overall construction process by either removal of open intervals or IFS is continued to obtain the quaternary Cantor set.

Dividing the closed intervals in four and keeping the first and third subintervals imply the coefficients of the expansion in base 4 would be 0's and 2's, similar to ternary Cantor set where the coefficients of the expansions in base 3 are also 0 and 2(see Figure 2.5). For the expansions in base 4, these coefficients corresponds to the first and third subintervals that were kept. From Figure 2.5, we observe that $1/2$ is the common endpoint different from 0 among the first three construction levels and by construction, to all construction levels. $1/2$ equals 0.2_4 (subscript "4" means "base 4"). Normally, we should be able to express the endpoints $1/4$ and $3/4$ of $C_1^{(4)}$ by an expansion in base 4 with coefficients 0 and 2 such that we can generate the endpoints on the next construction levels using right shift only or right shift plus translation as done for the ternary Cantor set. However, in Appendix A we show this is not possible.

In Figure 2.5, part (a) illustrates few construction levels of the ternary Cantor set with the value of the endpoints included and similarly, part (b), for the quaternary Cantor set. We

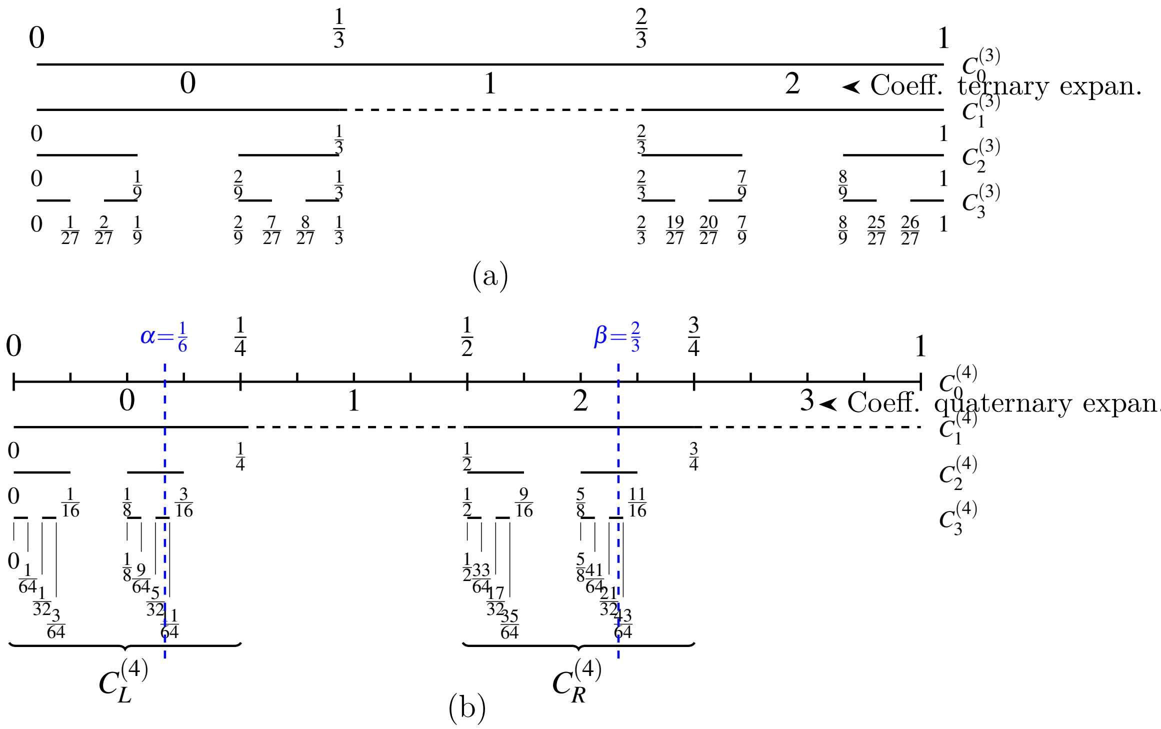

Figure 2.5: Few construction levels of the ternary and quaternary Cantor sets

observe:

(i) For the ternary Cantor set the left and right endpoints of the closed intervals remain when going from one construction level to the next. Whereas, for the quaternary Cantor set only the left endpoints remain but the right endpoints generated in the construction process do not. These right endpoints are not in $C^{(4)}$.

(ii) From Figure 2.5 the quaternary Cantor set ends up being skewed to the left of the interval $[0,1]$.

(iii) Referring to Figure 2.5, define:

- $C_L^{(4)} = \{x \in \mathbb{R} : x \in C^{(4)} \cap [0,1/4]\}$,
- $C_R^{(4)} = \{x \in \mathbb{R} : x \in C^{(4)} \cap [1/2,3/4]\}$,
- $\alpha = \sup_{x \in C^{(4)}}(C_L^{(4)})$,

- $\beta = \sup_{x \in C^{(4)}}(C_R^{(4)})$.

Appendix A shows that $\alpha = 1/6$ and $\beta = \alpha + 1/2 = 2/3$. Thus, the left and right parts of the quaternary Cantor set $C^{(4)}$ spread respectively over $[0, 1/6]$ and $[1/2, 2/3]$.

2.2.2.1 Cantor-Lebesgue function for quaternary Cantor set $W(x)$ as a fixed point

As observed above, in the construction of the quaternary Cantor set, we see the right endpoints replaced by right endpoints of smaller value. One could think with that phenomenon, it may appears more difficult to construct a Cantor-Lebesgue function for the quaternary Cantor set that can be extended over $[0, 1]$. However, using the same technique as in Sec. 2.1.3.1 for F in the ternary Cantor set case, we can arrive at that goal matching, for instance, the requirement that for $1/6 \le x \le 1/2$, $W(x) = 1/2$ and for $2/3 \le x \le 1$, $W(x) = 1$. To achieve this we have:

Definition 2.2.1. Let $(\mathcal{B}([0, 1], \|\cdot\|_\infty)$ be the Banach space of all uniformly bounded real-valued functions on $[0, 1]$ with the supremum norm. Define a sequence of functions $h_n : [0, 1] \longrightarrow \mathbb{R}$ as follows:

$$h_{n+1}(x) = \begin{cases} \frac{1}{2}h_n(4x) & \text{for} \quad 0 \le x \le \frac{1}{4}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}h_n(4x-2) & \text{for} \quad \frac{1}{2} \le x \le \frac{3}{4}, \\ 1 & \text{for} \quad \frac{3}{4} \le x \le 1. \end{cases} \tag{2.70}$$

where the functions $h_n \in (\mathcal{B}([0, 1], \|\cdot\|_\infty)$ $n = \{1, 2, 3, \ldots\}$ and $h_0 : [0, 1] \longrightarrow \mathbb{R}$ is arbitrary.

Proposition 2. If $h_0 \in (\mathcal{B}([0, 1], \|\cdot\|_\infty)$, then the sequence $\{h_n\}_{n=0}^{\infty}$ converges uniformly to a unique fixed point u_0 which we define to be W, the Cantor-Lebesgue function for quaternary Cantor set.

Proof.

Step 1 Similar as for the ternary Cantor set, we define a map $H : (\mathcal{B}([0,1], \|\cdot\|_\infty) \to (\mathcal{B}([0,1], \|\cdot\|_\infty)$ by

$$H(g)(x) = \begin{cases} \frac{1}{2}g(4x) & \text{for} \quad 0 \le x \le \frac{1}{4}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}g(4x-2) & \text{for} \quad \frac{1}{2} \le x \le \frac{3}{4}, \\ 1 & \text{for} \quad \frac{3}{4} < x < 1. \end{cases} \tag{2.71}$$

We need to show that $H(g)(x)$ is bounded on $[0,1]$, that is $\|H(g)\|_\infty < \infty$. This is done as follows:

(i) let $g \in \mathcal{B}([0,1], \|\cdot\|_\infty)$ so $\|g\|_\infty < \infty$ on $[0,1]$.

(ii) if $x \in [0, 1/4]$ we have

$$\begin{aligned} H(g)(x) &= \tfrac{1}{2}g(4x) \\ |H(g)(x)| &= \tfrac{1}{2}|g(4x)| \le \tfrac{1}{2}\|g\|_\infty \end{aligned} \tag{2.72}$$

(iii) if $x \in (1/4, 1/2)$ we have

$$H(g)(x) = \frac{1}{2} \tag{2.73}$$

(iv) if $x \in [1/2, 3/4]$ we have

$$\begin{aligned} H(g)(x) &= \tfrac{1}{2} + \tfrac{1}{2}g(4x-2) \\ |H(g)(x)| &\le \tfrac{1}{2} + \tfrac{1}{2}|g(4x-2)| \le \tfrac{1}{2} + \tfrac{1}{2}\|g\|_\infty \end{aligned} \tag{2.74}$$

(v) if $x \in (3/4, 1]$ we have

$$H(g)(x) = 1 \tag{2.75}$$

From eqs. (2.72), (2.73), (2.74) and (2.75) we obtain $\|H(g)\|_\infty \le \max\{1, 1/2 + 1/2\|g\|_\infty\} < \infty$. For $1/2 + 1/2\|g\|_\infty \ge 1$, it implies that $\|g\|_\infty \ge 1$. It results that we take $\|H(g)\|_\infty \le \max\{1, \|g\|_\infty\} < \infty$ to conclude that $H(g)$ is bounded on $[0,1]$ that is $H(g) \in \mathcal{B}([0,1], \|\cdot\|_\infty)$.

Step 2 We need to show that H is a contraction and this done by applying the mapping H to any two $g_1, g_2 \in \mathcal{B}([0,1], \|\cdot\|_\infty)$ as follows:

(i) if $x \in [0, 1/4]$ we have

$$\begin{aligned} H(g_1)(x) - H(g_2)(x) &= \tfrac{1}{2}(g_1(4x) - g_2(4x)) \\ |H(g_1)(x) - H(g_2)(x)| &= \tfrac{1}{2}|(g_1(4x) - g_2(4x))| \le \tfrac{1}{2}\|g_1 - g_2\|_\infty \end{aligned} \tag{2.76}$$

(ii) if $x \in (1/4, 1/2)$ we have

$$H(g_1)(x) - H(g_2)(x) = \frac{1}{2} - \frac{1}{2} = 0 \tag{2.77}$$

(iii) if $x \in [1/2, 3/4]$ we have

$$\begin{aligned} H(g_1)(x) - H(g_2)(x) &= \tfrac{1}{2}(g_1(4x-2) - g_2(4x-2)) \\ |H(g_1)(x) - H(g_2)(x)| &= \tfrac{1}{2}|(g_1(4x-2) - g_2(4x-2))| \le \tfrac{1}{2}\|g_1 - g_2\|_\infty \end{aligned} \tag{2.78}$$

(iv) if $x \in (3/4, 1]$ we have

$$H(g_1)(x) - H(g_2)(x) = 1 - 1 = 0 \tag{2.79}$$

to obtain that

$$\|H(g_1) - H(g_2)\|_\infty \le \frac{1}{2}\|g_1 - g_2\|_\infty \tag{2.80}$$

showing that the mapping H is a contraction. Applying the Banach contraction principle (see Appendix F) to the mapping H, we have that there is a unique fixed point $u_0 \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$ such that $H(u_0) = u_0$. We define W to be the fixed point.

To have that $W = u_0$ we need to show that the following identity holds:

$$W(x) = \begin{cases} \frac{1}{2}W(4x) & \text{for} \quad 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}W(4x-2) & \text{for} \quad \frac{1}{2} \leq x \leq \frac{3}{4}, \\ 1 & \text{for} \quad \frac{3}{4} \leq x \leq 1. \end{cases} \tag{2.81}$$

Step (i) if $0 \leq x \leq 1/4$, then $0 \leq 4x \leq 1$, so $0 \leq W(4x) \leq 1$ and we get that $0 \leq W(4x)/2 \leq 1/2$ and $0 \leq W(x) \leq 1/2$

Step (ii) if $1/4 < x < 1/2$, then $W(x) = 1/2$

Step (iii) if $1/2 \leq x \leq 3/4$, then $0 \leq 4x-2 \leq 1$, so $0 \leq W(4x-2) \leq 1$, $0 \leq W(4x-2)/2 \leq 1/2$, and we get that $0 + 1/2 \leq 1/2 + W(4x-2)/2 \leq 1/2 + 1/2$, simplifying, $1/2 \leq 1/2 + W(4x-2)/2 \leq 1$ and $1/2 \leq W(x) \leq 1$.

Step (iv) if $3/4 < x < 1$, then $W(x) = 1$

We conclude the identity (2.81) holds and $u_0 = W$.

□

Consider the sequence $\{h_n\} \in (\mathcal{B}([0,1], \|\cdot\|_\infty)$ in Definition 2.70. In the proof of the Banach contraction principle (see Appendix F), the choice of the initial function $h_0(x)$ is arbitrary as long as $h_0 \in \mathcal{B}([0,1], \|\cdot\|_\infty$. Usually, $h_0(x) = x$ and Figure 2.6 presents graphs of $h_n(x)$ for increasing n. We observe that $h_n(x)$ converges relatively quickly to a function $h_n(x)$ for $n = 100000$ graphically close to the Cantor-Lebesgue function W. In particular, we see the left endpoints of the first plateau at $W(x) = 1/2$ converge quickly to $\alpha = 1/6$ as it should since it is the supremum of all the right endpoints of the closed intervals arising in the construction of the quaternary Cantor set. Similarly, the left endpoints of the second plateau at $W(x) = 1$ converge quickly to $\beta = 2/3$ as it should. Since h_0 is arbitrary, Appendix F presents results when h_0 is a bounded step function on $[0,1]$ and we can observe the same convergence as in Figure 2.6.

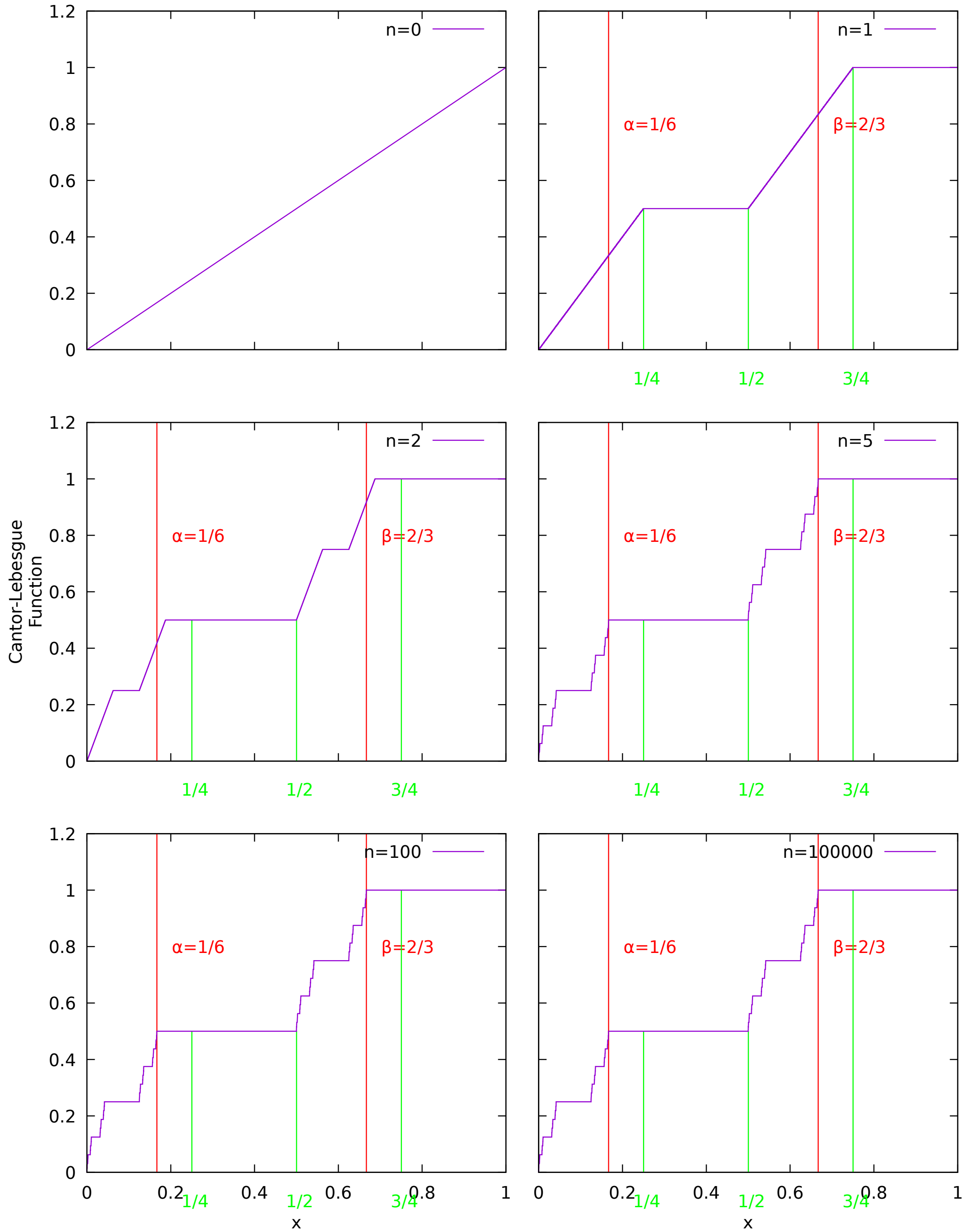

Figure 2.6: Graph of the functions $h_n(x)$ for $n = 0, 1, 2, 5, 100, 100000$ for the quaternary Cantor set

Remark 2.2.2. Adding the condition $W(y)=0$ for $y<0$ and $W(y)=1$ for $y>3/4$ to the identity (2.81) for the Cantor-Lebesgue function W, allows $W(x)$ to be written as a single expression:

$$W(x) = \frac{1}{2}W(4x) + \frac{1}{2} + \frac{1}{2}W(4x-2) \quad \forall x \in [0,1]. \tag{2.82}$$

We observe the difference between F and W for the second condition: for F, $F(y)=0$ for $y>1$ whereas for W, $W(y)=1$ for $y>3/4$

Chapter 3

Measure and dimension on Cantor sets

In this chapter, we consider four measures with support contained in a Cantor set C. For any Borel set $A \subset \mathbb{R}$, these are:

- mass distribution measure $\mu_m(A)$
- Hausdorff measure restricted to a Cantor set C, $\mathscr{H}^s(A \cap C)$ with dimension s
- unique measure $\mu_H(A)$ from Hutchinson's theorem for self-similar sets.
- unique Lebesgue-Stieltjes measure $\mu_F(A)$ where F is a Cantor-Lebesgue function on a Cantor set C extended to $\mathbb{R}$

We show the following theorem:

Theorem 3.0.1. Let F, be a Cantor-Lebesgue function on a Cantor set C extended to $\mathbb{R}$, then for every Borel set $A \in \mathscr{B}(\mathbb{R})$ we have the following equivalence:

$$\mu_m(A) = \mathscr{H}^s(A \cap C) = \mu_F(A) = \mu_H(A) \tag{3.1}$$

While this result may be known, we have not found it proved in the literature. Here $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$ [19, p. 34].

The steps followed to prove Theorem 3.0.1 are best seen in the following flowchart:

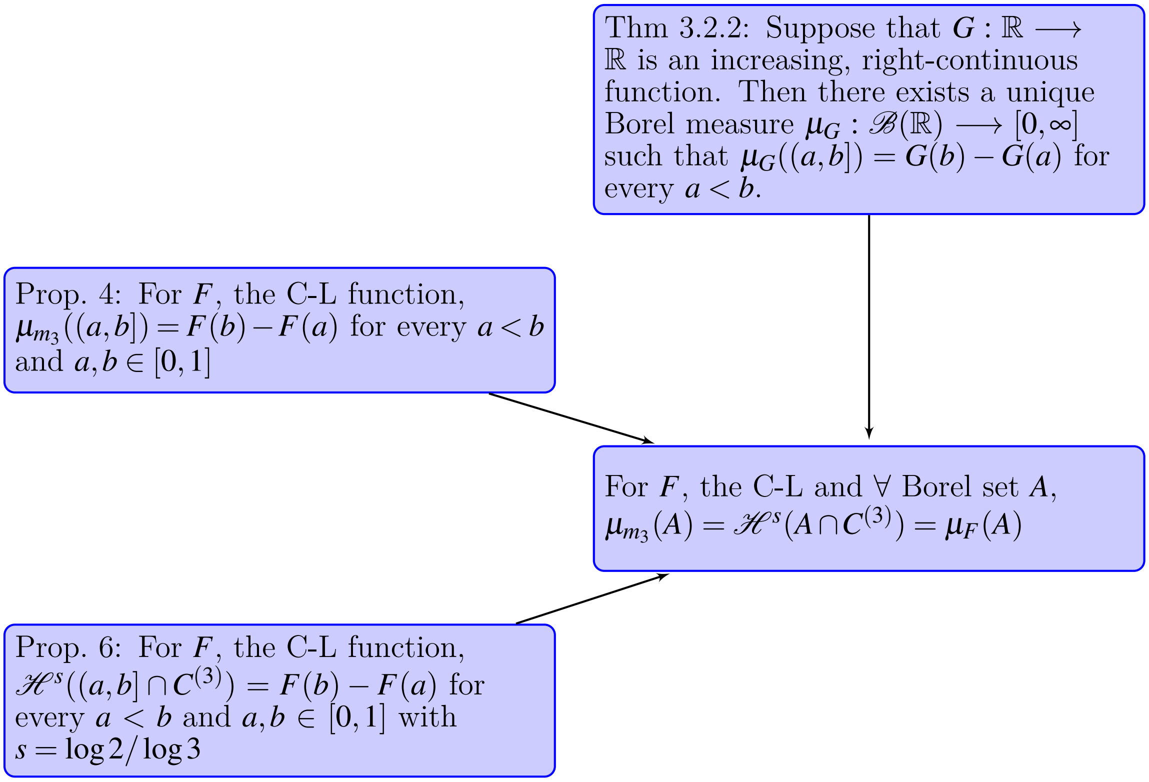

Figure 3.1: Relation between $\mu_{m_3}(A)$, $\mathscr{H}^s(A\cap C^{(3)})$ and $\mu_F(A)$ for every Borel set $A\in\mathscr{B}(\mathbb{R})$
.

We present Hutchinson's Theorem [18] that states that there exists a unique Borel measure μ_H with support contained in a Cantor set C such that for any Borel set $A\in\mathscr{B}(\mathbb{R})$, $\mu_H(A)=1/2\sum_{j=1}^{2}\mu_H(T_j^{-1}(A))$. We then show that $\mu_m(A)$ satisfies the recursive relation for $\mu_H(A)$. Therefore, by the uniqueness of μ_H we have that $\mu_m(A)=\mu_H(A)$ for every Borel set $A\in\mathscr{B}(\mathbb{R})$. This is illustrated in Figure 3.2 below. That result enable us to complete the proof of Theorem 3.0.1. Since we showed that $\mu_m(A)=\mathscr{H}^s(A\cap C)=\mu_F(A)$, we obtain by using $\mu_m(A)=\mu_H(A)$ the desired result: $\mu_m(A)=\mathscr{H}^s(A\cap C)=\mu_F(A)=\mu_H(A)$ for every Borel set $A\in\mathscr{B}(\mathbb{R})$.

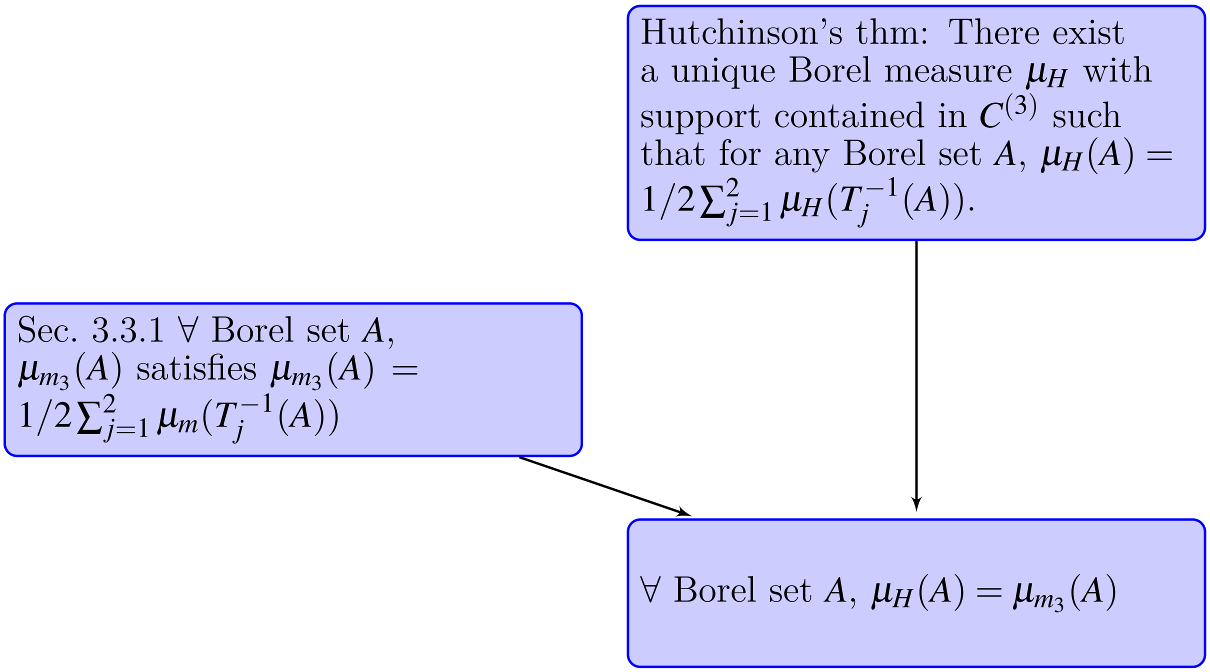

Figure 3.2: Relation between $\mu_{m_3}(A)$ and $\mu_H(A)$ for every Borel set $A \in \mathscr{B}(\mathbb{R})$

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3.1 Hausdorff measure and dimension

3.1.1 Hausdorff measure

This section presents definitions of Hausdorff measure and dimension taken from references [14, 15].

Definition 3.1.1. Let A be any non-empty subset of $\mathbb{R}$. We define its diameter $|A|$, by $|A| = \sup\{|x-y| : x,y \in A\}$.

Definition 3.1.2. A δ-cover of a set F is a countable (or finite) collection of sets $\{U_i\}$ with diameters $0 < |U_i| \leq \delta$ that cover F.

Definition 3.1.3. Let $F \subset \mathbb{R}$ and $s \in \mathbb{R}$ with $s \geq 0$. For each $\delta > 0$, we define

$$\mathscr{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\} \tag{3.2}$$

Definition 3.1.3 expresses a process that looks at all covers of F by sets of diameter at most δ to find the infimum of the sum of the sth powers of the diameters. As δ decreases, the family of permissible covers of F in eq. (3.2) is reduced. Therefore, the infimum $\mathscr{H}^s_\delta(F)$ increases or at least does not decreases, as $\delta \to 0$ and so approaches a limit. To see this:

Step 1 Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \ldots \geq \delta_j \geq \ldots$, assume $\mathscr{F}_j$ is a family of all covers of F by sets of diameter at most δ_j.

Step 2 Since the δ_j's are decreasing we have that $\mathscr{F}_1 \supset \mathscr{F}_2 \supset \ldots \supset \mathscr{F}_j \supset \ldots$.

Step 3 For any non-empty sets $A \subset B \subset \mathbb{R}$, we have that $\inf(B) \leq \inf(A)$ [6, p. 26].

Step 4 Since $\mathscr{F}_{j+1} \subset \mathscr{F}_j \ \forall j \geq 1$ then $\mathscr{H}^s_{\delta_j}(F) \leq \mathscr{H}^s_{\delta_{j+1}}(F)$.

Step 5 $\mathscr{H}^s_\delta(F)$ is monotone increasing as $\delta \longrightarrow 0$ and so approaches a limit in $[0, \infty]$.

Definition 3.1.4. $\mathscr{H}^s(F)$, s-dimensional Hausdorff measure of F, is defined by:

$$\mathscr{H}^s(F) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(F). \tag{3.3}$$

The limit in eq.(3.3) exists for any subset F of $\mathbb{R}$ although the limiting value can be 0 or ∞. In fact, as we shall see below that limiting value is usually 0 or ∞. We note that $\mathscr{H}^s(\emptyset) = 0$ and for any non-empty set F, $\mathscr{H}^s(F) \geq 0$.

3.1.2 Hausdorff dimension

We can see from Definition 3.1.3 that for any given set $F \subset \mathbb{R}$ and $\delta < 1$, $\mathscr{H}^s_\delta(F)$ is non-increasing with s. It follows by eq. (3.3) that $\mathscr{H}^s(F)$ is also non-increasing. More information on the behaviour of $\mathscr{H}^s(F)$ as a function of s can be obtained by the following:

Step 1: Consider s fixed.

Step 2: If $t > s$ and $\{U_i\}$ is a δ-cover of F, then

$$\sum_i |U_i|^t = \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{t-s}\sum_i |U_i|^s. \tag{3.4}$$

Step 3: Taking infima over all δ-covers, we get:

$$\mathscr{H}^t_\delta(F) \leq \delta^{t-s}\mathscr{H}^s_\delta(F) \tag{3.5}$$

Step 4: If we assume $\mathscr{H}^s_\delta(F) < \infty$ then taking the limit $\delta \to 0$ on both sides gives:

$$0 \leq \mathscr{H}^t(F) \leq 0 \tag{3.6}$$

and $\mathscr{H}^t(F) = 0$ for $t > s$

Step 5: If $t < s$ and

$$\begin{aligned} \textstyle\sum_i |U_i|^s &= \textstyle\sum_i |U_i|^{s-t}|U_i|^t \leq \delta^{s-t}\sum_i |U_i|^t \\ \textstyle\sum_i |U_i|^t &\geq \frac{\sum_i |U_i|^s}{\delta^{s-t}} \end{aligned} \tag{3.7}$$

Step 6: Taking infima over all δ-covers, we get:

$$\mathscr{H}^t_\delta(F) \geq \frac{\mathscr{H}^s_\delta(F)}{\delta^{t-s}} \tag{3.8}$$

Step 7: If we assume $0 < \mathscr{H}^s_\delta(F) < \infty$ then taking the limit $\delta \to 0$ on both sides gives:

$$\mathscr{H}^t(F) \geq \infty \tag{3.9}$$

and $\mathscr{H}^t(F) = \infty$ for $t < s$

Figure 3.3 illustrates the above discussion where we clearly see the jump of $\mathscr{H}^t(F)$ from ∞ down to 0 when t goes across s. This brings the following definition:

Definition 3.1.5. The Hausdorff dimension for any set $F \subset \mathbb{R}$ is given by [15, p. 48]:

$$\dim_H(F) = \inf\{s \geq 0 : \mathscr{H}^s(F) = 0\} = \sup\{s : \mathscr{H}^s(F) = \infty\} \tag{3.10}$$

so that

$$\mathscr{H}^s(F) = \begin{cases} \infty & \text{if} \quad 0 \leq s < \dim_H(F) \\ 0 & \text{if} \quad s > \dim_H(F) \end{cases} \tag{3.11}$$

and if $s = \dim_H(F)$, then $\mathscr{H}^s(F)$ may be 0 or ∞ or may satisfy

$$0 < \mathscr{H}^s(F) < \infty \tag{3.12}$$

The discussion on the behaviour of $\mathscr{H}^s(F)$ as a function of s shows that if ineq. (3.12) holds then $\dim_H(F) = s$.

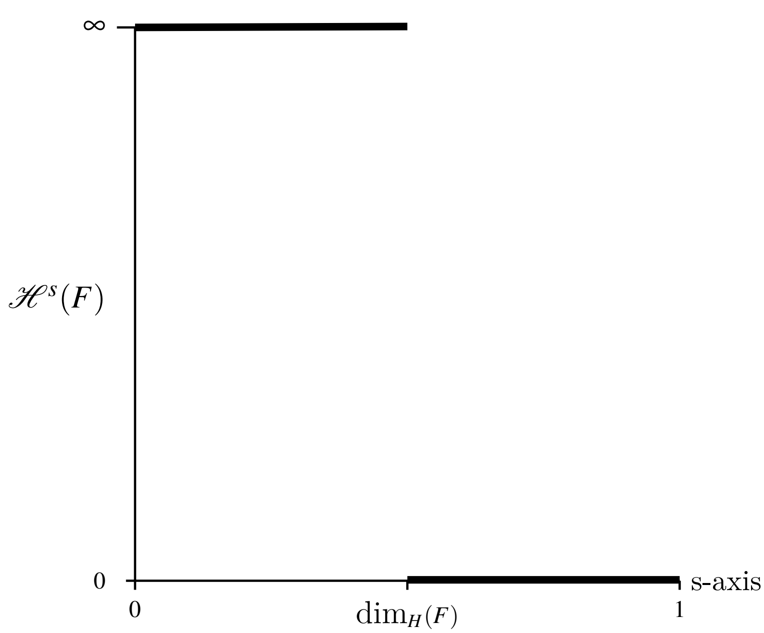


Figure 3.3: Graph of $\mathscr{H}^s(F)$ against s for a set $F \subset \mathbb{R}$.

The discussion on the behaviour of $\mathscr{H}^s(F)$ as a function of s brings the following questions. How do we know that such a finite s exist? What if $\mathscr{H}^s(F) = \infty \ \ \forall s$ or $\mathscr{H}^s(F) = 0 \ \ \forall s$? These questions can be answered by proving the following claim in the form of a Lemma.

Lemma 3.1.6. If $F \subset \mathbb{R}$ then there exists $s \leq 1$ such that $\mathscr{H}^s(F) = 0 \ \forall s > 1$.

Proof.

It is enough to show that $\mathscr{H}^s(\mathbb{R}) = 0 \ \forall s > 1$.

Step 1 Consider the δ-cover of $\mathbb{R}$ by $\bigcup_{-\infty}^{\infty} U_n$ with $|U_n| = \delta/|n|$ as illustrated in Figure 3.4. By definition of $\mathscr{H}^s_\delta(\cdot)$ we have:

$$\mathscr{H}^s_\delta(\mathbb{R}) \leq \sum_{-\infty}^{\infty} |U_n|^s = \delta^s \sum_{-\infty}^{\infty} \frac{1}{|n|^s}. \tag{3.13}$$

Step 2 Taking the limit $\delta \longrightarrow 0$ in eq.(3.13), we obtain that

$$\mathscr{H}^s(\mathbb{R}) \leq 0. \tag{3.14}$$

Step 3 By definition $\mathscr{H}^s(\cdot) \geq 0$, therefore we get that $\mathscr{H}^s(\mathbb{R}) = 0 \ \forall s > 1$.

□

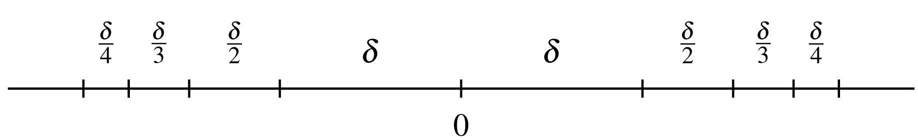


Figure 3.4: Covering of $\mathbb{R}$ by sets of diameter δ/n, $n = 1, 2, 3, \ldots$

3.1.3 Hausdorff measure and dimension of the ternary Cantor set

This section presents a proof that the Hausdorff dimension of $C^{(3)}$ is $s = \log 2/\log 3$ and the Hausdorff measure of $C^{(3)}$, $\mathscr{H}^s(C^{(3)}) = 1$. The present proof adds many details to the original one presented in [14, pp. 14-15] and claim to some originality by:

(i) including cases where one or both the endpoints of the intervals in the cover is an element of $C^{(3)}$

(ii) extending the reduction and methodology of the original proof in [14, pp. 14-15] used to obtain a lower bound for $\mathscr{H}^s(C^{(3)})$.

Theorem 3.1.7. The Hausdorff measure of the ternary Cantor set $C^{(3)}$, $\mathscr{H}^s(C^{(3)}) = 1$ with $s = \log 2/\log 3$.

The proof consists of two parts:

Part (a) Upper bound: prove that $\mathscr{H}^s(C^{(3)}) \leq 1$

Part (b) Lower bound: prove that $\mathscr{H}^s(C^{(3)}) \geq 1$

to conclude that $\mathscr{H}^s(C^{(3)}) = 1$

Proof.

We do a heuristic calculation [15, p. 52] to justify an assumption used in the proof that the Hausdorff dimension of $C^{(3)}$ is $s = \log 2/\log 3$. The Cantor set spreads over two equal and disjoint parts: $C_L = C^{(3)} \cap [0,1/3]$ and $C_R = C^{(3)} \cap [2/3,1]$. Each of these parts is geometrically similar to $C^{(3)}$ but scaled down by a factor of $1/3$. So, by the scaling property of Hausdorff measure [15, p. 46] and since $C^{(3)} = C_L \cup C_R$, we have:

$$\begin{aligned} \mathscr{H}^s(C^{(3)}) &= \mathscr{H}^s(C_L) + \mathscr{H}^s(C_R) \\ &= \tfrac{1}{3^s}\mathscr{H}^s(C^{(3)}) + \tfrac{1}{3^s}\mathscr{H}^s(C^{(3)}) \end{aligned} \tag{3.15}$$

Assuming $\mathscr{H}^s(C^{(3)}) \in (0,\infty)$, we can divide eq. (3.15) by $\mathscr{H}^s(C^{(3)})$, to obtain $s = \log 2/\log 3$. In fact, this heuristic calculation gives a "guess" for the value of the Hausdorff dimension of $C^{(3)}$. That "guess" is then used to obtain upper and lower bounds for $\mathscr{H}^s(C^{(3)})$.

Upper bound for $\mathscr{H}^s(C^{(3)})$

For $k \in \mathbb{N}$, since $C^{(3)}$ may be covered by the 2^k closed intervals of length 3^{-k} that form $C_k^{(3)}$, we have that

$$\mathscr{H}^s_{3^{-k}}(C^{(3)}) \leq 2^k(3^{-k})^s = 2^k(3^s)^{-k} = 2^k 2^{-k} = 1. \tag{3.16}$$

Letting $k \to \infty$ we have $\mathscr{H}^s(C^{(3)}) \leq 1$

Lower bound for $\mathscr{H}^s(C^{(3)})$

We need to prove that $\mathscr{H}^s(C^{(3)}) \geq 1$ (lower bound). This is equivalent to proving that for any cover $\{U_i\}$ of the Cantor set $C^{(3)}$, we have $\sum_i |U_i|^s \geq 1$. The proof consists of two parts:

1. Obtaining from $\{U_i\}$ a finite collection $\mathscr{G}$ of closed intervals that covers $C^{(3)}$.

2. Replacing each of the closed intervals in $\mathscr{G}$ by a finite set of closed intervals arising from the construction of the ternary Cantor set that offer the same covering.

Finite collection $\mathscr{G}$

We assume that $\{U_i\}$ to be any countable collection $\mathscr{F}$ of intervals covering $C^{(3)}$. So, $\mathscr{F} = \{U_i\}_{i=1}^{\infty}$ and $C^{(3)} \subset \bigcup_{i=1}^{\infty} U_i$.

The Cantor set $C^{(3)}$ is closed and bounded so $C^{(3)}$ is compact. This latter property of $C^{(3)}$ is used in obtaining from $\mathscr{F}$ a finite collection $\mathscr{G}$ of closed intervals that covers $C^{(3)}$ as follows:

Step 1: Expand slightly each interval in $\mathscr{F}$ to obtain an open cover of $C^{(3)}$:

(i) For each $i \in \mathbb{N}$ we consider $U_i \in \mathscr{F}$ with endpoints $a_i \leq b_i$.

(ii) Let $\varepsilon > 0$ and define $\widetilde{U}_i = (a_i - \varepsilon/2^{i+1}, b_i + \varepsilon/2^{i+1})$. Then $\{\widetilde{U}\}_{i=1}^{\infty}$ is an open cover of $C^{(3)}$ and $|\widetilde{U}_i|^s = (|U_i| + \varepsilon/2^i)^s$.

(iii) Since $s = \log 2/\log 3 < 1$, $2^s > 1$ so $1/2^s < 1$; recalling for $\forall x, y \in \mathbb{R}^+ \cup \{0\}$ and $0 \le \alpha < 1$, $(x+y)^\alpha \le x^\alpha + y^\alpha$ then

$$\sum_{i=1}^{\infty} |\widetilde{U}_i|^s = \sum_{i=1}^{\infty} (|U_i| + \varepsilon/2^i)^s \le \sum_{i=1}^{\infty} (|U_i|^s + \left(\frac{\varepsilon}{2^i}\right)^s) \tag{3.17}$$

we have

$$\sum_{i=1}^{\infty} \left(\frac{\varepsilon}{2^i}\right)^s = \varepsilon^s \sum_{i=1}^{\infty} \left(\frac{1}{2^s}\right)^i = \frac{\varepsilon^s 2^{-s}}{1 - 2^{-s}} \tag{3.18}$$

to obtain

$$\sum_{i=1}^{\infty} |\widetilde{U}_i|^s \le \sum_{i=1}^{\infty} |U_i|^s + \varepsilon^s \frac{1}{2^s - 1} \tag{3.19}$$

Step 2: Since $C^{(3)}$ is compact and $\{\widetilde{U}_{i=1}^{\infty}\}$ is an open cover of $C^{(3)}$, there exists a finite sub-cover of $C^{(3)}$, $\{V_i\}_{i=1}^n$ with $C^{(3)} \subset \cup_{i=1}^n V_i$ and each $V_i = \widetilde{U}_i$ for some i.

Step 3: Since $V_i \subset \overline{V}_i$ for $i = 1, 2, \ldots, n$ and $|\overline{V}_i| = |V_i|$, we take $\mathscr{G} = \{\overline{V}_i\}_{i=1}^n$, a finite collection of closed intervals that covers $C^{(3)}$. We observe that nothing precludes these closed intervals to overlap each other. That is, some pairwise intersections of these intervals may contain more that one element. From ineq. (3.19) we have that:

$$\sum_{i=1}^{\infty} |U_i|^s + \varepsilon^s \frac{1}{2^s - 1} \ge \sum_{i=1}^{\infty} |\widetilde{U}_i|^s \ge \sum_{i=1}^{n} |\overline{V}_i|^s \tag{3.20}$$

Step 4: Thus, we have to show that

$$\sum_{i=1}^{n} |\overline{V}_i|^s \ge 1 \tag{3.21}$$

Step 5: Since $\mathscr{G}$ originates from an arbitrary countable open cover of $C^{(3)}$, we have no information concerning the endpoints of each of these closed intervals. In this step, the left and right endpoints of each of the closed intervals in $\mathscr{G}$ are further adjusted using the following methodology, best explained by considering a generic example:

Step 5.1 Figure 3.5 shows the five first steps in the construction of the Cantor set. The collections $C_i^{(3)}$ of closed intervals that stem from the construction of $C^{(3)}$ form a net. That is, any two such intervals are either disjoint or else one is contained in the other. This net exists due to the self-similar pattern in the collections $C_i^{(3)}$ of closed intervals. Let $\cup_i C_i^{(3)}$ be the finite union of the closed interval in the collection $C_i^{(3)}$. By definition, the Cantor set $C^{(3)} = \bigcap_{i=1}^{\infty} \cup_i C_i^{(3)}$ and the line at the bottom of Figure 3.5 illustrates $C^{(3)}$ as the "dust" remaining from taking $\bigcap_{i=1}^{\infty} \cup_i C_i^{(3)}$.

Step 5.2 Consider $\overline{V}_i = [1/18, 1/4]$ as illustrated in Figure 3.5. That $\overline{V}_i$ defines a "slice" of the net but its left endpoint $1/18$ belongs to the complement of $C^{(3)}$. Its right endpoint $1/4$ is an element of the Cantor set and it does not coincide with any of the right endpoints of the closed intervals in all of the $C_i^{(3)}$ [14].

Step 5.3 The endpoints are adjusted without reducing the actual covering of $C^{(3)}$ by $\overline{V}_i$.

- The left endpoint of $\overline{V}_i$ is moved right to the left endpoint of some of the closed intervals in the $C_i^{(3)}$ as illustrated in Figure 3.6.
- The endpoint $1/4$ of $\overline{V}_i$ is modified: starting from $C_4^{(3)}$ and a given $\varepsilon > 0$, there exists a j such that endpoint $1/4 \in C^{(3)}$ is increased slightly to coincide with the right endpoint of one of the children closed intervals on level $C_j^{(3)}$ (j could be much larger than $i = 4$ and the smaller ε is, the larger j will be). In this instance, the right endpoint of $\overline{V}_i$ goes from $1/4$ to $547/2187$ which implies the "given" $\varepsilon > 1/8748$. Since every element of $C^{(3)}$ is approached by a sequence of endpoints, we can find such a right endpoint arbitrary closed to $1/4$ no matter how small ε is.

Step 5.4 Steps 5.1 to 5.3 are applied to each interval $\overline{V}_i \in \mathscr{G}$. This results in a refined covering of $C^{(3)}$ by a finite collection $\mathscr{G}$ of closed intervals $\overline{V}_i$ each having their left and right endpoints coincide respectively with the left and right endpoints of some closed intervals in all of the $C_i^{(3)}$.

Step 5.5 The generic example used to illustrate Steps 5.1 to 5.4 is indeed generic as it includes most if not all the possible cases that we could have. For instance:

- the endpoints of a given $\overline{V}_i$ coincide with the left and right endpoints of some closed intervals in all of the $C_i^{(3)}$ so there is no need to apply Steps 5.1 to 5.4 .
- the endpoints of a given $\overline{V}_i$ are such that the left endpoint is in the complement of $C^{(3)}$ but the right endpoint coincide with the right endpoint of some closed intervals in all of the $C_i^{(3)}$ and vice versa.
- the endpoints of a given $\overline{V}_i$ are such that the left endpoint is in $C^{(3)}$ but it is not a left endpoint of any of the closed intervals in all of the $C_i^{(3)}$ but the right endpoint of $\overline{V}_i$ coincide with the right endpoint of some closed intervals in all of the $C_i^{(3)}$ and vice versa.

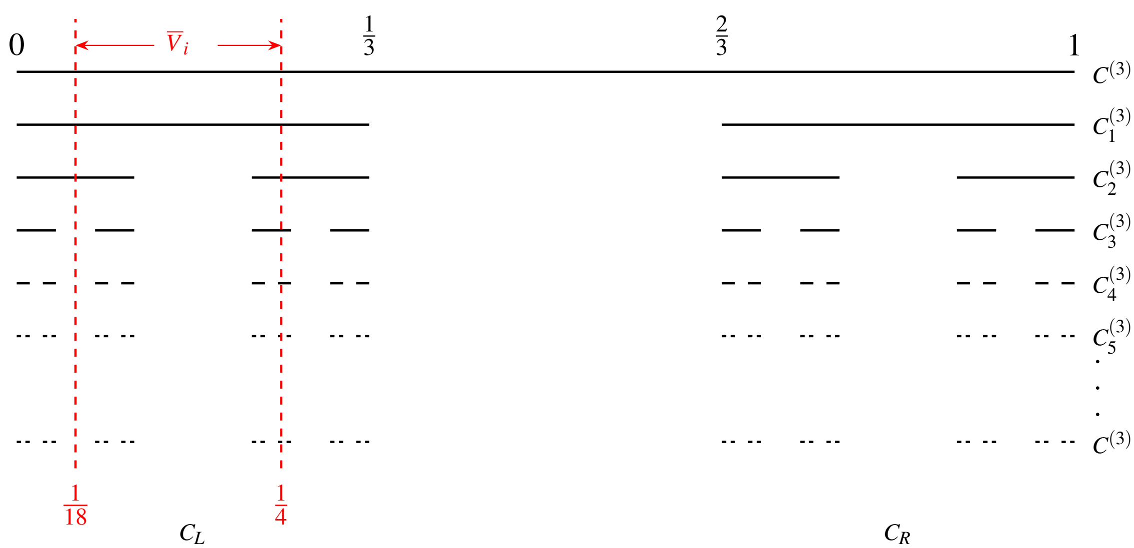


Figure 3.5: Ternary Cantor Set

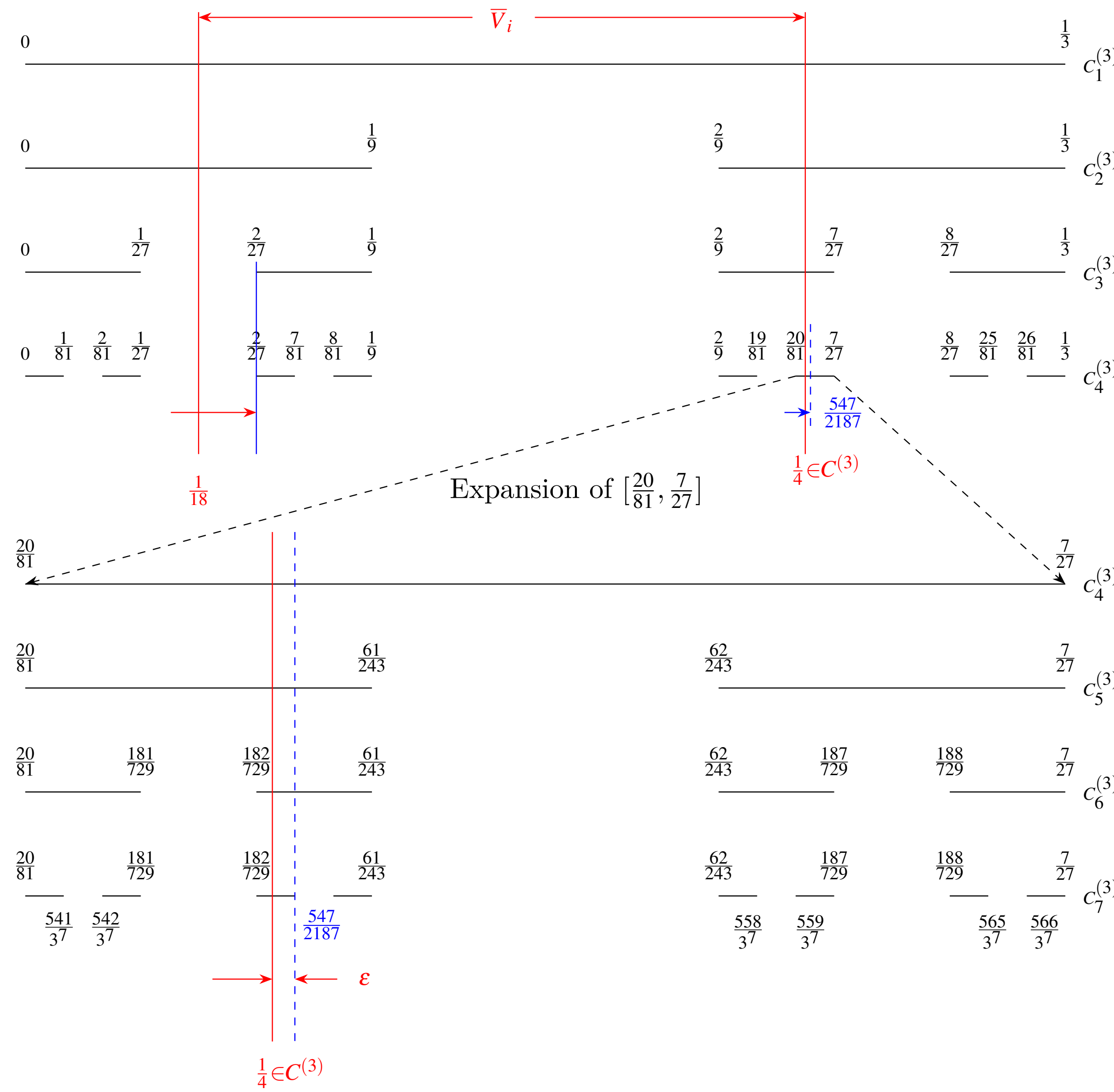


Figure 3.6: Adjustment of endpoints of $\overline{V}_i$ without affecting the cover of $C^{(3)}$

Reduction and replacement methodology

Up to now, we have that each $\overline{V}_i$ is a finite union of some of the closed intervals from the construction of $C^{(3)}$, perhaps from different construction levels, together with open intervals between them. In proving inequality (3.21), the goal is to get rid of these "gaps" (open intervals) without increasing the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$. We use a reduction and replacement methodology based on the approach of Falconer given in [14, p. 15] to reach that goal follows:

Step 1: Referring to Figure 3.5, the closed intervals are labelled as "J" and the opened intervals as "K". In any $C_i^{(3)}$, consider any closed interval J which is of length $1/3^i$.

- By construction of the net, if J is not at one of the extremities of $\overline{V}_i$, an opened interval, say K_1 precedes J and another opened interval, say K_2 follows J with the property that the length of K_1 and K_2 is greater or equal to the length of J.
- The construction of the ternary Cantor yields a self-similar net with the unique particularity observed in Figure 3.5 that no J is immediately preceded and followed by two $K's$ of exactly same length as J.
- If J is at the left extremity then it is followed by a K or if at the right extremity it is preceded by a K with the length of K equal to the length of J.

Step 2: Any of the $\overline{V}_i$ defines a "slice" of the net as shown in Figure 3.5. The endpoints of $\overline{V}_i$ coincide with endpoints of some $J's$ in that "slice" and can be represented by a finite union of $J's$ and $K's$. Amongst the $K's$ in that union, there exists a largest K that is unique. Proof (by contradiction):

Step 2.1 Assumption: that union contains two largest $K's$ of same length equal to $3^{-(i+1)}$.

Step 2.2 By construction of the net and the properties of the net intervals (see Figure 3.5), the "slice" contains either the full length of a J of a parent at level $C_i^{(3)}$, part of it or parts of two $J's$ separated by one K of length greater than

3^{-i}. These yield or define one K larger than any other $K's$ at the children levels that is for $C_j^{(3)}$ with $j > i$. This contradicts the assumption made in Step 2.1

Step 2.3 The latter is supported by the fact that no J is immediately preceded and followed by two $K's$ of exactly the same length as J as one side is always bigger.

Step 2.4 Therefore, in the above union representing $\overline{V}_i$ there exists a largest K that is unique .

Remark 3.1.8. The above proof relies on the uniqueness of a larger K if the finite union of $J's$ and $K's$ representing $\overline{V}_i$. This is particular to the ternary Cantor set as this uniqueness is lost when constructing Cantor set by removing two or more open intervals of equal length from a parent closed interval. For instance, a Cantor set can be constructed by dividing a parent closed interval in five equal parts and removing the second and fourth open intervals. So the parent closed interval is replaced by the union of three $J's$ and two $K's$ of equal length. This gives one J preceded and followed by $K's$ of same length as J.

Step 3: By construction of the net,

$$\overline{V}_i = (\cup_{i=1}^{N} J_i) \cup (\cup_{i=1}^{N-1} K_i) \tag{3.22}$$

with one of the $K_i's$, say K_j being the largest in that union. K_j divides the remaining $J's$ and $K's$ into two unions I_1 and I_2 respectively on the left and the right of K_j with $|K_j| \geq |I_1|, |I_2|$. I_1 and I_2 are closed intervals since their respective union starts and ends with a J. We use a reduction and replacement methodology to show that

$$|\overline{V}_i|^s \geq \sum_{i=1}^{N} |J_i|^s \tag{3.23}$$

giving that replacing $\overline{V}_i$ by the J_i does not increase the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$ in inequalities

(3.20) and (3.21) . Also, removing the $K's$ does not change the covering property as they do not contain any part of $C^{(3)}$.

Step 4: Basic case: for $N = 2$ (smallest value for N) we have $\overline{V}_i = J_1 \cup K_1 \cup J_2$. That is $I_1 = J_1$ and $I_2 = J_2$ and $K_j = K_1$. Then

$$|\overline{V_i}|^s = (|J_1| + |K_1| + |J_2|)^s \tag{3.24}$$

Since $|K_1| \geq |J_1|, |J_2|$ we have that $|K_1| \geq |J_1|/2 + |J_2|/2$ to obtain

$$|\overline{V}_i|^s \geq \left(\frac{2+1}{2}(|J_1| + |J_2|)\right)^s = (2+1)^s \left(\frac{1}{2}|J_1| + \frac{1}{2}|J_2|\right)^s \tag{3.25}$$

Since $f(t) = t^s \; \forall t > 0$, $0 < s < 1$ is concave we obtain with $(2+1)^s = 3^s = 2$ that

$$|\overline{V}_i|^s \geq 2\left(\frac{1}{2}|J_1|^s + \frac{1}{2}|(|J_2)^s\right) = |J_1|^s + |J_2|^s \tag{3.26}$$

Thus replacing $\overline{V}_i$ by J_1 and J_2 does not increase the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$.

Step 5: General case: Let $N > 2$ and from Steps 3 and 4 we can write

$$|\overline{V_i}|^s = (|I_1| + |K_j| + |I_2|)^s \tag{3.27}$$

Since $|K_j| \geq |I_1|, |I_2|$ we have that $|K_j| \geq |I_1|/2 + |I_2|/2$ to obtain

$$|\overline{V}_i|^s \geq \left(\frac{2+1}{2}(|I_1| + |I_2|)\right)^s = (2+1)^s \left(\frac{1}{2}|I_1| + \frac{1}{2}|I_2|\right)^s \tag{3.28}$$

Since $f(t) = t^s \; \forall t > 0$, $0 < s < 1$ is concave we obtain with $(2+1)^s = 3^s = 2$ that

$$|\overline{V}_i|^s \geq 2\left(\frac{1}{2}|I_1|^s + \frac{1}{2}|(|I_2)^s\right) = |I_1|^s + |I_2|^s \tag{3.29}$$

Thus replacing $\overline{V}_i$ by I_1 and I_2 does not increase the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$. Inner steps follow to reduce I_1 and I_2 to one net interval (i.e. one J_i) or to a union of two J_i's and one K_j that is addressed using Step 3. In the union(3.22) representing $|\overline{V}_i|$ there are $2N-1$ (odd number) J_i's and K_i's (including K_j) and for I_1 and I_2 they contain together $2N-2$ (even number) J_i's and K_i's. By construction of the net interval and since the sum of two odd integers is even then I_1 and I_2 contain each an odd number of J_i's and K_i's. Then this reduction is accomplished as follows

Step 5.1 Let I_1 and I_2 contain respectively $2M_1-1$ and $2M_2-1$ (odd numbers) of J_i's and K_i's. Then by Step 2.4, I_1 and I_2 contain respectively a unique and largest K_{j_1} and K_{j_2}.

Step 5.2 By Step 3 we obtain $I_1 = I_{11} \cup K_{j_1} \cup I_{12}$ and $I_2 = I_{21} \cup K_{j_2} \cup I_{22}$.

Step 5.3 By Step 4 we have

$$|I_1|^s \geq |I_{11}|^s + |I_{12}|^s \tag{3.30}$$

and

$$|I_2|^s \geq |I_{21}|^s + |I_{22}|^s \tag{3.31}$$

to obtain

$$|\overline{V}_i|^s \geq |I_{11}|^s + |I_{12}|^s + |I_{21}|^s + |I_{22}|^s \tag{3.32}$$

Thus replacing $\overline{V}_i$ by I_{11}, I_{12}, I_{21} and I_{22} does not increase the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$.

Step 5.4 Then with I_{11}, I_{12}, I_{21} and I_{22} we go back to Step 5.1 to obtain the new collection I_{111}, I_{112}, I_{121}, I_{122}, I_{211}, I_{212}, I_{221} and I_{222} to obtain

$$|\overline{V}_i|^s \geq |I_{111}|^s + |I_{112}|^s + |I_{121}|^s + |I_{122}|^s + |I_{211}|^s + |I_{212}|^s + |I_{221}|^s + |I_{222}|^s \tag{3.33}$$

Thus replacing $\overline{V}_i$ by $I_{111}, I_{112}, I_{121}, I_{122}, I_{211}, I_{212}, I_{221}$, and I_{222} does not increase the sum $\sum_{i=1}^{n} |\overline{V}_i|^s$. We continue this reduction and replacement process until all the $I_{iii\ldots}$ reduced to either one net interval (i.e. one J_i) or to a union of two

J_i's and one K_j which is further reduced using Step 4. This results in $\overline{V}_i$ being replaced in a finite number of steps by the N J_i's in the union for $\overline{V}_i$ in (3.22) with

$$|\overline{V}_i|^s \geq \sum_{i=1}^{N} |J_i|^s \tag{3.34}$$

which is the desired result. This completes the proof of inequality (3.23).

Step 6: Using the same for all the other $\overline{V}_i' s$ in the cover $\mathscr{G}$ of F are replaced using the same methodology.

Step 6.1 Since the J_i's replacing the $\overline{V}_i$'s are net intervals, there is a k such that all $|J_i| \geq 3^{-k}$. Using the same methodology of replacement, we reach, in a finite number of steps, a covering of $C^{(3)}$ by equal intervals of length 3^{-k}, which does not increase the sum $\sum_i |J_i|^s$.

Step 6.2 These must include, at least, all the closed intervals of J_k and possibly more since in the finite cover $\mathscr{G}$ of $C^{(3)}$ some of the $\overline{V}_i$ could overlap each other. Then the number of closed intervals of length 3^{-k} in this new cover of $C^{(3)}$ will be greater of equal to the number of closed intervals of same length in $C_k^{(3)}$.

Step 6.3 This implies that

$$\sum_i |\overline{V}_i|^s \geq 2^k(3^{-k})^s = 2^k((3^s)^{-k}) = 2^k(s^{-k}) = 1 \tag{3.35}$$

This completes the proof of (3.21).

For the original covering $\{U_i\}$, by (3.20), this gives

$$\sum_{i=1}^{\infty} |U_i|^s + \varepsilon^s \frac{1}{2^s - 1} \geq 1 \tag{3.36}$$

since $\varepsilon > 0$ was arbitrary we get that

$$\sum_{i=1}^{\infty} |U_i|^s \geq 1 \tag{3.37}$$

Taking the infimum over all δ-covers gives

$$\mathscr{H}^s_\delta(C^{(3)}) = \inf\left\{\sum_i |U_i|^s\right\} \geq 1 \tag{3.38}$$

Letting $\delta \to 0$ we obtain that $\mathscr{H}^s(C^{(3)}) \geq 1$.

In this section, we have shown that for $s = \log 2/\log 3$, $1 \leq \mathscr{H}^s(C^{(3)}) \leq 1$ which gives at once that the ternary Cantor set $C^{(3)}$ has dimension $s = \log 2/\log 3$ and Hausdorff measure $\mathscr{H}^s(C^{(3)}) = 1$. □

3.2 Measure of subset of $\mathbb{R}$ by mass distribution

This section presents the definition of a measure supported on a subset of $\mathbb{R}$ by mass distribution which is then applied to the ternary Cantor set. This is followed by giving a proof that the mass distribution measure of any subset of $[0,1]$ of the form $(a,b]$, $(a,b \in [0,1$ and $a < b)$ is given in terms of the extended Cantor-Lebesgue function as $F(b) - F(a)$. Similarly, we obtain the same result for the Hausdorff measure restricted to $C^{(3)}$.

3.2.1 Mass distribution on a subset of $\mathbb{R}$

The construction of a mass distribution [15, pp. 14-15] involves repeated subdivisions of a mass between parts of a bounded Borel subset $E \subset \mathbb{R}$.

Step 1 Preliminaries:

(i) Let $\mathscr{E}_0$ be a collection consisting of the single set E.

(ii) For $k = 1, 2, \ldots (k \in \mathbb{N})$, let each $\mathscr{E}_k$ be a finite collection of disjoint Borel subsets $\{U_i^{(k)}\}_{i=1}^{N_k}$ of E such that each set $U_i^{(k)}$ in $\mathscr{E}_k$ is contained in one of the sets of $\mathscr{E}_{k-1}$.

(iii) We assume that the maximum diameter of the sets $U_i^{(k)}$ in $\mathscr{E}_k$ tends to 0 as $k \to \infty$.

(iv) We define a mass distribution on these sets by repeated subdivision.

Step 2 We let $\mu(E)$ satisfy $0 < \mu(E) < \infty$ and we split this mass between the sets $\{U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, \ldots, U_{N_1}^{(1)}\}$ in $\mathscr{E}_1$ by defining $\mu(\cdot)$ in such a way that

$$\sum_{i=1}^{3} \mu(U_i^{(1)}) = \mu(E) \tag{3.39}$$

It should noted that the mass distribution among $\{U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, \ldots, U_{N_1}^{(1)}\}$ need not be uniform.

Step 3 Similarly, we assign masses to the sets of $\mathscr{E}_2$ so that if $\{U_1^{(2)}, U_2^{(2)}, \ldots, U_{k_j}^{(2)}\}$ are the sets of $\mathscr{E}_2$ contained in $U_j^{(1)}$ of $\mathscr{E}_1$ then (see Figure 3.7)

$$\sum_{i=1}^{k_j} \mu(U_i^{(2)}) = \mu(U_j^{(1)}), \quad j = 1, 2, \ldots, N_1 \tag{3.40}$$

In general, we assign masses so that

$$\sum_{i} \mu(U_i) = \mu(U) \tag{3.41}$$

for each set U in $\mathscr{E}_k$, where $\{U_i\}$ are the disjoint set in $\mathscr{E}_{k+1}$ contained in U.

Step 4 For each k, let E_k be the union of the sets in $\mathscr{E}_k$, that is:

$$E_k = U_1^{(k)} \cup U_2^{(k)} \cup U_3^{(k)} \cup \ldots \cup U_{N_k}^{(k)} \tag{3.42}$$

The subscript $N \in \mathbb{N}$ in the last term of the union in eq.(3.42) emphasizes the mass distribution to be over a finite number of sets.

Step 5 We define $\mu(A) = 0$ for all A with $A \cap E_k = \emptyset$

To complete the above method, let $\mathscr{E}$ denote the collection of sets that belong to $\mathscr{E}_k$ for some k together with the subsets of $E_k^c = \mathbb{R} \setminus E_k$ which is the complement of E_k relative to $\mathbb{R}$. That is:

$$\mathscr{E} = \{E : \exists k, E \subset \mathscr{E}_k \quad \text{or} \quad E \subset E_k^c\} \tag{3.43}$$

In eq.(3.43), the "$\exists k$" means "some k" and eq.(3.43) can be written as follows:

$$\mathscr{E} = \bigcup_{k=0}^{\infty} (\mathscr{E}_k \quad \cup \quad \mathscr{P}(E_k^c)) \tag{3.44}$$

Since E_k is the union of the sets in $\mathscr{E}_k$ then $\mu(E_k)$ is equal to the total mass being distributed. We can employ the above method to define the mass $\mu(A)$ for every set A in $\mathscr{E}$. This leads to the claim that by building sets from the sets in the collection $\mathscr{E}$, it specifies enough about the distribution of the mass μ across $\mathscr{E}$ to determine $\mu(A)$ for any Borel set A. Falconer states that this is indeed the case and we present a proposition in [15, p. 15] with a slight clarification:

Proposition 3. Let μ be defined on a collection of sets $\mathscr{E}$ as above. Then the definition of μ may be extended to all Borel subsets of $\mathbb{R}^n$ so that μ becomes a measure with its value $\mu(A)$ uniquely determined where $A \subset \mathbb{R}^n$ is a Borel set. The support of μ is contained in $E_\infty = \cap_{k=1}^{\infty} \overline{E}_k$.

with the following notes:

1. If A is any subset of $\mathbb{R}^n$, let

$$\mu(A) = \inf \left\{ \sum_{i=1}^{\infty} \mu(U_i) : A \cap E_\infty \subset \bigcup_{i=1}^{\infty} U_i \in \mathscr{E} \right\} \tag{3.45}$$

   To obtain $\mu(A)$, we take the greatest lower bound of the set of possible values of $\sum_{i=1}^{\infty} \mu(U_i)$ where the sets U_i are in $\mathscr{E}$ and cover $A \cap E_\infty$. In the above, $\mu(U_i)$ is defined for such U_i.

2. If $A \in \mathscr{E}$ then eq. (3.45) reduces to the mass $\mu(A)$ as specified in the construction.

3. As $\mu(\mathbb{R}^n \setminus E_k) = 0$, $\mu(A) = 0$ if A is a set that $A \cap E_k = \emptyset$ for some k. This implies the support of μ is in $\overline{E}_k$ for all k.

Remark 3.2.1. The mass distribution measure μ can also be obtain as the weak-limit of a sequence of measures. We will not touch on this here.

3.2.2 Mass distribution on the ternary Cantor set

This section presents how the above methodology can be used to define a mass distribution on the ternary Cantor set in such a way that it is as uniform as possible. The mass distribution is accomplished as follows:

Step 1 Let $[0,1]$ be the bounded Borel subset of $\mathbb{R}$ considered in this definition of a mass distribution. Let $\mathscr{C}_0^{(3)}$ be a collection consisting of the single set $[0,1]$.

Step 2 By construction, for $k = 1,2,\ldots (k \in \mathbb{N})$, let each $\mathscr{C}_k^{(3)}$ be a collection of 2^k closed and disjoint intervals as illustrated in Figure 3.7. The union of these intervals in $\mathscr{C}_k^{(3)}$ corresponds to E_k in the general mass distribution process above. These intervals are the required 2^k Borel subsets $U_i^{(k)}$ of $[0,1]$ with $i = 1,2,\ldots,2^k$. The construction of the Cantor set leads to a self-similar pattern and the generated closed intervals form a net such that each consecutive pair of intervals $U_i^{(k)}$ in $\mathscr{C}_k^{(3)}$ is contained in one of the sets of $\mathscr{C}_{k-1}^{(3)}$. In turn, each $U_i^{(k)}$ contains a consecutive pair of the sets in $\mathscr{C}_{k+1}^{(3)}$. For example, if $U_1^{(2)} = [0,1/3]$ then it contains the two consecutive pairs $[0,1/9]$ and $[2/9]$.

Step 3 Each of these closed intervals in $\mathscr{C}_k^{(3)}$ has a length (diameter) of $1/3^k$ which clearly tends to 0 as $k \to \infty$.

Step 4 Figure 3.7 shows the repeated subdivision generated by construction. For instance, $\mathscr{C}_k^{(3)}$ contains 2^k closed intervals then by construction, the next subdivision gives for $\mathscr{C}_{k+1}^{(3)}$ 2^{k+1} closed intervals and 2^k consecutive pairs where each pair are subsets of one closed interval in $\mathscr{C}_k^{(3)}$.

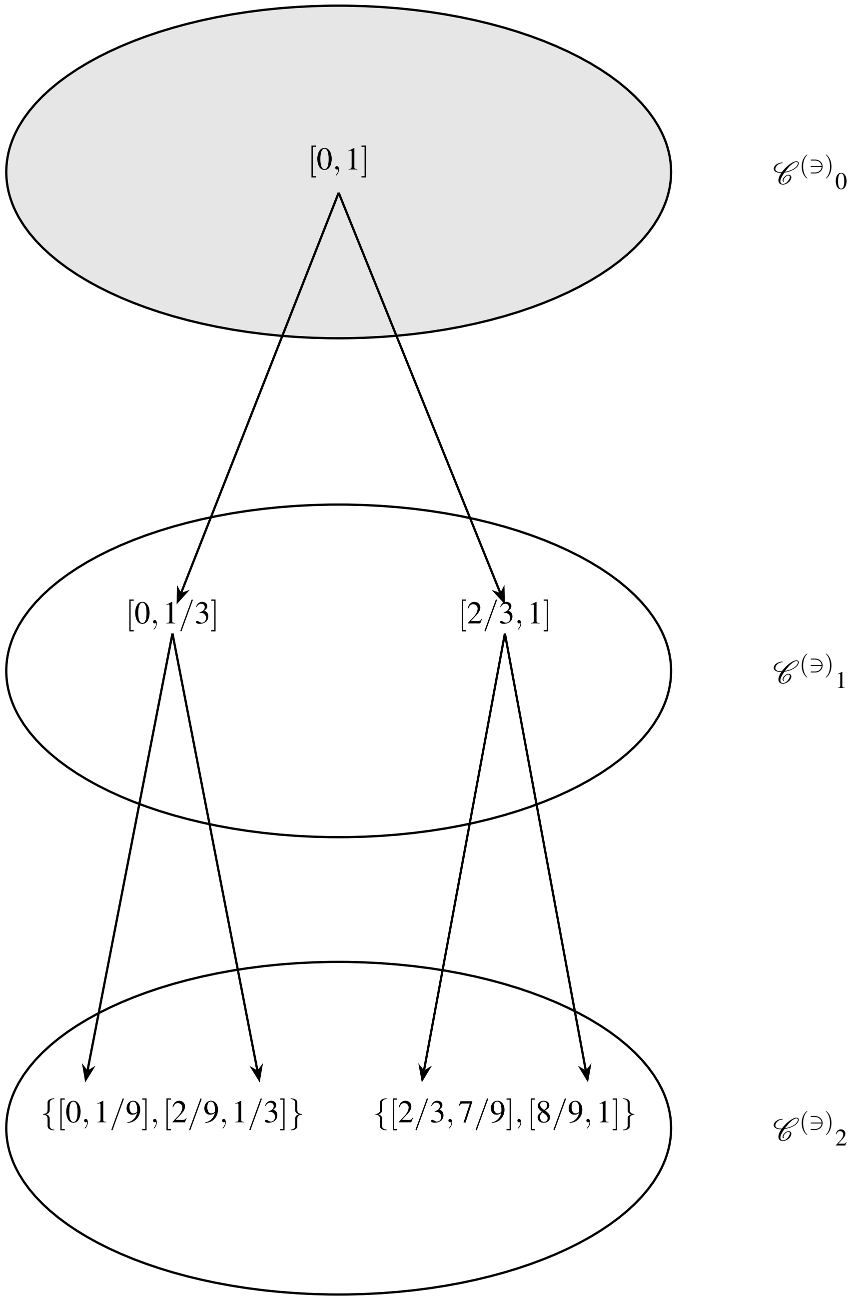


Figure 3.7: Steps in the construction of a mass distribution μ by repeated subdivision.

Step 5 Let $\mu([0,1]) = 1$ and we split this mass uniformly between the 2^k closed intervals in $\mathscr{C}_k^{(3)}$, giving a mass of $1/2^k$ for each closed interval in $\mathscr{C}_k^{(3)}$. Clearly,

$$\sum_{i=1}^{2^k} \mu(U_i^{(k)}) = \mu([0,1]) = 1 \ \forall k \in \mathbb{N} \tag{3.46}$$

Step 6 For each k, let C_k be the union of the sets in $\mathscr{C}_k^{(3)}$, that is:

$$C_k = U_1^{(k)} \cup U_2^{(k)} \cup U_3^{(k)} \cup \ldots \cup U_{2^k}^{(k)} \tag{3.47}$$

Step 7 Define $\mu(A) = 0$ for all A with $A \cap E_k = \emptyset$

Step 8 Let $\mathscr{C}$ denote the collection of sets that belongs to $\mathscr{C}_k^{(3)}$ for some k together with the subsets of $[0,1] \setminus C_k$ which is the complement of C_k relative to $[0,1]$. Since C_k is the union of the sets in $\mathscr{C}_k^{(3)}$ then by eq. (3.46) $\mu(C_k) = 1$. Then, we can define the mass $\mu(A)$ for every set A in $\mathscr{C}$, noting that if $A \in [0,1] \setminus C_k$ then $\mu(A) = 0$ since the mass is distributed only over $\mathscr{C}_k^{(3)}$.

Step 9 The preceding step specifies the distribution of the mass μ across $\mathscr{C}$ and Proposition 3 gives us the measure $\mu(A)$ for any Borel set $A \subset [0,1]$.

Example 1: Let $A = [0, 5/12]$, we then have that $5/12 \in (1//3, 2/3) \in [0,1] \setminus C_k$. We have $5/12 > 1/3$ but $\mu((1/3, 5/12]) = 0$ since $(1/3, 5/12] \cap C_k = \emptyset$ and $\mu([0,1/3]) = 1/2$ so, $\mu([0,5/12] = \mu([0,1/3]) + \mu((1/3,5/12]) = 1/2 + 0 = 1/2$.

Example 2: What is the mass distribution measure of a singleton like $\{0\} = [0,0]$? We know that $\bigcap_{i=1}^{\infty}[0, 0+1/3^i] = [0,0] = \{0\}$. So, by the theorem on measure of a decreasing intersection [2, p. 44] we have

$$\begin{aligned} \mu\left(\bigcap_{i=1}^{\infty}[0,0+1/3^i]\right) &= \lim_{i\to\infty}\mu([0,0+1/3^i]) \\ &= \lim_{i\to\infty} 1/2^i = 0 \end{aligned} \tag{3.48}$$

Therefore, the mass distribution measure $\mu(\{0\}) = 0$ and it also means that the mass of a single point is 0

3.2.3 Relation between mass distribution measure and the ternary Cantor-Lebesgue function

This section presents the relation between the mass distribution measure and the ternary Cantor-Lebesgue function by giving a proof of Proposition 4:

Proposition 4. For any closed interval $[0,a] \subset [0,1]$, the mass distribution measure is given by $\mu_m([0,a]) = F(a)$ and for half open interval $(a,b]$, we have $\mu_m((a,b]) = \mu_m([0,b]) - \mu_m([0,a]) = F(b) - F(a)$ for every $0 \leq a < b \leq 1$.

Proof. We prove the proposition by induction:

Step 1 $k = 0$: Trivial cases: $a = 0$ and $a = 1$

(i) Since $[0,0]$ is a singleton, we showed by eq.(3.48) in 92 the mass of a single point is 0. So, $\mu([0,0]) = 0 = F(0) = 0$.

(ii) $\mu([0,1]) = 1 = F(1) = 1$ by definition of unit mass.

(iii) We can write for these two trivial cases that $|\mu([0,a]) - F(a)| \leq 1 = 1/2^0$

Step 2 $k = 1$: endpoints are $\{0, 1/3, 2/3, 1\}$, $\mu_m([0,1/3]) = F(1/3) = 1/2$ and $\mu_m([0,2/3]) = F(2/3) = 1/2$. By monotonicity of μ_m and F, and that $\mu_m = 0$ on $\mathscr{E}_1^c$, we have: From Table 3.1 we conclude that $|\mu_m([0,a]) - F(a)| \leq 1/2^1 = 1/2$.

Step 3 From the above, we can then formulate the Induction Hypothesis: For $k = n-1$ and for any interval from construction $E = [a_0, b_0] \in \mathscr{E}_{n-1}$ (at level $k = n-1$) with positive mass, we suppose true that $q_0 = \mu_m([0,a_0]) = F(a_0)$ and $\forall a \in [0,1], |\mu_m([0,a]) - F(a)| \leq 1/2^{n-1}$.

Step 4 Induction Step: For $k = n$ and for any interval from construction $E = [a_0, b_0] \in \mathscr{E}_{k-1}$ (level $k = n-1$) and having a positive mass, we have that when a is an endpoint at the n^{th}

Construction level 1 ($k=1$)			
Sub-case No.	Interval considered	Implications for $\mu_m(\cdot)$	Implications for $F(\cdot)$
1	$a \in (0,1/3)$	$0 \le \mu_m([0,a]) \le 1/2$	$0 \le F(a) \le 1/2$
2	$a \in [1/3,2/3]$	$\mu_m([0,a]) = \mu_m([0,1/3] = 1/2$	$F(a) = 1/2$
3	$a \in (2/3,1)$	$1/2 \le \mu_m([0,a]) \le 1$	$1/2 \le F(a) \le 1$

Table 3.1: Mass distribution measure and Cantor function for construction level $k=1$.

level which is not a_0 or b_0, then $a = a_0 + 1/3^n$ or $a = a_0 + 2/3^n$ and by construction $\mu_m([0,a]) = q_0 + 1/2^n$. This can also been seen in Table 3.2 where by construction, $\mu_m([a_0,b_0])$ is decomposed equally by mass distribution as follows:

$$\begin{aligned} \mu_m([a_0,b_0]) &= \mu_m([a_0,a_0+1/3^n]) + \mu_m([a_0+2/3^n, a_0+3/3^n = b_0]) \\ \frac{1}{2^{n-1}} &= \frac{1}{2^n} + \frac{1}{2^n} \end{aligned} \tag{3.49}$$

By Induction Hypothesis, $\mu_m([0,a_0]) = q_0 = F(a_0)$ and $F(a) = F(a_0) + 1/2^n$ from construction of F, so $\mu_m([0,a]) = F(a)$. We conclude that $|\mu_m([0,a]) - F(a)| \le 1/2^n$ and

Construction level n ($k=n$)			
Sub-case No.	Interval considered	Implications for $\mu_m(\cdot)$	Implications for $F(\cdot)$
n_1	$a \in (a_0, a_0+1/3^n)$	$q_0 \le \mu_m([0,a]) \le q_0 + 1/2^n$	$q_0 \le F(a) \le q_0 + 1/2^n$
n_2	$a \in [a_0+1/3^n, a_0+2/3^n]$	$\mu_m([0,a]) = q_0 + 1/2^n$	$F(a) = q_0 + 1/2^n$
n_3	$a \in (a_0+2/3^n, a_0+1/3^{n-1} = b_0)$	$q_0 + 1/2^n \le \mu_m([0,a]) \le q_0 + 1/2^{n-1}$	$q_0 + 1/2^n \le F(a) \le q_0 + 1/2^{n-1}$

Table 3.2: Mass distribution measure and Cantor function for one group of sub-cases at construction level $k=n$.

equality holds at the endpoints of level $k=n$. Now, as $n \to \infty$, $\mu_m([0,a]) = F(a)\ \ \forall a \in [0,1]$ and $\mu_m((a,b]) = \mu_m([0,b]) - \mu_m([0,a]) = F(b) - F(a)$ for every $0 \le a < b \le 1$. This completes the proof of Proposition 4.

□

Appendix C presents a proof of Proposition 5 for the quaternary Cantor-Lebesgue function $W(x)$.

Proposition 5. For any closed interval $[0,a] \subset [0,1]$, the mass distribution measure is given by $\mu_m([0,a]) = W(a)$ and for a half open interval $(a,b]$, we have $\mu_m((a,b]) = \mu_m([0,b]) - \mu_m([0,a]) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$.

3.2.4 Relation between Hausdorff measure and the ternary Cantor-Lebesgue function

In this section we establish a relation between the Hausdorff measure of dimension $s = \log 2/\log 3$, restricted to $C^{(3)}$, of $[0,a] \subset [0,1]$ and the extended Cantor-Lebesgue function F. More precisely, we prove the claim that for every $0 \leq a \leq 1$, $\mathscr{H}^s([0,a] \cap C^{(3)}) = F(a)$. Then, the section concludes by the proof of relation between the Hausdorff measure restricted to $C^{(3)}$ of $(a,b] \subset [0,1]$ and F, $\mathscr{H}^s((a,b] \cap C^{(3)}) = F(b) - F(a)$ for every $0 \leq a < b \leq 1$.

By Theorem 3.1.7, $C^{(3)}$ has dimension $s = \log 2/\log 3$ and Hausdorff measure $\mathscr{H}^s(C^{(3)}) = 1$. The ternary Cantor set can be constructed using the IFS given in eq. (2.65). $T_0(x)$ and $T_1(x)$ are similarity transformations of scale factor $0 < \lambda = 1/3 < 1$. The value of λ makes them contractions and also Lipschitz mappings:

$$\begin{aligned} |T_0(x) - T_0(y)| &\leq \tfrac{1}{3}|x-y| \quad \forall x,y \in [0,1] \\ |T_1(x) - T_1(y)| = |T_0(x) + \tfrac{2}{3} - T_0(y) - \tfrac{2}{3}| = |T_0(x) - T_0(y)| &\leq \tfrac{1}{3}|x-y| \end{aligned} \tag{3.50}$$

Then, by the scaling property of Hausdorff measure [15, p. 46] we have for every $0 \leq a < b \leq 1$:

$$\mathscr{H}^s(T_k([a,b])) = \left(\frac{1}{3}\right)^s \mathscr{H}^s([a,b]) = \frac{1}{2}\mathscr{H}^s([a,b]) \quad k = \{0,1\} \tag{3.51}$$

Then, the heuristic calculation performed in proving Theorem 3.1.7 can be extended as follows:

(a) At construction level 2, define $C_{L_1} = [0,1/9] \cap C^{(3)}$, $C_{L_2} = [2/9,1/3] \cap C^{(3)}$, $C_{R_1} = [2/3,7/9] \cap C^{(3)}$, $C_{R_2} = [8/9,1] \cap C^{(3)}$ all disjoint, to obtain: $C^{(3)} = C_{L_1} \cup C_{L_2} \cup C_{R_1} \cup C_{R_2}$.

(b) We can write

$$\begin{aligned}\mathscr{H}^s(C^{(3)}) &= \mathscr{H}^s(C_{L_1})+\mathscr{H}^s(C_{L_2})+\mathscr{H}^s(C_{R_1})+\mathscr{H}^s(C_{R_2}) \\ &= 4\times\frac{1}{(3^2)^s}\mathscr{H}^s(C^{(3)}) \\ 1 &= 4\times\frac{1}{(3^s)^2}.\end{aligned} \tag{3.52}$$

Eq. (3.52) implies that $\mathscr{H}^s(C_{L_1})=\mathscr{H}^s(C_{L_2})=\mathscr{H}^s(C_{R_1})=\mathscr{H}^s(C_{R_2})=1/2^2$.

(c) Continuing inductively, then at construction level n, the Hausdorff measure of the intersection of each closed interval in the union $C_n^{(3)}$ with $C^{(3)}$ equals $1/2^n$ since the scaling ratio is $1/3^n$.

At construction level n, section 3.2.2 concluded to the uniform distribution of the unit mass, assigning a mass of $1/2^n$ to each closed interval giving mass distribution measure of $1/2^n$. Clearly, the mass of each of these closed intervals U in $C_n^{(3)}$ equals the Hausdorff measure of the intersection of each of these closed intervals with $C^{(3)}$, $1/2^n$ and this $\forall n\in\mathbb{N}$. So, we obtain that $\mathscr{H}(U\cap C^{(3)})=\mu_m(U)$ for every closed interval U arising in the construction of $C^{(3)}$ where by the definition of $\mathscr{E}$ in eq.(3.44, $U\in\mathscr{E}$ Also, the mass of the subsets of the complement of $C^{(3)}$ is 0 which implies the Hausdorff measure of these subsets is 0. This leads to the following proposition:

Proposition 6. For any closed interval $[0,a]\subset[0,1]$, the Hausdorff measure, restricted to $C^{(3)}$, is given by $\mathscr{H}([0,a]\cap C^{(3)})=F(a)$ and for half open interval $(a,b]$, we have $\mathscr{H}((a,b]\cap C^{(3)})=\mathscr{H}([0,b]\cap C^{(3)})-\mathscr{H}([0,a]\cap C^{(3)})=F(b)-F(a)$ for every $0\le a<b\le 1$.

Proof.
With every closed interval U arising in the construction of $C^{(3)}$ contained in $\mathscr{E}$, we have established that $\mathscr{H}(U\cap C^{(3)})=\mu_m(U)$ $\forall U\in\mathscr{E}$. So the proof of Proposition 6 is exactly the same as that of Proposition 4, since all that proof uses is the knowledge of $\mu_m(U)$ for such U. □

Appendix D presents a proof of Proposition 6 for the quaternary Cantor-Lebesgue function which formulation is as follows:

Proposition 7. For any closed interval $[0,a] \subset [0,1]$, the Hausdorff measure, restricted to $C^{(4)}$, is given by $\mathscr{H}([0,a] \cap C^{(4)}) = W(a)$ and for half open interval $(a,b]$, we have $\mathscr{H}((a,b] \cap C^{(4)}) = \mathscr{H}([0,b] \cap C^{(4)}) - \mathscr{H}([0,a] \cap C^{(4)}) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$.

3.2.5 Lebesgue-Stieltjes measures

From [19, p. 30-31], we briefly present the Lebesgue-Stieltjes measures on $\mathbb{R}$. These are a generalization of the Lebesgue measure. To obtain these Lebesgue-Stieltjes measures on $\mathbb{R}$ we first take an increasing, right-continuous function $G : \mathbb{R} \to \mathbb{R}$, and second, assign to a half-open interval $(a,b]$ the measure:

$$\mu_G((a,b]) = G(b) - G(a) \tag{3.53}$$

Four reasons motivates the use of half-open intervals:

(a) a non-zero measure could be assign to a single point (see 2 below)

(b) we observe that the complement of a half-open interval of the type $(a,b]$ can be a finite union of half-open intervals of the same type where some of the half-open intervals could be infinite.

(c) the Borel σ-algebra on $\mathbb{R}$, $\mathscr{B}(\mathbb{R})$, can be generated from the algebra of unions of disjoint half-opened intervals [16, Prop. 1.2(c), p. 22; p. 33]. We use that proposition from Folland to arrive at the conclusion of Theorem 3.0.1

(d) intervals half-open at the left offer consistency when verifying the right-continuity of G, since

$$\bigcap_{k=1}^{\infty} (a, a + 1/k] = \emptyset \tag{3.54}$$

so, from the measure of a decreasing intersection [2, p. 44] and if G is to define a measure we need

$$\mu_G(\bigcap_{k=1}^{\infty} (a, a + 1/k]) = \lim_{k \to \infty} \mu_G((a, a + 1/k]) = 0 \tag{3.55}$$

or by direct right continuity

$$\lim_{k\to\infty}[G(a,a+1/k)-G(a)] = \lim_{x\to a^+}G(x)-G(a) = 0 \tag{3.56}$$

Conversely, as stated in the next theorem, any such function G defines a unique Borel measure on $\mathbb{R}$.

Theorem 3.2.2. (Ref:[19, p.30], [16, Thm 1.16, p. 35]) Suppose that $G:\mathbb{R}\to\mathbb{R}$ is an increasing, right-continuous function. Then there is a unique Borel measure $\mu_G:\mathscr{B}(\mathbb{R})\to[0,]$ such that

$$\mu_G((a,b]) = G(b)-G(a) \tag{3.57}$$

for every $a<b$.

Here $\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$ [19, p. 34].

We have the following relevant examples:

Example 1: If $G(x)=x$ then μ_G is Lebesgue measure on $\mathbb{R}$ with $\mu_G((a,b]) = G(b)-G(a) = b-a = m((a,b])$.

Example 2: If G monotone function on $\mathbb{R}$, so G has only points of continuity or jump discontinuity. Then $\forall x\in\mathbb{R}$,

$$\mu_G(\{x\}) = \lim_{n\to\infty}\mu_G((a_n,b_n]) = \lim_{n\to\infty}(G(b_n)-G(a_n)) \tag{3.58}$$

for all sequences $a_n<b_n$ with $a_n\to x-$ and $b_n\to x+$. Therefore,

$$\mu_G(\{x\}) = G(x+)-G(x-) = \begin{cases} 0 & \text{if } \; G \text{ is continuous at } x \\ h & \text{otherwise, } \; h\text{: height of jump at } x \end{cases} \tag{3.59}$$

This example illustrates the significance of using half-open intervals because by eq. (3.59), the Lebesgue-Stieltjes measure assigns non-zero measure to a single point where G has a finite jump.

The Cantor-Lebesgue functions F and W are both an increasing, right-continuous functions. From Propositions 4 and 6, and Theorem 3.2.2, the question arises whether there is a relation between $\mu_{m_3}, \mathscr{H}^s$ and μ_F (G replaced by F)? By Theorem 3.2.2 any such increasing, right-continuous function G defines a unique Borel measure on $\mathbb{R}$. F is increasing and continuous function on $[0,1]$ but that can be further extended to the whole $\mathbb{R}$ by requiring that $F(x) = 0$ if $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, preserving the continuity of F on $\mathbb{R}$. Then using the proposition by Folland cited above, that $\mathscr{B}(\mathbb{R})$, can be generated from the algebra of unions of disjoint half-opened intervals, we get that F defines a unique Borel Measure μ_F such that for every Borel set $A \in \mathscr{B}$ we have:

$$\mu_{m_3}(A) = \mathscr{H}^s(A \cap C^{(3)}) = \mu_F(A) \tag{3.60}$$

answering the question in the affirmative. The above is illustrated in Figure 3.1. Similarly for the quaternary Cantor set, we have that W defines a unique Borel Measure μ_W such that for every Borel set $A \in \mathscr{B}$ we have:

$$\mu_{m_4}(A) = \mathscr{H}^s(A \cap C^{(4)}) = \mu_W(A) \tag{3.61}$$

This completes the first part of the proof of Theorem 3.0.1.

3.2.5.1 Measure on an interval as a Cantor set

In Section 2.1.2.1, the subdivision process used to create a binary tree of disjoint half-open intervals (see Figure 2.2) is in fact a process to obtain the binary Cantor set which is $[0,1) \cup 1 = [0,1]$. To see this, we take the intersection of all the unions of half-open intervals created at each level k that gives $[0,1)$ then we add 1 to obtain the binary Cantor set $[0,1]$. We have shown in Example 1 above that for $G(x) = x$ that Lebesgue measure m is a unique measure such that $\mu_G((a,b]) = G(b) - G(a) = b - a = m((a,b])$ for every $0 \leq a < b \leq 1$. The binary Cantor set can be constructed using the following IFS:

$$\begin{aligned} B_0(x) &= \frac{x}{2} \\ B_1(x) &= \frac{x+1}{2} \end{aligned} \tag{3.62}$$

Since $B_0(x)$ and $B_1(x)$ are well-defined and continuous over $\mathbb{R}$, their inverses

$$\begin{aligned} B_0^{-1}(x) &= 2x \\ B_1^{-1}(x) &= 2x-1 \end{aligned} \tag{3.63}$$

have the same properties. The binary Cantor set $[0,1]$ enjoys results similar to what is found in Propositions 4 and 6 for the ternary and quaternary Cantor sets with similar proofs that will not be repeated here. These results are ($s=1$):

$$\begin{aligned} \mu_{m_2}((a,b]) &= \mu_{m_2}([0,b]) - \mu_{m_2}([0,a]) \\ &= G(b)-G(a) = (b-a) \text{ for every } 0 \le a < b \le 1 \\ \mathscr{H}^s((a,b]) &= \mathscr{H}^s([0,b]) - \mathscr{H}^s([0,a]) \\ &= G(b)-G(a) = (b-a) \text{ for every } 0 \le a < b \le 1. \end{aligned} \tag{3.64}$$

3.3 Hutchinson's theorem: measure on self-similar sets

In this section, we present a theorem by Hutchinson [18] that there exists a unique measure with support on the ternary and quaternary Cantor sets and which satisfies recursive relations. This is followed by a proof that the mass distribution measure satisfies these recursive expressions.

Following [14, p. 119], a mapping $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is called a contraction if $|\psi(x)-\psi(y)| \le c|x-y|$ for all $x,y \in \mathbb{R}$, where $c<1$. Clearly, the IFS's for the ternary and quaternary Cantor sets are contraction. They are also continuous over $\mathbb{R}$ as well as their respective inverse. The infimum of c in each of the IFS's is the contraction ratio r: $1/3$ and $1/4$ for the ternary and quaternary

Cantor sets respectively. A contraction that transforms every subset of $\mathbb{R}$ to a geometrically similar set is called a similitude. In the present case, each of the IFS's represents a similitude being a composition of a dilation and a translation. The contraction ratio is the scale factor of the similitude.

Definition 3.3.1. A set $E \subset \mathbb{R}$ is called invariant for a set of contractions $\psi_1, \ldots, \psi_m$ if $E = \cup_{j=1}^{m} \psi_j(E)$.

The ternary and quaternary Cantor sets, $C^{(3)}$ and $C^{(4)}$, are invariant for their respective IFS. We adopted Falconer [14, p. 120] statement of Hutchinson's theorem [18], in the following form:

Theorem 3.3.2. There exist a unique Borel measure μ with support contained in E, such that for any Borel set F,

$$\mu(F) = \sum_{j=1}^{m} r_j^s \mu(\psi_j^{-1}(F)) \tag{3.65}$$

where each IFS equation ψ_j has a contraction ratio r_j. For the ternary and quaternary the contraction ratio r_j is the same in both equations of the IFS so the infimum of the $\{r_j\}$ defining r is 1/3 and 1/4.

For the ternary and quaternary Cantor sets with respective Hausdorff dimension of $s_3 = \log 2/\log 3$ and $s_4 = \log 2/\log 4 = 1/2$, eq.(3.65) takes the form for any Borel set F:

$$\mu_3(F) = \frac{1}{2}(\mu_3(T_0^{-1}(F)) + \mu_3(T_1^{-1}(F))) \text{ ternary} \tag{3.66}$$

$$\mu_4(F) = \frac{1}{2}(\mu_4(\tau_0^{-1}(F)) + \mu_4(\tau_1^{-1}(F))) \text{ quaternary} \tag{3.67}$$

For the ternary Cantor set, $r_j = 1/3$ for both $j = \{0,1\}$ and $(1/3)^{s_3} = 1/2$. Similarly, for the quaternary Cantor set $r_j = 1/4$ for both $j = \{0,1\}$ and $(1/4)^{s_4} = 1/2$. This explains the 1/2 factor on the right-hand side of both eqs.(3.66) and (3.67). We observe the absence of an explicit expression for μ_3 and μ_4, rather the expressions for μ_3 and μ_4 are recursive.

3.3.1 Ternary and Quaternary Cantor Measure by mass distribution

We present a verification that for any Borel set $A \subset [0,1]$, the mass distribution measures $m_{m_3}(\cdot)$ and $m_{m_4}(\cdot)$ satisfy the recursive relations in eq. (3.66) and eq. (3.67) respectively. The IFS's and their inverse for the ternary and quaternary Cantor sets are given by the eqs. (2.65), (2.66), (2.68) and (2.69). The main difference between these is the contraction ratio and this enable writing these in a common form for $C^{(p)}$ where $p \in \{3,4\}$ (ternary when $p=3$ and quaternary when $p=4$):

$$\begin{aligned}\gamma_0(x) &= \frac{x}{p} \\ \gamma_1(x) &= \frac{x+2}{p}\end{aligned} \tag{3.68}$$

$$\begin{aligned}\gamma_0^{-1}(x) &= px \\ \gamma_1^{-1}(x) &= px-2\end{aligned} \tag{3.69}$$

and the recursive expression for the measure takes the form

$$\mu_{m_p}(A) = \frac{1}{2}(\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A))) \tag{3.70}$$

Step 1 Note that $\gamma_0(x)$, $\gamma_1(x)$, $\gamma_0^{-1}(x)$ and $\gamma_1^{-1}(x)$ are linear transformations, straight lines in $\mathbb{R}^2$, continuous over all of $\mathbb{R}$.

Step 2 Additional properties of γ_0, γ_1,γ_0^{-1} and γ_1^{-1} when domain is the set of closed intervals in construction of $C^{(p)}$:

(i) Figure 3.8 shows the trees of application of γ_0, γ_1, γ_0^{-1} and γ_1^{-1} on the set of closed intervals - represented as $\bullet$ - in construction of $C^{(p)}$

(ii) Base case $k=0$, let $J_{0,0}=[0,1]$ then $\gamma_0(J_{0,0})=[0,1/p]=J_{1,0}\in\mathscr{E}$ and $\gamma_1(J_{0,0})=[2/p,3/p]=J_{1,1}\in\mathscr{E}$.

(iii) For $k\geq 1$ (see Figure 3.8):

$$\begin{aligned}\mathscr{E}_k^L &= \left\{J_{k,\ell}:J_{k,\ell}\subset[0,1/2]\right\} &= \left\{J_{k,\ell}:0\leq\ell\leq 2^{k-1}-1\right\}\\ \mathscr{E}_k^R &= \left\{J_{k,\ell}:J_{k,\ell}\subset[1/2,1]\right\} &= \left\{J_{k,\ell}:2^{k-1}\leq\ell\leq 2^{k}-1\right\}\end{aligned} \tag{3.71}$$

(iv) As linear maps in $\mathbb{R}$, γ_0 and γ_1 are bijections since:

(a) From Figure 3.8(a), γ_0 maps the whole tree at level $k-1$ onto the left-half tree at level k and γ_1 maps the whole tree at level $k-1$ onto the right-half tree at level k

(b) From Figure 3.8(b), γ_0^{-1} maps the left-half tree at level k onto the whole tree at level $k-1$ and γ_1^{-1} maps the right-half tree at level k onto the whole tree at level $k-1$

(c) From Figure 3.8(a) and (b), these mappings are clearly one-to-one.

(v) Consider the 2^{k-1} closed intervals $\left\{J_{k-1,\ell}\right\}_{\ell=0}^{2^{k-1}-1}$ contained in $\mathscr{E}_{k-1}$.

(a) Each $J_{k-1,\ell}=\gamma_0^{-1}(\widetilde{J}_{k,n})$ for some $\widetilde{J}_{k,n}\in\mathscr{E}_k^L$ since γ_0 is a bijection between $\mathscr{E}_{k-1}$ and $\mathscr{E}_k^L$. Hence, for $\widetilde{J}_{k,n}\in\mathscr{E}_k^R$, that is $\widetilde{J}_{k,n}\subset[2/p,3/p]$, $\gamma_0^{-1}(\widetilde{J}_{k,n})\subset[2,3]$ by eq. (3.69) since γ_0^{-1} is continuous over all of $\mathbb{R}$. Therefore, $\widetilde{J}_{k,n}\in\mathscr{E}_k^R$ implies $\gamma_0^{-1}(\widetilde{J}_{k,n})$ disjoint from any $J_{k-1,\ell}\in\mathscr{E}_{k-1}$.

(b) Each $J_{k-1,\ell}=\gamma_1^{-1}(\widetilde{J}_{k,n})$ for some $\widetilde{J}_{k,n}\in\mathscr{E}_k^R$ since γ_1 is a bijection between $\mathscr{E}_{k-1}$ and $\mathscr{E}_k^R$. Hence, for $\widetilde{J}_{k,n}\in\mathscr{E}_k^L$, that is $\widetilde{J}_{k,n}\subset[0,1/p]$, $\gamma_1^{-1}(\widetilde{J}_{k,n})\subset[-2,-1]$ by eq. (3.69) since γ_1^{-1} is continuous over all of $\mathbb{R}$. Therefore, $\widetilde{J}_{k,n}\in\mathscr{E}_k^L$ implies $\gamma_1^{-1}(\widetilde{J}_{k,n})$ disjoint from any $J_{k-1,\ell}\in\mathscr{E}_{k-1}$.

(c) Thus, $\gamma_0^{-1}(\mathscr{E}_k^R)$ does not intersect any $J_{k,\ell}\in\mathscr{E}_{k-1}$ so $\forall\widetilde{J}_{k,n}\in\mathscr{E}_k^R$, $\gamma_0^{-1}(\widetilde{J}_{k,n})\subset\mathscr{E}_{k-1}^c\subset(C^{(p)})^c$ and the mass $\mu_{m_p}(\gamma_0^{-1}(\widetilde{J}_{k,n}))=0$. Similarly, $\gamma_1^{-1}(\mathscr{E}_k^L)$ does not intersect any $J_{k,\ell}\in\mathscr{E}_{k-1}$, so the mass $\mu_{m_p}(\gamma_1^{-1}(\widetilde{J}_{k,n}))=0$ $\forall\widetilde{J}_{k,n}\in\mathscr{E}_k^L$.

(d) For all $\mathscr{U} \subset (-\infty, 0] \subset (C^{(p)})^c$, the mass $\mu_{m_p}(\mathscr{U}) = 0$. Since γ_0^{-1} is continuous on all of $\mathbb{R}$ then by eq.(3.69), $\gamma_0^{-1}(\mathscr{U}) \subset (-\infty, 0]$ and the mass $\mu_{m_p}(\gamma_0^{-1}(\mathscr{U})) = 0$. Similarly, since γ_1^{-1} is continuous on all of $\mathbb{R}$ then by eq.(3.69), $\gamma_1^{-1}(\mathscr{U}) \subset (-\infty, -2]$ and the mass $\mu_{m_p}(\gamma_1^{-1}(\mathscr{U})) = 0$.

(e) For all $\mathscr{U} \subset (3/p, \infty) \subset (C^{(p)})^c$, the mass $\mu_{m_p}(\mathscr{U}) = 0$. Since γ_0^{-1} is continuous on all of $\mathbb{R}$ then by eq.(3.69), $\gamma_0^{-1}(\mathscr{U}) \subset (3, \infty)$ and the mass $\mu_{m_p}(\gamma_0^{-1}(\mathscr{U})) = 0$. Similarly, since γ_1^{-1} is continuous then by eq.(3.69), $\gamma_1^{-1}(\mathscr{U}) \subset (2, \infty)$ and the mass $\mu_{m_p}(\gamma_1^{-1}(\mathscr{U})) = 0$.

(vi) In terms of mass distribution we have

Case 1: Consider $J_{k,\ell} \in \mathscr{E}_k^L$ then

$$\begin{aligned} \frac{1}{2^k} = \mu_{m_p}(J_{k,\ell}) &= \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(J_{k,\ell})) + \mu_{m_p}(\gamma_1^{-1}(J_{k,\ell}))\right] \\ &= \frac{1}{2}\left[\frac{1}{2^{k-1}} + 0\right] = \frac{1}{2^k} \end{aligned} \tag{3.72}$$

Case 2: Consider $J_{k,\ell} \in \mathscr{E}_k^R$ then

$$\begin{aligned} \frac{1}{2^k} = \mu_{m_p}(J_{k,\ell}) &= \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(J_{k,\ell})) + \mu_{m_p}(\gamma_1^{-1}(J_{k,\ell}))\right] \\ &= \frac{1}{2}\left[0 + \frac{1}{2^{k-1}}\right] = \frac{1}{2^k} \end{aligned} \tag{3.73}$$

Case 3: For all $\mathscr{U} \subset (-\infty, 0] \subset (C^{(p)})^c$

$$\begin{aligned} 0 = \mu_{m_p}(\mathscr{U}) &= \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(\mathscr{U})) + \mu_{m_p}(\gamma_1^{-1}(\mathscr{U}))\right] \\ &= \frac{1}{2}[0 + 0] = 0 \end{aligned} \tag{3.74}$$

Case 4: For all $\mathscr{U} \subset [1,\infty) \subset (C^{(p)})^c$

$$\begin{aligned} 0 \;=\; \mu_{m_p}(\mathscr{U}) &= \tfrac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(\mathscr{U})) + \mu_{m_p}(\gamma_1^{-1}(\mathscr{U}))\right] \\ &= \tfrac{1}{2}\left[0+0\right] = 0 \end{aligned} \tag{3.75}$$

Therefore, in terms of mass distribution, we have shown that $\forall \mathscr{U} \in \mathscr{E}$:

$$\mu_{m_p}(\mathscr{U}) = \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(\mathscr{U})) + \mu_{m_p}(\gamma_1^{-1}(\mathscr{U}))\right] \tag{3.76}$$

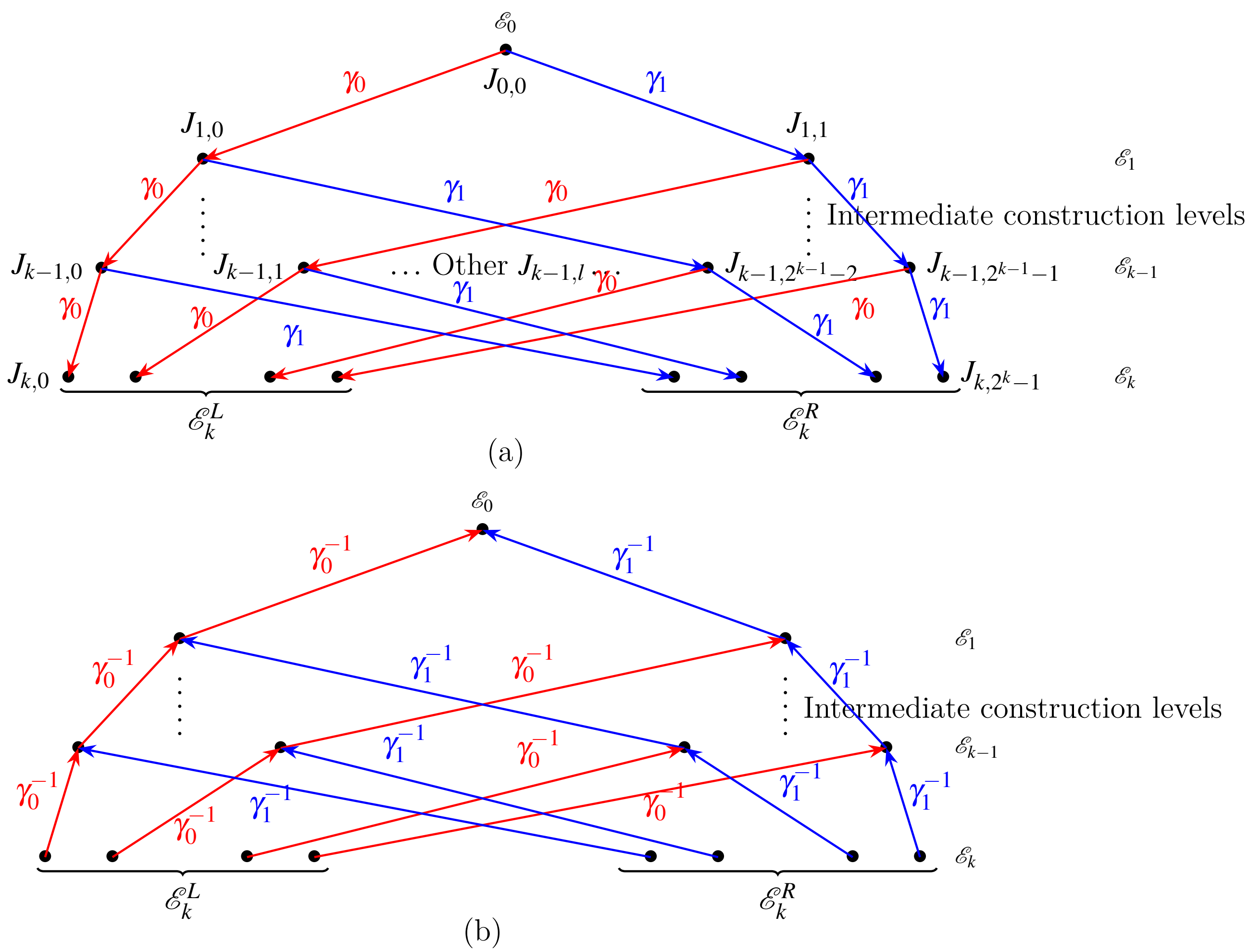


Figure 3.8: Trees of application of γ_0, γ_1, γ_0^{-1} and γ_1^{-1} on the set of closed intervals in construction of $C^{(p)}$

Step 3 Lower bound for $\mu_{m_p}(A)$:

(i) Consider a cover $\{V_i\}$ of $A \cap C^{(p)}$ with $V_i \in \mathscr{E}$. That is, $A \cap C^{(p)} \subset \bigcup_i V_i$. Since γ_0^{-1} and γ_1^{-1} are continuous on all of $\mathbb{R}$, we have that $\{\gamma_0^{-1}(V_i)\}$ and $\{\gamma_1^{-1}(V_i)\}$ are in $\mathscr{E}$. Now consider:

$$\gamma_0(\gamma_0^{-1}(A) \cap C^{(p)}) = \gamma_0(\gamma_0^{-1}(A)) \cap \gamma_0(C^{(p)}) \subset A \cap C^{(p)} \subset \bigcup_i V_i \tag{3.77}$$

where $\gamma_0(C^{(p)})$ is the left part of $C^{(p)}$ with $\gamma_0(C^{(p)}) \subset C^{(p)}$. Then from eq. (3.77) we have that

$$A \cap \gamma_0(C^{(p)}) \subset \bigcup_i V_i \tag{3.78}$$

Then we obtain from eq. (3.78) that

$$\begin{aligned} \gamma_0^{-1}(A \cap \gamma_0(C^{(p)})) &\subset \gamma_0^{-1}\left(\bigcup_i V_i\right) \\ \gamma_0^{-1}(A) \cap C^{(p)} &\subset \gamma_0^{-1}\left(\bigcup_i V_i\right) \end{aligned} \tag{3.79}$$

(ii) By definition:

$$\begin{aligned} \mu_{m_p}(\gamma_0^{-1}(A)) &= \inf\left\{\textstyle\sum_i \mu_{m_p}(\mathscr{U}_i) : \gamma_0^{-1}(A) \cap C^{(p)} \subset \bigcup_i \mathscr{U}_i,\ \mathscr{U}_i \in \mathscr{E}\right\} \\ &\leq \textstyle\sum_i \mu_{m_p}(\gamma_0^{-1}(V_i)) \end{aligned} \tag{3.80}$$

where from eq. (3.79), we take $\mathscr{U}_i = \gamma_0^{-1}(V_i)$. Similarly,

$$\mu_{m_p}(\gamma_1^{-1}(A)) \leq \sum_i \mu_{m_p}(\gamma_1^{-1}(V_i)) \text{ where we take } \mathscr{U}_i = \gamma_1^{-1}(V_i) \tag{3.81}$$

(iii) Combining ineq. (3.80) and (3.81), we obtain:

$$\frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A))\right] \leq \frac{1}{2}\left[\sum_i \mu_{m_p}(\gamma_0^{-1}(V_i)) + \sum_i \mu_{m_p}(\gamma_1^{-1}(V_i))\right] \tag{3.82}$$

By eq.(3.76) we have:

$$\frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A))+\mu_{m_p}(\gamma_1^{-1}(A))\right] \leq \sum_i \mu_{m_p}(V_i) \tag{3.83}$$

and eq.(3.83) is true for any cover of $A \cap C^{(p)} \neq \emptyset$ by elements of $\mathscr{E}$.

(iv) Taking the infimum over all such covers on the right-hand side of ineq.(3.83), keeping in mind the left-hand side of ineq.(3.83) is already an infimum (fixed real number) , we get the lower bound for $\mu_{m_p}(A)$:

$$\frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A))+\mu_{m_p}(\gamma_1^{-1}(A))\right] \leq \mu_{m_p}(A) \tag{3.84}$$

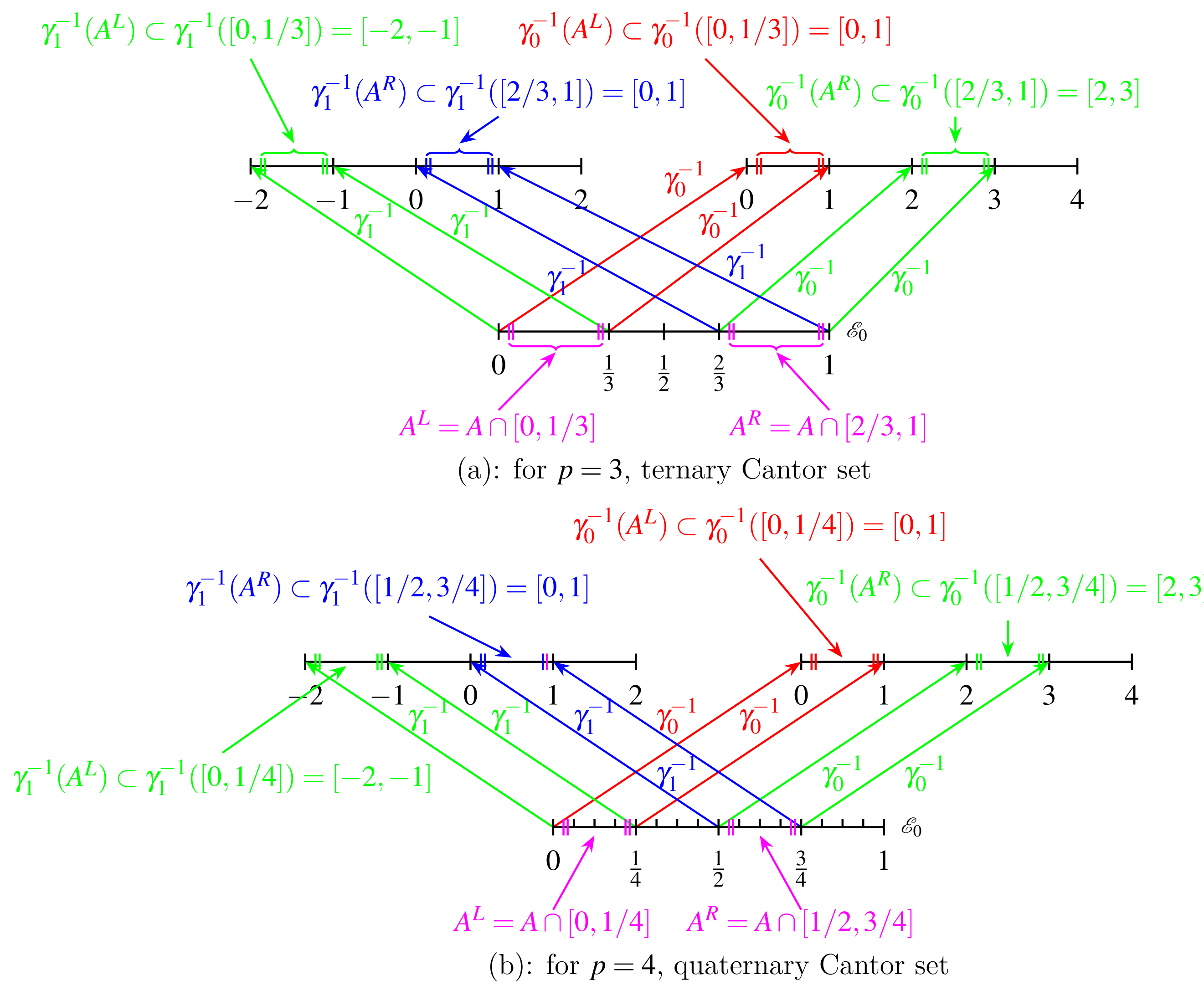


Figure 3.9: Sets A^L and A^R and location of $\gamma_0^{-1}(A^L)$, $\gamma_0^{-1}(A^R)$, $\gamma_1^{-1}(A^L)$, $\gamma_1^{-1}(A^R)$ with respect to $[0,1]$ for both $p = \{3,4\}$.

Step 4 Upper bound for $\mu_{m_p}(A)$:

(i) Given $\varepsilon > 0$, the aim of this step is to show:

$$\mu_{m_p}(A) < \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A))\right] + \varepsilon \tag{3.85}$$

and since ε is arbitrary, we would obtain:

$$\mu_{m_p}(A) \leq \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A))\right], \tag{3.86}$$

an upper bound for $\mu_{m_p}(A)$.

(ii) With $A \subset [0,1]$, the mass of $A \cap (1/p, 2/p)$ is equal to 0. Thus, we define $A^L = A \cap [0, 1/p]$ and $A^R = A \cap [2/p, 3/p]$ as illustrated in Figure 3.9. Clearly, A^L and A^R are disjoint and we have in terms of mass distribution:

$$\mu_{m_p}(A) = \mu_{m_p}(A^L) + \mu_{m_p}(A^R) \tag{3.87}$$

(iii) Since γ_0 and γ_1 are bijection, referring to Figure 3.9 we can write:

$$\gamma_i^{-1}(A) = \gamma_i^{-1}(A^L) \cup \gamma_i^{-1}(A^R) \;\; i = \{0,1\}. \tag{3.88}$$

Since $\gamma_i^{-1}(A^L)$ and $\gamma_i^{-1}(A^R)$ are disjoint, we have in terms of mass distribution:

$$\mu_{m_p}(\gamma_i^{-1}(A)) = \mu_{m_p}(\gamma_i^{-1}(A^L)) + \mu_{m_p}(\gamma_i^{-1}(A^R)) \;\; i = \{0,1\} \tag{3.89}$$

(iv) From Figure 3.9 we make the key observation that $\gamma_0^{-1}(A^R) \cap C^{(p)} = \emptyset$ and $\gamma_1^{-1}(A^L) \cap C^{(p)} = \emptyset$. So, $\mu_{m_p}(\gamma_0^{-1}(A^R)) = 0$, $\mu_{m_p}(\gamma_1^{-1}(A^L)) = 0$ and eq. (3.89) becomes:

$$\mu_{m_p}(\gamma_0^{-1}(A)) = \mu_{m_p}(\gamma_0^{-1}(A^L)) \tag{3.90}$$

and

$$\mu_{m_p}(\gamma_1^{-1}(A)) = \mu_{m_p}(\gamma_1^{-1}(A^R)). \tag{3.91}$$

(v) By using eqs. (3.90) and (3.91) in ineq. (3.85), it suffices to show that:

$$\mu_{m_p}(A) < \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A^L)) + \mu_{m_p}(\gamma_1^{-1}(A^R))\right] + \varepsilon \tag{3.92}$$

(vi) By definition:

$$\mu_{m_p}(\gamma_1^{-1}(A^R)) = \inf\left\{\sum_i \mu_{m_p}(\mathscr{U}_i) : \gamma_1^{-1}(A^R) \cap C^{(p)} \subset \bigcup_i \mathscr{U}_i,\ \mathscr{U}_i \in \mathscr{E}\right\} \tag{3.93}$$

We choose for cover, a collection of closed intervals J_i from the construction of $C^{(p)}$. For those we obtain:

$$\mu_{m_p}(\gamma_1^{-1}(A^R)) \leq \sum_i \mu_{m_p}(J_i) \tag{3.94}$$

Note: As illustrated in Figure 3.9, $\gamma_1^{-1}(A^R)$ "spreads" over $[0,1]$.

(vii) γ_1 is a bijection and a contraction. In addition, it is continuous. So $\mu_{m_p}(\gamma_1(J_i)) < \mu_{m_p}(J_i)$ and all $\gamma_1(J_i)$ are closed subsets $[2/p, 3/p]$ and $\gamma(J_i) \in \mathscr{E}$. Taking for cover these $\gamma_1(J_i)$ we have by definition of $\mu_{m_p}(\cdot)$ (a set function):

$$\mu_{m_p}(A^R) \leq \sum_i \mu_{m_p}(\gamma_1(J_i)) \tag{3.95}$$

(viii) For closed sets (intervals) J in $\mathscr{E}$ we showed in Step 3, (vi) that

$$\mu_{m_p}(J) = \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(J)) + \mu_{m_p}(\gamma_1^{-1}(J))\right] \tag{3.96}$$

So, using eq. (3.96), we can write ineq. (3.95) as follows:

$$\mu_{m_p}(A^R) \leq \sum_i \mu_{m_p}(\gamma_1(J_i)) = \frac{1}{2}\sum_i \left[\mu_{m_p}(\gamma_0^{-1}(\gamma_1(J_i))) + \mu_{m_p}(\gamma_1^{-1}(\gamma_1(J_i)))\right] \tag{3.97}$$

Since all $\gamma_1(J_i) \subset [2/p, 3/p]$, $\mu_{m_p}(\gamma_0^{-1}(\gamma_1(J_i))) = 0 \; \forall i \in \mathbb{N}$, then ineq. (3.97) becomes:

$$\mu_{m_p}(A^R) \leq \frac{1}{2}\sum_i \mu_{m_p}(\gamma_1^{-1}(\gamma_1(J_i))) = \frac{1}{2}\sum_i \mu_{m_p}(J_i) \tag{3.98}$$

(ix) Let $\varepsilon > 0$. By Archimedean Principle [6, p. 27] and recalling that $\mu_{m_p}(\gamma_1^{-1}(A^R))$ is an infimum, there exists a cover of closed intervals from the construction of $C^{(p)}$ such that:

$$\begin{array}{rcccl} \mu_{m_p}(\gamma_1^{-1}(A^R)) & \leq & \sum_i \mu_{m_p}(J_i) & < & \mu_{m_p}(\gamma_1^{-1}(A^R)) + \varepsilon \\ \frac{1}{2}\mu_{m_p}(\gamma_1^{-1}(A^R)) & \leq & \frac{1}{2}\sum_i \mu_{m_p}(J_i) & < & \frac{1}{2}\mu_{m_p}(\gamma_1^{-1}(A^R)) + \frac{\varepsilon}{2} \end{array} \tag{3.99}$$

(x) Using the $<$ part of ineq. (3.99), ineq. (3.98) becomes:

$$\mu_{m_p}(A^R) \leq \frac{1}{2}\sum_i \mu_{m_p}(J_i) < \frac{1}{2}\mu_{m_p}(\gamma_1^{-1}(A^R)) + \frac{\varepsilon}{2} \tag{3.100}$$

(xi) By definition:

$$\mu_{m_p}(\gamma_1^{-1}(A^L)) = \inf\left\{\sum_i \mu_{m_p}(\mathscr{U}_i) : \gamma_0^{-1}(A^L) \cap C^{(p)} \subset \bigcup_i \mathscr{U}_i,\; \mathscr{U}_i \in \mathscr{E}\right\} \tag{3.101}$$

We choose for cover, a collection of closed intervals J_i from the construction of $C^{(p)}$. For those we obtain:

$$\mu_{m_p}(\gamma_0^{-1}(A^L)) \leq \sum_i \mu_{m_p}(J_i) \tag{3.102}$$

Note: As illustrated in Figure 3.9, $\gamma_0^{-1}(A^L)$ "spreads over $[0,1]$. That is $\gamma_0^{-1}(A^L)$ is not confined to $[0,1/4]$

(xii) γ_0 is a bijection and a contraction. In addition, it is continuous. So $\mu_{m_p}(\gamma_0(J_i)) < \mu_{m_p}(J_i)$ and all $\gamma_0(J_i)$ are closed subsets $[0,1/4]$ and $\gamma(J_i) \in \mathscr{E}$. Taking for cover these $\gamma_0(J_i)$ we have by definition of $\mu_{m_p}(\cdot)$ (a set function):

$$\mu_{m_p}(A^L) \leq \sum_i \mu_{m_p}(\gamma_0(J_i)) \tag{3.103}$$

(xiii) Using eq. (3.96), we can write ineq. (3.103) as follows:

$$\mu_{m_p}(A^L) \leq \sum_i \mu_{m_p}(\gamma_0(J_i)) = \frac{1}{2}\sum_i \left[\mu_{m_p}(\gamma_0^{-1}(\gamma_0(J_i))) + \mu_{m_p}(\gamma_0^{-1}(\gamma_1(J_i)))\right] \tag{3.104}$$

Since all $\gamma_0(J_i) \subset [0,1/p]$, $\mu_{m_p}(\gamma_0^{-1}(\gamma_1(J_i))) = 0 \ \forall i \in \mathbb{N}$, then ineq. (3.104) becomes:

$$\mu_{m_p}(A^L) \leq \frac{1}{2}\sum_i \mu_{m_p}(\gamma_0^{-1}(\gamma_0(J_i))) = \frac{1}{2}\sum_i \mu_{m_p}(J_i) \tag{3.105}$$

(xiv) Let $\varepsilon > 0$. By Archimedean Principle [6, p. 27] and recalling that $\mu_{m_p}(\gamma_0^{-1}(A^L))$ is an infimum, there exists a cover of closed intervals from the construction of $C^{(p)}$ such that:

$$\begin{array}{rcccl} \mu_{m_p}(\gamma_0^{-1}(A^L)) & \leq & \sum_i \mu_{m_p}(J_i) & < & \mu_{m_p}(\gamma_0^{-1}(A^L)) + \varepsilon \\ \frac{1}{2}\mu_{m_p}(\gamma_0^{-1}(A^L)) & \leq & \frac{1}{2}\sum_i \mu_{m_p}(J_i) & < & \frac{1}{2}\mu_{m_p}(\gamma_0^{-1}(A^L)) + \frac{\varepsilon}{2} \end{array} \tag{3.106}$$

(xv) Using the $<$ part of ineq. (3.106), ineq. (3.105) becomes:

$$\mu_{m_p}(A^L) \leq \frac{1}{2}\sum_i \mu_{m_p}(J_i) < \frac{1}{2}\mu_{m_p}(\gamma_0^{-1}(A^L)) + \frac{\varepsilon}{2} \tag{3.107}$$

(xvi) Adding the $<$ part of ineqs. (3.100) and (3.107) and using eq. (3.87) we obtain:

$$\mu_{m_p}(A) = \mu_{m_p}(A^L) + \mu_{m_p}(A^R) < \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A^L)) + \mu_{m_p}(\gamma_1^{-1}(A^R))\right] + \varepsilon \tag{3.108}$$

Since $\varepsilon > 0$ is arbitrary, ineq. (3.108) becomes:

$$\mu_{m_p}(A) \leq \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A^L)) + \mu_{m_p}(\gamma_1^{-1}(A^R))\right] \tag{3.109}$$

Ineq.(3.109) with eqs. (3.90) and (3.91) imply that:

$$\mu_{m_p}(A) \leq \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A)\right] \tag{3.110}$$

which is the desired result for this step.

Step 5 From ineqs. (3.84) and (3.110) we obtain that:

$$\mu_{m_p}(A) = \frac{1}{2}\left[\mu_{m_p}(\gamma_0^{-1}(A)) + \mu_{m_p}(\gamma_1^{-1}(A)\right] \tag{3.111}$$

This completes the proof by mass distribution for $\mu_{m_p}(\cdot)$ of eq. (3.70) for any Borel set $A \subset [0,1]$.

As defined in Section 3.2.5.1, the binary Cantor set $[0,1]$ enjoys results similar to what is found in eq. (3.70) and (3.114) with similar proof that will not be repeated here. We then have :

$$\mu_{m_2}(A) = \frac{1}{2}\left[\mu_{m_2}(B_0^{-1}(A)) + \mu_{m_2}(B_1^{-1}(A)\right] \text{ for any Borel set } A \subset [0,1] \tag{3.112}$$

3.4 Relation between measures on Cantor sets

In Section 3.2.5 gives the proof of the first part of Theorem 3.0.1. The proof follows the steps in the flowchart of Figure 3.1 and shows the results given in eq. (3.1).

In the above we have shown the following results:

1. for ternary Cantor set ($s = \log 2/\log 3$):

$$\begin{aligned}
\mu_{m_3}((a,b]) &= \mu_{m_3}([0,b]) - \mu_{m_3}([0,a]) \\
&= F(b) - F(a) \text{ for every } 0 \leq a < b \leq 1 \\
\mathscr{H}^s((a,b] \cap C^{(3)}) &= \mathscr{H}^s([0,b] \cap C^{(3)}) - \mathscr{H}^s([0,a] \cap C^{(3)}) \\
&= F(b) - F(a) \text{ for every } 0 \leq a < b \leq 1. \\
\mu_{m_3}(A) &= \tfrac{1}{2}\left[\mu_{m_3}(T_0^{-1}(A)) + \mu_{m_3}(T_1^{-1}(A)\right] \text{ for any Borel set } A \subset [0,1]
\end{aligned} \tag{3.113}$$

2. for quaternary Cantor set ($s = \log 2/\log 4 = 1/2$):

$$\begin{aligned}
\mu_{m_4}((a,b]) &= \mu_{m_4}([0,b]) - \mu_{m_4}([0,a]) \\
&= W(b) - W(a) \text{ for every } 0 \leq a < b \leq 1 \\
\mathscr{H}^s((a,b] \cap C^{(4)}) &= \mathscr{H}^s([0,b] \cap C^{(4)}) - \mathscr{H}^s([0,a] \cap C^{(4)}) \\
&= W(b) - W(a) \text{ for every } 0 \leq a < b \leq 1. \\
\mu_{m_4}(A) &= \tfrac{1}{2}\left[\mu_{m_4}(\tau_0^{-1}(A)) + \mu_{m_4}(\tau_1^{-1}(A)\right] \text{ for any Borel set } A \subset [0,1]
\end{aligned} \tag{3.114}$$

Note: Slight change in nomenclature: $\mu_{m_3}(\cdot)$ is mass distribution measure on $C^{(3)}$ and $\mu_{m_4}(\cdot)$ is mass distribution measure on $C^{(4)}$.

The Cantor-Lebesgue functions F and W are both an increasing, right-continuous function. From Propositions and Theorem 3.2.2, the question arises whether there is a relation between μ_{m_3},$\mathscr{H}^s$ and μ_F (G replaced by F)? By Theorem 3.2.2 any such increasing, right-continuous function G defines a unique Borel measure on $\mathbb{R}$. F is increasing and continuous function on $[0,1]$ but that can be further extended to the whole $\mathbb{R}$ by requiring that $F(x) = 0$ if $x \leq 0$ and $F(x) = 1$ for $x \geq 1$, preserving the continuity of F on $\mathbb{R}$. Then F defines a unique Borel Measure μ_F such that for every Borel set $A \in \mathscr{B}$ we have:

$$\mu_{m_3}(A) = \mathscr{H}^s(A \cap C^{(3)}) = \mu_F(A) \tag{3.115}$$

answering the question in the affirmative. The above is illustrated in Figure 3.1. Similarly for the quaternary Cantor set, we have that W defines a unique Borel Measure μ_W such that for every Borel set $A \in \mathscr{B}$ we have:

$$\mu_{m_4}(A) = \mathscr{H}^s(A \cap C^{(4)}) = \mu_W(A) \tag{3.116}$$

where the function $W(x)$ is further extended to the whole $\mathbb{R}$ by requiring that $W(x) = 0$ if $x \leq 0$ and $W(x) = 1$ for $x \geq 3/4$, preserving the continuity of W on $\mathbb{R}$.

Hutchinson's Theorem states that there exists a unique Borel measure μ_H with support contained in $C^{(3)}$ such that for any Borel set $A \in \mathscr{B}(\mathbb{R})$, $\mu_H(A) = 1/2 \sum_{j=1}^{2} \mu_H(T_j^{-1}(A))$. We showed that $\mu_{m_3}(A)$ satisfies the recursive relation for $\mu_H(A)$. Therefore, by the uniqueness of μ_H we have that:

$$\mu_{m_3}(A) = \mu_H(A) \tag{3.117}$$

for every Borel set $A \in \mathscr{B}(\mathbb{R})$. This is illustrated in Figure 3.2. Similarly, for the quaternary Cantor set we have:

$$\mu_{m_4}(A) = \mu_H(A) \tag{3.118}$$

for every Borel set $A \in \mathscr{B}(\mathbb{R})$.

This completes the proof of the second part of Theorem 3.0.1 and the theorem is completely proved.

Chapter 4

Frames for measures on Cantor sets

4.1 Background on Frames

Duffin and Schaeffer [9] were the first to introduce frames and present their general definition. However, the core subject of their paper [9] is non-harmonic Fourier series in the context of sequences of complex exponential functions. The monograph of Young [28] presents non-harmonic Fourier series including frames of exponentials.

In studying vector spaces, basis arises to be of a notion of paramount importance. Having a basis represents an ideal tool to represent every vector in a given vector space by a linear expansion in terms of basis elements. However, that ideal tool often imposes requirements on the basis elements such as to be linearly independent and orthogonal with respect to an inner product. If additional requirements need to be satisfied, then that ideal tool becomes difficult or sometimes impossible to sustain. So, to obtain a more flexible tool, we have to relax these requirements. That is, new elements are added to the original basis to satisfy additional requirements but these new elements need not be independent and perhaps be orthogonal with respect to the inner product. This gives an "extended basis" that is called a frame and it has the same property as a basis which is that every vector in a given vector space can be represented by a linear expansion in terms of frame elements.

On top of linear independence and orthogonality, uniqueness of representation of vectors is also a requirement that comes in for any practical use of representations, may they be linear combinations or series. Again, this requirement can cause problem rather than help. For example, let e_n be an orthonormal basis for a Hilbert space. Then $f = \sum_n \langle f, e_n \rangle e_n$ is unique and the sequence of coefficients $\{\langle f, e_n \rangle\}_{n\in\mathbb{N}}$ characterise f. In data transmission, prior "sending" f as a "signal", f is represented by a unique series $f = \sum_n \langle f, e_n \rangle e_n$ where sequence of coefficients $\{\langle f, e_n \rangle\}_{n\in\mathbb{N}}$ characterise f. Ideally the data transmission consists in actually sending only the sequence of coefficients and at the receiving point, the "signal" f is recovered by using its series representation. However, the lost of only one coefficient $\langle f, e_n \rangle$ has a profound consequence. It makes the recovery of the "signal" f a very difficult, if not impossible, task. There could be a lost of more than one coefficient and redundancy in the sequence of coefficients could represent a viable way to recover the "signal" f from the coefficients that reach the receiving point. Having redundancy points to an "extended basis" where, for example, elements of the basis would be repeated but multiplied by a scalar making the basis no longer linearly independent and orthogonal. Frames provide such basis-like but usually redundant series representations of vectors in a Hilbert space. Although frames found many applications in engineering, pure mathematics sees them as important tools.

In this section we present the elements of frames in Hilbert spaces with some examples.

4.2 Elements of frames

Each orthonormal basis $\{e_n\}$ for a Hilbert space $\mathscr{H}$ satisfies the Plancherel equality. It states that $\sum_n \|\langle x, e_n \rangle\|^2 = \|x\|^2 \ \forall x \in \mathscr{H}$. However, a sequence $\{x_n\}$ can satisfy Plancherel equality without being orthonormal or a basis. For example, let $\mathscr{H} = \mathbb{R}^2$ and

$$x_1 = (1,0), \quad x_2 = (0,1), \quad x_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \quad x_4 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{4.1}$$

We observe that $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are each an orthonormal basis for $\mathbb{R}^2$ so we have

$$\sum_{i=1}^{4} \|\langle x, x_n \rangle\|^2 = 2\|x\|^2 \tag{4.2}$$

Therefore the sequence $\{x_n/\sqrt{2}\}_{i=1}^{4}$ satisfies the Plancherel equality i.e.

$$\sum_{i=1}^{4} \frac{\|\langle x, x_n \rangle\|^2}{2} = \frac{2\|x\|^2}{2} = \|x\|^2; \tag{4.3}$$

however, it is neither orthogonal nor a basis for $\mathbb{R}^2$. Such sequences that satisfy the Plancherel equality are called Parseval frames.

Let us look at another, less trivial, example of a Parseval frame defined by:

$$x_1 = (0,1), \quad x_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}), \quad x_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2}) \tag{4.4}$$

then

$$\sum_{i=1}^{3} \|\langle x, x_n \rangle\|^2 = \frac{3}{2}\|x\|^2 \; \forall x \in \mathbb{R}^2 \tag{4.5}$$

So, if we set $a = \sqrt{2/3}$ then the sequence $\{ax_1, ax_2, ax_3\}$ is a Parseval frame. To see this let $x = (u, v) \in \mathbb{R}^2$ then

$$\begin{aligned}
\sum_{i=1}^{3} \|\langle x, \sqrt{\frac{2}{3}} x_n \rangle\|^2 &= \left|0 + \sqrt{\frac{2}{3}} v\right|^2 + \left|-\sqrt{\frac{2}{3}} u \frac{\sqrt{3}}{2} - \sqrt{\frac{2}{3}} \frac{v}{2}\right|^2 + \left|\sqrt{\frac{2}{3}} u \frac{\sqrt{3}}{2} - \sqrt{\frac{2}{3}} \frac{v}{2}\right|^2 \\
&= \frac{2v^2}{3} + \frac{u^2}{2} + \frac{uv}{\sqrt{3}} + \frac{v^2}{2 \cdot 3} + \frac{u^2}{2} - \frac{uv}{\sqrt{3}} + \frac{v^2}{2 \cdot 3} \\
&= \frac{u^2}{2} + \frac{u^2}{2} + \frac{2v^2}{3} + \frac{v^2}{2 \cdot 3} + \frac{v^2}{2 \cdot 3} \\
&= u^2 + v^2 = \|x\|^2
\end{aligned} \tag{4.6}$$

Although it is a Parseval frame, it is not a union of orthogonal bases in $\mathbb{R}^2$ as in the previous example. This frame is called the Mercedes frame as shown in Figure 4.1:

Using three-dimensional visualization, it can seen the Mercedes frame is the orthogonal

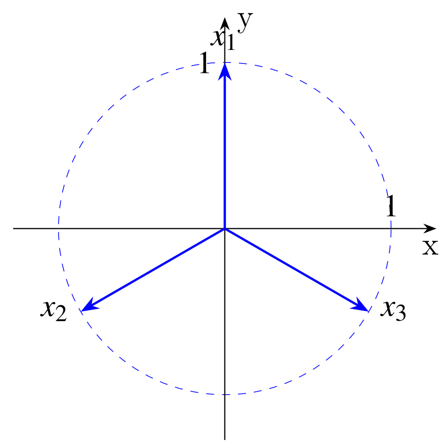


Figure 4.1: The three vectors of the Mercedes frame with blue dashed unit circle for comparison.

projection of a certain orthonormal basis for $\mathbb{R}^3$ onto a two-dimensional plane (see Appendix G).

The definition of a generic frame imposes a less stringent requirement than for a Parseval frame where the Plancherel equality must satisfied.

Definition of Frame: A sequence $\{x_n\}$ in a Hilbert space $\mathscr{H}$ is a frame for $\mathscr{H}$ if there exist constants $A, B > 0$ such that the following pseudo-Plancherel inequality holds:

$$A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2 \;\; \forall x \in \mathscr{H} \tag{4.7}$$

The constants A and B are called the frame bounds. These constants A and B are of paramount importance not only for the existence of the frame itself but also for the reconstruction of the original function f from the transmitted coefficients of the representation of the function f. "Unfortunately, in many cases of interest only a crude upper bound B and only the existence of a lower bound A are known. The determination of frame bounds is often a difficult mathematical problem and for instance there is only a handful of situations in wavelet theory where explicit estimates are known" [17].

The following special type of frames are important:

Definition Let $\{x_n\}$ be a frame for a Hilbert space $\mathscr{H}$, then:

- $\{x_n\}$ is a tight frame if we can choose A = B as frame bounds.

- $\{x_n\}$ is a Parseval frame if A = B = 1 are frame bounds.

- $\{x_n\}$ is an exact frame if it ceases to be a frame whenever any single element is deleted from the sequence.

Note: One-sided inequalities gives Bessel sequences and Riesz-Fischer sequence [25, pp. 75-76].

4.3 Fourier transform of Binary, Ternary and Quaternary Measures

While the theory of frames is well developed (see [4],[5]), the literature on frames on Cantor sets is recent and limited, see [10],[11],[12],[23],[24]. Central in defining frames of exponentials on Cantor sets is the set of integers obtained from the Fourier transform of each measure supported on the corresponding Cantor set. That set of integers is hereafter called a spectrum. Jorgensen and Pedersen [20] and [21] were among the first authors to discuss the spectrum of a measure supported on the corresponding Cantor set. Following some parts of their work, this section provides an analysis of the Fourier transform of the binary, ternary and quaternary measures with the aim of defining a frame for each one of them on their respective Cantor set. Theorem 3.0.1 shows an equivalence between four measures considered in this work for the ternary and quaternary Cantor sets. These are self-similar measures and they are denoted by μ_p $(p = \{2, 3.4\})$ with support respectively in:

- the binary Cantor set $[0,1]$, since μ_2 is the Lebesgue measure on $[0,1]$,

- the ternary Cantor set $C^{(3)}$,

- the quaternary Cantor set $C^{(4)}$.

Defining $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we show the analytic functions $\{e_n = e^{i2\pi nx} : n \in \mathbb{N}_0,\ x \in \mathbb{R}\}$ contain an orthonormal basis in $L^2(\mu)$ for each of the binary and quaternary measures, but not for the ternary measure. A distinct subset $P \subset \mathbb{N}_0$ identifies each of these orthonormal bases such that $\{e_n : n \in P\}$ form an orthonormal basis for $L^2(\mu)$.

We start by the definition of the Fourier transform:

Definition 4.3.1. For $t \in \mathbb{R}$, the Fourier transform of μ_p $(p = \{2,3,4\})$ is given by

$$\hat{\mu}_p(t) = \int_A e^{i2\pi tx} d\mu_p(x) \tag{4.8}$$

where A is a Borel set that contains the support of the measure.

Taking the Fourier Transform of these measures implies integration of $e^{i2\pi tx}$ with respect with these measures.

We analyse first the Fourier transform of quaternary measure followed by the analysis of the binary and ternary measures.

4.3.1 Fourier transform of the Quaternary Measure

Jorgensen and Pedersen developed in [20] what they call "a unique probability measure μ_4 on $\mathbb{R}$ of compact support", such that:

$$\int f d\mu_4 = \frac{1}{2}\left(\int f\left(\frac{x}{4}\right) d\mu_4(x) + \int f\left(\frac{x}{4} + \frac{1}{2}\right) d\mu_4(x)\right) \tag{4.9}$$

for all continuous functions f. The support of μ_4 is the quaternary Cantor set $C^{(4)}$ obtained by dividing $I = [0,1]$ into four equal subintervals, and retaining only the first and third. The quaternary Cantor set can also be constructed as done in Chapter 3 using the following Iterated Function System (IFS) (see Figure 4.2):

$$\begin{aligned} \tau_0(x) &= \frac{x}{4} \qquad (4.10) \\ \tau_1(x) &= \frac{x+2}{4} \end{aligned}$$

where $\tau_0(x)$ and $\tau_1(x)$ are the argument of the continuous function f in the first and second integrals respectively, on the right-hand side of eq. (4.9).

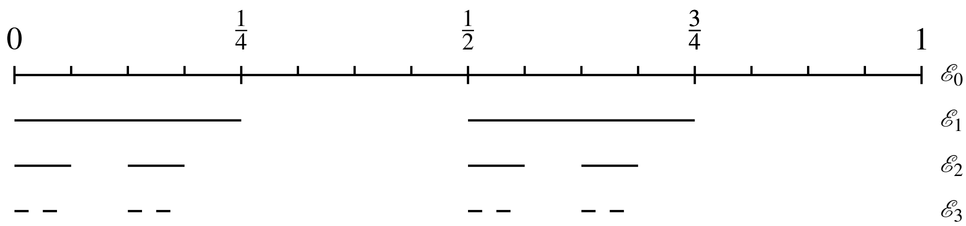


Figure 4.2: Few construction levels of the quaternary Cantor set for the support of μ_4

μ_4 defined by Jorgensen and Pedersen in [20] is in fact our μ_4. The measure μ_4 on $C^{(4)}$ assigns a measure of 1/2 to each of the sets $C^{(4)} \cap [0,1/4]$ and $C^{(4)} \cap [1/2,3/4]$, measure 1/4 to each of the four closed intervals at the next stage, etc. Eq. (4.9), developed by Jorgensen and Pedersen in [20], gives in fact a formula to obtain $\int f d\mu_4$ for all continuous functions f. Although we could have use that formula straight away, in this work, we take the different approach of developing eq. (4.9) by combining the recursive relations for μ_p $(p = \{2,3,4\})$ with a change of variable as documented in Appendix B.

Since $e^{i2\pi tx}$ is continuous, then by eq.(4.9) we can write:

$$\begin{aligned}
\hat{\mu}_4(t) &= \int e^{i2\pi tx} d\mu_4(x) = \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{4}x} d\mu_4(x) + \int e^{i2\pi \frac{t}{4}x + i2\pi t\frac{1}{2}} d\mu_4(x)\right) \\
&= \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{4}x} d\mu_4(x) + e^{i\pi t}\int e^{i2\pi \frac{t}{4}x} d\mu_4(x)\right) \\
&= \tfrac{1}{2}\left(1 + e^{i\pi t}\right)\int e^{i2\pi \frac{t}{4}x} d\mu_4(x) \\
&= \tfrac{1}{2}\left(1 + e^{i\pi t}\right)\hat{\mu}_4(\tfrac{t}{4})
\end{aligned} \tag{4.11}$$

Appendix I gives an alternate way to arrive at eq. (4.11) by considering the C-L function W as the distribution function of μ_4. From eq.(4.11) (last line), we define for the quaternary Cantor set:

$$\chi_4(t) = \frac{1}{2}\left(1 + e^{i\pi t}\right). \tag{4.12}$$

Then eq.(4.11) can be written as follows:

$$\hat{\mu}_4(t) = \chi_4(t)\hat{\mu}_4(\frac{t}{4}) \tag{4.13}$$

From eq. (4.11) we can write:

$$\hat{\mu}_4(\frac{t}{4}) = \frac{1}{2}\left(1+e^{i\pi\frac{t}{4}}\right)\hat{\mu}_4(\frac{t}{4^2}) \tag{4.14}$$

For $t = 0$, eq. (4.8) becomes

$$\hat{\mu}_4(0) = \int 1 d\mu_4 = \int \chi_{C^{(4)}} d\mu_4 = \mu_4(C^{(4)}) \tag{4.15}$$

By mass distribution $\mu_4(C^{(4)}) = 1$ and we get that $\hat{\mu}_4(0) = 1$. Then, we iterate the relation in eq. (4.14) N times to obtain:

$$\hat{\mu}_4(t) = \left[\prod_{n=0}^{N} \chi_4(\frac{t}{4^n})\right]\hat{\mu}_4(\frac{t}{4^{N+1}}) \tag{4.16}$$

Taking $N \to \infty$ and using the continuity of $\hat{\mu}_4(t)$ at $t = 0$, we can write

$$\begin{aligned} \hat{\mu}_4(t) &= \left[\prod_{n=0}^{\infty} \chi_4(\tfrac{t}{4^n})\right] \lim_{N\to\infty} \hat{\mu}_4(\tfrac{t}{4^{N+1}}) \\ &= \prod_{n=0}^{\infty} \tfrac{1}{2}\left(1+e^{i\frac{\pi t}{4^n}}\right) \end{aligned} \tag{4.17}$$

Eq. (4.17) can be written as follows:

$$
\begin{aligned}
\hat{\mu}_4(t) &= \prod_{n=0}^{\infty} \frac{1}{2}\left(1+e^{i\frac{\pi t}{4^n}}\right) = \prod_{n=0}^{\infty} \frac{1}{2}\left(e^{i\frac{\pi t}{2\cdot 4^n}-i\frac{\pi t}{2\cdot 4^n}} + e^{i\frac{\pi t}{2\cdot 4^n}+i\frac{\pi t}{2\cdot 4^n}}\right) \\
&= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2\cdot 4^n}} \frac{\left(e^{i\frac{\pi t}{2\cdot 4^n}} + e^{-i\frac{\pi t}{2\cdot 4^n}}\right)}{2} \\
&= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2\cdot 4^n}} \cos\left(\frac{\pi t}{2\cdot 4^n}\right) \\
&= e^{\sum_{n=0}^{\infty} i\frac{\pi t}{2\cdot 4^n}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2\cdot 4^n}\right) \\
&= e^{i\frac{\pi t}{2}\sum_{n=0}^{\infty}\frac{1}{4^n}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2\cdot 4^n}\right) \\
&= e^{i\frac{\pi 2t}{3}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2\cdot 4^n}\right)
\end{aligned} \tag{4.18}
$$

Clearly, with $t \in \mathbb{R}$, $\hat{\mu}_4(t)$ in eq.(4.18) is a continuous function in t. Also, in Appendix H, we show that the product in eq.(4.18) converges $\forall t \in \mathbb{R}$. Then, how do we find the spectrum of μ_4? To answer that question, we follow Jorgensen and Pedersen [20] and present two of their Lemmas adapted to the case of $C^{(4)}$ with some original enhancements in their respective proof.

Lemma 4.3.2. Let $P_4 = \{\ell_0 + 4\ell_1 + 4^2\ell_2 + \cdots + 4^k\ell_k : \ell_j \in L_4, \text{ finite sums }\}$ with $L_4 = \{0,1\}$, then the functions $\{e_\lambda : \lambda \in P_4\}$ are mutually orthogonal in $L^2(\mu_4)$ where

$$
e_\lambda(x) := e^{i2\pi\lambda x} \tag{4.19}
$$

Proof.

Let $\lambda = \sum_j 4^j\ell_j$, $\lambda' = \sum_j 4^j\ell'_j$ be elements of P_4 and assume that $\lambda \neq \lambda'$. Then

$$
\begin{aligned}
\langle e_\lambda | e_{\lambda'} \rangle &= \int \overline{e_\lambda} e_{\lambda'} d\mu_4 \\
&= \int e^{i2\pi(\lambda'-\lambda)x} d\mu_4(x) \\
&= \hat{\mu}_4(\lambda'-\lambda)
\end{aligned} \tag{4.20}
$$

Let

$$
t = (\lambda'-\lambda) = \ell'_0 - \ell_0) + 4(\ell'_1 - \ell_1) + 4^2(\ell'_2 - \ell_2) + \cdots + 4^k(\ell'_k - \ell_k) \tag{4.21}
$$

Using eq.(4.11), we have

$$\hat{\mu}_4(t)=\frac{1}{2}\left(1+e^{i\pi\left(\Sigma_{j=0}^{k}4^j(\ell'_j-\ell_j)\right)}\right)\hat{\mu}_4(\frac{\ell'_0-\ell_0}{4}+(\ell'_1-\ell_1)+4(\ell'_2-\ell_2)+4^2(\ell'_3-\ell_3)+\cdots+4^{k-1}(\ell'_k-\ell_k)) \tag{4.22}$$

In eq. (4.22, we can simplify the first factor as follows:

$$\begin{aligned} 1+e^{i\pi\left(\Sigma_{j=0}^{k}4^j(\ell'_j-\ell_j)\right)} &= 1+\prod_{j=0}^{k}e^{i\pi4^j(\ell'_j-\ell_j)} \\ &= 1+e^{i\pi(\ell'_0-\ell_0)}\prod_{j=1}^{k}e^{i\pi4^j(\ell'_j-\ell_j)} \end{aligned} \tag{4.23}$$

Since $\ell_j=\{0,1\}$, then if $\ell'_j\neq\ell_j$ then $\ell'_j-\ell_j=1 \quad \text{or} \quad -1$, otherwise $\ell'_j-\ell_j=0$. Then, for each j, $1\leq j\leq k$ the argument of the product in eq. (4.23) becomes

$$\begin{aligned} e^{i\pi4^j(\ell'_j-\ell_j)} &= 1 \quad \text{if } \ell'_j-\ell_j=0 \\ &= e^{-i\pi4^j}=\cos(\pi4^j)-i\sin(\pi4^j)=1 \quad \text{if } \ell'_j-\ell_j=-1 \\ &= e^{i\pi4^j}=\cos(\pi4^j)+i\sin(\pi4^j)=1 \quad \text{if } \ell'_j-\ell_j=1 \end{aligned} \tag{4.24}$$

So eq. (4.23) becomes using eq. (4.24)

$$1+e^{i\pi(\ell'_0-\ell_0)}\prod_{j=1}^{k}e^{i\pi4^j(\ell'_j-\ell_j)}=1+e^{i\pi(\ell'_0-\ell_0)} \tag{4.25}$$

Introducing eq (4.25) in eq. (4.22) we get

$$\hat{\mu}_4(t)=\frac{1}{2}\left(1+e^{i\pi(\ell'_0-\ell_0)}\right)\hat{\mu}_4(\frac{\ell'_0-\ell_0}{4}+(\ell'_1-\ell_1)+4(\ell'_2-\ell_2)+4^2(\ell'_3-\ell_3)+\cdots+4^{k-1}(\ell'_k-\ell_k)) \tag{4.26}$$

Since $\ell_j=\{0,1\}$, then if $\ell'_0\neq\ell_0$ then $\ell'_0-\ell_0=1 \quad \text{or} \quad -1$ and we obtain that $\chi_4(1)=(1+e^{i\pi})=0$, $\chi_4(-1)=(1+e^{-i\pi})=0$ and $\chi_4(\ell'_0-\ell_0)=0$. If $\ell'_0=\ell_0$ then $\ell'_0-\ell_0=0$ and we would have $\chi_4(0)=2$. In this case we go back to the expression for t in eq. (4.21) to search for the least

$n \leq k$ such that $\ell'_n \neq \ell_n$ to obtain a new t:

$$t = (\lambda' - \lambda) = 4^n(\ell'_n - \ell_n) + 4^{n+1}(\ell'_{n+1} - \ell_{n+1}) + 4^{n+2}(\ell'_{n+2} - \ell_{n+2}) + \cdots + 4^k(\ell'_k - \ell_k) \tag{4.27}$$

since for all $0 \leq j \leq n-1$, $\ell'_j - \ell_j = 0$. Using eq. (4.11) to perform n iterations we get with the new t

$$\begin{aligned} \hat{\mu}_4(t) = \hat{\mu}_4(\lambda' - \lambda) &= \hat{\mu}_4(4^n(\ell'_n - \ell_n) + 4^{n+1}(\ell'_{n+1} - \ell_{n+1}) + 4^{n+2}(\ell'_{n+2} - \ell_{n+2}) + \cdots + 4^k(\ell'_k - \ell_k)) \\ &= \left[\prod_{j=0}^{n-1} \chi_4\left(\tfrac{t}{4^j}\right)\right] \chi_4\left(\tfrac{t}{4^n}\right) \hat{\mu}_4\left(\tfrac{t}{4^{n+1}}\right) \\ &= \chi_4(\ell'_n - \ell_n)\hat{\mu}_4\left(\tfrac{\ell'_n - \ell_n}{4} + (\ell'_{n+1} - \ell_{n+1}) + 4(\ell'_{n+2} - \ell_{n+2}) + \cdots + 4^k(\ell'_k - \ell_k)\right) \\ &= 0 \end{aligned} \tag{4.28}$$

since $\chi_4(\ell'_n - \ell_n) = 0$. To arrive at eq. (4.28) the following simplifications have been used:

(a) Simplification of first factor $\prod_{j=0}^{n-1} \chi_4\left(\frac{t}{4^j}\right)$. For $0 \leq j \leq n-1$ we have $\ell'_j - \ell_j = 0$. Then

$$\prod_{j=0}^{n-1} \chi_4\left(\frac{t}{4^j}\right) = \frac{1}{2}\left(1 + \prod_{q=n}^{k} e^{i\pi 4^{q-j}(\ell'_q - \ell_q)}\right) = 1 \tag{4.29}$$

(b) Simplification of second factor $\chi_4(t/4^n)$:

$$\begin{aligned} \chi_4\left(\tfrac{t}{4^n}\right) &= \tfrac{1}{2}\left(1 + e^{i\pi\left(\sum_{j=n}^{k} 4^{j-n}(\ell'_j - \ell_j)\right)}\right) \\ &= \tfrac{1}{2}\left(1 + e^{i\pi(\ell'_n - \ell_n)}\right) \prod_{j=n+1}^{k} e^{i\pi 4^{j-n}(\ell'_j - \ell_j)} \\ &= \chi_4(\ell'_n - \ell_n) \end{aligned} \tag{4.30}$$

where $\prod_{j=n+1}^{k} e^{i\pi 4^{j-n}(\ell'_j - \ell_j)} = 1$ as shown for a similar case in eq. (4.24).

This completes the proof. □

We have the following remarks on Lemma 4.3.2:

(a) in the statement of the Lemma, that the set $L_4 \subset \mathbb{Z}$, is a requirement for the proof to show that $P_4 = \{\ell_0 + 4\ell_1 + 4^2\ell_2 + \cdots : \ell_j \in L = \{0, 1\},$ finite sums $\}$ is the spectrum of $\mu_4(\cdot)$.

(b) Clearly $P_4 \subset \mathbb{N} \cup \{0\}$

(c) In the proof of Lemma, we observe that the difference between elements of P_4 are in the set of zeros of $\hat{\mu}_4(\cdot)$.

(d) the Lemma only shows that $\{e_\lambda : \lambda \in P\}$ is an orthonormal subset of $L^2(\mu_4)$ and not an orthonormal basis for $L^2(\mu_4)$.

The following Lemma from [20, p. 190] gives the criterion for $\{e_\lambda : \lambda \in P_4\}$ to be an orthonormal basis for $L^2(\mu_4)$.

Lemma 4.3.3. Let

$$Q_4(t) := \sum_{\lambda \in P_4} |\hat{\mu}_4(t-\lambda)|^2, \quad t \in \mathbb{R}. \tag{4.31}$$

Then $\{e_\lambda : \lambda \in P_4\}$ is an orthonormal basis for $L^2(\mu_4)$ if and only if $Q_4 \equiv 1$ on $\mathbb{R}$.

Proof.

Step 1 ($\Rightarrow$): If $\{e_\lambda : \lambda \in P_4\}$ is an orthonormal basis for $L^2(\mu_4)$, then the Bessel inequality [25, p. 20] becomes an identity when applied to any $e^t = e^{2\pi i t x} \in L^2(\mu_4)$. That is

$$1 = \|e_t\|^2 = \sum_{\lambda \in P_4} |\langle e_\lambda | e_t \rangle_{\mu_4}|^2 = \sum_{\lambda \in P_4} |\hat{\mu}_4(t-\lambda)|^2 \tag{4.32}$$

Step 2 ($\Leftarrow$): At the beginning of this step $\{e_\lambda : \lambda \in P_4\}$ is an orthonormal set and we need to show that it is an orthonormal basis. By assumption, $Q_4 \equiv 1$ on $\mathbb{R}$, that is for any $t \in \mathbb{R}$ the following series converges to 1:

$$\sum_{\lambda \in P_4} |\langle e_\lambda, e_t \rangle_{\mu_4}|^2 = 1 \tag{4.33}$$

For any h in an Hilbert space $\mathscr{H}$ and for an orthonormal set $\{e_i\}_{i=1}^{\infty}$, we show the claim that $h \in \overline{\text{span}\{e_i\}_{i=1}^{\infty}} \iff \|h\|^2 = \sum_{i=1}^{\infty} |\langle h, e_i \rangle|^2$ with the following proof:

(a) The projection of h onto $\text{span}\{e_1, e_2, \ldots, e_n\}$ is $\sum_{i=1}^n \langle h, e_i\rangle e_i$. Then, $(h - \sum_{i=1}^n \langle h, e_i\rangle e_i) \perp \sum_{i=1}^n \langle h, e_i\rangle e_i$.

(b) By Pythagoras we get

$$\begin{aligned} \|h\|^2 &= \|h - \textstyle\sum_{i=1}^n \langle h, e_i\rangle e_i + \sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 \\ &= \|h - \textstyle\sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 + \|\sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 \\ &= \|h - \textstyle\sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 + \sum_{i=1}^n |\langle h, e_i\rangle e_i|^2 \end{aligned} \tag{4.34}$$

(c) The second term on the last line of eq.(4.34) comes from the application of Pythagoras multiple times to $\|\sum_{i=1}^n \langle h, e_i\rangle e_i\|^2$: since $\{\langle h, e_i\rangle e_i\}_{i=1}^n$ are pairwise orthogonal vectors in $\mathscr{H}$, by Pythagoras we have that

$$\begin{aligned} \|\textstyle\sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 &= \textstyle\sum_{i=1}^n \|\langle h, e_i\rangle e_i\|^2 \\ &= \textstyle\sum_{i=1}^n |\langle h, e_i\rangle|^2 \end{aligned} \tag{4.35}$$

To obtain

$$\|h\|^2 - \sum_{i=1}^n |\langle h, e_i\rangle|^2 = \|h - \sum_{i=1}^n \langle h, e_i\rangle e_i\|^2 \tag{4.36}$$

(d) Therefore, we have that:

$$h = \lim_{n\to\infty} \sum_{i=1}^n \langle h, e_i\rangle e_i \text{ in the norm, } \iff \|h\|^2 = \lim_{n\to\infty} \sum_{i=1}^n |\langle h, e_i\rangle|^2 \tag{4.37}$$

(e) Now, let $s_n = \sum_{i=1}^n \langle h, e_i\rangle e_i$, since h is arbitrary, $h = \lim_{n\to\infty} \sum_{i=1}^n \langle h, e_i\rangle e_i$ in the norm implies that for every $h \in \mathscr{H}$ there is a sequence $\{s_n\}$ in $\mathscr{H}$ such that $\|s_n - h\| \to 0$. Hence, $\text{span}\{e_i\}_{i=1}^\infty$ is dense in $\mathscr{H}$.

(f) So, we have shown $h \in \overline{\text{span}\{e_i\}_{i=1}^\infty}$ is equivalent to $\|h\|^2 = \sum_{i=1}^\infty |\langle h, e_i\rangle|^2$

By assumption, $\sum_{\lambda\in P_4} |\langle e_\lambda, e_t\rangle_{\mu_4}|^2 = 1$. Since for any $t \in \mathbb{R}$, $\|e_t\|^2 = 1$, we have that $\sum_{\lambda\in P_4} |\langle e_\lambda, e_t\rangle_{\mu_4}|^2 = \|e_t\|^2$ and by the above claim we get that $e_t \in \overline{\text{span}\{e_\lambda : \lambda \in P_4\}}$

Step 3 Let $f \in L^2(\mu_4) \ominus \{e_\lambda : \lambda \in P_4\}$, that is, $f \in \{e_\lambda : \lambda \in P_4\}^\perp = (\text{span}\{e_\lambda\})^\perp = \overline{(\text{span}\{e_\lambda\})}^\perp$

by the continuity of the inner product. By the conclusion of Step 2, this implies that $\langle e_t, f\rangle_{\mu_4} = 0$ for all $t \in \mathbb{R}$.

Step 4 Recall that the continuous functions are dense in $L^2(\mu_4)$, that is, $\forall u \in L^2(\mu_4)$ and $\forall \varepsilon > 0$, there exists g continuous such that $\|u - g\|_{L^2(\mu_4)} < \varepsilon/2$.

Step 5 Consider $\mathscr{P}$, the collection of all the trigonometric polynomials:

$$\begin{aligned} P_N(\theta) &= \textstyle\sum_{n=-N}^{N} c_n e^{in\theta} = \sum_{n=-N}^{N} c_n z^n \text{ with } z = e^{i\theta} \\ &= \textstyle\sum_{n=0}^{N} c_n z^n + \sum_{n=1}^{N} c_{-n} \bar{z}^n \text{ with } \bar{z} = e^{-i\theta}. \end{aligned} \tag{4.38}$$

There are polynomial in z and $\bar{z}$ and they form an algebra $\mathscr{A}$ which is self-adjoint, vanishes at no point of $[0,1]$ and separate points on $[0,1]$. Then the Stone-Weierstrass theorem tells us that $\mathscr{P}$ is dense in $C([0,1])$ i.e., $\forall g \in C([0,1])$, $\forall \varepsilon > 0$, there exists a trigonometric polynomial P such that

$$\|g - P\|_\infty = \sup_{\theta \in [0,1]} |g(e^{i\theta}) - P(e^{i\theta})| < \varepsilon/2. \tag{4.39}$$

Therefore, we have:

$$\|g - P\|_{L^2(\mu_4)} = \left(\int |g - P|^2 d\mu_4 \right)^{\frac{1}{2}} \leq \|g - P\|_\infty < \varepsilon/2. \tag{4.40}$$

For every $f \in L^2(\mu_4)$, by the triangle inequality we obtain

$$\|f - P\|_{L^2(\mu_4)} \leq \|f - g\|_{L^2(\mu_4)} + \|g - P\|_{L^2(\mu_4)} < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{4.41}$$

So, the collection $\mathscr{P}$ of all the trigonometric polynomials P_N as in eq. (4.38) is dense in $L^2(\mu_4)$ and $f \in \overline{\mathscr{P}}$. But, for $f \in \{e_\lambda\}^\perp$ we showed that $\langle e_t, f\rangle_{\mu_4} = 0$, that is, $e_t \perp f$ for all $t \in \mathbb{R}$. Letting $t = n/2\pi$, $n \in \mathbb{Z}$, we have that $\mathscr{P} = \text{span}\{e_t : t = n/2\pi, n \in \mathbb{Z}\}$ and with $0 = \int f \overline{e_t} d\mu_4 = \int f e^{-in\theta} d\mu_4$ gives that $f \in \mathscr{P}^\perp$. The only function f that can be at the same time $f \in \overline{\mathscr{P}}$ and $f \in \mathscr{P}^{\perp}$ is $f = 0$. This implies that $\{e_\lambda : \lambda \in P_4\}^{\perp} = \{0\}$. Therefore, $\{e_\lambda : \lambda \in P_4\}$ is an orthonormal basis in $L^2(\mu_4)$.

□

Table 4.1 gives the first 15 elements of P_4 where for example, for λ_5, $\{\ell_0,\ell_1,\ell_2,\ell_3\} = \{1,0,1,0\}$ and $\lambda_5 = 4^0 \cdot 1 + 4^1 \cdot 0 + 4^2 \cdot 1 + 4^3 \cdot 0 = 17$

n	$\{\ell_0,\ell_1,\ell_2,\ell_3\}$	λ_n
0	$\{0,0,0,0\}$	0
1	$\{1,0,0,0\}$	1
2	$\{0,1,0,0\}$	4
3	$\{1,1,0,0\}$	5
4	$\{0,0,1,0\}$	16
5	$\{1,0,1,0\}$	17
6	$\{0,1,1,0\}$	20
7	$\{1,1,1,0\}$	21
8	$\{0,0,0,1\}$	64
9	$\{1,0,0,1\}$	65
10	$\{0,1,0,1\}$	68
11	$\{1,1,0,1\}$	69
12	$\{0,0,1,1\}$	80
13	$\{1,0,1,1\}$	81
14	$\{0,1,1,1\}$	84
15	$\{1,1,1,1\}$	85

Table 4.1: Value of $\lambda_n \in P_4$ for finite sums of four elements $(\ell_i : i = 0,1,2,3)$

Jorgensen and Pedersen [20, p. 215] show that for P_4, $Q_4(t) \equiv 1$. This presents an interesting results because as discussed in Section 4.2, we have that $\{e_\lambda : \lambda \in P_4\}$ is an orthonormal basis.

From Eq.(4.18), the set of zeros of $\hat{\mu}_4(t)$ is:

$$\mathbf{Z}(\hat{\mu}_4) = \{4^n(1+2\mathbb{Z})\} \subset \mathbb{Z} \tag{4.42}$$

In the proof of Lemma 4.3.2, we observe that the difference between elements of P_4 are in $\mathbf{Z}(\hat{\mu}_4)$. In fact all those differences are in $\mathbf{Z}(\hat{\mu}_4)$ and since $0 \in P_4$, we have that $\{P_4 \setminus \{0\}\} \subset \mathbf{Z}(\hat{\mu}_4) \subset \mathbb{Z}$. An illustration of that fact is given in Table J.2.

Operations on elements of P_4	Some elements of $\mathbf{Z}(\hat{\mu}_4)$ as differences between elements of P_4
λ_1 minus each of λ_0 down to λ_0	$\{1\}$
λ_2 minus each of λ_1 down to λ_0	$\{3,4\}$
λ_3 minus each of λ_2 down to λ_0	$\{1,4,5\}$
λ_4 minus each of λ_3 down to λ_0	$\{11,12,15,16\}$
λ_5 minus each of λ_4 down to λ_0	$\{1,12,13,16,17\}$
λ_6 minus each of λ_5 down to λ_0	$\{3,4,15,16,19,20\}$
λ_7 minus each of λ_6 down to λ_0	$\{1,4,5,16,17,20,21\}$
λ_8 minus each of λ_7 down to λ_0	$\{43,44,47,48,59,60,63,64\}$
λ_9 minus each of λ_8 down to λ_0	$\{1,44,45,48,49,60,61,64,65\}$
λ_{10} minus each of λ_9 down to λ_0	$\{3,4,47,48,51,52,63,64,67,68\}$
λ_{11} minus each of λ_{10} down to λ_0	$\{1,4,5,48,49,52,53,64,65,68,69\}$
λ_{12} minus each of λ_{11} down to λ_0	$\{11,12,15,16,59,60,63,64,75,76,79,80\}$
λ_{13} minus each of λ_{12} down to λ_0	$\{1,12,13,16,17,60,61,64,65,76,77,80,81\}$
λ_{14} minus each of λ_{13} down to λ_0	$\{3,4,15,16,19,20,63,64,67,68,79,80,83,84\}$
λ_{15} minus each of λ_{14} down to λ_0	$\{1,4,5,16,17,20,21,64,65,68,69,80,81,84,85\}$

Table 4.2: λ's (in red) among the elements of the Zero Set

4.3.2 Spectrum of the Binary Measure

(a) Appendix B shows that we can arrive at eq. (4.9) using a change of variables using the recursive relations for μ_p ($p = \{2,3,4\}$). Since $e^{i2\pi tx}$ is continuous, then by eq.(4.9) we can write:

$$\begin{aligned} \hat{\mu}_2(t) &= \int e^{i2\pi tx} d\mu_2(x) = \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{2}x} d\mu_2(x) + \int e^{i2\pi \frac{t}{2}x + i2\pi t \frac{1}{2}} d\mu_2(x)\right) \\ &= \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{2}x} d\mu_2(x) + e^{i\pi t}\int e^{i2\pi \frac{t}{2}x} d\mu_2(x)\right) \\ &= \tfrac{1}{2}\left(1 + e^{i\pi t}\right)\int e^{i2\pi \frac{t}{2}x} d\mu_2(x) \\ &= \tfrac{1}{2}\left(1 + e^{i\pi t}\right)\hat{\mu}_2(\tfrac{t}{2}) \end{aligned} \tag{4.43}$$

From eq.(4.43) (last line), we define for the binary Cantor set:

$$\chi_2(t) = \frac{1}{2}\left(1 + e^{i\pi t}\right). \tag{4.44}$$

Then eq.(4.11) can be written as follows:

$$\hat{\mu}_2(t) = \chi_2(t)\hat{\mu}_2(\frac{t}{2}) \tag{4.45}$$

From eq. (4.43) we can write:

$$\hat{\mu}_2(\frac{t}{2}) = \frac{1}{2}\left(1 + e^{i\pi \frac{t}{2}}\right)\hat{\mu}_4(\frac{t}{2^2}) \tag{4.46}$$

For $t = 0$, eq. (4.8) becomes for $p = 2$

$$\hat{\mu}_2(0) = \int 1 d\mu_2 = \int \chi_{C^{(2)}} d\mu_2 = \mu_2(C^{(2)}) \tag{4.47}$$

By mass distribution $\mu_2(C^{(2)}) = 1$ and we get that $\hat{\mu}_2(0) = 1$. Then, we iterate the relation in eq. (4.46) N times to obtain:

$$\hat{\mu}_2(t) = \left[\prod_{n=0}^{N} \chi_2(\frac{t}{2^n})\right] \hat{\mu}_2(\frac{t}{2^{N+1}}) \tag{4.48}$$

Taking $N \to \infty$ and using the continuity of $\hat{\mu}_2(t)$ at $t = 0$, we can write

$$\begin{aligned} \hat{\mu}_2(t) &= \left[\prod_{n=0}^{\infty} \chi_2(\tfrac{t}{2^n})\right] \lim_{N\to\infty} \hat{\mu}_2(\tfrac{t}{2^{N+1}}) \\ &= \prod_{n=0}^{\infty} \tfrac{1}{2}\left(1 + e^{i\frac{\pi t}{2^n}}\right) \end{aligned} \tag{4.49}$$

Eq. (4.49) can be written as follows:

$$\begin{aligned} \hat{\mu}_2(t) &= \prod_{n=0}^{\infty} \tfrac{1}{2}\left(1 + e^{i\frac{\pi t}{2^n}}\right) = \prod_{n=0}^{\infty} \tfrac{1}{2}\left(e^{i\frac{\pi t}{2\cdot 2^n} - i\frac{\pi t}{2\cdot 2^n}} + e^{i\frac{\pi t}{2\cdot 2^n} + i\frac{\pi t}{2\cdot 2^n}}\right) \\ &= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2\cdot 2^n}} \frac{\left(e^{i\frac{\pi t}{2\cdot 2^n}} + e^{-i\frac{\pi t}{2\cdot 2^n}}\right)}{2} \\ &= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2^{n+1}}} \cos\left(\tfrac{\pi t}{2^{n+1}}\right) \\ &= \prod_{n=1}^{\infty} e^{i\frac{\pi t}{2^n}} \cos\left(\tfrac{\pi t}{2^n}\right) \\ &= e^{\sum_{n=1}^{\infty} i\frac{\pi t}{2^n}} \prod_{n=1}^{\infty} \cos\left(\tfrac{\pi t}{2^n}\right) \\ &= e^{i\pi t \sum_{n=1}^{\infty} \frac{1}{2^n}} \prod_{n=1}^{\infty} \cos\left(\tfrac{\pi t}{2^n}\right) \\ &= e^{i\pi t} \prod_{n=1}^{\infty} \cos\left(\tfrac{\pi t}{2^n}\right) \end{aligned} \tag{4.50}$$

With $t \in \mathbb{R}$, $\hat{\mu}_2(t)$ in eq.(4.50) is a continuous function in t.

(b) To find the spectra of μ_2, we go through Lemma 4.3.2 but with $P_2 = \{\ell_0 + 2\ell_1 + 2^2\ell_2 + \cdots + 2^k\ell_k : \ell_j \in L_2, \text{ finite sums }\}$ with $L_2 = \{0, 1\}$ to obtain $P_2 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$. Since the proof is similar the one for μ_4, it is not repeated here.

(c) Using the trigonometric identity $\sin 2u = 2\sin u \cos u$, Eq.(4.50) can be simplified to a form that easily yields the zero set for $\hat{\mu}_2$. We have:

$$\begin{aligned}
\hat{\mu}_2(t) &= \lim_{N\to\infty} e^{i\pi t}\prod_{n=1}^{N}\cos\left(\frac{\pi t}{2^n}\right)\\
&= \lim_{N\to\infty} e^{i\pi t}\prod_{n=1}^{N}\frac{\sin\left(\frac{\pi t}{2^{n-1}}\right)}{2\sin\left(\frac{\pi t}{2^n}\right)}\\
&= \lim_{N\to\infty} e^{i\pi t}\frac{\sin(\pi t)}{2\sin\left(\frac{\pi t}{2}\right)}\cdot\frac{\sin\left(\frac{\pi t}{2}\right)}{2\sin\left(\frac{\pi t}{2^2}\right)}\cdots\frac{\sin\left(\frac{\pi t}{2^{N-1}}\right)}{2\sin\left(\frac{\pi t}{2^N}\right)}\\
&= \lim_{N\to\infty} e^{i\pi t}\frac{\sin(\pi t)}{2^N\sin\left(\frac{\pi t}{2^N}\right)}\\
&= \lim_{N\to\infty} e^{i\pi t}\frac{\sin(\pi t)}{\pi t}\frac{\frac{\pi t}{2^N}}{\sin\left(\frac{\pi t}{2^N}\right)}\\
&= e^{i\pi t}\frac{\sin(\pi t)}{\pi t}
\end{aligned} \tag{4.51}$$

where in the last step in eq.(4.51) we use the fact that $\lim_{u\to 0}\sin u/u = 1$. Eq.(4.51) readily gives the zero set of $\hat{\mu}_2(t)$ to be $\mathbf{Z}(\hat{\mu}_2) = \mathbb{Z}\setminus\{0\}$. Similar as for the quaternary measure, we have that $\{P_2\setminus\{0\}\}\subset\mathbf{Z}(\hat{\mu}_2)$ since $0\in P_2$

Since μ_2 is supported on $[0,1]$, it is in fact Lebesgue measure. Because $L^2(\mu_2)$ could be identified with L^2 on the circle, we are back to Fourier series. As discussed in Section 4.2, the functions $\{e_\lambda : \lambda\in P_2\}$ are orthonormal basis.

4.3.3 Spectrum of the ternary measure

(a) Appendix B shows that we can arrive at eq. (4.9) using a change of variables using the recursive relations for μ_3. Since $e^{i2\pi tx}$ is continuous, then by eq.(4.9) we can write:

$$\begin{aligned}
\hat{\mu}_3(t) &= \int e^{i2\pi tx}d\mu_3(x) = \frac{1}{2}\left(\int e^{i2\pi\frac{t}{3}x}d\mu_3(x) + \int e^{i2\pi\frac{t}{3}x + i2\pi t\frac{2}{3}}d\mu_3(x)\right)\\
&= \frac{1}{2}\left(\int e^{i2\pi\frac{t}{3}x}d\mu_3(x) + e^{i\pi\frac{4t}{3}}\int e^{i2\pi\frac{t}{3}x}d\mu_3(x)\right)\\
&= \frac{1}{2}\left(1+e^{i\pi\frac{4t}{3}}\right)\int e^{i2\pi\frac{t}{3}x}d\mu_3(x)\\
&= \frac{1}{2}\left(1+e^{i\pi\frac{4t}{3}}\right)\hat{\mu}_3\left(\frac{t}{3}\right)
\end{aligned} \tag{4.52}$$

Appendix I gives an alternate way to arrive at eq. (4.52) by considering the C-L function F as the distribution function of μ_3. From eq.(4.52) (last line), we define for the ternary Cantor set:

$$\chi_3(t) = \frac{1}{2}\left(1 + e^{i\pi\frac{4t}{3}}\right). \tag{4.53}$$

Then eq.(4.11) can be written as follows:

$$\hat{\mu}_3(t) = \chi_3(t)\hat{\mu}_3(\frac{4t}{3}) \tag{4.54}$$

From eq. (4.52) we can write:

$$\hat{\mu}_3(\frac{t}{3}) = \frac{1}{2}\left(1 + e^{i\pi\frac{4t}{3^2}}\right)\hat{\mu}_4(\frac{t}{3^2}) \tag{4.55}$$

For $t = 0$, eq. (4.8) becomes for $p = 3$

$$\hat{\mu}_3(0) = \int 1 d\mu_3 = \int \chi_{C^{(3)}} d\mu_3 = \mu_3(C^{(3)}) \tag{4.56}$$

By mass distribution $\mu_3(C^{(3)}) = 1$ and we get that $\hat{\mu}_3(0) = 1$. Then, we iterate the relation in eq. (4.55) N times to obtain:

$$\hat{\mu}_3(t) = \left[\prod_{n=0}^{N} \chi_3(\frac{t}{3^n})\right]\hat{\mu}_3(\frac{t}{3^{N+1}}) \tag{4.57}$$

Taking $N \to \infty$ and using the continuity of $\hat{\mu}_3(t)$ at $t = 0$, we can write

$$\begin{aligned} \hat{\mu}_3(t) &= \left[\prod_{n=0}^{\infty} \chi_3(\tfrac{4t}{3^n})\right] \lim_{N\to\infty} \hat{\mu}_3(\tfrac{t}{3^{N+1}}) \\ &= \prod_{n=0}^{\infty} \chi_3(\tfrac{t}{3^n}) \\ &= \prod_{n=1}^{\infty} \tfrac{1}{2}\left(1 + e^{i\frac{4\pi t}{3^n}}\right) \end{aligned} \tag{4.58}$$

where we used the definition of $\chi_3(t)$ (eq. (4.53)) to obtain the last line of eq. (4.58) which can be written as follows:

$$\begin{aligned}
\hat{\mu}_3(t) &= \prod_{n=1}^{\infty}\frac{1}{2}\left(1+e^{i\frac{4\pi t}{3^n}}\right) = \prod_{n=1}^{\infty}\frac{1}{2}\left(e^{i\frac{4\pi t}{2\cdot 3^n}-i\frac{4\pi t}{2\cdot 3^n}}+e^{i\frac{4\pi t}{2\cdot 3^n}+i\frac{4\pi t}{2\cdot 3^n}}\right)\\
&= \prod_{n=1}^{\infty} e^{i\frac{2\pi t}{3^n}}\frac{\left(e^{i\frac{2\pi t}{3^n}}+e^{-i\frac{2\pi t}{3^n}}\right)}{2}\\
&= \prod_{n=1}^{\infty} e^{i\frac{2\pi t}{3^n}}\cos\left(\tfrac{2\pi t}{3^n}\right)\\
&= e^{\sum_{n=1}^{\infty} i\frac{2\pi t}{3^n}}\prod_{n=1}^{\infty}\cos\left(\tfrac{2\pi t}{3^n}\right)\\
&= e^{i2\pi t\sum_{n=1}^{\infty}\frac{1}{3^n}}\prod_{n=1}^{\infty}\cos\left(\tfrac{2\pi t}{3^n}\right)\\
&= e^{i\pi t}\prod_{n=1}^{\infty}\cos\left(\tfrac{2\pi t}{3^n}\right)
\end{aligned} \tag{4.59}$$

With $t \in \mathbb{R}$, $\hat{\mu}_3(t)$ in eq.(4.59) is a continuous function in t.

(b) the spectrum of μ_3 with $L_3 \subset \mathbb{Z}$ (a requirement) would take the form:

$$P_3 = \{\ell_0 + 3\ell_1 + 3^2\ell_2 + \cdots : \ell_j \in L_3,\ \text{finite sums}\ \} \tag{4.60}$$

but going through the proof of Lemma 4.3.2 for P_3, we come to the step where we would show that $\chi_3(\ell_0' - \ell_0) = 0$. From the definition of $\chi_3(t)$ in eq. (4.53), we see that with $t = \pm 3/4$, $\chi_3(\pm 3/4) = 0$. This implies, for example, that L_3 would have be equal to $\{0, 3/4\}$. This is incompatible with the requirement that $L_3 \subset \mathbb{Z}$. So, we cannot define the set P_3. Therefore, the proof of Lemma 4.3.2 for P_3 with $L_3 = \{0,1\}$ fails.

(c) Additional support to this last statement arises from the following considerations. From eq. (4.59), the zero set of $\hat{\mu}_3(t)$ is given by

$$\mathbf{Z}(\hat{\mu}_3) = \left\{\frac{3^n}{4}(1+2\mathbb{Z}) : n = 1, 2, \ldots\right\} \tag{4.61}$$

which indicates that if a set of functions $\{e_\lambda = e^{i2\pi\lambda x} : \lambda \in U\}$ would be mutually orthogonal in $L^2(\mu_3)$, then the set U would contain values whose mutual differences would be in the zero set of $\hat{\mu}_3(t)$. However, following Jorgensen and Pedersen [20, p. 217] we have the following theorem specific to $\hat{\mu}_3$ that proves the contrary, confirming that we cannot have a set of more than two functions that are mutually orthogonal in $L^2(\mu_3)$.

Theorem 4.3.4. Any set of μ_3-orthogonal exponentials contains at most two elements.

Proof.

Step 1 Recall that

$$\mathbf{Z}(\hat{\mu}_3) = \left\{ \frac{3^n}{4}(1+2\mathbb{Z}) : n = 0,1,2,\ldots \right\} \tag{4.62}$$

Step 2 Assume that λ_j, $j = 1,2,3$ are such that the e_{λ_j}'s are mutually orthogonal in $L^2(\mu_3)$, then the differences $\lambda_i - \lambda_j$ $(i \neq j)$ are in $\mathbf{Z}(\hat{\mu}_3)$.

Step 3 Let $\gamma_1 = \lambda_1 - \lambda_2$, $\gamma_2 = \lambda_2 - \lambda_3$ and $\gamma_0 = \lambda_1 - \lambda_3$ with

$$\gamma_j = \frac{3^{n_j}}{4}(1+2z_j) \quad z_j \in \mathbb{Z} \tag{4.63}$$

Step 4 Since

$$\gamma_1 + \gamma_2 = \gamma_0 \tag{4.64}$$

we obtain that

$$3^{n_1}(1+2z_1) + 3^{n_2}(1+2z_2) = 3^{n_0}(1+2z_0). \tag{4.65}$$

This is a contradiction since in eq. (4.65), the left-hand side is even but the right-hand side is odd. This completes the proof.

□

Now, assume that we can generate a countable sequence $\{e_\lambda\}_{\lambda \in U}$ of functions ($\{e_\lambda = e^{i2\pi\lambda x} : \lambda \in U\}$) where U is an index set. For example, using $L_3 = \{0, 3/4\}$ and $3/4$, the smallest positive element of $\mathbf{Z}(\hat{\mu}_3)$, we can calculate a set $U = \{0, 3/4, 3/2, 9/4, 3, 15/4, \ldots\} \equiv \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \ldots\}$ giving that $\langle e_{\lambda_0}, e_{\lambda_1}\rangle = 0$, $\langle e_{\lambda_1}, e_{\lambda_2}\rangle = 0$, $\langle e_{\lambda_2}, e_{\lambda_3}\rangle = 0 \ldots \langle e_{\lambda_k}, e_{\lambda_{k+1}}\rangle = 0$ but $\langle e_{\lambda_0}, e_{\lambda_2}\rangle \neq 0$, since $(3/2 - 0) \notin \mathbf{Z}(\hat{\mu}_3)$. Similarly, $\langle e_{\lambda_1}, e_{\lambda_3}\rangle \neq 0 \ldots \langle e_{\lambda_k}, e_{\lambda_\ell}\rangle \neq 0$ because by inspection, $(\lambda_\ell - \lambda_k) \notin \mathbf{Z}(\hat{\mu}_3) \quad \forall \ell \geq k+2$ So, for U as defined above, only consecutive pairs of exponentials are μ_3-orthogonal giving an illustration of the statement of Theorem 4.3.4.

$\{e_\lambda\}_{\lambda \in U}$ having the characteristic that any of its subsets μ_3-orthogonal exponentials contains at most two elements, cannot be a basis for $L^2(\mu_3)$ but $\{e_\lambda\}_{\lambda \in U} \subset L^2(\mu_3)$. Since $\{e_\lambda\}_{\lambda \in U}$ is countable then, can $\{e_\lambda\}_{\lambda \in U}$ be a frame for $L^2(\mu_3)$? That is, by the definition of a frame [4, p. 3], $\{e_\lambda\}_{\lambda \in U}$ is a frame for $L^2(\mu_3)$ if there exist constants $A, B > 0$ such that:

$$A\|f\|^2 \leq \sum_{\lambda \in U} |\langle f, e_\lambda\rangle|^2 \leq B\|f\|^2 \quad \forall f \in L^2(\mu_3) \tag{4.66}$$

The key to obtain such a frame is to find the constants $A, B > 0$ such that eq. (4.66) is satisfied $\forall f \in L^2(\mu_3)$ but as indicated in Section 4.2 this not a simple task.

4.4 Concluding remarks

The question of finding a frame for $L^2(\mu_3)$ was considered by Lev [23] and by Picioroaga and Weber [24] but, at of this writing, the question remains an open problem. Staying within the framework of identifying, among self-similar measures that satisfy the Hutchinson's recursive relationship, the ones that leads to an orthonormal basis, we have

(a) in Appendix J, we extend in general form the derivations in one dimension of the Fourier transform of the ternary and quaternary measure, μ_3 and μ_4 to odd and even scales higher than 3 and 4 . The key element in doing so is to establish general Iterative Function Systems (IFSs) that each leads to a Cantor set $C \subset [0, 1]$ of Lebesgue measure $m(C) = 0$. We show that we can obtain the Fourier transform of $(2k+1)$-ary measure (odd scale), $k \in \mathbb{N}$, for scale larger than 3, but from Jorgensen and Pedersen [20, Thm 6.1, p. 217], any set of $(2k+1)$-

ary μ-measure orthogonal exponentials contains at most two elements. We conclude that for $(2k+1)$-ary measure (odd scale) we cannot have an orthonormal basis of $L^2(\mu_{2k+1})$. For even scales, we conjecture that the quaternary results of Jorgensen and Pedersen [20] can be extended to a scale of 6 or $(2k)$-ary measure (even scale).

(b) Laba and Wang [22] studied self-similar measures that satisfy the Hutchinson's recursive relationship where the self-similar sets are generated by IFS having more than two equations. The aim of their work is identifying among these self-similar measures the ones that leads to an orthonormal basis. Their work extends the one of Jorgensen and Pedersen [20].

(c) Wang and Yin [27] studied the equal-weighted Moran measures. They first characterize all the maximal orthogonal sets in an L^2 space via a tree mapping. With this characterization, they arrive at sufficient conditions to identify an orthonormal basis.

Appendix A

Details of the construction of a quaternary Cantor set by IFS

In this Appendix, we give the details of the construction of a particular quaternary Cantor set. This means that there are others but the focus is on the one presented here as it becomes central later in this work. Although, we should normally be able to express the right endpoint of the closed intervals generated by construction endpoints by an expansion in base 4 with coefficients 0 and 2. This would enable the generation of these endpoints on the next construction levels using right shift only or right shift plus translation as done for the ternary Cantor set. However, this Appendix shows this to be impossible.

The construction levels come from the repeated application of the following Iterated Function System (IFS):

$$\begin{aligned} \tau_0(x) &= \frac{x}{4} \\ \tau_1(x) &= \frac{x+2}{4} \end{aligned} \tag{A.1}$$

starting with interval $[0,1]$ we have the first three construction levels of the quaternary Cantor set $C^{(4)}$:

$$C_0^{(4)} = \left[\frac{0}{1}, \frac{1}{1}\right] \tag{A.2}$$

$$C_1^{(4)} = \left[\frac{0}{4}, \frac{1}{4}\right] \cup \left[\frac{1}{2}, \frac{3}{4}\right] \tag{A.3}$$

$$C_2^{(4)} = \left[\frac{0}{16}, \frac{1}{16}\right] \cup \left[\frac{1}{8}, \frac{3}{16}\right] \cup \left[\frac{1}{2}, \frac{9}{16}\right] \cup \left[\frac{5}{8}, \frac{11}{16}\right] \tag{A.4}$$

$$C_3^{(4)} = \left[\frac{0}{64}, \frac{1}{64}\right] \cup \left[\frac{1}{32}, \frac{3}{64}\right] \cup \left[\frac{1}{8}, \frac{9}{64}\right] \cup \left[\frac{5}{32}, \frac{11}{64}\right] \cup \left[\frac{1}{2}, \frac{33}{64}\right] \cup \left[\frac{17}{32}, \frac{35}{64}\right] \cup \left[\frac{5}{8}, \frac{41}{64}\right] \cup \left[\frac{21}{32}, \frac{43}{64}\right], \tag{A.5}$$

illustrated in Figure A.1 below.

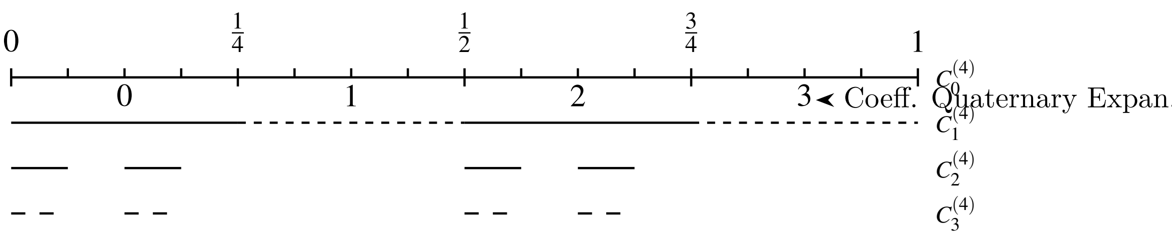


Figure A.1: Few construction levels of the quaternary Cantor set.

Dividing the closed intervals in four and keeping the first and third subintervals imply the expansion in base 4 would be with 0's and 2's as coefficients similar to ternary Cantor set where expansions are in base 3 with also 0 and 2 as coefficients (see Figure A.2 below). These coefficients corresponds to the first and third subintervals that were kept. The common endpoint among the three construction levels in eqs. (A.3, A.4, A.5) is 1/2 which is equal to 0.2_4 (subscript 4 indicates a base 4 number). Normally, we should be able to express the endpoints 1/4 and 3/4 of $C_1^{(4)}$ by an expansion in base 4 with coefficients 0 and 2 such that we can generate the endpoints on the next construction levels using right shift only or right shift plus translation as done for the ternary Cantor set. However, we observe that

$$0.00\overline{2}_4 = \frac{2}{4^3} + \frac{2}{4^4} + \frac{2}{4^5} + \cdots$$
$$= \frac{2}{4^3}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)$$
$$= \frac{1}{24} < \frac{1}{4} \tag{A.6}$$

$$0.0\overline{2}_4 = \frac{2}{4^2} + \frac{2}{4^3} + \frac{2}{4^4} + \cdots$$
$$= \frac{2}{4^2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)$$
$$= \frac{1}{6} < \frac{1}{4} \tag{A.7}$$

$$0.\overline{2}_4 = \frac{2}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \frac{2}{4^4} + \cdots$$
$$= \frac{2}{4}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)$$
$$= \frac{2}{3} > \frac{1}{4} \tag{A.8}$$

From eqs (A.6),(A.7) and (A.8) we obtain:

$$0.00\overline{2}_4 < 0.0\overline{2}_4 < \frac{1}{4} < 0.2_4 = \frac{1}{2} < 0.\overline{2} \tag{A.9}$$

Thus, starting from the second position of the quaternary expansion, any infinite combination of 0's and 2's always sum up to a value less than 1/4. Any quaternary expansion starting from the first position always sum up to a value greater than 1/4. We conclude that 1/4 cannot be expressed as a quaternary expansion with 0's and 2's as coefficients. The same conclusion applies to the other endpoint 3/4 since $0.\overline{2}_4 = 2/3 < 3/4$. Therefore, on construction level 1, only the left endpoints of the closed intervals can be expressed as a finite quaternary expansion with coefficients 0's and 2's. The same occurs at deeper construction levels. For instance, for Construction level 2 we have:

1. $0 = 0.0_4,\ 1/8 = 0.02_4,\ 1/2 = 0.2_4,\ 5/8 = 0.22_4$

2. $0.00\overline{2}_4 = 1/24 < 1/16 < 0.\overline{02}_4 = 2/15$

3. $0.\overline{02}_4 = 2/15 < 3/16 < 0.2_4 = 1/2$

4. $0.2_4 = 1/2 < 9/16 < 0.\overline{2}_4 = 2/3$

5. $0.\overline{2}_4 < 11/16$

Clearly, all the left endpoints of the closed intervals in construction level 2 can be expressed as a finite quaternary expansion with coefficients 0's and 2's, the respective right endpoints cannot be.

However, consider Figure A.2, part (a) illustrates few construction levels of the ternary Cantor set with the value of the endpoints included and similarly, part (b), for the quaternary Cantor set. We observe:

(i) For the ternary Cantor set that left and right endpoints of the closed intervals remain when going from one construction level to the next. Whereas, for the quaternary Cantor set only the left endpoints remain and implies that $\boldsymbol{C}^{(4)}$ contains none of the right endpoints generated in the construction process.

- Indeed, define $\boldsymbol{C}_L^{(4)} = \{x \in \mathbb{R} : x \in \boldsymbol{C}^{(4)} \cap [0, 1/4]\}$ and $\boldsymbol{C}_R^{(4)} = \{x \in \mathbb{R} : x \in \boldsymbol{C}^{(4)} \cap [1/2, 3/4]\}$. From Table A.1, with increasing level comes an increased value of the left endpoints of the right-most closed intervals subset of $[0, 1/4]$ and $[1/2, 3/4]$ respectively and these endpoints are in the quaternary Cantor set $\boldsymbol{C}^{(4)}$. Whereas, the value of right endpoints decrease for the corresponding intervals.

- From Figure A.1 and Table A.1, the quaternary Cantor set ends up being skewed to the left of the interval $[0, 1]$.

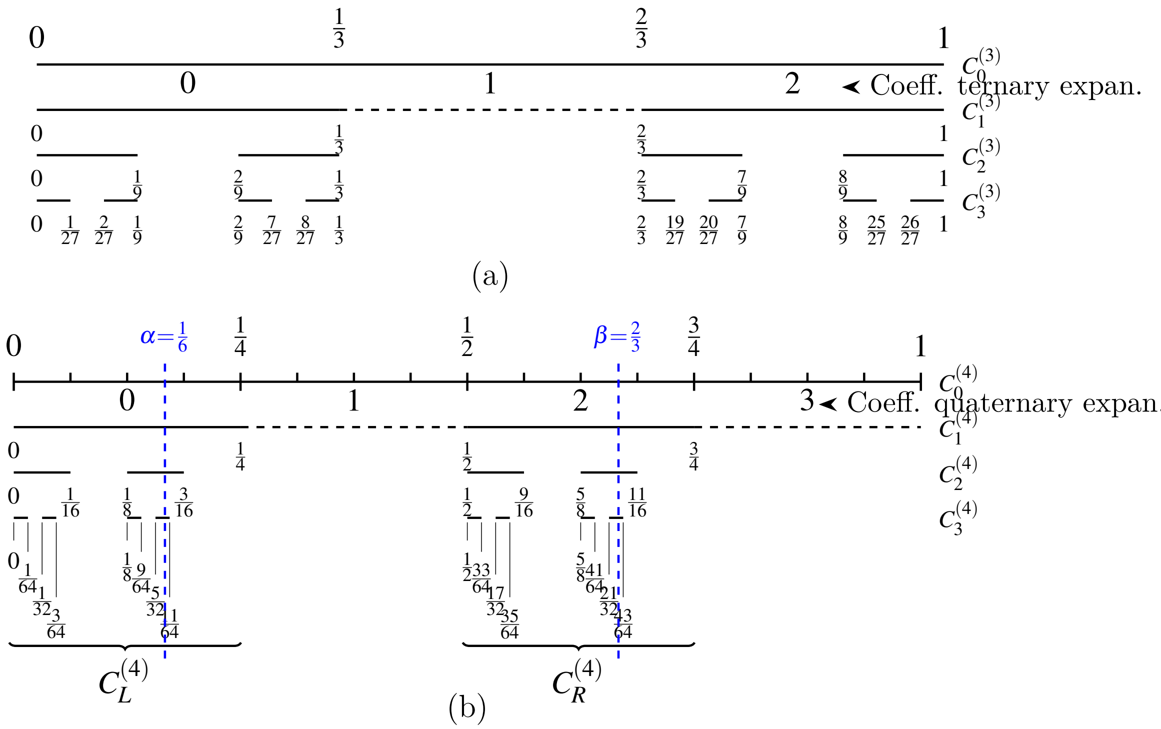


Figure A.2: Few construction levels of the ternary and quaternary Cantor sets

(ii) Define $\alpha = \sup_{x \in C^{(4)}}(C_L^{(4)})$ and $\beta = \sup_{x \in C^{(4)}}(C_R^{(4)})$ and consider the following claim that $\alpha = 1/6$ and $\beta = \alpha + 1/2 = 2/3$. From Table A.1, we can write:

(iii) We concluded that only each of the left endpoints of the closed intervals in the unions in eqs. (A.3), (A.4) and (A.5) can be expressed by a finite quaternary expansion with "0″ and "2″ for coefficients, as it must. However, to express the right endpoints as finite quaternary expansions, the coefficients will have to include "1″ and "3″ on top of "0″ and "2″. That is, the set of coefficients is $\{0,1,2,3\}$. Since at a given construction level k, the length of any of the closed intervals is $1/4^k$, we obtain the right endpoints in base 4 by adding $1/4^k$ to the last coefficient of the expansion for the left endpoint of the closed interval:

$$C_1^{(4)} = [0.0_4, 0.1_4] \cup [0.2_4, 0.3_4] \tag{A.10}$$

Constr. Level	Right-most subset of	
	$[0,1/4]$	$[1/2,3/4]$
1	$[0,1/4]$	$[1/2,3/4]$
2	$[1/8,3/16]$	$[5/8,11/16]$
3	$[5/32,11/64]$	$[21/32,43/64]$

Table A.1: Right-most subsets of $[0,1/4]$ and $1/2,3/4$ that cover some parts of $C^{(4)}$

Constr. Level	Left and right parts of $C^{(4)}$	
	$C_L^{(4)}$	$C_R^{(4)}$
1	$0<\alpha<1/4$	$1/2<\beta<3/4$
2	$1/8<\alpha<3/16$	$5/8<\beta<11/16$
3	$5/32<\alpha<11/64$	$21/32<\beta<43/64$

Table A.2: Upper and lower bounds for α and β

$$C_2^{(4)} = [0.00_4, 0.01_4] \cup [0.02_4, 0.03_4] \cup [0.20_4, 0.21_4] \cup [0.22_4, 0.23_4] \tag{A.11}$$

$$\begin{aligned} C_3^{(4)} &= [0.000_4, 0.001_4] \cup [0.002_4, 0.003_4] \cup [0.020_4, 0.021_4] \cup [0.022_4, 0.023_4] \quad (A.12) \\ &\cup [0.200_4, 0.201_4] \cup [0.202_4, 0.203_4] \cup [0.220_4, 0.221_4] \cup [0.222_4, 0.223_4] \end{aligned}$$

(iv) In quaternary representation, $0 = 0.0_4$ and $1 = 0.\overline{3}_4 = 0.33333\ldots_4$ then $C_0^{(4)} = \left[0.0_4, 0.\overline{3}_4\right]$. Let σ_4 be defined as the right shift equivalent to τ_0 in eq. (2.65). Since $\tau_1(x) = \tau_0(x) + 1/2$ and $1/2 = 0.2_4$ then adding 0.2_4 to a right shift σ_4 is equivalent to $\tau_1(x)$ in eq. (2.65), a combination of a contraction and translation. That is:

$$\begin{aligned} \sigma_4(0.x_1x_2x_3\ldots) &= 0.0x_1x_2x_3\ldots \quad (A.13) \\ \sigma_4(0.x_1x_2x_3\ldots) + 0.2_4 &= 0.2x_1x_2x_3\ldots \end{aligned}$$

Applying the right shift σ_4 to each endpoint at level k gives the left half of level $k+1$. Similarly, applying the right shift σ_4 with the translation by 0.2_3 to each endpoint at level k gives the right half of level $k+1$. Starting with $C_0^{(4)} = \left[0.0_4, 0.\overline{3}_4\right]$, application of eq. (A.13) three times results in eqs. (A.10), (A.11) and (A.13).

(v) From eqs. (A.10), (A.11) and (A.13), we write Table A.3 with endpoints in base 4:

Constr. Level	Left and right parts of $C^{(4)}$	
	$C_L^{(4)}$	$C_R^{(4)}$
1	$0.0_4 < \alpha < 0.1_4$	$0.2_4 < \beta < 0.2_4 + 0.1_4 = 0.3_4$
2	$0.02_4 < \alpha < 0.02_4 + 0.01_4 = 0.03_4$	$0.22_4 < \beta < 0.22_4 + 0.01_4 = 0.23_4$
3	$0.022_4 < \alpha < 0.022_4 + 0.001_4 = 0.023_4$	$0.222_4 < \beta < 0.222_4 + 0.001_4 = 0.223_4$
$\vdots$	$\vdots$	$\vdots$
k	$0.0\underbrace{2222\ldots2_4}_{k-1 \text{ times}} < \alpha < 0.0\ \underbrace{2222\ldots2}_{k-2 \text{ times}}\ 3_4$	$0.\underbrace{2222\ldots2_4}_{k \text{ times}} < \beta < 0.\ \underbrace{2222\ldots2}_{k-1 \text{ times}}\ 3_4$

Table A.3: Upper and lower bounds for α and β in base 4

(vi) For construction level k, we write the inequalities for α and β in terms of quaternary expansions:

$$\begin{aligned} \textstyle\sum_{j=2}^{k} \frac{2}{4^j} \quad &< \quad \alpha \quad < \quad \textstyle\sum_{j=2}^{k} \frac{2}{4^j} + \frac{1}{4^k} \\ \textstyle\sum_{j=1}^{k} \frac{2}{4^j} \quad &< \quad \beta \quad < \quad \textstyle\sum_{j=1}^{k} \frac{2}{4^j} + \frac{1}{4^k} \end{aligned} \tag{A.14}$$

(vii) Using eqs. (A.7) and (A.8) and applying the Squeeze Theorem with $k \to \infty$, eq. (A.14) becomes :

$$\begin{aligned} \tfrac{1}{6} \quad &\leq \quad \alpha \quad \leq \quad \tfrac{1}{6} \\ \tfrac{2}{3} \quad &\leq \quad \beta \quad \leq \quad \tfrac{2}{3} \end{aligned} \tag{A.15}$$

giving that $\alpha = 1/6$ and $\beta = 2/3$. Proving the claim that $\alpha = 1/6$ and $\beta = \alpha + 1/2 = 2/3$. Thus, the left and right parts of the quaternary Cantor set $C^{(4)}$ spread respectively over $[0, 1/6]$ and $[1/2, 2/3]$.

Appendix B

Change of Variable for Integration with Respect to a Measure

The binary, ternary and quaternary Cantor sets, $[0,1]$, $C^{(3)}$ and $C^{(4)}$ respectively, can be constructed using the following IFS's:

(i) for binary Cantor set $[0,1]$

$$\begin{aligned} B_0(x) &= \frac{x}{2} \\ B_1(x) &= \frac{x+1}{2} \end{aligned} \tag{B.1}$$

Since $B_0(x)$ and $B_1(x)$ are well-defined and continuous over $\mathbb{R}$, their inverses

$$\begin{aligned} B_0^{-1}(x) &= 2x \\ B_1^{-1}(x) &= 2x-1 \end{aligned} \tag{B.2}$$

have the same properties.

(ii) for ternary Cantor set $\boldsymbol{C}^{(3)}$

$$\begin{aligned} T_0(x) &= \frac{x}{3} \\ T_1(x) &= \frac{x+2}{3} \end{aligned} \tag{B.3}$$

Since $T_0(x)$ and $T_1(x)$ are well-defined and continuous over $\mathbb{R}$, their inverses

$$\begin{aligned} T_0^{-1}(x) &= 3x \\ T_1^{-1}(x) &= 3x-2 \end{aligned} \tag{B.4}$$

have the same properties.

(iii) for quaternary Cantor set $\boldsymbol{C}^{(4)}$

$$\begin{aligned} \tau_0(x) &= \frac{x}{4} \\ \tau_1(x) &= \frac{x+2}{4} \end{aligned} \tag{B.5}$$

Since $\tau_0(x)$ and $\tau_1(x)$ are well-defined and continuous over $\mathbb{R}$, their inverses

$$\begin{aligned} \tau_0^{-1}(x) &= 4x \\ \tau_1^{-1}(x) &= 4x-2 \end{aligned} \tag{B.6}$$

have the same properties.

We showed that for any Borel set A that

(i) for binary Cantor set $[0,1]$

$$\mu_{m_2}(A) = \frac{1}{2}\left(\mu_{m_3}(B_0^{-1}(A)) + \mu_{m_2}(B_1^{-1}(A))\right) \tag{B.7}$$

(ii) for ternary Cantor set $C^{(3)}$

$$\mu_{m_3}(A) = \frac{1}{2}\left(\mu_{m_3}(T_0^{-1}(A)) + \mu_{m_3}(T_1^{-1}(A))\right) \tag{B.8}$$

(iii) for quaternary Cantor set $C^{(4)}$

$$\mu_{m_4}(A) = \frac{1}{2}\left(\mu_{m_4}(\tau_0^{-1}(A)) + \mu_{m_4}(\tau_1^{-1}(A))\right) \tag{B.9}$$

To analyse the spectrum of μ_{m_2}, μ_{m_3} or μ_{m_4}, we need to take their Fourier transform which implies the need to be able to integrate a continuous function f on either $[0,1]$, $C^{(3)}$ or $C^{(4)}$ with respect to μ_{m_2}, μ_{m_3} or μ_{m_4} respectively. Since $B_0, B_1, T_0, T_1, \tau_0, \tau_1$ are all well-behaved, bijective and continuous functions, we do the integration of f with respect to μ_{m_3}, the integration with respect to μ_{m_2} and μ_{m_4} is very similar not to say identical. Using eq. (B.8), we have for any Borel set A

$$\int_A f d\mu_{m_3} = \frac{1}{2}\left(\int_{T_0(A)} f d(\mu_{m_3} \circ T_0^{-1}) + \int_{T_1(A)} f d(\mu_{m_3} \circ T_1^{-1})\right) \tag{B.10}$$

Do we have a way to verify that eq. (B.10) make sense? We can answer that question by the affirmative by considering $f \equiv 1$, a continuous function over all of $\mathbb{R}$, to obtain by [2, (3.4), p. 75] that

$$\begin{aligned}
\textstyle\int_A f d\mu_{m_3} &= \textstyle\int_A \chi_A d\mu_{m_3} &&= \mu_{m_3}(A) \\
\textstyle\int_{T_0(A)} f d(\mu_{m_3} \circ T_0^{-1}) &= \textstyle\int_{T_0(A)} \chi_{T_0(A)} d(\mu_{m_3} \circ T_0^{-1}) &&= \mu_{m_3} \circ T_0^{-1}(T_0(A)) = \mu_{m_3}(A) \\
\textstyle\int_{T_1(A)} f d(\mu_{m_3} \circ T_1^{-1}) &= \textstyle\int_{T_1(A)} \chi_{T_1(A)} d(\mu_{m_3} \circ T_1^{-1}) &&= \mu_{m_3} \circ T_1^{-1}(T_1(A)) = \mu_{m_3}(A)
\end{aligned} \tag{B.11}$$

so we get

$$\begin{aligned}
\textstyle\int_A f d\mu_{m_3} &= \tfrac{1}{2}\left(\textstyle\int_{T_0(A)} f d(\mu_{m_3} \circ T_0^{-1}) + \textstyle\int_{T_1(A)} f d(\mu_{m_3} \circ T_1^{-1})\right) \\
\mu_{m_3}(A) &= \tfrac{1}{2}(\mu_{m_3}(A) + \mu_{m_3}(A)) = \mu_{m_3}(A)
\end{aligned} \tag{B.12}$$

To evaluate the integrals on the right-hand side of eq. (B.10) we need to do a change of variables. That is, we need to prove the following Lemma for the first integral. The proof for the second integral is similar.

Lemma B.0.1. $T_0(\cdot)$ is a well-behave, bijective and continuous function. We then have for any Borel set A that:

$$\int_{T_0(A)} f d(\mu_{m_3} \circ T_0^{-1}) = \int_A f \circ T_0 \ d\mu_{m_3} \tag{B.13}$$

Proof.

Step 1 Let f be a non-negative function, then there exists a sequence of simple measurable functions:

$$\phi_n = \sum_{i=1}^{m_n} c_{n,i} \chi_{B_{n,i}} \tag{B.14}$$

such that $\phi_n \nearrow f$ and where for each n, $\{B_{n,i}\}$ is a sequence of disjoints set in the σ-algebra on A.

Step 2 By the Monotone Convergence Theorem [2, p. 78] we can write:

$$\begin{aligned}
\int_{T_0(A)} f d(\mu_{m_3} \circ T_0^{-1}) &= \lim_{n\to\infty} \int_{T_0(A)} \phi_n d(\mu_{m_3} \circ T_0^{-1}) \\
&= \lim_{n\to\infty} \int_{T_0(A)} \textstyle\sum_{i=1}^{m_n} c_{n,i} \chi_{B_{n,i}} d(\mu_{m_3} \circ T_0^{-1}) \\
&= \lim_{i\to\infty} \textstyle\sum_{i=1}^{m_n} c_{n,i} \int_{T_0(A)} \chi_{B_{n,i}} d(\mu_{m_3} \circ T_0^{-1}) \\
&= \lim_{i\to\infty} \textstyle\sum_{i=1}^{m_n} c_{n,i} \mu_{m_3}(T_0^{-1}(\chi_{B_{n,i}})) \\
&= \lim_{i\to\infty} \textstyle\sum_{i=1}^{m_n} c_{n,i} \int_A \chi_{T_0^{-1}(B_{n,i})} \ d\mu_{m_3} \\
&= \lim_{i\to\infty} \textstyle\sum_{i=1}^{m_n} c_{n,i} \int_A \chi_{B_{n,i}} \circ T_0 \ d\mu_{m_3} \\
&= \lim_{i\to\infty} \int_A \left(\textstyle\sum_{i=1}^{m_n} c_{n,i} \chi_{B_{n,i}}\right) \circ T_0 \ d\mu_{m_3} \\
&= \lim_{i\to\infty} \int_A \phi_n \circ T_0 \ d\mu_{m_3} \\
&= \int_A f \circ T_0 \ d\mu_{m_3}
\end{aligned} \tag{B.15}$$

The last equality is established because $\phi_n \circ T_0$ is a simple measurable function such that $\phi_n \circ T_0 \nearrow f \circ T_0$.

Step 3 The equality between $\chi_{T_0^{-1}(B_{n,i})}$ and $\chi_{B_{n,i}} \circ T_0$ in the above step, can be deduced as follows:

(i) if $x \in T_0^{-1}(B_{n,i})$ then $\chi_{T_0^{-1}(B_{n,i})} = 1$, that is, if x is in the pre-image of $B_{n,i}$ by T_0 then $T_0(x) \in B_{n,i}$ and we have $\chi_{B_{n,i}}(T_0(x)) = 1$

(ii) if $x \notin T_0^{-1}(B_{n,i})$ then $\chi_{T_0^{-1}(B_{n,i})} = 0$, that is, if x is NOT in the pre-image of $B_{n,i}$ by T_0 then $T_0(x) \notin B_{n,i}$ and we have $\chi_{B_{n,i}}(T_0(x)) = 0$

This completes the proof of the Lemma.

□

Applying Lemma B.0.1 to eq.(B.10) we obtain for

(a) for the binary Cantor set

$$\int_A f d\mu_{m_2} = \frac{1}{2}\left(\int_A f(B_0(x))d\mu_{m_2} + \int_A f(B_1(x))d\mu_{m_2}\right) \tag{B.16}$$

(b) for the ternary Cantor set

$$\int_A f d\mu_{m_3} = \frac{1}{2}\left(\int_A f(T_0(x))d\mu_{m_3} + \int_A f(T_1(x))d\mu_{m_3}\right) \tag{B.17}$$

(c) for the quaternary Cantor set

$$\int_A f d\mu_{m_4} = \frac{1}{2}\left(\int_A f(\tau_0(x))d\mu_{m_4} + \int_A f(\tau_1(x))d\mu_{m_4}\right) \tag{B.18}$$

Eq.(B.16) is obtained by B_0 playing the role of T_0 and B_1 playing the role of T_1 as B_0 and B_1 have the same properties as T_0 and T_1, respectively. Eq.(B.18) is obtained by τ_0 playing the role of T_0 and τ_1 playing the role of T_1 as τ_0 and τ_1 have the same properties as T_0 and T_1, respectively.

Appendix C

Relation between mass distribution measure and the quaternary Cantor-Lebesgue function

This Appendix presents the relation between mass distribution measure and quaternary Cantor-Lebesgue function by giving a proof of Proposition 5 which statement is repeated here for convenience:

For any closed interval $[0,a] \subset [0,1]$, the mass distribution measure is given by $\mu_m([0,a]) = W(a)$ and for a half open interval $(a,b]$, we have $\mu_m((a,b]) = \mu_m([0,b]) - \mu_m([0,a]) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$.

Proof.

Step 1 We recall that W is defined as the limit of the sequence defined by the recursive relation (2.70). The properties W we will use below can be derived by induction starting with $h_0(x) = x$. We start by considering the values for h_1 and h_2 for construction level 1 and 2 respectively

(a) Referring to Figure 2.5(b) for the quaternary Cantor set, we observe that all left endpoints of closed intervals generated by the construction process remain when go-

Construction level 1 $(k=1)$		
Sub-case No.	Interval considered	$h_1(x)$
1	$x \in (0,1/4)$	$0 \leq h_1(x) \leq 1/2$
2	$x \in [1/4,1/2]$	$h_1(x) = 1/2$
3	$a \in (1/2,3/4)$	$1/2 \leq h_1(x) \leq 1$
4	$a \in [3/4,1]$	$h_1(x) = 1$

Table C.1: Values for h_1 at construction level $k = 1$.

Construction level 2 $(k=2)$		
Sub-case No.	Interval considered	$h_2(x)$
1	$x \in (0,1/16)$	$0 \leq h_2(x) \leq 1/2^2$
2	$x \in [1/16,1/8]$	$h_2(x) = 1/2^2$
3	$a \in (1/8,3/16)$	$1/2^2 \leq h_1(x) \leq 1/2$
4	$a \in [3/16,1/2]$	$h_2(x) = 1/2$
5	$x \in (1/2,9/16)$	$1/2 \leq h_2(x) \leq 3/2^2$
6	$x \in [9/16,5/8]$	$h_2(x) = 3/2^2$
7	$a \in (5/8,11/16)$	$3/2^2 \leq h_2(x) \leq 1$
8	$a \in [11/16,1]$	$h_2(x) = 1$

Table C.2: Values for h_2 at construction level $k = 2$.

ing from one construction level to the next. For instance, $1/2$ is the left endpoint of $[1/2,3/4]$ at construction level 1,the left endpoint of $[1/2,9/16]$ at construction level 2 and the left endpoint of $[1/2,33/64]$ at construction level 3. The same occurs for left endpoint $1/8$ of $[1/8,3/16]$ and left endpoint $5/8$ of $[5/8,11/16]$ but starting at construction level 2. So, at construction level 1 we have 2 left endpoints $\{0,1/2\}$ repeated at construction level 2 (in red) where we have 2^2 left endpoints $\{0,1/8,1/2,5/8\}$. At construction level 3 we have 2^3 left endpoints $\{0,1/32,1/8,5/32,1/2,17/32,5/8,21/32\}$. Also, by construction all left endpoints eventually start to repeat from one level to the next.

(b) Tables C.1 and C.2 indicates using the recursive relation (2.70), that $h_1(h_0(1/2)) = 1/2 + h_0(4(1/2)-2) = 1/2$ and $h_2(h_1(h_0(1/2))) = h_2(1/2) = 1/2 + (1/2)h_1(4(1/2) - 2) = 1/2$ to get that for a repeated left endpoint like $1/2$, $h_2(1/2) = h_1(1/2) = h_0(1/2) = 1/2$. Continuing inductively we obtain that at construction level $k = n$, $h_n(a/b) = h_{n-1}(a/b) = \ldots = h_\ell(a/b) = p_\ell$ for repeated left endpoint a/b where ℓ is the construction level at which a/b first appears. So, $\forall n \in \mathbb{N}$, $h_n(a/b) = p_\ell$ (a constant). Then $\lim_{n\to\infty} h_n(a/b) = W(a/b) = p_\ell$.

(c) We can then write the following properties of W:

- for a repeated left endpoint a/b, we have that $W(a/b) = h_\ell(a/b) = p_\ell$ where ℓ is the construction level at which a/b first appears.
- by definition of W, $W(x) = p_\ell$ (a constant) for $a_l \le x \le a/b$ where a_l is the first right endpoint of a closed interval on the left of a/b.

We now prove the proposition by induction:

Step 2 $k = 0$: Trivial cases: $a = 0$ and $a = 1$

(i) Since $[0,0]$ is a singleton, we showed by eq.(3.48) in 92 the mass of a single point is 0. So, $\mu([0,0]) = 0 = W(0) = 0$.

(ii) $\mu([0,1]) = 1 = W(1) = 1$ by definition of unit mass.

(iii) We can write for these two trivial cases that $|\mu([0,a]) - W(a)| \le 1 = 1/2^0$

Step 3 $k = 1$: endpoints are $\{0, 1/4, 1/2, 3/4\}$, $W(1/2) = 1/2$ and by definition of W, $W(x) = 1/2$ for $1/4 \le x \le 1/2$. Then, $\mu_m([0,1/4]) = W(1/4) = 1/2$ and $\mu_m([0,1/2]) = W(1/2) = 1/2$. Similarly, for $x = 1$, $h_n(1) = 1, \quad \forall n \in \mathbb{N}$ so $W(1) = 1$. By extension of W, $W(x) = 1$ for $3/4 \le x \le 1$. By monotonicity of μ_m and by the properties of W, and that $\mu_m = 0$ on $\mathscr{E}_1^c$, we have: From Table C.3 we conclude that $|\mu_{m_4}([0,a]) - W(a)| \le 1/2^1 = 1/2$.

Construction level 1 ($k = 1$)			
Sub-case No.	Interval considered	Implications for $\mu_m(\cdot)$	Implications for $W(\cdot)$
1	$a \in (0, 1/4)$	$0 \leq \mu_m([0, a]) \leq 1/2$	$0 \leq W(a) \leq 1/2$
2	$a \in [1/4, 1/2]$	$\mu_m([0, a]) = \mu_m([0, 1/4] = 1/2$	$W(a) = 1/2$
3	$a \in (1/2, 3/4)$	$1/2 \leq \mu_m([0, a]) \leq 1$	$1/2 \leq W(a) \leq 1$
4	$a \in [3/4, 1]$	$\mu_m([0, a]) = \mu_m([0, 3/4] = 1$	$W(a) = 1$

Table C.3: Mass distribution and Cantor function for construction level $k = 1$.

Step 4 From the above, we can then formulate the Induction Hypothesis: For $k = n - 1$ and for any interval from construction $E = [a_0, b_0] \in \mathscr{E}_{n-1}$ (at level $k = n - 1$) with positive mass, we suppose true that $q_0 = \mu_m([0, a_0]) = W(a_0)$ by properties of W and $\forall a \in [0, 1], |\mu_m([0, a]) - W(a)| \leq 1/2^{n-1}$.

Step 5 Induction Step: For $k = n$ and for any interval from construction $E = [a_0, b_0] \in \mathscr{E}_{k-1}$ (level $k = n - 1$) and having a positive mass, we have that when a is an endpoint at the n^{th} level which is not a_0 or b_0, then $a = a_0 + 1/4^n$ or $a = a_0 + 2/4^n$ and by construction $\mu_m([0, a]) = q_0 + 1/2^n$. This can also been seen in Table C.4 where by construction, $\mu_m[a_0, b_0]$ is decomposed equally by mass distribution as follows:

$$\begin{aligned} \mu_m([a_0, b_0]) &= \mu_m([a_0, a_0 + 1/4^n]) + \mu_m([a_0 + 2/4^n, a_0 + 4/4^n = b_0]) \\ \frac{1}{2^{n-1}} &= \frac{1}{2^n} + \frac{1}{2^n} \end{aligned} \tag{C.1}$$

By Induction Hypothesis, $\mu_m([0, a_0]) = q_0 = W(a_0)$ and $W(a) = W(a_0) + 1/2^n$ by properties of W, so $\mu_m([0, a]) = W(a)$. We conclude that $|\mu([0, a]) - W(a)| \leq 1/2^n$ and equality holds at the endpoints of level $k = n$. Now, as $n \to \infty$, $\mu_m([0, a]) = W(a) \quad \forall a \in [0, 1]$ and $\mu_m((a, b]) = \mu_m([0, b]) - \mu_m([0, a]) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$. This completes the proof of Proposition 5.

□

Construction level n ($k=n$)			
Sub-case No.	Interval considered	Implications for $\mu_m(\cdot)$	Implications for $W(\cdot)$
n_1	$a \in (a_0, a_0 + 1/4^n)$	$q_0 \leq \mu_m([0,a]) \leq q_0 + 1/2^n$	$q_0 \leq W(a) \leq q_0 + 1/2^n$
n_2	$a \in [a_0 + 1/4^n, a_0 + 2/4^n]$	$\mu_m([0,a]) = q_0 + 1/2^n$	$W(a) = q_0 + 1/2^n$
n_3	$a \in (a_0 + 2/4^n, a_0 + 3/4^n)$	$q_0 + 1/2^n \leq \mu_m([0,a]) \leq q_0 + 1/2^{n-1}$	$q_0 + 1/2^n \leq W(a) \leq q_0 + 1/2^{n-1}$
n_4	$a \in [a_0 + 3/4^n, a_0 + 1/4^{n-1} = b_0]$	$\mu_m([0,a]) = q_0 + 1/2^{n-1}$	$W(a) = q_0 + 1/2^{n-1}$

Table C.4: Mass distribution and Cantor function for one group of sub-cases at construction level $k = n$.

Appendix D

Relation between Hausdorff measure and the quaternary Cantor-Lebesgue function

In this Appendix we establish a relation between the Hausdorff measure of dimension $s = 1/2$, restricted to $C^{(4)}$, of $[0,a] \subset [0,1]$ and the extended Cantor-Lebesgue function W for $C^{(4)}$. More precisely, we prove the claim that for every $0 \leq a \leq 1$, $\mathscr{H}^s([0,a] \cap C^{(4)}) = W(a)$. Then, the Appendix concludes by the proof of relation between the Hausdorff measure restricted to $C^{(4)}$ of $(a,b] \subset [0,1]$ and W, $\mathscr{H}^s((a,b] \cap C^{(4)}) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$.

We have that $C^{(4)}$ has dimension $s = \log 2/\log 4 = 1/2$ and we make the assumption that Hausdorff measure $\mathscr{H}^s(C^{(4)}) = 1$. The quaternary Cantor set can be constructed using the IFS given in eq. (2.68). $\tau_0(x)$ and $\tau_1(x)$ are similarity transformations of scale factor $0 < \lambda = 1/4 < 1$. The value of λ makes them contractions and also Lipschitz mappings:

$$\begin{aligned} |\tau_0(x) - \tau_0(y)| &\leq \tfrac{1}{4}|x-y| \quad \forall x,y \in [0,1] \\ |\tau_1(x) - \tau_1(y)| = |\tau_0(x) + \tfrac{1}{2} - \tau_0(y) - \tfrac{1}{2}| = |\tau_0(x) - \tau_0(y)| &\leq \tfrac{1}{4}|x-y| \end{aligned} \tag{D.1}$$

Then, by the scaling property of Hausdorff measure [15, p. 46] we have for every $0 \leq a < b \leq 1$:

$$\mathscr{H}^s(\tau_k([a,b])) = \left(\frac{1}{4}\right)^s \mathscr{H}^s([a,b]) = \frac{1}{2}\mathscr{H}^s([a,b]) \quad k = \{0,1\} \tag{D.2}$$

Then, the heuristic calculation performed in proving Theorem 3.1.7 can be extended as follows:

(a) At construction level 2, define $C_{L_1} = [0, 1/16] \cap C^{(4)}$, $C_{L_2} = [1/8, 3/16] \cap C^{(4)}$, $C_{R_1} = [1/2, 9/16] \cap C^{(4)}$, $C_{R_2} = [5/8, 11/16] \cap C^{(4)}$ all disjoint, to obtain: $C^{(4)} = C_{L_1} \cup C_{L_2} \cup C_{R_1} \cup C_{R_2}$.

(b) We can write

$$\begin{aligned} \mathscr{H}^s(C^{(4)}) &= \mathscr{H}^s(C_{L_1}) + \mathscr{H}^s(C_{L_2}) + \mathscr{H}^s(C_{R_1}) + \mathscr{H}^s(C_{R_2}) \\ &= 4 \times \tfrac{1}{(4^2)^s} \mathscr{H}^s(C^{(4)}) \\ 1 &= 4 \times \tfrac{1}{(4^s)^2}. \end{aligned} \tag{D.3}$$

Eq. (D.3) implies that $\mathscr{H}^s(C_{L_1}) = \mathscr{H}^s(C_{L_2}) = \mathscr{H}^s(C_{R_1}) = \mathscr{H}^s(C_{R_2}) = 1/2^2$.

(c) At construction level 3, the Hausdorff measure of the intersection of each closed interval with $C^{(4)}$ equals $1/2^3$ since the scaling ratio is $1/3^3$.

(d) Continuing inductively, then at construction level n, the Hausdorff measure of the intersection of each closed interval in the union $C_n^{(4)}$ with $C^{(4)}$ equals $1/2^n$ since the scaling ratio is $1/4^n$.

At construction level n, section 3.2.2 concluded to the uniform distribution of the unit mass, assigning a mass of $1/2^n$ to each closed interval giving mass distribution measure of $1/2^n$. Clearly, the mass of each of these closed intervals U in $C_n^{(4)}$ equals the Hausdorff measure of the intersection of each of these closed intervals with $C^{(4)}$, $1/2^n$ and this $\forall n \in \mathbb{N}$. So, we obtain that $\mathscr{H}(U \cap C^{(4)}) = \mu_m(U)$ for every closed interval U arising in the construction of $C^{(4)}$ where by definition of $\mathscr{E}$ in eq.(3.44), $U \in \mathscr{E}$ Also, the mass of the subsets of the complement of $C^{(4)}$ is 0 which implies the Hausdorff measure of these subsets is 0. This leads to the following proposition:

Proposition 8. For any closed interval $[0, a] \subset [0, 1]$, the Hausdorff measure, restricted to $C^{(4)}$, is given by $\mathscr{H}([0, a] \cap C^{(4)}) = W(a)$ and for half open interval $(a, b]$, we have $\mathscr{H}((a, b] \cap C^{(4)}) = \mathscr{H}([0, b] \cap C^{(4)}) - \mathscr{H}([0, a] \cap C^{(4)}) = W(b) - W(a)$ for every $0 \leq a < b \leq 1$.

Proof.

With every closed interval U arising in the construction of $C^{(4)}$ contained in $\mathscr{E}$, we have established that $\mathscr{H}(U \cap C^{(4)}) = \mu_m(U)\ \forall U \in \mathscr{E}$. So the proof of Proposition 8 is exactly the same as that of Proposition 4, since all that proof uses is the knowledge of $\mu_m(U)$ for such U. $\square$

Appendix E

Completeness of $(\mathcal{B}([0,1], \|\cdot\|_\infty)$

In this appendix, we show the completeness of $(\mathcal{B}([0,1], \|\cdot\|_\infty)$, the space of all uniformly bounded real-valued functions on $[0,1]$.

Theorem E.0.1. The space of bounded functions on $[0,1]$ with the supremum norm $\|\cdot\|_\infty$ is complete. That is

$$\begin{aligned} \mathcal{B}([0,1], \|\cdot\|_\infty) &= \{f : [0,1] \to \mathbb{R} \quad \text{with} \quad f \text{ bounded}\} \\ \|f\|_\infty &= \sup_{x\in[0,1]}|f(x)| < \infty \end{aligned} \tag{E.1}$$

is complete.

Proof.

A set X is said to be complete if every Cauchy sequence in X converges in X.

Step 1 We assume the sequence $\{f_n\}$ in $\mathcal{B}$ is Cauchy in $\|\cdot\|_\infty$, that is $\|f_n - f_m\|_\infty \to 0$ as $n,m \to \infty$ with $\|f_n - f_m\|_\infty = \sup_{x\in[0,1]}|f_n(x) - f_m(x)|$.

Step 2 By assumption, we have that $\forall x \in \mathbb{R}$, $|f_n(x) - f_m(x)| \to 0$ pointwise as $n,m \to \infty$. Therefore, for each $x \in [0,1]$, the sequence $\{f_n(x)\}$ is a sequence of numbers that is Cauchy in $\mathbb{R}$.

Step 3 Since $\mathbb{R}$ is complete, it implies that $\forall x$, $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}$. From this, let $f(x) = \lim_{n\to\infty} f_n(x)$, then $f_n \to f$ pointwise on $[0,1]$.

Step 4 We need to show that $f_n \to f$ uniformly on $[0,1]$, that is, in $\|\cdot\|_\infty$. Explicitly, we need to show that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$

Step 5 From the assumption in Step 1, let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that $\forall n, m \geq N$ we have $\|f_n - f_m\|_\infty < \varepsilon$. This implies that $\forall x$, $|f_n(x) - f_m(x)| < \varepsilon$ since by definitions of $\|\cdot\|_\infty$ and of supremum we have

$$\forall x, \quad |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon \quad \forall n, m \geq N \tag{E.2}$$

Step 6 Key step: We fix x and n and we let $m \to \infty$ to obtain

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \leq \varepsilon \tag{E.3}$$

The "$< \varepsilon$" in eq. (E.2), becomes "$\leq \varepsilon$" in taking the limit $m \to \infty$. That limit can be taken inside by the continuity of the absolute function.

Step 7 In Step 6, the choice of $x \in [0,1]$ is arbitrary. Therefore $\forall n \geq N$ and $\forall x \in [0,1]$, $|f_n(x) - f(x)| \leq \varepsilon$ giving that $\|f_n - f\|_\infty \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown that $f_n - f$ converges to 0 in the norm $\|\cdot\|_\infty$ so, $f_n \to f$ converges uniformly.

Step 8 There remains to show that $f \in \mathcal{B}([0,1], \|\cdot\|_\infty)$. Let $\varepsilon = 1$, then we choose $N \in \mathbb{N}$ such that for $n \geq N$, $\|f - f_n\|_\infty \leq \varepsilon = 1$. Taking $n = N$, we have $\|f - f_N\|_{infty} \leq 1$. Then, by the triangle inequality we can write that $\|f\|_\infty \leq \|f - f_N\|_\infty + \|f_N\|_\infty$. Since $f_N \in \mathcal{B}([0,1], \|\cdot\|_\infty)$, there exists a positive real constant A such that $\|f_N\|_\infty \leq A$. Then, we can write that $\|f\|_\infty \leq 1 + A$. Therefore f is bounded and $f \in \mathcal{B}([0,1], \|\cdot\|_\infty)$.

□

Appendix F

Banach contraction principle

F.1 Banach contraction principle

Theorem F.1.1. (Banach contraction principle) If (X,d) is a complete metric space and $f: X \to X$ is a contraction, i.e. for some $0 < \alpha < 1$, $d(f(x); f(y)) \leq \alpha d(x;y)$ for all $x, y \in X$, then f has a unique fixed point in X.

Proof.

Step 1 We take $x_1 \in X$, (x_1 arbitrary), $x_2 = f(x_1)$, $x_3 = f(x_2)$, ..., $x_{n+1} = f(x_n)$.

Step 2 Let $a > 0$ and let $d(x_1, x_2) = a$ where

$$\begin{array}{rcccl} d(x_2,x_3) & = & d(f(x_1),f(x_2)) & \leq & \alpha d(x_1,x_2) = \alpha \cdot a \\ d(x_3,x_4) & = & d(f(x_2),f(x_3)) & \leq & \alpha d(x_2,x_2) = \alpha^2 \cdot a \\ \vdots & \vdots & & & \vdots \\ d(x_n,x_{n+1}) & \leq & \alpha^{n-1} \cdot a & & \end{array} \tag{F.1}$$

Step 3 Applying the triangular inequality many times to $d(x_n, x_{n+k})$ we have:

$$\begin{aligned} d(x_n, x_{n+k}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &\leq a(\alpha^{n-1} + \alpha^n + \alpha^{n+1} + \cdots + \cdot\alpha^{n+k-2}) \\ &\leq a\alpha^{n-1}(1 + \alpha + \alpha^2 + \cdots + \alpha^{k-1}) < \tfrac{a\alpha^{n-1}}{1-\alpha} \end{aligned} \tag{F.2}$$

Let $\varepsilon > 0$. With $0 < \alpha < 1$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $a\alpha^{n-1}/(1-\alpha) < \varepsilon$. Given that $\varepsilon > 0$ is arbitrary, then from ineq. (F.2) we have that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ and $n + k \geq n_0$ $(k > 0)$, we get that $d(x_n, x_{n+k}) < \varepsilon$ Therefore $\{x_n\}$ is a Cauchy sequence.

Step 4 Then, we get that $x_n \underset{n\to\infty}{\longrightarrow} x_0$ where by completeness of X, $x_0 \in X$.

Step 5 Since f is a contraction, it is continuous and

$$\begin{array}{ccc} x_{n+1} & = & f(x_n) \\ \downarrow & & \downarrow \\ x_0 & = & f(x_0) \end{array} \tag{F.3}$$

giving that x_0 is a fixed point.

Step 6 We have to show that x_0 is unique, that is, there are no more fixed points. We assume that x_0, x_1 are fixed points. Thus

$$d(x_0, x_1) = d(f(x_0), f(x_1)) \leq \alpha d(x_0, x_1) \tag{F.4}$$

Since $0 < \alpha < 1$, the only value for $d(x_0, x_1)$ to be $\leq \alpha d(x_0, x_1)$ is $d(x_0, x_1) = 0$ so $x_0 = x_1$ and x_0 is unique.

□

F.2 Applications of Banach contraction principle

In the applications of the Banach contraction principle (Theorem F.1.1), in Propositions 1 and 2(Chapter 2) we have that that $X = \mathcal{B}([0,1])$ and d is the supremum norm $\|\cdot\|_\infty = \sup_{x\in[0,1]}|\cdot|$. They take the form of applications of contraction mappings given in Chapter 2, by eqs. (2.55) and (2.71) for respectively the ternary and quaternary Cantor sets where in both cases the initial function $f_0(x)$ is the following bounded and increasing step function:

$$f_0(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < \frac{1}{2}, \\ 1 & \text{for} \quad \frac{1}{2} \le x \le 1 \end{cases} \tag{F.5}$$

We observe that in both cases, the graphs of the functions $f_n(x)$ in the sequence resemble each other more and more, which is another way of saying that it is a Cauchy sequence. This quick convergence of $f_n(x)$ implies that $f_n(x)$ for $n = 100,000$ is graphically close to the limit functions, namely the Cantor-Lebesgue functions, F (Figure F.1) and W (Figure F.2), respectively for the ternary and quaternary Cantor sets. F and W cannot actually be graphed as each of them is a theoretical limit.

We include the additional case to contrast the increasing step function in eq. (F.5) used for f_0. We use the following decreasing step function for f_0:

$$f_0 = \begin{cases} 1 & \text{for} \quad 0 \le x < \frac{1}{2}, \\ 0 & \text{for} \quad \frac{1}{2} \le x \le 1 \end{cases} \tag{F.6}$$

Again we observe convergence in both cases similar to an increasing step function for f_0. This convergence of $f_n(x)$ implies that $f_n(x)$ for $n = 100,000$ is graphically close to the theoretical limit functions, namely the Cantor-Lebesgue functions, F (Figure F.3) and W (Figure F.4), respectively for the ternary and quaternary Cantor sets. Being theoretical limit functions, F and W cannot actually be graphed. However, the rate of convergence appears to be slower than with the increasing step function used for f_0. This could be explained by observing in the proof of Theorem F.1.1, how far in the metric, namely the $\sup$ norm, the initial function f_0 and $f_1 = H(f_0)$ are from each other. For the increasing step function, the initial distance $d(f_0, f_1 = H(f_0)) = a = 1/2$. Similarly, for the decreasing step function, the initial distance $d(f_0, f_1 = H(f_0)) = a = 1$. Thus, for a given $\varepsilon > 0$, the inequality $\frac{a\alpha^{n-1}}{1-\alpha} < \varepsilon$ would be satisfied with a smaller value of n for $a = 1/2$ than the value of n for $a = 1$. Another argument for this slower convergence with the step down initial function f_0 could be that it has a discontinuity, a sharp bounded decrease, of the first kind at $x = 1/2$ but the theoretical limit functions F and W are increasing over $[0, 1]$.

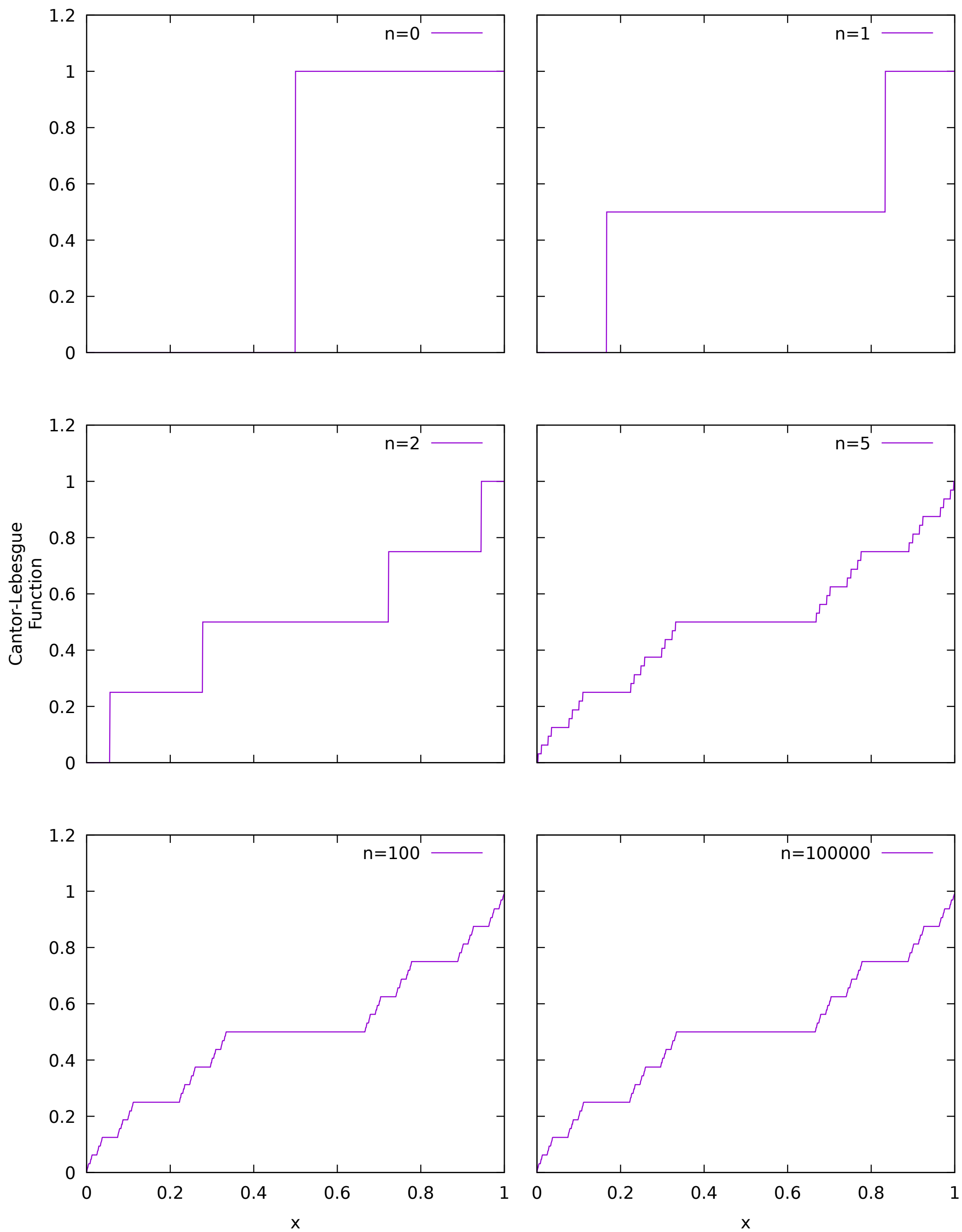


Figure F.1: Graph of the functions $f_n(x)$ for $n = 0, 1, 2, 5, 100, 100{,}000$ for the ternary Cantor set

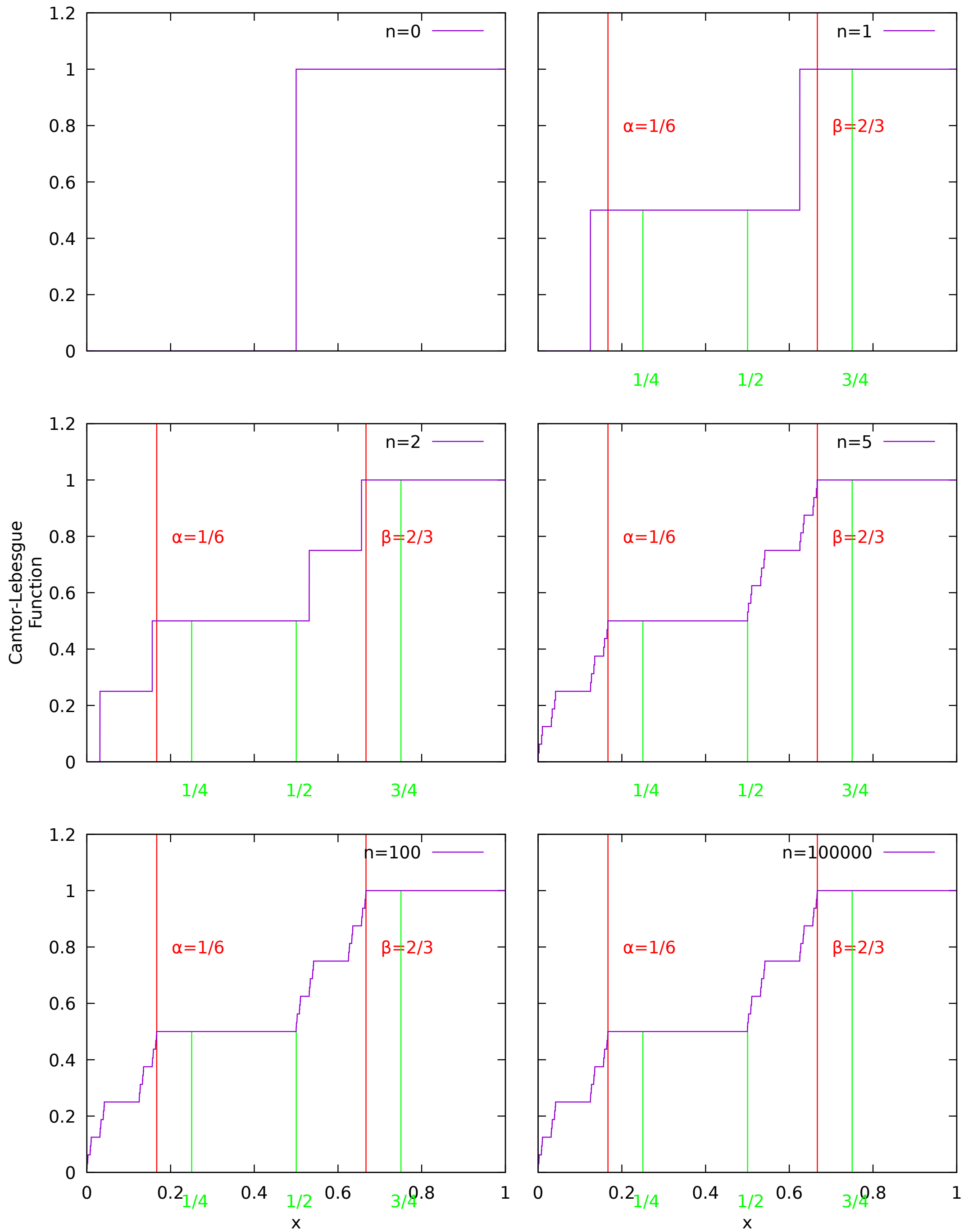


Figure F.2: Graph of the function $f_n(x)$ for $n = 0, 1, 2, 5, 100, 100{,}000$ for the quaternary Cantor set

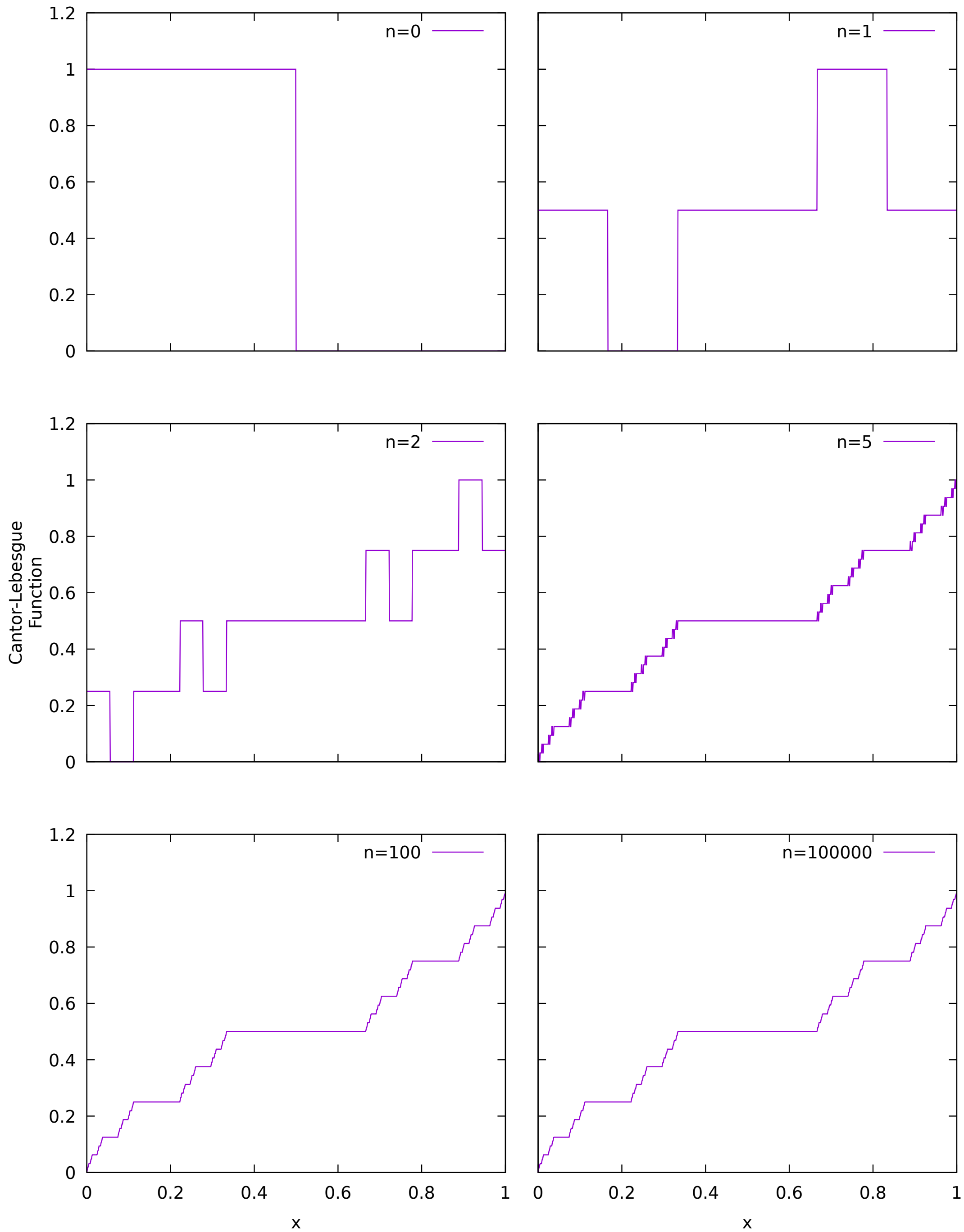


Figure F.3: Graph of the functions $f_n(x)$ for $n = 0, 1, 2, 5, 100, 100,000$ for the ternary Cantor set

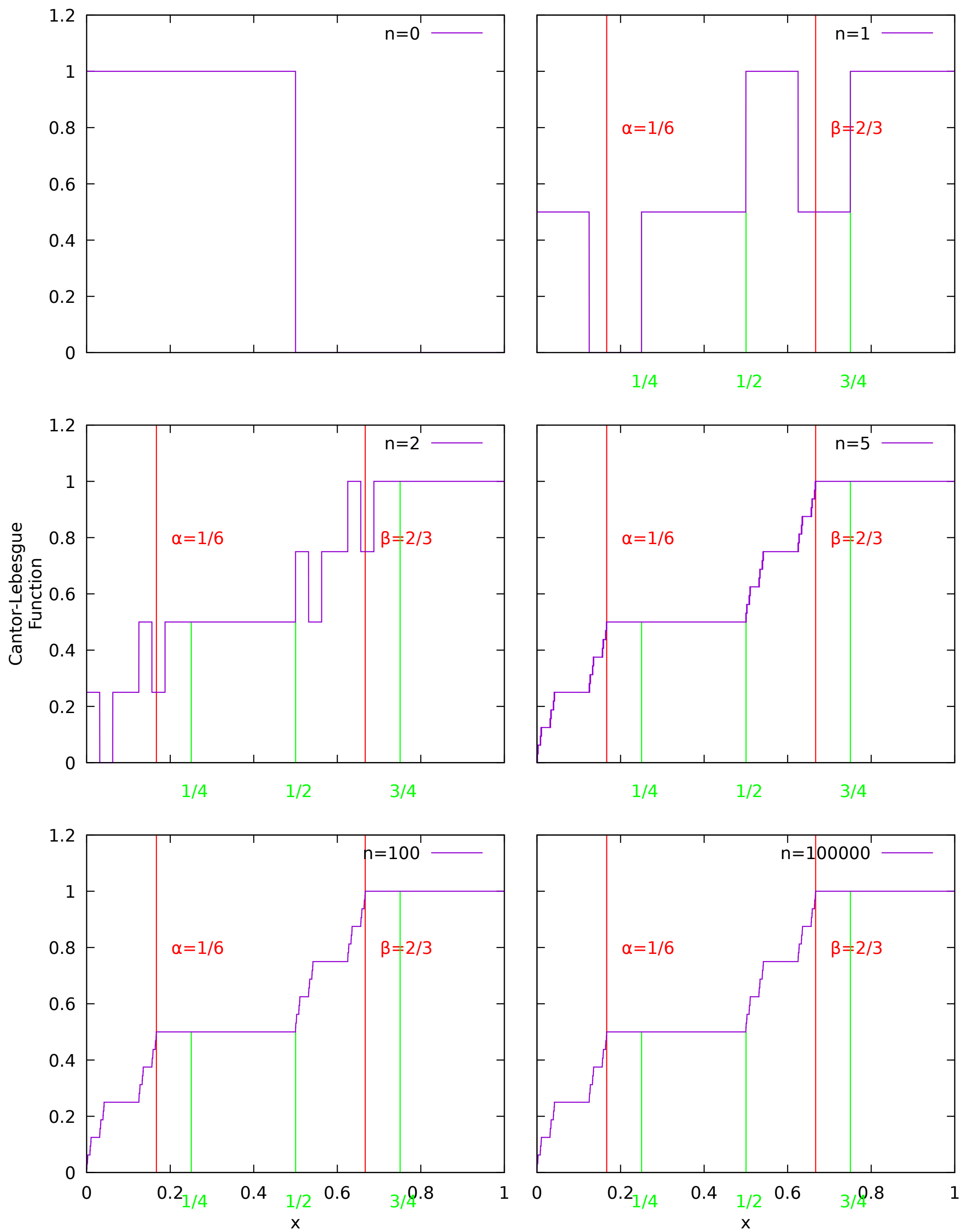


Figure F.4: Graph of the function $f_n(x)$ for $n = 0, 1, 2, 5, 100, 100,000$ for the quaternary Cantor set

Appendix G

Orthogonal projection for Mercedes frame

This appendix gives details on the meaning of the Mercedes frame being “the orthogonal projection of a certain orthonormal basis for $\mathbb{R}^3$ onto a two-dimensional plane”. The Mercedes frame is defined by

$$x_1 = (0,1), \quad x_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}), \quad x_3 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}) \tag{G.1}$$

and lies in the xy-plane as shown in Figure G.1.

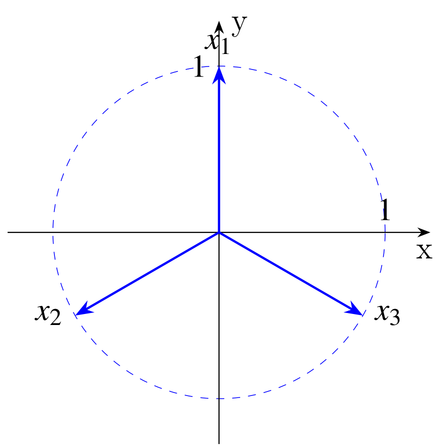


Figure G.1: The three vectors of the Mercedes frame with blue dashed unit circle for comparison.

We observe that $\|x_1\| = \|x_2\| = \|x_3\| = 1$. The orthonormal basis for $\mathbb{R}^3$ whose orthogonal projection yields the Mercedes frame has to be at an angle θ with respect to the xy-plane. So, we start by finding two orthogonal vectors, both at angle θ with respect to the xy-plane, in $\mathbb{R}^3$ whose projection on the xy-plane has the same direction as $x_2 = (\sqrt{3}/2, -1/2)$ and $x_3 = (-\sqrt{3}/2, -1/2)$ respectively. Being orthogonal, their dot product must be equal to 0:

$$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}), \tan\theta\right) \bullet \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \tan\theta\right) = 0 \tag{G.2}$$

to obtain that $\tan\theta = 1/\sqrt{2}$ giving two orthogonal vectors in $\mathbb{R}^3$. Using the definition of $\tan\theta$ we can calculate that $\sin\theta = 1/\sqrt{3}$ and $\cos\theta = \sqrt{2/3}$.

$$\begin{aligned} v_2 &= \left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right) \\ v_3 &= \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}, \tfrac{1}{\sqrt{2}}\right) \end{aligned} \tag{G.3}$$

where the specifications of the first two coordinates of v_2 and v_3 ensure that their respective orthogonal projection on the xy-plane has the same direction than their corresponding Mercedes frame elements x_2 and x_3 respectively. From eq.(G.3) we have that $\|v_2\| = \|v_3\| = \sqrt{3/2}$. Using $\sqrt{3/2}$ as normalization factor, we obtain, e_2 and e_3, the two first orthonormal elements of the basis in $\mathbb{R}^3$ we are looking for:

$$\begin{aligned} e_2 = \sqrt{\tfrac{2}{3}}v_2 &= \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{6}}, 1/\sqrt{3}\right) \\ e_3 = \sqrt{\tfrac{2}{3}}v_3 &= \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{6}}, 1/\sqrt{3}\right) \end{aligned} \tag{G.4}$$

with $\|e_2\| = \|e_3\| = 1$.

The cross product of $e_2 \times e_3$ gives e_1, the third orthogonal element of the basis in $\mathbb{R}^3$: $e_1 = (0, \sqrt{2/3}, 1/\sqrt{3})$ with $\|e_1\| = 1$. If we do the orthogonal projection onto the xy-plane of the orthonormal basis $\{e_1, e_2, e_3\}$, we would obtain vectors in the xy-plane with the same direction as the one in the Mercedes frames but with a length of $\sqrt{2/3}$ which is not what is required as the length of the orthogonal projection must be 1. As illustrated in the figure below, it results that the Mercedes frame can be recovered by the orthogonal projection from the orthogonal basis $\{v_1, v_2, v_3\}$. We obtain the third element v_1 of the orthogonal basis $\{v_1, v_2, v_3\}$ by multiplying

e_1 by $\sqrt{3/2}$ giving $v_1 = (0.0, 1.0, 1/\sqrt{2})$. So, the phrase "certain orthogonal projection from the orthonormal basis in $\mathbb{R}^3$" needs to be amended to read "certain orthogonal projection from the orthogonal basis in $\mathbb{R}^3$"

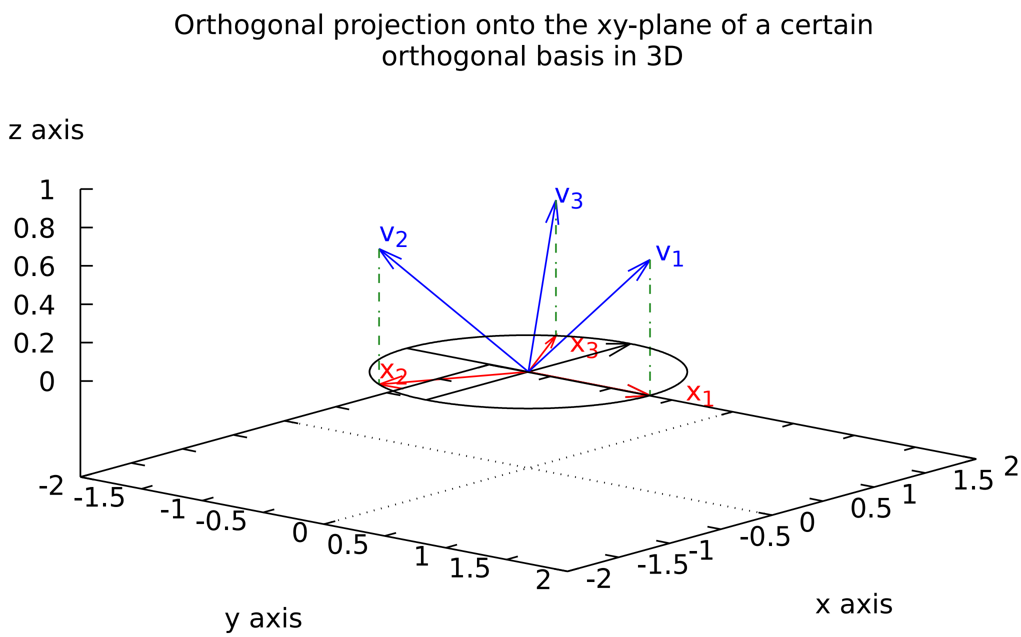


Figure G.2: Mercedes frame and the orthogonal basis $\{v_1, v_2, v_3\}$ in $\mathbb{R}^3$

This concludes the explanation of the meaning of "a certain orthogonal projection from the orthonormal basis in $\mathbb{R}^3$" for the recovery of the Mercedes frame.

Appendix H

Convergence of infinite products

H.1 Introduction

In this Appendix we show that the infinite product in the formula of $\hat{\mu}_4$ converges. For convenience, we give again the formula for $\hat{\mu}_4$ below:

$$\hat{\mu}_4(t) = e^{i\frac{\pi 2t}{3}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2 \cdot 4^n}\right) \tag{H.1}$$

We established in Chapter 4, that the zero set of $\hat{\mu}_{m_4}(t)$ is $\mathbf{Z}(\hat{\mu}_4) = \{4^n(1+2\mathbb{Z})\} \subset \mathbb{Z}$ with $n \in \mathbb{N} \cup \{0\}$. The existence of this zero set precludes using the $\log$ function on the infinite product to create a series and study its convergence. Moreover, in a neighbourhood of each of these these zeros, the corresponding factor in the product will take positive and negative values with the latter incompatible with the $\log$ function.

Apostol[1, p. 207] gives a useful definition of infinite product that we quote:

Definition H.1.1. Given an infinite product $\prod_{n=1}^{\infty} u_n$, let $p_n = \prod_{k=1}^{n} u_k$.

(a) If infinitely many factors u_n are zero, we say the product diverges to zero.

(b) If no factor u_n is zero, we say the product converges if there exists a number $p \neq 0$ such that $\{p_n\}$ converges to p. In this case, p is called the value of the product and we write $p = \prod_{n=1}^{\infty} u_n$. If $\{p_n\}$ converges to zero, we say the product diverges to zero.

(c) If there exists an N such that $n > N$ implies $u_n \neq 0$, we say $\prod_{n=1}^{\infty} u_n$ converges, provided that $\prod_{n=N+1}^{\infty} u_n$ converges as described in (b). In this case, the value of the product $\prod_{n=1}^{\infty} u_n$ is

$$u_1 u_2 \cdots u_N \prod_{n=N+1}^{\infty} u_n. \tag{H.2}$$

(d) $\prod_{n=1}^{\infty} u_n$ is called divergent if it does not converge as described in (b) and (c).

Also, Apostol[1, p. 207] provides the following note that we quote: The value of a convergent infinite product can be zero. But this happens if and only if, a finite number of factors are zero. The convergence of an infinite product is not affected by inserting or removing a finite number of factors, zero or not.

H.2 Convergence of the infinite product

Apostol[1, p. 207] gives the Cauchy condition in the form of theorem that we quote:

Theorem H.2.1. The infinite product $\prod_{n=1}^{\infty} u_n$ converges if, and only if , for every $\varepsilon > 0$ there is an N such that $n > N$ implies

$$|u_{n+1} u_{n+2} \cdots u_{n+k} - 1| < \varepsilon, \quad \text{for } k = 1, 2, 3, \ldots \tag{H.3}$$

Apostol[1, p. 208] offers the following observation, useful in the sequel that we quote: Taking $k = 1$ in eq. (H.3), we find that convergence of $\prod_{n=1}^{\infty} u_n$ implies that $\lim_{n\to\infty} u_n = 1$. For this reason, the factors of a product are written as $u_n = 1 + a_n$. Thus convergence of $\prod_{n=1}^{\infty}(1 + a_n)$ implies that $\lim_{n\to\infty} a_n = 0$.

According to Apostol[1, p. 209], for some products, the factors could be written as $u_n = 1 - a_n$ and Apostol[1, p. 209] gives the following theorem on the convergence of such products that we quote:

Theorem H.2.2. Assume that each $a_n \geq 0$. Then the product $\prod(1-a_n)$ converges if, and only if, the series $\sum a_n$ converges.

It is this latter theorem that we use to show the convergence of the product in the formula for $\hat{\mu}_4$. Using the identity $\cos 2\alpha = 1 - 2\sin^2\alpha$, each factor of the product in eq. (H.1) can be written as follows:

$$u_n(t) = \cos\left(\frac{\pi t}{2 \cdot 4^n}\right) = 1 - 2\sin^2\left(\frac{\pi t}{4^{n+1}}\right) \tag{H.4}$$

where we note the notation $u_n(t)$ for the n^{th} factor in the product. Each $n \in \mathbb{N} \cup \{0\}$ corresponds to a factor in the product and to a subset of the zero set $\mathbf{Z}(\hat{\mu}_4)$. For instance, $n = 1$ corresponds to the factor $\cos\left(\frac{\pi t}{2 \cdot 4}\right)$ and to $\{4(1+2\mathbb{Z})\} \subset \mathbf{Z}(\hat{\mu}_4)$. We observe that the subset $\{4(1+2\mathbb{Z})\}$ is not the subset of zeros for any of the factors for $n \neq 1$. Then, the study of the convergence of the product reduces to two cases:

(a) t fixed with $t \in \mathbb{R} \setminus \{4^n(1+2\mathbb{Z})\}\ \forall n \in \mathbb{N} \cup \{0\}$ so in eq.(H.4), $2\sin^2\left(\frac{\pi t}{4^{n+1}}\right) > 0$ and by Theorem H.2.2, we only need to study the convergence of the series

$$\sum_{n=0}^{\infty} 2\sin^2\left(\frac{\pi t}{4^{n+1}}\right). \tag{H.5}$$

For all $x > 0$, we have the inequality $\sin x < x = |x|$. Since $|\sin x|$ and $|x|$ are even functions, we can write that $|\sin x| < |x|\ \forall x \in \{\mathbb{R} \setminus \{0\}\}$. We have that $|\sin x| \leq 1$ then $|\sin x|^2 = \sin^2 x \leq |\sin x| < |x|$ to get that $\sin^2 x < |x|\ \forall x \in \{\mathbb{R} \setminus \{0\}\}$ since at $x = 0$, $\sin^2 0 = 0$ and we have that $\hat{\mu}_4(0) = 1$. That is, the product in eq. (H.1) converges to 1. We then focus on the convergence $\forall t \in \{\mathbb{R} \setminus \{0\}\}$. We can then write

$$\sum_{n=0}^{\infty} 2\sin^2\left(\frac{\pi t}{4^{n+1}}\right) < \sum_{n=0}^{\infty} |\frac{\pi t}{4^{n+1}}| = 2\frac{\pi|t|}{4}\sum_{n=0}^{\infty}\frac{1}{4^n} = 2\frac{\pi|t|}{3} < \infty \tag{H.6}$$

So the series in (H.5) converges by the Comparison test Apostol[1, p. 190] and by Theorem H.2.2 the product converges. Since t is arbitrary, the product converges $\forall t \in$

$\mathbb{R} \setminus \{4^n(1+2\mathbb{Z})\}\ \forall n \in \mathbb{N} \cup \{0\}$.

(b) For a given $N \in \mathbb{N} \cup \{0\}$, N corresponds to the factor $u_N(t) = \cos\left(\frac{\pi t}{2\cdot 4^N}\right)$. t fixed with $t \in \{4^N(1+2\mathbb{Z})\}$, then $u_N(t) = 0$ with all the other factors of the product different from 0. So, Part (c) of Definition H.1.1 applies and we have

$$u_1 u_2 \cdots u_{N-1} \cdot 0 \cdot \prod_{n=N+1}^{\infty} u_n. \tag{H.7}$$

where the product converges to 0, provided $\prod_{n=N+1}^{\infty} u_n$ converges. Similarly as in Case 1, we have

$$\sum_{n=N+1}^{\infty} 2\sin^2\left(\frac{\pi t}{4^{n+1}}\right) < 2\sum_{n=N+1}^{\infty} \left|\frac{\pi t}{4^{n+1}}\right| = 2\frac{\pi|t|}{4^{N+1}} \sum_{n=0}^{\infty} \frac{1}{4^n} = 2\frac{\pi|t|}{3\cdot 4^N} < \infty \tag{H.8}$$

So the series in (H.8) converges by the Comparison test Apostol[1, p. 190] and by Theorem H.2.2 the product converges. Since t and N are arbitrary, the product converges to 0 $\forall t \in \{4^n(1+2\mathbb{Z})\}$ with $n \in \mathbb{N} \cup \{0\}$.

We then conclude that the product in eq. (H.1) converges $\forall t \in \mathbb{R}$.

Appendix I

Relation between Fourier transforms of measure and Cantor-Lebesgue function

I.1 Introduction

In this Appendix, we derive the relation between the Fourier transform of the measures μ_3 and μ_4, and the Fourier transform of their corresponding Cantor-Lebesgue function. This gives an alternate way for the following formulae obtained in Ch. 4, repeated here for convenience:

1. for the ternary Cantor set (eq. (4.52))

$$\hat{\mu}_3(t) = \frac{1}{2}\left(1 + e^{i\pi\frac{4t}{3}}\right)\hat{\mu}_3(\frac{t}{3}) \tag{I.1}$$

2. for the quaternary Cantor set (eq. (4.11)):

$$\hat{\mu}_4(t) = \frac{1}{2}\left(1 + e^{i\pi t}\right)\hat{\mu}_4(\frac{t}{4}) \tag{I.2}$$

These formulae are the respective starting point of an iterative process leading to the formulae for the Fourier transform of ternary and quaternary measures in the form of an infinite product.

Proposition 4 in Ch. 3 shows for the ternary measure that for any closed interval $[0,x] \subset [0,1]$, $\mu_3([0,x]) = F(x)$ where F is the C-L function for the ternary Cantor set. Similarly, Proposition 5, shows for the quaternary measure that $\mu_4([0,x]) = W(x)$ where W is the C-L function for the quaternary Cantor set. With these relations, F and W can be called as distribution functions of respectively μ_3 and μ_4 [16, p. 33]. Both F and W are monotone increasing, bounded and continuous on the closed interval $[0,1]$. Also, the function $e_t = e^{2\pi tx}$ is continuous and bounded on $[0,1]$.

Following Apostol [1, Ch. 7], the Riemann-Stieltjes integral calls for two bounded functions f and α on a closed interval $[a,b]$ and is denoted by the symbol $\int_a^b f(x)d\alpha(x)$. If such an integral exists, we quote Apostol [1, p. 141]: we say that f is Riemann-integrable with respect to α on $[a,b]$, and we write $f \in R(\alpha)$ on $[a,b]$. For integrating by parts $\int_a^b f(x)d\alpha(x)$, we quote Apostol [1, p. 144, Thm 7.6]: if $f \in R(\alpha)$ on $[a,b]$, then $\alpha \in R(f)$ on $[a,b]$ and we have

$$\begin{aligned} \int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) &= f(b)\alpha(b) - f(a)\alpha(a) \\ \int_a^b f(x)d\alpha(x) &= -\int_a^b \alpha(x)df(x) + f(b)\alpha(b) - f(a)\alpha(a) \end{aligned} \tag{I.3}$$

Using Rudin [26, Thm. 6.30, p. 122], we can apply eq.(I.3) in the case of $f \in C[a,b]$ and α of bounded variation on $[a,b]$ or more specifically, α monotone on $[a,b]$. We use the formula in eq.(I.3) with the continuous function $f = e_t = e^{2\pi itx}$ and the monotone functions, F and W to obtain a relation between the Fourier transforms of the measure and its corresponding C-L function. These three functions are defined and bounded on $[0,1]$. In fact they are more than that, which is even better.

I.1.1 Ternary measure

By calling F a distribution function of μ_3 and Proposition 4, giving $\mu_3([0,x]) = F(x)$, we have that $d\mu_3 = dF$. Using integration by parts (eq.(I.3)), we can write the formula for the Fourier transform of μ_3 as

$$\begin{aligned}
\hat{\mu}_3(t) = \int_0^1 e^{2\pi itx} d\mu_3 &= \int_0^1 e^{2\pi itx} dF \\
&= -2\pi it \int_0^1 F(x) e^{2\pi itx} dx + F(x) e^{2\pi itx}\Big|_0^1 \\
&= -2\pi it \hat{F}(t) + e^{2\pi it}
\end{aligned} \tag{I.4}$$

where $t \in \mathbb{Z}$ and by definition of Fourier transform

$$\hat{F}(t) = \int_0^1 F(x) e^{2\pi itx} dx \tag{I.5}$$

with $t \in \mathbb{Z}$.We then have the relation between the Fourier transforms of μ_3 and the corresponding C-L function F. Can we obtain eq. (I.1) without having to use the methodology exposed in Ch. 4 i.e. eq. (4.9)? We can answer in the affirmative as shown below.

By Lemma 2.1.15, the following identity holds:

$$F(x) = \begin{cases} \frac{1}{2}F(3x) & \text{for} \quad 0 \le x \le \frac{1}{3}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2}F(3x-2) + \frac{1}{2} & \text{for} \quad \frac{2}{3} \le x \le 1. \end{cases} \tag{I.6}$$

with $F(y) = 0$ for $y < 0$ and $y > 1$. Then we write

$$\begin{aligned}
\hat{F}(t) &= \int_0^1 F(x) e^{2\pi itx} dx \\
&= \int_0^{\frac{1}{3}} \frac{1}{2} F(3x) e^{2\pi itx} dx + \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{2} e^{2\pi itx} dx + \int_{\frac{2}{3}}^1 \left(\frac{1}{2} + \frac{1}{2}F(3x-2)\right) e^{2\pi itx} dx \\
&= \frac{\hat{F}(t/3)}{6} + \frac{1}{4\pi it}\left(e^{\frac{4\pi it}{3}} - e^{\frac{2\pi it}{3}}\right) \\
&\quad + \frac{1}{4\pi it}\left(e^{2\pi it} - e^{\frac{4\pi it}{3}}\right) + e^{\frac{4\pi it}{3}} \frac{\hat{F}(t/3)}{6} \\
&= \left(1 + e^{\frac{4\pi it}{3}}\right) \frac{\hat{F}(t/3)}{6} + \frac{1}{4\pi it}\left(e^{2\pi it} - e^{\frac{2\pi it}{3}}\right)
\end{aligned} \tag{I.7}$$

but from eq. (I.4) we have

$$\hat{F}(t/3) = \frac{e^{\frac{2\pi it}{3}} - \hat{\mu}_3(t/3)}{\frac{2\pi it}{3}} \tag{I.8}$$

Introducing eq. (I.8) in eq. (I.7) we obtain:

$$\hat{F}(t) = \left(\frac{1+e^{\frac{4\pi it}{3}}}{6}\right)\left(\frac{e^{\frac{2\pi it}{3}} - \hat{\mu}_3(t/3)}{\frac{2\pi it}{3}}\right) + \frac{1}{4\pi it}\left(e^{2\pi it} - e^{\frac{2\pi it}{3}}\right) \tag{I.9}$$

Introducing eq. (I.9) in eq. (I.4), we obtain the desired result:

$$\hat{\mu}_3(t) = \frac{1}{2}\left(1+e^{i\pi\frac{4t}{3}}\right)\hat{\mu}_3(\frac{t}{3}) \tag{I.10}$$

This confirms the answer by the affirmative the question asked above.

I.1.2 Quaternary measure

By calling W a distribution function of μ_4 and Proposition 5, giving $\mu_4([0,x]) = W(x)$, we have that $d\mu_4 = dW$. Using integration by parts (eq.(I.3)), we can write the formula for the Fourier transform of μ_4 as

$$\begin{aligned} \hat{\mu}_4(t) = \int_0^1 e^{2\pi itx} d\mu_4 &= \int_0^1 e^{2\pi itx} dW \\ &= -2\pi it \int_0^1 W(x)e^{2\pi itx}dx + W(x)e^{2\pi itx}\Big|_0^1 \\ &= -2\pi it\hat{W}(t) + e^{2\pi it} \end{aligned} \tag{I.11}$$

where $t \in \mathbb{Z}$ and by definition of Fourier transform

$$\hat{W}(t) = \int_0^1 W(x)e^{2\pi itx}dx \tag{I.12}$$

with $t \in \mathbb{Z}$. We then have the relation between the Fourier transforms of μ_4 and the corresponding C-L function F. Can we obtain eq. (I.2) without having to use the methodology exposed in Ch. 4 i.e. eq. (4.9)? We can answer in the affirmative as shown below.

By Proposition 2, the following identity holds:

$$W(x) = \begin{cases} \frac{1}{2}W(4x) & \text{for} \quad 0 \le x \le \frac{1}{4}, \\ \frac{1}{2} & \text{for} \quad \frac{1}{4} < x < \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}W(4x-2) & \text{for} \quad \frac{1}{2} \le x \le \frac{3}{4}, \\ 1 & \text{for} \quad \frac{3}{4} \le x \le 1. \end{cases} \tag{I.13}$$

with $W(y) = 0$ for $y < 0$ and $y > 1$. Then we write

$$\begin{aligned} \hat{W}(t) &= \int_0^1 W(x) e^{2\pi i t x} dx \\ &= \int_0^{\frac{1}{4}} \frac{1}{2} W(4x) e^{2\pi i t x} dx + \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2} e^{2\pi i t x} dx \\ &\quad + \int_{\frac{1}{2}}^{\frac{3}{4}} \left(\frac{1}{2} + \frac{1}{2}W(4x-2)\right) e^{2\pi i t x} dx + \int_{\frac{3}{4}}^{1} e^{2\pi i t x} dx \\ &= \frac{\hat{W}(t/4)}{8} + \frac{1}{4\pi i t}\left(e^{\pi i t} - e^{\frac{\pi i t}{2}}\right) \\ &\quad + e^{\pi i t}\frac{\hat{W}(t/4)}{8} + \frac{1}{2\pi i t}\left(e^{2\pi i t} - e^{\frac{3\pi i t}{2}}\right) \\ &= \left(1 + e^{\pi i t}\right)\frac{\hat{W}(t/4)}{8} + \frac{1}{4\pi i t}\left(e^{\frac{3\pi i t}{2}} - e^{\frac{\pi i t}{2}}\right) \\ &\quad + \frac{1}{2\pi i t}\left(e^{2\pi i t} - e^{\frac{3\pi i t}{2}}\right) \end{aligned} \tag{I.14}$$

but from eq. (I.11) we have

$$\hat{W}(t/4) = \frac{4}{2\pi i t}\left(e^{\frac{\pi i t}{2}} - \hat{\mu}_4(t/4)\right) \tag{I.15}$$

Introducing eq. (I.15) in eq. (I.14) we obtain:

$$\begin{aligned} \hat{W}(t) &= \frac{\left(1+e^{\pi i t}\right)}{8}\frac{4}{2\pi i t}\left(e^{\frac{\pi i t}{2}} - \hat{\mu}_4(t/4)\right) \\ &\quad + \frac{1}{4\pi i t}\left(e^{\frac{3\pi i t}{2}} - e^{\pi i t}\right) + \frac{1}{2\pi i t}\left(e^{2\pi i t} - e^{\frac{3\pi i t}{2}}\right) \end{aligned} \tag{I.16}$$

Introducing eq. (I.16) in eq. (I.11), we obtain the desired result:

$$\hat{\mu}_4(t) = \frac{1}{2}\left(1 + e^{i\pi t}\right)\hat{\mu}_4\left(\frac{t}{4}\right) \tag{I.17}$$

This confirms the answer by the affirmative the question asked above.

Appendix J

One dimension: Fourier transform of measures of odd and even scales

J.1 Introduction

Chapter 4 presents the derivations in one dimension of the Fourier transform of the ternary and quaternary measure, μ_3 and μ_4. That is of scale 3 and 4 respectively. In this Appendix, we extend these results to odd and even scales higher than 3 and 4. The key element in doing so is to establish general Iterative Function Systems (IFSs) that each leads to a Cantor set $C \subset [0,1]$ of Lebesgue measure $m(C) = 0$.

J.2 IFSs for higher odd and even scales Fourier transform

The IFSs for the

(i) ternary Cantor set:

$$\begin{aligned} T_0(x) &= \tfrac{x}{3} \\ T_1(x) &= \tfrac{x}{3} + \tfrac{2}{3} \end{aligned} \tag{J.1}$$

(ii) quaternary Cantor set:

$$\begin{aligned} \tau_0(x) &= \tfrac{x}{4} \\ \tau_1(x) &= \tfrac{x}{4} + \tfrac{1}{2} \end{aligned} \tag{J.2}$$

From these IFSs, we propose the following general IFSs applicable for all odd and even scales:

(i) odd scale:

$$\begin{aligned} O_0(x) &= \tfrac{x}{2k+1} \\ O_1(x) &= \tfrac{x}{2k+1} + \tfrac{2}{2k+1} \end{aligned} \tag{J.3}$$

for $k \in \mathbb{N}$,

(ii) even scale:

$$\begin{aligned} E_0(x) &= \tfrac{x}{2k} \\ E_1(x) &= \tfrac{x}{2k} + \tfrac{k}{2k} = \tfrac{x}{2k} + \tfrac{1}{2} \end{aligned} \tag{J.4}$$

for $k \in \mathbb{N}$.

For example, for $k = 1$ we recover the IFS for the ternary (odd) Cantor set given in eq. (J.1) and for the binary (even) Cantor set in eq. (B.1. Also, for $k = 2$ we recover the IFS for the quaternary (even) Cantor set eq. (J.2). We are interested in odd scales for $k \geq 2$ and even scales for $k \geq 3$, for instance,

(i) odd scale $k = 2$, we obtain the IFS for a particular quinary Cantor set:

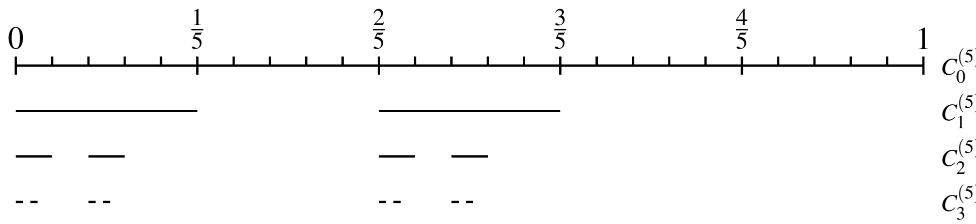
$$\begin{aligned} O_0(x) &= \tfrac{x}{5} \\ O_1(x) &= \tfrac{x}{5} + \tfrac{2}{5} \end{aligned} \tag{J.5}$$

(ii) even scale $k = 3$, we obtain the IFS for a particular senary Cantor set:

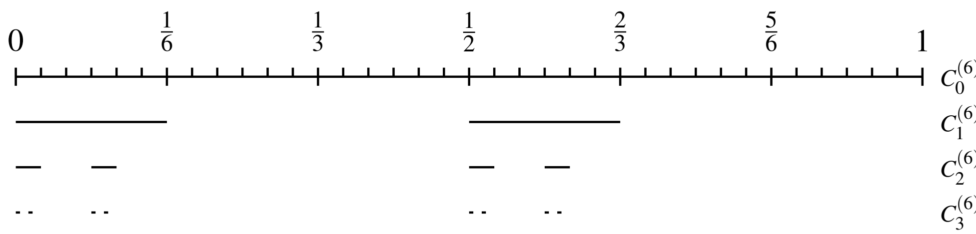
$$\begin{aligned} E_0(x) &= \tfrac{x}{6} \\ E_1(x) &= \tfrac{x}{6} + \tfrac{3}{6} \end{aligned} \tag{J.6}$$

although the fraction $3/6$ could be simplified it is kept intact. It ensure the proper form of the inverse IFS.

This results in Figure J.1 displaying few steps in the construction of this quinary Cantor set and this senary Cantor set. We observe that in both cases, the construction will lead to Cantor sets skewed towards left as for the quaternary Cantor set. Also, the construction is according to a self similar binary tree as for the ternary and quaternary Cantor sets.



(a) Quinary



(b) Senary

Figure J.1: Few construction levels of a quinary and a senary Cantor sets

Total length of the intervals removed:

(a) for the quinary Cantor set each parent closed interval is divided in five equal subintervals. The two child closed subintervals result from removing the second open subinterval and the last two open subintervals for a total of three open subintervals of equal length. Total length removed is then given by

$$
\begin{aligned}
\tfrac{3}{5}+2\tfrac{3}{5^2}+4\tfrac{3}{5^3}+\dots &= \tfrac{3}{5}+2^1\tfrac{3}{5^2}+2^2\tfrac{3}{5^3}+\dots \\
&= \tfrac{3}{5}\left(1+\tfrac{2}{5}+\tfrac{2^2}{5^2}+\dots\right) \\
&= \tfrac{3}{5}\tfrac{1}{1-\frac{2}{5}} = \tfrac{3}{5}\tfrac{5}{3} = 1.
\end{aligned} \tag{J.7}
$$

Since the total length removed is 1, the construction process leads to a quinary Cantor set $C^{(5)} \subset [0,1]$ of Lebesgue measure $m(C^{(5)}) = 0$. We note the quinary Cantor set corresponds to $k = 2$ and the construction process implies the removal of $k+1 = 3$ equal subintervals from each parent interval to obtain the two closed child subintervals.

(b) for the senary Cantor set each parent closed interval is divided in six equal subintervals. The two child closed subintervals result from removing the second and third open subintervals and the last two open subintervals for a total of four open subintervals of equal length. Total length removed is then given by

$$
\begin{aligned}
2\tfrac{2}{6}+2^2\tfrac{2}{6^2}+2^3\tfrac{2}{6^3}+\dots &= \tfrac{2}{3}+\tfrac{2}{3^2}+\tfrac{2}{3^3}+\dots \\
&= \tfrac{2}{3}\left(1+\tfrac{1}{3}+\tfrac{1}{3^2}+\dots\right) \\
&= \tfrac{2}{3}\tfrac{1}{1-\frac{1}{3}} = \tfrac{2}{3}\tfrac{3}{2} = 1
\end{aligned} \tag{J.8}
$$

Since the total length removed is 1, the construction process leads to a senary Cantor set $C^{(6)} \subset [0,1]$ of Lebesgue measure $m(C^{(6)}) = 0$. We note the senary Cantor set corresponds to $k = 3$ and the construction process implies the removal of $k+1 = 4$ equal subintervals from each parent interval to obtain the two closed child subintervals.

We then repeat the above for the general cases of odd and even scale where it suffices to show the total length removed is 1. We have:

(a) for the $(2k+1)$-ary Cantor set, given $k \in \mathbb{N}$, each parent closed interval is divided in $2k+1$ equal subintervals. The two child closed subintervals result from removing the second open subinterval and the last $2k-2$ open subintervals for a total of $2k-1$ open subintervals of equal length removed. Total length removed is then given by

$$
\begin{aligned}
\tfrac{2k-1}{2k+1}+2\tfrac{2k-1}{(2k+1)^2}+4\tfrac{2k-1}{(2k+1)^3}+\dots &= \tfrac{2k-1}{2k+1}+2^1\tfrac{2k-1}{(2k+1)^2}+2^2\tfrac{2k-1}{(2k+1)^3}+\dots \\
&= \tfrac{2k-1}{2k+1}\left(1+\tfrac{2}{2k+1}+\tfrac{2^2}{(2k+1)^2}+\dots\right) \\
&= \tfrac{2k-1}{2k+1}\tfrac{1}{1-\frac{2}{2k+1}}=\tfrac{2k-1}{2k+1}\tfrac{2k+1}{2k-1}=1
\end{aligned}
\tag{J.9}
$$

Since the total length removed is 1, the construction process does lead to a $2k+1$ (odd scale) Cantor set $C^{(2k+1)} \subset [0,1]$ of Lebesgue measure $m(C^{(2k+1)})=0$.

(b) for the $2k$ (even scale) Cantor set, given $k \in \mathbb{N}$, each parent closed interval is divided in $2k$ equal subintervals. The two child closed subintervals result from removing $k-1$ open subinterval(s) after the very first subinterval and the last $k-1$ open subintervals for a total of $2k-2$ open subintervals of equal length removed. Total length removed is then given by

$$
\begin{aligned}
2\tfrac{k-1}{2k}+2^2\tfrac{k-1}{(2k)^2}+2^3\tfrac{k-1}{(2k+1)^3}+\dots &= \tfrac{k-1}{k}+\tfrac{k-1}{k^2}+\tfrac{k-1}{k^3}+\dots \\
&= \tfrac{k-1}{k}\left(1+\tfrac{1}{k}+\tfrac{1}{k^2}+\dots\right) \\
&= \tfrac{k-1}{k}\tfrac{1}{1-\frac{1}{k}}=\tfrac{k-1}{k}\tfrac{k}{k-1}=1
\end{aligned}
\tag{J.10}
$$

Since the total length removed is 1, the construction process does lead to a $2k$ (even scale) Cantor set $C^{(2k)} \subset [0,1]$ of Lebesgue measure $m(C^{(2k)})=0$.

Since the IFSs in eq. (J.3) (odd scale) and in eq. (J.4) lead to Cantor sets of Lebesgue measure $m(C^{(2k+1)})=0$ and $m(C^{(2k)})=0$ ($k \in \mathbb{N}$)), we can obtain the Fourier transforms of measure of odd and even scales.

J.3 Fourier transform of measures of odd and even scales

J.3.1 Odd scale measure

We denote the odd scale measure by μ_{2k+1} with $k \in \mathbb{N}$. Hutchinson's theorem (Thm 3.3.2) states the existence of the measure μ_{2k+1} with support in $C^{(2k+1)}$ with respect to the IFS in eq. (J.3) and the corresponding integral recursive relation, analogous to eq. (4.9). Since $e^{i2\pi tx}$ is continuous, we can write using that analogous relation:

$$\begin{aligned}
\hat{\mu}_{2k+1}(t) &= \int e^{i2\pi tx} d\mu_{2k+1}(x) = \tfrac{1}{2}\left(\int e^{i2\pi\frac{t}{2k+1}x} d\mu_{2k+1}(x) + \int e^{i2\pi\frac{t}{2k+1}x + i2\pi t\frac{2}{2k+1}} d\mu_{2k+1}(x)\right) \\
&= \tfrac{1}{2}\left(\int e^{i2\pi\frac{t}{2k+1}x} d\mu_{2k+1}(x) + e^{i\pi\frac{4t}{2k+1}} \int e^{i2\pi\frac{t}{2k+1}x} d\mu_{2k+1}(x)\right) \\
&= \tfrac{1}{2}\left(1 + e^{i\pi\frac{4t}{2k+1}}\right) \int e^{i2\pi\frac{t}{2k+1}x} d\mu_{2k+1}(x) \\
&= \tfrac{1}{2}\left(1 + e^{i\pi\frac{4t}{2k+1}}\right) \hat{\mu}_{2k+1}\left(\tfrac{t}{2k+1}\right)
\end{aligned} \tag{J.11}$$

From eq.(J.11) (last line), we define for the $2k+1$-ary Cantor set:

$$\chi_{2k+1}(t) = \frac{1}{2}\left(1 + e^{i\pi\frac{4t}{2k+1}}\right). \tag{J.12}$$

Then eq.(J.19) can be written as follows:

$$\hat{\mu}_{2k+1}(t) = \chi_{2k+1}(t)\hat{\mu}_{2k+1}\left(\frac{4t}{2k+1}\right) \tag{J.13}$$

From eq. (J.11) we can write:

$$\hat{\mu}_{2k+1}\left(\frac{t}{2k+1}\right) = \frac{1}{2}\left(1 + e^{i\pi\frac{4t}{(2k+1)^2}}\right)\hat{\mu}_{2k+1}\left(\frac{t}{(2k+1)^2}\right) \tag{J.14}$$

For $t = 0$, eq. (4.8) becomes for $p = 2k+1$

$$\hat{\mu}_{2k+1}(0) = \int 1 d\mu_{2k+1} = \int \chi_{C^{(2k+1)}} d\mu_{2k+1} = \mu_{2k+1}(C^{(2k+1)}) \tag{J.15}$$

From the fact that $\mu_{2k+1}(C^{(2k+1)}) = 1$, we get that $\hat{\mu}_{2k+1}(0) = 1$. Then, we iterate the relation in eq. (J.14) N times to obtain:

$$\hat{\mu}_{2k+1}(t) = \left[\prod_{n=0}^{N} \chi_{2k+1}(\frac{t}{(2k+1)^n})\right] \hat{\mu}_{2k+1}(\frac{t}{(2k+1)^{N+1}}) \tag{J.16}$$

Taking $N \to \infty$ and using the continuity of $\hat{\mu}_{2k+1}(t)$ at $t = 0$, we can write

$$\begin{aligned} \hat{\mu}_{2k+1}(t) &= \left[\prod_{n=0}^{\infty} \chi_{2k+1}(\tfrac{4t}{(2k+1)^n})\right] \lim_{N\to\infty} \hat{\mu}_{2k+1}(\tfrac{t}{(2k+1)^{N+1}}) \\ &= \prod_{n=0}^{\infty} \chi_{2k+1}(\tfrac{t}{(2k+1)^n}) \\ &= \prod_{n=1}^{\infty} \tfrac{1}{2}\left(1 + e^{i\frac{4\pi t}{(2k+1)^n}}\right) \end{aligned} \tag{J.17}$$

where we used the definition of $\chi_{2k+1}(t)$ (eq. (J.12)) to obtain the last line of eq. (J.17) which can be written as follows:

$$\begin{aligned} \hat{\mu}_{2k+1}(t) &= \prod_{n=1}^{\infty} \tfrac{1}{2}\left(1 + e^{i\frac{4\pi t}{(2k+1)^n}}\right) = \prod_{n=1}^{\infty} \tfrac{1}{2}\left(e^{i\frac{4\pi t}{2\cdot(2k+1)^n} - i\frac{4\pi t}{2\cdot(2k+1)^n}} + e^{i\frac{4\pi t}{2\cdot(2k+1)^n} + i\frac{4\pi t}{2\cdot(2k+1)^n}}\right) \\ &= \prod_{n=1}^{\infty} e^{i\frac{2\pi t}{(2k+1)^n}} \frac{\left(e^{i\frac{2\pi t}{(2k+1)^n}} + e^{-i\frac{2\pi t}{(2k+1)^n}}\right)}{2} \\ &= \prod_{n=1}^{\infty} e^{i\frac{2\pi t}{(2k+1)^n}} \cos\left(\tfrac{2\pi t}{(2k+1)^n}\right) \\ &= e^{\sum_{n=1}^{\infty} i\frac{2\pi t}{(2k+1)^n}} \prod_{n=1}^{\infty} \cos\left(\tfrac{2\pi t}{(2k+1)^n}\right) \\ &= e^{\frac{i2\pi t}{2k+1}\sum_{n=0}^{\infty} \frac{1}{(2k+1)^n}} \prod_{n=1}^{\infty} \cos\left(\tfrac{2\pi t}{(2k+1)^n}\right) \\ &= e^{\frac{i\pi t}{k}} \prod_{n=1}^{\infty} \cos\left(\tfrac{2\pi t}{(2k+1)^n}\right) \end{aligned} \tag{J.18}$$

With $t \in \mathbb{R}$, $\hat{\mu}_{2k+1}(t)$ in eq.(J.18) is a continuous function in t. We have shown that we can obtain the Fourier transform of $(2k+1)$-ary measure (odd scale) for scale larger than 3.

J.3.2 Even scale measure

In the same way, we denote the even scale measure, given by Hutchinson's theorem for the IFS in eq. (J.4), by μ_{2k} with $k \in \mathbb{N}$. Since $e^{i2\pi tx}$ is continuous, then by the analogue of eq.(4.9) we can write:

$$\begin{aligned}\hat{\mu}_{2k}(t) &= \int e^{i2\pi tx} d\mu_{2k}(x) = \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{2k}x} d\mu_{2k}(x) + \int e^{i2\pi \frac{t}{2k}x + i2\pi t\frac{1}{2}} d\mu_{2k}(x)\right) \\ &= \tfrac{1}{2}\left(\int e^{i2\pi \frac{t}{2k}x} d\mu_{2k}(x) + e^{i\pi t}\int e^{i2\pi \frac{t}{2k}x} d\mu_{2k}(x)\right) \\ &= \tfrac{1}{2}\left(1+e^{i\pi t}\right)\int e^{i2\pi \frac{t}{2k}x} d\mu_{2k}(x) \\ &= \tfrac{1}{2}\left(1+e^{i\pi t}\right)\hat{\mu}_{2k}(\tfrac{t}{2k})\end{aligned} \tag{J.19}$$

From eq.(J.19) (last line), we define for the $2k$-ary Cantor set:

$$\chi_{2k}(t) = \frac{1}{2}\left(1+e^{i\pi t}\right). \tag{J.20}$$

Then eq.(J.19) can be written as follows:

$$\hat{\mu}_{2k}(t) = \chi_{2k}(t)\hat{\mu}_{2k}(\frac{t}{2k}) \tag{J.21}$$

From eq. (J.19) we can write:

$$\hat{\mu}_{2k}(\frac{t}{2k}) = \frac{1}{2}\left(1+e^{i\pi\frac{t}{2k}}\right)\hat{\mu}_{2k}(\frac{t}{(2k)^2}) \tag{J.22}$$

For $t=0$, eq. (4.8) becomes for $p=2k$

$$\hat{\mu}_{2k}(0) = \int 1 d\mu_{2k} = \int \chi_{C^{(2k)}} d\mu_{2k} = \mu_{2k}(C^{(2k)}) \tag{J.23}$$

By mass distribution $\mu_{2k}(C^{(2k)}) = 1$ and we get that $\hat{\mu}_{2k}(0) = 1$. Then, we iterate the relation in eq. (J.22) N times to obtain:

$$\hat{\mu}_{2k}(t) = \left[\prod_{n=0}^{N} \chi_{2k}(\frac{t}{(2k)^n})\right] \hat{\mu}_{2k}(\frac{t}{(2k)^{N+1}}) \tag{J.24}$$

Taking $N \to \infty$ and using the continuity of $\hat{\mu}_{2k}(t)$ at $t=0$, we can write

$$\begin{aligned} \hat{\mu}_{2k}(t) &= \left[\prod_{n=0}^{\infty} \chi_{2k}(\tfrac{t}{(2k)^n})\right] \lim_{N\to\infty} \hat{\mu}_{2k}\left(\tfrac{t}{(2k)^{N+1}}\right) \\ &= \prod_{n=0}^{\infty} \tfrac{1}{2}\left(1+e^{i\frac{\pi t}{(2k)^n}}\right) \end{aligned} \tag{J.25}$$

Eq. (J.25) can be written as follows:

$$\begin{aligned} \hat{\mu}_{2k}(t) &= \prod_{n=0}^{\infty} \tfrac{1}{2}\left(1+e^{i\frac{\pi t}{(2k)^n}}\right) = \prod_{n=0}^{\infty} \tfrac{1}{2}\left(e^{i\frac{\pi t}{2\cdot(2k)^n}-i\frac{\pi t}{2\cdot(2k)^n}} + e^{i\frac{\pi t}{2\cdot(2k)^n}+i\frac{\pi t}{2\cdot(2k)^n}}\right) \\ &= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2\cdot(2k)^n}} \frac{\left(e^{i\frac{\pi t}{2\cdot(2k)^n}} + e^{-i\frac{\pi t}{2\cdot(2k)^n}}\right)}{2} \\ &= \prod_{n=0}^{\infty} e^{i\frac{\pi t}{2\cdot(2k)^n}} \cos\left(\tfrac{\pi t}{2\cdot(2k)^n}\right) \\ &= e^{\sum_{n=0}^{\infty} i\frac{\pi t}{2\cdot(2k)^n}} \prod_{n=0}^{\infty} \cos\left(\tfrac{\pi t}{2\cdot(2k)^n}\right) \\ &= e^{i\frac{\pi t}{2}\sum_{n=0}^{\infty}\frac{1}{(2k)^n}} \prod_{n=0}^{\infty} \cos\left(\tfrac{\pi t}{2\cdot(2k)^n}\right) \\ &= e^{i\frac{\pi t}{2}\frac{2k}{2k-1}} \prod_{n=0}^{\infty} \cos\left(\tfrac{\pi t}{2\cdot(2k)^n}\right) \\ &= e^{i\pi t\frac{k}{2k-1}} \prod_{n=0}^{\infty} \cos\left(\tfrac{\pi t}{2\cdot(2k)^n}\right) \end{aligned} \tag{J.26}$$

Clearly, with $t \in \mathbb{R}$, $\hat{\mu}_{2k}(t)$ in eq.(J.26) is a continuous function in t. For $k=3$ in eq.(J.26), we get the Fourier transform of senary measure:

$$\hat{\mu}_6(t) = e^{i\pi t\frac{3}{5}} \prod_{n=0}^{\infty} \cos\left(\frac{\pi t}{2\cdot 6^n}\right) \tag{J.27}$$

Similar as for $\hat{\mu}_4(t)$, we apply the same steps as in Lemma 4.3.2 and Lemma 4.3.3 for $\hat{\mu}_6(t)$ where we let $P_6 = \{\ell_0 + 6\ell_1 + 6^2\ell_2 + \cdots + 6^k\ell_k : \ell_j \in L_6,$ finite sums $\}$ with $L_6 = \{0,1\}$, then the functions $\{e_\lambda : \lambda \in P_6\}$ are mutually orthogonal in $L^2(\mu_6)$ where

$$e_\lambda(x) := e^{i2\pi\lambda x} \tag{J.28}$$

Table J.1 gives the first 15 elements of P_6 where for example, for λ_5, $\{\ell_0, \ell_1, \ell_2, \ell_3\} = \{1,0,1,0\}$ and $\lambda_5 = 6^0 \cdot 1 + 6^1 \cdot 0 + 6^2 \cdot 1 + 6^3 \cdot 0 = 37$

n	$\{\ell_0, \ell_1, \ell_2, \ell_3\}$	λ_n
0	$\{0,0,0,0,\}$	0
1	$\{1,0,0,0,\}$	1
2	$\{0,1,0,0,\}$	6
3	$\{1,1,0,0,\}$	7
4	$\{0,0,1,0,\}$	36
5	$\{1,0,1,0,\}$	37
6	$\{0,1,1,0,\}$	42
7	$\{1,1,1,0,\}$	43
8	$\{0,0,0,1,\}$	216
9	$\{1,0,0,1,\}$	217
10	$\{0,1,0,1,\}$	222
11	$\{1,1,0,1,\}$	223
12	$\{0,0,1,1,\}$	252
13	$\{1,0,1,1,\}$	253
14	$\{0,1,1,1,\}$	258
15	$\{1,1,1,1,\}$	259

Table J.1: Value of $\lambda_n \in P_6$ for finite sums of four elements $(\ell_i : i = 0,1,2,3)$

From Eq.(J.27), the set of zeros of $\hat{\mu}_6(t)$ is:

$$\mathbf{Z}(\hat{\mu}_6) = \{6^n(1+2\mathbb{Z})\} \subset \mathbb{Z} \tag{J.29}$$

Similar as for $\hat{\mu}_4$, we observe that the difference between elements of P_6 are in $\mathbf{Z}(\hat{\mu}_6)$. In fact all those differences are in $\mathbf{Z}(\hat{\mu}_6)$ and since $0 \in P_6$, we have that $\{P_6 \setminus \{0\}\} \subset \mathbf{Z}(\hat{\mu}_6) \subset \mathbb{Z}$. An illustration of that fact is given in Table J.2.

We note the similarities between Table 4.2 for μ_4 and Table J.2 for μ_6 where the elements of the respective spectrum are at the same position. For the entry "λ_9 minus each of λ_8 down to

Operations on elements of P_6	Some elements of $\mathbf{Z}(\hat{\mu}_6)$ as differences between elements of P_6
λ_1 minus each of λ_0 down to λ_0	$\{1\}$
λ_2 minus each of λ_1 down to λ_0	$\{5,6\}$
λ_3 minus each of λ_2 down to λ_0	$\{1,6,7\}$
λ_4 minus each of λ_3 down to λ_0	$\{29,30,35,36\}$
λ_5 minus each of λ_4 down to λ_0	$\{1,30,31,36,37\}$
λ_6 minus each of λ_5 down to λ_0	$\{5,6,35,36,41,42\}$
λ_7 minus each of λ_6 down to λ_0	$\{1,6,7,36,37,42,43\}$
λ_8 minus each of λ_7 down to λ_0	$\{173,174,179,180,209,210,215,216\}$
λ_9 minus each of λ_8 down to λ_0	$\{1,174,175,180,181,210,211,216,217\}$
λ_{10} minus each of λ_9 down to λ_0	$\{5,6,179,180,185,186,215,216,221,222\}$
λ_{11} minus each of λ_{10} down to λ_0	$\{1,6,7,180,181,186,187,216,217,222,223\}$
λ_{12} minus each of λ_{11} down to λ_0	$\{29,30,35,36,209,210,215,216,245,246,251,252\}$
λ_{13} minus each of λ_{12} down to λ_0	$\{1,30,31,36,37,210,211,216,217,246,247,252,253\}$
λ_{14} minus each of λ_{13} down to λ_0	$\{5,6,35,36,41,42,215,216,221,222,251,252,257,258\}$
λ_{15} minus each of λ_{14} down to λ_0	$\{1,6,7,36,37,42,43,216,217,222,223,252,253,258,259$

Table J.2: λ's (in red) among the elements of the Zero Set

λ_0" in both tables, we start with one λ ($\lambda_1 = 1$), followed by six zeros that are in turn followed by two λ's. While Jorgensen and Pedersen prove Theorem 3.4 [20, p. 190] in one dimension and only for $R = 4$, their general approach and tools deployed for that proof, [20, Sec. 4, p. 192], apply in much generality. So, we conjecture that the same proof as for $R = 4$ will hold for $R = 6$ or more generally for even R.

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