The Impact of Structured Practice on Fifth Grader's Fraction Knowledge

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Abstract

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The present study investigated whether Grade 5 students, who are learning a procedure for a whole number by a fraction using repeated addition, would develop conceptual knowledge of fractions. The study aimed to answer three questions: 1 – Does practicing a procedure to multiply a whole number by a fraction through repeated addition result in acquiring greater procedural accuracy in fraction multiplication? 2 - Does practicing a procedure to multiply a whole number by a fraction through repeated addition result in greater conceptual knowledge? and 3 - Does self-explaining contribute to developing conceptual knowledge when practicing a procedure to multiply a whole number by a fraction through repeated addition?

For this pretest-intervention-posttest study, 56 Grade 5 students were selected from two schools located in the greater Montreal area (two classes from each school) Three randomly-selected groups were created within each class. Two groups received instruction on how to use the procedure: one was prompted while the other was not. The third group explored geometry notions without receiving any instruction on fractions.

As expected, the results demonstrate that after the 4-week intervention period, practicing exercises involving the multiplication of whole numbers by fractions enhanced procedural accuracy, whereas those who did not receive such instruction did not. In contrast to expectations, students' conceptual understanding of fraction magnitude was not enhanced by practicing a procedure for multiplying whole number by fractions. To clearly identify the mechanisms that would facilitate this transfer, more research is necessary.

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This projected started a long time ago from a chance encounter during a math training workshop given by Dr. Helen Osana. Our common interest for the relation between procedural knowledge and conceptual knowledge motivated me to embark on this academic challenge. Little did I know at the time that life would throw a few challenges on the path to completing this degree (a world pandemic, an education strike etc.) as seven years have passed since its beginning.

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Dedication

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Mathematical proficiency is said to be composed of five strands acting in a mutually reinforcing fashion, with no single strand being more important than the other (Kilpatrick et al., 2001). The components of this intertwined relationship are adaptive reasoning, strategic competence, conceptual understanding, a productive disposition, and procedural fluency. Such an interconnected relationship comes with many nuances and interpretations, which becomes a potential source of tension between those who promote the development of conceptual understanding prior to procedural understanding and those who promote approaching instruction through a procedural lens.

Fortunately, a growing number of studies have supported the iterative and bidirectional relationship that unites conceptual and procedural knowledge (Hansen et al., 2017; Rittle-Johnson et al., 2017). A consensus is slowly emerging, and the question has evolved from "if" the two types of knowledge interact to "how" conceptual and procedural knowledge interact, and how they develop (Rittle-Johnson, 2017). Nonetheless, researchers have predominantly focused their attention on the impact of conceptual knowledge on procedural knowledge, and far fewer have researched the impact from the opposite direction (Rittle-Johnson & Schneider, 2015; Star, 2005). If procedural knowledge is to be considered as a key component of mathematical proficiency, more research is required and would assist in moving closer to an optimal sequence of instruction that has not yet been found (Rittle-Johnson et al., 2017).

This is not to deny the importance of conceptual knowledge. As its presence in the definition of mathematical proficiency shows (Kilpatrick et al., 2001), conceptual knowledge is important and is key to developing a profound understanding of mathematics. As we combine more pieces of information or add new pieces to existing ones, we build concepts (Hiebert & Lefevre, 1986). The importance of concepts, and by extension conceptual knowledge, is not disputed. However, procedures provide something that concepts do not; without the presence of procedures, concepts would not be visible (Hiebert & Lefevre, 1986). Because of this, the path to better conceptual understanding may, at times, begin with procedures (Rittle-Johnson et al., 2015). Therefore, we need to focus more research on studying and assessing procedural knowledge (Star, 2005) and its impact on conceptual knowledge. Not doing so would be a mistake (Rittle-Johnson, 2015).

Multiple approaches have supported procedural instruction to promote conceptual understanding (Canobi, 2009; McNeil et al., 2012). Research has demonstrated the efficiency of sequencing practice problems, supporting self-explanation, and providing prompts in developing conceptual knowledge from procedural work. In essence, using approaches that more clearly link the procedures to their underlying concepts may be as effective as conceptual instruction alone (Rittle- Johnson & Alibali, 1999).

The goal of every math teacher is to bring their students as close to mathematical proficiency as possible in the time they are afforded within the classroom. It is therefore important for teachers to possess the tools and the knowledge to address the needs of each individual learner. To do so, they must be equipped with all of the supported research, as children will only learn when they are provided with the opportunity to do so (Hiebert & Grouws, 2007).

My interest in math instruction lies in clarifying and deepening the understanding of the relationship between conceptual and procedural knowledge to impact both student learning and teacher instructional strategies. The purpose of the following study will focus on the latter aspect of math instruction and answer the following question: Can students performing repeated procedural work with and without reflective prompts develop greater conceptual understanding? To achieve this, the students participating in this study were assigned to one of three conditions. The first focused on an instructional method proposed by Rittle-Johnson and Schneider (2015) – prompting for self-explanation during procedural instruction. I hope to show that combining reflection with procedural work will lead students to develop a greater conceptual understanding of fractions. The students in the second condition will receive the same treatment as the first, except that no reflective prompts will be made. The results will help to indicate if doing repeated procedural work on its own still leads to greater conceptual understanding or if reflection is a key component of the process. Finally, the students in the third condition will serve as the control group and will not participate in mathematics activities related to fractions but will take part in activities that will engage their spatial sense.

Literature Review

In 2001, the National Research Council (NRC) published *Adding it Up: Helping Children Learn Mathematics* (Kilpatrick et al., 2001). To describe what it means to do mathematics, and to encompass all the intricacies required to teach math, the authors settled on the term "mathematical proficiency." More specifically, Kilpatrick et al. (2001) defined mathematical proficiency as being composed of five distinct strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. To represent these ideas, the authors proposed to view mathematical proficiency as a combination of intertwined strands producing one single rope. Through this metaphor, Kilpatrick et al. (2001) clearly established that mathematical proficiency could only be as strong as its individual strands and supported that its interwoven status served to strengthen each of its parts as well as the whole. In other words, all strands must be present and strong for maximum math proficiency. Note that even though an individual may have, and probably does have, a "weaker" strand, it does not prevent individuals from attaining high levels of mathematical understanding as each strand simply assists in making that understanding deeper and prepares the mind for greater mathematical explorations.

To investigate each of these relationships would require a much larger study than the scope of this research allows. The focus will therefore be placed squarely on the relationship between two of its five strands - conceptual and procedural knowledge. The groundwork of this study will first be established by describing the historical evolution of the relationship between these two strands of mathematical proficiency and will be followed by a review of the theoretical components justifying the present research. As such, the theoretical framework will be divided into five sections: Defining conceptual and procedural knowledge, the development of conceptual and procedural knowledge, the role of fractions in mathematics, organizing arithmetic practice, and developing conceptual understanding through procedural activities.

Historical Background

Prior to Kilpatrick et al. (2001), the conversation in academic settings focused on the relationship between the conceptual and procedural components of mathematical understanding. Hiebert and Lefevre (1986) reported an on-going tug-of-war between conceptual and procedural knowledge, citing sources that placed emphasis on one or the other strand dating back as far as 1895. However, the authors noted that “due to the new language of cognitive science” (p. 2), the dialogue evolved as research considered conceptual and procedural knowledge less as individual and separate notions, but rather as existing in a mutually-dependent relationship. Research has continued to evolve since Hiebert and Lefevre’s (1986) work, and a growing number of studies now underline the bidirectional (i.e., each affects the other) and iterative (i.e., gains occur gradually over time) nature of the relationship between the two types of knowledge (Canobi, 2008; DeCaro, 2016; Fyfe et al., 2014; Rittle-Johnson et al., 2015).

As more research into the relationship between these two strands of mathematical proficiency developed and a greater understanding of their relationship was unveiled, data supporting its bidirectional and iterative relationship emerged. However, most of the research on the relationship between the two strands has been from the conceptual-first point-of-view and few studies have been pursued from a procedural-first point of view. Much research is needed from this other perspective (Rittle-Johnson et al., 2016; Star, 2005).

When tasked with teaching mathematics, teachers come to the table with varying degrees of knowledge, abilities, and experience, much in the same way that students do. Kilpatrick et al.

(2001) showed that multiple pathways exist to achieve mathematical understanding and have unveiled some of them. Perhaps there are more, but these five strands of mathematical proficiency have the virtue of being the primary ones identified so far.

Pasteur once said that “chance favours the prepared mind” (“La chance ne sourit qu’aux esprits bien préparés” (LeFigaro, n.d.)). I offer that for individuals (and here I refer to both students and teachers) to become proficient in mathematics, they must explore its multiple layers to be well prepared for learning and teaching. I believe that this is exactly what Kilpatrick et al. (2001) were referring to when they speak about the five strands of mathematical proficiency – the multiple layers of mathematics. Understanding the relationship between conceptual and procedural knowledge, how they are related, and how the development of one impacts the development of the other has yet to be completely understood (Star, 2005; Rittle-Johnson et al, 2015). As research in this field increases and common definitions of its primary terms are developed, it is likely that more effective instructional approaches will be developed.

So, how can the mind be prepared to develop and integrate mathematical knowledge? Besides the five strands of mathematical proficiency, Sweller’s Cognitive Load Theory (CLT) might help explain how we learn and remember general knowledge, including math knowledge (Lovell, 2020). The mind’s ability to acquire knowledge is limited by its working memory. To retain information, the knowledge being absorbed must be broken down and stored in its long-term memory. CLT tells us that the retention of knowledge can be improved by looking at similar, or worked-out, examples (Lovell, 2020, p. 104) that would lessen cognitive load and allow for understanding to follow. It also proposes that knowledge can be strengthened through a learned algorithm that develops an automated approach, reducing the load on the working memory, consequently allowing the learner to develop greater understanding. Combining CLT and the development of mathematical proficiency as described by Kilpatrick et al. (2001) suggests that the intertwined and interdependent nature of mathematical proficiency can be strengthened.

Defining Conceptual and Procedural Knowledge

Star and Stylianides (2013) raised an important aspect of research on conceptual and procedural knowledge. They underlined the need to clearly define what one means when referring to either conceptual or procedural understanding in their research. This is required as the terminological framework used in one study may not be the same as in another, potentially leading to a misleading comparison. For example, are the studies referring to conceptual and procedural knowledge as "types" of knowledge (i.e., what is known) or as "qualities" of knowledge (i.e., how well something is known)? The accuracy of the terms used and a shared understanding of what conceptual and procedural knowledge mean are very important for building on what is learned from earlier research. Therefore, a definition of conceptual and procedural knowledge is warranted and provides a solid footing to allow "other researchers [to] use, test and build on existing findings" (Star & Stylianides, 2013, p. 180). The definition of conceptual knowledge and procedural knowledge used in this paper, albeit not unique, will hopefully have the virtue of being clear and actionable.

Prior to providing definitions, a clarification on their origins may be warranted. Hiebert and Lefevre's (1986) seminal work provided a definition that was foundational to many studies that followed (e.g., Fyfe et al., 2014; Osana & Pitsolantis, 2013; Rittle-Johnson & Schneider, 2015; Star & Stylianides, 2013). Hiebert and Lefevre (1986) defined conceptual knowledge as "… characterized most clearly as knowledge that is rich in relationships" (p. 3) and procedural knowledge as "the formal language, or symbol representation system of mathematics" and as "algorithms, or rules, for completing mathematical tasks" (p. 6).

In a research commentary, Star (2005) proposed a reconceptualization of procedural knowledge. It is important to note that Star did not claim that procedural knowledge was more essential than conceptual knowledge but merely underlined the notion that little research had focused on the true impact of procedural knowledge on mathematical proficiency. He continued by observing that an individual may have both a deep and a superficial knowledge of different concepts, just as they may have a deep or superficial knowledge of various procedures. In a later article, Star and Stylianides (2013) remarked on the ongoing confusion created by the lack of distinction between knowledge type and knowledge quality, as noted by Star (2005).

Rittle-Johnson and Schneider (2015) provided a definition of conceptual and procedural knowledge that may satisfy Star and Stylianides' (2013) call for clarity on "which aspect(s) of conceptual and procedural knowledge we are focusing on" (p. 180) and is clearly grounded in the "common usage of procedural knowledge" (Star, 2005, p. 408) where the knowledge quality is superficial. Rittle-Johnson and Schneider thus defined conceptual knowledge "as knowledge of concepts" (p. 1119) and procedural knowledge as "the ability to execute action sequences (i.e., procedures) to solve problems" (p. 1120). It will be with these definitions in mind that these two terms will be used in the present study.

The Role of Fractions in Mathematics

As high school teachers will make it clear to anyone willing to listen, one of the greatest challenges teachers face in teaching mathematics is to have their students develop a clear understanding of fractions. This is true because understanding fractions is hard (Braithwaite & Siegler, 2017; Siegler et al., 2012). One of the reasons that fractions are challenging comes from the fact that whole numbers, which are learned from a very young age, and fractions behave differently. For example, in the addition algorithm for whole numbers, the sum is found by simply adding each number, so 4 + 4 = 8. However, when adding two fractions with the same denominator, that rule no longer applies (e.g., 1/4 + 1/4 ≠ 2/8). Similarly, when whole numbers are multiplied, their product is always larger than either of the factors (e.g., 2 x 4 = 8). This may not always be the case with fractions (e.g., 1/4 x 3/4 = 3/16). These examples illustrate some of the challenges inherent in teaching fractions. However, it is important to note that bridging the gap between whole numbers and fractions involves a uniting property: their respective magnitudes can be located on a number line.

Fraction magnitude has typically been evaluated by using both fraction number line and fraction magnitude comparison activities (Bailey et al., 2012; Bailey et al., 2017; Schneider et al., 2017). When a child is asked to place a fraction on a number line, they have to identify where it should appear on the number line. For example, if a child is asked to place the fraction 1/4 on a 0–1 number line, they would place the fraction approximately 1/4th of the distance between 0 and 1 away from 0. When a child participates in a fraction magnitude activity, they simply determine which of the two fractions is the larger of the two. Using either of these approaches is deemed to be a valid way to evaluate magnitude (Schneider et al., 2017)

Research on the development of mathematical understanding has revealed the importance of learning fractions (Bailey et al., 2012; Braithwaite & Siegler, 2017; Hamdan & Gunderson, 2017; Siegler et al., 2011; Siegler et al., 2012). Fraction knowledge at the elementary level has been shown to be predictive of a student's ability to understand mathematical notions at the secondary level and beyond, including algebraic understanding (Booth et al., 2015; Siegler et al., 2012). Further, research has shown that developmental trends in mathematical understanding begin with whole number line understanding (i.e., successfully locating where a whole number should appear on a number line) and slowly move towards fraction line understanding (i.e., successfully locating where a fraction number should appear on a fraction number line), which affects how they develop their understanding of fraction magnitude (Siegler et al., 2011). Even though improvements in understanding occur throughout elementary school and into high school, the challenges linked to the understanding of fractions have been observed throughout adulthood (Fazio et al., 2016). Consequently, the place of fractions in mathematics and its importance in the development of more advanced mathematical knowledge make it a rich–and critical–concept to study.

The Bidirectional Relationship of Conceptual and Procedural Knowledge

Now that conceptual and procedural knowledge have been defined, consider how they are developed and how their development is mutually impactful. Rittle-Johnson and Alibali (1999) observed a causal effect between the two types of knowledge in children in

Grades 4 and 5. Their study analyzed whether students understood the meaning of the equal sign in equivalence addition and multiplication equations. The authors studied non-standard equivalence problems of the form a + b + c = a + _. The goal of the study was to determine if either conceptual or procedural instruction would lead to an increase in the other type of knowledge. Conceptually speaking, did the children understand that the purpose of the equal sign is not to simply indicate where the "answer" is located, but rather to understand that both sides of an equation represent the same amount? From the procedural perspective, did they apply the procedure correctly to solve non-standard equivalence problems? The results revealed a causal effect between the two types of knowledge that further supports an iterative view of their relationship. Specifically, a better understanding of the concept of equivalence improves the understanding of its procedures and inversely, a better understanding of the procedures involved improves its conceptual understanding.

In a later study performed with Grade 5 and Grade 6 students, Rittle-Johnson et al.'s (2001) observations further supported the iterative relationship between conceptual and procedural knowledge. The authors evaluated the students' conceptual understanding of five decimal fraction concepts: relative magnitude, relations to fixed values, continuous quantities, equivalent values, and plausible addition solutions. The overall conclusions of these experiments showed that conceptual and procedural knowledge developed in a bidirectional and iterative fashion. It is also important to note that the authors added that the development of one type of knowledge to the detriment of the other may lead to incomplete understanding of the target mathematical notion because both types of knowledge are "mutually supportive" (p. 358).

A study by Hecht and Vagi (2010) also showed support for the bidirectional development of conceptual and procedural knowledge in the domain of fractions, which, together with Rittle-Johnson's et al.'s (2001) study, is central to the present research. The fourth- and fifth-grade students (N = 181) in this study were met at two time points. The authors used three fraction skills measures targeting procedural knowledge - a fraction computation test (adding and multiplying fractions), a word problem test (writing the correct equation that would result in the correct answer to the problem), and a fraction estimation test (identifying the nearest whole number to a fraction addition). The conceptual knowledge measures consisted of four tasks. The picture-symbol/symbol-picture tasks were used to assess part-whole understanding in which students had to either shade in a figure given a fraction or write a fraction that represented a shaded figure.

The third task required students to identify which of two fractions was the larger fraction. The last task required students to add pictorial representations of fractions and provide the sum using a similar drawing. The results further supported a bidirectional relationship between the researchers' conceptual and procedural measures. Further supporting the bidirectional relationship, Rittle-Johnson et al. (2015) noted that the evidence supporting such a claim was clear. As noted by others before (e.g., Canobi, 2008; Star, 2005), much work is needed to understand the relationship between these two strands of mathematical proficiency.

Rittle-Johnson et al. (2015) made two observations that provided support for further research into procedural knowledge that are worthy of noting, despite having some members of the research community posit that conceptual knowledge must always precede procedural knowledge (Boaler, 2016, p.71). The first, as mentioned earlier, is the lack of clarity in the terminology used in describing these two strands of knowledge. The second is the belief that starting instruction with procedural knowledge will bring us back to "old ways" of teaching.

To associate deep understanding with conceptual knowledge and the simple application of an algorithm to procedural knowledge can be misleading (Rittle-Johnson et al., 2015). Such an association takes away the possibility that procedural knowledge can be rich and meaningfully complex. Essentially, Rittle-Johnson et al. (2015) proposed that both types of knowledge be seen as continuously evolving and growing. Consequently, the concept-first view can lead to “misunderstandings and myths” and to a false belief that procedural knowledge does not lead to conceptual knowledge (Rittle-Johnson et al., 2015, p. 594). As a mounting number of studies confirming the iterative relationship between conceptual and procedural knowledge is being published (Canobi, 2009; Hecht & Vagi, 2010, McNeil et al., 2015), the authors concluded that an optimal order of instruction, conceptual first or procedural first, probably does not exist placing the emphasis on its iterative nature. As such, they proposed that more research in the effectiveness of ordering instruction is warranted.

Combining Instructional Approaches with Procedural Knowledge

In spite of their low number, the studies that have focused on the impact of procedural knowledge on the development of conceptual knowledge have produced interesting results but have done so by using different approaches. Structuring arithmetic practice in a way that supports conceptual connections (McNeil et al., 2012; McNeil et al., 2015), sequencing practice problems in line with the target concept (Canobi, 2009), supporting self-explaining (Fuchs et al., 2016; Rittle-Johnson, 2006), and providing students with prompts to reflect during procedural instruction (Lobato et al., 2005) have all demonstrated a positive effect of procedures on the understanding of the concepts attached to those procedures. Further, the aforementioned studies have shown that the more procedural knowledge is developed, the greater its impact on conceptual knowledge. Each approach is reviewed below.

Canobi (2009) observed the impact of conceptually sequencing addition practice worksheets. In her study, 72 children aged between 7 and 8 years old were assigned to one of three addition and subtraction practice groups: conceptually-sequenced practice, randomly- ordered practice, and no-practice. Prior to treatment, all students completed a pretest on a computer to assess their procedural knowledge and took part in a puppet game to assess their conceptual knowledge of addition and subtraction. The conceptually-sequenced practice problems consisted of five addition and five subtraction questions grouped in conceptually sequenced pairs (e.g., 1 + 2 = _; 2 + 1 = _). The randomly-ordered practice group saw the same problems but not conceptually sequenced and the no-practice group received nonmathematical activities. The results obtained supported Canobi's (2009) predictions and showed that procedural practice improved performance on procedures (assessed by measuring accuracy on practiced problems) but, more importantly, that practice using conceptually-sequenced problems positively impacted conceptual understanding of the principles of addition and subtraction, such as commutativity.

In a study evaluating performance on addition problems, McNeil et al. (2012) also considered the role of practice in the development of mathematical equivalence using addition practice. The authors recruited 104 Grade 2 and Grade 3 students who were randomly assigned to three conditions. Children in the first condition received practice on solving canonical (i.e., standard) addition problems (a + b = _) that were grouped by equivalent sums (i.e., 2 + 3 = _; 1 + 4 = _), hypothesized to support a relational understanding of the equal sign (e.g., 2 + 3 = 1 + 4). The students in the second condition also practiced problems, but these were grouped iteratively with the same first addend but with different sums, not reinforcing the notion of equivalence or sum as in the first condition (i.e., 2 + 3 = _; 2 + 4 = _). Finally, the students in the control group received no extra practice. The results showed that grouping practice questions by equivalent sums provided children with a better understanding of mathematical equivalence than either the random approach or the no-practice approach and that organizing practice problems conceptually made the conceptual underpinnings of equivalence salient.

Emphasizing mathematical equivalence through intentionally structured procedural work was further supported by the same authors in a later study (i.e., McNeil et al., 2015), in which 166 second-grade children were randomly assigned to one of two groups. One group received instruction using a modified workbook to target the development of math equivalence understanding while the control group received instruction using a standard workbook in which addition problems were presented in canonical form. The modified workbook contained three distinct features not seen in the regular workbook. The problems had most operations on the right side of the equal sign, some equal signs were replaced by their word equivalents (i.e., “is equal to” or “is the same amount as”), and some of the problems were organized by equivalent sums such that the same sum appeared in several problems in a row. The results showed that children who practiced using the modified workbook performed better than the children in the control group on all of the reported equivalence measures. Among other observations, the children in the modified-workbook group showed better understanding of non-canonical addition problems, solved equations more successfully, made fewer conceptual errors, and were able to define the equal sign relationally. In conclusion, the study serves to support the idea that well-crafted procedural activities can develop conceptual knowledge in children without explicit conceptual instruction.

Fuchs et al. (2016) distinguished three types of self-explaining: spontaneous self-explaining (i.e., when a student simply makes sense of the problem at hand without being prompted); elicited self-explaining (i.e., when a student is prompted to self-explain without any guidance); and finally, supported self-explaining (i.e., when the student is prompted to self-explain using rich explanations that were modeled for them and that they practiced). For their study, the authors decided to focus on the last type of self-explaining defined; that is, supported self-explaining.

The design of their study, in which 212 Grade 4 students participated, was composed of two intervention groups, an intervention with explanation group, an intervention with word-problem group, and a control. The instruction was delivered by tutors. The control group received classroom instruction that did not target any specific explanation process.

The instruction delivered in both intervention groups was divided between procedural work (about 40% of the time) and conceptual work (about 60% of the time). The intervention with explanation group focused on providing supported self-explanation through a four-step analysis process. The first step required students to identify if fractions had common denominators, common numerators, or had different numerators and denominators. The second step required students to comment on the quality of a drawing representing fractions (e.g., same size parts, correct number of parts shaded). In the third step, students had to identify the image of a fraction with its numerical value and describe if the fractions drawn had same-size parts or not, thus indicating that the fraction with fewer parts had the largest parts. Finally, the fourth step consisted of having students provide a written explanation indicating why one fraction was larger than another. The intervention with word-problem group received instruction consisting of word problems designed to introduce fraction concepts. The students were first taught to identify word problems according to two schemas - either a division story (e.g., cutting fruits into pieces) or a grouping story (e.g., what is the total length of string needed if a character in the story wants to make 5 necklaces that are each 1/3 of a meter long?). Students were also shown a series of steps to help structure and solve the questions. However, they were not prompted or taught to reflect on the work completed. Fuchs et al. (2016) found that the students who engaged in supported self- explaining significantly improved their accuracy of fraction magnitude comparisons and the quality of their explanations, outperforming the word-problem intervention group.

In another study on the impact of self-explanation on learning, Rittle-Johnson (2006) evaluated if providing prompts to self-explain improved procedural learning, procedural transfer, and conceptual understanding of equivalence, and if self-explanation had a lasting effect. Forty-two children from Grade 3 to Grade 5 were recruited for the study. They were randomly divided into 4 groups: direct instruction with or without explanation or discovery learning with or without explanation. The direct-instruction groups received explicit instruction on how to add and subtract numbers, while the discovery-learning groups received none. For the self-explain groups, the children were asked to explain verbally how an answer was found and why it was, or was not, correct. The no-explanation groups were simply provided with the answer.

The intervention questions consisted of equivalence problems of the form a + b + c = a +_ (e.g., 2 + 3 + 4 = 2 + _) or a + b + c = _ + c (e.g., 2 + 3 + 4 = _ + 4). The pretest and posttest questions contained two equivalence problems similar to the intervention questions to assess the learned procedure. To assess procedural transfer, two problems with no repeated addends were given (e.g., 4 + 3 + 6 = 9 + _ ; 7 + 9 + 3 = _ + 6), another two with the unknown on the left side of the equation were given (e.g., 9 + _ = 4 + 3 + 6 ; _ + 6 = 7 + 9 + 3), and two more where subtraction was included (e.g., 2 + 4 – 3 = 2 + _; 2 + 3 - 4 = _ - 4). Rittle-Johnson's (2006) results showed that self-explaining had a positive effect on learning and transfer and that combining direct instruction with self-explaining had a greater impact on procedural learning than the other conditions. Self-explaining was found to make learning correct procedures easier, facilitate the transfer of the procedure to other problems, allow for the application of the procedure to new problems, and enhance retention of the procedure.

Finally, Lobato et al. (2005) proposed another pathway to elicit conceptual understanding through procedural activities by readdressing what it means to "tell." In their article, the authors proposed that telling is not an ill-fated process that takes away from constructivism, but one that can be used to stimulate thought by introducing new ideas within the mathematics conversation. Telling needs to be viewed as a combination of a teacher's intention in presenting information, a student's interpretation of this new information, the conceptual nature of the new information, and how the action of telling is combined with other pedagogical interventions (Lobato et al., 2005).

The teaching act is not unidirectional. It requires two parties to interact, and these interactions can lead to greater understanding as they can serve to bring ideas together. In their article, Lobato et al. (2005) reported their interactions with 17 high school students from Grades 8 to 10 and analyzed three different teaching actions. The mathematical target for all lessons was to teach students the notion of rate of change, but the purpose of the three instructional sessions was to analyze the function of telling during instruction. As such, the authors redefined telling as the actions that trigger student engagement in mathematics by introducing new ideas through instruction. During the three teaching sessions, the teacher either initiated reflection about the notion being taught by describing a new concept, provided new information by summarizing the student’s work, or provided information that would allow the students to test their understanding of the concept taught. During these interventions, the teacher sometimes tried to elicit information on the student’s understanding of the concept presented and at other times, initiated reflection by asking questions that served to elicit the underlying concept creating a type of initiating-eliciting framework. Although limited in scope, the resulting analysis of the teacher-student interactions showed that a teacher’s intentions and actions in telling, and the students’ interpretations of these actions prompted student reflection on the procedures used that could bridge the procedural and conceptual divide.

In conclusion, evidence exists to show that students will not develop conceptual knowledge through procedures alone or without some form of prompting (Rittle-Johnson et al., 2015). The research described in this section demonstrated that students need to be prompted in some way – either implicitly or explicitly – to think about the concepts that underlie the procedures they practice. This can be done through conceptual sequencing (Canobi, 2009) or explicit prompts to reflect (think about why a procedure works or on the conceptual meaning of procedures) and self-explain (in this case, verbalising their thinking)

(Fuchs et al., 2016; Lobato et al., 2005).

When Procedural Knowledge of Fractions Leads to Conceptual Understanding

In sum, the leading question of the current research is: Can repeated procedural activities with fractions lead to the development of the procedure's underlying concepts and in essence make students understand why they work? The following two research papers will assist in making the case for studies that considered the impact of procedural knowledge of fractions on the development of conceptual knowledge.

Bailey et al. (2015) proposed that fraction arithmetic proficiency (i.e., procedural knowledge) is a factor that predicts the ability to place fractions on a number line (i.e., conceptual knowledge). Their study involved 44 Grade 6 and 39 Grade 8 students from China, and 24 Grade 6 and 24 Grade 8 students from the U.S. These two countries were selected because whole number knowledge in Chinese students had been shown to be greater than that of American students, especially where procedural knowledge was concerned (Siegler & Mu, 2008). The authors hypothesized that, for these two countries, if the discrepancy holds for whole numbers, then it should persist when it comes to fractions as well. This led the authors to hypothesize that, compared to American children: (a) Chinese children would have greater procedural and conceptual knowledge of fraction procedures; (b) the greater knowledge of fractions would be moderately to highly correlated with overall mathematics achievement; and (c) the difference in conceptual knowledge between the two groups would be mediated by their procedural knowledge. The authors also predicted that for both countries, fraction knowledge would be greater between low achieving and average students.

The Chinese children performed the tasks individually on a computer in a quiet room while the American children's data were collected in a previous study (Siegler & Mu, 2008). The procedural tasks consisted of adding, subtracting, multiplying, and dividing fractions. There were three fraction conceptual knowledge tasks consisting of two fraction line estimation tasks (i.e., a 0–1 and a 0-5 number line) and a fraction magnitude comparison task where eight fractions were compared to 3/5.

The results supported four of the five hypotheses and showed that Chinese students had greater procedural and conceptual knowledge of fractions than their American peers, with the difference being largest for procedural knowledge. Bailey et al. (2015) also found that the difference in conceptual knowledge of fractions between the two countries was fully mediated by their procedural knowledge of fractions. Also, the results provided correlational evidence that fraction arithmetic knowledge mediated the students' conceptual knowledge of fractions.

In Bailey et al. (2017), the authors investigated the path linking fraction arithmetic skills and fraction magnitude understanding. To do so, they used the state-trait model (based on Steyer, 1987) where trait (e.g., socioeconomic status, attention span, etc.) and status (e.g., how previous mathematical knowledge impacts future mathematical knowledge) to statistically analyse the data they gathered at four time points from which they developed their own model of fraction magnitude and fraction arithmetic skills. Specifically, their objective was to determine if and how much transfer takes place between fraction arithmetic knowledge and fraction magnitude knowledge as children progressed from Grade 4 to Grade 6. They proposed that the simple act of practicing arithmetic exercises could help students to make links between the fraction magnitudes they already understand and the arithmetic problems they are performing. For example, if students were asked to add 1/2 and 2/3, the sum would have to be greater than 1 but less than 2. A student who learns the procedure to add fractions would get a total of 7/6, which in turn, would contribute to their current knowledge of fraction magnitude (i.e., that 7/6 would be reasonable as a sum for ½ and ⅔).

The authors measured student fraction arithmetic knowledge using a pencil-and-paper assessment containing fraction addition and subtraction items with like and unlike denominators (adapted from Hecht, 1998). Fraction magnitude was measured using a well-known Fraction Number Line estimation task (Siegler et al., 2011). The data gathered were part of a larger longitudinal study of mathematics containing several other measures of fraction arithmetic skill and fraction magnitude understanding in children and were collected at four time points between the fourth and sixth grade: spring of Grade 4; fall of Grade 5; spring of Grade 5; and winter of Grade 6. The authors found that fraction arithmetic skills predicted fraction magnitude understanding in the second half of the study, between fall and spring of Grade 5, and again between spring of Grade 5 and winter of Grade 6.

The Present Study

The purpose of the present study was to further clarify the path between procedural and conceptual knowledge. It investigated whether teaching procedures alone can improve students' procedural accuracy in fraction multiplication, as well as enhance their conceptual understanding of fractions. It also examined whether encouraging students to reflect on the procedures they used would lead to a deeper conceptual understanding of fractions compared to simply practicing the multiplication procedure without reflection. Specifically, students in the reflection group were asked to compare and observe the difference between the fraction at the beginning of the equation and their final answer. They were also asked to determine which of the two fractions was greater and to justify their response. These reflective questions were not addressed to the practice intervention without reflection group.

Through repeated procedural fraction multiplication activities combined with reflective prompts (prompting participants to reflect on the size of the product relative to the original fraction in the problem), I hoped to show that combining reflection with procedural work would lead those students to develop a greater conceptual understanding of fractions. As stated above, the second group was treated similarly to the first but did not receive any reflective prompts. The results helped to determine if doing repeated procedural work on its own still led to greater conceptual understanding or if reflection is a key component of the process. Finally, a third group of participants served as the control group by engaging in mathematics activities that did not involve fractions. The activities they participated in targeted the development of their spatial sense through geometry activities.

The study followed a three-group pretest-intervention-posttest design. Grade 5 students were randomly assigned to each group and met with the researcher or his assistants three times a week over a period of four weeks. The measures I administered evaluated the participants' fraction multiplication knowledge, their fraction number line estimation knowledge (Siegler et al., 2011), and their fraction magnitude knowledge by analyzing the fraction magnitude strategies they used along with the quality of their justifications (Fazio et al., 2016) to justify why they identified one fraction as larger than the other. All measures were pencil-and-paper tests. Except for the Fraction Multiplication task, which I developed, the other measures were adapted from earlier studies (Bailey et al., 2012; Fazio et al., 2016; Hansen et al., 2017, Siegler et al., 2011).

Three research questions follow:

1. Will practicing a procedure to multiply a whole number by a fraction (i.e., performing repeated addition) result in acquiring greater procedural accuracy in fraction multiplication (as measured by the accuracy of their answers)?
2. Will practicing a procedure to multiply a whole number by a fraction (i.e., performing repeated addition) result in greater conceptual knowledge of fraction magnitude, as measured by the accuracy of their answers on two fraction lines (0–1 and 0–2), and a fraction magnitude comparison activity which measured the accuracy of the answers provided and the quality of the students' justifications?
3. Will reflecting on how multiplying a whole number by a fraction (i.e., performing repeated addition) result in greater conceptual knowledge of fraction magnitude than those who do not reflect on their multiplication, as measured by the accuracy of their answers on two fraction lines (0–1 and 0–2), and a fraction magnitude comparison activity which measured

the accuracy of the answers provided and the quality of the students' justifications?

Hypotheses

Practicing repeated fraction multiplication exercises will improve the accuracy of the students' procedure when multiplying a whole number by a fraction. Practicing a procedure for multiplying a whole number by a fraction will support the children's development of conceptual knowledge of fraction magnitude. Prompting students to reflect on how multiplying a whole number by a fraction affects the magnitude of a fraction will further augment the children's conceptual knowledge of fraction magnitude. Practicing a procedure for multiplying a whole number by a fraction, with or without prompting, will have a greater impact on conceptual knowledge of fraction magnitude than no procedural practice at all.

I hypothesized that the students who took part in the fraction multiplication intervention, with or without reflection prompts, would develop greater conceptual knowledge than the control group. I also predicted that the practice group that was prompted to reflect on the procedure would gain greater conceptual knowledge than the group that did not receive any prompts. Finally, I predicted that the students who participated in the control group would not develop either their procedural or their conceptual knowledge of fractions.

It is my hope that this research will serve to build on the work of those who have studied the iterative relationship between procedural and conceptual knowledge and will assist teachers in moving students closer to acquiring mathematical proficiency.

Method

Participants

Participants were 57 Grade 5 participants (M_{age}= 10.97 years; SD_{age} = 0.49 years) from four different classrooms (two classrooms in each school) within two schools (School 1 and School 2) located in a suburb of a large metropolitan area (population < 50,000) in the province of Quebec, Canada. Of these, 31 (54.4%) were girls and 26 (45.6%) were boys. Prior to the beginning of the study, the school board, schools, and teachers gave me permission to approach parents/guardians to seek their participation in the study. As such, I secured informed consent from the parents/guardians and assent from the participants. One (1) male participant withdrew from the study after the pretest, bringing the final number of participants to 56 (31 girls and 25 boys). For the purpose of this study, because one of the classes in one of the schools provided only four students, I integrated those few students within the other class from the same school (of that group of four, two were randomly selected for the PI+ group, and two for the Control group).

The socioeconomic status of the population in the two participating schools, as reported in the “Indices de défavorisation” document published by the Ministry of Education of Quebec (MEQ, 2024), each had decile ranks of 6 out of a maximum of 10. A ranking of 10 identifies children from the most underprivileged areas of the province. These rankings are obtained by combining the rank of each child based on the mother’s education level and the parents’ employment status. Consequently, a school whose population has a lower economic status would have a higher decile rank. Schools are considered to be located in an underprivileged area when their ranking varies between 7 and 10. Therefore, a ranking of 6 places the participating schools in the top half of the scale, suggesting that the schools are located in a moderately to highly underprivileged area.

Each school in the study followed the same provincial mathematics program. The provincial program for elementary mathematics is competency-based and was published in 2001 (Québec Education Program [QEP], Ministère de l’Éducation du Québec [MEQ], 2001). The *competencies* are called: “To solve a situational problem. To reason using mathematical concepts and processes [and] To communicate by using mathematical language” (p. 141). The mathematical content of the program is divided into five branches of mathematics: arithmetic, geometry, measurement, statistics, and probability.

Design

The study used a three-group pretest-intervention-posttest design (See Figure 1). The same four measures were used at both pretest and posttest and administered in the same order. One test targeted the participants' procedural understanding of fraction multiplication and required them to multiply a single-digit whole number by a fraction. The other three tests targeted the participants' conceptual understanding of fractions. Two of these tests required participants to place selected fractions on a number line, while the third consisted of comparing two fractions and identifying the larger of the two and justifying their choice using words or drawings. The practice interventions required participants to multiply a whole number by a fraction through repeated addition.

Figure 1

Phases of the Study

Pretests		Intervention		Posttests
Fraction Number Lines (0–1, 0–2)	→	PI+	→	Fraction Number Lines (0–1, 0–2)
Fraction Magnitude Comparison		PI-		Fraction Magnitude Comparison
Fraction Multiplication		No training		Fraction Multiplication
		9 class periods 4 weeks ←→		

Note. PI+ (Practice Intervention with Reflection Prompts), PI- (Practice Intervention without Reflection Prompts).

Participants within each school and each class were randomly assigned to one of three conditions: Practice Intervention with Reflection Prompts (PI+), Practice Intervention without Reflection Prompts (PI-), and Control (see Table 1). Random assignment within each class was performed using Excel's random number assignment function. First, each student was given a randomly-assigned number. The students were then ranked in numerical order from low to high and divided evenly, when possible, among the three groups from the lowest to highest number. For example, in a group of 18, the top six would be selected for the PI+ group, the following six for the PI- group, and the final six for the Control group. This was repeated for each class. This resulted in an overall distribution of 20 participants in the PI+ group, 19 in the PI- group, and 17 in the Control group (see Table 1). In one situation (Class B1), a participant remained in the classroom where the PI+ group met for the second day of the intervention without a research assistant noticing. He was kept with that group for the remainder of the interventions.

Table 1

Distribution of Participants by Intervention and Class.

Groups	Class A1	Class B1	Class B2	Total
PI+	4	9	7	20
PI-	4	8	7	19
Control	4	7	6	17
Total	12	24	20	56

Note. Students in class A1 were from School 1. Students in classes B1 and B2 where from School 2.

Intervention

The three experimental conditions were: (1) *Practice Intervention with Reflection Prompts* (PI+), (2) *Practice Intervention without Reflection Prompts* (PI-), and (3) *Control*. In the PI+ condition, participants were taught how to multiply a whole number by a fraction using a repeated addition procedure and were prompted to reflect on what the impact of the procedure was on the fraction being multiplied (i.e., how is the magnitude of a fraction affected when you multiply whole number by a fraction) will further augment the children's conceptual knowledge of fraction magnitude. Participants in the PI- condition were also shown how to multiply fractions using the same repeated addition procedure but were not asked to reflect on their answers. Participants in the Control condition did not receive fraction multiplication training; instead, they completed geometry activities that did not involve fractions and were instructed on labelling the parts of solids (ex., faces, edges, vertices, apex), along with being shown how to construct solids and their nets. This was achieved through worksheets and guided activities obtained from an Ontario-based math program (JUMP Math, 2019) whose content was similar to the program used in the province of Quebec and with which participants were familiar.

Working within the constraints of school-based research, each group participating in the intervention (PI+, PI-) completed nine intervention sessions at a rate of two to three sessions per week over a period of four weeks. Five research assistants participated in the intervention sessions. All received individual training and were provided with a strict script to follow (see Appendix A). The research assistants were instructed to follow the script, with periodic reminders provided throughout the intervention sessions. Strict adherence to the implementation script could not be fully ensured, as I was actively involved in the intervention. Both intervention groups, PI+ and PI-, completed their respective activities in separate rooms under my supervision or that of one of the research assistants, and completed their activity sheets individually (see Appendix B).

During the first three of the nine intervention sessions. participants in the PI+ and the PI- groups practiced multiplying a whole number by a fraction after observing a demonstration performed by me or the research assistant assigned to the group. To support learning, I (or the research assistant) began each of the first three sessions with the same two examples before the participants practiced the procedure on their own (the examples are provided in the procedures section below). In the first example, I or the assigned research assistant, showed participants how to multiply a whole number by a fraction through repeated addition. In a second example, a fourth step that required simplifying the product was introduced. That is, participants were shown how to reduce the fraction to its simplest form by dividing the numerator and denominator by a common factor whenever possible. Participants were informed that simplification of fractions would only involve dividing by 2, 3, 4, or 5 and were allowed to use a multiplication chart if they found it necessary.

Following the demonstration, participants completed two practice exercises, one requiring simplification and the other not. During this time, I or the research assistant, circulated to ensure proper execution of the procedure. Once all participants completed the practice exercises, they were asked to complete the activity sheets containing 10 multiplication problems similar to the examples students had just seen. The questions found in the activity sheets contained fractions that were selected to elicit three specific strategies that can be used to compare fractions (Fazio et al., 2016) – the larger fraction is the fraction with the larger numerator when they have common denominators; the larger fraction is the fraction with the smaller denominator when the numerators are the same; and the larger fraction is the fraction that has both the larger numerator and the smaller denominator when the other fraction has both a smaller numerator and a larger denominator. Each activity sheet was composed of three fractions from each fraction type,(i.e., three Common Denominators (CD) types, three Common Numerators (CN) types, and three Larger Numerator/Smaller Denominator (LNSD) types), plus another fraction that did not fit in the other three groups. All questions were randomly selected from a bank of questions I developed, for a total of 10 questions per worksheet (see Appendix C).

I, or the research assistant, picked up the activity sheets after the participants confirmed they were finished. From the fourth intervention onward, the same two examples were demonstrated to remind participants of the procedure (no support or feedback was provided). The answers to the questions they had just performed were then written on the board without explanations. The activity was untimed because time taken to completion was not of interest in this present study. On average, participants took between 30 to 40 minutes to complete the activity sheets and by the end of the intervention, participants took less than 30 minutes. In total, participants completed 9 intervention activity sheets each with a total of 10 practice exercises. Participants therefore completed 96 fraction multiplication exercises over the course of the nine intervention sessions, six during the introductory phase (Sessions 1 to 3) and 90 on the activity sheets (Sessions 4 to 9). All interventions were conducted in separate rooms under my supervision or that of one of the research assistants.

While the PI+ and PI- groups were practicing their fraction multiplication exercises on their activity sheets, the students in the Control group were learning notions of geometry through worksheets and guided activities in a separate room. At no time during the intervention periods were concepts or procedures related to fractions discussed.

Measures

Three measures were administered at pretest and posttest. They were selected to assess if fraction magnitude understanding (FNL 0–1, FNL 0–2 and FMC) had developed and assess if their ability to properly execute the procedure taught during the intervention had improved. I alternated with the five research assistants throughout the intervention period to ensure result objectivity and reliability.

Fraction Multiplication

The Fraction Multiplication task assessed the participants' ability to multiply a whole number by a fraction (see sample questions in Figure 2). There were five fraction multiplication questions on the pretest and four similar questions on the posttest. A printing error on the posttest required the removal of one of the questions. An accuracy score was calculated using the mean correct score for the task. The fractions were randomly selected from the same bank of questions used during the intervention portion of the study (see Appendix C).

Figure 2

Sample Questions for the Fraction Multiplication Task

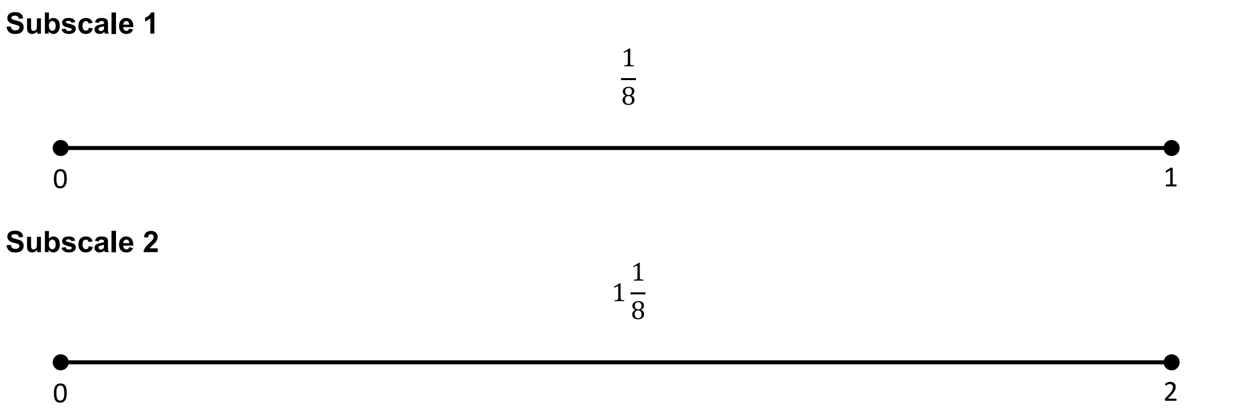
1) 4 x 2/10	Show your work.
2) 3 x 4/7	Show your work.

Fraction Number Line Tasks

The Fraction Number Line tasks (FNL; with subscale 1 [0 to 1 number line], and subscale 2 [0 to 2 number line]), adapted from Bailey et al. (2012), assessed participants' conceptual knowledge of fraction magnitude. Specifically, it measured participants' accuracy in locating a given fraction on a number line between 0 and 1 (booklet 1) and 0 and 2 (booklet 2). A number line is a linear visual representation where numbers are positioned according to their magnitude, with greater values placed further to the right of zero (see Figure 3; Schneider et al., 2017). The task required participants to understand the relation between fractions on a ratio scale level; that is, where on the number line are each fraction located given the range of the line. Based on previous work (Bailey et al., 2012; Fazio et al., 2016; Hansen et al., 2017; Siegler et al., 2011), two booklets were created for that purpose. The Fraction Number Line 0–1 (FNL 0–1) booklet contained nine items, each of which presented a 0 to 1 number line with a fraction placed above and centered on the number line. The participants placed a hash mark on the number line to represent where the fraction was located. The items in the FNL 0–1 booklet were: 1/5, 13/14, 2/13, 3/7, 5/8, 1/3, 1/2, 1/19, and 5/6. The second booklet, FNL 0–2, consisted of 11 items. Each item was presented in the same way as for the 0–1 number line except that the line ranged from 0 to 2. The students placed a hash mark on the number line to represent where the fraction was located (see Figure 3 for demonstration samples). The 11 questions required placement of the following fractions: 1/3, 7/4, 12/13, 1 11/12, 3/2, 5/6, 5/5, 1/2, 7/6, 1 2/4, 1, 3/8, 1 5/8, 2/3, 1 1/5, 7/9, 1/19, 1 5/6 and 4/3.

Figure 3

Demonstration Fractions for the Fraction Number Line Tasks for Each Subscale



Scoring for both Fraction Number Line tasks (Subscales 1 and 2) was obtained by calculating the degree of precision of the participant's placement of the fraction on the number line as related to its actual position on the number line called the *Percent Absolute Error* (PAE). The PAE is calculated by finding the absolute difference between the actual location of the fraction on the number line (in cm) and the position estimated by the participant (in cm) and dividing the absolute value of the result by the total length of the line (in cm) and multiplying the quotient by 100. The closer the PAE is to zero, the more accurate the participant's estimate.

Fraction Magnitude Comparison

The Fraction Magnitude Comparison task (FMC) is another task that also assessed the participant's conceptual understanding of fraction. The FMC contained 18 items (see Table 2) and required participants to identify which of two fractions was the larger fraction and then justify their selection. Three types of fractions were presented, 6 fractions for each fraction type: (1) fractions that had equal denominators but different numerators (e.g., 2/5 and 3/5), (2) fractions with equal numerators but different denominators (e.g., 4/7 and 4/9), (3) and fractions where the numerator of one fraction was larger, but its denominator was smaller than the fraction it was being compared to (ex.: 4/5 vs 3/7). The fractions in this measure served to elicit three conceptually-based strategies of fraction magnitude. First, when comparing fractions that have a common denominator, the fraction with the greater numerator is the larger fraction. Second, when comparing fractions with a common numerator, the fraction with the smaller denominator is the larger fraction. And last, when fractions have different numerators and denominators, the fraction with the larger numerator and the smaller denominator is the larger fraction. The pairs of fractions were randomly distributed in the booklet such that the larger fraction appeared on the right side eight times, and on the left side 10 times.

Table 2

Fraction Types for Magnitude Comparison

Equal Denominators	Equal Numerators	Large Numerator and Small Denominator
4/9 and 2/9	3/4 and 3/5	2/9 and 3/7
2/7 and 3/7	7/9 and 7/8	3/8 and 2/9
3/5 and 4/5	1/4 and 1/3	5/9 and 7/8
3/19 and 9/19	4/15 and 4/13	10/13 and 9/14
9/14 and 13/14	13/14 and 13/17	2/15 and 3/11
9/17 and 13/17	2/13 and 2/17	10/17 and 13/15

Note. All values obtained from Fazio et al., 2016.

Quantitative and qualitative data were obtained from the Fraction Magnitude Comparison task (see Figure 4). First, the proportion of correct responses was calculated by counting the number of times the bigger fraction was properly identified by the participant and dividing the answer by 18, the number of questions in the task. Second, the quality of the participants' strategies used to justify their choice was analyzed (see Table 3). To do so, I adapted a scale developed by Fazio et al. (2016), which has four levels of strategy quality: *logical necessity*, *intermediate steps*, *usually correct*, *questionable* (Fazio et al., 2016, pp. 46-47), to which I added an *uninterpretable* category because some answers provided by participants where unclear or offered circular logic (e.g., explaining that the identified fraction was the larger fraction simply because it is the larger fraction). To determine if participants improved the quality of their responses from pretest to posttest, these five levels of strategy quality were rank ordered (see ranking in Table 3) and points were assigned to each response based on the rank. A greater number of points were assigned to more conceptually complete strategies. For example, participants who used a strategy identified as *Logical Necessity* would receive 4 points, while participants who used a *Usually Correct* strategy would receive a score of 2. A quality score was computed for each participant by taking the mean number of points assigned across all six items for each fraction type.

Figure 4

Demonstration Fractions for the Fraction Magnitude Comparison Task

$\frac{3}{7}$	$\frac{2}{7}$

Explain why you think the circled fraction is bigger.

The distinction between all five levels rests on the strength of the strategy and its relation to conceptual understanding of fractions. To be identified as a *Logical Necessity* strategy, the participant had to show that the numerator, the denominator, and the relationship between the two were considered. For *Intermediate Steps*, the participant adequately compared the fraction to another known fraction (ex.: 0, ½, or a whole). Participants who used the *Usually Correct* strategy only considered the relationship between the denominators and the numerators of the two fractions or attempted to explain their choice through drawings or number lines. When the *Questionable* strategy was used, participants typically identified the larger fraction by using aspects of a fraction that are true only in particular situations, like picking the fraction in which the difference between the numerator and the denominator is the greatest. Finally, *Uninterpretable* strategies did not allow for a conclusion about the strategy used, such as , simply indicating that the selected fraction was the bigger fraction.

Table 3

Fraction Magnitude Comparison Strategies and *Scoring*

General Strategy Group	Scoring	Strategies Included	Strategy Description
Logical necessity: Strategy yields correct answer on all applicable problems.	**4**	Equal denominators	If both fractions have equal denominators, the fraction with the larger numerator is larger.
		Equal numerators	If both fractions have equal numerators, the fraction with the smaller denominator is larger.
		Larger numerator and smaller denominator	The larger fraction has a larger numerator and a smaller denominator.
Intermediate steps: Strategy yields correct answer on all applicable problems if intermediate steps are executed correctly.	**3**	Halves reference	The larger fraction is greater than ½ and the smaller fraction is smaller than ½.
		General magnitude reference	Compare one or both fractions to a nearby known magnitude, such as 0, ½, or 1.
Usually correct: Strategies that yield better than chance results, but do not guarantee correct answers.	**2**	Visualization	Using a pie chart, number line or other visual representation of a fraction to compare magnitudes.
		Smaller denominator	The larger fraction has a smaller denominator.
		Difference between numerator and denominator within each fraction is smaller	The difference between the numerator and denominator of the larger fraction is smaller than the difference between the numerator and denominator of the smaller fraction.
		Larger numerator	The larger fraction has a larger numerator.
Questionable: Strategies do not guarantee to yield above chance performance.	**1**	Larger denominator	The larger fraction has a larger denominator.

General Strategy Group	Scoring	Strategies Included	Strategy Description
		Larger numerator and denominator	The larger fraction has a larger numerator and denominator.
		Smaller numerator	The larger fraction has a smaller numerator.
		Difference between numerator and denominator within each fraction is larger	The difference between the numerator and the denominator of the larger fraction is larger than the difference between the numerator and the denominator of the smaller fraction.
Uninterpretable: The strategy used is inadequately explained and does not allow to conclude which of the two fractions is the larger fraction	**0**	Drawing	The drawing used does not identify which of the two fractions is the larger fraction (i.e., both fractions are not represented within the drawing)
		Number line	The number line drawn does not identify which of the two fractions is the larger fraction (i.e., both fractions do not appear on the number line)
		Words	The explanations are incomplete or do not make sense (i.e., this fraction is bigger because it is bigger.)
		Nothing	No answer provided

Note. Adapted from Fazio et al., 2016

Procedure

I administered the two Fraction Number Line tasks (Subscales 1 and 2), the Fraction Magnitude Comparison task, and the Fraction Multiplication task at pretest, with the help of a research assistant. The participants in each class completed the pretests together in their respective classroom settings or in a cafeteria. I provided the instructions to the students on how to answer the test questions to all three classes at pretest and posttest. All tasks were pencil-and-paper tests. Pretests and posttests were identical and administered in the same order at both time points.

Pretest

To begin the first estimation task, I showed the participants a fraction (1/8) on the Smartboard or bristle board. I began by saying, “Today we’re going to play a game using fraction number lines. Open your activity book called Fraction Number Line Activity – to page 1. Notice there is a line with 0 on the left end and 1 on the right end, and a fraction above it in the center. Now look at the Smartboard (or board). The same line and fraction appear here [I pointed the board]. Next, I will ask you to mark where you think the fraction is on the number line. Here is an example [I showed Figure 3, Subscale 1] I will place a hash mark where I think 1/8 goes on the number line. [I placed a hash mark near the 1/8th position on the number line and confirmed that all participants had understood what he had done]. Now it is your turn. Turn to page 2. On the page in front of you, there is a line with 0 on the left end and 1 on the right end. Above it, in the center, is the fraction 1/4. Place a hash mark where you think 1/4 goes on the number line.”

I waited for the children to place their hash mark and provided feedback on their performance, with the help of a research assistant, as the participants showed them their answers. The participants were told that the rest of the workbook had similar problems and were instructed to answer every question in the booklet (the example [1/4] completed by the students was not included in the analysis). Participants completed the booklet at their own pace. Participants were provided with reading material or crossword puzzles to avoid disrupting those who needed more time to complete the task if they finished early. From this time forward, all information provided to the children aimed to clarify administrative aspects of the task only and did not address the underlying conceptual components of the activity, nor were any strategies to answer the questions provided. As the participants completed their booklets, the research assistant and I verified that all questions were answered and that the booklets were properly identified. If a booklet was not completed, it was given back to the participant, and they were asked if they could complete it.

After completing the 0–1 number line booklet, I demonstrated a similar procedure for the 0–2 number line booklet: The participants watched as I placed 1/8 and then 1 1/8 on the 0–2 fraction line (see Figure 3 – Subscales 1 and 2). As before, I provided the participants with a practice item asking them to place ¼ on the 0–2 number line. Again, no feedback was provided except to confirm that the participants understood the task. Following the practice item, I provided the same instructions as for the activity with the Fraction Number Line 0–1 and then testing began.

For the third booklet, the Fraction Magnitude Comparison task, I began by saying, “This third booklet has a different look from the first two [I showed an image like the one in Figure 4]. As you can see on page 1, there are two fractions separated by a dark line above a box.”

I continued by saying, “For this activity, I will ask you to circle the fraction that you believe is the biggest fraction of the two. Once this is done, you will write down, or draw, why you think it is the biggest of the two fractions in the box below the fractions”. To practice, I demonstrated comparing 3/7 and 2/7 and instructed the participants to circle the bigger fraction. I said, “Now, I would circle whichever of these two fractions I predict to be the bigger fraction of the two”. I then pointed to the box below the fractions and said, “this is where I would like you to explain why you circled the fraction that you did. You can do this by writing words, using a number line, or drawing. Make sure that you clearly identify which of the two fractions is the biggest one”. I did not fill in the box on the demonstration item.

The participants were given a practice question where they compared 1/3 and ½ and were asked to justify their answer. Following the practice item, participants were reminded to justify their answers and began the test. I again provided reading material and crosswords to keep the participants occupied if they finished early. If a booklet was not properly completed, it was given back to the participants, and they were asked to complete it.

The final task on the pretest was the Fraction Multiplication task. I began by saying, “This last task will be a little trickier than the other three. I will give you a sheet on which there are five fraction multiplication questions. I know you may not have learned how to multiply fractions yet, and that's okay. I just need you to try your best. What I need you to do is to show how you think you can get to the answer. It doesn't matter how you do it. If you are really stuck, you can write a question mark where the answer goes, but you can't leave it blank." Participants then received the activity sheets on which the five multiplication tasks appeared. Access to reading material and crosswords was again provided. Activity sheets that were not properly identified or completed were handed back to the participants to be completed. If students confirmed they were done, the activity sheets were retrieved.

Intervention Phase

Five trained researcher assistants (RA1 to RA5) met with the intervention and control groups for a total of nine sessions. Due to scheduling issues, school 1 completed the pretest 11 days before beginning the sessions. Students were met two to three times a week over a four-week period every second school day. School 2 completed the pretests four days before the beginning of the sessions. Students from both classes in school 2 were met on alternating days and, as for school 1, were tested over four weeks every second school day. For both schools, posttests were completed 5 days after the last intervention session (see Figure 5). The same research assistant (RA1) and I were present during both pre- and posttests. All three groups (PI+, PI-, Control) from each class completed their activities in their respective classrooms and at the same time.

Figure 5

Research Project Schedule

Intervention Types	School	Date of session	Research Team Members and Conditions		
			PI+	**PI-**	**Control**
Pretest	School 1	Oct. 17	Researcher / RA1		
	School 2	Oct. 24	Researcher / RA1		
		Oct. 25			
Session 1	School 1	Oct. 28	Researcher	RA1	RA3
	School 2	Oct. 28	Researcher	RA1	RA2
		Oct 29	Researcher	RA1	RA2
Session 2	School 1	Oct. 30	RA1	RA3	Researcher
	School 2	Oct. 30	RA1	RA2	Researcher
		Oct. 31	RA1	RA2	Researcher
Session 3	School 1	Nov. 1	RA3	Researcher	RA1
	School 2	Nov. 1	RA2	Researcher	RA1
		Nov. 4	RA2	Researcher	RA1
Session 4	School 1	Nov. 5	Researcher	RA1	RA3
	School 2	Nov. 5	Researcher	RA1	RA2
		Nov. 6	Researcher	RA1	RA2
Session 5	School 1	Nov. 7	RA5	RA3	Researcher
	School 2	Nov. 7	RA5	RA2	Researcher
		Nov. 8	RA1	RA2	Researcher
Intervention Types	**School**	**Date of session**	**Research Team Members and Conditions**		
			PI+	**PI-**	**Control**
Session 6	School 1	Nov 12	RA3	Researcher	RA4
	School 2	Nov 12	RA2	Researcher	RA4
		Nov. 13	RA2	Researcher	RA1
Session 7	School 1	Nov. 14	Researcher	RA1	RA4
	School 2	Nov. 14	Researcher	RA1	RA2
		Nov. 15	Researcher	RA1	RA2
Session 8	School 1	Nov. 18	RA1	RA3	Researcher
	School 2	Nov. 18	RA1	RA2	Researcher
		Nov. 19	RA1	RA2	Researcher
Session 9	School 1	Nov. 20	RA3	Researcher	RA1
	School 2	Nov. 20	RA2	Researcher	RA1
		Nov. 21	RA2	Researcher	RA1
Posttest	School 1	Nov. 25	Researcher / RA1		
	School 2	Nov. 25	Researcher / RA1		
		Nov. 26			

Note. One Researcher and five different Research Assistants (RA1 to RA5) took part in the data collection and intervention. Only one research assistant (RA1) was present during the pretest and posttest sessions.

Participants in the PI+ group completed the practice exercises for multiplying fractions. Participants were prompted at the beginning and at the end of the interventions to reflect on the relationship between the fraction in the problem and the fraction obtained following the solution (i.e., the product). Specifically, participants were taught how to multiply a whole number by a fraction using a repeated addition procedure and were prompted to reflect on how the fraction in the problem and the product compared. Three reflection prompts were asked, "When you multiply 'a by b/c' (e.g., 5 x 1/2) and find the product to be 'ab/c' (e.g., 5/2), what do you notice that is different between the fraction at the beginning of the equation (i.e., 1/2) and the final answer (i.e., 5/2)? Which of the two fractions is greater (or larger)? Think about why that might be."

The first three practice sessions began with two demonstration problems, which were completed by me or a research assistant. In the first demonstration, participants were shown how to multiply 3 x 1/7 by performing repeated addition on a Smartboard (or a blackboard) in front of the class (see Figure 6).

Figure 6

Procedure Taught for Multiplying a Whole Number by a Fraction

	Example problem	Explanation
Step 1	$3 \times \frac{1}{7} =$	
Step 2	$3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$	Multiplying 3 by 1/7th is like adding 1/7th 3 times
Step 3	$3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}$	When adding fractions, when the denominator is the same (and we will only work with denominators that are the same for now), all we have to do is add the numerators together.

In the second demonstration, $4 \times 2/6$, a fourth step was introduced where the product would need to be simplified (see Figure 7). For Step 3, the answer obtained was 8/6 (e.g., $2/6 + 2/6 + 2/6 + 2/6 = 8/6$). Participants were shown that, whenever possible, the simplest fraction needs to be found by dividing the numerator and denominator by a common factor. In this case, both the numerator and denominator can be divided by 2 to get 4/3 (e.g., (8 ÷ 2)/ (6 ÷ 2) = 4/3; Step 4). The participants in the two practice conditions were told that simplification of fractions could be performed by dividing by 2, 3, 4, or 5. They were also asked to practice the following two exercises on their own: 3 x 2/7 (no simplifying needed) and 3 x 5/10 (with simplifying) one at a time, using the script presented below. Each of the first three sessions ended at this point.

Figure 7

Procedure taught for Multiplying a Whole Number by a Fraction and Simplifying

Step 1: $4 \times \frac{2}{6} =$	
Step 2: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$	Multiplying 4 by 2/6th is like adding 2/6th 4 times
Step 3: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8}{6}$	When adding fractions, when the denominator is the same, all we have to do is add the numerators together.
Step 4: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$	Sometimes you have to simplify the answer you get by dividing it by a common factor. We will only use 2, 3, 4, or 5. So here, what number can we use to simplify the fraction? I will try 2. If I divide 8 by 2 and 6 by 2, what do I get?

For the remaining six practice sessions (i.e., from the fourth practice session onward), the two original procedural demonstrations (e.g., 3 x 1/7 and 4 x 2/6) were rewritten on the board at the beginning of each session. Verbal support was limited to the mechanics of the procedure required to solve the problems such that when questions were raised, the answers were limited to how to perform the procedure. At no point was any conceptual notion addressed. Unlike the first three sessions, no practice problem was provided. Participants were reminded that they would only need to simplify using 2, 3, 4, or 5 for the rest of the exercises. Each practice session ended when all participants had completed their activity sheets.

Participants in the control group took part in guided activities and lessons that introduced geometric vocabulary associated to 3-shapes (cubes, prisms, pyramids), their construction, and their nets. These activities did not require any knowledge of fraction concepts or procedures. All participants met with me or one of the research assistants approximately three times a week over a three-week period, for a total of nine sessions. Each practice session lasted approximately 30 minutes.

Posttests

The posttests followed the 3-week intervention with all participants completing identical versions to the four pretests. Participants in each class performed the posttests again in their respective classroom settings or in the cafeteria. The same research assistant and I, who were present at pretest, were again present at posttest. All administration procedures were identical to those implemented at pretest. Later that same week, I tested the individuals who had missed the posttest intervention in the library. The following week, two more students were tested by their teacher in their own classrooms.

Analysis Plan

Three different one-way ANOVAs were conducted on the gain scores (posttest - pretest) from the procedural knowledge measure (the Fraction Multiplication task) and the conceptual knowledge measures (the number line estimation tasks, subscales 1 and 2). Fisher's Least Significant Difference (LSD) post hoc pairwise comparisons (Levin et al., 1994) were used to decompose any main effects of condition. Specifically, for each significant effect of condition, three pairwise comparisons were used to answer the research questions: (a) Practice Intervention with Prompt (PI+) vs Control, (b) Practice Intervention without Prompt (PI-) vs Control, and (c) Practice Intervention with Prompt (PI+) vs Practice Intervention without Prompt (PI-).

A mixed ANOVA was performed on the gain scores (posttest – pretest) from the fraction magnitude measure (the Fraction Magnitude Comparison task) on accuracy and the quality of the participants' justifications for their responses. I evaluated the quality of the strategies applied for three categories of fractions (CD, CN, LNSD) to ascertain whether conditions were a contributing factor. Fisher's Least Significant Difference (LSD) post hoc pairwise comparisons (Levin et al., 1994) were used to decompose any main effects of condition. For each significant effect of condition, three pairwise comparisons were used to answer the research questions: (a) Practice Intervention with Prompt (PI+) vs Control, (b) Practice Intervention without Prompt (PI-) vs Control, and (c) Practice Intervention with Prompt (PI+) vs Practice Intervention without Prompt (PI-).

Results

The research questions that guided the analyses are threefold: (1) Will practicing a procedure to multiply a whole number by a fraction (i.e., performing repeated addition) result in acquiring greater procedural accuracy in fraction multiplication (as measured by the accuracy of their answers)? (2) Will practicing a procedure to multiply a whole number by a fraction (i.e., performing repeated addition) result in greater conceptual knowledge of fraction magnitude (as measured by the accuracy of their answers on two fraction lines (0–1 and 0–2) activities, a fraction magnitude comparison activity which measured the accuracy of their answers, and the quality of the students' justifications)? (3) Will reflecting on how the magnitude of a fraction is affected when multiplying a whole number by a fraction develop greater knowledge of fraction concepts compared to practicing the multiplication procedure without reflection (measured in the same way as for question 2)?

Fraction Multiplication

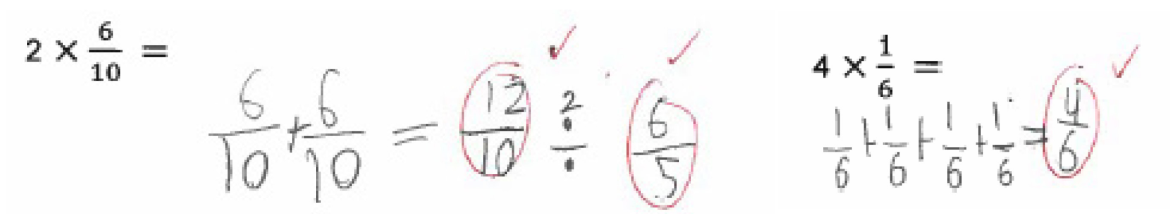
To address the first research question, a one-way analysis of variance (ANOVA) was conducted to determine if there were any condition differences on participants' accuracy when performing the multiplication procedure. The dependent variable was mean proportion correct at posttest and the independent variable was *condition* with three levels. The three levels were practice intervention with reflection prompts (PI+), practice intervention without reflection prompts (PI-), and control. Pretest scores were not used because participants were at floor at pretest. The ANOVA revealed a statistically significant main effect of condition, $F(2, 53) = 1270.29$, $p < .001$, $\eta_p^2 = .98$).

Post hoc comparisons using the Fisher's Least Significant Difference (LSD) test indicated that both the PI+ group ($M = .98$, $SD = .08$) and the PI- group ($M = .99$, $SD = .06$) scored significantly higher than the Control group ($M = .01$, $SD = .06$), $p < .001$. No significant difference was observed between the PI+ and PI- groups ($p = .58$). The results indicate that performing fraction multiplication activities through regular practice results in higher accuracy than no practice, whether they reflect on their actions or not. Examples of typical answers for the three conditions can be seen in Figure 8

Figure 8

Sample Student Multiplication Solutions at Posttest.

Practice with Intervention



Control

Note. The samples were selected from all three conditions and are examples of the vast variations of answers collected. No notable differences were observed between the two intervention groups.

The following three ANOVA analyses were conducted to answer the remaining two research questions: Will practicing a procedure to multiply a whole number by a fraction result in greater conceptual knowledge of fraction magnitude? Will reflecting on how the magnitude of a fraction is affected when multiplying a whole number by a fraction help students to develop greater knowledge of fraction concepts compared to practicing the multiplication procedure without reflection?

Fraction Number Line 0–1

The *Percent Absolute Error* (PAE) means and gains scores of the two subscales of the fraction estimation tasks (at pre- and posttest are presented as a function of condition in Table 4 and the gains in PAE mean scores in Table 5. A one-way ANOVA examined the effect of the interventions on the participants' abilities to properly locate a given fraction on a 0–1 number line. To account for the small sample size, gain scores were computed and used in the analyses. The dependent variable was *Percent Absolute Error* (PAE; a lower score indicates a more accurate response), and the independent variable was condition with the same three levels as named above (PI+, PI-, and Control). The ANOVA revealed no statistically significant effect of condition, $F(2, 53) = 0.32$, $p = .73$

Table 4

Percent Absolute Error Mean Scores on the Fraction Number Line Tasks

	Pretest FNL 0–1		Posttest FNL 0–1		Pretest FNL 0–2		Posttest FNL 0–2	
Condition	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
PI+	.21	.12	.23	.15	.23	.10	.24	.11
PI-	.15	.14	.18	.18	.27	.08	.23	.09
Control	.23	.13	.22	.15	.27	.09	.23	.09

Note. N= 56 (n = 20 for PI+, n = 19 for PI-, n = 17 for Con). PI+ = Practice Intervention with Reflection Prompts; PI- = Practice Intervention without Reflection Prompts (PI-); Control; FNL = Fraction Number Line (Subscales 1 and 2).

Table 5

Gains in Percent Absolute Error Mean Scores on the Fraction Number Line Tasks

	FNL 0–1		FNL 0–2	
Condition	*M*	*SD*	*M*	*SD*
PI+	.02	.12	.01	.08
PI-	.03	.11	-.03	.08
Control	.00	.13	-.04	.06

Note. N= 56 (n = 20 for PI+, n = 19 for PI-, n = 17 for Con). PI+ = Practice Intervention with Reflection Prompts; PI- = practice Intervention without Reflection Prompts (PI-); Control; FNL = Fraction Number Line (Subscales 1 and 2).

Fraction Number Line 0–2

Here again, the *Percent Absolute Error* (PAE) means and gains scores of the fraction estimation tasks (Subscale 2) at pre- and post-test are presented as a function of condition in Table 4, and the gains in PAE mean scores in Table 5. This one-way ANOVA was conducted to examine the effect of the interventions on the participants' abilities to properly locate a given fraction on a 0–2 number line. The dependent variable remained PAE, and the independent variable was condition with the same three levels as named above (PI+, PI-, Control). No statistically significant main effect of condition was revealed, F (2, 53) = 2.19, p = .12.

Fraction Magnitude Comparison

The means and gains scores of the Fraction Magnitude Comparison task at pre- and post-test are presented as a function of condition in Table 6 and 7. This mixed-design ANOVA was conducted to examine the effect of the intervention on the participants' abilities to improve the quality of their justification for selecting the larger of the two fractions. The test was a 3 (condition: PI+, PI-, Control) x 3 (fraction type: CD, CN, LNSD) mixed design ANOVA on the mean gain strategy score. As in the previous ANOVA, the between-subject variable was condition and the within-subjects variable was fraction type with three levels (CD, CN, and LNSD). The dependent variable was the gain strategy score. The higher the mean gain score observed, the greater the improvement in the quality of the participants' response. A negative number indicates that the participants' score at pretest was greater than at posttest. The analysis revealed no statistically significant main effect of condition, F (2, 52) = 0.20, p = .82.

Table 6

Fraction Magnitude Mean Strategy Scores

	CD Pretest		CD Posttest		CN Pretest		CN Posttest		LNSD Pretest		LNSD Posttest	
Condition	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
PI+	1.63	0.72	1.28	0.87	1.21	0.74	1.17	0.79	1.37	0.81	1.31	1.07
PI-	1.37	0.80	1.46	0.79	1.13	0.79	1.05	0.83	1.29	0.75	1.29	0.87
Control	1.88	0.62	1.69	0.84	1.49	0.54	1.52	0.68	1.48	0.70	1.57	0.82

Note. N= 55 (n = 20 for PI+, n = 18 for PI-, n = 17 for Con). PI+ = Practice Intervention with Reflection Prompts; PI- = Practice Intervention without Reflection Prompts (PI-); Control; CD = common denominators; CN = common numerators; LNSD = large numerator and small denominator.

Table 7

Gains in Fraction Magnitude Strategy Scores

Condition	CD		CN		LNSD	
	M	*SD*	*M*	*SD*	*M*	*SD*
PI+	- .34	1.02	- .04	1.02	- .06	1.12
PI-	.12	0.97	- .04	0.90	- .02	0.91
Control	- .20	0.97	.03	0.67	.10	0.87

Note. N= 55 (n = 20 for PI+, n = 18 for PI-, n = 17 for Con). PI+ = Practice Intervention with Reflection Prompts; PI- = practice Intervention without Reflection Prompts (PI-); Control; CD = common denominators; CN = common numerators; LNSD = large numerator and small denominator

Changes in Justification Types at Pretest and Posttest

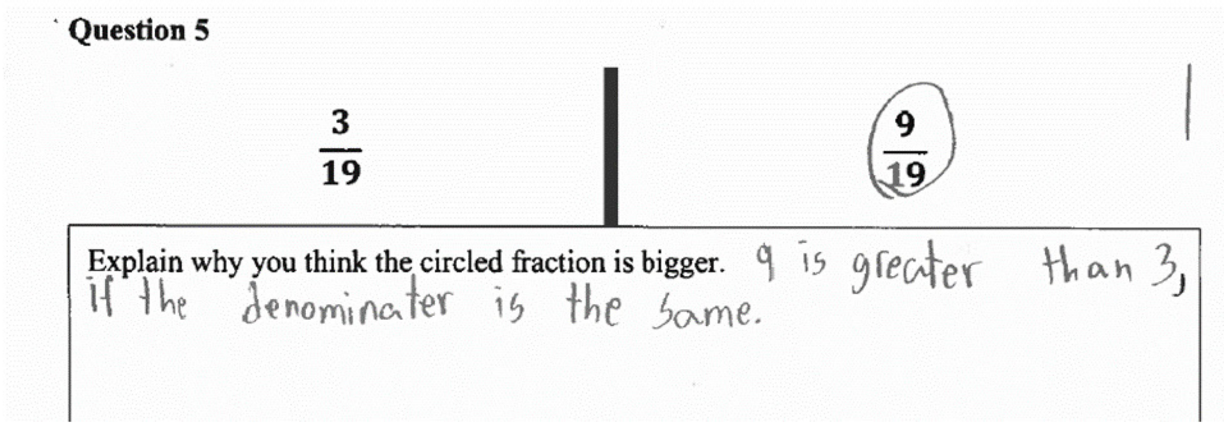
Common Denominator

To evaluate the trends in justification types within and between the conditions over time for fractions with common denominators, a stacked bar graph was used to compare the mean proportion of each strategy type in the three conditions at pretest and posttest (see Figure 9). Five justification types were identified for this study: *logical necessity* (LN), *intermediate steps* (IS), *usually correct* (UC), *questionable* (Q; Fazio et al., 2016) and *uninterpretable* (UN; see Table 3). The predominant strategy used at pretest for all conditions was the Usually Correct strategy (PI+ [*M* = .75], PI- [*M* = .67], Control [*M* = .76]). Even though the most used strategy justification type remained UC at posttest, other justification types shifted to the UN strategy. For example, the mean proportion of the UN strategy in the reflection with prompts group (PI+) grew from a proportion of 7% at pretest to 34% at posttest (see a student example in Figure 10).

Figure 10

Example of a Student's Justification on the Fraction Magnitude Task

Pretest – Student A

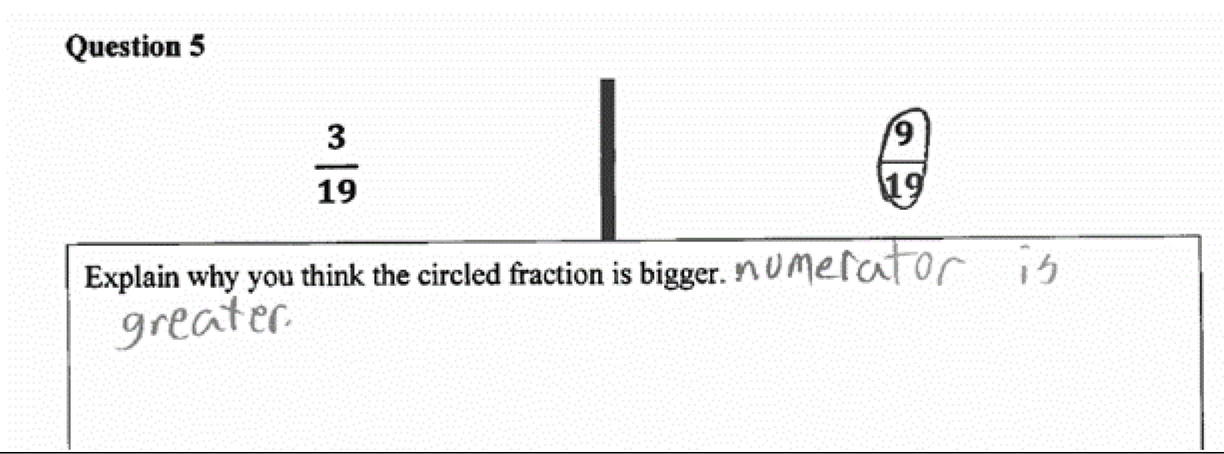


Question 5

$\frac{3}{19}$ | $\frac{9}{19}$

Explain why you think the circled fraction is bigger. 9 is greater than 3, if the denominater is the same.

Posttest – Student A



Question 5

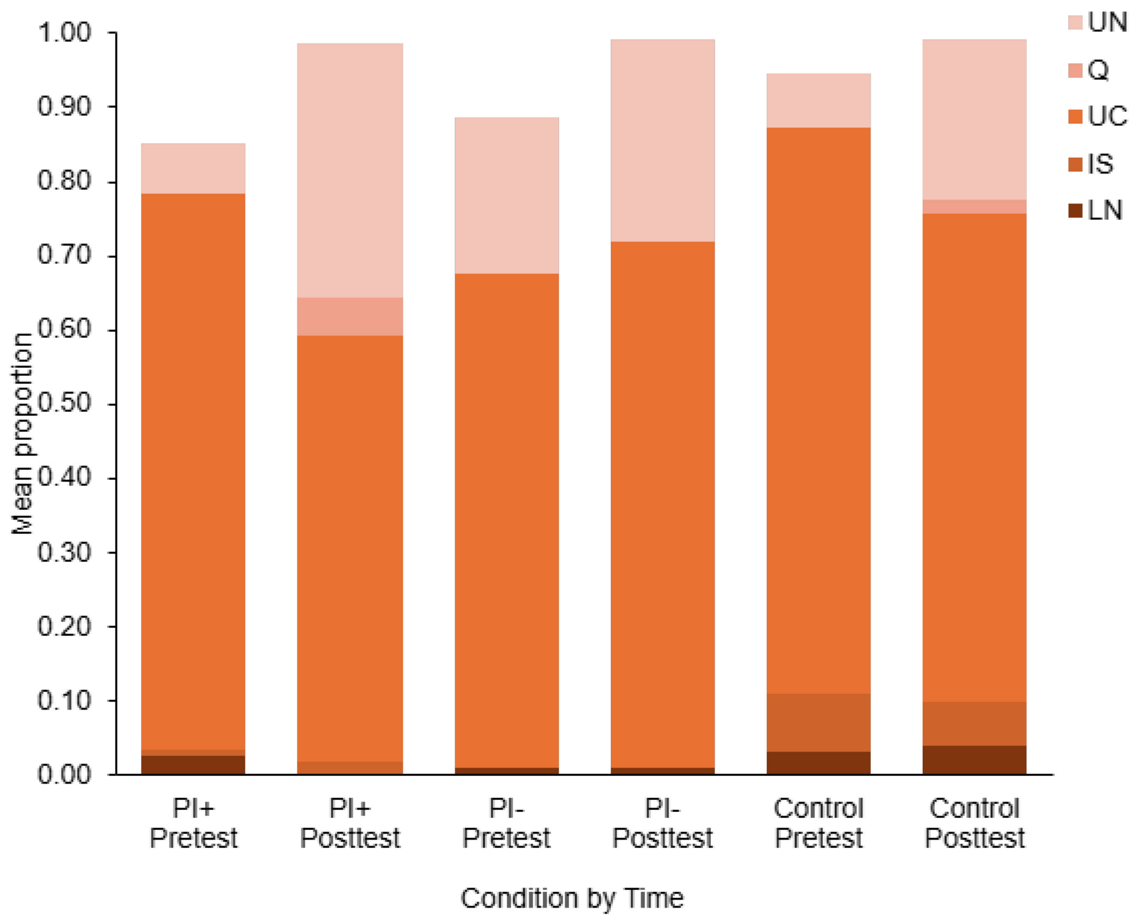
$\frac{3}{19}$ | $\frac{9}{19}$

Explain why you think the circled fraction is bigger. numerator is greater.

Note. Both examples were collected from the same student.

Figure 9

Mean Proportions of Justification Types for Fractions with Common Denominators



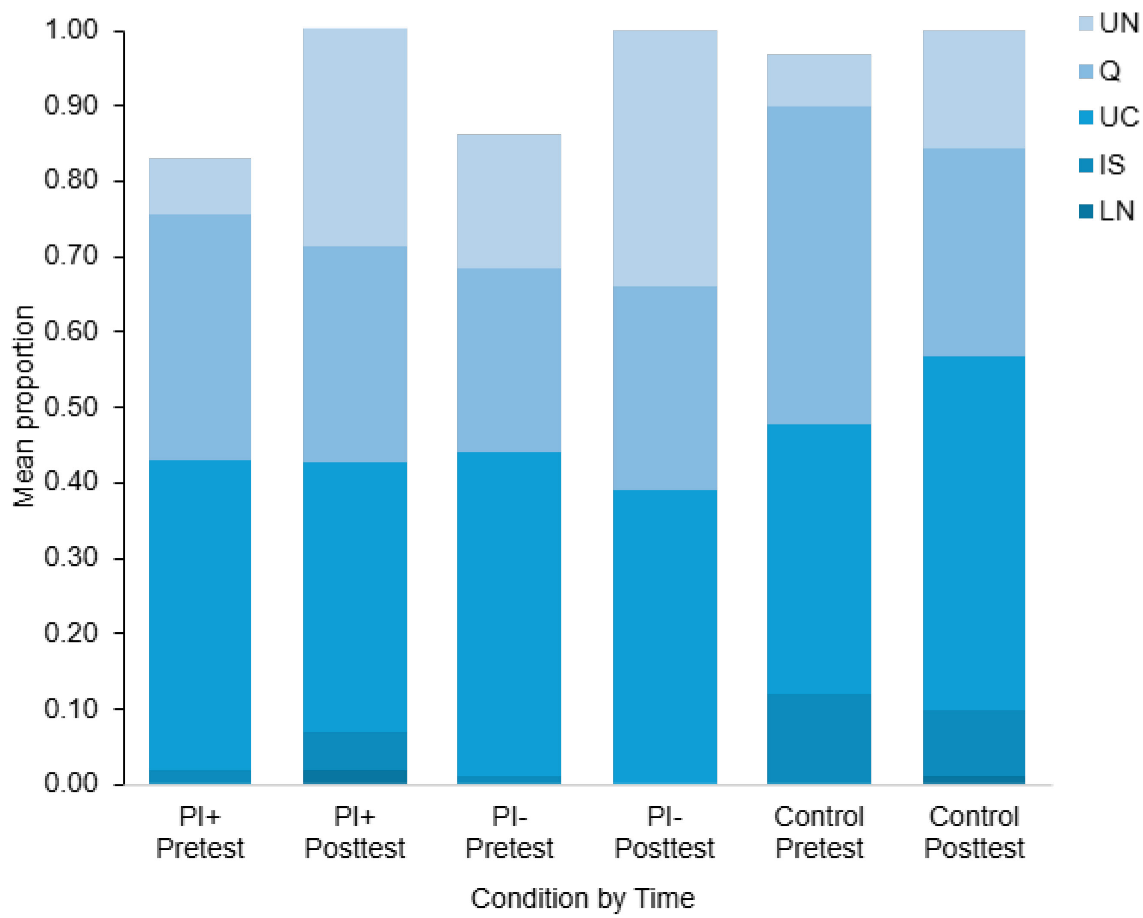
Note. LN = Logical Necessity; IS = Intermediate Steps; UC = Usually Correct; Q = Questionable; UN = Uninterpretable (UN).

Common Numerators

To evaluate the trends in use of the justification types within and between the conditions at pretest and posttest for fractions with common numerators, a stacked bar graph was used to compare the mean proportion of each strategy type in the three conditions (see Figure 11). The predominant strategy used at pretest for the practice with and without reflection groups was again the Usually Correct strategy, with respective mean proportions of 41% and 43%. The preferred strategy in the control group at pretest was the *Questionable* strategy (*M* = 0.42). All three conditions saw an increase in the mean proportion of the Uninterpretable strategy from pretest (PI+ [*M* = .08], PI- [*M* = .18], Control [*M* = .07]) to posttest (PI+ [*M* = .34], PI- [*M* = .34], Control [*M* = .16]).

Figure 11

Mean Proportions of Justification Types for Fractions with Common Numerators



Note. LN = Logical Necessity; IS = Intermediate Steps; UC = Usually Correct; Q = Questionable; UN = Uninterpretable (UN).

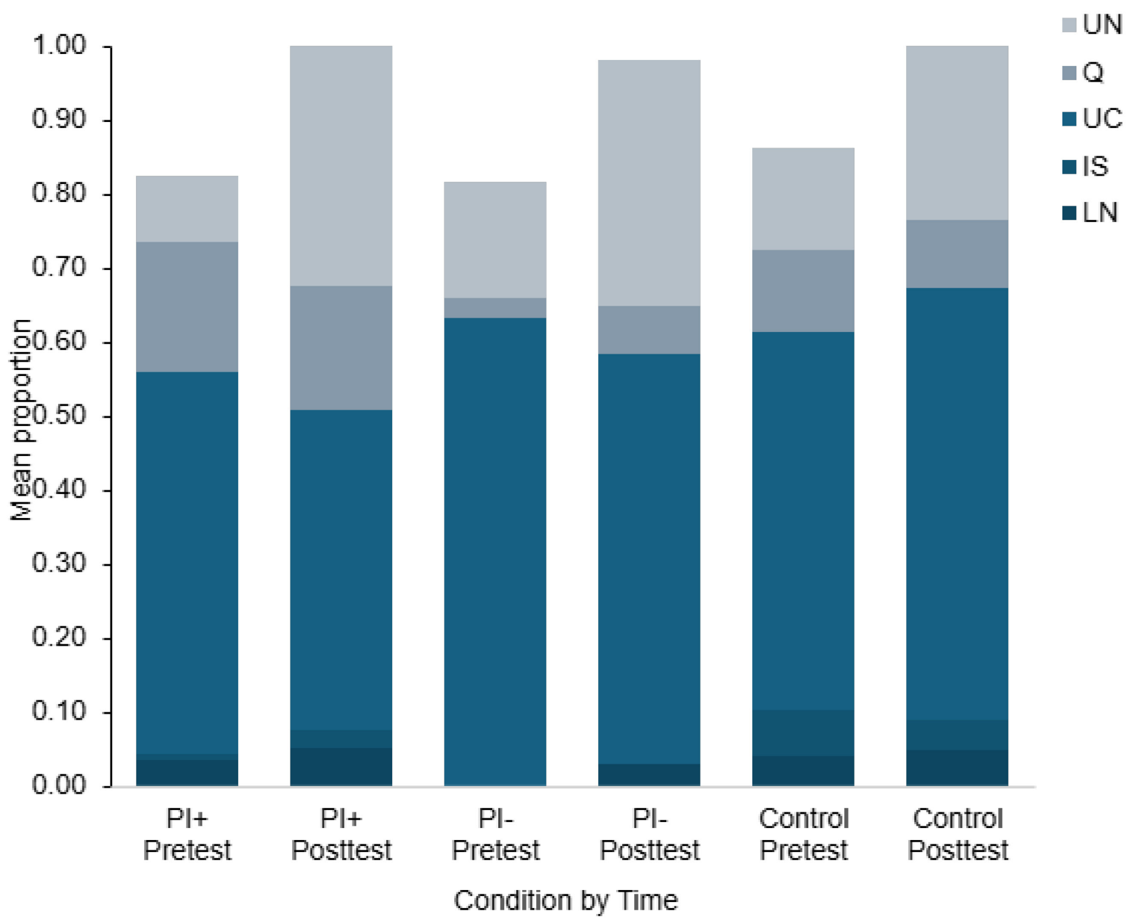
Large Numerators and Small Denominators

To evaluate the trends in the use of justification types within and between the conditions over time for fractions with a large numerator and a small denominator, a stacked bar graph was used to compare the mean proportion of each strategy type in the three conditions (see

Figure 12). The predominant strategy used at pretest for all conditions was once again the Usually Correct strategy (PI+ [*M* = .52], PI- [*M* = .63], Control [*M* = .51]). The most commonly used strategy justification type remained UC at posttest but, as observed with the fractions with common denominators, many justification types shifted to the UN strategy. For example, the mean proportion of justification type for the reflection with prompts group grew from a mean proportion of 9% at pretest to 33% at posttest.

Figure 12

Mean Proportions of Justification Types for Fractions with Large Numerators and Small Denominators



Note. LN = Logical Necessity; IS = Intermediate Steps; UC = Usually Correct; Q = Questionable; UN = Uninterpretable (UN).

Discussion

Objectives of the Study

The main objective of this study was to investigate whether Grade 5 students, who are learning a procedure for multiplying a whole number by a fraction using repeated addition, would develop conceptual knowledge of fractions, and if self-explaining, which cannot take place without reflection, was a contributing factor. First, I considered whether practicing a procedure to multiply a whole number by a fraction, on its own, would improve students' learning of the procedure taught. Second, I investigated the impact of practice on the development of conceptual understanding of fraction magnitude. Finally, I measured whether reflecting on how the multiplication procedure modified the product improved the students' conceptual understanding of fraction magnitude. The interventions were performed without the students receiving any instruction on the conceptual nature of the procedure.

According to Schneider et al. (2017), number line and magnitude comparison activities are deemed to be complementary in their analysis of fraction magnitude on two scales as they assess related but different aspects of magnitude understanding. Number lines do so in a spatial fashion, enabling students to visually locate the number they seek, in this case a fraction, using relational factors or landmark strategies that are part of the number line. On the other hand, magnitude comparison activities do so on an ordinal level, where the size of the numbers involved is considered. For those reasons, I decided to use both magnitude assessment activities to measure students' fraction magnitude understanding before and after the intervention in three ways.

First, I assessed students' ability to locate a fraction on a 0 to 1 number line, and then again on a 0 to 2 number line. Second, I assessed their ability to choose the larger fraction from a pair of fractions. Third, I measured the students' accuracy on this measure, but also whether they had learned three key concepts of fraction magnitude. Students were asked to determine: which of two fractions was the larger when they had a common denominator (the fraction with the greater numerator is the larger fraction; when they had common numerators (the fraction with the smaller denominator is the larger fraction); and when one fraction had a larger numerator but a smaller denominator than the other (the fraction that has the larger numerator and the smaller denominator is the larger fraction).

To investigate if repeated practice of fraction multiplication procedures can affect learning these concepts, students met with me or a research assitant two to three times per week over a four-week period. Three groups were randomly formed for this study. One group received instruction to on a fraction multiplication procedure and then practiced the procedure over the four-week period with reflective prompts; that is, students in this condition were prompted to reflect on the initial fraction and the product obtained. In the other condition, the students did not receive any reflective prompts. Student in both instructional conditions practiced the procedure on over 90 fraction multiplication questions. In the control condition, students engaged in geometry activities and were not exposed to any concepts of fractions or fraction multiplication procedures.

Predictions

As predicted, my results showed that practicing exercises on multiplying a whole number by a fraction resulted in improved accuracy over the 4-week intervention period. As an experienced teacher such a high degree of success (nearing 100%) for a group of students randomly selected was quite surprising, notwithstanding the fact that this result is supported by other similar findings indicating that repeated practice allows students to master the procedure taught (Canobi, 2009; McNeil et al., 2012; Osana & Pitsolantis, 2013). None of the students in the control group learned the procedure, which was expected because they did not receive any instruction on how to multiply a whole number by a fraction.

Contrary to what was expected, practicing a procedure for multiplying a whole number by a fraction did not support the students in developing their conceptual knowledge of fraction magnitude. Performance on both Fraction Number Line tasks and the Fraction Magnitude

Comparison task reported no significant differences among the three conditions. In essence, the repeated procedural activities performed in this study did not lead students to develop greater conceptual understanding of fractions.

This result supports what had been reported by Rittle-Johnson (2006) who had also observed that self-explanation had not improved conceptual knowledge when students were encouraged to self-explain when learning about mathematical equivalence. As in this present study, elicited self-explanation had a positive impact on learning procedures but not on conceptual knowledge. Furthermore, her study showed that students who had acquired the procedure with self-explanation were also able to apply it in various, new situations and self-explanation also enabled longer retention of the procedure. These other findings were not replicated within this study as it went beyond the scope of my research.

Because reflection is an invisible process, determining its impact can only be confirmed when differences between two distinct groups, one being prompted to reflect and the other not, express different behaviors on similar tests. Given the lack of differences observed between the two intervention groups related to their conceptual understanding, and despite the reflection questions provided, it is difficult to determine whether the lack of a self-explanation effect resulted from an inability to reflect, a lack of motivation, or even not engaging in self-explanation at all. Even more surprising is that the students in the intervention group (PI+ and PI-) did not perform any better than the control group who never received any procedural instruction on fraction multiplication, thus supporting the idea that the procedure learned had little effect on the students' conceptual development in the short term. The lack of difference between the three groups on the fraction magnitude tasks was observed through the descriptive analysis of students' application of the three specific concepts, assessed through their justifications on the fraction magnitude measure, as no significant differences were observed between the three groups.

The cause for such little difference between all three groups may stem from the students' lack of familiarity with the justification process. In their study, Fuchs et al. (2016) found that supporting self-explanation improved the students' fraction magnitude comparisons and the quality of their explanations. This finding is critical as it distinguishes the self-explaining approach used in this present study, elicited self-explaining, from the one used by Fuchs: The students in her study were taught how to self-explain through a four-step process. Students had to identify if fractions had common denominators, common numerators, or different numerators and denominators; comment on the quality of a drawing representing fractional quantities; associate an image of a fraction with its numerical value and describe if the fractions drawn had same-size parts or not; and provide a written explanation indicating why one fraction was larger than another. Such an approach was not possible in this current study, as the line between conceptual and procedural knowledge would have been blurred. That is, using Fuchs et al.'s (2016) approach would have potentially introduced conceptual notions during the instructional activity inadvertently introducing a confounding variable. To ensure that self-explaining occurred without providing conceptual knowledge to the students, I used elicited self-explaining (prompting a student by asking them a chosen question). Given that students were not familiar with this type of activity, perhaps it would have been more beneficial to develop self-explaining within another procedural-conceptual framework instead of the one that was used in this study.

Contributions

This study adds further evidence to the existing literature supporting the value of developing procedural skills in children through practice, as evidenced by the students in my study acquiring procedural mastery after four weeks of practice and also supports Rittle-Johnson (2016)'s finding on the impact of elicited self-explaining on the development of procedural learning. It demonstrates the effectiveness of repeatedly practicing activities with minimal guidance and its contribution towards the development of mathematical proficiency, as demonstrated in other studies (Canobi, 2009; McNeil et al., 2012; Osana & Pitsolantis, 2013). However, our results did not show, as in Bailey et al. (2015), that practicing arithmetic exercises assisted in linking the arithmetic problems they performed and fraction magnitude. As such, conceptual development appears to require a more explicit approach, beyond elicited self-explaining, to connect it to procedural knowledge. The supported self-explaining approach used by Fuchs et al. (2016) may sit on the cusp of conceptual and procedural learning and enable students and teachers alike to bridge that divide.

My results show that the students did not improve in their knowledge of the three concepts as the instances of justification types observed from pretest to posttest did not change. The students' explanations often lacked clarity, which posed a key issue in understanding their justifications. This may be the result of what Star (2005) referred to as a "superficial knowledge of procedures." Students can do the procedure but do not know why the procedure works and cannot explain how multiplying a whole number by a fraction changes the magnitude of the resulting fraction.

One element of the experimental protocol that may have impacted the results of the study is my suggestion to the students in all condition to use drawings to justify their answers. At pretest, students were instructed to use any means to justify their answers, such as drawings (e.g., number lines, pie charts, etc.), and many used the same strategy at posttest because students often rely on the same strategy when faced with more demanding cognitive challenges (Lovell, 2020). This process may have resulted in the *Usually Correct* strategy (in which drawings were included) becoming the dominant strategy used by the students at both time points. Perhaps having the opportunity to ask the students to clarify their answers in a one-on-one interview would have resolved the issue and provided clearer information before the data were coded. An unexpected consequence of both the lack of clarity of the students' responses and the coding used can be seen in the following example. A student in the practice with reflection prompts group identified 9/19 as being larger than 3/19, at pretest, with the following justification "9 is greater than three, if the denominater (sic) is the same". However, at posttest for the same question, he wrote "numerator is greater". The first answer was coded as *Logical*

Necessity with an associated scoring of 4 but was coded as *Usually Correct* with a scoring of 2 for his posttest answer. This resulted in a drop in the quality of justification for that student as the scoring fell from 4 to 2 which would erroneously indicate a drop in conceptual knowledge. I speculate that the second answer was of a lesser score because the student inferred that I knew what he meant. Had the opportunity to clarify his answer been present, perhaps his second answer would have been more complete.

Strengths and limitations

First, to enhance internal validity, student selection for the three groups, PI+, PI- and Control, was randomized within each classroom. Second, to control for experimenter bias, supervision of the students during the study was structured such that no one individual supervised any group on a regular basis (see Figure 5); I alternated between all three groups with the research assistants throughout the study. The only times the same individuals supervised the three groups was during the administration of the measures at pre- and posttest to limit bias and improve the credibility of the results. A third element of the data collection process that aimed to strengthen the conclusions was including a justification component to the fraction magnitude comparison task. The purpose was to understand why students selected one fraction over another and move beyond the simple accuracy component of the task. Even though this may not have provided the anticipated clarity to the results obtained, it did reveal that this extra step should be incorporated in the research process to truly understand the quality of student thinking as explained above.

The study's lack of power, duration, and coding system may have contributed to its limitations. The number of students who participated in this study was far below the required number to achieve 80% power. Fifty-six students participated in the study (N = 56) but a power analysis determined that 53 participants per group (N = 159) were required to achieve 80% power with a medium effect size. The number of students I was able to recruit therefore limits the confidence with which conclusions can be drawn from this study, as the risk of Type II errors was increased. Unfortunately, the lack of responses from potential participants limited the sample size to 56.

Another weakness of this study is that the intervention phase only lasted nine days which may not have been long enough for transfer of understanding to take place between procedural and conceptual knowledge. Comparatively, Fuchs et al.'s (2016) study included a 12-week intervention period and consisted of a total of 36 intervention sessions. This span of time is significantly longer than the four weeks and nine interventions performed in this study. I speculate that the short duration of the intervention in the present study contributed to limiting the effect of the procedure on the development of the students' conceptual knowledge.

As mentioned earlier, the coding system used to identify the quality of the justification strategies used may have underreported any positive impacts of the interventions and especially missed the differences between the two practice intervention groups. To improve the quality of the qualitative information gathered, working directly with the students to clarify their answers would have helped reduce unclear responses and yield data that more accurately reflected their thinking. Despite the limitations mentioned above, the results of this study offer an interesting insight into the development of procedural knowledge in children. The findings can also serve as a starting point for further investigation into the relation between procedural and conceptual knowledge and the tools needed to clarify this relation.

Educational implications

An important implication of this study is that when students are taught a procedure and are allowed to practice it on a regular basis, they will develop procedural proficiency over a relatively short period of time. In other words, "practice makes perfect." It does not, however, mean that they will automatically develop an understanding of the underlying concepts supporting the procedure or explain why it works. It appears to be the responsibility of the teacher to bridge the gap between procedures and concepts. To act on this responsibility is important if we want as this study shows that students require procedural work, repetition, and proper teacher guidance to move towards mathematical proficiency. As such, teachers themselves need procedural knowledge, but they also need to understand the concepts that underlie them and how to assist students in connecting procedures to the concepts behind them. Students do not appear to connect procedural knowledge to conceptual understanding on their own (Rittle-Johnson, 2006), so teacher training should include instructional strategies that develop teaching techniques encouraging student thinking and help bridge the gap between these two types of knowledge.

Another observation resulting from this study is the significance of whole-number bias in students' transition from numerical knowledge to the comprehension of fractions and may be pertinent for teachers who are tasked to introduce fractions concepts as it can help them identify the difficulties that students are facing in understanding them. The whole-number bias has been well researched and has been described as having an adverse effect on learning fractions (Siegler et al., 2012; Braithwaite & Siegler, 2017). This bias may have had an undue and unfortunate influence on the results of this study. I chose three fraction types to trigger the development of fraction concepts in students in the creation of the multiplication activities used during the interventions. The students compared fractions that had common denominators or common numerators or fractions where one fraction had a larger numerator but a smaller denominator than the other. It is possible that when comparing fractions with common denominators, students would pick the correct answer, that is, the fraction with the larger numerator, simply because it is the larger of the two numerators without truly understanding why. In essence, students with a whole-number bias might simply ignore the denominator because they do not recognize the purpose of the number “below” the numerator. Replicating that analysis with fractions that have different numerators and denominators, without consideration for the size of the denominator, would yield the same result and students would correctly identify the larger fraction simply by finding the larger numerator and disregarding the denominator. As for fractions with common numerators, if whole number bias were at play, children would systematically be incorrect in their selection of the greater fraction as they would select the fraction with the largest denominator. It is unclear if whole number bias had any role in the students' accuracy on the fraction magnitude measure, but such a bias may have affected the transferability of knowledge from procedure to concepts.

Despite limitations resulting from such things as a small sample size, the short intervention period, and potential effects of whole-number bias, the findings offer important implications for mathematics instruction. For students to reach mathematical proficiency, educators must intentionally connect procedures to concepts and employ teaching strategies that foster this connection (Fuchs et al., 2016; Hansen et al., 2015; Rittle-Johnson & Schneider, 2014). This research suggests that while "practice makes perfect" in terms of procedural fluency, meaningful conceptual understanding of mathematics requires targeted instructional support and careful attention to operational abilities, like the students' abilities to clearly express their thinking, as well as cognitive challenges, like whole-number bias. Future studies should further explore effective ways to support the development of conceptual knowledge, ensuring that procedural mastery is accompanied by genuine mathematical understanding.

Conclusion

This study aimed to further clarify the path between procedural and conceptual knowledge in mathematics education among Grade 5 students. My results showed that repeated practice with fraction multiplication procedures significantly improved students' procedural mastery, but that there was no corresponding development in their conceptual understanding of fraction magnitude. Further, adding reflective prompts did not result in noticeable improvements in the students' conceptual knowledge, suggesting that procedural proficiency alone does not naturally lead to a deeper understanding of underlying mathematical concepts. The lack of transfer between practiced procedures and conceptual knowledge in this study does not negate the results observed in previous research (Hansen et al., 2017; Rittle-

Johnson et al., 2017) but underscores the need for further research to clarify the relation that exists between procedures and concepts.

References

Bailey, D. H., Hansen, N., & Jordan, N. C. (2017). The codevelopment of children’s fraction arithmetic skill and fraction magnitude understanding. *Journal of Educational Psychology*, *109*(4), 509–519. https://doi.org/10.1037/edu0000152

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology*, *113*(3), 447–455. https://doi.org/10.1016/j.jecp.2012.06.004

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., Gersten, R., & Siegler, R. S. (2015). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68–83. https://doi.org/10.1016/j.jecp.2014.08.006

Boaler, J. (2015). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. John Wiley & Sons.

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2013). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118. https://doi.org/10.1016/j.jecp.2013.09.001

Braithwaite, D. W., & Siegler, R. S. (2017). Developmental changes in the whole number bias. *Developmental Science*, *21*(2). https://doi.org/10.1111/desc.12541

Canobi, K. H. (2008). Concept–procedure interactions in children’s addition and subtraction. *Journal of Experimental Child Psychology*, *102*(2), 131–149. https://doi.org/10.1016/j.jecp.2008.07.008

DeCaro, M. S. (2016). Inducing mental set constrains procedural flexibility and conceptual understanding in mathematics. *Memory & Cognition*, *44*(7), 1138–1148. https://doi.org/10.3758/s13421-016-0614-y

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2015). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology Learning Memory and Cognition*, *42*(1), 1–16. https://doi.org/10.1037/xlm0000153

Fuchs, L. S., Malone, A. S., Schumacher, R. F., Namkung, J., Hamlett, C. L., Jordan, N. C., Siegler, R. S., Gersten, R., & Changas, P. (2015). Supported self-explaining during fraction intervention. *Journal of Educational Psychology*, *108*(4), 493–508. https://doi.org/10.1037/edu0000073

Fyfe, E. R., DeCaro, M. S., & Rittle-Johnson, B. (2014). An alternative time for telling: When conceptual instruction prior to problem solving improves mathematical knowledge. *British Journal of Educational Psychology*, *84*(3), 502–519. https://doi.org/10.1111/bjep.12035

Hamdan, N., & Gunderson, E. A. (2016). The number line is a critical spatial-numerical representation: Evidence from a fraction intervention. *Developmental Psychology*, *53*(3), 587–596. https://doi.org/10.1037/dev0000252

Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development*, *35*, 34–49. https://doi.org/10.1016/j.cogdev.2015.02.001

Hansen, N., Rinne, L., Jordan, N. C., Ye, A., Resnick, I., & Rodrigues, J. (2017). Co-development of fraction magnitude knowledge and mathematics achievement from fourth through sixth grade. *Learning and Individual Differences*, *60*, 18–32. https://doi.org/10.1016/j.lindif.2017.10.005

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology*, *102*(4), 843–859. https://doi.org/10.1037/a0019824

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In Lester, F. K. (Ed) *Second handbook of research on mathematics teaching* *and learning: A project of the National Council of Teachers of Mathematics*. (pp.371-404). IAP

LeFigaro. (n.d.) Citations. Retrieved from https://tinyurl.com/9h6h4s66

Lobato, J., Clarke, D., & Ellis, A. B. (2005). Initiating and Eliciting in Teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, *36*(2), 101–136. https://doi.org/10.2307/30034827

Lovell, O. (2020). *Sweller's cognitive load theory in action*. John Catt Educational.

McNeil, N. M., Chesney, D. L., Matthews, P. G., Fyfe, E. R., Petersen, L. A., Dunwiddie, A. E., & Wheeler, M. C. (2012). It pays to be organized: Organizing arithmetic practice around equivalent values facilitates understanding of math equivalence. *Journal of Educational Psychology*, *104*(4), 1109–1121. https://doi.org/10.1037/a0028997

McNeil, N. M., Fyfe, E. R., & Dunwiddie, A. E. (2014). Arithmetic practice can be modified to promote understanding of mathematical equivalence. *Journal of Educational Psychology*, *107*(2), 423–436. https://doi.org/10.1037/a0037687

Ministère de l'Éducation du Québec. (2001). *Québec Education Program: Elementary Education.* Gouvernement du Québec. https://www.education.gouv.qc.ca

Osana, H. P., & Pitsolantis, N. (2011). Addressing the struggle to link form and understanding in fractions instruction. *British Journal of Educational Psychology*, *83*(1), 29–56. https://doi.org/10.1111/j.2044-8279.2011.02053.x

Rittle-Johnson, B. (2006). Promoting transfer: Effects of Self-Explanation and direct instruction. *Child Development*, *77*(1), 1–15. https://doi.org/10.1111/j.1467-8624.2006.00852.x

Rittle-Johnson, B. (2017). Developing mathematics knowledge. *Child Development Perspectives*, *11*(3), 184–190. https://doi.org/10.1111/cdep.12229

Rittle-Johnson, B., Fyfe, E. R., & Loehr, A. M. (2016). Improving conceptual and procedural knowledge: The impact of instructional content within a mathematics lesson. *British Journal of Educational Psychology*, *86*(4), 576–591. https://doi.org/10.1111/bjep.12124

Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. C. Kadosh & A. Dowker (Eds.), *The Oxford handbook of numerical cognition* (pp. 1118–1134). Oxford University Press. https://doi.org/10.1093/oxfordhb/9780199642342.013.014

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review*, *27*(4), 587–597. https://doi.org/10.1007/s10648-015-9302-x

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology*, *47*(6), 1525–1538. https://doi.org/10.1037/a0024997

Schneider, M., Thompson, C. A., & Rittle-Johnson, B. (2017). Associations of magnitude comparison and number line estimation with mathematical competence. In Lemaire, P. (Ed.). *Cognitive Development from a Strategy Perspective: A Festschrift for Robert Siegler* (1st ed., pp. 100–119). Routledge. https://doi.org/10.4324/9781315200446-7

Siegler, R. S., & Mu, Y. (2008). Chinese children excel on novel mathematics problems even before elementary school. *Psychological Science*, *19*(8), 759–763. https://doi.org/10.1111/j.1467-9280.2008.02153.x

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*(5), 404–411. https://doi.org/10.2307/30034943

Star, J. R., & Stylianides, G. J. (2013). Procedural and Conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science Mathematics and Technology Education*, *13*(2), 169–181. https://doi.org/10.1080/14926156.2013.784828

Appendix A

Script for the Research Assistants

Treatments 1, 2 and 3 – Practice Intervention with Reflection Prompts (PI+)

(Remember nothing conceptual can be hinted to. You are NOT allowed to use drawings to help the students. Students are allowed to use multiplication tables to reduce fractions, if needed.)

Example 1: 3 x 1/7

Researcher: Today we will be learning how to multiply fractions by a whole number. I will do two examples first, and two others with you and then you will practice on your own on with the worksheet you are given.

Researcher: Here is your first example, so look at the board. For now, I just want you to watch and listen. [The researcher writes…]:

$$3 \times \frac{1}{7} =$$

Researcher: Multiplying 3 by 1/7th is like adding 1/7th 3 times [The researcher writes…]:

$$3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$$

Researcher: When adding fractions, when the denominator is the same (and we will only work with denominators that are the same for now), all we have to do is add the numerators together. [The researcher writes…]:

$$3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}$$

Researcher: Notice that the denominator does not change. It stays the same.

FOR THE PROMPT WITH REFLECTION GROUP ONLY

Now I want you to pay attention to a few things. When you multiply 3 and 1/7 and find the product to be 3/7. What do you notice that is different between the fraction at the beginning of the equation (i.e.: 1/7) and the product obtained (i.e.: 3/7)? Which of the two fractions is greater [or larger/bigger]? Think about why that might be.”

Example 2: 4 x 2/6

Researcher: Now, let's try it with 4 x 2/6. Look at the board. For now, I just want you to watch and listen. [The researcher writes…]:

$$4 \times \frac{2}{6} =$$

Researcher: Multiplying 4 by 2/6th is like adding 2/6th 4 times. [The researcher writes…]:

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$$

Researcher: When adding fractions, when the denominator is the same (and we will only work with denominators that are the same for now), all we have to do is add the numerators together. [The researcher writes…]:

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8}{6} =$$

Researcher: Sometimes you have to simplify the answer you get by dividing it by a common factor. We will only use 2, 3, 4, or 5. So here, what number can we use to simplify the fraction? I will try 2. If I divide 8 by 2 and 6 by 2, what do I get?
[The researcher writes…]:

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$$

Do you think I can make 4/3 into even smaller whole numbers? No, so that is our simplified answer.

FOR THE PROMPT WITH REFLECTION GROUP ONLY

Researcher: Now I want you to pay attention to a few things again. When you multiply 4 and 2/6 and find the product to be 4/3. What do you notice that is different between the fraction at the beginning of the equation (i.e.: 2/6) and the product obtained (i.e.: 4/3)? Which of the two fractions is greater [or larger/bigger]? Think about why that might be."

Researcher: Now I will let you practice two questions on your own and we will go over them together. [The researcher presents the equations 3 x 2/7 (no simplifying needed) and 3 x 5/10 (with simplifying) one at a time using the script presented above; he tells the students that, for all the exercises we do from now on, if they need to simplify, they will only need to divide by 2, 3, 4, or 5.]

FOR THE PROMPT WITH REFLECTION GROUP ONLY

[For both exercises, remind the students to pay attention to a few things. Ask them, “What do you notice that is different between the fraction at the beginning of the equation and the final answer? Which of the two fractions is greater [or larger/bigger]? Think about why that might be.” NO ANSWERS ARE PROVIDED.

Treatments 1, 2 and 3 – Practice Intervention without Reflection Prompts (PI-)
(Remember nothing conceptual can be hinted to. You are not allowed to use drawings to help the students. Students are allowed to use multiplication tables to reduce fractions, if needed.)

Example 1: 3 x 1/7
Researcher: Today we will be learning how to multiply fractions. I will do two examples first, and two others with you and then you will practice on your own on with the worksheet you are given.
Researcher: Here is your first example, so look at the board. For now, I just want you to watch and listen. [The researcher writes…]:

$$\mathbf{3} \times \frac{\mathbf{1}}{\mathbf{7}} =$$

Researcher: Multiplying 3 by 1/7th is like adding 1/7th 3 times. [The researcher writes…]:

$$\mathbf{3} \times \frac{\mathbf{1}}{\mathbf{7}} = \frac{\mathbf{1}}{\mathbf{7}} + \frac{\mathbf{1}}{\mathbf{7}} + \frac{\mathbf{1}}{\mathbf{7}}$$

Researcher: When adding fractions, when the denominator is the same (and we will only work with denominators that are the same for now), all we have to do is add the numerators together. [The researcher writes…]:

$$\mathbf{3} \times \frac{\mathbf{1}}{\mathbf{7}} = \frac{\mathbf{1}}{\mathbf{7}} + \frac{\mathbf{1}}{\mathbf{7}} + \frac{\mathbf{1}}{\mathbf{7}} = \frac{\mathbf{3}}{\mathbf{7}}$$

Researcher: Notice that the denominator does not change. It stays the same.

Example 2: 4 x 2/6
Researcher: Now, let’s try it with 4 x 2/6. Look at the board. For now, I just want you to watch and listen. [The researcher writes…]:

$$4 \times \frac{2}{6} =$$

Researcher: Multiplying 4 by 2/6th is like adding 2/6th 4 times. [The researcher writes…]:

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$$

Researcher: When adding fractions, when the denominator is the same (and we will only work with denominators that are the same for now), all we have to do is add the numerators together. [The researcher writes…]

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8}{6} =$$

Researcher: Sometimes you have to simplify the answer you get by dividing it by a common factor. We will only use 2, 3, 4, or 5. So here, what number can we use to simplify the fraction? I will try 2. If I divide 8 by 2 and 6 by 2, what do I get?
[The researcher writes…]:

$$4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$$

Do you think I can make 4/3 into even smaller whole numbers? No, so that is our simplified answer.

Researcher: Now I will let you practice two questions on your own and we will go over them together. [The researcher presents the equations 3 x 2/7 (no simplifying needed) and 3 x 5/10 (with simplifying) one at a time using the script presented above; he tells the students that, for all the exercises we do from now on, if they need to simplify, they will only need to divide by 2, 3, 4, or 5.]

Control Group (CG)

No restrictions exist in what the researcher says to the students, but do not discuss anything about fractions.

Treatment 4 onwards – Practice Intervention with Reflection Prompts (PI+)

(Remember nothing conceptual can be hinted to. You are NOT allowed to use drawings to help the students. Students are NOT allowed to use multiplication tables.)

From now on we will limit ourselves to the following description of the task with some visual support:

Researcher says: "Now, we have been multiplying fractions by a whole number for a few days. I will review with you how to do that. Watch as I go through the steps for you again. [The bolded portions of the equations are added at each step. There is no need to rewrite the entire equations each time.]

Researcher says, "Step 1, and writes": $3 \times \frac{1}{7} =$

Researcher says, "Step 2, and writes": $3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$

Researcher says, "Step 3, and writes": $3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}$

Researcher says, "Sometimes that is the final step, but if you can, you divide the numerator and the denominator by the same number (either 2, 3, 4, or 5) to simplify your answer. [The bolded portions of the equations are added at each step. There is no need to rewrite the entire equations each time.] Here is an example:

Researcher says, "Step 1, and writes": $4 \times \frac{2}{6} =$

Researcher says, "Step 2, and writes": $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$

Researcher says, "Step 3, and writes": $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8}{6}$

Researcher says, "Step 4, and writes": $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$

Don't forget to show all your work so we can see if you make any mistakes."

Remind the students to pay attention to a few things by asking, "What do you notice that is different between the fraction at the beginning of the equation and the final answer? Which of the two fractions is greater [or larger/bigger]? Think about why that might be."

NO ANSWERS ARE PROVIDED.

“Okay, you may begin.”

IMPORTANT NOTE

Remind the students to compare the first and final fraction as you see them completing the first page, and again mid way during the second page. When they are done, ask them to take the time to do the comparison again and repeat the questions, “What do you notice that is different between the fraction at the beginning of the equation and the final answer? Which of the two fractions is greater [or larger/bigger]? Think about why that might be.”

Treatment 4 onwards – Practice Intervention without Reflection Prompts (PI-)

(Remember nothing conceptual can be hinted to. You are NOT allowed to use drawings to help the students. Students are NOT allowed to use multiplication tables.)

From now on we will limit ourselves to the following description of the task with some visual support:

Researcher says: “We have been multiplying fractions by a whole number for a few days. I will review with you how to do that. Watch as I go through the steps for you again. [The bolded portions of the equations are added at each step. There is no need to rewrite the entire equations each time.]

Researcher says, “Step 1, and writes”: $3 \times \frac{1}{7} =$

Researcher says, “Step 2, and writes”: $3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$

Researcher says, “Step 3, and writes”: $3 \times \frac{1}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}$

Researcher says, “Sometimes that is the final step, but if you can, you divide the numerator and the denominator by the same number (either 2, 3, 4, or 5) to simplify your answer. [The bolded portions of the equations are added at each step. There is no need to rewrite the entire equations each time.] Here is an example:

Researcher says, “Step 1, and writes”: $4 \times \frac{2}{6} =$

Researcher says, “Step 2, and writes”: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6}$

Researcher says, “Step 3, and writes”: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8}{6}$

Researcher says, “Step 4, and writes”: $4 \times \frac{2}{6} = \frac{2}{6} + \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = \frac{8 \div 2}{6 \div 2} = \frac{4}{3}$

“Researcher says, “Don’t forget to show all your work. You may begin.”

Appendix B

Sample Multiplication Activity Sheet

(Questions were provided on two pages, back to front)

1)	4 x 2/4	Show your work.
2)	4 x 2/10	Show your work.
3)	3 x 4/7	Show your work.
4)	3 x 4/9	Show your work.
5)	2 x 3/4	Show your work.
6)	4 x 1/4	Show your work.
7)	4 x 3/10	Show your work.
8)	2 x 8/9	Show your work.
9)	4 x 8/10	Show your work.
10)	2 x 4/8	Show your work.

Appendix C

Bank of Questions for the Multiplication Activities

Multiplier	Fraction	Product	Type of comparison
2	1/3	2/3	CD
4	1/3	4/3	CD
2	2/3	4/3	CD
4	2/3	8/3	CD
2	1/4	2/4	CD
3	1/4	3/4	CD
2	3/4	6/4	CD
3	3/4	9/4	CD
2	1/5	2/5	CD
3	1/5	3/5	CD
4	1/5	4/5	CD
2	2/5	4/5	CD
3	2/5	6/5	CD
4	2/5	8/5	CD
2	3/5	6/5	CD
3	3/5	9/5	CD
4	3/5	12/5	CD
2	4/5	8/5	CD
3	4/5	12/5	CD
4	4/5	16/5	CD
3	1/6	3/6	CD
2	1/7	2/7	CD
3	1/7	3/7	CD
4	1/7	4/7	CD
2	2/7	4/7	CD
3	2/7	6/7	CD
4	2/7	8/7	CD
2	3/7	6/7	CD
3	3/7	9/7	CD
4	3/7	12/7	CD
2	4/7	8/7	CD
3	4/7	12/7	CD
4	4/7	16/7	CD

Multiplier	Fraction	Product	Type of comparison
2	5/7	10/7	CD
3	5/7	15/7	CD
4	5/7	20/7	CD
2	6/7	12/7	CD
3	6/7	18/7	CD
4	6/7	24/7	CD
3	1/8	3/8	CD
3	3/8	9/8	CD
3	5/8	15/8	CD
3	7/8	21/8	CD
2	1/9	2/9	CD
4	1/9	4/9	CD
2	2/9	4/9	CD
4	2/9	8/9	CD
2	4/9	8/9	CD
4	4/9	16/9	CD
2	5/9	10/9	CD
4	5/9	20/9	CD
2	7/9	14/9	CD
4	7/9	28/9	CD
2	8/9	16/9	CD
4	8/9	32/9	CD
3	1/10	3/10	CD
3	3/10	9/10	CD
3	7/10	21/10	CD
3	9/10	27/10	CD
3	7/9	21/9 = 7/3	CD
3	8/9	24/9 = 8/3	CD
3	1/3	1/1	CN
3	2/3	2/1	CN
4	1/4	4/4 = 1	CN
4	2/4	8/4	CN
2	1/6	2/6 = 1/3	CN

Multiplier	Fraction	Product	Type of comparison
2	2/6	4/6 = 2/3	CN
3	3/6	9/6 = 3/2	CN
2	1/8	2/8 = 1/4	CN
4	1/8	4/8 = 2/4 = 1/2	CN
2	3/8	6/8 = 3/4	CN
4	3/8	12/8 = 3/2	CN
2	5/8	10/8 = 5/4	CN
4	5/8	20/8 =5/2	CN
2	7/8	14/8 = 7/4	CN
4	7/8	28/8 = 7/2	CN
3	1/9	3/9 = 1/3	CN
3	2/9	6/9 = 2/3	CN
3	4/9	12/9 = 4/3	CN
3	5/9	15/9 = 5/3	CN
2	1/10	2/10 = 1/5	CN
2	2/10	4/10 = 2/5	CN
2	3/10	6/10 = 3/5	CN
2	4/10	8/10 = 4/5	CN
2	7/10	14/10 = 7/5	CN
2	8/10	16/10 = 8/5	CN
2	6/10	12/10 = 6/5	CN
2	9/10	18/10 = 9/5	CN
3	2/4	6/4 = 3/2	LN/SD

Multiplier	Fraction	Product	Type of comparison
4	2/6	8/6 = 4/3	LN/SD
3	4/8	12/8 = 3/2	LN/SD
2	6/8	12/8 = 3/2	LN/SD
3	6/8	18/8 = 9/4	LN/SD
4	3/9	12/9 = 4/3	LN/SD
2	6/9	12/9 = 4/3	LN/SD
4	6/9	24/9 = 8/3	LN/SD
3	4/10	12/10 = 6/5	LN/SD
4	4/10	16/10 = 8/5	LN/SD
3	5/10	15/10 = 3/2	LN/SD
3	6/10	18/10 = 9/5	LN/SD
4	6/10	24/10 = 12/5	LN/SD
4	7/10	28/10 = 14/5	LN/SD
3	8/10	24/10 = 12/5	LN/SD
4	8/10	32/10 = 16/5	LN/SD
4	9/10	36/10 = 18/5	LN/SD
2	2/4	4/4 = 1	One
3	2/6	6/6 = 1	One
2	3/6	6/6 = 1	One
4	2/8	8/8 = 1	One
2	4/8	8/8 = 1	One
3	3/9	9/9 = 1/1	One
2	5/10	10/10 = 1/1	One

Multiplier	Fraction	Product	Type of comparison
4	6/8	24/8 = 6/2 = 3/1	Three
4	3/6	12/6 = 6/3 = 2	Two
4	4/8	16/8 = 2/1	Two

Multiplier	Fraction	Product	Type of comparison
3	6/9	18/9 = 2/1	Two
4	5/10	20/10 = 2/1	Two