

Dimensional Broadcast Schemes and Their Applications in Broadcast Graphs

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Abstract

Dimensional Broadcast Schemes and Their Applications in Broadcast Graphs

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In a graph $G = (V, E)$, where V and E are the sets of vertices and edges of G , respectively, broadcasting is an information dissemination process of distributing a message from one vertex, which is called the originator, to all other vertices of the graph. Spreading information is one of the important tasks of a modern interconnection network. This process is called *broadcasting*. A huge amount of research was conducted in this area.

In this thesis, we will mainly focus on structural and algorithmic aspects of broadcasting in special families of graphs, particularly hypercubes and Knödel graphs.

We investigate hypercubes as canonical examples of minimum broadcast graphs, where we develop an algorithm to generate and enumerate all minimum-time broadcast schemes. Our analysis confirms that every such scheme on a hypercube is also a minimum-distance broadcast scheme.

Subsequently, we shift our attention to Knödel graphs, exploring their potential as broadcast graphs under both standard and reversed dimensional broadcast schemes with cyclic shifts and arbitrarily dimension repetitions. We determine all values of n for which these schemes are valid and identify structural properties that maximize broadcast efficiency.

A new general upper bound on the broadcast function $B(n)$ for odd n (excluding $n = 2^k - 1$) is then established using a vertex deletion approach. In this context, we also resolve an open question concerning the existence of minimum average time broadcast graphs.

Finally, we extend our investigations to generalized Knödel graphs, characterizing the values of

n for which they admit certain valid dimensional broadcast schemes. Our results contribute new insights toward understanding the relationship between graph structure and information dissemination efficiency.

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Chapter 1

Introduction

In recent years, various social networks such as LinkedIn, Facebook, Twitter, and Telegram have emerged, enabling more efficient and accessible communication among individuals based on their specific needs. One approach to studying these networks is through graph modeling, where vertices represent individuals participating in the network and edges represent the connections or relationships among them. The term complex network is commonly used to describe real-world networks (modeled as graphs) with a large number of vertices. Examples include Internet graphs, phone networks, email networks, social networks, and many others. Despite their diversity, these networks often share common structural and functional properties. One such property is the capability to disseminate information from a single source (originator) to all connected vertices within the network. This feature is particularly relevant in applications such as product advertising or broadcasting notifications across the network. Another key property is the time required for all vertices to receive the information, which serves as an important measure of the efficiency of information propagation within the network.

Data mining, a fundamental aspect of data science, is used to analyze large data sets and extract relevant patterns and insights. It involves techniques for extracting patterns and

knowledge from data to support analytical and decision-making processes. In many modern data mining applications, the data is represented in the form of networks, such as social graphs derived from platforms like Facebook or Twitter. Analyzing these networks and understanding their properties is essential for many data mining tasks. Such structural insights contribute to a more effective analysis of networked data. For instance, understanding the speed of message propagation within a network can improve the design and evaluation of data mining algorithms. Understanding network dynamics supports the optimization of information dissemination and the development of more effective communication strategies in data-driven applications.

Solving broadcasting problems in graphs contributes to the analysis and enhancement of communication efficiency and overall network performance.

Computer systems and their processors provide a more technical example of networked structures. Due to the increasing complexity of computational problems, single-processor systems, even when equipped with advanced technologies, require significant time to solve tasks, particularly when multiple problems must be processed concurrently.

Multi-processor systems address this issue by decomposing a problem into smaller tasks that can be executed in parallel. Each processing unit performs its assigned portion of the algorithm concurrently once the problem has been decomposed. During execution, however, independent processing elements must exchange data. Parallel computing is critical because it reduces completion times and lowers costs. This significance is reflected in the widespread use of multi-core processors in personal computers. Consequently, parallel algorithms are designed to optimally distribute data and workload among processing elements to maximize utilization of the underlying hardware.

Extensive research has been conducted to analyze interconnected networks to optimize communication structures for parallel and distributed computing. The four main collective operations in parallel computing are summarized as follows:

- One-to-one **routing**
- One-to-all **broadcasting**
- One-to-many **multicasting**
- All-to-all **gossiping**

This thesis focuses on classical broadcasting (or telephone broadcasting), a fundamental process for information dissemination in connected networks, which are commonly modeled as undirected graphs. But other methods exist based on the number of originators, receivers, and senders at each time unit. We refer the interested reader to [1, 6, 10, 11, 12, 59, 60, 61, 68, 69] for a detailed discussion of alternative approaches.

In the classical broadcasting problem, a single vertex initially receives a message, referred to as the originator, and begins the dissemination process. At each time unit, every informed vertex may inform at most one of its uninformed neighbors. The process continues until all vertices in the graph are informed. The graph is assumed to be connected, as broadcasting cannot be completed otherwise. A key metric in this context is the *broadcast time*. Numerous studies over the past three decades have investigated broadcasting on various graph classes and models. For instance, variations in the number of originators, the number of receivers per time unit, and other graph properties have led to different broadcasting models. This thesis focuses on the classical broadcasting model.

The following definitions are relevant to this thesis.

Each network is represented as an undirected graph $G = (V, E)$, where V is the set of vertices and E the set of edges. The assumptions of the classical broadcasting model are as follows:

- (1) The broadcasting process is split into discrete time units.

- (2) The only vertex that is informed at the beginning of the first time unit is called *originator*.
- (3) In each time unit, an informed vertex called *sender* can *call* at most one of its uninformed neighbor called *receiver*, which hence becomes informed by the end of the time unit.
- (4) During one time unit, all calls are performed in parallel.
- (5) The process ends as soon as all the vertices in the graph are informed.

Sometimes we show a call from u to v by the ordered pair (u, v) . An informed vertex u is called *idle* at time unit i if u does not send the message to any other vertices at this time unit, otherwise it is called *active*. The set of calls that are placed in time unit i is denoted by C_i .

Definition 1.0.1. *Broadcast scheme* is the order of calls made by each vertex during a broadcasting process and can be represented as a sequence $(C_1, C_2, \dots, C_t)$, where C_i is the set of calls performed in time unit i .

Definition 1.0.2. A **Broadcast tree** is a directed spanning tree rooted at the originator according to the broadcast scheme, assuming each vertex is informed by exactly one other vertex. If multiple vertices can inform a given vertex, one must be selected as the sender. For simplicity, the direction of edges in the broadcast tree may be omitted.

Definition 1.0.3. The **broadcast time** for a vertex v in a given graph G is the minimum number of time units required for broadcasting in G if v is the originator and is denoted by $b(G, v)$. The broadcast time for a given graph G denoted by $b(G)$, is the maximum broadcast time from any originator in G , that is $b(G) = \max_{v \in V(G)} \{b(G, v)\}$.

By the assumption (3), the number of informed vertices after each time unit can be at most doubled. Therefore, the number of informed vertices after time unit i is at most 2^i . Hence, to inform all vertices of G with n vertices, at least $\lceil \log n \rceil$ time units are needed. It means that $b(G, v) \geq \lceil \log n \rceil$ for every vertex $v \in V$. Thus $b(G) \geq \lceil \log n \rceil$. It is worth mentioning that an additional time unit can substantially affect the number of informed vertices during the broadcasting process. We focus on graphs where the minimum broadcast time is achieved, i.e., where the equality condition holds. Such graphs support optimal broadcasting.

Definition 1.0.4. *broadcast graph*: Let G be a graph of order n . If $b(G) = \lceil \log n \rceil$, then G is called a *broadcast graph*, or simply *bg*. In this case, for every vertex v , we have $b(G, v) = \lceil \log n \rceil$.

Obviously, in a broadcast graph G with sufficient symmetry, calculating the broadcast time for a single vertex is enough to determine $b(G)$.

In graph-based models of real networks, each edge represents a physical or logical connection. Thus, a graph with fewer edges corresponds to a network with fewer connections, which typically implies lower implementation or maintenance costs. For example, consider a network topology of routers in the local internet of a company. Given the significant cost associated with connecting two routers through wires, it is prudent to minimize wire usage. Therefore, our interest lies in broadcast graphs with the fewest possible edges, aiming for the lowest cost in terms of the number of edges used.

Definition 1.0.5. A broadcast graph G of order n is called **minimum broadcast graph (mbg)**, if there does not exist any other broadcast graph G' of order n such that $|E(G')| < |E(G)|$. The number of edges in a minimum broadcast graph is called **broadcast function of n** and is denoted by $B(n)$.

The study of the minimum broadcast graphs (mbgs) has a rich history, originating with

Farley et al. [34] in 1979. In their work, they introduced the concept of mbgs and identified the first infinite family of such graphs, consisting of hypercube graphs with $n = 2^k$ vertices, where k is a positive integer. Subsequently, Park and Chwa [85] proposed another family of mbgs for graphs on 2^k vertices. Knödel graphs are shown to be mbgs on $n = 2^k$ [72] and $n = 2^k - 2$ vertices by Khachatryan and Haroutunian [70] and Dinneen et al. [23], independently. The reader is referred to foundational works [5, 16, 15, 24], as well as to subsequent developments and technical insights [43, 50, 51, 52, 55, 70, 88]. In addition, we recommend surveys [39, 63, 65, 68, 69] for a complete overview of the topic.

Generally, one can distinguish two main research directions on broadcasting in graphs.

- (1) *Minimum broadcast graph problem:* For a given integer n , construct a minimum broadcast graph on n vertices, determine the value of the broadcast function $B(n)$, or give an algorithm to show that a given graph G is broadcast graph.
- (2) *Broadcast time problem:* For a given graph G , find an optimal broadcast scheme for G , give an algorithm to broadcast a message over G , or determine the broadcast time $b(G)$.

This problem naturally leads to two categories of graphs: a) those for which the exact broadcast time can be determined, and b) those for which only approximate broadcast times can be obtained by adapting existing algorithms. The first category will be addressed in Section 1.1, while the second, which leads us to the hardness of the problem, will be discussed in Section 1.2.

Further in this thesis, the problems related to minimum broadcast graphs are discussed.

There are several extensions of the defined broadcasting, such as enabling each informed vertex to inform k or fewer of its uninformed neighbors at each time unit, instead of just one vertex. It is called k -broadcasting [62, 73]. All the previously defined concepts can be adapted to incorporate this extension. In this scenario, we replace $b(G)$ with

$b_k(G) = \max\{b_k(G, u) \mid u \in V(G)\}$, where $b_k(u)$, for a vertex u , represents the minimum number of time units required to accomplish a k -broadcast starting from vertex u . It is easy to see that $b_k(u) \geq \lceil \log_{k+1} n \rceil$. Similarly, the minimum k -broadcast graph and k -broadcast function are defined. Results from k -broadcast problems can be informative for our broadcast problem by setting $k = 1$.

1.1 The Graphs for Which Their Broadcast Time Has Been Investigated

In this section, we present several well-known graph families for which the broadcast time has been fully determined.

1.1.1 Path Graph P_n

A path P_n is a graph on $n \geq 1$ vertices, $V = \{1, \dots, n\}$, and the set $E = \{(i, i + 1) \mid 1 \leq i \leq n - 1\}$. P_n has $n - 1$ edges, diameter $n - 1$ and $\Delta(P_n) = 2$ when $n \geq 3$. The first and the last vertices have a degree of 1 and all other vertices have a degree of 2. The vertex with the minimum broadcast time in a path graph is the midpoint (either of the midpoints if n is even), denoted by u , with $b(u, P_n) = \lceil \frac{n}{2} \rceil$ (when $n \geq 2$). However, $b(P_n) = n - 1$. Path graphs have the maximum sequential delay of broadcasting. Figure 1.1 demonstrates P_7 .

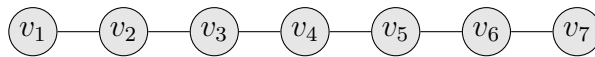


Figure 1.1: Path graph P_7 .

1.1.2 Cycle Graph C_n

A cycle C_n is a graph on $n \geq 3$ vertices, $V = \{1, \dots, n\}$, and the edge set $E = \{(i, i+1) | 1 \leq i \leq n-1\} \cup \{(1, n)\}$. A cycle C_n has n edges, is a 2-regular graph with diameter $D(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$, and $b(C_n) = \left\lceil \frac{n}{2} \right\rceil$. Figure 1.2 demonstrates C_6 .

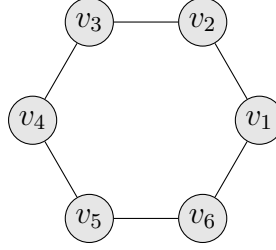


Figure 1.2: Cycle graph C_6 .

1.1.3 Star Graph S_n

A star S_n is a graph on $n \geq 1$ vertices, $V = \{0, \dots, n-1\}$, and the edges set $E = \{(0, i) | 0 < i \leq n-1\}$. A star S_n has $n-1$ edges with $\Delta(S_n) = n-1$, $b(S_n) = n-1$, and diameter $D(S_n) = 2$ when $n \geq 3$. Star graphs have the maximum parallel delay of broadcasting. Figure 1.3 demonstrates S_7 .

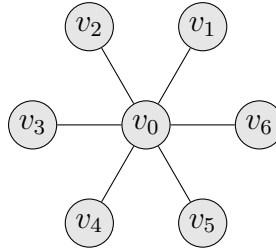


Figure 1.3: Star graph S_7 .

1.1.4 Complete Graph K_n

A complete graph (clique) K_n is a graph on $n \geq 1$ vertices, $V = \{1, \dots, n\}$, and all possible edges. A complete graph K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, is an $(n-1)$ -regular graph, with diameter $D(K_n) = 1$, and $b(K_n) = \lceil \log n \rceil$. Figure 1.4 demonstrates K_5 .

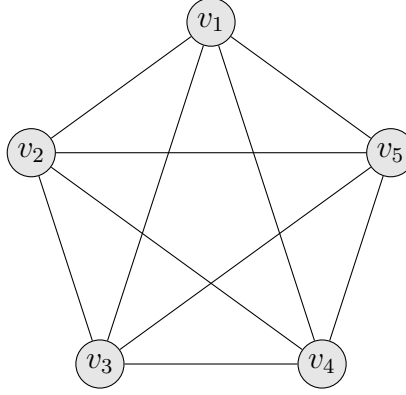


Figure 1.4: Complete graph K_5 .

1.1.5 Complete Bipartite Graph $K_{m,n}$

A complete bipartite graph $K_{m,n}$ consists of two vertex partitions A and B with $|A| = m \geq 1$ and $|B| = n \geq 1$ vertices, respectively. There are no edges between vertices from the same partition. For any $x \in A$ and $y \in B$, $(x, y) \in E(K_{m,n})$. Also, $\deg(x) = n$, and $\deg(y) = m$. Figure 1.5 demonstrates $K_{4,3}$. The broadcast time from any originator $u \in V(K_{m,n})$ is given as follows:

- 1) If $\lceil \log m \rceil < \lceil \log n \rceil$, then $b(u, K_{m,n}) = \lceil \log m \rceil + 1 + \left\lceil \frac{n - 2^{\lceil \log m \rceil}}{m} \right\rceil$.
- 2) If $\lceil \log m \rceil = \lceil \log n \rceil$, then $b(u, K_{m,n}) = \lceil \log m \rceil + 1 = \lceil \log n \rceil + 1$.
- 3) If $\lceil \log m \rceil > \lceil \log n \rceil$, then $b(u, K_{m,n}) = \lceil \log n \rceil + 1 + \left\lceil \frac{m - 2^{\lceil \log n \rceil}}{n} \right\rceil$.

Note that after the first time unit, each part of G contains exactly one informed vertex. Therefore, it does not matter whether the originator belongs to A or B . With no loss of

generality, assume that $m \leq n$. If $\lceil \log m \rceil = \lceil \log n \rceil$, after time unit $\lceil \log m \rceil$, exactly $2^{\lceil \log m \rceil - 1}$ vertices of each part are informed. Thus, the rest of the vertices will be informed in time unit $\lceil \log m \rceil + 1$. Hence, $b(K_{m,n}) \leq \lceil \log m \rceil + 1$. On the other hand, one can see that $b(K_{m,n}) \geq \lceil \log(m+n) \rceil \geq \lceil \log 2m \rceil = \lceil \log m \rceil + 1$. Therefore, $b(K_{m,n}) = \lceil \log m \rceil + 1$. Now if $\lceil \log m \rceil < \lceil \log n \rceil$, after time unit $\lceil \log m \rceil + 1$, exactly $2^{\lceil \log m \rceil}$ vertices of B are informed while all m vertices of A are informed. So, there are $n - 2^{\lceil \log m \rceil}$ uninformed vertices in B . From time unit $2^{\lceil \log m \rceil + 1}$ onward, exactly m vertices in B can be informed in each time unit. Therefore, we need an additional $\left\lceil \frac{n - 2^{\lceil \log m \rceil}}{m} \right\rceil$ time units to inform the remaining vertices.

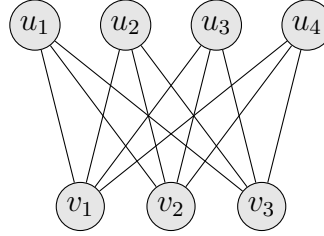


Figure 1.5: Complete bipartite graph $K_{4,3}$.

1.1.6 Fork Graph $F_{n,k}$

A Fork $F_{n,k}$, where $1 \leq k < n$, has n vertices and contains a path on $n - k$ vertices such that k isolated vertices are connected to one of endpoints of the path. A fork $F_{n,k}$ has $n - 1$ edges with diameter $D(F_{n,k}) = n - k$, and $b(F_{n,k}) = n - 1$. Figure 1.6 demonstrates $F_{9,5}$.

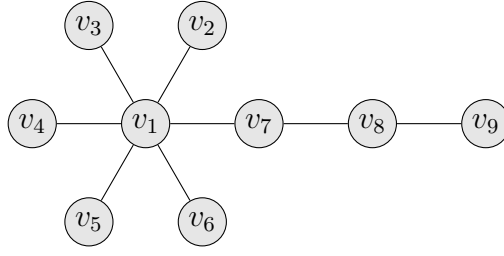


Figure 1.6: Fork graph with $n = 9$ and $k = 5$.

1.1.7 Wheel Graph W_n

A wheel W_n is a graph on $n \geq 4$ vertices, $V = \{1, \dots, n\}$, and the edge set $E = \{(1, i) | 1 < i \leq n\} \cup \{(i, i+1) | 2 \leq i \leq n-1\} \cup \{(2, n)\}$. A wheel W_n has $2n-1$ edges and diameter $D(W_n) = 2$ if $n > 4$. Whereas, $\Delta(W_n) = n-1$, and $b(W_n) = \left\lceil \frac{\sqrt{4n-3}+1}{2} \right\rceil$. Figure 1.7 demonstrates W_6 .

Note that if the originator is v_i for an $i \geq 2$, then in the first time unit, v_i sends the message to v_1 . So, without loss of generality, suppose that v_1 is the originator. Now, if in time unit j , vertex v_1 sends the message to v_i for some $i \geq 2$, then v_i can inform at most $2(t-j)$ vertices, where $t = b(W_n)$. Therefore, after t time units, at most $2 + \sum_{i=1}^{t-1} 2(t-i) = t(t-1) + 2$ vertices will be informed. Hence, $n \leq t(t-1) + 2$, and so $t = \left\lceil \frac{\sqrt{4n-3}+1}{2} \right\rceil$.

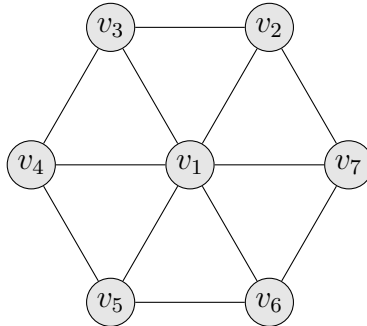


Figure 1.7: Wheel graph W_8 .

1.1.8 Grid Graph $G_{m \times n}$

A two-dimensional Grid (Lattice), $G_{m \times n}$, is a graph on $m \times n$ vertices, where $m, n \geq 1$. Each of the vertices is on integer coordinates of a Cartesian plane and there is an edge between vertices if they have Euclidean distance one. A grid $G_{m \times n}$ has $(m - 1)n + (n - 1)m = 2mn - (m + n)$ edges, diameter $D(G_{m \times n}) = m + n - 2$, $b(G_{m \times n}) = m + n - 2$ [33], and $\Delta(G_{m \times n}) = 4$ when $m, n \geq 3$. Figure 1.8 demonstrates $G_{3 \times 4}$.

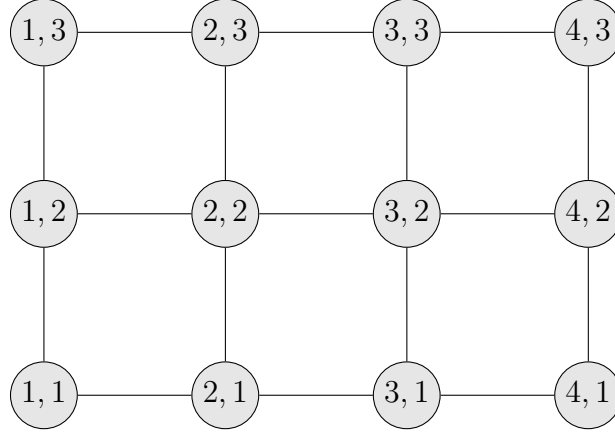


Figure 1.8: Grid graph $G_{3 \times 4}$.

1.1.9 Torus Graph $T_{m \times n}$

A two-dimensional Toroidal Grid (Torus), $T_{m \times n}$, is a graph on $m \times n$ vertices, where $m, n \geq 2$. It includes all edges of the grid and edges $((i, 1), (i, n))$ and $((1, j), (m, j))$, for all i, j such that $1 \leq i \leq m$ and $1 \leq j \leq n$. $T_{m \times n}$ is a 4 regular graph and has $2mn$ edges, diameter $D(T_{m \times n}) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor$. If both m and n are even, then $b(T_{m \times n}) = \frac{m + n}{2}$, otherwise, $b(T_{m \times n}) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + 1$ [33]. Figure 1.9 demonstrates $T_{3 \times 4}$.

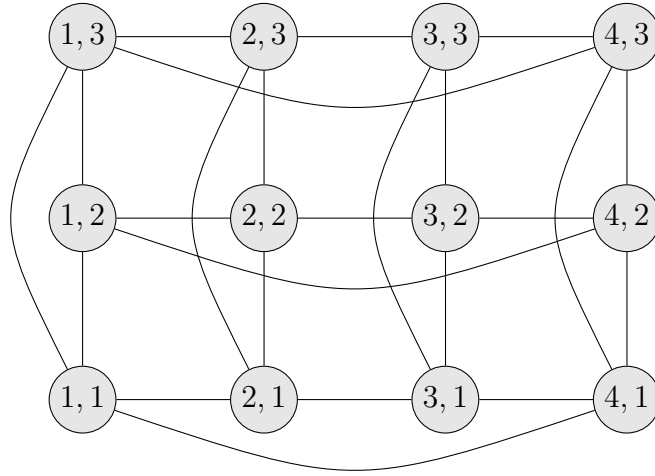


Figure 1.9: Torus graph $T_{3 \times 4}$.

1.1.10 Hypercube Graph H_n

The n -dimensional hypercube, Q_n , is a graph on 2^n vertices. Each vertex is a n -bit binary string, and two vertices are adjacent if and only if they differ at only one bit. Alternatively, Q_n can be defined recursively as a union of two copies of Q_{n-1} , where the corresponding vertices in two copies are connected by an edge. A hypercube Q_1 is a single-edge graph on two vertices. A hypercube Q_n is n regular and so has $n \times 2^{n-1}$ edges. This is bipartite, and has diameter $D(Q_n) = n$. Hypercube is a minimum broadcast graph and $b(Q_n) = n$. Figure 1.10 demonstrates Q_3 .

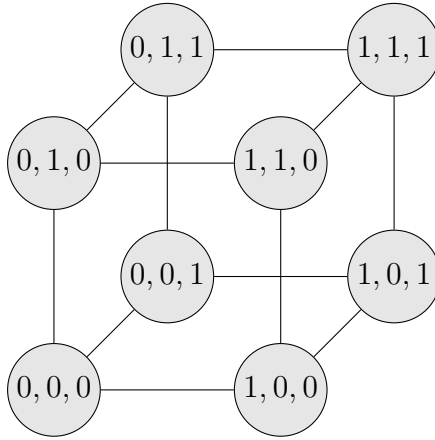


Figure 1.10: Hypercube Q_3 .

1.1.11 Complete k -Ary Tree $T_{k,h}$

A k -ary tree is a rooted tree in which the number of children of each internal vertex is k . The degree of the root is k , the degrees of internal vertices are $k + 1$, and the leaves degrees are 1. A complete k -ary tree of height h , (denoted by $T_{k,h}$), is a rooted k -ary tree in which all leaves are on the same level h . A tree $T_{k,h}$ has $\frac{k^{h+1} - 1}{k - 1}$ vertices, diameter $D(T_{k,h}) = 2h$, $\Delta(T_{k,h}) = k + 1$, and $b(T_{k,h}) = kh + h - 1$. Figure 1.11 demonstrates $T_{5,2}$.

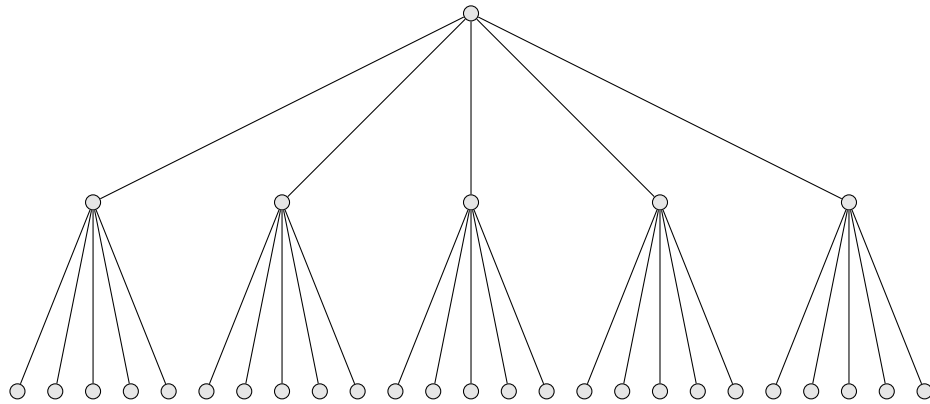


Figure 1.11: Tree $T_{5,2}$.

1.1.12 Recursive Circulant Graph $G(n, d)$

The recursive circulant graph [85] has $n = cd^m$ vertices $\{0, \dots, n-1\}$, where $d \geq 2$, $m \geq 1$, and $1 \leq c < d$. The edge set is defined as $E = \{(v, w) \mid \text{there exists } i, 0 \leq i \leq \lceil \log_d n \rceil - 1, \text{ such that } v + d^i \equiv w\}$.

Interestingly, $G(2^m, 2)$ has the same number of nodes, edges ($m2^{m-1}$), and the broadcast time $b(G(2^m, 2)) = \lceil \log 2^m \rceil = m$ as the hypercube Q_m . A recursive circulant graph $G(n, d)$ is $m = \lceil \log n \rceil$ -regular, and has diameter $D(G(n, d)) = \left\lceil \frac{3m-1}{4} \right\rceil$. Figure 1.12 demonstrates $G(2^3, 2)$.

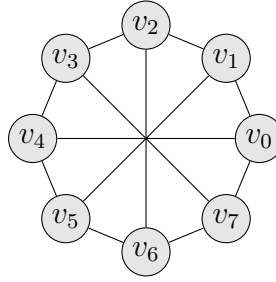


Figure 1.12: recursive circulant graph $G(2^3, 2)$.

1.1.13 Butterfly Graphs BF_d

The butterfly graph BF_d is a graph with $n = d \cdot 2^d$ vertices, where each vertex is labeled by a pair (l, x) . Here, l is called the level, where $0 \leq l \leq d-1$, and $x = x_0x_1 \dots x_{d-1}$ is a d -bit binary string referred to as the position within the level.

Each vertex (l, x) has two kinds of edges:

- a *straight edge* to $(l+1 \bmod d, x)$,
- and a *cross edge* to $(l+1 \bmod d, x_0x_1 \dots x_{l-1} \overline{x_l} x_{l+1} \dots x_{d-1})$, where $\overline{x_l}$ denotes the bitwise complement of x_l .

The butterfly graph BF_d has degree four and diameter $\left\lfloor \frac{3d}{2} \right\rfloor$. Stöhr was the first one to investigate broadcasting in BF_d , [94, 95]. Her result has been improved in [71]. The authors showed that:

$$1.7417d \leq b(BF_d) \leq 2d - 1.$$

Figure 1.13 illustrates the graph BF_2 . Note that the four open circles on the bottom row appear to represent the same vertices as do the four filled circles on the top row.

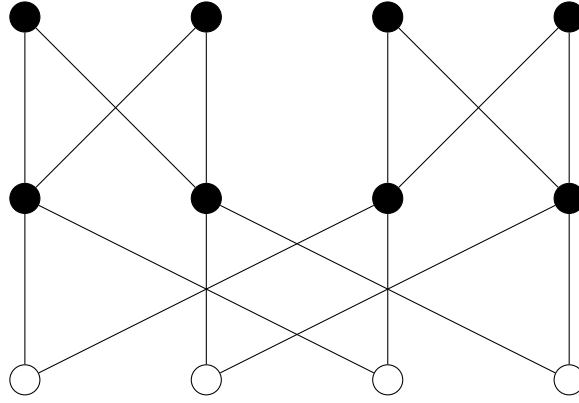


Figure 1.13: Butterfly Graphs BF_2 .

1.1.14 De Bruijn Graph DB_n

The de Bruijn graph $B(d, D)$ is a directed graph with in-degree and out-degree d and diameter D , where the number of vertices is $n = d^D$. Each vertex represents a word of length D over an alphabet of d symbols. There is a directed edge from a vertex $u = (u_0, \dots, u_{D-1})$ to a vertex $v = (u_1, \dots, u_{D-1}, \lambda)$ if and only if the last $D - 1$ symbols of u match the first $D - 1$ symbols of v , and λ is any symbol in the alphabet.

Figure 1.14 illustrates the graph $B(2, 3)$.

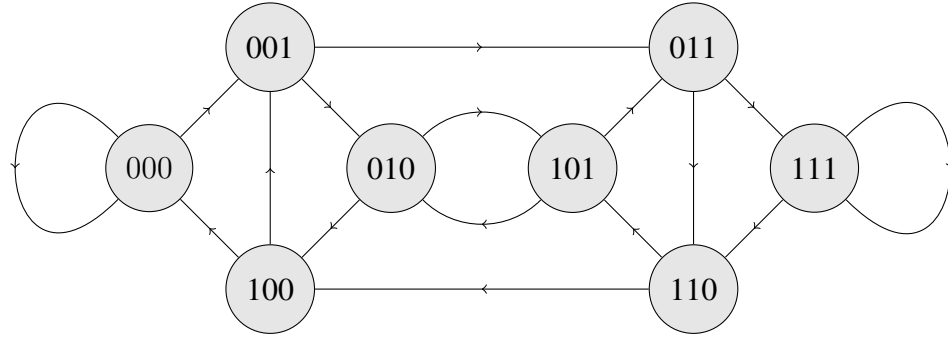


Figure 1.14: de Bruijn graph $B(2, 3)$.

We shall consider the undirected version of the de Bruijn graph, denoted $UB(d, D)$, which is obtained by removing the edge orientations from $B(d, D)$. The graph $UB(d, D)$ has degree $2d$, diameter D , and contains nd edges. The de Bruijn graph has been proposed as a strong competitor to the hypercube for various interconnection network applications [13]. In [14, 71], the following is proved:

$$1.13171D \leq b(UB(2, D)) \leq \frac{3}{2}(D + 1).$$

1.1.15 Cube Connected Cycle CCC_d

Cube Connected Cycles CCC_d is a modification of the hypercube H_d , where the d -dimensional graph CCC_d is constructed from the d -dimensional hypercube by replacing each vertex of the hypercube with a cycle of d vertices. For $d > 3$, the diameter of CCC_d is given by

$$D = 2d + \left\lfloor \frac{d}{2} \right\rfloor - 2$$

as shown in [81]. A straightforward algorithm described in [76] achieves a broadcast time of $\left\lceil \frac{5d}{2} \right\rceil - 1$ which shows that

$$b(CCC_d) \leq \left\lceil \frac{5d}{2} \right\rceil - 1.$$

In this algorithm, the message is first sent to the hypercube neighbor, then to the right neighbor on the cycle, followed by the left neighbor. Figure 1.15 demonstrates CCC_3 .

1) If d is even, then $\left\lceil \frac{5d}{2} \right\rceil - 1 = D + 1$. Hence, $D \leq b(CCC_d) \leq D + 1$ including $d = 4$, where by Lemma 3.1 of [39], we have $b(CCC_d) = D + 1$.

2) If d is odd, then $\left\lceil \frac{5d}{2} \right\rceil - 1 = D + 2$. Therefore, by Lemma 3.1 of [39], $D + 1 \leq b(CCC_d) \leq D + 2$.

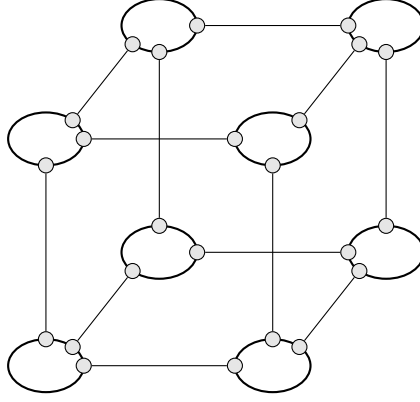


Figure 1.15: Cube Connected Cycles CCC_3 .

1.2 Hardness of the Problem

Despite the minimum broadcast graph problem is widely believed to be quite hard, formally, the complexity of this problem remains open.

The broadcast time decision problem is NP-complete in an arbitrary graph [42, 93]. The problem remains NP-complete even for 3-regular planar graphs [82], or when restricted to graphs with the feedback vertex set number one [96]. The study of the parameterized complexity of the broadcast time problem was initiated in [37] and has progressed in the following years [38, 97]. Furthermore, only a few graph classes are known to admit exact polynomial-time algorithms for determining broadcast time. In particular, linear-time algorithms have been developed for specific families such as trees [86, 93], unicyclic connected

graphs [56, 57], necklace graphs (chains of rings) [64], and Harary-like graphs [20, 19]. Ravi [87] also investigated the minimum broadcast time problem under the classical model to derive an $\mathcal{O}(\frac{\log^2 n}{\log \log n})$ -approximation algorithm for the minimum broadcast time problem of a graph on n vertices. This approximation has improved to $\mathcal{O}(\log n)$ [7, 8]. The current best approximation ratio of $\mathcal{O}(\frac{\log n}{\log \log n})$ for broadcasting was introduced by Elkin and Kortsarz [26, 29]. Additionally, the general broadcast time problem is known to be $\frac{3}{2}$ -inapproximable [91]. This was increased to a $(3 - \epsilon)$ -inapproximability result [28, 27].

Cactus graphs are a class of connected graphs in which any two simple cycles share at most one vertex. These graphs have attracted considerable attention in algorithmic graph theory due to their structural simplicity and the fact that many NP-hard problems admit polynomial-time solutions when restricted to them. An exact algorithm for solving the broadcast time problem on cactus graphs was presented in [98], with a time complexity of $O(kk!m + n)$, where m, n , and k are the number of edges, vertices, and the maximum number of cycles sharing a vertex, respectively. When k is fixed—i.e., in k -restricted cactus graphs—the algorithm becomes polynomial. However, the computational complexity of the broadcast time problem in general cactus graphs remained unresolved until recently.

Interestingly, the problem is still open even for relatively simple subfamilies such as k -cycle graphs (also known as flower graphs), where k cycles are joined at a single common vertex. These graphs are central to the present study. Recent advances have nearly resolved the complexity of the broadcast time problem in such structures. Notably, Aminian et al. [3] proved that the problem is NP-complete in *snowflake graphs*, a subclass of cactus graphs with pathwidth at most 2. This result implies that the broadcast time problem is NP-complete not only for general cactus graphs but also for all graphs with pathwidth at most 2. Recently, Egami et al. proved that the broadcast time problem is NP-complete in cactus graphs that are distance-1 to path forest [25].

The class of cactus graphs that are distance-1 to a path forest forms a strict subclass of

cactus graphs and bears a strong resemblance to k -cycle graphs. Specifically, such graphs can be viewed as k -cycle graphs with additional paths of arbitrary length attached to the central vertex. This result highlights that the broadcast time problem is NP-hard even within highly restricted subclasses of cactus graphs. Consequently, this raises further interest in k -cycle graphs—a closely related subclass—for which the complexity of the broadcast time problem remains unresolved.

The current best-known approximation ratio for k -cycle graphs is 2 [18]. In [4], the authors present a simple polynomial-time $(1.5 - \epsilon)$ -approximation algorithm for the broadcast time problem in k -cycle graphs.

In addition to these algorithms, there also exist several heuristic approaches for investigating broadcasting in graphs [12, 58, 61, 47, 90].

For a more detailed introduction to broadcasting, we refer the reader to [39, 54, 63, 65, 68, 74, 78, 80, 89].

Although the broadcast time problem is directly related to this area, we will not focus on that.

1.3 Integers for Which Their mbg Has Been Investigated

A long history is behind the problem of the minimum broadcast graph (mbg) and broadcast function $B(n)$. Other than the families of numbers mentioned above (for $n = 2^k$ and $n = 2^k - 2$), minimum broadcast graphs or $B(n)$ are known for some specific small values of n . In particular, minimum broadcast graphs are known for all values $n \leq 32$ except $n = 23, 24$ and 25 . The values of $B(n)$ for $1 \leq n \leq 15$ are presented in [34], also, $B(17) = 22$ [83], $B(18) = 23$, $B(19) = 25$ [15, 100], $B(20) = 26$, $B(21) = 28$, $B(22) = 31$ [79], $B(23) = 33$ or 34 [21, 83], $B(26) = 42$ [88, 102], $B(27) = 44$, $B(28) = 48$, $B(29) = 52$ [88], $B(30) = 60$ [15, 70], $B(31) = 65$ [15]. Note that for these

values of n , $\lceil \log n \rceil \leq 5$. Indeed, it means that verifying at most 5 time units is sufficient to find minimum broadcast graphs (mbgs). However, the difficulty of finding mbg on larger graphs with $n \geq 32$ vertices not only arises from the larger number of vertices but also due to the need to check 6 time units. Due to this fact, the number of integers $n \geq 33$ (excluding 2^k and $2^k - 2$) for which $B(n)$ is known is limited. For instance, $B(58) = 121$, $B(59) = 124$, $B(60) = 130$, $B(61) = 136$ [88], $B(63) = 162$ [75], $B(127) = 389$ [49], $B(1023) = 4650$, $B(4095) = 22680$ [92].

To determine the value of $B(n)$, both an upper bound and lower bound are necessary. Constructing a broadcast graph on n vertices is a typical method to establish an upper bound, wherein the number of edges represents the upper limit. However, proving that a lower bound matches the upper bound, even for small values of n , is exceedingly challenging, as most lower bound proofs rely solely on vertex degrees, and usually minimum broadcast graphs except hypercubes and Knödel graphs on $2^m - 2$ vertices are not regular.

A broadcast graph on 24 vertices and 36 edges was constructed by Bermond et al. [15]. This gives $B(24) \leq 36$. In [9], By proving a set of inequalities based on degrees and employing a program to check all graphs of order 24, it was established that $B(24) \geq 35$. This highlights the difficulty in determining lower bounds for the broadcast function, as it involves rigorous analysis and computational efforts. Therefore, $B(24) = 35$ or 36 . Using the same method and program, it was demonstrated that $B(25) \geq 38$. Additionally, a broadcast graph on 25 vertices and 40 edges, constructed by Bermond et al. [15], allows us to conclude that $B(25) = 38, 39$, or 40 . From 2014 till now, the values of $B(23)$, $B(24)$ and $B(25)$ are unknown.

1.4 The Monotonicity of The Broadcast Function

By the above, however, the exact values of $B(23)$, $B(24)$ and $B(25)$ are unknown, one can see that the function $B(n)$ is monotone for $1 \leq n \leq 16$, and $17 \leq n \leq 32$. Indeed, comparing $B(n)$ and $B(n+1)$ directly when $n = 2^k$ may not be fair since the addition of one vertex also adds one time unit. It is reasonable to expect that $B(n+1) < B(n)$ in such cases. Therefore, the following conjecture appears natural:

Conjecture 1.4.1. *Let m be a positive integer. The broadcast function $B(n)$ is non-decreasing for n in the range $2^{m-1} + 1 \leq n \leq 2^m$.*

Indeed, this conjecture is mentioned in [15, 41], but it is likely that researchers in this field were aware of it even before these papers. An evident extension of this conjecture is as follows:

Conjecture 1.4.2. *Let m be a positive integer. The k -broadcast function $B_k(n)$ is non-decreasing for n in the range $(k+1)^{m-1} + 1 \leq n \leq (k+1)^m$.*

Harutyunyan and Liestman [54] showed that Conjecture 1.4.1 holds for all n in the range $2^{m-1} + 1 \leq n \leq 2^{m-1} + 2^{m-3}$.

In some applications, $b(G) = \lceil \log n \rceil + t$, where t is a constant positive integer, is acceptable if graph G has fewer edges. Such a graph G on n vertices is called a t -relaxed k -broadcast graph if $b_k(G) \leq \lceil \log(k+1)n \rceil + t$, where $t \geq 0$. $B_k^t(n)$ is used to denote the minimum number of edges in any t -relaxed k -broadcast graph on n vertices. A minimum t -relaxed k -broadcast graph is a t -relaxed k -broadcast graph on n vertices with $B_k^t(n)$ edges. Moreover, Harutyunyan and Liestman [54] proved that if $t \geq 1$, then the function $B_k^t(n)$ is monotone in the range $(k+1)^{m-1} + 1 \leq n \leq (k+1)^m$. That means if $k = t = 1$, then

$B^1(n)$ is monotone in the range $2^{m-1} + 1 \leq n \leq 2^m$. Here, the significant effects of just one extra time unit become evident.

By the above, exploring these conjectures becomes more difficult as n approaches $2^{\lceil \log n \rceil}$. In addition, it is clear that $B_k^t(n) \geq B_k^{t+1}(n)$, that is, that $B_k^t(n)$ is monotone for all $t \geq 0$. Whether $B_k^t(n)$ is monotone with respect to k is a complicated question. In particular, if $\lceil \log_{k+1} n \rceil \neq \lceil \log_{l+1} n \rceil$ the relationship between $B_k^t(n)$ and $B_l^t(n)$ is unclear. However, if $\lceil \log_{k+1} n \rceil = \lceil \log_{k+2} n \rceil$, then $B_k^t(n) \geq B_{k+1}^t(n)$ for $k \geq 1$ and for any $t \geq 0$.

1.5 Minimum Broadcast Graphs and Broadcast Function

In this section, we present various techniques for constructing broadcast graphs and computing upper bounds for broadcast functions. Farley [32] constructs broadcast graphs recursively by combining two or three smaller broadcast graphs and shows that $B(n) \leq \frac{n}{2} \lceil \log n \rceil$. This construction was generalized in [22] by incorporating up to seven smaller broadcast graphs. Let $L(n)$ denote the number of consecutive leading 1s in the binary representation of $n - 1$. Grigoryan and others [44] established the asymptotically tight bound $B(n) = \theta(L(n) \cdot n)$ by proving that $\frac{L(n) - 1}{2}n \leq B(n) \leq (L(n) + 2)n$.

Compounding is a key method for constructing new broadcast graphs. Introduced in [70], it relies on the vertex cover of graphs, a set of vertices that contains at least one of the endpoints of each edge. The method was extended in [16] using solid vertex covers to cover arbitrary n . An equivalent formulation using center vertices is given in [99] and shown to be equivalent to the solid vertex cover method in [24]. A merging technique for vertices is introduced in [52]. Another construction involves compounding binomial trees with hypercubes, as presented in [5, 50].

Modifying existing graphs through the addition or deletion of vertices and edges has also been explored [15]. One such method adds a vertex and specific edges to a Knödel

graph on $2^k - 2$ vertices, as proposed in [49]. The new vertex is connected to a dominating set of the Knödel graph. This construction is generalized in [55] to broader classes of Knödel graphs. Ad hoc constructions that yield effective upper bounds typically involve adding edges to a binomial tree [44, 52, 70], while vertex deletion methods have also been studied [15]. Additional constructions can be found in [15, 44, 52, 99, 100]. For more precise upper bounds, we refer the reader to Chapter 4.

Lower bounds on $B(n)$ are also studied in the literature. In [43], it is shown that $B(n) \geq \frac{n}{2}(\lfloor \log n \rfloor - \log(1 + 2^{\lfloor \log n \rfloor} - n))$, for any n . also, $B(n) \geq \frac{n}{2}(m - k - 1)$ is proved in [53], where m is the length of the binary representation $a_{m-1}a_{m-2} \dots a_1a_0$ of n and k is the index of the leftmost 0 bit. This bound is improved to be $B(n) \geq \frac{n}{2}(m - k - 1 + b)$, where $b = 0$ if $k = 0$ or $a_0 = a_1 = \dots = a_{k-1} = 0$ and $b = 1$ otherwise. In [75], it is proved that $B(n) \geq \frac{m^2(2^m - 1)}{2(m + 1)}$, for $n = 2^m - 1$. Tight lower bounds on $B(2^m - 3)$, $B(2^m - 4)$, $B(2^m - 5)$ and $B(2^m - 6)$ are given in [88]. The lower bounds on $B(2^m - 2^k)$ and $B(2^m - 2^k + 1)$, where $3 \leq k \leq m$ are presented in [45].

Chapter 2

Broadcast Schemes of Hypercubes

Hypercube graphs form a family of graphs whose orders are powers of 2 and are examples of minimum broadcast graphs (mbgs). Their properties, such as regularity and vertex and edge transitivity, make them strong candidates for applications in many scientific fields, especially computer science.

In this chapter, we consider broadcasting in the hypercube graphs. In [34], the authors demonstrated the existence of specific broadcasting called dimensional broadcast schemes in the hypercube H_k . These schemes require that if $(\alpha_1, \dots, \alpha_k)$ is a permutation of dimensions $(1, 2, \dots, k)$ of hypercube H_k , every informed vertex at time unit i must inform its neighbor along dimension α_i .

In this chapter, we extend these schemes and propose an algorithm that generates all possible minimum-time broadcast schemes for hypercubes. We also, give an enumeration of all valid minimum-time broadcast schemes. The results presented in this chapter have been published in [2].

2.1 Notations

Definition 2.1.1. The **hypercube** of dimension k , denoted by H_k , is a simple graph with vertices representing 2^k binary strings of length $k \geq 1$ such that adjacent vertices have binary strings differing in exactly one-bit position, called a dimension, numbered $1, \dots, k$. An edge lies along dimension i if the differing bit is in position i . Also, H_k is defined recursively as follows (for any $k \in \mathbb{N}$):

- H_0 is a single vertex with no edges.
- H_k consists of two copies of H_{k-1} with edges joining corresponding vertices with a new dimension k , for any $k \geq 1$.

We use both definitions in the proofs. Since hypercubes are vertex and edge transitive [66], with no loss of generality, throughout this chapter, vertex 00...00 is considered to be the originator. For simplicity, we refer to it as 0.

Definition 2.1.2. A **binomial tree** B_k of dimension k on 2^k vertices is defined recursively as follows:

B_0 is a single vertex with no edges.

B_k consists of two copies of B_{k-1} connecting their roots by an edge and considering one of them as the root of B_k (see Figure 2.1).

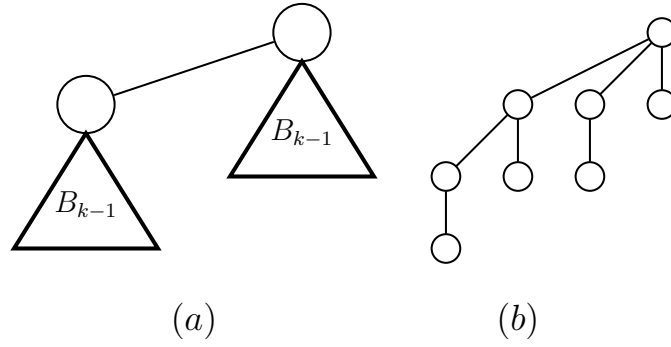


Figure 2.1: a) Construction of B_k from two copies of B_{k-1} . b) An example of binomial tree B_3 .

Let u be the root of a binomial tree B_k . Then B_k has 2^k vertices and $b(B_k, u) = k$. Note that if every informed vertex is active (i.e., sends the message) during each time unit of the process, then the resulting broadcast scheme after time unit t forms a binomial tree B_t .

For a hypercube H_k , if all informed vertices use dimension t in time unit t , for $1 \leq t \leq k$, then every vertex will be informed after time unit k . Hence, $b(H_k) = k$, and so H_k is a broadcast graph (bg).

Note that if G is a broadcast graph on 2^k vertices, then, regardless of the originator, every vertex must be active in all time units to ensure that all vertices are informed after time unit k . Consequently, for every vertex $v \in V(G)$, we must have $\deg(v) \geq k$. Therefore, the total number of edges in G is at least $\frac{1}{2}k2^k = k2^{k-1}$.

Since the hypercube H_k has exactly $k2^{k-1}$ edges, it achieves this minimum, and hence H_k is an mbg.

The distance between two vertices u and v in a graph G is the shortest path length between them, denoted by $d(u, v)$.

2.2 Main Results

In this section, we provide our algorithm to generate valid minimum-time broadcast schemes in hypercubes. Furthermore, we prove its correctness and enumerate all algorithm-generated broadcast schemes. Finally, we show that the algorithm gives all valid minimum-time broadcast schemes in hypercubes.

We use the next two combinatorial lemmas to prove the main theorem.

Lemma 2.2.1. *For integers t and k such that $2 \leq k \leq t$ the following holds:*

$$\binom{k-2}{k-2} + \binom{k-1}{k-2} + \binom{k}{k-2} + \cdots + \binom{t-2}{k-2} = \binom{t-1}{k-1}.$$

Proof. We provide a combinatorial proof. RHS represents the total number of ways to select $k-1$ elements from a set of $t-1$ elements. We evaluate this by examining cases based on the choices involving specific elements from the set:

- There are $\binom{t-2}{k-2}$ ways to select the subset such that the last element is chosen.
- In $\binom{t-3}{k-2}$ ways, the second-to-last element is selected while the last element is excluded.
- Similarly, $\binom{t-4}{k-2}$ counts the cases where the third-to-last element is chosen, with the last two elements excluded.

This process continues until only the first $k-1$ elements are chosen which is exactly one way to choose them. The summation of these terms (LHS) thus equates to the RHS. $\square$

Lemma 2.2.2. *For positive integers n and k such that $k < n$ the following holds:*

$$\binom{k-1}{k-1}(n-k) + \binom{k}{k-1}(n-k-1) + \binom{k+1}{k-1}(n-k-2) + \cdots + \binom{n-2}{k-1} = \binom{n}{k+1}.$$

Proof. As a combinatorial proof, observe that the first term on the left-hand side counts the number of ways to choose $k + 1$ elements from the set $\{1, \dots, n\}$ such that the second-largest element is k . Similarly, the second term counts the number of such subsets where the second-largest element is $k + 1$, and so on, up to the last term, which corresponds to the case where the second-largest element is $n - 1$. Summing over all possible values of the second-largest element yields the total number of ways to choose $k + 1$ elements from n , as given by the right-hand side. $\square$

We present our algorithm, whose core idea is based on the recursive definition of hypercubes. Let H_n be a hypercube of dimension n . When the originator selects an arbitrary dimension i in the first time unit, the hypercube H_n is effectively divided into two smaller hypercubes H_{n-1} by deleting all i -th dimensional edges, each containing an informed vertex. These informed vertices can then independently broadcast within their respective H_{n-1} subhypercubes. In the second time unit, the informed vertices in the two H_{n-1} subhypercubes are free to use different, arbitrary dimensions for broadcasting, with the restriction that neither they nor their descendants can reuse dimension i in subsequent time units. This process continues, utilizing the recursive structure of H_{n-1} in each time unit to advance the broadcasting process.

The algorithm operates as follows: Initially, the originator has a set of all dimensions, referred to as the *allowed* set. When an informed vertex u uses arbitrary dimension j from its own allowed set to inform a neighbor v , vertex u removes dimension j from its own allowed set and transfers the remaining dimensions to v . This ensures that each vertex broadcasts

without violating the dimensional constraints dictated by the structure of hypercube.

Algorithm 1: Broadcasting in a Hypercube Using Allowed Dimensions

Input: Hypercube $G = (V, E)$ of dimension n , originator o , message M

Output: All vertices in G receive message M

```

1 foreach  $v \in V$  do
2   | Mark  $v$  as uninformed;
3 Mark  $o$  as informed;
4 initialize  $t \leftarrow 1$ ;
5 Initialize allowed dimensions for  $o$ :  $allowed(o) \leftarrow \{1, 2, \dots, n\}$ ;
6 while  $t \leq n$  do
7   |  $t \leftarrow t + 1$ ;
8   |  $curr\_informed \leftarrow \{v \in V \mid \text{vertex } v \text{ is marked as informed}\}$ ;
9   | foreach vertex  $v \in curr\_informed$  do
10    | Arbitrarily choose a dimension  $i \in allowed(v)$ ;
11    | Remove  $i$  from  $allowed(v)$ ;
12    | Let  $u$  be the neighbor of  $v$  along dimension  $i$ ;
13    |  $M(u) \leftarrow M$  // vertex  $u$  receives the message;
14    |  $allowed(u) \leftarrow allowed(v)$  // Pass remaining allowed
    | dimensions;
15    | Mark  $u$  as informed;

```

Since each call at time unit t uses a dimension, say i , it informs two vertices belonging to two different smaller hypercubes. Moreover, none of the subsequent broadcasts initiated from these vertices reuse dimension i , ensuring that their descendants remain in two disjoint sub-hypercubes and do not overlap. As a result, the broadcast processes are entirely separated, and the number of informed vertices before time unit t is exactly 2^{t-1} . Therefore, the `for` loop in line 9 iterates exactly 2^{t-1} times for each t . Consequently, this results

in a total of $\sum_{i=1}^n 2^{i-1} = 2^n - 1$ iterations. Hence, the time complexity of the algorithm is $\Theta(2^n)$.

The recursive structure of hypercubes is instrumental in our approach. In the following theorem, we utilize this structure to prove the correctness of the proposed algorithm.

Theorem 2.2.3. *Let H_n be a hypercube of dimension n . Then any broadcast scheme generated by Algorithm 1 is a minimum time broadcast scheme in H_n .*

Proof. We prove the validity by induction on n . The base case $n = 1$ is straightforward. Suppose that $n \geq 1$. Without loss of generality, assume that vertex $0 = (0, \dots, 0)$ uses the first dimension in time unit 1 to inform vertex $u = (1, 0, \dots, 0)$. By the recursive structure of hypercubes, H_n consists of two copies, H_{n-1}^0 and H_{n-1}^1 , which contain vertices 0 and u , respectively. Thus, in each H_{n-1}^i for $i = 0, 1$, there is an initially informed vertex.

According to Algorithm 1, no vertex will use dimension 1 again. By the induction hypothesis, any broadcast scheme in Algorithm 1 for H_{n-1}^0 and H_{n-1}^1 is valid and completes in $n-1$ time units. Consequently, combining any two schemes for H_{n-1}^0 and H_{n-1}^1 coming from Algorithm 1 with the use of the first dimension in time unit 1 forms a valid minimum-time broadcast scheme for H_n . $\square$

Remark 2.2.1. *After each time unit, when a vertex informs its neighbor of dimension i , all its descendant vertices in this branch are prevented from using dimension i in next time units. Thus, it appears that by following Algorithm 1, many possible valid minimum-time broadcast schemes are ignored.*

To illustrate this with a simple observation, consider the following scenario. Without loss of generality, suppose that the first dimension is used in time unit 1. Assume that two vertices, $0a_2 \dots a_{n-2}11$ and $1a_2 \dots a_{n-2}10$, are informed. Imagine that in one of the time units, instead of informing vertices $0a_2 \dots a_{n-2}10$ and $1a_2 \dots a_{n-2}11$, respectively, these vertices inform $1a_2 \dots a_{n-2}11$ and $0a_2 \dots a_{n-2}10$ by using dimension 1, respectively, as shown in

Figure 2.2. At first glance, it appears that this broadcast scheme can be completed in the minimum number of time units, yet Algorithm 1 does not generate this broadcast scheme. However, the next theorem, surprisingly, shows that not only can this scenario not occur, but also that Algorithm 1 gives us all valid minimum-time broadcast schemes.

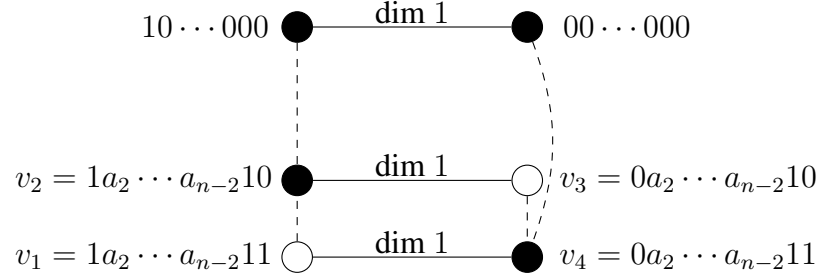


Figure 2.2: Informed vertices v_2 and v_4 use dimension 1 and inform vertices v_3 and v_1 , respectively, instead of informing v_1 and v_3 , respectively.

In the next theorem, we show that the previous remark is not happening, and Algorithm 1 gives us all valid minimum-time broadcast schemes.

Theorem 2.2.4. *Let H_n be a hypercube of dimension n . All n -time unit valid broadcast schemes of H_n can be generated by Algorithm 1.*

Proof. Note that if B_n is a binomial spanning subtree of H_n rooted at an arbitrary vertex u , then starting the broadcast from the originator u gives a valid minimum-time broadcast scheme. Conversely, since H_n has 2^n vertices, every informed vertex must be active in each time unit of any valid minimum-time broadcast scheme (consisting of n time units). Hence, the tree resulting after n time units, by starting from any originator, is a binomial tree B_n . Therefore, every valid minimum-time broadcast scheme corresponds to a spanning binomial subtree of H_n of dimension n , and vice versa. Note that by Theorem 2.2.3, all broadcast schemes produced by Algorithm 1 are valid. We show that the number of binomial trees of H_n is equal to the total number of broadcast schemes produced by Algorithm

1.

First, we analyze Algorithm 1. Since in each time unit, one dimension is removed from the allowed set of each informed vertex, in time unit t , each informed vertex has $n - t + 1$ available dimensions to send the message. Thus, because there are 2^{t-1} informed vertices before time unit t , the broadcast scheme of Algorithm 1 has $(n - t + 1)^{2^{t-1}}$ ways to extend the scheme in time unit t . Consequently, the total number of broadcast schemes produced by Algorithm 1 is

$$n(n-1)^{2^1}(n-2)^{2^2} \cdots 3^{2^{n-3}} 2^{2^{n-2}} = \prod_{i=0}^{n-1} (n-i)^{2^i}.$$

Now, we show that the number of spanning binomial trees of H_n , denoted by $f(n)$, also equals $\prod_{i=0}^{n-1} (n-i)^{2^i}$.

To compute the distances of all vertices from the originator 0, we employ a breadth-first search (BFS) algorithm. It is worth mentioning that the distance between two vertices is equal to the number of different bits in their binary representation. We classify vertices based on this distance, denoting all vertices at a distance d from 0 as level d vertices.

We demonstrate, by using strong induction on d , that in every valid minimum-time broadcast scheme, all level $d+1$ vertices must be informed just by level d vertices, $0 \leq d \leq n-1$.

This means that every broadcast scheme of H_n is a shortest-path broadcast scheme. In other words, every vertex receives the message via a shortest path from the originator. Note that every informed vertex cannot be idle at all time units. If not, then the number of informed vertices after time unit n will be less than 2^n . So, the originator, vertex 0, is responsible for informing all its neighbors, which shows the correctness of induction's base case for $d = 0$. Now, assume as the inductive hypothesis that for $1 \leq l \leq d$, all level l vertices are informed just by level $l-1$ vertices. We aim to show that this property holds for $\binom{n}{d+1}$ vertices of level $d+1$. It is worth mentioning that once a level d vertex u is informed in

time unit t , it is obligated to inform $n - t$ of its neighbors at the level $d + 1$ because there is no idle vertex as mentioned above and u cannot inform any level $d - 1$ vertices by IH.

Next, we use a second induction on d to claim that the number of level d vertices that become informed during time unit t is $\binom{t-1}{d-1}$. If $d = 1$, then $\binom{t-1}{d-1} = 1$. It means that in time unit t , just one of the neighbors of 0 receives the message from 0, which is correct. So the base case is done. The claim is correct because $\binom{t-1}{d-1}$ count matches the number of informed vertices at level $d - 1$ by the end of time unit $t - 1$, which by IH is

$$\binom{d-2}{d-2} + \binom{d-1}{d-2} + \binom{d}{d-2} + \cdots + \binom{t-2}{d-2}.$$

It is equal to $\binom{t-1}{d-1}$ by Lemma 2.2.1. Hence, all level d vertices must be able to inform

$$\binom{d-1}{d-1}(n-d) + \binom{d}{d-1}(n-d-1) + \binom{d+1}{d-1}(n-d-2) + \cdots + \binom{n-2}{d-1}$$

level $d+1$ vertices by time n . But, by Lemma 2.2.2, this sum equals $\binom{n}{d+1}$ which means that all level $d+1$ vertices must be informed just by level d vertices.

Assume, without loss of generality, that the originator vertex $0 = 00 \cdots 0$ informs vertex $u = 100 \cdots 0$ by using dimension 1 in the first time unit. Let $V_0 = \{0a_2 \cdots a_n \mid a_i \in \{0, 1\}, 2 \leq i \leq n\}$ and $V_1 = \{1a_2 \cdots a_n \mid a_i \in \{0, 1\}, 2 \leq i \leq n\}$. We claim that no vertex can use dimension 1 in other time units.

If $v \in V_1$ is a vertex such that $d(u, v) = d$ and uses dimension 1 in time unit $t > 1$ to inform $v' \in V_0$, then $d(0, v) = d + 1$ and $d(0, v') = d$. This would mean a level $d + 1$ vertex informs a level d vertex, as illustrated in Figure 2.3, a contradiction. If $v \in V_0$ uses dimension 1 in time unit $t > 1$ to inform a vertex in V_1 , then since no vertex of V_1 can send back to V_0 , implying that vertex v cannot contribute to informing the maximum number of vertices within V_0 , and hence the broadcast within V_0 cannot be completed, a contradiction.

Therefore, the claim is proved.

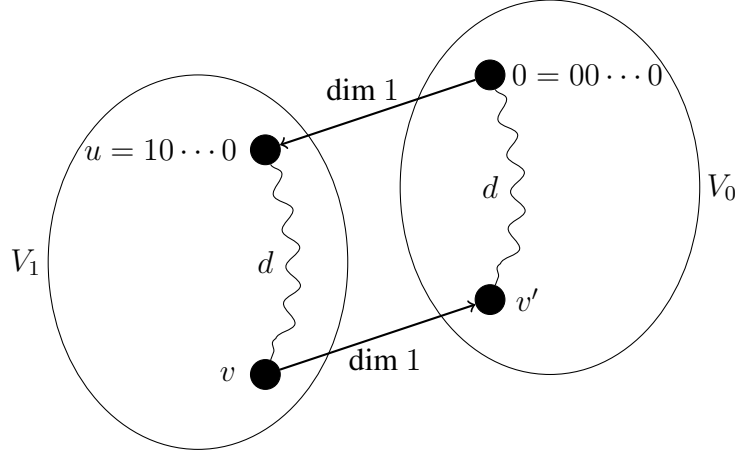


Figure 2.3: A vertex $v \in V_1$, with $d(u, v) = d$, uses dimension 1 in time unit $t > 1$ to inform $v' \in V_0$.

Since there is a one-to-one correspondence between valid minimum-time broadcast schemes and binomial trees, and the originator has n dimensions to use in the first time unit, one can see that $f(n) = nf^2(n-1)$, with $f(1) = 1$, where $f(n)$ is the number of all spanning binomial trees of H_n . By induction on n , we show that $f(n) = \prod_{i=0}^{n-1} (n-i)^{2^i}$. Assuming by IH that this holds for $n-1$, we have

$$f(n) = n \left(\prod_{i=0}^{n-2} (n-1-i)^{2^i} \right)^2 = n \prod_{i=0}^{n-2} (n-(i+1))^{2^{i+1}} = \prod_{j=0}^{n-1} (n-j)^{2^j}.$$

Therefore, the number of binomial trees of H_n is equal to the total number of broadcast schemes produced by Algorithm 1. □

As an immediate consequence of the previous proof, we have the following corollary.

Corollary 2.2.5. *The number of binomial subtrees rooted from vertex 0 as well as all valid*

minimum-time broadcast schemes of a hypercube H_n is

$$n(n-1)^{2^1}(n-2)^{2^2} \cdots 3^{2^{n-3}} 2^{2^{n-2}} = \prod_{i=0}^{n-1} (n-i)^{2^i}.$$

As highlighted in Corollary 2.2.5, the number of valid minimum-time broadcast schemes for a hypercube H_n is in $\Omega(2^{2^{n-2}})$. This exponential growth underscores the inherent difficulty of identifying all valid minimum-time broadcast schemes, even for a highly structured graph such as a hypercube. When considering general graphs, which often lack the regularity and symmetry of hypercubes, the complexity of the problem becomes even more pronounced.

2.3 Enumerating All Broadcast Schemes of Algorithm 1

Algorithm 1 not only provides a valid minimum-time broadcast scheme but also enables the generation of all possible minimum-time broadcasting schemes (discussed in Theorem 2.2.4). Since Line 7 of the algorithm selects an arbitrary number from the set of allowed vertices, we can systematically enumerate all broadcast schemes generated by this algorithm. Furthermore, the algorithm facilitates iteration through these schemes and allows unique identification by assigning a distinct number to each scheme. For instance, a hypercube of dimension 3 has 12 possible broadcasting schemes. In this section, we discuss how to determine the details of a scheme corresponding to a given broadcast number (ranging from 0 to number of possible schemes $- 1$) and vice versa.

For an informed vertex u , let $H^{u,t}$ denote the largest sub-hypercube containing u as its only informed vertex after time unit t . When u uses dimension i to send the message to vertex v at time unit t , $H^{u,t-1}$ is divided into two smaller hypercubes, $H^{u,t}$ and $H^{v,t}$. Because dimension i is removed from the allowed sets of both u and v , the hypercubes

$H^{u,t}$ and $H^{v,t}$ become distinct and separated. We call $H^{u,t}$ the first hypercube containing u (the sender) and $H^{v,t}$ the second hypercube containing v (the receiver).

Let H_n be a hypercube for $n \geq 3$. Suppose that $0 \leq s \leq f(n)-1$ is an arbitrary number, where $f(n) = \prod_{i=0}^{n-1} (n-i)^{2^i}$ is the number of valid minimum-time broadcast schemes of H_n by Corollary 2.2.5. We show that s uniquely corresponds to a specific broadcast scheme.

Define the cluster size $c(n) = \frac{f(n)}{n}$, which represents the total number of minimum-time broadcast schemes after the first time unit, where the originator has sent the message to one of its neighbors. For simplicity, we denote it by c . Now, consider the representation of s in base c , $s = (a_1, a_2)_c$, which contains at most two numbers denoted by a_1 and a_2 because $a_1 = \lfloor \frac{s}{c} \rfloor < n \leq c$ for $n \geq 3$. Obviously, $a_1 + 1$ equals the first dimension that the originator u uses in the first time unit to inform v . We show that the remainder a_2 represents two broadcast schemes of the two smaller hypercubes H_{n-1} . It is worth mentioning that $c' = \sqrt{c}$ equals the number of minimum-time broadcast schemes in H_{n-1} . Similarly, the representation of the remainder a_2 in base c' has at most two numbers, denoted $a_2 = (b_1, b_2)_{c'}$. Now, the $(b_1 + 1)$ -th scheme of the first hypercube and the $(b_2 + 1)$ -th scheme of the second hypercube, together with the edge uv , form a binomial tree, which constitutes a broadcast scheme for H_n .

To prove its correctness, note that if s and s' are two different numbers, then their representations in the same base (c or c') are different. Therefore, the resulting broadcast schemes are distinct.

For the inverse, suppose that B is a minimum-time broadcast scheme of H_n . Assume the originator u uses dimension i to inform vertex v in the first time unit, and the broadcast scheme in $H^{u,1}$ is the b_1 -th broadcast scheme, while the broadcast scheme in $H^{v,1}$ is the b_2 -th broadcast scheme. Then $s = (i - 1, a_2)_c$, where $a_2 = (b_1 - 1, b_2 - 1)_{c'}$, is the enumeration of B in Algorithm 1. Again, it is easy to verify that two different schemes yield two different numbers.

For instance, in Figure 2.4, let vertex A be the originator, and define the dimensions as follows: $A \rightarrow B$ corresponds to dimension 1, $A \rightarrow D$ to dimension 2, and $A \rightarrow E$ to dimension 3. After sending along $A \rightarrow B$ in dimension 1, this dimension is removed from the set of allowed dimensions, dividing the hypercube H_3 into two separate smaller hypercubes H_2 . As discussed earlier, the first hypercube is designated as the one containing the sender A . Thus, $ADHE$ forms the first hypercube, and $BCGF$ forms the second. The broadcasting scheme used within these smaller hypercubes determines the final number corresponding to the complete broadcasting scheme.

There are two possible schemes for H_2 . Using the same logic, if sending the message occurs along $A \rightarrow D$, corresponding to the smaller-numbered dimension, this is designated as $scheme_1$. Similarly, sending the message along $A \rightarrow E$ is assigned as $scheme_2$.

Now, consider the scenario where the first hypercube follows $scheme_2$ ($A \rightarrow E$), and the second hypercube follows $scheme_1$ ($B \rightarrow C$). Combining these two schemes yields the binary number (10). The binary (10) corresponds to decimal 2, which represents this particular broadcasting scheme for the smaller hypercubes. Finally, we incorporate the initial step, $A \rightarrow B$, which occurred along the first dimension. In base 4, this corresponds to (02), resulting in the final scheme number being 2.

Thus, the following scheme corresponds to

$$\begin{aligned} t_1 : A &\rightarrow B, \\ t_2 : A &\rightarrow E, B \rightarrow C, \\ t_3 : A &\rightarrow D, B \rightarrow F, E \rightarrow H, C \rightarrow G. \end{aligned} \tag{1}$$

Conversely, if we are given scheme number 2 and need to determine the corresponding broadcast scheme, we proceed as follows:

First, we express the number into base 4, which is equal to $(02)_4$. The first digit (0) indicates the action selected by the originator, meaning the originator chose its first dimension,

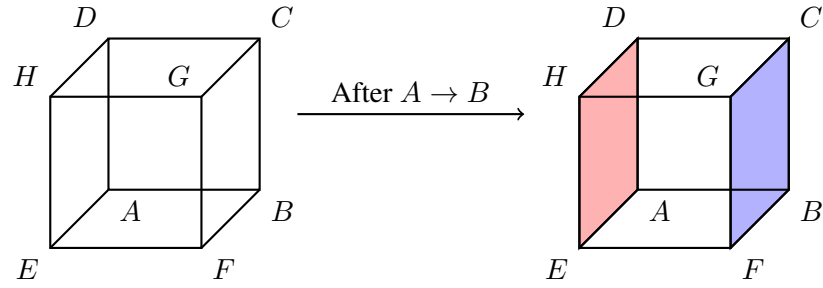


Figure 2.4: Transformation of the hypercube to two smaller hypercubes after the first transmission.

$A \rightarrow B$.

Next, the numbers corresponding to the two hypercubes are represented as $(10)_2$. This indicates that the first hypercube has chosen its second broadcast scheme, $A \rightarrow E$, because we defined the order of the dimensions earlier. The second hypercube has selected its first broadcast scheme, $B \rightarrow C$. Finally, the remaining scheme leads us directly to Relations (1).

2.4 Conclusion

In this chapter, we studied the minimum-time broadcast schemes in hypercube graphs. We introduced an algorithm that generates all valid optimal broadcast schemes. Our analysis revealed the enumeration and compact encoding/decoding of every possible scheme for minimum-time broadcasting in hypercubes. While constructing a naive minimum-time broadcast scheme is straightforward, our main contribution lies in the complete characterization of all such schemes and the demonstration that no other variants exist beyond those generated by our algorithm.

Chapter 3

All sequential dimensional broadcast schemes in Knödel graphs

In this chapter, we study another family of graphs known as *Knödel graphs*, which are also broadcast graphs. Their significance lies in the fact that they are defined for every even number of vertices. Therefore, for any even integer n , there exists a broadcast graph of order n , and so an upper bound for the broadcast function $B(n)$. The results presented in Sections 3.2 and 3.3 have been published in [30].

3.1 Notation

Throughout this chapter, for a set A and real numbers b and c , by $bA + c$, we mean $\{bx + c \mid x \in A\}$. Furthermore, when we refer to the second (d -th) dimension in a permutation $(\alpha_1, \dots, \alpha_n)$, where $\{\alpha_1, \dots, \alpha_n\} = \{1, \dots, n\}$, we mean α_2 (respectively α_d), rather than dimension 2 (dimension d) itself. Otherwise, we explicitly mention dimension 2 (or dimension d).

Definition 3.1.1. A **Knödel graph** of even order n and degree $\lfloor \log n \rfloor$, first introduced by

W. Knödel [72], is denoted by $KG(n) = (V, E)$, where the vertex set is $V = \{i \mid 0 \leq i \leq n-1\}$ and the edge set is $E = \{(i, j) \mid i+j \equiv 2^s-1 \pmod{n}, 1 \leq s \leq \lfloor \log n \rfloor, \text{ and } i, j \in V\}$. We refer to vertex j as the s dimensional neighbor of vertex i whenever $i+j \equiv 2^s-1 \pmod{n}$.

The Knödel graph, widely recognized for its numerous applications in broadcasting, has been extensively studied in the literature. The Knödel graph $KG(2^k)$ is one of three non-isomorphic infinite mbg graph families. The other two families are the well-known hypercube [34] and the recursive circulant graph [85]. $KG(2^k-2)$ is also an mbg [23, 70]. A detailed description of various graph-theoretic and communication properties of these three graph families, as well as a comparison among them, can be found in [35]. Therefore, in this thesis, we consider $KG(n)$ for $n \neq 2^k$.

As demonstrated in [17], the edges of the Knödel graph $KG(n)$ can be partitioned into $\lfloor \log n \rfloor$ perfect matchings, referred to as dimensions, which can be utilized similarly to the hypercube for broadcasting. Unlike the hypercube, which is defined only for powers of 2, the Knödel graph is defined for any even number of vertices. Properties such as vertex transitivity as a Cayley graph [67], high vertex and edge connectivity, dimensionality, and embedding properties [35], and a small diameter [46] make the Knödel graph a suitable candidate for different kinds of networks.

The symmetry of $KG(n)$ ensures minimum time broadcasting from any originator, thereby qualifying it as a broadcast graph [17, 36, 72]. Thus, throughout this chapter, the originator is considered as vertex 0. Moreover, $KG(n)$ is utilized in the construction of sparse broadcast graphs of larger sizes by interconnecting multiple smaller broadcast graphs, or by adding vertices and edges, deleting vertices, and modifying the edges of another broadcast graph [16, 24, 49, 48, 52, 54, 55, 70]. Additional insights into broadcasting and related topics can be found in [15, 34, 39, 63, 65, 83, 68, 69, 84].

Similar to hypercubes of dimension k , which have dimensions $1, \dots, k$, Knödel graphs

of order n have dimensions $1, \dots, \lceil \log n \rceil$. Consequently, the simplest broadcast algorithm for Knödel graphs involves using these dimensions, a method referred to as the *dimensional broadcast scheme* [17]. Given a permutation $D = (\alpha_1, \alpha_2, \dots, \alpha_k)$, in time unit i , every informed vertex sends the message to its α_i -dimensional neighbor. The permutation D is called *valid* if every vertex is informed after k time units by utilizing D . Note that the notion of dimensions in Knödel graphs is weaker compared to hypercubes because any permutation $D = (\alpha_1, \dots, \alpha_k)$ is a valid broadcast scheme in hypercubes, while it is not true in Knödel graphs. Also, since the number of dimensions in Knödel graphs is less than the number of time units by one, we have to repeat one of the dimensions in any dimensional broadcast scheme. In [17], it has been shown that, similar to hypercubes, broadcasting in $KG(n)$ can be done by using dimensions in some particular order. In particular, the authors proved that

Theorem 3.1.2. [17, Theorem 1] *The permutation $(1, 2, \dots, k-1, 1)$ is a valid permutation for $KG(n)$ and any cyclic shift (or the reversal of such a cyclic shift) of $(1, 2, \dots, k-1, 1)$ is also a valid permutation for $KG(n)$, where $k = \lceil \log n \rceil$ and $n \neq 2^k$.*

We refer to the permutation $1, 2, \dots, k-1$ as the *standard dimensional broadcast scheme*. Note that by Theorem 3.1.2 in the last time unit, the first dimension must be repeated; however, in general, any other dimension can be repeated. In general, knowing all possible broadcasting algorithms, particularly dimensional broadcast schemes, is beneficial due to the flexibility it provides in broadcasting on Knödel graphs. This is especially important in some broadcast graph constructions, like [52], where a specific vertex must be informed at a particular time unit. Unfortunately, there is no necessary and sufficient condition for a permutation to be valid in general. The problem of finding all dimensional broadcast schemes is very difficult. In fact, there are only two other dimensional broadcast schemes known in the literature. Theorems 1 and 2 of [51] show that if $2^{k-1} < n \leq 2^k - 4$

(or $n \neq 2^k - 2$) for an integer $k \geq 3$, then the standard dimensional broadcast scheme and the scheme with one right cyclic shift, both with the second dimension repeated (namely $(1, \dots, k-1, 2)$ and $(k-1, 1, \dots, k-2, 1)$) are valid permutations.

Figure 3.1(a) illustrates $KG(10)$ with a dimensional broadcast scheme $(3, 1, 2, 1)$, one right cyclic shift with repeating the second dimension in this permutation, which is valid. However, Figure 3.1(b) shows that two right cyclic shifts with repeating the second dimension, i.e., $2, 3, 1, 3$, is not a valid dimensional broadcast scheme. Therefore, the generalization of Theorem 2 of [51] with more right cyclic shifts is not always true. In this chapter, we show that more narrowing the number of vertices n can generate more right cyclic shifts by repeating the second dimension as a valid dimensional broadcast scheme. These valid dimensional broadcast schemes investigated in this chapter allow us to broadcast graphs in various ways, ensuring that certain vertices are informed at the last time unit. This approach provides the potential to create new broadcast graphs by leveraging the properties of Knödel graphs and other similar structures.

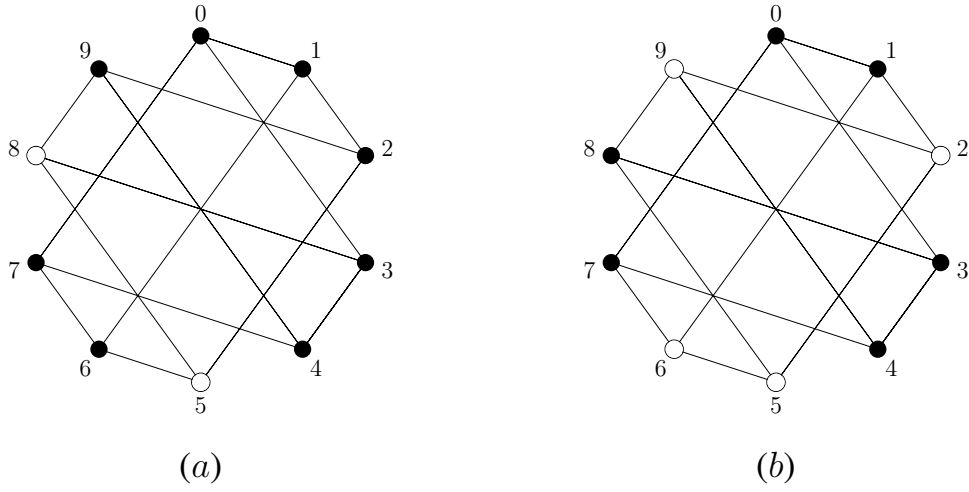


Figure 3.1: In these two Knödel graphs of order 10, black and white vertices are the informed and uninformed vertices before the last time unit. In (a), dimensional broadcast scheme is $(3, 1, 2, 1)$, one right cyclic shift of 1, 2, 3 with repeating the second dimension, gives a valid dimensional broadcast scheme. However, in (b), two right cyclic shifts of 1, 2, 3 with repeating the second dimension, i.e. dimensional broadcast scheme $(2, 3, 1, 3)$, is not valid. Note that 2 and 5 will be uninformed after the last time unit.

3.2 Arbitrary Cyclic Shifts of Standard Dimensional Broadcast Scheme With Repeating the Second Dimension

In this section, we answer the question of finding all valid cyclic shifts of dimensional broadcast schemes where the second dimension is repeated.

Let $KG(n)$ be a Knödel graph of order n , where $k = \lceil \log n \rceil$. According to the above, two right cyclic shifts of the standard dimensional broadcast scheme $(1, 2, \dots, k-1)$ with repeating the second dimension is not always valid. Note that by repeating the second dimension in a dimensional broadcast scheme, the chances of getting a valid broadcast scheme for arbitrary n are limited.

Examining the i right-cyclic shifts, where i denotes the number of such shifts, of dimensional broadcast schemes with repeated dimensions is a compelling research question that will be addressed later in this chapter. Instead, we will begin by concentrating on the two right cyclic shifts ($i = 2$), as their proof is more straightforward and may provide insights into the investigation of an arbitrary i right cyclic shifts. This approach can facilitate a clearer understanding of the proof of the general case.

In the next theorem, we specify the complete range of values for n where such a dimensional broadcast scheme remains valid.

Theorem 3.2.1. *Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil \geq 3$. Then the dimensional broadcast scheme $(k-2, k-1, 1, 2, \dots, k-3, k-1)$ is a valid broadcast scheme if and only if $n \leq 2^{k-1} + 2^{k-3} - 2$.*

Proof. In the first time unit, 0 informs $2^{k-2} - 1$. In the second time unit, 0 and $2^{k-2} - 1$ send the message to $2^{k-1} - 1$ and 2^{k-2} , respectively. In the third time unit, 0, $2^{k-2} - 1$, 2^{k-2} , and $2^{k-1} - 1$ inform 1, $n - 2^{k-2} + 2$, $n - 2^{k-2} + 1$, and $n - 2^{k-1} + 2$, respectively. Therefore, the informed vertices after time unit 3 are

$$0, 1, n - 2^{k-1} + 2, 2^{k-2} - 1, 2^{k-2}, n - 2^{k-2} + 1, n - 2^{k-2} + 2, 2^{k-1} - 1. \quad (2)$$

We claim that after time unit t , $3 \leq t \leq k-1$, the following vertices will be informed.

$$A = \{i \mid 0 \leq i \leq 2^{t-2} - 1\},$$

$$B = \{n - 2^{k-1} + 2i \mid 1 \leq i \leq 2^{t-3}\}$$

$$I_1 = \{2^{k-2} + i \mid -1 \leq i \leq 2^{t-2} - 2\},$$

$$I_1' = \{n - 2^{k-2} + i \mid 1 \leq i \leq 2^{t-2}\},$$

$$C = \{2^{k-1} + 2i - 1 \pmod{n} \mid 0 \leq i \leq 2^{t-3} - 1\}.$$

We prove this claim by induction on t . By (2), the claim is true for $t = 3$. Suppose that the claim is true for $t - 1$. During time unit t , $4 \leq t \leq k - 1$, vertices $0, 1, \dots, 2^{t-3} - 1$, which are already informed by induction hypothesis, inform $2^{t-2} - 1, 2^{t-2} - 2, \dots, 2^{t-3}$, respectively. Also, every informed vertex $n - 2^{k-1} + 2i \in B$, for $1 \leq i \leq 2^{t-4}$, informs $2^{k-1} + 2^{t-2} - 2i - 1 = 2^{k-1} + 2j - 1 \pmod{n}$, for $2^{t-4} \leq j \leq 2^{t-3} - 1$. Similarly, every vertex $2^{k-1} + 2i - 1 \in C$, for $0 \leq i \leq 2^{t-4} - 1$, informs $n - 2^{k-1} + 2j$ for $2^{t-4} + 1 \leq j \leq 2^{t-3}$. In parallel, every vertex $2^{k-2} + i \in I_1$, for $-1 \leq i \leq 2^{t-3} - 2$, informs $n - 2^{k-2} + j$ for $2^{t-3} + 1 \leq j \leq 2^{t-2}$, and vertex $n - 2^{k-2} + i \in I'_1$, for $1 \leq i \leq 2^{t-3}$, informs $2^{k-2} + j$, for $2^{t-3} - 1 \leq j \leq 2^{t-2} - 2$. The claim is proved.

Therefore, after time unit $(k - 1)$, the following vertices will be informed.

$$A = \{i | 0 \leq i \leq 2^{k-3} - 1\},$$

$$B = \{n - 2^{k-1} + 2i | 1 \leq i \leq 2^{k-4}\}$$

$$I_1 = \{2^{k-2} + i | -1 \leq i \leq 2^{k-3} - 2\},$$

$$I'_1 = \{n - 2^{k-2} + i | 1 \leq i \leq 2^{k-3}\},$$

$$C = \{2^{k-1} + 2i - 1 \pmod{n} | 0 \leq i \leq 2^{k-4} - 1\}.$$

Note that in each time unit, informed vertices of B extends the set C and inverse. It is the same for I_1 and I'_1 .

Now, suppose that $n \geq 2^{k-1} + 2^{k-3}$. We show that $(k - 2, k - 1, 1, 2, \dots, k - 3, k - 1)$ is not a valid dimensional broadcast scheme. For that, we claim that two $(k - 1)$ dimensional neighbors 2^{k-3} and $2^{k-1} - 2^{k-3} - 1$ are uninformed after time unit $(k - 1)$, and so they will remain uninformed after time unit k .

Note that 2^{k-3} is even, and so $2^{k-3} \notin C$.

Since $2^{k-3} > \max A = 2^{k-3} - 1$, then $2^{k-3} \notin A$.

If $2^{k-3} \in B$, then $2^{k-3} \stackrel{n}{\equiv} n - 2^{k-1} + 2i$ for an integer i , $1 \leq i \leq 2^{k-4}$. So, $2^{k-1} + 2^{k-3} - 2i \stackrel{n}{\equiv}$

0. But

$$0 < 2^{k-1} \leq 2^{k-1} + 2^{k-3} - 2i \leq 2^{k-1} + 2^{k-3} - 2 < n,$$

a contradiction. Hence, $2^{k-3} \notin B$.

If $2^{k-3} \in I_1$, then $2^{k-3} \equiv 2^{k-2} + i$ for an integer i , $-1 \leq i \leq 2^{k-3} - 2$. So, $2^{k-3} + i \equiv 0$.

But

$$0 < 2^{k-2} - 1 \leq 2^{k-2} + i \leq 2^{k-2} - 2 < n,$$

a contradiction. Hence, $2^{k-3} \notin I_1$.

If $2^{k-3} \in I'_1$, then $2^{k-3} \equiv n - 2^{k-2} + i$ for an integer i , $1 \leq i \leq 2^{k-3}$. So, $-2^{k-3} - 2^{k-2} + i \equiv 0$.

0. But

$$-n < -2^{k-3} - 2^{k-2} + 1 \leq -2^{k-3} - 2^{k-2} + i \leq -2^{k-2} < 0,$$

a contradiction. Hence, $2^{k-3} \notin I'_1$.

Therefore, 2^{k-3} is not informed after time unit $(k-1)$.

Now, consider $\alpha = 2^{k-1} - 2^{k-3} - 1$.

Note that α is odd, and so $\alpha \notin B$.

Since $\alpha > \max A = 2^{k-3} - 1$, then $\alpha \notin A$.

If $\alpha \in C$, then $\alpha \equiv 2^{k-1} + 2i - 1$ for an integer i , $0 \leq i \leq 2^{k-4} - 1$. So, $2^{k-3} + 2i \equiv 0$.

But

$$0 < 2^{k-3} \leq 2^{k-3} + 2i \leq 2^{k-2} - 2 < n,$$

a contradiction. Hence, $\alpha \notin C$.

If $\alpha \in I_1$, then $\alpha \equiv 2^{k-2} + i$ for an integer i , $-1 \leq i \leq 2^{k-3} - 2$. So, $2^{k-3} - i - 1 \equiv 0$. But

$$0 < 1 \leq 2^{k-3} - i - 1 \leq 2^{k-3} < n,$$

a contradiction. Hence, $\alpha \notin I_1$.

If $\alpha \in I'_1$, then $\alpha \equiv n - 2^{k-2} + i$ for an integer i , $1 \leq i \leq 2^{k-3}$. So, $2^{k-1} + 2^{k-3} - i - 1 \equiv 0$.

But

$$0 < 2^{k-1} - 1 \leq 2^{k-1} + 2^{k-3} - i - 1 \leq 2^{k-1} + 2^{k-3} - 2 < n,$$

a contradiction. Hence, $\alpha \notin I'_1$.

Therefore, α is not informed after time unit $(k - 1)$.

The claim is proved.

Consider $2^{k-1} + 2 \leq n \leq 2^{k-1} + 2^{k-3} - 2$. Since $n \leq 2^{k-1} + 2^{k-3} - 2$, one can see that $\min I'_1 \leq \max I_1 + 1$, and so $I_1 \cup I'_1$ contains all vertices from $2^{k-2} - 1$ to $n - 2^{k-2} + 2^{k-3}$. Now, since $\max C = 2^{k-1} + 2^{k-3} - 3 \geq n - 1$, all vertices i , for $0 \leq i \leq 2^{k-3} - 1$ and $2^{k-2} - 1 \leq i \leq n - 2^{k-2} + 2^{k-3}$, even vertices i , for $n - 2^{k-1} + 2 \leq i \leq n - 2^{k-1} + 2^{k-3}$, and odd vertices of i , for $2^{k-1} - 1 \leq i \leq 0$, are informed after time unit $(k - 1)$. Consider an even vertex between $2^{k-1} - 1$ and $n - 1$, say $2^{k-1} + 2j$, where $0 \leq j \leq \frac{1}{2}(n - 2^{k-1}) - 1$. Its $k - 1$ dimensional neighbor is $n - 2j - 1$. It is easy to check that $2^{k-1} - 1 \leq n - 2j - 1 \leq n - 1$, and so $n - 2j - 1 \in C$. Hence, $2^{k-1} + 2j$ will be informed by $n - 2j - 1$ in time unit k . Furthermore, uninformed vertices between $\max I'_1 = n - 2^{k-2} + 2^{k-3}$ and $2^{k-1} - 1 \in C$ will be informed in the last time unit by vertices in A , because their $k - 1$ dimensional neighbors fall between the $k - 1$ dimensional neighbors of $n - 2^{k-2} + 2^{k-3}$ and $2^{k-1} - 1$, which are $2^{k-1} + 2^{k-3} - 1 - n$ and 0 , respectively. It is worth noting that $0 \leq 2^{k-1} + 2^{k-3} - 1 - n \leq 2^{k-3} - 1$. Finally, suppose that $1 + \max A = 2^{k-3} \leq i \leq 2^{k-2} - 2 = -1 + \min I_1$. Then its $k - 1$ dimensional neighbor, denoted as j , belongs to $I_1 \cup I'_1$. This is because j falls between the $k - 1$ dimensional neighbors of 2^{k-3} and $2^{k-2} - 2$, which are $2^{k-1} - 2^{k-3} - 1$ and $2^{k-2} + 1$, respectively, and both are in $I_1 \cup I'_1$. Therefore, every vertex will become informed after the last time unit. $\square$

The previous theorem shows that when n is far enough from $2^{\lceil \log n \rceil}$, then there are other valid dimensional broadcast schemes (two right cyclic shifts with repeating the second dimension). This observation is confirmed again by the next theorem, which proves the validity of more dimensional broadcast schemes with right cyclic shifts. In the next

theorem, which is a generalization of the previous one, we specify the complete range of values n for which i right cyclic shifts, $2 \leq i \leq k - 2$, of the standard dimensional broadcast scheme $(1, 2, \dots, k - 1)$ with repeating the second dimension is valid. Note that the cases where $i = 0$ and $i = 1$ are covered by Theorems 1 and 2 of [51], respectively.

Theorem 3.2.2. *Let $KG(n)$ be a Knödel graph on $n \neq 2^k$ vertices and $k = \lceil \log n \rceil \geq 4$. Then, for $2 \leq i \leq k - 2$, dimensional broadcast scheme $(k - i, k - i + 1, \dots, k - 1, 1, \dots, k - i - 1, k - i + 1)$ is a valid broadcast scheme if and only if $n \leq 2^{k-1} + 2^{k-i-1} - 2$.*

Proof. First, consider the case $i < k - 2$.

Similar to the proof of Theorem 3.2.1, by induction on time unit t , we show that informed vertices after time unit $1 \leq t \leq i$ are as follows:

$$A = \{0\}, \quad I_p = \{p2^{k-i} - 1, p2^{k-i}\}, \quad \text{and} \quad C = \{2^{k-i+t-1} - 1\},$$

where $p = 1, 2, \dots, 2^{t-1} - 1$.

During time unit 1, 0 informs $2^{k-i} - 1$, and so the base case is true. We show that the correctness for t follows by considering the correctness for $t - 1$. During time unit $2 \leq t \leq i$, 0, $p2^{k-i} - 1$, $p2^{k-i}$, and $2^{k-i+t-2} - 1$ inform $2^{k-i+t-1} - 1$, $(2^{t-1} - p)2^{k-i}$, $(2^{t-1} - p)2^{k-i} - 1$, and $2^{k-i+t-2}$, respectively, where $p = 1, 2, \dots, 2^{t-2} - 1$. Note that $2^{t-2} + 1 \leq 2^{t-1} - p \leq 2^{t-1} - 1$ and $2^{k-i+t-2} = 2^{t-2}2^{k-i}$. Thus, these vertices together with informed vertices after time unit $t - 1$ (by IH) form all the vertices above. Hence, induction is complete. Therefore, after time unit i the informed vertices are as follows:

$$A = \{0\}, \quad I_p = \{p2^{k-i} - 1, p2^{k-i}\}, \quad \text{and} \quad C = \{2^{k-1} - 1\},$$

where $p = 1, 2, \dots, 2^{i-1} - 1$.

Again, by using another induction on t , we claim that the following vertices will be informed after time unit $i + 1 \leq t \leq k - 1$:

$$A_t = \{j | 0 \leq j \leq 2^{t-i} - 1\}$$

$$B_t = \{n - 2^{k-1} + 2j \mid 1 \leq j \leq 2^{t-i-1}\}$$

$$I_{p,t} = \{p2^{k-i} + j \mid -1 \leq j \leq 2^{t-i} - 2\}$$

$$I'_{p,t} = \{n - p2^{k-i} + j \mid 1 \leq j \leq 2^{t-i}\}$$

$$C_t = \{2^{k-1} + 2j - 1 \pmod{n} \mid 0 \leq j \leq 2^{t-i-1} - 1\},$$

where $p = 1, 2, \dots, 2^{i-1} - 1$.

Note that during the time unit $i + 1$, $0, p2^{k-i} - 1, p2^{k-i}$ and $2^{k-1} - 1$ inform $1, n - p2^{k-i} + 2, n - p2^{k-i} + 1$, and $n - p2^{k-1} + 2$, respectively. Hence, the base case is true. Suppose that this is true also for $t - 1$ (IH). We show that this is true for $i + 2 \leq t \leq k - 1$. Note that in time unit t , the dimension $t - i$ is used.

If $j \in A_{t-1}$, then j informs $\beta = 2^{t-i} - j - 1$. But $2^{t-i-1} \leq \beta \leq 2^{t-i} - 1$, and so together with A_{t-1} we have A_t .

If $\alpha = n - 2^{k-1} + 2j \in B_{t-1}$, where $1 \leq j \leq 2^{t-i-2}$, then α informs $\beta = 2^{k-1} + 2^{t-i} - 2j - 1 = 2^{k-1} + 2j' - 1$, where $2^{t-i-2} \leq j' = 2^{t-i-1} - j \leq 2^{t-i-1} - 1$. So, these vertices together with C_{t-1} form C_t .

If $\alpha = 2^{k-1} + 2j - 1 \in C_{t-1}$, where $0 \leq j \leq 2^{t-i-2} - 1$, then α informs $\beta = n - 2^{k-1} + 2^{t-i} - 2j = n - 2^{k-1} + 2j'$, where $2^{t-i-2} + 1 \leq j' = 2^{t-i-1} - j \leq 2^{t-i-1}$. So, these vertices together with B_{t-1} form B_t .

If $\alpha = p2^{k-i} + j \in I_{p,t-1}$, where $-1 \leq j \leq 2^{t-i-1} - 2$, then α informs $\beta = n - p2^{k-i} + 2^{t-i} - j - 1 = n - p2^{k-i} + j'$, where $2^{t-i-1} + 1 \leq j' = 2^{t-i} - j - 1 \leq 2^{t-i}$. So, these vertices together with $I'_{p,t-1}$ form $I'_{p,t}$.

If $\alpha = n - p2^{k-i} + j \in I'_{p,t-1}$, where $1 \leq j \leq 2^{t-i-1}$, then α informs $\beta = p2^{k-i} + 2^{t-i} - j - 1 = p2^{k-i} + j'$, where $2^{t-i-1} - 1 \leq j' = 2^{t-i} - j - 1 \leq 2^{t-i} - 2$. So, these vertices together with $I_{p,t-1}$ form $I_{p,t}$.

Therefore, the claim is proved. Note that it means that the set A_t is responsible for growing itself, while B_t and C_t are responsible for each other, and $I_{p,t}$ and $I'_{p,t}$ are responsible for each other's growth.

Therefore, the following vertices will be informed after time unit $(k - 1)$:

$$A = \{j | 0 \leq j \leq 2^{k-i-1} - 1\}$$

$$B = \{n - 2^{k-1} + 2j | 1 \leq j \leq 2^{k-i-2}\}$$

$$I_p = \{p2^{k-i} + j | -1 \leq j \leq 2^{k-i-1} - 2\}$$

$$I'_p = \{n - p2^{k-i} + j | 1 \leq j \leq 2^{k-i-1}\}$$

$$C = \{2^{k-1} + 2j - 1 \pmod{n} | 0 \leq j \leq 2^{k-i-2} - 1\},$$

where $p = 1, 2, \dots, 2^{i-1} - 1$.

Now, suppose that $n > 2^{k-1} + 2^{k-i-1} - 2$.

We show that the dimensional broadcast scheme $(k - i, k - i + 1, \dots, 1, 2, \dots, k - i - 1, k - i + 1)$ is not valid. For that, we claim that none of the $k - i + 1$ dimensional neighbors 2^{k-i-1} and $2^{k-i+1} - 2^{k-i-1} - 1$ are informed after time unit $(k - 1)$, and so they will remain uninformed after time unit k .

Obviously, $2^{k-i-1} \notin A \cup C$.

If $2^{k-i-1} \in B$, then $2^{k-i-1} \stackrel{n}{\equiv} n - 2^{k-i} + 2j$, for $1 \leq j \leq 2^{k-i-2}$. Thus, $2^{k-1} + 2^{k-i-1} - 2j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} \leq 2^{k-1} + 2^{k-i-1} - 2j \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction. Hence, $2^{k-i-1} \notin B$.

If $2^{k-i-1} \in I_p$, then $2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$, for $-1 \leq j \leq 2^{k-i-1} - 2$ and $1 \leq p \leq 2^{i-1} - 1$.

Thus, $p2^{k-i} - 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k-i-1} + j \leq 2^{k-1} - 2^{k-i} - 2 < n,$$

a contradiction. Hence, $2^{k-i-1} \notin I_p$.

If $2^{k-i-1} \in I'_p$, then $2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$, for $1 \leq j \leq 2^{k-i-1}$ and $1 \leq p \leq 2^{i-1} - 1$.

Thus, $p2^{k-i} + 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} \leq p2^{k-i} + 2^{k-i-1} - j \leq 2^{k-1} - 2^{k-i-1} - 1 < n,$$

a contradiction. Hence, $2^{k-i-1} \notin I'_p$.

Thus, 2^{k-i-1} is uninformed after time unit $(k-1)$.

Now, let $\alpha = 2^{k-i+1} - 2^{k-i-1} - 1$.

Obviously, $\alpha > \max A = 2^{k-i-1} - 1$, and so $\alpha \notin A$. Since α is odd, $\alpha \notin B$.

If $\alpha \in C$, then $\alpha \equiv_n 2^{k-1} + 2j - 1$, for $0 \leq j \leq 2^{k-i-2} - 1$. Thus, $2^{k-1} - 2^{k-i+1} + 2^{k-i-1} + 2j \equiv_n 0$. But

$$0 < 2^{k-1} - 2^{k-i+1} + 2^{k-i-1} \leq 2^{k-1} - 2^{k-i+1} + 2^{k-i-1} + 2j \leq 2^{k-1} - 2^{k-i} - 2 < n,$$

a contradiction. Hence, $\alpha \notin C$.

If $\alpha \in I_p$, then $\alpha \equiv_n p2^{k-i} + j$, for $-1 \leq j \leq 2^{k-i-1} - 2$ and $1 \leq p \leq 2^{i-1} - 1$. Thus, $p2^{k-i} - 2^{k-i+1} + 2^{k-i-1} + j + 1 \equiv_n 0$. If $p \geq 2$, then

$$0 < 2^{k-i-1} \leq p2^{k-i} - 2^{k-i+1} + 2^{k-i-1} + j + 1 \leq 2^{k-1} - 2^{k-i+1} - 1 < n,$$

a contradiction. If $p = 1$, then $2^{k-i} - 2^{k-i+1} + 2^{k-i-1} + j + 1 = 0$. So, $j = 2^{k-i-1} - 1 > 2^{k-i-1} - 2$, a contradiction. Hence, $\alpha \notin I_p$.

If $\alpha \in I'_p$, then $\alpha \equiv_n n - p2^{k-i} + j$, for $1 \leq j \leq 2^{k-i-1}$ and $1 \leq p \leq 2^{i-1} - 1$. Thus, $p2^{k-i} + 2^{k-i+1} - 2^{k-i-1} - j - 1 \equiv_n 0$. But

$$0 < 2^{k-i+1} - 1 \leq p2^{k-i} + 2^{k-i+1} - 2^{k-i-1} - j - 1 \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction. Hence, $\alpha \notin I'_p$.

Thus, $2^{k-i+1} - 2^{k-i-1} - 1$ is uninformed after time unit $(k-1)$. Therefore, the claim is proved. Hence, the dimensional broadcast scheme $(k-i, k-i+1, \dots, 1, 2, \dots, k-i-1, k-i+1)$ is not a valid broadcast scheme when $n > 2^{k-1} + 2^{k-i-1} - 2$.

Now, suppose that $n \leq 2^{k-1} + 2^{k-i-1} - 2$.

We prove that the dimensional broadcast scheme $(k-i, k-i+1, \dots, 1, 2, \dots, k-i-1, k-i+1)$ is valid.

For a fixed integer p , $1 \leq p \leq 2^{i-1} - 1$, suppose that $\alpha = \max I_p = p2^{k-i} + 2^{k-i-1} - 2$

and $\beta = \min I'_{2^{i-1}-p} = n - (2^{i-1} - p)2^{k-i} + 1 = n - 2^{k-1} + p2^{k-i} + 1$. It is easy to check that $\beta \leq \alpha + 1$. Hence, for every integer p , $1 \leq p \leq 2^{i-1} - 1$, $A_p = I_p \cup I'_{2^{i-1}-p}$ contains all vertices between $\min I_p = p2^{k-i} - 1$ and $\max I'_{2^{i-1}-p} = n - 2^{k-1} + p2^{k-i} + 2^{k-i-1}$. Let $U_p = \{p2^{k-i} + 2^{k-i} - 1 - s \mid s = 1, \dots, \gamma\}$, where $\gamma = \min A_{p+1} - \max A_p - 1 = 2^{k-1} + 2^{k-i-1} - 2 - n \geq 0$, be all uninformed vertices between A_p and A_{p+1} after time unit $(k-1)$. Consider $U'_p = \{(1-p)2^{k-i} + s \pmod{n} \mid s = 1, \dots, \gamma\}$ is $(k-i+1)$ th dimensional neighbor of U_p 's elements. If $p = 1$, then $U'_p = \{s \mid 1 \leq s \leq \gamma\} \subseteq A$. Therefore, U'_1 will be informed by A in the last time unit. If $p > 1$, then $U'_p = \{n - (p-1)2^{k-i} + s \mid 1 \leq s \leq \gamma\} \subseteq I'_{p-1}$. Thus, U_p will be informed by I'_{p-1} in the last time unit. So, all vertices between $\min A_1 = 2^{k-i} - 1$ and $\max A_{2^{i-1}-1} = n - 2^{k-i} + 2^{k-i-1}$ will be informed after time unit k . Now, note that the $(k-i+1)$ th dimensional neighbors of 2^{k-i-1} and $2^{k-i} - 2$ are $3 \times 2^{k-i-1} - 1$ and $2^{k-i} + 1$, respectively. Obviously, $2^{k-i} + 1 \in I_1 \subset A_1$. Also, $3 \times 2^{k-i-1} - 1 \in I'_{2^{i-1}-1} \subset A_1$ since $3 \times 2^{k-i-1} - 1 = n - (2^{i-1} - 1)2^{k-i} + j$, for non-negative $j = 2^{k-1} + 2^{k-i-1} - 1 - n \leq 2^{k-i-1}$. Therefore, all vertices between 2^{k-i-1} and $2^{k-i} - 2$ can be informed by A_1 in the last time unit. Similarly, the $(k-i+1)$ th dimensional neighbors of $n - 2^{k-i} + 2^{k-i-1} + 1$ and $n - 1$ are $5 \times 2^{k-i-1} - 2$ and 2^{k-i+1} , respectively. Obviously, $2^{k-i+1} \in I_2 \subset A_2$. Also, $5 \times 2^{k-i-1} - 2 \in I'_{2^{i-1}-2} \subset A_2$ since $5 \times 2^{k-i-1} - 2 = n - (2^{i-1} - 2)2^{k-i} + j$, for non-negative $j = 2^{k-1} + 2^{k-i-1} - 2 - n \leq 2^{k-i-1}$. Thus, all vertices between $n - 2^{k-i} + 2^{k-i-1} + 1$ and $n - 1$ can be informed by A_2 in the last time unit. Hence, the broadcast can be completed in k time units.

The remaining case $i = k - 2$ is easy.

It suffices to prove that the scheme is invalid, since the condition $n \leq 2^{k-1} + 2^{k-i-1} - 2$ does not hold when $i = k - 2$.

By the first induction used in the previous part, informed vertices after time unit $(k-2)$ are

$$\{0\} \cup \{4p - 1, 4p | 1 \leq p \leq 2^{k-3} - 1\} \cup \{2^{k-1} - 1\}.$$

During the time unit $k - 1$ by using dimension 1, $0, 4p - 1, 4p$, and $2^{k-1} - 1$ inform $1, n - 4p + 2, n - 4p + 1$, and $n - 2^{k-1} + 2$, respectively. Thus, informed vertices after time unit $(k - 1)$ are

$$\{0, 1\} \cup \{n - 2^{k-1} + 2, 2^{k-1} - 1\} \cup \{4p - 1, 4p, n - 4p + 1, n - 4p + 2 | 1 \leq p \leq 2^{k-3} - 1\}.$$

We show that two 2 dimensional neighbors 2 and 5 are not among them, and so will be uninformed after k time units.

Obviously, $2 \notin \{n - 2^{k-1} + 2, 2^{k-1} - 1, 4p - 1, 4p, n - 4p + 1\}$. If $2 = n - 4p + 2$, then $p = \frac{n}{4} > 2^{k-3}$, a contradiction. Hence, 2 is not informed.

Clearly, $5 \notin \{n - 2^{k-1} + 2, 2^{k-1} - 1, 4p - 1, 4p, n - 4p + 2\}$. If $5 = n - 4p + 1$, then $p = \frac{n - 4}{4} > 2^{k-3} - 1$, a contradiction. $\square$

Remark 3.2.1. Although Theorem 3.2.2 generalizes Theorems 1 and 2 of [51], it does not extend them directly because when considering $i = 0$ or 1, the results of Theorem 3.2.2 differ from those in Theorems 1 and 2 of [51]. This is due to the variations between 0, 1, and i cyclic shifts when $i \geq 2$. Similarly, the same distinction even more applies to the reverse dimensional broadcast scheme, as discussed in the next theorem.

3.3 Arbitrary Cyclic Shifts of Reversed Dimensional Broadcast Scheme With Repeating the Second Dimension

In Theorem 3.1.2, two types of broadcast schemes are discussed: standard and reverse dimensional. It is naturally compelling to explore reverse dimensions. In the next theorem, we specify the complete range of values for n where i right cyclic shifts, $0 \leq i \leq k - 2$, of the reverse dimensional broadcast scheme $(k - 1, k - 2, \dots, 1)$ with repeating the second dimension is valid.

Theorem 3.3.1. *Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil \geq 3$. Then, for $0 \leq i \leq k - 2$, i right cyclic shifts of the dimensional broadcast scheme $(k - 1, k - 2, \dots, 1)$ with repeating the second dimension is a valid broadcast scheme if and only if*

$$\left\{ \begin{array}{ll} i = 0 & \text{and } 2^{k-1} + 2 \leq n \leq 2^{k-1} + 2^{k-2} - 2 \\ i = 1 & \text{and } 4 \mid n \\ i = 2 & \text{and } n = 2^{k-1} + 2 \\ 3 \leq i \leq k - 2 & \text{and } 2^{k-1} + 2^{i-1} - 2 \leq n \leq 2^{k-1} + 2^i - 2. \end{array} \right.$$

Proof. First consider $i = 0$, i.e., the broadcast scheme is $(k - 1, k - 2, \dots, 1, k - 2)$. We prove that this is valid if and only if $n \leq 2^{k-1} + 2^{k-2} - 2$.

By induction on t , we show that informed vertices after time unit t , $1 \leq t \leq k - 1$, are

$$\{0\} \cup \{p2^{k-t} - 1 \mid 1 \leq p \leq 2^{t-1}\} \cup \{n - s2^{k-t} \mid 1 \leq s \leq 2^{t-1} - 1\}.$$

During the first time unit, vertex 0 informs $2^{k-1} - 1$, and so the base case is true for $t = 1$. Suppose that it holds for $t - 1$. We show its correctness for $2 \leq t \leq k - 1$. During time unit t , by using dimension $k - t$, vertices $p2^{k-t+1} - 1$ for $1 \leq p \leq 2^{t-2}$ inform $n - p2^{k-t+1} + 2^{k-t} = n - (2p - 1)2^{k-t}$, where $1 \leq 2p - 1 \leq 2^{t-1} - 1$. Note that these vertices, together with previously informed vertices (by IH) $n - s2^{k-t+1} = n - 2s2^{k-t}$, where $2 \leq 2s \leq 2^{t-1} - 2$, form all vertices of the form $n - s2^{k-t}$ for $1 \leq s \leq 2^{t-1} - 1$. Also, vertices $n - s2^{k-t+1}$ for $0 \leq s \leq 2^{t-2} - 1$ inform $s2^{k-t+1} + 2^{k-t} - 1 = (2s + 1)2^{k-t} - 1$, where $1 \leq 2s + 1 \leq 2^{t-1} - 1$. Note that these vertices, together with previously informed vertices (by IH) $p2^{k-t+1} - 1 = 2p2^{k-t} - 1$, where $2 \leq 2p \leq 2^{t-1}$, form all vertices of the form $p2^{k-t} - 1$ for $1 \leq p \leq 2^{t-1}$. Therefore, induction is complete.

Thus, after time unit $(k - 1)$, informed vertices are

$$I = \{0\} \cup \{2p - 1 \mid 1 \leq p \leq 2^{k-2}\} \cup \{n - 2s \mid 1 \leq s \leq 2^{k-2} - 1\}.$$

Suppose that $n > 2^{k-1} + 2^{k-2} - 2$.

We claim that two $(k - 2)$ dimensional neighbors 2^{k-2} and $n - 1$ are uninformed after time

unit $(k - 1)$.

If $2^{k-2} \in I$, then $2^{k-2} \stackrel{n}{\equiv} n - 2s$, for some $s = 0, \dots, 2^{k-2} - 1$. Hence, $2^{k-2} + 2s \stackrel{n}{\equiv} 0$. But in this case, $0 < 2^{k-2} + 2 \leq 2^{k-2} + 2s \leq 2^{k-2} + 2^{k-1} - 2 < n$, a contradiction. So $2^{k-2} \notin I$. Now, if $n - 1 \in I$, then $n - 1 \stackrel{n}{\equiv} 2p - 1$, for some $p = 1, \dots, 2^{k-2}$. Hence, $2p \stackrel{n}{\equiv} 0$. But in this case, $0 < 2 \leq 2p \leq 2^{k-1} < n$, a contradiction. So $n - 1 \notin I$.

Therefore, the claim is proved, and broadcast scheme $(k - 1, k - 2, \dots, 1, k - 2)$ is not valid.

Now, suppose that $n \leq 2^{k-1} + 2^{k-2} - 2$.

We prove that dimensional broadcast scheme $(k - 1, k - 2, \dots, 1, k - 2)$ is valid. According to the above, after time unit $(k - 1)$ all odd vertices between 0 and 2^{k-1} and all even vertices between $n - 2^{k-1} + 2$ and $n - 1$ are informed. Since $n - 2^{k-1} + 2 \leq 2^{k-2}$, all vertices between 2^{k-2} and 2^{k-1} are informed. Now, if $x = 2r$, for $1 \leq r < 2^{k-3}$, is an even vertex between 1 and 2^{k-2} , then since its $(k - 2)$ dimensional neighbor is an odd vertex $\alpha = 2^{k-2} - 1 - 2r < 2^{k-1}$, α informs x in the last time unit. Now, suppose that $2^{k-1} < x \leq n - 1$ is a vertex. Its $(k - 2)$ dimensional neighbor is $\alpha = n + 2^{k-2} - 1 - x$. Since $2^{k-2} \leq \alpha < 2^{k-1} - 3$, α will inform x in the last time unit. Therefore, the dimensional broadcast scheme $(k - 1, k - 2, \dots, 1, k - 2)$ is valid.

Now, consider $i = 1$, i.e., the broadcast scheme is $(1, k - 1, k - 2, \dots, 2, k - 1)$.

We prove that this is valid if and only if $4|n$.

Note that after the first time unit, 0 and 1 are two informed vertices. Similar to the previous part, by induction, one can see that informed vertices after time unit t , $2 \leq t \leq k - 1$, are

$$\begin{aligned} & \{0, 1\} \cup \{p2^{k-t+1} - 1, p2^{k-t-1} - 2 | 1 \leq p \leq 2^{t-2}\} \\ & \cup \{n - s2^{k-t+1}, n - s2^{k-t+1} + 1 | 1 \leq s \leq 2^{t-2} - 1\}. \end{aligned}$$

Thus, after time unit $(k - 1)$, informed vertices are

$$I = I_1 \cup I_2 \cup I_3 = \{0, 1\} \cup \{4p - 1, 4p - 2 | 1 \leq p \leq 2^{k-3}\}$$

$$\cup \{n - 4s, n - 4s + 1 | 1 \leq s \leq 2^{k-3} - 1\}.$$

Now, suppose that $4 \nmid n$.

In this case, we claim that two $(k - 1)$ dimensional neighbors 2^{k-1} and $n - 1$ are not in I , and so become uninformed after time unit k .

Note that $\max I_2 = 2^{k-1} - 1 < n - 1$ and $\max I_3 = n - 3$, and so $n - 1 \notin I$.

By contrary, suppose that $2^{k-1} \in I$. Thus, $2^{k-1} \stackrel{n}{\equiv} 4p - 2$ or $2^{k-1} \stackrel{n}{\equiv} n - 4s$. If $2^{k-1} \stackrel{n}{\equiv} 4p - 2$, then $2^{k-1} - 4p + 2 \stackrel{n}{\equiv} 0$. But $0 < 2^{k-1} - 4p + 2 < n$, a contradiction. Suppose that $2^{k-1} \stackrel{n}{\equiv} n - 4s$. In this case, $2^{k-1} + 4s \stackrel{n}{\equiv} 0$. Note that $0 < 2^{k-1} + 4s < 2n$. Thus, $2^{k-1} + 4s = n$, and so $4|n$, a contradiction. Therefore $2^{k-1} \notin I$. The claim is proved.

Hence, the dimensional broadcast scheme $(1, k - 1, k - 2, \dots, 2, k - 1)$ is not valid.

Now, suppose that $4|n$, and so $n \leq 2^k - 4$.

Note that $n - 2^{k-1} + 4 = \min I_3 \leq \max I_2 + 1 = 2^{k-1} - 1 + 1$. Hence, it is enough to show that all uninformed vertices between the vertices of I_2 and I_3 will be informed in the last time unit.

Suppose that $x = 4r + j$ where $1 \leq r \leq 2^{k-3} - 1$ and $j = 0, 1$. Its $(k - 1)$ dimensional neighbor is $\alpha = 4p - 1 - j$ where $p = 2^{k-3} - r$. Since $1 \leq p \leq 2^{k-3} - 1$, $\alpha \in I$. Therefore, x will be informed by α . Now, suppose that $x = n - 4r + j \geq 2^{k-1}$ where $1 \leq r \leq \frac{n}{4} - 2^{k-3} + \frac{1}{2}$ and $j = 2, 3$. Its $k - 1$ dimensional neighbor is $\alpha = n - 4p - j + 3$ where $p = \frac{n}{4} - r + 1 - 2^{k-3}$. Since $1 \leq p \leq 2^{k-3} - 1$, $\alpha \in I$. Therefore, x will be informed by α . Hence, the broadcast scheme $(1, k - 1, k - 2, \dots, 2, k - 1)$ is valid.

Now, consider $i = 2$, i.e., the broadcast scheme is $(2, 1, k - 1, k - 2, \dots, 3, 1)$.

We prove that this is valid if and only if $n = 2^{k-1} + 2$.

Note that after the first two time units, vertices $0, 1, 3, n - 2$ are informed. By induction, one can see that informed vertices after time unit t are

$$\{0, 1, 3, n - 2\} \cup \{p2^{k-t+2} + j | 1 \leq p \leq 2^{t-3}, j \in \{-1, -2, -4, 1\}\}$$

$$\cup\{n - s2^{k-t+1} + j | 1 \leq s \leq 2^{t-3} - 1, j \in \{0, 1, 3, -2\}\}.$$

Thus, after time unit $(k - 1)$, informed vertices are

$$I = \{0, 1, 3, n - 2\} \cup \{8p + j | 1 \leq p \leq 2^{k-4}, j \in \{-1, -2, -4, 1\}\} \\ \cup\{n - 8s + j | 1 \leq s \leq 2^{k-4} - 1, j \in \{0, 1, 3, -2\}\}.$$

Now, suppose that $n \geq 2^{k-1} + 4$.

In this case, we show that two 1 dimensional neighbors 2 and $n - 1$ are not in I , and so become uninformed after time unit k . Hence, the dimensional broadcast scheme $(2, 1, k - 1, k - 2, \dots, 3, 1)$ is not valid.

To see this, suppose that $2 \in I$. If $2 \stackrel{n}{\equiv} 8p + j$, where $1 \leq p \leq 2^{k-4}$ and $j \in \{-1, -2, -4, 1\}$, then $8p + j - 2 \stackrel{n}{\equiv} 0$. But $0 < 2 \leq 8p + j - 2 \leq 2^{k-1} - 1 < n$, a contradiction. If $2 \stackrel{n}{\equiv} n - 8s + j$, where $1 \leq s \leq 2^{k-4} - 1$ and $j \in \{0, 1, 3, -2\}$, then $n - 8s + j - 2 \stackrel{n}{\equiv} 0$. But

$$0 < n - 2^{k-1} + 6 \leq n - 8s + j - 2 \leq n - 7 < n,$$

a contradiction. Thus, $2 \notin I$.

If $n - 1 \stackrel{n}{\equiv} 8p + j$, where $1 \leq p \leq 2^{k-4}$ and $j \in \{-1, -2, -4, 1\}$, then $8p + j + 1 \stackrel{n}{\equiv} 0$. But $0 < 5 \leq 8p + j + 1 \leq 2^{k-1} + 2 < n$, a contradiction. If $n - 1 \stackrel{n}{\equiv} n - 8s + j$, where $1 \leq s \leq 2^{k-4} - 1$ and $j \in \{0, 1, 3, -2\}$, then $8s - j - 1 \stackrel{n}{\equiv} 0$. But $0 < 4 \leq 8s - j - 1 \leq 2^{k-1} - 7 < n$, a contradiction. Thus, $n - 1 \notin I$.

Now, suppose that $n = 2^{k-1} + 2$.

In this case, it is not hard to see that

$$I = \{0, 1, 3, 2^{k-1}\} \\ \cup\{8s', 8p + 1, 8s' + 2, 8s' + 3, 8p' + 4, 8s' + 5, 8p' + 6, 8p' + 7 | 1 \leq p \leq 2^{k-4}, p' = \\ p - 1, s' = 2^{k-4} - s\}.$$

Therefore, all vertices between 6 to $n - 1$ and 0, 1, 3 and 4 are in I . So there are just two uninformed vertices 2 and 5 which will be informed in the last time unit by $n - 1$ and $n - 4$, respectively. Hence, the dimensional broadcast scheme $(2, 1, k - 1, k - 2, \dots, 3, 1)$ is valid.

The last case, consider $3 \leq i \leq k - 2$, i.e., the dimensional broadcast scheme is $(i, i - 1, \dots, 1, k - 1, k - 2, \dots, i + 1, i - 1)$.

Informed vertices after time unit i , similar to the case $i = 0$, are

$$A = \{0\} \cup \{2p - 1 | 1 \leq p \leq 2^{i-1}\} \cup \{n - 2s | 1 \leq s \leq 2^{i-1} - 1\}.$$

By induction, we show that all informed vertices after time unit t , $i + 1 \leq t \leq k - 1$, are

$$\begin{aligned} &A \cup \{p2^{k-t+i} + j \pmod{n} | 1 \leq p \leq 2^{t-i-1}, j \in -A - 1\} \\ &\cup \{n - s2^{k-t+i} + j \pmod{n} | 1 \leq s \leq 2^{t-i-1} - 1, j \in A\}. \end{aligned}$$

It is worth mentioning that if $s = 0$ in $n - s2^{k-t+i} + j$, we get A . Note that during the time unit $i + 1$, by using dimension $k - 1$, vertices $2p - 1$ for $1 \leq p \leq 2^{i-1}$ inform $2^{k-1} - 2p$ while vertices $n - 2s$ for $0 \leq s \leq 2^{i-1} - 1$ inform $2^{k-1} + 2s - 1$. Hence, the base case is true.

Now, we prove for $i + 2 \leq t \leq k - 1$ by considering IH. During the time unit $t = i + (t - i)$, by using dimension $k - (t - i) = k - t + i$, vertices $p2^{k-t+i+1} + j$ for $1 \leq p \leq 2^{t-i-2}$ and $j \in -A - 1$ inform $n + 2^{k-t+i} - p2^{k-t+i+1} - j - 1 = n - (2p - 1)2^{k-t+i} + j'$, where $1 \leq 2p - 1 \leq 2^{t-i-1} - 1$ and $j' = -j - 1 \in A$. These vertices, together with the previously informed vertices (by IH), $n - s2^{k-t+i+1} + j = n - 2s2^{k-t+i} + j$, where $0 \leq 2s \leq 2^{t-i-1} - 2$ and $j \in A$, form all vertices of the form $\{n - s2^{k-t+i} + j \pmod{n} | 1 \leq s \leq 2^{t-i-1} - 1, j \in A\}$.

On the other hand, vertices $n - s2^{k-t+i+1} + j$ for $0 \leq s \leq 2^{t-i-2} - 1$ and $j \in A$ inform $2^{k-t+i} + s2^{k-t+i+1} - j - 1 = (2s + 1)2^{k-t+i} + j'$, where $1 \leq 2s + 1 \leq 2^{t-i-1} - 1$ and

$j' = -j - 1 \in -A - 1$. These vertices, together with the previously informed vertices (by IH), $p2^{k-t+i+1} + j = 2p2^{k-t+i} + j$, where $2 \leq 2p \leq 2^{t-i-1} - 2$ and $j \in -A - 1$, form all vertices of the form $\{p2^{k-t+i} + j \pmod n | 1 \leq p \leq 2^{t-i-1}, j \in -A - 1\}$. The proof of the induction is complete.

Therefore, by considering $t = k - 1$, informed vertices after time unit $(k - 1)$ are

$$\begin{aligned} I &= A \cup \{p2^{i+1} + j \pmod n | 1 \leq p \leq 2^{k-i-2}, j \in -A - 1\} \\ &\cup \{n - s2^{i+1} + j \pmod n | 1 \leq s \leq 2^{k-i-2} - 1, j \in A\} = \\ &\quad \{p2^{i+1} + j \pmod n | 1 \leq p \leq 2^{k-i-2}, j \in -A - 1\} \\ &\cup \{n - s2^{i+1} + j \pmod n | 0 \leq s \leq 2^{k-i-2} - 1, j \in A\} = I_2 \cup I_3, \end{aligned}$$

where

$$\begin{aligned} I_2 &= \{p2^{i+1} + j \pmod n | 1 \leq p \leq 2^{k-i-2}, j \in -A - 1\} \text{ and} \\ I_3 &= \{n - s2^{i+1} + j \pmod n | 0 \leq s \leq 2^{k-i-2} - 1, j \in A\}. \end{aligned}$$

Now, we prove that the dimensional broadcast scheme $(i, i - 1, \dots, 1, k - 1, k - 2, \dots, i + 1, i - 1)$ is valid if and only if $2^{k-1} + 2^{i-1} - 2 \leq n \leq 2^{k-1} + 2^i - 2$.

First suppose that $n \geq 2^{k-1} + 2^i$.

In this case, we claim that two $(i - 1)$ dimensional neighbors $n - 1$ and 2^{i-1} are not in I , and so become uninformed after time unit $k - 1$.

If $n - 1 \in I_2$, then $n - 1 \stackrel{n}{\equiv} p2^{i+1} + j$, for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Thus, $p2^{i+1} + j + 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^i + 1 \leq p2^{i+1} + j + 1 \leq 2^{k-1} + 2^i - 2 < n,$$

a contradiction. Hence, $n - 1 \notin I_2$.

If $n - 1 \in I_3$, then $n - 1 \stackrel{n}{\equiv} n - s2^{i+1} + j$, for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Thus, $s2^{i+1} - j - 1 \stackrel{n}{\equiv} 0$. Note that $n - 1 \notin A$. So, let $1 \leq s \leq 2^{k-i-2} - 1$. But

$$0 < 2^i \leq s2^{i+1} - j - 1 \leq 2^{k-1} - 2^i - 3 < n,$$

a contradiction. Therefore, $2^{i-1} \notin I$.

If $2^{i-1} \in I_2$, then $2^{i-1} \stackrel{n}{\equiv} p2^{i+1} + j$, for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Thus, $p2^{i+1} + j - 2^{i-1} \stackrel{n}{\equiv} 0$. But

$$0 < 2^{i-1} \leq p2^{i+1} + j - 2^{i-1} \leq 2^{k-1} + 2^{i-1} - 1 < n,$$

a contradiction.

If $2^{i-1} \in I_3$, then $2^{i-1} \stackrel{n}{\equiv} n - s2^{i+1} + j$, for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Thus, $s2^{i+1} + 2^{i-1} - j \stackrel{n}{\equiv} 0$. Note that if $2^{i-1} \in A$, then $2^{i-1} = n - 2s$ and so $n = 2^{i-1} + 2s \leq 2^i + 2^{i-1} - 2$, a contradiction. Thus, $2^{i-1} \notin A$. So, let $1 \leq s \leq 2^{k-i-2} - 1$. But

$$0 < 2^i + 2^{i-1} + 1 \leq s2^{i+1} + 2^{i-1} - j \leq 2^{k-1} - 2^{i-1} - 2 < n,$$

a contradiction. Therefore, $2^{i-1} \notin I$.

The claim is proved. Therefore, the dimensional broadcast scheme $(i, i-1, \dots, 1, k-1, k-2, \dots, i+1, i-1)$ is not valid.

Now, consider $n \leq 2^{k-1} + 2^{i-1} - 4$.

In this case, we claim that two $(i-1)$ dimensional neighbors $2^i - 2$ and $n - 2^{i-1} + 1$ are not in I , and so become uninformed after time unit k .

If $2^i - 2 \in I_2$, then $2^i - 2 \stackrel{n}{\equiv} p2^{i+1} + j$, for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Thus, $p2^{i+1} + j - 2^i + 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2 \leq p2^{i+1} + j - 2^i + 2 \leq 2^{k-1} - 1 < n,$$

a contradiction.

If $2^i - 2 \in I_3$, then $2^i - 2 \stackrel{n}{\equiv} n - s2^{i+1} + j$, for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Thus, $s2^{i+1} + 2^i - 2 - j \stackrel{n}{\equiv} 0$. Note that if $2^i - 2 \in A$, then $2^i - 2 = n - 2s$ and so $n = 2^i + 2s - 2 \leq 2^{i+1} - 4$, a contradiction. Thus, $2^i - 2 \notin A$. So, let $1 \leq s \leq 2^{k-i-2} - 1$. But

$$0 < 2^{i+1} - 1 \leq s2^{i+1} + 2^i - 2 - j \leq 2^{k-1} - 4 < n,$$

a contradiction. Therefore, $2^i - 2 \notin I$.

If $n - 2^{i-1} + 1 \in I_3$, then $n - 2^{i-1} + 1 \stackrel{n}{\equiv} n - s2^{i+1} + j$, for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$.

Thus, $s2^{i+1} - 2^{i-1} - j + 1 \stackrel{n}{\equiv} 0$. Note that if $n - 2^{i-1} + 1 \in A$, then $n - 2^{i-1} + 1 = 2p - 1$ and so $n = 2^{i-1} + 2p - 2 \leq 2^i + 2^{i-1} - 2$, a contradiction. Thus, $n - 2^{i-1} + 1 \notin A$. So, let $1 \leq s \leq 2^{k-i-2} - 1$. But

$$0 < 2^{i-1} + 2 \leq s2^{i+1} - 2^{i-1} - j + 1 \leq 2^{k-1} - 2^i - 2^{i-1} + 1 < n,$$

a contradiction. Therefore, $n - 2^{i-1} + 1 \notin I_3$.

If $n - 2^{i-1} + 1 \in I_2$, then $n - 2^{i-1} + 1 \stackrel{n}{\equiv} p2^{i+1} + j$, for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$.

Thus, $p2^{i+1} + 2^{i-1} + j - 1 \stackrel{n}{\equiv} 0$. Note that

$$0 < 2^i + 2^{i-1} - 1 \leq p2^{i+1} + 2^{i-1} + j - 1 \leq 2^{k-1} + 2^i + 2^{i-1} - 4 < 2n.$$

Hence, $p2^{i+1} + 2^{i-1} + j - 1 = n$. Then $j = n - p2^{i+1} - 2^{i-1} + 2 - 1 \in -A - 1$. Hence,

$p2^{i+1} + 2^{i-1} - 2 - n \in A$ and so $p2^{i+1} + 2^{i-1} - 2 - n = n - 2s$ for $0 \leq s \leq 2^{i-1} - 1$.

Hence, $2^k < 2n = p2^{i+1} + 2^{i-1} + 2s - 2 \leq 2^{k-1} + 2^i + 2^{i-1} - 4 < 2^k$, a contradiction.

Therefore, $n - 2^{i-1} + 1 \notin I$ and the claim is proved. So, the dimensional broadcast scheme $(i, i-1, \dots, 1, k-1, k-2, \dots, i+1, i-1)$ is not valid.

Now, consider $2^{k-1} + 2^{i-1} - 2 \leq n \leq 2^{k-1} + 2^i - 2$.

In this case, we prove that the dimensional broadcast scheme $(i, i-1, \dots, 1, k-1, k-2, \dots, i+1, i-1)$ is valid.

First, suppose that $v, v+1 \notin I$, for $0 \leq v \leq n-1$.

Note that all odd vertices between 0 and 2^i are in $A \subset I$. Also, all even vertices between $2^i - 1$ and $2^{i+1} + 1$ are in I_2 since they have the form $2^{i+1} - 2p + 1 - 1$, for $1 \leq p \leq 2^{i-1}$.

Furthermore, $n - 2p \in A$, for $0 \leq p \leq 2^{i-1} - 1$, contains all even vertices between $2^{k-1} - 1$ and $0 \stackrel{n}{\equiv} n$, and since in time unit $(i+1)$, 0 and 1 inform $2^{k-1} - 1$ and $2^{k-1} - 2$, respectively,

all even vertices between $2^{k-1} - 1$ and 0 inform all odd vertices between $2^{k-1} - 1$ and 0.

Hence, all vertices between $2^{k-1} - 3$ and 0 are in I .

Therefore, $2^{i+1} \leq v \leq 2^{k-1} - 4$. Suppose that $p2^{i+1} - 1 \leq v \leq (p+1)2^{i+1} - 1$ where $1 \leq p \leq 2^{k-i-2} - 1$. If v is odd, then since all vertices of the form $p2^{i+1} - n + 2p - 1 \stackrel{n}{\equiv} p2^{i+1} + 2p - 1$, for $0 \leq p \leq 2^{i-1} - 1$, are in I_2 , $v - p2^{i+1} + 1 \geq 2^i$. Since all vertices of the form $(p+1)2^{i+1} - 2p - 1 - 1$, for $1 \leq p \leq 2^{i-1}$, are in I_2 , then $(p+1)2^{i+1} - 2 - (v+1) \geq 2^i$. Thus, $p2^{i+1} + 2^i - 1 \leq v \leq (p+1)2^{i+1} - 2^i - 3$, a contradiction. Now, suppose that v is even. With a similar argument, one can see that $(p+1)2^{i+1} - 2 - v \geq 2^i$ and $v+1 - p2^{i+1} - 1 \geq 2^i$. Thus, $p2^{i+1} + 2^i - 2 \leq v \leq (p+1)2^{i+1} - 2^i - 2$. But we have LHS = RHS, and so $v = p2^{i+1} + 2^i - 2$. Now, consider $\alpha = n - (2^{k-i-2} - p)2^{i+1} \in I_3$. It is easy to check that $0 < v+1 - (\alpha+1) \leq 2^{i-1}$, and so $v+1 \in I_3$, a contradiction. Therefore, if $v \notin I$, then $v-1, v+1 \in I$.

Suppose that $v+1 \notin I$. We have two cases:

Case 1. $v \in I_2$. So $v = p2^{i+1} + j \pmod{n}$ where $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. There are two sub-cases.

sub-case 1. $j = -2p' + 1 - 1$ where $1 \leq p' \leq 2^{i-1}$. The $(i-1)$ dimensional neighbor of $v+1 \stackrel{n}{\equiv} p2^{i+1} - 2p' + 1$ is $\alpha = n + 2^{i-1} - 1 - p2^{i+1} + 2p' - 1$.

If $p' = 1$, then $v+1 = p2^{i+1} - 1 \in I_2$.

If $2 \leq p' \leq 2^{i-2} + 1$, then $\alpha \stackrel{n}{\equiv} p''2^{i+1} + j'$, for $p'' = 2^{k-i-2} - p + 1$ and $j' = -2(2^{k-2} + 2^{i+1} - \frac{n}{2} - 2^{i-2} - p' + 1)$. It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$. Thus, $p''2^{i+1} + j' \in I_2$, and so $v+1$ will be informed by its neighbor in the last time unit.

If $2^{i-2} + 2 \leq p' \leq 2^{i-1}$, then $\alpha \stackrel{n}{\equiv} n - p''2^{i+1} + j'$, for $p'' = p - 1$ and $j' = n - 2(2^i - 2^{i-2} - p' + 1)$. It is easy to check that $0 \leq p'' \leq 2^{k-i-2} - 1$ and $j' \in A$. Thus, $n - p''2^{i+1} + j' \in I_3$, and so $v+1$ will be informed by its neighbor in the last time unit.

sub-case 2. $j = -n + 2p' - 1$ where $0 \leq p' \leq 2^{i-1} - 1$. The $(i-1)$ dimensional neighbor of $v+1 \stackrel{n}{\equiv} p2^{i+1} + 2p'$ is $\alpha = n + 2^{i-1} - 1 - p2^{i+1} - 2p'$.

First suppose that $p \neq 2^{k-i-2}$.

If $2^{i-2} \leq p' \leq 2^{i-1} - 1$, then $\alpha \stackrel{n}{\equiv} p''2^{i+1} + j'$, for $p'' = 2^{k-i-2} - p$ and $j' = -n + 2(\frac{n}{2} - 2^{k-2} + 2^{i-2} - p') - 1$. It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$. Thus, $p''2^{i+1} + j' \in I_2$, and so $v + 1$ will be informed by its neighbor in the last time unit.

If $2^{i-2} + 2 \leq p' \leq 2^{i-1}$, then $\alpha \stackrel{n}{\equiv} n - p''2^{i+1} + j'$, for $p'' = p - 1$ and $j' = n - 2(2^i - 2^{i-2} - p' + 1)$. It is easy to check that $0 \leq p'' \leq 2^{k-i-2} - 1$ and $j' \in A$. Thus, $n - p''2^{i+1} + j' \in I_3$, and so $v + 1$ will be informed by its neighbor in the last time unit.

Now, suppose that $p = 2^{k-i-2}$.

If $0 \leq p' \leq 2^{i-2} - 2$, then $2^{k-1} \leq v + 1 \stackrel{n}{\equiv} 2^{k-1} + 2p' \leq 2^{k-1} + 2^{i-1} - 4 < n$ and so according to above, $v + 1 \in A \subset I$.

If $2^{i-2} - 1 \leq p' \leq 2^{i-1} - 2$, then $\alpha \stackrel{n}{\equiv} n - p''2^{i+1} + j'$, for $p'' = 0$ and $j' = 2(\frac{n}{2} + 2^{i-2} - 2^{k-2} - p') - 1$. It is easy to check that $0 \leq p'' \leq 2^{k-i-2} - 1$ and $j' \in A$. Thus, $n - p''2^{i+1} + j' \in I_3$, and so $v + 1$ will be informed by its neighbor in the last time unit.

Now, suppose that $p' = 2^{i-1} - 1$. First, assume that $n = 2^{k-1} + 2^{i-1} - 2$. In this case, $\alpha \stackrel{n}{\equiv} n - 1 \in I$, and so $v + 1$ will be informed by its neighbor in the last time unit. If $n \geq 2^{k-1} + 2^{i-1}$, then $\alpha \stackrel{n}{\equiv} n - p''2^{i+1} + j'$, for $p'' = 0$ and $j' = 2(\frac{n}{2} + 2^{i-2} - 2^{k-2} + 1) - 1$. It is easy to check that $j' \in A$. Thus, $n - p''2^{i+1} + j' \in I_3$, and so $v + 1$ will be informed by its neighbor in the last time unit.

Case 2. $v \in I_3$. So $v = n - p2^{i+1} + j \pmod{n}$ where $0 \leq p \leq 2^{k-i-2} - 1$ and $j \in A$.

There are two sub-cases.

sub-case 1. $j = 2p' - 1$ where $1 \leq p' \leq 2^{i-1}$. The $(i - 1)$ dimensional neighbor of $v + 1 \stackrel{n}{\equiv} n - p2^{i+1} + 2p'$ is $\alpha = 2^{i-1} - 1 + p2^{i+1} - 2p'$.

If $2^{i-2} + 1 \leq p' \leq 2^{i-1}$, then $v + 1 \stackrel{n}{\equiv} p''2^{i+1} + j'$, for some $p'' = 2^{k-i-2} - p + 1$ and $j' = -2(2^{k-2} + 2^i - \frac{n}{2} - p')$. It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$.

Thus, $v + 1 \in I_2$.

If $1 \leq p' \leq 2^{i-2}$ and $p \neq 0$, then $\alpha \stackrel{n}{\equiv} p''2^{i+1} + j'$, for $p'' = p$ and $j' = -n + 2(2^{i-2} - p') - 1$.

It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$. Thus, $p''2^{i+1} + j' \in I_2$, and so $v + 1$ will be informed by its neighbor in the last time unit.

If $1 \leq p' \leq 2^{i-2}$ and $p = 0$, then $\alpha \equiv \frac{n}{2}(2^{i-2} - p') - 1$. If $p' \neq 2^{i-2}$, then we have $2(2^{i-2} - p') - 1 \in A$, and so $v + 1$ will be informed by its neighbor in the last time unit. If $p' = 2^{i-2}$, then $\alpha \equiv \frac{n}{2}n - 1 \in I$, and so $v + 1$ will be informed by its neighbor in the last time unit.

sub-case 2. $j = n - 2p'$ where $0 \leq p' \leq 2^{i-1} - 1$. The $(i - 1)$ dimensional neighbor of $v + 1 \equiv \frac{n}{2}n - p2^{i+1} - 2p' + 1$ is $\alpha = 2^{i-1} - 1 + p2^{i+1} + 2p' - 1 \pmod{n}$.

If $p' = 0$, then $v + 1 = n - p2^{i+1} + 1 \in I_3$.

If $1 \leq p' \leq 2^{i-2}$, then $v + 1 \equiv \frac{n}{2}p''2^{i+1} + j'$, for $p'' = 2^{k-i-2} - p$ and $j' = -n + 2(\frac{n}{2} - 2^{k-2} - p' + 1) - 1$. It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$. Thus, $p''2^{i+1} + j' \in I_2$, and so $v + 1 \in I$.

If $2^{i-2} + 1 \leq p' \leq 2^{i-1} - 1$, then $\alpha \equiv \frac{n}{2}p''2^{i+1} + j'$, for some $p'' = p + 1$ and $j' = -2(2^i - 2^{i-2} - p' + 1)$. It is easy to check that $1 \leq p'' \leq 2^{k-i-2}$ and $j' \in -A - 1$. Thus, $p''2^{i+1} + j' \in I_2$, and so $v + 1$ will be informed by its neighbor in the last time unit.

Therefore, the dimensional broadcast scheme $(i, i - 1, \dots, 1, k - 1, k - 2, \dots, i + 1, i - 1)$ is valid for $3 \leq i \leq k - 2$. $\square$

Since Knödel graphs are symmetric, it seems that different dimensional broadcast schemes have approximately the same properties, especially the standard and reverse ones with repeating the second dimension in the last time unit. For instance, by Theorem 3.2.2, i right cyclic shifts of standard dimensional broadcast scheme, $2 \leq i \leq k - 2$, is valid for $2^{k-i-2} - 1$ values of n while, by Theorem 3.3.1, i right cyclic shifts of reverse dimensional broadcast scheme, $3 \leq i \leq k - 2$, is valid for $2^{i-2} + 1$ values of n . Therefore, the difference between these numbers for i and $k - i$ right cyclic shifts of standard and reverse dimensional broadcast schemes is just 2 for any $3 \leq i \leq k - 2$. However, there are big differences between the

standard and reverse dimensional broadcast schemes. A notable observation is that while the 0 and 1 right cyclic shifts of the standard dimensional broadcast scheme are valid for all even values of n except 2^k and $2^k - 2$, the same shifts in the reverse dimensional broadcast scheme are only valid for $n \leq 2^{k-1} + 2^{k-2} - 2$ and only for values of n divisible by 4, as shown in Theorem 3.3.1. This shows that different dimensional broadcast schemes have significant differences.

3.4 Arbitrary Cyclic Shifts of standard Dimensional Broadcast Scheme With Repeating an Arbitrary Dimension

In the following, we are interested in delving deeper into the broadcast schemes mentioned in this chapter, particularly identifying all dimensional broadcast schemes by repeating any other dimension except dimensions 1 and 2.

Figure 3.2 illustrates $KG(20)$ with dimensional broadcast scheme $(4, 1, 2, 3)$, one cyclic shift of $(1, 2, 3, 4)$. It is clear that repeating the second dimension in the last time unit gives us a valid dimensional broadcast scheme, while repeating the third one is not valid. Therefore, the generalization of Theorems 1 and 2 of [51], 3.2.2, and 3.3.1 with repeating a dimension other than the second one, in the last time unit, is not always true.

cyclic shift is $(c + 1, c + 2, c + 3, \dots, k - 1, 1, \dots, c, c + 3)$.

Theorem 3.4.1. *Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil$. Then for two integers $0 \leq c \leq k - 2$ and $1 \leq d \leq k - 1$, the dimensional broadcast scheme $(c + 1, \dots, k - 1, 1, 2, \dots, c)$ repeating the d -th dimension is a valid broadcast scheme if and only if*

$$\left\{ \begin{array}{ll} d = 1 & \text{and } n \leq 2^k - 2 \\ d = 2, c = 0, k - 2 & \text{and } n \leq 2^k - 4 \\ d = 2, 1 \leq c \leq k - 3 & \text{and } n \leq 2^{k-1} + 2^c - 2 \\ d \geq 3, c = 0 & \text{and } n \leq 2^k - 2^d \\ d \geq 3, c = k - d & \text{and } n \leq 2^{k-1} + 2^{c+1} - 4 \\ d \geq 3, c = k - d - 1 & \text{and } n \leq 2^{k-1} + 2^c - 2 \\ d \geq 3, 1 \leq c \leq k - d - 2 & \text{and } 2^{k-1} + p2^{c+1} \leq n \leq 2^{k-1} + p2^{c+1} + 2^c - 2, \end{array} \right.$$

where $0 \leq p \leq 2^{d-2} - 1$ (if $p = 0$, $2^{k-1} + 2 \leq n \leq 2^{k-1} + 2^c - 2$).

Proof. Since each left cyclic shift is equivalent to $k - 2$ right cyclic shifts of $(1, \dots, k - 1)$, we adopt right cyclic shifts for computational simplicity. Consequently, for c left shifts of $(1, \dots, k - 1)$, we instead consider $i = k - c - 1$ right cyclic shifts of $(1, \dots, k - 1)$. Notably, when $c = 0$, the case $i = 0$ coincides with $i = k - 1$. Accordingly, after reformulating the statement, we proceed to establish the following result:

Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil$. Then for two integers $0 \leq i \leq k - 2$ and $1 \leq d \leq k - 2$, the dimensional broadcast scheme $(k - i, k - i + 1, \dots, k - 1, 1, 2, \dots, k - i - 1)$ repeating the d -th dimension is a valid broadcast scheme if and only if

$$\left\{ \begin{array}{ll} d = 1 & \text{and } n \leq 2^k - 2 \\ d = 2, i = 0, 1 & \text{and } n \leq 2^k - 4 \\ d = 2, 2 \leq i \leq k - 2 & \text{and } n \leq 2^{k-1} + 2^{k-i-1} - 2 \\ d \geq 3, i = 0 & \text{and } n \leq 2^k - 2^d \\ d \geq 3, i = d - 1 & \text{and } n \leq 2^{k-1} + 2^{k-i} - 4 \\ d \geq 3, i = d & \text{and } n \leq 2^{k-1} + 2^{k-i-1} - 2 \\ d \geq 3, d + 1 \leq i \leq k - 2 & \text{and } 2^{k-1} + p2^{k-i} \leq n \leq 2^{k-1} + p2^{k-i} + 2^{k-i-1} - 2, \end{array} \right.$$

where $0 \leq p \leq 2^{d-2} - 1$ (if $p = 0$, $2^{k-1} + 2 \leq n \leq 2^{k-1} + 2^{k-i-1} - 2$).

It is evident that if $d = k - 1$, no vertex will be informed during the last time unit. Therefore, we assume $d \leq k - 2$.

If $d = 1$, then the assertion follows directly from Theorem 1 in [17].

For $d = 2$, the cases $i = 0$ and $i = 1$ are validated by Theorems 1 and 2 in [51], respectively.

When $d = 2$ and $i \geq 2$, the result is supported by Theorem 3.2.2.

If $d \geq 3$ and $i = 0$, then the result follows from Theorem 1 in [51].

Now, suppose that $d \geq 3$, $i \geq 1$, and vertex 0 is the originator. Similar to the proof of the Theorem 3.2.2, by induction, one can see that informed vertices after time unit t , $1 \leq t \leq i$ are

$$\{0\} \cup \{p2^{k-i} + j | 1 \leq p \leq 2^{t-1} - 1, -1 \leq j \leq 0\} \cup \{2^{k+t-i-1} - 1\}.$$

Hence, informed vertices after time unit i are

$$\{0\} \cup \{p2^{k-i} + j | 1 \leq p \leq 2^{i-1} - 1, -1 \leq j \leq 0\} \cup \{2^{k-1} - 1\}.$$

Again, by using induction, we claim that the set of informed vertices after time unit t , $i + 1 \leq t \leq k - 1$ is the union of the following sets:

$$A = \{j | 0 \leq j \leq 2^{t-i} - 1\}$$

$$B = \{n - 2^{k-1} + 2j | 1 \leq j \leq 2^{t-i-1}\}$$

$$I_p = \{p2^{k-i} + j | -1 \leq j \leq 2^{t-i} - 2\}$$

$$I'_p = \{n - p2^{k-i} + j | 1 \leq j \leq 2^{t-i}\}$$

$$C = \{2^{k-1} + 2j - 1 \pmod{n} | 0 \leq j \leq 2^{t-i-1} - 1\},$$

where $p = 1, 2, \dots, 2^{i-1} - 1$.

Therefore, the set of informed vertices I after time unit $(k-1)$ is the union of the following sets:

$$A = \{j | 0 \leq j \leq 2^{k-i-1} - 1\}$$

$$B = \{n - 2^{k-1} + 2j | 1 \leq j \leq 2^{k-i-2}\}$$

$$I_p = \{p2^{k-i} + j | -1 \leq j \leq 2^{k-i-1} - 2\}$$

$$I'_p = \{n - p2^{k-i} + j | 1 \leq j \leq 2^{k-i-1}\}$$

$$C = \{2^{k-1} + 2j - 1 \pmod{n} | 0 \leq j \leq 2^{k-i-2} - 1\},$$

where $p = 1, 2, \dots, 2^{i-1} - 1$.

First, we prove that $1 \leq i \leq d-2$ are not valid for any n .

Assume that $d \geq i+2$ and so d -th dimension has value s , where $2 \leq s \leq k-i-2$. We claim that $\alpha = 2^{k-i-1} + 1$ and its s dimensional neighbor $\beta = 2^s - 1 - 2^{k-i-1} - 1 \stackrel{n}{\equiv} n + 2^s - 2^{k-i-1} - 2$ are not in I . Obviously, $\alpha \notin A \cup B$.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} + 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} + j - 2^{k-i-1} - 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} \leq p2^{k-i} + j - 2^{k-i-1} - 1 \leq 2^{k-1} - 2^{k-i} - 2 < n,$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} + 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$.

Then $p2^{k-i} + 2^{k-i-1} + 1 - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} + 1 \leq p2^{k-i} + 2^{k-i-1} + 1 - j \leq 2^{k-1} - 2^{k-i-1} < n,$$

a contradiction.

If $\alpha \in C$, then $2^{k-i-1} + 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then $2^{k-1} -$

$2^{k-i-1} + 2j - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} - 2^{k-i-1} - 2 \leq 2^{k-1} - 2^{k-i-1} + 2j - 2 \leq 2^{k-1} - 4 < n,$$

a contradiction. Therefore, $\alpha \notin I$.

If $\beta \in A$, then $n + 2^s - 2^{k-i-1} - 2 \leq 2^{k-i-1} - 1$, and so $n \leq 2^{k-i} - 2^s + 1 < 2^{k-1}$, a contradiction.

Since β is even, $\beta \notin C$.

If $\beta \in B$, then $n + 2^s - 2^{k-i-1} - 2 \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} - 2^{k-i-1} + 2^s - 2j - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} - 2^{k-i} + 2 \leq 2^{k-1} - 2^{k-i-1} + 2^s - 2j - 2 \leq 2^{k-1} - 2^{k-i-2} - 4 < n,$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n + 2^s - 2^{k-i-1} - 2 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} - 2^s + 2^{k-i-1} + j + 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} + 2^{k-i-2} + 1 \leq p2^{k-i} - 2^s + 2^{k-i-1} + j + 2 \leq 2^{k-1} - 6 < n,$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n + 2^s - 2^{k-i-1} - 2 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} + 2^s - 2^{k-i-1} - j - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2 \leq p2^{k-i} + 2^s - 2^{k-i-1} - j - 2 \leq 2^{k-1} - 2^{k-i} - 2^{k-i-2} - 3 < n,$$

a contradiction.

Therefore, $\beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme $(k - i, k - i + 1, \dots, 1, 2, \dots, k - i - 1, s)$ is not valid when $i \leq d - 2$.

Now, consider $i = d - 1$.

Thus, the dimensional broadcast scheme is $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, 1)$.

Suppose that $n \geq 2^{k-1} + 2^{k-i} - 2$. We claim that $\alpha = 2^{k-i-1}$ and its 1 dimensional neighbor $\beta = n - 2^{k-i-1} + 1$ are not in I , and so this dimensional broadcast scheme is not valid.

Obviously, $\alpha \notin A \cup C$.

If $\alpha \in B$, then $2^{k-i-1} \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} + 2^{k-i-1} - 2j \stackrel{n}{\equiv} 0$.

But

$$0 < 2^{k-1} \leq 2^{k-1} + 2^{k-i-1} - 2j \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} - 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k-i-1} + j \leq 2^{k-1} - 2^{k-i} - 2 < n,$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then

$p2^{k-i} + 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} \leq p2^{k-i} + 2^{k-i-1} - j \leq 2^{k-1} - 2^{k-i-1} - 1 < n,$$

a contradiction.

Obviously, $\beta \notin B$.

If $\beta \in A$, then $n - 2^{k-i-1} + 1 \leq 2^{k-i-1} - 1$. Hence, $n \leq 2^{k-i} - 2$, a contradiction.

If $\beta \in C$, then $n - 2^{k-i-1} + 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then

$2^{k-1} + 2^{k-i-1} + 2j - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} + 2^{k-i-1} - 2 \leq 2^{k-1} + 2^{k-i-1} + 2j - 2 \leq 2^{k-1} + 2^{k-i} - 4 < n,$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n - 2^{k-i-1} + 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} + 2^{k-i-1} + j - 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} + 2^{k-i-1} - 2 \leq p2^{k-i} + 2^{k-i-1} + j - 1 \leq 2^{k-1} - 4 < n,$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n - 2^{k-i-1} + 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$.

Then $p2^{k-i} - 2^{k-i-1} - j + 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} \leq p2^{k-i} - 2^{k-i-1} - j + 1 \leq 2^{k-1} - 2^{k-i+1} - 1 < n,$$

a contradiction.

Therefore, $\beta \notin I$, and the claim is proved.

Now, suppose that $n \leq 2^{k-1} + 2^{k-i} - 4$.

We show that the the dimensional broadcast scheme $(k-i, k-i+1, \dots, 1, 2, \dots, k-i-1, 1)$ is valid for these values of n .

First suppose that $n \leq 2^{k-1} + 2^{k-i-1} - 2$. As shown in Figure 3.3, define $A_p = I_p \cup I'_{2^{i-1}-p}$ and $U_p = \{(p+1)2^{k-i} - 1 - s \mid 1 \leq s \leq \gamma\}$, where $1 \leq p \leq 2^{i-1} - 2$ and $\gamma = \min A_{p+1} - \max A_p - 1 = 2^{k-1} + 2^{k-i-1} - n - 2 \geq 0$. Since $\max I_p \geq \min I_{2^{i-1}-p} - 1$, A_p contains consecutive elements and U_p are all vertices between A_p and A_{p+1} . Note that $\gamma \leq 2^{k-i-1} - 4$. $U'_p = \{n - (p+1)2^{k-i} + s + 2 \mid 3 \leq s + 2 \leq \gamma + 2 \leq 2^{k-i-1} - 2\} \subseteq I'_{p+1} \subseteq I$ is the set of first dimensional neighbors of U_p . Hence, U_p will be informed by I'_{p+1} in the last time unit.

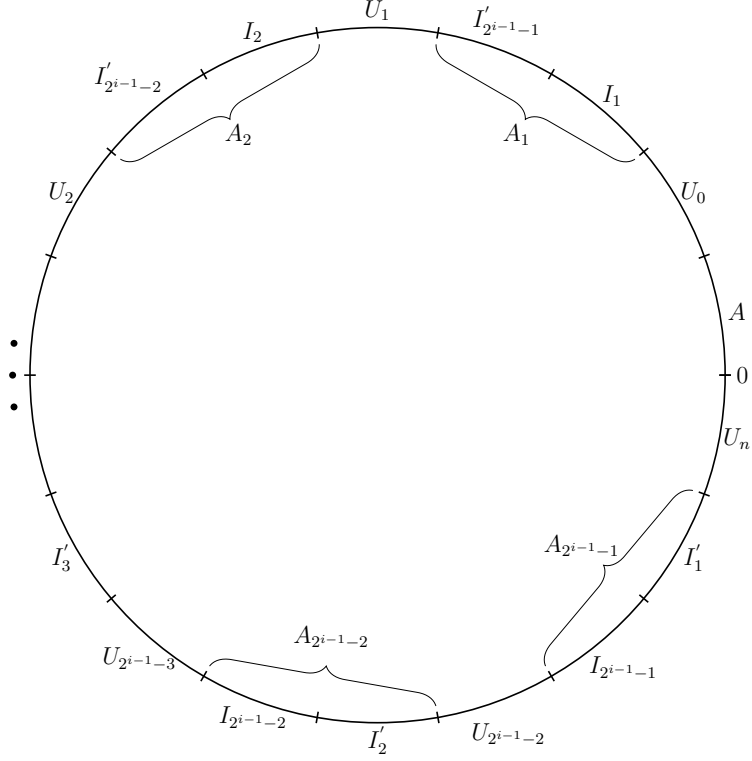


Figure 3.3: Sets of informed and uninformed vertices after time unit $(k - 1)$.

Let $U_0 = \{2^{k-i-1} + j' | 0 \leq j' \leq 2^{k-i-1} - 2\}$ and $U_n = \{n - 2^{k-i-1} + j' | 1 \leq j' \leq 2^{k-i-1} - 1\}$

be the sets of all vertices between A and I_1 , and I'_1 and A , respectively.

Since $2^{k-i-1} = n - 2^{k-1} + 2j$ for $j = \frac{2^{k-1} + 2^{k-i-1} - n}{2}$, $2^{k-i-1} \in B \subseteq I$. In this case,

note that $1 \leq j \leq 2^{k-i-2} - 1$. Let $2^{k-i-1} + j'$ for $1 \leq j' \leq 2^{k-i-1} - 2$ be another element of U_0 . Its first dimensional neighbor is $n - 2^{k-i-1} - j' + 1 = n - 2^{k-i} + 2^{k-i-1} - j' + 1$.

But $3 \leq 2^{k-i-1} - j' + 1 \leq 2^{k-i-1}$, and so $n - 2^{k-i-1} - j' + 1 \in I'_1$. Thus, U_0 will be informed by I'_1 in the last time unit.

For U_n , one can see that vertex $n - 2^{k-i-1} + 1$ will be informed by vertex $2^{k-i-1} \in B$ in the last time unit. Let $n - 2^{k-i-1} + j'$ for $2 \leq j' \leq 2^{k-i-1} - 1$ be another vertex of U_n . Its first dimensional neighbor is $2^{k-i-1} - j' + 1 \in A$. So, U_n will be informed by $A \cup \{2^{k-i-1}\}$.

Therefore, every vertex will be informed after time unit k .

Now, suppose that $2^{k-1} + 2^{k-i-1} \leq n \leq 2^{k-1} + 2^{k-i} - 4$. It is not hard to see that $1 + \min I_{p+1} \leq \max I'_{2^{i-1}-p} \leq \max I_{p+1} - 2$. So $A_{p+1} = I_{p+1} \cup I_{2^{i-1}-p}$ for $1 \leq p \leq 2^{i-1} - 3$ contains all consecutive vertices from $\min I'_{2^{i-1}-p}$ to $\max I_{p+1}$. As shown in Figure 3.4, define $A_1 = I_1$ and $A_{2^{i-1}-1} = I'_1$. Suppose that U_p for $1 \leq p \leq 2^{i-1} - 2$ is the set of vertices between A_p and A_{p+1} . Thus, $U_p = \{p2^{k-i} + 2^{k-i-1} - 2 + s \mid 1 \leq s \leq \gamma\}$, where $\gamma = n - 2^{k-1} - 2^{k-i-1} + 2 \geq 2$. Note that $\gamma \leq 2^{k-i-1} - 2$. Hence, $U'_p = \{n - p2^{k-i} - 2^{k-i-1} - s + 3 \mid 1 \leq s \leq \gamma\} = \{(2^{i-1} - p)2^{k-i} + j \mid j = n - 2^{k-1} - 2^{k-i-1} - s + 3, 1 \leq s \leq \gamma\} \subseteq I_{2^{i-1}-p}$ is the set of first dimensional neighbors of U_p . Hence, U_p will be informed by $I_{2^{i-1}-p}$.

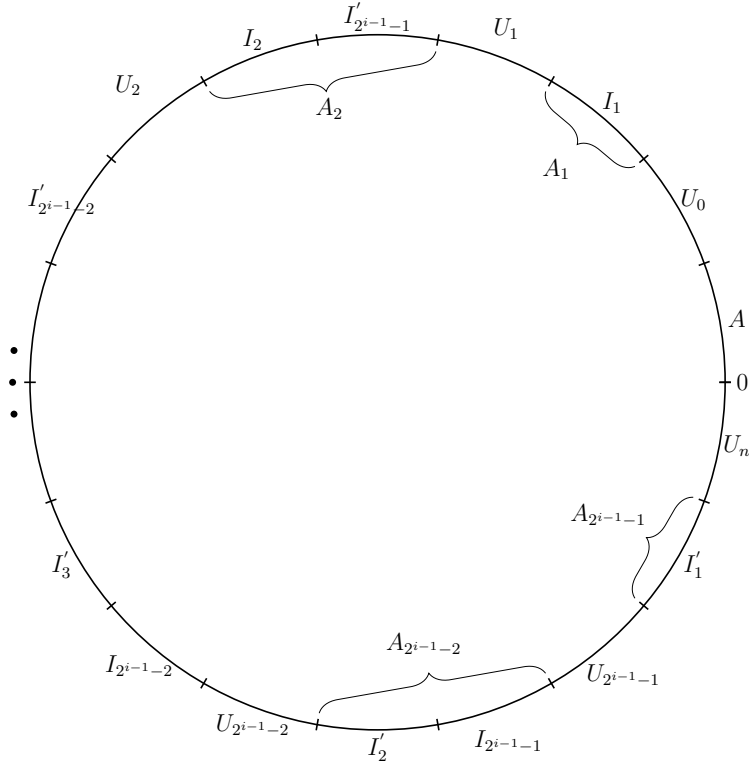


Figure 3.4: Sets of informed and uninformed vertices after time unit $(k - 1)$.

Let $U_0 = \{2^{k-i-1} + j' \mid 0 \leq j' \leq 2^{k-i-1} - 2\}$ and $U_n = \{n - 2^{k-i-1} + j' \mid 1 \leq j' \leq 2^{k-i-1} - 1\}$ be the sets of all vertices between A and I_1 , and I'_1 and A , respectively.

$2^{k-i-1} \in U_0$ will be informed in the last time unit since its first dimensional neighbor is

$n - 2^{k-i-1} + 1$ which is equal to $2^{k-1} + 2j - 1 \in C$ for $j = \frac{n - 2^{k-1} - 2^{k-i-1} + 2}{2}$. In this case, note that $1 \leq j \leq 2^{k-i-2} - 1$. Let $2^{k-i-1} + j'$ for $1 \leq j' \leq 2^{k-i-1} - 2$ be another vertex of U_0 . Its first dimensional neighbor is $n - 2^{k-i-1} - j' + 1 = n - 2^{k-i} + 2^{k-i-1} - j' + 1$. But $3 \leq 2^{k-i-1} - j' + 1 \leq 2^{k-i-1}$, and so $n - 2^{k-i-1} - j' + 1 \in I'_1$. Thus, U_0 will be informed by I'_1 in the last time unit.

For U_n , as we showed before, $n - 2^{k-i-1} + 1 \in C$. Let $n - 2^{k-i-1} + j'$ for $2 \leq j' \leq 2^{k-i-1} - 1$ be another vertex of U_n . Its first dimensional neighbor is $2^{k-i-1} - j' + 1 \in A$. So, U_n will be informed by A . Therefore, every vertex will be informed after time unit k .

Now, consider $i = d$.

The dimensional broadcast scheme is $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-1)$.

Suppose that $n \geq 2^{k-1} + 2^{k-i-1}$.

We show that for $n \geq 2^{k-1} + 2^{k-i-1}$, this dimensional broadcast scheme is not valid.

First, if $n \geq 2^{k-1} + 2^{k-2}$, then we claim that two $(k-1)$ dimensional neighbors $\alpha = 2^{k-2} + 2^{k-i-1} - 1$ and $\beta = 2^{k-2} - 2^{k-i-1}$ are not in I .

Obviously, $\alpha \notin A \cup B$.

If $\alpha \in C$, then $2^{k-2} + 2^{k-i-1} - 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then $2^{k-2} - 2^{k-i-1} + 2j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-2} - 2^{k-i-1} \leq 2^{k-2} - 2^{k-i-1} + 2j \leq 2^{k-2} - 2 < n,$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} + 2^{k-i-1} - 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k-2} - 2^{k-i-1} + j + 1 \stackrel{n}{\equiv} 0$. But

$$-n < 2^{k-i-1} - 2^{k-2} \leq p2^{k-i} - 2^{k-2} - 2^{k-i-1} + j + 1 \leq 2^{k-2} - 2^{k-i} - 1 < n.$$

Hence, $p2^{k-i} - 2^{k-2} - 2^{k-i-1} + j + 1 = 0$, and so $p2^{k-i} = 2^{k-2} + 2^{k-i-1} - j - 1$. Thus,

$$2^{i-2} + \frac{1}{2^{k-i}} \leq p \leq 2^{i-2} + \frac{1}{2},$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} + 2^{k-i-1} - 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$.

Then $p2^{k-i} + 2^{k-2} + 2^{k-i-1} - j - 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} + 2^{k-2} - 1 \leq p2^{k-i} + 2^{k-2} + 2^{k-i-1} - j - 1 \leq 2^{k-1} + 2^{k-2} - 2^{k-i} - 2^{k-i-1} - 2 < n,$$

a contradiction.

Obviously, $\beta \notin C$.

If $\beta \in A$, then $2^{k-2} - 2^{k-i-1} \leq 2^{k-i-1} - 1$, a contradiction.

If $\beta \in B$, then $2^{k-2} - 2^{k-i-1} \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} + 2^{k-2} - 2^{k-i} \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2j \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2 < n,$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} - 2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} - 2^{k-2} + 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$-n < -2^{k-2} + 2^{k-i} + 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k-2} + 2^{k-i-1} + j \leq 2^{k-2} - 2 < n,$$

Hence, $p2^{k-i} - 2^{k-2} + 2^{k-i-1} + j = 0$, and so $p2^{k-i} = 2^{k-2} - 2^{k-i-1} - j$. Thus,

$$2^{i-2} - 1 + \frac{2}{2^{k-i}} \leq p \leq 2^{i-2} - \frac{1}{2} + \frac{1}{2^{k-i}},$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} - 2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$.

Then $p2^{k-i} + 2^{k-2} - 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-2} \leq p2^{k-i} + 2^{k-2} - 2^{k-i-1} - j \leq 2^{k-1} + 2^{k-2} - 2^{k-i+1} - 1 < n,$$

a contradiction.

Therefore, $\alpha, \beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme

$(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-1)$ is not valid.

Second, if $2^{k-1} + 2^{k-i-1} + p'2^{k-i} \leq n \leq 2^{k-1} + (p' + 1)2^{k-i} - 2$ for $0 \leq p' \leq 2^{i-2} - 1$, then

we claim that $\alpha = 2^{k-i-1}$ and its $(k-1)$ dimensional neighbor $\beta = 2^{k-1} - 2^{k-i-1} - 1$ are not in I .

Obviously, $\alpha \notin A \cup C$.

If $\alpha \in B$, then $2^{k-i-1} \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} + 2^{k-i-1} - 2j \stackrel{n}{\equiv} 0$.

But

$$0 < 2^{k-1} \leq 2^{k-1} + 2^{k-i-1} - 2j \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} - 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k-i-1} + j \leq 2^{k-1} - 2^{k-i} - 2 < n.$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then

$p2^{k-i} + 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} \leq p2^{k-i} + 2^{k-i-1} - j \leq 2^{k-1} - 2^{k-i-1} - 1 < n,$$

a contradiction.

Obviously, $\beta \notin B$.

If $\beta \in A$, then $2^{k-1} - 2^{k-i-1} - 1 \leq 2^{k-i-1} - 1$, a contradiction.

If $\beta \in C$, then $2^{k-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then

$2^{k-i-1} + 2j \stackrel{n}{\equiv} 0$. But $0 < 2^{k-i-1} \leq 2^{k-i-1} + 2j \leq 2^{k-i} - 2 < n$, a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq$

$2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k-1} + 2^{k-i-1} + j + 1 \stackrel{n}{\equiv} 0$. But

$$-n < -2^{k-1} + 2^{k-i} + 2^{k-i-1} - 2 \leq p2^{k-i} - 2^{k-1} + 2^{k-i-1} + j + 1 \leq -1 < 0,$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$.

Then $p2^{k-i} + 2^{k-1} - 2^{k-i-1} - j - 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} - 1 \leq p2^{k-i} + 2^{k-1} - 2^{k-i-1} - j - 1 \leq 2^k - 2^{k-i} - 2^{k-i-1} - 2 < 2n.$$

Hence, $p2^{k-i} + 2^{k-1} - 2^{k-i-1} - j - 1 = n$, and so

$$\begin{aligned} 2^{k-1} + 2^{k-i-1} + p'2^{k-i} &\leq p2^{k-i} + 2^{k-1} - 2^{k-i-1} - j - 1 \leq 2^{k-1} + (p' + 1)2^{k-i} - 2 \Rightarrow \\ (p - p' - 1)2^{k-i} - 2^{k-i-1} + 1 &\leq j \leq (p - p' - 1)2^{k-i} - 1. \end{aligned}$$

Since $j \geq 1$, $p - p' - 1 > 0$. Hence,

$$2^{k-i-1} + 1 = 2^{k-i} - 2^{k-i-1} + 1 \leq (p - p' - 1)2^{k-i} - 2^{k-i-1} + 1 \leq j \leq 2^{k-i-1},$$

a contradiction.

Therefore, $\alpha, \beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme

$(k - i, k - i + 1, \dots, k - 1, 1, 2, \dots, k - i - 1, k - 1)$ is not valid.

Third, if $2^{k-1} + (p' + 1)2^{k-i} \leq n \leq 2^{k-1} + (p' + 1)2^{k-i} + 2^{k-i-1} - 2$ for $0 \leq p' \leq 2^{i-2} - 2$,

then we claim that $\alpha = 2^{k-1}$ and its $(k - 1)$ dimensional neighbor $\beta = n - 1$ are not in I .

Obviously, $\alpha \notin A \cup C$.

If $\alpha \in B$, then $2^{k-1} \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^k - 2j \stackrel{n}{\equiv} 0$. Since $n \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$ and $2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2 < 2^k - 2^{k-i-1}$, we have

$$n < 2^k - 2^{k-i-1} \leq 2^k - 2j \leq 2^k - 2 < 2n,$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k-1} + j \stackrel{n}{\equiv} 0$. But

$$-n < -2^{k-1} + 2^{k-i} - 1 \leq p2^{k-i} - 2^{k-1} + j \leq -2^{k-i-1} - 2 < 0.$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} + 2^{k-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} + 2^{k-i-1} \leq p2^{k-i} + 2^{k-1} - j \leq 2^k - 2^{k-i} - 1 < 2n,$$

Hence, $p2^{k-i} + 2^{k-1} - j = n$, and so

$$\begin{aligned} 2^{k-1} + (p' + 1)2^{k-i} &\leq p2^{k-i} + 2^{k-1} - j \leq 2^{k-1} + (p' + 1)2^{k-i} + 2^{k-i-1} - 2 \Rightarrow \\ p' + 1 + \frac{1}{2^{k-i}} &\leq p \leq p' + 2 - \frac{2}{2^{k-i}} \end{aligned}$$

a contradiction.

Obviously, $\beta \notin A \cup B$.

If $\beta \in C$, then $n - 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then $2^{k-1} + 2j \stackrel{n}{\equiv} 0$.

But since $2^{k-1} + 2^{k-i} \leq n$, we have

$$0 < 2^{k-1} \leq 2^{k-1} + 2j \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n - 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} + j + 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} \leq p2^{k-i} + j + 1 \leq 2^{k-1} - 2^{k-i-1} - 1 < n,$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n - 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} - j - 1 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i-1} - 1 \leq p2^{k-i} - j - 1 \leq 2^{k-1} - 2^{k-i} - 2 < n.$$

a contradiction.

Therefore, $\alpha, \beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme $(k - i, k - i + 1, \dots, k - 1, 1, 2, \dots, k - i - 1, k - 1)$ is not valid for $n \geq 2^{k-1} + 2^{k-i-1}$.

Now, suppose that $n \leq 2^{k-1} + 2^{k-i-1} - 2$.

We show that the dimensional broadcast scheme $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-1)$ is valid.

As denoted above, define $A_p = I_p \cup I'_{2^{i-1}-p}$ and $U_p = \{(p+1)2^{k-i} - 1 - s | 1 \leq s \leq \gamma\}$, where $1 \leq p \leq 2^{i-1} - 2$ and $\gamma = \min A_{p+1} - \max A_p - 1 = 2^{k-1} + 2^{k-i-1} - n - 2 \geq 0$.

Since $\max I_p \geq \min I'_{2^{i-1}-p} - 1$, A_p contains consecutive elements and U_p are all vertices between A_p and A_{p+1} . Note that $\gamma \leq 2^{k-i-1} - 4$. $U'_p = \{2^{k-1} - (p+1)2^{k-i} + s | 1 \leq s \leq \gamma \leq 2^{k-i-1} - 4\} = \{(2^{i-1} - p - 1)2^{k-i} + s | 1 \leq s \leq \gamma\} \subseteq I_{2^{i-1}-p-1} \subseteq I$ is the set of first dimensional neighbors of U_p . Hence, U_p will be informed by $I_{2^{i-1}-p-1}$ in the last time unit.

Let $U_0 = \{2^{k-i-1} + j' | 0 \leq j' \leq 2^{k-i-1} - 2\}$ and $U_n = \{n - 2^{k-i-1} + j' | 1 \leq j' \leq 2^{k-i-1} - 1\}$ be the sets of all vertices between A and I_1 , and I'_1 and A , respectively.

Since $2^{k-i-1} = n - 2^{k-1} + 2j$ for $j = \frac{2^{k-1} + 2^{k-i-1} - n}{2}$, $2^{k-i-1} \in B$. Let $2^{k-i-1} + j'$ for $1 \leq j' \leq 2^{k-i-1} - 2$ be another vertex of U_0 . Its $(K-1)$ dimensional neighbor is $2^{k-1} - 2^{k-i-1} - j' - 1 = (2^{i-1} - 1)2^{k-i} + 2^{k-i-1} - j' - 1$. But $1 \leq 2^{k-i-1} - j' - 1 \leq 2^{k-i-1} - 2$, and so $2^{k-1} - 2^{k-i-1} - j' - 1 \in I_{2^{i-1}-1}$. Thus, U_0 will be informed by $I_{2^{i-1}-1}$ in the last time unit.

For U_n , note that C covers all odd vertices between $2^{k-1} - 1$ and 0. Hence, all even vertices between $2^{k-1} - 1$ and 0 will be informed by C . Since the $(k-1)$ dimensional neighbor of $2^{k-1} - 2$ and $n - 2^{k-i-1} + 1$ are $1 \in A$ and $2^{k-1} - n + 2^{k-i-1} - 2 \in A$, respectively, U_n will be informed by $A \cup C$. Therefore, the dimensional broadcast scheme $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-1)$ is valid.

Now, consider $i \geq d+1$ or equivalently $3 \leq d \leq i-1$.

The dimensional broadcast scheme is $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-i+d-1)$.

First, suppose that $n \geq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1}$.

We claim that two $(k - i + d - 1)$ dimensional neighbors $\alpha = 2^{k+d-i-2} - 2^{k-i-1}$ and $\beta = 2^{k+d-i-2} + 2^{k-i-1} - 1$ are not in I .

Obviously, $\alpha \notin C$.

If $\alpha \in A$, then $2^{k+d-i-2} - 2^{k-i-1} \leq 2^{k-i-1} - 1$, and so $2^{k+d-i-2} \leq 2^{k-i} - 1$. Hence, $k + d - i - 2 < k - i$. Thus, $d < 2$, a contradiction.

If $\alpha \in B$, then $2^{k+d-i-2} - 2^{k-i-1} \stackrel{n}{\equiv} n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 2j \stackrel{n}{\equiv} 0$. But

$$\begin{aligned} 0 < 2^{k-1} + 2^{k+d-i-2} - 2^{k-i} &\leq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 2j \leq \\ &2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 2 < n, \end{aligned}$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k+d-i-2} - 2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k+d-i-2} + 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$-n < 2^{k-i} - 2^{k-3} + 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k+d-i-2} + 2^{k-i-1} + j \leq 2^{k-1} - 2^{k+d-i-2} - 2 < n.$$

Hence, $p2^{k-i} - 2^{k+d-i-2} + 2^{k-i-1} + j = 0$, and so $p2^{k-i} = 2^{k+d-i-2} - 2^{k-i-1} - j$. Thus,

$$2^{d-2} - 1 + \frac{2}{2^{k-i}} \leq p \leq 2^{d-2} - \frac{1}{2} + \frac{1}{2^{k-i}},$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k+d-i-2} - 2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} + 2^{k+d-i-2} - 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k+d-i-2} \leq p2^{k-i} + 2^{k+d-i-2} - 2^{k-i-1} - j \leq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 2^{k-i} - 1 < n,$$

a contradiction.

Obviously, $\beta \notin A \cup B$.

If $\beta \in C$, then $2^{k+d-i-2} + 2^{k-i-1} - 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then $2^{k-1} - 2^{k+d-i-2} - 2^{k-i-1} + 2j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} - 2^{k+d-i-2} - 2^{k-i-1} \leq 2^{k-1} - 2^{k+d-i-2} - 2^{k-i-1} + 2j \leq 2^{k-1} - 2^{k+d-i-2} - 2 < n,$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k+d-i-2} + 2^{k-i-1} - 1 \equiv_p 2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k+d-i-2} - 2^{k-i-1} + j + 1 \equiv 0$. But

$$-n < 2^{k-i-1} - 2^{k-3} \leq p2^{k-i} - 2^{k+d-i-2} - 2^{k-i-1} + j + 1 \leq 2^{k-1} - 2^{k+d-i-2} - 2^{k-i} - 1 < n,$$

Hence, $p2^{k-i} - 2^{k+d-i-2} - 2^{k-i-1} + j + 1 = 0$, and so $p2^{k-i} = 2^{k+d-i-2} + 2^{k-i-1} - j - 1$.

Thus,

$$2^{d-2} + \frac{1}{2^{k-i}} \leq p \leq 2^{d-2} + \frac{1}{2},$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k+d-i-2} + 2^{k-i-1} - 1 \equiv_n n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} + 2^{k+d-i-2} + 2^{k-i-1} - j - 1 \equiv 0$. But

$$0 < 2^{k-i} + 2^{k+d-i-2} - 1 \leq p2^{k-i} + 2^{k+d-i-2} + 2^{k-i-1} - j - 1 \leq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 2 < n.$$

a contradiction.

Therefore, $\alpha, \beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-i+d-1)$ is not valid for $n \geq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1}$.

Second, assume $2^{k-1} + p'2^{k-i} + 2^{k-i-1} \leq n \leq 2^{k-1} + (p'+1)2^{k-i} - 2$ for $0 \leq p' \leq 2^{d-2} - 2$.

In this case, we claim that $\alpha = 2^{k-i-1}$ and its $(k-i+d-1)$ dimensional neighbor $\beta = 2^{k-i+d-1} - 2^{k-i-1} - 1$ are not in I .

Obviously, $\alpha \notin A \cup C$.

If $\alpha \in B$, then $2^{k-i-1} \equiv_n n - 2^{k-1} + 2j$ where $1 \leq j \leq 2^{k-i-2}$. Then $2^{k-1} + 2^{k-i-1} - 2j \equiv 0$.

But

$$0 < 2^{k-1} \leq 2^{k-1} + 2^{k-i-1} - 2j \leq 2^{k-1} + 2^{k-i-1} - 2 < n,$$

a contradiction.

If $\alpha \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq 2^{k-i-1} - 2$.

Then $p2^{k-i} - 2^{k-i-1} + j \stackrel{n}{\equiv} 0$. But

$$-n < 2^{k-i-1} - 1 \leq p2^{k-i} - 2^{k-i-1} + j \leq 2^{k-1} - 2^{k-i} - 2 < n.$$

a contradiction.

If $\alpha \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i-1} \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then

$p2^{k-i} + 2^{k-i-1} - j \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-i} \leq p2^{k-i} + 2^{k-i-1} - j \leq 2^{k-1} + 2^{k+d-i-2} - 2^{k-i-1} - 1 < n,$$

a contradiction.

Obviously, $\beta \notin B$.

If $\beta \in A$, then $2^{k-i+d-1} - 2^{k-i-1} - 1 \leq 2^{k-i-1} - 1$, and so $2^{k-i+d-1} \leq 2^{k-i}$. Thus,

$k - i + d - 1 \leq k - i$. Hence, $d \leq 1$, a contradiction.

If $\beta \in C$, then $2^{k-i+d-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} 2^{k-1} + 2j - 1$ where $0 \leq j \leq 2^{k-i-2} - 1$. Then

$2^{k-1} - 2^{k-i+d-1} + 2^{k-i-1} + 2j \stackrel{n}{\equiv} 0$. But

$$\begin{aligned} 0 < 2^{k-1} - 2^{k-i+d-1} + 2^{k-i-1} &\leq 2^{k-1} - 2^{k-i+d-1} + 2^{k-i-1} + 2j \leq \\ &2^{k-1} - 2^{k-i+d-1} + 2^{k-i} - 2 < 2n. \end{aligned}$$

Hence $2^{k-1} - 2^{k-i+d-1} + 2^{k-i-1} + 2j = n$. Thus,

$$\begin{aligned} 2^{k-1} + p'2^{k-i} + 2^{k-i-1} &\leq 2^{k-1} - 2^{k-i+d-1} + 2^{k-i-1} + 2j \Rightarrow \\ 0 \leq p'2^{k-i} &\leq 2j - 2^{k-i+d-1} \Rightarrow 2^{k-i+d-1} \leq 2j = 2^{k-i-1} - 2 \Rightarrow \\ k - i + d - 1 &< k - i - 1 \Rightarrow d < 0, \end{aligned}$$

a contradiction.

If $\beta \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i+d-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} p2^{k-i} + j$ where $-1 \leq j \leq$

$2^{k-i-1} - 2$. Then $p2^{k-i} - 2^{k-i+d-1} + 2^{k-i-1} + j + 1 \stackrel{n}{\equiv} 0$. But

$$-n < 2^{k-i} - 2^{k-3} + 2^{k-i-1} \leq p2^{k-i} - 2^{k-i+d-1} + 2^{k-i-1} + j + 1 \leq 2^{k-1} - 2^{k-i+d-1} - 1 < n,$$

Hence, $p2^{k-i} - 2^{k-i+d-1} + 2^{k-i-1} + j + 1 = 0$, and so $p2^{k-i} = 2^{k-i+d-1} - 2^{k-i-1} - j - 1$.

Thus,

$$2^{d-1} - 1 + \frac{1}{2^{k-i}} \leq p \leq 2^{d-1} - \frac{1}{2},$$

a contradiction.

If $\beta \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-i+d-1} - 2^{k-i-1} - 1 \stackrel{n}{\equiv} n - p2^{k-i} + j$ where $1 \leq j \leq 2^{k-i-1}$. Then $p2^{k-i} + 2^{k-i+d-1} - 2^{k-i-1} - j - 1 \stackrel{n}{\equiv} 0$. But

$$\begin{aligned} 0 < 2^{k+d-i-2} - 1 &\leq p2^{k-i} + 2^{k-i+d-1} - 2^{k-i-1} - j - 1 \leq \\ 2^{k-1} - 2^{k-i} + 2^{k+d-i-2} - 2^{k-i-1} - 2 &< 2n. \end{aligned}$$

Hence, $p2^{k-i} + 2^{k-i+d-1} - 2^{k-i-1} - j - 1 = n$. Thus,

$$\begin{aligned} 2^{k-1} + p'2^{k-i} + 2^{k-i-1} &\leq p2^{k-i} + 2^{k-i+d-1} - 2^{k-i-1} - j - 1 \leq 2^{k-1} + (p' + 1)2^{k-i} - 2 \Rightarrow \\ 2^{i-1} - 2^{d-1} + p' + 1 + \frac{2}{2^{k-i}} &\leq p \leq 2^{i-1} - 2^{d-1} + p' + 2 - \frac{1}{2^{k-i}}, \end{aligned}$$

a contradiction.

Therefore, $\alpha, \beta \notin I$, and the claim is proved. Thus, the dimensional broadcast scheme $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-i+d-1)$ is not valid.

Now, suppose that $2^{k-1} + p'2^{k-i} \leq n \leq 2^{k-1} + p'2^{k-i} + 2^{k-i-1} - 2$ for $0 \leq p' \leq 2^{d-2} - 1$.

In this case, we show that the dimensional broadcast scheme $(k-i, k-i+1, \dots, k-1, 1, 2, \dots, k-i-1, k-i+d-1)$ is valid.

Similar to the case $d = i$, as shown in Figure 3.5, define $A_p = I_p \cup I'_{2^{i-1}-p+p'}$, for $p'+1 \leq p \leq 2^{i-1}-1$, and $U_p = \{(p+1)2^{k-i}-1-s \mid 1 \leq s \leq \gamma\}$, where $p'+1 \leq p \leq 2^{i-1}-2$ and $\gamma = \min A_{p+1} - \max A_p - 1 = 2^{k-1} + 2^{k-i-1} + p'2^{k-i} - n - 2 \geq 0$. Since $\max I_p \geq \min I'_{2^{i-1}-p+p'} - 1$, A_p contains consecutive elements and U_p are all vertices between A_p and A_{p+1} . Note that $\gamma \leq 2^{k-i-1} - 2$. Until now, all vertices belong to

$A \cup (\bigcup_{p=1}^{p'} I_p) \cup (\bigcup_{p=p'+1}^{2^{i-1}-1} A_p) \cup (\bigcup_{p=1}^{p'} I'_p)$ are informed after time unit $k-1$. We show that the following five remaining sets of vertices will be informed in the last time unit.

$U_0 = \{2^{k-i-1}, \dots, 2^{k-i} - 2\}$, vertices between A and I_1 .

$U_p = \{p2^{k-i} + 2^{k-i-1} - 1, \dots, (p+1)2^{k-i} - 2 \mid 1 \leq p \leq p'\}$, vertices between I_p and I_{p+1} .

$U_p = \{(p+1)2^{k-i} - 1 - s \mid 1 \leq s \leq \gamma, p'+1 \leq p \leq 2^{i-1} - 2\}$, vertices between A_p and A_{p+1} .

$V_p = \{n - (p+1)2^{k-i} + 2^{k-i-1} + 1, \dots, n - p2^{k-i} \mid 1 \leq p \leq p'\}$, vertices between I'_{p+1} and v_p .

$V_0 = \{n - 2^{k-i} + 2^{k-i-1} + 1, \dots, n - 1\}$, vertices between I'_1 and 0.

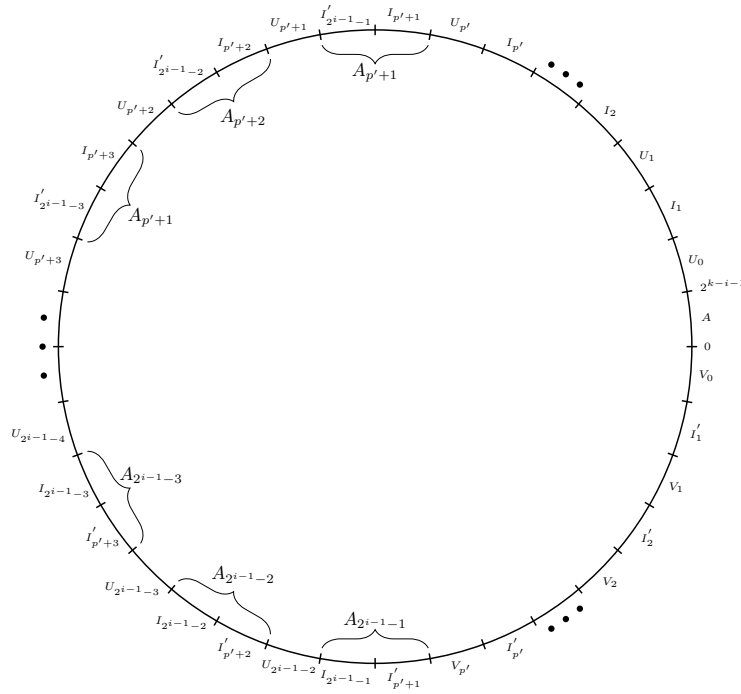


Figure 3.5: Sets of informed and uninformed vertices after time unit $(k-1)$.

U_0 : $(k+d-i-1)$ dimensional neighbor of $\min U_0 = 2^{k-i-1}$ is $\beta = 2^{k+d-i-1} - 2^{k-i-1} - 1 = n - p2^{k-i} + j \in I'_p$, for $p = 2^{i-1} + p' - 2^{d-1} + 1$ and $j = 2^{k-1} + p'2^{k-i} + 2^{k-i-1} - n - 1$. Note that $1 \leq j \leq 2^{k-i-1} - 1$. Hence, $\beta \in A_{2^{d-1}-1}$. Also, $(k+d-i-1)$ dimensional neighbor

of $\max U_0 = 2^{k-i} - 2$ is $2^{k+d-i-1} - 2^{k-i} + 1 = (2^{d-1} - 1)2^{k-i} + 1 \in I_{2^{d-1}-1} \subseteq A_{2^{d-1}-1}$.

Therefore, U_0 will be informed by $A_{2^{d-1}-1}$.

U_p for $1 \leq p \leq p' \leq 2^{d-2} - 1$: $(k + d - i - 1)$ dimensional neighbor of $\min U_p = p2^{k-i} + 2^{k-i-1} - 1$ is $\beta = 2^{k+d-i-1} - p2^{k-i} - 2^{k-i-1} = n - r2^{k-i} + j \in I'_r$, for $r = 2^{i-1} + p' + p - 2^{d-1} + 1$ and $j = 2^{k-1} + p'2^{k-i} + 2^{k-i-1} - n$. Note that $2 \leq j \leq 2^{k-i-1}$. Hence, $\beta \in A_{2^{d-1}-p-1}$. Also, $(k + d - i - 1)$ dimensional neighbor of $\max U_p = (p+1)2^{k-i} - 2$ is $2^{k+d-i-1} - (p+1)2^{k-i} + 1 = (2^{d-1} - p - 1)2^{k-i} + 1 \in I_{2^{d-1}-p-1} \subseteq A_{2^{d-1}-p-1}$. Therefore, U_p will be informed by $A_{2^{d-1}-p-1}$.

U_p for $p' + 1 \leq p \leq 2^{i-1} - 2$: There are three cases.

Case 1. $p < 2^{d-1} - 1$. If $(p+1)2^{k-i} - s - 1 \in V_p$, for $1 \leq s \leq \gamma$, then its $(k + d - i - 1)$ dimensional neighbor is $2^{k+d-i-1} - 1 - (p+1)2^{k-i} + s + 1 = (2^{d-1} - p - 1)2^{k-i} + s \in I_{2^{d-1}-p-1}$. Therefore, V_p will be informed by $I_{2^{d-1}-p-1}$.

Case 2. $p = 2^{d-1} - 1$. If $(p+1)2^{k-i} - s - 1 = 2^{k+d-i-1} - s - 1 \in V_p$, for $1 \leq s \leq \gamma$, then its $(k + d - i - 1)$ dimensional neighbor is $s \in A$. Therefore, V_p will be informed by A .

Case 3. $p > 2^{d-1} - 1$. If $(p+1)2^{k-i} - s - 1 \in V_p$, for $1 \leq s \leq \gamma$, then its $(k + d - i - 1)$ dimensional neighbor is $2^{k+d-i-1} - 1 - (p+1)2^{k-i} + s + 1 \stackrel{n}{\equiv} n + 2^{k+d-i-1} - (p+1)2^{k-i} + s = n - (p+1 - 2^{d-1})2^{k-i} + s \in I'_{p+1-2^{d-1}}$. Therefore, V_p will be informed by $I'_{p+1-2^{d-1}}$.

V_p for $1 \leq p \leq p' \leq 2^{d-2} - 1$: If $n - (p+1)2^{k-i} + 2^{k-i-1} + j' \in V_p$, for $1 \leq j' \leq 2^{k-i-1}$, then its $(k + d - i - 1)$ dimensional neighbor is $2^{k+d-i-1} - 1 - n + (p+1)2^{k-i} - 2^{k-i-1} - j' \stackrel{n}{\equiv} 2^{k+d-i-1} + (p+1)2^{k-i} - 2^{k-i-1} - j' - 1 = (p + 2^{d-1})2^{k-i} + 2^{k-i-1} - j' - 1 \in I_{p+2^{d-1}}$.

Therefore, V_p will be informed by $I_{p+2^{d-1}}$.

V_0 : If $n - 2^{k-i} + 2^{k-i-1} + j' \in V_0$, for $1 \leq j' \leq 2^{k-i-1} - 1$, then its $(k + d - i - 1)$ dimensional neighbor is $2^{k+d-i-1} - 1 - n + 2^{k-i} - 2^{k-i-1} - j' \stackrel{n}{\equiv} 2^{k+d-i-1} + 2^{k-i-1} - j' - 1 = (2^{d-1})2^{k-i} + 2^{k-i-1} - j' - 1 \in I_{2^{d-1}}$. Thus, V_0 will be informed by $I_{2^{d-1}}$.

Therefore, the broadcast scheme is valid. $\square$

3.5 Arbitrary Cyclic Shifts of Reversed Dimensional Broadcast Scheme With Repeating an Arbitrary Dimension

In the next theorem which is a generalization of Theorem 3.3.1, we specify the complete range of values for n where c left cyclic shifts, $0 \leq c \leq k - 2$, of the reverse dimensional broadcast scheme $(k - 1, k - 2, \dots, 1)$ with repetition the d -th dimension, $3 \leq d \leq k - 1$, is valid.

Theorem 3.5.1. *Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil$. Then for two integers $0 \leq c \leq k - 2$ and $3 \leq d \leq k - 1$, c left cyclic shifts of the dimensional broadcast scheme $(k - 1, k - 2, \dots, 1)$ repeating the d -th dimension is a valid broadcast scheme if and only if*

$$\left\{ \begin{array}{llll} c = 0 & \text{and } n \leq 2^{k-1} + 2^{k-d} - 2 & & \\ c = k - 2 & \text{and } n < 2^{k-1} + 2^{k-d+1} - 2 & \text{and } n \equiv 0^4 & \\ k - d \leq c \leq k - 3 & \text{and } n < 2^{k-1} + 2^{2k-c-d-2} & \text{and } n \equiv 2^{k-c} 2^{k-c-1} - 2 & \\ 1 \leq c \leq k - d - 1 & \text{and } 2^{k-1} + 2^{k-c-1} - 2^{k-c-d} - 2 \leq n \leq 2^{k-1} + 2^{k-c-1} - 2, & & \end{array} \right.$$

Proof. Since each left cyclic shift is equivalent to $k - 2$ right cyclic shifts of $(1, \dots, k - 1)$, we adopt right cyclic shifts for computational simplicity. Consequently, for c left shifts of $(1, \dots, k - 1)$, we instead consider $i = k - c - 1$ right cyclic shifts of $(1, \dots, k - 1)$. Notably, when $c = 0$, the case $i = 0$ coincides with $i = k - 1$. Accordingly, after reformulating the statement, we proceed to establish the following result:

Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil$. Then for two integers $0 \leq i \leq k - 2$ and $3 \leq d \leq k - 1$, i cyclic shifts of the dimensional broadcast scheme $(k - 1, k - 2, \dots, 1)$ repeating the d -th dimension is a valid broadcast scheme if and only if

$$\left\{ \begin{array}{lll} i = 0 & \text{and } n \leq 2^{k-1} + 2^{k-d} - 2 \\ i = 1 & \text{and } n < 2^{k-1} + 2^{k-d+1} - 2 & \text{and } n \equiv 0 \\ 2 \leq i \leq d-1 & \text{and } n < 2^{k-1} + 2^{k+i-d-1} & \text{and } n \equiv 2^i - 2 \\ d \leq i \leq k-2 & \text{and } 2^{k-1} + 2^i - 2^{i-d+1} - 2 \leq n \leq 2^{k-1} + 2^i - 2, \end{array} \right.$$

It is evident that if $d = k-1$, no vertex will be informed during the last time unit. Therefore, we assume $d \leq k-2$.

First, consider $i = 0$.

Then the broadcast scheme is $(k-1, \dots, 1, k-d)$. Similar to the proof of Theorem 3.3.1, by induction, one can see that informed vertices after time unit $(k-1)$ are

$$I = \{0\} \cup \{2p-1 | 1 \leq p \leq 2^{k-2}\} \cup \{n-2p | 1 \leq p \leq 2^{k-2}-1\}.$$

Suppose that $n \geq 2^{k-1} + 2^{k-d}$.

We claim that $\alpha = 2^{k-d}$ and its $(k-d)$ dimensional neighbor, vertex $n-1$, are not in I , and so will not be informed in the last time unit.

If $\alpha \in I$, then $2^{k-d} = n-2p$ for $0 \leq p \leq 2^{k-2}-1$. Hence, $n = 2^{k-d} + 2p \leq 2^{k-1} + 2^{k-d} - 2$, a contradiction.

If $\beta \in I$, then $n-1 = 2p-1$ for $1 \leq p \leq 2^{k-2}$. Hence, $n = 2p \leq 2^{k-1}$, a contradiction.

Now, suppose that $n \leq 2^{k-1} + 2^{k-d} - 2$.

It is not hard to see that $\{2^{k-d}, 2^{k-d} + 1, \dots, 2^{k-1}\} \subseteq I$. If $2 \leq 2r \leq 2^{k-d} - 2$ is an even vertex between 0 and 2^{k-d} , then since all odd vertices between 0 to 2^{k-d} are in I , its $(k-d)$ neighbor is $2^{k-d} - 1 - 2r \in I$. Hence, all even vertices in this range will be informed in the last time unit. Also, if $n-2r+1$ is an odd vertex between 2^{k-1} and 0, then its $(k-d)$ neighbor is $n + 2^{k-d} - 1 - n + 2r - 1 = 2^{k-d} + 2r - 2 \leq 2^{k-1}$. Therefore, all odd vertices between 2^{k-1} and 0 will be informed in the last time unit.

Now, consider $i = 1$.

Then the broadcast scheme is $(1, k-1, \dots, 2, k-d+1)$. Similar to the proof of Theorem 3.3.1, by induction, one can see that informed vertices after time unit t , $2 \leq t \leq k-1$, will be

$$\begin{aligned} & \{0, 1\} \cup \{p2^{k-t+1} - 1, p2^{k-t-1} - 2 \mid 1 \leq p \leq 2^{t-2}\} \\ & \cup \{n - p2^{k-t+1}, n - p2^{k-t+1} + 1 \mid 1 \leq p \leq 2^{t-2} - 1\}. \end{aligned}$$

Thus, after time unit $(k-1)$, informed vertices will be

$$I = I_2 \cup I_3 = \{4p-1, 4p-2 \mid 1 \leq p \leq 2^{k-3}\} \cup \{n-4p, n-4p+1 \mid 0 \leq p \leq 2^{k-3}-1\},$$

where $I_2 = \{4p-1, 4p-2 \mid 1 \leq p \leq 2^{k-3}\}$ and $I_3 = \{n-4p, n-4p+1 \mid 0 \leq p \leq 2^{k-3}-1\}$.

suppose that $n \not\equiv 0 \pmod{4}$ or $n \geq 2^{k-1} + 2^{k-d+1}$.

We claim that $\alpha = 2^{k-d+1}$ and its $(k-d)$ dimensional neighbor $n-1$ are not in I , and so will not be informed in the last time unit.

Obviously $n-1 \notin I$.

If $\alpha \in I$ and $2^{k-d+1} = 4p-2$ for $1 \leq p \leq 2^{k-3}$, then $4 \mid 4p-2^{k-d+1} = 2$, a contradiction. If $\alpha \in I$ and $2^{k-d+1} = n-4p$ for $0 \leq p \leq 2^{k-3}-1$, then $n = 2^{k-d+1} + 4p \leq 2^{k-1} + 2^{k-d+1} - 4$ and $4 \nmid n$, a contradiction.

Now, suppose that $n \leq 2^{k-1} + 2^{k-d+1} - 4$ and $n \equiv 0 \pmod{4}$.

It is easy to see that all consecutive vertices between $\min I_3 = n - 2^{k-1} + 4 \leq 2^{k-d+1}$ and $\max I_2 = 2^{k-1} - 1$ belong to I , i.e. $\{2^{k-d+1}, 2^{k-d+1} + 1, \dots, 2^{k-1} - 1\} \subseteq I$. Let $0 < 4r+j < 2^{k-d+1}$, for $j = 0$ or 1 , be a vertex out of I . Its $(k-d+1)$ dimensional neighbor is $2^{k-d+1} - 4r - j - 1 = 4(2^{k-d-1} - r) - j - 1 \in I_2$. Thus, all vertices less than 2^{k-d+1} will be informed by I_2 . Also, $(k-d+1)$ dimensional neighbor of vertex $n - 4r + j \geq 2^{k-1}$, for $j = 2$ or 3 , is $2^{k-d+1} - 1 + 4r - j = n - 4(\frac{n}{4} - 2^{k-d-1} - r + 1) - j + 3 \in I_3$. Thus, all vertices bigger than 2^{k-1} will be informed in the last time unit. Therefore, the broadcast scheme is valid.

Now, consider $2 \leq i \leq d-1$.

Then the broadcast scheme is $(i, i-1, \dots, 1, k-1, \dots, i+1, k-d+i)$. It is easy to check by induction that Informed vertices after time unit i , like the case $i=0$, are

$$A = \{0\} \cup \{2p-1 | 1 \leq p \leq 2^{i-1}\} \cup \{n-2p | 1 \leq p \leq 2^{i-1}-1\}.$$

By induction, one can see that all informed vertices after time unit t , $i+1 \leq t \leq k-1$, is

$$\begin{aligned} & A \cup \{p2^{k-t+i} + j \pmod{n} | 1 \leq p \leq 2^{t-i-1}, j \in -A-1\} \cup \\ & \{n - s2^{k-t+i} + j \pmod{n} | 1 \leq s \leq 2^{t-i-1}-1, j \in A\}. \end{aligned}$$

Therefore, by considering $t = k-1$, informed vertices after time unit $(k-1)$ are

$$\begin{aligned} I &= A \cup \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\} \cup \\ & \{n - s2^{i+1} + j \pmod{n} | 1 \leq s \leq 2^{k-i-2}-1, j \in A\} = \\ & \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\} \cup \\ & \{n - s2^{i+1} + j \pmod{n} | 0 \leq s \leq 2^{k-i-2}-1, j \in A\} = I_2 \cup I_3, \end{aligned}$$

where

$$\begin{aligned} A &= \{2p-1 | 1 \leq p \leq 2^{i-1}\} \cup \{n-2p \pmod{n} | 0 \leq p \leq 2^{i-1}-1\}, \\ I_2 &= \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\}, \text{ and} \\ I_3 &= \{n - s2^{i+1} + j \pmod{n} | 0 \leq s \leq 2^{k-i-2}-1, j \in A\}. \end{aligned}$$

We prove that the dimensional broadcast scheme is valid if and only if $n < 2^{k-1} + 2^{k+i-d-1}$ and $n \equiv 2^i - 2$.

First, assume $2^{k-1} + p'2^{i+1} \leq n \leq 2^{k-1} + p'2^{i+1} + 2^i - 4$, for $0 \leq p' \leq 2^{k-d-2} - 1$.

In this case, we claim that the $(k-d+i)$ dimensional neighbors 2 and $2^{k-d+i} - 3$ are not in I , and so remain uninformed after the last time unit.

If $2 \in I_2$, then $2 \equiv p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A-1$. Since 2 is even, $j \equiv -2p''$, where $1 \leq p'' \leq 2^{i-1}$. Then $p2^{i+1} + j - 2 \equiv 0$. But $0 < 2^i - 2 \leq p2^{i+1} + j - 2 \leq 2^{k-1} - 4 < n$, a contradiction.

If $2 \in I_3$, then $2 \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since 2 is even, $j \stackrel{n}{\equiv} n - 2p''$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $s2^{i+1} - j + 2 \stackrel{n}{\equiv} 0$. But $0 < 2 \leq s2^{i+1} - j + 2 \leq 2^{k-1} - 2^i < n$, a contradiction.

If $2^{k-d+i} - 3 \in I_2$, then $2^{k-d+i} - 3 \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since $2^{k-d+i} - 3$ is odd, $j \stackrel{n}{\equiv} -n + 2p'' - 1$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $p2^{i+1} - 2^{k-d+i} + 2p'' + 2 \stackrel{n}{\equiv} 0$. But

$$-n < 2^{i+1} - 2^{k-1} + 2 \leq p2^{i+1} - 2^{k-d+i} + 2p'' + 2 \leq 2^{k-1} - 2^{i+2} + 2^i < n.$$

Hence, $p2^{i+1} - 2^{k-d+i} + 2p'' + 2 = 0$, and so $2^{k-d+i} = p2^{i+1} + 2p'' + 2$. Since $k-d+i \geq i+1$, $2^{i+1} \mid 2^{k-d+i}$, and so $2^{i+1} \mid 2p'' + 2 \leq 2^i$, a contradiction.

If $2^{k-d+i} - 3 \in I_3$, then $2^{k-d+i} - 3 \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since $2^{k-d+i} - 3$ is odd, $j = 2p'' - 1$, where $1 \leq p'' \leq 2^{i-1}$. Then $s2^{i+1} + 2^{k-d+i} - 2p'' - 2 \stackrel{n}{\equiv} 0$.

But

$$0 < 2^i - 2 \leq s2^{i+1} + 2^{k-d+i} - 2p'' - 2 \leq 2^k - 2^{i+1} - 4 < 2n.$$

Hence, $s2^{i+1} + 2^{k-d+i} - 2p'' - 2 = n$, and so $2^{k-d+i} + s2^{i+1} = n + 2p'' + 2$. Since $k-d+i \geq i+1$, $2^{i+1} \mid n + 2p'' + 2$. But

$$2^{k-1} + p'2^{i+1} + 4 \leq n + 2p'' + 2 \leq 2^{k-1} + p'2^{i+1} + 2^{i+1} - 2.$$

Since $2^{k-1} + p'2^{i+1}$ and $2^{k-1} + p'2^{i+1} + 2^{i+1}$ are two consecutive multiples of 2^{i+1} , $2^{i+1} \nmid n + 2p'' + 2$, a contradiction.

Therefore, $2, 2^{k-d+i} - 3 \notin I$.

Second, assume $2^{k-1} + p'2^{i+1} + 2^i \leq n \leq 2^{k-1} + (p' + 1)2^{i+1} - 2$, for $0 \leq p' \leq 2^{k-d-2} - 1$.

In this case, we claim that two $(k-d+i)$ dimensional neighbors 2^{k-d+i} and $n-1$ are not in I , and so remain uninformed after the last time unit.

If $2^{k-d+i} \in I_2$, then $2^{k-d+i} \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since 2^{k-d+i} is even, $j \stackrel{n}{\equiv} -2p''$, where $1 \leq p'' \leq 2^{i-1}$. Then $p2^{i+1} - 2^{k-d+i} + j \stackrel{n}{\equiv} 0$. But

$$-n < 2^i - 2^{k-1} - 2 \leq p2^{i+1} - 2^{k-d+i} - 2p'' \leq 2^{k-1} - 2^{k-d+i} - 2 < n.$$

Hence, $p2^{i+1} - 2^{k-d+i} - 2p'' = 0$, and so $p2^{i+1} = 2^{k-d+i} + 2p''$. Thus,

$$2^{k-d-1} + \frac{2}{2^{i+1}} \leq p \leq 2^{k-d-1} + \frac{2}{2},$$

a contradiction.

If $2^{k-d+i} \in I_3$, then $2^{k-d+i} \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since 2^{k-d+i} is even, $j \stackrel{n}{\equiv} n - 2p''$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $s2^{i+1} + 2^{k-d+i} + 2p'' \stackrel{n}{\equiv} 0$.

But

$$0 < 2^{k-d+i} \leq s2^{i+1} + 2^{k-d+i} + 2p'' \leq 2^k - 2^i - 2 < 2n.$$

Hence, $s2^{i+1} + 2^{k-d+i} + 2p'' = n$. Since $k - d + i \geq i + 1$, we have $2^{i+1} \mid s2^{i+1} + 2^{k-d+i}$, and so $2^{i+1} \mid n - 2p''$. But

$$2^{k-1} + p'2^{i+1} + 2 \leq n - 2p'' \leq 2^{k-1} + (p' + 1)2^{i+1} - 2.$$

Since $2^{k-1} + p'2^{i+1}$ and $2^{k-1} + (p' + 1)2^{i+1}$ are two consecutive multiples of 2^{i+1} , $2^{i+1} \nmid n - 2p''$, a contradiction.

If $n - 1 \in I_2$, then $n - 1 \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since $n - 1$ is odd, $j \stackrel{n}{\equiv} -n + 2p'' - 1$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $p2^{i+1} + 2p'' \stackrel{n}{\equiv} 0$. But

$$0 < 2^{i+1} \leq p2^{i+1} + 2p'' \leq 2^{k-1} + 2^i - 2 < n,$$

a contradiction.

If $n - 1 \in I_3$, then $n - 1 \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since $n - 1$ is odd, $j = 2p'' - 1$, where $1 \leq p'' \leq 2^{i-1}$. Then $s2^{i+1} - 2p'' \stackrel{n}{\equiv} 0$. But

$$-n < -2^i \leq s2^{i+1} - 2p'' \leq 2^{k-1} - 2^{i+1} - 2 < n.$$

Hence, $s2^{i+1} - 2p'' = 0$, and so $s2^{i+1} = 2p''$. Thus, $2^{i+1} \mid 2p'' \leq 2^i$, a contradiction.

Therefore, $2^{k-d+i}, n - 1 \notin I$.

Third, suppose that $n \geq 2^{k-1} + 2^{k-d+i-1}$.

In this case, we claim that two $(k-d+i)$ dimensional neighbors $2^{k-d+i-1}+2$ and $2^{k-d+i-1}-3$ are not in I , and so remain uninformed after the last time unit.

If $2^{k-d+i-1}+2 \in I_2$, then $2^{k-d+i-1}+2 \stackrel{n}{\equiv} p2^{i+1}+j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A-1$. Since $2^{k-d+i-1}+2$ is even, $j \stackrel{n}{\equiv} -2p''$, where $1 \leq p'' \leq 2^{i-1}$. Then $p2^{i+1}-2^{k-d+i-1}-2p''-2 \stackrel{n}{\equiv} 0$. But

$$-n < 2^i - 2^{k-1} - 2 \leq p2^{i+1} - 2^{k-d+i-1} - 2p'' - 2 \leq 2^{k-1} - 2^{k-d+i-1} - 4 < n.$$

Hence, $p2^{i+1} - 2^{k-d+i-1} - 2p'' - 2 = 0$, and so $p2^{i+1} = 2^{k-d+i-1} + 2p'' + 2$. Thus,

$$2^{k-d-2} + \frac{4}{2^{i+1}} \leq p \leq 2^{k-d-2} + \frac{1}{2} + \frac{2}{2^{i+1}},$$

a contradiction.

If $2^{k-d+i-1}+2 \in I_3$, then $2^{k-d+i-1}+2 \stackrel{n}{\equiv} n-s2^{i+1}+j$ for $0 \leq s \leq 2^{k-i-2}-1$ and $j \in A$. Since $2^{k-d+i-1}+2$ is even, $j \stackrel{n}{\equiv} n-2p''$, where $0 \leq p'' \leq 2^{i-1}-1$. Then $s2^{i+1}+2^{k-d+i-1}+2p''+2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-d+i-1}+2 \leq s2^{i+1}+2^{k-d+i-1}+2p''+2 \leq 2^{k-1}-2^i+2^{k-d+i-1} < n,$$

a contradiction.

If $2^{k-d+i-1}-3 \in I_2$, then $2^{k-d+i-1}-3 \stackrel{n}{\equiv} p2^{i+1}+j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A-1$. Since $2^{k-d+i-1}-3$ is odd, $j \stackrel{n}{\equiv} -n+2p''-1$, where $0 \leq p'' \leq 2^{i-1}-1$. Then $p2^{i+1}-2^{k-d+i-1}+2p''+2 \stackrel{n}{\equiv} 0$. But

$$-n < 2^{i+1} - 2^{k-d+i-1} + 2 \leq p2^{i+1} - 2^{k-d+i-1} + 2p'' + 2 \leq 2^{k-1} - 2^{k-d+i-1} + 2^i < n,$$

Hence, $p2^{i+1} - 2^{k-d+i-1} + 2p'' + 2 = 0$ and so $p2^{i+1} = 2^{k-d+i-1} - 2p'' - 2$. Thus,

$$2^{k-d-2} - \frac{1}{2} \leq p \leq 2^{k-d-2} - \frac{2}{2^{i+1}},$$

a contradiction.

If $2^{k-d+i-1}-3 \in I_3$, then $2^{k-d+i-1}-3 \stackrel{n}{\equiv} n-s2^{i+1}+j$ for $0 \leq s \leq 2^{k-i-2}-1$

and $j \in A$. Since $2^{k-d+i-1} - 3$ is odd, $j = 2p'' - 1$, where $1 \leq p'' \leq 2^{i-1}$. Then $s2^{i+1} + 2^{k-d+i-1} - 2p'' - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-d+i-1} - 2^i - 2 \leq s2^{i+1} + 2^{k-d+i-1} - 2p'' - 2 \leq 2^{k-1} + 2^{k-d+i-1} - 2^{i+1} - 4 < n,$$

a contradiction.

Therefore, $2^{k-d+i-1} + 2, 2^{k-d+i-1} - 3 \notin I$.

Now, suppose that $n < 2^{k-1} + 2^{k-d+i-1}$ and $n \stackrel{2^{i+1}}{\equiv} 2^i - 2$.

In this case, $n \leq 2^{k-1} + 2^{k-d+i-1} - 2^i - 2$, $2^{i+1} | n - 2^i + 2$, and so $2^{i+1} | n + 2^i + 2$.

For a fixed integer p , $1 \leq p \leq 2^{k-i-2}$, let $I_{2(p)} = \{p2^{i+1} + j | j \in -A - 1\} \subseteq I_2$. One can see that

$$I_{2(p)} = \{p2^{i+1} - 2p'' | 1 \leq p'' \leq 2^{i-1}\} \cup \{p2^{i+1} + 2p'' - 1 \pmod{n} | 0 \leq p'' \leq 2^{i-1} - 1\}.$$

Note that between every two consecutive elements of $I_{2(p)}$, other than $p2^{i+1} - 1$ and $p2^{i+1} - 2$, there is exactly one vertex not belonging in I_2 . Also, since $\min I_{2(p+1)} - \max I_{2(p)} = 3$, there are exactly two vertices between $I_{2(p)}$ and $I_{2(p+1)}$ not belonging in I_2 . Therefore, the set $\{p2^{i+1} - 2p'' - 1 | 1 \leq p'' \leq 2^{i-1}\} \cup \{p2^{i+1} + 2p'' | 0 \leq p'' \leq 2^{i-1} - 1\}$, for $1 \leq p \leq 2^{k-i-2}$, contains all vertices out of I_2 from $\min I_2 - 1 = 2^i - 1$ to $\max I_2 + 1 = 2^{k-1} + 2^i - 2$. We show that which one of these vertices are in I_3 . It is easy to check that if $j = 2^i - 2p'' + 1$ and $s = \frac{n + 2^i + 2}{2^{i+1}} - p$ for $2^{k-d-2} + 1 \leq p \leq 2^{k-i-2}$ and $1 \leq p'' \leq 2^{i-1}$, then $p2^{i+1} - 2p'' - 1 = n - s2^{i+1} + j \in I_3$. Note that in this case, $1 \leq s \leq 2^{k-i-2}$ and $j = 2(2^{i-1} - p'' + 1) - 1 \in A$. Also, if $j = n + 2p'' + 2 - 2^i$ and $s = \frac{n - 2^i + 2}{2^{i+1}} - p$ for $2^{k-d-2} \leq p \leq 2^{k-i-2}$ and $0 \leq p'' \leq 2^{i-1} - 1$, then $p2^{i+1} + 2p'' \stackrel{n}{\equiv} n - s2^{i+1} + j \in I_3$. Note that in this case, $0 \leq s \leq 2^{k-i-2} - 1$ and $j = n - 2(2^{i-1} - p'' - 1) \in A$. Therefore, $B = \{2^{k-d+i-1} - 2, 2^{k-d+i-1} - 1, \dots, 2^{k-1} + 2^i - 1\} \subseteq I$ contains consecutive vertices in I . Now, for every vertex $v < 2^{k-d+i-1} - 2$, its $(k - d + i)$ dimensional neighbor is $2^{k-d+i} - 1 - v > 2^{k-d+i-1} + 1 \in B$. Thus, all vertices between 0 to $2^{k-d+i-1} - 2$ will be informed in the last time unit by vertices of B . Also, for every vertex $v > 2^{k-1} + 2^i - 1$, its

$(k - d + i)$ dimensional neighbor is $n + 2^{k-d+i} - 1 - v < 2^{k-d+i} + 2^{k-d+i-1} - 2^{i+1}$. But the last is less than 2^{k-1} , and so belongs to B if $d \geq i + 2$. Therefore, if $d \geq i + 2$, then the broadcast scheme is valid.

Now, suppose that $d = i + 1$. Hence the broadcast scheme is $(i, i - 1, \dots, 1, k - 1, \dots, i + 1, k - 1)$ and $n \leq 2^{k-1} + 2^{k-2} - 2^i - 2$.

For a fixed integer s , $0 \leq s \leq 2^{k-i-2} - 1$, let $I_{3(s)} = \{n - s2^{i+1} + j | j \in A\} \subseteq I_3$. One can see that

$$I_{3(s)} = \{n - s2^{i+1} + 2p'' - 1 | 1 \leq p'' \leq 2^{i-1}\} \cup \{n - s2^{i+1} - 2p'' \pmod{n} | 0 \leq p'' \leq 2^{i-1} - 1\}.$$

Note that between every two consecutive elements of $I_{3(s)}$, other than $n - s2^{i+1}$ and $n - s2^{i+1} + 1$, there is exactly one vertex not belonging in I_3 . Also, since $\min I_{3(s)} - \max I_{3(s+1)} = 3$, there are exactly two vertices between $I_{3(s)}$ and $I_{3(s+1)}$ not belonging in I_3 . Hence, for some $0 \leq s \leq \frac{n - 2^{k-1} - 2^i + 2}{2^{i+1}}$, all vertices between $2^{k-1} - 1$ and 0 are in the following forms $n - s2^{i+1} + 2p''$ for some $1 \leq p'' \leq 2^{i-1}$, or $n - s2^{i+1} - 2p'' - 1$ for some $0 \leq p'' \leq 2^{i-1} - 1$. Now, it is easy to check that if $j = 2^i - 2p'' + 1$ and $s' = \frac{n - 2^{k-1} - 2^i + 2}{2^{i+1}} - s + 1$ for $0 \leq s \leq \frac{n - 2^{k-1} - 2^i + 2}{2^{i+1}}$ and $1 \leq p'' \leq 2^{i-1}$, then the $(k-1)$ dimensional neighbor of $n - s2^{i+1} + 2p''$ is $2^{k-1} - 1 + s2^{i+1} - 2p'' = n - s'2^{i+1} + j \in I_3$.

Note that in this case, $1 \leq s' \leq 2^{k-i-3}$ and $j = 2(2^{i-1} - p'' + 1) - 1 \in A$. Also, if $j = n + 2p'' + 2 - 2^i$ and $s' = \frac{n - 2^{k-1} - 2^i + 2}{2^{i+1}} - s$ for $0 \leq s \leq \frac{n - 2^{k-1} - 2^i + 2}{2^{i+1}}$ and $0 \leq p'' \leq 2^{i-1} - 1$, then the $(k-1)$ dimensional neighbor of $n - s2^{i+1} - 2p'' - 1$ is $2^{k-1} + s2^{i+1} + 2p'' = n - s'2^{i+1} + j \in I_3$. Note that in this case, $0 \leq s' \leq 2^{k-i-3} - 1$ and $j = n - 2(2^{i-1} - p'' - 1) \in A$. Therefore, all vertices will be informed in the last time unit. So the broadcast scheme is valid.

Now, consider $3 \leq d \leq i \leq k - 2$.

Then the broadcast scheme is $(i, i - 1, \dots, 1, k - 1, \dots, i + 1, i - d + 1)$. In this case,

similarly, the set of informed vertices after time unit $(k-1)$ are

$$\begin{aligned} I &= A \cup \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\} \cup \\ &\quad \{n - s2^{i+1} + j \pmod{n} | 1 \leq s \leq 2^{k-i-2} - 1, j \in A\} = \\ &\quad \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\} \cup \\ &\quad \{n - s2^{i+1} + j \pmod{n} | 0 \leq s \leq 2^{k-i-2} - 1, j \in A\} = I_2 \cup I_3, \end{aligned}$$

where

$$\begin{aligned} A &= \{2p-1 | 1 \leq p \leq 2^{i-1}\} \cup \{n-2p \pmod{n} | 0 \leq p \leq 2^{i-1}-1\}, \\ I_2 &= \{p2^{i+1} + j \pmod{n} | 1 \leq p \leq 2^{k-i-2}, j \in -A-1\}, \text{ and} \\ I_3 &= \{n - s2^{i+1} + j \pmod{n} | 0 \leq s \leq 2^{k-i-2} - 1, j \in A\}. \end{aligned}$$

We prove that the dimensional broadcast scheme is valid if and only if $2^{k-1} + 2^i - 2^{i-d+1} - 2 \leq n \leq 2^{k-1} + 2^i - 2$.

First, suppose that $n \leq 2^{k-1} + 2^i - 2^{i-d+1} - 4$.

In this case, we claim that $2^{k-1} - 3$ and its $(i-d+1)$ dimensional neighbor $n + 2^{i-d+1} - 2^{k-1} + 2$ are not in I , and so they remain uninformed after the last time unit.

If $n + 2^{i-d+1} - 2^{k-1} + 2 \in I_2$, then $n + 2^{i-d+1} - 2^{k-1} + 2 \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A-1$. Since $n + 2^{i-d+1} - 2^{k-1} + 2$ is even, $j \stackrel{n}{\equiv} -2p''$, where $1 \leq p'' \leq 2^{i-1}$. Then $p2^{i+1} - 2^{i-d+1} + 2^{k-1} - 2 - 2p'' \stackrel{n}{\equiv} 0$. But

$$n < 2^i + 2^{k-1} - 2^{i-d+1} - 2 \leq p2^{i+1} - 2^{i-d+1} + 2^{k-1} - 2 - 2p'' \leq 2^k - 2^{i-d+1} - 4 < 2n,$$

a contradiction.

If $n + 2^{i-d+1} - 2^{k-1} + 2 \in I_3$, then $n + 2^{i-d+1} - 2^{k-1} + 2 \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since $n + 2^{i-d+1} - 2^{k-1} + 2$ is even, $j \stackrel{n}{\equiv} n - 2p''$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $s2^{i+1} + 2^{i-d+1} - 2^{k-1} + 2p'' + 2 \stackrel{n}{\equiv} 0$. But

$$-n < 2^{i-d+1} - 2^{k-1} + 2 \leq s2^{i+1} + 2^{i-d+1} - 2^{k-1} + 2p'' + 2 \leq 2^{i-d+1} - 2^i < 0,$$

a contradiction.

If $2^{k-1} - 3 \in I_2$, then $2^{k-1} - 3 \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since $2^{k-1} - 3$ is odd, $j \stackrel{n}{\equiv} -n + 2p'' - 1$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $p2^{i+1} - 2^{k-1} + 2p'' + 2 \stackrel{n}{\equiv} 0$.

But

$$-n < 2^{i+1} - 2^{k-1} + 2 \leq p2^{i+1} - 2^{k-1} + 2p'' + 2 \leq 2^i < n.$$

Hence, $p2^{i+1} - 2^{k-1} + 2p'' + 2 = 0$ and so $p2^{i+1} = 2^{k-1} - 2p'' - 2$. Thus,

$$2^{k-i-2} - \frac{1}{2} \leq p \leq 2^{k-i-2} - \frac{2}{2^{i+1}},$$

a contradiction.

If $2^{k-1} - 3 \in I_3$, then $2^{k-1} - 3 \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since $2^{k-1} - 3$ is odd, $j = 2p'' - 1$, where $1 \leq p'' \leq 2^{i-1}$. Then $s2^{i+1} + 2^{k-1} - 2p'' - 2 \stackrel{n}{\equiv} 0$. But

$$0 < 2^{k-1} - 2^i - 2 \leq s2^{i+1} + 2^{k-1} - 2p'' - 2 \leq 2^k - 2^{i+1} - 4 < 2n.$$

Hence, $s2^{i+1} + 2^{k-1} - 2p'' - 2 = n$ and so $s2^{i+1} = n - 2^{k-1} + 2p'' + 2$. Thus,

$$\frac{6}{2^{i+1}} \leq s \leq 1 - \frac{4}{2^{i+1}},$$

a contradiction.

Therefore, $n + 2^{i-d+1} - 2^{k-1} + 2, 2^{k-1} - 3 \notin I$.

Second, suppose that $n \geq 2^{k-1} + 2^i$.

In this case, we claim that two $(i - d + 1)$ dimensional neighbors 2^{i-d+1} and $n - 1$ are not in I , and so they remain uninformed after the last time unit.

If $2^{i-d+1} \in I_2$, then $2^{i-d+1} \stackrel{n}{\equiv} p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since 2^{i-d+1} is even, $j \stackrel{n}{\equiv} -2p''$, where $1 \leq p'' \leq 2^{i-1}$. Then $p2^{i+1} - 2^{i-d+1} - 2p'' \stackrel{n}{\equiv} 0$. But

$$0 < 2^i - 2^{i-d+1} \leq p2^{i+1} - 2^{i-d+1} - 2p'' \leq 2^{k-1} - 2^{i-d+1} - 2 < n,$$

a contradiction.

If $2^{i-d+1} \in I_3$, then $2^{i-d+1} \stackrel{n}{\equiv} n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since

2^{i-d+1} is even, $j \equiv n - 2p''$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $s2^{i+1} + 2^{i-d+1} + 2p'' \equiv 0$.

But

$$0 < 2^{i-d+1} \leq s2^{i+1} + 2^{i-d+1} + 2p'' \leq 2^{k-1} - 2^i + 2^{i-d+1} - 2 < n,$$

a contradiction.

If $n - 1 \in I_2$, then $n - 1 \equiv p2^{i+1} + j$ for $1 \leq p \leq 2^{k-i-2}$ and $j \in -A - 1$. Since $n - 1$ is odd, $j \equiv -n + 2p'' - 1$, where $0 \leq p'' \leq 2^{i-1} - 1$. Then $p2^{i+1} + 2p'' \equiv 0$. But

$$0 < 2^{i+1} \leq p2^{i+1} + 2p'' \leq 2^{k-1} + 2^i - 2 < n,$$

a contradiction.

If $n - 1 \in I_3$, then $n - 1 \equiv n - s2^{i+1} + j$ for $0 \leq s \leq 2^{k-i-2} - 1$ and $j \in A$. Since $n - 1$ is odd, $j = 2p'' - 1$, where $1 \leq p'' \leq 2^{i-1}$. Then $s2^{i+1} - 2p'' \equiv 0$. But

$$-n < -2^i \leq s2^{i+1} - 2p'' \leq 2^{k-1} - 2^{i+1} - 2 < n.$$

Hence, $s2^{i+1} - 2p'' = 0$ and so $s2^{i+1} = 2p''$. Thus,

$$\frac{1}{2^i} \leq s \leq \frac{1}{2},$$

a contradiction.

Therefore, $2^{i-d+1}, n - 1 \notin I$.

Now, suppose that $2^{k-1} + 2^i - 2^{i-d+1} - 2 \leq n \leq 2^{k-1} + 2^i - 2$.

According to above, the set $\{p2^{i+1} - 2p'' - 1 | 1 \leq p'' \leq 2^{i-1}\} \cup \{p2^{i+1} + 2p'' | 0 \leq p'' \leq 2^{i-1} - 1\}$, for $1 \leq p \leq 2^{k-i-2}$, contains all vertices out of I_2 from $\min I_2 - 1 = 2^i - 1$ to $\max I_2 + 1 = 2^{k-1} + 2^i - 2$. We show which one of these vertices is in I_3 . It is easy to check that if $j = 2^i - 2p'' + 1$ and $s = 2^{k-i-2} - p + 1$ for $2 \leq p \leq 2^{k-i-2}$ and $1 \leq p'' \leq 2^{i-1}$, then $p2^{i+1} - 2p'' - 1 = n - s2^{i+1} + j \in I_3$. Note that in this case, $1 \leq s \leq 2^{k-i-2} - 1$ and $j = 2(2^{i-1} - p'' + 1) - 1 \in A$. Also, if $j = n + 2p'' + 2 - 2^i$ and $s = 2^{k-i-2} - p$ for $1 \leq p \leq 2^{k-i-2}$ and $0 \leq p'' \leq 2^{i-1} - 1$, then $p2^{i+1} + 2p'' \equiv n - s2^{i+1} + j \in I_3$. Note that in this case, $0 \leq s \leq 2^{k-i-2} - 1$ and $j = n - 2(2^{i-1} - p'' - 1) \in A$. Therefore,

$B = \{2^{i+1} - 2, 2^{i+1} - 1, \dots, 2^{k-1} + 2^i - 1\} \subseteq I$ contains consecutive vertices in I . Note that $2^{k-1} + 2^i - 1 \geq n$. Now, since A contains all odd vertices between 0 and 2^i , all even vertices between 0 and $2^{i-d+1} - 1$ will be informed by A in the last time unit. Also, $n - 1 \in B$ informs 2^{i-d+1} in the last time unit, and the $(i - d + 1)$ dimensional neighbor of $2^{i+3} - 3$ is $n + 2^{i-d+1} - 2^{i+1} + 2$. But the last is bigger than $2^{i+1} - 2$, and so it belongs to B if $i \leq k - 3$. Therefore, if $i \leq k - 3$, then the broadcast scheme is valid.

Now, suppose that $i = k - 2$. Hence, the broadcast scheme is $(k - 2, k - 3, \dots, 1, k - 1, k - d - 1)$,

$$2^{k-1} + 2^{k-2} - 2^{k-d-1} - 2 \leq n \leq 2^{k-1} + 2^{k-2} - 2,$$

$$I_2 = \{2^{k-1} - 2p'' | 1 \leq p'' \leq 2^{k-3}\} \cup \{2^{k-1} + 2p'' - 1 \pmod{n} | 0 \leq p'' \leq 2^{k-3} - 1\},$$

$$I_3 = A = \{2p'' - 1 | 1 \leq p'' \leq 2^{k-3}\} \cup \{n - 2p'' \pmod{n} | 0 \leq p'' \leq 2^{k-3} - 1\},$$

$$\text{and } B = \{2^{k-1} - 2, \dots, 2^{k-1} + 2^{k-2} - 1\} \subseteq I.$$

Since A contains all odd vertices between 0 and 2^{k-2} , all even vertices between 0 and $2^{k-d-1} - 1$ will be informed by vertices of A in the last time unit. Also, B informs all vertices from 2^{k-d-1} to $\beta = n - 2^{k-1} + 2^{k-d-1} + 1$. Note that $\min I_2 = 2^{k-1} - 2^{k-2} \leq 2^{k-2} \leq \beta + 1$. Thus, I_2 contains all even vertices between β and $2^{k-1} - 2$. Therefore, all odd vertices in this range will be informed by I_2 in the last time unit. Therefore, the broadcast scheme is valid. $\square$

Given the symmetry of Knödel graphs, it appears that various dimensional broadcast schemes exhibit similar properties, particularly the standard and reverse ones. However, significant differences exist. Through Theorems 3.4.1 and 3.5.1, these distinctions are further emphasized. From Table 3.1, one can see these differences. For example, when $k - d + 1 \leq c \leq k - 2$ there is no n admits c cyclic shifts of the standard dimensional broadcast scheme with repeating d -th dimension while there are at least $2^{k-d-2} - 1$ values of n that permit this in the reverse scheme. This fact is further evident when examining

other rows of the table.

	Standard	reverse
$c = 0$	$2^{k-2} - 2^{d-1}$	$2^{k-d-1} - 1$
$c = k - 2$	0	$2^{k-d-1} - 1$
$k - d + 1 \leq c \leq k - 3$	0	$2^{k-d-2} - 1$
$c = k - d$	$2^{k-d} - 2$	$2^{k-d-2} - 1$
$c = k - d - 1$	$2^{k-d-2} - 1$	2
$c \leq k - d - 2$	2^{c+d-3}	$2^{k-c-d-1} + 1$

Table 3.1: For a fix d , the second (third) column is the number of values of n such that c cyclic shifts of the standard (reverse) dimensional broadcast scheme with repeating d -th dimension in the last time unit gives a valid dimensional broadcast scheme for $KG(n)$.

In the following theorem, we analyze the dimensional broadcast scheme $(1, 2, \dots, i - 1, d, i, \dots, k - 1)$, where the d -th dimension is repeated in the i -th position, with $d < i$. We demonstrate that the choice $d = 1$ maximizes the number of values of n for which the dimensional broadcast scheme is valid. Furthermore, the best position for repeating the d th dimension is in the last two positions.

Theorem 3.5.2. *Let $KG(n)$ be a Knödel graph on n vertices and $k = \lceil \log n \rceil$. Then for two integers $3 \leq i \leq k - 1$ and $1 \leq d \leq i - 2$, the dimensional broadcast scheme $(1, 2, \dots, i - 1, d, i, \dots, k - 1)$ is a valid broadcast scheme if and only if $n \leq 2^{k-1} + 2^i - 2^{d+1}$.*

if w is informed by some vertex in S at time t , then

Proof. Similar to the previous proofs, all vertices between 0 and $2^{i-1} - 1$ are informed after time unit $i - 1$. In time unit i , these vertices send the message to all vertices between $u = n - 2^{i-1} + 2^d$ and 0, denoting their set as S , as illustrated in Figure 3.6 (a).

Note that if a vertex w is informed by one of the vertices in S during time unit $t < k$, then it is adjacent to some vertex between 0 and $2^t - 1$ that alternatively can inform w at time $t + 1$. Hence, the role of S can be ignored until the last time unit.

In the last time unit, S informs all vertices between 2^{k-1} and $v = 2^{k-1} + 2^{i-1} - 2^d - 1$, denoted by S' . Thus, after time unit k , all vertices $S \cup S'$ together with vertices between 0 and $2^{k-1} - 1$ are informed.

Therefore, the dimensional broadcast scheme is valid if and only if $\max S' = v \geq u - 1 = \min S - 1$, or equivalently, $n \leq 2^{k-1} + 2^i - 2^{d+1}$.

□

Remark 3.5.1. *With the notation of Theorem 3.5.2, at first glance, it might seem that the two dimensional broadcast schemes $D_1 = (1, 2, \dots, i - 1, d, i, \dots, k - 1)$ and $D_2 = (1, 2, \dots, k - 1, k - i + d)$ are equivalent. However, this is not true. As shown in Figure 3.6 (a), applying D_1 results in the uninformed vertices between $2^{k-1} + 2^{i-1} - 2^d$ and $n - 2^{i-1} + 2^d - 1$ after time unit k . Therefore, D_1 is valid if and only if $n \leq 2^{k-1} + 2^i - 2^{d+1}$. Consider $p_1 = 2^{k-1} + 2^i - 2^{d+1}$.*

On the other hand, after applying D_2 , as shown in Figure 3.6 (b), we observe that the vertices S' between 2^{k-1} and $n + 2^{k-i+d} - 2^{k-1}$ remain uninformed after time unit k . Hence, D_2 is valid if and only if $n \leq 2^k - 2^{k-i+d}$. Consider $p_2 = 2^k - 2^{k-i+d}$.

When $i \leq k - 2$, we have $p_2 \geq 2^{k-1} - 2^{k-2} \geq 2^i$, and thus $p_2 > p_1$. This implies that the number of values of n for which D_2 is valid is greater than for D_1 . Therefore, D_1 and D_2 are not equivalent.

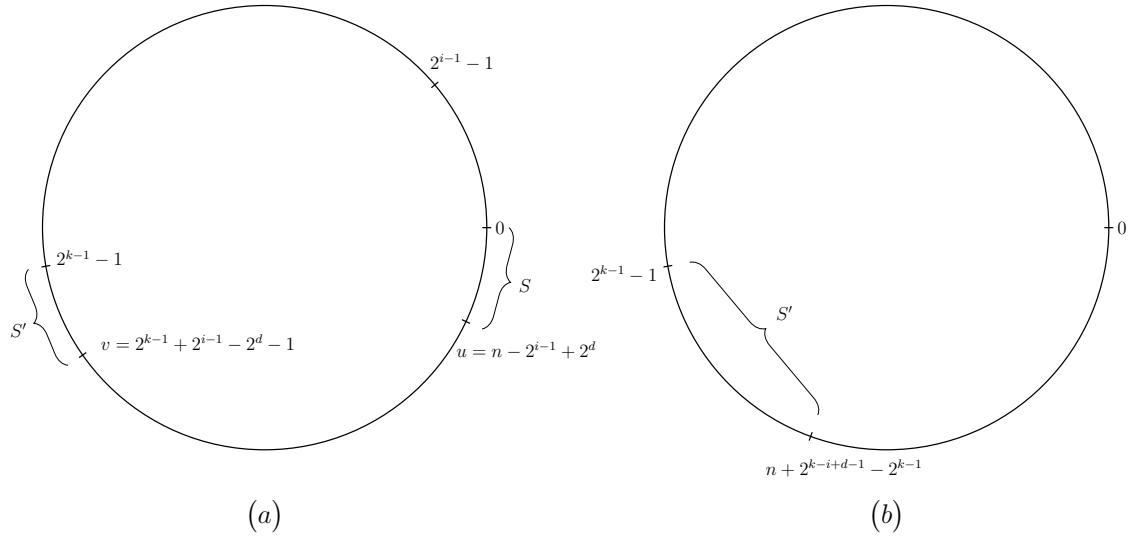


Figure 3.6: Two dimensional broadcast schemes D_1 and D_2 are illustrated in (a) and (b), respectively.

3.6 Conclusion

Determining all possible sequential broadcast schemes of the form $(j, j + 1, \dots, k - 1, 1, \dots, j - 1, d)$, including their shifts and reversals, for Knödel graphs has been a long-standing open problem. In this work, we provide a complete solution to this question. Furthermore, for sequential broadcast schemes that are not universally valid, we precisely characterize the range of values for n in which the given scheme remains valid. A significant challenge that remains is identifying all valid permutations of dimensions for broadcasting in Knödel graphs. While in hypercubes, any permutation of dimensions is valid, the situation for Knödel graphs is considerably more complex. Consequently, determining all valid broadcasting permutations in Knödel graphs presents a highly challenging open problem.

Chapter 4

A general upper bound on broadcast function $B(n)$ using Knödel graph

This chapter presents a new general upper bound on $B(n)$ for all odd n except when $n = 2^k - 1$ for an integer k . In order to achieve this bound, a vertex deletion method is applied to Knödel graphs on $n + 1$ vertices. To prove our result, we characterize all informed vertices in a Knödel graph after every time unit by applying the standard and reverse dimensional broadcast scheme with any cyclic shifts from an arbitrary originator. Furthermore, we completely resolve the open problem about the existence of minimum average time broadcast graphs. The results presented in this chapter have been submitted to Discrete Applied Mathematics.

Since identifying mbgs seems to be particularly challenging [101], a series of studies have introduced various techniques for constructing broadcast graphs and establishing upper bounds on broadcast function $B(n)$ for specific intervals of n (see [32, 43, 52, 65, 70, 72, 75]). Farley [32] has given the first general upper bound on $B(n)$ using a recursive construction.

$$B(n) \leq \frac{n \lceil \log n \rceil}{2}, \text{ for all } n.$$

An even better general upper bound on $B(n)$ is a direct result of the number of edges of Knödel graphs [72].

$$B(n) \leq \frac{n \lfloor \log n \rfloor}{2} \text{ for all even } n.$$

This result was slightly improved in [49] for odd values of n by using vertex addition method. In this chapter, we prove that for all $n \neq 2^k - 1$, $B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil$ by extending the upper bound derived from Knödel graphs. This upper bound for an odd n is obtained by applying a vertex deletion method to $KG(n+1)$ and demonstrating that the resulting graph remains a broadcast graph. The new upper bound represents an improvement of approximately $\frac{n}{2}$ over the best known general upper bound from [49]. Another advantage of our upper bound is that it is based on a direct (ad hoc) construction. This explicit and straightforward construction contrasts with Farley's upper bound [32], which is derived from a recursive construction of broadcast graphs and lacks an immediate combinatorial interpretation for arbitrary n .

Note that another general upper bound [52] shows that

$$B(n) \leq (\lceil \log n \rceil - \log(2^{\lceil \log n \rceil} - n) + 1) n. \quad (3)$$

While (3) is a good bound when $\log(2^{\lceil \log n \rceil} - n) = n - c$, where c is constant, but in general it is as loose as $B(n) \leq \lceil \log n \rceil n$, which is twice the bound obtained by Farley. Note that the construction used to derive inequality (3) includes some vertices with high degree $n - 1$, while the remaining vertices have relatively small degrees. As a result, the graph is far from being regular. In contrast, in our construction, all vertices except possibly one when $\lceil \log n \rceil$ is even have degree $\lfloor \log n \rfloor$, making it a regular graph.

It is worth mentioning that there are better bounds on $B(n)$ in [16, 43, 51, 52, 55] but only for a small interval or specific values of n , the best among them is $B(n) \leq \frac{n}{2}(m - \frac{k+1}{2})$ for $n = 2^m - 2^k$. [16, 52, 70]

4.1 Auxiliary results

Our method is to demonstrate that by removing vertex 0 from $KG(n)$ for $n \neq 2^k$, where $k = \lceil \log n \rceil$, and introducing new edges, we construct another broadcast graph on $n - 1$ vertices. To achieve this, based on the originator, we utilize a specific dimensional broadcast scheme in $KG(n)$, ensuring that vertex 0 receives the message at time unit $(k - 1)$ or k . If 0 receives at time unit $(k - 1)$, then 0 is idle during the last time unit k , or instead of vertex 0, one of its neighbors sends the message to the vertex that 0 would have sent in the last time unit k .

In the next lemmas, we introduce all informed vertices of $KG(n)$ after time unit $t \leq k - 1$ from an arbitrary originator by utilizing the standard and reverse dimensional broadcast scheme with i cyclic shifts, where $0 \leq i \leq k - 2$. Note that from now on, when integers outside the range $\{0, \dots, n-1\}$ are referred to as vertices, then their congruential equivalent $(\cdot \bmod n)$ within the range is meant.

Lemma 4.1.1. *Let $KG(n)$ be a Knödel graph of order $n < 2^k$, where $k = \lceil \log n \rceil$, and let u , an arbitrary vertex, be the originator. The set of informed vertices after time unit $t \leq k - 1$ by using the dimensional broadcast scheme $(1, \dots, k - 1, 1)$ is*

$$\{2p + u \mid 0 \leq p \leq 2^{t-1} - 1\} \cup \{2s - u - 1 \mid 1 \leq s \leq 2^{t-1}\}.$$

Proof. Let $A_t = \{2p + u \mid 0 \leq p \leq 2^{t-1} - 1\}$ and $B_t = \{2s - u - 1 \mid 1 \leq s \leq 2^{t-1}\}$ for $1 \leq t \leq k - 1$. In the first time unit, by utilizing dimension 1, u informs $1 - u$. During the second time unit, these two vertices inform $3 - u$ and $2 + u$, respectively. We prove by induction on t .

The base case $t = 1$ is easily verified.

Assume the claim holds for some $t < k - 1$. We show that it holds for $t + 1$.

Each vertex $v \in A_t$ has the form $v = 2p + u$ for $0 \leq p \leq 2^{t-1} - 1$. Using dimension $t + 1$, such a vertex informs:

$$2^{t+1} - v - 1 = 2^{t+1} - (2p + u) - 1 = 2(2^t - p) - u - 1 = 2s - u - 1,$$

where $s = 2^t - p$ ranges from $2^{t-1} + 1$ to 2^t . These vertices, together with the vertices of B_t , form B_{t+1} .

Similarly, each vertex in B_t has the form $v = 2s - u - 1$ for $1 \leq s \leq 2^{t-1}$. Using dimension $t + 1$, such a vertex informs:

$$2^{t+1} - 2s + u = 2(2^t - s) + u = 2p + u,$$

where $p = 2^t - s$ ranges from 2^{t-1} to $2^t - 1$. These vertices, together with the vertices of A_t , form A_{t+1} .

Thus, $I_{t+1} = A_{t+1} \cup B_{t+1}$, completing the induction. $\square$

In the following lemma, we analyze the standard dimensional broadcast scheme with right cyclic shifts.

Lemma 4.1.2. *Let $KG(n)$ be a Knödel graph of order $n < 2^k$, where $k = \lceil \log n \rceil$, and let u , an arbitrary vertex, be the originator. Let $1 \leq i \leq k - 2$ be an integer. Consider the dimensional broadcast scheme $(i + 1, \dots, k - 1, 1, \dots, i, i + 1)$. Then the following hold:*

1) *The set of informed vertices after time unit $t \leq k - i - 1$ is*

$$\{p2^{i+1} + u | 0 \leq p \leq 2^{t-1} - 1\} \cup \{s2^{i+1} - u - 1 | 1 \leq s \leq 2^{t-1}\}.$$

2) The set of informed vertices after time unit $t \geq k - i$ is the union of the following sets:

$$\begin{aligned} A_t &= \{p2^{i+1} + 2j + u \mid 0 \leq p \leq 2^{k-i-2} - 1, 0 \leq j \leq 2^{t+i-k} - 1\}, \\ B_t &= \{s2^{i+1} + 2j - u - 1 \mid 1 \leq s \leq 2^{k-i-2}, 0 \leq j \leq 2^{t+i-k} - 1\}, \\ A'_t &= \{-p2^{i+1} + 2j - u - 1 \mid 0 \leq p \leq 2^{k-i-2} - 1, 1 \leq j \leq 2^{t+i-k}\}, \\ B'_t &= \{-s2^{i+1} + 2j + u \mid 1 \leq s \leq 2^{k-i-2}, 1 \leq j \leq 2^{t+i-k}\}. \end{aligned}$$

Proof. In the first time unit, by utilizing dimension $i + 1$, u informs $2^{i+1} - u - 1$. These two vertices during the second time unit inform $2^{i+2} - u - 1$ and $2^{i+1} + u$, respectively. One can see by induction that after time unit $1 \leq t \leq k - i - 1$, the set of informed vertices I_t equals $A_t \cup B_t$, where

$$A_t = \{p2^{i+1} + u \mid 0 \leq p \leq 2^{t-1} - 1\} \text{ and } B_t = \{s2^{i+1} - u - 1 \mid 1 \leq s \leq 2^{t-1}\}.$$

This holds because A_t and B_t contribute to each other's growth by informing new vertices at each time unit. Therefore, Part 1) is complete.

By setting $t = k - i - 1$, the set of informed vertices after time unit $(k - i - 1)$ is

$$\begin{aligned} I_{k-i-1} &= A_{k-i-1} \cup B_{k-i-1} = \{p2^{i+1} + u \mid 0 \leq p \leq 2^{k-i-2} - 1\} \cup \\ &\quad \{s2^{i+1} - u - 1 \mid 1 \leq s \leq 2^{k-i-2}\}. \end{aligned}$$

In time unit $(k - i)$, the vertices of the form $p2^{i+1} + u \in A_{k-i-1} \pmod{n}$ and $s2^{i+1} - u - 1 \in B_{k-i-1}$ inform vertices of the form $-p2^{i+1} - u + 1$ and $-s2^{i+1} + u + 2$, respectively. Let A'_{k-i} and B'_{k-i} be two sets containing these newly informed vertices, respectively. By induction, one can show that for all $t \geq k - i$, the sets A_t and A'_t mutually contribute to each other's growth in each time unit $(t + 1)$; similarly, this holds for B_t and B'_t . Furthermore, by the same inductive reasoning, the correctness of Part 2) follows. $\square$

In the next two lemmas, we investigate similar results for the reverse dimensional broadcast schemes. We provide brief proofs of the next two lemmas, as their proofs are similar to the previous ones.

Note that next lemma can be proved by an induction technique analogous to the proof of Lemma 4.1.1.

Lemma 4.1.3. *Let $KG(n)$ be a Knödel graph of order $n < 2^k$, where $k = \lceil \log n \rceil$, and let u , an arbitrary vertex, be the originator. The set of informed vertices after time unit $t \leq k - 1$ by using the dimensional broadcast scheme $(k - 1, \dots, 1, k - 1)$ is*

$$\{-p2^{k-t} + u \mid 0 \leq p \leq 2^{t-1} - 1\} \cup \{s2^{k-t} - u - 1 \mid 1 \leq s \leq 2^{t-1}\}.$$

In the next lemma, we show the informed vertices after any time unit $1 \leq t \leq k - 1$ by using the reverse dimensional broadcast scheme $(k - 1, \dots, 1)$ with $1 \leq i \leq k - 2$ cyclic shifts.

Lemma 4.1.4. *Let $KG(n)$ be a Knödel graph of order $n < 2^k$, where $k = \lceil \log n \rceil$, and let u , an arbitrary vertex, be the originator. Let $1 \leq i \leq k - 2$ be an integer. Consider the dimensional broadcast scheme $(k - i - 1, \dots, 1, k - 1, \dots, k - i, k - i - 1)$. Then the following hold:*

1) *The set of informed vertices after time unit $t \leq k - i - 1$ is*

$$\{-p2^{k-i-t} + u \mid 0 \leq p \leq 2^{t-1} - 1\} \cup \{s2^{k-i-t} - u - 1 \mid 1 \leq s \leq 2^{t-1}\}.$$

2) *The set of informed vertices after time unit $t \geq k - i$ is the union of the following sets:*

$$A_t = \{-j2^{2k-t-i-1} - 2p + u \mid 0 \leq j \leq 2^{t+i-k} - 1, 0 \leq p \leq 2^{k-i-2} - 1\},$$

$$B_t = \{-j2^{2k-t-i-1} + 2s - u - 1 \mid 0 \leq j \leq 2^{t+i-k} - 1, 1 \leq s \leq 2^{k-i-2}\},$$

$$A'_t = \{j2^{2k-t-i-1} + 2p - u - 1 \mid 1 \leq j \leq 2^{t+i-k}, 0 \leq p \leq 2^{k-i-2} - 1\},$$

$$B'_t = \{j2^{2k-t-i-1} - 2s + u \mid 1 \leq j \leq 2^{t+i-k}, 1 \leq s \leq 2^{k-i-2}\}.$$

Proof. In the first time unit, by utilizing dimension $k - i - 1$, u informs $2^{k-i-1} - u - 1$. These two vertices during the second time unit inform $2^{k-i-2} - u - 1$ and $-2^{k-i-2} + u$, respectively. One can see by induction that after time unit $1 \leq t \leq k - i - 1$, the set of informed vertices I_t equals $A_t \cup B_t$, where

$$A_t = \{-p2^{k-i-t} + u | 0 \leq p \leq 2^{t-1} - 1\} \text{ and } B_t = \{s2^{k-i-t} - u - 1 | 1 \leq s \leq 2^{t-1}\}.$$

This holds because A_t and B_t contribute to each other's growth by informing new vertices at each time unit. Therefore, Part 1 is complete.

By setting $t = k - i - 1$, the set of informed vertices after time unit $(k - i - 1)$ is

$$I_{k-i-1} = A_{k-i-1} \cup B_{k-i-1} = \{-2p + u | 0 \leq p \leq 2^{k-i-2} - 1\} \cup \{2s - u - 1 | 1 \leq s \leq 2^{k-i-2}\}.$$

In time unit $(k - i)$, the vertices of the form $-2p + u \in A_{k-i-1}$ and $2s - u - 1 \in B_{k-i-1}$ inform vertices of the form $2^{k-1} + 2p - u - 1$ and $2^{k-1} - 2s + u$, respectively. Let A'_{k-i} and B'_{k-i} be two sets containing these newly informed vertices, respectively. By induction on t , one can show that for all $t \geq k - i$, the sets A_t and A'_t mutually contribute to each other's growth in each time unit $(t + 1)$; similarly, this holds for B_t and B'_t . Furthermore, by the same inductive reasoning, the correctness of Part 2 follows. $\square$

In the next proposition, we consider the case where the originator is $2^i - 1$, an i -dimensional neighbor of vertex 0, where $1 \leq i \leq k - 1$. We prove that by utilizing i cyclic shifts of the reverse dimensional broadcast scheme, vertex 0 remains uninformed after time unit $(k - 2)$.

Proposition 4.1.5. *Let $KG(n)$ be a Knödel graph of order n with $k = \lceil \log n \rceil$ and $0, u = 2^i - 1 \in V(KG(n))$ be two i dimensional neighbors for an integer $1 \leq i \leq k - 1$. If the originator and the dimensional broadcast scheme are u and $(i + 1, i + 2, \dots, k - 1, 1, \dots, i, i + 1)$, respectively, then vertex 0 is uninformed after time unit $k - 2$ and will be*

informed by u in time unit $k - 1$. (In case of $i = k - 1$, the dimensional broadcast scheme is $(1, \dots, k - 1, 1)$).

Proof. First, suppose that $i \neq k - 1$. In Lemma 4.1.2, let $u = 2^i - 1$ and $t = k - 2$. If $i = 1$, then by Part 1, the set of informed vertices I after time unit $(k - 2)$ is

$$\{4p + 1 | 0 \leq p \leq 2^{k-3} - 1\} \cup \{4s - 2 | 1 \leq s \leq 2^{k-3}\}.$$

If $i \geq 2$, then by Part 2 of Lemma 4.1.2, the set of informed vertices I after time unit $(k - 2)$ is the union of the following sets:

$$A = \{(2p + 1)2^i + j | -1 \leq j \leq 2^{i-1} - 2\}$$

$$B = \{-(2p + 1)2^i + j \pmod{n} | 1 \leq j \leq 2^{i-1}\},$$

where $0 \leq p \leq 2^{k-i-2} - 1$.

Now, we check if $0 \stackrel{n}{\equiv} n \in I$.

Obviously, $0 \notin I$ when $i = 1$.

Let $i \geq 2$. Note that $0 < \min A = 2^i - 1 < \max A = 2^{k-1} - 2^{i-1} - 2 < n$. Hence, $0 \notin A$.

Since $i \neq k - 1$, then $-n < \min B = -2^{k-1} + 2^i + 1 < \max B = -2^{i-1} < 0$. Thus, all members of $B \pmod{n}$ are between 0 and n and so $0 \notin B$. Therefore, $0 \notin I$ and the proof is complete.

Now, suppose that $i = k - 1$. In this case, $u = 2^{k-1} - 1$ and the dimensional broadcast scheme is $(1, \dots, k - 1, 1)$. By Lemma 4.1.1, the set of informed vertices I after time unit $(k - 2)$ is the union of the following sets:

$$A = \{2^{k-1} + 2p - 1 | -1 \leq p \leq 2^{k-3} - 1\}$$

$$B = \{2p - 2^{k-1} \pmod{n} | 1 \leq p \leq 2^{k-3}\}.$$

Since all members of A are odd vertices, $0 \notin A$.

It is worth mentioning that $-n < \min B = -2^{k-1} + 2 < \max B = -2^{k-2} < 0$. Thus, all members of B modulo n are between 0 and n . Therefore, $0 \notin B$ and so $0 \notin I$.

Finally, it is obvious that u informs 0 in time unit $k - 1$ by using dimension i . □

In the next proposition, we consider the case where the originator is an i -dimensional neighbor of vertex 0, with $1 \leq i \leq k-1$. We prove that by utilizing $k-i$ cyclic shifts of the reverse dimensional broadcast scheme, vertex 0 remains uninformed after time unit $(k-2)$.

Proposition 4.1.6. *Let $KG(n)$ be a Knödel graph of order n with $k = \lceil \log n \rceil$ and $0, u = 2^i - 1 \in V(KG(n))$ be two i dimensional neighbors for an integer $1 \leq i \leq k-1$. If the originator and the dimensional broadcast scheme are u and $(i-1, i-2, \dots, 1, k-1, k-2, \dots, i, i-1)$, respectively, then vertex 0 is uninformed after time unit $k-2$ and will be informed by u in time unit $k-1$. (In case of $i = 1$, the dimensional broadcast scheme is $(k-1, \dots, 1, k-1)$).*

Proof. First, suppose that $i \neq 1$. In Lemma 4.1.4, consider $k-i$ cyclic shifts instead of i and let $u = 2^i - 1$ and $t = k-2$.

If $i = k-1$, then by Part 1, the set of informed vertices I after time unit $(k-2)$ is

$$\{2^{k-1} - 2p - 1 \mid 0 \leq p \leq 2^{k-3} - 1\} \cup \{2s - 2^{k-1} \mid 1 \leq s \leq 2^{k-3}\}.$$

If $i \leq k-2$, then by Part 2 of Lemma 4.1.4, the set of informed vertices I after time unit $(k-2)$ is the union of the following sets:

$$\begin{aligned} A_{k-2} &= \{-j2^{i+1} - 2p + 2^i - 1 \mid 0 \leq j \leq 2^{k-i-2} - 1, 0 \leq p \leq 2^{i-2} - 1\}, \\ B_{k-2} &= \{-j2^{i+1} + 2s - 2^i \mid 0 \leq j \leq 2^{k-i-2} - 1, 1 \leq s \leq 2^{i-2}\}, \\ A'_{k-2} &= \{j2^{i+1} + 2p - 2^i \mid 1 \leq j \leq 2^{k-i-2}, 0 \leq p \leq 2^{i-2} - 1\}, \\ B'_{k-2} &= \{j2^{i+1} - 2s + 2^i - 1 \mid 1 \leq j \leq 2^{k-i-2}, 1 \leq s \leq 2^{i-2}\}. \end{aligned}$$

Now, we check if $0 \stackrel{n}{\equiv} n \in I$.

Obviously, $0 \notin I$ when $i = k-1$.

Let $i \leq k - 2$. Note that $-n < \min B_{k-2} = -2^{k-1} + 2^i + 2 < \max B_{k-2} = -2^{i-1} < 0$.

Hence, all members of $B_{k-2} \pmod n$ are between 0 and n and so $0 \notin B_{k-2}$.

Also, $0 < \min A'_{k-2} = 2^i < \max A'_{k-2} = 2^{k-1} - 2^{i-1} - 2 < n$. Thus, $0 \notin A'_{k-2}$. Therefore, $0 \notin I$ and the proof is complete.

Now, suppose that $i = 1$. In this case, $u = 1$ and the dimensional broadcast scheme is $(k-1, \dots, 1, k-1)$. By Lemma 4.1.3, the set of informed vertices I after time unit $(k-2)$ is

$$\{-4p + 1 \mid 0 \leq p \leq 2^{k-3} - 1\} \cup \{4s - 2 \mid 1 \leq s \leq 2^{k-3}\}.$$

It is easy to check that $0 \notin I$.

Finally, it is obvious that u informs 0 in time unit $k-1$ by using dimension i . $\square$

In the next claim, we consider the case where the originator u is a vertex that is not a neighbor of vertex 0. We prove that based on u , there exists a dimensional broadcast scheme such that the vertex 0 is either uninformed after time unit $(k-1)$ or remains uninformed after time unit $(k-2)$, receives the message in time unit $(k-1)$, and remains idle during the last time unit k .

Proposition 4.1.7. *Let $KG(n)$ be a Knödel graph of order n with $k = \lceil \log n \rceil$. If u , the originator, is an arbitrary vertex that is not adjacent to vertex 0, then there exists a dimensional broadcast scheme such that vertex 0 is either uninformed after time unit $(k-1)$ or remains uninformed after time unit $(k-2)$, receives the message in time unit $(k-1)$, and remains idle during the last time unit k .*

Proof. There are two cases:

Case 1. Assume that u is odd. In this case, we show that there exists a dimensional broadcast scheme, dependent on u , such that vertex 0 remains uninformed after time unit $(k-1)$.

If $u \geq 2^{k-1}$, then consider dimensional broadcast scheme $(1, \dots, k-1, 1)$. By Lemma 4.1.1, the set of informed vertices I_{k-1} after time unit $(k-1)$ equals to the union of the following sets:

$$A_{k-1} = \{2p + u \mid 0 \leq p \leq 2^{k-2} - 1\} \text{ and } B_{k-1} = \{2s - u - 1 \mid 1 \leq s \leq 2^{k-2}\}.$$

Since u is odd, $0 \notin A_{k-1}$. Also, $-n < \min B_{k-1} = 1 - u < \max B_{k-1} = 2^{k-1} - u - 1 < 0$.

Hence, $0 \notin B_{k-1}$ and so $0 \notin I_{k-1}$.

If $2^i \leq u \leq 2^{i+1} - 3$ for some integer $2 \leq i \leq k-2$, then consider the broadcast scheme obtained by applying i cyclic shifts to the standard dimensional broadcast order, i.e., $(i+1, \dots, k-1, 1, \dots, i, i+1)$. By Lemma 4.1.2, the set of informed vertices I_{k-1} after time unit $(k-1)$ is the union of the following sets:

$$\begin{aligned} A_{k-1} &= \{p2^{i+1} + 2j + u \mid 0 \leq p \leq 2^{k-i-2} - 1, 0 \leq j \leq 2^{i-1} - 1\}, \\ B_{k-1} &= \{s2^{i+1} + 2j - u - 1 \mid 1 \leq s \leq 2^{k-i-2}, 0 \leq j \leq 2^{i-1} - 1\}, \\ A'_{k-1} &= \{-p2^{i+1} + 2j - u - 1 \mid 0 \leq p \leq 2^{k-i-2} - 1, 1 \leq j \leq 2^{i-1}\}, \\ B'_{k-1} &= \{-s2^{i+1} + 2j + u \mid 1 \leq s \leq 2^{k-i-2}, 1 \leq j \leq 2^{i-1}\}. \end{aligned}$$

Since u is odd, $0 \notin A_{k-1}, B'_{k-1}$.

Suppose that $0 \in B_{k-1}$. Thus, there exist $1 \leq s \leq 2^{k-i-2}$ and $0 \leq j \leq 2^{i-1} - 1$ such that $s2^{i+1} + 2j - u - 1 \stackrel{n}{=} 0$. If $s2^{i+1} + 2j - u - 1 = 0$, then $u \geq 2^{i+1} - 1$, a contradiction. If $s2^{i+1} + 2j - u - 1 = n$, then $u \leq 2^{k-1} + 2^i - n - 3 < 2^i$, a contradiction. Hence, $0 \notin B_{k-1}$.

Assume that $0 \in A'_{k-1}$. Then there exist $0 \leq p \leq 2^{k-i-2} - 1$ and $1 \leq j \leq 2^{i-1}$ such that $-p2^{i+1} + 2j - u - 1 \stackrel{n}{=} 0$. If $-p2^{i+1} + 2j - u - 1 = 0$, then $u \leq 2^i - 1$, a contradiction. If $-p2^{i+1} + 2j - u - 1 = -n$, then $u \geq n - 2^{k-1} + 2^{i+1} - 1 > 2^{i+1}$, a contradiction. Hence, $0 \notin A'_{k-1}$.

Thus, $0 \notin I_{k-1}$. Note that since u is not adjacent to vertex 0, then $u \neq 2^i - 1$ for $1 \leq i \leq k - 1$. Therefore, for every odd vertex u (which is not adjacent to 0), there exists a dimensional broadcast scheme such that 0 remains uninformed after time unit $(k - 1)$.

Case 2. Assume that u is even. There are 5 sub-cases.

Sub-case 1. $u \leq n - 2^{k-1}$. Consider the dimensional broadcast scheme $(1, \dots, k - 1, 1)$. By Lemma 4.1.1, the set of informed vertices I_{k-1} after time unit $(k - 1)$ equals to the union of the following sets:

$$A_{k-1} = \{2p + u \mid 0 \leq p \leq 2^{k-2} - 1\} \text{ and } B_{k-1} = \{2s - u - 1 \mid 1 \leq s \leq 2^{k-2}\}.$$

Since u is even, $0 \notin B_{k-1}$. Also, $0 < \min A_{k-1} = u < \max A_{k-1} = 2^{k-1} + u - 2 < n$. Hence, $0 \notin A_{k-1}$ and so $0 \notin I_{k-1}$.

Sub-case 2. $u < 2^{k-2}$. Consider the broadcast scheme obtained by applying 1 cyclic shift to the standard dimensional broadcast scheme, i.e. $(k - 1, 1, \dots, k - 2, k - 1)$. By setting $i = k - 2$ in Lemma 4.1.2, the set of informed vertices I_{k-1} after time unit $(k - 1)$ equals to the union of the following sets:

$$\begin{aligned} A_{k-1} &= \{2j + u \mid 0 \leq j \leq 2^{k-3} - 1\}, \\ B_{k-1} &= \{2^{k-1} + 2j - u - 1 \mid 0 \leq j \leq 2^{k-3} - 1\}, \\ A'_{k-1} &= \{2j - u - 1 \mid 1 \leq j \leq 2^{k-3}\}, \\ B'_{k-1} &= \{-2^{k-1} + 2j + u \mid 1 \leq j \leq 2^{k-3}\}. \end{aligned}$$

Since u is even, $0 \notin B_{k-1}, A'_{k-1}$.

Suppose that $0 \in A_{k-1}$. Thus, there exists $0 \leq j \leq 2^{k-3} - 1$ such that $u + 2j \stackrel{n}{=} 0$. So, $u + 2j = n$. Hence, $u \geq n - 2^{k-2} + 2 > 2^{k-2}$, a contradiction. Therefore, $0 \notin A_{k-1}$.

Assume that $0 \in B'_{k-1}$. Then there exists $1 \leq j \leq 2^{k-3}$ such that $-2^{k-1} + 2j + u \stackrel{n}{=} 0$. So,

$-2^{k-1} + 2j + u = 0$. Hence, $u \geq 2^{k-2}$, a contradiction. Thus, $0 \notin B'_{k-1}$.

Therefore, $0 \notin I_{k-1}$.

Sub-case 3. $u \geq 2^{k-1}$. Consider the dimensional broadcast scheme $(k-1, \dots, 1, k-1)$. By Lemma 4.1.3, the set of informed vertices I_{k-1} after time unit $(k-1)$ equals to the union of the following sets:

$$A_{k-1} = \{-2p + u \mid 0 \leq p \leq 2^{k-2} - 1\} \text{ and } B_{k-1} = \{2s - u - 1 \mid 1 \leq s \leq 2^{k-2}\}.$$

Since u is even, $0 \notin B_{k-1}$. Also, $0 < \min A_{k-1} = -2^{k-1} + u + 2 < \max A_{k-1} = u < n$.

Hence, $0 \notin A_{k-1}$ and so $0 \notin I_{k-1}$.

Sub-case 4. $u > n - 2^{k-2}$. Consider the broadcast scheme obtained by applying 1 cyclic shift to the reverse dimensional broadcast scheme, i.e. $(k-2, \dots, 1, k-1, k-2)$. By setting $i = 1$ in Lemma 4.1.4, the set of informed vertices I_{k-1} after time unit $(k-1)$ equals to the union of the following sets:

$$\begin{aligned} A_{k-1} &= \{-2p + u \mid 0 \leq p \leq 2^{k-3} - 1\}, \\ B_{k-1} &= \{2s - u - 1 \mid 1 \leq s \leq 2^{k-3}\}, \\ A'_{k-1} &= \{2^{k-1} + 2p - u - 1 \mid 0 \leq p \leq 2^{k-3} - 1\}, \\ B'_{k-1} &= \{2^{k-1} - 2s + u \mid 1 \leq s \leq 2^{k-3}\}. \end{aligned}$$

Since u is even, $0 \notin B_{k-1}, A'_{k-1}$.

Suppose that $0 \in A_{k-1}$. Thus, there exists $0 \leq p \leq 2^{k-3} - 1$ such that $u - 2p \stackrel{n}{=} 0$. So, $u - 2p = 0$. Hence, $u \leq 2^{k-2} - 2 < n - 2^{k-2}$, a contradiction. Therefore, $0 \notin A_{k-1}$.

Assume that $0 \in B'_{k-1}$. Then there exists $1 \leq s \leq 2^{k-3}$ such that $2^{k-1} - 2s + u \stackrel{n}{=} 0$. So, $2^{k-1} - 2s + u = n$. Hence, $u \leq n - 2^{k-2}$, a contradiction. Thus, $0 \notin B'_{k-1}$.

Therefore, $0 \notin I_{k-1}$.

Note that in all previous subcases, there exists a dimensional broadcast scheme, depending on u , under which vertex 0 remains uninformed after time unit $(k - 1)$. However, for the next subcase, we provide a dimensional broadcast scheme such that 0 remains uninformed after time unit $(k - 2)$, receives the message during time unit $(k - 1)$, and remains idle during the last time unit k .

Sub-case 5. $n - 2^{k-1} + 2 \leq u \leq \min\{2^{k-1} - 2, n - 2^{k-2}\}$. In this case, consider the dimensional broadcast scheme $(1, \dots, k - 1, 1)$. By Lemma 4.1.1, the set of informed vertices I_{k-2} after time unit $(k - 2)$ equals to the union of the following sets:

$$A_{k-2} = \{2p + u \mid 0 \leq p \leq 2^{k-3} - 1\} \text{ and } B_{k-2} = \{2s - u - 1 \mid 1 \leq s \leq 2^{k-3}\}.$$

Since u is even, $0 \notin B_{k-2}$. Also, $0 < \min A_{k-2} = u < \max A_{k-2} = 2^{k-2} + u - 2 < n$. Hence, $0 \notin A_{k-2}$ and so $0 \notin I_{k-2}$. It is worth mentioning that for $s = \frac{u - n + 2^{k-1}}{2}$, $2s - u - 1 \stackrel{n}{=} 2^{k-1} - 1$. Thus, $2^{k-1} - 1 \in I_{k-1}$, and so $0 \in I_{k-1}$. Now, we claim that $1 \in I_{k-1}$. Let $s = \frac{u+2}{2}$. We have $2s - u - 1 = 1$ and $s \leq 2^{k-2}$. Hence, $1 \in I_{k-1}$. Thus, 0 remains idle during the last time unit.

Hence, for every even originator vertex, there exists a dimensional broadcast scheme such that 0 remains idle after time unit $(k - 2)$ or remains idle after time unit $(k - 2)$, receives the message during time unit $(k - 1)$, and remains idle during the last time unit. Therefore, the proof is complete. $\square$

4.2 Construction of a Broadcast Graph

In this section, we remove one vertex, say vertex 0, from $KG(n)$ and add $\lceil \frac{k-1}{2} \rceil$ edges, where $k = \lceil \log n \rceil$, to construct a new broadcast graph on $n - 1$ vertices. Using this method, we construct a broadcast graph for every odd number of vertices. Consequently, we derive an upper bound for $B(n)$ for all integers n .

In the worst case, when a vertex v is deleted from a broadcast graph, it may be possible to add $\binom{\deg(v)}{2}$ edges (forming a clique among v 's neighbors) so that the resulting graph on $n - 1$ vertices is still a broadcast graph. However, in $KG(n)$, it suffices to add only $\left\lceil \frac{\deg(v)}{2} \right\rceil$ edges, thanks to its good structural properties.

Let $KG(n)$ be a Knödel graph of order n with $k = \lceil \log n \rceil$ and $n \neq 2^k$. We introduce a new graph $KG'(n - 1)$ by removing vertex 0 and all its incident edges, and adding edges $(2^{i-1} - 1, 2^i - 1)$ for all even $2 \leq i \leq k - 1$. If $k - 1$ is odd, then we also add edge $(2^{k-2} - 1, 2^{k-1} - 1)$ as shown in Figures 4.1 and 4.2.

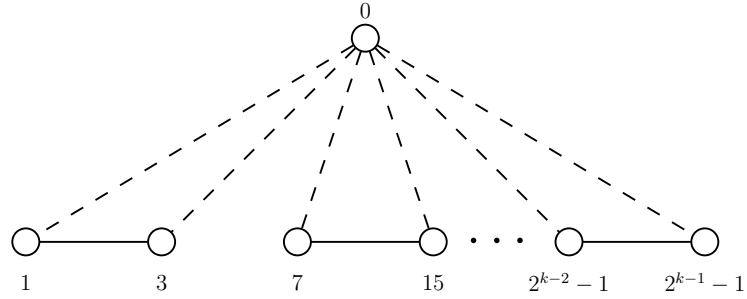


Figure 4.1: Subgraph of $KG'(n - 1)$ when $k - 1$ is even. Dashed lines represent removed edges, and solid lines represent newly added edges.

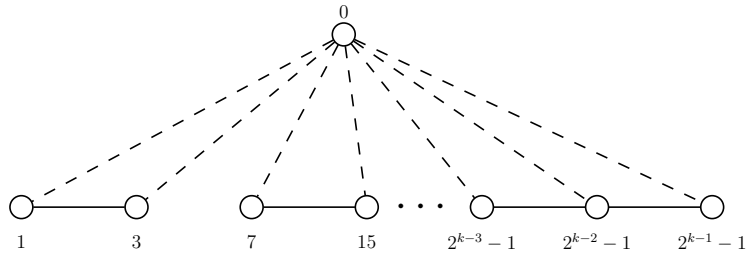


Figure 4.2: Subgraph of $KG'(n - 1)$ when $k - 1$ is odd. Dashed lines represent removed edges, and solid lines represent newly added edges.

In the next theorem, we prove that $KG'(n - 1)$ is a broadcast graph.

Theorem 4.2.1. *Let $KG(n)$ be a Knödel graph of order n with $k = \lceil \log n \rceil$ such that $n \neq 2^k$. The constructed graph $KG'(n-1)$ from $KG(n)$ is a broadcast graph.*

Proof. If the originator is $2^i - 1$ for some odd integer $1 \leq i \leq k-2$, we claim that the dimensional broadcast scheme $(i+1, \dots, k-1, 1, \dots, i, i+1)$ is valid for graph $KG'(n-1)$. This holds because, in $KG(n)$, by Proposition 4.1.5, vertex 0 remains uninformed after time unit $(k-2)$, receives the message from the originator at time unit $(k-1)$, and then informs vertex $2^{i+1} - 1$ in the last time unit. In the constructed graph $KG'(n-1)$, the originator directly informs vertex $2^{i+1} - 1$ at time unit $(k-1)$, replacing the role of vertex 0 in graph $KG(n)$. The rest of the broadcast scheme remains unchanged. Therefore, every vertex will be informed after the last time unit k .

If the originator is $2^i - 1$ for some even integer $1 \leq i \leq k-1$, a similar argument applies using the dimensional broadcast scheme $(i-1, \dots, 1, k-1, \dots, i, i-1)$. In $KG(n)$, by Proposition 4.1.6, vertex 0 receives the message at time unit $(k-1)$ and informs vertex $2^{i-1} - 1$ at time k . In $KG'(n-1)$, the originator directly informs $2^{i-1} - 1$ at time unit $(k-1)$, ensuring all vertices are informed after the last time unit k .

If $k-1$ is odd, then similar to the previous case, the dimensional broadcast scheme $(k-2, \dots, 1, k-1, k-2)$ is valid for $KG'(n-1)$.

If the originator is a vertex that is not adjacent to 0, then, in $KG(n)$, by Proposition 4.1.7, there exists a dimensional broadcast scheme D such that 0 remains uninformed after time unit $(k-1)$ or remains uninformed after time unit $(k-2)$, receives the message in time unit $(k-1)$ and remains idle during the last time unit. Since vertex 0 does not contribute to the broadcast by using D , removing it does not affect the dissemination of the message in $KG'(n-1)$. Therefore, $KG'(n-1)$ is a broadcast graph. $\square$

Theorem 4.2.2. *For all $n \neq 2^k - 1$ where $k \geq 2$*

$$B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil.$$

Proof. If n is even, then the number of edges in $KG(n)$ is $\frac{n \lfloor \log n \rfloor}{2}$, and there is nothing to prove. Suppose that n is odd. Consider graph $KG'(n)$ obtained from $KG(n+1)$. The number of edges in $KG'(n)$ is $\frac{n(k-1)}{2}$ if $k-1$ is even because every vertex has degree $k-1$, and $\frac{n(k-1)+1}{2}$ if $k-1$ is odd because all vertices have degree $k-1$ except vertex $2^{k-2}-1$ which has degree k . Since by Theorem 4.2.1, $KG'(n)$ is a broadcast graph, then

$$B(n) \leq \left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil.$$

□

In the case $n = 2^k - 1$, it is proved in [75] that for all $k \geq 2$

$$B(2^k - 1) \geq \frac{(k-1)(2^k - 1)}{2} + \frac{2^k - 1}{2(k+1)}.$$

This lower bound is greater than $\left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil$. Hence the general upper bound $\left\lceil \frac{n \lfloor \log n \rfloor}{2} \right\rceil$ cannot be expanded to the case $n = 2^k - 1$.

4.3 Minimum Average Broadcast Graphs

The concept of *minimum average broadcast graphs* has been investigated in [77]. Its definition is as follows: A broadcast scheme can be represented by a spanning tree T_u of a graph G , rooted at the originator u , where each vertex v is assigned a value $t(v)$ indicating the time at which v receives the message. We assume $t(u) = 0$. The *average broadcast time* of a scheme T_u is defined as

$$b_a(T_u) = \frac{1}{|V|} \sum_{v \in V} t(v).$$

For a fixed originator u , the *minimum average broadcast time for u* is defined as

$$b_a(u) = \min\{b_a(T_u)\},$$

where the minimum is taken over all broadcast trees rooted at u . The *minimum average broadcast time for a graph G* is then defined as

$$b_a(G) = \max_{u \in V(G)} \{b_a(u)\}.$$

Since the complete graph K_n has the minimum $b_a(G)$ among all graphs on n vertices, we define $\beta_a(n) = b_a(K_n)$, and refer to it as the *minimum average broadcast time for n vertices*.

A *minimum average time broadcast graph* (matbg) is a graph G on n vertices such that $b_a(G) = \beta_a(n)$, and for any other graph G' on n vertices with $b_a(G') = \beta_a(n)$, the graph G has no more edges than G' . In other words, matbg's are graphs on n vertices with the fewest edges that still allow broadcasting in minimum average time from any originator. Note that in order to achieve the minimum average broadcast time $\beta_a(n)$, the number of informed vertices must double in each time unit except the last time unit. Consequently, the degree of every vertex v in a minimum average time broadcast graph (matbg) on n vertices must satisfy $\deg(v) \geq \lfloor \log n \rfloor$. We denote by $B_a(n)$ the number of edges in a matbg on n vertices.

Note that Knödel graphs $KG(n)$ are matbgs for all even values of n . In Theorems 3.9 and 3.11 of [77], it was shown that for $2^k + 1 \leq n \leq 2^k + 2^{k-1}$ with $k \geq 5$, we have $B_a(n) = \lceil \frac{kn}{2} \rceil$, by explicitly constructing a matbg on n vertices using the multiple fixed step graph method. This construction cannot be extended for the second half of the interval $(2^{k-1}, 2^k)$. While the authors successfully addressed the case for the first half of the interval $(2^{k-1}, 2^k)$, the broader problem of finding matbg graphs throughout the entire interval remained an open problem. Theorem 4.2.2 provides a new construction that also yields a matbg. This is because the graphs $KG'(n)$ constructed from $KG(n+1)$ have the same subtrees as $KG(n+1)$, with one pendant vertex less, as well as the minimum degree sequence possible for matbgs. As a result, we completely resolve the open problem by

extending the characterization of matbg graphs to the full interval.

Chapter 5

Dimensional Broadcast Schemes for Generalized Knödel Graphs

Definition 5.0.1. A **generalized Knödel graph** of even order n and degree $\Delta \leq \lfloor \log n \rfloor$ is $W_{n,\Delta} = (V, E)$, where $V = \{i | 0 \leq i \leq n - 1\}$ and $E = \{(i, j) | i + j \equiv 2^s - 1 \pmod{n}, 1 \leq s \leq \Delta \text{ and } i, j \in V\}$. We say that the s dimensional neighbor of vertex i is vertex j when $i + j \equiv 2^s - 1 \pmod{n}$ [40].

When $\Delta = \lfloor \log n \rfloor$, the graph $W_{n,\lfloor \log n \rfloor}$ is a Knödel graph $KG(n)$.

According to its definition, the generalized Knödel graph $W_{n,\Delta}$, for an even n and $\Delta \leq \lfloor \log n \rfloor$, has dimensions $1, \dots, \Delta$. By a standard dimensional broadcast scheme, we refer to the scheme $(1, 2, \dots, \Delta, \alpha_1, \dots, \alpha_c)$, where $c = \lfloor \log n \rfloor - \Delta$ and $\alpha_i \in \{1, \dots, \Delta\}$ for $1 \leq i \leq c$.

By applying a dimensional broadcast scheme $(\alpha_1, \alpha_2, \dots, \alpha_k)$ in a generalized Knödel graph, we mean that in time unit i , all informed vertices inform their α_i dimensional neighbors.

Due to the extensive applications of Knödel graphs in constructing other broadcast graphs, we aim to utilize generalized Knödel graphs for similar constructions. First, we

seek to determine the values of n and Δ for which the generalized Knödel graph $W_{n,\Delta}$ functions as a broadcast graph. Second, we identify various valid dimensional broadcast schemes. Given the complexity of investigating all possible broadcast schemes in generalized Knödel graphs, and the fact that dimensional broadcast schemes present a strong candidate among them, we restrict our focus to the study of dimensional broadcast schemes. In this chapter, we determine all values of n (the number of vertices) and Δ for which a generalized Knödel graph $W_{n,\Delta}$ is a broadcast graph under the standard dimensional broadcast scheme with i cyclic shifts and arbitrary repetitions of the remaining dimensions. The results presented in this chapter have been published in [31].

Note that for two sets A and B of real numbers, we say $A \geq B$ ($A > B$) if for every $a \in A$ and every $b \in B$ we have $a \geq b$ ($a > b$). Since generalized Knödel graphs are vertex transitive, throughout this chapter, we consider vertex 0 as the originator.

5.1 Generalized Knödel Graph $W_{n,k-c}$

In the next theorem, we determine which of the standard dimensional broadcast schemes is the most efficient in terms of broadcasting to all vertices in $W_{n,k-c}$, where $k = \lceil \log n \rceil$.

Theorem 5.1.1. *Let $D = (1, 2, \dots, k - c, \alpha_1, \dots, \alpha_c)$ be a dimensional broadcast scheme for generalized Knödel graph $W_{n,k-c}$, where $k = \lceil \log n \rceil$, $c < k$ is a positive integer, and $\alpha_i \in \{1, 2, \dots, k - c\}$ for $1 \leq i \leq c$. Then the following hold:*

- i) *If $\alpha_i = 1$ for odd i and $\alpha_i = k - c$ for even i , then the number of informed vertices after time unit k reaches its maximum when using D among all dimensional broadcast schemes.*
- ii) *If $c \geq 3$ and the number of informed vertices after time unit k reaches its maximum by using D , then $\alpha_i = 1$ for odd i and $\alpha_i = k - c$ for even i .*

Proof. First, suppose that the dimensional broadcast scheme is $D^* = (1, 2, \dots, k - c, 1, k -$

$c, 1, k - c, \dots$). Note that informed vertices after time unit $(k - c)$ are $\{0, 1, \dots, 2^{k-c} - 1\}$.

By induction, one can see that informed vertices after time unit $(k - c + j)$, $1 \leq j \leq c$ are

$$\begin{cases} \{0, 1, \dots, (p+1)2^{k-c} - (2p+1)\} \cup \{n - (p+1)2^{k-c} + 2p + 2, \dots, n - 1\} & j = 2p + 1 \\ \{0, 1, \dots, (p+1)2^{k-c} - (2p+1)\} \cup \{n - p2^{k-c} + 2p, \dots, n - 1\} & j = 2p. \end{cases}$$

Now, suppose that by using dimensional broadcast scheme $D = (1, 2, \dots, k - c, \alpha_1, \dots, \alpha_c)$, the number of informed vertices after time unit k reaches its maximum among all dimensional broadcast schemes. We show that D cannot inform more vertices than D^* after time unit k . Assume that $l = 2p + 1$ is the smallest odd integer for which $\alpha_l \neq 1$, and for $j < l$, α_j is equal to its corresponding component of D^* . Then informed vertices after time unit $(k - c + l)$ are $\{0, 1, \dots, (p+1)2^{k-c} - (2p+1)\} \cup \{n + 2^{\alpha_l} - (p+1)2^{k-c} + 2p, \dots, n - 1\}$. But $n + 2^{\alpha_l} - (p+1)2^{k-c} + 2p > n - (p+1)2^{k-c} + 2p + 2$, and so by using D , informed vertices after time unit $(k - c + l)$ is a proper subset of informed vertices after the same time unit by using D^* . Hence, changing α_l to 1 does not decrease the number of informed vertices after time unit k . Thus, suppose that $\alpha_l = 1$. The argument is similar when l is the smallest even integer for which $\alpha_l \neq k - c$, and for $j < l$, α_j is equal to its corresponding component of D^* . Therefore, D^* gives the maximum number of informed vertices after time unit k among all dimensional broadcast schemes.

Now, suppose that $c \geq 3$ and the number of informed vertices after time unit k reaches its maximum using D . We will show that $D = D^*$. As demonstrated above, D^* informs the maximum number of vertices after time unit k among all dimensional broadcast schemes. Moreover, if $D \neq D^*$, then the set of informed vertices after time unit k using D is a proper subset of those informed by D^* , unless both schemes inform all vertices.

When using D^* , the informed vertices after time unit $(k - c)$ are $0, 1, \dots, 2^{k-c} - 1$. In time unit $(k - c + 1)$, 0 and 1 are idle while the rest of vertices can inform $2^{k-c} - 2$ vertices.

From now on, since the dimensions 1 and $l - c$ is repeating, in each subsequent time unit,

exactly $2^{k-c} - 2$ vertices that became informed in the previous time unit send the message to new vertices. Therefore, after time unit k , at most $2^{k-c} + c(2^{k-c} - 2)$ vertices will be informed. Thus, if D^* informs all vertices, then

$$2^{k-c} + c(2^{k-c} - 2) \geq 2^{k-1} + 2 \Rightarrow (c+1)2^{k-c} > 2^{k-1} \Rightarrow \log(c+1) > c-1 \Rightarrow c \leq 2.$$

It is a contradiction. Therefore, D^* , and consequently D , are not valid dimensional broadcast schemes. Thus, $D = D^*$. $\square$

As demonstrated in the proof of the previous theorem, if $c \geq 3$, then $D^* = (1, 2, \dots, k-c, 1, k-c, 1, k-c, \dots)$ is not a valid dimensional broadcast scheme, however, it gives the maximum number of informed vertices after time unit k . Based on this, the following corollary demonstrates that $W_{n,k-c}$, where $k = \lceil \log n \rceil$ and $c \geq 3$, is not a broadcast graph under the standard dimensional broadcast scheme when any of the last c dimensions are repeated.

Corollary 5.1.2. *Let $W_{n,k-c}$, for some positive integer c , be a generalized Knödel graph of order n such that $k = \lceil \log n \rceil \geq 4$. If $c \geq 3$, then the dimensional broadcast scheme $(1, 2, \dots, k-c, \alpha_1, \dots, \alpha_c)$ is not valid for any $\alpha_1, \dots, \alpha_c$ in $\{1, 2, \dots, k-c\}$.*

Remark 5.1.1. *We conjecture that the standard dimensional broadcast scheme (without any cyclic shift) is likely to be the most efficient scheme for broadcasting in a Knödel graph. The latter means that $W_{n,k-c}$ is not a broadcast graph when $c \geq 3$. It is worth mentioning that for small values of n, k , $W_{n,k-3}$ is not a broadcast graph. For example, when $k = 5$, $W_{n,2}$ is the cycle on n vertices, for all $18 \leq n \leq 30$, and obviously not a broadcast graph. For another example, consider the case where $n = 36$ and $k = 6$. By Theorem 6 in [46], the diameter of $W_{n,k-3} = W_{36,6}$ is at least 7, thus $b(W_{36,6}) \geq 7$. Hence, $W_{36,6}$ is not a broadcast graph. Therefore, based on the previous theorem, we concentrate on generalized*

Knödel graphs $W_{n,k-2}$, thereby omitting all $k - 1$ dimensional edges within the Knödel graph $KG(n)$.

5.2 Generalized Knödel Graph $W_{n,k-2}$

Our goal is to classify all values of n for which the generalized Knödel graph $W_{n,k-2}$ is a broadcast graph under a specific cyclic shift of the standard dimensional broadcast scheme. We initiate our investigation with the simplest variant of the standard dimensional broadcast scheme, with 0 cyclic shifts, whose results differ from those of other dimensional broadcast schemes that include more cyclic shifts.

Theorem 5.2.1. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n such that $k = \lceil \log n \rceil$. Then the dimensional broadcast scheme $(1, 2, \dots, k - 2, 1, k - 2)$ is valid if and only if $n \leq 2^{k-1} + 2^{k-2} - 4$.*

Proof. After time unit $(k - 2)$, by using dimensional broadcast scheme $(1, 2, \dots, k - 2, 1, k - 2)$, vertices $A = \{0, 1, \dots, 2^{k-2} - 1\}$ are informed. In time unit $(k - 1)$, 0 and 1 are idle while $2, 3, \dots, 2^{k-2} - 1$ inform $n - 1, n - 2, \dots, n - 2^{k-2} + 2$, respectively. In the last time unit, vertices of A will be idle, while $n - 1, \dots, n - 2^{k-2} + 2$ inform $2^{k-2}, \dots, 2^{k-1} - 3$, respectively. Hence, the vertex set $\{2^{k-1} - 2, \dots, n - 2^{k-2} + 1\}$ is uninformed. It is easy to check that this set is empty if and only if $n \leq 2^{k-1} + 2^{k-2} - 4$. $\square$

The following result is an immediate corollary of the previous theorem.

Corollary 5.2.2. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n where $k = \lceil \log n \rceil$. If $n \leq 2^{k-1} + 2^{k-2} - 4$, then $W_{n,k-2}$ is a broadcast graph.*

Note that in generalized Knödel graph $W_{n,k-2}$ such that $k = \lceil \log n \rceil$, if $n > 2^{k-1} + 2^{k-2} - 4$, then by Theorem 5.1.1 and Theorem 5.2.1, the dimensional broadcast scheme

$(1, 2, \dots, k-2, \alpha_1, \alpha_2)$ is not valid. Now, we begin our investigation into dimensional broadcast schemes that incorporate cyclic shifts. First, since the last two dimensions are unknown, we demonstrate the informed vertices after time unit $(k-2)$.

Lemma 5.2.3. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n , $k = \lceil \log n \rceil$ and i be an integer such that $1 \leq i \leq k-3$. Then informed vertices after time unit $(k-2)$ by applying the dimensional broadcast scheme $(k-i-1, k-i, \dots, k-2, 1, \dots, k-i-2, \alpha_1, \alpha_2)$, where $1 \leq \alpha_1, \alpha_2 \leq k-2$, are the union of the following sets:*

$$\left\{ \begin{array}{l} A = \{0, 1, \dots, 2^{k-i-2} - 1\} \\ I_p = \{p2^{k-i-1} + j \mid -1 \leq j \leq 2^{k-i-2} - 2\} \\ B = \{2^{k-2} + 2j - 1 \pmod{n} \mid 0 \leq j \leq 2^{k-i-3} - 1\} \\ C = \{n - 2^{k-2} + 2j \mid 1 \leq j \leq 2^{k-i-3}\} \\ I'_p = \{n - p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-2}\}, \end{array} \right.$$

where $1 \leq p \leq 2^{i-1} - 1$.

Proof. By induction, one can see that informed vertices after time unit i are $0, p2^{k-i-1} - 1, p2^{k-i-1}, 2^{k-2} - 1$, where $1 \leq p \leq 2^{i-1} - 1$. Since, by utilizing other dimensions after time unit i , vertices of A, B, I_p, I'_p and C are responsible for adding vertices to A, C, I'_p, I_p and B , respectively, and vice versa, by using another inductive argument, it is straightforward to verify the correctness of the assertion. $\square$

5.2.1 $W_{n,k-2}$ With 1 Cyclic Shift

The set of values of n for which the standard dimensional broadcast scheme with $i = 0$ and $i = 1$ cyclic shifts is valid for $W_{n,k-2}$ is the same, but it differs from those for $i > 1$. Therefore, in the following theorem, we examine the case $i = 1$.

Theorem 5.2.4. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n with $k = \lceil \log n \rceil$. Then the dimensional broadcast scheme $(k-2, 1, 2, \dots, k-3, k-2, 1)$ is valid if and only*

if $n \leq 2^{k-1} + 2^{k-2} - 4$.

Proof. Suppose that the dimensional broadcast scheme is $D = (k-2, 1, 2, \dots, k-3, k-2, 1)$. By using Lemma 5.2.3 for $i = 1$ and applying dimension $(k-2)$ in time unit $(k-1)$, the set of informed vertices I is the union of the following sets:

$$\begin{cases} A = \{0, \dots, 2^{k-2} - 1\} \\ B = \{2^{k-2} + 2j - 1 | 0 \leq j \leq 2^{k-3} - 1\} \\ C = \{n - 2^{k-2} + 2j | 1 \leq j \leq 2^{k-3}\}. \end{cases}$$

In the last time unit, vertices 0 and 1 will be idle. But $2, \dots, 2^{k-2} - 1$ inform $n-1, \dots, n-2^{k-2} + 2$, respectively. Also, the first dimensional neighbors of vertices in C are in set A .

Now, if $n \leq 2^{k-1} + 2^{k-2} - 4$, then $\max B = 2^{k-1} - 3 \geq n - 2^{k-2} + 1 = \min C - 1$.

Therefore, all uninformed vertices between $2^{k-2} - 1$ and $n - 2^{k-2} + 2$ are even and so will be informed by B in the last time unit. Hence, D is valid.

Now, if $n \geq 2^{k-1} + 2^{k-2} - 2$, then $\min C - 1 = n - 2^{k-2} + 1$ and 2^{k-2} are first dimensional neighbors which are not among informed vertices after time unit $(k-1)$. Thus, D is not valid. $\square$

In the next theorem, we show that if $n > 2^{k-1} + 2^{k-2} - 4$, then 1 cyclic shift does not give us a valid dimensional broadcast scheme.

Theorem 5.2.5. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n such that $k = \lceil \log n \rceil$. If $n > 2^{k-1} + 2^{k-2} - 4$, then the dimensional broadcast scheme $(k-2, 1, 2, \dots, k-3, \alpha_1, \alpha_2)$ is not valid for any $\alpha_1, \alpha_2 \in \{1, \dots, k-2\}$.*

Proof. Assume that $D = (k-2, 1, 2, \dots, k-3, \alpha_1, \alpha_2)$. First, suppose that $\alpha_1 = k-2$. Since $n \geq 2^{k-1} + 2^{k-2} - 2$, following the proof of the previous theorem, $u = \min C - 1 = n - 2^{k-2} + 1$ is not in the set of informed vertices I after time unit $(k-1)$. α_2 dimensional

neighbor of u is $x_{\alpha_2} = 2^{k-2} + 2^{\alpha_2} - 2$. But it is easy to see that $\max A = 2^{k-2} - 1 < x_{\alpha_2} < n - 2^{k-2} + 2 = \min C$. Hence, $x_{\alpha_2} \notin I$. Therefore, u becomes uninformed after time unit k . So D is not valid.

Now, suppose that $\alpha_1 \neq k - 2$. By Lemma 5.2.3, the set of informed vertices I after time unit $(k - 2)$ is the union of the following sets:

$$\begin{cases} A = \{0, \dots, 2^{k-3} - 1\} \\ B = \{2^{k-2} + 2j - 1 | 0 \leq j \leq 2^{k-4} - 1\} \\ C = \{n - 2^{k-2} + 2j | 1 \leq j \leq 2^{k-4}\}. \end{cases}$$

Consider $u = \max B + 1 = 2^{k-2} + 2^{k-3} - 2$. Note that $A < B < C$. Since $\max B < u < \min C$, $u \notin I$. We show that all u 's neighbors are uninformed after time unit $(k - 2)$. α dimensional neighbor of u is $x_\alpha = n - 2^{k-2} - 2^{k-3} + 2^\alpha + 1$ for $\alpha \in \{1, \dots, k - 2\}$. Note that $x_\alpha \notin C$ because x_α is odd, and $x_\alpha > \max B$, and so $x_\alpha \notin I$. Therefore, u will be uninformed after time unit $(k - 1)$. Now, we show that x_α are uninformed after time unit $(k - 1)$, for $1 \leq \alpha \leq k - 2$. Since $\alpha_1 \neq k - 2$, we need to check all x_α 's neighbors except $(k - 2)$ dimensional neighbor. Suppose that $y_{\alpha,\beta} = 2^{k-2} + 2^{k-3} - 2^\alpha + 2^\beta - 2$ is β dimensional neighbor of x_α for $1 \leq \beta \leq k - 3$. One can see that $\max A = 2^{k-3} - 1 < y_{\alpha,\beta} < n - 2^{k-2} + 2 = \min C$. $y_{\alpha,\beta} \notin B$ since it is even. Thus, $y_{\alpha,\beta} \notin I$. Therefore, u remains uninformed after time unit k . $\square$

5.2.2 $W_{n,k-2}$ With an Arbitrary Cyclic Shift

By Theorem 5.2.1, it is apparent that repeating the first dimension and the greatest dimension $(k - 2)$, in a dimensional broadcast scheme with i cyclic shifts, yields a valid scheme for most values of n . This is confirmed by the following theorem.

Theorem 5.2.6. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n , $k = \lceil \log n \rceil$ and i be an integer such that $2 \leq i \leq k - 3$. Then the dimensional broadcast scheme $(k - i - 1, k -$*

$i, \dots, k-2, 1, \dots, k-i-2, k-i-1, k-2)$ is valid if and only if $n \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$.

Proof. Suppose that the dimensional broadcast scheme is $D = (k-i-1, k-i, \dots, k-2, 1, \dots, k-i-2, k-i-1, k-2)$. By Lemma 5.2.3 and applying dimension $(k-i-1)$ in time unit $(k-1)$, the set of informed vertices I equals the union of following sets:

$$\begin{cases} A = \{0, \dots, 2^{k-i-1} - 1\} \\ I_p = \{p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-1} - 2\} \\ B = \{2^{k-2} + 2j - 1 \mid 0 \leq j \leq 2^{k-i-2} - 1\} \\ C = \{n - 2^{k-2} + 2j \mid 1 \leq j \leq 2^{k-i-2}\} \\ I'_p = \{n - p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-1}\}, \end{cases}$$

where $1 \leq p \leq 2^{i-1} - 1$.

Note that $A < I_1 < \dots < I_{2^{i-1}-1} < B < C < I'_{2^{i-1}-1} < \dots < I'_1$. Furthermore, for $1 \leq p \leq 2^{i-1} - 1$, I_p and I'_p contain consecutive vertices and also, $\max I_p + 1 = \min I_{p+1}$ and $\max I'_p + 1 = \min I'_{p-1}$. Hence,

$$(\bigcup_p I_p \cup A) \cup (\bigcup_p I'_p) = \{0, \dots, 2^{k-2} - 1\} \cup \{n - 2^{k-2} + 2^{k-i-1}, \dots, n - 1\} \subset I.$$

Now, suppose that D is valid. Consider vertex $u = \max C - 1 = n - 2^{k-2} + 2^{k-i-1} - 1 \notin I$. Its $(k-2)$ dimensional neighbor is $x = 2^{k-1} - 2^{k-i-1}$. Since D is valid, $x \in I$. It is easy to check that $x > 2^{k-2}$. So, $x \notin A, I_p, B$. Hence, $x \geq \min C = n - 2^{k-2} + 2$. Therefore, $n \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$.

Now, suppose that $n \leq 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$. In this case, $x \in \bigcup_p I'_p \cup C$. If x belongs to one of I'_p , then since $n - 1$ is adjacent to 2^{k-2} , all vertices between $2^{k-2} - 1$ and $n - 2^{k-2} + 2^{k-i-1}$ will be informed by vertices of $\bigcup_p I'_p$, and so D is valid. If $x \in C$, then all vertices less than x will be informed by $\bigcup_p I'_p$ and vertices between x and u will be informed by C . Therefore, D is valid. $\square$

Next theorem shows that there is no valid dimensional broadcast scheme with i cyclic shifts, $2 \leq i \leq k-3$, when $n > 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$.

Theorem 5.2.7. *Let $W_{n,k-2}$ be a generalized Knödel graph of order n , $k = \lceil \log n \rceil$ and i be an integer such that $2 \leq i \leq k-3$. If $n > 2^{k-1} + 2^{k-2} - 2^{k-i-1} - 2$, then the dimensional broadcast scheme $(k-i-1, k-i, \dots, k-2, 1, \dots, k-i-2, \alpha_1, \alpha_2)$ is not valid for any $\alpha_1, \alpha_2 \in \{1, \dots, k-2\}$.*

Proof. Suppose that the dimensional broadcast scheme is $D = (k-i-1, k-i, \dots, k-2, 1, \dots, k-i-2, \alpha_1, \alpha_2)$. By Lemma 5.2.3, the set of informed vertices $I_{(k-2)}$ after time unit $(k-2)$ equals the union of the following sets:

$$\left\{ \begin{array}{l} A = \{0, \dots, 2^{k-i-2} - 1\} \\ I_p = \{p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-2} - 2\} \\ B = \{2^{k-2} + 2j - 1 \mid 0 \leq j \leq 2^{k-i-3} - 1\} \\ C = \{n - 2^{k-2} + 2j \mid 1 \leq j \leq 2^{k-i-3}\} \\ I'_p = \{n - p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-2}\}, \end{array} \right.$$

where $1 \leq p \leq 2^{i-1} - 1$.

Note that $A < I_1 < \dots < I_{2^{i-1}-1} < B < C < I'_{2^{i-1}-1} < \dots < I'_1$. We have four cases.

Case 1. $\alpha_1 < k-i-1$. Consider $u = \max B + 1 = 2^{k-2} + 2^{k-i-2} - 2$. Its d dimensional neighbor is $x_d = n - 2^{k-2} + 2^d - 2^{k-i-2} + 1$ for $1 \leq d \leq k-2$. Note that $u < \min C$ and $\max B < x_d$. So $u \notin I_{(k-2)}$ and $x_d \notin \bigcup_p I_p \cup A \cup B \cup C$. If $x_d \in I'_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $n - 2^{k-2} + 2^d - 2^{k-i-2} + 1 = n - p2^{k-i-1} + j$ where $1 \leq j \leq 2^{k-i-2}$. Thus,

$$2^{i-1} - 2^{d-k+i+1} + \frac{1}{2} \leq p \leq 2^{i-1} - 2^{d-k+i+1} + 1 - \frac{1}{2^{k-i-1}}.$$

If $d - k + i + 1 \geq 0$ or $d - k + i + 1 \leq -2$, then p is not integer, a contradiction. If $d - k + i + 1 = -1$, then $p = 2^{i-1}$, a contradiction. Therefore, $x_d \notin I_{(k-2)}$. Hence,

in time unit $(k-1)$, u remains uninformed. To inform u in the last time unit, at least one of its neighbors must be informed in time unit $(k-1)$ by using dimension α_1 . α_1 dimensional neighbor of x_d is $y_{d,\alpha_1} = 2^{k-2} + 2^{\alpha_1} + 2^{k-i-2} - 2^d - 2$. One can see that $\max A = 2^{k-i-2} - 1 < y_{d,\alpha_1} < \min C$. Thus, $y_{d,\alpha_1} \notin \bigcup_p I'_p \cup C \cup B \cup A$. If $y_{d,\alpha_1} \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} + 2^{\alpha_1} + 2^{k-i-2} - 2^d - 2 = p2^{k-i-1} + j$, where $-1 \leq j \leq 2^{k-i-2} - 1$. Thus,

$$2^{i-1} - 2^{d-k+i+1} + 2^{\alpha_1-k+i+1} - \frac{1}{2^{k-i-1}} \leq p \leq 2^{i-1} - 2^{d-k+i+1} + 2^{\alpha_1-k+i+1} + \frac{1}{2} - \frac{1}{2^{k-i-1}}.$$

Note that $\alpha_1 \leq k-i-2$, and so $\alpha_1 - k + i + 1 \leq -1$. Thus,

$$2^{i-1} - 2^{d-k+i+1} + \frac{1 - 2^{-\alpha_1}}{2^{k-i-1-\alpha_1}} \leq p \leq 2^{i-1} - 2^{d-k+i+1} + \frac{1 - 2^{-\alpha_1}}{2^{k-i-1-\alpha_1}} + \frac{1}{2}.$$

It is worth mentioning that $0 < \frac{1 - 2^{-\alpha_1}}{2^{k-i-1-\alpha_1}} < \frac{1}{2}$. If $d - k + i + 1 \geq 0$, then $2^{i-1} - 2^{d-k+i+1}$ is an integer, and so, p is not an integer, a contradiction. Hence, $d - k + i + 1 \leq -1$. Therefore, $2^{i-1} - 1 < 2^{i-1} + \frac{1 - 2^{-\alpha_1}}{2^{k-i-1-\alpha_1}} - \frac{1}{2^{k-d-i-1}} \leq p$, a contradiction. So, $y_{d,\alpha_1} \notin I_{(k-2)}$. Hence, none of u 's neighbors will be informed in time unit $(k-1)$ by using dimension α_1 . Therefore, D is not valid.

Case 2. $\alpha_1 = k-i-1$. By applying dimension $(k-i-1)$ in time unit $(k-1)$, the set of informed vertices $I_{(k-1)}$ after time unit $(k-1)$ is the union of the following sets:

$$\left\{ \begin{array}{l} A = \{0, \dots, 2^{k-i-1} - 1\} \\ I_p = \{p2^{k-i-1} + j \mid -1 \leq j \leq 2^{k-i-1} - 2\} \\ B = \{2^{k-2} + 2j - 1 \mid 0 \leq j \leq 2^{k-i-2} - 1\} \\ C = \{n - 2^{k-2} + 2j \mid 1 \leq j \leq 2^{k-i-2}\} \\ I'_p = \{n - p2^{k-i-1} + j \mid 1 \leq j \leq 2^{k-i-1}\}. \end{array} \right.$$

There are two sub-cases to consider.

sub-Case 1. $\alpha_2 \leq k-i-2$. Consider $u = \min C - 1 = n - 2^{k-2} + 1$. Its α_2 dimensional neighbor is $x_{\alpha_2} = 2^{k-2} + 2^{\alpha_2} - 2$. Note that $u > \max B$ and $2^{k-2} \leq x_{\alpha_2} \leq 2^{k-2} + 2^{k-i-2} -$

$2 < n - 2^{k-2} + 2 = \min C$. So, $u, x_{\alpha_2} \notin I_{(k-1)}$. Therefore, u remains uninformed after time unit k . Hence, D is not valid.

sub-Case 2. $\alpha_2 \geq k - i$. Consider $u = \max C - 1 = n - 2^{k-2} + 2^{k-i-1} - 1$. Its α_2 dimensional neighbor is $x_{\alpha_2} = 2^{k-2} + 2^{\alpha_2} - 2^{k-i-1}$. Note that $u > \max B$ and $2^{k-2} \leq x_{\alpha_2} \leq 2^{k-2} + 2^{k-2} - 2^{k-i-1} < n - 2^{k-2} + 2 = \min C$. So, $u, x_{\alpha_2} \notin I_{(k-1)}$. Therefore, u remains uninformed after time unit k . Hence, D is not valid.

Case 3. $k - i - 1 < \alpha_1 < k - 2$. There are two sub-cases.

sub-Case 1. $\alpha_2 < k - 2$. Consider $u = 2^{k-2} + 2^{k-3} - 1$. Note that $\max B < u < \min C$. So $u \notin I_{(k-2)}$. Its d dimensional neighbor is $x_d = n - 2^{k-2} - 2^{k-3} + 2^d$ for $d < k - 2$. Since x_d is even and $\min B < x_d < \min C$, for $d < k - 2$, $x_d \notin I_{(k-2)}$. We show that x_d will not be informed in time unit $(k - 1)$ by using dimension $\alpha_1 \in \{k - i - 1, \dots, k - 3\}$. $y_{d, \alpha_1} = 2^{k-2} + 2^{k-3} - 2^d + 2^{\alpha_1} - 1$ is α_1 dimensional neighbor of x_d . Since $d \neq k - 2$, $y_{d, \alpha_1} > \max B$. As y_{d, α_1} is odd, $y_{d, \alpha_1} \notin \bigcup_p I_p \cup A \cup B \cup C$. If $y_{d, \alpha_1} \in I'_p$, for $1 \leq p \leq 2^{i-1} - 1$, then $y_{d, \alpha_1} = n - p2^{k-i-1} + j$, where $1 \leq j \leq 2^{k-i-2}$. Hence, $p \geq 2^{i-1} - 1 + \frac{4}{2^{k-i-1}}$, a contradiction. Thus, $y_{d, \alpha_1} \notin I_{(k-2)}$. Therefore, x_d and u cannot be informed in time units $(k - 1)$ and k , respectively. Hence, D is not valid.

sub-Case 2. $\alpha_2 = k - 2$. Consider $u = 2^{k-1} - 1$.

Note that $\max B < u < n - 2^{k-2} + 2^{k-i-1} + 1 = \min I'_{2^{i-1}-1}$. Since u is odd, $u \notin I_{(k-2)}$. Its d dimensional neighbor, for $k - i - 1 < d \leq k - 2$, is $x_d = n - 2^{k-1} + 2^d$. It is worth to mention that $\max B < x_d < \min C$. Hence, $x_d \notin I_{(k-2)}$. Thus, u remains uninformed after time unit $(k - 1)$. Since the last dimension is $(k - 2)$, we show that $x_{k-2} = n - 2^{k-2}$ remains uninformed after time unit $(k - 1)$. This is because its α_1 dimensional neighbor, $k - i - 1 < \alpha_1 < k - 2$, is $y_{\alpha_1} = 2^{k-2} + 2^{\alpha_1} - 1$. It is easy to see that $\max B < y_{\alpha_1} < \min C$. So, $y_{\alpha_1} \notin I_{(k-2)}$. Therefore, u remains uninformed after time unit k , and so D is not valid.

Case 4. $\alpha_1 = k - 2$. Again there are two sub-cases.

sub-Case 1. $\alpha_2 < k - i - 1$. In this case, consider $u = 2^{k-i-1}$. Note that $\max A < u < \min I_1$. So, $u \notin I_{(k-2)}$. Its $(k-2)$ dimensional neighbor is $x_{k-2} = 2^{k-2} - 2^{k-i-1} - 1$. One can see that $\max A < x_{k-2} < \min B$. Hence, $x_{k-2} \notin \bigcup_p I'_p \cup A \cup B \cup C$. If $x_{k-2} \in I_p$ for $1 \leq p \leq 2^{i-1} - 1$, then $2^{k-2} - 2^{k-i-1} - 1 = p2^{k-i-1} + j$, where $-1 \leq j \leq 2^{k-i-2} - 2$. But $p \geq 2^{i-1} - 1 + \frac{1}{2^{k-i-1}}$, a contradiction. Thus, $x_{k-2} \notin I_{(k-2)}$, and so u remains uninformed after time unit $(k-1)$. Now, we show that u 's d dimensional neighbors for $d < k - i - 1$ are uninformed after time unit $(k-1)$, and so u cannot be informed in time unit k . Suppose that $x_d = n - 2^{k-i-2} + 2^d - 1$, for $d < k - i - 1$, is d dimensional neighbor of u . Note that $x_d > \max I'_1$ and since $d < k - i - 1$, then $x_d < n$. Thus, $x_d \notin I_{(k-2)}$. We show that x_d remains uninformed after time unit $(k-1)$. The $(k-2)$ dimensional neighbor of x_d is $y_d = 2^{k-2} + 2^{k-i-2} - 2^d$. One can see that $\min B < y_d < \min C$. Since y_d is even, $y_d \notin I_{(k-2)}$. So, D is not valid.

sub-Case 2. $\alpha_2 \geq k - i - 1$. In this case, $\alpha_2 \neq k - 2$ because otherwise, all vertices are idle in the last time unit. Consider $u = 2^{k-2} = \min B + 1$. Hence $u \notin I_{(k-2)}$. Its d dimensional neighbor, for $d \geq k - i - 1$, is $x_d = n - 2^{k-2} + 2^d - 1$. Note that $x_d > \max C$. So $x_d \notin \bigcup_p I_p \cup A \cup B \cup C$. If $x_d \in I'_p$, for $1 \leq p \leq 2^{i-1} - 1$, then $n - 2^{k-2} + 2^d - 1 = n - p2^{k-i-1} + j$, where $1 \leq j \leq 2^{k-i-2}$. Thus,

$$2^{i-1} - 2^{d-k+i+1} + \frac{2}{2^{k-i-1}} \leq p \leq 2^{i-1} - 2^{d-k+i+1} + \frac{2}{2^{k-i-1}} + \frac{1}{2}.$$

If $d > k - i - 1$, then $2^{i-1} - 2^{d-k+i+1}$ is an integer, and so p is not an integer, a contradiction. So, suppose that $d = k - i - 1$. In this case, $2^{i-1} - 1 + \frac{2}{2^{k-i-1}} \leq p \leq 2^{i-1} - 1$, a contradiction. Thus, $x_d \notin I_{(k-2)}$, and so u remains uninformed after time unit $(k-1)$. Now, we show that u 's d dimensional neighbors for $d \geq k - i - 1$ are uninformed after time unit $(k-1)$, and so u cannot be informed in time unit k . The $(k-2)$ dimensional

neighbor of x_d is $y_d = 2^{k-1} - 2^d$. If $d = k - 2$, then $y_d = u \notin I_{(k-2)}$. One can see that $\max B < y_d < \min C$, for $k - i - 1 \leq d < k - 2$. Therefore, $y_d \notin I_{(k-2)}$. So, D is not valid. $\square$

It is important to note that the upper bound $n \leq 2^{k-1} + 2^{k-2} - 4$, in Corollary 5.2.2 is attained when our dimensional broadcast scheme involves 0 or 1 cyclic shift. This indicates that the standard dimensional broadcast scheme provides the most effective approach in terms of allowing a greater number of values of n for broadcasting in $W_{n,k-2}$. Therefore, this observation strengthens the conjecture that generalized Knödel graphs $W_{n,k-c}$, for $c \geq 3$, are unlikely to be broadcast graphs as mentioned in Remark 5.1.1.

As observed in all the previous theorems and their proofs, particularly the last one, the dimension $(k - 2)$ plays a crucial role and appears to be more complex than other dimensions. This complexity might be due to the fact that, at this dimension, although messages are transmitted only to neighbors, an informed vertex u may eventually reach and inform a vertex v with large $|u - v|$ through successive transmissions. It indicates that different dimensions possess various degrees of influence. If this is correct, then the dimension $(k - 1)$ has the greatest influence in the Knödel graph $KG(n)$, where degree is $k - 1$. Consequently, we conjecture that if, instead of the dimension $(k - 1)$, another dimension is removed to obtain a generalized Knödel graph of degree $k - 2$, the same number of edges will be eliminated, but the values of n for which a valid dimensional broadcast scheme with i cyclic shifts exists may increase.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

In Chapter 1, we introduce the concept of broadcasting in graphs and define key notions such as broadcast schemes, broadcast time of a graph, broadcast graphs, and minimum broadcast graphs. We also discuss the computational hardness associated with broadcasting problems. Furthermore, we present foundational results regarding the broadcast function, including its values for small integers and its monotonicity. Finally, we outline known upper and lower bounds on the broadcast function.

In Chapter 2, we investigate hypercubes as a prominent family of minimum broadcast graphs. We develop an algorithm that generates all valid minimum time broadcast schemes for hypercubes and provides a numerical enumeration of these schemes. Furthermore, we demonstrate that every broadcast scheme on a hypercube corresponds to a minimum distance scheme.

In Chapter 3, we determine all values of n (the number of vertices) for which the Knödel graph $KG(n)$ admits, as valid broadcast schemes, both standard and reversed dimensional broadcast schemes with $0 \leq i \leq k - 2$ cyclic shifts, including repetitions of an arbitrary

dimension. We begin our investigation by considering the repetition of the second dimension and then generalize our results to the case of any arbitrary dimension repeated in the last time unit. We further show that choosing the first dimension maximizes the number of values of n for which the dimensional broadcast scheme is valid. Additionally, we demonstrate that the most effective position for repeating the d -th dimension is in the last two positions of the dimensional scheme.

In Chapter 4, we present a new general upper bound on $B(n)$ for all odd n except when $n = 2^k - 1$ for an integer k . In order to achieve this bound, a vertex deletion method is applied to Knödel graphs on $n + 1$ vertices. To prove our result, we characterize all informed vertices in a Knödel graph after every time unit by applying the standard and reverse dimensional broadcast scheme with any cyclic shifts from an arbitrary originator. Furthermore, we completely resolve the open problem about the existence of minimum average time broadcast graphs.

In Chapter 5, we investigate various valid dimensional broadcast schemes for generalized Knödel graphs. Specifically, we determine all values of n (the number of vertices) for which a generalized Knödel graph $W_{n,\Delta}$ is a broadcast graph under the standard dimensional broadcast scheme with i cyclic shifts and arbitrary repetitions of the remaining dimensions.

6.2 Future Plan

All of the dimensional broadcast schemes investigated in this thesis are based on standard and reversed dimensional broadcast schemes combined with cyclic shifts. However, determining all valid broadcasting permutations in Knödel graphs remains a highly challenging open problem. One promising direction for further investigation is to consider jumps instead of shifts. Specifically, let $G = KG(n)$ be a Knödel graph on n vertices, with

$k = \lceil \log n \rceil$, and let j be an integer that is relatively prime to $k - 1$. Then, for which values of n is the dimensional broadcast scheme $(\alpha, \alpha + j, \alpha + 2j, \dots)$ (modulo k) a valid broadcast scheme? This generalization opens the door to new families of broadcast schemes with potentially distinct combinatorial and structural properties.

Another challenging direction of investigation is to identify all valid broadcast schemes that are not derived from dimensional broadcasting. This problem is significantly more difficult than the study of dimensional broadcast schemes, as it lacks the inherent structure provided by dimension-based transmission patterns. Unlike dimensional schemes, where the broadcast behavior can often be described through cyclic patterns, non-dimensional schemes may follow irregular patterns. Characterizing and classifying such schemes requires deeper combinatorial insights and possibly new techniques beyond current frameworks.

An interesting direction for future research is to explore the influence of different dimensions on the feasibility of broadcast schemes in Knödel and generalized Knödel graphs. Our results suggest that the standard (unshifted) dimensional broadcast scheme yields the largest range of valid values of n , particularly when the number of cyclic shifts is limited to zero or one. This observation supports the conjecture that generalized Knödel graphs $W_{n,k-c}$ are unlikely to be broadcast graphs for $c \geq 3$.

Moreover, our findings indicate that the last dimension plays a more critical role in the broadcast process. It would be valuable to formally analyze the relative influence of dimensions and investigate whether removing a dimension other than $(k - 1)$ in $KG(n)$ leads to generalized Knödel graphs that admit broadcast schemes for a broader range of n , despite having the same number of edges. Such insights could lead to a deeper understanding of structural properties that enable efficient broadcasting.

Finally, identifying a broadcast graph of order $2^k - 1$ for a positive integer k remains an intriguing open problem, as it would yield an upper bound for the broadcast function

$B(2^k - 1)$. One possible approach is to investigate deletion-based methods, where more than one vertex is removed from known broadcast graphs (such as Knödel graphs) in a controlled way to construct new broadcast graphs. This strategy may lead to improved upper bounds on the broadcast function for various values of n .

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