

Wavelets characterization of $h^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$

Mohamed El Aoumari

A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science (Mathematics) at

Concordia University

Montreal, Quebec, Canada

November 2025

©Mohamed El Aoumari, 2025

CONCORDIA UNIVERSITY
School of Graduate Studies

This is to certify that the thesis prepared

By: Mohamed El Aoumari

Entitled: Wavelets characterization of $h_1(R_n)$ and $bmo(R_n)$

and submitted in partial fulfillment of the requirements for the degree of

Master of Science

complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

_____ Chair
Dr. Pawel Gora

_____ Examiner
Dr. Robert Raphael

_____ Examiner

_____ Thesis Supervisor(s)
Dr. Galia Dafni

_____ Thesis Supervisor(s)

Approved by _____
Dr. Cody Hyndman Chair of Department or Graduate Program Director

Dr. Pascale Sicotte Dean of Faculty of Arts and Science

Abstract

Wavelets characterization of $h^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$

Mohamed El Aoumari

This dissertation presents a rigorous and comprehensive wavelet-based framework for the characterization of function spaces, focusing in depth on the local Hardy space $h^1(\mathbb{R}^n)$ as well as the non-homogeneous bounded mean oscillation space $\text{bmo}(\mathbb{R}^n)$. Building on the foundational work of Meyer, we develop a localized version of the theory of dyadic tent spaces, introducing a new space t_1 reflecting, for arbitrary L^1 functions, the behavior of their wavelet coefficients at small scales and marked by an abstraction of their large scale features. We prove that t_1 is a Banach space, establish its atomic decomposition theory, and construct a continuous embedding from Meyer's dyadic tent space T_1 , viewed as a quotient space, into t_1 .

Using this framework, we provide a complete wavelet-based characterization of $h^1(\mathbb{R}^n)$, including equivalent norms and necessary conditions for membership. At the same time, we revisit and expand upon Meyer's characterizations of $H^1(\mathbb{R}^n)$, offering detailed proofs and new insights. Furthermore, we extend Goldberg's results on $h^1(\mathbb{R}^n)$ to include the case of test functions which are bounded with compact support and normalized integral, beyond the classical normalized Schwartz class.

In the final chapter, we establish a complete wavelet-based characterization of $\text{bmo}(\mathbb{R}^n)$, the dual of $h^1(\mathbb{R}^n)$, including equivalent norms, and extend Goldberg's non-homogeneous bmo results to include explicit constructions involving bounded functions with compact support and normalized integral. To our knowledge, the use of the theory of tent spaces t_1 to achieve the wavelet characterizations of $h^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$, and the extensions of Meyer's and Goldberg's theorems have not appeared in the literature.

Throughout the different sections, we put emphasis on the regularity, localization, and cancellation properties inherent in wavelet systems, demonstrating their analytical power in both homogeneous and non-homogeneous settings.

Acknowledgments

This dissertation marks the completion of an immensely enriching learning process and I am delighted to express my appreciation to all the people who have provided their assistance during its course.

First of all, I must express my deepest thanks to Dr. Galia Dafni, my thesis supervisor, whose leadership, insight, and encouragement have been irreplaceable throughout this research project. The scholarly distinction and intellectual openness of Dr. Galia Dafni have influenced not only the direction of the current work but also the personal evolution of a trained engineer turned a “young” mathematician. I am particularly grateful for the time and energy devoted to the interactions, the excellence of the feedback given, and the prompting to look for both rigor and originality along the way.

I also thank Dr. Robert Raphael and Dr. Pavel Gora for accepting to be members of my examining committee. Their keen attention to my work, as well as their perceptive commentary, have been invaluable in enriching this thesis considerably and compelled me to think more widely and more deeply. I am thankful to have their participation and I am particularly grateful for their time and expertise.

My academic experience has been deeply affected by the lively community of students at Concordia University and I take the liberty to express appreciation to faculty members and staff of the Department of Mathematics and Statistics for cultivating an environment of curiosity, cooperation, and excellence. I take the liberty of also expressing appreciation to the administrative staff, whose often unseen labor makes the graduate academic experience efficient and supportive. I also extend appreciation to the general campus community for making the necessary facilities and resources available and the intellectual environment critical to the completion of this research work. Finally, I wish to extend great appreciation to my colleagues and fellow graduate students for their friendship, as well as the enriching discussions, seminars, and talks, and our shared passion for mathematics. The encouragement and camaraderie directed to me have made the entire journey not only intellectually enriching but also personally fulfilling.

Lastly, I want to dedicate my deepest appreciation to my family. Your unwavering faith in me, your patience during my long years of study and work, and your ever-firm encouragement have been your contributions to me. I want to sincerely thank my mother and my late father whose sacrifices and integrity continue to inspire me, and also to my loved ones and friends who encouraged me during both times of doubt and triumph. This achievement is clearly a testament to your sacrifices as it is to mine.

Contents

List of Figures	viii
List of Tables	ix
Nomenclature	x
1 Introduction	1
1.1 Motivation	1
1.2 Statement of originality	3
2 Wavelets	5
2.1 History and evolution	5
2.2 Multiresolution Analysis (MRA)	8
2.3 Orthonormal Wavelets	14
2.3.1 Orthonormal Wavelets in $\mathbb{R}$	14
2.3.2 Construction of Orthonormal Wavelets with compact support in $\mathbb{R}^n$ by Tensor Product	17
2.4 Variation on Dyadic cubes	24
3 Function Spaces of interest	29
3.1 The Hardy Space $H^1(\mathbb{R}^n)$	29
3.1.1 The maximal characterization of $H^1(\mathbb{R}^n)$	30

3.1.2	The atomic decomposition of $H^1(\mathbb{R}^n)$	33
3.1.3	Other characterizations of $H^1(\mathbb{R}^n)$	38
3.2	The Bounded Mean Oscillation Space $\text{BMO}(\mathbb{R}^n)$	40
3.3	The Local Hardy Space $h^1(\mathbb{R}^n)$	47
3.4	The non-homogeneous space $\text{bmo}(\mathbb{R}^n)$	53
4	Results	57
4.1	The Tent Space T_1 and its Local version t_1	57
4.2	Wavelets Characterization of the local Hardy space $h^1(\mathbb{R}^n)$	70
4.3	Wavelets Characterization of the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space	91
	Bibliography	107

List of Figures

4.1	Tent over $B(x, r)$: $\mathcal{T}(B(x, r)) = \{(y, t) \in \mathbb{R}^n \times [0, \infty) : B(y, t) \subseteq B(x, r)\}$	58
4.2	Tent over R , with dyadic layers: coarse at top, fine at bottom, labeled by scale $\ell(Q)$	59

List of Tables

2.1 Comparison of wavelet families and their trade-offs 18

Nomenclature

Function Spaces

$\mathbb{R}^n$	Euclidean space of dimension n .
$\mathcal{D}'(\mathbb{R}^n)$	Space of distributions (dual of $\mathcal{D}$).
$\mathcal{D}(\mathbb{R}^n)$	Space of smooth compactly supported test functions.
$\mathcal{S}'(\mathbb{R}^n)$	Tempered distributions (dual of $\mathcal{S}$).
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space of rapidly decreasing functions.
$\mathcal{S}^r(\mathbb{R}^n)$	Space of differentiable functions whose derivatives up to order r decay rapidly at infinity.
$\text{BMO}(\mathbb{R}^n)$	Space of functions of bounded mean oscillation.
$\text{bmo}(\mathbb{R}^n)$	The non-homogeneous space of functions of bounded mean oscillation.
$C^r(\mathbb{R}^n)$	Space of r -times continuously differentiable functions.
$C_c^\infty(\mathbb{R}^n)$	Smooth functions with compact support.
$H^1(\mathbb{R}^n)$	Real Hardy space.

$h^1(\mathbb{R}^n)$	Local Hardy space.
$H_a^1(\mathbb{R}^n)$	Atomic Hardy space.
$L^\infty(\mathbb{R}^n)$	Space of essentially bounded functions.
$L^p(\mathbb{R}^n)$	Lebesgue space of p -integrable functions.

Operators

Mf	Hardy–Littlewood maximal operator.
$R_j f$	The j -th Riesz transform of f .

Sets, Domains and functions

χ_E	Characteristic function of a set E .
$\text{supp}(f)$	Support of a function f .
$B(x, r)$	Ball centered at x of radius r .
Q	Cube in $\mathbb{R}^n$.

Symbols and Relations

$\gtrsim$	Reverse inequality up to a constant factor.
$\langle f, g \rangle$	Duality pairing between distributions and test functions.
$\lesssim$	Inequality up to a constant factor.

$\ f\ _p$	The L^p -norm of f .
p.v.	Cauchy principal value.
$\simeq$	Equivalence up to multiplicative constants.

Chapter 1

Introduction

1.1 Motivation

The theory of wavelets has emerged lately as a powerful framework or tool to treat the local, multi-scale structure of functions. Compared to classical Fourier analysis, which is mostly based on global frequency information, wavelet theory presents a localized basis capable, at the same time, of describing both spatial and frequency features. The dual localization property makes wavelet theory a prominent approach to treat function spaces, like the classical Lebesgue spaces, the real Hardy spaces, Besov spaces, or bounded mean oscillation space and its likes, as well as for the analysis of singular integral operators.

In the early 1990s, Yves Meyer in [44] pioneered the use of wavelet decompositions to characterize function spaces, providing simultaneously a geometric and scale sensitive alternative to classical atomic decomposition and maximal function approaches/characterizations.

Building on this foundation, mathematicians such as Bonami, Feuto, Grellier, and Ky (see, for instance [37], [4] and [5]) extended wavelet techniques to the study of Calderón–Zygmund operators, gaining insights into results such as new boundedness criteria and also structural findings in both homogeneous and non-homogeneous settings.

In this thesis, for the second chapter, we present the theory of wavelets, tracing its history, evolution, and its emergence from multi-resolution analysis (MRA). Then, we dive into the world of the construction of orthonormal bases of wavelets in $\mathbb{R}$, and after that in $\mathbb{R}^n$ for $n > 1$ through tensor product. For the third chapter, we introduce our four function spaces of interest, namely: the Hardy space $H^1(\mathbb{R}^n)$, the space of functions of bounded mean oscillation $\text{BMO}(\mathbb{R}^n)$, the local Hardy space $h^1(\mathbb{R}^n)$, and the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space, with their respective definitions and different characterizations along with the most interesting results and theorems with their proofs or proof sketches. And finally in the fourth and last chapter, we state and prove our results along the lines of Meyer's results, which present a rigorous exposition of wavelet-based characterizations of our four function spaces of interest. These results lead to several applications in singular integral theory. We emphasize the modularity, scale invariance, and cancellation properties inherent in wavelet systems, and we show how these features enable precise control over the wavelet coefficients of the above-mentioned spaces.

While both $H^1(\mathbb{R}^n)$ and $\text{BMO}(\mathbb{R}^n)$ are well-studied in the literature, we note that $h^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$ are, quite the opposite, under-studied because often mathematicians assume that results from the homogeneous spaces would naturally extend to their non-homogeneous counterparts, but this assumption seems to be far from being sound as results for the latter spaces have more subtleties than would have been previously anticipated.

We note that wavelet methods offer several advantages over classical techniques as summarized below:

- **Modularity:** Wavelet systems can be tailored to specific geometries (dyadic, etc...), measures (doubling measures, etc..), or boundary conditions.
- **Localization:** Operators can be analyzed locally, avoiding cumbersome global assumptions.
- **Numerical tractability:** Wavelet coefficients can be efficiently computed by exploiting the localization property of wavelets, enabling algorithmic applications.

- **Structural clarity:** Function space norms become more “palpable” in terms of scale and location.

These advantageous features make wavelets essential in modern harmonic analysis, particularly in settings where classical tools fail or become inefficient and often overcomplicated.

1.2 Statement of originality

The original contributions of this thesis can be summed up as follows:

- (1) We develop a theory of a local dyadic tent space t_1 similar to the dyadic tent space T_1 of Meyer (see [44]). We state and prove many results related to it such as proving it to be a Banach space in Theorem 4.1.1, provide a continuous embedding from T_1 , seen as a quotient space, into t_1 in Proposition 4.1.2 and a theorem that provides t_1 with a complete characterization mainly in terms of a suitable atomic decomposition that would later be used to prove some important results in Theorem 4.1.2.
- (2) We detail, in complete steps, proofs of Meyer’s results from [44, Theorem 2, p.p 145–148] on the T_1 space.
- (3) We provide a complete characterization of the local Hardy space $h^1(\mathbb{R}^n)$ in Theorem 4.2.1 in terms of wavelet-based decompositions using our newly defined local dyadic tent space t_1 . To the best of our knowledge, we have not found this set of results stated nor proved in the literature.
- (4) We provide equivalent norms to the h^1 norm based on these wavelet-based decompositions in Corollary 4.2.2. To the best of our knowledge, we have not found this result stated nor proved in the literature.
- (5) We provide necessary conditions for functions to belong to $h^1(\mathbb{R}^n)$ space in Corollary 4.2.1.

- (6) We provide more details on the proofs of Meyer's characterization of the Hardy space $H^1(\mathbb{R}^n)$ in terms of wavelet-based decompositions from [44, Theorem 1, p.p 143–149].
- (7) We provide a new result in part (G) of Theorem 4.2.1 extending Goldberg's result from [29, Lemma 4, p.p 34–35] to include the space of bounded functions with compact support with normalized integral over the whole domain besides the space of normalized Schwartz functions, through a novel way of proof. To the best of our knowledge, we have not found this result stated nor proved in the literature.
- (8) We provide a complete characterization of the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ in Theorem 4.3.1 in terms of wavelet-based decompositions. To the best of our knowledge, we have not found this set of results stated nor proved in the literature.
- (9) We provide a new result in part (B) of Theorem 4.3.1 extending Goldberg's result from [29, Corollary 4, p. 36] to include some bounded function with compact support, explicitly shown to be the normalized father wavelet, beside some normalized Schwartz function. To the best of our knowledge, we have not found this result stated nor proved in the literature.
- (10) We provide equivalent norms to the bmo norm based on these wavelet-based decompositions in Corollary 4.3.1. To the best of our knowledge, we have not found this result stated nor proved in the literature.

Chapter 2

Wavelets

2.1 History and evolution

Mathematicians had been struggling with the limitations of the classical tools for the analysis of functions long before the term ‘wavelet’ came into existence. The use of Fourier analysis, first introduced by Joseph Fourier in his 1822 treatise [27], which consists of the decomposition of functions into infinite sums of cosines and sines, was very common in the 19th century. That was revolutionary in its own right as Fourier realized that the heat equation, vibrations, and signals could all be expressed in terms of frequency.

However, Fourier analysis had one major drawback : the spectrum was global. One sine wave extends in both directions to negative and positive infinity. If we were to analyze a sudden spike in a signal, such as a seismogram tremor or a heartbeat anomaly, Fourier would analyze the data across the whole domain. It was lacking a crucial property, which is localization.

This limitation had been worrying applied physicists and mathematicians over decades. They needed a tool that could vary in scale, one that could efficiently cover both time/frequency and space location. The precursors to wavelet theory had been developed piecemeal out of frustration. In 1910, Alfréd Haar, in his paper [32], introduced what is now called the Haar

wavelet, as simple as it may seem : no more than a discontinuous function that alternates between two different values. His wavelet was used to construct an orthonormal basis of the space of square-integrable functions $L^2([0,1])$. Although Haar's approach was rudimentary, it had one significant advantage over Fourier analysis, which is the localization property in the form of compact support. Haar's wavelet was capable of detecting local variations. It was discontinuous and non-smooth, but it was the precursor to the developments that would follow. Haar's work rested quietly in the background for many decades, mostly used in numerical analysis and probability theory. Came the mid-20th century, a time when there was much activity going on, and that would much later be synthesized into wavelet theory, although no one realized it at the time. The fractal geometry emerged with Benoît Mandelbrot exploring scaling [43]. Multiresolution analysis had its advent in the area of signal processing, more particularly in what relates to the field of filters design [42]. Mathematicians specialized in functional analysis also were interested in the subject of bases and frames on Hilbert spaces [11], which, later on, became one of the foundations of wavelet decompositions. But these threads were not connected. The mathematicians who worked on orthonormal bases were not coordinating with the engineers working on filters. Physicists exploring the topic of quantum localization hadn't been aware of numerical analysts who developed efficient algorithms. But the ingredients began slowly to come together.

The 1980s went on to be a turning point. This is where the contributions of Ingrid Daubechies, a Belgian physicist, turned mathematician, come in bringing elegance and rigor to wavelet theory. Through her groundbreaking work presented in [21], She constructed the first family of compactly supported orthonormal wavelets that were also as regular as one wanted them to be, a feat that had eluded researchers for decades.

Daubechies' wavelets came as a revelation. They could handle signals with sudden transitions, yet remain sufficiently regular. They possessed the localization property. And they could be computed in a reasonably efficient manner.

Her brilliant research integrated the following disparate elements:

- Haar's discrete intuition
- Fourier's frequency domain
- Multiresolution analysis from engineering
- Functional analysis from pure mathematics

Wavelets completely transformed from being simple mathematical curiosities to being firmly entrenched with their own theory. After they became formally defined, their use flourished rapidly in many fields:

- Image compression: JPEG2000 standard [35] allows the compression of images using wavelets, thus avoiding the bulkiness of traditional methods.
- Signal denoising: wavelets are able to extract the noise from useful data in areas such as biomedical signaling, seismic readings, sound...
- Numerical PDEs: wavelets provide adaptive grids used in the numerical methods for solving partial differential equations efficiently.
- Quantum Mechanics: wavelets provide localized basis functions for solving Schrödinger equations.

At the same time, mathematicians continued to develop and work on the subject:

- Frame theory generalized orthonormal bases into redundant systems [11].
- The wavelet packets also provided the ability for more versatile analysis and decomposition [18].
- Continuous wavelet transforms filled the gap that existed between the discrete and the analog approaches [31].

Wavelets became a language that provided a jargon to simplify the talk about structure, scale, and transformation.

The most profound effect of wavelets, perhaps, can be perceived to be philosophical. On the one hand, Fourier analysis taught us to think in terms of frequency, but on the other hand wavelets made us to think in terms of scale. And this is a deep shift in perspective and view that mimics what happens in nature. Indeed, in nature, patterns occur at vastly different scales such as branch division in tree branching, rhythmic structures of speech, turbulence in fluids, and wavelets strike at the right place as they mirror this reality. They don't just analyze, they adapt.

Wavelets are not purely mathematical concepts either but rather tools that let us perceive complexity and simplify it. Wavelet theory continues to improve in many directions and researchers now explore subjects such as:

- Curvelets and shearlets for directional sensitivity [10, 38]
- Graph wavelets for network data [47]
- Deep wavelets, i.e, wavelet transformation integrated with neural network schemes [8]

2.2 Multiresolution Analysis (MRA)

On our way to delving deep into the world of wavelets, we come across, first of all, the very important topic of multiresolution analysis that we will discuss in this section. At its core, multiresolution analysis is a sort of procedure used for the construction of function spaces that describe the behavior of the information at different scales. It may also be viewed as a mathematical microscope or tool that allows us to observe the function at low resolution and then keep on sharpening the vision.

In the context of $\mathbb{R}^n$, MRA becomes a framework providing the means for decomposing functions into elementary components that reside on various spatial scales and locations. It's not just

about the wavelets themselves but it's about organizing the entire space of square-integrable functions $L^2(\mathbb{R}^n)$, in our context, into nested layers of detail. Now, let's provide the formal definition of an MRA on $\mathbb{R}^n$ as presented by Meyer in [44, Definition 1, p. 21].

Definition 2.2.1. *A multiresolution analysis (MRA) in $\mathbb{R}^n$ is a sequence of closed subspaces $\{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^n)$ satisfying the five following properties:*

- (i) $V_j \subset V_{j+1}$ for any $j \in \mathbb{Z}$
- (ii) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$
- (iii) $f(x) \in V_j \iff f(2x) \in V_{j+1}, \forall f \in L^2(\mathbb{R}^n)$ and $\forall j \in \mathbb{Z}$
- (iv) $f(x) \in V_0 \iff f(x-k) \in V_0, \forall f \in L^2(\mathbb{R}^n)$ and $\forall k \in \mathbb{Z}^n$
- (v) $\exists \phi \in L^2(\mathbb{R}^n)$, called the scaling function, such that the family $\{\phi_k(x) = \phi(x-k)\}_{k \in \mathbb{Z}^n}$ is an orthonormal basis for V_0 .

A multiresolution analysis $\{V_j\}$ is said to be r -regular, for some $r \in \mathbb{N} \cup \{0\}$, if the function ϕ satisfies : $|D^\alpha \phi(x)| \leq C_l (1+|x|)^{-l}$, for every $l \in \mathbb{N} \cup \{0\}$ and every multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ s.t: $|\alpha| = \alpha_1 + \dots + \alpha_n \leq r$.

Each subspace V_j is called the approximation space at scale 2^{-j} .

For any $j \in \mathbb{Z}$, we let W_j denote the orthogonal complement of V_j in V_{j+1} , that is, $V_{j+1} = V_j \oplus W_j$, and we call it the detail subspace at scale 2^{-j} .

The intuition for each of the properties listed above may now be summarized as follows:

- (i) The strictly increasing nature of the sequence $\{V_j\}$ corresponds to the observation that while V_j detects detail at resolution 2^{-j} , V_{j+1} detects detail at higher resolution.
- (ii) The density and triviality constraints guarantee that no information gets lost or repeated on different scales.
- (iii) The scaling invariance corresponds to the fact that zooming in means going to a higher resolution.

- (iv) The translation invariance property guarantees the uniformity of the resolution over the whole space.
- (v) The scaling function ϕ may also be viewed as the seed for the resolution levels. It is the “Adam” of the wavelets. Later on we will also see that it is called the “Father Wavelet,” from which we derive the “Mother Wavelet” ψ that we can perceive as the “Eve” of the wavelets.

Remark 2.2.1. *Built-in Definition 2.2.1, for an r -regular MRA, we can see that the scaling function ϕ belongs to $S^r(\mathbb{R}^n)$, the space of r -times differentiable functions with rapid decay for derivatives up to order r , which is a proper subspace of $C^r(\mathbb{R}^n)$.*

Example 2.2.1. *We give here the most elementary example based on the Haar scaling function on $\mathbb{R}$, defined as:*

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

This is the characteristic function of the interval $[0, 1)$. It is simple, compactly supported, and piecewise continuous. It will be the basis for the construction of the approximation spaces V_j .

For each integer $j \in \mathbb{Z}$, define: $V_j = \text{span}\{\phi_{j,k}(x) = 2^{\frac{j}{2}} \phi(2^j x - k) \mid k \in \mathbb{Z}\}$.

By the above definition of the V_j ’s, we see that the translates of ϕ form a basis of V_0 and the scaling and translation invariances are also covered. The sequence is also increasing. Indeed: if $j, k \in \mathbb{Z}$, then: $\phi_{j,k} \in V_j$ and for any $x \in \mathbb{R}$, we have:

$$\begin{aligned} \phi_{j,k}(x) &= 2^{\frac{j}{2}} \phi(2^j x - k) \\ &= 2^{\frac{j}{2}} X_{[k2^{-j}, (k+1)2^{-j})}(x) \\ &= 2^{\frac{j}{2}} (X_{[2k2^{-j-1}, (2k+1)2^{-j-1})}(x) + X_{[(2k+1)2^{-j-1}, 2(k+1)2^{-j-1})}(x)) \\ &= \frac{1}{\sqrt{2}} (\phi_{j+1, 2k+1}(x) + \phi_{j+1, 2k}(x)) \in V_{j+1} \end{aligned}$$

, exploiting the fact that each dyadic interval in $\mathbb{R}$ of a given scale can be split up into two dyadic intervals of the upper scale.

Now for the trivial intersection, if $f \in \bigcap_{j \in \mathbb{Z}} V_j$, then for every j , f is a piecewise constant function on dyadic intervals of length 2^{-j} . As $j \rightarrow -\infty$, the intervals become arbitrarily large and eventually covering any two different points in the real line, thus forcing global constancy of f . But $f \in L^2(\mathbb{R})$, so it must be that f is zero almost everywhere.

For the density part, it follows immediately from the classical result of the density of step functions in $L^2(\mathbb{R})$.

Other examples that use the same procedure to construct their MRAs, i.e., generating them by dilation and translation of their scaling functions ϕ , include Daubechies wavelets and Battle–Lemarié wavelets (details can be found in [3], [40] and [39]). However, examples such as the Alpert Multiwavelets [1], use multiple scaling functions for their MRA construction, and are designed to reproduce polynomials and achieve high-order accuracy. It is especially influential in numerical analysis and PDE solvers.

The theory of multiresolution analysis (MRA) has evolved through several foundational contributions (see the historical overviews in [42, 23, 44, 12, 53]):

- **Yves Meyer** was the first to construct an orthonormal wavelet basis of $L^2(\mathbb{R})$ using smooth functions with infinite support in the early 1980s in a series of conference proceedings and internal reports, not formally published as a journal article. His earliest most cited paper on the subject was [45]. His work laid the analytical foundation for wavelet theory [44].
- **Stéphane Mallat** introduced in [41] the formal framework of multiresolution analysis (MRA), showing how nested subspaces and scaling functions lead to wavelet bases. His work connected wavelets to signal processing and filter banks, and his monograph [42] provides a comprehensive historical account.
- **Ingrid Daubechies**, in her groundbreaking paper [21], constructed the first family of

compactly supported orthonormal wavelets with arbitrary regularity r . Her wavelets satisfy refinement equations and are central to both theory and applications. A historical perspective is given in [23].

- **Robert Strichartz**, in his paper [51], extended MRA theory to $\mathbb{R}^n$ using self-affine tilings and expansive dilation matrices, constructing r -regular wavelets in higher dimensions. This development is also surveyed in [53].
- **Marcin Bownik**, in [7], removed the tiling assumption and proved the existence of r -regular MRAs in $\mathbb{R}^n$ for arbitrary expansive dilations, completing the general theory.

These contributions collectively established the existence and construction of smooth, localized wavelet bases in both one-dimensional and multi-dimensional cases.

One may refer to [44, Chapter 2] for a concrete example of the construction of a scaling function $\phi \in V_0$ that is C^∞ but not compactly supported.

Now, we will state some of the properties that follow from the definition of an MRA.

Proposition 2.2.1. *For an MRA $\{V_j\}_{j \in \mathbb{Z}}$, and its corresponding detail subspaces $\{W_j\}_{j \in \mathbb{Z}}$, we have the following properties:*

- (1) $V_j = \text{span}\{\phi_{j,k}(x) = 2^{\frac{nj}{2}} \phi(2^j x - k) \mid k \in \mathbb{Z}^n\}$ and $\{\phi_{j,k}\}_{k \in \mathbb{Z}^n}$ forms an orthonormal basis for V_j
- (2) $\bigoplus_{-\infty}^{+\infty} W_j = L^2(\mathbb{R}^n)$
- (3) $f(x) \in W_0 \iff f(2^j x) \in W_j$, for any $j \in \mathbb{Z}$

Proof. Part (1) can be proved simply by noticing that by Definition 2.2.1:

$$V_j = \{f \in L^2(\mathbb{R}^n) : f(x) = g(2^j x) \text{ for some } g \in V_0, \text{ for any } x \in \mathbb{R}^n\}$$

So, the generating property, linear independence, and orthonormality of $\{\phi_{j,k}\}$ stem from that of $\{\phi_k\}$. Part (2) can also be proved by exploiting the density of the union of the V_j 's in $L^2(\mathbb{R}^n)$

and their trivial intersection. Finally, part (3) is an immediate consequence of the scaling invariance property of the V_j 's. $\square$

Now, we can define notions of projections on those subspaces that would allow us to establish many useful identities.

Definition 2.2.2. *For any $j \in \mathbb{Z}$, we define E_j and D_j as the orthogonal projection of $L^2(\mathbb{R}^n)$ onto V_j and W_j , respectively.*

We now give some important properties related to these projections.

Proposition 2.2.2. *We have the following properties:*

- (1) *For any $j \in \mathbb{Z}$, we have: $E_{j+1} = E_j + D_j$ and $D_j = E_{j+1} - E_j$*
- (2) *For any $f \in L^2(\mathbb{R}^n)$, we have: $f = \lim_{j \rightarrow \infty} E_j f = \sum_{j=-\infty}^{\infty} D_j f$ in $L^2(\mathbb{R}^n)$.*

Proof. Part (1) follows directly from the Definition 2.2.2 of the two orthogonal projections E_j and D_j and the Definition 2.2.1 of V_j and W_j .

For the first half of part (2), since $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$, for any $\epsilon > 0$, there exists $g \in V_J$ for some $J \in \mathbb{Z}$ st: $\|f - g\|_{L^2} < \epsilon$. But $V_J \subset V_j$ for all $j \geq J$, and E_j is the orthogonal projection onto V_j , using the fact that orthogonal projections minimize distance to the given subspace, we have: $\|f - E_j f\|_{L^2} \leq \|f - g\|_{L^2} < \epsilon$ for all $j \geq J$. Then, it follows that: $\lim_{j \rightarrow \infty} \|f - E_j f\|_{L^2} = 0$ or $E_j f \rightarrow f$ in $L^2(\mathbb{R}^n)$.

For the second half of part (2), let $j \in \mathbb{Z}$ and let $J > j$. Using part (1) and iterating, we get: $E_J = E_j + \sum_{k=j}^{J-1} D_k$. Letting $J \rightarrow \infty$ and using $\lim_{J \rightarrow \infty} E_J f = f$ in $L^2(\mathbb{R}^n)$, we obtain: $f = E_j f + \sum_{k=j}^{\infty} D_k f$.

Now, letting $j \rightarrow -\infty$ and since $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, we have: $\lim_{j \rightarrow -\infty} E_j f = 0$ in $L^2(\mathbb{R}^n)$. hence:

$$f = \sum_{k=-\infty}^{\infty} D_k f \text{ in } L^2(\mathbb{R}^n). \quad \square$$

Remark 2.2.2. *A key takeaway from the preceding proof is that:*

$$\text{For any } f \in L^2(\mathbb{R}^n), \text{ for any } j \in \mathbb{Z}: f = E_j f + \sum_{k=j}^{\infty} D_k f \quad (2.1)$$

This will be useful in what is to come.

2.3 Orthonormal Wavelets

2.3.1 Orthonormal Wavelets in $\mathbb{R}$

So far, we have been using the term wavelet loosely but without defining it concretely. Now, let us tackle this task. But first let's define the notion of dyadic intervals.

Definition 2.3.1. *An interval I on $\mathbb{R}$ is called dyadic if it is in the form:*

$$I_{j,k} = [k 2^{-j}, (k+1) 2^{-j})$$

for some indices $j, k \in \mathbb{Z}$ with j determining the scale and k the location. We denote by $\mathcal{I}$ the collection of all dyadic intervals on $\mathbb{R}$, and for any $i \in \mathbb{Z}$, we denote:

$$\mathcal{I}_i := \{I \in \mathcal{I} : \ell(I) \leq 2^{-i}\} = \{I_{j,k} : (j,k) \in (\mathbb{Z} \cap [i, \infty)) \times \mathbb{Z}\}$$

Where $\ell(I)$ is the length of the interval I .

Remark 2.3.1. *$\mathcal{I}$ is countable, so we may index any sequence by intervals $I \in \mathcal{I}$ interchangeably with their corresponding indices $j, k \in \mathbb{Z}$.*

Another observation is that the dyadic intervals are defined as half-open intervals to make sure that for any fixed scale j , they form a partition of $\mathbb{R}$.

Now, we proceed to define a wavelet.

Definition 2.3.2. *Let $r \in \mathbb{N} \cup \{0\}$. A wavelet of class r (in dimension 1), is a function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ that verifies the following four properties:*

(i) *Regularity: For any $0 \leq i \leq r$, we have: $\psi^{(i)}$ exists and is in $L^\infty(\mathbb{R})$;*

(ii) *Localization:* For any $0 \leq i \leq r$, we have: $\psi^{(i)}$ has rapid decay as $x \rightarrow \pm\infty$ i.e: for any $N \geq 1$, we have: $\left| \left(\frac{d}{dx} \right)^i \psi(x) \right| \leq C_N (1 + |x|)^{-N}$ for some constant $C_N > 0$;

(iii) *Cancellation:* For any $0 \leq i \leq r$, we have: $\int_{\mathbb{R}} x^i \psi(x) dx = 0$;

(iv) $\{\psi_I\}_{I \in \mathcal{I}} := \{\psi_{j,k} = 2^{\frac{j}{2}} \psi(2^j x - k)\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

ψ is called the mother wavelet and the $\psi_I = 2^{\frac{j}{2}} \psi(2^j x - k)$, for $I \in \mathcal{I}$ and their corresponding indices $j, k \in \mathbb{Z}$ are the wavelets generated by ψ .

Remark 2.3.2. By a change of variable, the generated wavelets also satisfy conditions (i) through (iii) with the localization property given as follows:

$$\left| \left(\frac{d}{dx} \right)^i \psi_I(x) \right| \leq C_N 2^{j(i+\frac{1}{2})} (1 + |2^j x - k|)^{-N}$$

for any $N \geq 1$ and for any $0 \leq i \leq r$, for some $C_N > 0$.

Now, based on property (iv) in Definition 2.3.2, we can see that for any $f \in L^2(\mathbb{R})$, we have:

$$f = \sum_{I \in \mathcal{I}} \alpha(I) \psi_I \text{ where: } \alpha(I) = (f, \psi_I) \quad (2.2)$$

the question whether (2.2) extends to other $L^p(\mathbb{R})$ spaces for $1 \leq p \leq \infty$ will be discussed later on.

Now, as we defined formally the notion of a wavelet, the question of the existence of a mother wavelet that is r -regular for any non-negative integer r should be addressed. And as we hinted to in the previous section, the construction of such a wavelet arises from an r -regular MRA and its scaling function ϕ that we will call from now on the father wavelet. So we follow Meyer's construction in [44, Theorem 1, p. 72].

Theorem 2.3.1. $\exists$ a function ψ in the detail subspace W_0 satisfying:

(i) For any $0 \leq i \leq r$ and for any $N \geq 1$: $\left| \left(\frac{d}{dx} \right)^i \psi(x) \right| \leq C_N (1 + |x|)^{-N}$ for some constant $C_N > 0$;

(ii) $\{\psi(x-k)\}_{k \in \mathbb{Z}}$ forms an orthonormal basis of the detail subspace W_0 .

Proof. See [44, proof of Theorem 1, p.p 71–73]. □

From the proof above by Meyer ([44, proof of Theorem 1, p.p 71–73]), the mother wavelet ψ is constructed from the father wavelet (scaling function) ϕ , but most importantly, it inherits the same regularity in terms of its order of smoothness and also the same order of decay.

Now, an important result following from this theorem and previous results on the projections E_j and D_j will be stated in the next proposition.

Proposition 2.3.1. *Let $f \in L^2(\mathbb{R})$. If ϕ and ψ are the father and mother wavelets, respectively, with $\phi_{j,k}(x) = \phi(2^j x - k)$, $\beta_{j,k} = 2^j (f, \phi_{j,k})$ and $\alpha(I) = (f, \psi_I)$, then for any $j \in \mathbb{Z}$ we have:*

$$f = \sum_{k \in \mathbb{Z}} \beta_{j,k} \phi_{j,k} + \sum_{I \in \mathcal{I}_j} \alpha(I) \psi_I \quad (2.3)$$

Proof. The proof follows directly from Theorem 2.3.1 and equation (2.1). □

Remark 2.3.3. *First, we remark that we can obtain Formula (2.2) from Formula (2.3) by letting $j \rightarrow -\infty$.*

By setting $\phi_k = \phi_{0,k}$ and $\beta_k = \beta_{0,k}$, we also get the interesting equation:

$$f = \sum_{k \in \mathbb{Z}} \beta_k \phi_k + \sum_{I \in \mathcal{I}_0} \alpha(I) \psi_I \quad (2.4)$$

Finally, if $p \geq 1$, and the interval pI is defined to be the interval with the same center as the interval I and length $\ell(pI) = p\ell(I)$, the localization property of the wavelets tells us that:

$$\left(\int_{(pI)^c} |\psi_I|^2 \right)^{\frac{1}{2}} \text{ decays rapidly to } 0 \text{ as } p \rightarrow \infty$$

The following result is attributed to I. Daubechies in her paper [21]. It establishes the existence of wavelets with compact support, however its modern formulation appears explicitly in [22].

Theorem 2.3.2. *For any $r \in \mathbb{N} \cup \{0\}$, there exists an r -regular MRA $\{V_j\}_{j \in \mathbb{Z}}$ of $L^2(\mathbb{R})$ for which its scaling function (father wavelet) ϕ and its corresponding mother wavelet ψ have compact support.*

Proof. See [21] for the original proof or Meyer's proof in [44, p.p 97–107] relying on Mallat's program. $\square$

Remark 2.3.4. *Daubechies shows that $\exists c, C > 0$ s.t: $\text{supp}(\phi) \subseteq [-Cr, Cr]$, $\text{supp}(\psi) \subseteq [-Cr, Cr]$ and $\min(\text{diam}(\text{supp}(\phi)), \text{diam}(\text{supp}(\psi))) \geq cr$. So, the dependence of the compact support on the regularity r is linearly optimized.*

Also, Meyer shows in [44, p. 97] that for sufficiently large $m \geq 1$, we can choose ϕ and ψ so that their supports are contained in the interval $m[0, 1] = [\frac{1}{2} - \frac{m}{2}, \frac{1}{2} + \frac{m}{2}]$, this is due in fact to the orthogonality and some construction conditions that require more degrees of freedom as the compact support in only one dyadic interval would be too restrictive to encode the necessary oscillations for the wavelets, so enlarging that by a factor of m ensures proper frequency tiling, overlap control and other approximation properties and achieves a better interaction between the wavelets and the scaling function and its translates.

Finally, we list some of the wavelet constructions and their properties in the following table 2.3.1.

2.3.2 Construction of Orthonormal Wavelets with compact support in $\mathbb{R}^n$ by Tensor Product

Now, we jump into the construction of wavelets, whose definition remains the same as Definition 2.3.2, just extending it to the $\mathbb{R}^n$ setting.

We define the notion of dyadic cubes in this context and some other associated notions.

Definition 2.3.3. *A cube Q on $\mathbb{R}^n$ is called dyadic if it is in the form:*

$$Q_{j,k} = \{x \in \mathbb{R}^n : 2^j x - k \in [0, 1]^n\}$$

Wavelet Family	Compact Support	Regularity	Vanishing Moments	Orthonormal	Construction Domain
Haar	Yes	Piecewise continuous	No	Yes	Time
Daubechies	Yes	C^r	Yes	Yes	Time
Symlets	Yes	C^r	Yes	Yes	Time
Coiflets	Yes	C^r	Yes	Yes	Time
Meyer	No	C^∞	Yes	Yes	Frequency
Morlet	No	C^∞	No	No	Time
Shannon	No	Discontinuous	Yes	Yes	Frequency
Battle–Lemarié	No	C^∞	Yes	Yes	Time

Table 2.1: Comparison of wavelet families and their trade-offs

for some indices $(j, k) \in \mathbb{Z} \times \mathbb{Z}^n$ determining the scale and the location of the cubes, respectively.

We denote:

$$Q_k = Q_{0,k} = \{x \in \mathbb{R}^n : x - k \in [0, 1)^n\}$$

We denote by $\mathcal{Q}$ the collection of all dyadic cubes on $\mathbb{R}^n$, and for any $i \in \mathbb{Z}$, we denote:

$$\mathcal{Q}_i := \{Q \in \mathcal{Q} : \ell(Q) \leq 2^{-i}\} = \{Q_{j,k} : (j, k) \in (\mathbb{Z} \cap [i, \infty)) \times \mathbb{Z}^n\}$$

where $\ell(Q)$ is the side-length of the cube Q . We denote the direction set of cardinality $2^n - 1$ by:

$$E := \{0, 1\}^n \setminus \{0_{\mathbb{R}^n}\}$$

and the full index set of all wavelet translates across all scales and directions by:

$$\Lambda := \{\lambda = \frac{k}{2^j} + \frac{e}{2^{j+1}} : k \in \mathbb{Z}^n, j \in \mathbb{Z}, e \in E\}$$

And for any $i \in \mathbb{Z}$:

$$\Lambda^i := \{\lambda \in \Lambda : j \in \mathbb{Z} \cap [i, \infty)\}$$

We will follow Meyer's construction in [44]. Our starting point is the existence of an orthonormal wavelet basis of $L^2(\mathbb{R})$ of compact support as established in the previous subsection. Assume that ϕ is a father wavelet for the one-dimensional case with corresponding MRA $\{V_j\}$, associated with the mother wavelet ψ , both having compact support $m[0, 1]$ for some $m \geq 1$, and both with regularity r . This makes the family of generated wavelets given by $\{\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k) : j, k \in \mathbb{Z}\}$ into an orthonormal basis of $L^2(\mathbb{R})$.

To extend this to the multi-dimensional case $L^2(\mathbb{R}^n)$, we use tensor products of the one-dimensional constructions. The key idea serving as the guiding principle of such construction is that the Hilbert space $L^2(\mathbb{R}^n)$ is naturally isomorphic to the tensor product of n copies of $L^2(\mathbb{R})$. Indeed:

Lemma 2.3.1. *There exists a canonical unitary isomorphism:*

$$L^2(\mathbb{R})^{\otimes n} \cong L^2(\mathbb{R}^n),$$

given on simple tensors by:

$$f_1 \otimes \cdots \otimes f_n \longmapsto \left[(x_1, \dots, x_n) \mapsto f_1(x_1) \cdots f_n(x_n) \right].$$

Proof. Let $H = L^2(\mathbb{R})$. The algebraic tensor product $H^{\otimes n}$ consists of finite linear combinations of simple tensors:

$$f_1 \otimes \cdots \otimes f_n, \quad f_i \in H.$$

Define a map T on simple tensors by:

$$T(f_1 \otimes \cdots \otimes f_n)(x_1, \dots, x_n) := f_1(x_1) \cdots f_n(x_n),$$

and extend T linearly to finite sums.

For simple tensors, we can compute:

$$\begin{aligned}
\langle T(f_1 \otimes \cdots \otimes f_n), T(g_1 \otimes \cdots \otimes g_n) \rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \prod_{i=1}^n f_i(x_i) \overline{g_i(x_i)} dx \\
&= \prod_{i=1}^n \int_{\mathbb{R}} f_i(x) \overline{g_i(x)} dx \\
&= \prod_{i=1}^n \langle f_i, g_i \rangle_{L^2(\mathbb{R})},
\end{aligned}$$

which is exactly the defining inner product of the Hilbert tensor product. Thus T is an isometry on the algebraic tensor product.

But the space of finite linear combinations of separable functions: $(x_1, \dots, x_n) \mapsto f_{1,k}(x_1) \cdots f_{n,k}(x_n)$ is dense in $L^2(\mathbb{R}^n)$. This follows from the density of step functions in L^2 and the fact that step functions factor into products of one-dimensional step functions.

Since the image of the algebraic tensor product under T is dense in $L^2(\mathbb{R}^n)$ and T is an isometry, it extends uniquely to a unitary isomorphism:

$$T : L^2(\mathbb{R})^{\otimes n} \longrightarrow L^2(\mathbb{R}^n).$$

□

This identification justifies the construction of multidimensional wavelets by taking the tensor products of one-dimensional wavelets. As such, we start by defining the scaling function in this context.

Definition 2.3.4. *We define the n -dimensional scaling function (father wavelet) by:*

$$\phi^{((n))}(x_1, \dots, x_n) = \phi(x_1) \cdots \phi(x_n)$$

and its integer translates by:

$$\phi_k^{((n))}(x) = \phi^{((n))}(x - k)$$

Also:

$$\phi_{j,k}^{((n))}(x) = 2^{\frac{jn}{2}} \phi^{((n))}(2^j x - k)$$

Proposition 2.3.2. *The $\phi_k^{((n))}$'s have compact support in mQ_k ; moreover, they span the base approximation space $V_0^{((n))}$.*

Proof. It follows from Definition 2.3.4, Remark 2.3.4 and the fact that $V_0^{((n))}$ is the tensor product of the one-dimensional V_0 spaces by Lemma 2.3.1. $\square$

Corollary 2.3.1. *For any $j \in \mathbb{Z}$, the collection $\phi_{j,k}^{((n))}$ spans the approximation space $V_j^{((n))}$.*

Proof. It follows from Proposition 2.3.2 by scaling. $\square$

Now, for each coordinate direction, we can replace ϕ by ψ . Thus, in n -dimensions, there are $2^n - 1$ (excluding the scaling function itself) distinct wavelet types, each corresponding to non-trivial choices of ϕ or ψ in each coordinate.

Example 2.3.1. *For example, in two dimensions, we define the following:*

- *Horizontal wavelet:* $\psi(x_1)\phi(x_2)$.
- *Vertical wavelet:* $\phi(x_1)\psi(x_2)$.
- *Diagonal wavelet:* $\psi(x_1)\psi(x_2)$.

Now, we generalize to the multi-dimensional case.

Definition 2.3.5. *In n dimensions, we define, for any $e = (e_1, \dots, e_n) \in E$, the base wavelets:*

$$\psi^e(x) = \prod_{i=1}^n \phi_i(x_i)$$

Where:

$$\phi_i = \begin{cases} \phi & \text{if } e_i = 0 \\ \psi & \text{if } e_i = 1 \end{cases}$$

We list one of the most important results regarding the wavelets generated by these base wavelets.

Theorem 2.3.3. *The collection:*

$$\{\psi_{j,k}^e(x) = 2^{\frac{jn}{2}} \psi^e(2^j x - k) : e \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^n)$.

Proof. Orthonormality follows from the tensor product orthonormality in each coordinate direction, and completeness is ensured by the density of the tensor product spaces and the dyadic refinement property. $\square$

Remark 2.3.5. *The above generated wavelets can be denoted, if it's more convenient, as:*

$$\psi_{j,k}^e = \psi_\lambda = \psi_Q^e$$

Where: $\lambda = \frac{k}{2^j} + \frac{e}{2^{j+1}} \in \Lambda$ and $Q = Q_{j,k} \in \mathcal{Q}$.

Now, we list some of the properties of these wavelets.

Proposition 2.3.3. *For any $e \in E, j \in \mathbb{Z}, k \in \mathbb{Z}^n$, we have:*

- (i) *Localization:* $\psi_{j,k}^e$ is supported in the cube $mQ_{j,k}$, which has the same center as the dyadic cube $Q_{j,k}$ and side-length $\ell(mQ_{j,k}) = m\ell(Q_{j,k})$.
- (ii) *Regularity:* $D^\alpha \psi_{j,k}^e$ exists and $|D^\alpha \psi_{j,k}^e| \leq C 2^{j(|\alpha| + \frac{n}{2})}$, for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ of order $|\alpha| \leq r$, for some $C > 0$.
- (iii) *Cancellation:* $\int_{\mathbb{R}^n} x^\alpha \psi_{j,k}^e(x) dx = 0$, whenever $|\alpha| \leq r$.

Proof. The three properties follow from their counter-part in one-dimension in each coordinate direction by tensor product. $\square$

Now, we define the notion of a distribution of order less than r .

Definition 2.3.6. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is said to be of order less than r if for every compact $K \subseteq \mathbb{R}^n$, there exists a constant $C_K > 0$ such that:

$$|\langle T, \varphi \rangle| \leq C_K \max_{|\alpha| \leq r} \sup_{x \in K} |D^\alpha \varphi(x)|, \quad \forall \varphi \in C_c^\infty(K).$$

Example 2.3.2. A locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$ defines a distribution of order 0, since for any compact $K \subseteq \mathbb{R}^n$, we have:

$$|\langle f, \varphi \rangle| \leq \int_K |f| |\varphi| \leq C_K \sup_{x \in K} |\varphi(x)|, \quad \forall \varphi \in C_c^\infty(K).$$

So this extends to any sub-space of $L^1_{loc}(\mathbb{R}^n)$, as follows:

- if it controls basic integrability: $L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$, or
- if it controls oscillation such as BMO, BV (functions of bounded variation), or
- it controls derivatives in some integrable sense such as: Sobolev, Besov, Triebel-Lizorkin, or
- if it refines integrability such as: Lorentz, Orlicz.

Remark 2.3.6. From now on, we drop the superscript $((n))$ from the notation of the father wavelet if from the context we know we are in the n dimensions setting for some $n \geq 1$.

Now, an important theorem that encompasses the one-dimensional and the multi-dimensional cases is given in what follows.

Theorem 2.3.4. For any distribution f of order less than r , we have, in the sense of distributions:

$$f = \sum_{k \in \mathbb{Z}^n} (f, \phi_k) \phi_k + \sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} (f, \psi_Q^e) \psi_Q^e$$

Proof. See [44, p.p 108–109]. □

Remark 2.3.7. *In what follows, we adopt the notation $\beta_k = (f, \phi_k)$ and $\alpha(\lambda) = (f, \psi_Q^e) = (f, \psi_\lambda)$, for any $k \in \mathbb{Z}^n$ and any $\lambda \in \Lambda$.*

Corollary 2.3.2. *For any distribution f of order less than r , we have, in the sense of distributions:*

$$f = \sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda$$

Proof. It follows from Theorem 2.3.4 by scaling and taking $j \rightarrow -\infty$. □

2.4 Varia on Dyadic cubes

Recall that we defined the dyadic cubes $Q_{j,k}$ for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ as follows:

$$x \in Q_{j,k} \iff 2^j x - k \in [0, 1)^n$$

A cube was chosen to be the cross product of half-open intervals so that for each fixed scale j , the $Q_{j,k}$'s would be pairwise disjoint and thus constitute a partition of $\mathbb{R}^n$.

Each cube is centered at the point $2^{-j}(k + \frac{1}{2} \mathbf{1}_{\mathbb{R}^n})$ (where $\mathbf{1}_{\mathbb{R}^n} = (1, 1, \dots, 1)$) and is of side-length $\ell(Q_{j,k}) = 2^{-j}$.

We defined $\mathcal{Q}$ as the collection of all dyadic cubes in $\mathbb{R}^n$, and $\mathcal{Q}_i = \{Q_{j,k} \in \mathcal{Q} : j \geq i\}$, the collection of all dyadic cubes of side-length smaller than or equal to 2^{-i} . In the following proposition we list, without proof, some of the properties of dyadic cubes.

Proposition 2.4.1. *Let $j, j' \in \mathbb{Z}$ and $k, k' \in \mathbb{Z}^n$. We have the following:*

- (i) *If $Q_{j,k} \cap Q_{j',k'} \neq \emptyset$, then: either $Q_{j,k} \subseteq Q_{j',k'}$ or $Q_{j',k'} \subseteq Q_{j,k}$.*
- (ii) *$\forall j \in \mathbb{Z}, \forall k \in \mathbb{Z}^n, \exists! k' \in \mathbb{Z}^n$ such that $Q_{j,k} \subset Q_{j-1,k'}$.*
- (iii) *If $j' > j$, then each $Q_{j,k}$ is the disjoint union of $2^{n(j'-j)}$ cubes of scale j' .*

Now, we define boundedness in the context of dyadic cubes.

Definition 2.4.1. *A non-empty sub-collection $\mathcal{C} \subset \mathcal{Q}$ is said to be bounded above if there exists $j_0 \in \mathbb{Z}$ s.t:*

$$\sup_{Q \in \mathcal{C}} \ell(Q) \leq 2^{-j_0}.$$

Or equivalently, all cubes in $\mathcal{C}$ have side-length at most 2^{-j_0} .

We then define the notion of maximality in this context.

Definition 2.4.2. *For a non-empty sub-collection $\mathcal{C} \subset \mathcal{Q}$, a cube $Q \in \mathcal{C}$ is called maximal (with respect to inclusion) if there is no $\tilde{Q} \in \mathcal{C}$ such that $Q \subsetneq \tilde{Q}$.*

Now, we state and prove an important theorem related to the disjoint maximal decomposition in the context of bounded sub-collections of dyadic cubes.

Theorem 2.4.1. *Let $\mathcal{C}$ be a bounded non-empty sub-collection of dyadic cubes. Then:*

- (i) The set $\mathcal{M}$ of maximal cubes in $\mathcal{C}$ is pairwise disjoint.*
- (ii) Every cube $Q \in \mathcal{C}$ is contained in some $M \in \mathcal{M}$.*
- (iii) Consequently,*

$$\mathcal{C} = \bigcup_{M \in \mathcal{M}} \{Q \in \mathcal{C} : Q \subseteq M\},$$

i.e. the maximal cubes partition the sub-collection into disjoint fibers.

Proof. For part (i), suppose that $M_1, M_2 \in \mathcal{M}$ intersect. By dyadic structure, either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$. Maximality forbids strict inclusion, hence $M_1 = M_2$. Thus, distinct maximal cubes are disjoint.

For part (ii), for any $Q \in \mathcal{C}$, consider the chain of dyadic parents within $\mathcal{C}$. Since side-lengths are bounded above by 2^{-j_0} for some $j_0 \in \mathbb{Z}$, this chain must terminate at or before the scale j_0 . Therefore, Q is consequently contained in that maximal parent which, by dyadic structure, is shown to be a maximal cube.

Part (iii) follows immediately as each cube belongs to exactly one maximal cube, because distinct maximal cubes are disjoint. $\square$

Now, we need to define the notion of summability of a series indexed by the collection of dyadic cubes or one of its non-empty sub-collections. We start first by defining the notion of an exhaustion.

Definition 2.4.3. A sequence $\{F_m\}_{m \in \mathbb{N}}$ of sub-collections of $\mathcal{Q}$ is said to be an exhaustion of $\mathcal{Q}$ if:

- (i) For any $m \in \mathbb{N}$, F_m is finite.
- (ii) For any $m \in \mathbb{N}$, we have: $F_m \subseteq F_{m+1}$.

$$(iii) \bigcup_{m \in \mathbb{N}} F_m = \mathcal{Q}.$$

Example 2.4.1. A very simple example of such exhaustion is the sequence defined by:

$$F_m = \{Q_{j,k} \in \mathcal{Q} : -m \leq j \leq m, |k| \leq m\}, \text{ for any } m \in \mathbb{N}$$

F_m is finite, the sequence is increasing and its image covers $\mathcal{Q}$ as for any $Q_{j,k} \in \mathcal{Q}$, we have: $Q_{j,k} \in F_m$ where $m = \max(|j|, |k|) + 1$.

Now, we are in a position to define the desired notion of summability.

Definition 2.4.4. Let $\{\alpha(Q)\}_{Q \in \mathcal{Q}}$ be a sequence of non-negative scalars indexed by the collection of all dyadic cubes. We say that the series $\sum_{Q \in \mathcal{Q}} \alpha(Q)$ is summable or converges, and write:

$$\sum_{Q \in \mathcal{Q}} \alpha(Q) < \infty, \text{ if } \sup_{\substack{F \subseteq \mathcal{Q} \\ F \text{ finite}}} \sum_{Q \in F} \alpha(Q) < \infty \text{ exists.}$$

Remark 2.4.1. A sequence $\{\alpha(Q)\}_{Q \in \mathcal{Q}}$ of non-negative scalars indexed by the collection of all dyadic cubes converges in the sense of Definition 2.4.4 $\iff \sup_{m \in \mathbb{N}} \sum_{Q \in F_m} \alpha(Q) = \lim_{m \rightarrow \infty} \sum_{Q \in F_m} \alpha(Q) < \infty$ exists, for some exhaustion $\{F_m\}_{m \in \mathbb{N}}$ of $\mathcal{Q}$ and $\sum_{Q \in \mathcal{Q}} \alpha(Q) = \sup_{\substack{F \subseteq \mathcal{Q} \\ F \text{ finite}}} \sum_{Q \in F} \alpha(Q) = \sup_{m \in \mathbb{N}} \sum_{Q \in F_m} \alpha(Q) =$

$$\lim_{m \rightarrow \infty} \sum_{Q \in F_m} \alpha(Q).$$

In the basic theory of series of non-negative scalars, monotonicity of the partial sums simplifies the analysis, and convergence amounts to the boundedness of these partial sums. We now extend the discussion to arbitrary real scalars.

Definition 2.4.5. Let $\{\alpha(Q)\}_{Q \in \mathcal{Q}}$ be a sequence of real scalars indexed by the collection of all dyadic cubes. We define:

$$\alpha(Q)^+ = \max\{\alpha(Q), 0\}, \quad \alpha(Q)^- = \max\{-\alpha(Q), 0\}.$$

Then $\alpha(Q) = \alpha(Q)^+ - \alpha(Q)^-$ and $|\alpha(Q)| = \alpha(Q)^+ + \alpha(Q)^-$.

Proposition 2.4.2. For any real series $\sum_{Q \in \mathcal{Q}} \alpha(Q)$, we have:

$$\sum_{Q \in \mathcal{Q}} \alpha(Q) = \sum_{Q \in \mathcal{Q}} \alpha(Q)^+ - \sum_{Q \in \mathcal{Q}} \alpha(Q)^-,$$

whenever at least one of the two series on the right converges in the sense of Definition 2.4.4.

We define the notion of absolute convergence or absolute summability.

Definition 2.4.6. A series $\sum_{Q \in \mathcal{Q}} \alpha(Q)$ is said to converge absolutely if:

$$\sum_{Q \in \mathcal{Q}} |\alpha(Q)| < \infty.$$

An expected yet important result follows.

Proposition 2.4.3. If $\sum_{Q \in \mathcal{Q}} |\alpha(Q)|$ converges, then $\sum_{Q \in \mathcal{Q}} \alpha(Q)$ converges, and:

$$\left| \sum_{Q \in \mathcal{Q}} \alpha(Q) \right| \leq \sum_{Q \in \mathcal{Q}} |\alpha(Q)|.$$

We now define the notion of conditional convergence.

Definition 2.4.7. *If $\sum_{Q \in \mathcal{Q}} \alpha(Q)$ converges but $\sum_{Q \in \mathcal{Q}} |\alpha(Q)|$ diverges, then the series is said to converge conditionally. In this case, rearrangements of the terms may alter the sum.*

For non-negative series, convergence is governed by boundedness of partial sums, but for general real series, one decomposes them into positive and negative parts. On one hand, absolute convergence guarantees stability, while conditional convergence introduces subtleties such as dependence on rearrangements.

Chapter 3

Function Spaces of interest

3.1 The Hardy Space $H^1(\mathbb{R}^n)$

The classical Lebesgue function spaces $L^p(\mathbb{R}^n)$ (for $1 \leq p \leq \infty$) offer a robust framework for generalized integration based on measure theory and are central to modern analysis as complete normed function spaces, i.e, Banach spaces, that allow for orthogonality (in the case of $p = 2$), projections, and many other desirable properties. The duality between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ (for $1 < p, q < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$) is also crucial when it comes to operator and distribution theories, partial differential equations, and variational methods. They also interpolate in a natural way, since if an operator (sublinear weak-type(1,1) or strong-type(∞, ∞)) is bounded on $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$, it is bounded on all intermediate $L^p(\mathbb{R}^n)$. Their applications across mathematics are palpable from Fourier analysis to probability theory, but extend also to physics in areas such as quantum mechanics, signal processing, and image analysis.

Nevertheless, they present some limitations when $p \leq 1$ or $p = \infty$. In particular, $L^1(\mathbb{R}^n)$ lacks certain harmonic analytic properties, such as boundedness of singular integral operators. Such difficulties were overcome with the introduction of Hardy spaces $H^p(\mathbb{R}^n)$ instead of $L^p(\mathbb{R}^n)$, mainly in the range $0 < p \leq 1$, preserving many desirable analytic features.

The concept of Hardy spaces originated in complex analysis in the early 20th century, primarily through the work of Frigyes Riesz (in [46]) and named after G.H. Hardy, based on Hardy's basic, yet foundational work (in [33]). Through his study of the boundary behavior of holomorphic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$, Riesz was able to define $H^p(\mathbb{D})$ as the space consisting of the holomorphic functions whose p -th integral means on circles centered at the origin remain uniformly bounded as the radius approaches 1. Thus, through his work, the foundation for modern harmonic analysis was laid.

In the late 1960s and early 1970s, the concept was generalized to real-variable settings by E. Stein and G. Weiss (in [50]). Fefferman and Stein in their foundational paper [26], liberated real hardy spaces from the complex boundary values framework and introduced maximal function, atomic, and singular integral characterizations. By this transition in the real context, Hardy spaces became a natural alternative for $L^p(\mathbb{R}^n)$ spaces in harmonic analysis, especially for $p \leq 1$, where classical tools such as Calderón–Zygmund theory fail. C. Fefferman's duality result ([25]), showing that the dual of $H^1(\mathbb{R}^n)$ is $\text{BMO}(\mathbb{R}^n)$, was pivotal in terms of connecting Hardy spaces to the study of singular integrals (see [49]) and partial differential equations.

3.1.1 The maximal characterization of $H^1(\mathbb{R}^n)$

Before formally introducing the space $H^1(\mathbb{R})$, we properly define some notions that would help us in this task, and we follow Stein's notation (see [48]).

Definition 3.1.1. *We denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of all tempered distributions with the weak*-topology, i.e., the space of all linear functionals on the Fréchet space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ that are bounded in the topology defined by the countable family of semi-norms $\|\cdot\|_{\alpha,\beta}$ given by:*

$$\|\phi\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| \text{ for any } \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and any multi-indices } \alpha, \beta \in (\mathbb{N} \cup \{0\})^n, \text{ meaning}$$
that if T is such a functional:

$\exists$ *finitely many semi-norms $\|\cdot\|_{\alpha,\beta}$ st: $\forall \phi \in \mathcal{S}(\mathbb{R}^n) : |T(\phi)| \leq C \sum_{j=1}^N \|\phi\|_{\alpha_j, \beta_j}$ for some constant $C > 0$.*

The weak*-topology associated to it is the topology induced by the dual pairing with the Schwartz functions. That is: a net $T_\lambda \in \mathcal{S}'(\mathbb{R}^n)$ converges to $T \in \mathcal{S}'(\mathbb{R}^n) \iff \lim_\lambda \langle T_\lambda, \phi \rangle = \langle T, \phi \rangle$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

If f is a tempered distribution, we say that f is bounded if $f * \phi \in L^\infty(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Remark 3.1.1. If f is a tempered distribution and ϕ is a Schwartz function, then $f * \phi$ is a well-defined $C^\infty(\mathbb{R}^n)$ function. See [34, Section 2.3, pp. 66–68].

Throughout this section, if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then ϕ_t will denote the function defined as:

$$\phi_t(x) = t^{-n} \phi(t^{-1}x), \text{ for } x \in \mathbb{R}^n, t > 0.$$

This family of Schwartz functions can be thought of as an approximation to the identity.

We also consider the Poisson kernel:

$$P(x) = \frac{c_n}{(1 + |x|^2)^{\frac{(n+1)}{2}}} \text{ with } c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{(n+1)}{2}}} \text{ chosen so that: } \int_{\mathbb{R}^n} P = 1,$$

and define P_t as:

$$P_t(x) = t^{-n} P(t^{-1}x), \text{ for } x \in \mathbb{R}^n, t > 0.$$

Furthermore, if $\mathcal{F}$ is a finite collection of semi-norms $\|\cdot\|_{\alpha,\beta}$ on $\mathcal{S}(\mathbb{R}^n)$, we let:

$$\mathcal{S}_{\mathcal{F}} = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \|\phi\|_{\alpha,\beta} \leq 1, \forall \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}.$$

Definition 3.1.2. If ϕ is a Schwartz function, f is a tempered distribution, and $\mathcal{F}$ is a finite collection of semi-norms on the space of Schwartz functions, we call the maximal function of f (based on ϕ), the function defined by:

$$M_\phi f(x) := \sup_{t>0} |(f * \phi_t)(x)|$$

We also call the grand maximal function of f (based on $\mathcal{F}$), the function defined by:

$$M_{\mathcal{F}}f(x) := \sup_{\phi \in \mathcal{S}_{\mathcal{F}}} M_{\phi}f(x)$$

If, moreover, f is bounded, we call the non-tangential maximal function of the Poisson integral of f , the function defined by:

$$u^*(x) := \sup_{|x-y| \leq t} |u(y, t)|$$

where the Poisson integral of f is the function defined by:

$$u(x, t) := (f * P_t)(x).$$

Now, if $f \in L^1_{Loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function is defined by:

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy.$$

After defining the necessary machinery and before providing the maximal characterization of $H^1(\mathbb{R}^n)$, we state an important theorem ([26]) that would help us in this endeavor and refer to one of its detailed proofs.

Theorem 3.1.1. *If f is a tempered distribution, then T.F.A.E:*

- (i) $\exists \phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi \neq 0$ s.t. $M_{\phi}f \in L^1(\mathbb{R}^n)$.
- (ii) $\exists \mathcal{F}$ a finite collection of semi-norms on the space of Schwartz functions s.t. $M_{\mathcal{F}}f \in L^1(\mathbb{R}^n)$.
- (iii) f is bounded and $u^* \in L^1(\mathbb{R}^n)$.

Proof. See [48, proof of Theorem 1, pp. 92-93] in the case $p = 1$. □

Definition 3.1.3. *We define the real Hardy space $H^1(\mathbb{R}^n)$ as the space of tempered distributions for which one of the equivalent conditions of Theorem 3.1.1 holds. For $f \in H^1(\mathbb{R}^n)$, we also define: $\|f\|_{H^1} := \|M_{\phi}f\|_1$.*

Proposition 3.1.1. $(H^1(\mathbb{R}^n), \|\cdot\|_{H^1})$ is a normed vector space.

Proof. It follows directly from the sub-additivity of the maximal function and the definition of the norm. $\square$

Remark 3.1.2. The proof of Theorem 3.1.1 yields several important observations, which we can summarize as follows. First: we can choose the finite collection of semi-norms $\mathcal{F}$ from condition (ii) not to depend on f but only on the dimension n , so that for sufficiently large $m \in \mathbb{N}$, $\mathcal{F} = \mathcal{F}_m = \{\|\cdot\|_{\alpha,\beta} : |\alpha|, |\beta| \leq m\}$, which also means that we can choose ϕ from (i) to be independent of f and supported in the unit ball. The second takeaway is that for sufficiently large m , we have (not depending on f): $\|M_\phi f\|_1 \simeq \|M_{\mathcal{F}} f\|_1 \simeq \|u^*\|_1$, so we may choose any of the mentioned expressions to be a norm on $H^1(\mathbb{R}^n)$.

Another remark is that Uchiyama showed in [52] that if f is a distribution and $\phi \in C_c^\infty$ with $\int \phi \neq 0$ s.t. $M_\phi f \in L^1(\mathbb{R}^n)$, then f can be extended to a tempered distribution with $f \in H^1(\mathbb{R}^n)$.

From now on and throughout this section, we are fixing some choice of ϕ and $\mathcal{F}$. So if we talk about the operators M_ϕ and $M_{\mathcal{F}}$, it is going to be for the above fixed choices.

Before ending this sub-section, we state an important result that would be used in the next one.

Lemma 3.1.1. the maximal function (M_ϕ) is pointwise dominated by the Hardy–Littlewood maximal function everywhere. Moreover, the latter is a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. see [30, proof of Theorem 2.1.4, pp. 60–72]. $\square$

3.1.2 The atomic decomposition of $H^1(\mathbb{R}^n)$

Now, we introduce the important notion of atoms that constitute a sort of building block for the $H^1(\mathbb{R}^n)$ space.

Definition 3.1.4. An $H^1(\mathbb{R}^n)$ -atom in the $L^q(\mathbb{R}^n)$ sense, for $1 < q \leq \infty$, is a function a satisfying the following three properties :

(i) compact support: $\text{supp}(a) \subseteq B$ for some ball B .

(ii) size condition: $\|a\|_q \leq |B|^{1/q-1}$ (for the ball B chosen in (i)).

(iii) cancellation: $\int_{\mathbb{R}^n} a = 0$.

Remark 3.1.3. The standard approach uses $L^\infty(\mathbb{R}^n)$ atoms (see [13] for $n = 1$ and [17] for $n > 1$), but $L^q(\mathbb{R}^n)$ atoms, as an extension proposed in the work of Coifman and Weiss in [17], provide more flexibility in interpolation theory, as well as for the study of singular integral operators that are bounded on $L^q(\mathbb{R}^n)$ but not necessarily on $L^2(\mathbb{R}^n)$. In their paper, they demonstrate that the atomic Hardy space given by $L^q(\mathbb{R}^n)$ atoms remains norm-equivalent to the classical $H^1(\mathbb{R}^n)$, independently of the choice of q . It should be noted that if a is an L^∞ , it is an L^q atom for any $1 < q < \infty$.

Another observation is that we can replace balls with cubes in the definition of atoms and still arrive at an equivalent atomic decomposition as can be seen proved much later on in Theorem 4.2.1.

In what follows, if we invoke H^1 -atoms it will be in the L^2 sense unless stated otherwise explicitly. Following the formal definition, we derive some important properties of these atoms.

Proposition 3.1.2. If a is an $H^1(\mathbb{R}^n)$ -atom, then we have:

(a) $\|a\|_1 \leq 1$.

(b) $\|a\|_{H^1} = \|M_\phi a\|_1 \lesssim 1$.

(c) $a \in H^1(\mathbb{R}^n)$.

Proof. Part (a) follows immediately from the size condition in Definition 3.1.4 and the Cauchy-Schwarz Inequality.

For part (b), let $B = B(x_0, r)$ be the ball that contains the support of a . Let $\tilde{B} = B(x_0, 2r)$.

Then: $\int_{\mathbb{R}^n} M_\phi a = \int_{\tilde{B}} M_\phi a + \int_{\mathbb{R}^n \setminus \tilde{B}} M_\phi a$.

For the first term, by the Cauchy-Schwarz Inequality, Lemma 3.1.1 and the size condition in

Definition 3.1.4, we have:

$$\int_{\tilde{B}} M_{\phi} a \leq |\tilde{B}|^{1/2} \|M_{\phi} a\|_2 \lesssim |B|^{1/2} \|Ma\|_2 \lesssim |B|^{1/2} \|a\|_2 \leq 1.$$

For the second term, by the cancellation property of the atom a , for a fixed $t > 0$ and for $x \notin \tilde{B}$, we have:

$$|a * \phi_t(x)| = \left| \int_B a(y) [\phi_t(x-y) - \phi_t(x-x_0)] dy \right| \leq \int_B |a(y)| |\phi_t(x-y) - \phi_t(x-x_0)| dy.$$

by the smoothness of ϕ_t and its rapid decay, using Taylor expansion estimate, we get for some $N \gg n$ sufficiently large:

$$|\phi_t(x-y) - \phi_t(x-x_0)| \lesssim \frac{|y-x_0|}{t^{n+1}} \frac{1}{(1 + \frac{|x-x_0|}{t})^N}.$$

But by part (a), we have: $\int_B |a(y)| |y-x_0| dy \leq r \|a\|_1 \leq r$. Hence: $|a * \phi_t(x)| \lesssim \frac{r}{t^{n+1}} \frac{1}{(1 + \frac{\rho}{t})^N}$ with $\rho := |x-x_0|$.

As $N > n+1$, we get:

$$M_{\phi} a(x) \leq r \sup_{t>0} \frac{1}{t^{n+1}} \frac{1}{(1 + \frac{\rho}{t})^N} = r \sup_{u>0} \frac{u^{n+1}}{\rho^{n+1}} \frac{1}{(1+u)^N} \lesssim \frac{r}{\rho^{n+1}}.$$

Using spherical coordinates, we get:

$$\int_{\mathbb{R}^n \setminus \tilde{B}} M_{\phi} a \lesssim \int_{\rho>2r} \frac{r}{\rho^{n+1}} dx = r \int_{2r}^{\infty} \frac{1}{\rho^{n+1}} \rho^{n-1} d\rho \lesssim 1.$$

Now part (c) follows directly from part (b) and Definition 3.1.3. □

Now, we state, without proof, the central result of this sub-section, which is the Atomic Decomposition Theorem for the real Hardy space $H^1(\mathbb{R}^n)$.

Theorem 3.1.2. $f \in H^1(\mathbb{R}^n) \iff f = \sum_k \lambda_k a_k$ where $\{a_k\}_k$ is a sequence of $H^1(\mathbb{R}^n)$ -atoms

and $\{\lambda_k\}_k$ is a sequence of scalars s.t: $\sum_k |\lambda_k| < \infty$.

Moreover $\|f\|_{H^1} \simeq \|f\|_{At} := \inf \left\{ \sum_k |\lambda_k| < \infty : f = \sum_k \lambda_k a_k, a_k \text{'s are } H^1\text{-atoms} \right\}$.

Using the atomic decomposition and the equivalence of the norm it induces to the established H^1 norm, we can state and directly prove a number of results.

Corollary 3.1.1. H_a^1 , the subspace of finite linear combinations of H^1 -atoms, $\mathcal{S}_{(0)}(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \int x^\alpha \varphi(x) dx = 0, \forall \alpha \in (\mathbb{N} \cup \{0\})^n\}$, and $L_{c,(0)}^\infty := \{\varphi \in L_c^\infty : \int x^\alpha \varphi(x) dx = 0, \forall |\alpha| < N\}$ (for any $N \in \mathbb{N}$), are dense in $H^1(\mathbb{R}^n)$.

Proof. the result for H_a^1 follows directly from the atomic decomposition Theorem 3.1.2. The proof for $\mathcal{S}_{(0)}(\mathbb{R}^n)$ relies on approximating each H^1 -atom by its convolution with a suitable mollifier belonging to $\mathcal{S}_{(0)}(\mathbb{R}^n)$ itself, since the convolution preserves vanishing moments and smoothness. Finally, the proof for $L_{c,(0)}^\infty$ follows the same strategy but now choosing a suitable mollifier belonging to $C_{c,(0)}^\infty := \{\varphi \in C_c^\infty : \int x^\alpha \varphi(x) dx = 0, \forall |\alpha| < N\}$ instead. $\square$

Corollary 3.1.2. $H^1(\mathbb{R}^n) \subsetneq L_0^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \int f = 0\} \subsetneq L^1(\mathbb{R}^n)$ and $\|\cdot\|_1 \lesssim \|\cdot\|_{H^1}$.

Proof. If $\sum_k \lambda_k a_k$ is an atomic decomposition of f , then by the Monotone Convergence Theorem and part (a) of Proposition 3.1.2, we get:

$$\|f\|_1 \leq \sum_k |\lambda_k| \|a_k\|_1 \leq \sum_k |\lambda_k| < \infty$$

So $f \in L^1(\mathbb{R}^n)$, and by the same token taking the infimum over all atomic decomposition, we would have: $\|f\|_1 \leq \|f\|_{At} \simeq \|f\|_{H^1}$.

The series $\sum_k \lambda_k a_k$ then also converges in $L^1(\mathbb{R}^n)$, so integrating term by term and using the cancellation property of the H^1 -atoms, we get: $\int f = \sum_k \lambda_k \int a_k = 0$.

To conclude the proof, for $n = 1$, consider the function:

$$f(x) = \frac{\chi_{[-\frac{1}{2}, 0) \cup (0, \frac{1}{2}]}(x)}{x (\log|x|)^2}$$

A simple verification yields: $f \in L_0^1(\mathbb{R})$, but $f \notin H^1(\mathbb{R})$ as for any $\phi \in \mathcal{S}(\mathbb{R})$, in a neighborhood of 0, we have: $M_\phi f(x) \geq \frac{c}{x \log|x|}$ for some $c > 0$, so $M_\phi f \notin L^1(\mathbb{R})$. Generalize this counter-example to the n dimensional case by defining:

$$f(x) = \frac{\chi_{B(0, \frac{1}{2}) \setminus \{x_1=0\}}(x)}{x_1 (\log|x_1|)^2}$$

Concluding the proof. □

Now, one of the main results of the atomic decomposition is that we can naturally prove the completeness of $H^1(\mathbb{R}^n)$ as a normed vector space.

Proposition 3.1.3. *$(H^1(\mathbb{R}^n), \|\cdot\|_{H^1})$ is a Banach space.*

Proof. As established in Proposition 3.1.1, $(H^1(\mathbb{R}^n), \|\cdot\|_{H^1})$ is a normed vector space, so it suffices to prove it is complete and we will so following the approach in the argument following [24, Theorem 4.3, pp. 102–104].

Let $\{f_i\}_{i \in \mathbb{N}}$ be a Cauchy sequence of elements of $H^1(\mathbb{R}^n)$. By equivalence of norms established in the atomic decomposition Theorem 3.1.2, we can find a subsequence $\{f_{i_j}\}$ s.t: $\forall j \in \mathbb{N}$, we have: $\|f_{i_{j+1}} - f_{i_j}\|_{At} < \frac{1}{2^j}$.

Now, by the domination of the L^1 norm by the H^1 norm proved in Corollary 3.1.2, that subsequence converges in $L^1(\mathbb{R}^n)$ (and pointwise a.e) to a function f .

For any $j \in \mathbb{N} \cup \{0\}$, let $\sum_{k=1}^{\infty} \lambda_k^{i_j} a_k^{i_j}$ be an atomic decomposition of $f_{i_{j+1}} - f_{i_j}$ with $\sum_{k=1}^{\infty} |\lambda_k^{i_j}| < \frac{1}{2^j}$ and $f_{i_0} = 0$. Therefore, for any $m \in \mathbb{N}$, we have:

$$f_{i_m} = \sum_{j=0}^{m-1} (f_{i_{j+1}} - f_{i_j}) = \sum_{j=0}^{m-1} \sum_{k=1}^{\infty} \lambda_k^{i_j} a_k^{i_j}$$

As $\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} |\lambda_k^{i_j}| \leq \sum_{j=0}^{\infty} \frac{1}{2^j} < \infty$, letting $m \rightarrow \infty$, we get: $f = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \lambda_k^{i_j} a_k^{i_j}$ in $L^1(\mathbb{R}^n)$. We also deduce that $f \in H^1(\mathbb{R}^n)$.

It remains to show that the convergence occurs in the H^1 norm. We have that:

$$\|f - f_{i_m}\|_{At} = \left\| \sum_{j=m}^{\infty} \sum_{k=1}^{\infty} \lambda_k^{i_j} a_k^{i_j} \right\|_{At} \leq \sum_{j=m}^{\infty} \sum_{k=1}^{\infty} |\lambda_k^{i_j}| \leq \sum_{j=m}^{\infty} \frac{1}{2^j} = 2^{1-m} \rightarrow 0 \text{ as } m \rightarrow \infty$$

Since $\{f_i\}$ is Cauchy in H^1 with a sub-sequence converging to f in H^1 , this means that $\lim f_i = f$ in H^1 as well. $\square$

3.1.3 Other characterizations of $H^1(\mathbb{R}^n)$

Now, we present another useful characterization of the real hardy space $H^1(\mathbb{R}^n)$, which is the Riesz Transform characterization, which we would be referring to in our Results Chapter 4.

Definition 3.1.5. For $1 \leq j \leq n$, the Riesz transform R_j is defined by:

$$R_j f(x) := \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy.$$

Remark 3.1.4. The Riesz transform is well-defined for Schwartz class functions because the kernel is smooth away from the origin and the cancellation makes the principal value converge. To extend this definition to general L^p spaces using the standard Calderón–Zygmund extension mechanism.

Theorem 3.1.3. A function $f \in L^1(\mathbb{R}^n)$ belongs to $H^1(\mathbb{R}^n)$ if and only if each $R_j f \in L^1(\mathbb{R}^n)$. Moreover, $\|f\|_{H^1} \simeq \|f\|_1 + \sum_{j=1}^n \|R_j f\|_1$.

Proof. See [48, proof of Proposition 3, pp. 123–124] for the case $p = 1$. $\square$

Several other characterizations of $H^1(\mathbb{R}^n)$ exist, famously, the characterization by the square function (see [48, pp. 124–127]) and the characterization in terms of systems of conjugate harmonic functions (see [48, pp. 120–123]) which was the original one for defining the real Hardy space. But one of the important characterizations and the one most related to the subject of

this thesis remains the wavelets characterization proved by Meyer in [44, Theorem 1, p. 143] that we present in the next theorem. But first let's present some notations: we denote:

$$E = \{0, 1\}^n \setminus \{0_{\mathbb{R}^n}\},$$

$$\Lambda = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{Z}, e \in E\}$$

and:

$$\Lambda^{j_0} = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{Z}, j \geq j_0, e \in E\} \text{ for any } j_0 \in \mathbb{Z}.$$

So, in particular:

$$\Lambda^0 = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{N} \cup \{0\}, e \in E\}.$$

For $f \in L^1(\mathbb{R}^n)$, we denote $\alpha(\lambda) = (f, \psi_\lambda)$ and $\beta(k) = (f, \phi_k)$, where $\psi_\lambda, \lambda \in \Lambda$ and $\phi_k, k \in \mathbb{Z}^n$, are derived from the mother and father wavelets, respectively, as shown in the wavelet construction section where we introduced them.

Theorem 3.1.4 (Meyer). *For $f \in L^1(\mathbb{R}^n)$, T.F.A.E definitions of $f \in H^1(\mathbb{R}^n)$:*

- (A) $\sup_{\substack{F \subseteq \Lambda \\ \text{finite}}} \sup_{\varepsilon(\lambda) = \pm 1} \left\| \left(\sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right) \right\|_1 < \infty$
- (B) $\left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$
- (C) $\left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$
- (D) $\left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$ (where $R(\lambda) \subseteq Q(\lambda)$ and $|R(\lambda)| \geq \gamma |Q(\lambda)|$ for some constant $\gamma > 0$ for any $\lambda \in \Lambda$)
- (E) $f = \sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda$ has an atomic decomposition.

Proof. See [44, proof of Theorem 1, pp. 143–149] and details of the proof of Theorem 4.2.1 in the next chapter. □

Corollary 3.1.3. *In the notation of Theorem 3.1.4, for $f \in H^1(\mathbb{R}^n)$, the following norms are equivalent:*

$$\begin{aligned} \|f\|_{At} &\simeq \sup_{\substack{F \subseteq \Lambda \\ \text{finite}}} \sup_{\varepsilon(\lambda)=\pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda \right\|_1 \simeq \left\| \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{1/2} \right\|_1 \\ &\simeq \left\| \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)} \right)^{1/2} \right\|_1 \simeq \left\| \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)} \right)^{1/2} \right\|_1 \end{aligned}$$

with: $\|f\|_{At} = \inf \left\{ \sum_j |\lambda_j| < \infty : f = \sum_j \lambda_j a_j, a_j \text{'s are } H^1\text{-atoms} \right\}.$

Proof. See [44, proof of Theorem 1, pp. 143–149]. □

The last thing to discuss about $H^1(\mathbb{R}^n)$ is its duality with $\text{BMO}(\mathbb{R}^n)$, but this discussion will be left until the next section, where we first introduce the $\text{BMO}(\mathbb{R}^n)$ space.

3.2 The Bounded Mean Oscillation Space $\text{BMO}(\mathbb{R}^n)$

The space $\text{BMO}(\mathbb{R}^n)$, or briefly BMO which is short for “bounded mean oscillation”, was first developed by F. John & L. Nirenberg in 1961 in their paper [36] not as a pure mathematical curiosity, but because it answered a geometric question associated with understanding the “average deviation” of a function with respect to its local means without the need for pointwise control. Unlike $L^\infty(\mathbb{R}^n)$, which requires uniform boundedness, $\text{BMO}(\mathbb{R}^n)$ can tolerate local “wild” behavior as long as it is averaged out. This can be made precise with the following definition.

Definition 3.2.1. *For $f \in L^1_{loc}(\mathbb{R}^n)$, we say that $f \in \text{BMO}(\mathbb{R}^n)$ if:*

$$\|f\|_{\text{BMO}} := \sup_{\substack{\text{Over all balls} \\ B \subseteq \mathbb{R}^n}} \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty$$

where $f_B := \int_B f(x) dx$ is called the mean oscillation of f over the ball B . The quantity $\|f\|_{\text{BMO}}$

is called the BMO norm of f .

Remark 3.2.1. *The above definition sparks several remarks:*

- (1) *if we replace f_B by any constant c_B depending only on the ball B in the above definition, it yields an equivalent definition of $\text{BMO}(\mathbb{R}^n)$. Indeed: for each ball B , if c_B is a constant then:*

$$\begin{aligned} \oint_B |f(x) - f_B| dx &\leq \oint_B |f(x) - c_B| + \oint_B |c_B - f_B| = \oint_B |f(x) - c_B| + |f_B - c_B| \\ &= 2 \oint_B |f(x) - c_B| \end{aligned}$$

Thus, we get: $\|f\|_{\text{BMO}} \leq 2 \sup_{\substack{\text{Over all balls} \\ B \subseteq \mathbb{R}^n}} \frac{1}{|B|} \int_B |f(x) - c_B| dx < \infty.$

- (2) *All constant functions have zero BMO norm. So $\|\cdot\|_{\text{BMO}}$ is not, per say, a norm unless we consider BMO as a quotient space modulo the closed subspace of constant functions. And from now on, this is what we intend when we invoke the BMO space.*
- (3) *As pointed out in [48, p 140], in the above definition, we could replace the collection of all balls by the collection of all cubes contained in $\mathbb{R}^n$, and this would yield an equivalent definition of the space $\text{BMO}(\mathbb{R}^n)$ (see [30, p.p 155-156]).*
- (4) *$\text{BMO}(\mathbb{R}^n)$ is scaling and translation invariant, i.e, $f(x) \longrightarrow f(ax+b)$ for $a > 0$ and $b \in \mathbb{R}^n$, maps $\text{BMO}(\mathbb{R}^n)$ into $\text{BMO}(\mathbb{R}^n)$, and the BMO norm remains unchanged (see [30, p.p 155-156]).*

Now, we start listing some important results and properties of the $\text{BMO}(\mathbb{R}^n)$ space following the book of Grafakos (see [30]).

Proposition 3.2.1. *We have the following properties:*

- (i) $(\text{BMO}(\mathbb{R}^n), \|\cdot\|_{\text{BMO}})$ *is a normed vector space.*
- (ii) $L^\infty(\mathbb{R}^n) \subsetneq \text{BMO}(\mathbb{R}^n)$ *and $\|f\|_{\text{BMO}} \leq 2\|f\|_\infty$.*

(iii) If $f \in \text{BMO}(\mathbb{R}^n)$, then $|f| \in \text{BMO}(\mathbb{R}^n)$.

(iv) If $f, g \in \text{BMO}(\mathbb{R}^n)$, then $\max(f, g), \min(f, g) \in \text{BMO}(\mathbb{R}^n)$.

(v) If $f \in \text{BMO}(\mathbb{R}^n)$, B is a ball contained in $\mathbb{R}^n$ and $B_k = 2^k B$ for any $k \in \mathbb{N} \cup \{0\}$, we have:

$$|f_B - f_{B_k}| \leq 2^n k \|f\|_{\text{BMO}}.$$

(vi) $\forall \epsilon > 0, \exists C_{\epsilon, n} > 0$ a constant s.t for any ball $B(x_0, r)$, we have:

$$\int_{\mathbb{R}^n} \frac{|f(x) - f_B|}{(1 + \frac{|x - x_0|}{r})^{n+\epsilon}} dx \leq C_{\epsilon, n} r^n \|f\|_{\text{BMO}}.$$

Proof. We start with part (i) and $\text{BMO}(\mathbb{R}^n)$ is a vector space by additivity of the integral, and by part (2) of Remark 3.2.1, one can show it to also be normed. For part (ii), the inclusion and the bound on the BMO norm are from the fact that for any $f \in L^\infty(\mathbb{R}^n)$ and any ball $B \subseteq \mathbb{R}^n$, we have: $|f| \leq \|f\|_\infty$ and $|f_B| \leq \|f\|_\infty$. To prove the strict inclusion, the function $\log|x| \in \text{BMO}(\mathbb{R}^n)$ although it is unbounded (see [30, p.p 156-157]). Part (iii) follows from the inequality:

$$||f(x)| - |f|_B| \leq ||f(x)| - |f_B|| + ||f|_B - |f_B|| \leq |f(x) - f_B| + \int_B |f(x) - f_B| dx.$$

Now part (iv) follows from part (iii) as $\max(f, g) = \frac{f + g + |f - g|}{2}$ and $\min(f, g) = \frac{f + g - |f - g|}{2}$.

For part (v), it follows from the facts that for any k : $B_k \subseteq B_{k+1}$ and:

$$|f_{B_k} - f_{B_{k+1}}| = \left| \int_{B_k} f - f_{B_{k+1}} \right| \leq 2^n \int_{B_{k+1}} |f - f_{B_{k+1}}| \leq 2^n \|f\|_{\text{BMO}}$$

Then using induction we obtain the required result. Finally, for part (vi), notice that we can take $B = B(0, 1)$ and then replace f by $f(rx + x_0)$ exploiting the scaling and translation invariance

of the BMO norm. So using (iv), we have:

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x) - f_B|}{(1+|x|)^{n+\epsilon}} dx &\leq \int_B \frac{|f(x) - f_B|}{(1+|x|)^{n+\epsilon}} dx + \sum_{k=0}^{\infty} \int_{B_{k+1} \setminus B_k} \frac{|f(x) - f_{B_{k+1}}| + |f_{B_{k+1}} - f_B|}{(1+|x|)^{n+\epsilon}} dx \\
&\leq \int_B |f - f_B| + \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \int_{B_{k+1}} |f - f_{B_{k+1}}| + |f_{B_{k+1}} - f_B| \\
&\leq |B(0,1)| \left(1 + \sum_{k=0}^{\infty} 2^{n-k\epsilon} (1 + (k+1)2^n)\right) \|f\|_{\text{BMO}}
\end{aligned}$$

□

We follow with an important result showing the completeness of the space $\text{BMO}(\mathbb{R}^n)$ with its defined norm again following Grafakos hints (see [30]).

Proposition 3.2.2. $(\text{BMO}(\mathbb{R}^n), \|\cdot\|_{\text{BMO}})$ is a Banach space.

Proof. By part (i) of Proposition 3.2.1, $(\text{BMO}(\mathbb{R}^n), \|\cdot\|_{\text{BMO}})$ is a normed vector space. It remains to show that it is complete.

Let $\{b_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence of BMO functions. And for any k define $\tilde{b}_k := b_k - (b_k)_{B(0,1)}$.

Using part (vi) of Proposition 3.2.1, for any ball $B(0, R)$ and $i, j \in \mathbb{N}$, we have:

$$\begin{aligned}
\|\tilde{b}_i - \tilde{b}_j\|_{L^1(B(0,R))} &= \int_{B(0,R)} |\tilde{b}_i - \tilde{b}_j| \leq (1+R)^{n+\epsilon} \int_{\mathbb{R}^n} \frac{|b_i(x) - b_j(x) - (b_i - b_j)_{B(0,1)}|}{(1+|x|)^{n+\epsilon}} \\
&\leq (1+R)^{n+\epsilon} C_{\epsilon,n} \|b_i - b_j\|_{\text{BMO}}
\end{aligned}$$

So $\{\tilde{b}_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence of elements of $L^1(B(0,R))$ which is complete. So: $\tilde{b}_k \rightarrow b$ for some $b \in L^1(B(0,R))$. Now b is well-defined. Indeed: if $R' \geq R$ and $\tilde{b}_k \rightarrow b'$ for some $b' \in L^1(B(0,R'))$, then as $B(0,R) \subseteq B(0,R')$ and by uniqueness of the limit, we would have: $b' = b$ on $B(0,R)$. This shows that $\tilde{b}_k \rightarrow b$ in $L^1(K)$ for any compact $K \subseteq \mathbb{R}^n$.

Now constants do not change the BMO norm, so $\{\tilde{b}_k\}_{k \in \mathbb{N}}$ is also Cauchy in $\text{BMO}(\mathbb{R}^n)$. Fixing $\epsilon > 0$, we can choose $N > 0$ s.t for any $i, j > N$, for all balls B , we get: $\int_B |\tilde{b}_i - \tilde{b}_j| < \frac{\epsilon}{2}$. Letting $j \rightarrow \infty$, we obtain: $\int_B |\tilde{b}_i - b| \leq \frac{\epsilon}{2}$. Taking the supremum over all balls contained in $\mathbb{R}^n$, we get:

$\|\tilde{b}_i - b\|_{\text{BMO}} \leq \frac{\epsilon}{2} < \epsilon$. This means $\tilde{b}_k \rightarrow b$ in $\text{BMO}(\mathbb{R}^n)$. And again, constants do not change the BMO norm, which means $\|b_i - b\|_{\text{BMO}} \rightarrow 0$. Concluding the proof. $\square$

We now state and provide a reference for the proof of a foundational result for the $\text{BMO}(\mathbb{R}^n)$ space known as the John-Nirenberg Inequality. This inequality shows that wild deviations from the mean are exponentially rare for BMO functions, and implying that the latter are exponentially integrable.

Theorem 3.2.1 (John-Nirenberg Inequality). *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then: $f \in \text{BMO}(\mathbb{R}^n) \iff \exists C_1, C_2 > 0$ s.t. for any ball $B \subseteq \mathbb{R}^n$ and for any $\lambda > 0$, we have:*

$$\mu_B(\lambda) := |\{x \in B : |f(x) - f_B| > \lambda\}| \leq C_1 |B| \exp(-C_2 \lambda).$$

Proof. See [36]. $\square$

An important direct consequence of the John-Nirenberg Inequality is given in the following corollary (see [30, Corollary 3.1.9, p. 165]), which is one of the key properties used implicitly in proving the duality with the real Hardy space..

Corollary 3.2.1. *If $f \in \text{BMO}(\mathbb{R}^n)$ then:*

$$\sup_{\substack{\text{Over all balls} \\ B \subseteq \mathbb{R}^n}} \left(\frac{1}{|B|} \int_B |f(x) - f_B|^p dx \right)^{\frac{1}{p}} \simeq \|f\|_{\text{BMO}}, \text{ for any } 1 < p < \infty$$

Proof. First, the above supremum is larger or equal to the same supremum with $p = 1$ by Jensen's Inequality. In the other direction, we have by the John-Nirenberg Inequality 3.2.1:

$$\frac{1}{|B|} \int_B |f(x) - f_B|^p dx = \frac{p}{|B|} \int_0^\infty \lambda^{p-1} \mu_B(\lambda) d\lambda \leq \frac{1}{|B|} C_1 |B| \int_0^\infty \lambda^{p-1} e^{-\frac{C_2 \lambda}{\|f\|_{\text{BMO}}}} d\lambda = C_{p,n} \|f\|_{\text{BMO}}^p.$$

Concluding the proof. $\square$

Several other interesting results can be found in [30], [48] and [28] regarding the space of functions of bounded mean oscillation.

Now, near the end of the discussion on the BMO space, we shall talk about the duality between that space and the real Hardy space. The duality between the two spaces was first established by Fefferman in the 1970s (see [25]), who showed that every bounded linear functional on $H^1(\mathbb{R}^n)$ arises from pairing with a $\text{BMO}(\mathbb{R}^n)$ function. Fefferman and Stein's work (see [26]) also revealed that Calderón–Zygmund operators, such as the Riesz transforms, and more general convolution-type operators, map $L^\infty(\mathbb{R}^n)$ into $\text{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Later, Coifman and Rochberg (see [16] and [15]) extended the theory to commutators of singular integrals with multiplication by $\text{BMO}(\mathbb{R}^n)$ functions, showing that such commutators are bounded on $L^p(\mathbb{R}^n)$ spaces for $1 < p < \infty$.

We will now present the duality result following Stein's steps in [48, p.p 142–144]. Before stating the theorem, let's walk through a subtlety first. Consider a continuous linear functional T on $H^1(\mathbb{R}^n)$. We would be tempted to characterize it as:

$$T_b(h) = \int b h, \text{ for any } h \in H^1(\mathbb{R}^n), \text{ for some } b \in \text{BMO}(\mathbb{R}^n).$$

But the integral that defines the linear functional described above does not necessarily converge for any $b \in \text{BMO}(\mathbb{R}^n)$ and any $h \in H^1(\mathbb{R}^n)$. So, we need to define an appropriate dense subspace of $H^1(\mathbb{R}^n)$, which we choose to be H_a^1 , the subspace of finite linear combinations of H^1 -atoms, for which that integral converges.

Now, let $b \in \text{BMO}(\mathbb{R}^n)$ and assume further that it is bounded. Let $\sum_k \lambda_k a_k$ be an atomic decomposition of $h \in H^1(\mathbb{R}^n)$ (with the atoms a_k having support in the balls B_k). Since $\sum_k \lambda_k a_k$ converges in the L^1 norm, we can write:

$$T_b(h) = \int_{\mathbb{R}^n} b h = \int_{\mathbb{R}^n} b \sum_k \lambda_k a_k = \sum_k \lambda_k \int_{B_k} b a_k.$$

Fixing k , by the cancellation property of the atoms, we have: $\int_{B_k} b a_k = \int_{B_k} (b - b_{B_k}) a_k$.

So that:

$$|T_b(h)| \leq \sum_k |\lambda_k| \left| \int_{B_k} (b - b_{B_k}) a_k \right| \leq \sum_k |\lambda_k| \|a_k\|_2 \left(\int_{B_k} |b - b_{B_k}|^2 \right)^{\frac{1}{2}} \leq \sum_k |\lambda_k| \left(\frac{1}{|B_k|} \int_{B_k} |b - b_{B_k}|^2 \right)^{\frac{1}{2}}.$$

Hence, by Corollary 3.2.1, for some $c > 0$, we have: $T_b(h) \leq c \|h\|_{H^1} \|b\|_{\text{BMO}}$.

Now, assume that $b \in \text{BMO}(\mathbb{R}^n)$ and $h \in H_a^1$. Let:

$$b_k(x) = \begin{cases} k & \text{if } b(x) \geq k \\ b(x) & \text{if } -k \leq b(x) \leq k \\ -k & \text{if } b(x) \leq -k \end{cases}$$

By [19, Corollary 6.5], we have that: $\|b_k\|_{\text{BMO}} \leq 2 \|b\|_{\text{BMO}}$. Moreover b_k is bounded, so by the previous result, we have: $\left| \int_{\mathbb{R}^n} b_k h \right| \leq 2c \|b\|_{\text{BMO}} \|h\|_{H^1}$.

Now, we have that: $b_k \rightarrow b$ a.e as $k \rightarrow \infty$, so by the Dominated Convergence Theorem, we obtain:

$$\left| \int_{\mathbb{R}^n} b h \right| \leq 2c \|b\|_{\text{BMO}} \|h\|_{H^1}.$$

Therefore, by density, T_b extends uniquely to a bounded linear functional on $H^1(\mathbb{R}^n)$.

We will now present the duality result.

Theorem 3.2.2. . $\text{BMO}(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$, i.e, each continuous linear functional T on $H^1(\mathbb{R}^n)$ can be characterized as: $T_b(h) = \int b h$, for any $h \in H^1(\mathbb{R}^n)$ (when defined appropriately), for some $b \in \text{BMO}(\mathbb{R}^n)$.

Proof. See [48, p.p 142–144]. □

Now we give Meyer's result (see [44, Theorem 4, p.p 154–155]) related to the subject of this thesis, which gives a wavelets characterization of the $\text{BMO}(\mathbb{R}^n)$ space. But first we set some notation as we did in the Hardy space case. For $f \in L_{loc}^1(\mathbb{R}^n)$, we denote $\alpha(\lambda) = (f, \psi_\lambda)$ where $\psi_\lambda, \lambda \in \Lambda$ are derived from the mother wavelet, respectively, as shown in the wavelet

construction section where we introduced them. For $\frac{k}{2^j} + \frac{e}{2^{j+1}} = \lambda \in \Lambda$, we define $Q(\lambda) := \{x \in \mathbb{R}^n : 2^j x - k \in [0, 1)^n\}$.

Theorem 3.2.3. *If $f \in \text{BMO}(\mathbb{R}^n)$, then its wavelet coefficients $\alpha(\lambda)$ satisfy Carleson's condition, i.e:*

$$\exists C > 0, \text{ a constant, s.t.: } \forall Q \in \mathcal{Q}, \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq C|Q| \quad (3.1)$$

Conversely, if the coefficients $\alpha(\lambda)$ satisfy the above condition, then f is the limit of its wavelet decomposition series $\sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda$ in the weak-topology of $\text{BMO}(\mathbb{R}^n)$ (seen as the dual of $H^1(\mathbb{R}^n)$).*

Moreover, we have: $\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} \simeq \|f\|_{\text{BMO}}$.

Proof. See [44, proof of Theorem 4, p.p 154–155]), [48, p.p 166–169], and for more details on the proof steps, one can refer to the proof of Theorem 4.3.1. $\square$

3.3 The Local Hardy Space $h^1(\mathbb{R}^n)$

The real Hardy space $H^1(\mathbb{R}^n)$, presented previously in Section 3.1, was introduced in the mid-20th century as a convenient substitute for the Lebesgue space $L^1(\mathbb{R}^n)$ broadly in harmonic analysis to account for the lack of boundedness of singular integral operators on it. However, by its global essence, it was not very useful for localized analysis and presented limitations in the study of partial differential equations and in geometric measure theory. In 1979, David Goldberg in his PhD dissertation [29] suggested a localized version of the classical Hardy space, denoted $h^1(\mathbb{R}^n)$, that behaves like $H^1(\mathbb{R}^n)$ on small scales but with behavior aligning with that of $L^1(\mathbb{R}^n)$ at the tails near infinity. Goldberg's construction relied on truncated or localized versions of maximal functions, as well as on a variant of the classical atomic decomposition. The hybrid nature of the local Hardy space $h^1(\mathbb{R}^n)$ makes it a suitable setting for problems related to local regularity and compact embeddings, among others.

Following the notation adopted in Section 3.1, we present some definitions useful for the introduction of the $h^1(\mathbb{R}^n)$ space.

Definition 3.3.1. *If ϕ is a Schwartz function, f is a tempered distribution, and $\mathcal{F}$ is a finite collection of semi-norms on the space of Schwartz functions, we call the local maximal function of f (based on ϕ), the function defined by:*

$$m_\phi f(x) := \sup_{0 < t < 1} |(f * \phi_t)(x)|$$

We also call the grand local maximal function of f (based on $\mathcal{F}$), the function defined by:

$$m_{\mathcal{F}} f(x) := \sup_{\phi \in \mathcal{S}_{\mathcal{F}}} m_\phi f(x)$$

If $\Gamma(x) = \{(y, t) : |x - y| < t, 0 < t < \frac{1}{2}\}$ and the Poisson integral of f is the function defined by: $u(x, t) := (f * P_t)(x)$, we call the non-tangential local maximal function of the Poisson integral of f , the function defined by:

$$u_l^*(x) := \sup_{(y, t) \in \Gamma(x)} |u(y, t)|$$

Now, we present an important theorem before formally defining the $h^1(\mathbb{R}^n)$ space.

Theorem 3.3.1. *If f is a tempered distribution, then T.F.A.E:*

- (i) $\exists \phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int \phi \neq 0$ st: $m_\phi f \in L^1(\mathbb{R}^n)$.
- (ii) $\exists \mathcal{F}$ (or for some $m \in \mathbb{N}$, $\exists \mathcal{F}_m$) a finite collection of semi-norms on the space of Schwartz functions st: $m_{\mathcal{F}} f \in L^1(\mathbb{R}^n)$.
- (iii) $u_l^* \in L^1(\mathbb{R}^n)$.

Proof. See [29, proof of Theorem 1, p.p 31-33] in the case $p = 1$. □

Remark 3.3.1. *Similar to what we pointed out in Remark 3.1.2, the proof of Theorem 3.3.1 shows us that we can choose the finite collection of semi-norms $\mathcal{F}$ from condition (ii) not to depend on f but only on the dimension n , so that for sufficiently large $m \in \mathbb{N}$, $\mathcal{F} = \mathcal{F}_m = \{\|\cdot\|_{\alpha, \beta} : |\alpha|, |\beta| \leq m\}$, which also means that we can choose ϕ from (i) to be independent of*

f and supported in the unit ball. It also proves that for sufficiently large m , we have (not depending on f): $\|m_\phi f\|_1 \simeq \|m_{\mathcal{F}} f\|_1 \simeq \|u_i^*\|_1$.

From now on and throughout this section, if we talk about the operators m_ϕ and $m_{\mathcal{F}}$, it is going to be for the above fixed choices of ϕ and $\mathcal{F}$ completely independent of any function f . Now, we introduce the local Hardy space $h^1(\mathbb{R}^n)$.

Definition 3.3.2. We define the local Hardy space $h^1(\mathbb{R}^n)$ as the space of tempered distributions for which one of the equivalent conditions of Theorem 3.3.1 holds. For $f \in h^1(\mathbb{R}^n)$, we also define: $\|f\|_{h^1} := \|m_\phi f\|_1$.

Proposition 3.3.1. $(h^1(\mathbb{R}^n), \|\cdot\|_{h^1})$ is a normed vector space.

Proof. It follows from the sub-additivity of the local maximal function and the definition of the norm. □

Now, we introduce the important notion of atoms that are the building blocks of the $h^1(\mathbb{R}^n)$ space. In what follows throughout the entire dissertation, we restrict ourselves to the h^1 -atoms in the L^2 sense.

Definition 3.3.3. An $h^1(\mathbb{R}^n)$ -atom (in the $L^2(\mathbb{R}^n)$ sense) is a function a satisfying the following three (or two) properties :

- (i) compact support: $\text{supp}(a) \subseteq B$ for some ball B .
- (ii) size condition: $\|a\|_2 \leq \frac{1}{|B|^{1/2}}$.
- (iii) cancellation: $\int_{\mathbb{R}^n} a = 0$ required only if $|B| < 1$.

Remark 3.3.2. Following Definition 3.1.4 and Definition 3.3.3, every $H^1(\mathbb{R}^n)$ -atom is an $h^1(\mathbb{R}^n)$ -atom, and every $h^1(\mathbb{R}^n)$ -atom with $|B| < 1$ is an $H^1(\mathbb{R}^n)$ -atom.

Now, we state and prove some important properties of these atoms.

Proposition 3.3.2. If a is an $h^1(\mathbb{R}^n)$ -atom, then we have:

$$(a) \quad \|a\|_1 \leq 1.$$

$$(b) \quad \|a\|_{h^1} = \|m_\phi a\|_1 \lesssim 1.$$

$$(c) \quad a \in h^1(\mathbb{R}^n).$$

Proof. The proof of part (a) is exactly the same as the proof of part (a) of Proposition 3.1.2.

For part (b), if $|B| < 1$, then, by part (b) of Proposition 3.1.2, we have:

$$\|a\|_{h^1} = \|m_\phi a\|_1 \leq \|M_\phi a\|_1 = \|a\|_{H^1} \lesssim 1$$

Now, if $|B| \geq 1$, we split the integral in a similar manner to what we have done in the proof of part (b) of Proposition 3.1.2. The estimate of the integral on the doubled ball $2B$ is the same.

Now outside of $2B$, for some $C > 0$, we have:

$$|(a * \phi_t)(x)| \leq \int_B |a(y)| |\phi_t(x-y)| dy = t^{-n} \int_B |a(y)| \left| \phi\left(\frac{x-y}{t}\right) \right| dy \leq C t^{-n} \int_B |a(y)| \left(1 + \frac{|x-y|}{t}\right)^{-N} dy$$

Choosing $N \gg n$, using $|x-y| \geq |x-x_0| - r \gtrsim |x-x_0|$ for $x \notin 2B$, and $\|a\|_1 \leq 1$, we get:

$$|(a * \phi_t)(x)| \leq C t^{-n} \|a\|_1 \left(1 + \frac{|x-x_0|}{t}\right)^{-N} \leq C t^{-n} \left(1 + \frac{|x-x_0|}{t}\right)^{-N}$$

Taking the supremum over $0 < t < 1$, we get: $m_\phi a(x) \leq \frac{C}{|x-x_0|^n}$ for $x \notin 2B$.

Hence: $\int_{\mathbb{R}^n \setminus 2B} m_\phi a \leq C \int_{\mathbb{R}^n \setminus 2B} |x-x_0|^{-n} \lesssim 1$. This allows us to conclude with the required result.

Part (c) then follows from part (b). □

We now introduce the local Riesz transform and state the local Riesz transform characterization of $h^1(\mathbb{R}^n)$.

Definition 3.3.4. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and any $j \in \llbracket 1, n \rrbracket$, we define the j^{th} local Riesz transform of f as: $r_j^{(\varphi)} f := R_j(f - \varphi * f)$.

Theorem 3.3.2. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, a tempered distribution f is in $h^1(\mathbb{R}^n) \iff f \in L^1(\mathbb{R}^n)$ and $r_j^{(\varphi)} f \in L^1(\mathbb{R}^n)$ for any $j \in [1, n]$.

Proof. See [29, proof of Theorem 2, p. 33]. □

As a corollary, we get another characterization of the local Hardy space $h^1(\mathbb{R}^n)$.

Corollary 3.3.1. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ s.t.: $\int \varphi = 1$, then: $f \in h^1(\mathbb{R}^n) \iff (f - \varphi * f) \in H^1(\mathbb{R}^n)$.

Proof. See [29, proof of Lemma 4, p.p 34–36] for the case $p = 1$. □

Now, we are ready to state the Atomic Decomposition Theorem for the local Hardy space $h^1(\mathbb{R}^n)$ in terms of h^1 -atoms.

Theorem 3.3.3. $f \in h^1(\mathbb{R}^n) \iff f = \sum_k \lambda_k a_k$ where $\{a_k\}_k$ is a sequence of $h^1(\mathbb{R}^n)$ -atoms and $\{\lambda_k\}_k$ is a sequence of scalars s.t.: $\sum_k |\lambda_k| < \infty$.

Moreover: $\|f\|_{h^1} \simeq \|f\|_{at} = \inf \left\{ \sum_k |\lambda_k| < \infty : f = \sum_k \lambda_k a_k, a_k \text{'s are } h^1\text{-atoms} \right\}$.

Proof. We will follow Goldberg's proof in [29, proof of Lemma 5, p. 37] for the case $p = 1$.

Assume $f \in h^1(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with its support contained in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$, which we will denote C_0 , $\varphi = 1$ in a neighborhood of the origin, and s.t.: $\int \varphi = 1$, and let $C_k = k + C_0$, B_k be the smallest ball containing C_k and $K > 1$ be a constant depending only on n s.t.: $|B_k| = K |C_k| = K$, for any $k \in \mathbb{Z}^n$.

Then, by Corollary 3.3.1, we can write: $f - \varphi * f = \sum_{j=1}^{\infty} \lambda_j a_j \in H^1(\mathbb{R}^n)$ where $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ and the a_j 's are $H^1(\mathbb{R}^n)$ -atoms (associated with balls).

Denoting: $\tilde{A}_k = (\varphi * f)|_{C_k}$, we have: $\text{supp}(\tilde{A}_k) = C_k \subseteq B_k$ for any $k \in \mathbb{Z}^n$. Also, as $\varphi \in L^2(\mathbb{R}^n)$, in particular, and $f \in L^1(\mathbb{R}^n)$, by Young's Inequality, we have: $(\varphi * f) \in L^2(\mathbb{R}^n)$ and so are all the $\tilde{A}_k$'s.

Dropping potentially any $k \in \mathbb{Z}^n$ for which $\|\tilde{A}_k\|_2 = 0$, let $A_k = \frac{1}{K^{\frac{1}{2}} \|\tilde{A}_k\|_2} \tilde{A}_k$ and $\gamma_k = K^{\frac{1}{2}} \|\tilde{A}_k\|_2$, so that we have: $\|A_k\|_2 = \frac{1}{K^{\frac{1}{2}}} = \frac{1}{|B_k|^{\frac{1}{2}}}$, $\text{supp}(A_k) = \text{supp}(\tilde{A}_k) \subseteq B_k$ and: $\varphi * f = \sum_{k \in \mathbb{Z}^n} \tilde{A}_k = \sum_{k \in \mathbb{Z}^n} \gamma_k A_k$.

Setting $\Gamma(x) = \{(y, t) : |x - y| < t, 0 < t < \frac{1}{2}\}$ for any $x \in \mathbb{R}^n$ and based on the maximal function characterization of $h^1(\mathbb{R}^n)$, we then have:

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} |\gamma_k| &\lesssim \sum_{k \in \mathbb{Z}^n} \|\tilde{A}_k\|_2 \leq \sum_{k \in \mathbb{Z}^n} \sup_{x \in C_k} |(\varphi * f)(x)| \leq \int_{\mathbb{R}^n} \sup_{(y, t) \in \Gamma(x)} |(\varphi_t * f)(y)| dx \\ &\lesssim \int_{\mathbb{R}^n} \sup_{0 < t < 1} |(\varphi_t * f)(x)| dx = \|f\|_{h^1} < \infty \end{aligned}$$

So, we arrived at an atomic decomposition similar to what we have done for the $H^1(\mathbb{R}^n)$ part. Conversely, if $f = \sum_k \lambda_k a_k$ where $\{a_k\}_k$ is a sequence of $h^1(\mathbb{R}^n)$ -atoms and $\{\lambda_k\}_k$ is a sequence of scalars s.t. $\sum_k |\lambda_k| < \infty$, then by Triangle Inequality and part (b) of Proposition 3.3.2, we have that:

$$\|f\|_{h^1} \leq \sum_k |\lambda_k| \|a_k\|_{h^1} \lesssim \sum_k |\lambda_k| < \infty$$

Concluding the proof. □

We now deduce and derive many results from the above decomposition theorem.

Corollary 3.3.2. *We have the following results:*

(i) $H^1(\mathbb{R}^n) \subsetneq h^1(\mathbb{R}^n)$.

(ii) $h_a^1 :=$ the space of finite linear combinations of h^1 -atoms, $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense subspaces of $h^1(\mathbb{R}^n)$.

Proof. For part (i), the inclusion follows from Remark 3.3.2, the atomic decomposition Theorem 3.1.2 and the atomic decomposition Theorem 3.3.3. For the strict inclusion, we take a multiple of an h^1 -atom without cancellation requirement like $\chi_{B(0,2)}$, which is not in $H^1(\mathbb{R}^n)$ as $\int \chi_{B(0,2)} \neq 0$.

For part (ii), the density of h_a^1 in $h^1(\mathbb{R}^n)$ follows from the Atomic Decomposition Theorem 3.3.3. For the remaining subspaces, it suffices to show that $\mathcal{S}(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$ and that $C_c^\infty(\mathbb{R}^n)$ is dense in $h^1(\mathbb{R}^n)$. The first claim follows from the local maximal characterization of $h^1(\mathbb{R}^n)$ and exploiting the rapid decay and smoothness of Schwartz functions. The second one follows from

the approximation of any h^1 -atom with a convenient sequence of smooth compactly supported functions (which are the atom itself convoluted with a standard mollifier) and the result follows by an application of the Atomic Decomposition Theorem 3.3.3. $\square$

Proposition 3.3.3. *3.4.1 $(h^1(\mathbb{R}^n), \|\cdot\|_{h^1})$ is a Banach space.*

Proof. The proof follows from Proposition 3.3.1 and an application of the atomic decomposition Theorem 3.3.3 as in the proof of Proposition 3.1.3. $\square$

Many other characterizations exist for $h^1(\mathbb{R}^n)$ such as the characterization by duality with $\text{bmo}(\mathbb{R}^n)$ (to be addressed in the next Section 3.4 after formally introducing the non-homogeneous bmo space), the characterization by molecules, the characterization by local square functions, and the characterization as conjugate harmonic function boundary values in the strip $\{(x, t) : 0 < t < 1\}$ (see [29]).

Our original contribution is a wavelet characterization of the local Hardy space $h^1(\mathbb{R}^n)$ as presented in Theorem 4.2.1.

3.4 The non-homogeneous space $\text{bmo}(\mathbb{R}^n)$

The function space of the bounded mean oscillation $\text{BMO}(\mathbb{R}^n)$, formally introduced by John and Nirenberg in the 1960s, plays a fundamental role in harmonic analysis in general as the dual space of the real Hardy space $H^1(\mathbb{R}^n)$. However, David Goldberg in his 1979 PhD thesis ([29]), defined a localized or non-homogeneous version of BMO that he denoted $\text{bmo}(\mathbb{R}^n)$, crafted to the local Hardy space $h^1(\mathbb{R}^n)$ that he introduced in the same dissertation. In contrast to the classical BMO space that is sensitive to the global behavior of its functions, Goldberg's bmo is more useful particularly in the study of local oscillations, and thus better-suited to fit non-homogeneous settings. In this section, we will present a rigorous treatment of $\text{bmo}(\mathbb{R}^n)$, starting with its formal definition, its norm(s), its different characterizations, and any other functional analysis related properties.

Definition 3.4.1. If $f \in L^1_{loc}(\mathbb{R}^n)$, we say that: $f \in \text{bmo}(\mathbb{R}^n)$ if and only if:

$$\|f\|_{\text{bmo}} := \sup_{\substack{\text{over all balls } B \\ |B| < 1}} \int_B |f - f_B| + \sup_{\substack{\text{over all balls } B \\ |B| \geq 1}} \int_B |f| < \infty.$$

We call $\|f\|_{\text{bmo}}$ the bmo norm of f .

Remark 3.4.1. As pointed out in [9], we can replace balls with cubes in the above definition and still arrive at an equivalent definition of the bmo space. Also we can change the size limit to any other finite number, switch the strict limit of the size to either one of the two supremums or even choose both to be strict, and again we will still hold an equivalent definition of the said space.

Another crucial observation is that the bmo functions are not taken as equivalence classes modulo the constant functions, in contrast with BMO functions.

Finally, similar to BMO space, bmo is also invariant under translation with the bmo norm of the translated function equal to the bmo norm of the original one. Similarly, bmo is invariant under dilation, with the bmo norm scaling appropriately.

Proposition 3.4.1. $(\text{bmo}(\mathbb{R}^n), \|\cdot\|_{\text{bmo}})$ is a normed vector space.

Proof. The fact that $\text{bmo}(\mathbb{R}^n)$ is a vector space comes from Definition 3.4.1 and by linearity of the integral. For the norm, Triangle Inequality, absolute homogeneity and non-negativity follow from its definition, whereas the positive definiteness follows from the fact that $\int_B |f| = 0$, for any ball B s.t: $|B| \geq 1 \implies f = 0$ a.e. $\square$

Now, we list some of the important properties of the $\text{bmo}(\mathbb{R}^n)$ space mainly following Galia's work in [20].

Proposition 3.4.2. We have the following properties:

- (i) $L^\infty(\mathbb{R}^n) \subsetneq \text{bmo}(\mathbb{R}^n) \subsetneq \text{BMO}(\mathbb{R}^n)$ and the space of BMO functions with compact support $\text{BMO}_c(\mathbb{R}^n) \subsetneq \text{bmo}(\mathbb{R}^n)$ (where the inclusion are taken in the set inclusion sense and not in the space inclusion sense).

(ii) $f \in \text{bmo}(\mathbb{R}^n) \implies |f| \in \text{bmo}(\mathbb{R}^n)$ and $\|f\|_{\text{bmo}} \leq 2\|f\|_{\text{bmo}}$

(iii) $f, g \in \text{bmo}(\mathbb{R}^n) \implies \min(f, g), \max(f, g) \in \text{bmo}(\mathbb{R}^n)$

Proof. For part (i), if f is bounded, then: $\sup_{\substack{\text{over all balls } B \\ |B| < 1}} \int_B |f - f_B| < \infty$ and $\sup_{\substack{\text{over all balls } B \\ |B| \geq 1}} \int_B |f| < \infty$, so it is a bmo function. Now, if f is a bmo function, similar to the proof of (C) $\implies$ (A) direction of Theorem 4.3.1, we have: $\sup_{\text{over all balls } B} \int_B |f - f_B| < \infty$, so f is BMO. Now, if f is a BMO function with compact support, we would have: $\sup_{\substack{\text{over all balls } B \\ |B| < 1}} \int_B |f - f_B| \leq \sup_{\text{over all balls } B} \int_B |f - f_B| < \infty$ and by compact support: $\sup_{\substack{\text{over all balls } B \\ |B| \geq 1}} \int_B |f| < \infty$, so f is a bmo function. For the strict inclusions, $\ln|x| \in \text{BMO}(\mathbb{R}^n) \setminus \text{bmo}(\mathbb{R}^n)$ and (an example from [20]) $\sum_{k \in \mathbb{Z}} \log|x - 2k| \chi_{(2k-1, 2k+1)}(x) \in \text{bmo}(\mathbb{R}^n) \setminus (L^\infty(\mathbb{R}^n) \cup \text{BMO}_c(\mathbb{R}^n))$ (for the case $n = 1$). For part (ii) see [20, p. 5]. And finally, part (iii) follows from part (ii) in a similar manner as part (iv) follows from part (iii) in Proposition 3.2.1. $\square$

Now, we give a characterization of the space $\text{bmo}(\mathbb{R}^n)$ in a theorem similar to the John-Nirenberg Inequality for the space $\text{BMO}(\mathbb{R}^n)$.

Theorem 3.4.1. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then:*

$$f \in \text{bmo}(\mathbb{R}^n) \iff \exists C_1, C_2 > 0 \text{ s.t. for any ball } B: |\{x \in B : |f(x) - c_B| > \lambda\}| \leq C_1 |B| \exp(-C_2 \lambda)$$

Where:

$$c_B = \begin{cases} f_B & \text{if } |B| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. See [20, proof of Theorem 3.1]. $\square$

A corollary of this John-Nirenberg-type Inequality is given in what follows.

Corollary 3.4.1. *For $1 \leq p < \infty$, we have: $\sup_{\text{over all balls } B} \left(\int_B |f - c_B|^p \right)^{\frac{1}{p}} \simeq \|f\|_{\text{bmo}}$ where c_b is the same quantity as defined in Theorem 3.4.1.*

Proof. See [20, proof of Corollary 3.2]. □

Now, we turn our direction to state an important theorem that gives some interesting characterization of the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space, among these characterizations we will see the duality with the local hardy space $h^1(\mathbb{R}^n)$.

Theorem 3.4.2. *The spaces listed below are equivalent:*

- (1) $(h^1(\mathbb{R}^n))^*$ i.e, the dual of the local Hardy space $h^1(\mathbb{R}^n)$
- (2) $\{f = f_0 + \sum_{j=1}^n r_j f_j, f_j \in L^\infty(\mathbb{R}^n), \forall j \in [1, n]\}$ where r_j is the j^{th} local Riesz transform
- (3) $\text{bmo}(\mathbb{R}^n)$
- (4) $\{f \in \text{BMO}(\mathbb{R}^n) : \exists \varphi \in \mathcal{S}(\mathbb{R}^n), \text{ s.t.: } \int \varphi = 1 \text{ and } (\varphi * f) \in L^\infty(\mathbb{R}^n)\}$

Proof. See [29, Section 4, proof of Corollary 1]. □

Many other interesting results regarding the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space can be found in [29] and [20].

Our original contribution in this thesis is a wavelet characterization of $\text{bmo}(\mathbb{R}^n)$ given in Theorem 4.3.1 in Section 4.3 of Chapter 4, as well as extending the φ in part (4) of Theorem 3.4.2 from only Schwartz functions to bounded functions with compact support, in what seems that giving up on smoothness is compensated by going from rapid decay at infinity to completely compact support.

Chapter 4

Results

In this chapter, we present our main results related to the wavelets characterization of $h^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$, but first we establish a theory of the local dyadic tent space t_1 similar and in contrast to that of the dyadic tent space T_1 presented by Meyer in [44, pp. 144–148], that is, in its essence mirrors the relationship between the Hardy space $H^1(\mathbb{R}^n)$ and its local counterpart $h^1(\mathbb{R}^n)$, and that would help us establish an atomic decomposition for $h^1(\mathbb{R}^n)$ involving the wavelets coefficients.

4.1 The Tent Space T_1 and its Local version t_1

Tent spaces were first introduced by Coifman, Meyer, and Stein [14]. The concept stemmed from a need to rigorously analyze Carleson measures and square functions in settings where classical Lebesgue theory and Hardy spaces were not the best choice, for instance, particularly to study the boundary behavior of harmonic functions and also singular integrals. Tent spaces are defined by areas set up like vertical tents (see figure 4.1 for instance) placed above subsets of Euclidean space and so allow precise control of function behavior in both space and scale. Their geometrical arrangement is actually tailored to capture local oscillation and decay properties that make it especially useful in Littlewood–Paley theory, paraproducts, and non-tangential

maximal functions. Tent spaces have since become a core tool of modern harmonic analysis theory following their introduction, even further developments stretching to vector-valued settings as well as metric measure spaces, and with applications to both partial differential equations and geometric measure theory. We begin first by defining some important notions that will be used throughout this section.

Definition 4.1.1. *For any cube R (not necessarily dyadic), we call the “Tent over R ”, the collection of all dyadic cubes contained in R , and denote it by: $\mathcal{T}(R) = \{Q \in \mathcal{Q} : Q \subseteq R\}$. And we call the “Local Tent over R ”, the collection of all small dyadic cubes contained in R , and denote it by: $\mathcal{T}_0(R) = \{Q \in \mathcal{Q}_0 : Q \subseteq R\}$.*

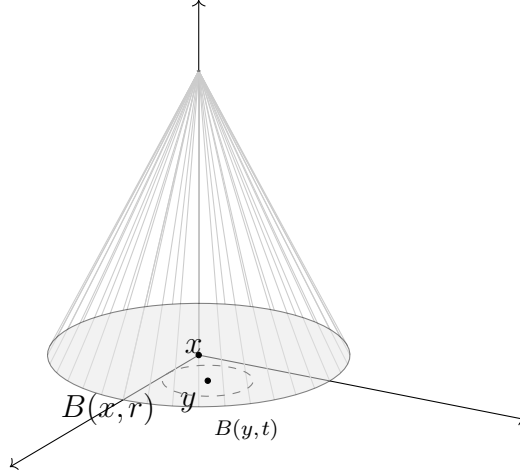


Figure 4.1: Tent over $B(x, r)$: $\mathcal{T}(B(x, r)) = \{(y, t) \in \mathbb{R}^n \times [0, \infty) : B(y, t) \subseteq B(x, r)\}$

Collections of such tents (see figure 4.2) are used to discretize and analyze function behavior across scales and are especially used in dyadic and martingale-type decompositions.

Now, we turn our attention to define some notions related to sequences of scalars indexed by any subset of the set of dyadic cubes $\mathcal{Q}$.

Definition 4.1.2. *Let $\mathcal{Q}' \subseteq \mathcal{Q}$ be a subset of the set of dyadic cubes and $\alpha = \{\alpha(Q)\}_{Q \in \mathcal{Q}'}$ be a sequence of scalars. We define the support of α by: $\text{supp}(\alpha) = \{Q \in \mathcal{Q}' : \alpha(Q) \neq 0\}$ and its socle by: $\text{socle}(\alpha) = \cup_{Q \in \text{supp}(\alpha)} Q$.*

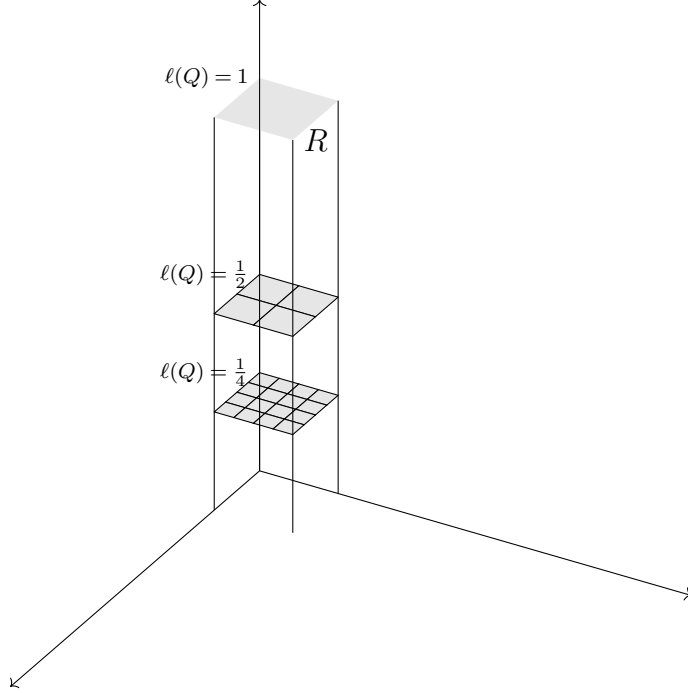


Figure 4.2: Tent over R , with dyadic layers: coarse at top, fine at bottom, labeled by scale $\ell(Q)$

The characterization of a sequence indexed by dyadic cubes as having its support, in one hand, as the set of dyadic cubes where it does not vanish allows for precise and systematic localization of its spatial activity. The notion of a socle, as an aggregate of these supporting cubes, allows, on the other hand, for a geometric representation of the sequence that captures the region in $\mathbb{R}^n$ where its effect is most strongly concentrated. This two-eyed perspective that combines algebraic sparsity along with spatial distribution is especially useful in dyadic analysis itself, where such sequences arise most commonly in the context of projections, wavelet coefficients, or martingale differences. The socle acts as a way to bridge the gap between the discrete structural properties and the geometric intuition and thus enables better control of localization and interactions at different scales.

Now, we are ready to define our dyadic tent spaces.

Definition 4.1.3. *For any sequences $\alpha = \{\alpha(Q)\}_{Q \in \mathcal{Q}_0}$ and $\beta = \{\beta(Q)\}_{Q \in \mathcal{Q}}$, we assign the*

$$\begin{aligned}
\text{functions: } S_0(\alpha) : \mathbb{R}^n &\rightarrow [0, \infty] & \text{and: } S(\beta) : \mathbb{R}^n &\rightarrow [0, \infty] \\
x &\mapsto \left(\sum_{x \in Q} |\alpha(Q)|^2 |Q|^{-1} \right)^{1/2} & x &\mapsto \left(\sum_{x \in Q} |\beta(Q)|^2 |Q|^{-1} \right)^{1/2}
\end{aligned}$$

We call $S(\beta)$ the square function of the sequence β , and we call $S_0(\alpha)$ the local square function of the sequence α .

We say that the sequences $\alpha \in t_1$ and $\beta \in T_1$, if the functions $S_0(\alpha), S(\beta) \in L^1(\mathbb{R}^n)$, and we denote: $\|\alpha\|_{t_1} = \|S_0(\alpha)\|_1$ and $\|\beta\|_{T_1} = \|S(\beta)\|_1$.

Now, we can dive into some of the properties of the tent space and its local counterpart.

Proposition 4.1.1. $(T_1, \|\cdot\|_{T_1})$ and $(t_1, \|\cdot\|_{t_1})$ are both normed vector spaces.

Proof. First of all, it is clear from the Definition 4.1.3 that both T_1 and t_1 are vector spaces. Also $\|\cdot\|_{T_1}$ and $\|\cdot\|_{t_1}$ are both norms, as their non-negativity, positive definiteness, absolute homogeneity and Triangle Inequality follow from that of the L^1 -norm of the square function and local square function, respectively. $\square$

Remark 4.1.1. For any sequences $\alpha = \{\alpha(Q)\}_{Q \in \mathcal{Q}_0}$ and $\beta = \{\beta(Q)\}_{Q \in \mathcal{Q}}$, the Monotone Convergence Theorem allows us to show that: $\|S(\beta)\|_2 = \|\beta\|_{l^2(\mathcal{Q})}$ and $\|S_0(\alpha)\|_2 = \|\alpha\|_{l^2(\mathcal{Q}_0)}$. It should also be noted that the sums defining the square function and its local version in Definition 4.1.3 are over the whole set of dyadic cubes and the set of small dyadic cubes, respectively, which should be understood from the context.

Having introduced the square functions and defined the tent spaces, we now investigate their relationship with $l^2(F)$ spaces for any finite subset of $\mathcal{Q}$ and $\mathcal{Q}_0$, respectively. So, we state and prove the important following lemma.

Lemma 4.1.1. $l^2_{Finite}(\mathcal{Q}) = \bigcup_{\substack{F \subseteq \mathcal{Q} \\ \text{finite}}} l^2(F)$ and $l^2_{Finite}(\mathcal{Q}_0) = \bigcup_{\substack{F \subseteq \mathcal{Q}_0 \\ \text{finite}}} l^2(F)$ are dense in T_1 and t_1 , respectively.

Proof. We will only prove that: $l^2_{Finite}(\mathcal{Q}_0) = \bigcup_{\substack{F \subseteq \mathcal{Q}_0 \\ \text{finite}}} l^2(F)$ is dense in t_1 , since the other claim can be proved in exactly the same manner.

Let F be a finite subset of $\mathcal{Q}_0$ and let $\alpha \in l^2(F)$.

Using Hölder's Inequality and observing that: $S_0(\alpha) = \left(\sum_{Q \in F} |\alpha(Q)|^2 |Q|^{-1} \chi_Q \right)^{1/2}$, so that: $\text{supp}(S_0(\alpha)) \subseteq \bigcup_{Q \in F} Q$, giving us: $|\text{supp}(S_0(\alpha))| \leq \sum_{Q \in F} |Q|$, we get by exploiting Remark 4.1.1:

$$\|\alpha\|_{t_1} = \|S_0(\alpha)\|_1 \leq |\text{supp}(S_0(\alpha))|^{1/2} \|S_0(\alpha)\|_2 \leq \left(\sum_{Q \in F} |Q| \right)^{1/2} \|\alpha\|_{l^2(F)}.$$

On the other hand, using Hölder's Inequality and by Remark 4.1.1 again, we get:

$$\|\alpha\|_{l^2(F)}^2 = \|S_0(\alpha)\|_2^2 \leq \|S_0(\alpha)\|_\infty \|S_0(\alpha)\|_1 = \|S_0(\alpha)\|_\infty \|\alpha\|_{t_1}$$

But we have:

$$\begin{aligned} \|S_0(\alpha)\|_\infty &= \sup_{x \in \mathbb{R}^n} \left(\sum_{Q \in F} |\alpha(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2} \\ &\leq \left(\inf_{Q \in F} |Q| \right)^{-1/2} \left(\sum_{Q \in F} |\alpha(Q)|^2 \right)^{1/2} \\ &= \left(\inf_{Q \in F} |Q| \right)^{-1/2} \|\alpha\|_{l^2(F)} \end{aligned}$$

Simplifying and combining the results, we get:

$$\left(\inf_{Q \in F} |Q| \right)^{1/2} \|\alpha\|_{l^2(F)} \leq \|\alpha\|_{t_1} \leq \left(\sum_{Q \in F} |Q| \right)^{1/2} \|\alpha\|_{l^2(F)} \quad (4.1)$$

Now, let $\alpha \in t_1$ and let $\{F_m\}_{m \in \mathbb{N}}$ be an “exhaustion” of $\mathcal{Q}_0$, i.e. a sequence of finite subsets of $\mathcal{Q}_0$ st: for any $m \in \mathbb{N}$, we have: $F_m \subseteq F_{m+1}$ and: $\bigcup_{m \in \mathbb{N}} F_m = \mathcal{Q}_0$.

We can see that $\{\alpha \mathbb{1}_{F_m}\}_{m \in \mathbb{N}}$ is a sequence of elements of $l_{Finite}^2(\mathcal{Q}_0)$. We then observe also that: $\{S_0(\alpha \mathbb{1}_{F_m})\}_{m \in \mathbb{N}}$ is a non-decreasing sequence of nonnegative functions converging pointwise to $S_0(\alpha)$ which is integrable (as $\alpha \in t_1$) and dominates the said sequence. Thus, the Monotone Convergence Theorem tells us that: $\lim_{m \rightarrow \infty} \|S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m})\|_1 = 0$.

Now, fixing $m \in \mathbb{N}$ and observing that: $(S_0(\alpha - \alpha \mathbb{1}_{F_m}))^2 = (S_0(\alpha))^2 - (S_0(\alpha \mathbb{1}_{F_m}))^2$, we get that:

$$\begin{aligned}
\|\alpha - \alpha \mathbb{1}_{F_m}\|_{t_1} &= \|S_0(\alpha - \alpha \mathbb{1}_{F_m})\|_1 \\
&\leq \int_{\mathbb{R}^n} (S_0(\alpha) + S_0(\alpha \mathbb{1}_{F_m}))^{\frac{1}{2}} (S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m}))^{1/2} \\
&\leq \left(\int_{\mathbb{R}^n} S_0(\alpha) + S_0(\alpha \mathbb{1}_{F_m}) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m}) \right)^{1/2} \quad ((\text{by Hölder's Inequality})) \\
&\leq \|S_0(\alpha) + S_0(\alpha \mathbb{1}_{F_m})\|_1^{\frac{1}{2}} \|S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m})\|_1^{1/2} \\
&\leq \sqrt{2 \|S_0(\alpha)\|_1} \|S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m})\|_1^{1/2} \quad (\text{Triangle Inequality} + \|S_0(\alpha \mathbb{1}_{F_m})\|_1 \leq \|S_0(\alpha)\|_1) \\
&\leq \sqrt{2 \|\alpha\|_{t_1}} \|S_0(\alpha) - S_0(\alpha \mathbb{1}_{F_m})\|_1^{1/2}
\end{aligned}$$

Hence: $\lim_{m \rightarrow \infty} \|\alpha - \alpha \mathbb{1}_{F_m}\|_{t_1} = 0$. Concluding the proof of the density as required. $\square$

This establishes a desired result that would allow us to proceed and state an important theorem central to this section.

Theorem 4.1.1. $(T_1, \|\cdot\|_{T_1})$ and $(t_1, \|\cdot\|_{t_1})$ are Banach spaces.

Proof. We will prove that $(t_1, \|\cdot\|_{t_1})$ is a Banach space, as the proof for $(T_1, \|\cdot\|_{T_1})$ is exactly similar. We will borrow ideas from the work of the authors in [2] and adapt them from the continuous realm to the discrete context.

Now, let $\{\alpha_j\}_{j \in \mathbb{N}}$ be a Cauchy sequence of elements of t_1 .

From Lemma 4.1.1 and Inequality (4.1), for any finite subset F of $\mathcal{Q}_0$, $\{\alpha_j \mathbb{1}_F\}_{j \in \mathbb{N}}$ is also a Cauchy sequence of elements of $l^2(F)$, which is a Banach space, thus converging to some $\alpha_F \in l^2(F)$.

Now, let $\{F_m\}_{m \in \mathbb{N}}$ be an “exhaustion” of $\mathcal{Q}_0$. For any $Q \in \mathcal{Q}_0$, we define: $\alpha(Q) = \alpha_{F_m}(Q)$, if $Q \in F_m$, for some $m \in \mathbb{N}$, which can be written as: $\alpha_{F_m} = \alpha \mathbb{1}_{F_m}$, for any $m \in \mathbb{N}$. α is well-defined since if we choose any $p > q \in \mathbb{N}$, we also have the consistent definition: $\alpha_{F_p} \mathbb{1}_{F_q} = \alpha_{F_q}$.

First, we claim that: $\alpha \in t_1$. Indeed, fix an $m \in \mathbb{N}$ and take any $j \in \mathbb{N}$. We have $\alpha_{F_m} = \lim(\alpha_j \mathbb{1}_{F_m})$ in $l^2(F_m)$. So, combined with Inequality (4.1) and the fact that $\{\alpha_j\}_{j \in \mathbb{N}}$ is a Cauchy sequence in a normed vector space (thus metric) which shows that it is uniformly

bounded by a constant $M > 0$ in $\|\cdot\|_{t_1}$, this also tells us that: $\|\alpha_{F_m}\|_{t_1} = \lim \|\alpha_j \mathbb{1}_{F_m}\|_{t_1} \leq \sup \|\alpha_j\|_{t_1} < M$. Now using the proof of Lemma 4.1.1, for $\epsilon = 1$, choosing sufficiently large m , we have that: $\|\alpha\|_{t_1} < \|\alpha_{F_m}\|_{t_1} + 1 < M + 1 < \infty$.

Now, we have that: $\lim_{m \rightarrow \infty} \|\alpha - \alpha_{F_m}\|_{t_1} = 0$, for any $j \in \mathbb{N}$: $\lim_{m \rightarrow \infty} \|\alpha_j \mathbb{1}_{F_m} - \alpha_j\|_{t_1} = 0$, and for any $m \in \mathbb{N}$: $\lim_{j \rightarrow \infty} \|\alpha_{F_m} - \alpha_j \mathbb{1}_{F_m}\|_{l^2(F_m)} = 0$.

So, fixing $\epsilon > 0$, we can choose $N, m \in \mathbb{N}$, s.t: $\forall j \geq N$, we have: $\|\alpha - \alpha \mathbb{1}_{F_m}\|_{t_1} = \|\alpha - \alpha_{F_m}\|_{t_1} < \frac{\epsilon}{3}$, also $\|\alpha_j \mathbb{1}_{F_m} - \alpha_j\|_{t_1} < \frac{\epsilon}{3}$ and finally: $\|\alpha_{F_m} - \alpha_j \mathbb{1}_{F_m}\|_{l^2(F_m)} < \frac{\epsilon}{3} \implies \|\alpha_{F_m} - \alpha_j \mathbb{1}_{F_m}\|_{t_1} \prec_{F_m} \frac{\epsilon}{3}$. Hence, we get: $\|\alpha - \alpha_j\|_{t_1} \leq \|\alpha - \alpha_{F_m}\|_{t_1} + \|\alpha_{F_m} - \alpha_j \mathbb{1}_{F_m}\|_{t_1} + \|\alpha_j \mathbb{1}_{F_m} - \alpha_j\|_{t_1} \prec_{F_m} \epsilon$.

Concluding the proof of completeness. $\square$

We now define an equivalence relation that would allow us to redefine the relationship between t_1 and T_1 .

Definition 4.1.4. Two sequences $\alpha = \{\alpha(Q)\}_{Q \in \mathcal{Q}}$ and $\beta = \{\beta(Q)\}_{Q \in \mathcal{Q}}$ in T_1 are said to be equivalent and denoted $\alpha \sim \beta$, if and only if $\{\alpha(Q)\}_{Q \in \mathcal{Q}_0} = \{\beta(Q)\}_{Q \in \mathcal{Q}_0}$. We denote by $\tilde{T}_1$ the quotient space defined by the equivalence relation mentioned above.

We can verify that $\tilde{T}_1 = T_1/M$ where M is the subspace of sequences vanishing on large dyadic cubes, which is closed in T_1 . This induces a quotient norm $\|\cdot\|_{\tilde{T}_1}$ on $\tilde{T}_1$ that makes it into a Banach space itself. And this suggests the following.

Proposition 4.1.2. $\tilde{T}_1 \hookrightarrow t_1$.

Proof. We define the map:

$$\begin{aligned} \tilde{\mathbb{1}}_{\mathcal{Q}_0} : \tilde{T}_1 &\longrightarrow t_1 \\ [\alpha] &\longmapsto \{\alpha(Q)\}_{Q \in \mathcal{Q}_0} \end{aligned}$$

It is linear and injective. Moreover, it is continuous because if α is an element of T_1 and m is any element of T_1 supported only on small dyadic cubes, then: $\|\alpha \mathbb{1}_{\mathcal{Q}_0}\|_{t_1} \leq \|\alpha + m\|_{T_1}$, and thus by taking the infimum over all such m 's, we get: $\|\alpha \mathbb{1}_{\mathcal{Q}_0}\|_{t_1} \leq \|[\alpha]\|_{\tilde{T}_1}$. So, indeed, $\tilde{\mathbb{1}}_{\mathcal{Q}_0}$ is a continuous embedding from $\tilde{T}_1$ into t_1 . $\square$

Remark 4.1.2. We identify a sequence $\alpha \in T_1$ with its equivalence class $[\alpha] \in \tilde{T}_1 = T_1/M$ where $M \subseteq T_1$ denotes the subspace of sequences vanishing on dyadic cubes of sidelength greater than or equal to 1. In essence, this quotient structure represses the large-scale structure of T_1 , retaining only the fine-scale behavior encoded by the coefficients on the set of small dyadic cubes. The resulting space may be understood as a refined tent space, in which sequences are considered indistinguishable when they agree on the set of small cubes. Here, the quotient describes the microscopic characteristics of the said sequence, thus abstracting away its macroscopic behavior. Thus, Proposition 4.1.2 allows us, by abuse of language, to consider that the tent space T_1 can be continuously embedded in the local tent space t_1 by truncating its entries to $\mathcal{Q}_0$ and thus can be seen as one of its proper subspaces in this regard; and naturally $\|\cdot\|_{t_1} \leq \|\cdot\|_{T_1}$.

Next, we continue to define some important elements of our two tent spaces, which we will see as the building blocks of these spaces.

Definition 4.1.5. We say that the sequences $\alpha = \{\alpha(Q)\}_{Q \in \mathcal{Q}_0}$ and $\beta = \{\beta(Q)\}_{Q \in \mathcal{Q}}$ are t_1 -atom and T_1 -atom, respectively, if there exists a small dyadic cube R containing $\text{socle}(\alpha)$ and there exists a dyadic cube S containing $\text{socle}(\beta)$, respectively, s.t:

$$\sum_{Q \in \mathcal{Q}_0} |\alpha(Q)|^2 \leq \frac{1}{|R|}$$

and:

$$\sum_{Q \in \mathcal{Q}} |\beta(Q)|^2 \leq \frac{1}{|S|}$$

We call the smallest such dyadic cubes R and S , the base of the atoms α and β , respectively.

Remark 4.1.3. From Definition 4.1.5, if α and β are a t_1 -atom and a T_1 -atom with bases R and S , respectively, then: $\text{supp}(\alpha) \subseteq \mathcal{T}_0(R)$ and $\text{supp}(\beta) \subseteq \mathcal{T}(S)$.

Notice also, the parallel between t_1 -atoms and T_1 -atoms in one side, and h^1 -atoms and H^1 -atoms in the other side, in terms of their l^2 and L^2 definitions.

We now state and prove some important properties about these newly defined atoms.

Proposition 4.1.3. *If α is a t_1 -atom and β is a T_1 -atom, then: $\alpha \in t_1$, $\beta \in T_1$, moreover $\|\alpha\|_{t_1}, \|\beta\|_{T_1} \leq 1$.*

Proof. We are going to prove the case for α a t_1 -atom with base R , as the other case is similar in terms of proof.

By Cauchy-Schwarz Inequality, Remark 4.1.1, Remark 4.1.3 and Definition 4.1.5, we have:

$$\int_{\mathbb{R}^n} S_0(\alpha)(x) dx = \int_R S_0(\alpha)(x) dx \leq |R|^{\frac{1}{2}} \left(\int_R \left(S_0(\alpha)(x) \right)^2 dx \right)^{\frac{1}{2}} = |R|^{\frac{1}{2}} \|\alpha\|_{l^2} \leq 1.$$

Concluding the argument. □

Remark 4.1.2 allows us, by abuse of language, to note that every T_1 -atom is a t_1 -atom, but the converse is not true, in general.

Before diving into the next important topic of this section, we need to define an important notion related to measurable subsets of $\mathbb{R}^n$.

Definition 4.1.6. *For $f \in L^1_{Loc}(\mathbb{R}^n)$, the dyadic maximal function $M^d f$ and the local dyadic maximal function $M_0^d f$ are defined by $M^d f(x) := \sup_{\substack{Q \in \mathcal{Q} \\ x \in Q}} \int_Q |f(y)| dy$ and $M_0^d f(x) := \sup_{\substack{Q \in \mathcal{Q}_0 \\ x \in Q}} \int_Q |f(y)| dy$, for any $x \in \mathbb{R}^n$, respectively.*

These dyadic maximal functions are simplified versions of the Hardy-Littlewood maximal function, but still retain some key properties such as weak-type $(1,1)$ boundedness, pointwise control over singular integrals, yet offering complete compatibility with dyadic analysis and martingale theory.

If $f = \chi_E$ where E is a measurable subset of $\mathbb{R}^n$ with $|E| < \infty$, then these dyadic maximal functions measure the “dyadic density” of E around a point $x \in \mathbb{R}^n$, in other terms, how much of the nearby dyadic cubes are occupied by E (measured by normalized averages). If the dyadic density is close to 1, then E occupies almost all of every dyadic cube containing x . On the other hand, if it is close to 0, then E is sparse near x .

Definition 4.1.7. Fixing $\beta \in (0, 1)$, for any measurable subset E of $\mathbb{R}^n$ with $|E| < \infty$, we define the “magnification” and the “local magnification” of E by $E^\star := \{x \in \mathbb{R}^n : M^d \chi_E(x) > \beta\}$ and $E_0^\star := \{x \in \mathbb{R}^n : M_0^d \chi_E(x) > \beta\}$, respectively.

We proceed to outline and prove the main properties of these magnifications which will be useful in the proof of the next theorem.

Proposition 4.1.4. Fixing $\beta \in (0, 1)$, for measurable subsets E and F of $\mathbb{R}^n$ with $|E|, |F| < \infty$, we have the following:

- (a) If $\mathcal{C} = \{Q \in \mathcal{Q} : |Q \cap E| > \beta |Q|\}$ and $\mathcal{C}_0 = \{Q \in \mathcal{Q}_0 : |Q \cap E| > \beta |Q|\}$ then $E^\star = \cup_{Q \in \mathcal{C}} Q$ and $E_0^\star = \cup_{Q \in \mathcal{C}_0} Q$
- (b) $|E^\star| < \beta^{-1} |E|$ and $|E_0^\star| < \beta^{-1} |E|$
- (c) $E_0^\star \subseteq E^\star$
- (d) $E_0^\star$ contains E up to a set of zero measure
- (e) If $E \subseteq F$ then $E_0^\star \subseteq F_0^\star$ and $E^\star \subseteq F^\star$

Proof. we will prove only the local part as the proof of the other is similar.

(a) If $Q \in \mathcal{C}_0$ and if $x \in Q$. Then: $\int_Q \chi_E = \frac{|Q \cap E|}{|Q|} > \beta$ with $x \in Q$. Thus: $x \in \mathbb{R}^n$ and

$\sup_{\substack{Q \in \mathcal{Q}_0 \\ x \in Q}} \int_Q \chi_E = M_0^d \chi_E(x) > \beta$. So: $x \in E_0^\star$.

On the other hand, if $x \in E_0^\star$, then: $\sup_{\substack{Q \in \mathcal{Q}_0 \\ x \in Q}} \int_Q \chi_E = \sup_{\substack{Q \in \mathcal{Q}_0 \\ x \in Q}} \frac{|Q \cap E|}{|Q|} > \beta$. Therefore: $\frac{|Q \cap E|}{|Q|} > \beta$

for some $Q \in \mathcal{Q}_0$ with $x \in Q$.

(b) $\mathcal{C}_0$ is a collection of small dyadic cubes of a bounded size, so each cube $Q \in \mathcal{C}_0$ is contained in a maximal cube $Q_{max} \in \mathcal{C}_0$. These maximal cubes are pairwise disjoint and form a partition of $\mathcal{C}_0$ and thus of $E_0^\star$. So we get: $|E_0^\star| = \sum |Q_{max}| < \beta^{-1} \sum |Q_{max} \cap E| \leq \beta^{-1} |E|$.

(c) Immediate from part (a) and $\mathcal{C}_0 \subseteq \mathcal{C}$.

(d) Applying the Lebesgue Differentiation Theorem to the measurable function χ_E , we get the following:

$$\lim_{\substack{|Q| \rightarrow 0 \\ x \in Q}} \frac{|Q \cap E|}{|Q|} = \lim_{\substack{|Q| \rightarrow 0 \\ x \in Q}} \frac{1}{|Q|} \int_Q \chi_E = \chi_E(x) = 1, \text{ for a.e. } x \in E.$$

So for a.e. $x \in E$, there exists $Q \in \mathcal{Q}_0$ st: $x \in Q$ and: $|Q \cap E| > \beta |Q|$. Hence: $Q \subseteq E_0^\star$ and thus: $x \in E_0^\star$.

(e) Immediate from part (a) and $|Q \cap F| \geq |Q \cap E|$ for any small dyadic cube Q . $\square$

Next, we turn our attention to an important theorem of this section in terms of the atomic decomposition of the tent spaces that would also enable us to rigorously prove the central theorem of the next section in terms of the wavelets atomic decomposition. We followed here the footsteps of Meyer in [44, Theorem 2, p. 145].

Theorem 4.1.2. *T.F.A.E for α :*

(i) $\alpha \in t_1$ ($\alpha \in T_1$)

(ii) *There exists $\gamma \in (0, 1]$ st: for any $Q \in \mathcal{Q}_0$ ($Q \in \mathcal{Q}$), there exists a measurable subset*

$$R \subseteq Q \text{ with: } |R| \geq \gamma |Q| \text{ st: } \sigma_0(x) := \left(\sum_{Q \in \mathcal{Q}_0} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n) \quad \left(\sigma(x) := \left(\sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n) \right)$$

(iii) *There exist a sequence of t_1 -atoms (T_1 -atoms) $\{\alpha_j\}$ and a sequence of scalars $\{\lambda_j\}$ st:*

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty \text{ and } \alpha = \sum_{j=1}^{\infty} \lambda_j \alpha_j$$

Proof. we will prove only the local part as the proof of the other is similar.

(i) $\implies$ (ii)

Suppose that: $\alpha \in t_1$. Then, taking $\gamma = 1$ and Q as a measurable subset of itself for any small dyadic cube, we have, by definition of an element of the local tent space, that:

$$S_0(\alpha) = \left(\sum_{Q \in \mathcal{Q}_0} |\alpha(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2} \in L^1(\mathbb{R}^n).$$

(ii) $\implies$ (iii)

Suppose that there exists $\gamma \in (0, 1]$ st: for any $Q \in \mathcal{Q}_0$, there exists a measurable subset $R \subseteq Q$ with: $|R| \geq \gamma |Q|$ st: $\sigma_0(x) = \left(\sum_{Q \in \mathcal{Q}_0} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$.

Fix $k \in \mathbb{Z}$ and $\beta \in (0, \gamma)$, and define: $E_k = \{x \in \mathbb{R}^n : \sigma_0(x) > 2^k\}$. First thing: E_k is a subset of $\mathbb{R}^n$ of finite measure, because otherwise $\sigma_0 \notin L^1(\mathbb{R}^n)$. Second thing: $E_{k+1} \subseteq E_k$. And finally, noticing that $(E_j \setminus E_{j+1})_{j \in \mathbb{Z}}$ is a partition of $\mathbb{R}^n$ and that $|E_j \setminus E_{j+1}| = |E_j| - |E_{j+1}|$, we get:

$$\sum_{j=-\infty}^{\infty} 2^j |E_j| \leq 2 \int_{\mathbb{R}^n} \sigma_0 = 2 \|\sigma_0\|_1.$$

Also, define $\mathcal{C}_{k,0} = \{Q \in \mathcal{Q}_0 : |Q \cap E_k| > \beta |Q|\}$. From what we showed earlier in Proposition 4.1.4, we have: $E_{k,0}^* = \cup_{Q \in \mathcal{C}_{k,0}} Q$ and: $|E_{k,0}^*| < \beta^{-1} |E_k|$. So that: $\sum_{j=-\infty}^{\infty} 2^j |E_{j,0}^*| < 2 \beta^{-1} \|\sigma_0\|_1$. Now, let $(Q(k, l))_l$ denote the small dyadic cubes that are maximal in $\mathcal{C}_{k,0}$. They form a partition of $E_{k,0}^*$.

We claim: $\text{supp}(\alpha) \subseteq \cup_{j \in \mathbb{Z}} \mathcal{C}_{j,0}$. Indeed: if $\alpha(Q) \neq 0$, then for some $j \in \mathbb{Z}$: $|\alpha(Q)| |Q|^{-\frac{1}{2}} > 2^j$, so that: $\sigma_0(x) > 2^j$ for all $x \in R \subseteq Q$. So, by the definition of E_j , we get: $R \subseteq Q \cap E_j$, from which we obtain: $|Q \cap E_j| \geq |R| \geq \gamma |Q| > \beta |Q|$. Thus: $Q \in \mathcal{C}_{j,0}$.

Now, let: $\Delta_k = \mathcal{C}_{k,0} \setminus \mathcal{C}_{k+1,0}$, and subsequently define: $\Delta(k, l) = \{Q : Q \in \Delta_k \text{ and } Q \subseteq Q(k, l)\}$. We can see then that the sets $\Delta(k, l)$ form a partition of Δ_k , though some of them may be empty, and in that case should be omitted. Hence: $\text{supp}(\alpha)$ is the disjoint union of the sets $(\Delta(k, l))_{(k, l) \in \mathbb{Z} \times F(k)}$, where we define $F(k) = \{l : \Delta(k, l) \neq \emptyset\}$.

Now that we have reached a convenient partition of $\text{supp}(\alpha)$, we turn our attention to an important inequality. We claim that for fixed $(k, l) \in \mathbb{Z} \times F(k)$:

$$\sum_{Q \in \Delta(k, l)} |\alpha(Q)|^2 \leq \frac{1}{\gamma - \beta} \int_{Q(k, l) \setminus E_{k+1}} \sigma_0^2(x) dx \leq \frac{4^{k+1} |Q(k, l)|}{\gamma - \beta}$$

Indeed: if $Q \in \Delta(k, l)$, then: $Q \subseteq Q(k, l)$ and $Q \notin \mathcal{C}_{k+1,0}$. Thus: $|Q \cap E_{k+1}| \leq \beta |Q|$, so as a result: $|R \cap E_{k+1}| \leq \beta |Q| \leq \frac{\beta}{\gamma} |R|$. Hence, we get: $|R \setminus E_{k+1}| \geq (1 - \frac{\beta}{\gamma}) |R| \geq (\gamma - \beta) |Q|$.

But for any $x \in \mathbb{R}^n$, we have:

$$\sigma_0^2(x) \geq \sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x)$$

So, we get the following, which proves the left-hand side of the inequality:

$$\int_{Q(k,l) \setminus E_{k+1}} \sigma_0^2(x) dx \geq \sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 |Q|^{-1} |R \setminus E_{k+1}| \geq (\gamma - \beta) \sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2$$

Now, when $x \notin E_{k+1} \implies \sigma_0(x) \leq 2^{k+1} \implies \int_{Q(k,l) \setminus E_{k+1}} \sigma_0^2(x) dx \leq 4^{k+1} |Q(k,l)|$, which proves the right-hand side.

Now, denote: $\lambda(k,l) = |Q(k,l)|^{\frac{1}{2}} \left(\sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 \right)^{\frac{1}{2}}$

$$\text{And set: } \alpha_{k,l}(Q) = \begin{cases} \frac{1}{\lambda(k,l)} \alpha(Q) & \text{when: } Q \in \Delta(k,l) \\ 0 & \text{otherwise} \end{cases}$$

By construction, we see from the above that the $\alpha_{k,l}$'s are t_1 -atoms with bases contained in $Q(k,l)$'s, and that: $\alpha|_{\Delta(k,l)} = \lambda(k,l) \alpha_{k,l}$. Hence: $\alpha = \sum_{k \in \mathbb{Z}} \sum_{l \in F(k)} \lambda(k,l) \alpha_{k,l}$.

Furthermore, we have the following:

$$\sum_{k \in \mathbb{Z}} \sum_{l \in F(k)} \lambda(k,l) \leq \frac{2}{(\gamma - \beta)^{\frac{1}{2}}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} 2^k |Q(k,l)| = \frac{2}{(\gamma - \beta)^{\frac{1}{2}}} \sum_{k \in \mathbb{Z}} 2^k |E_{k,0}^*| \leq \frac{4}{\beta(\gamma - \beta)^{\frac{1}{2}}} \|\sigma_0\|_1 < \infty.$$

This concludes the proof of this direction.

(iii) $\implies$ (i)

Assume now that there exist a sequence of t_1 -atoms $\{\alpha_j\}$ and a sequence of scalars $\{\lambda_j\}$ s.t:

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty \text{ and } \alpha = \sum_{j=1}^{\infty} \lambda_j \alpha_j.$$

$$\text{This means that: } \lim_{N \rightarrow \infty} \left\| \alpha - \sum_{j=1}^N \lambda_j \alpha_j \right\|_{t_1} = 0.$$

$$\text{But, for any } N \in \mathbb{N}, \text{ we have that: } \left| \|\alpha\|_{t_1} - \left\| \sum_{j=1}^N \lambda_j \alpha_j \right\|_{t_1} \right| \leq \left\| \alpha - \sum_{j=1}^N \lambda_j \alpha_j \right\|_{t_1}.$$

Therefore, by the Triangle Inequality, uniform boundedness of the t_1 -atoms proven in Proposi-

tion 4.1.3, and passing to the limit, we have:

$$\|\alpha\|_{t_1} = \lim_{N \rightarrow \infty} \left\| \sum_{j=1}^N \lambda_j \alpha_j \right\|_{t_1} \leq \lim_{N \rightarrow \infty} \sum_{j=1}^N |\lambda_j| \|\alpha_j\|_{t_1} \leq \sum_{j=1}^{\infty} |\lambda_j| < \infty.$$

Thus, by definition, $\alpha \in t_1$. □

4.2 Wavelets Characterization of the local Hardy space $h^1(\mathbb{R}^n)$

In what follows, we will adapt the work of Meyer [44, pp. 143–149] to provide a complete characterization of $h^1(\mathbb{R}^n)$ and some equivalent norms associated with it.

Recall that we denote: $E = \{0, 1\}^n \setminus \{0_{\mathbb{R}^n}\}$, $\Lambda = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{Z}, e \in E\}$ and: $\Lambda^{j_0} = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{Z}, j \geq j_0, e \in E\}$ for any $j_0 \in \mathbb{Z}$. So, in particular: $\Lambda^0 = \{\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}, p \in \mathbb{Z}^n, j \in \mathbb{N} \cup \{0\}, e \in E\}$.

And recall that for $f \in L^1(\mathbb{R}^n)$, we denote $\alpha(\lambda) = (f, \psi_\lambda)$ and $\beta(k) = (f, \phi_k)$, where $\psi_\lambda, \lambda \in \Lambda$ and $\phi_k, k \in \mathbb{Z}^n$, are derived from the mother and father wavelets, respectively, as shown in the wavelet construction section where we introduced them.

For $\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}$ where $p \in \mathbb{Z}^n, j \in \mathbb{Z}$ and $e \in E$, we can explicitly write for $x \in \mathbb{R}^n$: $\psi_\lambda(x) = 2^{\frac{nj}{2}} \psi^e(2^j x - p)$. In what follows, we may assume that: $\int_{[0,1]^n} |\psi^e(x)|^2 dx \neq 0$, replacing, if necessary, the basic wavelets by convenient translates by some $k \in \mathbb{Z}^n$. This readily means that: $|\psi^e(x)| \geq c > 0$ when x is in some cube $A^e \subseteq [0,1]^n$ that satisfies: $|A^e| \geq \gamma > 0$, where c and γ are some constants.

For any $\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}}$, we denote by $Q(\lambda)$ the dyadic cube defined by $2^j x - p \in [0,1]^n$ and by $R(\lambda)$, the subcube defined by: $2^j x - p \in A^e$, so that we always have: $|R(\lambda)| \geq \gamma |Q(\lambda)|$.

Now, we are ready to state our main theorem.

Theorem 4.2.1. *For $f \in L^1(\mathbb{R}^n)$, T.F.A.E definitions of $f \in h^1(\mathbb{R}^n)$:*

$$(A) \sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\varepsilon(\lambda) = \pm 1} \left\| \left(\sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right) \right\|_1 < \infty$$

$$(B) \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

$$(C) \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n) \quad (\text{where } |Q(\lambda)| \leq 1, R(\lambda) \subseteq Q(\lambda) \text{ and } |R(\lambda)| \geq \gamma |Q(\lambda)| \text{ for some constant } \gamma > 0 \text{ for any } \lambda \in \Lambda^0)$$

$$(D) \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

$$(E) f = \sum_{j=1}^{\infty} \lambda_{1,j} A_j + \sum_{j=1}^{\infty} \lambda_{2,j} a_j \quad (\text{where the } A_j \text{'s and the } a_j \text{'s are } h^1(\mathbb{R}^n)\text{-atoms associated to large and small cubes, respectively, and } \sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty, \text{ as well as } \sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty)$$

$$(F) f = \sum_{j=1}^{\infty} \lambda_{1,j} A_j + \sum_{j=1}^{\infty} \lambda_{2,j} a_j \quad (\text{where the } A_j \text{'s and the } a_j \text{'s are } h^1(\mathbb{R}^n)\text{-atoms associated to large and small balls, respectively, and } \sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty, \text{ as well as } \sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty)$$

$$(G) \text{ For any } \varphi \in L_c^\infty(\mathbb{R}^n) \cup \mathcal{S}(\mathbb{R}^n) \text{ st: } \int \varphi = 1, \text{ we have that: } (f - \varphi * f) \in H^1(\mathbb{R}^n)$$

Proof.

(A) $\implies$ (B)

$$\text{Suppose that: } \sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\varepsilon(\lambda) = \pm 1} \left\| \left(\sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right) \right\|_1 < \infty$$

This means that $\exists C_1 > 0$ st: for any finite subset $F \subseteq \Lambda^0$ and for every sequence $\varepsilon(\lambda), \lambda \in F$, taking the values ± 1 , we have:

$$\left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right\|_1 \leq C_1 \quad (4.2)$$

On the other hand, applying the Khinchin's Inequality to any finite subset $F \subseteq \Lambda^0$ and any $x \in \mathbb{R}^n$, we get that:

$$\left(\sum_{\lambda \in F} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \leq \frac{C_2}{2^{|F|}} \sum_{\text{sequences } \varepsilon = \pm 1} \left| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right|, \text{ for some } C_2 > 0.$$

Hence, passing to the $L^1(\mathbb{R}^n)$ norm, we get:

$$\left\| \left(\sum_{\lambda \in F} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \right\|_1 \leq \frac{C_2}{2^{|F|}} \sum_{\text{sequences } \varepsilon = \pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right\|_1$$

Using Inequality (4.2), we obtain the following:

$$\left\| \left(\sum_{\lambda \in F} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \right\|_1 \leq \frac{C_2}{2^{|F|}} \sum_{\text{sequences } \varepsilon = \pm 1} C_1$$

Since there are $2^{|F|}$ sequences of exactly $|F|$ elements taking the values ± 1 , we get the following:

$$\left\| \left(\sum_{\lambda \in F} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \right\|_1 \leq C_1 C_2$$

Therefore, by the Monotone Convergence Theorem, we get:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \right\|_1 \leq C_1 C_2$$

Allowing us to conclude:

$$\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

(B) $\implies$ (C)

Now, assume that: $\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$

From how $Q(\lambda)$ and $R(\lambda)$ were defined, for any $\lambda \in \Lambda^0$ and any $x \in R(\lambda)$, we get the following:

$$|\psi_\lambda(x)| \geq \frac{c}{|Q(\lambda)|^{\frac{1}{2}}}, \text{ for some constant } c > 0.$$

This means that for any $\lambda \in \Lambda^0$ and any $x \in \mathbb{R}^n$, we get: $|\psi_\lambda(x)| \geq \frac{c}{|Q(\lambda)|^{\frac{1}{2}}} \chi_{R(\lambda)}(x)$.

So we obtain that:

$$\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

(C) $\implies$ (D)

Assume: $\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$ (where $|Q(\lambda)| \leq 1$, $R(\lambda) \subseteq Q(\lambda)$ and $|R(\lambda)| \geq \gamma |Q(\lambda)|$ for some constant $\gamma > 0$ for any $\lambda \in \Lambda^0$).

This can be rewritten as:

$$\left(\sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} |\alpha^e(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

In particular, for any $e \in E$, we have that:

$$\left(\sum_{Q \in \mathcal{Q}_0} |\alpha^e(Q)|^2 |Q|^{-1} \chi_{R(Q)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

So, by Theorem 4.1.2, for any $e \in E$, we have that: $\{\alpha^e(Q)\}_{Q \in \mathcal{Q}_0} \in t_1$

Again, by Theorem 4.1.2, for any $e \in E$, we get:

$$\left(\sum_{Q \in \mathcal{Q}_0} |\alpha^e(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

We conclude:

$$\left(\sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} |\alpha^e(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{\frac{1}{2}} = \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

(D) $\implies$ (E)

Assume now that: $\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$

Again, in particular, for any $e \in E$, we have that:

$$\left(\sum_{Q \in \mathcal{Q}_0} |\alpha^e(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$$

Thus, again, for any $e \in E$, we have that: $\{\alpha^e(Q)\}_{Q \in \mathcal{Q}_0} \in t_1$

So, by the Theorem 4.1.2, for any $e \in E$, we see that $\{\alpha^e(Q)\}_{Q \in \mathcal{Q}_0}$ admits an atomic decom-

position (in the sense of the t_1 space) in the following form:

$$\alpha^e(Q) = \sum_{j=1}^{\infty} \lambda_j^e \alpha_j^e(Q), \forall Q \in \mathcal{Q}_0 \text{ with } \sum_{j=1}^{\infty} |\lambda_j^e| < \infty \text{ and } \{\alpha_j^e\}_{j \in \mathbb{N}} \text{ a sequence of } t_1\text{-atoms.} \quad (4.3)$$

As shown in [44, Formula 10.2, p. 110], in the sense of distributions:

$$f = \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k + \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda$$

Rewriting the above expression, we have that in the sense of distributions:

$$f - \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k = \sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} \alpha^e(Q) \psi_Q^e$$

Meaning, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$:

$$\langle f - \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k, \varphi \rangle = \langle \sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} \alpha^e(Q) \psi_Q^e, \varphi \rangle = \sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} \alpha^e(Q) \langle \psi_Q^e, \varphi \rangle$$

Using the decomposition (4.3), for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have:

$$\langle f - \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k, \varphi \rangle = \sum_{e \in E} \sum_{Q \in \mathcal{Q}_0} \sum_{j=1}^{\infty} \lambda_j^e \alpha_j^e(Q) \langle \psi_Q^e, \varphi \rangle$$

Now, fixing a $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and as E is finite, fixing an $e \in E$, we want to show that:

$$\sum_{Q \in \mathcal{Q}_0} \sum_{j=1}^{\infty} \lambda_j^e \alpha_j^e(Q) \langle \psi_Q^e, \varphi \rangle = \sum_{j=1}^{\infty} \sum_{Q \in \mathcal{Q}_0} \lambda_j^e \alpha_j^e(Q) \langle \psi_Q^e, \varphi \rangle$$

It suffices then to show that the right-hand side is absolutely convergent, that is:

$$\sum_{j=1}^{\infty} \sum_{Q \in \mathcal{Q}_0} |\lambda_j^e| |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| = \sum_{j=1}^{\infty} |\lambda_j^e| \sum_{Q \in \mathcal{Q}_0} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| < \infty$$

Indeed, let $j \in \mathbb{N}$. By the definition of the socle of the t_1 -atom $(\alpha_j^e(Q))_{Q \in \mathcal{Q}_0}$, denoted $\text{socle}(\alpha_j^e)$, and its base, denoted Q_j^e , we have the following:

$$\sum_{Q \in \mathcal{Q}_0} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| = \sum_{Q \subseteq \text{socle}(\alpha_j^e)} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| = \sum_{Q \subseteq Q_j^e} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle|$$

Now, using the Cauchy-Schwarz Inequality, we have the following:

$$\sum_{Q \subseteq Q_j^e} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| \leq \left(\sum_{Q \subseteq Q_j^e} |\alpha_j^e(Q)|^2 \right)^{\frac{1}{2}} \left(\sum_{Q \subseteq Q_j^e} |\langle \psi_Q^e, \varphi \rangle|^2 \right)^{\frac{1}{2}}$$

By the definition of a t_1 -atom and its base, we have: $\sum_{Q \in \mathcal{Q}_0} |\alpha_j^e(Q)|^2 = \sum_{Q \subseteq Q_j^e} |\alpha_j^e(Q)|^2 \leq \frac{1}{|Q_j^e|}$

On the other hand, as $\varphi \in \mathcal{D}(\mathbb{R}^n)$, it also means, in particular, that $\varphi \in \text{BMO}(\mathbb{R}^n)$. Also, since Q_j^e is a dyadic cube, φ satisfies Carleson's condition proved in [44, Theorem 4, p. 154], that is:

$$\sum_{Q \subseteq Q_j^e} |\langle \psi_Q^e, \varphi \rangle|^2 \leq \sum_{Q(\lambda) \subseteq Q_j^e} |\langle \psi_\lambda, \varphi \rangle|^2 \leq K |Q_j^e|, \text{ for some uniform constant } K > 0.$$

Since $\sum_{j=1}^{\infty} |\lambda_j^e| < \infty$ by decomposition 4.3, we get:

$$\sum_{j=1}^{\infty} |\lambda_j^e| \sum_{Q \in \mathcal{Q}_0} |\alpha_j^e(Q)| |\langle \psi_Q^e, \varphi \rangle| \leq K \sum_{j=1}^{\infty} |\lambda_j^e| < \infty$$

Therefore, in the sense of distributions, we have:

$$f = \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k + \sum_{e \in E} \sum_{j=1}^{\infty} \lambda_j^e \sum_{Q \in \mathcal{Q}_0} \alpha_j^e(Q) \psi_Q^e$$

Now, let: $a_j^e = \sum_{Q \in \mathcal{Q}_0} \alpha_j^e(Q) \psi_Q^e$, for any $e \in E$ and $j \in \mathbb{N}$.

Let us establish that these are, in fact (up to a uniform constant which can be seen in the following to be $m^{-\frac{n}{2}}$), $h^1(\mathbb{R}^n)$ atoms with the cancellation property.

Fix $e \in E$ and $j \in \mathbb{N}$. It has already been established that the support of ψ_Q^e is contained in mQ for some $m \geq 1$. Then, by the definition of Q_j^e the base of the t_1 -atom $(\alpha_j^e(Q))_{Q \in \mathcal{Q}_0}$ that contains its socle, we see that the support of a_j^e is contained in mQ_j^e .

Moreover, combining the fact that $(\psi_\lambda)_{\lambda \in \Lambda}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$ and $(\alpha_j^e(Q))_{Q \in \mathcal{Q}_0}$ is a t_1 -atom, we get:

$$\|a_j^e\|_2 = \left(\sum_{Q \in \mathcal{Q}_0} |\alpha_j^e(Q)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{|Q_j^e|^{\frac{1}{2}}}$$

Using the fact that a_j^e has a compact support, we get $a_j^e \in L^1(\mathbb{R}^n)$. So we can integrate the series defining it term by term, from which we get: $\int a_j^e = 0$ (as $\int \psi_Q^e = 0$).

So, we proved the 3 defining properties of an $H^1(\mathbb{R}^n)$ atom as intended.

Given that E is finite, we also get:

$$\sum_{e \in E} \sum_{j=1}^{\infty} |\lambda_j^e| < \infty$$

Now, for the non-cancellation part, $\phi_k(x) = \phi(x - k)$ for any $x \in \mathbb{R}^n$ and any $k \in \mathbb{Z}^n$. As $\phi \in L^2(\mathbb{R}^n)$ with compact support $m[0,1]^n = mQ_0$, so are the ϕ_k 's with support mQ_k (where the latter are defined by $x \in Q_k \iff x - k \in Q_0$). Thus, they are $h^1(\mathbb{R}^n)$ atoms (up to the common multiplicative constant $m^{-n} \|\phi\|_{\infty}^{-1}$) with no cancellation property requirement (as $m \geq 1$).

Now, since the ϕ_k 's are uniformly bounded and $\{Q_k\}_{k \in \mathbb{Z}^n}$ form a partition of $\mathbb{R}^n$, we have:

$$\sum_{k \in \mathbb{Z}^n} |\beta(k)| = \sum_{k \in \mathbb{Z}^n} \left| \int_{mQ_k} f(x) \phi_k(x) dx \right| \lesssim \sum_{k \in \mathbb{Z}^n} \int_{Q_k} |f(mx)| dx \lesssim \|f\|_1 < \infty$$

It should be noted that the above is just a necessary condition for f to be in $L^1(\mathbb{R}^n)$.

So, we arrive at an atomic decomposition of f (associated with cubes) in the sense of $h^1(\mathbb{R}^n)$ as follows:

$$f = \sum_{k \in \mathbb{Z}^n} \beta(k) \phi_k + \sum_{e \in E} \sum_{j=1}^{\infty} \lambda_j^e a_j^e$$

Concluding the proof of this direction.

(E) $\implies$ (F)

Assume that: $f = \sum_{j=1}^{\infty} \lambda_{1,j} A_j + \sum_{j=1}^{\infty} \lambda_{2,j} a_j$ (where the A_j 's and the a_j 's are $h^1(\mathbb{R}^n)$ -atoms

associated to large and small cubes, respectively, and $\sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty$, as well as $\sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty$).

Let a be one of the atoms in the above decomposition of f , associated with the cube Q , and let

B be the smallest ball that contains Q .

Therefore: $\text{supp}(a) \subseteq Q \subseteq B$, and $|B| = K_n|Q|$, where $K_n > 1$ is a uniform constant that depends only on n .

Then we verify that: for any $x \notin B$, $a(x) = 0$ as $\text{supp}(a) \subseteq B$. And: $\|a\|_2 \leq |Q|^{-\frac{1}{2}} = K_n^{-1}|B|^{-\frac{1}{2}}$.

So, adjusting the atoms with the uniform constant K_n , we still arrive at properly defined atoms associated to balls, and the same atomic decomposition only requiring the adjustment of the coefficients with the uniform constant K_n^{-1} .

And if a has the cancellation property, then: $\int_{\mathbb{R}^n} K_n a = \int_Q K_n a = \int_B K_n a = 0$, the same applies to $K_n a$ that preserves that property.

Also, if a is not required to have the cancellation property, this means that $|Q| \geq 1$, then: $|B| \geq 1$ also, and the atoms $K_n a$ are not required to have it neither.

This concludes the proof of this direction.

(F) $\implies$ (G)

Assume that: $f = \sum_{j=1}^{\infty} \lambda_{1,j} A_j + \sum_{j=1}^{\infty} \lambda_{2,j} a_j$ (where the A_j 's and the a_j 's are $h^1(\mathbb{R}^n)$ -atoms

associated to large and small balls, respectively, and $\sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty$, as well as $\sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty$).

For the proof in the case $\varphi \in \mathcal{S}(\mathbb{R}^n)$ only, by Goldberg, see [29, Lemma 4, p. 34] and in [29, Theorem 4, p. 38] (where the moment conditions of Lemma 4 are relaxed, and in the case $p = 1$ does not apply).

Let $\varphi \in L_c^\infty(\mathbb{R}^n)$ st: $\int \varphi = 1$. And let $\text{supp}(\varphi) \subseteq B_\varphi = B(0, r_\varphi)$. Fix an $x \in \mathbb{R}^n$.

We have the following:

$$\begin{aligned} (f - \varphi * f)(x) &= \sum_{j=1}^{\infty} \lambda_{1,j} A_j(x) + \sum_{j=1}^{\infty} \lambda_{2,j} a_j(x) - \int_{\mathbb{R}^n} \left[\sum_{j=1}^{\infty} \lambda_{1,j} A_j(y) + \sum_{j=1}^{\infty} \lambda_{2,j} a_j(y) \right] \varphi(x-y) dy \\ &= \sum_{j=1}^{\infty} \lambda_{1,j} A_j(x) + \sum_{j=1}^{\infty} \lambda_{2,j} a_j(x) - \int_{\mathbb{R}^n} \left[\sum_{j=1}^{\infty} \lambda_{1,j} A_j(y) \right] \varphi(x-y) dy \\ &\quad - \int_{\mathbb{R}^n} \left[\sum_{j=1}^{\infty} \lambda_{2,j} a_j(y) \right] \varphi(x-y) dy \end{aligned}$$

Now, we need to justify interchanging the order of summation and integration. For that it

suffices to prove absolute summability, and to show that, notice that for any $h^1(\mathbb{R}^n)$ atom a , we have that: $\|a\|_1 \leq 1$ (by the Cauchy-Schwarz Inequality). So we get:

$$\sum_{j=1}^{\infty} |\lambda_{1,j}| \int_{\mathbb{R}^n} |A_j(x)| |\varphi(x-y)| dy \leq \sum_{j=1}^{\infty} |\lambda_{1,j}| \|\varphi\|_{\infty} \|A_j\|_1 \leq \|\varphi\|_{\infty} \sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty$$

The same holds for the a_j 's (which also justifies the splitting of the two sums in the first place).

So, we obtain the following:

$$\begin{aligned} f - \varphi * f &= \sum_{j=1}^{\infty} \lambda_{1,j} A_j + \sum_{j=1}^{\infty} \lambda_{2,j} a_j - \sum_{j=1}^{\infty} \lambda_{1,j} (\varphi * A_j) - \sum_{j=1}^{\infty} \lambda_{2,j} (\varphi * a_j) \\ &= \sum_{j=1}^{\infty} \lambda_{1,j} (A_j - \varphi * A_j) + \sum_{j=1}^{\infty} \lambda_{2,j} a_j - \sum_{j=1}^{\infty} \lambda_{2,j} (\varphi * a_j) \end{aligned}$$

The latter regrouping is allowed for now, at least in the sense of distributions.

Denoting $A_{\varphi,j} = A_j - \varphi * A_j$ and $a_{\varphi,j} = \varphi * a_j$ for any $j \in \mathbb{N}$, we can write:

$$f - \varphi * f = \sum_{j=1}^{\infty} \lambda_{1,j} A_{\varphi,j} + \sum_{j=1}^{\infty} \lambda_{2,j} a_j - \sum_{j=1}^{\infty} \lambda_{2,j} a_{\varphi,j}$$

We will show that the right-hand series converges in $H^1(\mathbb{R}^n)$.

We claim that if A is an $h^1(\mathbb{R}^n)$ atom associated with the large ball $B_A = B(x_A, r_A)$ and the coefficient λ in the atomic decomposition, then $A_{\varphi} = (A - \varphi * A)$ is an $h^1(\mathbb{R}^n)$ atom with the cancellation property, up to a multiplicative constant.

First, we see that $\text{supp}(A_{\varphi}) \subseteq \text{supp}(A) \cup \text{supp}(\varphi * A)$. But: $\text{supp}(A) \subseteq B_A$ and $\text{supp}(\varphi * A) \subseteq \text{supp}(\varphi) + \text{supp}(A) = B_{\varphi} + B_A$, where $+$ denotes the Minkowski sum of sets.

But: $B_{\varphi} + B_A = B(x_A, r_{\varphi} + r_A)$. Hence: $\text{supp}(A_{\varphi}) \subseteq B_A \cup B(x_A, r_{\varphi} + r_A) \subseteq B(x_A, r_{\varphi} + r_A) = B_{A_{\varphi}}$. So, the first property of $h^1(\mathbb{R}^n)$ -atoms checks out.

Now, we have $\varphi \in L^1(\mathbb{R}^n)$ and $A \in L^2(\mathbb{R}^n)$, so by Young's Convolution Inequality, we have the following: $\|\varphi * A\|_2 \leq \|\varphi\|_1 \|A\|_2$

From which and also from the fact that $|B_A| \geq 1$, we have:

$$\|A_\varphi\|_2 = \|A - \varphi * A\|_2 \leq \|A\|_2 + \|\varphi * A\|_2 = (1 + \|\varphi\|_1) \|A\|_2 \leq \frac{1 + \|\varphi\|_1}{|B_A|^{\frac{1}{2}}} \leq 1 + \|\varphi\|_1$$

If $r_\varphi \geq r_A$, then: $B_{A_\varphi} \subseteq 2B(x_A, r_\varphi)$ with $|B(x_A, r_\varphi)| = |B_\varphi|$.

So, if we normalize by setting $\tilde{A}_\varphi = \frac{1}{2^{\frac{n}{2}}(1+\|\varphi\|_1)|B_\varphi|^{\frac{1}{2}}} A_\varphi$ (and in the meantime setting the coefficient $\tilde{\lambda} = 2^{\frac{n}{2}}(1+\|\varphi\|_1)|B_\varphi|^{\frac{1}{2}}\lambda$), we get the following:

$$\|\tilde{A}_\varphi\|_2 = \frac{1}{2^{\frac{n}{2}}(1+\|\varphi\|_1)|B_\varphi|^{\frac{1}{2}}} \|A_\varphi\|_2 \leq \frac{1}{2^{\frac{n}{2}}|B_\varphi|^{\frac{1}{2}}} \leq \frac{1}{|B_{A_\varphi}|^{\frac{1}{2}}}$$

If $r_\varphi < r_A$, then $B_{A_\varphi} \subseteq 2B_A$.

So, if we normalize this time by setting $\tilde{A}_\varphi = \frac{1}{2^{\frac{n}{2}}(1+\|\varphi\|_1)} A_\varphi$ (and in the meantime, setting the coefficient $\tilde{\lambda} = 2^{\frac{n}{2}}(1+\|\varphi\|_1)\lambda$), we get:

$$\|\tilde{A}_\varphi\|_2 = \frac{1}{2^{\frac{n}{2}}(1+\|\varphi\|_1)} \|A_\varphi\|_2 \leq \frac{1}{2^{\frac{n}{2}}|B_A|^{\frac{1}{2}}} \leq \frac{1}{|B_{A_\varphi}|^{\frac{1}{2}}}$$

$\tilde{A}_\varphi$ as defined have the same support as A_φ , and the second property of $h^1(\mathbb{R}^n)$ -atoms also checks out.

Now, for the cancellation part, it suffices to show it for A_φ . We have that:

$$\begin{aligned} \int_{\mathbb{R}^n} A_\varphi(z) dz &= \int_{\mathbb{R}^n} A(z) dz - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(y) \varphi(z-y) dy dz \\ &= \int_{\mathbb{R}^n} A(z) dz - \int_{\mathbb{R}^n} A(y) \left(\int_{\mathbb{R}^n} \varphi(z-y) dz \right) dy \\ &= \int_{\mathbb{R}^n} A(z) dz - \int_{\mathbb{R}^n} A(y) dy \quad (\text{as we have: } \int \varphi = 1) \\ &= 0 \end{aligned}$$

Interchanging the order of integration in the second equality above is justified by Fubini's Theorem, as both A and φ are in $L^2(\mathbb{R}^n)$ and have compact support.

Now, we claim that if a is an $h^1(\mathbb{R}^n)$ -atom associated with the small ball $B_a = B(x_a, r_a)$ and the coefficient λ in the atomic decomposition, then $a_\varphi = \varphi * a$ is an $h^1(\mathbb{R}^n)$ -atom with the cancellation property, up to a multiplicative constant.

First, we see that $\text{supp}(a_\varphi) = \text{supp}(\varphi * a) \subseteq \text{supp}(\varphi) + \text{supp}(a) = B_\varphi + B_a$, where $+$ denotes the Minkowski sum of sets.

But: $B_\varphi + B_a = B(x_a, r_\varphi + r_a)$. Hence: $\text{supp}(a_\varphi) \subseteq B(x_a, r_\varphi + r_a) = B_{a_\varphi}$. So, the first property of $h^1(\mathbb{R}^n)$ atoms checks out.

Now, we have $\varphi \in L^2(\mathbb{R}^n)$ and $a \in L^1(\mathbb{R}^n)$, so by Young's Convolution Inequality, we have the following estimate:

$$\|a_\varphi\|_2 = \|\varphi * a\|_2 \leq \|\varphi\|_2 \|a\|_1 \leq \|\varphi\|_2$$

Similarly, we have $\varphi \in L^1(\mathbb{R}^n)$ and $a \in L^2(\mathbb{R}^n)$, so by Young's Convolution Inequality we have this other estimate:

$$\|a_\varphi\|_2 = \|\varphi * a\|_2 \leq \|\varphi\|_1 \|a\|_2 \leq \frac{\|\varphi\|_1}{|B_a|^{\frac{1}{2}}}$$

If $r_\varphi \geq r_a$, then: $B_{a_\varphi} \subseteq 2B(x_a, r_\varphi)$ with $|B(x_a, r_\varphi)| = |B_\varphi|$.

So, if we normalize by setting $\tilde{a}_\varphi = \frac{1}{2^{\frac{n}{2}} \|\varphi\|_2 |B_\varphi|^{\frac{1}{2}}} a_\varphi$ (and in the meantime setting the coefficient $\tilde{\lambda} = 2^{\frac{n}{2}} \|\varphi\|_2 |B_\varphi|^{\frac{1}{2}} \lambda$), we get:

$$\|\tilde{a}_\varphi\|_2 = \frac{1}{2^{\frac{n}{2}} \|\varphi\|_2 |B_\varphi|^{\frac{1}{2}}} \|a_\varphi\|_2 \leq \frac{1}{2^{\frac{n}{2}} |B_\varphi|^{\frac{1}{2}}} \leq \frac{1}{|B_{a_\varphi}|^{\frac{1}{2}}}$$

If $r_\varphi < r_a$, then $B_{a_\varphi} \subseteq 2B_a$.

So, if we normalize this time by setting $\tilde{a}_\varphi = \frac{1}{2^{\frac{n}{2}} \|\varphi\|_1} a_\varphi$ (and in the meantime, setting the coefficient $\tilde{\lambda} = 2^{\frac{n}{2}} \|\varphi\|_1 \lambda$), we get:

$$\|\tilde{a}_\varphi\|_2 = \frac{1}{2^{\frac{n}{2}} \|\varphi\|_1} \|a_\varphi\|_2 \leq \frac{1}{2^{\frac{n}{2}} |B_a|^{\frac{1}{2}}} \leq \frac{1}{|B_{a_\varphi}|^{\frac{1}{2}}}$$

$\tilde{a}_\varphi$ as defined have the same support as a_φ , and the second property of $h^1(\mathbb{R}^n)$ -atoms also checks

out.

Now, for the cancellation part, it suffices to show it for a_φ . We have that:

$$\begin{aligned}
\int_{\mathbb{R}^n} a_\varphi(z) dz &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(y) \varphi(z-y) dy dz \\
&= \int_{\mathbb{R}^n} a(y) \left(\int_{\mathbb{R}^n} \varphi(z-y) dz \right) dy \\
&= \int_{\mathbb{R}^n} a(y) dy && \text{(as we have: } \int \varphi = 1) \\
&= 0 && \text{(as the atom } a \text{ have the cancellation property)}
\end{aligned}$$

Interchanging the order of integration in the second equality above is justified by Fubini's Theorem, as both a and φ are in $L^2(\mathbb{R}^n)$ and have compact support.

So, now we can write:

$$f - \varphi * f = \sum_{j=1}^{\infty} \tilde{\lambda}_{1,j} \tilde{A}_{\varphi,j} + \sum_{j=1}^{\infty} \lambda_{2,j} a_j - \sum_{j=1}^{\infty} \tilde{\lambda}_{2,j} \tilde{a}_{\varphi,j}$$

Here, all the $\tilde{A}_{\varphi,j}$'s, the a_j 's and the $\tilde{a}_{\varphi,j}$'s are $H^1(\mathbb{R}^n)$ -atoms. As $\sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty$, it remains to

show that: $\sum_{j=1}^{\infty} |\tilde{\lambda}_{1,j}| < \infty$ and $\sum_{j=1}^{\infty} |\tilde{\lambda}_{2,j}| < \infty$.

Indeed, if we set $F_1 = \{j \in \mathbb{N} : r_\varphi \geq r_{1,j}\}$, $G_1 = \{j \in \mathbb{N} : r_\varphi < r_{1,j}\}$, $F_2 = \{j \in \mathbb{N} : r_\varphi \geq r_{2,j}\}$ and $G_2 = \{j \in \mathbb{N} : r_\varphi < r_{2,j}\}$ where the $r_{1,j}$'s and the $r_{2,j}$'s are the radii of the balls associated with the A_j 's and the a_j 's, respectively. we then have:

$$\begin{aligned}
\sum_{j=1}^{\infty} |\tilde{\lambda}_{1,j}| &= \sum_{j \in F_1} |\tilde{\lambda}_{1,j}| + \sum_{j \in G_1} |\tilde{\lambda}_{1,j}| = 2^{\frac{n}{2}} (1 + \|\varphi\|_1) |B_\varphi|^{\frac{1}{2}} \sum_{j \in F_1} |\lambda_{1,j}| + 2^{\frac{n}{2}} (1 + \|\varphi\|_1) \sum_{j \in G_1} |\lambda_{1,j}| \\
&\leq 2^{\frac{n}{2}} (1 + \|\varphi\|_1) (1 + |B_\varphi|^{\frac{1}{2}}) \sum_{j=1}^{\infty} |\lambda_{1,j}| < \infty
\end{aligned}$$

Similarly:

$$\begin{aligned} \sum_{j=1}^{\infty} |\tilde{\lambda}_{2,j}| &= \sum_{j \in F_2} |\tilde{\lambda}_{2,j}| + \sum_{j \in G_2} |\tilde{\lambda}_{2,j}| = 2^{\frac{n}{2}} \|\varphi\|_2 |B_\varphi|^{\frac{1}{2}} \sum_{j \in F_2} |\lambda_{2,j}| + 2^{\frac{n}{2}} \|\varphi\|_1 \sum_{j \in G_2} |\lambda_{2,j}| \\ &\leq 2^{\frac{n}{2}} (\|\varphi\|_1 + \|\varphi\|_2 |B_\varphi|^{\frac{1}{2}}) \sum_{j=1}^{\infty} |\lambda_{2,j}| < \infty \end{aligned}$$

So, indeed, the series converges in $H^1(\mathbb{R}^n)$ and thus: $(f - \varphi * f) \in H^1(\mathbb{R}^n)$.

We conclude the proof of this direction.

(G) $\implies$ (B)

Assume now that for any $\varphi \in L_c^\infty(\mathbb{R}^n) \cup \mathcal{S}(\mathbb{R}^n)$ st: $\int \varphi = 1$, we have that: $(f - \varphi * f) \in H^1(\mathbb{R}^n)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with its support contained in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$, which we will denote C_0 , $\varphi = 1$ in a neighborhood of the origin, and s.t: $\int \varphi = 1$, and let $C_k = k + C_0$, B_k be the smallest ball containing C_k and $K > 1$ be a constant depending only on n s.t: $|B_k| = K |C_k| = K$, for any $k \in \mathbb{Z}^n$.

Then, by our assumption, we can write: $f - \varphi * f = \sum_{j=1}^{\infty} \lambda_j a_j \in H^1(\mathbb{R}^n)$ where $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ and the a_j 's are $H^1(\mathbb{R}^n)$ -atoms (associated with balls).

Now, let: $S_0(f)(x) = \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}}$, for any $x \in \mathbb{R}^n$.

First, we claim that S_0 is sublinear. Indeed, let $g_1, g_2 \in L^1(\mathbb{R}^n)$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. We have the following:

$$\begin{aligned} \left(S_0(g_1 + g_2)(x) \right)^2 &= \sum_{\lambda \in \Lambda^0} |(g_1 + g_2, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 = \sum_{\lambda \in \Lambda^0} |(g_1, \psi_\lambda) + (g_2, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \\ &\leq \sum_{\lambda \in \Lambda^0} \left(|(g_1, \psi_\lambda)|^2 + 2 |(g_1, \psi_\lambda)| |(g_2, \psi_\lambda)| + |(g_2, \psi_\lambda)|^2 \right) |\psi_\lambda(x)|^2 \\ &= \left(S_0(g_1)(x) \right)^2 + \left(S_0(g_2)(x) \right)^2 + 2 \sum_{\lambda \in \Lambda^0} |(g_1, \psi_\lambda)| |(g_2, \psi_\lambda)| |\psi_\lambda(x)|^2 \end{aligned}$$

On the other hand, by the Cauchy-Schwarz Inequality, we have:

$$\begin{aligned} S_0(g_1)(x) S_0(g_2)(x) &= \left(\sum_{\lambda \in \Lambda^0} |(g_1, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{1/2} \left(\sum_{\lambda \in \Lambda^0} |(g_2, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{1/2} \\ &\geq \sum_{\lambda \in \Lambda^0} |(g_1, \psi_\lambda)| |(g_2, \psi_\lambda)| |\psi_\lambda(x)|^2 \end{aligned}$$

Hence: $\left(S_0(g_1 + g_2)(x) \right)^2 \leq \left(S_0(g_1)(x) + S_0(g_2)(x) \right)^2$. So: $S_0(g_1 + g_2)(x) \leq S_0(g_1)(x) + S_0(g_2)(x)$.

We also have:

$$S_0(tg_1)(x) = \left(\sum_{\lambda \in \Lambda^0} |(tg_1, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{1/2} = |t| \left(\sum_{\lambda \in \Lambda^0} |(g_1, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{1/2} = |t| S_0(g_1)(x)$$

Concluding the argument for the sublinearity of the above operator.

From it we get that: $S_0(f) \leq S_0(f - \varphi * f) + S_0(\varphi * f)$

Now: $\sum_{j=1}^N \lambda_j a_j \xrightarrow[N \rightarrow \infty]{} f - \varphi * f$ in the sense of distributions and more strongly in $H^1(\mathbb{R}^n)$.

So, we also get the following convergence: $\left(\sum_{j=1}^N \lambda_j a_j, \psi_\lambda \right) \xrightarrow[N \rightarrow \infty]{} (f - \varphi * f, \psi_\lambda)$, for any $\lambda \in \Lambda^0$.

Fixing $x \in \mathbb{R}^n$ and applying the discrete analog of Fatou's Lemma (since all the terms are non-negative), we have that:

$$\begin{aligned} \left(S_0(f - \varphi * f)(x) \right)^2 &= \sum_{\lambda \in \Lambda^0} |(f - \varphi * f, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 = \sum_{\lambda \in \Lambda^0} \liminf_{N \rightarrow \infty} \left| \left(\sum_{j=1}^N \lambda_j a_j, \psi_\lambda \right) \right|^2 |\psi_\lambda(x)|^2 \\ &\leq \liminf_{N \rightarrow \infty} \sum_{\lambda \in \Lambda^0} \left| \left(\sum_{j=1}^N \lambda_j a_j, \psi_\lambda \right) \right|^2 |\psi_\lambda(x)|^2 \end{aligned}$$

But for any $N \in \mathbb{N}$, by the sublinearity of S_0 :

$$\sum_{\lambda \in \Lambda^0} \left| \left(\sum_{j=1}^N \lambda_j a_j, \psi_\lambda \right) \right|^2 |\psi_\lambda(x)|^2 = \left(S_0 \left(\sum_{j=1}^N \lambda_j a_j \right)(x) \right)^2 \leq \left(\sum_{j=1}^N |\lambda_j| S_0(a_j)(x) \right)^2$$

Hence: $S_0(f - \varphi * f)(x) \leq \liminf_{N \rightarrow \infty} \sum_{j=1}^N |\lambda_j| S_0(a_j)(x)$

Another application of Fatou's Lemma gives us:

$$\int_{\mathbb{R}^n} S_0(f - \varphi * f) \leq \int_{\mathbb{R}^n} \liminf_{N \rightarrow \infty} \sum_{j=1}^N |\lambda_j| S_0(a_j) \leq \liminf_{N \rightarrow \infty} \sum_{j=1}^N |\lambda_j| \int_{\mathbb{R}^n} S_0(a_j)$$

Similarly, by [29, Theorem 2, p. 33], as $r_j(f) = R_j(f - \varphi * f) \in L^1(\mathbb{R}^n)$ for any $j \in [1, n]$, where the R_j 's are the Riesz transforms and the r_j 's their localized counterparts, given the assumption and the Riesz transform characterization of $H^1(\mathbb{R}^n)$, we have $f \in h^1(\mathbb{R}^n)$.

Proceeding as in [29, proof of Lemma 5, p. 37] and denoting: $\tilde{A}_k = (\varphi * f)|_{C_k}$, we have: $\text{supp}(\tilde{A}_k) = C_k \subseteq B_k$ for any $k \in \mathbb{Z}^n$. Also, as $\varphi \in L^2(\mathbb{R}^n)$, in particular, and $f \in L^1(\mathbb{R}^n)$, by Young's Inequality, we have: $(\varphi * f) \in L^2(\mathbb{R}^n)$ and so are all the $\tilde{A}_k$'s.

Dropping potentially any $k \in \mathbb{Z}^n$ for which $\|\tilde{A}_k\|_2 = 0$, let $A_k = \frac{1}{K^{\frac{1}{2}} \|\tilde{A}_k\|_2} \tilde{A}_k$ and $\gamma_k = K^{\frac{1}{2}} \|\tilde{A}_k\|_2$, so that we have: $\|A_k\|_2 = \frac{1}{K^{\frac{1}{2}}} = \frac{1}{|B_k|^{\frac{1}{2}}}$, $\text{supp}(A_k) = \text{supp}(\tilde{A}_k) \subseteq B_k$ and: $\varphi * f = \sum_{k \in \mathbb{Z}^n} \tilde{A}_k = \sum_{k \in \mathbb{Z}^n} \gamma_k A_k$.

Setting $\Gamma(x) = \{(y, t) : |x - y| < t, 0 < t < \frac{1}{2}\}$ for any $x \in \mathbb{R}^n$ and based on the maximal function characterization of $h^1(\mathbb{R}^n)$, we then have:

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} |\gamma_k| &\lesssim \sum_{k \in \mathbb{Z}^n} \|\tilde{A}_k\|_2 \leq \sum_{k \in \mathbb{Z}^n} \sup_{x \in C_k} |(\varphi * f)(x)| \leq \int_{\mathbb{R}^n} \sup_{(y, t) \in \Gamma(x)} |(\varphi_t * f)(y)| dx \\ &\lesssim \int_{\mathbb{R}^n} \sup_{0 < t < 1} |(\varphi_t * f)(x)| dx = \|f\|_{h^1} < \infty \end{aligned}$$

So, we arrived at an atomic decomposition similar to what we have done for the $H^1(\mathbb{R}^n)$ part.

Given that: $\sum_{\substack{1 \leq |k| \leq N \\ k \in \mathbb{Z}^n}} \gamma_k A_k \xrightarrow{N \rightarrow \infty} \varphi * f$ in the sense of distributions and more strongly in $h^1(\mathbb{R}^n)$,

as we argued again for the $H^1(\mathbb{R}^n)$ part and using Fatou's Lemma twice, we have the following:

$$\int_{\mathbb{R}^n} S_0(\varphi * f) \leq \int_{\mathbb{R}^n} \liminf_{N \rightarrow \infty} \sum_{\substack{1 \leq |k| \leq N \\ k \in \mathbb{Z}^n}} |\gamma_k| S_0(A_k) \leq \liminf_{N \rightarrow \infty} \sum_{\substack{1 \leq |k| \leq N \\ k \in \mathbb{Z}^n}} |\gamma_k| \int_{\mathbb{R}^n} S_0(A_k).$$

So, given that $\sum_{j=1}^{\infty} \lambda_j < \infty$ and $\sum_{k \in \mathbb{Z}^n} |\gamma_k| < \infty$, it suffices to prove that if we fix a an $h^1(\mathbb{R}^n)$ atom

associated with a ball $B = B(x_0, r)$, then: $\int_{\mathbb{R}^n} S_0(a) \lesssim 1$ to conclude the proof in this direction. Notice that what we showed so far with what follows can be used as well to prove the $(F) \implies (B)$ direction.

Let $C > 1$ be a constant depending only on m and n , large enough (we will see how large would we require it to be in what follows), and decompose $\mathbb{R}^n$ into an inflated ball $\tilde{B} = B(x_0, Cr)$ and concentric dyadic annuli $A_k = \{x \in \mathbb{R}^n : 2^k \leq \frac{|x-x_0|}{Cr} < 2^{k+1}\}$, for $k \in \mathbb{N} \cup \{0\}$. Thus:

$$\int_{\mathbb{R}^n} S_0(a) = \int_{\tilde{B}} S_0(a) + \sum_{k=0}^{\infty} \int_{A_k} S_0(a).$$

For the first integral, using the Cauchy-Schwarz Inequality, we get: $\int_{\tilde{B}} S_0(a) \leq |\tilde{B}|^{\frac{1}{2}} \|S_0(a)\|_2$. But applying the Monotone Convergence Theorem, noticing that $a \in L^2(\mathbb{R}^n)$ (which means $a = \sum_{\lambda \in \Lambda} (a, \psi_\lambda) \psi_\lambda$) and the $(\psi_\lambda)_{\lambda \in \Lambda}$ are an orthonormal basis for $L^2(\mathbb{R}^n)$, and finally that a is an $h^1(\mathbb{R}^n)$ -atom, we get:

$$\|S_0(a)\|_2^2 \leq \int_{\mathbb{R}^n} \sum_{\lambda \in \Lambda} |(a, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 dx = \|a\|_2^2 \leq \frac{1}{|B|}.$$

On the other hand: $|\tilde{B}|^{\frac{1}{2}} = C^{\frac{n}{2}} |B|^{\frac{1}{2}}$. So: $\int_{\tilde{B}} S_0(a) \leq C^{\frac{n}{2}} \lesssim 1$.

Now, let us turn our attention to the second sum. Let $k \in \mathbb{N} \cup \{0\}$ and let $\lambda \in \Lambda^0$. The support of ψ_λ is contained in $mQ(\lambda)$ where the dyadic cube $Q(\lambda)$ has side length $\frac{1}{2^j}$ for some $j \in \mathbb{N} \cup \{0\}$. If $mQ(\lambda) \cap B = \emptyset$, then: $(a, \psi_\lambda) = 0$.

Similarly, if $mQ(\lambda) \cap A_k = \emptyset$, then: $\psi_\lambda = 0$ on A_k .

This motivates us to reduce the sum in $S_0(a)$, by defining: $\Lambda_k^0 = \{\lambda \in \Lambda^0 : mQ(\lambda) \cap B \neq \emptyset \text{ and } mQ(\lambda) \cap A_k \neq \emptyset\}$. In doing so, we have: $\int_{A_k} S_0(a) = \int_{A_k} \left(\sum_{\lambda \in \Lambda_k^0} |(a, \psi_\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} dx$.

Now, a necessary condition for $\lambda \in \Lambda_k^0$, is that $mQ(\lambda)$ intersects both B and A_k , which is only possible if the maximum distance between two points of that cube (which is its diameter) is strictly greater than the distance between the ball B and the annulus A_k . But this means that:

$$\frac{m\sqrt{n}}{2^j} > Cr2^k - r = (C - \frac{1}{C2^k})r2^k \geq (C - \frac{1}{C})r2^k.$$

Then, we can write: $1 \geq \frac{1}{2^j} > cr2^k \geq cr$ where $c = (C - \frac{1}{C2^k}) \frac{1}{m\sqrt{n}}$. Choosing C large enough, so we get: $c > \max(1, \frac{\sqrt{\pi}}{(\Gamma(1 + \frac{n}{2}))^{\frac{1}{n}}})$. This means that we only have finitely many cube sizes (or j 's) that intersect both the annulus A_k and the ball B .

Another observation is that if a is an atom without cancellation property, then necessarily $|B| \geq 1$, which means $r \geq \frac{(\Gamma(1 + \frac{n}{2}))^{\frac{1}{n}}}{\sqrt{\pi}} > \frac{1}{c}$. So by the above, there are no cubes satisfying the required expression for this type of atoms, and thus we will get a common bound for their local square function integral over the whole domain, as it only consists of the integral over $\tilde{B}$ contribution, thus concluding the result directly for this type of atoms.

In the following, assume that a has the cancellation property and that we have k such that $\Lambda_k^0 \neq \emptyset$.

Now, using the cancellation property of atom a and the Cauchy-Schwarz Inequality, we get:

$$\begin{aligned} |(a, \psi_\lambda)| &= \left| \int_{\mathbb{R}^n} a(x) \psi_\lambda(x) dx - \int_{\mathbb{R}^n} a(x) \psi_\lambda(x_0) dx \right| = \left| \int_B a(x) (\psi_\lambda(x) - \psi_\lambda(x_0)) dx \right| \\ &\leq \|a\|_2 \left(\int_B |\psi_\lambda(x) - \psi_\lambda(x_0)|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

Exploiting the regularity property of ψ_λ applied to its first derivative, namely: $|D\psi_\lambda| \leq C' 2^j 2^{\frac{nj}{2}}$ for some uniform constant $C' > 0$ and applying the Generalized Mean Value Theorem, we get:

$$|\psi_\lambda(x) - \psi_\lambda(x_0)| \leq C' 2^j 2^{\frac{nj}{2}} |x - x_0| \leq C' 2^j 2^{\frac{nj}{2}} r.$$

Since $\|a\|_2 \leq \frac{1}{|B|^{\frac{1}{2}}}$ and setting $C'' = \frac{C'}{c^{1+\frac{n}{2}}}$, we get:

$$|(a, \psi_\lambda)| \leq C' 2^j 2^{\frac{nj}{2}} r \leq C'' 2^{-k(1+\frac{n}{2})} r^{-\frac{n}{2}}.$$

Now, we have that: $\left(\sum_{\lambda \in \Lambda_k^0} |\psi_\lambda(x)|^2 \right)^{\frac{1}{2}} \leq \sum_{\lambda \in \Lambda_k^0} |\psi_\lambda(x)|$.

Fix $\lambda = \frac{p}{2^j} + \frac{e}{2^{j+1}} \in \Lambda_k^0$, such that: $p \in \mathbb{Z}^n$, $e \in E$ and j satisfying $1 \geq \frac{1}{2^j} > cr2^k$.

The regularity property of ψ_λ , applied to itself, namely that: $|\psi_\lambda| \leq C_0 2^{\frac{nj}{2}}$ for some constant

$C_0 > 0$ and their localization property, namely that: for any $x \in \mathbb{R}^n$, there are at most M^n dyadic cubes of a given size such that their corresponding ψ_λ does not vanish at x , where $M > 0$ is a constant that depends only on m , tells us that, in particular, for any $x \in A_k$:

$$\sum_{\lambda \in \Lambda_k^0} |\psi_\lambda(x)| \leq C_0 M^n \sum_{j=0}^{-k-\log_2(cr)} 2^{\frac{nj}{2}} \leq C_0 M^n \sum_{j=k+\log_2(cr)}^{\infty} (2^{\frac{-n}{2}})^{-j} = \frac{C_0 M^n 2^{\frac{-nk}{2}}}{c^{\frac{n}{2}} r^{\frac{n}{2}} (1 - 2^{\frac{-n}{2}})}$$

Thus, for some constant $C''' > 0$ depending only on m and n , we have, for any $x \in A_k$:

$$S_0(a)(x) \leq C''' 2^{-k(n+1)} r^{-n}.$$

Therefore, since the volume of the annulus A_k is: $D c^n r^n 2^{n(k+1)} - D c^n r^n 2^{nk} = D c^n (2^n - 1) r^n 2^{nk}$, for some constant $D > 0$ depending only on the dimension n , we have:

$$\sum_{k=0}^{\infty} \int_{A_k} S_0(a) \leq D' \sum_{k=0}^{\infty} 2^{-k} \lesssim 1 \quad (\text{for some constant } D' \text{ depending only on } m \text{ and } n).$$

Concluding the proof for this direction.

(D) $\implies$ (A)

Assume that: $\left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \in L^1(\mathbb{R}^n)$.

In the way of proving (D) $\implies$ (E), we showed that:

$$\sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda(x) = \sum_{e \in E} \sum_{j \in \mathbb{N}} \lambda_j^e a_j^e(x).$$

Here, the a_j^e 's are $H^1(\mathbb{R}^n)$ -atoms (up to a multiplicative constant) with support $m Q_j^e$, satisfying $\|a_j^e\|_2 \leq \frac{1}{|Q_j^e|^{\frac{1}{2}}}$. So applying the Cauchy-Schwarz Inequality, we get: $\|a_j^e\|_1 \leq m^{\frac{n}{2}}$. Hence, by the triangle inequality, we get:

$$\left\| \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda(x) \right\|_1 \leq m^{\frac{n}{2}} \sum_{e \in E} \sum_{j \in \mathbb{N}} |\lambda_j^e|.$$

But the λ_j^e 's came from the decomposition by Theorem 4.1.2, where we established when proving the direction (ii) $\implies$ (iii) (which applies here with $\gamma = 1$ and β a fixed constant in $(0, 1]$), that:

$$\sum_{j \in \mathbb{N}} |\lambda_j^e| \leq \frac{4}{\beta(1-\beta)^{\frac{1}{2}}} \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \right\|_1.$$

Hence, we obtain:

$$\left\| \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda(x) \right\|_1 \leq \frac{4m^{\frac{n}{2}}(2^n - 1)}{\beta(1-\beta)^{\frac{1}{2}}} \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \right\|_1.$$

Notice that the right-hand side of the last inequality does not change if we replace $\alpha(\lambda)$ with $\pm \alpha(\lambda)$. Thus, for any fixed finite subset $F \subseteq \Lambda'$, and for all sequences $\epsilon(\lambda) = \pm 1$ with $|F|$ elements, we have the following:

$$\left\| \sum_{\lambda \in F} \epsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right\|_1 \leq \frac{4m^{\frac{n}{2}}(2^n - 1)}{\beta(1-\beta)^{\frac{1}{2}}} \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \right\|_1$$

Hence, taking the supremum over all such sequences, then the supremum over all finite subsets of Λ^0 , we get:

$$\sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\epsilon(\lambda) = \pm 1} \left\| \sum_{\lambda \in F} \epsilon(\lambda) \alpha(\lambda) \psi_\lambda(x) \right\|_1 \leq \frac{4m^{\frac{n}{2}}(2^n - 1)}{\beta(1-\beta)^{\frac{1}{2}}} \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)}(x) \right)^{\frac{1}{2}} \right\|_1 < \infty$$

Concluding the proof of the theorem. □

Remark 4.2.1. *We now give two important observations from the above proof.*

- (1) *There is nothing special about the limit value 1 for the volume of the dyadic cubes, we could change it to any other fixed value bigger than 1, and the theorem still holds with slight to no changes to the proof.*
- (2) *we assumed and required that the father wavelet ϕ and mother wavelet ψ are $C_c^r(\mathbb{R}^n)$ functions with $r \geq 1$, but as seen throughout the proofs, we could relax this condition to both being in $Lip_c(\mathbb{R}^n)$, i.e Lipschitz with compact support.*

Now, we present two important corollaries. The first corollary provides us with necessary conditions for a function to be in $h^1(\mathbb{R}^n)$. The second one provides us with equivalent $h^1(\mathbb{R}^n)$ norms.

Corollary 4.2.1. *If $f \in h^1(\mathbb{R}^n)$ then: $\sum_{k \in \mathbb{Z}^n} |\beta_k| < \infty$ and $\exists C > 0$ s.t: $|\alpha(\lambda)| \leq C |Q(\lambda)|^{-\frac{1}{2}}$, for any $\lambda \in \Lambda^0$.*

Proof. The first necessary condition follows directly from the fact that $f \in L^1(\mathbb{R}^n)$ as we showed in the proof of Theorem 4.2.1 (direction $(D) \implies (E)$). The second one follows also from part (A) of Theorem 4.2.1 by taking the singleton $\{\lambda\}$ as a finite subset of Λ^0 and noticing that: $\|\psi(\lambda)\|_1 \geq c\gamma |Q(\lambda)|^{\frac{1}{2}}$ (the constants follow the notation of the theorem). $\square$

Corollary 4.2.2. *In the notation of Theorem 4.2.1, for $f \in h^1(\mathbb{R}^n)$, the following norms are equivalent:*

$$\begin{aligned} \|f\|_{at} &\simeq \sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\varepsilon(\lambda)=\pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k| \simeq \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{1/2} \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k| \\ &\simeq \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)} \right)^{1/2} \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k| \simeq \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)} \right)^{1/2} \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k| \end{aligned}$$

where $\|f\|_{at} = \inf \left\{ \sum_j |\lambda_j| < \infty : f = \sum_j \lambda_j a_j, a_j \text{'s are } h^1\text{-atoms} \right\}$.

Proof. From the Theorem 4.2.1's proof of direction $(A) \implies (B)$, we get:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{1/2} \right\|_1 \lesssim \sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\varepsilon(\lambda)=\pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda \right\|_1$$

Also, from the $(D) \implies (A)$ direction proof, we get:

$$\sup_{\substack{F \subseteq \Lambda^0 \\ \text{finite}}} \sup_{\varepsilon(\lambda)=\pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda) \alpha(\lambda) \psi_\lambda \right\|_1 \lesssim \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)} \right)^{1/2} \right\|_1$$

And it is clear that we have:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)} \right)^{\frac{1}{2}} \right\|_1 \lesssim \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{\frac{1}{2}} \right\|_1$$

And:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)} \right)^{\frac{1}{2}} \right\|_1 \leq \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{Q(\lambda)} \right)^{\frac{1}{2}} \right\|_1$$

Now what we showed in the proof of the $(D) \implies (E)$ direction, can also be showed for the $(C) \implies (E)$ direction, and would give us for the $H^1(\mathbb{R}^n)$ part of f i.e: $\sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda$:

$$\sum_j |\gamma_j| \lesssim \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)} \right)^{\frac{1}{2}} \right\|_1 \text{ for some atomic decomposition of } \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda.$$

As $\sum_{k \in \mathbb{Z}^n} \beta_k \phi_k$ is the pure $h^1(\mathbb{R}^n)$ part of f and give rise to an atomic decomposition up to a uniform constant, then by definition of $\|\cdot\|_{at}$, we have:

$$\|f\|_{at} \lesssim \left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)} \right)^{\frac{1}{2}} \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k|$$

Now, by what we showed in $(G) \implies (B)$ direction proof, for any atomic decomposition of f :

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{\frac{1}{2}} \right\|_1 \lesssim \sum_j |\gamma_j|.$$

Hence:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{\frac{1}{2}} \right\|_1 \lesssim \|f\|_{at}.$$

On the other hand, we showed that: $\sum_{k \in \mathbb{Z}^n} |\beta_k| \lesssim \|f\|_1$.

But we have: $\|f\|_1 \lesssim \|f\|_{h^1} \cong \|f\|_{at}$ where $\|\cdot\|_{h^1}$ is the $h^1(\mathbb{R}^n)$ norm arising from its maximal

characterization. Hence, we get:

$$\left\| \left(\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)|^2 |\psi_\lambda|^2 \right)^{\frac{1}{2}} \right\|_1 + \sum_{k \in \mathbb{Z}^n} |\beta_k| \lesssim \|f\|_{at}$$

Concluding the proof of this corollary. □

Remark 4.2.2. *In the above corollary, we can replace $\sum_{k \in \mathbb{Z}^n} |\beta_k|$ by $\|f\|_1$ in the listed equivalent norms and it will still apply as we noted in the proof that these two quantities are equivalent.*

4.3 Wavelets Characterization of the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space

Following the notations of the previous section on the wavelet characterization of the $h^1(\mathbb{R}^n)$ space, we can also characterize the non-homogeneous $\text{bmo}(\mathbb{R}^n)$ space in terms of the wavelet coefficients of functions belonging to it. We will follow closely the footsteps of Meyer in [44, Theorem 4, pp. 154–155], for the $\text{BMO}(\mathbb{R}^n)$ space, which we will build on and meticulously detail its steps in our following theorem and its proof.

For $f \in L^1_{loc}(\mathbb{R}^n)$, recall the notation $\alpha(\lambda) = (f, \psi_\lambda)$ and $\beta_k = (f, \phi_k)$, where $\psi_\lambda, \lambda \in \Lambda$ and $\phi_k, k \in \mathbb{Z}^n$, are derived from the mother and father wavelets, respectively, as shown in the wavelets construction section where we introduced them. If B is a ball and Q is a cube, then:

$$f_B = \int_B f \text{ and } f_Q = \int_Q f.$$

We first start by stating and proving a useful lemma.

Lemma 4.3.1. *For $f \in L^1_{loc}(\mathbb{R}^n)$, if its wavelet coefficients $\alpha(\lambda)$ satisfy Carleson's condition, i.e:*

$$\forall Q \in \mathcal{Q}, \quad \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq C |Q| \quad \text{for some constant } C > 0$$

Then, for any cube Q , not necessarily dyadic, we have:

$$\sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq 4^n C |Q|$$

Proof. Let Q be a cube not necessarily dyadic.

The cube Q can be covered by at most 2^n dyadic cubes $Q^{(i)}$, $1 \leq i \leq 2^n$, all of them of same side-length less or equal than the double of the sidelength of the cube Q (it can be shown for the case $n = 1$ for intervals, then generalized to any dimension n by cartesian product).

Noticing that if any dyadic cube $Q(\lambda)$ that are included in Q intersects one of these larger dyadic cubes $Q^{(i)}$, it means that it is, in fact, fully included in it, we have by the Carleson's condition in the assumption:

$$\sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq \sum_{i=1}^{2^n} \sum_{Q(\lambda) \subseteq Q^{(i)}} |\alpha(\lambda)|^2 \leq 2^n C |Q^{(i)}| \leq 4^n C |Q|$$

□

Now, we are ready to state our main theorem.

Theorem 4.3.1. For $f \in L^1_{loc}(\mathbb{R}^n)$, T.F.A.E:

$$(A) \sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \int_Q |f - f_Q| < \infty \text{ and } \sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \int_Q |f| < \infty.$$

(B) $\exists$ a constant $C > 0$, s.t:

(i) The wavelet coefficients $\alpha(\lambda)$ satisfy Carleson's condition, i.e:

$$\forall Q \in \mathcal{Q}, \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq C |Q|$$

(ii) The wavelet coefficients β_k are uniformly bounded, i.e, $\sup_{k \in \mathbb{Z}^n} |\beta_k| \leq C^{\frac{1}{2}}$

(C) We have the following:

(i) f is the limit of its wavelet decomposition series $\sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda$ in the weak*-topology of $\text{BMO}(\mathbb{R}^n)$ (seen as the dual of $H^1(\mathbb{R}^n)$)

(ii) $\varphi * f \in L^\infty(\mathbb{R}^n)$ for some $\varphi \in \mathcal{S}(\mathbb{R}^n) \cup L_c^\infty(\mathbb{R}^n)$ st: $\int \varphi = 1$

(D) f is the limit of its wavelet decomposition series $\sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda + \sum_{k \in \mathbb{Z}^n} \beta_k \phi_k$ in the weak*-topology of $\text{bmo}(\mathbb{R}^n)$ (seen as the dual of $h^1(\mathbb{R}^n)$).

Proof.

(A) $\implies$ (B)

Assume that: $\sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \int_Q |f - f_Q| < \infty$ and $\sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \int_Q |f| < \infty$.

First, if Q_0 is a cube of size 1, then it means: $Q_0 \subseteq 2Q_0$ and $|2Q_0| = 2^n > 1$, so that: $\int_{Q_0} |f| \leq \int_{2Q_0} |f|$. Hence:

$$\int_{Q_0} |f| = \int_{Q_0} |f| \leq \frac{2^n}{2^n |Q_0|} \int_{2Q_0} |f| = 2^n \int_{2Q_0} |f| \leq 2^n \sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \int_Q |f|.$$

Therefore:

$$\sup_{\substack{\text{cubes } Q \\ |Q| = 1}} \int_Q |f| \leq 2^n \sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \int_Q |f| < \infty.$$

This means that: $\sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f| < \infty$.

Now, if Q_0 is any cube of size larger than or equal to 1, we have that:

$$\int_{Q_0} |f - f_{Q_0}| \leq \int_{Q_0} |f| + \int_{Q_0} |f_{Q_0}| = \int_{Q_0} |f| + |f_{Q_0}| = \int_{Q_0} |f| + \left| \int_{Q_0} f \right| \leq 2 \int_{Q_0} |f| \leq 2 \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f|$$

Therefore:

$$\sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f - f_Q| \leq 2 \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f| < \infty.$$

And this allows us to deduce that: $\sup_{\text{cubes } Q} \int_Q |f - f_Q| < \infty$.

Now, let $Q \in \mathcal{Q}$. We can write: $f = f_1 + f_2 + f_{mQ}$, where for any $x \in \mathbb{R}^n$, we have:

$$f_1(x) = \begin{cases} f(x) - f_{mQ} & \text{when: } x \in mQ \\ 0 & \text{otherwise} \end{cases}$$

And, similarly:

$$f_2(x) = \begin{cases} f(x) - f_{mQ} & \text{when: } x \notin mQ \\ 0 & \text{otherwise} \end{cases}$$

Now, if $Q(\lambda) \subseteq Q$, then: $\text{supp}(\psi_\lambda) \subseteq mQ(\lambda) \subseteq mQ$, thus: $(f_2, \psi_\lambda) = 0$, combined with the fact that: $\int \psi_\lambda = 0$, which makes: $(f_{mQ}, \psi_\lambda) = 0$, gives us that: $(f, \psi_\lambda) = (f_1, \psi_\lambda)$.

But: $f_1 \in L^2(\mathbb{R}^n)$, hence we get:

$$\sum_{Q(\lambda) \subseteq Q} |(f, \psi_\lambda)|^2 \leq \sum_{\lambda \in \Lambda} |(f_1, \psi_\lambda)|^2 = \|f_1\|_2^2 = \int_{mQ} |f - f_{mQ}|^2 \leq m^n |Q| \left(\sup_{\text{cubes } Q} \int_Q |f - f_Q|^2 \right)^{\frac{1}{2}})^2$$

But by [20, Corollary 3.2], which uses the John-Nirenberg Inequality and by appropriately changing balls by cubes, we have that:

$$\sup_{\text{cubes } Q} \left(\int_Q |f - f_Q|^2 \right)^{\frac{1}{2}} \leq K \sup_{\text{cubes } Q} \int_Q |f - f_Q|, \text{ for some uniform constant } K > 0.$$

Setting: $C = m^n K^2 \left(\sup_{\text{cubes } Q} \int_Q |f - f_Q| + \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f| \right)^2$, we get:

$$\sum_{Q(\lambda) \subseteq Q} |(f, \psi_\lambda)|^2 \leq C |Q|.$$

Now, let $k_0 \in \mathbb{Z}^n$. Decompose $f = f_1 + f_2$ st:

$$f_1(x) = \begin{cases} f(x) & \text{when: } x \in mQ_{k_0} \\ 0 & \text{otherwise} \end{cases}$$

And, similarly:

$$f_2(x) = \begin{cases} f(x) & \text{when: } x \notin mQ_{k_0} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, by the orthogonality of the ϕ_k 's and the fact that $f_1 \in L^2(\mathbb{R}^n)$, we get:

$$\begin{aligned} |\beta_{k_0}|^2 &= |(f, \phi_{k_0})|^2 = |(f_1, \phi_{k_0})|^2 \leq \sum_{k \in \mathbb{Z}^n} |(f_1, \phi_k)|^2 = \sum_{\lambda \in \Lambda \setminus \Lambda^0} |(f_1, \psi_\lambda)|^2 \\ &\leq \sum_{\lambda \in \Lambda} |(f_1, \psi_\lambda)|^2 = \|f_1\|_2^2 = \int_{mQ_{k_0}} |f|^2 \end{aligned}$$

But, similarly to what we did previously using [20, Corollary 3.2], and as $|Q_{k_0}| = 1$, we have:

$$\int_{mQ_{k_0}} |f|^2 = m^n \int_{mQ_{k_0}} |f|^2 \leq m^n K^2 \left(\sup_{\text{cubes } Q} \int_Q |f - f_Q| + \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f| \right)^2$$

This means: $|\beta_{k_0}|^2 \leq C$. Therefore: $\sup_{k \in \mathbb{Z}^n} |\beta_k| \leq C^{\frac{1}{2}}$.

(B) $\implies$ (C)

Suppose that there exists a constant $C > 0$, s.t: $\forall Q \in \mathcal{Q}$, $\sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq C|Q|$ and $\sup_{k \in \mathbb{Z}^n} |\beta_k| \leq C^{\frac{1}{2}}$.

First, recall that for $f \in L^1_{loc}(\mathbb{R}^n)$, we have: $f = \sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda$ in the sense of distributions. Let us then prove the weak*-convergence.

Detailing Meyer's proof of [44, Theorem 4, p.p 154–155], for integers $p \leq q$, we define the

following partial sums:

$$S_P := \sum_{\lambda \in \mathcal{F}_{p,q}} \alpha(\lambda) \psi_\lambda \text{ where: } \mathcal{F}_{p,q} = \{\lambda \in \Lambda : p \leq j \leq q\}$$

Let $B = B(x_B, r_B)$ be the ball with center x_B and radius $r_B > 0$. Let $j_B = -\lfloor \log_2(r_B) \rfloor \in \mathbb{Z}$ be so that: $2^{-j_B} \leq r < 2^{-j_B+1}$.

Now, since $\text{supp}(\psi_\lambda) \subseteq mQ(\lambda)$, if $mQ(\lambda) \cap B = \emptyset$ for some $\lambda \in \Lambda$, we have $\psi_\lambda = 0$ on B .

Thus, on B , we can decompose S_P as follows:

$$S_P = S_S + S_L \text{ with: } S_S := \sum_{\lambda \in \mathcal{F}_S(B)} \alpha(\lambda) \psi_\lambda \text{ and: } S_L := \sum_{\lambda \in \mathcal{F}_L(B)} \alpha(\lambda) \psi_\lambda$$

where: $\mathcal{F}_S(B) = \{\lambda \in \mathcal{F}_{p,q} : mQ(\lambda) \cap B \neq \emptyset, j \geq j_B\}$ and: $\mathcal{F}_L(B) = \{\lambda \in \mathcal{F}_{p,q} : mQ(\lambda) \cap B \neq \emptyset, j < j_B\}$

If either $\mathcal{F}_S(B) = \emptyset$ or $\mathcal{F}_L(B) = \emptyset$ or both, the corresponding sum(s) are zero. So, WLOG, assume that: $\mathcal{F}_S(B) \neq \emptyset$ and $\mathcal{F}_L(B) \neq \emptyset$.

Now, for small cubes $Q(\lambda)$ (for which $j \geq j_0$) that have the property that $mQ(\lambda) \cap B \neq \emptyset$, we can geometrically observe that for some constant $K_1 > 0$ that depends only on m and n , we have $Q(\lambda) \subseteq K_1 B$. Let Q_B be the smallest cube containing $K_1 B$ and $K_2 > 0$ a constant depending only on n for which $Q_B \subseteq K_2 B$. Therefore, $S_S \in L^2(Q_B) \cap \text{BMO}$ and we can write:

$$\|S_S\|_2^2 \leq \sum_{Q(\lambda) \subseteq Q_B} |\alpha(\lambda)|^2$$

Lemma 4.3.1 allows us to see that:

$$\|S_S\|_2^2 \leq \sum_{Q(\lambda) \subseteq Q_B} |\alpha(\lambda)|^2 \leq 4^n C |Q_B| \leq C (4K_2)^n |B|$$

Denoting $C_1 = \sqrt{C(4K_2)^n}$ and using the Cauchy-Schwarz Inequality, we get:

$$\int_B |S_S| \leq \|S_S\|_2 |B|^{\frac{1}{2}} \leq C_1 |B| \tag{4.4}$$

For large cubes $Q(\lambda)$ (for which $j < j_0$), $S_L \in \text{BMO}$ as all the wavelets are bounded, so we can use the localization and regularity properties of wavelets (as used in the proof of the $(G) \implies (B)$ implication of Theorem 4.2.1), that is, for a given scale 2^{-j} of large cubes $Q(\lambda)$ only M^n at most wavelets ψ_λ do not vanish on B , for some constant $M > 0$ that depends only on m , and that for those wavelets and for some constant $C' > 0$, we have for any $x \in B$:

$$|\psi_\lambda(x) - \psi_\lambda(x_B)| \leq C' 2^j 2^{\frac{nj}{2}} |x - x_B|.$$

Now again, by the Carleson's condition, for such λ 's, we have:

$$|\alpha(\lambda)| \leq C^{\frac{1}{2}} |Q(\lambda)|^{\frac{1}{2}} = C^{\frac{1}{2}} 2^{-\frac{nj}{2}}.$$

Therefore, for any $x \in B$:

$$\begin{aligned} |S_L(x) - S_L(x_B)| &= \left| \sum_{\lambda \in \mathcal{F}_L(B)} \alpha(\lambda) (\psi_\lambda(x) - \psi_\lambda(x_B)) \right| \leq \sum_{\lambda \in \mathcal{F}_L(B)} |\alpha(\lambda)| |\psi_\lambda(x) - \psi_\lambda(x_B)| \\ &\leq M^n C' C^{\frac{1}{2}} 2^{-j_0+1} \sum_{j < j_0} 2^j = 2 M^n C' C^{\frac{1}{2}} \end{aligned}$$

Denoting $C_2 = 2 M^n C' C^{\frac{1}{2}}$, we get the following:

$$\int_B |S_L - S_L(x_B)| \leq C_2 |B| \tag{4.5}$$

Combining estimate (4.4) and estimate (4.5) and denoting $c_B = S_L(x_B)$, we get:

$$\oint_B |S_P - c_B| \leq \oint_B |S_S| + \oint_B |S_L - c_B| \leq C_1 + C_2$$

Therefore:

$$\|S_P\|_{\text{BMO}} \leq C_1 + C_2 \text{ independent of the indices } p \text{ and } q$$

Showing the uniform boundedness of the sequence of partial sums.

By applying the Banach–Alaoglu Theorem, there exists a subsequence of partial sums $\{S_{P_i}\}_{i \in \mathbb{N}}$ converging to a function g in the weak*-topology of BMO.

As $S_P \rightarrow f$ as $p \rightarrow -\infty$ and $q \rightarrow \infty$ in the distributional sense, both limits must agree.

Concluding the weak*-convergence proof.

Now, the father wavelet $\phi \in \mathcal{S}(\mathbb{R}^n) \cup L_c^\infty(\mathbb{R}^n)$. WLOG, dividing by some constant coefficient, we may assume that: $\int \phi = 1$. Fix $x \in \mathbb{R}^n$. We then have:

$$|(\phi * f)(x)| = \left| (\phi * \sum_{k \in \mathbb{Z}^n} \beta_k \phi_k)(x) + (\phi * \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda)(x) \right|$$

For the second part of the right-hand side of the above expression, we can interchange the order of summation and integration as the absolute summability is guaranteed as follows:

$$\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)| |(\phi * \psi_\lambda)(x)| = \sum_{\lambda \in \Lambda^0} |\alpha(\lambda)| \left| \int_{(x-mQ_0) \cap mQ(\lambda)} \phi(x-t) \psi_\lambda(t) dt \right|$$

If $\Lambda_x^0 = \{\lambda \in \Lambda^0 : (x-mQ_0) \cap mQ(\lambda) \neq \emptyset\}$, then: for any $\lambda \in \Lambda_x^0$, we have: $Q(\lambda) \subseteq (x-2mQ_0)$.

So, by the Cauchy-Schwarz Inequality, we have:

$$\sum_{\lambda \in \Lambda^0} |\alpha(\lambda)| |(\phi * \psi_\lambda)(x)| \leq \left(\sum_{Q(\lambda) \subseteq (x-2mQ_0)} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda \in \Lambda_x^0} \left| \int_{mQ(\lambda)} \phi(x-t) \psi_\lambda(t) dt \right|^2 \right)^{\frac{1}{2}}$$

For the first part of the right-hand side, we have an estimate by the Carleson's condition even if $(x-2mQ_0)$ is not a dyadic cube by Lemma 4.3.1:

$$\left(\sum_{Q(\lambda) \subseteq (x-2mQ_0)} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} \leq 4^n C |(x-2mQ_0)| = C (8m)^n$$

For its second part, if we denote $\phi_{-x} = \phi(-(x - \cdot))$, then as $\phi \in L^2(\mathbb{R}^n)$, so is ϕ_{-x} . Therefore:

$$\begin{aligned} \left(\sum_{\lambda \in \Lambda_x^0} \left| \int_{mQ(\lambda)} \phi(x-t) \psi_\lambda(t) dt \right|^2 \right)^{\frac{1}{2}} &\leq \left(\sum_{\lambda \in \Lambda} \left| \int_{mQ(\lambda)} \phi_{-x}(t) \psi_\lambda(t) dt \right|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{\lambda \in \Lambda} |(\phi_{-x}, \psi_\lambda)|^2 \right)^{\frac{1}{2}} = \|\phi_{-x}\|_2 = \|\phi\|_2 \end{aligned}$$

This, by the same token, means that the second part $(\phi * \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda)$ is uniformly bounded. For its first part, we can also interchange the order of summation and integration since absolute summability is guaranteed as follows:

$$\sum_{k \in \mathbb{Z}^n} |\beta_k| |(\phi * \phi_k)(x)| = \sum_{k \in \mathbb{Z}^n} |\beta_k| \left| \int_{(x-mQ_0) \cap mQ_k} \phi(x-t) \phi(t-k) dt \right|$$

Noticing further that for any given $x \in \mathbb{R}^n$, there are only at most D^n number of k 's (for some $D > 0$ depending only on m) st: $(x-mQ_0) \cap mQ_k \neq \emptyset$, and using the upper bound on the $|\beta_k|$'s, we can write:

$$\sum_{k \in \mathbb{Z}^n} |\beta_k| |(\phi * \phi_k)(x)| = \sum_{i=1}^{D^n} |\beta_{k_i}| |(\phi * \phi_{k_i})(x)| \leq m^n \|\phi\|_\infty^2 \sum_{i=1}^{D^n} |\beta_{k_i}| \leq C^{\frac{1}{2}} (mD)^n \|\phi\|_\infty^2 < \infty.$$

This, by the same token, means that the first part is also uniformly bounded. Hence, the Triangle Inequality allows us to conclude that, indeed, we have: $(\phi * f) \in L^\infty(\mathbb{R}^n)$, concluding the proof of this direction.

(C) $\implies$ (A)

Suppose that f is the limit of its wavelet decomposition series $\sum_{\lambda \in \Lambda} \alpha(\lambda) \psi_\lambda(x)$ in the weak*-topology of $\text{BMO}(\mathbb{R}^n)$ (seen as the dual of $H^1(\mathbb{R}^n)$) and $\varphi * f \in L^\infty(\mathbb{R}^n)$ for some $\varphi \in \mathcal{S}(\mathbb{R}^n) \cup L_c^\infty(\mathbb{R}^n)$ st: $\int \varphi = 1$.

For the proof of the case where $\varphi \in \mathcal{S}(\mathbb{R}^n)$, it can be found in [29, Corollary 1, Section 4, p. 36].

For the other case, we will once again adopt a more direct approach.

Let $\varphi \in L_c^\infty(\mathbb{R}^n)$ and Q_φ be a cube centered at the origin (and by the same token symmetric)

st: $|Q_\varphi| > 1$ and $\text{supp}(\varphi) \subseteq Q_\varphi$. Let Q be any cube st: $1 < |Q| \leq |Q_\varphi|$ and x_Q its center.

We have that:

$$\begin{aligned} \int_Q |f| &\leq \int_Q |f - f_Q| + \int_Q |f_Q - f * \varphi(x_Q)| + \int_Q |f * \varphi(x_Q)| \\ &= \int_Q |f - f_Q| + |f_Q - f * \varphi(x_Q)| + |f * \varphi(x_Q)| \end{aligned}$$

But as $f \in \text{BMO}(\mathbb{R}^n)$ and: $f * \varphi = \varphi * f$, then: $\int_Q |f - f_Q| \leq \|f\|_{\text{BMO}}$ and: $|f * \varphi(x_Q)| \leq \|\varphi * f\|_\infty$. So, we get:

$$\int_Q |f| \leq \|f\|_{\text{BMO}} + |f_Q - f * \varphi(x_Q)| + \|\varphi * f\|_\infty.$$

Now, as $\int \varphi = 1$, we have:

$$\begin{aligned} |f_Q - f * \varphi(x_Q)| &= \left| \int_{\mathbb{R}^n} f(y) \varphi(x_Q - y) dy - \int_{\mathbb{R}^n} f_Q \varphi(x_Q - y) dy \right| \\ &\leq \int_{x_Q + Q_\varphi} |f(y) - f_Q| |\varphi(x_Q - y)| dy \\ &\leq \|\varphi\|_\infty \int_{x_Q + Q_\varphi} |f(y) - f_Q| dy \end{aligned}$$

Using the results of [28, Lemma 1.1 and Theorem 1.2, p. 217], denote: $1 \leq K = \lfloor \log_2(\frac{|Q_\varphi|}{|Q|}) \rfloor + 1 \leq K_\varphi = \lfloor \log_2(|Q_\varphi|) \rfloor + 1$.

It means $Q \subseteq x_Q + Q_\varphi$ and it takes K -times doubling of Q to get: $x_Q + Q_\varphi \subseteq 2^K Q$.

For any $k \in \{0, \dots, K\}$, let $Q_k = 2^k Q$. We can then write:

$$\begin{aligned}
\int_{x_Q + Q_\varphi} |f(y) - f_Q| dy &\leq \int_Q |f(y) - f_Q| dy + \sum_{k=1}^K \int_{Q_k \setminus Q_{k-1}} |f(y) - f_Q| dy \\
&\leq \int_Q |f(y) - f_Q| dy + \sum_{k=1}^K \int_{Q_k} |f(y) - f_{Q_k}| dy + \sum_{k=1}^K \int_{Q_k \setminus Q_{k-1}} |f_{Q_k} - f_Q| dy \\
&\leq |Q| \int_Q |f(y) - f_Q| dy + \sum_{k=1}^K |Q_k| \int_{Q_k} |f(y) - f_{Q_k}| dy + \sum_{k=1}^K |Q_k \setminus Q_{k-1}| |f_{Q_k} - f_Q| \\
&\leq |Q_\varphi| \|f\|_{\text{BMO}} + |Q_\varphi| \|f\|_{\text{BMO}} \left(\sum_{k=1}^{K_\varphi} 2^{nk} \right) + \sum_{k=1}^K |Q_k \setminus Q_{k-1}| |f_{Q_k} - f_Q|
\end{aligned}$$

But:

$$|Q_k \setminus Q_{k-1}| = |Q_k| - |Q_{k-1}| = 2^{nk} |Q| - 2^{n(k-1)} |Q| \leq (2^{nk} - 2^{n(k-1)}) |Q_\varphi|.$$

And as a property of $\text{BMO}(\mathbb{R}^n)$ functions, for any $k \in \{1, \dots, K\}$, as $Q_{k-1} \subseteq Q_k$ and $|Q_k| = 2^{nk} |Q| = 2^n |Q_{k-1}| \geq 2 |Q_{k-1}|$, for some universal constant $c > 0$, we have: $|f_{Q_k} - f_{Q_{k-1}}| \leq c \|f\|_{\text{BMO}}$.

So, by a simple induction, we get: $|f_{Q_k} - f_Q| \leq ck \|f\|_{\text{BMO}}$. Therefore:

$$\int_{x_Q + Q_\varphi} |f(y) - f_Q| dy \leq |Q_\varphi| \|f\|_{\text{BMO}} + |Q_\varphi| \|f\|_{\text{BMO}} \left(\sum_{k=1}^{K_\varphi} 2^{nk} \right) + c |Q_\varphi| \|f\|_{\text{BMO}} \left(\sum_{k=1}^{K_\varphi} k (2^{nk} - 2^{n(k-1)}) \right).$$

Denoting: $C_\varphi = |Q_\varphi| \|\varphi\|_\infty \left(1 + \sum_{k=1}^{K_\varphi} 2^{nk} + \sum_{k=1}^{K_\varphi} k (2^{nk} - 2^{n(k-1)}) \right)$, we get:

$$|f_Q - f * \varphi(x_Q)| \leq C_\varphi \|f\|_{\text{BMO}}.$$

Concluding that:

$$\sup_{\substack{\text{cubes } Q \\ 1 < |Q| \leq |Q_\varphi|}} \int_Q |f| < \infty.$$

Now, similarly to [6, proof of Lemma 7], let Q_0 be a cube st: $|Q_0| > |Q_\varphi|$.

If Q_0 has side-length equal to an integer $N (\in \mathbb{N})$ times the side-length of Q_φ , then Q_0 is the

union of N^n non-overlapping cubes Q_j of side-lengths equal to the side-length of Q_φ . Hence:

$$\int_{Q_0} |f| = \frac{1}{N^n} \sum_{j=1}^{N^n} \int_{Q_j} |f| \leq \sup_{\substack{\text{cubes } Q \\ 1 < |Q| \leq |Q_\varphi|}} \int_Q |f|$$

Now, if Q_0 has a non-integer side-length L times the side-length of Q_φ (with $L > 1$), then: $Q_0 \subseteq Q'_0$, where the side-length of Q'_0 is $\lfloor L \rfloor + 1$. Thus, we have:

$$\int_{Q_0} |f| \leq \frac{|Q'_0|}{|Q_0|} \int_{Q'_0} |f| \leq 2^n \int_{Q'_0} |f| \leq 2^n \sup_{\substack{\text{cubes } Q \\ 1 < |Q| \leq |Q_\varphi|}} \int_Q |f|$$

From which, we conclude that: $\sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \int_Q |f| < \infty$.

Now, as $f \in \text{BMO}(\mathbb{R}^n)$, we have, by definition: $\sup_{\text{cubes } Q} \int_Q |f - f_Q| < \infty$.

So, in particular: $\sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \int_Q |f - f_Q| < \infty$. Concluding the proof of this direction.

(B) $\implies$ (D)

Assume that there exists a constant $C > 0$, s.t: $\forall Q \in \mathcal{Q}$, $\sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \leq C|Q|$ and $\sup_{k \in \mathbb{Z}^n} |\beta_k| \leq C^{\frac{1}{2}}$.

Recall that, in the sense of distributions, we have:

$$f = \sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda + \sum_{k \in \mathbb{Z}^n} \beta_k \phi_k$$

Let's start proving the uniform boundedness.

For non-negative integers q and N and in the notation of the proof of the direction (B) $\implies$ (C), we define the following partial sums:

$$S_P := S_T + S_N \text{ where: } S_T := \sum_{\lambda \in \mathcal{F}_{0,q}} \alpha(\lambda) \psi_\lambda \text{ and: } S_N = \sum_{0 \leq |k| \leq N} \beta_k \phi_k$$

Let $B = B(x_B, r_B)$ be the ball with center x_B and radius $r_B > 0$. Let $j_B = -\lfloor \log_2(r_B) \rfloor \in \mathbb{Z}$ be

so that: $2^{-j_B} \leq r < 2^{-j_B+1}$.

Now, since $\text{supp}(\psi_\lambda) \subseteq mQ(\lambda)$ and $\text{supp}(\phi_k) \subseteq mQ_k$, if $mQ(\lambda) \cap B = \emptyset$ and $mQ_k \cap B = \emptyset$ for some $\lambda \in \Lambda$ and some $k \in \mathbb{Z}^n$, we have $\psi_\lambda = 0$ and $\phi_k = 0$ on B .

Thus, on B , we can write:

$$S_T = S_S + S_L \text{ with: } S_S := \sum_{\lambda \in \mathcal{F}_S(B)} \alpha(\lambda) \psi_\lambda, \quad S_L := \sum_{\lambda \in \mathcal{F}_L(B)} \alpha(\lambda) \psi_\lambda, \text{ and: } S_N = \sum_{k \in \mathcal{F}_N(B)} \beta_k \phi_k$$

where: $\mathcal{F}_S(B) = \{\lambda \in \mathcal{F}_{0,q} : mQ(\lambda) \cap B \neq \emptyset, j \geq j_B\}$, $\mathcal{F}_L(B) = \{\lambda \in \mathcal{F}_{0,q} : mQ(\lambda) \cap B \neq \emptyset, j < j_B\}$,

$$\text{and: } \mathcal{F}_N(B) = \{k \in \mathbb{Z}^n : 0 \leq |k| \leq N, mQ_k \cap B \neq \emptyset\}$$

If either $\mathcal{F}_S(B) = \emptyset$, $\mathcal{F}_L(B) = \emptyset$ or $\mathcal{F}_N(B) = \emptyset$, the corresponding sum(s) are zero.

Now, for sufficiently large N , since $\#\{k \in \mathbb{Z}^n : 0 \leq |k| \leq N \text{ and } mQ_k \cap B \neq \emptyset\} \simeq |B|$, with a constant $C_{m,n} > 0$ depending only on m and n for proportionality, we can write:

$$\int_B |S_N| \leq C_{m,n} \sup_{k \in \mathbb{Z}^n} |\beta_k| \|\phi\|_1 |B| \leq C_3 |B| \text{ with: } C_3 = C_{m,n} C^{\frac{1}{2}} \|\phi\|_1$$

If $r_B \geq 1$, then $j_B \leq 0$, thus: $\mathcal{F}_L(B) = \emptyset$ and hence: $S_L = 0$. For the small cubes part, we proceed exactly as we did in the proof of $(B) \implies (C)$ direction for the uniform boundedness and get:

$$\int_B |S_S| \leq C_1 |B|$$

Allowing us to conclude that by the Triangle Inequality:

$$\sup_{B \text{ with radius} \geq 1} \int_B |S_P| \leq C_1 + C_3$$

Now, if $r_B < 1$, then $j_B > 0$, so again proceeding as we did in the proof of $(B) \implies (C)$ direction for the uniform boundedness part for small and large cubes, we get:

$$\int_B |S_T - S_L(x_B)| \leq (C_1 + C_2) |B|$$

Thus:

$$\oint_B |S_P - S_L(x_B)| \leq (C_1 + C_2 + C_3) |B|$$

Hence:

$$\sup_{B \text{ with radius} < 1} \oint_B |S_P - c_B| \leq C_1 + C_2 + C_3 \quad \text{with: } c_B = S_L(x_B)$$

Which tells us that:

$$\|S_P\|_{\text{bmo}} \leq 2(C_1 + C_3) + C_2$$

And the uniform boundedness is proved.

By applying the Banach–Alaoglu Theorem, there exists a subsequence of partial sums $\{S_{P_i}\}_{i \in \mathbb{N}}$ converging to a function g in the weak*-topology of bmo .

As $S_P \rightarrow f$ as $q \rightarrow \infty$ and $N \rightarrow \infty$ in the distributional sense, both limits must agree.

Concluding the weak*-convergence proof.

Concluding the weak*-convergence proof.

(D) $\implies$ (A)

Assume that f is the limit of its wavelet decomposition series $\sum_{\lambda \in \Lambda^0} \alpha(\lambda) \psi_\lambda + \sum_{k \in \mathbb{Z}^n} \beta_k \phi_k$ in the weak*-topology of $\text{bmo}(\mathbb{R}^n)$ (seen as the dual of $h^1(\mathbb{R}^n)$). Then:

$$\|f\|_{\text{bmo}} \simeq \sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \oint_Q |f - f_Q| + \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \oint_Q |f| < \infty$$

So, in particular: $\sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \oint_Q |f - f_Q| < \infty$ and $\sup_{\substack{\text{cubes } Q \\ |Q| > 1}} \oint_Q |f| < \infty$.

Concluding the proof of this theorem. □

Remark 4.3.1. *We now give two important observations from the above proof similar to what we observed following the proof of Theorem 4.2.1.*

- (1) *There is nothing special about the limit value 1 for the volume of the dyadic cubes, we could change it to any other fixed value bigger than 1, and the theorem still holds with slight to no changes to the proof.*

(2) we assumed and required that the father wavelet ϕ and mother wavelet ψ are $C_c^r(\mathbb{R}^n)$ functions with $r \geq 1$, but as seen throughout the proofs, we could relax this condition to both being in $Lip_c(\mathbb{R}^n)$, i.e Lipschitz with compact support.

Corollary 4.3.1. *In the notation of Theorem 4.3.1, for $f \in \text{bmo}(\mathbb{R}^n)$, the following norms are equivalent:*

$$\|f\|_{\text{bmo}} \simeq \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} + \sup_{k \in \mathbb{Z}^n} |\beta_k| \simeq \|f\|_{\text{BMO}} + \|\phi * f\|_{\infty} \text{ where } \phi \text{ is the father wavelet.}$$

Proof. From the proof of Theorem 4.3.1, specifically the direction $(A) \implies (B)$, we have:

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} \lesssim \sup_{\text{cubes } Q} \int_Q |f - f_Q|$$

Also, we saw that:

$$\sup_{k \in \mathbb{Z}^n} |\beta_k| \lesssim \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f|$$

And:

$$\sup_{\substack{\text{cubes } |Q| \\ |Q| \geq 1}} \int_Q |f - f_Q| \leq 2 \sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f|$$

So:

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} + \sup_{k \in \mathbb{Z}^n} |\beta_k| \lesssim \|f\|_{\text{bmo}}$$

Also, from the proof of the $(B) \implies (C)$ direction, we showed that:

$$\|\phi * f\|_{\infty} \lesssim \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} + \sup_{k \in \mathbb{Z}^n} |\beta_k|$$

And in the proof of the direction $(C) \implies (A)$, we showed that:

$$\sup_{\substack{\text{cubes } Q \\ |Q| \geq 1}} \int_Q |f| \lesssim \|f\|_{\text{BMO}} + \|\phi * f\|_{\infty}$$

(Actually, we showed it for the supremum taken over all cubes of size strictly larger than 1, but a small modification of the argument would lead us to the above result).

We also have:

$$\sup_{\substack{\text{cubes } Q \\ |Q| < 1}} \int_Q |f - f_Q| \lesssim \|f\|_{\text{BMO}}$$

So:

$$\|f\|_{\text{bmo}} \lesssim \|f\|_{\text{BMO}} + \|\phi * f\|_{\infty}$$

Finally, from Theorem 3.2.3, we have:

$$\sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} \simeq \|f\|_{\text{BMO}}$$

Thus, we have:

$$\|f\|_{\text{BMO}} + \|\phi * f\|_{\infty} \lesssim \sup_{Q \in \mathcal{Q}} \left(\frac{1}{|Q|} \sum_{Q(\lambda) \subseteq Q} |\alpha(\lambda)|^2 \right)^{\frac{1}{2}} + \sup_{k \in \mathbb{Z}^n} |\beta_k|$$

Concluding the proof. □

Remark 4.3.2. *In the above corollary above, we could replace the father wavelet ϕ in the term $\|\phi * f\|_{\infty}$ by any other φ satisfying the conditions of part (C) of Theorem 4.3.1.*

Bibliography

- [1] Bradley K. Alpert. “A class of bases in L^2 for the sparse representation of integral operators”. *SIAM Journal on Mathematical Analysis* 24.1 (1993), pp. 246–262.
- [2] Pascal Auscher, Alan McIntosh, and Emmanuel Russ. “Tent spaces and applications to Hardy spaces”. *Journal of the London Mathematical Society* 67.2 (2003), pp. 326–348.
- [3] Guy Battle. “A block spin construction of ondelettes. I. Lemarié functions”. *Communications in Mathematical Physics* 110.4 (1987), pp. 601–615.
- [4] Aline Bonami and Justin Feuto. “Products of Functions in Hardy and Lipschitz or BMO Spaces”. In: *Recent Developments in Real and Harmonic Analysis*. Ed. by Carlos Segovia and Javier Duoandikoetxea. Applied and Numerical Harmonic Analysis. Springer, 2009, pp. 1–28.
- [5] Aline Bonami, Sandrine Grellier, and Luong Dang Ky. “Paraproducts and products of functions in $BMO(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ through wavelets”. *Mathematische Zeitschrift* 272.3–4 (2012), pp. 993–1011.
- [6] G. Bourdaud, M. Lanza de Cristoforis, and W. Sickel. “Functional Calculus on BMO and Related Spaces”. *Journal of Functional Analysis* 192.1 (2002), pp. 1–33.
- [7] Marcin Bownik. “The Construction of r -Regular Wavelets for Arbitrary Dilation”. *Journal of Geometric Analysis* 11.3 (2001), pp. 471–491.
- [8] Joan Bruna and Stéphane Mallat. “Invariant Scattering Convolution Networks”. *IEEE Transactions on Pattern Analysis and Machine Intelligence* 35.8 (2013), pp. 1872–1886.

- [9] Almaz Butaev and Galia Dafni. “Locally uniform domains and extension of bmo functions”. *Annales Fennici Mathematici* 48 (2023), pp. 567–594.
- [10] Emmanuel J. Candès and David L. Donoho. “Curvelets: A Surprisingly Effective Non-adaptive Representation for Objects with Edges”. *Curves and Surfaces* (2000), pp. 105–120.
- [11] Ole Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser, 2016.
- [12] Charles K. Chui. *An Introduction to Wavelets*. Academic Press, 1992.
- [13] Ronald R. Coifman. “A real variable characterization of H^p ”. *Studia Mathematica* 51 (1974), pp. 269–274.
- [14] Ronald R. Coifman, Yves Meyer, and Elias M. Stein. “Some new function spaces and their applications to Harmonic analysis”. *Journal of Functional Analysis* 62.2 (1985), pp. 304–335.
- [15] Ronald R. Coifman and Richard Rochberg. “Another characterization of BMO”. In: *Proceedings of the American Mathematical Society* 79.2 (1980), pp. 249–254.
- [16] Ronald R. Coifman, Richard Rochberg, and Guido Weiss. “Factorization theorems for Hardy spaces in several variables”. *Annals of Mathematics* 103.3 (1976), pp. 611–635.
- [17] Ronald R. Coifman and Guido Weiss. *Extensions of Hardy spaces and their use in analysis*. Vol. 83. Bulletin of the American Mathematical Society, 1977, pp. 569–645.
- [18] Ronald R. Coifman and Mladen V. Wickerhauser. “Entropy-Based Algorithms for Best Basis Selection”. *IEEE Transactions on Information Theory* 38.2 (1992), pp. 713–718.
- [19] Galia Dafni and Ryan Gibara. “BMO on Shapes and the John–Nirenberg Inequality”. *Canadian Mathematical Bulletin* 63.4 (2020), pp. 794–812.
- [20] Galia Dafni and Hong Yue. “Some characterizations of local bmo and h^1 on metric measure spaces”. *Analysis and Mathematical Physics* 2.4 (2012), pp. 285–318.

- [21] Ingrid Daubechies. “Orthonormal bases of compactly supported wavelets”. *Communications on Pure and Applied Mathematics* 41.7 (1988), pp. 909–996.
- [22] Ingrid Daubechies. *Ten Lectures on Wavelets*. SIAM, 1992.
- [23] Ingrid Daubechies. “Where do wavelets come from? A personal point of view”. In: *Proceedings of the IEEE* 78.4 (1990), pp. 510–513.
- [24] Javier Duoandikoetxea. *Fourier Analysis*. Vol. 29. Graduate Studies in Mathematics. American Mathematical Society, 2001.
- [25] Charles Fefferman. “Characterizations of bounded mean oscillation”. *Bulletin of the American Mathematical Society* 77.4 (1971), pp. 587–588.
- [26] Charles Fefferman and Elias M. Stein. “ H^p spaces of several variables”. *Acta Mathematica* 129 (1972), pp. 137–193.
- [27] Joseph Fourier. *Théorie analytique de la chaleur*. French. Original treatise introducing Fourier series and Fourier analysis. Paris: Chez Firmin Didot, père et fils, 1822.
- [28] John B. Garnett. *Bounded Analytic Functions*. Vol. 236. Graduate Texts in Mathematics. New York: Springer, 2007. ISBN: 978-0-387-49763-1.
- [29] David Goldberg. “A local version of real Hardy spaces”. *Duke Mathematical Journal* 46.1 (1979), pp. 27–42.
- [30] Loukas Grafakos. *Modern Fourier Analysis*. 3rd. Vol. 250. Graduate Texts in Mathematics. Springer, 2014.
- [31] Alexander Grossmann and Jean Morlet. “Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape”. *SIAM Journal on Mathematical Analysis* 15.4 (1984), pp. 723–736.
- [32] Alfréd Haar. “Zur Theorie der orthogonalen Funktionensysteme”. German. *Mathematische Annalen* 69.3 (1910), pp. 331–371.

- [33] G. H. Hardy. “On the mean value of the modulus of an analytic function”. *Proc. London Math. Soc.* 14 (1915), pp. 269–277.
- [34] Lars Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Vol. 256. Grundlehren der Mathematischen Wissenschaften. Berlin: Springer, 1990.
- [35] *Information Technology – JPEG 2000 Image Coding System*. International Organization for Standardization, 2000.
- [36] Fritz John and Louis Nirenberg. “On functions of bounded mean oscillation”. *Communications on Pure and Applied Mathematics* 14.3 (1961), pp. 415–426.
- [37] Luong Dang Ky. “Bilinear decompositions and commutators of singular integral operators”. *Transactions of the American Mathematical Society* 365.6 (2013), pp. 2931–2958.
- [38] Demetrio Labate et al. “Sparse Multidimensional Representation Using Shearlets”. *Wavelets XI* 5914 (2005), pp. 254–262.
- [39] Ming-Jun Lai. “On the computation of Battle-Lemarié’s wavelets”. *Mathematics of Computation* 63.208 (1994), pp. 689–707.
- [40] Pierre-Gilles Lemarié. “Bases d’ondelettes sur les groupes de Lie”. *Bulletin de la Société Mathématique de France* 116 (1988), pp. 211–232.
- [41] Stéphane Mallat. “A Theory for Multiresolution Signal Decomposition: The Wavelet Representation”. *IEEE Transactions on Pattern Analysis and Machine Intelligence* 11.7 (1989), pp. 674–693.
- [42] Stéphane Mallat. *A Wavelet Tour of Signal Processing*. 2nd. Academic Press, 1999.
- [43] Benoit Mandelbrot. *The Fractal Geometry of Nature*. New York: W. H. Freeman, 1982.
- [44] Yves Meyer. *Wavelets and Operators*. Vol. 37. Cambridge Studies in Advanced Mathematics. Translated from the French by D. H. Salinger. Cambridge: Cambridge University Press, 1992. ISBN: 9780521426043.

- [45] Yves Meyer and Pierre-Gilles Lemarié-Rieusset. *Ondelettes et bases orthonormées*. Séminaire de l'École Polytechnique. First formal construction of orthonormal wavelet bases. 1986.
- [46] Frigyes Riesz. “Sur les fonctions conjuguées”. *Math. Z.* 18 (1923), pp. 258–263.
- [47] David I. Shuman et al. “The Emerging Field of Signal Processing on Graphs”. *IEEE Signal Processing Magazine* 30.3 (2013), pp. 83–98.
- [48] Elias M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Vol. 43. Princeton Mathematical Series. Princeton, NJ: Princeton University Press, 1993. ISBN: 9780691034097.
- [49] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Vol. 30. Princeton Mathematical Series. Princeton University Press, 1972. ISBN: 9780691080798.
- [50] Elias M. Stein and Guido Weiss. “On the theory of harmonic functions of several variables. I. The theory of H^p -spaces”. *Acta Mathematica* 103 (1960), pp. 25–62.
- [51] Robert S. Strichartz. “Wavelets and Self-Affine Tilings”. *Constructive Approximation* 9 (1993), pp. 327–346.
- [52] Akihito Uchiyama. “Hardy spaces on the Euclidean space”. *Kodai Mathematical Journal* 3.2 (1986), pp. 239–258.
- [53] Przemysław Wojtaszczyk. *A Mathematical Introduction to Wavelets*. Cambridge University Press, 1997.