Static and Dynamic Matching with Dichotomous Preferences

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Abstract

Static and Dynamic Matching with Dichotomous Preferences

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This thesis consists of three studies in matching theory where agents have dichotomous preferences, while institutions have strict rankings over agents, interpreted as preferences or priorities depending on the setting. Dichotomous preferences simply divide institutions according to whether they are acceptable or not. This preference domain is natural in applications where the primary concern is to match as many agents as possible to any acceptable institution, such as centralized daycare assignments and other allocation problems with scarcity, as well as when preferences are naturally binary due to compatibility or eligibility criteria.

In Chapter 2, I study a two-sided matching problem with dichotomous agent preferences and institutions with strict rankings of agents. I design a new class of mechanisms called SAFE mechanisms (Sequential Allocation for Fairness and Efficiency), equivalently characterized as Rank-Maximal mechanisms, that produce a maximum-size matching while respecting individual rationality, institutional rankings, and satisfying all standard fairness and incentive properties, including strategyproofness and nonbossiness. These mechanisms are computationally tractable and can be implemented in polynomial time.

Chapter 3 extends the analysis to a two-period overlapping dynamic setting motivated by daycare allocation. Children participate for two periods, report dichotomous acceptance over daycares, and daycares have strict priority orderings over children. I study whether one can simultaneously achieve maximum size together with dynamic fairness, efficiency, and incentive properties. I propose two dynamic mechanisms, tailored to two distinct policy objectives: a history-dependent (HD) policy and a childcare-guarantee (CG) policy. Under HD, a dynamic SAFE mechanism attains maximum size and the entire set of targeted dynamic properties. Under CG, I show that no reasonable mechanism can satisfy these dynamic axioms. However, the proposed SAFE mechanism retains strong period-by-period properties (maximum and fair in each period), and impossibility results justify it as

a best possible compromise under the CG policy.

In Chapter 4, I analyze matching under institutional constraints that limit how many schools an agent may list as acceptable, which is a typical practice in school choice and centralized university admissions. I ask whether dichotomous preferences mitigate the manipulation and fairness issues known from the strict-preference setting. I show that if the constraint binds, then any individually rational and maximum constrained mechanism is manipulable. Nevertheless, for any constraint level, all Nash equilibria of the induced preference revelation game of individually rational and maximum mechanisms yield fair outcomes. Finally, I show that manipulability and fairness cannot be compared in an unambiguous way across SAFE mechanisms with different constraint levels.

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Dedication

This dissertation is dedicated to my parents, my only sister (Rumi), and my wife (Eity).

Contribution of Authors

Chapter 2: joint work with my supervisor, Dr. Szilvia Pápai, and with Dr. Haris Aziz (Computer Science and Mathematics, University of New South Wales).

Chapter 3: joint work with my supervisor, Dr. Szilvia Pápai.

Chapter 4: joint work with my supervisor, Dr. Szilvia Pápai.

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Chapter 1

Introduction

There are many real-world assignment problems: each agent seeks an object (such as a seat at an institution), while objects (institutions) rank potential assignees. Classic examples include school choice, daycare assignments, public housing, and various rationing problems. A central theme in the matching literature is that, under strict preferences, desirable properties of mechanisms are often incompatible. There is a well-known conflict between stability (fairness) and Pareto efficiency: namely, no matching mechanism exists under agents' strict preferences that guarantees both efficiency and stability simultaneously. Stability is a specific formal notion that can be interpreted as fairness, depending on the context, and it requires that there is no priority or preference violation such that a specific pair of agent and object both would get a higher-ranked match if they were assigned to each other, rather than to their current matches.

This thesis explores a preference domain that is both practically motivated and theoretically revealing: I study agents with **dichotomous preferences**, meaning that they only distinguish between acceptable and unacceptable institutions, while institutions maintain strict rankings (preferences or priorities) over agents. This assumption is natural when there is substantial scarcity and the primary objective is to match as many agents as possible to any acceptable institution. It is also relevant in settings where acceptance is determined by a binary eligibility or compatibility criterion. The dichotomous domain allows for combining three objectives: (i) matching as many agents as possible, (ii) respecting institutional rankings (known as stability or fairness), and (iii) providing robust incentives so agents do not benefit from misreporting.

Chapter 2 (static, two-sided): I begin with a two-sided matching model in which agents report their acceptable institutions and institutions rank agents strictly. The fundamental research question I explore in this chapter is:

Can maximum size, efficiency, fairness, and strategyproofness be satisfied simultaneously in a two-sided matching model with dichotomous preferences on one side and strict rankings on the other side?

I answer this question constructively by designing a novel set of mechanisms (the SAFE / Rank-Maximal class) and establishing that it satisfies all the desired properties simultaneously. The findings are in stark contrast to well-established results on the strict preference domain.

Chapter 3 (dynamic, overlapping participation): Many assignment problems are inherently dynamic. Inspired by the daycare assignment application, I study an overlapping two-period environment in which agents participate for two consecutive periods: a new cohort arrives in each period, and institutions have priorities that apply in both periods. The central question becomes:

Does there exist a mechanism that is maximum and fair, and also satisfies meaningful dynamic incentive and efficiency requirements?

I show that the answer depends crucially on the policy environment. Under a history-dependent regime, where continuing children are prioritized for the daycare they attended previously, dynamic SAFE mechanisms achieve all the targeted dynamic properties. Under a childcare-guarantee regime that prioritizes initially unmatched children for their second-period assignment, I show that the dynamic desiderata cannot be satisfied simultaneously with static criteria, implying that the dynamic SAFE mechanisms provide a good compromise in this policy environment.

Chapter 4 (institutional constraints on preference reporting): Finally, I study a constraint that is common in practice: institutions restrict how many schools an applicant may list. Under strict preferences, such constraints create opportunities for manipulation and produce priority violations. The core research question is:

Do dichotomous preferences improve the incentive and fairness properties of constrained mechanisms?

I show that despite the simplification introduced by dichotomous preferences, key negative results persist: binding constraints rule out strategyproof, individually rational, and maximum mechanisms, and comparisons of manipulation and fairness remain fundamentally ambiguous across SAFE mechanisms with different constraints.

Overall, the thesis contributes to mechanism design and market design by clarifying when and how dichotomous preferences can resolve classical tradeoffs among fairness, efficiency, and incentives, and by identifying the limits of such resolution in dynamic and constrained settings.

Chapter 2

Strategyproof Maximum Matching under Dichotomous Agent Preferences

2.1 Introduction

We consider a fundamental matching problem in which agents are to be matched to institutions. Agents express dichotomous preferences over institutions by specifying acceptable institutions. Institutions express preferences among the set of agents but cannot declare any agent unacceptable. There is a shortage of the available spots at institutions, and agents care primarily about being assigned to an acceptable object rather than which institution they are assigned to. The system typically requires that agents are matched to institutions in a way that is fair with respect to the preferences of both sides. At the same time, the designer is concerned about assigning as many agents to acceptable spots as possible.

The problem that we consider is inspired by the centralized matching of children to daycare centres or schools. In these problems, parents of children express a subset of daycares as acceptable if they are near enough or provide the required facilities (see, e.g. Sun, Takenami, Moriwaki, Tomita, and Yokoo, 2023). On the other hand, each daycare may have its own preference ordering over children according to its own criteria. Such a problem also arises with an increasing number of applicants to various institutions, such as when there are too few school seats compared to the number of school-age children, or when there is a shortage of daycare capacity compared to the demand for daycare. Accommodating applicants in schools, daycares, and similar institutions is a frequent concern in many places worldwide, where applicants may stay on waiting lists for a long time. Depending on the setting, the spots at the institutions could be daycare spots, immigration slots, school seats, or healthcare treatments. In particular, our setting captures healthcare rationing problems in which healthcare patients are matched to limited healthcare resources. These problems have received tremendous interest in recent years (see, e.g. Aziz and Brandl, 2024; Pathak, Sönmez, Ünver, and Yenmez, 2021).

Our main goal is to maximize the number of placements of agents to acceptable slots in the centralized setting that we consider, and to do this in a fair and incentive compatible way. There are many decentralized daycare systems with a substantial shortage of daycare openings. There is often a major concern that many children are on the waiting lists of daycares for an extended period of time, and thus parents face difficulties when returning to work after parental leave. Surprisingly, there are still vacant spots remaining in daycares which typically means an inefficient usage of resources. However, matching the maximum number of applicants is unlikely under a decentralized allocation system. Che and Koh (2016) shows that decentralized matching mechanisms lead to unfairness and inefficiency. These might be unexpected consequences if applicants remain unassigned due to problems in the allocation system. Matching as many children as possible to daycares, therefore, is the primary objective. Secondarily, we also want to ensure that the allocation is fair, which means that preferences or priorities over applicants are respected. Finally, it is also important to use an allocation mechanism that provides the correct incentives when applicants report their preferences, in order to make sure that the allocation is indeed efficient and maximizes the number of matched applicants, and also respects the preferences over applicants. Many of the above concerns are captured by key axioms in market design that we discuss below.

When finding a desirable matching, we are guided by basic axiomatic properties that are well-established in the matching literature. A basic requirement is to find an *individually rational* matching in which each agent is matched to an acceptable institution and each institution is matched to an acceptable agent. A matching satisfies *fairness* (respects the preferences/priorities of institutions) if there is no unmatched agent i who wishes to be matched to an acceptable institution d and d is matched to an agent j that is ranked lower than i by d. We want to find a maximum size individually rational matching that respects preferences or priorities over agents. Interpreted in our setting, a result of Aziz and Brandl (2021) implies that a maximum size (individually rational) matching that respects priorities can be computed in polynomial time. They propose a rule, the Reverse Rejecting Rule (REV), that considers the agents in order of the baseline ordering from lowest to highest priority. It rejects an agent if the agents who have not been rejected thus far can form a maximum size matching for which none of the rejected agents has justified envy towards a matched agent. However, their rule has some limitations. It is not strategyproof in the classical sense for either of the two sides. It is also not necessarily Pareto-efficient from the institutions' side. Whether there exists a mechanism that is agent-strategyproof, maximum, fair and non-bossy was posed as an unresolved problem by Aziz and Brandl (2021). Non-bossiness requires that an agent cannot affect a result in the outcome without getting a different matching.

In this paper we explore the following fundamental question:

> *Can maximum size, efficiency, fairness and strategyproofness be satisfied simultaneously in a two-sided matching model with dichotomous preferences on one side and strict preferences (or priorities) on the other side?*

The combination of individual rationality, fairness, and non-wastefulness is typically called stability in classical two-sided matching problems. When both sides have strict or weak preferences/priorities, combining fairness, efficiency, and strategyproofness yields impossibility results. It is well-known that for the classical setting with strict preferences on both sides (1) there is no mechanism that satisfies stability and strategyproofness for both sides (Roth, 1982); (2) stability and Pareto-efficiency on one side are incompatible (Balinski and Sönmez, 1999); (3) stability/fairness and maximum size are incompatible (Afacan, Bó, and Turhan, 2023); and (4) stability and non-bossiness are incompatible (Kojima, 2010). Moreover, finding a maximum size stable matching is NP-hard when there can be ties in the preferences (Biró, Manlove, and Mittal, 2010). By contrast, for the setting that we consider with dichotomous agent preferences, the answer to our fundamental research question is surprisingly positive.

Contributions. We present mechanisms for two-sided matching with dichotomous preferences on one side and strict preferences on the other side that satisfy the following properties:

1. non-wastefulness;
2. individual rationality;
3. maximum size;
4. fairness;
5. Pareto-efficiency on both sides;
6. strategyproofness on both sides;
7. non-bossiness on both sides;
8. polynomial-time.

To the best of our knowledge, these are the first known mechanisms that satisfy all these properties simultaneously, including the striking properties of strategyproofness and Pareto-efficiency on both sides. We present two families of mechanisms, the SAFE and Rank-Maximal mechanisms. SAFE mechanisms are based on the idea of safe blocks that identify subsets of institutions that are not over-subscribed by agents and should be assigned first in order to reach a maximum matching. We also propose another class of mechanisms with a different perspective, the Rank-Maximal mechanisms, based on the graph-theoretic notion of rank maximality. We show that these two apparently quite different families of mechanisms are equivalent. We then show that, quite surprisingly, these mechanisms satisfy all the key properties listed above.

Having two different formulations of the family of mechanisms that we propose is beneficial for our understanding of these mechanisms. The multiple perspectives give different insights into the mechanisms and provide distinct tools to analyse them and establish their axiomatic properties. Whereas Rank-Maximal mechanisms are clearly polynomial-time and highlight the overall selection of matched institutions, SAFE mechanisms provide a different intuition about how these mechanisms work by identifying sets of institutions that need to be matched first if the matching is to be maximum size. Table 2.1 summarizes the properties satisfied by our mechanisms compared to other mechanisms in this setting, and highlights how our new mechanisms have striking advantages over existing mechanisms in the literature. In contrast to two prominent mechanisms for this setting, our mechanisms satisfy Pareto-efficiency for both sides, strategyproofness for both sides, and non-bossiness.[1]

Properties	DA	REV	**SAFE/Rank-Max**
Individual rationality	✓	✓	✓
Fairness	✓	✓	✓
Institution Pareto-efficiency	–	–	✓
Agent Pareto-efficiency	–	✓	✓
Both-sided Pareto-efficiency	–	–	✓
Agent strategyproofness	✓	–	✓
Institution strategyproofness	–	–	✓
Both-sided strategyproofness	–	–	✓
Both-sided non-bossiness	–	–	✓

Table 2.1: Axioms satisfied by various mechanisms. The SAFE/Rank-Max mechanisms are introduced in this paper. DA is the classical Deferred Acceptance rule applied in our context with tie-breakers. REV was introduced by Aziz and Brandl (2024).

[1] An example that the DA is bossy is provided in the appendix.

Multiple Capacities. We assume throughout the paper that each institution has unit capacity. This serves both to simplify the way we think about this problem and to simplify the presentation. A mechanism that works for institutions with unit capacities also gives a mechanism for institutions with higher capacity. Many of our results for unit-capacity institutions generalize to higher capacities since each institution can be viewed as being divided into smaller institutions with unit capacity each. The assumption that each slot at an institution is represented by a separate institution leads to a larger preference domain than the general multi-unit case, since with unit capacity each institution can be reported acceptable/unacceptable by each agent independently of other institutions, while for the multi-unit case all slots at the same institution are either acceptable or not. Thus, most of the properties of SAFE/Rank-Max mechanisms generalize to the many-to-one case immediately, given that this represents a smaller domain than the one-to-one domain that we are working with. The only exceptions are the properties for the institution side, namely Pareto-efficiency, strategyproofness, and non-bossiness for institutions, some of which require additional assumptions in order to hold when institutions may have multiple capacities.

2.2 Related Work

Two-sided matching under preferences has a long history (Manlove, 2013; Sotomayor, 1990). Classical results focus on strict preferences on both sides (Gale and Shapley, 1962; Roth, 2008). When both sides have strict preferences, stability and strategyproofness from both sides is impossible (Roth, 1982), and Pareto efficiency and stability are incompatible (Abdulkadiroğlu and Sönmez, 2003). The classical Deferred Acceptance algorithm can be applied to our problem as follows: break the ties in the dichotomous preferences of the agents to convert them into strict preferences and then run agent proposing Deferred Acceptance algorithm. However, the approach does not necessarily give a maximum size or Pareto optimal matching. It is also only strategyproof for one of the sides whereas we establish strategyproofness for both sides. Similarly, the Top Trading Cycles algorithm is another algorithm for matching under preferences but if applied to our context, it does not satisfy the maximum size property or fairness.

There is also work on two-sided matching where both sides have dichotomous preferences. For example, Bogomolnaia and Moulin (2004) presents several results on randomized matching under dichotomous preferences. The results do not apply to our setting where one side has strict preferences. For example, when both sides have dichotomous preferences, fairness or respect of priorities has very little bite. We also focus on deterministic matchings and are able to achieve several axiomatic properties without resorting to randomisation. Aziz (2016) proposes rules for exchange problems when agents have dichotomous preferences. There are also a few recent exchange papers with dichotomous preferences (see Andersson, Cseh, Ehlers, and Erlanson, 2021; Manjunath and Westkamp, 2021; Mishra, Sarkar, Sen, Sethuraman, and Yadav, 2025). One of the key properties that we focus on is computing a feasible *maximum size* matching. The assignment maximization problem is not only relevant for daycares, but also for schools (Abdulkadiroğlu, Pathak, and Roth, 2005; Basteck, Huesmann, and Nax, 2015), and for the allocation of any goods which are in shortage or need to be rationed, such as public housing, vaccines and organs. Roth, Sönmez, and Ünver (2005) study the kidney exchange problem to find a maximal and strategyproof mechanism. Ergin, Sönmez, and Ünver (2017), Andersson and Kratz (2020), and Ergin, Sönmez, and Ünver (2020) aim to maximize the number of patients receiving transplants in organ exchange including kidneys, lungs, and liver. Achieving a maximum and efficient matching between refugees and landlords is also an increasingly important problem in market design (Andersson and Ehlers, 2020; Delacrétaz, Kominers, Teytelboym, et al., 2020). Another application in which assignment maximization of objects to agents is of significant concern is the house allocation problem (Abraham, Cechlárová, Manlove, and Mehlhorn, 2005; Aziz, 2016; Krysta, Manlove, Rastegari, and Zhang, 2014). One particular problem for which assignment maximization is important is healthcare rationing where we want to utilize the maximum number of healthcare resources. We discuss the connections below.

Pathak et al. (2021) considers a healthcare rationing in which agents have types and they are matched to categories pertaining to particular types. An agent can be matched to a category if it satisfies some type that the category is dedicated to. The healthcare rationing problem can be abstracted to our model by ignoring the types and simply assuming that an agent and category find each other acceptable if they can be matched to each other in the healthcare problem. Pathak et al. (2021) focus on homogenous priorities whereas we allow institutions to have heterogeneous preferences. A standard approach for the problem is to treat reserves from categories in a sequential manner (Aygün and Bó, 2021; Dur, Pathak, and Sönmez, 2020; Kominers and Sönmez, 2016). These approaches violate axioms pertaining to neutrality or fairness towards categories. The myopic picks can also lead to outcomes that do not satisfy the maximum size property.

Aziz and Brandl (2021, 2024) studies a healthcare rationing problem with heterogeneous priorities. Aziz and Brandl (2021) shows that maximum size individually rational matching that satisfies respect for priorities can be computed in polynomial time. The result is in contrast to the fact that when both sides have weak preferences/priorities, then the problem of computing a maximum size fair matching is NP-hard. Their Reverse Rejecting (REV) rule works by considering agents in the reverse order of a baseline ordering of agents and iteratively deciding whether the agents are to be removed from consideration or not. In follow-up work, Banerjee, Eichhorn, and Kempe (2023) provide an algorithmic characterization of all valid allocations, exhibiting a bijection between sets of agents who can be allocated and maximum-weight matchings under carefully chosen rank-based weights.

Aziz and Brandl (2024) proves that their algorithm is strategyproof in the following sense: no agent can express some institution as unacceptable in order to get an advantage. Their strategy space also allows for agents to lower themselves in the priority ordering of the institutions. Since they examine a healthcare rationing setting, they do not allow agents to falsely make themselves eligible for an institution for which they are ineligible. In our setup, we allow agents to report unacceptable institutions as acceptable, in addition to reporting acceptable institutions as unacceptable. Therefore, strategyproofness amounts to the following: no agent wants to declare an acceptable institution unacceptable or an unacceptable institution acceptable. Another aspect that is overlooked in most of the previous work is Pareto-efficiency of the institutions. We design rules that have two additional advantages over previous rules: they are strategyproof for both agents and institutions, and they are Pareto-efficient for institutions. Finally, the REV rule of Aziz and Brandl violates non-bossiness. Aziz and Brandl posed the question whether there exists a rule satisfying maximum size, fairness, agent strategyproofness and non-bossiness. We show that our rules satisfy all these properties.

2.3 Model with Dichotomous Agent Preferences

Our model is a two-sided matching model between agents and institutions. Agents have dichotomous preferences over institutions, while institutions strictly rank all agents according to their preferences. Our setting has the following components.

- Set of n agents: N
- Set of m institutions: D
- Dichotomous agent preferences: for all $i \in N$, $A_i \subseteq D$ is the set of acceptable institutions for i. The set of acceptance reports is represented by the acceptance profile

$A = (A_1, \ldots, A_n)$. Each acceptance profile A has a simple graph representation. In the underlying *acceptance graph* $G = (N \cup D, E)$, for all pairs $i \in N$ and $d \in D$, $\{i, d\} \in E$ if and only if $d \in A_i$.

- Strict institution preferences: for all $d \in D$, $\succ_d$ denotes the preference list of institution d over agents N. Institutions are assumed to find all agents acceptable. The set of strict preference reports by institutions is represented by the preference profile $\succ = (\succ_{d_1}, \ldots, \succ_{d_m})$.

All the information is captured in a problem instance $(N, D, A, \succ)$. If we assume that N and D are fixed, then a problem is given by a *profile* $(A, \succ)$, consisting of an acceptance profile for the agents and a preference profile for the institutions.

Given a profile $(A, \succ)$, we call the preference list of institution $d \in D$ that consists of only the agents who find institution d acceptable, and which follows the priority ordering $\succ_d$, the *acceptance list* of institution d. Formally, the **acceptance list** of institution $d \in D$ is the ordered list $\succ_d^A$ of the agents in $\{i \in N : d \in A_i\}$ for which, for all $i, j \in N$ such that $d \in A_i \cap A_j$, we have $i \succ_d^A j$ if and only if $i \succ_d j$. Note that $\succ_d^A$ is a function of A in addition to $\succ_d$, since only agents that report an institution acceptable appear in the institution's acceptance list. The set of acceptance lists for all institutions is the *acceptance list profile* $\succ^A$.

We are interested in matching agents to institutions. Each agent is either matched to some institution or remains unmatched, and each institution is either matched to some agent or remains unmatched. A matching μ specifies which agent is matched to which institution (e.g., $\mu_i = d$), and which agents and institutions remain unmatched (denoted by $\mu_i = 0$ and $\mu_d = 0$).

We illustrate the matching problem by the next example.

Example 1. *Suppose there are four agents* $1, 2, 3, 4$ *and two institutions* d_1 *and* d_2. *So* $N = \{1, 2, 3, 4\}$ *and* $D = \{d_1, d_2\}$. *Each agent specifies its set of acceptable institutions:*

$$A_1 = \{d_1, d_2\},\ A_2 = \{d_2\},\ A_3 = \{d_1\},\ A_4 = \{d_1, d_2\}.$$

The two institutions have the following preferences in decreasing order of preferences from left to right:

$$\succ_{d_1}: \quad 2, 4, 1, 3$$
$$\succ_{d_2}: \quad 1, 3, 4, 2$$

The acceptance lists are:

$$\succ^A_{d_1}: \quad 4, 1, 3$$
$$\succ^A_{d_2}: \quad 1, 4, 2$$

One desirable matching is $\{(d_1, 4), (d_2, 1)\}$*, which matches agent* 4 *to* d_1 *and agent* 1 *to* d_2*. Agents 2 and 3 are unmatched.*

2.4 Axioms

A matching is *individually rational* if for each $i \in N$, $d \in D$ that are matched to each other, it must be that $d \in A_i$. Thus, a matching is *individually rational* if it is a matching in graph G.

A matching is *maximum size* if there is no other matching that matches more agents to institutions. A matching μ is *Pareto-efficient for agents* (or *N-efficient*, for short) if there is no matching μ' in which no agent is worse off and at least one agent strictly prefers μ' to μ. In other words, N-efficiency requires that the matching is not Pareto-dominated for agents. Under dichotomous preferences, it is well-known that a matching is N-efficient if and only if it is a maximum matching. Moreover, note that N-efficiency implies individual rationality, since a matching that includes any agent assigned to an unacceptable institution is N-Pareto-dominated by the matching which leaves such agents unmatched but otherwise makes the same matches.

If we want to find a matching that is individually rational and maximum-size, we can determine these from the acceptance graph. If in addition to individual rationality and maximum size, we also want the matching to be fair or Pareto-efficient for the institutions, we need more information than just G, we also need to know the institutions' preference profile $\succ$.

A matching is *fair* (or, equivalently, respects the preferences of the institutions) if the following scenario does not arise: there exists $i, j \in N$ and $d \in D$ such that i is unmatched, $d \in A_i$, $\mu_j = d$ and $i \succ_d j$. This axiom is in the same spirit as the standard notion of fairness (no justified envy) for strict preferences, adapted to the dichotomous preference domain.

A matching μ is *Pareto-efficient for institutions* (or *D-efficient*, for short) if there is no matching μ' in which no institution is worse off and at least one institution strictly prefers μ' to μ. Note that a matching that is Pareto-effient for institutions is fair, but the converse is not necessarily true. To illustrate the latter, consider matching $\{\{d_1, 1\}, \{d_2, 4\}\}$ in Example 1.

This matching is fair but not D-efficient.

A matching μ is *Pareto-efficient for both agents and institutions* (or *efficient,* for short) if there is no matching μ' in which no agent or institution is worse off and at least one agent or institution strictly prefers μ' to μ. While in general Pareto-efficiency for each set of a partition of the agents separately does not imply Pareto-efficiency for the entire set of agents, in our model the conjunction of N-efficiency and D-efficiency implies efficiency. This follows because N-efficiency implies individual rationality, and individually rational maximum matching cannot be Pareto-dominated by matching a different set of agents, and thus a D-efficient individually rational maximum matching is efficient.

A *mechanism* f is a function which matches agents to institutions at each profile $(A, \succ)$ such that no more than one agent is matched to an institution and each agent is matched to at most one institution. All of the above definitions of axioms for individual matchings are extended to mechanisms in the usual manner. For example, a matching mechanism is *fair* if it assigns a fair matching to each profile $(A, \succ)$.

We also aim for *strategyproofness*: no agent has an incentive to change its preference to get matched; and no institution has an incentive to change its preference to get a more preferred matched agent. A mechanism is *strategyproof for the agents* (or N-strategyproof, for short) if no agent can misreport its preferences to obtain a strictly better outcome. A mechanism is *strategyproof for the institutions* (or D-strategyproof, for short) if no institution can misreport its preferences to obtain a strictly better outcome.

Observe that any mechanism that yields a maximum size matching would not be D-strategyproof if institutions were allowed to report some agents unacceptable. To see this, consider a case in which two institutions have identical preferences over acceptable agents 1 and 2: $1 \succ_{d_1} 2$ and $1 \succ_{d_2} 2$. The institution that is assigned 2 can ensure it gets 1 by reporting 2 as unacceptable. Hence we are able to ensure that a mechanism is D-strategyproof only because institutions need to accept all agents. This is a standard and reasonable assumption for institutions, since daycares and schools typically cannot exclude any children from attending them. Similarly, if the institutions represent healthcare equipment or vaccines, patients and customers cannot be excluded from access based on the preferences of healthcare providers. Therefore, we require that all reported preference orderings $\succ_d$ by institution d rank all agents in N.

We also consider *non-bossiness*. A mechanism is *agent non-bossy* if no agent can change her acceptable set such that her match does not change but the matching changes. A mechanism is *institution non-bossy* if no institution can change its preferences such that its match remains the same but the matching changes. A mechanism is both-sided non-bossy by definition if it is both agent non-bossy and institution non-bossy.

Existence of a Maximum and Fair Matching

In matching models with strict preferences a fair and maximum matching does not necessarily exist. It is well known that in a one-to-one or many-to-one two-sided matching model all stable matchings have the same cardinality, which follows from the Rural Hospital Theorem (McVitie and Wilson, 1970; Roth, 1985). Moreover, one can easily construct examples where stable matching are not maximum matchings. For example, if higher-priority agents have more acceptable schools than lower-priority agents, and both priorities and preferences are homogeneous, it is possible that higher priority agents are assigned to all the institutions that are acceptable to lower-priority agents, based on the strict preferences of higher-priority agents, leaving lower-priority agents unassigned; this is not a maximum matching. Since stability corresponds to fairness in one-sided matching models, this means that fairness cannot be reconciled with maximum matchings when preferences are strict. By contrast, we point out that in our model with dichotomous preferences a maximum and fair matching always exists.

Observation 1. *There exists a maximum and fair matching at every profile.*

Proof. Fix a profile $(A, \succ) \in \mathcal{A} \times \Pi$. Fix a maximum matching μ^0 at $(A, \succ)$. If μ^0 is not fair then there exists an agent-institution pair (c_1, d_1) such that $d_1 \in A_{c_1}$, $\mu^0_{c_1} = 0$ and, given that μ^0 is a maximum matching, there exists an agent c_1' such that $\mu^0_{d_1} = c_1'$ and $c_1 \succ_{d_1} c_1'$. Assign c_1 to d_1, and let the other assignments be the same as in μ^0. Call this matching μ^1. If μ^1 is not fair then there is an agent-institution pair (c_2, d_2) such that $d_2 \in A_{c_2}$, and, given that μ^1 is a maximum matching, there exists an agent c_2' such that $\mu^1_{d_2} = c_2'$ and $c_2 \succ_{d_2} c_2'$. Assign agent c_2 to d_2, and let the other assignments be the same as in μ^1. Call this matching μ^2. Keep repeating the same argument and apply a similar modification to the matching iteratively. Observe that an agent-institution pair cannot be repeated in this sequence, since in each step the priority of the new agent assigned to an institution is higher than the priority of the previous agent who was assigned to this institution. Moreover, the number of such improvement steps is finite, given that there is a finite number of agents and a finite number of institutions. When we can no longer find such an agent-institution pair for some matching μ^k ($k \geq 0$), the matching μ^k is fair by definition. Moreover, since each institution that had an agent assigned to it in μ^0 still has an agent assigned to it in μ^k, implying that $|\mu^0| = |\mu^k|$, and given that μ^0 is a maximum matching, it follows that μ^k is also a maximum matching. Thus, we have shown that there exists a maximum and fair matching for an arbitrary profile. □

Given that a maximum matching is Pareto-efficient for agents, **??** implies that Pareto-efficiency for agents can be reconciled with fairness, another result which does not hold when preferences are strict. This positive result is possible due to the indifferences in dichotomous preferences. Observation 1 also follows from Theorem 1 and Theorem 3 together (see Section 2.7) and is implied by the results of Aziz and Brandl (2021) as well. We included an intuitive direct proof here to verify this important observation directly.

2.5 SAFE Mechanisms

We introduce a class of mechanisms called SAFE that is based on the idea of 'safe blocks'.

2.5.1 The Acceptance Graph and Safe Blocks

Given a profile $(A, \succ)$, we define the following concepts for sets of institutions for any k such that $1 \leq k \leq m$. In the definitions below, we exclude institutions that are not acceptable to any agent at $(A, \succ)$. We refer to such institutions as *null-institutions*, since they have an empty acceptance list.

Equal-acceptable set of institutions:

A set of k institutions that have exactly k agents on their acceptance lists collectively.

Under-acceptable set of institutions:

A set of k institutions that have less than k agents on their acceptance lists collectively.

Over-acceptable set of institutions:

A set of k institutions that have more than k agents on their acceptance lists collectively.

Safe k-block:

Given a profile $(A, \succ)$ and a set of institutions $\bar{D} \subseteq D$, a **safe k-block**, or simply a **safe block,** in $\bar{D}$ is an equal-acceptable set of k institutions $\tilde{D} \subseteq \bar{D}$, and there is no proper subset of $\tilde{D}$ which is also an equal-acceptable set.

There may not exist any equal-acceptable set of institutions in $\bar{D}$ at a specific profile, which would imply that there is no safe block in $\bar{D}$. On the other hand, there may be multiple safe blocks at a profile in $\bar{D}$, and safe blocks may even overlap. By definition, a safe block cannot have a proper subset which is also a safe block. We will show below that it is always feasible to match the k agents to the k institutions in a safe k-block in a one-to-one manner (see Lemma 2). However, it is easy to observe from the examples below that when safe blocks overlap it is not necessarily feasible to match some agent to each institution that is in a safe block, but all agents that are on the acceptance list of some institution in a safe block are matched to an institution in any maximum matching. Moreover, observe that safe blocks only depend on the underlying acceptance graph and are independent of the strict preferences of institutions, that is, they only depend on A and are independent of $\succ$.

Example 2. Safe blocks

Consider the following problems, given by their acceptance lists:

Problem 1:	**Problem 2:**	**Problem 3:**	**Problem 4:**
$d_1 : 1, 3$	$d_1 : 2, 1$	$d_1 : 2, 1, 3$	$d_1 : 1, 2$
$d_2 : 3, 2$	$d_2 : 1, 2$	$d_2 : 1, 2$	$d_2 : 2, 1, 3$
$d_3 : 1$	$d_3 : 1, 2, 3$	$d_3 : 1, 2, 3$	$d_3 : 3, 1, 2$
			$d_4 : 3, 1$

Problem 1: Institution $\{d_3\}$ is a safe 1-block. There are no other safe blocks. In particular, although $\{d_1, d_2, d_3\}$ and $\{d_1, d_3\}$ are both equal-acceptable, these are not safe blocks because $\{d_3\}$ is also equal-acceptable.

Problem 2: The set $\{d_1, d_2\}$ is a safe 2-block.

Problem 3: The three institutions $\{d_1, d_2, d_3\}$ together constitute a safe 3-block, since each proper subset is over-acceptable.

Problem 4: There are four safe blocks: any three institutions constitute a safe 3-block. Observe that all three agents can be assigned to an institution, but one of the institutions will be unmatched. $\diamond$

Example 3. No safe block

There are four agents $(n = 4)$ and three institutions $(m = 3)$ with the following acceptance lists:

$d_1 : 1, 2, 3$
$d_2 : 3, 2, 4$
$d_3 : 1, 3$

There is no set of institutions that is equal-acceptable in this problem, so there is no safe block. Each subset of the institutions is over-acceptable. $\diamond$

2.5.2 Definition of the SAFE Mechanisms

Let π be a fixed ordering (i.e., a permutation) of the set of institutions D. Given a profile $(A, \succ)$, a **π-sequential matching** at $(A, \succ)$ is the matching reached by iteratively assigning each institution in the order of π the highest-priority agent on its acceptance list who is still unassigned, if there is any. We will also say that, given a subset of the institutions $\bar{D} \subseteq D$, the **π-sequential $\bar{D}$-matching** at $(A, \succ)$ is the matching restricted to $\bar{D}$ which is reached by iteratively assigning each institution in $\bar{D}$ in the order of π the highest-priority agent on its acceptance list who is still unassigned, if there is any. A matching is a **sequential matching** at $(A, \succ)$ if there exists a permutation π of D such that it is the π-sequential matching at $(A, \succ)$. A **sequential mechanism** is a mechanism which assigns a sequential matching at $(A, \succ)$ to each profile $(A, \succ) \in \mathcal{A} \times \Pi$. Note that the permutation π may vary with the profile $(A, \succ)$, and thus sequential mechanisms are not simply serial dictatorships (Satterthwaite and Sonnenschein, 1981) that are constrained by the acceptance graph.

A **SAFE (Sequential Allocation for Fairness and Efficiency) mechanism** is a sequential mechanism for which a fixed baseline permutation $\bar{\pi}$ over D is used at each profile $(A, \succ)$ to determine the profile-dependent permutation $\pi(A, \succ)$, which leads to the π-sequential matching. Given a fixed a $\bar{\pi}$ and the acceptance lists for institutions at profile $(A, \succ)$, the first step is to determine if there is a safe block. If there is, we choose the first institution according to $\bar{\pi}$ that is in a safe block. If there is no safe block, we choose the first institution according to $\bar{\pi}$. Then we assign the highest-ranked agent on the selected institution's acceptance list to this institution. Then we update the acceptance lists by removing the just matched institution and the matched agent from all remaining institution lists and repeat these steps until no more assignments can be made. The mechanism is defined formally as Mechanism 1.

Each SAFE mechanism is specified by the fixed baseline ranking $\bar{\pi}$, and hence the class of SAFE mechanisms is given by the $m!$ permutations of D. We will denote the SAFE mechanism with baseline permutation $\bar{\pi}$ by $\varphi^{\text{SAFE}(\bar{\pi})}$.

The example below demonstrates how SAFE mechanisms work. The updated acceptance lists are displayed for each remaining institution in each step after the initial step.

Example 4. *Illustration of the SAFE mechanism.*

Let $\bar{\pi} = (d_1, d_2, d_3, d_4)$ and consider the following acceptance list profile.

Mechanism 1: SAFE

Input : $(N, D, A, \succ)$ and baseline permutation $\bar{\pi}$ over D
Output: A matching M
Initialize G as the acceptance graph for $(N, D, A, \succ)$.;
Initialize matching M to empty set.;
while *G is not the empty graph* **do**
  if *there is a safe block in G* **then**
    find the highest ranked (according to permutation $\bar{\pi}$) institution d in graph G that is in a safe block and match the most preferred agent i in G (according to $\succ_d$) to d: add $\{i, d\}$ to M. Remove i and d and all adjacent edges from G
  else
    take the highest ranked (according to permutation $\bar{\pi}$) institution d in graph G and match to the most preferred agent in G (according to $\succ_d$) to d: add $\{i, d\}$ to M. Remove i and d and all adjacent edges from G
return M

$$\begin{aligned} d_1 &: 1, 2, 3 \\ d_2 &: 1, 2, 3 \\ d_3 &: 1 \\ d_4 &: 2 \end{aligned}$$

There are two safe blocks in step 1, consisting of institution d_3 and d_4 respectively, and agent 1 is assigned to d_3 based on $\bar{\pi}$. After removing agent 1 and institution d_3 from the problem, the updated acceptance list profile yields a new safe block: $\{d_1, d_2\}$. Thus, agent 2 is assigned to institution d_1 in step 2, and thus agent 3 is assigned to institution d_2 in step 3. The steps are illustrated below.

Step 1:	**Step 2:**	**Step 3:**
$d_1 : 1, 2, 3$	$d_1 : \boxed{2}, 3$	
$d_2 : 1, 2, 3$	$d_2 : 2, 3$	$d_2 : \boxed{3}$
$d_3 : \boxed{1}$		
$d_4 : 2$	$d_4 : 2$	

The final matching is $\{(d_1, 2), (d_2, 3), (d_3, 1), (d_4, 0)$.

Note that in this problem the maximum matching is selected with the aid of safe blocks in each step, while the baseline ranking $\bar{\pi}$ plays a role in selecting the first agent in a safe block in each step. The result is the π-sequential matching with $\pi = (d_3, d_1, d_2, d_4)$. $\diamond$

2.5.3 SAFE Mechanisms are Maximum Mechanisms

We establish next that the SAFE mechanisms are maximum mechanisms. We prove four lemmas first. Lemmas 1-3 show that it is feasible to assign an agent to each institution within a given safe block, even after making an arbitrary first assignment within this safe block. This also means that each agent that finds at least one institution acceptable in a safe block is assigned by any maximum mechanism, but it does not imply that each institution that is in a safe block is assigned. These lemmas, including Lemma 4, are used in the proof of Theorem 1 which shows that SAFE mechanisms are maximum mechanisms, while the proof of Theorem 4, which demonstrates that SAFE mechanisms are strategyproof, relies on Lemmas 1 and 2. In the arguments of the proofs of the lemmas and theorems null-institutions are excluded by default.

We also note that we will use Hall's theorem (Hall, 1987) repeatedly in the proofs below. Hall's theorem states, in our terminology, that it is feasible to assign an agent to each of k institutions if and only if each subset of the k institutions is either equal-acceptable or over-acceptable. We state a series of lemmas. The proofs are in the appendix.

Lemma 1. *Each under-acceptable set of institutions contains a safe block.*

Lemma 2. *It is feasible to assign an agent to each institution within a safe block.*

Proof. By Lemma 1, each subset of a safe block is either equal-acceptable or over-acceptable. Then Hall's theorem implies that it is feasible to assign an agent to each institution in a safe block. □

It follows from Lemma 2 that in any maximum matching each agent is assigned who has any institution in its acceptance set that is in a safe block. However, note that it does not follow that each institution in a safe block is assigned in a maximum matching.

Lemma 3. *Let an acceptance list be given for each institution in $\bar{D}$, where $\bar{D} \subseteq D$. Assign an arbitrary institution from $\bar{D}$ to an arbitrary agent on this institution's acceptance list. If $\bar{D}$ is a safe block, it is feasible to assign each institution in $\bar{D} \setminus d$ an agent other than d on its acceptance list.*

Lemma 4. *Let an acceptance list be given for each institution in $\bar{D}$, where $\bar{D} \subseteq D$. Assign an arbitrary institution from $\bar{D}$ to an arbitrary agent on this institution's acceptance list. If $\bar{D}$ has no safe block, then it is feasible to have a maximum matching for $\bar{D}$ which includes this initial assignment.*

Theorem 1. *A SAFE mechanism is a maximum mechanism.*

Proof. Fix a permutation $\bar{\pi}$ of institutions in D, and fix a profile $(A, \succ) \in \mathcal{A} \times \Pi$. The SAFE mechanism $\varphi^{\bar{\pi}}$ leads to a permutation π at profile $(A, \succ)$. Without loss of generality, let $\pi = (d_1, \ldots, d_m)$. We will show that after each step of the procedure, that is, after assigning the first remaining agent, if there is any, to institution d_t on the acceptance list of d_t in each step t, for $t = 1, \ldots, m-1$, it is feasible to reach a maximum matching. Since this implies that selecting any remaining agent on the acceptance list of the last institution d_m in step m leads to a maximum matching (or, alternatively, if d_m is a null-institution in step m then the matching reached before step m is a maximum matching), this will ensure that at the end of the SAFE mechanism procedure applied to $(A, \succ)$ the assignment made iteratively in each step results overall in a maximum matching.

Fix $t \in \{1, \ldots, m-1\}$ and assume that after step $t-1$ in the procedure all previous assignments make it feasible to reach a maximum matching at the end. Note that this assumption holds vacuously for step 0, that is, prior to beginning the procedure, when no assignments have been made yet. Consider the following two cases based on the updated institution acceptance lists after step $t-1$: a) there is a safe block, b) there is no safe block.

Case a): *There is a safe block based on the updated institution acceptance lists after step* $t-1$.

In this case a yet unassigned institution d is selected from a safe block (the first institution in a safe block according to $\bar{\pi}$), and thus it follows from Lemma 3 that all agents on the acceptance lists of institutions in a safe block can be assigned to an institution in this safe block, regardless of the first institution that is chosen from this safe block. Since, for all k, a safe k-block is an equal-acceptable set of institutions by definition, only the k agents on the acceptance lists of the k institutions in the safe block can be assigned to these institutions by the individual rationality of the SAFE mechanism, and thus starting the assignment with any arbitrary institution in a safe block allows for reaching a maximum matching after making this assignment.

Case b): *There is no safe block based on the updated institution acceptance lists after step* $t-1$.

In this case Lemma 4 implies that starting the assignment with any arbitrary yet unassigned institution d allows for reaching a maximum matching after making this assignment.

Given the above arguments for both cases, it follows by induction that the SAFE mechanism selects a maximum matching at profile $(A, \succ)$. Since $(A, \succ)$ is an arbitrary profile in $\mathcal{A} \times \Pi$, this proves that the SAFE mechanism $\varphi^{\bar{\pi}}$ is a maximum mechanism for any fixed permutation $\bar{\pi}$ of the institutions. $\square$

2.6 Rank-Maximal Mechanisms

We now introduce another family of mechanisms, which we call Rank-Maximal, and show that it is equivalent to the family of SAFE Mechanisms.

2.6.1 Rank-Maximal Mechanisms and Their Basic Properties

The Rank-Maximal family of mechanisms is parametrized by a baseline permutation of institutions, just like the family of SAFE mechanisms. Based on a baseline permutation $\bar{\pi}$ over institutions, a set $S \subset D$ is *lexicographically better* than another set $T \subset D$ if for the earliest (according to $\bar{\pi}$) institution where S and T differ, it is the case that the corresponding institution in S comes earlier according to $\bar{\pi}$. The Rank-Maximal algorithm achieves a maximum matching by assigning the set of institutions that is lexicographically better than any other set of institutions with the same (maximum) cardinality. We call this set of institutions *lexi-optimal*. The mechanism follows the ordering of the baseline permutation $\bar{\pi}$ and iteratively assigns each institution to the most preferred agent such that the resulting matching can be extended to a matching that assigns all institutions in the lexi-optimal set of institutions. In the definition of the rule we first determine the set of institutions that will be matched, based on the acceptance graph of the problem. Then, given the corresponding restricted acceptance graph, we determine the assignments step-by-step such that each institution is assigned the most preferred agent that allows for matching all the institutions in the restricted graph. We will denote the Rank-Maximal mechanism with baseline permutation $\bar{\pi}$ by $\varphi^{\text{RankMax}(\bar{\pi})}$. The mechanism is formally specified as Mechanism 2.

For a permutation $\bar{\pi}$ over institutions, consider the *ranked acceptance graph* $(N \cup D, E, \ell)$ where the edge set represents the individual rationality relations and each edge is assigned a rank as follows. Suppose the edge set E is partitioned into r disjoint sets, i.e., $E = E_1 \cup E_2 \cup \cdots \cup E_{|D|}$ where E_i is the set of edges adjacent to institution $d = \bar{\pi}(i)$ which are given a rank i. The *signature* $\rho(M) = \langle x_1, x_2, ..., x_r \rangle$ of a matching M in G is a tuple of integers where each element x_i represents the number of edges of rank i in M.

For a ranked bipartite graph, we compare the signatures of matchings in a lexicographical manner. A matching M' with $\rho(M') = \langle x_1, \cdots, x_r \rangle$ is ***strictly better*** than another matching M'' with $\rho(M'') = \langle y_1, \cdots, y_r \rangle$, if there exists an index $1 \leq k \leq r$ s.t. for $1 \leq i < k$, $x_i = y_i$ and $x_k > y_k$. A matching M' is ***weakly better*** than another matching M'' if M'' is not strictly better than M'. Let $M' \succ_{lex} M''$ denote that M' is strictly better than M'' and let $M' \succsim_{lex} M''$ denote that M' is weakly better than M''.

A matching M in a ranked bipartite graph G is ***rank-maximal*** if M is weakly better than any other matching M' in G. A rank maximal matching can be computed in polynomial

Mechanism 2: Rank-Maximal

Input : $(N, D, A, \succ)$ and permutation π over D
Output: Matching M
Set W to empty set and let G be the acceptance graph for $(N, D, A, \succ)$.
while $\exists d \in D \setminus W$ *such that* $W \cup \{d\}$ *can be matched in some matching in* G **do**
    Find the highest priority (according to π) such d;
    Add to W the highest priority (according to π) institution d in W such that all elements of $W \cup \{d\}$ can be matched in some matching in G
Initialize G as the acceptance graph for $(N, W, A, \succ)$ restricted to $W \subseteq N$;
Initialize matching M to empty;
while $|M| < |W|$ **do**
    For the highest ranked institution $c \in D$ (according to π), find the highest priority agent (according to $\succ_c$) such that if we remove c and i from G, then the modified graph G admits a matching that matches every institution;
    Remove i and c from G;
    Permanently match i to c: add $\{i, c\}$ to M;
return M.

time (Manlove, 2013).

Proposition 1. *A set of institutions matched in a rank maximal matching of a ranked acceptance graph is the lexi-optimal set of institutions.*

Proof. By definition, a rank maximal matching's matched institutions are lexi-optimal. □

Although a rank maximal matching of a bipartite graph need not be a maximum size matching in general for our particular problem that has more structure, a rank maximal matching gives a maximum size matching.

Proposition 2. *A rank maximal matching of a ranked acceptance graph gives a maximum size matching. Equivalently, if a matching matches the lexi-optimal set of institutions, then the matching is maximum size.*

Proof. Suppose the rank maximal matching of graph $(N \cup D, E, \ell)$ is not of maximum size. By Berge's lemma, it admits an augmenting path p. Now suppose we switch the matching M to M' by removing the matched edges in the augmenting path p and selecting the complement of the edges in p. Note that those vertices that were matched in p under M continue to be matched under M' and there is at least one additional edge. Therefore M' is better than M in terms of rank maximality which is a contradiction. After this step, institutions may exchange agents but the size of the matching does not change. □

Next, we illustrate how a Rank-Maximal mechanism works.

Example 5 (Rank-Maximal Mechanism)**.**

Let $\bar{\pi} = (d_1, d_2, d_3, d_4)$ and consider the following acceptance list profile.

$$\begin{aligned} d_1 &: 1, 2, 3 \\ d_2 &: 1, 2, 3 \\ d_3 &: 1 \\ d_4 &: 2 \end{aligned}$$

The lexi-optimal set of institutions W that will be matched, as computed by Rank-Maximal in the initial step of the algorithm, is $\{d_1, d_2, d_3\}$.

$$W = \{d_1, d_2, d_3\}.$$

For d_1, it cannot be matched to 1 because if it does, then not all members of $\{d_1, d_2, d_3\}$ can be matched. Hence d_1 is matched to 2.

$$\begin{aligned} d_1 &: 1, \boxed{2}, 3 \\ d_2 &: 1, 2, 3 \\ d_3 &: 1 \\ d_4 &: 2 \end{aligned}$$

Institution d_2, cannot be matched to 1 while d_1 is matched to 2 as then d_3 cannot be matched. Also, d_2, cannot be matched to 2 as d_1 is already matched to 2. Hence d_2 is matched to 3.

$$\begin{aligned} d_1 &: 1, \boxed{2}, 3 \\ d_2 &: 1, 2, \boxed{3} \\ d_3 &: 1 \\ d_4 &: 2 \end{aligned}$$

Finally, d_3 is matched to 1.

Final matching:

$$\begin{aligned} d_1 &: 1, \boxed{2}, 3 \\ d_2 &: 1, 2, \boxed{3} \\ d_3 &: \boxed{1} \\ d_4 &: 2 \end{aligned}$$

Note that in this example both the SAFE and Rank-Maximal mechanisms give the same outcome. This is not a coincidence. We prove next that the two families of mechanisms that we introduced, SAFE and Rank-Maximal, are equivalent.

2.6.2 Equivalence of SAFE and Rank-Maximal Mechanisms

Theorem 2. *For all permutations $\bar{\pi}$ over D, mechanisms $\varphi^{SAFE(\bar{\pi})}$ and $\varphi^{RankMax(\bar{\pi})}$ are outcome equivalent.*

Proof. Fix a permutation $\bar{\pi}$ of D and a profile $(A, \succ)$. Let $\mu^{SAFE} := \varphi^{SAFE(\bar{\pi})}(A, \succ)$ and $\mu^{RM} := \varphi^{RankMax(\bar{\pi})}(A, \succ)$. We will show that $\mu^{SAFE} = \mu^{RM}$.

Suppose, for a contradiction, that both μ^{SAFE} and μ^{RM} make the same assignment to each institution $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$ (the first $t-1$ institutions according to $\bar{\pi}$), and $\bar{\pi}(t)$ is the first institution according to $\bar{\pi}$ for which the two assignments differ, where $t \geq 1$. For ease of notation, let $d := \bar{\pi}(t)$. We distinguish between the following two cases.

Case 1: *One of the matchings μ^{SAFE} and μ^{RM} matches d and the other one does not.*

Since Proposition 1 implies that the set of institutions that are matched by Rank-Maximal is lexi-optimal with respect to $\bar{\pi}$ at each profile $(A, \succ)$, if $\mu^{SAFE}_d \neq \emptyset$ then $\mu^{RM}_d \neq \emptyset$. Therefore, $\mu^{SAFE}_d = \emptyset$ and $\mu^{RM}_d \neq \emptyset$. Let $j \in N$ such that $\mu^{RM}_d = j$. Note that $d \in A_j$. Given our assumption that both μ^{SAFE} and μ^{RM} make the same assignments to each institution $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$, $\mu^{SAFE}_j \notin \{\bar{\pi}(1), \ldots, \bar{\pi}(t-1)\}$, and since $\mu^{SAFE}_d = \emptyset$, $\mu^{SAFE}_j \neq \bar{\pi}(t) = d$. Since $d \in A_j$, the SAFE algorithm ensures that $\mu^{SAFE}_j = \bar{\pi}(\tilde{t})$ with $\tilde{t} > t$. Then, without loss of generality, we can let $i \in N$ be the agent on d's acceptance list assigned in some step k of the SAFE algorithm at $(A, \succ)$ such that all remaining agents on d's acceptance list after step k are assigned by μ^{SAFE} to institutions in $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$ and $\mu^{SAFE}_i = \bar{\pi}(t')$ with $t' > t$.

Note that the institutions that are assigned the remaining agents on d's acceptance list to some institution in $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$ (if any) after step k, together with $\bar{\pi}(t) = d$, would have to constitute an over-acceptable set in step $k-1$ of the SAFE algorithm at $(A, \succ)$. Let's call this set D^t. D^t is an over-acceptable set in step $k-1$ since an equal-acceptable set would contain a safe block, by definition, and an under-acceptable set would contain a safe block by Lemma 1, and this would contradict the fact that $\bar{\pi}(t')$ with $t' > t$ is selected to be matched in step k. However, in subsequent steps after step k in the SAFE algorithm, as the institutions that are assigned the remaining agents on d's acceptance list to some institution in $D^t \setminus \{d\}$ become assigned, there will be a step $\hat{k} > k$ where the remaining set of unmatched institutions in D^t becomes under-acceptable, given that $\mu^{SAFE}_d = \emptyset$. This implies, however, that the remaining set of unmatched institutions in D^t is equal-acceptable and contains a safe block in step $\hat{k} - 1$, implying that the assignment in step $\hat{k}$ could not have been made. This is a contradiction, and therefore it is not the case that one of μ^{SAFE} and μ^{RM} matches d and the other one does not. Given that $\mu^{SAFE}_d \neq \mu^{RM}_d$, this only leaves the possibility of Case 2 below.

Case 2: *Both of the matchings μ^{SAFE} and μ^{RM} match d and $\mu_d^{SAFE} \neq \mu_d^{RM}$.*

There are two sub-cases to consider. Let $\mu_d^{SAFE} = i$ and $\mu_d^{RM} = j$, where $i, j \in N$. Since $i \neq j$, we have either $i \succ_d j$ or $j \succ_d i$. Since SAFE mechanisms are maximum mechanisms, as shown by Theorem 1, it cannot be the case that $i \succ_d j$, given that Rank-Maximal mechanisms assign the highest priority agent according to $\succ_d$ to d that still allows for a maximum matching, and here the assignments of institutions $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$ are the same in both μ^{SAFE} and μ^{RM}. Hence, $j \succ_d i$. However, given $j \succ_d i$ and that j is not assigned to any of the institutions $\bar{\pi}(1), \ldots, \bar{\pi}(t-1)$, the reason j is not assigned by μ^{SAFE} to d is that there is a safe block in the set of institutions in $\bar{\pi}(t+1), \ldots, \bar{\pi}(m)$, in the step of the SAFE algorithm when d is next in $\bar{\pi}$ to be matched.

This means that not every institution in such a safe block can be matched if j is assigned to d. However, μ^{SAFE} matches all agents in the first such safe block, where we find this first safe block as a lexi-optimal set. Therefore, $\mu_d^{RM} = j$ implies that the Rank-Maximal mechanism does not match the set of lexi-optimal daycares at $(A, \succ)$, which is a contradiction.

Note that the above arguments also hold for $t = 1$, when $\{\bar{\pi}(1), \ldots, \bar{\pi}(t-1)\} = \emptyset$, and thus, by induction on t, μ^{SAFE} and μ^{RM} make the same assignment for each $t = 1, \ldots, m$, which means that $\mu^{SAFE} = \mu^{RM}$. Since $\bar{\pi}$ and $(A, \succ)$ were chosen arbitrarily, this implies the result. □

2.7 Properties of SAFE/Rank-Maximal Mechanisms

In this section, we establish key axiomatic properties of SAFE/Rank-Maximal Mechanisms, in addition to being maximum mechanisms which has already been shown by Theorems 1 and 2. Given Theorem 2, we can prove the axiomatic properties by establishing them for either SAFE mechanisms or Rank-Maximal mechanisms.

Theorem 3. *A SAFE mechanism is fair.*

Proof. Fix a permutation $\bar{\pi}$ of institutions in D, and fix a profile $(A, \succ) \in \mathcal{A} \times \Pi$. The SAFE mechanism $\varphi^{\bar{\pi}}$ leads to a permutation π at profile $(A, \succ)$ such that $\varphi^{\bar{\pi}}(A, \succ)$ is the π-sequential matching at $(A, \succ)$. Let $\mu \equiv \varphi^{\bar{\pi}}(A, \succ)$. Suppose, by contradiction, that there is an agent-institution pair (c, d) such that i) $\mu_c = 0$, ii) $d \in A_c$, and iii) either $\mu_d = 0$ or $\mu_d = c'$ such that $c \succ_d c'$. Given $\mu_c = 0$ and $d \in A_c$, when we get to the step in the algorithm where institution d is next in permutation π, agent c is on d's acceptance list since c is unassigned in μ and d is acceptable to c, and thus $\mu_d = c'$ such that $c' \succ_d c$. This is a contradiction. Therefore, μ is a fair matching at $(A, \succ)$. The same argument holds for $\varphi^{\bar{\pi}}(A, \succ)$ at each profile $(A, \succ) \in \mathcal{A} \times \Pi$, and thus $\varphi^{\bar{\pi}}$ is fair for an arbitrary permutation $\bar{\pi}$ of the institutions. □

Theorem 4. *A SAFE mechanism is strategyproof for agents.*

Proof. Let $\psi^{\bar{\pi}}$ be a SAFE mechanism where $\bar{\pi}$ is the baseline permutation of the institutions. Suppose by contradiction that $\psi^{\bar{\pi}}$ is not strategyproof. Then there exist an agent $c \in C$, a profile $(A, \succ)$ and an alternative set of acceptable institutions A'_c for agent c such that $\psi^{\bar{\pi}}_c(A, \succ) = 0$ and $\psi^{\bar{\pi}}_c((A'_c, A_{-c}), \succ) \in A_c$. Let $\mu \equiv \psi^{\bar{\pi}}(A, \succ)$ and $\mu' \equiv \psi^{\bar{\pi}}_c((A'_c, A_{-c}), \succ)$. Note that $\mu_c = 0$.

Case 1: *There exists a permutation π of the institutions such that μ is the π-sequential matching at $(A, \succ)$ and μ' is the π-sequential matching at $((A'_c, A_{-c}), \succ)$.*

In this case subtracting any acceptable institutions by reporting A'_c instead of A_c would not have any impact, since c remains unassigned at $(A, \succ)$. Moreover, adding some acceptable institutions by reporting A'_c instead of A_c can only result in an assignment to an unacceptable institution at $((A'_c, A_{-c}), \succ)$ (i.e., $\mu_c \notin A_c$), which is a contradiction.

Case 2: *There does not exist a permutation π of the institutions such that μ is the π-sequential matching at $(A, \succ)$ and μ' is the π-sequential matching at $((A'_c, A_{-c}), \succ)$.*

In this case the difference between A_c and A'_c changes the permutation of institutions used in the sequential matching when we compare μ to μ'. This means that c either destroys at least one existing safe block at $(A, \succ)$, so that it is not a safe block at $((A'_c, A_{-c}), \succ)$, or c creates at least one new safe block at $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$ by misrepresenting her true preferences over the institutions (or both).

Subcase 2.1 *Only subtract institutions: $A'_c \subset A_c$*

Assume first that c only subtracts institutions from her acceptance set, that is, $A'_c \subset A_c$. Then if an existing safe block at $(A, \succ)$ is destroyed and thus it is no longer a safe block at $((A'_c, A_{-c}), \succ)$, this implies that agent c is on the acceptance list of at least one institution in this safe block at $(A, \succ)$. Then, by Lemma 2, c is assigned to an institution at $(A, \succ)$, since $\psi^{\bar{\pi}}$ is a maximum mechanism by Theorem 1. Therefore, $\mu_c \neq 0$, which is a contradiction.

This implies that c creates at least one new safe block at $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$. Let one such new safe block at $((A'_c, A_{-c}), \succ)$ be a safe k-block. Then it is an over-acceptable set of institutions at $(A, \succ)$ such that $k + 1$ agents have these institutions in their acceptance sets: the k agents in $C \setminus \{c\}$ who have them in their acceptance sets, as given by A_{-c}, in addition to agent c. If any of these k agents, say $\tilde{c}$, is assigned to an institution in a step t prior to reaching the first institution among these k institutions in the SAFE procedure at $(A, \succ)$, then these k institutions form an equal-acceptable set of institutions in step $t+1$. Suppose by contradiction that there is an under-acceptable subset of $k' < k$ institutions of these k institutions in step $t+1$. Then these k' institutions form an equal-acceptable set at step t, when $\tilde{c}$ was still on the acceptance lists of institutions, since by Lemma 1 it could not have been under-acceptable, given that the k institutions would constitute a safe block if we deleted c from the acceptance lists. This means that the set of these k' institutions is either a safe block at $(A, \succ)$ or contains a safe block, which contradicts the fact that after deleting c from the acceptance lists of the k institutions the set of $k > k'$ institutions becomes a safe k-block. Therefore, each subset of the k institutions in step $t+1$ is either equal-acceptable or over-acceptable, and Hall's theorem implies that it is feasible to assign an agent to each of the k institutions at $(A, \succ)$. This would mean that agent c was assigned to an institution at $(A, \succ)$, which is a contradiction. Therefore, we conclude that none of these k agents is assigned to an institution in a step t prior to reaching the first institution among the k institutions in the SAFE procedure at $(A, \succ)$. In this case, however, it makes no difference whether these institutions are in a safe block and whether agents are assigned to them earlier than $\bar{\pi}$ calls for (due to c reporting untruthfully that some of these institutions are not in her acceptance set), and the final matching remains the same at $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$. This is a contradiction.

In sum, since $A'_c \subset A_c$ and $\varphi_c((A'_c, A_{-c}), \succ) \notin A'_c$, since $\varphi_c((A'_c, A_{-c}), \succ)$ is a feasible matching, it follows that $\varphi_c((A'_c, A_{-c}), \succ) = 0$.

Subcase 2.2 *Only add institutions:* $A_c \subset A'_c$

Now assume that c only adds institutions to her acceptance set, that is, $A_c \subset A'_c$. Consider the case first where an existing safe block at $(A, \succ)$ is destroyed and thus it is no longer a safe block at $((A'_c, A_{-c}), \succ)$, given that agent c adds institutions to her acceptance set. Let this safe block at $(A, \succ)$ be a safe k-block. As shown by Lemma 2, c is not one of the k agents on the acceptance lists of the institutions in this safe block.

If any of the k agents who have these k institutions in their acceptance sets, as given by A_{-c}, say $\tilde{c}$, is assigned to an institution in a step t prior to reaching the first institution in the SAFE mechanism procedure at $((A'_c, A_{-c}), \succ)$ then the k institutions in the safe block constitute an equal-acceptable set of institutions in step $t+1$. Suppose by contradiction that there is an under-acceptable subset of these k institutions in this subsequent step. Then this strict subset of the k institutions was equal-acceptable in step t, when $\tilde{c}$ was still on the acceptance lists of institutions, since by Lemma 1 it could not have been under-acceptable, given that the k institutions constitute a safe block when we delete c

from the acceptance lists. This means that this strict subset of the k institutions is either a safe block at $((A'_c, A_{-c}), \succ)$ or contains a safe block, which contradicts the fact that after deleting c from the acceptance lists of the k institutions they become a safe k-block. Therefore, each subset of of these k institutions in step $t+1$ is either equal-acceptable or over-acceptable, and Hall's theorem implies that it is feasible to assign an agent to each of the k institutions at $((A'_c, A_{-c}), \succ)$. This would mean that agent c is assigned to an institution at $((A'_c, A_{-c}), \succ)$. If c is assigned to an institution in the safe k-block then $\mu'_c \notin A_c$, and if c is assigned prior to reaching the safe block then $\mu'_c = \mu_c$ and thus $\mu_c \in A_c$, implying a contradiction in both cases. Therefore, we conclude that none of these k agents is assigned to an institution in a step t prior to reaching the first institution in the SAFE mechanism procedure at $((A'_c, A_{-c}), \succ)$. In this case, however, it makes no difference whether these institutions are in a safe block and whether agents are assigned to them earlier than $\bar{\pi}$ calls for (due to c reporting untruthfully that some of these institutions are not in her acceptance set), and the final matching remains the same at $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$. This is a contradiction.

Finally, if c creates at least one new safe block at $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$ by adding institutions to her acceptance set, then Lemma 2 implies that c would be assigned to an institution in this safe block at $((A'_c, A_{-c}), \succ)$, unless c is assigned at a prior step at $((A'_c, A_{-c}), \succ)$. In the latter case, if this prior step is the same at profile $(A \succ)$, then $\mu_c \in A_c$, which is a contradiction. Thus, c is assigned to one of the k institutions in a new safe block $((A'_c, A_{-c}), \succ)$ compared to $(A, \succ)$. Note that since c was not assigned to μ'_c at $(A, \succ)$, it is not possible that if μ'_c is earlier in the institution permutation than at $(A, \succ)$, while otherwise the matching procedure remains the same, that c is assigned to μ'_c, except if c was not on the acceptance list of μ'_c at $(A, \succ)$, that is, c can only be assigned to μ'_c in this case if $\mu'_c \in A'_c \setminus A_c$. This means that according to the true preferences of agent c, μ'_c is unacceptable, that is $\mu'_c \notin A_c$. This means that the manipulation attempt was not successful, which is a contradiction.

Completion of Case 2:

We have shown in subcase 2.1 that agent c cannot manipulate the outcome by only subtracting institutions and remains unmatched. Let $A''_c = A_c \cap A'_c$. Then $A''_c \subseteq A_c$ and thus $\varphi_c((A''_c, A_{-c}), \succ) = 0$, by subcase 2.1. However, $A''_c \subseteq A'_c$ and since we have shown in subcase 2.2 that agent c cannot manipulate the outcome by only adding institutions, $\varphi_c((A'_c, A_{-c}), \succ) \notin A''_c$. Since it is a feasible matching, $\varphi_c((A'_c, A_{-c}), \succ) \notin (A_c \setminus A'_c)$, and therefore $\varphi_c((A'_c, A_{-c}) \succ) \notin A_c$, which means that the manipulation attempt was not successful. Therefore, subcases 2.1 and 2.2 together cover all possible cases of manipulation by agent c for Case 2.

In sum, since an arbitrarily chosen agent c cannot manipulate in either Case 1 or Case 2, a SAFE mechanism is N-strategyproof. □

Theorem 5. *The SAFE/Rank-Maximal Mechanisms are strategyproof for institutions.*

Proof. The set of lexi-optimal set of institutions that can be matched cannot be changed by any institution, as it is based on the acceptance graph.

Suppose for a contradiction that SAFE/Rank-Maximal Mechanisms are not strategyproof for institutions. Then there is an institution d that can change its preference over agents to be assigned a more preferred agent. Before d's turn comes, note that d cannot affect the matches of institutions before it in the order π as the decision about whether a previous institution d' can match to a particular agent depends on the acceptance graph. When d's turn comes in Algorithm 2, it is assigned the most preferred agent that it can get while ensuring that the remaining institutions in W are also matched. Since d cannot affect the acceptance graph, Algorithm 2 is strategyproof for institutions. □

We show that the outcome of the SAFE/Rank-Maximal Mechanisms can be computed in polynomial time.

Theorem 6. *The outcome of the SAFE/Rank-Maximal Mechanisms can be computed in polynomial time.*

Proof. Firstly, Algorithm 2 builds up a lexi-optimal set of institutions W that can all be matched by repeatedly calling an algorithm that computed a maximum size matching of a given a graph. A maximum size matching of a bipartite graph can be computed in polynomial time (more precisely, cubic time in the number of vertices of the graph) by any of several well-established methods such as Kuhn's algorithm (Kuhn, 1955) or the Hopcroft-Karp-Karzanov algorithm (Hopcroft and Karp, 1973; Karzanov, 1973). While ensuring that a matching that matches each element of W exists, it finds the highest priority agent i of the highest ranked institution c such that if i and c are matched, every element in W can be matched. Hence, if i and c are removed from the problem, all the remaining elements in W can still be matched. The process iterates to match each element in W and returns a matching that match each element of W. □

Theorem 7. *The SAFE/Rank-Maximal Mechanisms are Pareto-efficient for the institutions.*

Proof. This follows immediately from the fact that SAFE mechanism are sequential mechanisms. □

Next, we consider non-bossiness that has been studied in several allocation and matching contexts (Kojima, 2010; Pápai, 2001; Svensson, 1999). The (agent-proposing) Deferred Acceptance mechanism when applied to our setting by using arbitrary tie-breaking in agents' preferences does not satisfy agent non-bossiness. The REV rule of Aziz and Brandl (2024) also violates agent non-bossiness (Aziz and Brandl, 2021).[2]

In contrast, we show that our rules satisfy non-bossiness.

Theorem 8. *The SAFE/Rank-Maximal Mechanisms are agent non-bossy.*

Proof. Let $c \in C$, $(A, \succ) \in \mathcal{A} \times \Pi$, and $A'_c \subset D$ such that $\varphi_c(A, \succ) = \varphi_c((A'_c, A_{-c}), \succ)$ where φ is the Rank-Maximal Mechanism with some fixed permutation of institutions $\bar{\pi}$. We will show that then $\varphi(A, \succ) = \varphi((A'_c, A_{-c}), \succ)$. For notational ease, let $A' = (A'_c, A_{-c})$. Note that, although agent c may report some institutions acceptable and others unacceptable in A'_c compared to A_c, given that the two profiles $(A, \succ)$ and $(A', \succ)$ are symmetric in this setup, it is enough to consider $A'_c = A_c \cup \{d'\}$, where $d' \notin A_c$, since a repeated application of the result that $\varphi(A, \succ) = \varphi(A', \succ)$ holds for this specific case implies the general result. Let W and W' denote the lexi-optimal set of institutions at $(A, \succ)$ and $(A', \succ)$ respectively.

Step 1:
We first prove that $W' = W$. Suppose to the contrary that $W' \neq W$. Let $\hat{d}$ be the first institution in $\bar{\pi}$ such that either $\hat{d} \in W \setminus W'$ or $\hat{d} \in W' \setminus W$. That is, for all institutions d preceding $\hat{d}$ in $\bar{\pi}$ either $d \in W \cap W'$ or $d \notin W \cup W'$, and thus $\hat{d}$ is the first institution in the permutation $\bar{\pi}$ whose inclusion status is different with respect to the lexi-optimal set at the two profiles $(A, \succ)$ and $(A', \succ)$.

> *Case 1:* $\hat{d} \in W \setminus W'$
> Since for all institutions d preceding $\hat{d}$ in $\bar{\pi}$ either $d \in W \cap W'$ or $d \notin W \cup W'$ and, for all $\tilde{c} \in C$, $A'_{\tilde{c}} \supseteq A_{\tilde{c}}$, $\hat{d} \notin W'$ implies that $\hat{d} \notin W$, a contradiction.
>
> *Case 2:* $\hat{d} \in W' \setminus W$
> Since for all institutions d preceding $\hat{d}$ in $\bar{\pi}$ either $d \in W \cap W'$ or $d \notin W \cup W'$, $A'_c = A_c \cup \{d'\}$ and, for all $\tilde{c} \in C \setminus \{c\}$, $A'_{\tilde{c}} = A_{\tilde{c}}$, $\hat{d} \in W' \setminus W$ implies that $\hat{d} = d'$ and $\varphi_c(A', \succ) = d'$. However, this means that $d' \notin W$ and d' is unmatched at $(A, \succ)$. This is a contradiction to the assumption that $\varphi_c(A, \succ) = \varphi_c(A', \succ)$.

Therefore, $W' = W$.

[2] In REV, it is even possible that an unmatched agent remains unmatched but manages to change its preferences and change the set of matched agents (Aziz and Brandl, 2021).

Step 2:
Now we are ready to show that $\varphi(A, \succ) = \varphi(A', \succ)$. Since $W' = W$, at both $(A, \succ)$ and $(A', \succ)$ the Rank-Maximal algorithm assigns the same highest-priority agent to each institution in W' in the order of $\bar{\pi}$ subject to feasibility, with the only possible exception of the assignment of institution d'. Therefore, if the two matchings differ at $(A, \succ)$ and $(A', \succ)$, it must be that $d' \in W'$ and $\varphi_c(A', \succ) = d'$. Since $d' \notin A_c$ and hence $\varphi_c(A, \succ) \neq d'$, this would contradict the assumption that $\varphi_c(A, \succ) = \varphi_c(A', \succ)$. Consequently, the two matchings do not differ at $(A, \succ)$ and $(A', \succ)$ and $\varphi(A, \succ) = \varphi(A', \succ)$. Therefore, SAFE/Rank-Maximal Mechanisms are agent non-bossy. □

Theorem 9. *The SAFE/Rank-Maximal Mechanisms are institution non-bossy.*

Proof. Suppose that d is a bossy institution that reports differently, receives the same assignment at these two adjacent profiles, but the overall matching changes.

First note that the lexi-optimal set of institutions is independent of the institutions' preferences, so the same set of institutions will be matched at the two adjacent profiles (for this it is crucial that institutions find all agents acceptable, as we assume, since the lexi-optimal set only depends on the underlying acceptance graph).

Second, the selected institutions get their highest-ranked feasible agent subject to matching each institution in the lexi-optimal set. If this leaves institution d (the one that reports different preferences at the two profiles) with the same assignment, then the entire matching is unchanged. In more detail: institutions prior to d in the baseline permutation get the same assignment since their preferences are unchanged and the different preferences reported by d allow for the same assignments as before for these agents. If d is assigned the same as before reporting different preferences, then clearly all institutions after d can also be assigned the same as before, and the entire matching is unchanged. This contradicts the assumption that d is bossy. □

2.8 Conclusion

In a model with dichotomous preferences of agents and strict rankings of institutions we identify a set of mechanisms, the SAFE/Rank-Maximal mechanisms, which always maximize the matching size and do not violate the preferences or priorities over the agents. We also show a number of other properties that are satisfied, including strong incentive properties such as strategyproofness for both agents and institutions in the unit capacity settings. Most properties carry over immediately to the multiple-capacity setting, with the exceptions of D-Pareto-efficiency, D-strategyproofness and D-non-bossiness. We conjecture that D-strategyproofness and D-non-bossiness hold under the standard assumption of responsive preferences for institutions. While D-Pareto-efficiency does not hold even under responsiveness, we leave it to future work to find the exact additional properties of mechanisms that allow for D-Pareto-efficiency in the many-to-one setting.

Appendix

Proof of Lemma 1

Proof. Let an acceptance list be given for each institution in D. Let a set $\tilde{D} \subseteq D$ of institutions be under-acceptable. Suppose, by contradiction, that $\tilde{D}$ does not contain any safe block. Then there is no institution in $\tilde{D}$ with exactly one agent on its acceptance list. Thus, each institution in $\tilde{D}$ has at least two agents on its acceptance list. If there are two institutions with two agents only on their acceptance lists jointly, then these two institutions constitute a safe 2-block. Thus, each pair of institutions in $\tilde{D}$ is over-acceptable. Assume that $|\tilde{D}| > 2$. Let $k \geq 2$ be such that each set of less than or equal to k institutions is over-acceptable within $\tilde{D}$. Since any set of k institutions is over-acceptable, it follows that any set of $k+1$ institutions is either equal-acceptable or over-acceptable. If there exists a set of $k+1$ institutions which is equal-acceptable then it is a safe block, since all subsets of this set are over-acceptable. As $\tilde{D}$ contains no safe blocks, this is a contradiction. Thus, each set of $k+1$ institutions is over-acceptable. By induction, $\tilde{D}$ is over-acceptable, which is a contradiction, since it is assumed to be under-acceptable. Therefore, each under-acceptable set of institutions contains a safe block. □

Proof of Lemma 3

Proof. Let an acceptance list be given for each institution in $\bar{D}$, where $\bar{D} \subseteq D$. Let $D' \subseteq \bar{D}$ be a safe k-block in $\bar{D}$. Take any $k-1$ institutions from D', say $D' \setminus \{d\}$, where $d \in D'$. The acceptance lists of institutions in $D' \setminus \{d\}$ contain at most k agents collectively. Consider the following three cases. Collectively there are a) less than $k-1$ agents, b) $k-1$ agents, and c) k agents on the acceptance lists of the $k-1$ institutions in $D' \setminus \{d\}$. We will show that cases a) and b) lead to contradictions, and prove the statement in the lemma for case c).

In case a) $D' \setminus \{d\}$ is under-acceptable, and thus Lemma 1 implies that it contains a safe block. Since a safe block is equal-acceptable, any safe block in $D' \setminus \{d\}$ has to be a strict subset of $D' \setminus \{d\}$, and any safe block in $D' \setminus \{d\}$ is a safe block in $\bar{D}$. This contradicts the fact that D' is a safe block in $\bar{D}$.

In case b) $D' \setminus \{d\}$ is an equal-acceptable set and hence it is safe block in $\bar{D}$. This contradicts the fact that D' is a safe block in $\bar{D}$.

In case c), after assigning an agent to d from d's acceptance list and removing this agent from the acceptance lists in $D' \setminus \{d\}$, there are $k-1$ agents remaining collectively on the $k-1$ acceptance lists of institutions in $D' \setminus \{d\}$. Suppose by contradiction that it is not feasible to assign these remaining $k-1$ agents to the $k-1$ institutions in $D' \setminus \{d\}$. Then there exists a subset of $D' \setminus \{d\}$ which is an under-acceptable set, since otherwise the $k-1$ agents would be feasible to assign by Hall's theorem. Then Lemma 1 implies that there is a safe block in $D' \setminus \{d\}$, which is a contradiction since D' is a safe block. Thus, it is feasible to assign the remaining $k-1$ agents to the $k-1$ institutions in $D' \setminus \{d\}$.

Therefore, since d is an arbitrary institution in D', it is feasible to assign each agent on the acceptance list of at least one institution in D' to an institution in D' that is acceptable to this agent, regardless of the first institution that is assigned an agent from its acceptance list. Given that D' was an arbitrary safe block in $\bar{D}$ where $\bar{D} \subseteq D$, the proof is completed. □

Proof of Lemma 4

Proof. Let an acceptance list be given for each institution in $\bar{D}$, where $\bar{D} \subseteq D$ such that there is no safe block in $\bar{D}$. Since there is no safe block in $\bar{D}$, there is no equal-acceptable subset of institutions in $\bar{D}$, and thus Lemma 1 implies that each subset of institutions in $\bar{D}$ is over-acceptable. Let $d \in \bar{D}$ and assign an agent to d from d's acceptance list. After removing this agent from the acceptance lists in $D' \setminus \{d\}$, there are at least k agents remaining collectively on the $k-1$ acceptance lists of institutions in $D' \setminus \{d\}$. Moreover, note that each non-empty subset of $D' \setminus \{d\}$ is either equal-acceptable or over-acceptable. Then Hall's theorem implies that it is feasible to assign each institution in $D' \setminus d$ an agent other than d on its acceptance list, and the resulting matching is a maximum matching for $D' \setminus d$. This means that, together with the assignment of an agent to d from d's acceptance list, we have a maximum matching for D'. □

Example 6 (DA is bossy)**.** *Consider the following preferences with tie-breaking among the acceptable institutions as indicated below:*

$$A_1 : (d_2, d_1)$$
$$A_2 : (d_1, d_2)$$
$$A_3 : (d_1, d_3)$$
$$A_4 : (d_3)$$

The acceptance lists of the institutions are as follows:

$$d_1 : 1, 3, 2$$
$$d_2 : 2, 1$$
$$d_3 : 4, 3$$

The agent-optimal matching is $d_1 - 1$*,* $d_2 - 2$*,* $d_3 - 4$*. Note that 3 is unmatched.*

If 3 reports $A_3' : (d_3)$ *then the agent-optimal matching at* (A_3', A_{-3}) *is* $d_1 - 2$*,* $d_2 - 1$*,* $d_3 - 4$*, leaving 3 unmatched.*

Chapter 3

Overlapping Dynamic Matching with Dichotomous Preferences

3.1 Introduction

Many public daycare systems operate in a dynamic environment where children attend daycares for multiple periods. These systems, particularly in European countries, are centrally administered and attempt to balance parents' reported preferences over different daycares with the priorities of the daycares over the children. Children can start daycare at the age of six months, and when they turn a certain age they move to the next level of pre-schooling. Daycare assignment takes place regularly, based on parents' report of their top choices among daycares. Parents may additionally indicate whether they wish to opt for a guaranteed placement in the event that their child is not initially assigned. Admission priorities for daycare placements are determined by local governments and differ across municipalities (Kennes, Monte, and Tumennasan, 2014). The centralized daycare matching system may prioritizes children currently allocated to a daycare over other children in the subsequent period. This ensures that children currently enrolled in a daycare will not be displaced from this daycare involuntarily. This policy is called *history dependence* which is one of the key features of a dynamic matching environment. Another policy, used in the Danish daycare assignment system, is that older unassigned children are given high priority in daycares, a policy called *childcare guarantee* (Kennes et al., 2014). In Quebec each child may attend a daycare for two periods, but not necessarily in the same facility. In any given period children of different ages may be allocated to the same daycare. Every month a new group of young children starts daycare, while those children who have turned 18 months leave for the next stage of up to 5 years.

Using a dynamic school choice model where students have strict preferences over objects for two periods and schools have strict priority orderings over agents, Kennes et al. (2014) present an impossibility result that demonstrates that there is no dynamic matching mechanism that is both stable and strategyproof. They find that if the Deferred Acceptance (DA) algorithm (Gale and Shapley, 1962), which is introduced for a static problem, is used period-by-period with a simple modification to make it consistent with a dynamic environment, the mechanism gives a stable matching. This stability is conditional upon independence from the previous assignments, which means that there is no history-dependent priority update or childcare guarantee: daycares do not prioritize children currently attending them and they do not give high priority to a child who stays at home in the first period. Kennes et al. (2014) also find that there is no mechanism that is strategyproof even when this independence condition is satisfied. If the market gets larger, however, manipulation approaches zero in this same setting where children attend daycares for two periods (Kennes, Monte, and Tumennasan, 2015, 2019). Abdulkadiroglu and Loertscher (2007) analyze a dynamic house allocation setting in which the same group of agents participates across multiple periods. Emphasizing efficiency considerations, they introduce a randomized allocation mechanism that achieves higher efficiency than random serial dictatorship.. Kurino (2014) studies the centralized housing allocation problem with overlapping generations of agents. Unlike in the school choice model, since objects have no priorities in the house allocation problem, stability or fairness is not considered in either of these papers. Atay and Romero-Medina (2023) study the allocation of children to childcare facilities, assuming that children submit strict preferences over pairs of the same schools for two periods. Their setting involves yearly cohorts, with no competition among children from different cohorts, and demonstrates that stable matchings exist.

Further dynamic matching papers which study various dynamic models include Doval (2014) who studies a concept of dynamic stability and finds that dynamically stable matchings may fail to exist in two-sided economies such as in adoption markets or in dynamic object allocation problems including public housing. In addition, to guarantee that efficiency is achieved, the central clearing house needs to restrict agents' options to wait for a better match in both f these settings. In a dynamic assignment problem where objects are assigned to agents in a constant-size waiting lists, Bloch and Cantala (2017) study the optimal design of probabilistic queuing disciplines, punishment schemes and the optimal timing of applications. Kadam and Kotowski (2018) study a dynamic two-sided, one-to-one matching market in which participants on both sides interact over time. They characterize sufficient conditions under which a dynamically stable matching exists, allowing for the possibility that initial matchings may need to be revised. Kurino (2020) develops a dynamic framework to study two-sided matching environments that evolve over time, including applications such as teacher–student assignments and hospital–intern markets in the United Kingdom. The paper introduces a dynamic notion of credible group stability and demonstrates that implementing the proposer-optimal stable matching in each period satisfies this criterion. A dynamically credibly group-stable matching is individually rational and robust to all defensible group deviations under an appropriate notion of defensibility. Pereyra (2013) examines a dynamic model of teacher assignments to public schools, focusing on environments with rankable preferences and seniority-based priorities, and shows that the Deferred Acceptance mechanism is strategy-proof in this context. Dur (2011) studies a dynamic school choice setting that incorporates incentives related to sibling placement and demonstrates that no mechanism can simultaneously satisfy fairness (stability) and strategyproofness. Bloch and Cantala (2013) study a dynamic matching problem focusing on the long-run properties of different assignment rules. Andersson, Dur, Ertemel, and Kesten (2018) introduce straightforwardness as a notion of truthfulness in their dynamic sequential setting and give a set of rules that guarantees the existence of a subgame perfect Nash equilibrium and minimizes waste and untruthfulness.

In a static matching problem under strict preferences of agents, fairness (stability) always conflicts with both Pareto-efficiency and assignment maximization. The two most popular strategyproof matching mechanisms, the Deferred Acceptance (DA) algorithm, and the Top Trading Cycles (TTC) algorithm, do not provide a matching that is both fair and efficient for the agents, since these two requirements cannot be reconciled. In the case of strict preferences and priorities, the DA algorithm leads to a fair matching, but not a Pareto-efficient one (Gale and Shapley, 1962). The Top Trading Cycles (TTC) algorithm provides an efficient matching, but not a fair one (Abdulkadiroğlu and Sönmez, 2003). Neither the DA nor the TTC guarantees a maximum matching. Nevertheless, assignment maximization is a very important requirement, especially if there is a significant shortage of objects. Daycare seats are notoriously in short supply in some places, including Quebec, and thus our study focuses on maximum matchings. Moreover, under severe shortages it becomes mostly irrelevant how the daycares are ordered in agents' preferences, and the main objective becomes to be matched to an acceptable daycare. Accordingly, we assume that preferences over daycares are dichotomous (Bogomolnaia and Moulin, 2004), dividing daycares into acceptable and unacceptable daycares, which is not only a realistic assumption in our setting with major shortages, but also provides informational simplicity and allows for escaping some of the impossibility results that are present under strict preferences.

In a static problem with dichotomous agent preferences, Aziz and Brandl (2021) study the rationing of medical units to patients and allow for weak priorities of medical units over patients, while patients may or may not be eligible for medical units. Their algorithm sequentially eliminates agents from being assigned based on an agent permutation and provides a maximum and fair matching in a model that is closely related to the static model that we use in each period. Since patients cannot freely pretend to be eligible for a medical unit, their incentive axiom is weaker than standard strategyproofness. Assuming dichotomous agent preferences, Chapter 2 proposes a matching mechanism for a static centralized matching system called the SAFE mechanism that ensures three desired properties: maximum cardinal allocation (which is equivalent to Pareto-efficiency), fairness (stability), and strategyproofness.

In this chapter, I study a dynamic daycare allocation problem where each child participates in the assignment process twice, in period 1 and period 2, overlapping in periods as in Kennes et al. (2014). Children report their dichotomous preferences over objects only once, which apply to both periods. Objects have a fixed priority ordering of agents for all periods. We address the following question: does a maximum (Pareto-efficient), fair, and strategyproof matching mechanism exist in this dynamic matching problem with dichotomous preferences? I study two treatments of daycare priorities, the history-dependent updating policy and the childcare guarantee policy. I adapt the SAFE mechanism of Chapter 2 to both of these treatments and study the SAFE-HD and SAFE-CG mechanisms. Our results demonstrate that the history-dependent treatment allows for all the required properties to be satisfied simultaneously by showing that the SAFE-HD mechanism possesses all these properties. By contrast, the childcare-guarantee treatment leads to impossibility results, and we show that the SAFE-CG mechanism has the basic required attributes that are compatible.

3.2 Overlapping Dynamic Model with Dichotomous Preferences

Children can go to daycare when they are one and two years old, which we consider as two distinct periods in which they may or may not go to the same daycare, or they may not go to any daycare in one or both periods. Time is discrete and there are an infinite number of periods, denoted by $t = 1, \ldots, \infty$. There is a finite set of daycares D that are available in each period t. The capacity of each daycare $d \in D$ in each period is q_d, indicating the number of available seats at each daycare. We can work with the simpler case of unit-capacity daycares due to the indifferences that arise when preferences are dichotomous, and we assume without loss of generality that $q_d = 1$ for all $d \in D$, assuming implicitly that each daycare is represented by as many "copies" as its original capacity. Since the many-to-one domain is a subset of the one-to-one domain, this simplification does not matter for our results.

Daycare attendance is not mandatory and may not be feasible for each child, and we let 0 represent the option of staying home which is available to every child. In each period $t \geq 1$, a new set of one-year-old children C_t arrives. The set of daycare-age children follows the OLG (overlapping generations) model over two consecutive periods. Let C_t denote the set of children who enter the daycare matching market in period t. Letting $C_0 = \emptyset$, in any period $t \geq 1$, the set of daycare-age children is $C_{t-1} \cup C_t$, and the set of all children is given by $\hat{C} = \bigcup_{t=1,\ldots,\infty} C_t$. For ease of notation, we let $C^t = C_{t-1} \cup C_t$ for all $t \geq 1$.

3.2.1 Children's Preferences

Each child has *dichotomous* preferences over daycares in both periods in which the child is in the daycare matching market, that is, each child finds a daycare either acceptable or unacceptable in both periods, and children are indifferent among acceptable daycares. The reported preferences over the acceptance of daycares apply in both periods. Note that we do not elicit preferences over the two periods jointly, to ensure informational simplicity.

For all $c \in \hat{C}$, let $A_c \subseteq D$ denote the set of *acceptable* daycares for c. Then $D \setminus \ A_c$ is the set of *unacceptable* daycares for c. Thus, daycare d is acceptable for child c in either period if and only if $d \in A_c$. Let $A = (A_c)_{c \in \hat{C}}$ denote a preference profile for all children, including all different periods.

Children prefer to be matched to an acceptable daycare in both periods to remaining unmatched. Let '1' denote being matched to an acceptable daycare and let '0' denote being unmatched in either period. Then for any child $(0, 1)$ means, for example, that the child is unmatched in period 1 and matched in period 2. Given that for each period each child strictly prefers 1 to 0, since for all $d \in A_c,\ d \ P_c \ 0$, we can write $1 \ P_c \ 0$, where P_c denotes child c's strict preference. It follows that for two-period outcomes, the following two strict preference orderings are possible:

a) $(1, 1) \ P_c \ (1, 0) \ P_c \ (0, 1) \ P_c \ (0, 0)$

b) $(1, 1) \ P_c \ (0, 1) \ P_c \ (1, 0) \ P_c \ (0, 0)$

Some children may prefer $(1, 0)$ to $(0, 1)$ while some may prefer $(0, 1)$ to $(1, 0)$.

3.2.2 Daycare Priorities

In each period $t \geq 1$, each daycare ranks all the daycare-age children by priority; that is, they rank all children in C^t. Daycares are not considered strategic agents in this model and priorities do not represent daycare preferences. Instead, priorities are mandated and formed according to laws and regulations, and they may be idiosyncratic, based on characteristics of the children such as attendance of a sibling, proximity to daycare, medical condition, age, immigration status, etc., according to the context of the application which may be different from daycare allocation.

Formally, each daycare $d \in D$ has a strict priority ordering $\succ_d$ over all the children in $\hat{C}$, which we call the **master (priority) list of daycare *d*.** Let $\succ = (\succ_d)_{d \in D}$ denote the profile of master lists for daycares. In each period $t \geq 1$, the priorities over the children in C_t are consistent with the master list for each daycare. However, in period $t+1$ the master list may be modified over children in C_t according to these children's assignments in period t, and thus the subset of priorities over the children in C^t for $t \geq 2$ may differ from this master list for a daycare. We assume that the priority modification for the second period is systematic and is based on the first-period assignments of children in their second period.

3.2.3 Profiles

Given fixed $\hat{C}$ and D, an infinite-period overlapping childcare matching problem (or, in short, a market) is specified by a pair $(A, \succ)$, a profile of dichotomous preferences of the children and a profile of master lists for daycares. We will refer to $(A, \succ)$ henceforth as a **profile** or a **market**.

3.2.4 Matchings

For all $t \geq 1$, a **period-t matching μ^t** is a mapping from C^t to $D \cup \{0\}$, where 0 means being unassigned. Let $\mu^t(c) \in D \cup \{0\}$ denote the assignment of child c in period t and, with a slight abuse of notation, let $\mu^t(d) \in C \cup \{0\}$ denote the assignment of daycare d under μ^t in period t. If a daycare is assigned to 0, it means that no child is assigned to this daycare in this period (but recall that this only means one vacant seat in the original multiple-capacity model).

A period-t matching μ^t is a function indicating which daycare-age child in period t attends which daycare in period t, and a **matching μ** is a collection of all period-t matchings. Formally, $\mu = (\mu^1, \mu^2, \ldots, \mu^t, \ldots)$. We use the notation $\mu(c)$ to denote $(\mu^t(c), \mu^{t+1}(c))$, where t is the period in which c is a one-year-old child, that is, $c \in C_t$. Let $\mathcal{M}^t$ denote the set of all period-t matchings, and let $\mathcal{M}$ denote the set of all matchings.

3.2.5 Mechanisms

A *mechanism* is a systematic process that assigns a matching to each market. We consider direct mechanisms, i.e., each child c reports her dichotomous preferences A_c over daycares and, based on these reports, the mechanism returns a matching $\mu \in \mathcal{M}$. Thus, for each market $(A, \succ)$, a mechanism returns period-t matchings for all periods $t \geq 1$. Let $\mathcal{A}$ be the set of dichotomous preference profiles A for $\hat{C}$, and let Π denote the set of master list profiles $\succ$ for D. A **(matching) mechanism** φ assigns in each period $t \geq 1$ a matching μ^t to each profile $(A, \succ) \in \mathcal{A} \times \Pi$. Formally, in each period $t \geq 1$ period, $\varphi^t : \mathcal{A} \times \Pi \to \mathcal{M}^t$, and $\varphi : \mathcal{A} \times \Pi \to \mathcal{M}$. Let $\varphi_c(A, \succ) = (\varphi_c^t(A, \succ), \varphi_c^{t+1}(A, \succ))$ be the pair of daycares to which child $c \in C_t$ is assigned by mechanism φ at profile $(A, \succ)$.

3.3 Daycare Priority Treatments

We study two relevant scenarios when priority orderings are updated between periods, based on the previous matchings. Following Kennes et al. (2014) and Kennes et al. (2019), we consider two types of modifications from one period to the next in the priorities over children who participate in both periods. Specifically, we study the following two treatments for daycare priorities.

i) **HD: History-Dependence - priority for currently enrolled children in the next period**

   If $c \in C_t$ and $c \in \mu^t(d)$ for a daycare $d \in D$ then $c \succ_d^{t+1} c'$ for all $c' \notin \mu^t(d)$. This means that the currently assigned child gets the highest priority in the daycare in her second period, what we refer to as history-dependence (HD). All other priority orderings follow the master list for each daycare d, and the only change is that each daycare places the child who was assigned to the daycare in the current period at the top of its priority list in the next period, $t + 1$ (when the daycare is not unassigned in period t) if $c \in C^t$ and thus c is still in the daycare market in period $t + 1$.

ii) **CG: Childcare Guarantee - priority for unassigned children in the next period**

   If $c \in C_t$ and $\mu^t(c) = 0$ then, for all $d \in D$, $c \succ_d^{t+1} c'$ for all c' such that $\mu^t(c') \in D$. This means that the currently unassigned children get the highest priority in each daycare in their second period, what we refer to as childcare guarantee (CG).

All other priority orderings follow the master list for each daycare d, and the only change is that each daycare places the children at the top of their lists who remained unassigned in their first period.[1]

3.4 Axioms

We define next the axioms that we impose on mechanisms.

3.4.1 Individual Rationality

A matching $\mu \in \mathcal{M}$ is **individually rational** if, for all $c \in \hat{C}$, in any period $t \geq 1$, either $\mu_c^t \in A_c$ or $\mu_c^t = 0$ (the child is unassigned). This means that no child is assigned to a daycare that is unacceptable to the child in any period. A matching **mechanism** is **individually rational** if it assigns an individually rational matching to each profile $(A, \succ)$. From now on we will restrict attention to individually rational matchings and matching mechanisms, without explicitly stating that the mechanism is individually rational. That is, we assume that a child can only be assigned to a daycare in any period that is acceptable to her.

3.4.2 Fairness

Fairness in a static sense (i.e., specific to a period) requires that there is no priority violation for any child in the matching made in that period. Only unmatched children may have their priorities violated under dichotomous preferences, since each acceptable daycare is welfare-wise identical to the children.

A period-t matching $\mu^t \in \mathcal{M}^t$ is **fair** at profile $(A, \succ)$ if there is no child-daycare pair (c, d), with $c \in C^t$, such that the following three conditions hold:

i) $d \in A_c$;

ii) $\mu^t(c) = 0$;

iii) there exists a child c' such that $\mu^t(c') = d$ and $c \succ_d^t c'$.

For strict preferences this notion of fairness corresponds to the standard *no justified-envy* notion for one-sided matching models.

It is also possible for a child to have justified envy in a dynamic sense over two periods, which is distinct from the one-period static notion of justified envy. When preferences

[1] This kind of childcare guarantee is practiced in Denmark to ensure that the system is not unfair to children who go unassigned when they are one year old, and thus it is also known as the Danish priority (see Kennes et al. (2014) for more detail).

are dichotomous, having dynamic two-period justified envy implies in almost all cases that there is a one-period priority violation (or justified envy), since with dichotomous preferences a child can only be made strictly better off if he is unassigned, and a daycare that the child has justified envy for would have to be either unassigned or have higher priority for the child in question than for its assigned child. This means that when preferences are dichotomous it is sufficient to require that in each period the period-t matching is fair in order to rule out dynamic justified envy, with the exception of the scenario where $(0,1)\ P_c\ (1,0)$. Since a $(1,0)$ assignment cannot occur under the HD treatment, this requirement is only relevant under the CG treatment.

Dynamic Fairness is violated at profile $(A, \succ)$ if there is a child-daycare pair (c, d), with $c \in C^t$, such that the following four conditions hold:

i) $d \in A_c$;

ii) $\mu(c) = (1, 0)$;

iii) $(0,1)\ P_c\ (1,0)$ (in case of a single period matching of child c);

iv) there exists c' such that $\mu(c') = (0, d)$ and $c \succ_d c'$.

We will say in this case that child c has **dynamic justified envy** (with respect to daycare d and child c'). Under the assumptions of the model, the above definition captures the only case of dynamic justified envy that is not already implied by static justified envy. In particular, the only additional possibility arises when a child strictly prefers receiving a seat in period t to receiving one in period $t-1$, i.e., $(0,1)P_c(1,0)$, and the envy arises because the updated priority in period t differs from the original priority ranking at daycare d.

A **matching** μ is **dynamically fair** if

i) the period-t matching μ^t is fair for each period $t \geq 1$;

ii) there is no child who has dynamic justified envy.

A matching **mechanism** is **dynamically fair** if it assigns a dynamically fair matching μ to each profile $(A, \succ)$.

3.4.3 Pareto-efficiency

Let $|\mu^t|$ denote the total number of children who are assigned to a daycare in matching μ^t (i.e., the cardinality of μ^t) for any $t \geq 1$. A matching μ^t is a **maximum matching** in period $t \geq 1$ if there is no other matching which assigns more children to daycares in period t: for all $\tilde{\mu}^t \in \mathcal{M}^t$, $|\mu^t| \geq |\tilde{\mu}^t|$. A matching $\mu \in \mathcal{M}$ is a **maximum matching** if, in any period $t \geq 1$, there is no other matching which assigns more children to daycares: for all $\tilde{\mu} \in \mathcal{M}$, in each period $t \geq 1$, $|\mu^t| \geq |\tilde{\mu}^t|$. In other words, μ has maximum cardinality in each period. A matching mechanism is **maximum** if it assigns a maximum matching in $\mathcal{M}$ to each profile $(A, \succ)$.

For any period $t \geq 1$, a matching $\bar{\mu}^t \in \mathcal{M}^t$ **Pareto-dominates** another matching $\mu^t \in \mathcal{M}^t$ at profile $(A, \succ)$ if none of the children are worse off with $\bar{\mu}^t$ than with μ^t, and at least one child strictly prefers $\bar{\mu}^t$ to μ^t, given A. Thus, since preferences are dichotomous, Pareto-domination means that $\bar{\mu}^t$ assigns all the children to a daycare who are assigned in μ^t, and it assigns at least one more child to a daycare than μ^t does. For any period $t \geq 1$, a matching $\bar{\mu}^t \in \mathcal{M}^t$ is **Pareto-efficient** if it is not Pareto-dominated. We call $\mu \in \mathcal{M}$ **statically Pareto-efficient** if for all $t \geq 1$, $\mu^t \in \mathcal{M}^t$ is Pareto-efficient.

In a static model, Pareto-efficient matchings and maximum matchings are the same (Chapter 2), and thus for any period $t \geq 1$, matching $\bar{\mu}^t \in \mathcal{M}^t$ is Pareto-efficient if and only if it is maximum. When preferences are strict, Pareto-efficient and maximum matchings are distinct, and neither property implies the other. Nonetheless, in a static model with strict preferences a maximum and Pareto-efficient matching exists for every profile. With dichotomous preferences, on the other hand, not only a maximum and Pareto-efficient matching exists for each profile but the two notions coincide (see Chapter 2 for more details).

We define Pareto-domination similarly for a matching $\bar{\mu} \in \mathcal{M}$, to capture the notion of efficiency over multiple periods. A matching $\bar{\mu} \in \mathcal{M}$ **dynamically Pareto-dominates** another matching $\mu \in \mathcal{M}$ at profile $(A, \succ)$ if none of the children are worse off with $\bar{\mu}$ than with μ, and at least one child strictly prefers $\bar{\mu}$ to μ, given A. One illustrative form of intertemporal Pareto-domination is the following cycle: for any period $t \geq 1$, there are no $\{c_1, ..., c_k\} \subseteq C_t$ and daycares $d_1, ..., d_k$ such that $(d_2, 0) P_{c_1} (0, d_1)$, $(0, d_3) P_{c_2} (d_2, 0)$, . . . , $(0, d_1) P_{c_k} (d_k, 0)$. This cycle demonstrates how multi-period Pareto-domination can arise even when in each period the matching is Pareto-efficient. Therefore, even if a matching $\mu = (\mu^t)_{t \geq 1}$ is statically Pareto-efficient, μ may still be Pareto-dominated.

Note that these concepts only take into account the children's welfare, which is consistent with the fact that daycares have priorities over children instead of preferences, as the daycares' welfare is not affected by which children are assigned to them.

Given that a child's welfare does not only depend on her matching to one of her acceptable daycares but also on a matching that is consistent with her strict preference relations over two-period matchings ($(1, 0)\ P_c\ (0, 1)$ or $(0, 1)\ P_c\ (1, 0)$ are both possible), Pareto-efficiency in the dynamic model requires more than just static Pareto-efficiency. Thus, in our dynamic model with dichotomous preferences, **dynamic Pareto-efficiency** for a matching μ requires that μ is not Pareto-dominated. Note that dynamic Pareto-efficiency implies static Pareto-efficiency.

A **mechanism** φ is **dynamically Pareto-efficient** if it assigns a dynamic Pareto-efficient matching μ to each profile $(A, \succ)$.

3.4.4 Strategyproofness

A matching mechanism φ is **period-*t* strategyproof** if, for period $t \geq 1$, for all profiles $(A, \succ)$ and for all $c \in C^t$ such that $\varphi_c^t(A, \succ) = 0$, there is no $A'_c \subseteq D$ such that $\varphi_c^t((A'_c, A_{-c}), \succ) \in A_c$. If φ^t is individually rational, this means that $\varphi_c^t((A'_c, A_{-c}), \succ) = 0$ or $\varphi_c^t((A'_c, A_{-c}), \succ) \in A'_c \setminus A_c$.

Strategyproofness is more difficult to achieve in a dynamic matching problem than in a static problem. In static problems under dichotomous preferences a child has the incentive to misreport her preferences if she is unmatched and wants to get matched to one of the acceptable daycares, which applies to each period of a dynamic matching problem. In addition, a child might misrepresent her preferences to affect the daycares' priorities in the subsequent period when priorities change in response to the assignments in the previous period, namely under the assumptions of HD or CG daycare priorities. Under HD daycare priorities, a child may manipulate to get matched to a different daycare in the first period, in order to be matched to this daycare as well in the second period, instead of being matched to a different daycare in the first period and be unassigned in the second period. Under CG daycare priorities, a child may try to sacrifice her matching in the first period in order to get matched in the second period, if this outcome is preferred by the child. In general, we need to require more than just period-t strategyproofness.

A matching mechanism φ is **dynamically strategyproof** if it is period-t strategyproof in all periods $t \geq 1$ and, in addition, there is no profile $(A, \succ)$, period t, child $c \in C_t$ and an alternative preference report A'_c such that $\varphi_c((A'_c, A_{-c}), \succ)$ is preferred by child c with preferences A_c to $\varphi_c(A, \succ)$. In other words, children cannot manipulate dynamically over two periods either, not just in single periods.

3.5 Dynamic SAFE Matching Mechanisms and Their Properties

We now introduce the Dynamic SAFE mechanisms, which are based on the static SAFE mechanisms studied in Chapter 2 for a static one-sided model under dichotomous preferences.

3.5.1 Static SAFE Mechanisms

A SAFE (Sequential Allocation for Fairness and Efficiency) mechanism) is a sequential mechanism for which a fixed baseline permutation $\bar{\pi}$ of D is used at each profile $(A, \succ)$ to determine the profile-dependent permutation $\pi(A, \succ)$ which leads to the π-sequential matching. Given a profile $(A, \succ)$, **acceptance lists** are the daycare priority lists that consists of only those children who find the daycares acceptable. Permutation π is a specific daycare order that depends on the length and composition of the acceptance lists of daycares at $(A, \succ)$ and on the fixed $\bar{\pi}$.

A set of k daycares is called *equal-acceptable* if the k daycares have exactly k children on their acceptance lists collectively. Given a profile $(A, \succ)$, a **safe *k*-block**, or simply a **safe block,** is an equal-acceptable set of k daycares that has no equal-acceptable proper subset. There may not exist any equal-acceptable set of daycares at a specific profile, which would imply that there is no safe block. On the other hand, there may be multiple safe blocks, and safe blocks may even overlap.

Each SAFE mechanism is specified by the fixed baseline permutation $\bar{\pi}$, and hence the class of SAFE mechanisms is parameterized by the $m!$ permutations of D. The SAFE mechanism with baseline permutation $\bar{\pi}$ is denoted by $\varphi^{\bar{\pi}}$.

SAFE mechanisms:

Fix a baseline permutation $\bar{\pi}$ of the set of daycares D. Let $(A, \succ) \in \mathcal{A} \times \Pi$. The steps of the SAFE mechanism $\varphi^{\bar{\pi}}$ associated with $\bar{\pi}$ are the following:

> **Step 1.a:** If there is a safe block at $(A, \succ)$, choose the first daycare according to $\bar{\pi}$ that is in a safe block. If there is no safe block at $(A, \succ)$, choose the first daycare according to $\bar{\pi}$.
>
> **Step 1.b:** Assign the highest-ranked child on the selected daycare's acceptance list to this daycare.
>
> **Step 1.c:** Update the set of acceptance lists by removing the daycare that has just been assigned a child, and by removing this child from all the remaining daycares' lists.

Repeat Steps 1.a to 1.c iteratively in the reduced market until no more assignments can be made.

In each step of the iterative procedure there is either at least one safe block or no safe block, based on the remaining set of daycares with their acceptance lists containing only the remaining children.

Chapter 2 shows that in a static model under dichotomous preferences SAFE mechanisms are Pareto-efficient, fair, and strategyproof. Moreover, as already mentioned, Pareto-efficiency in the one-period static model is equivalent to always assigning a maximum matching.

3.5.2 Dynamic SAFE Mechanisms

A Dynamic SAFE mechanism is a mechanism which applies the SAFE mechanism in each period $t \geq 1$ to the set of children who are eligible for daycare in that period, namely to C^t. Since daycares may revise their priority lists, using either the HD or CG treatments, as specified in Section 3.3, we distinguish between Dynamic SAFE mechanisms accordingly, and we study the SAFE-HD and SAFE-GC mechanisms, along with the benchmark SAFE-IP mechanism.

Recall that each child reports her acceptance set only, as in the static setting, and the reported acceptance sets apply to both periods that the child is present in the daycare market. Despite the use of one-period preferences, the problem is dynamic since the daycare priority lists depend on the matching in the previous period.

3.5.3 Examples

Assume that there are two daycares, d_1 and d_2, and each daycare has one seat. Let $C_{t-2} = \emptyset$, $C_{t-1} = \{c_1\}$, $C_t = \{c_2, c_3, c_4\}$, $C_{t+1} = \{c_5, c_6\}$, and $C_T = \emptyset$ for all $T \geq t+2$. Let $A_{c_1} = \{d_1\}$, $A_{c_2} = \{d_1, d_2\}$, $A_{c_3} = \{d_1, d_2\}$, $A_{c_4} = \{d_2\}$, $A_{c_5}^{t+1} = \{d_2\}$, and $A_{c_6} = \{d_1\}$. The two master lists of daycares are the following:

$$d_1 : c_1, c_2, c_3, c_6$$

$$d_2 : c_5, c_3, c_2, c_4$$

Given the profile $(A, \succ)$, we can find the acceptance lists in each period $t \geq 1$. Let the fixed baseline permutation of daycares be $\bar{\pi} = \{d_1, d_2\}$.

We will use this setup to illustrate the SAFE-HD and SAFE-CG mechanisms.

SAFE-HD Mechanism

The acceptance lists for period t are as follows:

$$d_1 : c_1, c_2, c_3$$
$$d_2 : c_3, c_2, c_4$$

Assume that child c_1 was matched in period $t-1$ to daycare d_1 in matching μ^0. Note that there is no safe block in the acceptance lists in period t. Thus, child c_1 is matched to daycare d_1 and child c_3 is matched to daycare d_2. The matching in period t is the following:

$$d_1 : \boxed{c_1}, c_2, c_3$$
$$d_2 : \boxed{c_3}, c_2, c_4$$

The acceptance lists for period $t+1$ are shown in the following figure where child c_1 is absent, because she exits in period t. In period $t+1$, children c_2, c_3, and c_4 are in their second period and children c_5 and c_6 are in their first period. The acceptance lists in period $t+1$ are:

$$d_1 : c_2, c_3, c_6$$
$$d_2 : c_3, c_5, c_2, c_4$$

Note that d_2's acceptance list does not follow daycare d_2's master priority list since it has been modified based on history-dependence: child c_3 gets the first priority over child c_5 due to her assignment to d_2 in the previous period. Since there is no safe block, child c_2 is matched to daycare d_1 and child c_3 is matched to daycare d_2. The period $t+1$ matching in the SAFE-HD mechanism is the following:

$$d_1 : \boxed{c_2}, c_3, c_6$$
$$d_2 : \boxed{c_3}, c_5, c_2, c_4$$

The acceptance lists for period $t+2$ do not include children c_2, c_3, and c_4 since they exit in period $t+1$.

$$d_1 : c_6$$
$$d_2 : c_5$$

Daycare d_1 forms a safe 1-block, and daycare d_2 forms another safe 1-block. Child c_6 is matched to daycare d_1, and child c_5 is matched to daycare d_2. The period $t+2$ matching in the SAFE-HD mechanism is the following:

$$d_1 : \boxed{c_6}$$
$$d_2 : \boxed{c_5}$$

The final matching in different periods:

Period t :	**Period $t+1$:**	**Period $t+2$:**
$\mu_{c_1} = d_1$	$\mu_{c_2} = d_1$	$\mu_{c_6} = d_1$
$\mu_{c_3} = d_2$	$\mu_{c_3} = d_2$	$\mu_{c_5} = d_2$

Assume that the six children have the following strict preference orderings for two-period outcomes, where the boxes show when they were matched in the two periods:

$\boxed{(1,1)}\ P_{c_1}\ (1,0)\ P_{c_1}\ (0,1)\ P_{c_1}\ (0,0)$
$(1,1)\ P_{c_2}\ (1,0)\ P_{c_2}\ \boxed{(0,1)}\ P_{c_2}\ (0,0)$
$\boxed{(1,1)}\ P_{c_3}\ (0,1)\ P_{c_3}\ (1,0)\ P_{c_3}\ (0,0)$
$(1,1)\ P_{c_4}\ (1,0)\ P_{c_4}\ (0,1)\ P_{c_4}\ \boxed{(0,0)}$
$(1,1)\ P_{c_5}\ \boxed{(0,1)}\ P_{c_5}\ (1,0)\ P_{c_5}\ (0,0)$
$(1,1)\ P_{c_5}\ \boxed{(0,1)}\ P_{c_5}\ (1,0)\ P_{c_5}\ (0,0)$

In this dynamic matching problem SAFE-HD maximizes the number of matches in each period. In addition, history dependence (HD) implies that if a child is matched in the first period, she is also matched in the second period. Thus, the $(1,0)$ outcome does not occur in the SAFE-HD mechanism due to the history-dependent priority updates. This implies that dynamic Pareto-efficiency is satisfied by SAFE-HD. And since the SAFE-HD mechanism maximizes the number of matches in each period, it is also a maximum mechanism. The static SAFE mechanism is a sequential mechanism that assigns the highest-priority children to daycares according to some daycare ordering at each profile, and thus it is fair in static problems. We will also show that SAFE-HD mechanisms are dynamically fair and, moreover, dynamically strategyproof.

SAFE-CG Mechanism

The acceptance lists for period t are as follows:

$$d_1 : c_1, c_2, c_3$$
$$d_2 : c_3, c_2, c_4$$

Assume that child c_1 was matched in period $t-1$ to daycare d_1. Note that there is no safe block in period t. Thus, child c_1 is matched to daycare d_1 and child c_3 is matched to daycare

d_2. The matching in period t is the following:

$$d_1 : \boxed{c_1}, c_2, c_3$$
$$d_2 : \boxed{c_3}, c_2, c_4$$

The acceptance lists for period $t+1$ are shown in the following figure. Child c_1 exits in period t and children c_2, c_3, and c_4 are in the second period, while children c_5 and c_6 are in the first period in period $t+1$.

$$d_1 : c_2, c_3, c_6$$
$$d_2 : c_2, c_4, c_5, c_3$$

Note that d_2's acceptance list does not follow daycare d_2's master priority list as it has been modified based on the childcare guarantee principle. Children c_2 and c_4 are prioritized over children c_5 and c_3 because c_2 and c_4 were unmatched in their first period. Since there is no safe block, child c_2 is matched to daycare d_1 and child c_4 is matched to daycare d_2. According to the SAFE-CG mechanism, the $t+1$ period matching is the following:

$$d_1 : \boxed{c_2}, c_3, c_6$$
$$d_2 : c_2, \boxed{c_4}, c_5, c_3$$

The period $t+2$ acceptance lists do not include children c_2, c_3, and c_4 since they exit after period $t+1$:

$$d_1 : c_6$$
$$d_2 : c_5$$

Daycare d_1 forms a safe 1-block, and daycare d_2 forms another safe 1-block. Child c_6 is matched to daycare d_1, and child c_5 is matched to daycare d_2. The period $t+2$ matching of the SAFE-HD mechanism is the following:

$$d_1 : \boxed{c_6}$$
$$d_2 : \boxed{c_5}$$

The final matching in different periods:

Period t :	**Period $t+1$:**	**Period $t+2$:**
$\mu_{c_1} = d_1$	$\mu_{c_2} = d_1$	$\mu_{c_6} = d_1$
$\mu_{c_3} = d_2$	$\mu_{c_4} = d_2$	$\mu_{c_5} = d_2$

Assume that the six children have the following strict preference orderings for two-period outcomes, where the boxes show thier outcomes they were matched in the two periods:

$\boxed{(1,1)}\ P_{c_1}\ (1,0)\ P_{c_1}\ (0,1)\ P_{c_1}\ (0,0)$
$(1,1)\ P_{c_2}\ (1,0)\ P_{c_2}\ \boxed{(0,1)}\ P_{c_2}\ (0,0)$
$(1,1)\ P_{c_3}\ (0,1)\ P_{c_3}\ \boxed{(1,0)}\ P_{c_3}\ (0,0)$
$(1,1)\ P_{c_4}\ (1,0)\ P_{c_4}\ \boxed{(0,1)}\ P_{c_4}\ (0,0)$
$(1,1)\ P_{c_5}\ \boxed{(0,1)}\ P_{c_5}\ (1,0)\ P_{c_5}\ (0,0)$
$(1,1)\ P_{c_6}\ \boxed{(0,1)}\ P_{c_6}\ (1,0)\ P_{c_6}\ (0,0)$

The SAFE-CG mechanism maximizes the number of matches in each period $t \geq 1$. However, unlike SAFE-HD, outcome $(1,0)$ may occur under childcare guarantee, since previously unmatched children are prioritized over matched children in the following period. However, a SAFE-CG mechanism does not satisfy dynamic Pareto-efficiency. For example, children c_2 and c_3 can both be made better off by exchanging their outcomes, thus there is scope for Pareto-improvement. This proves that SAFE-CG mechanism are not dynamically Pareto-efficient. We also find that child c_3, for example, can manipulate and receive outcome $(0,1)$ instead of $(1,0)$ by reporting $A'_c = \{d_1\}$ instead of $A_c = \{d_1, d_2\}$. However, a child cannot manipulate $(1,0)$ to receive $(1,1)$ instead in a SAFE-GC mechanisms. This follows because a child cannot be bossy in the first period and change the assignments of some other children, and subsequently this child will remain unmatched in her second period. This follows because the SAFE mechanism in the static model is non-bossy.

3.6 Properties of the SAFE-HD Mechanism

We first define a weak non-bossiness concept which will be used in the proof of dynamic strategyproofness for SAFE-HD.

0-non-bossiness. A mechanism φ is 0-nonbossy if for all $c \in C$, $(A, \succ) \in \mathcal{A} \times \Pi$ and $A'_c \subset A$, $\varphi_c(A, \succ) = \varphi_c((A'_c, A_{-c}), \succ) = 0$ implies that $\varphi(A, \succ) = \varphi((A'_c, A_{-c}), \succ)$.

0-nonbossiness means intuitively that unassigned children are not bossy.

Proposition 3. *A SAFE mechanism is 0-nonbossy.*

Proof. Let $c \in C$, $(A, \succ) \in \mathcal{A} \times \Pi$. Moreover, let $A'_c \subset A$ such that $\varphi_c(A, \succ) = \varphi_c(A'_c, A_{-c}), \succ) = 0$. We will show that then $\varphi(A, \succ) = \varphi((A'_c, A_{-c}), \succ)$.

A child c can misreport and show A'_c, either by subtracting some daycares from her set of acceptable daycares, A_c, or by adding some daycares to A_c. By adding some daycares to A_c, she can break a safe k-block and form a new safe $k+1$-block as well. By subtracting daycares, she can break a safe k-block and form a new safe $k-1$-block as well.

Given $\varphi_c(A, \succ) = \varphi_c((A'_c, A_{-c}), \succ) = 0$, we need to show $\varphi(A, \succ) = \varphi((A'_c, A_{-c}), \succ)$ for the case where $A'_c = A_c \cup \{d\}$ for $d \notin A_c$. Since two preference profiles A_c and A'_c are symmetric, a repeated application of this gives the result.

Suppose, by contradiction, that $\varphi_c(A, \succ) = \varphi_c(A'_c, A_{-c}), \succ) = 0$, and $\varphi(A, \succ) \neq \varphi((A'_c, A_{-c}), \succ)$. We consider the following two cases:

(a) Adding daycares to A_c and breaking a safe k-block;

(b) Adding daycares to A_c and forming a new safe $k+1$-block.

(a) Adding daycares to A_c and breaking a safe block change σ^*_π, actual sequence that is used to select daycares throughout the assignment process at $(A, \succ)$ under the SAFE mechanism, except in the case where the first daycare in the safe block is the first daycare in π. If breaking a safe block does not change σ^*_π, $\varphi(A, \succ) = \varphi((A'_c, A_{-c}), \succ)$ for sure. Let's say, it changes σ^*_π, and the first daycare in the safe block is the r^{th} daycare in π. After breaking the safe block, the first daycare in π will take the first turn in the SAFE assignment process assuming that there is no more safe block. Assume that at $(A, \succ)$, the r^{th} daycare $\hat{d}$ matches with child $\hat{c}$. After breaking the safe k-block at $(A'_c, A_{-c}), \succ)$, no daycare upto $(r-1)$ daycares will receive one of the k children. If one of the daycares upto $(r-1)$ daycares receives one of the k children, $\varphi(A, \succ) \neq \varphi((A'_c, A_{-c}), \succ)$. If it happens, child c does become a required child for a maximum matching under the SAFE mechanism and thus must match to one of the k daycares irrationally, which violates the necessary part of the definition of 0 - non-bossiness, $\varphi_c(A, \succ) = \varphi_c((A'_c, A_{-c}), \succ) = 0$. This is a contradiction. To maintain $\varphi_c((A'_c, A_{-c}), \succ) = 0$, none of the k children gets matched to the daycares that are up to $(r-1)$ daycares in π. Therefore, each daycare up to $(r-1)$ daycares gets the same children as before given that σ^*_π remains the same for the part outside the safe k-block.

(b) Adding daycares to A_c, breaking a safe k-block and forming a new safe $k+1$-block mean that child c must get matched to a daycare at $(A'_c, A_{-c}), \succ)$ because a child in a safe block gets matched to $d \in D$ for sure, not necessarily to one of the $k+1$ daycares. This is a contradiction.

Therefore, we have shown that a SAFE mechanism is 0-nonbossy. □

Proposition 4. *A maximum dynamic mechanism is dynamically Pareto-efficient if it does not have a* $(1,0)$ *outcome for any child.*

Proof. According to the definition, a dynamically Pareto-efficient mechanism must be a maximum mechanism, and it cannot leave scope for Pareto-improvement where children can exchange their $(1,0)$ outcome for a $(0,1)$ outcome, and vice versa, in a Pareto-improving manner in an even cycle of children over two periods. If outcome $(1,0)$ does not occur, the relevant strict preference orderings over two-period outcomes become $(1,1)\ P_c$ $(0,1)\ P_c\ (0,0)$ for each child. This means that a Pareto-improving cycle over the two periods is not possible. Therefore, a maximum dynamic mechanism is dynamically Pareto-efficient if it does not have a $(1,0)$ outcome. □

Theorem 10. *A SAFE-HD mechanism is dynamically fair, dynamically Pareto-efficient, and dynamically strategyproof.*

Proof. The SAFE mechanism in a static model is a sequential mechanism that selects daycares sequentially and picks the highest priority child on the acceptance list of a daycare to assign to that daycare, and therefore it is a fair mechanism (Chapter 2). The SAFE-HD mechanism is an application of the SAFE mechanism period-by-period, and given that it does not allow for a $(1,0)$ outcome, it is a dynamically fair mechanism.

Since the SAFE mechanism is a maximum mechanism (Chapter 2), the SAFE-HD mechanism is a maximum mechanism given that it is an application of the SAFE mechanism period-by-period. Therefore, since the SAFE-HD mechanism does not have a $(1,0)$ outcome for any child, Proposition 4 implies that it is dynamically Pareto-efficient.

The SAFE mechanism is strategyproof (Chapter 2), which means that a child cannot manipulate in the same period. Thus, an unassigned child in period 1 remains unassigned in period 1 regardless of her reported preferences. Moreover, according to Proposition 3, the SAFE mechanism is 0-nonbossy. This means that an unassigned child cannot be bossy, and thus it implies that an unassigned child cannot change the outcome for other children in period 1 and consequently gain in period 2 under the SAFE-HD mechanism. As long as a child is unassigned in period 1, the child cannot change the outcome for anyone in period. Therefore, it is not possible to manipulate to receive $(0,1)$ instead of $(0,0)$. Thus, $(0,0)$ or $(0,1)$ cannot be successfully manipulated. We also know that $(1,0)$ does not occur under a SAFE-HD mechanism. This leaves $(1,1)$, which cannot be manipulated since it is already the best outcome for a child. Hence, the SAFE-HD mechanism is dynamically strategyproof.

Therefore, a SAFE-HD mechanism is dynamically fair, dynamically Pareto-efficient, and dynamically strategyproof. □

3.7 Impossibility Results and the Properties of the SAFE-CG Mechanism

First we show that under the CG treatment it is not possible for a mechanism to be both fair (in the static sense) and dynamically fair. Intuitively, if a child $c \in C_t$ prefers to be matched in the second period instead of the first one, assuming that c will only be matched in one of the two periods, then if there is another child $c' \in C_t$ with a lower priority for a daycare d than c, and if c is matched to d while c' remains unassigned in period t, then the CG treatment will prioritize c' over c for all daycares, including daycare d, and it may easily happen that c' is matched to d and c is unassigned in period 2. This causes child c to have dynamic justified envy, but it is precisely the static fairness of the mechanism that implies that c is matched to daycare d in period 1 instead of c', since c' has a lower priority at d than c. So a higher priority for a daycare may cause a child to be matched in period 1, but subsequently become unassigned in period 2 due to the childcare guarantee policy. This causes justified envy if the child prefers being matched in the second period instead of the first one.

Theorem 11. *There is no dynamically fair mechanism under the CG treatment.*

Proof. Let $C_{t-1} = \{c_1\}$ and $C_t = \{c_2, c_3, c_4\}$. Let $D = \{d_1, d_2\}$ and $q_{d_1} = q_{d_2} = 1$. The acceptance lists are as follows:

$$d_1 : c_1, c_4, c_3, c_2$$
$$d_2 : c_3, c_2, c_4$$

At profile $(A, \succ)$, there is only one matching that satisfies (static) fairness, as indicated below:

Period t :	**Period $t+1$:**
$d_1 : \boxed{c_1}, c_4, c_3, c_2$	$d_1 : \boxed{c_4}, c_2, c_3$
$d_2 : \boxed{c_3}, c_2, c_4$	$d_2 : \boxed{c_2}, c_4, c_3$

This matching satisfies fairness because no child has justified envy in either period t or in period $t+1$, based on the daycare priority orderings, while any other matching would violate at least one child's priorities. Let's denote this matching by μ.

Assume that the children have the following strict preference orderings for two-period outcomes, where the boxes show whether they were matched in the two periods in matching μ (assume that c_1 was unmatched in period $t-1$):

$(1,1)\ P_{c_1}\ \boxed{(0,1)}\ P_{c_1}\ (1,0)\ P_{c_1}\ (0,0)$
$(1,1)\ P_{c_2}\ \boxed{(0,1)}\ P_{c_2}\ (1,0)\ P_{c_2}\ (0,0)$
$(1,1)\ P_{c_3}\ (0,1)\ P_{c_3}\ \boxed{(1,0)}\ P_{c_3}\ (0,0)$
$(1,1)\ P_{c_4}\ (1,0)\ P_{c_4}\ \boxed{(0,1)}\ P_{c_4}\ (0,0)$

This demonstrates that (static) fairness is incompatible with dynamic fairness under the CG treatment. The CG treatment reverses priorities for children when we move from period t to period $t+1$ such that previously matched children become unassigned in period $t+1$. In this particular matching μ, priorities are reversed for children c_3 versus c_2. This leads to c_2 being assigned in period $t+1$ due to the CG treatment, while c_3 is unassigned in period $t+1$. Since $(0,1)\ P_{c_3}\ (1,0)$ and $c_3 \succ_{d_2} c_2$ according to the original priorities, child c_3 has dynamic justified envy, which occurs due to the reversal of priorities according to the CG treatment. This proves that static fairness is incompatible with dynamic fairness under the CG treatment. □

Theorem 12. *A SAFE-CG mechanism is not dynamically fair, not dynamically Pareto-efficient and not dynamically strategyproof.*

Proof. The SAFE mechanism in a static model is a sequential mechanism that selects daycares sequentially and picks the highest priority child on daycares' acceptance lists, and therefore it is a fair mechanism (Chapter 2). Hence, by Theorem 11, the SAFE-CG mechanism is not dynamically fair. Finally, as shown by the SAFE-CG example in section 3.5.3, the SAFE-CG mechanism is neither dynamically Pareto-efficient nor dynamically strategyproof. □

However, we can prove that under the CG treatment the dynamic versions of Pareto-efficiency and strategyproofness are incompatible with the static properties of assigning a maximum and fair matching in each period.

Theorem 13. *Under the CG treatment, if a mechanism is maximum and fair then it is neither dynamically Pareto-efficient nor dynamically strategyproof.*

Proof. Let $C_{t-1} = \{c_1\}$ and $C_t = \{c_2, c_3, c_4\}$. Let $D = \{d_1, d_2\}$ and $q_{d_1} = q_{d_2} = 1$.

The acceptance lists are as follows:

$$d_1 : c_1, c_3, c_2, c_4$$
$$d_2 : c_3, c_4$$

At profile $(A, \succ)$, the only maximum and fair matching under the CG treatment is the following:

Period t :	**Period $t+1$:**
d_1 : $\boxed{c_1}, c_3, c_2, c_4$	d_1 : $\boxed{c_2}, c_4, c_3$
d_2 : $\boxed{c_3}, c_4$	d_2 : $\boxed{c_4}, c_3$

Any other maximum matching at this profile $(A, \succ)$ is unfair since at least one child would have justified envy otherwise in either period. Let's denote this matching by μ.

Assume that the children have the following strict preference orderings for two-period outcomes, where the boxes show whether they were matched in the two periods (assume that c_1 was unmatched in period $t-1$):

$(1,1)\ P_{c_1}\ \boxed{(0,1)}\ P_{c_1}\ (1,0)\ P_{c_1}\ (0,0)$
$(1,1)\ P_{c_2}\ (1,0)\ P_{c_2}\ \boxed{(0,1)}\ P_{c_2}\ (0,0)$
$(1,1)\ P_{c_3}\ (0,1)\ P_{c_3}\ \boxed{(1,0)}\ P_{c_3}\ (0,0)$
$(1,1)\ P_{c_4}\ (1,0)\ P_{c_4}\ \boxed{(0,1)}\ P_{c_4}\ (0,0)$

Note that matching μ is not dynamically Pareto-efficient since $(0,1)\ P_{c_3}\ (1,0)$ and $(1,0)\ P_{c_4}\ (0,1)$. Specifically, the matching $\hat{\mu}$ such that $\hat{\mu}^t_{c_1} = d_1$, $\hat{\mu}^t_{c_4} = d_2$, $\hat{\mu}^{t+1}_{c_2} = d_1$, $\hat{\mu}^{t+1}_{c_3} = d_2$ dynamically Pareto-dominates μ. Thus, it follows that a mechanism that is maximum and fair is not dynamically Pareto-efficient.

Now consider the preference profile where child c_3 reports $A'_{c_3} = \{d_1\}$ instead of $A_{c_3} = \{d_1, d_2\}$, and assume that all other children report the same as at A. Let $A' = (A'_{c_3}, A_{-c_3})$. At $(A', \succ)$ consider matching $\bar{\mu}$ as follows:

Period t :	**Period $t+1$:**
d_1 : $\boxed{c_1}, c_3, c_2, c_4$	d_1 : $\boxed{c_3}, c_2, c_4$
d_2 : $\boxed{c_4}$	d_2 : $\boxed{c_4}$

Matching $\bar{\mu}$ is the only maximum and fair matching at $(A', \succ)$. In the strict preference orderings over two-period outcomes this matching corresponds to the following:

$(1,1)\ P_{c_1}\ \boxed{(0,1)}\ P_{c_1}\ (1,0)\ P_{c_1}\ (0,0)$
$(1,1)\ P_{c_2}\ (1,0)\ P_{c_2}\ (0,1)\ P_{c_2}\ \boxed{(0,0)}$
$(1,1)\ P_{c_3}\ \boxed{(0,1)}\ P_{c_3}\ (1,0)\ P_{c_3}\ (0,0)$

$\boxed{(1,1)}\ P_{c_4}\ (1,0)\ P_{c_4}\ (0,1)\ P_{c_4}\ (0,0)$

Now it is easy to see that the mechanism is not dynamically strategyproof. In particular, note that child c_3 can manipulate: instead of $\mu_{c_3} = (d_2, 0)$, c_3 receives $\bar{\mu}_{c_3} = (0, d_1)$. Given that $(0,1)\ P_{c_3}\ (1,0)$, $\bar{\mu}_{c_3}$ makes c_3 better off than μ_{c_3}. □

The SAFE mechanism in a static model ensures a maximum matching at each profile (Chapter 2). The SAFE-CG mechanism selects a maximum matching throughout the dynamic assignment process in each period and guarantees a maximum matching in each period. Thus, since the SAFE-CG mechanism is also fair (in the static sense), Theorem 13 implies that the SAFE-CG mechanism cannot be either dynamically Pareto-efficient or dynamically strategyproof, confirming these results in Theorem 12. Consequently, if we insist on the fundamental static properties of maximum size and fairness, the SAFE-CG mechanism is as good as it gets, due to Theorems 11 and 13.

3.8 Conclusion

In a dynamic matching problem with dichotomous preferences of children and strict priorities of daycares, we analyze the properties of dynamic SAFE mechanisms, namely, the SAFE-HD and SAFE-CG mechanisms, under the history-dependent and childcare guarantee treatments respectively. These are the first dynamic mechanisms that are proposed for dichotomous preferences. Dynamic SAFE mechanisms are sequential mechanisms that allow daycares in each period to choose their highest-priority child according to a specific sequence of daycares among all children who find this daycare acceptable. The dynamic SAFE mechanisms always produce a maximum matching and do not violate the priorities of children in the different periods. Thus, the proposed SAFE-HD and SAFE-CG mechanism are both maximum and fair in each period. However, while static and dynamic fairness are compatible under the history-dependent treatment, we show that they are incompatible under the childcare guarantee treatment. Thus, while the SAFE-HD mechanism is dynamically fair, this means that the fair SAFE-CG mechanism is not dynamically fair.

We also show that the SAFE-HD mechanism is dynamically Pareto-efficient and dynamically strategyproof. By contrast, the SAFE-CG mechanism is neither dynamically Pareto-efficent nor dynamically strategyproof. Under the SAFE-CG mechanism, a matched child can manipulate by sacrificing her first-period assignment to be assigned in the second period. This is an intuitive consequence of the childcare guarantee principle and seems unavoidable. We also show that under the childcare guarantee treatment a maximum and fair mechanism, such as the SAFE-CG mechanism, cannot satisfy either dynamic Pareto-efficiency or dynamic strategyproofness.

Compared to the previous results in the literature, mechanisms proposed for strict preferences in a dynamic overlapping matching environment do not have all the nice properties of the SAFE-HD mechanism, and impossibility results can be established with strict preferences even under the history-dependent treatment. Therefore, the informationally simple dichotomous preferences may provide a good alternative to eliciting strict preference orderings, especially in situations where strict preferences are less prominent, such as in the case of significant shortages, or less relevant, such as when preferences are driven by eligibility or compatibility.

If a centralized matching system is to achieve all the desired properties in a dynamic overlapping matching environment with dichotomous preferences, our results imply that the SAFE-HD mechanism should be applied when the history-dependent updating policy is deemed appropriate. Otherwise, if policy-makers wish to prioritize initially unmatched children for their second-period daycare assignment, no mechanism can satisfy both the static and dynamic properties that the SAFE-HD mechanism possesses. Therefore, despite its lack of desirable dynamic properties, the SAFE-CG mechanism represents a good solution if a childcare guarantee policy is to be enforced, since a compromise will be inevitable in that case. The contrast between the HD and CG results suggests a policy objective: encourage the use of the HD treatment and reduce the relevance of the CG treatment. Since HD ensures continuity across periods and CG compensates for missing first-period assignments, reducing shortages, and hence unmatched children in their first period, would advance this goal.

Chapter 4

Constrained Matching under Dichotomous Agent Preferences

4.1 Introduction

We study a one-sided matching problem where there is an institutional constraint. In many real-world school choice problems, applicants are required to submit preference lists of limited length. For instance, students in Boston could rank at most five schools until 2006. In addition, each year more than $90,000$ students were assigned to roughly 500 school programs in New York City, where parents were allowed to report only 12 school programs. Similar restrictions apply to higher-education admissions: students in Spain and Hungary may submit preference lists containing at most 8 and 4 academic programs, respectively. Such institutional constraints on preference reporting have been identified as a potential source of concern in the design of matching mechanisms (Haeringer and Klijn, 2009).

The matching literature typically assumes that participants can report their preferences truthfully. Under institutional constraints on preference submission, however, truthful reporting may be risky, as it can increase the likelihood of assignment to a less preferred school or even result in non-assignment. Consequently, students may have incentives to strategically change their reported preferences by reordering schools relative to their true preferences or by omitting highly preferred options in favor of safer, less desirable alternatives (Nesterov, Rospuskova, and Rubtcova, 2024). We refer to such strategic behaviors as manipulations and say that a mechanism is manipulable if a student can obtain an assignment to a more preferred school through misreporting than under truthful reporting. From a policy perspective, the prevalence of strategic behavior is particularly problematic. In earlier admissions systems, both policymakers and parents expressed concerns that strategic ability could outweigh academic merit in determining outcomes (Chen and Kesten, 2017). Manipulability thus elevated strategic sophistication to a central role in students' decision-making, contributing to inefficient and inequitable matchings, and was widely perceived as an undesirable feature of the allocation process (Bonkoungou and Nesterov, 2021).

As a result, many countries—including the United States, England, and China—have undertaken major reforms of their admissions systems, replacing older mechanisms with redesigned ones, with concerns about manipulability often cited as a key motivation for these changes (Bonkoungou and Nesterov, 2021; Pathak and Sönmez, 2013). In parallel, the literature has developed several approaches to measuring manipulability under preference constraints. Pathak–Sönmez (PS) manipulability compares mechanisms according to the set of preference profiles at which they are susceptible to profitable manipulation (Pathak and Sönmez, 2013). Bonkoungou–Nesterov (BN) manipulability instead evaluates vulnerability by identifying the set of schools that become strategically accessible to students under each mechanism (Bonkoungou and Nesterov, 2021). Arribillaga and Massó (2016) propose an alternative notion—AM manipulability—defined in terms of the set of matching problems for which truthful reporting constitutes a dominant strategy for students.

Concerns about fairness played a central role in motivating these policy changes. A prominent example is the major admissions reform implemented in England in 2007, which applied to 146 local school admissions systems. Among other provisions, the reform prohibited the practice of assigning admission priority based on the relative position of a school in a student's submitted preference list, commonly referred to as the first-preference-first principle. Under this rule, students who ranked a school higher received priority over those who ranked it lower. Prior to the reform, this principle was employed by up to one-third of schools in England (Bonkoungou and Nesterov, 2025). In 2009, education authorities in Chicago reformed the Selective High School admissions system by replacing the Boston mechanism, which applied the first-preference-first principle at each school. This change was motivated by concerns that the mechanism could lead to the rejection of high-scoring students solely because of the order in which schools were listed on their preference forms (Pathak and Sönmez, 2013). The same Boston mechanism has also been used in college admissions across several provinces in China, where it generated similar concerns and complaints regarding the fairness of assignment outcomes (Bonkoungou and Nesterov, 2025).

Ayoade and Pápai (2023) propose the Preference Rank Partitioned (PRP) rules, which extend the classic Deferred Acceptance framework by incorporating applicants' preference rankings into schools' selection decisions. When school priorities play only a minor role, often serving as tie-breakers only, the use of preference rank classes improves efficiency because assignments depend largely on students' stated preferences, similarly to the Boston mechanism. Conversely, when school priorities carry greater weight, the influence of preference rankings diminishes, and matchings tend to be more stable, as there are fewer instances of justified envy. In 2003, one-third of the students who participated in the Chinese college admission process were placed in colleges that were considerably less selective than their grades would have justified (Wu and Zhong, 2020). These outcomes highlight persistent fairness concerns associated with the older mechanisms. In particular, such systems can produce matchings that include so-called blocking students: individuals who are denied admission to a school despite the existence of an available seat that has been assigned to a student with lower grades or priority. Although the blocking student is entitled to that seat, the assignment system fails to allocate it properly (Bonkoungou and Nesterov, 2025).

Accordingly, concerns regarding both strategic manipulation and fairness motivated the replacement of older mechanisms with new designs (Nesterov et al., 2024). In some cases, the need for reform was particularly urgent. For instance, the Chicago Selective High School system implemented changes in the middle of its admissions cycle, raising questions about policymakers' concerns and the stakes of such midstream adjustments. Reports indicate that a primary motivation behind these reforms was the high susceptibility of the existing mechanisms to strategic manipulation (Bonkoungou and Nesterov, 2021). Similarly, the former superintendent of the Boston Public Schools observed that their mechanism should be replaced by a system that reduces incentives for gaming (Pathak and Sönmez, 2008). These observations suggest that one would expect the newly implemented mechanisms to be less prone to manipulation than their predecessors. Yet, an important question remains: did the reforms completely eliminate opportunities for strategic misreporting? Surprisingly, many of the reformed matching mechanisms continue to exhibit the same shortcomings that initially motivated policy changes. Although the reforms were largely driven by concerns over manipulability and fairness, evidence suggests that most of the newly implemented mechanisms still experience vulnerabilities in both areas (Bonkoungou and Nesterov, 2025).

In this paper, we study the manipulation and fairness properties of constrained mechanisms under dichotomous preferences. Specifically, we examine the role of SAFE mechanisms (Chapter 2) that resolve a long-standing conflict between Pareto-efficiency and fairness in the matching theory literature with strict agent preferences over objects. In addition to resolving this conflict, SAFE mechanisms provide many other desired properties in unconstrained matching problems under dichotomous preferences. Under dichotomous preferences, students need to report only which schools are acceptable for them. They partition the set of schools into two sets: a set of acceptable schools and a set of unacceptable schools, where each school in their acceptable set is welfare-wise identical to the students. We address the following question: Are constrained mechanisms less manipulable and more fair under dichotomous agent preferences? What we find, somewhat unexpectedly, is that neither of these predictions are confirmed, and our results in fact are in line with the findings of Haeringer and Klijn (2009) on the strict preference domain.

4.2 Related Work

Several approaches exist to measure and compare the manipulability of matching mechanisms. One method evaluates manipulability by identifying schools that students can access strategically (Bonkoungou and Nesterov, 2021). Another considers the set of preference profiles that are susceptible to strategic manipulation (Pathak and Sönmez, 2013). A third approach focuses on the set of matching problems for which truthful reporting constitutes a dominant strategy (Decerf and Van der Linden, 2021). Each of these concepts is meaningful, logically distinct, and provides a useful lens for assessing reforms. For example, using the preference-profile approach alone is insufficient to compare constrained versions of the First-Preference-First and Deferred Acceptance mechanisms, limiting its ability to evaluate certain reforms implemented in England (Bonkoungou and Nesterov, 2021)

Bonkoungou and Nesterov (2021) introduce BN-manipulability, which evaluates a mechanism's vulnerability by considering the set of schools that are strategically accessible to students. A school s is deemed inaccessible to student i under a mechanism φ if there exists no misreport that would allow i to obtain a seat at s. They show that, for $k > l$ and at least k schools available, the constrained Gale–Shapley mechanism GS^k is less strategically accessible—and therefore less BN-manipulable—than GS^l. Pathak and Sönmez (2013) define a preference profile as vulnerable if at least one student can benefit from misreporting. Based on this notion, a mechanism is considered more PS-manipulable than another when the set of vulnerable profiles under the first mechanism is strictly larger than under the second. For example, when students are limited to ranking at most k schools in the Deferred Acceptance mechanism (DA^k), it exhibits higher PS-manipulability than DA^{k+1}; in other words, allowing students to submit longer preference lists reduces susceptibility to strategic misreporting. Similar patterns are observed for the constrained Boston mechanism: $\mathcal{B}^k$ is more PS-manipulable than $\mathcal{B}^{k+1}$ (Decerf and Van der Linden, 2021). Decerf and Van der Linden (2021) define AM-manipulability using the concept of truthful dominant strategies. A student has a truthful dominant strategy if no preference profile exists in which misreporting could be beneficial. Following Arribillaga and Massó (2016), one mechanism is considered at least as AM-manipulable as another when the set of problems for which truthful reporting is dominant under the first mechanism is a subset of the corresponding set under the second. If this subset relation is strict for at least one student, the first mechanism is deemed more AM-manipulable. Using this framework, the constrained Boston mechanism $\mathcal{B}^k$ is more AM-manipulable than DA^k, which is in turn more AM-manipulable than DA^{k+1}. In contrast, different constrained versions of the Boston mechanism are equally AM-manipulable (Decerf and Van der Linden, 2021).

Several studies have evaluated fairness by counting the number of blocking agents or blocking pairs under different mechanisms (Eriksson and Häggström, 2008; Niederle and Roth, 2009; Roth and Xing, 1997). Many matching mechanisms generate blocking students when participants face limits on the number of schools they can rank (Bonkoungou and Nesterov, 2025). In such systems, each student is permitted to submit a preference list containing only a fixed number of schools (Pathak and Sönmez, 2013). The stability of Nash equilibrium outcomes under constrained mechanisms has also been analyzed. Romero-Medina (1998) and Haeringer and Klijn (2009) examined the games induced by the Deferred Acceptance (DA) and Boston mechanisms when students face ranking constraints. A key insight from this work is that, under constraints, all Nash equilibrium outcomes of the Boston mechanism remain stable, whereas some equilibrium outcomes of the constrained Gale–Shapley mechanism may fail to satisfy stability conditions. Building on this, Bonkoungou and Nesterov (2025) formalize the concept of fairness with respect to students' true preferences, rather than the preferences actually reported, highlighting that fairness evaluations must consider what students genuinely value. According to them, a mechanism φ' is considered more fair by counting blocking students than another mechanism φ if two conditions hold: (i) for every matching problem, the number of blocking students under φ' is at least as low as under φ, and (ii) there exists at least one problem in which φ' has strictly fewer blocking students than φ. Following this definition, for $k > l$, the constrained Gale–Shapley mechanism GS^k is more fair by this measure than GS^l.

4.3 Model and Definitions

4.3.1 Model

There is a finite set of m schools denoted by C, and a finite set of n students denoted by I. Each school has unit capacity.[1] Given dichotomous preferences, let $A_i \subseteq C$ denote the set of *acceptable* schools for student $i \in I$, which we call i's *acceptance set* in short. Then $C \setminus A_i$ is the set of *unacceptable* schools for i. A profile of acceptance sets is $A = (A_1, \ldots, A_n)$. Let $\mathcal{A}$ denote the set of acceptance set profiles. Each school $c \in C$ has a strict priority order $\succ_c$ over I. A profile of priority orders is $\succ = (\succ_1, \ldots, \succ_m)$. Let Σ denote the set of the priority order profiles. We will refer to a pair of an acceptance set profile and a priority order profile $(A, \succ)$ as a *profile*.

A *matching* is an injective function $\mu : I \to C$, which specifies which student is matched to which school such that no more than one student is matched to a school. Let $\mathcal{M}$ denote the set of matchings and let μ_i be student i's assignment. If a student i is unmatched then we write $\mu_i = 0$, and if a school c is unmatched then we write $\mu_c^{-1} = 0$. A *mechanism* f is a function $\mathcal{A} \times \Sigma \to \mathcal{M}$ which assigns a matching to each profile $(A, \succ)$.

4.3.2 Axioms

All the axioms in this section are either standard in the literature or are variations of standard axioms adapted to dichotomous preferences. The definitions of axioms that are defined for individual matchings are extended to mechanisms in the usual manner.

A matching μ is **individually rational** if, for all $i \in I$, $\mu_i \in A_i \cup \emptyset$, that is, no student is assigned to a school that is unacceptable to the student. A matching *mechanism* is *individually rational* if it assigns an individually rational matching to each profile.

A matching is **maximum (size)** if there is no other matching that matches more students to schools: μ is maximum if for all $\nu \in \mathcal{M},\ |\mu| \geq |\nu|$, where the cardinality of a matching denotes the number of matched students in the matching.

A matching is **Pareto-efficient** if it is not Pareto-dominated. A matching μ is *Pareto-dominated* if there is a matching μ' which at least one student strictly prefers to μ, while none of the students strictly prefer μ to μ'. In our model a matching is Pareto-efficient

[1] This is without loss of generality, as explained in previous chapters, so our results also apply to many-to-one matching applications.

if and only if it is a maximum matching (see Chapter 2). Moreover, note that Pareto-efficiency implies individual rationality, since a matching that includes any student assigned to an unacceptable school is Pareto-dominated by the matching which leaves such a student unmatched but otherwise makes the same matchings.

A matching is **fair** if there is no student-school pair such that the student is unassigned and finds the school acceptable, and there is another student matched to this school who has a lower priority for this school than the unassigned student. Formally, a matching $\mu \in \mathcal{M}$ is *fair* at profile $(A, \succ)$ if there is no student-school pair (i, c) such that the following three conditions hold:

i) $\mu_i = 0$

ii) $c \in A_i$

iii) $\mu_c^{-1} = j$ such that $i \succ_c j$

A *mechanism* is *fair* if it assigns a fair matching to each profile.

We also aim for *strategyproofness*: no student i has an incentive to change its submitted set of acceptable schools A_i to obtain a strictly better outcome. Under dichotomous preferences, a student has an incentive to misreport only when she is unmatched or matched to an unacceptable school.

A mechanism f is **strategyproof** if, for all profiles $(A, \succ)$ and for all $i \in I$ the following hold:

1. $f_i(A, \succ) \notin A_i$, there is no $\hat{A}_i \subseteq C$ such that $f_i((\hat{A}_i, A_{-i}), \succ) \in A_i$;

2. $f_i(A, \succ) \in C \setminus A_i$, there is no $\hat{A}_i \subseteq C$ such that $f_i((\hat{A}_i, A_{-i}), \succ) = 0$.

If f is individually rational, this simplifies to the following: for all profiles $(A, \succ)$, for all $i \in I$ such that $f_i(A, \succ) = 0$, and for all $A'_i \subseteq C$, $f_i((A'_i, A_{-i}), \succ) = 0$.

4.4 Incentives Under Constrained Matching

4.4.1 Constraints

We will say that a matching mechanism is ***k*-constrained** if students are allowed to submit at most k acceptable schools: for all $i \in I$, the reported acceptance set $Q_i^k \subset C$ is such that $|Q_i^k| \leq k$. We will denote an acceptance profile in which each agent i reports Q_i^k such that $|Q_i^k| \leq k$ by Q^k.

A reported acceptance set Q_i^k of student $i \in I$ is *k-constrained truthful*, given that the true (unconstrained) acceptance set for i is A_i, if $|Q_i^k| = min\{k, |A_i|\}$ and $Q_i^k \subseteq A_i$. We will say that a k-constrained mechanism is k-constrained strategyproof if, for all profiles

$(Q^k, \succ)$ such that $|Q^k_j| \leq k$ for all $j \in I$, for all $i \in I$ and k-constrained truthful report of i the following hold:

1. $f_i(Q^k, \succ) \notin A_i$, there is no $\hat{Q}^k_i \subseteq C$ such that $|\hat{Q}^k_i| \leq k$ and $f_i((\hat{Q}^k_i, A_{-i}), \succ) \in A_i$;
2. $f_i(Q^k, \succ) \in C \backslash A_i$, there is no $\hat{Q}^k_i \subseteq C$ such that $|\hat{Q}^k_i| \leq k$ and $f_i((\hat{Q}^k_i, A_{-i}), \succ) = 0$.

Note that if $k = m$ then k-constrained strategyproofness is equivalent to strategyproofness, while the case of $k < m$ requires that k-constrained truthful reports cannot be manipulated. We will refer to $\hat{Q}^k_i$ in the above definition as a *manipulation strategy.*

We will call k a **binding constraint** if $k < \min \{n, m\}$. It is clear why we require $k < m$, as in this case students cannot list all the schools. We also require $k < n$, since $k \geq n$ allows for each of the n students being matched by an individually rational matching even if $k < m$.

4.4.2 Constrained Strategyproofness and Truthful Manipulation

We show first an impossibility theorem when there is a binding constraint; namely, that k-constrained strategyproofness, individual rationality and maximum size cannot be reconciled under dichotomous preferences. This is in sharp contrast to the unconstrained case, since the unconstrained SAFE mechanism satisfies all three axioms.

Theorem 14. *If k is a binding constraint, there exists no k-constrained matching mechanism that is individually rational, maximum and k-constrained strategyproof.*

Proof. Suppose for a contradiction that k is a binding constraint and φ^k is a k-constrained matching mechanism that is individually rational, maximum and k-constrained strategyproof. Let $(A, \succ)$ and $(Q^k, \succ)$ be as follows: for all $j \in I$, $A_j = C$ and each student reports exactly the same k schools as acceptable in Q^k: for all $j, j' \in I$, $Q^k_j = Q^k_{j'}$, where $|Q^k_j| = k$. Since $k < m$ and φ^k is individually rational, at least one school c remains unmatched at $\varphi^k(Q^k, \succ)$. Note that $c \notin Q^k_j$ for any $j \in I$. Since $k < n$ and φ^k is individually rational, φ^k matches fewer than n students. Thus, there exists $i \in N$ such that

$$\varphi^k_i(Q^k, \succ) = 0.$$

Now consider a different reported acceptance set for student i, namely, $\hat{Q}^k_i = \{c\}$. Then, since φ^k is individually rational and maximum

$$\varphi^k_i((\hat{Q}^k_i, Q^k_{-i}), \succ) = c.$$

Note that $c \in A_i \setminus Q_i^k$ since each agent finds each school acceptable truthfully and $c \notin Q_i^k$. This implies that i can manipulate at profile $(Q^k, \succ)$ which contradicts the k-constrained strategyproofness of φ^k. Therefore, if k is a binding constraint, under dichotomous preferences an individually rational and maximum k-constrained mechanism cannot be k-constrained strategyproof. □

Given that the SAFE mechanism is individually rational and maximum, the following is an immediate corollary of Theorem 14.

Corollary 5. *The k-constrained SAFE mechanism ψ^k with binding constraint k is not k-constrained strategyproof.*

A **truthful manipulation** is a manipulation strategy which does not report any unacceptable school as acceptable. A truthful manipulation $\hat{Q}_i^k$ for agent i with acceptance set A_i is such that $\hat{Q}_i^k \subseteq A_i$. We will say that a k-constrained mechanism is **truthfully manipulable** if, for all true profiles $(A, \succ)$, reported constrained profiles $(Q^k, \succ)$ and student $i \in I$, whenever i can manipulate at $(Q^k, \succ)$, there exists a truthful manipulation strategy for i at $(Q^k, \succ)$. The next theorem shows that, despite Theorem 14, unconstrained strategyproofness does have some positive implications for k-constrained mechanisms. Namely, manipulation does not have to resort to reporting unacceptable schools as acceptable, implying that the opportunity to manipulate arises due to the constraint, not due to the general design of the mechanism.

Theorem 15. *A k-constrained mechanism that is unconstrained strategyproof is truthfully manipulable.*

Proof. Let φ^k be a k-constrained mechanism that is unconstrained strategyproof. Assume that there exist a true profile $(A, \succ)$, a reported constrained profile $(Q^k, \succ)$ and student $i \in I$, such that i can manipulate $(Q^k, \succ)$ with a manipulation strategy $\tilde{Q}_i^k$, where $\varphi_i^k((\tilde{Q}_i^k, Q_{-i}^k), \succ) = c$ for some $c \in A_i$. Let $\hat{Q}_i^k \subseteq A_i$ be such that $c \in \hat{Q}_i^k$ and $|\hat{Q}_i^k| = \min\{k, A_i\}$. Then, by the strategyproofness of φ, given that $c \in \hat{Q}_i^k$, $\varphi_i^k((\hat{Q}_i^k, Q_{-i}^k), \succ) \in A_i$. Since $\hat{Q}_i^k$ is a truthful manipulation strategy, it follows that φ^k is truthfully manipulable. □

Given that the SAFE mechanism is unconstrained strategyproof, the following is an immediate corollary of Theorem 6.

Corollary 6. *The k-constrained SAFE mechanism ψ^k is truthfully manipulable.*

4.4.3 Nash Equilibria

Given Theorem 14, we study the Nash equilibria of individually rational and maximum mechanisms next.

Nash Equilibrium in a Constrained Revelation Game $\Gamma^{\varphi}(A^k, \succ)$ of mechanism φ: An acceptance profile under constraint k, A^k, together with strict school priorities $\succ$, is a **Nash equilibrium** of the φ-revelation game $\Gamma^{\varphi}(\bar{A}^k, \bar{\succ})$, where $(\bar{A}^k, \bar{\succ})$ is the true acceptance profile, if for all $i \in N$ and all reports $\hat{A}_i^k$,

$$\varphi_i((A_i^k, A_{-i}^k), \succ) \; R_i \; \varphi_i((\hat{A}_i^k, A_{-i}^k), \succ).$$

Under dichotomous preferences, a Nash equilibrium of a revelation game indicates that there is no student who can successfully deviate from the NE outcome by gaining a better outcome.

Theorem 16. *For any constraint k, a Nash equilibrium outcome $\varphi^k(A^k, \succ)$ of a maximum and individually rational mechanism φ is fair.*

Proof. Let φ be a maximum and individually rational mechanism and fix k. Then φ^k is also maximum and individually rational. Suppose $(A^k, \succ)$ is a Nash equilibrium of Γ^{φ^k}, and there is $i \in I$ who has justified envy at $(A^k, \succ)$. Student i's justified envy implies that i is unmatched at this matching and has at least one acceptable school c such that $c \notin A_i^k$ and $i \succ_c \mu_c(A^k, \succ)$. Consider that student i reports $\hat{A}_i^k = \{c\}$, where other students' reports are the same as before. It makes student j unmatched, where $\mu_c(A^k, \succ) = j$). Because φ^k is a maximum mechanism, school c is matched to a student. Suppose for a contradiction that c is matched to student j', instead of student i. It implies that either $j' \succ_c i$ or j' is on another school c''s acceptance list. The former is a contradiction, since j was the highest priority student on c's acceptance list at A^k, and now it is the student i. The latter scenario indicates an individually irrational matching. Thus, being a maximum and individually rational matching mechanism, φ^k matches school c with student i at $\hat{A}_i^k$. This is a contradiction to the assumption that $(A^k, \succ)$ is a Nash equilibrium. Hence, it is proved that at a given constraint k, any Nash equilibrium of a maximum and individually rational $\Gamma^{\varphi}(A^k, \succ)$ is fair. Note that this proof is independent of the degree of constraint k. Therefore, regardless of the constraint, any Nash equilibrium $(A^k, \succ)$ of a maximum and individually rational mechanism φ^k is fair. □

Given that the SAFE mechanism is maximum and individually rational, the following is an immediate corollary of Theorem 6.

Corollary 7. *A Nash equilibrium of the revelation game $\Gamma^{SAFE}(A^k, \succ)$ is fair for any constraint k.*

4.5 Manipulation and Fairness Comparisons

4.5.1 Manipulation Comparison Criteria

When students report strict preferences, and the Deferred Acceptance (DA) algorithm is run with this preference and strict priorities, we call the algorithm DA^S.

Similarly, when DA is run with students' truthful preferences and schools strict priorities, we call the algorithm DA^T. We use DA^D when DA is run with students' dichotomous preferences where tie-breaking corresponds to the submitted strict orderings of preferences under P_i, S_i^k or Q_i^k.

We measure strategic manipulation by two separate terms: (a) in terms of the number of manipulating students, and (b) in terms of the number of strategically accessible schools.

Manipulating students are those who can gain by misrepresenting their preferences. In contrast, strategically accessible schools are those that students can achieve by misrepresenting their preferences, and these are preferred outcomes to the existing outcomes.

Let T^k be a k-constrained preference profile corresponding to P if, for all $i \in N$, T_i^k is one of the following:

i. S_i^k- top k-truncation of P_i (straightforward strict preferences);

ii. Q_i^k- at most k-acceptable schools ordered according to P_i (Truthful strict preferences);

iii. A_i^k- at most k acceptable schools where each acceptable school is welfare-wise identical to the student i (dichotomous preferences).

Manipulation comparison in terms of the number of manipulating students: Let $\varphi^k(T^k)$ be the matching assigned to reported k-constrained preference profiles T^k by the k-constrained mechanism φ^k. A student $i \in I$ is called a manipulating student who can manipulate $\varphi^k(T^k)$ if she can achieve a better outcome compared to her existing outcome by misrepresentation of her true preferences. Given constraint $k \geq 1$ and k-constrained mechanism φ^k, for all preference profiles P, let $M(\varphi^k(T^k), P)$ be the number of students who can manipulate $\varphi^k(T^k)$ under mechanism φ^k at preference profile P, given that the reported preference profile is T^k at P such that T^k is a k-constrained preference profile corresponding to P. Let $\hat{T}^k$ be an alternative to T^k and is said to be consistent with T^k if, for all $i \in I$, $\hat{T}_i^k$ and T_i^k report the same set of acceptable schools.

φ^k is weakly less $M-manipulable$ under T^k than under $\hat{T}^k$ if, for all P, $M(\varphi^k(T^k), P) \leq M(\varphi^k(\hat{T}^k), P)$ where (T^k, P) is consistent with $(\hat{T}^k, P)$ for all P.

Manipulation comparison in terms of the number of strategically accessible schools: Strategically accessible schools are those that students can achieve by misrepresenting their true preferences, and these schools are preferred outcomes to the existing outcomes. Given constraint $k \geq 1$ and k-constrained mechanism φ^k, for all preference profiles P, let $G(\varphi^k(T^k), P)$ be the number of strategically accessible schools under mechanism φ^k at preference profile P, given that the reported preference profile is T^k at P such that T^k is a k-constrained preference profile corresponding to P.

φ^k is weakly less $G-manipulable$ under T^k than under $\hat{T}^k$ if, for all P, $G(\varphi^k(T^k), P) \leq G(\varphi^k(\hat{T}^k), P)$ where (T^k, P) is consistent with $(\hat{T}^k, P)$ for all P.

Fairness comparison: Let's say, student i is prioritized over student j at school c. If student j is assigned to c, while student i is unassigned or assigned to a less-preferred school, i experiences justified envy. Given constraint $k \geq 1$ and k-constrained mechanism φ^k, for all preference profiles P, let $E(\varphi^k(T^k), P)$ be the number of students who has justified envy toward $\varphi^k(T^k)$ under mechanism φ^k at preference profile P, given that the reported preference profile is T^k at P such that T^k is a k-constrained preference profile corresponding to P.

φ^k is weakly more fair under T^k than under $\hat{T}^k$ if, for all P, $E(\varphi^k(T^k), P) \leq E(\varphi^k(\hat{T}^k), P)$ where (T^k, P) is consistent with $(\hat{T}^k, P)$ for all P.

4.5.2 Manipulation Comparison of Constrained SAFE Mechanisms

For all preference profiles $(A, \succ)$, let $M(\varphi^{\bar{\pi}}(A, \succ))$ be the number of manipulating students under the SAFE mechanism $\varphi^{\bar{\pi}}$ at preference profile $(A, \succ)$.

On the contrary, for all preference profiles $(A, \succ)$, let $G(\varphi^{\bar{\pi}}(A, \succ))$ be the number of strategically accessible schools under the SAFE mechanism $\varphi^{\bar{\pi}}$ at preference profile $(A, \succ)$.

A constrained SAFE mechanism $\varphi^{\bar{\pi}}$ is weakly less $M-manipulable$ under constraint l than under constraint k if, $l > k$ and for all profiles $(A, \succ)$, $M(\varphi^{\bar{\pi}}(A^l, \succ)) \leq M(\varphi^{\bar{\pi}}(A^k, \succ))$ where A^k and A^l are consistent with A such that $A^k \subset A^l \subset A$.

On the contrary, a constrained SAFE mechanism $\varphi^{\bar{\pi}}$ is weakly less $G - manipulable$ under constraint l than under constraint k if, $l > k$ and for all profiles $(A, \succ)$, $G(\varphi^{\bar{\pi}}(A^l, \succ)) \leq G(\varphi^{\bar{\pi}}(A^k, \succ))$ where A^k and A^l are consistent with A such that $A^k \subset A^l \subset A$.

M-Manipulability and G-Manipulability in Constrained SAFE Mechanism: Because the domain of unrevealed set of acceptable schools under binding constraints, $A_i \backslash A_i^k$, decreases as the binding constraints changes from k to l, where $l > k$, it is expected that both the number of manipulating students and the number of strategically accessible schools at A^l is less than that at A^k. However, the following counterexample (Example 1) reveals that we cannot compare M-manipulability and G-manipulability in constraint SAFE mechanism for $l > k$:

Example 7. *Assume five students i_1, i_2, i_3, i_4, i_5 and four schools c_1, c_2, c_3, c_4, where all students accept all four schools. Let the constraints be $k = 1$ and $l = 2$.*

Let school priorities be the following:

$$
\begin{aligned}
\succ_{c_1}&: \quad i_4, i_5, i_1, i_2, i_3 \\
\succ_{c_2}&: \quad i_4, i_1, i_2, i_3, i_5 \\
\succ_{c_3}&: \quad i_1, i_2, i_3, i_4, i_5 \\
\succ_{c_4}&: \quad i_1, i_2, i_3, i_4, i_5
\end{aligned}
$$

Consider $A_{i_1}^k = \{c_3\}$, $A_{i_2}^k = \{c_4\}$, $A_{i_3}^k = \{c_3\}$, $A_{i_4}^k = \{c_2\}$, and $A_{i_5}^k = \{c_1\}$.

The acceptance lists and the SAFE outcomes at constraint k are ($\pi = i_1, i_2, i_3, i_4, i_5$):

$$
\begin{aligned}
\succ_{c_1}&: \quad \boxed{i_5} \\
\succ_{c_2}&: \quad \boxed{i_4} \\
\succ_{c_3}&: \quad \boxed{i_1}, i_3 \\
\succ_{c_4}&: \quad \boxed{i_2}
\end{aligned}
$$

The $SAFE^k$ leaves school c_3 unmatched. However, c_3 is not strategically accessible via manipulation, and thus there are no strategically accessible schools.

Let us compare this outcome with that under $SAFE^l$. Consider $A_{i_1}^l = \{c_1, c_3\}$, $A_{i_2}^l = \{c_2, c_4\}$, $A_{i_3}^l = \{c_1, c_3\}$, $A_{i_4}^l = \{c_1, c_2\}$, and $A_{i_5}^l = \{c_1, c_4\}$. Note that these constrained acceptance sets are consistent with A_i^k in the sense that for each student i, $A_i^k \subset A_i^l$.

The acceptance lists and the SAFE outcomes at constraint l are ($\pi = i_1, i_2, i_3, i_4, i_5$):

$$\begin{aligned}
\succ_{c_1}&: \ \boxed{i_4},\ i_5,\ i_1,\ i_3 \\
\succ_{c_2}&: \ i_4,\ \boxed{i_2} \\
\succ_{c_3}&: \ \boxed{i_1},\ i_3 \\
\succ_{c_4}&: \ i_2,\ \boxed{i_5}
\end{aligned}$$

The $SAFE^l$ mechanism leaves c_3 unmatched. But now c_3 can be strategically accessed via manipulation, and in particular school c_4 becomes strategically accessible. Thus, a change in the binding constraint from $k = 1$ to $l = 2$ increases both the number of manipulating students and the number of strategically accessible schools. $\diamond$

4.5.3 Fairness Comparison of Constrained SAFE Mechanisms

For all preference profiles $(A, \succ)$, let $E(\varphi^{\bar{\pi}}(A, \succ))$ be the number of students who have justified envy under the SAFE mechanism $\varphi^{\bar{\pi}}$ at preference profile $(A, \succ)$.

A constrained SAFE mechanism $\varphi^{\bar{\pi}}$ is **weakly more fair** under constraint l than under constraint k if, $l > k$ and for all profiles $(A, \succ)$, $E(\varphi^{\bar{\pi}}(A^l, \succ)) \leq E(\varphi^{\bar{\pi}}(A^k, \succ))$ where A^k and A^l are consistent with A such that $A^k \subset A^l \subset A$.

Because the domain of unrevealed set of acceptable schools under binding constraints, $A_i \backslash A_i^k$, decreases as the binding constraints changes from k to l, where $l > k$, it is expected that the number of justified envious students at A^l is less than that at A^k. However, the counterexample above (Example 1) reveals that we cannot make a fairness comparison in constraint SAFE mechanism for $l > k$:

In Example 1, we find that the $SAFE^k$ leaves student i_3 unmatched but she does not have justified envy. On the contrary, $SAFE^l$ leaves student i_3 unmatched but she has now justified envy toward school c_4. Thus, a change in the binding constraint from $k = 1$ to $l = 2$ increases the number of students having justified envy.

4.6 Conclusion

We have shown that good incentive properties are not possible for an individually rational and maximum mechanism under binding constraints, as there is room for manipulation in this case. However, for strategyproof mechanisms the manipulation can always be carried out in a truthful manner, indicating that manipulation stems from the constraints.

We also illustrate by an example that manipulation and fairness comparisons are not possible in general for constrained SAFE mechanisms with constraints $l > k$, even if $A_i^k \subset A_i^l \subset A_i$ for all students i. However, with appropriate assumptions on how the constrained acceptance set is selected by students, it may be possible to make such comparisons. This is left for future research.

Chapter 5

Conclusion and Future Research

This thesis studies matching with dichotomous agent preferences and strict institutional rankings. Considering three settings, static, dynamic, and constrained, I identify both positive results, that is, mechanisms that achieve strong combinations of properties, and sharp limits, that is, impossibilities that persist even with this simplified preference domain.

In Chapter 2, I identify a set of mechanisms, the SAFE (equivalently, Rank-Maximal) mechanisms, which always maximize the matching size and respect the institutions' strict rankings over the agents (whether these are preferences or priorities). The SAFE mechanisms are sequential: in a predetermined order, each institution selects its highest-ranked applicant among those not yet matched and who find this institution acceptable. I also show that SAFE mechanisms are strategyproof, that is, agents have no incentive to manipulate by misrepresenting their preferences. SAFE mechanisms satisfy many other prominent and desirable properties, including individual rationality, maximum size, fairness, Pareto-efficiency on both sides, strategyproofness on both sides, and nonbossiness on both sides. They are also computable in polynomial time.

There is scope for further research concerning structured hybrid preference domains, where agents are indifferent over certain subsets but rank others strictly. In future research, I intend to explore mechanisms that combine the advantages of the dichotomous preference domain with the richness of the strict preference domain in applicable hybrid settings. Another direction for future work from Chapter 2 is to extend the SAFE mechanisms beyond strict institutional rankings. Many applications feature ties, coarse priority classes, affirmative-action type constraints, or other institutional policies that induce partial orders rather than strict rankings. Identifying when SAFE-style mechanisms and their properties can be preserved under such rankings would broaden the applicability of this approach.

Building on the novel SAFE mechanisms proposed in the previous chapter, I adapt these mechanisms to a dynamic setting in Chapter 3. I study an overlapping dynamic matching model in which agents participate in the assignment process for two periods and submit dichotomous preferences over objects. I introduce two dynamic SAFE mechanisms applicable under the history-dependent (HD) and childcare-guarantee (CG) policies. I show that a SAFE-HD mechanism is maximum, dynamically fair, dynamically Pareto-efficient, and dynamically strategyproof. By contrast, a SAFE-CG mechanism does not satisfy any of the desirable dynamic properties, while retaining strong static properties. I demonstrate that if policymakers wish to prioritize initially unmatched children for their second-period daycare assignment, a SAFE-CG mechanism, providing a maximum and fair matching in each period, is as good as possible, due to necessary tradeoffs.

Two natural extensions follow from Chapter 3. First, many real dynamic systems face uncertainty in arrivals and departures. Extending the analysis to stochastic participation would clarify robustness under uncertainty. Second, it would be useful to characterize more broadly which classes of dynamic policies admit mechanisms with full dynamic guarantees, and which necessarily force trade-offs.

In Chapter 4, I focus on constrained matching environments in which agents may report only a limited number of acceptable institutions. I show that if the constraint binds, then no mechanism can be simultaneously constrained strategyproof, individually rational, and maximum, even with dichotomous preferences. At the same time, I establish that Nash equilibria of the induced revelation game are fair for maximum and individually rational mechanisms (including SAFE mechanisms), and I demonstrate that manipulability and fairness cannot be compared in an unambiguous way across different constraint levels when SAFE mechanisms are used.

Future work suggested by Chapter 4 is to impose behavioral or informational structure on how agents choose constrained acceptance sets. With such predictable selection patterns in place, it may become possible to recover meaningful comparisons across different reporting limits.

Overall, the thesis shows that instead of merely serving as a simplification, the dichotomous preference domain provides a useful lens on feasibilities and impossibilities in matching theory. On this preference domain, some classical impossibilities are eliminated, notably in the static model and, under appropriate policies, in the dynamic model. At the same time, the thesis also demonstrates that with constrained reporting, as well as with policies in dynamic settings that prioritize previously unmatched agents, the negative results are not resolved even by the simple dichotomous preference domain.

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