Applications of Partial Differential Equations to Convex Geometry

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ABSTRACT. These notes were written following a summer mini-course given by the author at the XXIII School of Mathematics "Lluís Santaló" held at the Universidad Internacional Menéndez Pelayo (UIMP) in Santander, Spain, in 2024. The course comprised four one-hour lectures and introduced geometric equations, with particular emphasis on applications in convex geometry.

The notes start by exploring standard parametrizations of smooth convex bodies and their connection to various curvature measures. Fundamental tools such as the inverse Gauss map parametrization and the support function are introduced, resulting in the derivation of the curvature function of a convex body in spherical coordinates. These tools are employed to discuss problems of prescribing surface areas, framed as Monge-Ampère type equations and related elliptic partial differential equations on the sphere, and to present some characterizations of ellipsoids and Euclidean balls. The last part of the course turns to curvature flows as geometric parabolic partial differential equations. These equations are presented both as techniques for establishing the existence of convex bodies with prescribed curvature properties and as methods for proving geometric inequalities.

The material, intended to be accessible to students with only basic prior knowledge of differential equations, serves as an entry point to more advanced topics in geometric analysis and PDEs in convex geometry. References have been kept to a minimum, limited to those that are both essential and of broader relevance, and from which the interested reader may pursue more specialized developments.

1. Introduction

We begin by introducing some basic notions from the theory of partial differential equations and, respectively, the geometry of convex bodies. As stated in the

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abstract, we assume minimal prerequisite knowledge. Therefore, in order to highlight a few interesting directions of research within the area of geometric partial differential equations, relevant to the theory of convex bodies, we select certain essential concepts and results that will be employed throughout the text.

We will use this introduction in Section 2 to present several foundational facts about the classical Minkowski problem, which played a central role in the Brunn–Minkowski theory of convex bodies and led to many of its subsequent developments. From the point of view of partial differential equations, later developments unfold along two principal axes: elliptic partial differential equations and parabolic partial differential equations. We will illustrate these axes of development in Sections 3, 4 (for elliptic PDEs) and Section 5 (for parabolic PDEs) with some specific examples.

1.1. Partial Differential Equations at a Glance.

Definition 1.1. A **partial differential equation (PDE)** is an equation involving an unknown function $u(x_1, x_2, \dots, x_n)$ of several independent variables and its partial derivatives. A general form of a PDE of order k on some domain $\Omega \subseteq \mathbb{R}^n$ is

$$F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^k u}{\partial x_{i_1} \dots \partial x_{i_k}}\right) = 0, \quad \forall (x_1, \dots, x_n) \in \Omega.$$

A function $u \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is called a **classical solution** of a PDE in a domain Ω if u is continuously differentiable up to the required order (e.g., C^2 for second-order PDEs), and u satisfies the PDE *pointwise* at every $x \in \Omega$.

We will be mostly concerned with second-order PDEs which, in a very general form, can be described by

$$F\left(x, u, \nabla u, D^2 u\right) = 0,$$

for some function $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$, where ∇u denotes the gradient of the function u and $D^2 u$ the Hessian matrix of second derivatives of u.

To **linearize** a nonlinear second-order PDE of the form

$$F\left(x, u, \nabla u, D^2 u\right) = 0,$$

we consider a small perturbation of the solution u by introducing $u + \varepsilon v$, where v is a smooth test function and ε is a small parameter. Expanding $F(x, u + \varepsilon v, \nabla u + \varepsilon \nabla v, D^2 u + \varepsilon D^2 v)$ in ε and retaining only the terms linear in ε, we obtain the **linearized equation**:

$$L_u[v] := \sum_{i,j=1}^{n} \frac{\partial F}{\partial r_{ij}} \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial v}{\partial x_i} + \frac{\partial F}{\partial u} v = 0,$$

where $p_i = \frac{\partial u}{\partial x_i}$ and $r_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Note that the linearized equation is now in v and its coefficients will depend on u. This linear operator L_u reflects the behavior of the original nonlinear PDE near the solution u.

The **linearized form** of a second-order PDE is, essentially,

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \text{lower order terms} = 0,$$

where the matrix $A = (a_{ij})$ determines the type of the PDE. If the equation is linear, then it is already in the later form. Based on the eigenvalues of A, a second order PDE is said to be

- **Elliptic** if all eigenvalues of A are of the same sign (e.g., Laplace's equation).
- **Hyperbolic** if one eigenvalue has opposite sign to the rest (e.g., wave equation).
- **Parabolic** if, at least, one eigenvalue is zero, and the others have the same sign (e.g., heat equation).
- **Degenerate** if the classification changes depending on the point x, or if the matrix A loses rank.

Example: Monge–Ampère Equation in n Dimensions. The Monge–Ampère equation is a fully nonlinear second-order PDE of the form:

$$\det\left(D^2u(x)\right) = f(x, u(x), \nabla u(x)),$$

where, again, $D^2u(x)$ is the Hessian matrix:

$$D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}.$$

This equation is one of the most important partial differential equations that arises in differential geometry, optimal transport, and convex analysis. It is easy to see that the linearized Monge–Ampère operator L_u applied to v is

$$L_u[v] = \operatorname{trace}(UD^2v),$$

where $U = (\operatorname{cof} D^2u)$ is the matrix of cofactors of the Hessian matrix D^2u. Therefore, the equation is elliptic when u is convex and $f > 0$. □

Second Order Parabolic PDE of Heat-like (diffusive) Behavior. Since one of the classes of partial differential equations considered in this paper are evolution equations, it is important to note further that a second-order linear PDE is *parabolic* on $\Omega \subseteq \mathbb{R}^n$ if it can be written in the form

$$u_t = \sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2 u}{\partial x_i \partial x_j} \;+\; \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial u}{\partial x_i} \;+\; c(x,t)\,u \;+\; f(x,t),$$

where the matrix

$$A(x,t) = \left(a_{ij}(x,t)\right)_{ij}$$

is symmetric and positive definite (or sometimes positive semidefinite) for all $x \in \Omega$ and $t > 0$. The later is known as the parabolicity condition. Note that matrix A of this example is different than the matrix A used earlier to describe the general second order case. Nonetheless, we kept the same notation because steady-state solutions of a parabolic PDE are solutions of the corresponding elliptic equation in one less spatial dimension.

Thus, the equation is parabolic on Ω if, at each $(x,t) \in \Omega \times (0,\infty)$,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \geq 0 \quad \text{for all } \xi \in \mathbb{R}^n,$$

and it is **uniformly parabolic** if

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j > 0 \quad \text{for all } \xi \neq 0.$$

Geometric evolution equations are generally nonlinear parabolic PDEs that describe the evolution of geometric objects (such as curves, surfaces, and hypersurfaces). Their linearizations are typically uniformly parabolic when restricted to the space of smooth, strictly convex objects, highlighting the fundamental role of convexity. □

In these notes, we will consider certain elliptic and parabolic second order PDEs, albeit all nonlinear. For a standard entry point to second-order elliptic and parabolic theory, the reader should consult [**9**]. A more technical reference that provides one of the standard analytic treatments of geometric flows especially important for understanding parabolicity via geometry and how convexity controls behavior is [**8**].

1.2. Convex bodies, a first look.

Definition 1.2. A compact, convex set $K \subset \mathbb{R}^n$ is called a **convex body** if it has non-empty interior. A convex body $K \subset \mathbb{R}^n$ is **strictly convex** if the boundary ∂K contains no line segments; and K is a **smooth convex body** if its boundary ∂K is a smooth, closed hypersurface in $\mathbb{R}^n$.

Because of the nature of the topics, particularly the use of PDEs, we will require at least some differentiability for the convex bodies considered here. For simplicity, we will often assume that the bodies are smooth and strictly convex, even if considering bodies of class $\mathcal{C}^2_+$ (twice differentiable boundary and strictly positive Gauss curvature) may suffice. This will become clear shortly.

Furthermore, given a convex body K, we call its volume, $V(K)$, the Lebesgue measure of K as a subset of $\mathbb{R}^n$. Convex bodies have reasonably nice boundary properties even without assuming additional properties, and while we do not discuss this here, we would like to state, as a measure of the boundary, Minkowski's (definition of) surface area:

Definition 1.3. Let $K \subset \mathbb{R}^n$ be a convex body and let B denote the unit Euclidean ball in $\mathbb{R}^n$. The **surface area of** K, denoted $S(K)$, is defined by

$$S(K) := \lim_{\varepsilon \to 0^+} \frac{V(K+\varepsilon B) - V(K)}{\varepsilon},$$

where $V(\cdot)$ denotes the n-dimensional volume, and $K + \varepsilon B$ is the Minkowski sum

$$K + \varepsilon B = \{x + y : x \in K,\, y \in \varepsilon B\}.$$

This limit exists for all convex bodies and this definition of surface area of K agrees with certain integral forms that we will see later, some of which are undoubtedly familiar to the reader from other sources. Let us note that Minkowski's sum is not the only operation that can produce new convex bodies and that, later, different *sums* will be employed to produce different notions of surface areas.

1.3. A Brief Tour of Hypersurfaces in $\mathbb{R}^n$. To clarify what we mean by a smooth convex body, as well as to talk about the curvature of its boundary, we have to dwell into various possible parametrizations that can be considered for convex bodies. Intuitively, the smoothness of the convex body is the smoothness of its boundary parametrization, independent of the choice of parameters, and the (Gauss) curvature of its boundary is an intrinsic quantity that is also independent of the parametrization, measuring how bent the boundary is in $\mathbb{R}^n$ at a particular point.

In this section, we will briefly recall some basic notions from the theory of hypersurfaces in $\mathbb{R}^n$. A smooth hypersurface can be locally parametrized by immersions and is equipped with fundamental geometric objects such as the unit normal vector field, the Gauss map, and the second fundamental form. The associated shape operator encodes curvature, whose eigenvalues are the principal curvatures and whose determinant gives the Gauss curvature. In the convex setting, the support function and Gauss map provide a global parametrization that is particularly convenient. For further background, the reader may consider [**16**] and [**20**], from both the differential-geometric and convex-geometric perspectives, respectively.

The set $\Sigma = \partial K$ is a smooth $(n-1)$-dimensional submanifold (or hypersurface) of $\mathbb{R}^n$ if it can be covered by local charts

$$\phi_l : U_l \subset \mathbb{R}^{n-1} \to \Sigma \subset \mathbb{R}^n,$$

such that ϕ_l is a smooth immersion, and on any overlap $U_l \cap U_k$ the transitions maps $\phi_k^{-1} \circ \phi_l$ are smooth. Sometimes, we can use a global parametrization, e.g., via the inverse Gauss map, which will be explained shortly, or we may use spherical coordinates to view the convex body as a graph over the unit sphere:

$$u \in \mathbb{S}^{n-1} \mapsto \rho_K(u)u \in K,$$

where

$$\rho_K(u) = \max\{r > 0 \mid ru \in K\}$$

is the **radial function of** K that equals the distance from the origin to the boundary in the direction u, with respect to some interior point of K taken to be the origin.

Induced Metric. Given any fixed, local or global, parametrization

$$\phi : U \to \mathbb{R}^n,$$

the induced metric $g = g_{ij}\, dx^i\, dx^j$ (with the Einstein notation for summation) on $\phi(U) \subseteq M$ is given by:

$$g_{ij}(x) = \left\langle \frac{\partial \phi}{\partial x^i}(x), \frac{\partial \phi}{\partial x^j}(x) \right\rangle,$$

where $(x^1, \ldots, x^{n-1}) \in U$, the space of parameters (or local coordinates), and $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Note that the matrix $(g_{ij})_{i,j=1,\ldots,n-1}$ is positive definite.

In local coordinates $(x^1, \ldots, x^{n-1})$, with induced metric tensor g_{ij}, the area element on ∂K is:

$$dA_g = \sqrt{\det g_{ij}}\, dx^1 \ldots dx^{n-1}$$

and the surface area (agreeing with Minkowski's definition as stated before) has the integral representation:

$$S(K) = \int_{\partial K} \sqrt{\det g_{ij}}\, dx^1 \dots dx^{n-1}.$$

Second Fundamental Form. Let ν be the outer unit normal vector field along $\phi(U) \subseteq \Sigma$. The second fundamental form II is defined by:

$$\mathrm{II}_{ij}(x) = \left\langle \frac{\partial^2 \phi}{\partial x^i \partial x^j}(x), \nu(x) \right\rangle.$$

The **Gauss map** $\nu : \Sigma \to \mathbb{S}^{n-1}$ assigns to each point on a boundary of a convex body K its outer unit normal vector.

Weingarten Map (Shape Operator). The Weingarten map $\mathcal{W}$ (also called the shape operator) is the differential of the Gauss map at a point $p = \phi(x) \in \Sigma$:

$$\mathcal{W} = -d\nu : T_p\Sigma \to T_p\Sigma.$$

It is a linear map that satisfies $\mathrm{II}(X, Y) = \langle \mathcal{W}X, Y\rangle$, for all tangent vectors $X, Y \in T_p\Sigma$.

Principal Curvatures. The eigenvalues $\kappa_1, \dots, \kappa_{n-1}$ of $\mathcal{W}$ at a point $p \in \Sigma$ are what we call the **principal curvatures**. For strictly convex boundaries, which we also sometimes refer to as strongly convex or uniformly strictly convex boundaries, we have that all $\kappa_i > 0$, so II is positive definite and $\mathcal{W}$ equals $g^{-1}\mathrm{II}$ as matrices. The **Gauss curvature** $\kappa(p)$ at a point $p \in \Sigma$ is defined as the product of principal curvatures or, equivalently, the Jacobian determinant of the Gauss map.

1.4. The Parametrization by the Inverse of the Gauss Map. For a strictly convex, smooth hypersurface $\Sigma = \partial K$, the Gauss map is a diffeomorphism and we can define the parametrization of K by the inverse Gauss map:

$$u \in \mathbb{S}^{n-1} \mapsto x(u) \in \Sigma,$$

where $x(u)$ the position vector of the point of Σ where the unit outer normal to Σ is u. Again, use of spherical coordinates as $(x^1, \dots, x^{n-1})$ leads to the above first and second fundamental form, hence a notion of surface area and, respectively, curvature. In this setup, we can see the Gauss curvature in a natural way as the limit

$$\kappa(p) = \lim_{S(U)\to 0} \frac{S(\nu(U))}{S(U)},$$

where $U \subset \Sigma$ is a neighborhood of p, $S(U)$ is the $(n-1)$-dimensional volume, or surface area, of $U \subset \mathbb{R}^n$, and $S(\nu(U))$ is the $(n-1)$-dimensional volume of the image $\nu(U) \subset \mathbb{S}^{n-1}$.

Then, the surface area of K can also be expressed as

$$S(K) = \int_{\partial K} dA_g = \int_{\mathbb{S}^{n-1}} \frac{1}{\kappa(u)}\, d\sigma(u),$$

where $\kappa(u)$ is the Gauss curvature at the point with outer unit normal u, and $d\sigma(u)$ is the surface area element on the unit sphere $\mathbb{S}^{n-1}$. This will appear again shortly and will, hopefully, provide additional insight.

A (third) global parametrization of Σ via the inverse Gauss map can be given via the support function which will be presented in the next section. For now, let us mention an important problem of convex geometry.

2. Minkowski's Observation and Problem

It is attributed to Minkowski that the surface area measure of the convex hypersurface Σ, when pushed forward by the Gauss map, satisfies the following **closure condition**:

$$\int_{\mathbb{S}^{n-1}} \frac{u}{\kappa(u)}\, d\sigma(u) = \mathbf{0},$$

where $\kappa(u)$ denotes the Gauss curvature expressed as a function of the unit normal vector $u \in \mathbb{S}^{n-1}$, and $d\sigma$ is the standard measure on the sphere.

The converse question, said also to have been originally posed by Minkowski, is now known as the **classical Minkowski problem**: Given a positive function $f : \mathbb{S}^{n-1} \to (0, \infty)$ satisfying the closure condition

$$\int_{\mathbb{S}^{n-1}} \frac{u}{f(u)}\, d\sigma(u) = \mathbf{0},$$

does there exist a closed, convex hypersurface $\Sigma \subset \mathbb{R}^n$, such that the Gauss curvature at the point with normal u is $f(u)$, i.e., $\kappa(u) = f(u)$? If it exists, is the hypersurface unique? If f is positive but only integrable on the sphere, one can in general expect existence only in a weak sense. What additional conditions on f ensure the existence of a classical (in particular, smooth) solution? We will come back to this topic shortly.

2.1. The curvature function of a convex body. In convex geometry, the curvature function of a convex body $K \subset \mathbb{R}^n$ with sufficiently smooth boundary is a fundamental concept that plays an important role in the study of surface area measures. In particular, for smooth convex bodies, the *classical* surface area measure dS_K is absolutely continuous with respect to the spherical Lebesgue measure, and its Radon-Nikodym derivative is the reciprocal of the Gauss curvature

$$\frac{dS_K}{d\sigma}(u) = \frac{1}{\kappa(x(u))} = f_K(u),$$

where $d\sigma$ is the standard surface measure on the unit sphere $\mathbb{S}^{n-1}$. More precisely, for a strictly convex body K, of class C^2, the curvature function $f_K(u)$ viewed as a function on the unit sphere $\mathbb{S}^{n-1}$ is defined via the surface area measure and is given by

$$f_K(u) = \frac{1}{\kappa(x(u))},$$

where $x(u)$ is the unique point on the boundary of K with outer normal u and $\kappa(x(u))$ is the *Gauss curvature* at that point.

In Sections 1.3 and 1.4, we have seen that convex bodies can be parametrized naturally either over $\mathbb{R}^{n-1}$ or over the unit sphere $\mathbb{S}^{n-1}$, and many problems, such as Minkowski's problem, can be formulated in both frameworks. A key notion that enables the transition between these two parametrizations is **the support function of a convex body**.

Definition 2.1. Given a convex body $K \subset \mathbb{R}^n$, its support function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle,$$

for each $u \in \mathbb{S}^{n-1}$.

This function is the signed distance from the origin to the supporting hyperplane of K in direction u, and fully determines the convex body K.

The support function h_K can be naturally extended to a positively homogeneous function of degree one on $\mathbb{R}^n \setminus \{0\}$ by setting

$$h_K(x) := \|x\| \cdot h_K\left(\frac{x}{\|x\|}\right), \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

This extension enables analytic formulations of curvature and Monge–Ampère-type equations in Euclidean space rather than on the sphere.

Remark 2.2. Let $K \subset \mathbb{R}^n$ be a strictly convex body with smooth boundary, and let $h : \mathbb{S}^{n-1} \to \mathbb{R}$ denote its support function. The **curvature function** $f_K : \mathbb{S}^{n-1} \to \mathbb{R}$ can be expressed in terms of the support function and its second covariant derivatives on the unit sphere.

Fix an orthonormal frame $\{e_1, \ldots, e_{n-1}\}$ tangent to $\mathbb{S}^{n-1}$ at a point $u \in \mathbb{S}^{n-1}$. Denote by h_i and h_{ij} the first and second covariant derivatives of h with respect to this frame. Then the curvature function of ∂K at the point $x(u)$ is given by

$$f_K(u) = \det\left(h_{ij}(u) + h(u)\delta_{ij}\right). \tag{2.1}$$

Outline of the derivation: For a strictly convex, smooth body, the support function is $h(u) = \langle x(u), u \rangle$, where $x(u)$ the point of ∂K with outer normal u. Therefore, the boundary of the convex body is described by

$$x(u) = \nabla h(u) + h(u)u, \quad u \in \mathbb{S}^{n-1},$$

which is simultaneously the inverse of the Gauss map, ν^{-1}.

Hence, the differential of the inverse Gauss map, $(d\nu^{-1})$, is represented (on the tangent space of the sphere) by the matrix

$$\left(h_{ij}(u) + h(u)\delta_{ij}\right)_{i,j=1,\ldots n-1},$$

where h_{ij} denotes the covariant Hessian of h on $\mathbb{S}^{n-1}$.

Since the Weingarten map is $d\nu$, the determinant of the differential of the inverse Gauss map is given by

$$\det\left((d\nu)^{-1}\right) = \frac{1}{\kappa(u)}.$$

Consequently, the reciprocal of the Gauss curvature is given by

$$\frac{1}{\kappa(u)} = \det\left(h_{ij}(u) + h(u)\delta_{ij}\right).$$

We will prove directly (2.1) in the planar case. Let $K \subset \mathbb{R}^2$ be a smooth, strictly convex body, and let $\gamma(\theta) \in \partial K$ denote the boundary point whose outer unit normal is $n(\theta) = (\cos\theta, \sin\theta)$. The support function of K is defined by

$$h(\theta) = \langle \gamma(\theta), n(\theta) \rangle.$$

Proving (2.1) amounts to showing that the radius of curvature $\rho(\theta)$ of ∂K at the point with outer normal direction $n(\theta)$ is given by

$$\rho(\theta) = h''(\theta) + h(\theta).$$

Proof. Let $\gamma(s) \in \mathbb{R}^2$ be the positively oriented parametrization by arclength of the strictly convex, smooth boundary curve, thus $\|\gamma'(s)\| = 1$. Let $T(s) = \gamma'(s)$ be the unit tangent vector and $N(s)$ the unit normal vector such that $\{T(s), N(s)\}$ is a positively oriented orthonormal basis.

By the Frenet formulas for curves in $\mathbb{R}^2$, we have:

$$T'(s) = \kappa(s)N(s), \quad N'(s) = -\kappa(s)T(s),$$

where $\kappa(s)$ is the curvature at $\gamma(s)$, and the radius of curvature is $\rho(s) = 1/\kappa(s)$.

Let $\theta(s)$ be the angle between $-N(s)$, the unit outer normal vector, and the positive direction of the x-axis. Then θ increases with s because the curve is strictly convex, and we have

$$\frac{d\theta}{ds} = \kappa(s), \quad \text{so} \quad \frac{ds}{d\theta} = \rho(\theta).$$

Indeed, since $N(s) = -(\cos\theta(s), \sin\theta(s))$, we differentiate to obtain

$$N'(s) = -\theta'(s)\,(-\sin\theta(s), \cos\theta(s)) = \theta'(s)\,(\sin\theta(s), -\cos\theta(s)).$$

Noting that $T(s) = (-\sin\theta(s), \cos\theta(s))$, we have

$$(\sin\theta(s), -\cos\theta(s)) = -T(s),$$

hence

$$N'(s) = -\theta'(s)\,T(s).$$

Comparing with the Frenet formula $N'(s) = -\kappa(s)\,T(s)$, we conclude $\theta'(s) = \kappa(s)$. Recall now that $h(\theta)$, the support function of γ, is defined by

$$h(\theta) = \langle \gamma(s(\theta)), n(\theta) \rangle,$$

where $n(\theta) = (\cos\theta, \sin\theta) = -N(s(\theta))$. Since the curve is parametrized by arclength, $\gamma'(s) = T(s(\theta)) = (-\sin\theta, \cos\theta) =: t(\theta)$, Differentiate $h(\theta)$ with respect to θ using the chain rule

$$h'(\theta) = \frac{d}{d\theta}\langle \gamma(s(\theta)), n(\theta) \rangle = \left\langle \frac{d\gamma}{ds}\frac{ds}{d\theta}, n(\theta) \right\rangle + \langle \gamma(s(\theta)), n'(\theta) \rangle,$$

so

$$h'(\theta) = \langle T(s(\theta)), n(\theta) \rangle \rho(\theta) + \langle \gamma(s(\theta)), T(s(\theta)) \rangle = \langle \gamma(s(\theta)), t(\theta) \rangle.$$

Now differentiate $h'(\theta)$ with respect to θ again

$$\begin{aligned} h''(\theta) = \frac{d}{d\theta}\langle \gamma(s(\theta)), t(\theta) \rangle = \left\langle \frac{d\gamma}{ds}\frac{ds}{d\theta}, t(\theta) \right\rangle + \langle \gamma, t'(\theta) \rangle &= \rho(\theta) + \langle \gamma(s(\theta)), -n(\theta) \rangle \\ &= \rho(\theta) - h(\theta). \end{aligned}$$

Thus,

$$\rho(\theta) = h''(\theta) + h(\theta).$$

□

2.2. The Classical Minkowski Problem Revisited. Formula (2.1) is central in formulations of the Minkowski problem, and the study of Monge–Ampère equations on the sphere, where the curvature function is prescribed and one seeks a support function h satisfying this nonlinear PDE.

REMARK 2.3. The classical Minkowski problem asks, in its larger generality, the following: given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, does there exist a convex body $K \subset \mathbb{R}^n$ whose surface area measure is μ? A necessary condition for the existence of such a body is the *closure condition*:

$$\int_{\mathbb{S}^{n-1}} u \, d\mu(u) = 0.$$

If μ has a smooth, strictly positive density $f \in C^\infty(\mathbb{S}^{n-1})$ with respect to the standard measure of the unit sphere, then the Minkowski problem has a unique (up to translation) solution: there exists a strictly convex body K with smooth boundary $\partial K \in C^\infty$, whose support function h satisfies **the Monge–Ampère equation on the sphere**

$$\det \left(h_{ij}(u) + h(u)\delta_{ij} \right) = f(u).$$

Thus, under the closure condition and assuming smooth positive data, the Minkowski problem has a unique (up to translation), smooth, strictly convex solution. In the planar case ($n = 2$), this corresponds to reconstructing a convex, closed plane curve from its curvature function $\kappa(\theta)$, and the problem reduces to solving an ODE with periodicity and closure constraints.

The resolution of the Minkowski problem follows from work of many mathematicians. To name just a few, the existence and uniqueness of generalized convex solutions to Monge-Ampère equations were obtained by Minkowski, Lewy, Aleksandrov, Fenchel and Jessen. The existence of smooth solutions to the Minkowski problem was established independently by Nirenberg and Pogorelov in the 1950s, with further substantial regularity results by Shing-Tung Yau, and Shiu-Yuen Cheng in the 1970s. See Schneider's book, [**20**], particularly Chapters 8 and 9, for details and references.

Later, we will encounter some generalized surface areas measures, and thus generalized Minkowski problems, in particular the L_p surface area, where p is some arbitrary real number.

3. An application of the Monge-Ampère equation to the theory of convex bodies

The aim of this section is to present a remarkable result due to Petty, which plays a crucial role in affine differential geometry.

We start stating the following known facts that were proved in the theory of Monge-Ampère equations in connection to the resolution of the Minkowski problem:

REMARK 3.1 (Pogorelov [**18**], Chapter 3, and, independently, Cheng -Yau [**6**]). Let $f : \mathbb{S}^{n-1} \to \mathbb{R}$ be the curvature function of a strictly convex body in $\mathbb{R}^n$. If f extends to a real-analytic function on $\mathbb{R}^n \setminus \{0\}$, then the boundary ∂K of the corresponding convex body is a real-analytic hypersurface.

REMARK 3.2 (Pogorelov [**18**], Chapter 5). Let $\Omega \subset \mathbb{R}^n$ be an open domain in $\mathbb{R}^n$ and let $F : \Omega \to \mathbb{R}$ be a strictly convex function of class C^2 over Ω satisfying

$$\det\left(D^2 F(x)\right) = c > 0 \quad \text{for all } x \in \Omega.$$

Then F is real-analytic on Ω.
This result reflects the strong regularity of convex solutions to Monge–Ampère equations with constant determinant.

Before stating the result that is the focus of this section, we need the following two definitions, mentioning again as reference for all notions pertaining to convex geometry the book of R. Schneider [**20**].

DEFINITION 3.3 (Polar Body of a Convex Body). Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. The **polar body** K° of K is the convex body in $\mathbb{R}^n$ defined by

$$K^\circ = \{y \in \mathbb{R}^n \mid \langle x, y\rangle \le 1 \quad \text{for all } x \in K\}.$$

In other words, K° consists of all points whose inner product with every point of K is at most 1. The polar operation reverses inclusion: if $K_1 \subset K_2$, then $K_2^\circ \subset K_1^\circ$. Also, $(K^\circ)^\circ = K$ when K is closed, convex, and contains the origin in its interior.

DEFINITION 3.4 (Santaló Point of a Convex Body). Let $K \subset \mathbb{R}^n$ be a convex body. The **Santaló point** $s_K \in \mathbb{R}^n$ is the unique point in the interior of K such that the volume product

$$\mathrm{Vol}(K) \cdot \mathrm{Vol}((K - s_K)^\circ)$$

is minimized. Here, $K - s_K$ denotes the translation of K so that s_K becomes the origin, and the polar is then taken with respect to the origin.

The existence and uniqueness of the Santaló point are guaranteed for convex bodies, and the point plays an important role in the affine geometry of convex bodies, particularly in the context of the **Blaschke–Santaló inequality (3.1)** which is the object of the next proposition:

PROPOSITION 3.1 ([**20**]). *Let $K \subset \mathbb{R}^n$ be a convex body with Santaló point $s(K)$. Then, the following holds*

$$\mathrm{Vol}(K) \cdot \mathrm{Vol}\left((K - s(K))^\circ\right) \le \mathrm{Vol}(B)^2, \tag{3.1}$$

with equality if and only if K is an ellipsoid. As before, B is the unit ball in $\mathbb{R}^n$.

We are now ready to state the central result of this section:

THEOREM 3.5 (Petty's Lemma 1985, [**17**]). *Let K be a convex body in $\mathbb{R}^n$, $n \ge 2$, and if $n \ge 3$ assume that K is of class $\mathcal{C}^2$. If $f_K(u) = ch_K^{-n-1}(u)$ on $\mathbb{S}^{n-1}$, where h_K is the support function of K with respect to the Santaló point of K, f_K is its curvature function as a function on $\mathbb{S}^{n-1}$, and $c > 0$ is a constant, then K is an ellipsoid.*

PROOF. We will first prove this theorem in the case $n = 2$.

Let $u = (\cos\theta, \sin\theta)$ and thus regard $h_K(u) = \tilde{h}_K(\theta)$ as a function of $\theta \in [0, 2\pi]$, where for simplicity we will still use the notation h_K instead of $\tilde{h}_K$ for the function defined on $[0, 2\pi]$. Similarly, for the curvature function f_K.

Let $h(\theta) := c_1 \cos\theta + c_2 \sin\theta + \int_0^\theta \sin(\theta - t) f_K(t)\, dt$. Then h is a 2π-periodic function and $h_{\theta\theta} + h = f_K$ on $[0, 2\pi]$.

By Minkowski's existence theorem, there exists a planar convex body $\tilde{K}$ with curvature function f_K. Moreover, $\tilde{K}$ is of class $\mathcal{C}^2$.

This body is unique up to translation, i.e., up to some choice of constants $c_{1,2}$, we have that, up to translation, K is a copy of $\tilde{K}$ as they have the same curvature function.

Now, let us focus on the hypothesis $(h_K)_{\theta\theta}(\theta)+h_K(\theta)[= f_K(\theta)] = ch_K^{-3}(\theta)$, for all $\theta \in [0, 2\pi]$.

Rotate K if needed such that h_K has a maximum at $\theta = 0$. Thus, $(h_K)_\theta(0) = 0$ and $(h_K)_{\theta\theta}(0) \le 0$.

Denote by $a := h_K(0)$ the value of the maximum and by $e^2 := |(h_K)_{\theta\theta}(0)|/a$. For the simplicity of presentation, further we will use h', and h'' for the derivatives of h in θ. The equality $h'' + h = ch^{-3}$ at $\theta = 0$ gives $-e^2a + a = \frac{c}{a^3}$. Note that if h' is identically zero on $[0, 2\pi]$, then K is a disk which is validating the conclusion. Assume therefore the existence of a non-trivial periodic solution h on $[0, 2\pi]$ for which $h'' + h = ch^{-3}$, for all θ, and multiply this equation by h'. Integrating, we obtain

$$(h')^2 + ch^{-2} + h^2 = \frac{c}{a^2} + a^2 = a^2(2 - e^2).$$

It is not hard to check that $z(\theta) := h^2(\theta)$ satisfies the second order linear ODE with constant coefficients: $z'' + 4z = 2a^2(2 - e^2)$ that has the solution $h^2(\theta) = a^2(1 - e^2 \sin^2\theta)$ which is the equation of an ellipse of major axis a and eccentricity e. (For an ellipse E whose semi-axes are parallel to the axes of coordinates, it is easy to check using the above definition of the support function that $h_E(\theta) = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = a\sqrt{1 - \frac{a^2-b^2}{a^2}\,\sin^2\theta}$.)

Proof for $n > 2$: Let $h_K(u)$ be the support function of K with respect to its Santaló point which satisfies $f_K(u) = ch_K^{-n-1}(u) =: R_1(u)\dots R_{n-1}(u)$ where R_i is the i-th principal curvature at the point $y \in \partial K$ of normal vector $u \in \mathbb{S}^{n-1}$. Extend $h : \mathbb{S}^{n-1} \to (0,\infty)$, which takes $u \mapsto h(u)$, by homogeneity on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, to the function

$$H(x) = \|x\|\, h\left(\frac{x}{\|x\|}\right), \text{ and set } H(\mathbf{0}) = 0,$$

which thus becomes a positively homogeneous convex function of class $\mathcal{C}^2$ on $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Then, from the homogeneity of H, we obtain that, for every $x \in \mathbb{S}^{n-1}$, we have $\nabla H(x) = y$, where y is the point of the boundary of normal $u = \dfrac{x}{\|x\|}$. We also have that $\nabla H(x)\cdot x = H(x)$, by the definition of the support function and its extension. Differentiating the last identity with respect to x_i, we obtain that the matrix $[H_{ij}]_{ij}\cdot x = \mathbf{0}$. Hence, x is in the null-space of the Hessian at x and, by the Rodriguez formula, $dy_j = (H_{ij} - R_j)\,dx_j$, thus the eigenvalues of H_{ij} are zero and $R_1,\dots,R_{n-1}$.

Now, let $F = \frac{1}{2}\,H^2$. We will calculate $\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right]$ which it suffices to evaluate on unit vectors because the entries of the matrix are positively homogeneous of degree zero.

Note that

$$\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] = \det\left[y_j y_i + HH_{ij}\right]$$

and, recall that, H_{ij} has rank $(n-1)$ and yy^T has rank 1, thus the determinant depends on how y projects onto the nullspace of H_{ij}. More precisely,

$$\det\left[y_j y_i + HH_{ij}\right](x) = (x^T y)^2 \cdot h^{n-1}(x)\ \det(H_{n-1})(x),$$

where $x \in \ker(H_{ij})$ is a unit vector and $\det(H_{n-1})$ denotes the non-zero determinant of one of the $(n-1)\times(n-1)$ principal minors of HH_{ij} (only one of which is nonzero when $\operatorname{rank}(H_{ij}) = n-1$). Thus, at each point,

$$\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right] = h^2 \cdot h^{n-1} R_1 \dots R_{n-1} = c > 0, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Now, since on any (convex) domain, $s \mapsto \frac{1}{2}\, s^2$ is strictly convex, we will have that, if H is a convex function, then F will be strictly convex on any domain we fix for $s \mapsto \frac{1}{2}\, s^2$. Let Ω be an arbitrary open convex domain in $\mathbb{R}^n \setminus \{\mathbf{0}\}$, then F is analytic on Ω, and thus F is analytic on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Thus, F is smooth. Now, we can apply the following result of Brickell: If H is a positive function on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ of class $\mathcal{C}^4$, positively homogeneous of degree one, and $\det\left[\frac{\partial^2 F}{\partial x_i \partial x_j}\right]$ is constant on the domain, then F is a positive quadratic form.

This implies that K is an ellipsoid centred at the origin of $\mathbb{R}^n$. □

We end by mentioning that affine differential geometry is a branch of differential geometry that studies geometric properties of hypersurfaces in $\mathbb{R}^n$ that are invariant under the affine group of $\mathbb{R}^n$, that is, the group formed by compositions of linear transformations and translations. The study of this field was initiated by Wilhelm Blaschke in the 1920s. He introduced concepts like the affine normal vector and affine surface area. Blaschke's influential book series "Vorlesungen über Differentialgeometrie" (Lectures on Differential Geometry), particularly Volume III (1929), laid much of the groundwork. Unlike classical differential geometry, which emphasizes metric concepts like angles and distances, tied to the Euclidean structure, affine differential geometry focuses on properties that remain unchanged under volume-preserving or general affine transformations. The above Blaschke-Santaló inequality is an example of an affine invariant inequality.

4. Rigidity Results and Generalized Minkowski Problems

We will proceed somewhat in the spirit of Petty's lemma even if we move away from the realm of affine differential geometry. Our next aim is to investigate classification results for convex bodies whose curvatures and support functions satisfy certain identities. These type of results originated quite some time ago, but have recently attracted renewed attention, particularly within the framework of the L_p-Minkowski theory introduced by Lutwak, see, for instance, [**14**] and [**4**], as two of the seminal works in the field.

4.1. L_p Surface Area via Firey Sum and Volume Limit.

One may recall the definition of surface area relying on the Minkowski addition of a convex body with a small Euclidean ball. However, other notions of addition of convex bodies are also possible. A well-known example is the Firey sum, defined for any real number $p \geq 1$. Let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior, and let B denote the Euclidean unit ball. The Firey p-sum of K and $\varepsilon \cdot B$ is defined via support functions

$$h_{K+_p\varepsilon\cdot B}(u) = \left(h_K(u)^p + \varepsilon h_B(u)^p\right)^{1/p}, \quad u \in \mathbb{S}^{n-1},$$

where $h_K(u)$ is the support function of K, and $\epsilon \cdot B := \epsilon^{1/p} B$.

Lutwak proposed as an object of study the L_p surface area of K defined as the following limit

$$S_p(K) := p \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot B) - V(K)}{\varepsilon}.$$

This generalizes the classical surface area, which corresponds to the case $p = 1$. If K has a smooth, strictly convex boundary, then

$$S_p(K) = \int_{\mathbb{S}^{n-1}} h_K(u)^{1-p} \, f_K(u) \, du,$$

where $h_K(u)$ is the support function of K, and $f_K(u)$ is its curvature function. This definition was adopted later for all real numbers p, beyond the range $p \geq 1$.

In the smooth context, the L_p Minkowski problem seeks a convex body whose L_p surface area measure is absolutely continuous with respect to the Lebesgue measure on the unit sphere and its density coincides with a given smooth positive function. Here is a more rigorous statement using earlier notations: Let $p \in \mathbb{R}$, and let $f : \mathbb{S}^{n-1} \to (0, \infty)$ be a smooth positive function. The **smooth L_p Minkowski problem** seeks a strictly convex, smooth convex body $K \subset \mathbb{R}^n$, with the origin in its interior, such that its support function satisfies the Monge-Ampère type equation

$$h_K^{1-p}(u) \det \left(h_{ij}(u) + \delta_{ij} h(u)\right) = f(u) \quad \text{on } \mathbb{S}^{n-1}, \tag{4.1}$$

hence characterizing convex bodies whose curvature and support functions are related as above. The uniqueness and regularity of solutions have also been extensively studied. We will return to this topic in the last section.

We now step away from the L_p Minkowski problem to state several rigidity results in the spirit of Petty's lemma.

Theorem 4.1 (U. Simon, 1967, [**21**]). *Let Σ be a $\mathcal{C}^\infty$ strictly convex closed hypersurface in $\mathbb{R}^n$ (i.e., K is a smooth strictly convex body and $\partial K = \Sigma$). Then Σ is a sphere if any of the k-th standard elementary functions of the principal curvatures k_j of Σ, denoted by σ_k, $1 \leq k \leq n-1$, and its support function $h > 0$ with respect to an inner point of K are related by*

$$\sigma_k = g(h),$$

for a $\mathcal{C}^1$ function g with $\frac{dg}{dh} \leq 0$.

Let us make a few remarks:

Remark 4.2. (i) Note that for a constant function g, and $k = n-1$, or $k = 1$, these are known rigidity results for the Gauss curvature, respectively mean curvature, with the former being a particular case of the classical Minkowski problem.

(ii) Simon's result is sharp in the sense that there exists a convex body K with $\sigma_1 = g(h)$ and $\frac{dg}{dh} > 0$ that is not a Euclidean ball. Indeed, for any ellipsoid of revolution $E \subset \mathbb{R}^n$, the mean curvature σ_1 of its boundary can be written as a function of the support function h_E, defined with respect to the centre of symmetry of E. Specifically, there exist constants $c_1, c_2 > 0$ such that

$$\sigma_1 = c_1 h_E^3 + c_2 h_E.$$

These constants can be found explicitly in terms of the volume of E and the area of its sections passing through the axis of revolution.

(iii) Nonetheless, for certain special functions g satisfying $\frac{dg}{dh} > 0$, one can still conclude that K is a Euclidean ball. In particular, the fact that the Gauss curvature $\sigma_{n-1} = \kappa = c\,h$, with $c > 0$, implies that Σ is a sphere was established by Firey under the assumption that K is centrally symmetric. His proof is not difficult and makes use of the Blaschke-Santaló inequality. In the same paper, Firey conjectured that the result remains valid for non-symmetric convex bodies as well, provided that suitable minimal regularity assumptions on the boundary are imposed [**10**].

While Udo Simon's 1967 rigidity theorem employed a variety of tools and techniques, it falls within the broad area of elliptic partial differential equations. More recently, Ivaki-Milman and Hu-Ivaki obtained results of similar nature extended to identities involving the Gauss curvature and a function of the support function **and** the length of its gradient for smooth strictly convex bodies. Their methods are different and rely on studying the first eigenvalue of a certain differential operator.

THEOREM 4.3 (Ivaki-Milman, 2023, [**13**]). *Suppose* $\varphi : (0,\infty)\times(0,\infty) \to (0,\infty)$ *is a* C^1*-smooth function with*

$$\partial_1\varphi \geq 0, \quad \partial_2\varphi \geq 0,$$

and at least one of these inequalities is strict. Let Σ *be a smooth, strictly convex, origin-centred hypersurface in* $\mathbb{R}^n$*, with support function* h *and gradient* $Dh(u) = \nabla_{\mathbb{S}^{n-1}} h(u) + h(u)\,u, \quad \forall u \in \mathbb{S}^{n-1}$*, satisfying the equation*

$$\varphi(h, |Dh|)\kappa = h^{n+1} \quad \text{on} \quad \mathbb{S}^{n-1},$$

where κ *is the Gauss curvature of* Σ*. Then* Σ *is a sphere centred at the origin.*

THEOREM 4.4 (Hu-Ivaki, 2024, [**11**]). *Let* $n \geq 3$*, and assume* $-n \leq p$ *and* $q \leq n$*, with at least one of these inequalities being strict. Suppose* Σ *is a smooth, strictly convex, origin-centred hypersurface in* $\mathbb{R}^n$*, with support function* h*, such that:*

$$h^{p-1}|Dh|^{n-q}\kappa = c,$$

where $c > 0$ *is a constant,* Dh *denotes the gradient of* h *as above, and* κ *is the Gauss curvature of* Σ*. Then* Σ *is an origin-centred sphere.*

The interest of the later papers is their connection to the L_p dual Minkowski problem seeking positive, smooth solutions to the equation on $\mathbb{S}^{n-1}$:

$$h^{1-p}(h^2 + |\nabla_{\mathbb{S}^{n-1}} h|^2)^{(q-n)/2} \det\left(h_{ij}(u) + h(u)\delta_{ij}\right) = f,$$

for some positive smooth function $f : \mathbb{S}^{n-1} \to (0,\infty)$, see [**15**].

5. Interplay between Elliptic and Parabolic Partial Differential Equations - A Case Study

Firey's paper [**10**] named "Shapes of worn stones" is referential partly because it contains one of the earliest instances of a geometric curvature flow, even though Firey did not use this terminology at the time. He imagined the erosion of a pebble on the beach being guided by geometry, the larger the Gauss curvature at a point, the most erosion will occur there. The idea was later picked up by Andrews who studied the Gauss curvature flow in $\mathbb{R}^3$, [**2**]. See also Chapter 15 in [**3**] for a more comprehensive coverage and due credit to Firey.

Let us first explain a curvature flow intuitively. Imagine a shape, like a stretchy membrane, that evolves or changes over time, perhaps deforming under pressure.

A curvature flow is a rule that tells each point on the membrane's surface how to move depending on the shape's curvature, i.e., how "bent" the surface is at that point. For example, like in Firey's paper, places on the surface that are more curved move inward faster, while relatively flatter areas change more slowly. Over time, this process can smooth out the shape, shrink it, or transform it into something more regular, like a sphere.

Let us now get more precise.

5.1. Example: Gauss Curvature Flow of Convex Hypersurfaces. Let $\Sigma_t \subset \mathbb{R}^n$ be a family of smooth, closed, strictly convex hypersurfaces evolving in time t over some time interval. The *Gauss curvature flow* moves each point on the hypersurface opposite the outer unit normal vector, with speed equal to the Gauss curvature. The evolution equation is given by:

$$\frac{\partial x}{\partial t} = -\kappa(x,t)\,\nu(x,t),$$

where $x(\cdot,t)$ is the position vector of the hypersurface Σ_t with some parametrization, $\kappa(x,t)$ is the Gauss curvature at point $x \in \Sigma_t$, and $\nu(x,t)$ is the outer unit normal to Σ_t at x.

The Gauss curvature flow is a non-linear parabolic equation on strictly convex smooth hypersurfaces. This flow contracts the hypersurface to a point in finite time and, as it does so, the surface becomes asymptotically spherical, [**3**]. In general, parabolicity of the equation is an essential feature of curvature flows. Otherwise, even starting with a *nice* initial, smooth hypersurface, solutions to the flow may not exist for $t > 0$. This brings up also the fact that flows need to be defined on a certain class of objects called admissible, for example smooth, closed and strictly convex hypersurfaces.

A self-similar solution satisfies the condition that the hypersurface evolves only by scaling, thus $h(u,t) = \lambda(t)h(u,0)$ for some function of t, $\lambda(t)$. Self-similar solutions often describe the long-time (or blow-up) behaviour of more general solutions. For example, shrinking circles appear as attractors for convex curves under many flows. So, self-similar solutions are building blocks for understanding complex behaviours in curvature-driven evolutions. By zooming in near singularities or far out in time, the solution often converges to one of these simple forms.

Note that, for the Gauss curvature flow, if Σ_t evolves self-similarly, also said homothetically, then, up to some constant, $h(u,0) = \kappa(u,0)$. From Simon's theorem, we know that this implies that Σ_0, thus also Σ_t for each t, is a sphere centred at the origin. In fact, it was proved that the only closed, compact, strictly convex self-similar solutions to the Gauss curvature flow are spheres directly by studying the properties of the flow without employing Simon's result, [**7**].

In 1995, Leichtweiss noticed that one can apply known results on self-similar solutions to certain curvature flows to derive results in convex geometry. For example, he showed that, for K smooth strictly convex body in $\mathbb{R}^n$, each of the several cases considered by Simon among which $h = cH$, and $h = c\kappa^{\frac{1}{n-1}}$, $c > 0$, implies that K is a sphere by employing an appropriate flow: such as the flow by mean curvature studied by Gerhard Huisken for the first example and, respectively, the flow by the $(n-1)$-th root of the Gauss curvature studied by Ben Chow for the second example. Leichtweiss also provided a *simpler proof* of Petty's result assuming smoothness of K using an affine curvature flow defined by Ben Andrews for which the normal speed of each point was given by the power of Gauss curvature $\kappa^{1/(n+1)}$. An excellent reference for these results as well as on overall geometric evolution equations is the textbook by Andrews and co-authors, [**3**].

To give an idea of how flows are used to prove existence of solutions to problems of convex geometry, we will present the steps in the resolution of the even, smooth, planar, L_p-Minkowski problem for $0 < p < 1$ following the author's paper with S. Vikram. For higher dimensions, and other ranges of p, one may refer to the paper by Bryant-Ivaki-Scheuer, [**5**].

To understand what is an *appropriate* flow for this problem, we look first at:

5.2. The Variational Definition of the L_p Minkowski Problem due to Lutwak. Let $p > 0$ be a real number, and let $K \subset \mathbb{R}^n$ be a convex body with the origin in its interior. Recall that the general L_p *Minkowski problem* asks: Given a finite Borel measure μ on the unit sphere $\mathbb{S}^{n-1}$, find a convex body K such that

$$dS_p(K, u) = \mu,$$

where $dS_p(K, u)$ is the L_p surface area measure of K defined by

$$dS_p(K, u) = h_K(u)^{1-p}\, dS(K, u).$$

Lutwak [**15**] showed that the above formulation is equivalent to finding a support function of a convex body K with the origin in its interior which is minimizing the functional

$$\Phi_p(K) = \int_{\mathbb{S}^{n-1}} h_K(u)^p\, d\mu(u),$$

subject to the constraint that K has fixed volume. We may include the volume normalization to reformulate the problem as finding

$$\min_{h:\mathbb{S}^{n-1}\to(0,\infty)} \left\{ \frac{1}{V(K)^{p/n}} \int_{\mathbb{S}^{n-1}} h_K(u)^p\, d\mu(u) \right\} =: \min_{h_K} \tilde{\Phi}(h_K).$$

The associated Euler–Lagrange equation to the latter minimization problem is

$$d\mu(u) = \lambda\, h_K^{1-p}(u)\, dS_K(u), \tag{5.1}$$

where dS_K is the surface area measure of K, and λ is a scalar defined by

$$\lambda = \frac{1}{nV(K)} \int_{\mathbb{S}^{n-1}} h_K(u)^p\, d\mu(u).$$

Note that if μ has a smooth density f with respect to the Lebesgue measure on $\mathbb{S}^{n-1}$, the above equation (5.1) reduces to (4.1). In the next section, we will look at a low dimensional case of this problem.

5.3. A Planar Case Study.

THEOREM 5.1 (Stancu-Vikram, 2017, [**22**]). *Given $0 < p < 1$ a real number and f a π-periodic positive function of class $\mathcal{C}^2$, there exists a support function $h : [0, 2\pi] \to (0, \infty)$ that is a solution to the L_p-Minkowski problem*

$$h^{1-p}(h_{\theta\theta} + h) = f \quad on \quad [0, 2\pi]. \tag{5.2}$$

To show the existence of a support function solving the above equation, we define the following curvature flow as an initial value problem:

$$h_t(u,t) = -f(u)h(u,t)^p k(u,t), \quad h(u,0) = h_K(u), \tag{5.3}$$

where K is an arbitrary, smooth, strictly convex, origin-symmetric convex body.

The claim of Theorem 5.1 follows from the following result on the above curvature flow:

THEOREM 5.2 (Stancu-Vikram, 2017, [**22**]). *Let $0 < p < 1$ be a real number and let f be a π-periodic positive function of class $\mathcal{C}^2$. For any $K \subset \mathbb{R}^2$ strictly convex body of class $\mathcal{C}^2$, symmetric with respect to the origin, the solution to the flow (5.3) exists for finite time until it shrinks the evolving curves to a point such that, if properly renormalized to enclose constant area, they converge subsequentially in the Hausdorff metric to the boundary of a convex body which is a solution to the L_p-Minkowski problem of density f.*

A legitimate question is why this flow? More generally, how do we choose a flow tailored to a specific problem? Note that the self similar solutions of this flow are, up to rescaling, solutions to the L_p Minkowski problem prescribed by f. An alternate, yet equivalent, point of view, is to seek a curvature flow that decreases the functional $\tilde{\Phi}(K)$ in the variational definition of the problem and, indeed one can easily check using Cauchy-Schwarz that this is the case of this flow and the decrease is strict unless K is, up to rescaling, a solution to the L_p Minkowski problem prescribed by f.

Once these two observations have been established, they open the possibility of exploiting the flow to prove the existence of solutions to the L_p Minkowski problem prescribed by f. This can be achieved by analyzing the asymptotic shape of the flow, which, based on general properties of similar geometric flows, we conjecture a priori to be a self-similar solution.

5.3.1. *The study of the flow.* In these notes, we will outline the study of the asymptotic behaviour of the flow, giving some specific details, and refer the reader to [**22**] for the the full analysis. To rigorously study a flow and prove Theorem 5.2, we outline first the general strategy, common to geometric curvature flows:

Main Steps of the Proof

- **Setup:** For the geometric flow under consideration, specify the class of admissible initial data, such as smooth, closed, strictly convex, and origin symmetric curves. Among other things, this class insures the parabolicity of the flow, hence short time existence of solutions on some interval $[0, T)$, for some $T > 0$. An excellent reference, for a lot more than just this fact, are the course notes by Sinestrari in [**19**].
- **Preservation of convexity, long time existence of solutions:** Show that as long as the flow exists, it preserves key geometric properties of evolving curves such as convexity, smoothness, and embeddedness. Then, show that if the area enclosed by the curve is non-zero, the evolving curves Σ_t will approach at time T a smooth, closed, strictly convex, and origin symmetric curve. Hence, the flow can be extended until a singularity appears when the area enclosed by the curve vanishes. Denote the final time of existence to the flow, when the area vanishes, by ω.

- **Monotonicity:** Identify a geometric functional (often an isoperimetric-type ratio or a curvature integral) and prove that it is monotone (typically non-increasing) along the flow. For us, this role is played by a variant of Lutwak's functional from the variational definition of the L_p-Minkowski problem.
- **Asymptotic behaviour:** Analyze the long-time behaviour or singularity formation under the flow. For instance, demonstrate that a convex curve shrinks to *an* L_p*-point*, i.e., the support function will satisfy the equation (5.2).
- **Nondegeneracy:** We still have to prove that the above solution to the equation (5.2) is non-degenerate which amounts to showing that the L_p-*point*, when rescaled, results in a compact, convex set with non-empty interior. This ingredient is equivalent in the study of the flow to proving compactness of solutions to the normalized flow, because the blow up to study the singularity, when done via volume normalization, either results into a convex body or a non-compact convex set. To obtain a convex body solution to the L_p Minkowski problem, we want to rule out the second possibility.

We skipped a few technicalities mainly because to prove existence of solutions to the L_p-Minkowski problem it suffices to obtain a weak asymptotic behaviour of the flow. The solutions to the flow can converge smoothly to an L_p solution or only subsequentially as we will see in the next section. Showing smooth convergence would require considering the parabolic PDEs satisfied by all derivatives of the evolving support functions, or derivatives of curvatures, as well as establishing certain uniform estimates. Subsequential convergence is easier to achieve and relies on Blaschke's selection theorem which states that any sequence of convex sets of $\mathbb{R}^n$ contained in a compact contains a convergent subsequence, [**20**].

5.3.2. *The normalized flow and long time behaviour of solutions.* To study the singularity of the flow at time ω, we rescale the solution such that the normalized evolving curves enclose constant area 1. More precisely, we define the support function of the normalized curve by

$$\tilde{h}(\theta,t)=\frac{h(\theta,t)}{\sqrt{A(t)}},$$

and, thus, the curvature becomes

$$\tilde{k}(\theta,t)=k(\theta,t)\sqrt{A(t)},$$

where $A(t)$ is the area enclosed by the un-normalized curve at time $t\in[0,\omega)$. We will now pass from $t\in[0,\omega)$ to a new time variable, $\tau\in[0,\infty)$, via the change of variables

$$\tau=-\frac{1}{2}\ln\frac{A(t)}{A(0)}.$$

It is easy to check that the evolutions of the main normalized quantities are

$$\tilde{h}_\tau=\tilde{h}-\frac{f\tilde{h}^p\tilde{k}}{\frac{1}{2}\int_{S^1}f\tilde{h}^p\,\mathrm{d}\theta},$$

$$\tilde{k}_\tau=-\tilde{k}+\frac{\tilde{k}^2}{\frac{1}{2}\int_{S^1}f\tilde{h}^p\,\mathrm{d}\theta}\Big((f\tilde{h}^p\tilde{k})_{\theta\theta}+(f\tilde{h}^p\tilde{k})\Big).$$

We now define the *entropy* of the normalized flow, $\tilde{\mathcal{E}} : [0,\infty) \to \mathbb{R}$, by

$$\tilde{\mathcal{E}}(\tau) = \int_{\mathbb{S}^1} f\tilde{h}^p \,\mathrm{d}\theta, \tag{5.4}$$

which we ask the reader to compare with Lutwak's functional. Now, we will show that the entropy is monotone and uniformly bounded from both sides for all $\tau \in [0,\infty)$. This is in fact equivalent to the non-increase of Lutwak's functional under the un-normalized flow that we mentioned earlier.

LEMMA 5.3. *The entropy of the normalized flow is non-increasing for all time.*

PROOF. Note that

$$\begin{aligned}\frac{d\tilde{\mathcal{E}}}{d\tau} &= \int_{S^1} fp\tilde{h}^{p-1}\tilde{h}_\tau \,\mathrm{d}\theta \\ &= \frac{2p}{\int_{S^1} f\tilde{h}^p \mathrm{d}\theta}\left[\frac{1}{2}\left(\int_{S^1} f\tilde{h}^p \,\mathrm{d}\theta\right)^2 - \int_{S^1} f^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right].\end{aligned}$$

The normalized curves enclose constant area 1 for all $t \in [0,\infty)$, that is

$$\frac{1}{2}\int_{S^1} \frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta = 1,$$

and, thus,

$$\frac{d\tilde{\mathcal{E}}}{d\tau} = \frac{p}{\int_{S^1} f\tilde{h}^p \mathrm{d}\theta}\left[\left(\int_{S^1} f\tilde{h}^p \,\mathrm{d}\theta\right)^2 - \int_{S^1}\frac{\tilde{h}}{\tilde{k}}\,\mathrm{d}\theta \int_{S^1} f^2\tilde{h}^{2p-1}\tilde{k}\,\mathrm{d}\theta\right].$$

By the Cauchy-Schwarz inequality, we deduce that $\tilde{\mathcal{E}}_\tau \leq 0$ with equality if and only if

$$\frac{\tilde{h}}{\tilde{k}} = \lambda f^2 \tilde{h}^{2p-1}\tilde{k},$$

where λ is a nonnegative constant.

Rearranging the terms, note that equality occurs if and only if

$$\frac{\tilde{h}^{1-p}}{\tilde{k}} = \sqrt{\lambda} f \quad \Leftrightarrow \quad \tilde{h}^{1-p}(\tilde{h}'' + \tilde{h}) = \sqrt{\lambda} f,$$

or equivalently if, up to rescaling, $\tilde{K}$ is a solution to the L_p-Minkowski problem. □

PROPOSITION 5.1. *The entropy of the normalized solution is uniformly bounded from both above and below for all time* $\tau \geq 0$.

PROOF. From the previous lemma, $\tilde{\mathcal{E}}(\tau) \leq \tilde{\mathcal{E}}(0)$, so the entropy is bounded above by the initial data.

To obtain the lower bound, we apply Jensen's inequality:

$$\begin{aligned}\tilde{\mathcal{E}}(\tau) &\geq 2\pi \exp\left[\int_{S^1} \ln\left(f\tilde{h}^p\right)\frac{\mathrm{d}\theta}{2\pi}\right] \\ &= 2\pi \exp\left[\frac{1}{2\pi}\int_{S^1}\ln\left(f\right)\mathrm{d}\theta\right]\cdot \exp\left[p\int_{S^1}\ln\left(\tilde{h}\right)\frac{\mathrm{d}\theta}{2\pi}\right].\end{aligned}$$

Since f is independent of τ, the first integral is constant, and we get

$$\tilde{\mathcal{E}} = \tilde{C}_0 \exp\left[p\int_{S^1}\ln\left(\tilde{h}\right)\frac{\mathrm{d}\theta}{2\pi}\right],$$

for some positive constant $\tilde{C}_0$.

By a particular case of the logarithmic Minkowski inequality, [**4**], the second integral is bounded below as $\displaystyle\int_{S^1} \ln \tilde{h}\ \frac{\mathrm{d}\theta}{2\pi} \geq \frac{1}{2} \ln \frac{\tilde{A}}{\pi}$. Consequently, we have

$$\tilde{\mathcal{E}}(\tau) \quad \geq \tilde{C}_0 \exp\left[\tfrac{p}{2} \ln \tfrac{\tilde{A}}{\pi}\right] = \tilde{C}_0 \exp\left[\tfrac{p}{2} \ln \tfrac{1}{\pi}\right] =: \tilde{C}_1 > 0,$$

where the last part follows from the fact that the area enclosed by the evolving normalized curves remains constant for all $\tau \in [0, \infty)$.

Therefore the entropy is bounded below and above for all time by two positive constants, $\tilde{C}_1 \leq \tilde{\mathcal{E}}(\tau) \leq \tilde{C}_2$ where $\tilde{C}_2 = \tilde{\mathcal{E}}(0)$.

□

Finally, we will need the following technical lemma:

LEMMA 5.4. *As $\tau \to \infty$, we have that*

$$\limsup_{\tau\to\infty} \left(\frac{d\tilde{\mathcal{E}}}{d\tau}\right) = 0.$$

PROOF. Recall from equation (5.5) that

$$\tilde{\mathcal{E}}\,\frac{d\tilde{\mathcal{E}}}{d\tau} = p\left[\left(\int_{S^1} f\tilde{h}^p\ \mathrm{d}\theta\right)^2 - \int_{S^1} \frac{\tilde{h}}{\tilde{k}}\ \mathrm{d}\theta\ \int_{S^1} f^2\tilde{h}^{2p-1}\tilde{k}\ \mathrm{d}\theta\right].$$

As the entropy is non-increasing, suppose that $\limsup_{t\to\infty} \tilde{\mathcal{E}}_\tau < 0$. Then there exists an $\epsilon > 0$ such that for some $[t_0, \infty)$, with $0 < t_0 < \infty$, we have $\tilde{\mathcal{E}}\,\tilde{\mathcal{E}}_\tau \leq -p\epsilon$.

Integrating both sides of the previous inequality from t_0 to T, where $t_0 < T$, we obtain

$$\frac{1}{2}\left(\tilde{\mathcal{E}}^2(T) - \tilde{\mathcal{E}}^2(t_0)\right) \leq -p\epsilon\,(T - t_0).$$

By Proposition 5.1, the normalized entropy is bounded from both sides, hence the left-hand side of the above inequality is bounded uniformly from below for all time. However, the right-hand side goes to $-\infty$, as T tends to ∞, leading to a contradiction. □

Let us now assume compactness of solutions to the normalized flow (i.e., the evolving normalized curves are contained in an annulus of fixed radii): we do not justify this here, but let us hint that its proof would use both the bounds on the entropy and the π-periodicity of the problem. We can then prove the result establishing the asymptotic behaviour of solutions to the flow and, concomitantly, the existence of solution to the L_p-Minkowski problem for $0 < p < 1$.

PROOF. Let $\{\tau_j\}_{j\in\mathbb{N}}$ be a sequence of times diverging to infinity. Then, by compactness of solutions, the sequence of convex bodies with general term $\tilde{K}_j = \tilde{K}(\tau_j)$, $j \nearrow \infty$, the normalized solution of the flow (5.3) at times τ_j, satisfies the conditions of the Blaschke selection theorem and a subsequence of them, denoted for simplicity the same way, converges to a convex set $\tilde{K}$ satisfying the equation defining the L_p Minkowski problem associated to the function f.

Indeed, as $\limsup_{\tau\to\infty} \tilde{\mathcal{E}}_\tau = 0$, then

$$\limsup_{\tau\to\infty} \left[\left(\int_{S^1} f\tilde{h}^p \,\mathrm{d}\theta \right)^2 - \int_{S^1} \frac{\tilde{h}}{\tilde{k}} \,\mathrm{d}\theta \int_{S^1} f^2 \tilde{h}^{2p-1} \tilde{k} \,\mathrm{d}\theta \right) \right] = 0.$$

Thus, for the subsequence of times $\tau_j \nearrow \infty$, as $j \nearrow \infty$, we have in the limit

$$\left(\int_{S^1} f\tilde{h}^p \,\mathrm{d}\theta \right)^2 = \int_{S^1} \frac{\tilde{h}}{\tilde{k}} \,\mathrm{d}\theta \int_{S^1} f^2 \tilde{h}^{2p-1} \tilde{k} \,\mathrm{d}\theta.$$

Since the asymptotic shape of the normalized flow is non-degenerate, $\tilde{h}$ and $\tilde{k}$ are bounded from above and below for all time, and the equality is non-trivially satisfied by the limiting convex body $\tilde{K}$.

Due to the continuity of the functions involved, the above equality case of the corresponding Cauchy-Schwarz inequality occurs if and only if

$$\frac{\tilde{h}}{\tilde{k}} = \lambda f^2 \, \tilde{h}^{2p-1} \, \tilde{k},$$

where λ is positive constant. Rearranging the terms, we get $\tilde{h}^{1-p}(\tilde{h}'' + \tilde{h}) = \sqrt{\lambda} f$. By choosing $\nu = \lambda^{\frac{1}{2(2-p)}}$ and, rescaling again the limit body, $\tilde{K}$, by ν, we obtain a non-degenerate convex body satisfying $\tilde{h}^{1-p}(\tilde{h}_{\theta\theta} + \tilde{h}) = f$ as claimed. This is the final step that concludes the proof of both theorems. $\square$

Finally, we end by mentioning that another use of curvature flows is to establish isoperimetric-type inequalities. For example, given the monotonicity of the entropy functional, a consequence of the previous result can be stated as follows:

Corollary 5.1. *Let $h_K : [0, 2\pi] \to (0, \infty)$ be the support function of the convex body $K \subset \mathbb{R}^2$, which is a solution to the L_p Minkowski problem and let $L \subset \mathbb{R}^2$ be a smooth, strictly convex, origin-symmetric convex body with support function h_L. If $A(K), A(L)$ denote the area of K, respectively L, and $f_K : [0, 2\pi] \to (0, \infty)$ is the curvature function of K, then*

$$\frac{1}{A(L)^{p/2}} \int_0^{2\pi} h_L(\theta)^p h_K(\theta)^{1-p} f_K(\theta) \, d\theta \geq 2A(K)^{1-\frac{p}{2}},$$

with equality if and only if L is homothetic to K, i.e., there exists $\lambda > 0$ such that $L = \lambda K$.

This is an illustration of a general phenomenon. Classic geometric flows, such as the mean curvature flow (MCF), have been powerful tools in the analysis of geometric inequalities. In particular, the MCF has been employed to give an elegant proof of the isoperimetric inequality within the class of smooth convex bodies. The original proof belongs to Huisken, [**12**], but the reader is urged to check Ritoré's notes in [**19**]. Under the MCF, a hypersurface evolves in the direction of its inward normal with speed equal to its mean curvature. For smooth, strictly convex initial data, the flow preserves convexity and eventually contracts a compact closed hypersurface to a *round* point in finite time. By carefully analyzing the evolution of geometric quantities, such as volume and surface area, along the flow, one can show that the appropriate ratio of surface area to volume strictly decreases, unless the hypersurface is a sphere in which case is remains constant.

Similarly, the affine normal flow which contracts compact closed strictly convex hypersurfaces to ellipsoids was used to prove the affine isoperimetric inequality and to give a new proof of the Blaschke–Santaló inequality within the class of smooth convex bodies, [**1**]. The common ingredient of all these proofs is the monotonicity of appropriate functionals along solutions to the flow. This offers an analytic tool for proving inequalities in addition to a method for approaching equality cases. In fact, time did not allow us to explore the use of flows to study uniqueness of solutions to the L_p Minkowski problem, also considered in the literature, see for example some of the results in [**5**].

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