The Radon Split of Radially Acting Linear Integral Operators on $\mathfrak{H}_2$ with Uniformly Bounded Double Norms

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ABSTRACT

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This M. Sc. thesis treats the feasibility of the Radon Split for solving radial integral equation involving radially acting integral operator on the Hardy-Lebesgue Class $\mathfrak{H}_2$ of the half-upper plane Π_+. In this process, we take a scrutinizing look at $\mathfrak{H}_2$ by means of the conformal map $z \mapsto \log z$ taking $\Pi_+ \to \mathbb{R} + i(0,\pi)$. We demonstrate that $\mathfrak{H}_2$-functions f always possess a.e. unique boundary values $f(\pm r)$ $(r > 0)$ as $z \to \pm r$ from within Π_+. These boundary values are also angular limit functions in the $L_2(0,\infty)$-sense - i.e. $||f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})||_{L_2(0,\infty)} \to 0$ as $\phi \to \psi$ $(\psi = 0, \pi)$ from within $(0,\pi)$. Concomitantly, the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r,r',\phi) = e^{-i\phi}K_\phi(r,r')$ with uniformly bounded double norms, have unique angular limit $\mathcal{L}_2$-kernels $K(r,r',\psi) = e^{-i\psi}K_\psi(r,r')$ $(\psi = 0,\pi)$ in the $L_2((0,\infty)\times(0,\infty))$-sense - i.e.

$$||K_\phi - K_\psi||_{L_2(0,\infty)^2} = |||K_\phi - K_\psi||| \to 0 \text{ as } \phi \to \psi \ (\psi = 0,\pi) \text{ from within } (0,\pi).$$

These properties are consequences of the inverse Mellin-Transform, which transformation originates in Fourier-Plancherel Theorem for $L_2(\mathbb{R})$ and $L_2(\mathbb{R}^2)$. Because of this Mellin-Transform representation of $\mathfrak{H}_2$ and $\mathfrak{K}_2$, we may regard $\mathfrak{H}_2$ as the three

entities: $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$, $\langle\ \boldsymbol{\mathfrak{H}_2} \mid \boldsymbol{\mathfrak{H}_2}\ \rangle$ and $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$, where the first two are Hilbert spaces and the third is a dual system with

$$\langle\ f\ ,\ g\ \rangle \equiv \int_0^\infty f(re^{i\phi})g(re^{i\phi})e^{i\phi}dr\ (0 \leq \phi \leq \pi).$$

Consequently, we look upon $\mathfrak{K}_2$ as the Banach algebra $(\mathfrak{K}_2, |||\cdot|||_{s(2)})$ and further as the Hilbert space $\langle\ \mathfrak{K}_2 \mid \mathfrak{K}_2\ \rangle$. We successfully construct for every radial linear integral operator K of finite rank on $\mathfrak{K}_2$, its transpose K^T in $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ as well as its adjoint K^* in $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$, which leans heavily on the interaction of * and ^{T} in $\mathfrak{H}_2$. We prove a necessary and sufficient condition as to when an element of $\mathfrak{H}_2{}'$ is radially representable. And finally, we construct Fredholm Resolvents not only finite-dimensional $K \in \mathfrak{K}_2$ but also, by means of the Radon Split, the Fredholm Resolvents of any $K \in \mathfrak{K}_2$ and that of its transpose K^T in terms of $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$. Herein, the Fredholm Alternatives are induced by the derivations.

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" And say: Work! Soon Allah will see your Work,

and His Messenger, and the believers."

(Qur'an 9:105)

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CHAPTER 1

INTRODUCTION

This M. Sc. degree thesis has as one of its objectives the study of radially acting Volterra integral operators on the analytic function space $\mathfrak{H}_2$ of the upper-half plane $\Pi_+ = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$ of the set $\mathbb{C}$ of complex numbers, where

$$\mathfrak{H}_2 \equiv \left\{ f \in H(\Pi_+) : ||f||_{\mathfrak{H}_2} \equiv \sup_{y>0} \left[\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \right]^{1/2} < \infty \right\}$$

with $H(\Pi_+)$ denoting the set of all functions holomorphic on Π_+. The other more ambitious objective is the investigation of the feasibilities of the Radon splitting of a radially acting linear integral operator K on $\mathfrak{H}_2$ defined by

$$(1.0.1) \qquad (Kf)(re^{i\phi}) = \int_0^{\infty} K(r, r', \phi) f(r'e^{i\phi}) e^{i\phi} dr' \text{ a.e. in } r > 0 \ (0 < \phi < \pi),$$

determined by a ϕ-parameter family of $\mathcal{L}_2$-kernels $K_\phi(r, r') \equiv K(r, r', \phi)e^{i\phi}$ having uniformly bounded double norms $|||K_\phi|||$ ([8], [20]), also called Hille-Tamarkin norm ([8], pg. 168), where

$$(1.0.2) \quad |||K_\phi||| \equiv \left[\int_0^{\infty} \int_0^{\infty} |K(r, r', \phi)|^2 dr dr' \right]^{1/2}, \ |||K|||_{s(2)} \equiv \sup_{0<\phi<\pi} |||K_\phi||| < \infty.$$

Radon splitting for such an operator K means that for every $\varepsilon > 0$, there exist radially acting linear operators $P = P_\varepsilon$ and $Q = Q_\varepsilon$ on $\mathfrak{H}_2$, such that $K_\phi(r, r')$ may be split into the sum of two ϕ-parameter family of $\mathcal{L}_2$-kernels, namely

$$K_\phi(r, r') = P_\phi(r, r') + Q_\phi(r, r') \text{ with } P_\phi(r, r') = P(r, r', \phi)e^{i\phi}$$

$$\text{and } Q_\phi(r, r') = Q(r, r', \phi)e^{i\phi} \ (0 < \phi < \pi) \text{ with}$$

$$(1.0.3)\quad \begin{cases} (Pf)(re^{i\phi}) = \int_0^\infty P(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' \text{ a.e. in } r>0 \quad (0<\phi<\pi) \\ \text{and} \\ (Qf)(re^{i\phi}) = \int_0^\infty Q(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' \text{ a.e. in } r>0 \quad (0<\phi<\pi) \end{cases}$$

defining two operators $P, Q \in \mathcal{B}(\mathfrak{H}_2)$ satisfying

$$(1.0.4)\qquad |||Q|||_{s(2)} < \varepsilon \text{ and } rank(P) \equiv dim P(\mathfrak{H}_2) < \infty.$$

By means of this Radon split, we can reduce the radial integral equation

$$(1.0.5)\qquad f(re^{i\phi}) = g(re^{i\phi}) + \lambda \int_0^\infty K(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' \text{ a.e. in } r>0$$

$$(0<\phi<\pi),$$

where $g \in \mathfrak{H}_2$ is given and $f \in \mathfrak{H}_2$ is sought, to a matrix equation involving characteristic polynomials of matrices. These matrices allow us to calculate the radially acting Fredholm Resolvent operator $H_\lambda = H_\lambda(K)$ with kernel $H_\lambda(r,r',\phi)$ in terms of tensor products, and thereby we arrive at the solution of the radial integral equation (1.0.5), namely

$$(1.0.6)\qquad f(re^{i\phi}) = g(re^{i\phi}) + \lambda \int_0^\infty H_\lambda(r,r',\phi)g(r'e^{i\phi})e^{i\phi}dr' \text{ a.e. in } r>0$$

$$(0<\phi<\pi).$$

1.1. Results Achieved In $\mathfrak{H}_2$

Clasine van Winter, in her study of the continuous spectrum of the Hamiltonian operator of a two particle system ([18], [19]), examined radially acting linear integral operators on the Hardy-Lebesgue Class $\mathfrak{H}_2$. Every $\mathfrak{H}_2$-function f possesses a unique $L_2(\mathbb{R})$-"boundary value function", which is also denoted by f, in the sense of

$$\lim_{z\to u} f(z) = f(u) \text{ with } z \to u \text{ non-tangentially from within } \Pi_+ \text{ a.e. in } u \tag{1.1.1}$$

on $\mathbb{R}$, where $f(u)$ is an $L_2(\mathbb{R})$-function of variable u, as well as in the $L_2(\mathbb{R})$-sense

$$\lim_{y\to 0^+} ||f(\cdot + iy) - f(\cdot)||_{L_2(\mathbb{R})} = \lim_{y\to 0^+} \left[\int_{-\infty}^{\infty} |f(x+iy) - f(x)|^2 dx\right]^{1/2} = 0. \tag{1.1.2}$$

From this also follows that

$$||f||_{\mathfrak{H}_2} = ||f(\cdot)||_{L_2(\mathbb{R})} = \left[\int_{-\infty}^{\infty} |f(x)|^2 dx\right]^{1/2} \ (f \in \mathfrak{H}_2); \tag{1.1.3}$$

moreover, we can retrieve the values of $f(z)$ for $z \in \Pi_+$ from its boundary values $f(x)$ $(x \in \mathbb{R})$ by means of the Poisson Integral Formula

$$f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x-u)^2 + y^2} f(u)du \ (z = x+iy \in \Pi_+). \tag{1.1.4}$$

She defines the Banach algebra $\mathfrak{K}$ of radially acting linear integral operators with action on $\mathfrak{H}_2$ given by equation (1.0.1) and ϕ-parameter family of $\mathcal{L}_2$-kernels

$$K_\phi(r, r') \equiv K(r, r', \phi)e^{i\phi}$$

satisfying(1.0.2). Further, she noticed that not only $\mathfrak{H}_2$-functions f admit an inverse Mellin-transform representation

$$f(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \mathbf{f}(t)(re^{r\phi})^{-it-1/2}dt \ (re^{i\phi} \in \Pi_+) \tag{1.1.5}$$

with a.e. unique Lebesgue measurable $\mathbf{f}$ satisfying

$$\int_{-\infty}^{\infty} [1 + e^{2\pi t}]|\mathbf{f}(t)|^2 dt < \infty,$$

but every $K \in \mathfrak{K}$ possesses an inverse Mellin-transform representation in the sense of

$$(1.1.6) \qquad K(r,r',\phi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{K}(t,t')(re^{\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2} dt' dt$$

$$\text{for} \quad (0 < \phi < \pi),$$

where it is understood that r and r' always belong to $(0, \infty)$, by means of a $\mathbb{C}$-valued Lebesgue measurable function $\mathbf{K}(t,t')$ on $\mathbb{R}^2$ satisfying

$$(1.1.7) \qquad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}] |\mathbf{K}(t,t')|^2 dt' dt < \infty.$$

If K and L belong to Banach algebra $\mathfrak{K}$, then $M = KL \in \mathfrak{K}$ with kernel

$$(1.1.8) \quad M(r,r',\phi) = \int_0^{\infty} K(r,r'',\phi)L(r'',r',\phi)e^{i\phi} dr'' \text{ a.e. on } (0,\infty)^2 \ (0 < \phi < \pi),$$

where $M(r,r',\phi)$ has inverse Mellin-transform representation

$$(1.1.9) \qquad M(r,r',\phi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{M}(t,t')(re^{\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2} dt' dt$$

for $(r,r') \in (0,\infty)^2$ $(0 < \phi < \pi)$ by means of $\mathbf{M}(t,t')$ satisfying

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}] |\mathbf{M}(t,t')|^2 dt' dt < \infty.$$

Further, this kernel $\mathbf{M}(t,t')$ is expressible in terms of the kernels of $\mathbf{K}$ and $\mathbf{L}$ as follows:

$$(1.1.10) \qquad \mathbf{M}(t,t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{K}(t,t'')\mathbf{L}(t'',t') dt'' \text{ for almost all } (t,t') \in \mathbb{R}^2,$$

where $\mathbf{K}(t,t')$ and $\mathbf{L}(t,t')$ are the Lebesgue measurable functions on $\mathbb{R}^2$ giving the inverse Mellin-transform representation of $K(r,r',\phi)$ and $L(r,r',\phi)$ respectively in the sense of equation (1.1.6) and inequality (1.1.7). Because inequality (1.1.7) entails that $\mathbf{K}(t,t')$ and $\mathbf{L}(t,t')$ are both $\mathcal{L}_2$-kernels on $\mathbb{R}^2$ in the conventional sense ([16], pg. 11), namely their respective double norms are finite, that is

$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{K}(t,t')|^2dt'dt\right]^{1/2},\ \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{L}(t,t')|^2dt'dt\right]^{1/2}<\infty,$$

the trace

$$tr(M)\equiv\int_{-\infty}^{\infty}\mathbf{M}(t,t)dt= \tag{1.1.11}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{L}(t',t)dt'dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{L}(t,t')\mathbf{K}(t',t)dt'dt$$

of M exists, and this trace is independent of the parameter ϕ. On account of the uniqueness of the correspondence between $M(r,r',\phi)$ and the kernel $\mathbf{M}(t,t')$ giving the inverse Mellin-transform representation (1.1.9) of $M(r,r',\phi)$, she calls tr($\mathbf{M}$) plainly the trace tr(M) of operator M. Hence, for any $K\in\mathfrak{K}$

$$tr(K^n)\equiv tr(\mathbf{K}^n)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{K}^{n-1}(t',t)dt'dt\quad(n\geq 2) \tag{1.1.12}$$

shall always exist and moreover, the quantities

$$\delta_0(K)\equiv 1,\ \delta_1(K)\equiv 0,\ \delta_n(K)\equiv \tag{1.1.13}$$

$$\frac{(-1)^n}{n!}det\begin{pmatrix} 0 & n-1 & 0 & \cdots & 0 & 0\\ tr(K^2) & 0 & n-2 & \cdots & 0 & 0\\ tr(K^3) & tr(K^2) & 0 & \cdots & 0 & 0\\ tr(K^4) & tr(K^3) & tr(K^2) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ tr(K^{n-1}) & tr(K^{n-2}) & \cdots & tr(K^2) & 0 & 1\\ tr(K^n) & tr(K^{n-1}) & tr(K^{n-2}) & \cdots & tr(K^2) & 0\end{pmatrix}\quad(n\geq 2)$$

are always independent of ϕ, whereas all the operators $\Delta_n(K)$ having kernels

(1.1.14)
$$\Delta_n(K;r,r',\phi) \equiv$$
$$\frac{(-1)^n}{n!} det \begin{pmatrix} K(r,r',\phi) & n & 0 & \cdots & 0 & 0 \\ K^2(r,r',\phi) & 0 & n-1 & \cdots & 0 & 0 \\ K^3(r,r',\phi) & tr(K^2) & 0 & \cdots & 0 & 0 \\ K^4(r,r',\phi) & tr(K^3) & tr(K^2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K^n(r,r',\phi) & tr(K^{n-1}) & tr(K^{n-2}) & \cdots & 0 & 1 \\ K^{n+1}(r,r',\phi) & tr(K^n) & tr(K^{n-1}) & \cdots & tr(K^2) & 0 \end{pmatrix} \quad (n \geq 0)$$

belong to $\mathfrak{K}$. More than that, due to estimates ([18], pg. 133, (5.15)) on $|\delta_n(K)|$ and $|||\Delta_n(K)|||_{s(2)}$ not unlike to those derived in the book entitled "Integral Equations" by F. Smithies ([16], chapter VI, section 6.4 with heading "Modification of the Formulae"), namely

$$|\delta_n(K)| \leq |||K|||_{s(2)} \left[\sqrt{e/n}|||K|||_{s(2)}\right]^n \text{ and}$$
$$|||\Delta_n(K)|||_{s(2)} \leq e|||K|||_{s(2)} \left[\sqrt{e/n}|||K|||_{s(2)}\right]^n \ (n \geq 1),$$

she was able to construct the $\mathbb{C}$-valued and $\mathfrak{K}$-kernel-valued entire functions

(1.1.15)
$$\delta(K;\lambda) \equiv \sum_{n=0}^{\infty} \delta_n(K)\lambda^n \text{ and } \Delta_\lambda(K;r,r',\phi) \equiv \sum_{n=0}^{\infty} \lambda^n \Delta_n(K;r,r',\phi)$$

of complex variable λ respectively, which are designated as the modified Fredholm determinant and modified Fredholm first minor of the radially acting integral operator K with kernel $K(r,r',\phi)$. We note that $\delta(K;\lambda)$ is a modified Fredholm determinant independent of parameter ϕ, although the kernel $K(r,r',\phi)$ of the operator K is not. In terms of the expressions in equation (1.1.15), she showed that all the characteristic values of the operator K ((1.0.1)) are accounted for by the zeros of $\delta(K;\lambda)$, which do not accumulate in $\mathbb{C}$. On the other hand, if $\delta(K;\lambda) \neq 0$, then the Fredholm Resolvent $H_\lambda = H_\lambda(K)$ of the operator K exists, $H_\lambda \in \mathfrak{K}$ and H_λ has kernel

$$H_\lambda(r,r',\phi) = H_\lambda(K;r,r',\phi) = \Big[\delta(K;\lambda)\Big]^{-1}\Delta_\lambda(K;r,r',\phi)(r,r'>0;0<\phi<\pi).$$

Consequently, the unique solution of the radial integral equation (1.0.5) in the Hilbert space $\mathfrak{H}_2$ assumes the form

$$f(re^{i\phi}) = g(re^{i\phi}) + \lambda\int_0^\infty H_\lambda(r,r',\phi)g(r'e^{i\phi})e^{i\phi}dr' \quad (re^{i\phi}\in\Pi_+). \tag{1.1.16}$$

On the other hand, each $\mathcal{L}_2$-kernel

$$K_\phi(r,r') = K(r,r',\phi)e^{i\phi} \text{ of } L_2(0,\infty)$$

has finite double norm $|||K_\phi|||$ given by equation (1.0.2). Therefore, if

$$M_\phi(r,r') = (KL)_\phi(r,r') = \int_0^\infty K_\phi(r,r'')L_\phi(r'',r')dr'',$$

then the traces

$$tr(M_\phi) = \int_0^\infty\int_0^\infty K_\phi(r,r')L_\phi(r',r)dr'dr = \int_0^\infty\int_0^\infty L_\phi(r,r')K_\phi(r',r)dr'dr \tag{1.1.17}$$

of each integral operator M_ϕ on $L_2(0,\infty)$ exist. In consequence hereof, we can construct the modified Fredholm determinants $\delta(K_\phi;\lambda)$ and modified Fredholm first minors $\Delta_\lambda(K_\phi)$ with $\mathcal{L}_2$-kernels $\Delta_\lambda(K_\phi;r,r')$ of $K_\phi(0<\phi<\pi)$; however, these expressions are definitely dependent upon ϕ and therefore shed no light on the collective behaviour of the ϕ-parameter family of integral operators K_ϕ, when looked upon as generating an operator $K \in \mathcal{B}(\mathfrak{H}_2)$ with action (1.0.1) on $\mathfrak{H}_2$. Thus the approach by means of a ϕ-independent trace , and more so a ϕ-independent modified Fredholm determinant is inevitable. Furthermore, there is also no guarantee that the $\mathcal{L}_2$ kernel coefficients

$$\Delta_n(K_\phi; r, r') \equiv$$

$$\frac{(-1)^n}{n!} det \begin{pmatrix} K_\phi(r,r') & n & 0 & \cdots & 0 & 0 \\ K_\phi{}^2(r,r') & 0 & n-1 & \cdots & 0 & 0 \\ K_\phi{}^3(r,r') & tr(K_\phi{}^2) & 0 & \cdots & 0 & 0 \\ K_\phi{}^4(r,r') & tr(K_\phi{}^3) & tr(K_\phi{}^2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ K_\phi{}^n(r,r') & tr(K_\phi{}^{n-1}) & tr(K_\phi{}^{n-2}) & \cdots & 0 & 1 \\ K_\phi{}^{n+1}(r,r') & tr(K_\phi{}^n) & tr(K_\phi{}^{n-1}) & \cdots & tr(K_\phi{}^2) & 0 \end{pmatrix} \quad (n \geq 0)$$

are kernels of radially acting integral operators on $\mathfrak{H}_2$ and not to mention the question of the relationship between the ϕ-parameter family of Fredholm Resolvent kernels

$$H_\lambda(K_\phi; r, r') \equiv \Big[\delta(K_\phi; \lambda)\Big]^{-1} \Delta_\lambda(K_\phi; r, r')$$

of the integral operators $K_\phi \in \mathcal{B}(L_2(0,\infty))$ and the Fredholm Resolvent kernel $H_\lambda(K; r, r', \phi)$ defined by the ratio $\Delta_\lambda/\delta(\lambda)$ from relation (1.1.15) for the radially acting linear integral operator $K \in \mathcal{B}(\mathfrak{H}_2)$. Do adequate terms in $\delta(K_\phi; \lambda)$ and $\Delta_\lambda(K_\phi; r, r')$ cancel each other out, so that we may conclude

$$H_\lambda(K_\phi; r, r') = H_\lambda(K; r, r')e^{i\phi} \ (0 < \phi < \pi)?$$

The answer to this is affirmative, but only by the circuitous path of introducing a ϕ-independent trace and invoking the density of of the restriction of $\mathfrak{H}_2$-function to the rays $\{re^{i\phi} : r > 0\}$ in $L_2(0,\infty)$ $(0 < \phi < \pi)$

1.2. Results Achieved In $\mathfrak{H}_p$ $(1 < p < \infty)$

The inverse Mellin-transform is a special version of the Fourier-transform on $L_2(\mathbb{R})$, which determines a Hilbert space isomorphism on $L_2(\mathbb{R})$, as states by the Plancherel theorem ([15], pg. 186). The Plancherel theorem has diminished validity in $L_p(\mathbb{R})$ for $1 < p < 2$, as can readily be seen from the statement of the Hausdorff-Young theorem ([15], pg. 261), and fails totally in $L_p(\mathbb{R})$ for $p > 2$. Thus the inverse Mellin-transform representation (1.1.6) of the kernel $K(r, r', \phi)$ defining an operator $K \in \mathcal{B}(\mathfrak{H}_p)$ $(p \neq 2)$ by means of equation (1.0.1) is no longer valid, even though we require that the ϕ-parameter family of kernels $K_\phi(r, r') \equiv K(r, r', \phi)e^{i\phi}$ possess uniformly bounded double norms (Hille-Tamarkin norms) ([8], pg. 168) in the sense of

$$(1.2.1) \qquad |||K_\phi|||_{s(p)} \equiv \sup_{0<\phi<\pi} |||K_\phi|||_{p,p'}, \text{ where}$$

$$|||K_\phi|||_{p,p'} \equiv \left(\int_0^\infty \left[\int_0^\infty |K_\phi(r, r')|^{p'} dr' \right]^{p/p'} dr \right)^{1/p}.$$

In the absence of an inverse Mellin-transform representation (1.1.5) of $\mathfrak{H}_p$ $(1 < p < \infty,\ p \neq 2)$ and also of $\mathfrak{K}_p$ in the sense of equation (1.1.6), where $\mathfrak{K}_p$ in the Banach algebra of all radially acting linear integral operators $K \in \mathcal{B}(\mathfrak{H}_p)$ with action defined as in (1.0.1) by means of kernels $K(r, r', \phi)$ satisfying condition (1.2.1), my thesis advisor realized that the most crucial aspect to exploit was the fact that

$$\int_0^\infty f(re^{i\phi})g(re^{i\phi})e^{i\phi}dr \text{ is independent of } \phi \ (0 \leq \phi \leq \pi),$$

$$(f,g) \in \mathfrak{H}_p \times \mathfrak{H}'_p \ (p' = p\,[p-1]^{-1}).$$

He showed by means of the reflexivity of $L_p(X)$ $(X = \mathbb{R}, (0,\infty))$ $(1 < p < \infty)$ that $\mathfrak{H}_p$-functions f have angular limits in the sense of

$$\lim_{\phi \to \psi} f(re^{i\phi}) = f(re^{i\psi}) \text{ for } \phi \to \psi \text{ from within } (0,\pi) \text{ a.e. in } r > 0,$$

$$\text{as well as } \lim_{\phi \to \psi} ||f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})||_{L_p(0,\infty)} = 0 \ (\psi = 0, \pi).$$

Moreover, not only is the integral in the subsequent relation

$$\langle f, g\rangle \equiv \int_0^\infty f(re^{i\phi})g(re^{i\phi})e^{i\phi}dr \ (0 \leq \phi \leq \pi, (f,g) \in \mathfrak{H}_p \times \mathfrak{H}_{p'}) \tag{1.2.2}$$

for $p' = p[p-1]^{-1}$ independent of ϕ on $[0,\pi]$, but $\langle \cdot, \cdot \rangle : \mathfrak{H}_p \times \mathfrak{H}_{p'} \to \mathbb{C}$ is a bounded bilinear functional $\mathfrak{H}_p \times \mathfrak{H}_{p'}$ with the essential properties, that out of $\langle \mathfrak{H}_p, g\rangle = 0$ and $\langle f, \mathfrak{H}_{p'}\rangle = 0$ shall follow that g and f vanish identically on Π_+ - i.e. they are the zero functions. In short $\langle \mathfrak{H}_p, \mathfrak{H}_{p'}\rangle$ constitutes a dual system, albeit not the natural one.

He further develops "necessary and sufficient" conditions, as to when a continuous linear functional ℓ on $\mathfrak{H}_p$ is radially representable - i.e.

$$\ell(f) = \int_0^\infty f(re^{i\phi})g_\ell(re^{i\phi})e^{i\phi}dr \text{ for some unique } g_\ell \in \mathfrak{H}_{p'}$$

([10]). With these results, he examines radially acting linear integral operators K of finite rank and shows that they have kernels of the form

$$K(r,r',\phi) = \sum_{\mu=1}^{n} (f_\mu \otimes g_\mu)(r,r',\phi) \ (f_\mu \in \mathfrak{H}_p, g_\mu \in \mathfrak{H}_{p'}, \text{ for } 1 \leq \mu \leq n), \tag{1.2.3}$$

where $(f \otimes g)(r, r', \phi) \equiv f(re^{i\phi})g(r'e^{i\phi})$ for all $(f, g) \in \mathfrak{H}_p \times \mathfrak{H}_{p'}$. By further writing these f_μ and g_μ in Cauchy integral form

$$f_\mu(z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} f_\mu(u)(u - z)^{-1} du \text{ and } g_\mu(z) = (2\pi i)^{-1} \int_{-\infty}^{\infty} g_\mu(u)(u - z)^{-1} du,$$

where $f_\mu(u)$ anf $g_\mu(u)$ denote the respective $L_p(\mathbb{R})$- and $L_{p'}(\mathbb{R})$-"boundary value functions" of f_μ and g_μ calculated non-tangentially from within Π_+ of course, he arrived at the new a class of a Cauchy-Integral representable kernels

$$K(r, r', \phi) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (re^{i\phi} - s)^{-1} \tilde{K}(s, t)(r'e^{i\phi} - t)^{-1} ds dt$$

with $\tilde{K}$ and $\tilde{K}^T$ having finite Hille-Tamarkin norms in $L_p(\mathbb{R})$ and $L_{p'}(\mathbb{R})$ respectively - i.e.

$$|||\tilde{K}|||_{p,p'} = \left(\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |\tilde{K}(s, t)|^{p'} ds \right]^{p/p'} dt \right)^{1/p} < \infty,$$

$$|||\tilde{K}^T|||_{p',p} = \left(\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} |\tilde{K}(s, t)|^{p} ds \right]^{p'/p} dt \right)^{1/p'} < \infty.$$

For $K(r, r', \phi)$ and $L(r, r', \phi)$ Cauchy-Integral representable by means of kernels $\tilde{K}(s, t)$ and $\tilde{L}(s, t)$ respectively, the operator product KL has kernel

$$(KL)(r, r', \phi) = \int_0^{\infty} K(r, r'', \phi) L(r'', r', \phi) e^{i\phi} dr''$$

and is also Cauchy-Integral representable through

$$(\widetilde{KL})(s, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{K}(s, t')[s' - t']^{-1} \log(s'/t') \tilde{L}(s', t) ds' dt'$$

with $\log(s'/t')$ interpreted in the complex sense, although s' and t' are real variables. We thus have a new multiplication of the kernels $\tilde{K}(s, t)$, whereby we specifically construct the powers $\tilde{K}^{[n]}(s, t)$ of $\tilde{K}(s, t)$, and also obtain a new Banach algebra without a multiplicative identity. He also demonstrated the existence of a new kind of trace, called representation trace $Sp_R(K)$ (the subscript R standing for representation and Sp for the German word "Spur" meaning trace), therewith a new modified Fredholm determinant

$$\delta_R(K;\lambda) = \sum_{n=0}^{\infty} \delta_{Rn}(K)\lambda^n \text{ with } \delta_{R0}(K) = 1, \quad \delta_{R1}(K_0) = 0$$

$$\delta_{Rn}(K) = \frac{(-1)^n}{n!} det \begin{pmatrix} 0 & n-1 & \cdots & 0 & 0 \\ Sp_R(K^2) & 0 & \cdots & 0 & 0 \\ Sp_R(K^3) & Sp_R(K^2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 1 \\ Sp_R(K^n) & Sp_R(K^{n-1}) & \cdots & Sp_R(K^2) & 0 \end{pmatrix} (n \geq 2),$$

as well as new modified Fredholm first minor, particularly

$$\Delta_{R\lambda}(K;r,r',\phi) = \sum_{n=0}^{\infty} \lambda^n \Delta_{Rn}(K;r,r',\phi) \quad \text{with} \quad \Delta_{R0}(K;,r,r',\phi) = K(r,r',\phi)$$

$$\text{and } \Delta_{Rn}(K;r,r',\phi) = \frac{(-1)^n}{n!} \times$$

$$det \begin{pmatrix} K(r,r',\phi) & n & 0 & \cdots & 0 & 0 & 0 \\ K^2(r,r',\phi) & 0 & n-1 & \cdots & 0 & 0 & 0 \\ K^3(r,r',\phi) & Sp_R(K^2) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ K^n(r,r',\phi) & Sp_R(K^{n-1}) & Sp_R(K^{n-2}) & \cdots & Sp_R(K^2) & 0 & 1 \\ K^{n+1}(r,r',\phi) & Sp_R(K^n) & Sp_R(K^{n-1}) & \cdots & Sp_R(K^3) & Sp_R(K^2) & 0 \end{pmatrix} (n \geq 1).$$

The $\mathbb{C}$ and $\mathfrak{K}_p$-valued functions $\delta_R(K;\lambda)$ and $\Delta_{R\lambda}(K;r,r',\phi)$ of complex variable λ have the following properties. If λ is zero of $\delta_R(K;\lambda)$, then λ is a characteristic value of the opertor K. On the other hand, if $\delta_R(K;\lambda) \neq 0$, then

$$H_\lambda(K;r,r',\phi) \equiv \Big[\delta_R(K;\lambda)\Big]^{-1} \Delta_{R\lambda}(K;r,r',\phi)$$

is the kernel of the Fredholm Resolvent $H_\lambda(K)$ of K for regular value λ of K with $H_\lambda(K) \in \mathfrak{K}_p$. The rationale for examining the Radon Split for a radially acting linear integral operator is to find a direct, perhaps "brutal", alternative for constructing the Fredholm Resolvent kernel of $K(r,r',\phi)$, instead of approaching by way of the elegant, but cumbersome, means of modified Fredholm determinants and modified Fredholm first minors.

CHAPTER 2

FUNDAMENTAL NOTIONS

Our method of approach to the Radon Split of a radially acting integral operator $K \in \mathfrak{K}$ $(\mathfrak{K}_2)$ relies heavily upon: integral convergence theorems from real analysis; Fourier-Plancherel transform in inverse Mellin-Transform disguise; Volterra Integral Operators on $L_2(a, b)$; tensor product representation of the Fredholm Resolvent of a continuous integral operator in $L_2(a, b)$ of finite rank; and the Radon Split for integral operators on $L_2(a, b)$ with kernels of finite double norm ([8], pg. 168; [16], pg. 11). In the preceding, the a and b of the interval (a, b) may assume the values $-\infty$ and ∞ respectively. Therefore, we shall first describe these concepts of convergence of integrals from measure theory, followed thereafter by a brief discussion of the Fourier-Plancherel Theorem, a statement of the Minkowski Integral Inequality, and last by discourse on integral operators and equation on the Hilbert Space $L_2(a, b)$ ($a = -\infty$ and $b = \infty$ admissible).

2.1. Essential Convergence Concepts From Measure Theory

Although we shall be dealing primarily with Lebesgue Measure on $\mathbb{R}$, we shall assume μ to be a positive measure on a σ-algebra $\mathfrak{M}$ of subsets of set X. Integrals are initially defined for non-negative measurable simple functions

$s(x) \equiv \sum_{k=1}^{n} \chi_{E_k}(x)$, where χ_{E_k} denote the characteristic functions of the measurable disjoint subsets E_k of X, with

$$\int_X s(x)d\mu(x) \equiv \sum_{k=1}^{n} \alpha_k \mu(E_k).$$

Thereafter, the concept of an integral with suitable additivity properties is made meaningfull by the Lebesgue Monotone Convergence Theorem, which appears as

PROPOSITION 2.1.1. *If $\{f_n\}_{n=1}^{\infty}$ is a monotone sequence of measurable functions on X with $f_n \geq 0$, and $\lim_{n\to\infty} f_n(x) = f(x)$ μ-a.e. on X, then f is measurable and*

$$\lim_{n\to\infty} \int_X f_n(x)d\mu(x) = \int_X f(x)d\mu(x).$$

PROOF. ([15], Theorem 1.26, pg. 21) □

The next vital convergence result, where the monotonicity of the sequence is not stipulated and in a direction toward realizing the integral of a measurable non-negative function with respect to measure μ on X, is Fatou's Lemma, which we designate as

LEMMA 2.1.2. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence a non-negative measurable functions on X with ∞ allowed as value to be assumed, then*

$$\int_X (\lim_{n\to\infty} \inf f_n)(x)d\mu(x) \leq \lim_{n\to\infty} \inf \int_X f_n(x)d\mu(x).$$

PROOF. ([15], pg.23) □

Finally, because we want to consider the integrals of complex valued measurable functions and the integrals of convergent sequences of measurable functions, we need a theorem that dispenses with non-negativeness, namely the Lebesgue Dominated Convergence Theorem, which we formulate by

PROPOSITION 2.1.3. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence complex measurable functions with $\lim_{n\to\infty} f_n(x) = f(x)$ μ-a.e. on X and there exists a non-negative μ-integrable function g on X such that $|f_n(x)| \leq g(x)$ μ-a.e. on X for each $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu(x) = 0 \text{ and also } \lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x).$$

PROOF. ([15],Theorem 1.34, pg. 26). □

Not only it is possible to interchange the sumations in

$$\Big(\sum_{\mu=1}^{\infty}\Big|\sum_{\nu=1}^{\infty} a_\mu\Big|^p\Big)^{1/p} \leq \sum_{\nu=1}^{\infty}\Big(\sum_{\mu=1}^{\infty}|a_\mu|^p\Big)^{1/p}$$

as proved by H. Minkowski, but it has an integral version due to F. Riesz, ([7]) in particular

PROPOSITION 2.1.4. *(Minkowski Integral Inequality)*

If $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ are σ-finite measure spaces and $f(x, y)$ is a $(\mu \times \nu)$-measurable $\mathbb{C}$-valued function on $X \times Y$, then

$$\Big\{\int_Y \Big|\int_X f(x,y) d\mu(x)\Big|^p d\mu(y)\Big\}^{1/p} \leq \int_X \Big\{\int_Y \Big|f(x,y)\Big|^p d\nu(y)\Big\}^{1/p} d\mu(x).$$

We now turn our attention to the Fourier-transformation and its extension to a Hilbert space isomorphism on $L_2(\mathbb{R})$.

2.2. Essential Properties Of Fourier-Transformation

The Fourier-transform $\wedge : L_1(\mathbb{R}) \to C_0(\mathbb{R})$ defined by means of

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx = \int_{-\infty}^{\infty} f(x) e^{-ixt} dm(x) \ \Big(dm(x) = \frac{dx}{\sqrt{2\pi}}\Big) \tag{2.2.1}$$

determines a continuous injective linear map from $L_1(\mathbb{R})$ into $C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the set of all complex valued continuous functions g on $\mathbb{R}$ vanishing at ∞ in the sense of $g(t) \to 0$ as $|t| \to \infty$. Further, $||\hat{f}||_\infty \leq ||f||_{L_1(\mathbb{R})}$ and the image $\{\hat{f} : f \in L_1(\mathbb{R})\}$ under the Fourier-transform is a proper subset of $C_0(\mathbb{R})$ - i.e. $\hat{} : L_1(\mathbb{R}) \to C_0(\mathbb{R})$ fails to be surjective. The Fourier-transform carries multiplication of f by $e^{i\alpha}$ into translation of $\hat{f}$ by α, translation by α of f into multiplication by $e^{-i\alpha}$ of $\hat{f}$, and the convolution $f * g$ of f with g into the product $\hat{f}\hat{g}$ in the conventional sense. Due to the fact that $\hat{} : L_1(\mathbb{R}) \to C_0(\mathbb{R})$ is also a Banach algebra homomorphism satisfying

$$||f * g||_{L_1(\mathbb{R})} \leq ||f||_{L_1(\mathbb{R})}||g||_{L_1(\mathbb{R})},$$

where convolution plays the role of multiplication in $L_1(\mathbb{R})$, and the fact that $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ as well as its Fourier-transform image is dense in $L_2(\mathbb{R})$, we can prove with the aforementioned properties of $\hat{} : L_1(\mathbb{R}) \to C_0(\mathbb{R})$ the Plancherel Theorem, which is formulated as

PROPOSITION 2.2.1. *There exists a Hilbert space isomorphism on $L_2(\mathbb{R})$ denoted by $\hat{} : f \mapsto \hat{f}$ such that if $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, then $\hat{f}$ denotes the conventional Fourier-transform defined by (2.2.1). Moreover, $\hat{}$ possesses the following symmetry relationships: given $A > 0$ and*

$$\varphi_A(t) = (2\pi)^{-1/2} \int_{-A}^{A} f(x)e^{-ixt}dt \quad \text{and} \quad \psi_A(t) = (2\pi)^{-1/2} \int_{-A}^{A} f(x)e^{ixt}dt$$

$$\text{then} \quad \lim_{A\to\infty} ||\hat{f} - \varphi_A||_{L_2(\mathbb{R})} = \lim_{A\to\infty} ||f - \psi_A||_{L_2(\mathbb{R})} = 0.$$

PROOF. ([15], Theorem 9.13, pg. 186). $\square$

Due to the nature of the two-diminsional inverse Mellin-Transformation, that is it draws its existence from the Fourier-Plancherel Theorem in $L_2(\mathbb{R}^2)$, we therefore state without proof

PROPOSITION 2.2.2. *The Fourier Transformation* $\wedge : L_1(\mathbb{R}^2) \to C_0(\mathbb{R}^2)$ *defined*

$$by \;\; \hat{f}(t,t') = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v) e^{-i(ut+vt')} dudv,$$

restricted to the dense linear manifold $L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$ *of the Hilbert space* $L_2(\mathbb{R}^2)$*, has an extention* $\wedge$ *to all of* $L_2(\mathbb{R}^2)$*, which is a Hilbert space isomorphism* $\wedge : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ *and it possesses the following properties. If* $f \in L_1(\mathbb{R}^2) \cap L_2(\mathbb{R}^2)$*, then* $\hat{f}(t,t')$ *is the conventional Fourier-Transform of the* $L_1(\mathbb{R}^2)$*-function* f*. Given any* $A > 0$

$$and\ functions \;\; \varphi_A(t,t') = (2\pi)^{-1} \int_{-A}^{A} \int_{-A}^{A} f(u,v) e^{-i(ut+ivt')} dudv$$

$$and\ \psi_A(u,v) = (2\pi)^{-1} \int_{-A}^{A} \int_{-A}^{A} \hat{f}(t,t') e^{-i(ut+ivt')} dudv, \;\; then$$

$$\lim_{A\to\infty} ||\hat{f} - \varphi_A||_{L_2(\mathbb{R}^2)} = \lim_{A\to\infty} ||f - \psi_A||_{L_2(\mathbb{R}^2)} = 0.$$

The Fourier-Plancherel Theorem, combined with the Cauchy Integral Theorem, brings about the realization of the inverse Melin-Transform Representation (1.1.5) and (1.1.6) respectively of $\mathfrak{H}_2$-functions f and kernels $K(r,r',\phi)$ belonging to operators $K \in \mathfrak{K}_2$. These we shall see in the next chapter. Further, for the next two sections we shall again emphasize that the a and b appearing in the Hilbert space $L_2(a,b)$ can assume the values $-\infty$ and ∞ respectively.

2.3. Essential From Integral Operators on $L_2(a,b)$

In the process of investigating integral equations and integral operators, we encounter the concept of "*relative uniform*" and "*relative uniform absulute*" convergence. A sequence $\{x_n\}_{n=1}^{\infty}$ of $L_2(a,b)$ is said to converge to $x \in L_2(a,b)$ *relatively uniformly* on (a,b) if there exists a non-negative $L_2(a,b)$-function p and an integer $n(\varepsilon)$ such that

$$|x_n(s) - x(s)| \leq \varepsilon p(s) \quad \text{for all } s \in (a,b) \text{ and } n \geq n(\varepsilon).$$

Clearly, relative uniform convergence implies $L_2(a,b)$-convergence, that is

$$||x_n - x||_{L_2(a,b)} = [\int_a^b |x_n(s) - x(s)|^2 ds]^{1/2} \to 0 \text{ as } n \to \infty,$$

and also $\langle\, x_n \mid y \,\rangle \to \langle\, x \mid y \,\rangle$ as $n \to \infty$ for all $y \in L_2(a,b)$. Correspondingly, a series $\sum_{n=1}^{\infty} x_n$ is said to converge *relatively uniformly*, if the sequence of its partial sums $\left\{\sum_{\mu=1}^{n} x_\mu\right\}_{n=1}^{\infty}$ converges relatively uniformly in $L_2(a,b)$.

We further extend this notion by saying that the series in $\sum_{n=1}^{\infty} x_n$ of $L_2(a,b)$ functions converges *relatively uniformly absolutely*, if there exists a non-negative $L_2(a,b)$-function p and an integer $n(\varepsilon)$ such that

$$\sum_{\mu=n+1}^{n+p} |x_\mu(s)| \leq \varepsilon p(s) \text{ for all } s \in (a,b) \text{ and } n \geq n(\varepsilon) \text{ independent of integers } p \geq 1.$$

Parallel to the idea of "relative uniform" convergence in $L_2(a,b)$ is that of a sequence $\{K_n(s,t)\}_{n=1}^{\infty}$ of $\mathcal{L}_2$-kernels on (a,b) converging "relatively uniformly" to the $\mathcal{L}_2$-kernel $K(s,t)$. This means that there exists a non-negative $\mathcal{L}_2$-kernel $P(s,t)$ on $(a,b) \times (a,b)$ and an integer $n(\varepsilon)$ such that

$$|K_n(s,t) - K(s.t)| \leq \varepsilon P(s,t) \text{ for all } (s,t) \in (a,b)^2 \text{ and } n \geq n(\varepsilon).$$

Needless to say, if $K_n(s,t)$ converges "relatively uniformly" to $\mathcal{L}_2$-kernel $K(s,t)$ on $(a,b)\times(a,b)$, then

$$(LK)(s,t) = \int_a^b L(s,u)K_n(u,t)du \to \int_a^b L(s,u)K(u,t)du = (LK)(s,t),$$

$$(K_nL)(s,t) = \int_a^b K_n(s,u)L(u,t)du \to \int_a^b K(s,u)L(u,t)du = (KL)(s,t) \text{ and}$$

$$(K_nx)(s) = \int_a^b K_n(s,t)x(t)dt \to \int_a^b K(s,t)x(t)dt = (Kx)(s)$$

$(L(s,t)$ an $\mathcal{L}_2$-kernel on $(a,b)\times(a,b)$ and $x \in L_2(a,b))$ "relatively uniformly"

as $n \to \infty$ in the Hilbert space of $\mathcal{L}_2$-kernels and $L_2(a,b)$ respectively. Continuing this trend of thought, we say that the series

$$\sum_{n=1}^{\infty} K_n(s,t) \text{ of } \mathcal{L}_2\text{-kernels converges } \textit{relatively uniformly absolutely},$$

if the exist a non-negative $\mathcal{L}_2$-kernel $P(s,t)$ and an integer $n(\varepsilon)$ such that

$$\sum_{\mu=n+1}^{n+p} |K_\mu(s,t)| \le \varepsilon P(s,t) \text{ for all } (s,t) \in (a,b)\times(a,b)$$

and $n \ge n(\varepsilon)$ independent of the integers p.

If the above series of $\mathcal{L}_2$-kernels converges "relatively uniformly absolutely" towards the $\mathcal{L}_2$-kernel $K(s,t)$, then

$$\sum_{n=1}^{\infty}(LK_n)(s,t) = (LK)(s,t), \ \sum_{n=1}^{\infty}(K_nL)(s,t) = (KL)(s,t)$$

$$\text{and } \sum_{n=1}^{\infty}(K_nx)(s,t) = (Kx)(s)$$

with convergence in the sense of "relatively uniformly absolutely" in the space of $\mathcal{L}_2$-kernels and $L_2(a,b)$ respectively, assuming that $L(s,t)$ is an $\mathcal{L}_2$-kernel on (a,b) and $x \in L_2(a,b)$. This conclusion follows from the inequalities

$$\sum_{\mu=n+1}^{n+p} |(LK_n)(s,t)| \leq \varepsilon ||L(s,\cdot)||_{L_2(a,b)} ||P(\cdot,t)||_{L_2(a,b)} \text{ for all } (s,t) \in (a,b)^2,$$

$$\sum_{\mu=n+1}^{n+p} |(K_nL)(s,t)| \leq \varepsilon ||P(s,\cdot)||_{L_2(a,b)} ||L(\cdot,t)||_{L_2(a,b)} \text{ for all } (s,t) \in (a,b)^2, \text{ and}$$

$$\sum_{\mu=n+1}^{n+p} |(K_nx)(s,t)| \leq \varepsilon ||x||_{L_2(a,b)} ||P(s,\cdot)||_{L_2(a,b)} \text{ for all } s \in (a,b)$$

under the assumption that $n \geq n(\varepsilon)$, which inequalities are also independent of the choice of the positive integer p.

The Fredholm Resolvent of an integral operator K, having $\mathcal{L}_2$-kernel $K(s,t)$ on $L_2(a,b)$ with action $(Kx)(s) \equiv \int_a^b K(s,t)x(t)dt$ for all $s \in (a,b)$, can be obtained by means of the Neumann Series $H_\lambda = H_\lambda(K) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}$ for all λ satisfying $|\lambda| \times |||K||| < 1$. This is so, because $K(s,t)$ an $\mathcal{L}_2$-kernel on (a,b) means: $K(s,t)$ is Lebesgue measurable on $(a,b) \times (a,b)$;

$$\int_a^b \int_a^b |K(s,t)|^2 dtds = \int_a^b \left[||K(s,\cdot)||_{L_2(a,b)}\right]^2 ds = \int_a^b \left[||K(\cdot,t)||_{L_2(a,b)}\right]^2 dt < \infty;$$

$$||K(s,\cdot)||_{L_2(a,b)} < \infty \text{ for all } s \in (a,b) \text{ and } ||K(\cdot,t)||_{L_2(a,b)} < \infty \text{ for all } t \in (a,b).$$

Lebesgue measurable $K(s,t)$ satisfying only $|||K||| < \infty$ are called $\mathcal{L}_2$-kernels "in the wide sense" ([16], pg. 14); however, there always exists an $\mathcal{L}_2$-kernel $K_0(s,t)$ such that $|||K - K_0||| = 0$. This is in consequence of the Tonelli-Hobson Theorem, as indicated by F. Smithies in his text entitled *Integral Equations* ([16], pgs. 14 - 15). So, we shall assume in this sequel that $K(s,t)$ is an $\mathcal{L}_2$-kernel.

For special $\mathcal{L}_2$-kernels called Volterra $\mathcal{L}_2$-kernel $K(s,t)$, namely $K(s,t) = 0$ for all $t > s$, the Fredholm Resolvent kernel

$$H_\lambda(s,t) = H_\lambda(K;s,t) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(s,t)$$

is a "Volterra $\mathcal{L}_2$-kernel"-valued entire function of λ, because the estimates

$$|K^{n+1}(s,t)| \leq [(n-1)!]^{-1/2} |||K|||^{n-1} ||K(s,\cdot)||_{L_2(a,t)} ||K(\cdot,t)||_{L_2(t,b)}$$

guarantee that the radius of convergence of the Neumann Series is ∞. It is also clear, that Volterra $\mathcal{L}_2$-kernels do not have characteristic values.

Most $\mathcal{L}_2$-kernels on (a,b) are not of Volterra type, and require therefore another kind of treatment. To this end we note that $\mathcal{L}_2$-kernel of rank 1 are given in terms of the sesquilinear tensor product $u \otimes v$ from $L_2(a,b) \otimes L_2(a,b)$ with $(u \otimes v)(s,t) \equiv u(s)\overline{v(t)}$, wherefrom it follows that all finite rank operators $K \in \mathcal{B}(L_2(a,b))$ are of the form $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu$ with $\mathcal{L}_2$-kernel

$$K(s,t) = \sum_{\mu=1}^{n} a_\mu(s)\overline{b_\mu(t)}$$

having finite double norm

$$|||K||| \leq \sum_{\mu=1}^{n} ||a_\mu||_{L_2(a,b)} \times ||b_\mu||_{L_2(a,b)}.$$

Thus the integral equation $x = y + \lambda K x$, namely

$$x(s) = y(s) + \lambda \int_a^b K(s,t)x(t)dt \text{ for all } s \in (a,b)$$

with $y \in L_2(a,b)$ given and $L_2(a,b)$-function x sought after, reduces to

$$x(s) - \lambda \sum_{\nu=1}^{n} \langle\, x \mid b_\nu \,\rangle a_\nu(s) = y(s) \text{ for all } s \in (a,b).$$

After "inner-producting" this equation from the right with $b_\mu (1 \le \mu \le n)$, we obtain the equivalent linear system of n equations $[I - \lambda k]\vec{x} = \vec{y}$, where $k \in \mathbb{C}^{(n\times n)}$ having μ-th row and ν-th column entry $k_{\mu\nu} = \langle\, a_\nu \mid b_\mu \,\rangle$ and $\vec{x}$ and $\vec{y}$ both belong to $\mathbb{C}^{(n\times 1)}$ with μ-th entry $\langle\, x \mid b_\mu \,\rangle$ and $\langle\, y \mid b_\mu \,\rangle$ respectively. The integral equation $(I - \lambda K)x = y$ is solvable in $L_2(a, b)$, if and only if the induced linear equation $[I - \lambda k]\vec{x} = \vec{y}$, in which $\vec{y}$ denotes the column vector comprising of the entries $\langle y|\, b_\mu\rangle$ $(1 \le \mu \le n)$, is solvable for the column vector $\vec{x}$. Moreover,

$$x(s) = y(s) + \lambda \sum_{\mu=1}^{n} x_\mu a_\mu(s)$$

is the solution of the integral equation in terms of $\vec{x}$, whose entries are x_μ $(1 \le \mu \le n)$. Consequently, the determinant $d(\lambda) = det(I - \lambda k)$ and the classical adjoint $A_\lambda \equiv \mathfrak{Adj}(I - \lambda k)$ are polynomials with coefficients from $\mathbb{C}$ and $\mathbb{C}^{(n\times n)}$ respectively of degree at most n and $n-1$. Out of the relation $[I - \lambda k]A_\lambda = A_\lambda[I - \lambda k] = d(\lambda)I$ rewritten as $\lambda k A_\lambda = \lambda A_\lambda k = A_\lambda - d(\lambda)I$, shall follow that

$$H_\lambda = H_\lambda(K) = [d(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(a_\mu \otimes b_\mu)$$

is the Fredholm Resolvent of $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu$ provided $d(\lambda) \ne 0$, where $\alpha_{\mu\nu}(\lambda)$ stands for the μ-th row and ν-th column entry of $A_\lambda = \mathfrak{Adj}(I - \lambda k)$. $d(\lambda) \ne 0$ except for at most n such λ, and these zeros of $d(\lambda)$ account for all characteristic values of K.

2.4. The Radon Split For $\mathfrak{L}_2$-Kernels

For the case of an arbitrary $\mathfrak{L}_2$-kernel $K(s, t)$, we invoke the Weierstraß Approximation Theorem for many variables in integral format, as presented by R. Courant and D. Hilbert in their famous work *Methods of Mathematical Physics* ([3], Vol. I, pgs. 65 - 68), coupled with a compactness argument in case the interval (a,b) is unbounded ($a=-\infty$ or $b=\infty$), to obtain for an arbitrary small $\varepsilon>0$ a finite rank operator

$$P=\sum_{\mu=1}^{n} a_\mu \otimes b_\mu \in \mathcal{B}(L_2(a,b)) \text{ such that } |||K-P|||<\varepsilon.$$

We therefore arrive at the Radon Split of K, namely $K=P+Q$ with P an operator of finite rank on $L_2(a,b)$ and Q an integral operator on $L_2(a,b)$ with an $\mathcal{L}_2$-kernel $Q(s,t)$ of double norm less than ε - i.e. $|||Q|||<\varepsilon$. The integral equation $x=y+\lambda Kx$ in terms of the Radon Split becomes $x=y+\lambda(P+Q)x$ and is rewritten as $(I-\lambda Q)x=y+\lambda Px$, wherein Q has Neumann Series type Fredholm Resolvent

$$G_\lambda=\sum_{n=0}^{\infty}\lambda^n Q^{n+1} \quad (\sum_{n=0}^{\infty}|||\lambda^n Q^{n+1}|||\le\sum_{n=0}^{\infty}|\lambda|^n|||Q|||^{n+1}=|||Q|||[1-|\lambda||||Q|||]^{-1})$$

for all λ satisfying $|\lambda|\times|||Q|||<1$, especially for $|\lambda|<1/\varepsilon$. Acting from the left by $I+\lambda G_\lambda$ on the $L_2(a,b)$-"integral equation" $(I-\lambda Q)x=y+\lambda Px$ leads to the equivalent integral equation $x=(I+\lambda G_\lambda)y+\lambda(I+\lambda G_\lambda)Px$, after observing $\lambda Q G_\lambda=G_\lambda-Q=\lambda G_\lambda Q$ (Fredholm Resolvent Equation for Q) for all λ such that $|\lambda|<1/\varepsilon$. However,

$$(I+\lambda G_\lambda)P=\sum_{\mu=1}^{n}(a_\mu+\lambda G_\lambda a_\mu)\otimes b_\mu$$

is again an $\mathcal{L}_2$-kernel of finite rank, and thus

$$x(s)=(y+\lambda G_\lambda y)(s)+\lambda\sum_{\nu=1}^{n}\langle\, x \mid b_\nu\,\rangle(a_\nu+\lambda G_\lambda a_\nu)(s) \text{ for all } s\in(a,b).$$

Taking the inner product of this equation from the right with b_μ leads to

$$\langle\, x \mid b_\mu \,\rangle = \langle\, y + \lambda G_\lambda y \mid b_\mu \,\rangle + \lambda \sum_{\nu=1}^{n} \langle\, a_\nu + \lambda G_\lambda a_\nu \mid b_\mu \,\rangle \, \langle\, x \mid b_\nu \,\rangle \; (1 \le \mu \le n),$$

or equivalently

$$\sum_{\nu=1}^{n} \Big[\delta_{\mu\nu} - \lambda \langle\, a_\nu + \lambda G_\lambda a_\nu \mid b_\mu \,\rangle \Big] \langle\, x \mid b_\mu \,\rangle = \langle\, y + \lambda G_\lambda y \mid b_\mu \,\rangle \quad (1 \le \mu \le n).$$

This last linear system of equations we write in short as $[I - F_\lambda]\vec{x} = \vec{z}$, wherein F_λ is the $n \times n$ matrices with $\mu-$th row and $\nu-$th column entry being the holomorphic function $f_{\mu\nu}(\lambda) = \langle a_\nu + \lambda G_\lambda a_\nu | \, b_\mu \rangle$ of variable λ in a domain containing

$\{\lambda \in \mathbb{C} : |\lambda| < 1/\varepsilon\}$ $(1 \le \mu, \nu \le n)$, $\vec{x} \in \mathbb{C}^{(n\times 1)}$ with μ-th entry $x_\mu = \langle\, x \mid b_\mu \,\rangle$

$(1 \le \mu \le n)$ and $\vec{z} \in \mathbb{C}^{(n\times 1)}$ with μ-th entry being the holomorphic function

$\langle\, y + \lambda G_\lambda y \mid b_\mu \,\rangle$ $(1 \le \mu \le n)$ of variable λ in a domain containing the open

disc of radius $1/\varepsilon$ and centre 0.

Like before for $\mathcal{L}_2$-kernels of finite rank, we define instead of polynomials, the holomorphic functions $\delta(\lambda) \equiv det(I - \lambda F_\lambda)$ and $A_\lambda \equiv \mathfrak{Adj}(I - \lambda F_\lambda)$ of variable λ in a domain containing $\{\lambda \in \mathbb{C} : |\lambda| < 1/\varepsilon\}$ with values in $\mathbb{C}$ and $\mathbb{C}^{(n\times n)}$ respectively. Again as before, we let $\alpha_{\mu\nu}(\lambda)$ stand for the $\mu-$th row and $\nu-$th column entry of A_λ, which entries are no longer polynomials in λ, but holomorphic function in an open neighborhood of 0 containing the open disc of radius $1/\varepsilon$ centered at the origin for $1 \le \mu, \nu \le n$.

Every solution of the integral equation $x = y + \lambda Kx$ induces a solution the linear equation $[I - \lambda F_\lambda]\vec{x} = \vec{z}$, and every solution $\vec{x}$ with column entries x_μ $(1 \leq \mu \leq n)$ of the linear system of equations $[I - \lambda F_\lambda]\vec{x} = \vec{z}$ induces a solutions

$$x(s) = (y + \lambda G_\lambda y)(s) + \lambda \sum_{\mu=1}^{n} x_\mu (a_\mu + \lambda G_\lambda y)(s)$$

of the integral equation $x = y + \lambda Kx$. Taking note of the relationship between a square matrix and its classical adjoint, namely $A_\lambda(I - \lambda F_\lambda) = (I - \lambda F_\lambda)A_\lambda = det(I - \lambda F_\lambda)I$, we note that for $\delta(\lambda) = det(I - \lambda F_\lambda) \neq 0$ the vector $\vec{x}$ with entries

$$x_\mu = [\delta(\lambda)]^{-1} \sum_{\nu=1}^{n} \alpha_{\mu\nu}(\lambda) \langle\, y + \lambda G_\lambda y \mid b_\nu \,\rangle = [\delta(\lambda)]^{-1} \sum_{\nu=1}^{n} \alpha_{\mu\nu}(\lambda) \langle\, y \mid b_\nu + \bar{\lambda} G_\lambda{}^* b_\nu \,\rangle,$$

where $G_\lambda{}^*$ denotes the adjoint of the bounded linear operator G_λ on the Hilbert space $L_2(a, b)$, is the unique solution of the linear equation $[I - \lambda F_\lambda]\vec{x} = \vec{z}$. After utilizing the relation $\lambda A_\lambda F_\lambda = \lambda F_\lambda A_\lambda = A_\lambda - \delta(\lambda)I$, we see that

$$H_\lambda = H_\lambda(K) = G_\lambda + [\delta(\lambda)]^{-1} \sum_{\nu=1}^{n} (a_\mu + \lambda G_\lambda a_\nu) \otimes (b_\nu + \bar{\lambda} G_\lambda{}^* b_\nu)$$

is the Fredholm Resolvent of the operator K with $\mathcal{L}_2$-kernel $K(s,t)$ for all λ such that $|\lambda| < 1/\varepsilon$ and $d(\lambda) \neq 0$. The set of zeros of $\delta(\lambda)$ on the closed disc $\{\lambda \in \mathbb{C} : |\lambda| \leq 1/\varepsilon\}$ is finite and constitutes the set of characteristic values of K lying therein with characteristic functions $x(s) = \lambda \sum_{\mu=1}^{n} x_\mu (a_\mu + \lambda G_\lambda a_\mu)(s)$ for $\vec{x}$ a solution of $[I - \lambda F_\lambda]\vec{x} = \vec{0}$. This is what the Radon Split of $\mathcal{L}_2$-kernel $K(s,t)$, by means of $K = P + Q$ with $\mathcal{L}_2$-kernel $P(s,t)$ of finite rank and $\mathcal{L}_2$-kernel $Q(s,t)$ with $|||Q||| < \varepsilon$, is about.

A norm $||\cdot||$ on a vector space E is induced by an inner product $\langle \cdot \mid \cdot \rangle : E \times E \to \mathbb{C}$ if and only if the norm satisfies the parallelogram law - i.e.

$$||f+g||^2 + ||f-g||^2 = 2||f||^2 + 2||g||^2 \quad (f, g \in E).$$

As consequence hereof, it is vitally important to be able to retrive the inner product in terms of the norm only. This is accomplished by means of the *polarization identity*, which says

$$\langle f \mid g \rangle = \frac{1}{4}\left\{||f+g||^2 - ||f-g||^2\right\} + \frac{i}{4}\left\{||f+ig||^2 - ||f-ig||^2\right\} \quad (f, g \in E).$$

Consequently, E becomes an inner product space $\langle E \mid E \rangle$ and its completion with respect to this norm $||\cdot||$ is a Hilbert Space.

CHAPTER 3

INVERSE MELLIN-TRANSFORM REPRESENTATION OF $\mathfrak{H}_2$ AND $\mathfrak{K}_2$

In the process of examining $\mathfrak{H}_2$ from the radial point of view, Clasine van Winter ([18]) introduced the normed linear space

$$\mathfrak{S}_2 = \{f \in H(\Pi_+) : ||f||_{s(2)} \equiv \sup_{0<\phi<\pi} ||f(\cdot e^{i\phi})||_{L_2(0,\infty)} < \infty\} \tag{3.0.1}$$

and in an arduous way showed that $\mathfrak{S}_2 = \mathfrak{H}_2$, which proof inherently contained the equivalence of the two norms $||\cdot||_{s(2)}$ and $||\cdot||_{\mathfrak{H}_2}$ on $\mathfrak{H}_2$. We shall radically deviate from Clasine van Winter in this chapter and shall show the equivalence of $\mathfrak{S}_2$ and $\mathfrak{H}_2$ by means of: the Minkowski Integral Inequality, whose validity was shown by Minkowski for series only and its integral form is due to F. Riesz ([7]); the Mean-Value Property of harmonic functions in area form; the Fourier-Plancherel Theorem; and conformal mapping. Out of this shall also emerge the inverse Mellin-Transform representability of $\mathfrak{H}_2$ as well as that of the Banach algebra $\mathfrak{K}_2$ consisting of all ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$ having uniformly bounded double norms $|||K_\phi|||$, which also collectively induce an operator $K \in \mathcal{B}(\mathfrak{H}_2)$ defined by equation (1.0.1):

$$(Kf)(re^{i\phi}) = \int_0^\infty K(r, r', \phi) f(r'e^{i\phi}) e^{i\phi} dr' \text{ a.e. in } r > 0 \ (0 < \phi < \pi).$$

3.1. An Elementry Integral and $\mathfrak{H}_2 \subset \mathfrak{G}_2$

Prior to demonstrating that $\mathfrak{H}_2 \subset \mathfrak{G}_2$, we need to ascertain the values of the integral expression

$$(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}|u|^{-1/2}du,$$

which expression shall be inevitable in establishing the equivalence of the norms $||\cdot||_{s(2)}$ and $||\cdot||_{\mathfrak{H}_2}$ of $\mathfrak{H}_2$. In this integral, it is the absolute value of u, namely $|u|^{-1/2}$ that cause difficulty in its exact evaluation, and not the weak singularity at 0 due to the power $-1/2$. Therefore, we avail ourselves of the following well-established technique from residue calculus to deal with it. To this end, we slit the complex w-plane ($w = u + iv$) along the positive real axis and look at any path $\mathcal{C}$ that begins at $\infty e^{i2\pi}$ and circumscribes 0 counter-clockwise and returns to ∞e^{i0} in this split w-plane, so that $e^{i\phi}$ and $e^{i(2\pi-\phi)}$ lie in the component not containing the positive real axis. We note that $\mathcal{C}$ is a "Jordan-curve" in the compact $w-$plane (Riemann-Sphere) and we require the image of this "Jordan-curve" on the Riemann-Sphere to be rectifiable. For this path $\mathcal{C}$,

$$(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[w^2-2w\cos\phi+1]^{-1}w^{-1/2}dw = \sum \mathfrak{Res}\{(w-e^{i\phi})^{-1}(w-e^{-i\phi})^{-1}w^{-1/2}\},$$

where the summation in over all the poles at $[w^2 - 2w\cos\phi + 1]^{-1}w^{-1/2}$ lying in the component of the w-plane not containing the positive real axis. Moreover, it is understood that we choose that branch of $w^{-1/2}$ for which $u^{-1/2} = \frac{1}{\sqrt{u}}$ for $u > 0$. There are only two poles, namely $e^{i\phi}$ and $e^{i(2\pi-\phi)}$ in the component not containing the positive real axis, as is indicated in the immediately subsequent diagram labelled Figure 1, and

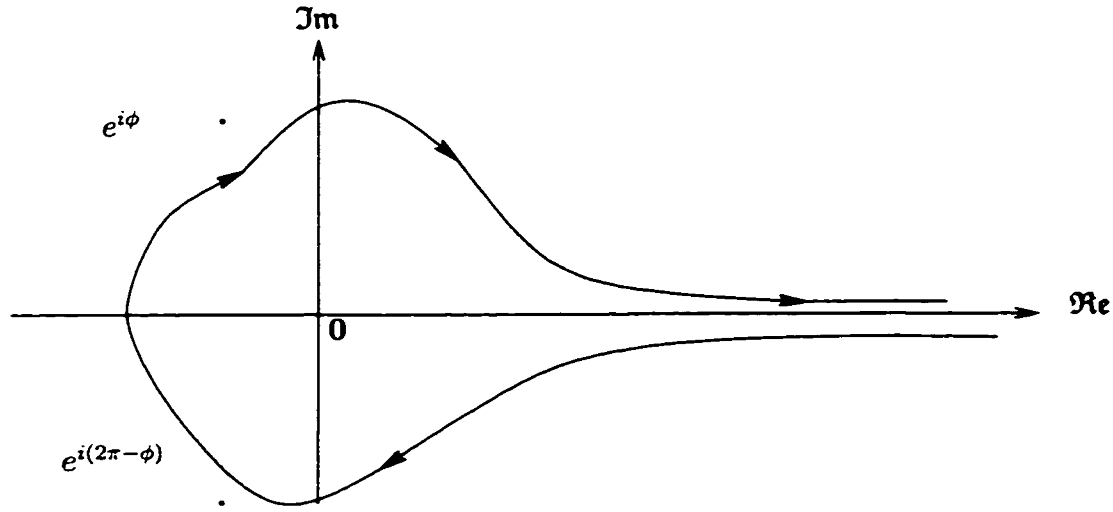

Figure 1.

$$\underset{w=e^{i\phi}}{\mathfrak{Res}}\{(w-e^{i\phi})^{-1}(w-e^{-i\phi})^{-1}w^{-1/2}\} = \lim_{w\to e^{i\phi}}(w-e^{i\phi})(w-e^{i\phi})^{-1}$$

$$(w-e^{-i\phi})^{-1}w^{-1/2} = \lim_{w\to e^{i\phi}}(w-e^{-i\phi})^{-1}w^{-1/2} = (2i\sin\phi)^{-1}e^{-i\phi/2}$$

and

$$\underset{w=e^{i(2\pi-\phi)}}{\mathfrak{Res}}\{(w-e^{i\phi})^{-1}(w-e^{-i\phi})^{-1}w^{-1/2}\} = \lim_{w\to e^{i(2\pi-\phi)}}(w-e^{i(2\pi-\phi)})(w-e^{i\phi})^{-1}$$

$$(w-e^{-i\phi})^{-1}w^{-1/2} = \lim_{w\to e^{i(2\pi-\phi)}}(w-e^{-i\phi})^{-1}w^{-1/2} = (-2i\sin\phi)^{-1}e^{-i\pi}e^{-i\phi/2} =$$

$$(2i\sin\phi)^{-1}e^{i\phi/2},$$

which in turn let us calculate the path integral over curve C as

$$(2\pi i)^{-1}\int_C [w^2-2w\cos\phi+1]^{-1}w^{-1/2}dw = (2i\sin\phi)^{-1}[e^{-i\phi/2}+e^{i\phi/2}] =$$

$$(i\sin\phi)^{-1}\cos(\phi/2).$$

On the other hand, by letting C be the path Γ_ε, where Γ_ε begins at $\infty = \infty e^{i2\pi}$ and goes along the lower edge of the positive real axis to $\varepsilon e^{i2\pi}$, then continues counter-clockwise from $\varepsilon e^{i2\pi}$ along the circle $|w| = \varepsilon$ to εe^{i0} and finally follows thereafter on the upper edge of the positive real axis to $\infty = \infty e^{i0}$, we obtain after letting $\varepsilon \to 0^+$ that

$$(2\pi i)^{-1} \int_C [w^2 - 2w\cos\phi + 1]^{-1} w^{-1/2} dw = 2(2\pi i)^{-1} \int_0^\infty [u^2 - 2u\cos\phi + 1]^{-1} (\sqrt{u})^{-1} du$$

$$= (\pi i)^{-1} \int_0^\infty [u^2 - 2u\cos\phi + 1]^{-1} (\sqrt{u})^{-1} du = (i \sin\phi)^{-1} \cos(\phi/2),$$

which after multiplication by $i \sin\phi$ leads to

$$(\pi^{-1} \sin\phi) \int_0^\infty [u^2 - 2u\cos\phi + 1]^{-1} (\sqrt{u})^{-1} du = \cos(\phi/2) \quad (0 < \phi < \pi).$$

We now look at

$$(\pi^{-1} \sin\phi) \int_{-\infty}^0 [u^2 - 2u\cos\phi + 1]^{-1} |u|^{-1/2} du = (\pi^{-1} \sin\phi) \int_0^\infty [u^2 + 2u\cos\phi + 1]^{-1} \times$$

$$(\sqrt{u})^{-1} du = (\pi^{-1} \sin(\pi - \phi)) \int_0^\infty [u^2 - 2u\cos(\pi - \phi) + 1]^{-1} (\sqrt{u})^{-1} du$$

$$= \cos(\frac{\pi - \phi}{2}) = \sin(\phi/2).$$

Combining these results, we arrive at

$$(\pi^{-1} \sin\phi) \int_{-\infty}^\infty [u^2 - 2u\cos\phi + 1]^{-1} |u|^{-1/2} du = (\pi^{-1} \sin\phi) \Big\{ \int_{-\infty}^0 + \int_0^\infty \Big\}$$

$$[u^2 - 2u\cos\phi + 1]^{-1} |u|^{-1/2} du = \cos\phi/2 + \sin\phi/2,$$

which function of ϕ has its maximum at $\phi = \pi/2$ and is thus bounded by $2/\sqrt{2} = \sqrt{2}$. We have therefore,

$$(3.1.2) \qquad (\pi^{-1} \sin\phi) \int_{-\infty}^\infty [u^2 - 2u\cos\phi + 1]^{-1} |u|^{-1/2} du \leq \sqrt{2} \text{ for } (0 < \phi < \pi)$$

and are now able to demonstrate the following

LEMMA 3.1.1. *$\mathfrak{H}_2 \subset \mathfrak{G}_2$ and for every $f \in \mathfrak{H}_2$ we have*

$$\sup_{0<\phi<\pi} ||f(\cdot e^{i\phi})||_{L_2(0,\infty)} \leq \sqrt{2}||f||_{\mathfrak{H}_2}.$$

PROOF. We let $f \in \mathfrak{H}_2$ with Poisson Integral Representation (1.1.4) in terms of its boundary value function $f(u) = f(u + i0)$. Writing $z = x + iy$ in polar form $z = re^{i\phi} = r\cos\phi + ir\sin\phi$ ($x = r\cos\phi,\ y = r\sin\phi$) yields

$$f(re^{i\phi}) = (\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}f(ru)du \ (re^{i\phi} \in \Pi_+) \tag{3.1.3}$$

after replacing the integration variable u by ru in equation (1.1.4). We estimate the $L_2(0,\infty)$-norm of $f(\cdot e^{i\phi})$ from equation (3.1.3) utilizing the Minkowski Integral Inequality as follows:

$$||f(\cdot e^{i\phi})||_{L_2(0,\infty)} = \left[\int_0^{\infty} |(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}f(ru)du|^2 dr\right]^{1/2} \leq$$

$$(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}\left[\int_0^{\infty}|f(ru)|^2 dr\right]^{1/2} du =$$

$$(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}|u|^{-1/2}\left[\int_0^{\infty}|f(r\ \mathrm{sgn}\,(u)|^2 dr\right]^{1/2} du \leq$$

$$\left[(\pi^{-1}\sin\phi)\int_{-\infty}^{\infty}[u^2 - 2u\cos\phi + 1]^{-1}|u|^{-1/2}du\right]\left[\int_{-\infty}^{\infty}|f(r)|^2 dr\right]^{1/2} =$$

$$[\cos(\phi/2) + \sin(\phi/2)]||f||_{\mathfrak{H}_2} \leq \sqrt{2}||f||_{\mathfrak{H}_2}$$

in consequence of equation (3.1.2), where $\mathrm{sgn}\,u = 1$ or -1 according as $u > 0$ or $u < 0$, and thereby completing the proof. □

3.2. Demonstrating $\mathfrak{G}_2 = \mathfrak{H}_2$.

We return to our normed liner space $\mathfrak{G}_2$ as defined by statement (3.0.1) and write for an $f \in \mathfrak{G}_2$ the square of the norm $||f(\cdot e^{i\phi})||_{L_2(0,\infty)}$ as the integral expression

$$\int_0^\infty |f(re^{i\phi})|^2 dr = \int_{-\infty}^\infty |e^{(u+iv)/2} f(e^{u+iv})|^2 du = \int_{-\infty}^\infty |F(u+i\phi)|^2 du$$

in terms of

$$F(u+i\phi) = e^{(u+i\phi)/2} f(e^{u+i\phi}) \text{ for all } (u,\phi) \in \mathbb{R} \times (0,\pi). \tag{3.2.4}$$

$F \in H(\mathbb{R}+i(0,\pi))$ - i.e. F is holomorphic on the infinite strip $\mathbb{R}\times(0,\pi) = \mathbb{R}+i(0,\pi)$ - and if we choose A arbitrary positive and $u+i\theta$ in this infinite strip, then the Mean-Value Theorem for harmonic functions in area form states that

$$F(u+iv) = (\pi\rho^2)^{-1} \int\int_{|u_1-u|^2+|\theta_1-\theta|^2 \le \rho^2} F(u_1+i\theta_1) du_1 d\theta_1 \tag{3.2.5}$$

$$\text{for all } \rho < \min\{\theta, \pi-\theta\}.$$

Thereafter we observe that the admissible ρ can not exceed $\pi/2$, which number is definitely less than 2, and we estimate the values of F, and indirectly those of f, from Mean-Value (3.2.5) as follows:

$$|F(u+i\theta)| \le (\pi\rho^2)^{-1} \int\int_{|u_1-u|^2+|\theta_1-\theta|^2 \le \rho^2} |F(u_1+i\theta_1)| du_1 d\theta_1 \le$$

$$(\pi\rho^2)^{-1} \int_{|\theta_1-\theta|\le\rho} \int_{u_1=u-2}^{u+2} |F(u_1+i\theta_1)| du_1 d\theta_1 \le$$

$$(\pi\rho^2)^{-1} \int_{\theta-\rho}^{\theta+\rho} \left[\int_{u-2}^{u+2} |F(u_1+i\theta_1)|^2 du_1\right]^{1/2} (4)^{1/2} d\theta_1$$

by means of the Schwarz Inequality applied to the term $\int_{u-2}^{u+2} |F(u_1 + i\theta_1)| du_1$. Thus we are able to write

$$(3.2.6) \qquad |F(u_1 + i\theta)| \leq 2(\pi\rho^2)^{-1} \int_{\theta-\rho}^{\theta+\rho} ||F(\cdot + i\theta_1)||_{L_2(u-2,u+2)} d\theta_1,$$

and because of

$$\lim_{|u|\to\infty} ||F(\cdot + i\phi)||_{L_2(u-2,u+2)} = 0 \text{ for each } \phi \in (0, \pi),$$

which follows out of

$$||F(\cdot + i\phi)||_{L_2(\mathbb{R})} \leq \infty \text{ and } \sup_{0<\phi<\pi} ||F(\cdot + i\phi)||_{L_2(\mathbb{R})} = \sup_{0<\phi<\pi} ||f(\cdot e^{i\phi})||_{L_2(0,\infty)} < \infty,$$

the Lebesgue Dominated Convergence Theorem (Proposition 2.1.3 of this thesis) lets us conclude for the interval $[\theta - \rho, \theta + \rho]$ that the limit of the integral expression of inequality (3.2.6) tends torwards 0 as $u \to \pm\infty$, and therefore

$$(3.2.7) \qquad \lim_{|u|\to\infty} F(u + i\theta) = 0 \text{ for each } \theta \in (0, \pi).$$

The validity of this result may be strengthened by stating that this limit is attained uniformly in θ on every closed subinterval $[\alpha, \beta]$ of $(0, \pi)$. If we let $\rho = (1/2)min\{\alpha, \pi - \beta\}$, then we directly derive from (3.2.6) that

$$(3.2.8) \qquad |F(u + i\theta)| \leq 8(\pi\rho^2)^{-1} \int_{\rho}^{\pi-\rho} ||F(\cdot + i\theta_1)||_{L_2(u-2,u+2)} d\theta_1 \ (\alpha \leq \theta \leq \beta).$$

Again, the Lebesgue Dominated Convergence Theorem on the closed interval $[\rho, \pi - \rho]$ applied to

$$\lim_{|u|\to\infty} ||F(\cdot + i\theta_1)||_{L_2(u-2.u+2)} = 0 \text{ for each } \theta_1 \in [\rho, \pi - \rho]$$

guarantees for us that limit (3.2.7) is attained uniformly in θ in the closed subinterval $[\alpha, \beta]$ of $(0, \pi)$. On the other hand, every compact subset C of the interval $(0, \pi)$ has positive distance from the set $\mathbb{R}\backslash(0, \pi) = (-\infty, 0] \cup [\pi, \infty)$ - i.e. $\gamma \equiv min\{|s-t| : s \in \mathbb{R}\backslash(0, \pi), t \in C\}$ - and consequently $C \subset [\gamma, \pi - \gamma] \subset (0, \pi)$. Hence, the limit (3.2.7) is attained uniformly an every compact subset of $(0, \pi)$. Returning to the function $f \in \mathfrak{G}_2$ from which F was constructed, we formulate these results as

LEMMA 3.2.1. *If $f \in \mathfrak{G}_2$, then on every compact subset of the interval $(0, \pi)$ the limits*

$$\lim_{r \to 0^+} (re^{i\phi})^{1/2} f(re^{i\phi}) = \lim_{r \to \infty} (re^{i\phi})^{1/2} f(re^{i\phi}) = 0 \tag{3.2.9}$$

are attained uniformly.

The effectiveness of Lemma 3.2.1 in the form (3.2.7) becomes evident, when we take the inverse Fourier-transform of the $L_2(\mathbb{R})$-function $F(\cdot + i\theta)$, that is to say

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u + i\theta) e^{itu} du,$$

and multiply it by $e^{-t\theta}$ and thereby have the integral expression

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u + i\theta) e^{it(u+i\theta)} du$$

standing for the integral of the holomorphic function $F(w)e^{itw}$ over the path $\mathfrak{Im}(z) = 0$ from $-\infty + i\theta$ to $\infty + i\theta$. Utilising this idea and the Fourier-Plancherel Theorem (Proposition 2.2.1), specifically that

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u + i\theta) e^{itu} du = \lim_{A \to \infty} (2\pi)^{-1/2} \int_{-A}^{A} F(u + i\theta) e^{itu} du$$

in the $L_2(\mathbb{R})$-sense, we look at the Cauchy-Integral Theorem applied to the holomorphic function $F(w)e^{itw} = e^{w/2}f(e^w)e^{itw}$ on the boundary $\partial\mathcal{R}$ of the rectangle

$$\mathcal{R} = \mathcal{R}(A;\theta_1,\theta_2) = \{w \in \mathbb{C} : |\mathfrak{Re}(w)| \leq A,\ \theta_1 \leq \mathfrak{Im}(w) \leq \theta_2\}$$

contained inside the infinite strip $\mathbb{R} + i(0,\pi)$ of the complex w-plane -i.e. $0 < \theta_1 < \theta_2 < \pi$ - as depicted in the subsequent diagram labelled Figure 2. Clearly,

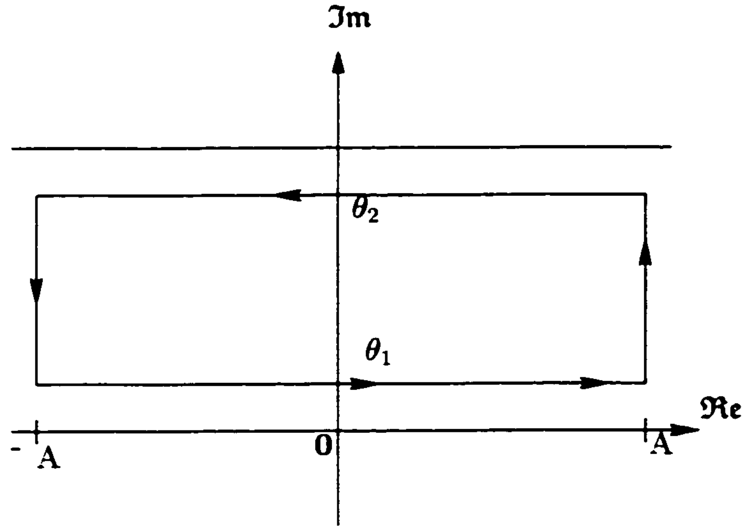

Figure 2.

$$0 = \oint_{\partial\mathcal{R}} F(w)e^{itw}dw = \int_{-A}^{A} F(u+i\theta_1)e^{it(u+i\theta_1)}du + i\int_{\theta_1}^{\theta_2} F(A+i\theta)e^{it(A+i\theta)}d\theta +$$
$$\int_{A}^{-A} F(u+i\theta_2)e^{it(u+i\theta_2)}du + i\int_{\theta_2}^{\theta_1} F(-A+i\theta)e^{it(-A+i\theta)}d\theta \ (A>0).$$

Rewritten, we have out of this that

$$\text{(3.2.10)} \qquad \int_{-A}^{A} F(u+i\theta_1)e^{it(u+i\theta_1)}du + ie^{iAt}\int_{\theta_1}^{\theta_2} F(A+i\theta)e^{-t\theta}d\theta =$$

$$\int_{-A}^{A} F(u+i\theta_2)e^{it(u+i\theta_2)}du + ie^{-iAt}\int_{\theta_1}^{\theta_2} F(-A+i\theta)e^{-t\theta}d\theta \text{ for } (A>0).$$

By invoking our Lemma 3.2.1 in the $F(w)$-form, specifically limit statement (3.2.7) - i.e. $\lim\limits_{A\to\infty} F(\pm A + i\theta) = 0$ uniformly in θ on every compact subset of $(0,\pi)$ - for the closed interval $[\theta_1, \theta_2]$, we have that both integrations with respect to θ on $[\theta_1, \theta_2]$ in equation (3.2.10) tend to zero as $A \to \infty$ and thus

$$(3.2.11) \quad \int_{-\infty}^{\infty} F(u+i\theta_1)e^{it(u+i\theta_1)}du = \int_{-\infty}^{\infty} F(u+i\theta_2)e^{it(u+i\theta_2)}du \ (0 < \theta_1, \theta_2 < \pi).$$

By fixing θ_1 as ϕ_0 $(0 < \phi_0 < \pi)$ and letting $\theta_2 = \phi$ vary freely on the interval $(0, \pi)$, we arrive at

$$(3.2.12) \qquad (2\pi)^{-1/2} \int_{-\infty}^{\infty} F(u+i\phi)e^{itu}du = e^{\phi t}\mathbf{f}(t) \quad \text{with}$$

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} |F(u+i\phi)|^2 du = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{2\phi t}|\mathbf{f}(t)|^2 dt \ (0 < \phi < \pi),$$

where the last equality is the Parseaval Equality for the inverse Fourier-Plancherel transform for $L_2(\mathbb{R})$-functions. However, by retrieving $F(u+i\phi)$ through the Fourier-Plancherel transform, we have that

$$e^{(u+i\phi)/2} f(e^{(u+i\phi)}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{\phi t}\mathbf{f}(t)e^{-iut}dt \ ((u, \phi) \in \mathbb{R} \times (0, \pi)),$$

which after the substitution $u = \ln r$ for the integration variable gives us the inverse Mellin-Transformation representation

$$(3.2.13) \qquad f(re^{i\phi}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{f}(t)(re^{i\phi})^{-it-1/2}dt \ (re^{i\phi} \in \Pi_+) \text{ with}$$

$$\int_{-\infty}^{\infty} e^{2\phi t}|\mathbf{f}(t)|^2 dt = \int_{-\infty}^{\infty} |f(re^{i\phi})|^2 dr \ (0 < \phi < \pi).$$

In the Parseaval Equality of statement (3.2.12) we note that as $\phi \nearrow \pi$ from within $(0, \pi)$, $e^{2\phi t}|\mathbf{f}(t)|^2$ defines on the subsets $(-\infty, 0)$ and $(0, \infty)$ a monotone decreasing and increasing ϕ-parameter family of $L_1(-\infty, 0)$- and $L_1(0, \infty)$-functions, which converge to $e^{2\pi t}||\mathbf{f}(t)|^2$ respectively. The Lebesgue Monotone Convergence Theorem guarantees through

$$\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} e^{2\phi t}|\mathbf{f}(t)|^2 dt = \int_{-\infty}^{\infty} e^{2\phi t}|\mathbf{f}(t)|^2 dt \leq ||f||_{s(2)}{}^2 \text{ that}$$

$$\int_{-\infty}^{\infty} e^{2\pi t}|\mathbf{f}(t)|^2 dt = \lim_{\phi\to\pi} \int_{-\infty}^{\infty} e^{2\phi t}|\mathbf{f}(t)|^2 dt \text{ as } \phi \to \pi \text{ from within } (0,\pi).$$

Correspondingly, as $\phi \searrow 0$ we have a ϕ-parameter family of $L_1(-\infty,0)$- and $L_1(-\infty,0)$-functions, which increase and decrease to $|\mathbf{f}(t)|^2$ respectively on the intervals $(-\infty,0)$ and $(0,\infty)$. Application of the Lebesgue Monotone Convergence Theorem like before, lets us conclude that

$$\int_{-\infty}^{\infty} |\mathbf{f}(t)|^2 dt = \lim_{\phi\to 0} \int_{-\infty}^{\infty} e^{2\phi t}|\mathbf{f}(t)|^2 dt \text{ as } \phi \to 0 \text{ from within } (0,\pi).$$

It is tacitly understood that we first showed that $|\mathbf{f}(t)|^2$ and $e^{2\pi t}|\mathbf{f}(t)|^2$ define two $L_1(\mathbb{R})$-functions and that the Lebesgue Dominated Convergence Theorem, in terms of the estimate

$$e^{2\phi t}|\mathbf{f}(t)|^2 \leq [1 + e^{2\pi t}]|\mathbf{f}(t)|^2$$

with $[1 + e^{2\pi \cdot}]|\mathbf{f}(\cdot)|^2$ being a non-negative $L_1(\mathbb{R})$-function, lets us replace the limits $\phi \searrow 0$ and $\phi \nearrow \pi$ from within $(0,\pi)$ by lim as $\phi \to 0$ and $\phi \to \pi$ from within $(0,\pi)$. Thus we have

$$\text{(3.2.14)} \qquad \int_{-\infty}^{\infty} |\mathbf{f}(t)|^2 dt, \int_{-\infty}^{\infty} e^{2\pi t}|\mathbf{f}(t)|^2 dt \leq ||f||_{s(2)} \ (f \in \mathfrak{G}_2),$$

and by means of these inequalities, we can extend the definition of a $\mathfrak{G}_2$-function f to the positive and negative real axis by

$$\text{(3.2.15)} \quad f(re^{i\psi}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{f}(t)(re^{i\psi})^{-it-1/2} dt \text{ for almost all } r > 0 \ (\psi = 0,\pi).$$

The limit functions $\mathbf{f}(t)$ and $e^{\pi t}\mathbf{f}(t)$ belong to $L_2(\mathbb{R})$ and allow the holomorphic function $F(w) = e^{w/2}f(e^w)$ of the infinite strip $\mathbb{R} + i(0,\pi)$ to have the property that

$$(3.2.16) \qquad \lim_{\phi \to \psi}(2\pi)^{-1/2}\int_{-\infty}^{\infty}|F(u+i\phi) - F(u+i\psi)|^2 du =$$

$$\lim_{\phi \to \psi}(2\pi)^{-1/2}\int_{-\infty}^{\infty}|e^{\phi t} - e^{\psi t}|^2|\mathbf{f}(t)|^2 dt = 0 \text{ for } \phi \to \psi \text{ from within } (0,\pi),$$

because:

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}F(u+i\phi)e^{iut}du = e^{\phi t}\mathbf{f}(t),\ e^{\phi t} \to e^{\psi t} \text{ as } \phi \to \psi$$

$$\text{from within } (0,\pi) \text{ and } |e^{\phi t} - e^{\psi t}| \le 2[1 + e^{2\pi t}] \text{ for all real } t.$$

As a few times already before, the Lebersgue Dominated Convergence Theorem lets us conclude that the two limits in statement (3.2.16) are zero. Substituting in the left limit statement for function F its expression in terms of f, namely

$$F(u+i\phi) = e^{(u+i\phi)/2}f(e^{(u+i\phi)}) \text{ and } F(u+i\psi) = e^{(u+i\psi)/2}f(e^{(u+i\psi)}),$$

and replacing the variable u by $\ln r$ leads to

$$(3.2.17) \qquad \lim_{\phi \to \psi}||e^{i\phi/2}f(\cdot e^{i\phi}) - e^{i\psi/2}f(\cdot e^{i\psi})||_{L_2(0,\infty)} = 0 \text{ as}$$

$$\phi \to \psi \text{ from within } (0,\pi)\ (\psi = 0,\pi),$$

and after using

$$||f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})||_{L_2(0,\infty)} \le$$

$$||e^{i\phi/2}f(\cdot e^{i\phi}) - e^{i\psi/2}f(\cdot e^{i\psi})||_{L_2(0,\infty)} + ||e^{i\phi/2} - e^{i\psi/2}|||f(\cdot e^{i\psi})||_{L_2(0,\infty)},$$

we obtain the following

PROPOSITION 3.2.2. *Every $\mathfrak{G}_2$-function possesses angular limit functions in the a.e. pointwise- as well as in the $L_2(0,\infty)$-sense as $\phi \to \psi$ from within $(0,\pi)$ - i.e.*

$$\lim_{\phi\to\psi} f(re^{i\phi}) = f(re^{i\psi}) \text{ for almost all } r > 0 \ (\psi = 0, \pi), \text{ and} \tag{3.2.18}$$

$$\lim_{\phi\to\psi} \|f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})\|_{L_2(0,\infty)} = 0 \ (\psi = 0, \pi). \tag{3.2.19}$$

The limit statement (3.2.17) suffers from the shortcoming that for any $u \in \partial\Pi_+$ (that is $u \in \mathbb{R}$), the value $f(u)$ has to be calculated by means of

$$f(u) = \lim_{\phi\to 0^+} f(|u|e^{i\phi}) \text{ or } \lim_{\phi\to\pi^-} f(|u|e^{i\phi}) \text{ according as } u > 0 \text{ or } u < 0,$$

in other words the limit is calculated along the circular path $|z| = |u|$ from above the real axis. This does not say anything about the limit of $f(z)$ as $z \to u$ from within Π_+ generally. Consequently, a stronger version of limit statement (3.2.17), which takes into account the conformality of the transformation $w \mapsto e^w$, mapping the closed infinite strip $\mathbb{R} + i[0,\pi]$ ($\mathbb{R} \times [0,\pi]$) onto the punctured closed upper half-plane

$$(\Pi_+)^- \setminus \{0\} = \{z \in \mathbb{C} : z \neq 0, \mathfrak{Im}(z) \geq 0\},$$

is given in

THEOREM 3.2.3. *If $f \in \mathfrak{G}_2$, then as $z \to re^{i\psi}$ from within Π_+*

$$\lim_{z\to re^{i\psi}} f(z) = f(re^{i\psi}) \text{ for almost all } r > 0 \ (\psi = 0, \pi). \tag{3.2.20}$$

PROOF. We choose for any $w \in \mathbb{R} + i(0,\pi)$ a rectangle $\mathcal{R} = \mathcal{R}(A;\theta_1,\theta_2)$ as depicted by Figure 2 such that $0 < \theta_1 < \mathfrak{Im}(w) < \theta_2 < \pi$ and $A > |\mathfrak{Re}(w)|$ and write the Cauchy-Integral Formula for $F(w)$ in this rectangle - i.e.

$$F(w) = (2\pi i)^{-1} \oint_{\partial\mathcal{R}} F(\zeta)[\zeta - w]^{-1} d\zeta = (2\pi i)^{-1} \int_{-A}^{A} F(\xi + i\theta_1)[\xi + i\theta_1 - w]^{-1} d\xi +$$

$$(2\pi)^{-1} \int_{\theta_1}^{\theta_2} F(A + i\eta)[A + i\eta - w]^{-1} d\eta + (2\pi i)^{-1} \int_{A}^{-A} F(\xi + i\theta_2)[\xi + i\theta_2 - w]^{-1} d\xi +$$

$$(2\pi)^{-1} \int_{\theta_2}^{\theta_1} F(-A + i\eta)[-A + i\eta - w]^{-1} d\eta.$$

We note that limit statement (3.2.7) is valid uniformly on every compact subset of $(0, \pi)$ (Proposition 3.2.2 in terms of $F(w) = e^{w/2} f(e^w)$) and then by letting $A \to \infty$ in the immediately preceeding formula for $F(w)$, we have

$$F(w) = (2\pi i)^{-1} \int_{-\infty}^{\infty} F(\xi + i\theta_1)[\xi + i\theta_1 - w]^{-1} d\xi - (2\pi i)^{-1} \int_{-\infty}^{\infty} F(\xi + i\theta_2)[\xi + i\theta_2 - w]^{-1} d\xi.$$

Here we used the facts that $F(\cdot + i\theta_k)$ and $[\cdot + i\theta_k - w]^{-1}$ are both $L_2(\mathbb{R})$-functions $(k = 1, 2)$. Since θ_1 and θ_2 are arbitrary as long as $0 < \theta_1 < \mathfrak{Im}(w) < \theta_2 < \pi$, we may let $\theta_1 \to 0^+$ and $\theta_2 \to \pi^-$, and thus achieve, by means of limit statements (3.2.16)

$$\lim_{\theta_1 \to 0^+} ||F(\cdot + i\theta_1) - F(\cdot + i0)||_{L_2(\mathbb{R})} = \lim_{\theta_2 \to \pi^-} ||F(\cdot + i\theta_2) - F(\cdot + i\pi)||_{L_2(\mathbb{R})} = 0$$

as well as the self-evident ones

$$\lim_{\theta_1 \to 0^+} ||[\cdot + i\theta_1 - w]^{-1} - [\cdot - w]^{-1}||_{L_2(\mathbb{R})} =$$

$$\lim_{\theta_2 \to \pi^-} ||[\cdot + i\theta_2 - w]^{-1} - [\cdot + i\pi - w]^{-1}||_{L_2(\mathbb{R})} = 0,$$

the fact that

$$(3.2.21) \qquad F(w) = (2\pi i)^{-1} \int_{-\infty}^{\infty} F(\xi + i0)[\xi - w]^{-1} d\xi - (2\pi i)^{-1} \int_{-\infty}^{\infty} F(\xi + i\pi)$$

$$[\xi + i\pi - w]^{-1} d\xi = F_0(w) - F_\pi(w).$$

The first integral expression herein is denoted by $F_0(w)$ and it represents an $H_2(\Pi_+)$-function of the upper half-plane (of the w-plane), whereas the second integral $F_\pi(w)$ is an $H_2(i\pi - \Pi_+)$-function of the lower half-plane $i\pi - \Pi_+ = \{w \in \mathbb{C} : \mathfrak{Im}(w) < \pi\}$. For these, the following limit statements hold on the lower edge $\mathbb{R}$ of the boundary of the strip $\mathbb{R} + i(0, \pi)$:

$$\lim_{w \to u} F_0(w) \text{ exists as } w \to u \text{ non-tangentially from within } \mathbb{R} + i(0,\pi) \text{ for almost}$$

$$\text{all } u \in \mathbb{R}, \lim_{w \to u} F_\pi(w) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(\xi + i\pi)[\xi - u + i\pi]^{-1} d\xi \text{ for all } u \in \mathbb{R}.$$

We have correspondingly for the upper edge $\mathbb{R} + i\pi$ of the infinite strip:

$$\lim_{w \to u + i\pi} F_\pi(w) \text{ exists as } w \to u + i\pi \text{ non-tangentially from within}$$

$$\mathbb{R} + i(0,\pi) \text{ for almost all } u \in \mathbb{R},$$

$$\lim_{w \to u + i\pi} F_0(w) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(\xi + i0)[\xi - u - i\pi]^{-1} d\xi \text{ for all } u \in \mathbb{R}.$$

These we summarize in terms of the limit statements (3.2.16) as

$$\lim_{w \to u + i\psi} F(w) = F(u + i\psi) \text{ as } w \to u + i\pi \text{ non-tangentially} \tag{3.2.22}$$

$$\text{from within the infinite strip } \mathbb{R} + i(0,\pi).$$

However, because of Proposition 3.2.2 and the fact that $w \mapsto e^w$ is a conformal map of the closed infinite strip $\mathbb{R} + i[0, \pi]$ onto the punctured closed upper half-plane $(\Pi_+)^- \setminus \{0\}$, the process "$w \to u + i\psi$ non-tangantially from within the infinite strip" maps onto the process "$z \to re^{i\psi}$ non-tangentially from within Π_+" ($\psi = 0, \pi$) for all $r > 0$ and vica versa with the exception of $r = 0$. This completes the proof. □

Theorem 3.2.3 implies that every $\mathfrak{G}_2$-function f belongs to $\mathfrak{H}_2$, because limit statement (3.2.20) combined with relation (3.2.19) of Proposition 3.2.2 lets us construct out of $f(\cdot e^{i\psi}) \in L_2(0,\infty)$ ($\psi = 0, \pi$) the $L_2(\mathbb{R})$-"boundary value function" $f(\cdot + i0)$ of f, namely

$$f(x + i0) = f(xe^{i0}) \text{ or } f((-x)e^{i\psi}) \text{ according as } x > 0 \text{ or } x < 0.$$

This $f(\cdot + i0)$ is a boundary value satisfying properties (1.1.1), (1.1.3) and especially (1.1.4), thereby affirming that $f \in \mathfrak{H}_2$ and further

$$||f||_{\mathfrak{H}_2}{}^2 = ||f(\cdot + i0)||_{L_2(\mathbb{R})}{}^2 = ||f(\cdot e^{i0})||_{L_2(0,\infty)}{}^2 + ||f(\cdot e^{i\pi})||_{L_2(0,\infty)}{}^2 \leq$$

$$2 \sup_{0<\phi<\pi} \int_0^\infty |f(re^{i\phi})|^2 dr = 2||f||_{s(2)}{}^2$$

on account of limit property (3.2.19). Therefore, we have proved

PROPOSITION 3.2.4. *$\mathfrak{G}_2 = \mathfrak{H}_2$ and every $\mathfrak{H}_2$-function f satisfies*

$$||f||_{\mathfrak{H}_2} \leq \sqrt{2}||f||_{s(2)}.$$

We may extend the Parseaval Equality of the Mellin-Transform representation (3.2.12) to all $\phi \in [0, \pi]$, because $f(\cdot e^{i\phi}) \to f(\cdot e^{i\psi})$ in the $L_2(0,\infty)$-sense as $\phi \to \psi$ ($\psi = 0, \pi$) from within $(0, \pi)$ (statement (3.2.19) of Proposition 3.2.2), and thus see that

$$||f(\cdot e^{i\phi})||_{L_2(0,\infty)}{}^2 \text{ is a convex function of } \phi \text{ on the closed interval } [0, \pi],$$

because $e^{2\phi t}$ is convex as a function of ϕ for each $t \in \mathbb{R}$. In particular

$$(3.2.23) \qquad \int_0^\infty |f(re^{i\phi})|^2 dr \leq (\phi/\pi) \int_0^\infty |f(re^{i\pi})|^2 dr + (1 - \phi/\pi) \int_0^\infty |f(re^{i0})|^2 dr$$

$$(0 \leq \phi \leq \pi)$$

and further, we have from this convexity that the supremum of $||f(\cdot e^{i\phi})||_{L_2(0,\infty)}$ is attained at $\phi = \psi$ ($\psi = 0, \pi$) - i.e.

$$||f||_{s(2)} = \sup_{0 \leq \phi \leq \pi} ||f(\cdot e^{i\phi})||_{L_2(0,\infty)} = max\{||f(\cdot e^{i\psi})||_{L_2(0,\infty)} : \psi = 0, \pi\}. \tag{3.2.24}$$

The best possible norm equivalence between $|| \cdot ||_{\mathfrak{H}_2}$ and $|| \cdot ||_{s(2)}$ is therefore

$$||f||_{s(2)} \leq ||f||_{\mathfrak{H}_2} \leq \sqrt{2}||f||_{s(2)}, \tag{3.2.25}$$

which is a consequence of

$$||f(\cdot e^{i\psi})||_{L_2(0,\infty)} \leq ||f(\cdot + i0)||_{L_2(\mathbb{R})} = ||f||_{\mathfrak{H}_2} \; (\psi = 0, \pi). \tag{3.2.26}$$

3.3. Inverse Mellin-Transform Representation of $\mathfrak{H}_2$

My thesis director demonstrated ([12]), that the totality $\mathfrak{K}_p$ $(1 < p < \infty)$ of all radially acting linear integral operators (Section 1.2) with action (1.0.1) on $\mathfrak{H}_p$ constitutes a Banach algebra; however, for the case of $p = 2$, we can say even more. Using the result of my thesis director for the special case of $p = 2$, in particular that the radial integral (1.2.2) defines the bounded bilinear functional

$$\langle \cdot \, , \, \cdot \rangle : \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathbb{C} \text{ is independent of } \phi \; (0 \leq \phi \leq \pi),$$

we have for $K \in \mathfrak{K}_2$ and its ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$ that

$$\langle g, Kf \rangle = \int_0^\infty g(re^{i\phi})(Kf)(re^{i\phi})e^{i\phi}dr = \tag{3.3.27}$$

$$\int_0^\infty \int_0^\infty g(re^{i\phi})K(r, r', \phi)f(r'e^{i\phi})e^{i2\phi}dr'dr \; (f, g \in \mathfrak{H}_2)$$

exists (because $(g(\cdot e^{i\phi}), (Kf)(\cdot e^{i\phi}) \in L_2(0,\infty)$ $(0 \leq \phi \leq \pi))$ and is independent of ϕ. Moreover, by expressing f and g in terms of their respective inverse Mellin-Transform representations

$$g(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\mathbf{g}(t)(re^{i\phi})^{-it-1/2}dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty}\mathbf{g}(-t)(re^{i\phi})^{it-1/2}dt$$

and relation (3.2.12), where the last integral was obtained by replacing t with $-t$ and writing in the integral (3.2.12) for $f(r'e^{i\phi})$ the variable of integration as t' instead of t, we convert equation (3.2.27) to

$$(3.3.28)\qquad \langle\, g\ ,\ Kf\,\rangle = \int_{0}^{\infty}\mathbf{g}(-t)\Big[(2\pi)^{-1/2}\int_{0}^{\infty}(Kf)(re^{i\phi})(re^{i\phi})^{it-1/2}e^{i\phi}dr\Big]dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{g}(-t)\Big[(2\pi)^{-1}\int_{0}^{\infty}\int_{0}^{\infty}K(r,r',\phi)(re^{i\phi})^{it-1/2}(r'e^{i\phi})^{-it-1/2}e^{i2\phi}dr'dr\Big]\mathbf{f}(t')dt'dt$$

after invoking the Fubini-Tonelli Theorem twice. This quadruple integral is independent of ϕ for all $\mathbf{g}$ and $\mathbf{f}$ belonging to the Hilbert space

$$(3.3.29)\qquad \mathfrak{H}_2 = \Big\{\mathbf{f} : ||f||_{\mathfrak{H}_2} \equiv \Big(\int_{-\infty}^{\infty}[1+e^{2\pi t}]|\mathbf{f}(t)|^2dt\Big)^{1/2} < \infty\Big\}$$

(these $\mathbf{f}$'s are tacitly assumed Lebesgue measurable on $\mathbb{R}$) with the inner product

$$\langle\, \mathbf{f} \mid \mathbf{g}\,\rangle = \int_{-\infty}^{\infty}[1+e^{2\pi t}]\mathbf{f}(t)\overline{\mathbf{g}(t)}dt$$

ensuing from polarization. This can only take place if

$$(3.3.30)\qquad (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(r,r',\phi)(re^{i\phi})^{it-1/2}(r'e^{i\phi})^{-it'-1/2}e^{2\pi}dr'dr =$$

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(e^u,e^v,\phi)(e^{u+i\phi})^{it+1/2}(e^{v+i\phi})^{-it'+1/2}dvdu = \mathbf{K}(t,t')$$

is a Lebesgue measurable function on $\mathbb{R}^2$ independent of ϕ.

In this equality we replaced (r, r') by (e^u, e^v) and the Fourier-Integral's existence is justfied by the Fourier-Plancherel Theorem on $L_2(\mathbb{R}^2)$ (Proposition 2.2.2). In particular, $K(e^u, e^v, \phi)e^{(u+v)/2}$ is an $L_2(\mathbb{R})$-function, because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K(e^u, e^v, \phi)|^2 e^{u+v} du dv = \int_0^{\infty}\int_0^{\infty} |K(r, r', \phi)|^2 dr dr' = |||K_\phi|||^2.$$

By factoring out $e^{-\phi(t-t'-i)}$ from the integrand in each double integral of equality (3.3.30), we obtain

$$(3.3.31) \qquad (2\pi)^{-1}\int_0^{\infty}\int_0^{\infty} K(r, r', \phi) r^{it-1/2} r'^{-it'-1/2} dr dr' =$$

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(e^u, e^v, \phi) e^{(u+v)/2} e^{i(ut-vt')} dv du =$$

$$e^{\phi(t-t'-i)}\mathbf{K}(t, t') \ ((t, t') \in \mathbb{R}^2; 0 < \phi < \pi).$$

Herein we replace t by $-t$ to change the second integral into the inverse Fourier-Transform format

$$(3.3.32) \qquad (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(e^u, e^v, \phi) e^{(u+v)/2} e^{i(ut+vt')} dv du = e^{-\phi(t+t'+i)}\mathbf{K}(-t, t')$$

in two variables on $\mathbb{R}^2$ for each ϕ $(0 < \phi < \pi)$

(Proposition 2.2.2), for which the Parseaval Equality states

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K(e^u, e^v, \phi)|^2 e^{(u+v)} dv du = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\phi(t+t')}|\mathbf{K}(-t, t')|^2 dt' dt \ (0 < \phi < \pi).$$

Noting herein that the first integral is the double norm of K_ϕ (let $u = \ln r$ and $v = \ln r'$) and the second is the square of the $L_2(\mathbb{R}^2)$-norm of $e^{\phi(t-t'-i)}\mathbf{K}(t, t')$ (just replace the integration variable t by $-t$), we can directly conclude that

$$(3.3.33) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\phi(t-t')}|\mathbf{K}(t, t')|^2 dt' dt = |||K_\phi|||^2 \leq |||K|||_{s(2)}{}^2 \ (0 < \phi < \pi).$$

We now show that the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r,r',\phi)$ possess boundary $\mathcal{L}_2$-kernels $K(r,r',\psi)$ $(\psi = 0,\pi)$ as $\phi \to \psi$ from within $(0,\pi)$. To this end, we let $\Delta^- = \{(t,t') \in \mathbb{R}^2 : t > t'\}$ and $\Delta^+ = \{(t,t') \in \mathbb{R}^2 : t < t'\}$, observe that Δ^- and Δ^+ are two disjoint Lebesgue measurable subsets of $\mathbb{R}^2$ (because Δ^- and Δ^+ are open in $\mathbb{R}^2$), $\partial\Delta^- = \partial\Delta^+ = \{(t,t) : t \in \mathbb{R}\}$ is also Lebesgue measurable with two dimensional Lebesgue measure equal to zero, and hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\phi(t-t')}|\mathbf{K}(-t,t')|^2 dt'dt = \int\int_{\Delta^-} e^{2\phi(t-t')}|\mathbf{K}(-t,t')|^2 dt'dt +$$

$$\int\int_{\Delta^+} e^{2\phi(t-t')}|\mathbf{K}(-t,t')|^2 dt'dt \leq |||K|||_{s(2)}{}^2.$$

As $\phi \searrow \pi$ from within $(0,\pi)$, the integrands of the integrals over domain Δ^- and Δ^+ monotonely increase and decrease respectively, in terms of parameter ϕ, to the function $exp(2\pi(t-t'))|\mathbf{K}(t,t')|^2$. On the other hand, as $\phi \nearrow \pi$ from within $(0,\pi)$, we have that the selfsame integrands become respectively monotone decreasing and increasing in parameter ϕ with limiting values $exp(0(t-t'))|\mathbf{K}(t,t')|^2 = |\mathbf{K}(t,t')|^2$. In consequence hereof, the Lebesgue Monotone Convergence Theorem lets us conclude that

$$\text{(3.3.34)} \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}]|\mathbf{K}(t,t')|^2 dtdt' \leq 2|||K|||_{s(2)}{}^2.$$

We return to equation (3.3.27) and take notice of the fact that

$$(2\pi)^{-1}\int_0^{\infty} (Kf)(re^{i\phi})(re^{i\phi})^{it-1/2}e^{i\phi}dr = (2\pi)^{-1}\int_0^{\infty} (Kf)(e^{u+i\phi})$$

$$e^{(u+i\phi)/2}(e^{u+i\phi})^{it}du = (2\pi)^{-1}\int_{-\infty}^{\infty} F(u+i\phi)(e^{u+i\phi})^{it}du \text{ with}$$

$$F(u+i\phi) = (Kf)(e^{u+i\phi})e^{(u+i\phi)/2}, \text{ where } Kf \in \mathfrak{H}_2.$$

Relations (3.2.11) and (3.2.12) for the specific holomorphic function F constructed from Kf of the infinite strip $\mathbb{R}+i(0,\pi)$, namely the independence of the immediately preceeding integral of ϕ, tells us that

$$(3.3.35) \qquad (2\pi)^{-1}\int_{-\infty}^{\infty}(Kf)(re^{i\phi})(re^{i\phi})^{it-1/2}e^{i\phi}dr = (\mathbf{Kf})(t) \text{ with}$$

$$\int_{-\infty}^{\infty}[1+e^{2\pi t}]|\mathbf{Kf}(t)|^2dt < \infty.$$

Therewith, we reduce equation (3.3.27) to

$$(3.3.36) \qquad \int_{-\infty}^{\infty}\mathbf{g}(-t)(\mathbf{Kf})(t)dt = \int_{-\infty}^{\infty}\mathbf{g}(-t)(\mathbf{K}(t,t')\mathbf{f})(t')dt'dt \ (\mathbf{f},\mathbf{g}\in\mathfrak{H}_2)$$

in particular for all $\mathbf{g} \in C_c(\mathbb{R})$ - i.e. the space of continuous functions on $\mathbb{R}$ with compact support. Out of this follows immediately

$$(3.3.37) \qquad (\mathbf{Kf})(t) = \int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{f}(t')dt' \ (t\in\mathbb{R}),$$

and the a.e equality implied by equation (3.3.36) may be dropped by redefining $\mathbf{K}(t,t')$ on a set of two dimensional measure zero in such way, that $\mathbf{K}(t,t')$ and $exp(\pi(t-t'))\mathbf{K}(t,t')$ are both $\mathcal{L}_2$-kernels on $\mathbb{R}^2$ ([16], pgs. 14 - 16). This is feasible, because inequality (3.3.33) entails that $\mathbf{K}(t,t')$ and $exp(\pi(t-t'))\mathbf{K}(t,t')$ are both $\mathcal{L}_2$-kernels in the wide sense. We have therefore almost demonstrated

PROPOSITION 3.3.1. *Every $K \in \mathfrak{K}_2$ is inverse Mellin-Transform representable kernel through an $\mathfrak{K}_2$-kernel $\mathbf{K}(t,t')$ on $\mathbb{R}^2$ in the sense of*

$$(3.3.38) \qquad K(r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}(t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dtdt'$$

$$\textit{with} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t'}]|\mathbf{K}(t,t')|^2dt'dt < \infty,$$

and the action of the radially acting linear integral operator $K \in \mathcal{B}(\mathfrak{H}_2)$ *on* $\mathfrak{H}_2$ *in terms of* $\mathbf{K}$ *and* $\mathbf{f}$ *is*

$$(3.3.39) \qquad (Kf)(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathbf{Kf})(t)(re^{i\phi})^{-it-1/2}dt$$

$$\textit{with } (\mathbf{Kf})(t) = \int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{f}(t')dt',$$

where $\mathbf{f}$ *gives the Mellin-Transform representation (3.2.13) of* f *in* Π_+.

PROOF. That Kf is Mellin-Transform representable through $\mathbf{Kf}$, as defined by equation (3.3.35), has just been shown. Consequently, relation (3.3.39) is nothing more than the Mellin-Transform representation of $Kf \in \mathfrak{H}_2$. To prove representation (3.3.38), we solve for the $L_2(\mathbb{R}^2)$-function $K(e^u, e^v, \phi)exp((u+v)/2)$ of variable (u,v) in equation (3.3.30) by utilizing the inverse Fourier-Transform and thereby have that

$$(3.3.40) \qquad K(e^u, e^v, \phi)e^{(u+v)/2} = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\phi(t+t'+i)}\mathbf{K}(-t,t')e^{i(ut+vt')}dt'dt.$$

This equation we multiply by $exp(-(u+v)/2)$ and thereafter replace (e^u, e^v) with (r, r') and t with $-t$, which yields representation (3.3.38) and completes the proof. □

3.4. Boundary Value $\mathcal{L}_2$-Kernels of $\mathfrak{K}_2$-Kernels

As a result of norm inequaltiy (3.3.33), we may introduce the boundary value $\mathcal{L}_2$-kernels through the use of the $L_2(\mathbb{R}^2)$-functions appearing in equation (3.3.40), where we bring out the fact that norm inequality (3.3.33) assumes the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{-2\pi(t+t')}]|\mathbf{K}(-t,t')|^2dt'dt \leq 2|||K|||_{s(2)}{}^2,$$

when we replace t by $-t$ in it. Consequently, by setting $\phi = \psi$ ($\psi = 0, \pi$) in equation (3.3.40), we immediately have the $L_2(\mathbb{R}^2)$-functions

$$(3.4.41) \qquad K(e^u, e^v, \psi)e^{(u+v)/2} \equiv (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\psi(t+t'+i)} \mathbf{K}(-t, t') e^{i(ut+vt')} dt' dt$$

$$(\psi = 0, \pi)$$

of variables (u, v) on $\mathbb{R}^2$, as guaranteed by the Parseaval Equality for the $L_2(\mathbb{R}^2)$-functions $e^{-\psi(t+t'+i)}\mathbf{K}(-t, t')$ ($\psi = 0, \pi$) respectively. We apply to the defining equations (3.4.41) the selfsame process that converted equation (3.3.40) to represention (3.3.38) in the proof of Proposition 3.3.1, and thus we have

$$(3.4.42) \qquad K(r, r', \psi) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{K}(t, t')(re^{i\psi})^{-it-1/2}(r'e^{i\psi})^{it'-1/2} dt' dt$$

$$(\psi = 0, \pi).$$

Again, the Parseaval Equality (the Fourier-Transform version thereof) allows us to conclude for the difference of the expressions (3.3.38) and (3.4.42), written in the forms (3.3.40) and (3.4.41) respectively, that

$$(3.4.43) \qquad \int_0^{\infty} \int_0^{\infty} |K(r, r', \phi) - K(r, r', \psi)|^2 dr' dr = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |e^{-\phi(t+t'+i)} -$$

$$e^{-\psi(t+t'+i)}|^2 |\mathbf{K}(-t, t')|^2 dt' dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |e^{\phi(t-t'-i)} - e^{\psi(t-t'-i)}|^2 |\mathbf{K}(t, t')|^2 dt' dt.$$

The Lebesgue Dominated Convergence Theorem applies to this last expression, because its integrand is bounded from above by the non-negative $L_1(\mathbb{R}^2)$-function $2[1 + e^{2\pi(t-t')}]|\mathbf{K}(t, t')|^2$. Therefore, the fact that $exp(\phi(t-t'-i)) \to exp(\psi(t-t'-i))$ as $\phi \to \psi$ from within $(0, \pi)$ lets us conclude

PROPOSITION 3.4.1. *Every $K \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)(0 < \phi < \pi)$ possesses boundary $\mathcal{L}_2$-kernels $K(r, r', \psi)$ $(\psi = 0, \pi)$ satisfying $|||K_\psi||| \leq |||K|||_{s(2)}$ $(\psi = 0, \pi)$ and*

$$(3.4.44) \qquad \lim_{\phi \to \psi} \int_0^\infty \int_0^\infty |K(r, r', \phi) - K(r, r', \psi)|^2 dr' dr = 0$$

as $\phi \to \psi$ from within $(0, \pi)$ $(\psi = 0, \pi)$.

Limit statement (3.4.44) can be expressed in terms of the $\mathcal{L}_2$-kernels $K_\phi(r, r')$ and $K_\psi(r, r')$ in the form

$$(3.4.45) \qquad \lim_{\phi \to \psi} |||K_\phi - K_\psi||| = 0 \text{ as } \phi \to \psi \text{ from within } (0, \pi) \ (\psi = 0, \pi),$$

because

$$|||K_\phi - K_\psi||| = ||K(\cdot, \cdot, \phi)e^{i\phi} - K(\cdot, \cdot, \psi)e^{i\psi}||_{L_2((0,\infty)^2)} \leq$$

$$|e^{i\phi} - e^{i\psi}||||K_\phi||| + ||K(\cdot, \cdot, \phi)e^{i\phi} - K(\cdot, \cdot, \psi)e^{i\psi}||_{L_2((0,\infty)^2)},$$

which tends to zero as $\phi \to \pi$ from within $(0, \pi)$ ($|||K_\phi||| \leq |||K|||_{s(2)}$ for $0 < \phi < \pi$). Proposition 3.4.1 lets us further extend the domain $(0, \pi)$ of the parameter ϕ appearing in the kernels of $K \in \mathfrak{K}_2$ to $[0, \pi]$ by extending to $K(r, r', \psi)$ $(\psi = 0, \pi)$, and thereby having the ϕ-parameter $K(r, r', \phi)$ $(0 \leq \phi \leq \pi)$ correspond to the radially acting integral operator $K \in \mathfrak{K}_2$. We shall henceforth use the closed interval $[0, \pi]$ as the domain for parameter ϕ whenever dealing with any $f \in \mathfrak{H}$ (Proposition 3.2.2) as well as any $K \in \mathfrak{K}_2$ (Proposition 3.4.1), and continue by formulating the next

THEOREM 3.4.2. *For any $K \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$ $(0 \leq \phi \leq \pi)$ and $f \in \mathfrak{H}_2$, we have that $Kf \in \mathfrak{H}_2$ possesses the following* *properties:*

$$(3.4.46)\quad (Kf)(re^{i\psi}) = \int_0^\infty K(r,r',\psi)f(r'e^{i\psi})e^{i\psi}dr' \text{ a.e. in } r \text{ on } (0,\infty)\ (\psi = 0,\pi),$$

$$(3.4.47)\quad \lim_{z\to re^{i\psi}}(Kf)(z) = (Kf)(re^{i\psi}) \text{ as } z \to re^{i\psi} \text{ non-tangentially from within } \Pi_+$$

$$\text{for almost all } r > 0\ (\psi = 0,\pi), \text{ and}$$

$$(3.4.48)\quad \lim_{\phi\to\psi}||(Kf)(\cdot e^{i\phi}) - (Kf)(\cdot e^{i\psi})||_{L_2(0,\infty)} = 0 \text{ as } \phi \to \psi \text{ from within } (0,\pi).$$

PROOF. Kf is an $\mathfrak{H}_2$-function and as such possesses the non-tangential limits (3.4.47) from within Π_+ as $z \to re^{i\psi}$ for almost all r in $(0,\infty)$ $(\psi = 0,\pi)$. To show statements (3.4.46) and (3.4.48), we observe that in terms of the $\mathcal{L}_2$-kernels $K_\phi(r,r') \equiv K(r,r',\phi)e^{i\phi}$ $(0 \le \phi \le \pi)$

$$\int_0^\infty K(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' = \int_0^\infty K_\phi(r,r')f(r'e^{i\phi})dr' = (K_\phi f(\cdot e^{i\phi}))(r)\ (0 \le \phi \le \pi)$$

and write

$$||(Kf)(\cdot e^{i\phi}) - \int_0^\infty K(\cdot,r',\psi)f(r'e^{i\psi})e^{i\psi}dr'||_{L_2(0,\infty)} =$$

$$||(K_\phi f(\cdot e^{i\psi})) - (K_\psi f(\cdot e^{i\psi}))||_{L_2(0,\infty)} \le$$

$$||(K_\phi f(\cdot e^{i\phi})) - (K_\psi f(\cdot e^{i\phi})||_{L_2(0,\infty)} + ||(K_\psi f(\cdot e^{i\phi})) - (K_\psi f(\cdot e^{i\psi})||_{L_2(0,\infty)} =$$

$$||([K_\phi - K_\psi]f(\cdot e^{i\phi}))||_{L_2(0,\infty)} + ||K_\psi(f(\cdot e^{i\phi}) - f(\cdot e^{i\psi}))||_{L_2(0,\infty)} \le$$

$$|||K_\phi - K_\psi|||\ ||f(\cdot e^{i\phi})||_{L_2(0,\infty)} + |||K_\psi|||\ ||f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})||_{L_2(0,\infty)} \le$$

$$|||K_\phi - K_\psi|||\ ||f||_{s(2)} + |||K|||_{s(2)}||f(\cdot e^{i\phi}) - f(\cdot e^{i\psi})||_{L_2(0,\infty)}.$$

Herein we applied the Schwarz Inequality twice in going from the expression before the first inequality to that after it, and we utilized the definitions of $||\cdot||_{s(2)}$ and

$|||\cdot|||_{s(2)}$ given in statements (3.0.1) and (1.0.2). Due to the limit values given by statements (3.4.45) and (3.2.19), we arrive at

$$(3.4.49) \qquad \lim_{\phi\to\psi} ||(Kf)(\cdot e^{i\phi}) - \int_0^\infty K(\cdot, r', \psi) f(r'e^{i\psi})e^{i\psi}dr'||_{L_2(0,\infty)} = 0$$

as $\phi \to \psi$ from within $(0, \pi)$.

Because $(Kf)(re^{i\psi})$ is the limit of $(Kf)(z)$ as $z \to re^{i\psi}$ non-tangentially from within Π_+ for almost all $r > 0$ ($\psi = 0, \pi$), in particular, $re^{i\phi} \to re^{i\psi}$ as $\phi \to \psi$ from within $(0, \pi)$ and the circle $|z| = r$ cuts the real axis at angles $\pm\pi/2$, we have out relation (3.4.48) that

$$||(Kf)(\cdot e^{i\psi}) - \int_0^\infty K(\cdot, r', \psi) f(r'e^{i\psi})e^{i\psi}dr'||_{L_2(0,\infty)} = 0$$

which implies statement (3.4.46). Statement (3.4.47) is a direct consequence of Theorem 3.2.3, and the last statement follows after writting the integral expression in limit statement (3.4.49) as $(Kf)(\cdot e^{i\psi})(\psi = 0, \pi)$. This completes the proof of our theorem. $\square$

3.5. Hilbert Space Properties of $\mathfrak{H}_2$ and $\mathfrak{K}_2$

Not only is $||f(\cdot e^{i\phi})||_{L_2(0,\infty)}{}^2$ a convex function of ϕ on $[0, \pi]$ as indicated by inequality (3.2.23) for every $f \in \mathfrak{H}_2$, but so is $|||K_\phi|||^2$ for every $K \in \mathfrak{K}_2$. This is so, because $exp(2\phi(t - t'))$ is convex in ϕ for all conceivable values of $t - t'$ - i.e.

$$exp(2[(1-\lambda)\phi_1 + \lambda\phi_2](t - t')) \leq (1-\lambda)exp(2\phi_1(t - t')) + \lambda exp(2\phi_2(t - t'))$$

$$(0 \leq \lambda \leq 1; 0 \leq \phi_1, \phi_2 \leq \pi; t, t' \in \mathbb{R}) \text{ -}$$

inspite of the two variables (t,t'). We multiply this convexity inequality by $|\mathbf{K}(t,t')|^2$, thereafter integrate over $\mathbb{R}^2$ and thereby arrive at

$$\int_0^\infty \int_0^\infty e^{2[(1-\lambda)\phi_1+\lambda\phi_2](t-t')}|\mathbf{K}(t,t')|^2 dt'dt \leq$$

$$(1-\lambda)\int_{-\infty}^\infty \int_{-\infty}^\infty e^{2\phi_1(t-t')}|\mathbf{K}(t,t')|^2 dt'dt + \lambda \int_{-\infty}^\infty \int_{-\infty} e^{2\phi_2(t-t')}|\mathbf{K}(t,t')|^2 dt'dt,$$

which translates by means of the Parseaval Equality (3.3.33) into

$$|||K_{(1-\lambda)\phi_1+\lambda\phi_2}|||^2 \leq (1-\lambda)|||K_{\phi_1}|||^2 + \lambda|||K_{\phi_2}|||^2 \ (0 \leq \lambda \leq 1; 0 \leq \phi_1, \phi_2 \leq \pi)$$

- i.e. $|||K_\phi|||^2$ is a convex function of ϕ on $[0,\pi]$. Convex functions on a closed interval attain their maximum at the end points, which lets us formulate

THEOREM 3.5.1. *If $K \in \mathfrak{K}_2$, then*

$$|||K|||_{s(2)} = \sup_{0\leq\phi\leq\pi} |||K_\phi||| = max\{|||K_\psi||| : \psi = 1,2\}. \tag{3.5.50}$$

We now turn to the fact that square root of the integrals

$$\int_{-\infty}^\infty |\mathbf{f}(t)|^2[1+e^{2\pi t}]dt \quad \text{and} \quad \int_{-\infty}^\infty \int_{-\infty}^\infty |\mathbf{K}(t,t')|^2[1+e^{2\pi(t-t')}]dt'dt$$

define norms on $\mathfrak{H}_2$ and $\mathfrak{K}_2$, which satisfy the parallelogram law. This is because both of these norms are $\mathfrak{L}_2$-norms with respect to the σ-finite measures

$d\mu(t) = [1+e^{2\pi t}]dt$ and $d\mu(t,t') = [1+e^{2\pi(t-t')}]dt'dt$ of $\mathbb{R}$ and $\mathbb{R}^2$ respectively.

Applying Theorem 3.2.3, we split the integral expression of the norm in equation (1.1.3) into

$$||f||_{\mathfrak{H}_2}{}^2 = ||f(\cdot+i0)||_{L_2(\mathbb{R})}{}^2 = ||f(\cdot e^{i\pi})||_{L_2(0,\infty)}{}^2 + ||f(\cdot e^{i0})||_{L_2(0,\infty)}{}^2 =$$

$$\int_{-\infty}^{\infty} e^{2\pi t}|\mathbf{f}(t)|^2 dt + \int_{-\infty}^{\infty} e^{2(0)t}|\mathbf{f}(t)|^2 dt$$

as consequence of the Parseaval Equality in relation (3.2.13) extended to $\phi = \psi$ ($\psi = 0, \pi$) with the aid of limit statement (3.2.19) and the Lebesgue Monotone (or Dominated) Convergence Theorem. If we write the two $L_2(0,\infty)$-norm expressions in integral form, then we readily see that

$$\int_0^{\infty} |f(-r)|^2 dr + \int_0^{\infty} |f(r)|^2 dr = \int_{-\infty}^{\infty} [1 + e^{2\pi t}]|\mathbf{f}(t)|^2 dt \text{ for all } f \in \mathfrak{H}_2$$

with f having inverse Mellin-Transform representation (3.2.13) through $\mathbf{f} \in \mathfrak{H}_2$. To this equality we apply polarization to determin the inner products that induce the specific norms, and therewith we obtain

$$\int_{-\infty}^{\infty} f(x+i0)\overline{g(x+i0)}dx = \int_0^{\infty} f(-r)\overline{g(-r)}dr + \int_0^{\infty} f(r)\overline{g(r)}dr =$$

$$\int_{-\infty}^{\infty} [1 + e^{2\pi t}]\mathbf{f}(t)\overline{g(t)}dt \; (f, g \in \mathfrak{H}_2),$$

where f and g are inverse Mellin-Transform representable through $\mathbf{f}$ and $\mathbf{g}$ respectively. Furthermore, $f \mapsto \mathbf{f}$ defines an injective linear transformation $\mathfrak{H}_2 \to \mathfrak{H}_2$, and because of the inverse Mellin-Transform (3.2.13), this transformation is bijective. We summarize these results as

Proposition 3.5.2. *The map $\mathbf{f} \mapsto f$ defines a Hilbert space isomorphism from $\mathfrak{H}_2 \to \mathfrak{H}_2$ with inner product relationship*

$$\langle f \mid g \rangle \equiv \int_{-\infty}^{\infty} f(x+i0)\overline{g(x+i0)}dx = \int_0^{\infty} f(-r)\overline{g(-r)}dr + \tag{3.5.51}$$

$$\int_0^{\infty} f(r)\overline{g(r)}dr = \langle \mathbf{f} \mid \mathbf{g} \rangle \equiv \int_{-\infty}^{\infty} [1 + e^{2\pi t}]\mathbf{f}(t)\overline{\mathbf{g}(t)}dt \text{ for all } f \text{ and } g \in \mathfrak{H}_2.$$

The fact that $\mathfrak{H}_2$ is a Hilbert space in terms of inner product $\langle \cdot \mid \cdot \rangle$ we express by writing $\langle \mathfrak{H}_2 \mid \mathfrak{H}_2 \rangle$ or $\langle \cdot \mid \cdot \rangle : \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathbb{C}$; whence $\langle \mathfrak{H}_2 \mid \mathfrak{H}_2 \rangle$ and $\langle \boldsymbol{\mathfrak{H}_2} \mid \boldsymbol{\mathfrak{H}_2} \rangle$. Analogous to $\mathfrak{H}_2$, we can also write $\langle \mathfrak{K}_2 \mid \mathfrak{K}_2 \rangle$, provided we indicate the inner product for $\mathfrak{K}_2$. We observe that out of the Parseaval Equality (3.3.33) extended, via limit statement (3.4.44) and the Lebesgue Dominated Convergence Theorem applied to the integral expression of relation (3.3.33) with $\phi \to \psi$ ($\psi = 0, \pi$) from within $(0, \pi)$, to the values $\phi = \psi$ ($\psi = 0, \pi$) follows

$$\int_0^\infty \int_0^\infty |K(r, r', 0)|^2 dr' dr + \int_0^\infty \int_0^\infty |K(r, r', \pi)|^2 dr' dr =$$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty [1 + e^{2\pi(t-t')}] |\mathbf{K}(t, t')|^2 dt' dt,$$

where $K \in \mathfrak{K}_2$ with inverse Mellin-Transform representation through $\mathbf{K} \in \boldsymbol{\mathfrak{K}_2}$ as realized by Proposition 3.3.1. It is also more than clear, that the two integral expressions to the left of the immediately preceeding equality denote $|||K_0|||^2$ and $|||K_\pi|||^2$ respectively, and are thus squares of $\mathfrak{L}_2$-norms with respect to Lebesgue measure $d\mu(r, r') = dr' dr$ on $[0, \infty]$; whereas the integral on the right denotes the square of the $\mathfrak{L}_2$-norms of $\mathbf{K}$ with respect to the measure $d\mu(t, t') = [1 + e^{2\pi(t-t')}] dt' dt$. Thus the parallelogram law applies to the square roots of all these three integral expressions, and polarization gives us that

$$\text{(3.5.52)} \quad \int_0^\infty \int_0^\infty K(r, r', 0)\overline{L(r, r', 0)} dr' dr + \int_0^\infty \int_0^\infty K(r, r', \pi)\overline{L(r, r', \pi)} dr' dr =$$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty [1 + e^{2\pi(t-t')}] \mathbf{K}(t, t') \overline{\mathbf{L}(t, t')} dt' dt \ (K, L \in \mathfrak{K}_2).$$

Herein, K and $L \in \mathfrak{K}_2$ have inverse Mellin-Transform representation (3.3.38) through $\mathbf{K}$ and $\mathbf{L} \in \boldsymbol{\mathfrak{K}_2}$ respectively. Not only are $\mathfrak{K}_2$ and $\boldsymbol{\mathfrak{K}_2}$ Banach algebras on their own, but they are also Hilbert spaces - i.e.

$$(3.5.53) \qquad \langle\ \mathfrak{K}_2 \mid \mathfrak{K}_2\ \rangle \text{ with } \langle\ K \mid L\ \rangle = \int_0^\infty \int_0^\infty K(r,r',0)\overline{L(r,r',0)}dr'dr +$$

$$\int_0^\infty \int_0^\infty K(r,r',\pi)\overline{L(r,r',\pi)}drdr' \text{ and}$$

$$(3.5.54) \qquad \langle\ \boldsymbol{\mathfrak{K}_2} \mid \boldsymbol{\mathfrak{K}_2}\ \rangle \text{ with } \langle\ \mathbf{K} \mid \mathbf{L}\ \rangle =$$

$$\int_{-\infty}^\infty \int_{-\infty}^\infty [1 + e^{2\pi(t-t')}]\mathbf{K}(t,t')\overline{\mathbf{L}(t,t')}dt'dt;$$

whence the validity of the following

PROPOSITION 3.5.3. *The map* $\mathbf{K} \mapsto K$, *by means of the inverse Mellin-Transform (3.3.38) , determines a Hilbert space isomorphism between* $\langle\ \boldsymbol{\mathfrak{K}_2} \mid \boldsymbol{\mathfrak{K}_2}\ \rangle$ *and* $\langle\ \mathfrak{K}_2 \mid \mathfrak{K}_2\ \rangle$ *- i.e.*

$$\langle\ K \mid L\ \rangle = \langle\ \mathbf{K} \mid \mathbf{L} \rangle \ (K, L \in K_2 \textit{ with } \mathbf{K} \mapsto K \textit{ and } \mathbf{L} \mapsto L).$$

PROOF. The map $\mathbf{K} \mapsto K$ is injective and the independence of the expression

$$(2\pi)^{-1} \int_0^\infty \int_0^\infty K(r,r',\phi)(re^{i\phi})^{it-1/2}(r'e^{i\phi})^{-it'-1/2}e^{i2\phi}dr'dr$$

of parameter ϕ $(0 < \phi < \pi)$ and inequality (3.3.33) lets us conclude that our rule of correspondence is surjective. The preceeding integral equality (3.5.52) implies $\langle\ \mathbf{K} \mid \mathbf{L}\ \rangle = \langle\ K \mid L\ \rangle$, and therefore completes the proof. $\square$

How does the $\boldsymbol{\mathfrak{K}_2}$-kernel $(\mathbf{KL})(t,t')$, yielding the inverse Mellin-Transform representation (3.3.38) of the product kernel $(KL)(r,r',\phi)$ for K and $L \in \mathfrak{K}_2$, relate to the $\mathcal{L}_2$-kernels $\mathbf{K}(t,t')$ and $\mathbf{L}(t,t')$, which give representation (3.3.38) of

$K(r,r',\phi)$ and $L(r,r',\phi)$ respectively? We do expect an answer that is compatible with the established fact, that the inverse Mellin-Transform representation of $(KL)(f)(re^{i\phi}) = (K(Lf))(re^{i\phi})$ in the sense of equation (3.3.39) comes on the one hand from $(\mathbf{KL})(\mathbf{f})(t)$, and on the other hand from $(\mathbf{K}(\mathbf{Lf}))(t)$, in other words

$$(\mathbf{KL})(\mathbf{f})(t) = \int_{-\infty}^{\infty}(\mathbf{KL})(t,t')\mathbf{f}(t')dt = \int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{Lf}(t'')dt'' =$$

$$\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\int_{-\infty}^{\infty}\mathbf{L}(t'',t')\mathbf{f}(t')dt'dt'' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\mathbf{L}(t'',t')dt''\mathbf{f}(t')dt',$$

where the interchange of order integration is justified by the Tonelli-Hobson Theorem and the property that $\mathbf{K}(t,t')$ and $\mathbf{L}(t,t')$ are $\mathcal{L}_2$-kernels on $(-\infty,\infty)$ and $\mathbf{f} \in L_2(\mathbb{R})$. Because $\mathfrak{H}_2$ is a dense linear manifold of $L_2(\mathbb{R})$, owing to the property that the set $C_c(\mathbb{R})$ of continuous functions on $\mathbb{R}$ with compact support lies dense in $\mathfrak{H}_2$ as well as in $L_2(\mathbb{R})$, we may conclude with some minor trepidations that

$$(3.5.55)\qquad (\mathbf{KL})(t,t') = \int_{-\infty}^{\infty}\mathbf{K}(t,t'')\mathbf{L}(t'',t')dt'' \text{ for almost all } (t,t') \in \mathbb{R}^2.$$

For all that however, we place ourselves on more solid ground by arguing from the relation

$$(3.5.56)\quad (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\mathbf{KL})(t,t')(r'e^{i\phi})^{it'-1/2}dt'dt = (KL)(r,r',\phi) =$$

$$\int_0^{\infty}\left[(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\mathbf{K}(t,t'')(r''e^{i\phi})^{-it''-1/2}dt''dt\right]L(r'',r',\phi)e^{i\phi}dr''$$

as follows. In the first and last integral expression we let $r' = e^u$ and thereby obtain out of equation (3.5.56) that

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\mathbf{KL})(t,t')(e^{u+i\phi})^{it'-1/2}dt'dt =$$

$$\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\left[(2\pi)^{-1}\int_0^{\infty}(r''e^{i\phi})^{-it''-1/2}L(r,e^u,\phi)e^{i\phi}dr''\right]dt''dt,$$

which after multiplication by $e^{(u+i\phi)/2}$ reduces to

$$(3.5.57)\qquad (2\pi)^{-1/2}\int_{-\infty}^{\infty}(2\pi)^{-1/2}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\mathbf{KL}(t,t')e^{-\phi t'}dte^{iut'}dt' =$$

$$\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\Big[(2\pi)^{-1}\int_{0}^{\infty}(r''e^{i\phi})^{it''-1/2}L(r'',e^{u},\phi)e^{(u+i\phi)/2}e^{i\phi}dr''\Big]dt''dt.$$

Thus we may utilize the Fourier-Plancherel Theorem (Proposition 2.2.2) and write

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\mathbf{KL})(t,t')dt\,e^{-\phi t'} = (2\pi)^{-1}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')$$

$$\Big[(2\pi)^{-1}\int_{0}^{\infty}\int_{-\infty}^{\infty}(r''e^{i\phi})^{it''-1/2}L(r'',e^{u},\phi)e^{(u+i\phi)/2}e^{i\phi}dr''e^{-iut'}du\Big]dt''dt \text{ a.e. in } t' \text{ on } \mathbb{R},$$

because the first integral expression is an $L_2(\mathbb{R})$-function of t' on account of $(\mathbf{KL})(t,t')$ being an $\mathcal{L}_2$-kernel on $(-\infty,\infty)$, and similarly for the quadruple integral expression. In this quadruple integral expression, the reshuffling of the order of integration is a direct consequence of the kernels $\mathbf{K}(t,t'')$ and $L(r'',r',\phi)$ having finite double norms respectively and the Fubini-Tonelli Theorem. Taking further note of the fact that $e^{(u+i\phi)/2}e^{i\phi}e^{-iut} = e^{(u+i\phi)/2}(e^{u+i\phi})^{-it'}e^{i\phi}$ and replacing u by $\ln r'$ shall lead to

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\mathbf{KL})(t,t')dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')dt$$

$$\Big[(2\pi)^{-1}\int_{0}^{\infty}\int_{0}^{\infty}(r''e^{i\phi})^{it''-1/2}L(r'',r',\phi)(r',e^{i\phi})^{-it'-1/2}e^{i2\phi}dr''dr'\Big]dt''dt \text{ a.e in } t' \text{ on } \mathbb{R}$$

and by means of relation (3.3.30) for $L(r,r',\phi)$ and $\mathbf{L}(t,t')$, we convert this to

$$(3.5.58)\qquad (2\pi)^{-1/2}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\mathbf{KL})(t,t')dt =$$

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\mathbf{L}(t'',t')dt''dt \text{ for almost all } t',$$

which we multiply by $(2\pi)^{-1/2}(r'e^{i\phi})^{it'-1/2}$ and integrate with respect to t' over $\mathbb{R}$ to obtain

$$(3.5.59) \qquad (KL)(r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}$$

$$\left[\int_{-\infty}^{\infty}\mathbf{K}(t,t')\mathbf{L}(t'',t)dt''\right](r'e^{i\phi})^{it'-1/2}dt'dt \ (r,r'>0; 0\leq\phi\leq\pi).$$

This indeed justifies relation (3.5.55). Needless to say, the vector space structures of $\mathfrak{K}_2$ and $\boldsymbol{\mathfrak{K}}_2$ are linked to each other through

$$(3.5.60) \qquad (\alpha K+\beta L)(r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(re^{i\phi})^{-it-1/2}(\alpha\mathbf{K}+\beta\mathbf{L})(t,t')$$

$$(r'e^{i\phi})^{it'-1/2}dt'dt \text{ with } (\alpha\mathbf{K}+\beta\mathbf{L})(t,t') = \alpha\mathbf{K}(t,t')+\beta\mathbf{L}(t,t')$$

and $\alpha\mathbf{K}+\beta\mathbf{L}\in\boldsymbol{\mathfrak{K}}_2$, if $\mathbf{K}$ and $\mathbf{L}$ both belong to $\boldsymbol{\mathfrak{K}}_2$. We also have for $\boldsymbol{\mathfrak{K}}_2$ the following

THEOREM 3.5.4. *$\langle\ \mathfrak{K}_2 \mid \mathfrak{K}_2\ \rangle$ and $\langle\ \boldsymbol{\mathfrak{K}}_2 \mid \boldsymbol{\mathfrak{K}}_2\ \rangle$ are not only isomorphic Hilbert spaces, but in terms of the norms induced by their respective inner products, they are also Banach algebras.*

PROOF. For multiplication in $\boldsymbol{\mathfrak{K}}_2$ defined by equation (3.5.55), we need only show that

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|(\mathbf{KL})(t,t')|^2dt'dt\right\}^{1/2}\leq$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{K}(t,t')|^2dt'dt\right\}^{1/2}\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{L}(t,t')|^2dt'dt\right\}^{1/2}.$$

This shall immediately follow by noticing that

$$1+e^{2\pi(t-t')}\leq[1+e^{2\pi(t-t'')}][1+e^{2\pi(t''-t')}] \text{ for all } t,t',t''\in\mathbb{R}$$

and that the Schwarz Inequality implies

$$\int_{-\infty}^{\infty} \sqrt{1+e^{2\pi(t-t')}}|\mathbf{K}(t,t'')||\mathbf{L}(t'',t')|dt'' \leq \int_{-\infty}^{\infty} |\mathbf{K}(t,t'')\sqrt{1+e^{2\pi(t-t')}}|\times$$

$$|\mathbf{L}(t'',t')|\sqrt{1+e^{2\pi(t''-t')}}|dt'' \leq \left\{\int_{-\infty}^{\infty} |\mathbf{K}(t,t'')|^2[1+e^{2\pi(t-t'')}]dt''\right\}^{1/2}\times$$

$$\left\{\int_{-\infty}^{\infty} |\mathbf{L}(t'',t')|^2[1+e^{2\pi(t''-t')}]dt''\right\}^{1/2}(t,t' \in \mathbb{R}).$$

We further continue our upward estimations as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]\left|\int_{-\infty}^{\infty}\mathbf{K}(t,t'')\mathbf{L}(t'',t')dt''\right|^2 dt'dt \leq$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|\sqrt{1+e^{2\pi(t-t'')}}\mathbf{K}(t,t'')\sqrt{1+e^{2\pi(t''-t')}}\mathbf{L}(t'',t')\right|dt''\right]^2 dt'dt \leq$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}[1+e^{2\pi(t-t'')}]|\mathbf{K}(t,t'')|^2dt''\right\}\left\{\int_{-\infty}^{\infty}[1+e^{2\pi(t''-t')}]|\mathbf{L}(t'',t')|^2dt''\right\}dt'dt =$$

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t'')}]|\mathbf{K}(t,t'')|^2dt''dt\right\}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t''-t')}]|\mathbf{L}(t'',t')|^2dt''dt'\right\}$$

and thus complete the proof of our theorem. □

The relationship between the norms $|||\cdot|||_{s(2)}$ of $\mathfrak{K}_2$ and that of the Hilbert space norm

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|(\cdot)(t,t')|^2dt'dt\right\}^{1/2} \text{ of } \mathfrak{K}_2 \text{ is}$$

(3.5.61) $$|||K|||_{s(2)}{}^2 \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{K}(t,t')|^2dt'dt \leq 2^2|||K|||_{s(2)}{}^2,$$

as is directly evident from Theorem 3.5.1 and relation (3.3.33) for $\phi = \psi(\psi = 0, \pi)$.

CHAPTER 4

FREDHOLM RESOLVENTS BY NEUMANN SERIES

Prior to embarking upon a discourse on Fredholm Resolvents, we need to fix some notions of convergence of a sequence of $\mathfrak{H}_2$-functions as well as of a ϕ-parameter family of $\mathcal{L}_2$ kernels belonging to operators from $\mathfrak{K}_2$. Relying on the concept of "*relative uniform*" and "*relative uniform absolute*" convergence for $L_2(a,b)$-functions and $\mathcal{L}_2$-kernels as presented by F. Smities in his text entitled *Integral Equations* ([16], pgs. 23 - 26) we first make for a sequence in $\mathfrak{H}_2$ the following

DEFINITION 4.0.1. *The sequence $\{f_n\}_{n=1}^{\infty}$ of $\mathfrak{H}_2$-functions is said to converge relatively uniformly to $f \in \mathfrak{H}_2$, if there exists a ϕ-parameter family of non-negative $L_2(0,\infty)$-functions p_ϕ $(0 \le \phi \le \pi)$ with uniformly bounded $L_2(0,\infty)$-norms on $[0,\pi]$ and an $n(\varepsilon) > 0$ such that, out of $n \ge n(\varepsilon)$ shall follow*

$$|f_n(re^{i\phi}) - f(re^{i\phi})| \le \varepsilon p_\phi(r) \ (r > 0;\ 0 \le \phi \le \pi). \tag{4.0.1}$$

From Definition 4.0.1 it becomes quite obvious, that if we square both sides of inequality (4.0.1), integrate with respect to r over $(0,\infty)$ and take the supremum over ϕ $(0 \le \phi \le \pi)$, then

$$||f_n - f||_{s(2)} \le \varepsilon \sup_{0 \le \phi \le \pi} ||p_\phi||_{L(0,\infty)} \ (n \ge n(\varepsilon)),$$

and thus f_n converges to f in $\mathfrak{H}_2$. In other words, "*relative uniform*"-convergence in $\mathfrak{H}_2$ implies convergence in $\mathfrak{H}_2$. Because $\mathfrak{H}_2$ is a Banach space, we need not worry about the limits of "*relatively uniform*"-Cauchy Sequences in $\mathfrak{H}_2$. A series $\sum_{n=1}^{\infty} f_n$ of $\mathfrak{H}_2$-

functions is said to be "*relatively uniformly*"-convergent in $\mathfrak{H}_2$, if the sequence of its partial sums is "*relatively uniformly*"-convergent in $\mathfrak{H}_2$. However, we shall say that the series $\sum_{n=1}^{\infty} f_n$ of $\mathfrak{H}_2$-functions converges "*relatively uniformly absolutely*" in $\mathfrak{H}_2$, if there exists a ϕ-parameter family of non-negative $L_2(0,\infty)$-functions p_ϕ $(0 \le \phi \le \pi)$ and an $n(\varepsilon)$ such that, out of $n \ge n(\varepsilon)$ shall follow

$$\sum_{\mu=n+1}^{n+p} |f_\mu(re^{i\theta})| \le \varepsilon p_\phi(r) \ (r > 0;\ p \ge 1;\ 0 \le \phi \le \pi).$$

It is quite clear that a "*relatively uniformly absolutely*"-convergent series in $\mathfrak{H}_2$ is "*relatively uniformly*"-convergent in $\mathfrak{H}_2$ as well as what its limit is in $\mathfrak{H}_2$, since $\mathfrak{H}_2$ is complete in both the norms $||\cdot||_{s(2)}$ and $||\cdot||_{\mathfrak{H}_2}$. Further, if $K \in \mathfrak{K}_2$, then Kf_n converges "*relatively uniformly*" to Kf, whenever f_n converges "*relatively uniformly*" to f in $\mathfrak{H}_2$, because out of $n \ge n(\varepsilon)$ follows $|(Kf_n)(re^{i\phi}) - (Kf)(re^{i\phi})| \le \varepsilon ||K_\phi(r,\cdot)||_{L_2(0,\infty)} ||p_\phi||_{L_2(o,\infty)}$ $(0 \le \phi \le \pi)$. On the other hand, the series $\sum_{n=1}^{\infty} Kf_n$ converges "*relatively uniformly absolutely*" to $K(\sum_{n=1}^{\infty} f_n)$, because $n \ge n(\varepsilon)$ implies

$$\sum_{\mu=n+1}^{n+p} |(Kf_n)(re^{i\phi})| \le \int_0^\infty |K(r,r',\phi)| \Big[\sum_{\mu=n+1}^{n+p} |f_n(r'e^{i\theta})| \Big] dr' \le$$

$$\varepsilon ||K_\phi(r,\cdot)||_{L_2(0,\infty)} \times ||p_\phi||_{L_2(0,\infty)},$$

where $||K_\phi(r,\cdot)||_{L_2(0,\infty)} \times ||p_\phi||_{L_2(0,\infty)}$ is a ϕ-parameter family of $L_2(0,\infty)$-functions with $L_2(0,\infty)$-norms that are uniformly bounded by

$$|||K|||_{s(2)} \times \sup_{0 \le \phi \le \pi} ||p_\phi||_{L_2(0,\infty)}.$$

4.1. Relative Uniform Absolute Convergence in $\mathfrak{K}_2$

Parallel to the concept of "*relative uniform*"-convergence in $\mathfrak{H}_2$, we introduce on $\mathfrak{K}_2$ the

DEFINITION 4.1.1. *The sequence $\{K_n\}_{n=1}^{\infty}$ of radially acting linear integral operators from $\mathfrak{K}_2$ is said to converge "relatively uniformly" to $K \in \mathfrak{K}_2$, if there exist a ϕ-parameter family of non-negative $\mathcal{L}_2$-kernels P_ϕ $(0 \leq \phi \leq \pi)$ with uniformly bounded double norms and $n(\varepsilon)$ such that, out of $n \geq n(\varepsilon)$ shall follow*

$$|K_n(r,r',\phi) - K(r,r',\phi)| \leq \varepsilon P_\phi(r,r') \ (r,r' > 0;\ 0 \leq \phi \leq \pi) \tag{4.1.2}$$

for the ϕ-parameter family of $\mathfrak{L}_2$-kernels inducing the operators K_n and K respectively.

In the selfsame evident way as in $\mathfrak{H}_2$, $K_n \to K$ "*relatively uniformly*" in $\mathfrak{K}_2$ implies $LK_n \to LK$ and $K_nL \to LK$ "*relatively uniformly*" in $\mathfrak{K}_2$, because

$$|(LK_n)(r,r',\phi) - (LK)(r,r',\phi)| \leq \varepsilon ||L_\phi(r,\cdot)||_{L_2(0,\infty)} \times ||P_\phi(\cdot,r')||_{L_2(0,\infty)}$$

$$(r,r' > 0;\ 0 \leq \phi \leq \pi)$$

holds for all $n \geq n(\varepsilon)$, and $||L_\phi(r,\cdot)||_{L_2(0,\infty)} \times ||P_\phi(\cdot,r')||_{L_2(0,\infty)}$ is a ϕ-parameter family of non-negative $\mathcal{L}_2$-kernels in variables (r,r') on $(0,\infty)^2$ $(0 \leq \phi \leq \pi)$ with uniformly bounded double norms. We have correspondingly for all $n \geq n(\varepsilon)$ that

$$|(K_nL)(r,r',\phi) - (KL)(r,r',\phi)| \leq \varepsilon ||P_\phi(r,\cdot)||_{L_2(0,\infty)} \times ||_\phi(\cdot,r')||_{L_2(0,\infty)}$$

$$(r,r' > 0;\ 0 \leq \phi \leq \pi)$$

implies for $K_n L \to KL$ "*relatively uniformly*" in $\mathfrak{K}_2$. We note that

$$\int_0^\infty \int_0^\infty \Big[||L_\phi(r,\cdot)||_{L_2(0,\infty)}\Big]^2 \Big[||P_\phi(\cdot,r')||_{L_2(0,\infty)}\Big]^2 dr'dr,$$

$$\int_0^\infty \int_0^\infty \Big[||P_\phi(r,\cdot)||_{L_2(0,\infty)}\Big]^2 \Big[||L_\phi(\cdot,r')||_{L_2(0,\infty)}\Big]^2 dr'dr \leq |||L|||_{s(2)}{}^2 \times \sup_{0\leq\phi\leq\pi} |||P_\phi|||^2.$$

Just as for $\mathfrak{H}_2$, we say that the series $\sum_{n=1}^\infty K_n$ of radially acting linear integral operators out of $\mathfrak{K}_2$ converges "*relatively uniformly*", if its sequence of partial sums does. We further continue by saying that the series $\sum_{n=1}^\infty K_n$ from $\mathfrak{H}_2$ converges "*relatively uniformly absolutely*", if there exists a ϕ-parameter family of non-negative $\mathcal{L}_2$-kernels P_ϕ $(0 \leq \phi \leq \pi)$ with uniformly bounded double norms and $n(\varepsilon)$ such that, out of $n \geq n(\varepsilon)$ shall follow

$$\text{(4.1.3)} \qquad \sum_{\mu=n+1}^{n+p} |K_n(r,r',\phi)| \leq \varepsilon P_\phi(r,r') \;\; (r,r' > 0;\; p \geq 1;\; 0 \leq \phi \leq \pi)$$

for the ϕ-parameter family of $\mathcal{L}_2$-kernels inducing the operators K_n belonging to $\mathfrak{K}_2$.

Owing to the fact that $\mathfrak{K}_2$ is complete ($\mathfrak{K}_2$ is a Banach algebra), it is quite clear what the limit in $\mathfrak{K}_2$ of a "*relatively uniformly*"-convergent series is, and even more so that of a "*relatively uniformly absolutely*"-convergent one. If $f \in \mathfrak{K}_2$, $L \in \mathfrak{K}_2$ and $\sum_{n=1}^\infty K_n = K$ in the sense of "*relative uniform absolute*"-convergence in $\mathfrak{K}_2$, then

$$\sum_{n=1}^\infty K_n f = Kf,\; \sum_{n=1}^\infty LK_n = LK \text{ and } \sum_{n=1}^\infty K_n L = KL$$

"*relatively uniformly absolutely*" in $\mathfrak{H}_2$ and $\mathfrak{K}_2$, because

$$\sum_{n=1}^\infty |(K_n f)(re^{i\phi})| \leq \int_0^\infty \Big[\sum_{n=n+1}^{n+p} |K_\mu(r,r',\phi)|\Big] |f(r'e^{i\phi})| dr' \leq \varepsilon \int_0^\infty P_\phi(r,r') |f(r'e^{i\phi})| dr',$$

$$\sum_{n=1}^\infty |(LK_n)(r,r',\phi)| \leq \int_0^\infty |L_\phi(r,r")| \Big[\sum_{n=n+1}^{n+p} |K_\mu(r'',r',\phi)|\Big] dr'' \leq$$

$$\varepsilon \int_0^\infty |L_\phi(r,r'')| P_\phi(r'',r') dr'' \text{ and further}$$

$$\sum_{n=n+1}^{n+p} |(K_\mu L)(r,r',\phi)| \leq \int_0^\infty \Big[\sum_{n=n+1}^{n+p} |K_\mu(r,r'',\phi)| \Big] |L_\phi(r'',r')| dr'' \leq$$

$$\varepsilon \int_0^\infty P_\phi(r,r'') |L_\phi(r'',r')| dr'' \;\; (r,r' > 0;\; n \geq n(\varepsilon);\; p \geq 1;\; 0 \leq \phi \leq \pi).$$

We are now in a position to begin to look at the Fredholm Resolvent $H_\lambda(K)$ of a radially acting linear integral operator $K \in \mathfrak{K}_2$ with the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r,r',\phi)$ $(0 \leq \phi \leq \pi)$, wherein the domain $(0,\pi)$ of the parameter ϕ has been extended to the compact interval $[0,\pi]$ by means of Proposition 3.4.1 and Theorem 3.4.2. Our notation shall be abused quite often in the sense of writing H_λ for the Fredholm Resolvent $H_\lambda(K)$ of $K \in \mathfrak{K}_2$. This shall occur, whenever it is clear that we are dealing with the Fredholm Resolvent of the operator K. For the purpose of examining Fredholm Resolvents, we note that $\mathfrak{K}_2$ is a Banach algebra - i.e. $(\mathfrak{K}_2, ||\cdot||_{s(2)})$ is a Banach space with the algebraic structure of $\alpha K + \beta L$ and KL belong to $\mathfrak{K}_2$ for all $K, L \in \mathfrak{K}_2$ and $\alpha, \beta \in \mathbb{C}$. Moreover, if the radially acting linear integral operators K and L possess the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r,r',\phi)$ and $L(r,r',\phi)$, then

$$(4.1.4) \qquad (\alpha K + \beta L)(r,r',\phi) = \alpha K(r,r,\phi) + \beta L(r,r',\phi) \quad (0 \leq \phi \leq \pi) \text{ and}$$

$$(KL)(r,r',\phi) = \int_0^\infty K(r,r'',\phi) L(r,r'',\phi) e^{i\phi} dr" = e^{-i\phi}(K_\phi L_\phi)(r,r') \;\; (0 \leq \phi \leq \pi)$$

are the respective ϕ-parameter family of $\mathcal{L}_2$-kernels belonging to the radially acting linear integral operator $\alpha K + \beta L$ and KL and the norm relation concerning "*operator products*" in $\mathfrak{K}_2$ is

$$(4.1.5) \qquad |||KL|||_{s(2)} \leq |||K|||_{s(2)} \times |||L|||_{s(2)} \;\; (K, L \in \mathfrak{K}_2).$$

In a very self-evident way, $\mathfrak{K}_2$ is a Banach algebra without a multiplicative identity, because the identity would have to be a radially acting integral operator I, and it is well know that I fails to be an integral operator. In terms of the $\mathcal{L}_2$-kernels $K_\phi(r, r') \equiv K(r, r', \phi)e^{i\phi}$ defined in Chapter 1, we have for the ϕ-parameter family $\mathcal{L}_2$ kernels $(KL)(r, r', \phi)$ $(K, L \in \mathfrak{K}_2)$ of $KL \in \mathfrak{K}_2$ the relation

$$(4.1.6) \qquad |(KL)(r, r', \phi)| \leq ||K_\phi(r, \cdot)||_{L_2(0,\infty)} \times ||L_\phi(\cdot, r')||_{L_2(0,\infty)}$$

$$(r, r' > 0;\ 0 \leq \phi \leq \pi),$$

which follows directly out of the Schwarz Inequality for inner products defined by means of integrals. Consequently, by integrating $|(KL)(r, r', \phi)|^2$ with respect to (r, r') on $(0, \infty)^2$, we obtain out of the immediately preceding inequality (4.1.6) that $|||K_\phi L_\phi||| \leq |||K_\phi||| \times |||L_\phi|||$, and after taking the supremum with respect to ϕ $(0 \leq \phi \leq \pi)$ our norm relation (4.1.5) for "operator products in $\mathfrak{K}_2$" follows.

Continuing this trend of thought, we consider another operator $M \in \mathfrak{K}_2$ with the ϕ-parameter family of $\mathcal{L}_2$-kernels $M(r, r', \phi)$ $(0 \leq \phi \leq \pi)$ and write

$$|(KML)(r, r', \phi)| \leq \int_0^\infty \int_0^\infty |(K_\phi(r, u)M_\phi(u, v)L_\phi(v, r')|^2 du dv \leq$$

$$||K_\phi(r, \cdot)||_{L(0,\infty)} \times |||M_\phi||| \times ||L_\phi(\cdot, r')||_{L_2(0,\infty)},$$

and by utilizing $|||M_\phi||| \leq |||M|||_{s(2)}$ we hereform arrive at

$$(4.1.7) \qquad |(KML)(r, r', \phi)| \leq ||K_\phi(r, \cdot)||_{L_2(0,\infty)} \times |||M|||_{s(2)} \times ||L_\phi(\cdot, r')||_{L_2(0,\infty)}$$

$$(K, M, L \in \mathfrak{K}_2).$$

With the aid of these results, we now are able to formulate

DEFINITION 4.1.2. *An operator $H_\lambda = H_\lambda(K)$ is said to be the Fredholm Resolvent of $K \in \mathfrak{K}_2$ for the regular value $\lambda \in \mathbb{C}$, if $H_\lambda \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels*

$$H_\lambda(r, r', \phi) = H_\lambda(K; r, r', \phi) \ (r, r' > 0;\ 0 \le \phi \le \pi) \text{ and satisfies}$$

$$\lambda H_\lambda K = \lambda K H_\lambda = H_\lambda - K \ (\textit{Fredholm Resolvent Equations}). \tag{4.1.8}$$

The Fredholm Resolvent Equations (4.1.8) allow us to construct the inverse of the operator $I - \lambda K$ in the Banach algebra $\mathcal{B}(\mathfrak{H}_2)$ with multiplicative identity I or even better, in the Banach algebra obtained by adjoining to $\mathfrak{K}_2$ the identity operator I ([8], pgs. 143, 200). Thus, if the Fredholm Resolvent $H_\lambda = H_\lambda(K)$ of $K \in \mathfrak{K}_2$ exists, then $I - \lambda K$ is invertible and $(I - \lambda K)^{-1} = I + \lambda H_\lambda$. In addition, we want to point out that the Fredholm Resolvent Equations (4.1.8) assume the form

$$\lambda(H_\lambda K)(r, r', \phi) = \lambda(K H_\lambda)(r, r', \phi) = H_\lambda(r, r', \phi) - K(r, r', \phi) \tag{4.1.9}$$

$$\text{a.e. on } (0, \infty)^2 \ (0 \le \phi \le \pi)$$

and the radial integral equation (1.0.5) for the unknown $\mathfrak{H}_2$-function f has the unique solution (1.0.6) in terms of the Fredholm Resolvent Kernel $H_\lambda(r, r', \phi) = H_\phi(K; r, r', \phi)$ belonging to the Fredholm Resolvent $H_\lambda = H_\lambda(K)$ of the element $K \in \mathfrak{K}_2$. Values λ, for which a Fredholm Resolvent of $K \in \mathfrak{K}_2$ fails to exist, are called "*characteristic values*" of K, and for these we expect the null-space of $I - \lambda K$ to be non-trivial - i.e.

$$N(I - \lambda K) = \{f \in \mathfrak{K}_2 :\ f - \lambda K f = 0\} \text{ has positive dimension.}$$

4.2. Neumann Series

Given a $K \in \mathfrak{K}_2$, how can we directly construct a Fredholm Resolvent $H_\lambda = H_\lambda(K)$ for regular values λ of K ? The most elementary way is through the *Neumann Series*

$$(4.2.10) \qquad H_\lambda = H_\lambda(K) = \sum_{n=0}^{\infty} \lambda^n K^{n+1} \ (|\lambda| \times |||K|||_{s(2)} < 1)$$

with the ϕ-parameter family of $\mathcal{L}_2$-kernels

$$(4.2.11) \qquad H_\lambda(r, r', \phi) = H_\lambda(K; r, r', \phi) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \phi)$$

$$\text{a.e. on } (0, \infty)^2 \ (0 \leq \phi \leq \pi),$$

where the convergence of the series (4.2.11) of $\mathcal{L}_2$-kernels is in the sense of "*relatively uniformly absolutely*" in variables (r, r', ϕ) on $(0, \infty)^2 \times [0, \pi]$. To see this, we first note that

$$\sum_{\mu=n+1}^{n+p} |||\lambda^\mu K^{\mu+1}|||_{s(2)} \leq \sum_{\mu=n+1}^{n+p} [|\lambda| \times |||K|||_{s(2)}]^\mu |||K|||_{s(2)} \leq$$

$$|||K|||_{s(2)} \times [|\lambda| \times |||K|||_{s(2)}]^{n+1} [1 - |\lambda| \times |||K|||_{s(2)}]^{-1}$$

and readily discern the condition $|\lambda| \times |||K|||_{s(2)} < 1$, which originates from geometric series, as well as the fact that $|||K|||_{s(2)} \times [\lambda||||K|||_{s(2)}]^{n+1}[1 - |ld| \times |||K|||_{s(2)}]^{-1}$ can be made arbitrarily small by choosing n sufficiently large (provided $|\lambda| \times |||K|||_{s(2)} < 1$) independent of the integer p. Therefore, the series converges to $H_\lambda \in \mathfrak{K}_2$. For the "*relative uniform absolute*"-convergence of the series $\sum_{\mu=n+1}^{n+p} \lambda^n K^{n+1}(r, r', \phi)$ we observe $K^{\mu+1}(r,r',\phi) = (KK^{\mu})(r,r',\phi) = (K^{\mu}K)(r,r',\phi)$ and by means of estimate (4.1.7) we have

$$|K^{\mu+1}(r,r',\phi)| \leq ||K_{\phi}(r,\cdot)||_{L_2(0,\infty)} \times |||K^{\mu-1}|||_{s(2)} \times ||K_{\phi}(\cdot,r')||_{L_2(0,\infty)}$$

$(r,r' > 0;\ 0 \leq \phi \leq \pi)$, or even better

$$|K^{\mu+1}(r,r',\phi)| \leq ||K_{\phi}(r,\cdot)||_{L_2(0,\infty)} \times [|||K|||_{s(2)}]^{\mu-1} \times ||K_{\phi}(\cdot,r')||_{L_2(0,\infty)}$$

$(r,r' > 0;\ 0 \leq \phi \leq \pi)$.

As result of this, we can write the inequalities

$$\sum_{\mu=n+1}^{n+p} |\lambda^{\mu} K^{\mu+1}(r,r',\phi)| \leq$$

$$\sum_{\mu=n+1}^{n+p} |\lambda| \times ||K_{\phi}(r,\cdot)||_{L_2(0,\infty)} \times [|\lambda| \times |||K|||_{s(2)}]^{\mu-1} \times ||K_{\phi}(\cdot,r')||_{L_2(0,\infty)} \leq$$

$$|\lambda| \times [|\lambda| \times |||K|||_{s(2)}]^{n}[1 - |\lambda| \times |||K|||_{s(2)}]^{-1} \times ||K_{\phi}(r,\cdot)||_{L_2(0,\infty)} \times ||K_{\phi}(\cdot,r')||_{L_2(0,\infty)},$$

where $P_{\phi}(r,r') \equiv ||K_{\phi}(r,\cdot)||_{L_2(0,\infty)} \times ||K_{\phi}(\cdot,r')||_{L_2(0,\infty)}$ is a ϕ-parameter family of non-negative $\mathcal{L}_2$-kernels $(0 \leq \phi \leq \pi)$, whose double norms on $(0,\infty)^2$ are uniformly bounded by $[|||K|||_{s(2)}]^2$, and $|\lambda| \times [|\lambda| \times |||K|||_{s(2)}]^{n}[1 - |\lambda| \times |||K|||_{s(2)}]^{-1}$ can be made arbitrarily small by choosing n sufficiently large (independent of integer p). That H_{λ} satisfies the Fredholm Resolvent Equations (4.1.9), is a direct consequence of observing that in $H_{\lambda} - K = \sum_{n=1}^{\infty} \lambda^{n} K^{n+1}$ the term λK can be factored out to the left as well as the right of the series, provided $|\lambda| \times |||K|||_{s(2)} < 1$, and the "*relative uniform absolute*"-convergence of $\sum_{n=0}^{\infty} \lambda^{n} K^{n+1}(r,r',\phi)$ in variables (r,r',ϕ) on $(0,\infty)^2 \times [0,\pi]$ guarantees, by means of equality (4.2.11) that

$$\lambda\Big(K\Big[\sum_{n=1}^{\infty} \lambda^{n} K^{n+1}\Big]\Big)(r,r',\phi) = \lambda\Big(\Big[\sum_{n=1}^{\infty} \lambda^{n} K^{n+1}\Big]K\Big)(r,r',\phi) = \tag{4.2.12}$$

$$\sum_{n=1}^{\infty} \lambda^n K^{n+1}(r,r',\phi) - K(r,r',\phi) \ (|\lambda| \times |||K|||_{s(2)} < 1).$$

In terms of ϕ-parameter family of $\mathcal{L}_2$-kernels $H_\lambda(r,r',\phi) = H_\lambda(K;r,r',\phi)$, this also means the validity of equations (4.1.9) in the sense of a.e in (r,r') on $(0,\infty)^2$ $(0 \leq \phi \leq \pi)$, provided again that $|\lambda| \times |||K|||_{s(2)} \leq 1$. Transfering these results to the radial integral equation (1.0.5), we unequivocally see that its solution (1.0.6) has the form

$$f(re^{i\phi}) = g(re^{i\phi}) + \lambda(H_\lambda g)(re^{i\phi}) = \sum_{n=0}^{\infty} \lambda^n (K^n g)(re^{i\phi}) \text{ with} \tag{4.2.13}$$

$$(K^n g)(re^{i\phi}) = \int_0^\infty K^n(r,r',\phi) g(r'e^{i\phi}) e^{i\phi} dr \text{ for almost all}$$

$$r > 0 \ (0 \leq \phi \leq \pi; \ n \geq 1) \text{ and } K^0 \equiv I,$$

wherein the series converges "*relatively uniformly absolutely*" in $\mathfrak{H}_2$.

Let us now consider the situation where λ_0 and λ are two regular values of $K \in \mathfrak{K}_2$ with $H_{\lambda_0} = H_{\lambda_0}(K)$ and $H_\lambda = H_\lambda(K)$ being the Fredholm Resolvent of K for regular values λ_0 and λ respectively of K - i.e.

$$\lambda_0 K H_{\lambda_0} = H_{\lambda_0} - K = \lambda_0 H_{\lambda_0} K \text{ and } \lambda K H_\lambda = H_\lambda - K = \lambda H_\lambda K. \tag{4.2.14}$$

We single out $\lambda_0 K H_{\lambda_0} = H_{\lambda_0} - K$ and $\lambda H_\lambda K = H_\lambda - K$, "*multiply*" the first equation by λH_λ from the left and the second equation by $\lambda_0 H_{\lambda_0}$ from the right and thereby achieve

$$\lambda\lambda_0 H_\lambda K H_{\lambda_0} = \lambda H_\lambda H_{\lambda_0} - \lambda H_\lambda K = \lambda_0 H_\lambda H_{\lambda_0} - \lambda_0 K H_{\lambda_0}$$

and by further substituting $H_\lambda - K$ and $H_{\lambda_0} - K$ respectively for $\lambda H_\lambda K$ and $\lambda_0 K H_{\lambda_0}$, we arrive at

$$\lambda H_\lambda H_{\lambda_0} - (H_\lambda - K) = \lambda_0 H_\lambda H_{\lambda_0} - (H_{\lambda_0} - K) \text{ or } (\lambda - \lambda_0) H_\lambda H_{\lambda_0} = H_\lambda - H_{\lambda_0}.$$

By considering the pair of equations $\lambda H_{\lambda_0} K = H_{\lambda_0} - K$ and $\lambda K H_\lambda = H_\lambda - K$, where the first is "*multiplied*" by λH_λ from right and the second by $\lambda_0 H_{\lambda_0}$ from the left, we immediately see that $\lambda_0 \lambda H_{\lambda_0} K H_\lambda = \lambda H_{\lambda_0} H_\lambda - \lambda K H_\lambda = \lambda_0 H_{\lambda_0} H_\lambda - \lambda_0 H_{\lambda_0} K$, which we convert in the same manner as before into $(\lambda - \lambda_0) H_{\lambda_0} H_\lambda = H_\lambda - H_{\lambda_0}$, or

$$(4.2.15) \qquad (\lambda - \lambda_0) H_{\lambda_0} H_\lambda = H_\lambda - H_{\lambda_0} = (\lambda - \lambda_0) H_\lambda H_{\lambda_0}.$$

This means that H_λ is the Fredholm Resolvent of H_{λ_0} for regular value $\lambda - \lambda_0$ and therefore, $H_\lambda = H_\lambda(K)$ has Nemann series expansion

$$(4.2.16) \quad H_\lambda = H_\lambda(K) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n {H_{\lambda_0}}^{n+1}, \text{ provided } |\lambda - \lambda_0| \times |||H_{\lambda_0}|||_{s(2)} < 1,$$

or in terms of its ϕ-parameter family of $\mathcal{L}_2$-kernels

$$(4.2.17) \qquad H_\lambda(r, r', \phi) = H_\lambda(K; r, r', \phi) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n {H_{\lambda_0}}^{n+1}(r, r', \phi)$$

$$(r, r' > 0;\ 0 \le \phi \le \pi),$$

whose convergence is "*relative uniform absolute*" in $\mathfrak{K}_2$ as long as $|\lambda - \lambda_0| \times |||H_{\lambda_0}|||_{s(2)} < 1$.

Let us now consider the converse of the situation just described - i.e. H_{λ_0} is the Fredholm Resolvent of $K \in \mathfrak{K}_2$ for regular value λ_0 and H_λ is the Fredholm Resolvent of H_{λ_0} for regular value $\lambda - \lambda_0$ of H_{λ_0}. Can we then conclude that H_λ is the Fredholm Resolvent of K for regular value λ of K? The answer is affirmative, and we procede as follows. We concentrate on the last two expressions of equations (4.2.14), "*multiply*" these two from the right by $\lambda\lambda_0 K$ in the sense of operator multiplication, and thus write

$$\lambda(\lambda-\lambda_0)H_\lambda(\lambda_0 H_{\lambda_0}K) = \lambda\lambda_0 H_\lambda K - \lambda(\lambda_0 H_{\lambda_0}K),$$

which by use of $\lambda_0 H_{\lambda_0}K = H_{\lambda_0} - K$ reduces to

$$\lambda(\lambda-\lambda_0)H_\lambda(H_{\lambda_0} - K) = \lambda\lambda_0 H_\lambda K - \lambda(H_{\lambda_0} - K).$$

We "*multiply*" this out and regroup as

$$\lambda[(\lambda-\lambda_0)H_\lambda H_{\lambda_0}] - \lambda^2 H_\lambda K + \lambda\lambda_0 H_\lambda K = \lambda\lambda_0 H_\lambda K - \lambda(H_{\lambda_0} - K)$$

$$\text{or } \lambda(H_\lambda - H_{\lambda_0}) - \lambda^2 H_\lambda K = -\lambda(H_{\lambda_0} - K),$$

which after a cancellation of λ yields

$$H_\lambda - H_{\lambda_0} - \lambda H_\lambda K = -H_{\lambda_0} + K \text{ or } \lambda H_\lambda K = H_\lambda - K.$$

If we single out only $(\lambda-\lambda_0)H_{\lambda_0}H_\lambda = H_\lambda - H_{\lambda_0}$ from the equations (4.2.14) and "*multiply*" it from the left by $\lambda\lambda_0 K$, then we shall have

$$\lambda(\lambda-\lambda_0)(\lambda_0 K H_{\lambda_0})H_\lambda = \lambda\lambda_0 K H_\lambda - \lambda(\lambda_0 K H_{\lambda_0}),$$

out of which shall follow

$$\lambda(\lambda-\lambda_0)(H_{\lambda_0} - K) = \lambda\lambda_0 K H_\lambda - \lambda(H_{\lambda_0} - K)$$

after we replace $\lambda_0 K H_{\lambda_0}$ with $H_{\lambda_0} - K$. This we regroup as

$$\lambda[(\lambda-\lambda_0)H_{\lambda_0}H_\lambda] - \lambda(\lambda-\lambda_0)KH_\lambda = \lambda\lambda_0 K H_\lambda - \lambda(H_{\lambda_0} - K)$$

and rewrite it as $\lambda(H_\lambda - H_{\lambda_0}) - \lambda^2 K H_\lambda = -\lambda(H_{\lambda_0} - K)$. Cancelling a λ leads to $H_\lambda - H_{\lambda_0} - \lambda K H_\lambda = -H_{\lambda_0} + K$ or $\lambda K H_\lambda = H_\lambda - K$. Combining both of these results, we obtain the Fredholm Resolvent Equations (4.1.8). This result entails the far reaching conclusion, that the set of regular values of any $K \in \mathfrak{K}_2$ is an open subset of $\mathbb{C}$, and if λ_0 is any regular value of K with $H_{\lambda_0} = H_{\lambda_0}(K)$ being its Fredholm Resolvent, then the $H_\lambda = H_\lambda(K)$ as given by the equations (4.2.15) and (4.2.16) is the Fredholm Resolvent of K for $|\lambda - \lambda_0| \times |||H_{\lambda_0}|||_{s(2)} < 1$, and hence $\{\lambda : |\lambda - \lambda_0| \times |||H_{\lambda_0}|||_{s(2)} < 1\}$ is a subset of the regular values of K.

The condition $|\lambda| \times |||K|||_{s(2)} < 1$ is sufficient for the $|||\cdot|||_{s(2)}$-norm convergence of the Neumann Series, namely $\sum_{n=0}^{\infty} |\lambda|^n |||K^{n+1}|||_{s(2)} < \infty$; however, it is not necessary. What is necessary and at the same time also sufficient, is that

$$|\lambda| \limsup_{n\to\infty} \sqrt[n]{|||K^n|||_{s(2)}} < 1,$$

which tells us that the radius of convergence of the Neumann Series is

$$\mathfrak{r}(K) = \liminf_{n\to\infty} \Big[|||K^n|||_{s(2)}\Big]^{-1/n}, \tag{4.2.18}$$

and hence H_λ, as given by equation (4.2.10), represents a $\mathfrak{K}_2$ valued holomorphic function in the open disk centered at 0 of radius $\mathfrak{r}(K)$. It becomes also evident, by means of some complicated calculations in dual spaces ([8], Satz 4.11, pgs 48 - 49) , that the holomorphy of H_λ in terms of λ must fail somewhere on the circle $\{\lambda \in \mathbb{C} : |\lambda| = \mathfrak{r}(K)\}$ of convergence, as is well established for $\mathbb{C}$-valued holomorphic functions in complex analysis. It is more than clear, that the radius of convergence $\mathfrak{r}(K)$ exceeds $|||K|||_{s(2)}$. Nonetheless, it may so happen on the one hand that $\mathfrak{r}(K) =$

$|||K|||_{s(2)}$, whereas on the other $\mathfrak{r}(K) = \infty$, which leads us to the next section of this chapter.

4.3. Radially Acting Volterra Integral Operators on $\mathcal{K}_2$

An integral operator defined by an $\mathcal{L}_2$-kernel on $L_2(a,b)$, wherein $a = -\infty$ and $b = \infty$ are admissible, of *Volterra Type* has the advantage that the Neumann Series not only represents the Fredholm Resolvent of the operator in a neighborhood about 0 in $\mathbb{C}$, but actually in all of $\mathbb{C}$. This means that the Fredholm Resolvent is an $\mathcal{L}_2$-valued entire function of the complex variable λ. Concomitantly, we shall look at radially acting linear integral operator on $\mathfrak{H}_2$ of *Volterra Type*, which leads us to

DEFINITION 4.3.1. *The class $\mathfrak{V}\mathfrak{K}_2$ of Volterra Type radially acting linear integral operators on $\mathfrak{H}_2$ is*

$$\mathfrak{V}\mathfrak{K}_2 = \Big\{K \in \mathfrak{K}_2 : \textit{with } \phi\textit{-parameter family of } \mathcal{L}_2\textit{-kernels } K(r,r',\phi) \tag{4.3.19}$$

$$\textit{satisfying } K(r,r',\phi) = 0 \ (r' > r;\ 0 < \phi < \pi)\Big\}.$$

Due to the property of $|||K_\phi - K_\psi||| \to 0$ as $\phi \to \psi$ ($\psi = 0, \pi$) from within $(0,\infty)$ (Proposition 3.4.1, equation (3.4.44) in the form (3.4.45)), we coclude out of the functional relationship

$$\lim_{\phi\to\psi} \int_0^\infty \int_0^\infty K_\phi(r,r')\chi_A(r,r')dr'dr = \int_0^\infty \int_0^\infty K_\psi(r,r')\chi_A(r,r')dr'dr = 0 \ (\psi = 0,\pi)$$

for a characteristic functions χ_A of Lebesgue measurable subsets A of $\{(r,r') : r' > r \geq 0\}$, that $K_\psi(r,r') = 0$ a.e. in $\{(r,r') : r' > r \geq 0\}$ if $K \in \mathfrak{V}\mathfrak{K}_2$; and therefore we have the simple

LEMMA 4.3.1. *If $K \in \mathfrak{V}\mathfrak{K}_2$, then its ϕ-parameter family of $\mathcal{L}_2$-kernels satisfies $K(r, r', \phi) = 0$ for all $r' > r$ $(0 \le \phi \le \pi)$.*

We further note that if K and L belong to $\mathfrak{V}\mathfrak{K}_2$, then $\alpha K + \beta L$ and $KL \in \mathfrak{V}\mathfrak{K}_2$. This is so, because if $K(r, r', \phi)$ and $L(r, r', \phi)$ are the respective ϕ-parameter family of $\mathcal{L}_2$-kernels belonging to K and L, then the ϕ-parameter family of $\mathcal{L}_2$-kernels $(\alpha K + \beta L)(r, r', \phi)$ and $(KL)(r, r', \phi)$ belonging to $\alpha K + \beta L$ and KL respectively $(\alpha, \beta \in \mathbb{C})$ satisfy

$$(\alpha K + \beta L)(r, r', \phi) \equiv \alpha K(r, r', \phi) + \beta L(r, r', \phi) = 0 \ (r' > r \ge 0;\ 0 \le \phi \le \pi) \text{ and}$$

$$(KL)(r, r', \phi) = \int_{u=r'}^{r} K(r, u, \phi) L(u, r', \phi) e^{i\phi} du = \tag{4.3.20}$$

$$\int_{[r', r]} K(r', u, \phi) L(u, r', \phi) e^{i\phi} = 0 \ (r' > r \ge 0;\ 0 \le \phi \le \pi),$$

where we note that the closed interval $[r', r] = \emptyset$ (empty set) if $r' > r$. Moreover, equality (4.3.20) implies

$$|(KL)(r, r', \phi)| = ||K_\phi(r, \cdot)||_{L_2(r', r)} \times ||L_\phi(\cdot, r')||_{L_2(r', r)} \ (r, r' > 0;\ 0 \le \phi \le \pi)$$

as a direct result of the Schwarz Inequality applied to the immediately preceding equality defining $(KL)(r, r', \phi)$ for K, $L \in \mathfrak{V}\mathfrak{K}_2$. However, if we replace the domain of integration (r', r) in the first norm by $(0, r)$ and note that

$$|(KL)(r, r', \phi)| = |(LK)_\phi(r, r')| = |L_\phi K_\phi(r, r')|,$$

then we obtain the following very useful estimate

$$|(KL)(r, r', \phi)| = |K_\phi L_\phi(r, r')| \le ||K_\phi(r, r')||_{L_2(0, \infty)} \times ||L_\phi(\cdot, r')||_{L_2(r', r)} \tag{4.3.21}$$

$$(r, r' > 0;\ 0 \le \phi \le \pi;\ K, L \in \mathfrak{V}\mathfrak{K}_2).$$

This result in the case of $K = L$, under the observation that $|K^2(r, r', \phi)| = |K^2_\phi(r, r')| = |K_\phi{}^2(r, r')|$ and replacement of r' by u, reduces to

$$(4.3.22) \qquad |K^2(r, u, \phi)| = |K_\phi{}^2(r, u)| \le ||K_\phi(r, \cdot)||_{L_2(,r)} \times ||K_\phi(\cdot, u)||_{L_2(u,r)}$$

$$(r, u > 0;\ 0 \le \phi \le \pi;\ K \in \mathfrak{V}\mathfrak{K}_2).$$

By taking the $L_2(r, r')$-norm with respect to variable u in expression to the left and right of the inequality sign in relation (4.3.22), we have

$$(4.3.23) \qquad ||K_\phi{}^2(r, \cdot)||_{L_2(r',r)} \le ||K_\phi(r, \cdot)||_{L_2(0,r)} \times \left(\int_{u=r'}^{r} ||K_\phi(\cdot, u)||_{L_2(u,r)}{}^2 du\right)^{1/2}$$

$$(r, r' > 0;\ 0 \le \phi \le \pi;\ K \in \mathfrak{V}\mathfrak{K}_2).$$

Utilizing $|K^3(r, r', \phi)| = |(K^2K)(r, r', \phi)|$ and estimate (4.2.10) for $L = K^2$, we see that

$$|K^3(r, u, \phi)| = ||(K^2)_\phi(r, \cdot)||_{L_2(u,r)} \times ||K_\phi(\cdot, u)||_{L_2(u,r)}\ (r, r' > 0;\ 0 \le \phi \le \pi;\ K \in \mathfrak{V}\mathfrak{K}_2),$$

where $||(K^2)_\phi(r, \cdot)||_{L_2(u,r)} = ||K_\phi{}^2(r, \cdot)||_{L_2(u,r)}$, is estimated by means of inequality (4.2.23) as

$$(4.3.24) \qquad |K^3(r, u, \phi)| = |K_\phi{}^3(r, u)| \le ||K_\phi(r, \cdot)||_{L_2(0,r)} \times$$

$$\left(\int_{v=u}^{r} ||K_\phi(\cdot, v)||_{L_2(\cdot,u)}{}^2 du\right)^{1/2} \times ||K_\phi(\cdot, u)||_{L_2(u,r)}\ (r, r' > 0;\ 0 \le \phi \le \pi;\ K \in \mathfrak{V}\mathfrak{K}_2).$$

Taking the $L_2(r', r)$-norm of both sides in terms of variable u, we thereby obtain

$$(4.3.25) \qquad ||K_\phi{}^3(r, \cdot)||_{L_2(r',r)} \le \frac{||K_\phi(r, \cdot)||_{L_2(0,r)}}{\sqrt{2!}} \times \left(\int_{u=r'}^{r} ||K_\phi(\cdot, u)||_{L_2(u,r)}{}^2 du\right)^{2/2}$$

$$(r, r' > 0;\ 0 \le \phi \le \pi;\ K \in \mathfrak{V}\mathfrak{K}_2).$$

Continuing this process, we formulate

LEMMA 4.3.2. *If* $K \in \mathfrak{VR}_2$ *with* ϕ*-parameter family of Volterra* $\mathcal{L}_2$*-kernels* $K(r, r', \phi)$*, then*

$$(4.3.26) \qquad |K^n(r, u, \phi)| \leq \frac{||K_\phi(r, \cdot)||_{L_2(0,r)}}{\sqrt{(n-2)!}} \times \left(\int_{v=u}^{r} ||K_\phi(\cdot, v)||_{L_2(v,r)}{}^2 du\right)^{(n-2)/2} \times$$

$$||K_\phi(\cdot, u)||_{L_2(u,r)} \ (r, u > 0; \ 0 \leq \phi \leq \pi; \ n \geq 2).$$

PROOF. We procede by induction on n, because we already proved the validity of estimate (4.3.26) for $n = 2$ from inequality (4.3.22) and for $n = 3$ from (4.3.24). Assuming it to be true for $n = m$, we write $|K^{m+1}(r, r', \phi)| = |(K^m K)(r, r', \phi)|$ and use inequality (4.3.21) for $L = K^m$ and achieve with this that

$$(4.3.27) \qquad |K^{m+1}(r, r', \phi)| \leq ||K_\phi{}^m(r, \cdot)||_{L_2(r',r)} \times ||K_\phi(\cdot, r')||_{L_2(r',r)}$$

$$(r, r' > 0; \ 0 \leq \phi \leq \pi; \ K \in \mathfrak{VR}_2).$$

We look at both sides of inequality (4.3.26) for $n = m$ as functions of variable u, take the $L_2(r', r)$-norm of both of these functions with respect to variable u and thus have, after noting $|K^m(r, u, \phi)| = |(K^m)_\phi(r, u)| = |(K_\phi)^m(r, u)|$, that

$$(4.3.28) \quad ||K_\phi{}^m(r, \cdot)||_{L_2(r',r)} \leq \frac{||K_\phi(r, \cdot)||_{L_2(0,r)}}{\sqrt{(m-1)!}} \times \left(\int_{v=r'}^{r} ||K_\phi(\cdot, v)||_{L_2(v,r)}{}^2 du\right)^{(m-1)/2}$$

$$(r, r' > 0; \ 0 \leq \phi \leq \pi; \ K \in \mathfrak{VR}_2).$$

Estimating the expression $||K_\phi{}^m(r, \cdot)||_{L_2(r',r)}$ in inequality (4.3.27) by means of norm relation (4.3.28) gives us

$$|K^{m+1}(r, r', \phi)| \leq \frac{||K_\phi(r, \cdot)||_{L_2(0,r)}}{\sqrt{(m-1)!}} \times \left(\int_{v=r'}^{r} ||K_\phi(\cdot, v)||_{L_2(v,r)}{}^2 du\right)^{(m-1)/2} \times$$

$$||K_\phi(\cdot,r)||_{L_2(r',r)}\ (r,r'>0;\ 0\le\phi\le\pi;\ K\in\mathfrak{V}\mathfrak{K}_2),$$

which actually is the estimation (4.3.26) for $n = m+1$ and $u = r'$; and therewith our inductive proof is completed. □

Lemma 4.3.2 has far reaching implication. In particular, by taking advantage of

$$||K_\phi(\cdot,v)||_{L_2(v,r)} \le ||K_\phi(\cdot,v)||_{L_2(v,\infty)}\ (v>0;\ 0\le\phi\le\pi),$$

on account of $(v,r)\subset(v,\infty)$, we have that out of

$$\int_{v=u}^{r} ||K_\phi(\cdot,v)||_{L_2(v,r)}{}^2dv \le \int_0^\infty ||K_\phi(\cdot,v)||_{L_2(v,\infty)}{}^2dv =$$

$|||K_\phi|||^2\ (0\le\phi\le\pi;\ K\in\mathfrak{V}\mathfrak{K}_2)$ follows that

$$(4.3.29)\quad |K^n(r,r',\phi)| \le \frac{[|||K|||_{s(2)}]^{(n-2)/2}}{\sqrt{(n-2)!}} \times ||K_\phi(r,\cdot)||_{L_2(0,r)} \times ||K_\phi(\cdot,r')||_{L_2(r',\infty)}$$

$(r,r'>0;\ 0\le\phi\le\pi;\ K\in\mathfrak{V}\mathfrak{K}_2;\ n\ge 2)$, where $P_\phi(r,r') =$

$$||K_\phi(r,\cdot)||_{L_2(r,\cdot)} \times ||K_\phi(\cdot,r')||_{L_2(r',\infty)}\ (r,r'>0;\ 0\le\phi\le\pi;\ K\in\mathfrak{V}\mathfrak{K}_2;\ n\ge 2)$$

is undoubtedly a non-negative ϕ-parameter family of $\mathcal{L}_2$-kernels on $(0,\infty)^2$. By defining the entire function Λ of complex variable z as

$$(4.3.30)\qquad \Lambda(z) = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}},$$

we immediately see from the relation (4.3.29) that

$$\sum_{\mu=n+1}^{n+p} |\lambda^n K^{n+1}(r,r',\phi)| \le \sum_{\mu=n+1}^{n+p} |\lambda|^\mu \times \frac{[|||K|||_{s(2)}]^{\mu-1}}{\sqrt{(\mu-1)!}} \times ||K_\phi(r,\cdot)||_{L_2(0,r)}\times$$

$$||K_\phi(\cdot,r')||_{L(r',\infty)} \le \frac{[|\lambda|\times|||K|||_{s(2)}]^n}{\sqrt{n!}} \times |\lambda| \times \Lambda(|\lambda|\times|||K|||_{s(2)})\times$$

$$||K_\phi(r,\cdot)||_{L_2(0,r)} \times ||K_\phi(\cdot,r')||_{L_2(r',\infty)}.$$

Because $|\lambda| \times \Lambda(|\lambda| \times |||K|||_{s(2)})$ is bounded whenever λ is confined to a compact subset of $\mathbb{C}$, $[|\lambda| \times |||K|||_{s(2)}]^n [n!]^{-1/2}$ can be made arbitrarily small by choosing n sufficiently large and $P_\phi(r, r') \equiv ||K_\phi(r, \cdot)||_{L_2(0,r)} \times ||K_\phi(\cdot, r')||_{L_2(r',\infty)}$ defines a ϕ-parameter family of $\mathcal{L}_2$-kernels with uniformly bounded double norms, we have that the Neumann Series (4.2.11) converges "*relatively uniformly absolutly*" in (r, r', ϕ) on $(0, \infty)^2 \times [0, \pi]$ for all $\lambda \in \mathbb{C}$. Further, the "*relative uniform absolute*"-convergence of the Neumann Series of $K \in \mathfrak{V}\mathfrak{K}_2$ for all $\lambda \in \mathbb{C}$ allows us to conclude that

$$(4.3.31) \qquad \lambda\Big(K\Big[\sum_{n=0}^{\infty} \lambda^n K^{n+1}\Big]\Big)(r, r', \phi) = \sum_{n=1}^{\infty} \lambda^n K^{n+1}(r, r', \phi) =$$

$$\lambda\Big(\Big[\sum_{n=0}^{\infty} \lambda^n K^{n+1}\Big] K\Big)(r, r', \phi) \text{ (for almost all } r, r' > 0;\ 0 \leq \phi \leq \pi;\ \lambda \in \mathbb{C})$$

and this proves the following

THEOREM 4.3.3. *If $K \in \mathfrak{V}\mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$ $(0 \leq \phi \leq \pi)$, then the Fredholm Resolvent $H_\lambda = H_\lambda(K)$ of K defines an*

$\mathfrak{V}\mathfrak{K}_2$-valued entire function of λ- i.e. $H_\lambda = H_\lambda(K) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}$ and

$$H_\lambda(r, r', \phi) = H_\lambda(K; r, r', \phi) = \sum_{n=0}^{\infty} \lambda^n K^{n+1}(r, r', \phi) \ (r, r' > 0;\ 0 \leq \phi \leq \pi;\ \lambda \in \mathbb{C}).$$

Turning toward the inverse Mellin-Transformation representation of $K(r, r', \phi)$ by means of $\mathbf{K}(t, t')$, as given by relation (3.3.38), we obtain out of equation (3.5.55) and (3.5.47) that

$$(4.3.32) \quad K^{n+1}(r, r', \phi) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (re^{i\phi})^{-it-1/2} \mathbf{K}^{n+1}(t, t') (re^{i\phi})^{it'-1/2} dt' dt$$

$$(r, r' > 0;\ 0 \leq \phi \leq \pi) \text{ with}$$

$$\mathbf{K}^{n+1}(t,t') = \int_{-\infty}^{\infty} \mathbf{K}(t,t')\mathbf{K}^{n}(t'',t')dt'' = \int_{-\infty}^{\infty} \mathbf{K}^{n}(t,t'')\mathbf{K}(t'',t')dt''$$

defined inductively on n ($n \geq 1$). Further, out of norm relationship (3.5.61) shall follow that

$$H_{\lambda}(\mathbf{K}) = \sum_{n=0}^{\infty} \lambda^{n}\mathbf{K}^{n+1} \text{ with } \mathcal{L}_2\text{-kernel } H_{\lambda}(\mathbf{K};t,t') = \sum_{n=0}^{\infty} \lambda^{n}\mathbf{K}^{n+1}(t,t')$$

represents the Fredholm Resolvent Kernel of $\mathbf{K} \in \mathfrak{K}_2$ and it has the same radius of convergnce as $H_{\lambda}(K) = \sum_{n=0}^{\infty} \lambda^{n}K^{n+1}$ for $K \in \mathfrak{K}_2$. Moreover, the convergence of the series for $H_{\lambda}(\mathbf{K};t,t')$ is "*relative uniform absolute*" for all λ such that $|\lambda| < \mathfrak{r}(K) = \mathfrak{r}(\mathbf{K})$ (norm relation (3.5.61) combined with defining equation (4.3.18)). Therefore, for all λ with $|\lambda| < \mathfrak{r}(K)$ we have, in consequence of relations (3.3.38) and (4.3.32), that the ϕ-parameter family of $\mathcal{L}_2$ kernels of the Fredholm Resolvent $H_{\lambda}(K)$ are given by

(4.3.33)
$$H_{\lambda}(K;r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_{\lambda}(\mathbf{K};t,t')(re^{i\phi})^{-it-1/2} \times$$
$$(r'e^{i\phi})^{it'-1/2}dt'dt = \sum_{n=0}^{\infty}\lambda^{n}K^{n+1}(r,r',\phi) =$$
$$\sum_{n=0}^{\infty}\lambda^{n}(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}^{n+1}(t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt.$$

If we invoke the inverse Mellin-Transform representation (3.3.39) with the accompaning formula for $\mathbf{Kf}$, then the solution (1.1.16) of the radial integral equation (1.0.5) assumes the form

(4.3.34)
$$f(re^{i\phi}) = \sum_{n=0}^{\infty}\lambda^{n}(K^{n+1}g)(re^{i\phi}) =$$
$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}([I+\lambda H_{\lambda}(\mathbf{K})]\mathbf{g})(t)(re^{i\phi})^{-it-1/2}dt =$$

$$\sum_{n=0}^{\infty} \lambda^n (2\pi)^{-1/2} \int_{-\infty}^{\infty} (\mathbf{K}^{n+1}\mathbf{g})(t)(re^{i\phi})^{-it-1/2} dt \ (r > 0;\ 0 \le \phi \le \pi),$$

where the series representing the solution $f(re^{i\phi})$ is "*relatively uniformly absolutely*"-convergent as long as $|\lambda| < \mathfrak{r}(K) = \mathfrak{r}(\mathbf{K})$. In the case of $K \in \mathfrak{V}\mathfrak{K}_2$, we have that $H_\lambda(\mathbf{K}; t, t')$ is an $\mathfrak{K}_2$-valued entire function of complex variable λ, because $\mathfrak{r}(\mathbf{K}) = \mathfrak{r}(K) = \infty$, and the series representations (4.3.33) and (4.3.34) are $\mathfrak{K}_2$-and $\mathfrak{H}_2$-valued entire functions of complex variable λ, as result of Theorem 4.3.3 and the discourse on "*relative uniform absolute*"-convergence in $\mathfrak{K}_2$ and $\mathfrak{H}_2$ at the begining of this chapter.

CHAPTER 5

RADIALLY ACTING LINEAR INTEGRAL OPERATORS OF FINTE RANK

Before delving into the topic of radially acting linear integral operators of finite rank, we first have to ascertain which continuous linear functionals on $\mathfrak{H}_2$ are radially representable in terms of the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ defined by equation (1.2.2) of chapter 1. Thereafter, we must also look at the adjoint K^* in the inner product space $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ of the operator $K \in \mathfrak{H}_2$ in terms of the inverse Mellin-Transform representation as well as the transposition operation "T" in the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$.

5.1. Radially Representable Continuous Linear Functionals on $\mathfrak{H}_2$.

Let ℓ denote a radially representable bounded linear functional on $\mathfrak{H}_2$ - i.e.

$$\ell(f) = \langle\ f\ ,\ g^\bullet\ \rangle = \int_0^\infty f(re^{i\phi})g^\bullet(re^{i\phi})e^{i\phi}dr \text{ is independent of } \phi \tag{5.1.1}$$

$$(f \in \mathfrak{H}_2;\ 0 \leq \phi \leq \pi)$$

with unique $g^\bullet \in \mathfrak{H}_2$ determined by ℓ, although $g^\bullet$ should be written as $g_\ell{}^\bullet$ to bring out the dependence of $g^\bullet$ on the bounded linear functional $\ell \in \mathfrak{H}_2{}'$ ([10]). The Riesz Representation of the dual space of a Hilbert Space, specifically $\mathfrak{H}_2$ and $\boldsymbol{\mathfrak{H}_2}$ in terms of $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ and $\langle\ \boldsymbol{\mathfrak{H}_2} \mid \boldsymbol{\mathfrak{H}_2}\ \rangle$ respectively, guarantees by means of Proposition 3.5.2, especially equation (3.3.51), that

$$\ell(f) = \langle\ f \mid g\ \rangle \equiv \int_0^\infty f(-r)\overline{g(-r)}dr + \int_0^\infty f(r)\overline{g(r)}dr = \langle\ \mathbf{f} \mid \mathbf{g}\ \rangle \equiv \tag{5.1.2}$$

$$\int_{-\infty}^{\infty} [1+e^{2\pi t}]\mathbf{f}(t)\overline{\mathbf{g}}(t)dt \ (f \in \mathfrak{H}_2, \mathbf{f} \mapsto f \text{ via inverse Mellin-Transform}),$$

where g is a unique element of $\mathfrak{H}_2$ and so is $\mathbf{g} \in \mathfrak{H}_2$ with $\mathbf{g} \mapsto g$. We return to equation (5.1.1), single out $\phi = 0$, write

$$(5.1.3) \qquad \ell(f) = \int_0^\infty f(r)g^\bullet(r)dr = \int_0^\infty f(r)\overline{\overline{g^\bullet}(r)}dr \ (\overline{g^\bullet}(r) \equiv \overline{g^\bullet(r)} \text{ for all } r > 0),$$

and note that the inverse Mellin-Transformation representation (3.2.13) for $f = g^\bullet$ extended to include $\phi = 0$ and π by means of equation (3.2.15) gives us, after taking the complex conjugate $\overline{\mathbf{g}^\bullet}(r) \equiv \overline{g^\bullet(r)}$ of the expression (3.2.15) written for $f = g^\bullet$ and $\psi = 0$, that

$$\overline{g^\bullet}(r) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \overline{\mathbf{g}^\bullet(t)}r^{it-1/2}dt = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \overline{\mathbf{g}^\bullet}(-t)r^{-it-1/2}dt,$$

where the integration variable t has been replaced by $-t$; therefore,

$$(5.1.4) \qquad \overline{g^\bullet}(e^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \overline{\mathbf{g}^\bullet}(-t)(re^{i\phi})^{-it-1/2}dt \text{ provided}$$

$$\int_{-\infty}^{\infty} [1+e^{2\pi t}]|\mathbf{g}^\bullet(-t)|^2 dt < \infty.$$

A word of caution, $\overline{g^\bullet}(t)$ as defined by equation (5.1.4) is not the complex conjugate of $g^\bullet(z)$; however, it must be regarded as a "*involution*" in the sense of a norm-preserving anti-linear bijective operation ([8], Section 4.7, pgs. 53 - 54) on $\mathfrak{H}_2$. Because of relations (5.1.2) and (5.1.3), we have from norm-equality of the inverse Mellin-Transform representation (3.2.13) extended to $\phi = 0$ and polarization, that

$$(5.1.5) \quad \ell(t) = \int_0^\infty f(t)\overline{\overline{g^\bullet}(r)}dr = \int_{-\infty}^{\infty} \mathbf{f}(t)\overline{\overline{\mathbf{g}^\bullet}(-t)}dt = \int_{-\infty}^{\infty} \mathbf{f}(t)[1+e^{2\pi t}]\overline{\mathbf{g}(r)}dt =$$

$$\langle\, \mathbf{f} \mid \mathbf{g}\,\rangle \text{ for all } f \in \mathfrak{H}_2, \text{ provided } \overline{\mathbf{g}^\bullet}(-\cdot) \in \mathfrak{H}_2.$$

In consequence hereof, we formulate

THEOREM 5.1.1. *A bounded linear functional $\ell = \langle \cdot \mid g \rangle$ on $\mathfrak{H}_2$ is radially representable if and only if the* $\mathbf{g}$ *in the inverse Mellin-Transform representation (3.2.13) of g has the property* $\mathbf{g}^\bullet(\cdot) \equiv [1 + e^{-2\pi \cdot}]\overline{\mathbf{g}}(-\cdot) \in \mathfrak{H}_2$. *In this case, we also have*

$$\ell(f) = \int_0^\infty f(re^{i\phi})g^\bullet(re^{i\phi})e^{i\phi}dr \text{ with} \tag{5.1.6}$$

$$g^\bullet(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}[1 + e^{-2\pi t}]\overline{\mathbf{g}(-t)}(re^{i\phi})^{-it-1/2}dt.$$

PROOF. That $\mathbf{g}^\bullet(t) = [1 + e^{-2\pi t}]\overline{\mathbf{g}}(-t)$ is evident from the equality of the second and third integral in relation (5.1.5), followed by replacement of t with $-t$. Conversely, if $[1 + e^{-2\pi \cdot}]\overline{\mathbf{g}}(-\cdot) \in \mathfrak{H}_2$, then defining $g^\bullet(re^{i\phi})$ by the equation on the right in statement (5.1.6) for $\mathbf{g}^\bullet(t) = [1 + e^{-2\pi t}]\overline{\mathbf{g}}(-t)$ lets us write the four equalities of relation (5.1.5). Therewith our theorem is proved. □

Since not all elements of $\mathfrak{H}_2{}'$ are radially representable, we may ask for a counterexample. To this end we let s be any complex number with $\mathfrak{Re}(s) > 1/4$ and define $\mathbf{g}_s(t) = [1 + e^{2\pi t}]^{-1/2}[1 + t^2]^{-s}$ ($t \in \mathbb{R}$), after choosing that branch for which $[1 + t^2]^{-\mathfrak{Re}(s)} > 0$. Clearly, $\mathbf{g}_s \in \mathfrak{H}_2$, because

$$\int_{-\infty}^{\infty}[1 + t^2]^{-2\mathfrak{Re}(s)}dt < \infty;$$

however $[1 + e^{-2\pi t}]\overline{\mathbf{g}_s(-t)} = [1 + e^{-2\pi t}]^{-3/2}[1 + t^2]^{-s}$ has the property that

$$\int_{-\infty}^{\infty}[1 + e^{2\pi t}][1 + e^{-2\pi t}]^{-3}[1 + t^2]^{-2\mathfrak{Re}(s)}dt \geq 2^{-3}\int_0^\infty [1 + e^{2\pi t}][1 + t^2]^{-2\mathfrak{Re}(s)}dt = \infty.$$

Moreover, we may further inquire about the density of radially representable bounded linear functionals in $\mathfrak{H}_2{}'$, and thereby arrive at the following

THEOREM 5.1.2. *The linear manifold of bounded linear functionals ℓ on $\mathfrak{H}_2$, which are radially representable in the sense of $\ell = \langle \cdot , g \rangle$ ($g \in \mathfrak{H}_2$), lies dense in $\mathfrak{H}_2{}'$. Furthermore, the set of bounded linear functionals on $\mathfrak{H}_2$ not radially representable is also dense in $\mathfrak{H}_2{}'$.*

PROOF. On account of the denisty of $C_c(\mathbb{R})$ in $\mathfrak{H}_2$, we may approximate every $\mathbf{g} \in \mathfrak{H}_2$ by an $[1 + e^{-2\pi\cdot}]\overline{h(-\cdot)} \in C_c(\mathbb{R})$ arbitrarily close in $\mathfrak{H}_2$-norm, whence the validity of the first assertion. By adding to anyone of these $C_c(\mathbb{R})$-functions the term $\varepsilon\mathbf{g}_s$, $\mathbf{g}_s(t) = [1+e^{2\pi t}]^{-1/2}[1+t^2]^{-s}(\mathfrak{Re}(s) > 1/4)$, demonstrates the second assertion; thereby completing the proof. □

We now turn our attention to the next pressing question, namely that of the Hilbert space adjoints of operators belonging to $\mathfrak{H}_2$.

5.2. Adjoints of $\mathfrak{K}_2$-Operators in $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$

To determine the Hilbert space adjoint of an operator $K \in \mathfrak{K}_2$, we invoke the inverse Mellin-Transform Representation (3.3.38) of its ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$, by means of the $\mathcal{L}_2$-kernel $\mathbf{K}(t, t')$, and with the aid of relation (3.3.39) and (3.3.48) we write

$$(5.2.7) \qquad \langle\ Kf \mid g\rangle = \langle\ \mathbf{Kf} \mid \mathbf{g}\ \rangle = \langle\ \mathbf{f} \mid \mathbf{K}^*\mathbf{g}\ \rangle = \langle\ f \mid K^*g\ \rangle \text{ for all } f, g \in \mathfrak{H}_2.$$

In particular, the last equality lets us conclude by means of Proposition 3.5.2 and the inverse Mellin-Transform Representation (3.2.13) of $\mathfrak{H}_2$-functions that

$$(5.2.8) \qquad (K^*g)(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathbf{K}^*\mathbf{g}(t)(re^{i\phi})^{-it-1/2}dt \ (re^{i\phi} \in \Pi_+; g \in \mathfrak{H}_2),$$

because $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ is a Hilbert space and $\mathbf{K} \in \mathcal{B}(\mathfrak{H}_2)$ implies that $\mathbf{K}^* \in \mathcal{B}(\mathfrak{H}_2)$.

Pursuing this concept further, we would like to know what $\mathbf{K}^* \in \mathcal{B}(\mathfrak{H}_2)$ looks like, and also if it is an integral operator belonging to $\mathfrak{K}_2$. We disentangle the second inner product in equation (5.2.7) as follows:

$$\langle\, \mathbf{Kf} \mid \mathbf{g}\,\rangle = \int_{-\infty}^{\infty}[1+e^{2\pi t'}](\mathbf{Kf})(t')\overline{\mathbf{g}(t')}dt' = \int_{-\infty}^{\infty}[1+e^{2\pi t'}]\int_{-\infty}^{\infty}\mathbf{K}(t,t')\times$$

$$\mathbf{f}(t)dt\overline{\mathbf{g}(t')}dt' = \int_{-\infty}^{\infty}[1+e^{2\pi t}]\mathbf{f}(t)\int_{-\infty}^{\infty}[1+e^{2\pi t'}]\overline{\overline{\mathbf{K}}(t',t)}[1+e^{2\pi t}]^{-1}\overline{\mathbf{g}(t')}dt'dt =$$

$$\int_{-\infty}^{\infty}[1+e^{2\pi t'}]\mathbf{f}(t)\overline{\left[\int_{-\infty}^{\infty}[1+e^{2\pi t'}]\overline{\mathbf{K}}(t',t)[1+e^{2\pi t}]^{-1}\mathbf{g}(t')dt'\right]}dt = \langle\, \mathbf{f} \mid \mathbf{K}^*\mathbf{g}\,\rangle =$$

$$\int_{-\infty}^{\infty}[1+e^{2\pi t}]\mathbf{f}(t)\overline{(\mathbf{K}^*\mathbf{g})(t)}dt \text{ for all } \mathbf{f} \text{ and } \mathbf{g} \in \mathfrak{H}_2.$$

Because of the positive nature of the measure $d\mu(t) = [1+e^{2\pi t}]dt$, we conclude that

$$(\mathbf{K}^*\mathbf{g})(t) = \int_{-\infty}^{\infty}\mathbf{K}^*(t,t')\mathbf{g}(t')dt', \tag{5.2.9}$$

$$\text{where } \mathbf{K}^*(t,t') = [1+e^{2\pi t'}]\times\overline{\mathbf{K}(t',t)}\times[1+e^{2\pi t}]^{-1};$$

however, by using the standard notation ^{T} for transposition and * for transposition followed by complex conjugation, we have that kernel $\mathbf{K}^*(t,t')$ may be written as

$$\mathbf{K}^*(t,t') = [1+e^{2\pi t'}]\times\overline{\mathbf{K}^T(t,t')}\times[1+e^{2\pi t}]^{-1} = \tag{5.2.10}$$

$$[1+e^{2\pi t'}]\times\mathbf{K}^*(t,t')\times[1+e^{2\pi t}]^{-1}.$$

The action of $\mathbf{K}^*$ is well defined by equation (5.2.9), because

$$\int_{-\infty}^{\infty}|\mathbf{K}^*(t,t')\mathbf{g}(t')|dt' = \int_{-\infty}^{\infty}[1+e^{2\pi t}]^{-1}|\mathbf{K}(t,t')||\mathbf{g}(t')|d\mu(t') \le [1+e^{2\pi t}]^{-1}\times$$

$$\left[\int_{-\infty}^{\infty}|\mathbf{K}(\mathbf{t},\mathbf{t}')|^2|d\mu(t')\right]^{1/2}||\mathbf{g}||_{\mathfrak{H}_2} < \infty \text{ for almost all } t\in\mathbb{R}\ (d\mu(t') = [1+e^{2\pi t'}]dt');$$

and by the very nature of the adjoint of continuous operators on a Hilbert space, in particular $\mathbf{K}^* \in \mathcal{B}(\mathfrak{H}_2)$.

THEOREM 5.2.1. *The adjoint K^* of $K \in \mathfrak{K}_2$ belongs to the Banach algebra $\mathfrak{K}_2$ if and only if the kernel $\mathbf{K}(t,t')$, in the inverse Mellin-Transform representation (3.3.38) of the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r,r',\phi)$, has the additional property that $(\cosh \pi t)(\cosh \pi t')^{-1}\mathbf{K}(t,t')$ is the kernel of an operator belonging to $\mathfrak{K}_2$ - i.e.*

$$(5.2.11) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}](\cosh \pi t)^2(\cosh \pi t')^{-2}|\mathbf{K}(t,t')|^2 dt' dt < \infty.$$

PROOF. If $K^* \in \mathfrak{K}_2$, then by the fact that $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ and $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ are isomorphic as Hilbert spaces, the kernel $\mathbf{K}^*(t,t')$ defined in relation (5.2.9) must yield the inverse Mellin-Transform representation (3.35) of $K^*(r,r',\phi)$ - i.e.

$$K^*(r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}^*(t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt$$

$(r,r'>0;\ 0\le\phi\le\pi)$ and $\mathbf{K}^*(t,t') = [1+e^{2\pi t'}]\overline{\mathbf{K}(t',t)}[1+e^{2\pi t}]^{-1}$ must satisfy

$$(5.2.12) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{K}^*(t,t')|^2 dt'dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{1+e^{2\pi(t-t')}}{1+e^{2\pi(t'-t)}}\right]\left[\frac{1+e^{2\pi t'}}{1+e^{2\pi t}}\right]^2[1+e^{2\pi(t'-t)}]|\mathbf{K}(t',t)|^2dt'dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{2\pi(t-t')}]\left[\frac{1+e^{2\pi t'}}{1+e^{2\pi t}}\right]^2[1+e^{2\pi(t'-t)}]|\mathbf{K}(t',t)|^2dt'dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t'-t)}]\left|\frac{\cosh \pi t'}{\cosh \pi t}\mathbf{K}(t',t)\right|^2 dtdt' < \infty.$$

Herein, we used the identity $(1+w)(1+1/w)^{-1} = w$ and took $e^{2\pi(t-t')}$ underneath the square of the first bracket. Clearly, condition (5.2.11) follows, if we interchange the integration variables t and t'. Conversely, condition (5.2.11) implies the finiteness of the first double integral in the chain of equations (5.2.12) and thus the proof is completed. □

Not all operators $K \in \mathfrak{K}_2$ have $K^* \in \mathfrak{K}_2$. Such a counter-example is the following operator $K_{s,s'}$ with ϕ-parameter family of $\mathcal{L}_2$-kernels

$$(5.2.13) \quad K_{s,s'}(r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}(s,s';t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt$$

$$(r,r'>0;\ 0\le\phi\le\pi) \text{ with } \mathbf{K}(s,s;t,t') \equiv$$

$$[1+e^{2\pi(t-t')}]^{-1/2}[1+t^2]^{-s}[1+t'^2]^{-s'} \ (\mathfrak{Re}(s),\ \mathfrak{Re}(s')>1/4), \text{ for which}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{K}(s,s';t,t')|^2dt'dt =$$

$$\left\{\int_{-\infty}^{\infty}[1+t^2]^{-2\mathfrak{Re}(s)}dt\right\}\left\{\int_{-\infty}^{\infty}[1+t'^2]^{-2\mathfrak{Re}(s')}dt'\right\} < \infty.$$

That the immediately preceding double integral with expression $\mathbf{K}(s,s';t,t')$ replaced by $(\cosh\pi t)(\cosh\pi t')^{-1}\mathbf{K}(s,s';t,t')$ is infinite, follows from

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|(\cosh\pi t)(\cosh\pi t')^{-1}\mathbf{K}(s,s';t,t')|^2dt'dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cosh\pi t)^2(\cosh\pi t')^{-2}[1+t^2]^{-2\mathfrak{Re}(s)}[1+t'^2]^{-2\mathfrak{Re}(s')}dt'dt =$$

$$\left\{\int_{-\infty}^{\infty}(\cosh\pi t)^2[1+t^2]^{-2\mathfrak{Re}(s)}dt\right\}\left\{\int_{-\infty}^{\infty}(\cosh\pi t')^{-2}[1+t'^2]^{-2\mathfrak{Re}(s')}dt'\right\} = \infty\times$$

$$\int_{-\infty}^{\infty}(\cosh\pi t')^{-2}[1+t'^2]^{-2\mathfrak{Re}(s')}dt' = \infty \text{ for all } s,\ s'\in\mathbb{C} \text{ with } \mathfrak{Re}(s),\ \mathfrak{Re}(s')>1/4.$$

Just as for the case of radially representable continuous linear functionals - i.e. those expressible in terms of the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ in the sense of

$$\ell(f) = \langle\ f\ ,\ g^\bullet\ \rangle = \int_0^{\infty} f(re^{i\phi})g^\bullet(re^{i\phi})e^{i\phi}dr \ (0\le\phi\le\pi) \text{ with unique } g^\bullet\in\mathfrak{H}_2 \text{ -}$$

being dense in $\mathfrak{H}_2'$, we expect a result similar to Theorem 5.1.2 to also hold for the adjoint operator K^* in the Hilbert Space $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ of an element $K\in\mathfrak{K}_2$. This is precisely the content of the next

THEOREM 5.2.2. *The linear manifold* $\{K \in \mathfrak{K}_2 : K^* \in \mathfrak{K}_2\}$ *is dense in the Banach algebra* $\mathfrak{K}_2$*, whereas the set* $\{K \in \mathfrak{K}_2 : K^* \notin \mathfrak{K}_2\}$ *is also dense in* $\in \mathfrak{K}_2$*.*

PROOF. The density of $\{K \in \mathfrak{K}_2 : K^* \in \mathfrak{K}_2\}$ follows immediately from the inverse Mellin-Transform representation (3.3.38) of the ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi)$ through $\mathbf{K}(t, t')$. We note that for the charactristic function $\chi_{[-A,A]^2}$ of the square $[-A, A]^2 = [-A, A] \times [-A, A]$ $(A > 0)$, we have

$$\lim_{A\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}]|[1 - \chi_{[-A,A]^2}(t, t')]\mathbf{K}(t, t')|^2 dtdt' = 0 \text{ and} \tag{5.2.14}$$

$\mathbf{K}_A(t, t') \equiv \chi_{[-A,A]^2}(t, t')\mathbf{K}(t, t')$ is a $\mathcal{L}_2$-kernel belonging to $C_0(\mathbb{R}^2)$ for all $A > 0$.

Consequently, $\mathbf{K}_A(t, t')$ and $(\cosh \pi t)(\cosh \pi t')^{-1}\mathbf{K}_A(t, t')$ satisfy the norm-condition set forth in relation (3.3.38), because

$$\int_{-A}^{A}\int_{-A}^{A} [1 + e^{2\pi(t-t')}]|\mathbf{K}(t, t')|^2 dt'dt \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}]|\mathbf{K}(t, t')|^2 dt'dt < \infty$$

$$\text{and } \int_{-A}^{A}\int_{-A}^{A} [1 + e^{2\pi(t-t')}]|(\cosh \pi t)(\cosh \pi t')^{-1}\mathbf{K}(t, t')|^2 dt'dt \leq$$
$$(\cosh A)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [1 + e^{2\pi(t-t')}]|\mathbf{K}(t, t')|^2 dt'dt.$$

By means of Theorem 5.2.1, we have that operator $K_A \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels

$$K_A(r, r', \phi) = (2\pi)^{-1} \int_{-A}^{A}\int_{-A}^{A} \mathbf{K}(t, t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2} dtdt'$$
$$(r, r' > 0;\ 0 \leq \phi \leq \pi)$$

and $K_A{}^* \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels

$$K_A{}^*(r, r', \phi) = (2\pi)^{-1} \int_{-A}^{A}\int_{-A}^{A} [1 + e^{2\pi t'}]\overline{\mathbf{K}(t', t)}[1 + e^{2\pi t}]^{-1} \times \tag{5.2.15}$$

$$(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it-1/2}dt'dt \ (r, r' > 0;\ 0 \leq \phi \leq \pi)$$

and norm relationship (3.5.61) guarantees that

$$|||K - K_A|||_{s(2)}{}^2 \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1 + e^{2\pi(t-t')}]|\chi_{\mathbb{R}^2\setminus[-A,A]^2}(t,t')\mathbf{K}(t,t')|^2 dtdt'.$$

Limit statement (5.2.14) immediately implies the density of $\{K \in \mathfrak{K}_2 : K^* \in \mathfrak{K}_2\}$ in $\mathfrak{K}_2$. If we take any $K \in \mathfrak{K}_2$ that has adjoint $K^* \in \mathfrak{K}_2$, then by adding $\varepsilon K_{s,s'}$ to K, where $K_{s,s'}$ has ϕ-parameter family of $\mathcal{L}_2$-kernels $K_{s,s'}(r, r', \phi)$ given by equation (5.2.13) with $\mathcal{L}_2$-kernels $\mathbf{K}(s, s'; t, t')$ in the variables (t, t') defined subsequent to expression (5.2.13), we have an operator $K + \varepsilon K_{s,s'}$, whose adjoint $(K + \varepsilon K_{s,s'})^* = K^* + \varepsilon K_{s,s'}{}^*$ no longer belongs to $\mathfrak{K}_2$ if $\varepsilon > 0$, because $K_{s,s'}{}^* \notin \mathfrak{K}_2$ and $|||K + \varepsilon K_{s,s'} - K|||_{s(2)} = \varepsilon|||K_{s,s'}|||_{s(2)}$ can be made arbitrarily small by choosing ε sufficiently small. Since the manifold $\{K \in \mathfrak{K}_2 : K^* \in \mathfrak{K}_2\}$ is dense in $\mathfrak{K}_2$, we conclude that the set $\{K \in \mathfrak{K}_2 : K^* \notin \mathfrak{K}_2\}$ is also dense in $\mathfrak{K}_2$, and thus our proof is completed. $\square$

5.3. Transposition in the Dual System $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$.

Because we shall be looking at radial acting linear integral operators of finite rank, we must return to the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ and examine the concept of transposition ^{T} in this dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$, where we emphatically point out that $\langle\ \cdot\ ,\ \cdot\ \rangle : \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathbb{C}$ is by no means a sesquilinear form. $\langle\ \cdot\ ,\ \cdot\ \rangle : \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathbb{C}$ is far from being an inner product, and complex conjugation of the entries to the right of the comma, shall not turn it into one, because we definitely lose the ϕ-independence of the defining integral. Nevertheless, things are not as hopeless as they seem.

We turn to the inverse Mellin-Transform representation of $K(r, r', \phi)$ as given in relation (3.3.38) and single out the norm condition on $\mathbf{K}(t, t')$ therein, because

$$(5.3.16) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1 + e^{2\pi(t-t')}]|\mathbf{K}(-t', -t)|^2 dt'dt =$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1 + e^{2\pi(t'-t)}]|\mathbf{K}(t, t')|^2 dt'dt < \infty,$$

when we replace the variable (t, t') by $(-t', -t)$ in the second double integral. We therefore define the $\mathcal{L}_2$-kernel

$$(5.3.17) \qquad \mathbf{K}^T(t, t') \equiv \mathbf{K}(-t', -t) \ ((t, t') \in \mathbb{R}^2) \text{ and note that}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1 + e^{2\pi(t-t')}]|\mathbf{K}^T(t, t')|^2 dt'dt < \infty,$$

which implies that the inverse Mellin-Transform applied to $\mathbf{K}^T(t, t')$ yields

$$(5.3.18) \qquad K^T(r, r', \phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{K}^T(t, t') \times$$
$$(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt \ (r, r' > 0; 0 \leq \phi \leq \pi),$$

which is the ϕ-parameter family of $\mathcal{L}_2$-kernels of the operator $K^T \in \mathfrak{K}_2$ defined by

$$(5.3.19) \qquad (K^T f)(re^{i\phi}) \equiv \int_0^{\infty} K^T(r, r', \phi) f(r'e^{i\phi})e^{i\phi}dr' \ (r > 0; \ 0 \leq \phi \leq \pi).$$

To see this, all we have to do is to calculate the integral expression in this defining equation by means of inverse Mellin-Transform representation (5.3.18), which gives us

$$(5.3.20) \qquad \int_0^{\infty} K^T(r, r', \phi) f(r'e^{i\phi})e^{i\phi}dr' = (2\pi)^{-1}\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}\mathbf{K}^T(t, t')$$
$$\Big\{(2\pi)^{-1/2}\int_0^{\infty} f(r'e^{i\phi})(r'e^{i\phi})^{it'-1/2}e^{i\phi}dr'\Big\}dt'\Big](re^{i\phi})^{-it-1/2}dt =$$
$$(2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathbf{K}^T\mathbf{f})(t)(re^{i\phi})^{-it-1/2}dt \text{ with}$$

$$(5.3.21) \qquad (\mathbf{K}^T\mathbf{f})(t) = \int_{-\infty}^{\infty} \mathbf{K}^T(t,t')\mathbf{f}(t')dt' = \int_{-\infty}^{\infty} \mathbf{K}(-t',-t)\mathbf{f}(t')dt'(t \in \mathbb{R}).$$

At this stage we momentarily stop to emphasize that $\mathbf{K}^T$ is definitely *not* the transpose of $\mathbf{K}$ in the sense of $\mathbf{K}^{\mathbf{T}}(t,t') = \mathbf{K}(t',t)$, and this clarifies the use of the symbol "T" instead of "$\mathbf{T}$" in $\mathbf{K}^T$. The second equality is justified by the fact that the defining equation (3.2.12) of $\mathbf{f}(t)$ came from $F(u+i\phi) = f(e^{u+i\phi})e^{(u+i\phi)/2}$ and the independence of the integral expression

$$(2\pi)^{-1/2}\int_{-\infty}^{\infty} f(e^{u+i\phi})e^{(u+i\phi)/2}e^{i(u+i\phi)/2}e^{i(u+i\phi)t}du = \mathbf{f}(t)$$

from ϕ $(0 \le \phi \le \pi)$. If we replace herein e^u by r' ($u = \ln r'$ and $du = (r')^{-1}dr'$), then we have

$$(5.3.22) \qquad \mathbf{f}(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(r'e^{i\phi})(r'e^{i\phi})^{it-1/2}e^{i\phi}dr' \ (0 \le \phi \le \pi),$$

after appealing to Proposition 3.2.2 for the case of $\phi = \psi$ $(\psi = 0, \pi)$. That $\mathbf{K}^T\mathbf{f} \in \mathfrak{K}_2$ follows from the estimates

$$\int_{-\infty}^{\infty} |\mathbf{K}^T\mathbf{f}(t)|^2 dt = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\mathbf{K}^T(t,t')\mathbf{f}(t')dt'\right|^2 dt \le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathbf{K}(t,t')|^2dt'dt \times$$
$$||\mathbf{f}||_{L_2(\mathbb{R})}{}^2, \quad \int_{-\infty}^{\infty} e^{2\pi t}|\mathbf{K}^T\mathbf{f}(t)|^2dt = \int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} e^{\pi(t-t')}\mathbf{K}^T(t,t')e^{\pi t'}\mathbf{f}(t')dt'\right|^2 dt \le$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi(t-t')}|\mathbf{K}^T(t,t')|^2dtdt' \times ||e^{\pi\cdot}\mathbf{f}||_{L_2(\mathbb{R})}{}^2,$$ which taken together yield

$$\int_{-\infty}^{\infty}[1+e^{2\pi t}]|(\mathbf{K}^T\mathbf{f})(t)|^2dt \le \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}]|\mathbf{K}(t,t')|^2dtdt'\right\}\times$$
$$\left\{\int_{-\infty}^{\infty}[1+e^{2\pi t}]|\mathbf{f}(t)|^2dt\right\}$$ and therefore the action (5.3.20) becomes

$$(5.3.23) \qquad \int_0^{\infty} K^T(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(\mathbf{K}^T\mathbf{f})(t)(re^{i\phi})^{-it-1/2}dt =$$
$$(K^Tf)(re^{i\phi}) \text{ with } K^Tf \in \mathfrak{H}_2 \text{ for all } f \in \mathfrak{H}_2.$$

The justification of the notation "$T(^T)$" for transpose stems from the following two arguments. If we replace in the inverse Mellin-Transform representation (5.3.18) the $\mathbf{K}^T(t,t')$ by $\mathbf{K}(-t',-t)$ and interchange (t,t') with $(-t',-t)$ again, then we have in terms of the inverse Mellin-Transform representation (3.3.38) of $K(r,r',\phi)$ that

(5.3.24) $$K^T(r,r',\phi) = K(r',r,\phi) \ (r,r' > 0;\ 0 \leq \phi \leq \pi).$$

On the other hand, by calculating in the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ the expression

$$\langle\ Kf\ ,\ g\ \rangle = \int_0^\infty (Kf)(re^{i\phi})g(re^{i\phi})e^{i\phi}dr = \int_{-\infty}^\infty \int_{-\infty}^\infty K(r,r',\phi)f(r'e^{i\phi})e^{i\phi}dr' \times$$

$$g(e^{i\phi})e^{i\phi}dr = \int_0^\infty f(r'e^{i\phi}) \int_0^\infty K^T(r',r,\phi)g(e^{i\phi})e^{i\phi}dre^{i\phi}dr' =$$

$$\int_0^\infty f(r'e^{i\phi})(K^Tg)(r'e^{i\phi})e^{i\phi}dr' = \langle\ f\ ,\ K^Tg\ \rangle,$$

we have the forumlation of

THEOREM 5.3.1. *Every $K \in \mathfrak{K}_2$ possesses is a transpose $K^T \in \mathfrak{K}_2$ in the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ in the sense of*

(5.3.25) $\langle\ Kf\ ,\ g\ \rangle = \langle\ f\ ,\ K^Tg\ \rangle$ *for all $f,g \in \mathfrak{H}_2$, where K^T has the*

ϕ-parameter family of $\mathcal{L}_2$-kernls $K^T(r,r',\phi) = K(r',r,\phi)\ (r,r' > 0;\ 0 \leq \phi \leq \pi)$.

If $K(r,r',\phi)$ has inverse Mellin-Transform representation through $\mathbf{K}(t,t')$, then $K^T(r,r',\phi)$ has inverse Mellin-Transform representation through $\mathbf{K}^T(t,t') \equiv \mathbf{K}(-t',-t)$. The operation of transposition $T(^T)$ is on the one hand linear - i.e. $(\alpha K + \beta L)^T = \alpha K^T + \beta L^T$ - and on the other hand contra-variant - i.e. $(KL)^T = L^T K^T$ - on $\mathfrak{K}_2$.

We now are able to address ourselves to the following question. What do radially acting linear integral operators of finite rank look like? Herewith we enter into the next section.

5.4. Radially Acting Linear Integral Operator of Finite Rank.

We start by recalling that a linear operator is of finite rank, if its image space is finite dimensional, and the rank of an operator is defined to be the dimension of its image space.

DEFINITION 5.4.1. *An operator $K \in \mathfrak{K}_2$ is said to be of finite rank, if it range $\mathcal{R}(K) \equiv \{Kf : f \in \mathfrak{H}_2\}$ has finite dimension and*

$$rank(K) \equiv dim(\mathcal{R}(K)). \tag{5.4.26}$$

An operator $K \in \mathfrak{K}_2$ not of finite rank, is said to be infinite dimensional and we shall not treat it in this chapter. What does a 1-dimensional operator $K \in \mathfrak{K}_2$ look like? It has the form

$$(Kf)(re^{i\phi}) = \langle\ f \mid g\ \rangle h(re^{i\phi}) \text{ for some } g, h \in \mathfrak{H}_2 \text{ and all } f \in \mathfrak{H}_2. \tag{5.4.27}$$

We take the bilinear product of this equation $Kf = \langle\ f \mid g\ \rangle h$ with respect to some $k \in \mathfrak{H}_2$, such that $\langle\ h \mid k\ \rangle \neq 0$ - i.e. $\langle\ Kf = \langle\ f \mid g\ \rangle h\ ,\ k\ \rangle$ - and thus obtain $\langle\ Kf\ ,\ k\ \rangle = \langle\ f \mid g\ \rangle\langle\ h\ ,\ k\ \rangle$ or $\langle\ f\ ,\ K^T k\ \rangle = \langle\ f \mid g\ \rangle\langle\ h\ ,\ k\ \rangle$ for all $f \in \mathfrak{H}_2$. This means that $\langle\ f \mid g\ \rangle = \langle\ h\ ,\ k\ \rangle^{-1}\langle\ f\ ,\ K^T k\ \rangle = \langle\ f\ ,\ \langle\ h\ ,\ k\ \rangle^{-1} K^T k\ \rangle$ for all $f \in \mathfrak{H}_2$, or the bounded linear functional $\langle\ \cdot \mid g\ \rangle : f \mapsto \langle\ f \mid g\ \rangle$ is radially representable through the $\mathfrak{H}_2$-function $\langle\ h\ ,\ k\ \rangle^{-1} K^T k$ - i.e.

(5.4.28) $\quad \langle\ \cdot \mid g\ \rangle = \langle\ \cdot\ ,\ g^\bullet\ \rangle$ with $g^\bullet = \langle\ h\ ,\ k\ \rangle^{-1} K^T k$ having inverse

Mellin-Transform representation through $\mathbf{g}^\bullet = \langle\ h\ ,\ k\ \rangle^{-1} \int_{-\infty}^{\infty} \overline{\mathbf{K}(-t', -\cdot)}\mathbf{k}(t')dt'$

as consequence of Theorem 5.3.1 The conclusion we draw from this, for our radially acting integral operator defined by equation (5.3.20), is that

(5.4.29) $\quad (Kf)(re^{i\phi}) = \int_0^\infty (h \otimes g^\bullet)(r, r', \phi) f(r'e^{i\phi}) e^{i\phi} dr \ (re^{i\phi} \in \Pi_+;\ f \in \mathfrak{H}_2),$

where the bilinear map $\otimes :\ \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathfrak{K}_2$ given by

(5.4.30) $\quad (a \otimes b)(r, r', \phi) \equiv a(re^{i\phi}) b(r'e^{i\phi}) \ (r, r' > 0;\ 0 \leq \phi \leq \pi;\ a, b \in \mathfrak{H}_2)$

is lifted to a map $\otimes :\ \mathfrak{H}_2 \otimes \mathfrak{H}_2 \to \mathfrak{K}_2$ of the tensor product space of $\mathfrak{H}_2$ with itself ([14], pg. 327). Continuing in this manner, we do know that if $K \in \mathfrak{K}_2$ has rank n, then

(5.4.31) $\quad (Kf)(re^{i\phi}) = \sum_{\mu=1}^{n} \langle\ f \mid b_\mu\ \rangle a_\mu(re^{i\phi})$ with $\{b_\mu\}_{\mu=1}^n$ and $\{a_\mu\}_{\mu=1}^n$ two linearly

independent subsets of $\mathfrak{H}_2$ containing precisely n elements.

We recall that a finite system of vectors of a Hilbert space is linearly independent, if and only if its Grammian $\mathfrak{Gr}$ is positive, which for our set $\{a_\mu\}_{\mu=1}^n$ from $\mathfrak{H}_2$ means that

(5.4.32) $$\mathfrak{Gr}\begin{pmatrix} a_1\ a_2\ a_3\ \cdots\ a_n \\ a_1\ a_2\ a_3\ \cdots\ a_n \end{pmatrix} \equiv$$

$$
det \begin{pmatrix} \langle a_1 \mid a_1 \rangle \langle a_1 \mid a_2 \rangle \cdots \langle a_1 \mid a_n \rangle \\ \langle a_1 \mid a_2 \rangle \langle a_2 \mid a_2 \rangle \cdots \langle a_r \mid a_2 \rangle \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ \langle a_1 \mid a_r \rangle \langle a_2 \mid a_r \rangle \cdots \langle a_r \mid a_r \rangle \end{pmatrix} > 0.
$$

Because of the continuity of the sesquilinear maps $\langle \cdot \mid \cdot \rangle : \ \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathbb{C}$ and $\langle \cdot \mid \cdot \rangle : \ \boldsymbol{\mathfrak{H}_2} \times \boldsymbol{\mathfrak{H}_2} \to \mathbb{C}$ and $\langle a_\mu \mid a_\nu \rangle = \langle \mathbf{a}_\mu \mid \mathbf{a}_\nu \rangle$ $(1 \le \mu, \nu \le n)$, we have that by choosing $\mathfrak{H}_2$-functions c_μ arbitrarily close to a_μ, which is the same as choosing the $\boldsymbol{\mathfrak{H}_2}$-functions $\mathbf{c}_\mu$ and $\mathbf{a}_\mu$ (giving the inverse Mellin-Transform representation (3.2.13) of c_μ and a_μ respectively) arbitrarily near to each other,

$$
(5.4.33) \quad \mathfrak{Gr} \begin{pmatrix} a_1 \; a_2 \; a_3 \; \cdots \; a_n \\ c_1 \; c_2 \; c_3 \; \cdots \; c_n \end{pmatrix} = det \begin{pmatrix} \langle a_1 \mid c_1 \rangle \langle a_1 \mid c_2 \rangle \cdots \langle a_1 \mid c_n \rangle \\ \langle a_2 \mid c_1 \rangle \langle a_2 \mid c_2 \rangle \cdots \langle a_2 \mid c_n \rangle \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ \langle a_n \mid c_1 \rangle \langle a_n \mid c_2 \rangle \cdots \langle a_n \mid c_n \rangle \end{pmatrix} =
$$

$$
\mathfrak{Gr} \begin{pmatrix} \mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n \\ \mathbf{c}_1 \; \mathbf{c}_2 \; \cdots \; \mathbf{c}_n \end{pmatrix} = det \begin{pmatrix} \langle \mathbf{a}_1 \mid \mathbf{c}_1 \rangle \langle \mathbf{a}_1 \mid \mathbf{c}_2 \rangle \cdots \langle \mathbf{a}_1 \mid \mathbf{c}_n \rangle \\ \langle \mathbf{a}_2 \mid \mathbf{c}_1 \rangle \langle \mathbf{a}_2 \mid \mathbf{c}_2 \rangle \cdots \langle \mathbf{a}_2 \mid \mathbf{c}_n \rangle \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ \langle \mathbf{a}_n \mid \mathbf{c}_1 \rangle \langle \mathbf{a}_n \mid \mathbf{c}_2 \rangle \cdots \langle \mathbf{a}_n \mid \mathbf{c}_n \rangle \end{pmatrix} \neq 0
$$

can be made sufficiently near to the Grammian of $\{a_1, a_2, \cdots, a_n\}$. In particular, we can always guarantee that there exists a $\delta > 0$ such that

$$
(5.4.34) \qquad \mathfrak{Re}\left(\mathfrak{Gr} \begin{pmatrix} \mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n \\ \mathbf{c}_1 \; \mathbf{c}_2 \; \cdots \; \mathbf{c}_n \end{pmatrix}\right) > (1/2)\mathfrak{Gr} \begin{pmatrix} a_1 \; a_2 \; a_3 \; \cdots \; a_n \\ a_1 \; a_2 \; a_3 \; \cdots \; a_n \end{pmatrix} \ \text{if}
$$

$$
||a_\mu - c_\mu||_{\mathfrak{H}_2} = ||\mathbf{a}_\mu - \mathbf{c}_\mu||_{\boldsymbol{\mathfrak{H}_2}} < \delta \ (1 \le \mu \le n).
$$

Theorem 5.1.2 permits us to to choose c_μ arbitararily close a_μ, so that $\mathbf{c}_\mu^\bullet = [1 + e^{-2\pi\cdot}]\overline{\mathbf{c}_\mu}(-\cdot) \in \mathfrak{H}_2$ $(1 \le \mu \le n)$, and for such $\mathbf{c}_\mu$ we have $\langle\, a_\mu \mid c_\nu \,\rangle = \langle\, \mathbf{a}_\mu \mid \mathbf{c}_\nu \,\rangle = \langle\, a_\mu \,,\, c_\nu{}^\bullet \,\rangle$ $(1 \le \mu, \nu \le n)$, and therefore the matrix $A =$

$$\begin{pmatrix} \langle\, a_1 \,,\, c_1{}^\bullet \,\rangle & \langle a_2 \,,\, c_1{}^\bullet \,\rangle & \cdots & \langle\, a_n \,,\, c_1{}^\bullet \,\rangle \\ \langle\, a_1 \,,\, c_2{}^\bullet \,\rangle & \langle a_2 \,,\, c_2{}^\bullet \,\rangle & \cdots & \langle\, a_n \,,\, c_2{}^\bullet \,\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\, a_1 \,,\, c_n{}^\bullet \,\rangle & \langle a_2 \,,\, c_n{}^\bullet \,\rangle & \cdots & \langle\, a_n \,,\, c_n{}^\bullet \,\rangle \end{pmatrix} \left(det(A) = \mathfrak{Gr}\begin{pmatrix} \mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n \\ \mathbf{c}_1 \; \mathbf{c}_2 \; \cdots \; \mathbf{c}_n \end{pmatrix} \neq 0\right)$$

is invertible provided $||\mathbf{a}_\mu - \mathbf{c}_\mu||_{\mathfrak{H}_2} < \delta$ $(1 \le \mu \le n)$ and hence, the system of equations

$$\langle\, Kf = \sum_{\nu=1}^{n} \langle\, f \mid b_\nu \,\rangle a_\nu \,,\, c_\mu{}^\bullet \,\rangle \text{ or } \langle\, Kf \,,\, c_\mu{}^\bullet \,\rangle = \sum_{\nu=1}^{n} \langle\, a_\nu \,,\, c_\mu{}^\bullet \,\rangle \langle\, f \mid b_\nu \,\rangle \; (1 \le \mu \le n)$$

is uniquely solvable for the unknowns $\langle\, f \mid b_\mu \,\rangle$ $(1 \le \mu \le n)$. Rewriting the terms $\langle\, Kf \,,\, c_\mu{}^\bullet \,\rangle$ as $\langle\, f,\; K^T c_\mu{}^\bullet \,\rangle$, because $K \in \mathfrak{K}_2$, we have that

$$\text{(5.4.35)} \qquad \langle\, f \mid b_\mu \,\rangle = \langle\, f \,,\, \sum_{\nu=1}^{n} \gamma_{\mu\nu} K^T c_\nu{}^\bullet \,\rangle = \langle\, f \,,\; K^T\Big(\sum_{\nu=1}^{n} \gamma_{\mu\nu} c_\nu{}^\bullet\Big) \,\rangle = \langle\, f \,,\, b_\mu{}^\bullet \,\rangle \; (1 \le \mu \le n),$$

where the $\gamma_{\mu\nu}$ denotes the μ-th row and ν-column entry of A^{-1} $(1 \le \mu, \nu \le n)$ and thus

$$b_\mu{}^\bullet = K^T\Big(\sum_{\nu=1}^{n} \gamma_{\mu\nu} c_\nu{}^\bullet\Big)(1 \le \mu \le n)$$

and in terms of the tensor product space gives us the following

THEOREM 5.4.1. *Every $K \in \mathfrak{K}_2$ with $rank(K) = n$ and action $(Kf)(z) = \sum_{\mu=1}^{n} \langle\, f \mid b_\mu \,\rangle a_\mu(z)$ has the ϕ-parameter family of $\mathcal{L}_2$-kernels*

$$\text{(5.4.36)} \qquad K(r, r', \phi) = \sum_{\mu=1}^{n} (a_\mu \otimes b_\mu{}^\bullet)(r, r', \phi) \text{ with } b_\mu{}^\bullet(r' e^{i\phi}) =$$

$$(2\pi)^{-1}\int_{-\infty}^{\infty}[1+e^{-2\pi t}]\overline{\mathbf{b}_{\mu}}(-t)(r'e^{i\phi})^{-it-1/2}dt \ (1\le\mu\le n)$$

and Mellin-Transform representation through

$$\mathbf{K}(t,t')=\sum_{\mu=1}^{n}\mathbf{a}_{\mu}(t)\mathbf{b}_{\mu}{}^{\bullet}(-t')=\sum_{\mu=1}^{n}\mathbf{a}_{\mu}(t)[1+e^{2\pi t'}]\overline{\mathbf{b}_{\mu}(t')}.$$

PROOF. The Mellin-Transform represenatation of $K(r,r',\phi)$ is a direct consequence of those of a_μ and b_μ $(1\le\mu\le n)$, thereby completing the proof. □

From the preceding is becomes quite apparent that the transpose K^T of the finite dimensional operator $K\in\mathfrak{K}_2$, appearing in Theorem 5.4.1, in the dual system $\langle\ \mathfrak{H}_2\ ,\ \mathfrak{H}_2\ \rangle$ is

(5.4.37) $$K^T(r,r',\phi)=\sum_{\mu=1}^{n}(b_\mu{}^{\bullet}\otimes a_\mu)(r,r',\phi)=\sum_{\mu=1}^{n}b_\mu{}^{\bullet}(re^{i\phi})a_\mu(r'e^{i\phi})(r,r'>0;$$

$0\le\phi\le\pi)$ with inverse Mellin-Transform representation through $\mathbf{K^T}(t,t')=$

$$\mathbf{K}(-t',-t)=\sum_{\mu=1}^{n}\mathbf{b}_\mu{}^{\bullet}(-t')\mathbf{a}_\mu(-t')=\sum_{\mu=1}^{n}[1+e^{2\pi t}]\overline{\mathbf{b}_\mu}(t)\mathbf{a}_\mu(-t')\ (t,t'\in\mathbb{R}).$$

5.5. Solving Radial Integral Equations with $K\in\mathfrak{K}_2$ of Finite Rank.

The radial integral equation (1.0.6) with the radial integral operator $K\in\mathfrak{K}_2$ of finite rank n, where $K(r,r',\phi)$ is given by relation (5.4.34), gives us

(5.5.38) $$f(re^{i\phi})=g(re^{i\phi})+\lambda\sum_{\nu=1}^{n}\langle\ f\ ,\ b_\nu{}^{\bullet}\ \rangle a_\nu(re^{i\phi})=$$

$$g(re^{i\phi})+\lambda\sum_{\nu=1}^{n}\langle\ f\mid b_\nu\ \rangle a_\nu(re^{i\phi})\ (re^{i\phi}\in\Pi_+)$$

for the expression $f=g+\lambda Kf$. Calculating $\langle\ f-\lambda Kf=g\ ,\ b_\mu{}^{\bullet}\ \rangle$ $(1\le\mu\le n)$ leads to the system of linear equations

(5.5.39) $$(\underline{I}-\lambda\underline{k})\vec{x}=\vec{y},\ \text{where}$$

$$\vec{x} = \begin{pmatrix} \langle\, f \;,\; b_1^\bullet \,\rangle \\ \langle\, f \;,\; b_2^\bullet \,\rangle \\ \vdots \\ \langle\, f \;,\; b_n^\bullet \,\rangle \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} \langle\, g \;,\; b_1^\bullet \,\rangle \\ \langle\, g \;,\; b_2^\bullet \,\rangle \\ \vdots \\ \langle\, g \;,\; b_n^\bullet \,\rangle \end{pmatrix} \in \mathbb{C}^{(n\times 1)} \text{ and } \underline{k} = (k_{\mu\nu}) \in \mathbb{C}^{(n\times n)}$$

$$(k_{\mu\nu} = \langle\, a_\nu \;,\; b_\mu^\bullet \,\rangle = \langle\, a_\nu \mid b_\mu \,\rangle = \langle\, \mathbf{a}_\nu \mid \mathbf{b}_\mu \,\rangle \;(1 \le \mu, \nu \le n))$$

with $\underline{I}$ standing for the identity of $\mathbb{C}^{(n\times n)}$. Therefore, every solution $f \in \mathfrak{H}_2$ of the radial integral equation (1.0.6) induces via relation (5.5.38) a solution $\vec{x} \in \mathbb{C}^{(n\times 1)}$ of the linear system (5.5.39) of n equations in the unknowns $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Conversely, every such solution $\vec{x}$ of the linear system (5.5.39) yields a solution

(5.5.40) $$f(re^{i\phi}) = g(re^{i\phi} + \lambda \sum_{\mu=1}^{n} x_\mu a_\mu(re^{i\phi}) \;(r > 0;\; 0 \le \phi \le \pi)$$

of the radial integral equation (1.0.6), because the linear equation (5.5.39) tells us that

$$x_\mu = \langle\, g \;,\; b_\mu^\bullet \,\rangle + \lambda \sum_{\nu=1}^{n} \langle\, a_\nu \;,\; b_\mu^\bullet \,\rangle x_\nu \;(1 \le \mu \le n),$$

and these x_μ sustituted into the expression (5.5.40) gives us

$$f(re^{i\phi}) = g(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} \Big[\langle\, g \;,\; b_\mu^\bullet \,\rangle + \lambda \sum_{\nu=1}^{n} \langle\, a_\nu \;,\; b_\mu^\bullet \,\rangle x_\nu \Big] a_\mu(re^{i\phi}) =$$

$$g(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} \Big[\langle\, g \;,\; b_\mu^\bullet \,\rangle + \langle\, \lambda \sum_{\nu=1}^{n} x_\nu a_\nu \;,\; b_\mu^\bullet \,\rangle \Big] a_\mu(re^{i\phi}) = g(re^{i\phi}) +$$

$$\lambda \sum_{\mu=1}^{n} \langle\, g + \lambda \sum_{\nu=1}^{n} x_\nu a_\nu \;,\; b_\mu^\bullet \,\rangle a_\mu(re^{i\phi}) = g(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} \langle\, f \;,\; b_\mu^\bullet \,\rangle a_\mu(re^{i\phi}) =$$

$$g(re^{i\phi}) + \lambda \int_0^\infty K(r, r', \phi) f(r'e^{i\phi}) e^{i\phi} dr' \;(r > 0;\; 0 \le \phi \le \pi).$$

Of course, all of this is valid only for $K \in \mathfrak{K}_2$ of finite rank n with a ϕ-parameter family of $\mathcal{L}_2$-kernels given by equation (5.4.36). For such $K \in \mathfrak{K}_2$ of finite rank n, we set

$$\text{(5.5.41)} \qquad \delta(\lambda) \equiv det(\underline{I} - \lambda\underline{k}) \text{ and } \underline{A}_\lambda = \mathfrak{Adj}(\underline{I} - \lambda\underline{k}) = (\alpha_{\mu\nu}(\lambda)) \; (\lambda \in \mathbb{C})$$

and note that $\delta(\lambda)$ is precisely a polynomial of degree n in λ and the *classical adjoint* $\underline{A}_\lambda$ of $\underline{I} - \lambda\underline{k}$ is an $n \times n$ matrix with polynomial entries in λ of degree at most $n - 1$. In general, the relation between a determinant of a matrix and the classical adjoint of the self-same matrix lets write for these expressions

$$\text{(5.5.42)} \qquad \underline{A}_\lambda(\underline{I} - \lambda\underline{k}) = d(\lambda)\underline{I} = (\underline{I} - \lambda\underline{k})\underline{A}_\lambda \; (\lambda \in \mathbb{C}).$$

If herein $d(\lambda) = 0$, then the null-space $N(\underline{I} - \lambda\underline{k})$ is non-trivial; and therefore, the homogeneous radial integral equation $(I - \lambda K)f = 0$ with g being the 0 function of $\mathfrak{H}_2$, has a non-trivial solution f. Moreover, the characteristic function space is given by

$$\text{(5.5.43)} \qquad N(I - \lambda K) = \{f \in \mathfrak{H}_2 : (I - \lambda K)f = 0\} = \{\lambda \sum_{\nu=1}^{n} x_\nu a_\nu(re^{i\phi}) :$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \vec{x} \in N(\underline{I} - \lambda\underline{k})\}$$

and due to the linear independence of the sets $\{b_\mu\}_{\mu=1}^n$ and $\{a_\mu\}_{\mu=1}^n$ in $\mathfrak{H}_2$,

$$\text{(5.5.44)} \qquad dim(N(I - \lambda K)) = dim N(\underline{I} - \lambda\underline{k}) \; (\lambda \in \mathbb{C}).$$

Turning toward the situation $d(\lambda) \neq 0$, the linear system (5.5.39) of equation has the unique solution

$$(5.5.45) \qquad \vec{x} = [d(\lambda)]^{-1}\underline{A}_{\lambda}\vec{y} \text{ - i.e. } x_{\mu} = [d(\lambda)]^{-1}\sum_{\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\langle\ g\ ,\ b_{\nu}^{\bullet}\ \rangle\ (1 \leq \mu \leq n)$$

and thus

$$(5.5.46) \qquad f(re^{i\phi}) = g(re^{i\phi}) + \lambda\sum_{\mu=1}^{n}[d(\lambda)]^{-1}\sum_{\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\langle\ g\ ,\ b_{\nu}^{\bullet}\ \rangle a_{\mu}(re^{i\phi}) =$$

$$g(re^{i\phi}) + \lambda\Big(\Big\{[d(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(a_{\mu}\otimes b_{\nu}^{\bullet})\Big\}g\Big)(re^{i\phi})\ (r > 0;\ 0 \leq \phi \leq \pi)$$

is the unique solution of the radial integral equation $(I - \lambda K)f = g$ induced by the linear system of equations (5.5.39). This leads us to formulate the following

THEOREM 5.5.1. *If $K \in \mathfrak{K}_2$ of finite rank n with ϕ-parameter family of $\mathcal{L}_2$-kernels $K(r, r', \phi) = \sum_{\mu=1}^{n}(a_{\mu}\otimes b_{\mu}^{\bullet})(r, r', \phi)$ in terms of statement (5.4.34), then $\{\lambda \in \mathbb{C} : d(\lambda) \equiv det(\underline{I} - \lambda\underline{k}) \neq 0\}$ constitutes the set of all regular values λ of K and the Fredholm Resolvent of K for these regular value λ has the ϕ-parameter family of $\mathcal{L}_2$-kernels*

$$(5.5.47) \qquad H_{\lambda}(r, r', \phi) = H_{\lambda}(K; r, r', \phi) = [d(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(a_{\mu}\otimes b_{\nu}^{\bullet})(r, r', \phi)$$

$$(r, r' > 0;\ 0 \leq \phi \leq \pi),\ \textit{where}\ (\alpha_{\mu\nu}(\lambda)) = \mathfrak{Adj}(\underline{I} - \lambda\underline{k})\ \textit{and}\ \underline{k} = (k_{\mu\nu}) \in \mathbb{C}^{(n\times n)}$$

$$\textit{with}\ k_{\mu\nu} = \langle\ a_{\nu}\ ,\ b_{\mu}^{\bullet}\ \rangle = \langle\ a_{\nu}\ |\ b_{\mu}\ \rangle\ (1 \leq \mu, \nu \leq n).$$

PROOF. We need only verify the Fredholm Resolvent Equation in either operator form (4.1.8) or ϕ-parameter family of $\mathcal{L}_2$-kernels form (4.1.9). We shall take the less cumbersome approach of form (4.1.8) and observe: for the tensor products elements

$(a \otimes b)$ and $(c \otimes d)$ of $\mathfrak{K}_2$ defined by equation (5.4.29) and any $K \in \mathfrak{K}_2$, we always have

(5.5.48) $$(a \otimes b)(c \otimes d) = \langle\ c\ ,\ b\ \rangle (a \otimes d),\ \ K(a \otimes b) = (Ka) \otimes b$$

$$\text{and } (a \otimes b)K = a \otimes (K^T b).$$

This is because,

$$\Big((a \otimes b)(c \otimes d)\Big)(r, r', \phi) = \int_0^\infty a(re^{i\phi})b(r''e^{i\phi})c(r''e^{i\phi})d(r'e^{i\phi})e^{i\phi}dr'' =$$

$$\int_0^\infty c(r''e^{i\phi})b(r''e^{i\phi})e^{i\phi}dr''(a \otimes d)(r, r', \phi),\quad (K(a \otimes b))(r, r', \phi) =$$

$$\int_0^\infty K(r, r'', \phi)a(r''e^{i\phi})e^{i\phi}dr''b(r'e^{i\phi}) = ((Ka) \otimes b)(r, r', \phi) \text{ and } ((a \otimes b)K)(r, r', \phi) =$$

$$a(re^{i\phi}) \int_0^\infty b(r''e^{i\phi})K(r'', r', \phi)e^{i\phi}dr'' = a(re^{i\phi}) \int_0^\infty K^T(r', r'', \phi)b(r''e^{i\phi})e^{i\phi}dr'' =$$

$$(a \otimes K^T b)(r, r', \phi).$$

Further, the $(n \times n)$-matrix equations (5.5.42), satisfied by classical adjoint $\underline{A}_\lambda$ of $\underline{I} - \lambda \underline{k}$, we rewrite in the form

(5.5.49) $$\lambda \underline{k}\ \underline{A}_\lambda = \underline{A}_\lambda - d(\lambda)\underline{I} = \lambda \underline{A}_\lambda\ \underline{k},\ \text{where } (\alpha_{\mu\nu}(\lambda)) = \underline{A}_\lambda \text{ and } \delta_{\mu\nu} = \underline{I},$$

or in terms of matrix entries

$$\lambda \sum_{\gamma=1}^n k_{\mu\gamma}\alpha_{\gamma\nu}(\lambda) = \alpha_{\mu\nu}(\lambda) - \alpha(\lambda)\delta_{\mu\nu} = \lambda \sum_{\gamma=1}^n \alpha_{\mu\gamma}(\lambda)k_{\gamma\mu}\ (1 \le \mu, \nu \le n).$$

By means of these, we calculate $\lambda K H_\lambda$ and $\lambda H_\lambda K$ by writing $K = \sum_{\mu=1}^n a_\mu \otimes b_\mu{}^\bullet$

$$\text{and } H_\lambda = [d(\lambda)]^{-1} \sum_{\nu=1}^n \alpha_{\mu\nu}(\lambda)(a_\mu \otimes b_\mu{}^\bullet);\ \text{in particular,}$$

$$(5.5.50) \qquad \lambda K H_\lambda = \lambda\Big[\sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet\Big]\Big[\frac{1}{d(\lambda)}\sum_{\gamma,\nu=1}^{n}\alpha_{\gamma\nu}(\lambda)(a_\gamma \otimes b_\nu{}^\bullet\Big] = \sum_{\mu,\nu=1}^{n}\frac{1}{d(\lambda)} \times$$

$$\Big[\lambda\sum_{\gamma=1}^{n}\langle\, a_\gamma \ , \ b_\mu{}^\bullet \,\rangle\alpha_{\gamma\nu}(\lambda)\Big](a_\mu \otimes b_\nu{}^\bullet) = \sum_{\mu,\nu=1}^{n}\frac{1}{d(\lambda)}\Big[\alpha_{\mu\nu}(\lambda) - d(\lambda)\delta_{\mu\nu}\Big](a_\mu \otimes b_\nu{}^\bullet) =$$

$$\frac{1}{d(\lambda)}\sum_{\mu,\nu=1}^{n}\alpha_{\mu,\nu}(a_\mu \otimes b_\mu{}^\bullet) - \frac{d(\lambda)}{d(\lambda)}\sum_{\mu,\nu=1}^{n}\delta_{\mu\nu}(a_\mu \otimes b_\mu{}^\bullet) = H_\lambda - \sum_{\nu=1}^{n} a_\mu \otimes b_\mu{}^\bullet = H_\lambda - K$$

$$(5.5.51) \quad \text{and } \lambda H_\lambda K = \lambda\Big[\frac{1}{d(\lambda)}\sum_{\mu,\gamma=1}^{n}\alpha_{\mu\gamma}(\lambda)(a_\mu \otimes b_\gamma{}^\bullet)\Big]\Big[\sum_{\nu=1}^{n} a_\nu \otimes b_\nu{}^\bullet\Big] = \sum_{\mu,\nu=1}^{n}\frac{1}{d(\lambda)} \times$$

$$\Big[\lambda\sum_{\gamma=1}^{n}\alpha_{\mu\gamma}(\lambda)\langle\, a_\nu \ , \ b_\gamma{}^\bullet\rangle\Big](a_\mu \otimes b_\nu{}^\bullet) = \sum_{\mu,\nu=1}^{n}\frac{1}{d(\lambda)}\Big[\lambda\sum_{\gamma=1}^{n}\alpha_{\mu\gamma}(\lambda)k_{\gamma\nu}\Big](a_\mu \otimes b_\nu{}^\bullet) =$$

$$\sum_{\mu,\nu=1}^{n}\frac{1}{d(\lambda)}\Big[\alpha_{\mu\nu}(\lambda) - d(\lambda)\delta_{\mu\nu}\Big](a_\mu \otimes b_\nu{}^\bullet) =$$

$$\frac{1}{d(\lambda)}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(a_\mu \otimes b_\nu{}^\bullet) - \sum_{\nu=1}^{n}(a_\mu \otimes b_\mu{}^\bullet) = H_\lambda - K,$$

out of which follows $\lambda K H_\lambda = H_\lambda - K = \lambda H_\lambda K$ and our proof is completed. □

By taking notice of the action of operator transposition T in the dual system $\langle\, \mathfrak{H}_2 \ , \ \mathfrak{H}_2 \,\rangle$ on the tensor product operator $(a \otimes b) \in \mathfrak{K}_2$, namely $(a \otimes b)^T = b \otimes a$, we need not repeat the selfsame proof for

$$K^T = \Big[\sum_{\mu=1}^{n}(a_\mu \otimes b_\mu{}^\bullet)\Big]^T = \sum_{\mu=1}^{n}(b_\mu{}^\bullet \otimes a_\mu),$$

but observe that transposition T is a contra-variant operation - i.e. $(KL)^T = L^T K^T$. Therefore, $(\lambda K H_\lambda = H_\lambda - K = \lambda K H_\lambda)^T$ leads to $\lambda K^T H_\lambda{}^T = H_\lambda{}^T - K^T = \lambda H_\lambda{}^T K^T$, which means that if $d(\lambda) \neq 0$, then $H_\lambda{}^T = [H_\lambda(K)]^T$ is the Fredholm Resolvent of the finite dimension radial integral operator $K^T = \sum_{\mu=1}^{n}(b_\mu{}^\bullet \otimes a_\mu)$ - i.e.

$$(5.5.52) \qquad H_\lambda(K^T) = [H_\lambda(K)]^T = \frac{1}{d(\lambda)}\sum_{\mu\nu=1}^{n}\alpha_{\mu,\nu}(\lambda)(b_\nu{}^\bullet \otimes a_\mu) \ \ (d(\lambda) \neq 0).$$

Let us for a moment turn to the realm of linear algebra and recall that if $A \in \mathbb{C}^{(m\times n)}$, then the range $\mathcal{R}(A)$ of matrix A has the property, that it is the annihilator of the nullspace of the transpose A^T of A - i.e. $\mathcal{R}(A) =$

$$[N(A^T)]^\perp = \left\{ \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{C}^{(n\times 1)} : \sum_{\mu=1}^{m} u_\mu w_\mu = 0 \text{ for } \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \in N(A^T) \right\}.$$

Therefore, the linear system of equations (5.4.36) is solvable for the unknown vector $\vec{x}$ ($\in \mathbb{C}^{(n\times 1)}$), if and only if

$$\vec{y} = \begin{pmatrix} \langle\, g \,,\, b_1{}^\bullet \,\rangle \\ \langle\, g \,,\, b_2{}^\bullet \,\rangle \\ \vdots \\ \langle\, g \,,\, b_n{}^\bullet \,\rangle \end{pmatrix} \in [N((\underline{I} - \lambda \underline{k})^T)]^\perp = [N(\underline{I} - \lambda \underline{k}^T)]^\perp.$$

However, the transpose of the finite-dimensional operator $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet$ is $K^T = \sum_{\mu=1}^{n} b_\mu{}^\bullet \otimes a_\mu$ and the radial integral equation

$$u(re^{i\phi}) = v(re^{i\phi}) + \lambda(K^T u)(re^{i\phi}) = v(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} \langle\, u \,,\, a_\mu \,\rangle b_\mu{}^\bullet(re^{i\phi})$$

is solvable for the unknown $\mathfrak{H}_2$-function u, if and only if the linear system of equations

$$(5.5.53)\qquad (\underline{I} - \lambda \underline{k}^T)\vec{u} = \vec{v}, \text{ where } \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} \langle\, g \,,\, a_1 \,\rangle \\ \langle\, g \,,\, a_2 \,\rangle \\ \vdots \\ \langle\, g \,,\, a_3 \,\rangle \end{pmatrix} \in \mathbb{C}^{(n\times 1)}$$

is solvable for $\vec{u} \in \mathbb{C}^{(n\times 1)}$. Like before for $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet$, we have that

$$(5.5.54) \qquad u(re^{i\phi}) = v(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} u_\mu b_\mu{}^\bullet(re^{i\phi})$$

is the solution of the transposed radial linear integral equation $(I - \lambda K^T)u = v$. Further,

$$N(I - \lambda K^T) = \left\{\lambda \sum_{\mu=1}^{n} u_\mu b_\mu{}^\bullet(re^{i\phi}) : \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in N(\underline{I} - \lambda \underline{k}^T)\right\} \text{ and}$$

$$\vec{y} = \begin{pmatrix} \langle g , b_1{}^\bullet \rangle \\ \langle g , b_2{}^\bullet \rangle \\ \vdots \\ \langle g , b_n{}^\bullet \rangle \end{pmatrix} \in [N(\underline{I} - \lambda \underline{k}^T)]^\perp \text{ if and only if } \lambda \sum_{\mu=1}^{n} \langle g , b_\mu{}^\bullet \rangle u_\mu = 0 \text{ for all}$$

$\vec{u} \in N(\underline{I} - \lambda \underline{k}^T)$, which is rewritten as $\langle g , \lambda \sum_{\mu=1}^{n} u_\mu b_\mu{}^\bullet \rangle = 0$ for all $\vec{u} \in N(\underline{I} - \lambda \underline{k}^T)$ yields the Fredholm Alternatives, as formulated by

THEOREM 5.5.2. *The radial integral equation $f = g + \lambda K f$ for the unknow $\mathfrak{H}_2$-function f, where $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{K}_2$ with $rank(K) = n$, is solvable if and only if*

$$(5.5.55) \qquad \langle g , N(I - \lambda K^T) \rangle = 0.$$

Moreover, the transposed radial integral equation $u = v + \lambda K^T u$ for the unknow $\mathfrak{H}_2$-function u, is solvable if and only if

$$\langle\ N(I - \lambda K)\ ,\ v\ \rangle = 0. \tag{5.5.56}$$

How do these results relate to the inverse Mellin-Transform representation of $\mathfrak{H}_2$ and $\mathfrak{K}_2$? Owing to Propostion 3.3.1, in particular equation (3.3.39) and further to equation (3.3.59), we have that our radial integral equation (1.0.6) in format (5.4.36) is solvable, if and only if the integral equation for the unknow $\mathfrak{H}_2$-function $\mathbf{f}$

$$\mathbf{f}(t) = \mathbf{g}(t) + \lambda \int_{-\infty}^{\infty} \mathbf{K}(t,t')\mathbf{f}(t')dt = \mathbf{g}(t) + \lambda \sum_{\nu=1}^{n} \langle\ \mathbf{f}\ |\ \mathbf{b}_\nu\ \rangle \mathbf{a}_\nu(t) \tag{5.5.57}$$

is solvable in $\mathfrak{H}_2$, where the $\mathfrak{H}_2$-function g occuring in relation (1.0.6) and (5.4.36) has inverse Mellin-Transform representation (3.2.13) through $\mathbf{g} \in \mathfrak{H}_2$ and the kernel $\mathbf{K}(t,t')$ is given by relation (5.4.36) of Theorem 5.4.1. Just as for the $K \in \mathfrak{K}_2$ appearing in Theorem 5.4.1, we obtain that the $\mathfrak{H}_2$-integral equation (5.5.57) with degenerate (finite rank) kernel

$$\mathbf{K}(t,t') = \sum_{\nu=1}^{n} \mathbf{a}_\nu(t)\mathbf{b}_\nu{}^{\bullet}(-t') = \sum_{\nu=1}^{n} a_\nu(t)[1 + e^{2\pi t'}]\overline{\mathbf{b}_\nu(t')}$$

is solvable in $\mathfrak{H}_2$ if and only if the linear system (5.5.39) is solvable for

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ with } \vec{y} = \begin{pmatrix} \langle\ g\ ,\ b_1{}^{\bullet}\ \rangle \\ \langle\ g\ ,\ b_2{}^{\bullet}\ \rangle \\ \vdots \\ \langle\ g\ ,\ b_n{}^{\bullet}\ \rangle \end{pmatrix} = \begin{pmatrix} \langle\ g \mid b_1\ \rangle \\ \langle\ g \mid b_2\ \rangle \\ \vdots \\ \langle\ g \mid b_n\ \rangle \end{pmatrix} = \begin{pmatrix} \langle\ \mathbf{g} \mid \mathbf{b}_1\ \rangle \\ \langle\ \mathbf{g} \mid \mathbf{b}_2\ \rangle \\ \vdots \\ \langle\ \mathbf{g} \mid \mathbf{b}_n \rangle \end{pmatrix},$$

because $\langle\ \mathbf{f} \mid \mathbf{b}_\mu\ \rangle = \langle\ f \mid b_\mu\ \rangle = \langle\ f\ , b_\mu{}^{\bullet}\ \rangle$ $(1 \leq \mu \leq n)$, $\langle\ \mathbf{g} \mid \mathbf{b}_\mu\ \rangle = \langle\ g \mid b_\mu\ \rangle = \langle\ g\ ,\ b_\mu{}^{\bullet}\ \rangle$ $(1 \leq \mu \leq n)$, and $\langle\ \mathbf{a}_\nu \mid \mathbf{b}_\mu\ \rangle = \langle\ a_\nu \mid b_\mu\ \rangle = \langle\ a_\nu\ ,\ b_\mu{}^{\bullet}\ \rangle = k_{\mu\nu}$ $(1 \leq \mu, \nu \leq n)$.

Without any difficulty, we conclude that if $d(\lambda) = 0$, then λ is a characteristic value of $\mathbf{K} \in \mathcal{B}(\mathfrak{H}_2)$ and

$$(5.5.58) \qquad N(I - \lambda K) = \{(2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{f}(t)(re^{i\phi})^{-it-1/2} dt : \mathbf{f} \in N(I - \lambda \mathbf{K})\},$$

as a direct consequence of equation (3.3.37) and representation (3.3.38) in Proposition 3.3.1. In case $d(\lambda) \neq 0$, the solution of the linear system (5.5.39) is unique with

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (\underline{I} - \lambda \underline{k})^{-1} \begin{pmatrix} \langle g | \mathbf{b}_1^\bullet \rangle \\ \langle g | \mathbf{b}_2^\bullet \rangle \\ \vdots \\ \langle g | \mathbf{b}_n^\bullet \rangle \end{pmatrix} = [d(\lambda)]^{-1} \underline{A}_\lambda \begin{pmatrix} \langle \mathbf{g} \mid \mathbf{b}_1^\bullet \rangle \\ \langle \mathbf{g} \mid \mathbf{b}_2^\bullet \rangle \\ \vdots \\ \langle \mathbf{g} \mid \mathbf{b}_n^\bullet \rangle \end{pmatrix},$$

$$\text{where } \underline{A}_\lambda = \mathfrak{Adj}(\underline{I} - \lambda \underline{k}) = (\alpha_{\mu\nu}(\lambda)),$$

and the unique solution of the $\mathfrak{H}_2$-"integral equation" (5.5.57) is given by

$$(5.5.59) \qquad \mathbf{f}(t) = \mathbf{g}(t) + \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda) \langle \mathbf{g} \mid \mathbf{b}_\nu \rangle \mathbf{a}_\mu(t) \ (t \in \mathbb{R}).$$

At this point we introduce the tensor product map $\odot : \mathfrak{H}_2 \times \mathfrak{H}_2 \to \mathfrak{K}_2$ by means of

$$(5.5.60) \qquad \Big((\mathbf{a} \odot \mathbf{b})\mathbf{f}\Big)(t) \equiv \langle \mathbf{f} \mid \mathbf{b} \rangle \mathbf{a}(t) = \int_{-\infty}^{\infty} (\mathbf{a} \odot \mathbf{b})(t, t')\mathbf{f}(t') dt \text{ with kernel}$$

$$(\mathbf{a} \odot \mathbf{b})(t, t') \equiv \mathbf{a}(t)[1 + e^{2\pi t'}]\overline{\mathbf{b}(t')},$$

and note that the rule of correspondence $\odot$ is linear in the first variable and anti-linear in the second - i.e. it is linear to the left of $\odot$ and anti-linear to the right of $\odot$. With the aid of $\otimes$ we translate the result (5.5.59) into saying that

$$(5.5.61) \qquad \mathbf{H}_\lambda = \mathbf{H}_\lambda(\mathbf{K}) = \frac{1}{d(\lambda)} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(\mathbf{a}_\mu \odot \mathbf{b}_\nu) \ (\text{ provided } d(\lambda) \neq 0)$$

is the Fredholm Resolvent of the $\mathfrak{H}_2$-"integral operator" $\mathbf{K}$, with the $\mathcal{L}_2$-kernel

$$(5.5.62)\qquad \mathbf{H}_\lambda(t,t') = \mathbf{H}_\lambda(\mathbf{K};t,t') = \frac{1}{d)(\lambda)}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\mathbf{a}_\mu(t)[1+e^{2\pi t'}]\overline{\mathbf{b}_\nu(t')} =$$

$$\frac{1}{d(\lambda)}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\mathbf{a}_\mu(t)\mathbf{b}_\nu{}^\bullet(-t') = \frac{1}{d(\lambda)}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(\mathbf{a}\odot\mathbf{b}_\nu{}^\bullet)(t,t')$$

as a consequence of defining equation (5.4.31) and the fact that $(a_\mu \otimes b_\nu{}^\bullet)(r,r',\phi)$ has inverse Mellin-Transform representation through $\mathbf{a}_\mu(t)\mathbf{b}_\nu{}^\bullet(-t') = \mathbf{a}_\mu(t)[1+e^{2\pi t'}]\overline{\mathbf{b}_\mu(-t)}$. Therefore, the Fredholm Resolvent kernels of $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{K}_2$ and $\mathbf{K} = \sum_{\mu=1}^{n} \mathbf{a}_\mu \odot \mathbf{b}_\mu$ are related via the inverse Mellin-Transform as follows:

$$(5.5.63)\quad H_\lambda(K;r,r',\phi) = (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{H}_\lambda(\mathbf{K};t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt$$

$$(r,r'>0;\ 0\le\phi\le\pi).$$

Solving the radial integral equation (1.0.6) in $\mathfrak{H}_2$ is therefore equivalent to solving the $\mathfrak{H}_2$-"integral equation" (5.5.59), because of the "Hilbert space isomorphism"-nature of the inverse Mellin-Transform $\mathfrak{H}_2 \to \mathfrak{H}_2$ and $\mathfrak{K}_2 \to \mathfrak{K}_2$.

What about the adjoint kernel K^* of $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{K}_2$? Is K^* also an element of $\mathfrak{K}_2$? Let us begin by considering the radially acting linear integral operator $a\otimes b^\bullet$ of rank 1, where $(a\otimes b^\bullet)(r,r',\phi) = a(re^{i\phi})b^\bullet(r'e^{i\phi})$ has inverse Mellin-Transform representation through $\mathbf{a}(t)\mathbf{b}^\bullet(-t') = \mathbf{a}(t)[1+e^{2\pi t'}]\overline{\mathbf{b}(t')}$. If $(a\otimes b^\bullet)^* \in \mathfrak{K}_2$, then it must also be of *rank* 1, because out of

$$\langle\, (a\otimes b^\bullet)f \mid g\,\rangle = \langle\, \langle\, f\mid b\,\rangle a \mid g\,\rangle = \langle\, f\mid b\,\rangle\langle\, a\mid g\,\rangle = \langle\, f \mid \langle\, g\mid a\,\rangle b\,\rangle = \langle\, f \mid (a\otimes b^\bullet)^* g\,\rangle$$

follows $(a\otimes b^\bullet)^* g = \langle\, g \mid a\,\rangle b$ for all $g \in \mathfrak{H}_2$. Therefore $(a\otimes b^\bullet)^* \in \mathfrak{K}_2$, if and only if

$$(5.5.64)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[1+e^{2\pi(t-t')}](\cosh \pi t)^2(\cosh \pi t')^{-2}|\mathbf{a}(t)[1+e^{2\pi t'}]\overline{\mathbf{b}(t')}|^2 dt dt' < \infty.$$

By choosing $\mathbf{a} \equiv \mathbf{g}_s$ ($\mathbf{g}_s(t) = [1 + e^{2\pi t}]^{-1/2}[1 + t^2]^{-s}, \mathfrak{Re}(s) > 1/4$) we clearly see that condition (5.5.64) must fail, otherwise $(a \otimes b^\bullet)^* \in \mathfrak{K}_2$ with ϕ-parameter family of $\mathcal{L}_2$-kernels $(a \otimes b^\bullet)^*(r, r', \phi)$ $(r, r' > 0;\ 0 \le \phi \le \pi)$ and $(a \otimes b^\bullet)^{*T}(r, r', \phi) = (a \otimes b^\bullet)^*(r', r, \phi)$ is the ϕ-parameter family of $\mathcal{L}_2$-kernels belonging to $(a \otimes b^\bullet)^* \in \mathfrak{K}_2$; hence,

$$\langle\ (a \otimes b^\bullet)^* g\ ,\ u\ \rangle = \langle\langle\ g \mid a\ \rangle b^\bullet\ ,\ u\ \rangle\ (g, u \in \mathfrak{H}_2).$$

By choosing a $u \in \mathfrak{H}_2$ such that $\langle\ b^\bullet\ ,\ u\ \rangle \neq 0$, we obtain that

$$\langle\ g \mid a\ \rangle = \langle\ b^\bullet\ ,\ u\ \rangle^{-1}\langle\ (a \otimes b^\bullet)^* g\ ,\ u\ \rangle = \langle\ g\ , \langle\ b^\bullet\ ,\ u\ \rangle^{-1}(a \otimes b^\bullet)^{*T} u\ \rangle\ (g \in \mathfrak{H}_2),$$

which means that $g \mapsto \langle\ g \mid a\ \rangle = \langle\ \mathbf{g} \mid \mathbf{a}\ \rangle$ is a radially representable bounded linear functional on $\mathfrak{H}_2$; however, $\mathbf{a}(t) = \mathbf{g}_s(t)$ was the counter-example of Section 5.1 demonstrating that not all elements of the dual space $\mathfrak{H}_2{}'$ of $\mathfrak{H}_2$ are radially representable.

The situation with $K^* \in \mathfrak{K}_2$ for a $K \in \mathfrak{K}_2$ of finite rank is not so disenchanting. We have the following

THEOREM 5.5.3. *If $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{K}_2$ and $rank(K) = n$, then $K^* \in \mathfrak{K}_2$ if and only if $a_\mu{}^\bullet \in \mathfrak{H}_2$ $(1 \le \mu \le n)$. If this is indeed the case, then*

$$K^* = \sum_{\mu=1}^{n} b_\mu \otimes a_\mu{}^\bullet,\ \textit{where}\ a_\mu{}^\bullet(re^{i\phi}) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}[1 + e^{-2\pi t}] \times \tag{5.5.65}$$

$$\overline{\mathbf{a}_\mu(-t)}(re^{i\phi})^{-it-1/2}dt\ (r > 0;\ 0 \le \phi \le \pi;\ 1 \le \mu \le n).$$

PROOF. The action $Kf = \sum_{\mu=1}^{n} \langle f , b_\mu^\bullet \rangle a_\mu = \sum_{\mu=1}^{n} \langle f \mid b_\mu \rangle a_\mu$, as result of Theorem 5.4.1, applies to each of the $b_\mu^\bullet$ individually $(1 \le \mu \le n)$; therefore, out of

$$\langle Kf \mid g \rangle = \sum_{\mu=1}^{n} \langle f \mid b_\mu \rangle \langle a_\mu \mid g \rangle = \langle f \mid \sum_{\mu=1}^{n} \langle g \mid a_\mu \rangle b_\mu \rangle = \langle f \mid K^* g \rangle \ (f, g \in \mathfrak{H}_2)$$

shall follow that

$$K^* g = \sum_{\mu=1}^{n} \langle g \mid a_\mu \rangle b_\mu \text{ - i.e. } (K^* g)(re^{i\phi}) = \sum_{\mu=1}^{n} \langle g \mid a_\mu \rangle b_\mu(re^{i\phi}) \tag{5.5.66}$$

$$(r > 0;\ 0 \le \phi \le \pi).$$

We note that the set of $\mathfrak{H}_2$-functions $\{b_\mu\}_{\mu=1}^{n}$ is linearly independent in $\mathfrak{H}_2$; thus, we can find n $\mathfrak{H}_2$-functions $\{d_\mu\}_{\mu=1}^{n}$, d_μ sufficiently near to b_μ in $\mathfrak{H}_2$-norm $(1 \le \mu \le n)$ such that $\mathbf{d}_\mu^\bullet \in \mathfrak{H}_2$ $(1 \le \mu \le n)$ and

$$\mathfrak{Gr}\begin{pmatrix} b_1\ b_2\ b_3\ \cdots\ b_n \\ d_1\ d_2\ d_3\ \cdots\ d_n \end{pmatrix} = \mathfrak{Gr}\begin{pmatrix} \mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \cdots\ \mathbf{b}_n \\ \mathbf{d}_1\ \mathbf{d}_2\ \mathbf{d}_3\ \cdots\ \mathbf{d}_n \end{pmatrix} =$$

$$det\begin{pmatrix} \langle b_1 , d_1 \rangle \langle b_1 , d_2 \rangle \langle b_1 , d_3 \rangle \cdots \langle b_1 , d_n \rangle \\ \langle b_2 , d_1 \rangle \langle b_2 , d_2 \rangle \langle b_2 , d_3 \rangle \cdots \langle b_2 , d_n \rangle \\ \vdots \qquad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ \langle b_n , d_1 \rangle \langle b_n , d_2 \rangle \langle b_n , d_3 \rangle \cdots \langle b_n , d_n \rangle \end{pmatrix} \neq 0,$$

as consequence of the comment made to justify relation (5.4.34). Let us try to find n $\mathfrak{H}_2$-functions u_μ $(1 \le \mu \le n)$ with $u_\mu^\bullet \in \mathfrak{H}_2$ $(1 \le \mu \le n)$ by means of setting $u_\mu = \sum_{\eta=1}^{n} \beta_{\eta\mu} d_\eta$ $(1 \le \mu \le n)$, which is equivelent to $u_\mu^\bullet = \sum_{\eta=1}^{n} \beta_{\eta\mu} d_\eta^\bullet$ $(1 \le \mu \le n)$, so that

$$\langle \mathbf{b}_\nu \mid \mathbf{u}_\mu \rangle = \langle b_\nu,\ u_\mu^\bullet \rangle = \delta_{\mu\nu}\ (1 \le \mu \le n), \tag{5.5.67}$$

where $\delta_{\mu\nu}$ stands for the *Kronecker delta* symbol. By replacing each $u_\mu^\bullet$ by its respective linear combination of $d_\eta^\bullet$ $(1 \le \eta \le n)$, we obtain the $\mathbb{C}^{(n\times n)}$ matrix equation

$$\begin{pmatrix} \langle\, b_1 \,,\, d_1^\bullet \,\rangle \langle\, b_1 \,,\, d_2^\bullet \,\rangle \cdots \langle\, b_1 \,,\, d_n^\bullet \,\rangle \\ \langle\, b_2 \,,\, d_1^\bullet \,\rangle \langle\, b_2 \,,\, d_2^\bullet \,\rangle \cdots \langle\, b_2 \,,\, d_n^\bullet \,\rangle \\ \vdots \qquad\qquad \vdots \qquad \ddots \qquad \vdots \\ \langle\, b_n \,,\, d_1^\bullet \,\rangle \langle\, b_n \,,\, d_2^\bullet \,\rangle \cdots \langle\, b_n \,,\, d_n^\bullet \,\rangle \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix} = \underline{I},$$

which is uniquely solvable for the unknown $(n \times n)$-matrix $(\beta_{\mu\nu})$ on account of $\mathfrak{Gr}\begin{pmatrix} b_1\, b_2\, b_3 \cdots b_n \\ d_1\, d_2\, d_3 \cdots d_n \end{pmatrix} \neq 0$. Hence, these n $\mathfrak{H}_2$-functions exist. We now calculate $\langle\, K^* g \,,\, u_\mu^\bullet \,\rangle$ $(1 \le \mu \le n)$ by use of equation (5.5.66) and thus conclude out of relation (5.5.67), that

$$\langle\, K^* g \,,\, d_\mu^\bullet \,\rangle = \langle\, \sum_{\nu=1}^{n} \langle\, g \mid a_\nu \,\rangle b_\nu \,,\, d_\mu^\bullet \,\rangle =$$

$$\sum_{\nu=1}^{n} \langle\, g \mid a_\nu \,\rangle \langle\, b_\nu \,,\, d_\mu^\bullet \,\rangle = \langle\, g \mid a_\mu \,\rangle \ (1 \le \mu \le n),$$

which is the same as saying $\langle\, g \mid a_\mu \rangle = \langle\, K^* g \,,\, d_\mu^\bullet \,\rangle$ $(1 \le \mu \le n)$. If $K^* \in \mathfrak{K}_2$, then $K^{*T} \in \mathfrak{K}_2$ by Theorem 5.3.1 and $\langle\, K^* g \,,\, d_\mu^\bullet \,\rangle = \langle\, g \,,\, (K^*)^T d_\mu^\bullet \,\rangle$ $(1 \le \mu \le n)$; in other words, $\langle\, g \mid a_\mu \,\rangle = \langle\, g \,,\, (K^*)^T d_\mu^\bullet \,\rangle$ $(g \in \mathfrak{H}_2;\ 1 \le \mu \le n)$, which actually means

$$\text{(5.5.68)} \quad \langle\, g \mid a_\mu \,\rangle = \langle\, g \,,\, a_\mu^\bullet \,\rangle \text{ for all } g \in \mathfrak{H}_2 \text{ and } a_\mu^\bullet = (K^*)^T d_\mu^\bullet \ (1 \le \mu \le n).$$

By writing each quantity $\langle\, g \,,\, a_\mu^\bullet \,\rangle$ in terms of its respective integral expression (1.2.2) for each fixed ϕ, we immediately obtain (5.5.65). Conversely, if each $a_\mu \in \mathfrak{H}_2$ appearing in the tensor product format of K, has the property that $a_\mu^\bullet \in \mathfrak{H}_2$ $(1 \le$

$\mu \leq n$), then $\langle g \mid a_\mu \rangle = \langle g , a_\mu{}^\bullet \rangle$ holds for all $g \in \mathfrak{H}_2$ and μ $(1 \leq \mu \leq n)$, out of which we immediately extrapolate that $K^\bullet$ has the tensor product representation (5.5.65), and hence $K^\bullet \in \mathfrak{H}_2$. Herewith our proof is completed. □

Let us now look at the radial integral equation $f = g + \lambda K^\bullet f$ for a $K \in \mathfrak{K}_2$ of finite rank n, possessing a tensor product representation as in Theorem 5.5.3, such that $K^\bullet \in \mathfrak{K}_2$. Because $K^\bullet$ has tensor product representation (5.5.65), our radial integral equation $f = g + \lambda K^\bullet f$ takes the form $f = g + \lambda \sum_{\mu=1}^{n} \langle f , a_\mu{}^\bullet \rangle b_\mu$, which admits a solution if and only if the linear system of equations

$$(5.5.69) \qquad (\underline{I} - \overline{\lambda}\underline{k})^\bullet \vec{x} = \vec{y} \text{ with } \vec{y} = \begin{pmatrix} \langle g , a_1{}^\bullet \rangle \\ \langle g , a_2{}^\bullet \rangle \\ \vdots \\ \langle g , a_2{}^\bullet \rangle \end{pmatrix} \text{ and } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^{(n \times 1)}$$

is solvable for $\vec{x}$, and the interaction between radial integral equation $(I - \lambda K^\bullet) f = g$ and linear equation (5.5.69) is that of

$$(5.5.70) \qquad f(re^{i\phi}) = g(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} x_\mu b_\mu(re^{i\phi}) \ (r > 0;\ 0 \leq \phi \leq \pi).$$

This comes about by redoing for $K^\bullet$, what was done for K in showing the interaction between equation (5.5.38) and (5.5.39), under the considerations that

$$\langle b_\nu , a_\mu{}^\bullet \rangle = \langle b_\nu \mid a_\mu \rangle = \overline{\langle a_\mu \mid b_\nu \rangle} = \overline{k_{\nu\mu}} = (\underline{k}^\bullet)_{\mu\nu} \ (1 \leq \mu, \nu \leq n).$$

Let us now look at $\mathbb{C}^{(n\times 1)}$ as an n-dimensional Hilbert Space with the conventional inner product $\langle\ \vec{x}\mid\vec{y}\ \rangle \equiv \sum_{\mu=1}^{n} x_\mu\overline{y_\mu}$ for $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$. Denoting by $\oplus$ the direct sum of orthogonal subspaces, we recall that for the matrices $\underline{I} - \lambda\underline{k}$ and $(\underline{I}-\lambda\underline{k})^* = \underline{I}-\overline{\lambda}\underline{k}^*$ we have: $\mathbb{C}^{(n\times 1)} = \mathcal{R}(\underline{I}-\lambda\underline{k})\oplus N(\underline{I}-\overline{\lambda}\underline{k}^*) = \mathcal{R}(\underline{I}-\overline{\lambda}\underline{k}^*)\oplus N(\underline{I}-\lambda\underline{k})$. Converting this to the interaction between radial integral equations $(I - \lambda K)f = g$ and $(I - \lambda K^*)f = g$ and their respective linear equations $(\underline{I} - \lambda\underline{k})\vec{x} = \vec{y}$ and $(\underline{I} - \overline{\lambda}\underline{k})^*\vec{x} = \vec{y}$, we have first that

$$dim\Big(N(I - \lambda K)\Big) = dim\Big(N(I - \overline{\lambda}K^*)\Big) \text{ for all } \lambda \in \mathbb{C}, \tag{5.5.71}$$

owing to the fact that the $n \times n$ matrices $\underline{I} - \lambda\underline{k}$ and $(\underline{I} - \lambda\underline{k})^*$ have the same rank, and second, the validity of the Fredholm alternatives - i.e.

(5.5.72) $\quad (I - \lambda K)f = g$ solvable, if and only if $\langle\ g \mid N(I - \overline{\lambda}K^*)\ \rangle = 0$ and

$(I - \overline{\lambda}K^*)f = g$ solvable, if and only if $\langle\ N(I - \lambda K) \mid g\ \rangle = 0.$

Therefore, the characteristic values of K^* are the complex conjugates of the characteristic values of K, and the regular values of K^* are also the complex conjugates of the regular values of K if $K \in \mathfrak{K}_2$ is of finite rank and $K^* \in \mathfrak{K}_2$.

For linear operators on finite dimensional Hilbert Spaces, we do not worry about the interchangability of the processes of taking adjoints and inverses - i.e.$(A^*)^{-1} = (A^{-1})^*$ for any such operator A as long as A or A^* is invertible. Therefore, we turn to the unique solution $\vec{x}$ of the linear the linear system (5.5.69), namely

$$\vec{x} = ((\underline{I} - \overline{\lambda}\underline{k})^{*})^{-1}\vec{y} = ((\underline{I} - \overline{\lambda}\underline{k})^{-1})^{*}\vec{y} = \left(\left[d(\overline{\lambda})\right]^{-1} \mathfrak{Adj}(\underline{I} - \overline{\lambda}\underline{k})\right)^{*}\vec{y} =$$

$$\left[\overline{d}(\lambda)\right]^{-1}\left(\alpha_{\mu\nu}(\overline{\lambda})\right)^{*}\vec{y} = \left[\overline{d}(\lambda)\right]^{-1}\left(\overline{\alpha_{\mu\nu}}(\lambda)\right)^{T}\vec{y},$$

where $\overline{d}(\lambda)$ is the polynomial in λ obtained by complex conjugating the coefficients of the polynomial $d(\lambda)$ and likewise for the n^2 polynomials $\overline{\alpha_{\mu\nu}}(\lambda)$. Hence,

$$x_\mu = \left[\overline{d}(\lambda)\right]^{-1} \sum_{\mu,\nu=1}^{n} \overline{\alpha_{\nu\mu}}(\lambda)\langle g \ , \ a_\nu{}^{\bullet} \ \rangle \ (1 \le \mu \le n; \ d(\overline{\lambda}) \neq 0) \tag{5.5.73}$$

and by repeating the proof of Theorem 5.5.1,

$$H_\lambda(K^{*}) \equiv \left[\overline{d}(\lambda)\right]^{-1} \sum_{\mu,\nu=1}^{n} \overline{\alpha_{\nu\mu}}(\lambda)(b_\mu \otimes a_\mu{}^{\bullet}) \ (\overline{\alpha}(\lambda) \neq 0) \tag{5.5.74}$$

is the Fredholm Resolvent of $K^{*} \in \mathfrak{K}_2$, in consequence of $\{\lambda \in \mathbb{C} : \overline{d}(\lambda) \neq 0\} = \{\overline{\lambda} \in \mathbb{C} : d(\overline{\lambda}) \neq 0\}$, with ϕ-parameter family of $\mathcal{L}_2$-kernels

$$H_\lambda(K^{*}; r, r', \phi) = \left[\overline{d}(\lambda)\right]^{-1} \sum_{\mu,\nu=1}^{n} \overline{\alpha_{\nu\mu}}(\lambda) b_\mu(re^{i\phi}) a_\mu{}^{\bullet}(r'e^{i\phi}) \tag{5.5.75}$$

$$(r > 0; \ 0 \le \phi \le \pi)$$

having inverse Mellin-Transform Representation

$$H_\lambda(K^{*}; r, r', \phi) = (2\pi)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{H}_\lambda(\mathbf{K}^{*}; t, t')(re^{i\phi})^{-it-1/2} \times \tag{5.5.76}$$

$$(r'e^{i\phi})^{it'-1/2} dt' dt \ (r, r' > 0; \ 0 \le \phi \le \pi), \text{ where}$$

$$\mathbf{H}_\lambda(\mathbf{K}^{*}; t, t') \equiv \left[\overline{d}(\lambda)\right]^{-1} \sum_{\mu,\nu=1}^{n} \overline{\alpha_{\nu\mu}}(\lambda)(\mathbf{b}_\mu \odot \mathbf{a}_\nu)(t, t').$$

In this closing paragraph of Chapter 5, we emphasize that: $K \in \mathfrak{K}_2$ is finite rank n with tensor product representation $K = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^{\bullet}$, its adjoint K^{*} also belongs to $\mathfrak{K}_2$, $\alpha_{\mu\nu}(\lambda)$ is the μ-th row and ν-th column entry of $\mathfrak{Adj}(\underline{I} - \lambda\underline{k})$, $\overline{\alpha_{\mu\nu}}(\lambda)$ is the polynomial obtained from $\alpha_{\mu\nu}(\lambda)$ by complex conjugating the coefficients of the polynomials $\alpha_{\mu\nu}(\lambda)$ $(1 \leq \mu, \nu \leq n)$, and $\overline{d}(\lambda)$ is obtained from $d(\lambda) = det(\underline{I} - \lambda\underline{k})$ by doing likewise.

CHAPTER 6

THE RADON SPLIT FOR $\mathfrak{K}_2$-OPERATORS

$\mathbb{R}^2$ is a locally compact Hausdorff Space and we define the σ-finite measure μ on $\mathbb{R}^2$ by means of

$$\mu(E) = \int\int_E [1+e^{2\pi(t-t')}]dt'dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_E(t,t')[1+e^{2\pi(t-t')}]dt'dt \; (\chi_E(t,t') = 1 \text{ or } 0$$

$$\text{according as } (t,t') \in \text{ or } \notin E, \text{ for } E \in \mathfrak{M}),$$

where the measure space $(\mathbb{R}^2, \mathfrak{M}, \mu)$ encompasses the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^2$. μ defines a "Radonizing measure " in the sense of determining a positive linear functional ([8], pg. 179; [15], pgs. 40 - 49) on the linear space $C_c(\mathbb{R}^2)$ of all $\mathbb{C}$-valued continuous functions with compact support in $\mathbb{R}^2$. Hence, $C_c(\mathbb{R}^2)$ lies dense in $L_p(\mathbb{R}^2, \mathfrak{M}, \mu)$ $(1 \leq p < \infty)$, especially for $p = 2$.

6.1. Approximation by Means Operators of Finite Rank

The density of $C_c(\mathbb{R}^2)$ in $L_2(\mathbb{R}^2, \mathfrak{M}, \mu)$ means, that for any $\mathbf{K} \in \mathfrak{K}_2$ and $\varepsilon > 0$ there exists a function $\mathbf{K}_\varepsilon \in C_c(\mathbb{R}^2)$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathbf{K}(t,t') - \mathbf{K}_\varepsilon(t,t')|^2 d\mu(t,t') < (\varepsilon/2)^2 \tag{6.1.1}$$

Moreover, because support $\mathfrak{trg}(\mathbf{K}_\varepsilon)$ ($\mathfrak{trg}$ standing for "Träger", carrier in German) is a compact subset $\mathbb{R}^2$, we can find an $A > 0$ such that

$$\mathfrak{trg}(\mathbf{K}_\varepsilon) \subset [-A, A]^2 = [-A, A] \times [-A, A].$$

Continuing this trend of thought, due to the Weierstraß-Approximation ([3], Vol. I, pgs. 65 - 68, especially pg. 68), there exist $\mathbf{a}_\mu$ and $\mathbf{b}_\mu{}^\bullet$ with $\mathbf{trg}(\mathbf{a}_\mu)$ and $\mathbf{trg}(\mathbf{b}_\mu{}^\bullet) \subset [-A, A]$ $(1 \leq \mu \leq n)$ satisfying

$$\text{(6.1.2)} \qquad \sup\Big\{|\mathbf{K}_\varepsilon - \sum_{\mu=1}^{n} \mathbf{a}_\mu(t)\mathbf{b}_\mu{}^\bullet(-t')|^2 : (t,t') \in [-A, A]^2\Big\} \leq \frac{(\varepsilon/2)^2}{\mu([-A, A]^2)}$$

with $\mathbf{b}_\mu{}^\bullet(-t') = [1 + e^{2\pi t'}]\overline{\mathbf{b}_\mu(t')}$ and $\mathbf{b}_\mu \in C_c(\mathbb{R})$ $(\mathbf{trg}(\mathbf{b}_\mu) \subset [-A, A])$ $(1 \leq \mu \leq n)$.

The relations (6.1.1) and (6.1.2) let us estimate as follows:

$$\text{(6.1.3)} \qquad |||K - \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet|||_{s(2)} \leq$$

$$\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathbf{K}(t,t') - \sum_{\mu=1}^{n}(\mathbf{a}_\mu \odot \mathbf{b}_\mu)(t,t')|^2 d\mu(t,t')\Big]^{1/2} \leq$$

$$\Big[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathbf{K}(t,t') - \mathbf{K}_\varepsilon(t,t')|^2 d\mu(t,t')\Big]^{1/2} +$$

$$\Big[\int_{-A}^{A}\int_{-A}^{A} |\mathbf{K}_\varepsilon(t,t') - \sum_{\mu=1}^{n}(\mathbf{a}_\mu \odot \mathbf{b}_\mu)(t,t')|^2 d\mu(t,t')\Big]^{1/2} < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

where we used the defining relation (5.5.60) for the sesqulinear tensor product $\odot$. However, we can say even more,

$$\text{(6.1.4)} \qquad |||K_\phi - \sum_{\mu=1}^{n}(a_\mu \otimes b_\mu{}^\bullet)_\phi|||^2 = |||(K - \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet)_\phi|||^2 =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\phi(t-t')}|\mathbf{K}(t,t') - \sum_{\mu=1}^{n} \mathbf{a}_\mu(t)\mathbf{b}_\mu{}^\bullet(-t')|^2 dt'dt =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\phi(t-t')}|\mathbf{K}(-t',-t) - \sum_{\mu=1}^{n} \mathbf{b}_\mu{}^\bullet(t)\mathbf{a}_\mu(-t')|^2 dt'dt =$$

$$|||(K^T - \sum_{\mu=1}^{n}(b_\mu{}^\bullet \otimes a_\mu)_\phi|||^2 \leq$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathbf{K}(t,t') - \sum_{\mu=1}^{n}(\mathbf{a}_\mu \odot \mathbf{b}_\mu{}^\bullet)(t,t')|^2 d\mu(t,t') \ (0 \leq \phi \leq \pi),$$

which in particular for $\phi = \psi$ ($\psi = 0, \pi$) combined with equation (3.5.50) of Theorem 3.5.1 yields

$$(6.1.5) \qquad |||K - \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet|||_{s(2)} = |||K^T - \sum_{\mu=1}^{n} b_\mu{}^\bullet \otimes a_\mu|||_{s(2)} < \varepsilon,$$

and leads into the next section.

6.2. The Radon Split of $\mathfrak{K}_2$-Operators

We use inequality (6.1.5) to make the following

DEFINITION 6.2.1. *Given an $\varepsilon > 0$ and $K \in \mathfrak{K}_2$, the Radon Split of K for $\varepsilon > 0$ is*

$$(6.2.6) \quad K = Q + P, \quad P = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \ \textit{satisfying}\ |||K - P|||_{s(2)} = |||Q|||_{s(2)} < \varepsilon.$$

For such a Radon Split, the radial integral equation (1.0.5) turns into

$$f = g + \lambda(Q + P)f \text{ or } (I - \lambda Q)f = g + \lambda P f.$$

Owing to $|||Q|||_{s(2)} < \varepsilon$, the Fredholm Resolvent G_λ of Q is given by the Neumann Series $G_\lambda = \sum_{n=0}^{\infty} \lambda^n Q^{n+1}$, whose radius of convergence as $\mathfrak{K}_2$-valued holomorphic function of λ is definitely not less than $1/\varepsilon$; in particular, $I + \lambda G_\lambda = \sum_{n=0}^{\infty} \lambda^n Q^n$ ($Q^0 \equiv I$) - i.e. $(I - \lambda Q)^{-1} = I + \lambda G_\lambda$. Therefore, out of $(I + \lambda G_\lambda)[(I - \lambda Q)f = g + \lambda P f]$ shall follow

$$(6.2.7) \qquad f = (I + \lambda G_\lambda)g + \lambda(I + \lambda G_\lambda)Pf =$$

$$(I + \lambda G_\lambda)g + \lambda \sum_{n=1}^{n} \langle\, f \,,\, b_\mu{}^\bullet \,\rangle (I + \lambda G_\lambda) a_\mu.$$

Proceding just like before for operators of finite rank belonging to $\mathfrak{K}_2$, namely bilinear producting these expressions and thereafter rewriting them, we obtain

$$\langle\ f = (I+\lambda G_\lambda)g + \lambda\sum_{\nu=1}^{n}\langle\ f\ ,\ b_\mu{}^\bullet\ \rangle(I+\lambda G_\lambda)a_\nu\ ,\ b_\mu{}^\bullet\ \rangle\ (1\le\mu\le n) \text{ or alternately}$$

$$\sum_{\nu=1}^{n}\Big[\delta_{\mu\nu} - \lambda k_{\mu\nu}(\lambda)\Big]\langle\ f,\ b_\mu{}^\bullet\ \rangle = \langle\ [I+\lambda G_\lambda]g\ ,\ b_\mu{}^\bullet\ \rangle\ (1\le\mu\le n), \tag{6.2.8}$$

$$\text{where } k_{\mu\nu}(\lambda) \equiv \langle\ [I+\lambda G_\lambda]a_\nu\ ,\ b_\mu{}^\bullet\ \rangle\ (1\le\mu\le n).$$

These n^2 functions $k_{\mu\nu}(\lambda)$ of complex variable λ are all holomorphic in an open discs centered at the orgin with radius not less than $1/\varepsilon$. Therefore, our radial integral equation (1.0.5) is solvable for λ ($|\lambda| < 1/\varepsilon$), if and only if the system of linear equations

$$[\underline{I} - \lambda\underline{k}_\lambda]\vec{x} = \vec{y}_\lambda \text{ is solvable for } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^{(n\times 1)}, \tag{6.2.9}$$

$$\text{where } \vec{y}_\lambda = \begin{pmatrix} \langle\ [I-\lambda G_\lambda]g\ ,\ b_1\ \rangle \\ \langle\ [I-\lambda G_\lambda]g\ ,\ b_2\ \rangle \\ \vdots \\ \langle\ [I-\lambda G_\lambda]g\ ,\ b_n\ \rangle \end{pmatrix},$$

$\underline{k}_\lambda$ is a $\mathbb{C}^{(n\times n)}$-valued holomorphic function of variable λ in a disc centered at 0 of radius not less than $1/\varepsilon$ and $k_{\mu\nu}(\lambda)$ is μ-th row and ν-th column entry of $\underline{k}_\lambda$.

After we write $[I - \lambda G_\lambda]g$ and the $[I + \lambda G_\lambda]a_\mu$ $(1 \le \mu \le n)$ as $g + \lambda G_\lambda g$ and $a_\mu + \lambda G_\lambda a_\mu (1 \le \mu \le n)$ respectively, the solution of radial integral equation (1.0.5) induced by solution $\vec{x}$ of the linear system (6.2.9) is

$$f(re^{i\phi}) = (g + \lambda G_\lambda g)(re^{i\phi}) + \lambda \sum_{\mu=1}^{n} x_\mu (a_\mu + \lambda G_\lambda a_\mu)(re^{i\phi}) \; (re^{i\phi} \in \Pi_+). \tag{6.2.10}$$

Let us now examine the linear system (6.2.9) more closely. As in Chapter 5, we define the $\mathbb{C}^{(n\times n)}$-valued holomorphic function

$$\underline{A}_\lambda \equiv \mathfrak{Adj}(\underline{I} - \lambda \underline{k}_\lambda) = (\alpha_{\mu\nu}(\lambda)) \; (|\lambda| < 1/\varepsilon) \text{ and note that} \tag{6.2.11}$$

$$(\underline{I} - \lambda \underline{k}_\lambda)\underline{A}_\lambda = \delta(\lambda)\underline{I} = \underline{A}_\lambda(\underline{I} - \lambda \underline{k}_\lambda) \text{ with } \delta(\lambda) \equiv det(\underline{I} - \lambda \underline{k}_\lambda), \tag{6.2.12}$$

where $\delta(\lambda)$ is no longer a polynomial in λ but a holomorphic function in a disc centered at 0 of radius not less than $1/\varepsilon$. We consider the case of $\delta(\lambda) = 0$, which can only occur for a finite number of λ in any compact subset of $\{\lambda \in \mathbb{C} : |\lambda| < 1/\varepsilon\}$, and note that the solution $\vec{x}$ of the linear system (6.2.9) for g being the identically vanishing $\mathfrak{H}_2$-function, yields the characteristic function

$$f(re^{i\phi}) = \lambda \sum_{\mu=1}^{n} x_\mu (a_\mu + \lambda G_\lambda a_\mu)(re^{i\phi}) \; (re^{i\phi} \in \Pi_+) \text{ or better}$$

$$N(I - \lambda K) = \Big\{ \lambda \sum_{\mu=1}^{n} x_\mu (a_\mu + \lambda G_\lambda a_\mu)(re^{i\phi}) : \tag{6.2.13}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in N(\underline{I} - \lambda \underline{k}_\lambda) \Big\} \text{ and}$$

$$(6.2.14) \qquad dim\Big(N(I-\lambda K)\Big) = dim\Big(N(\underline{I}-\lambda \underline{k}_\lambda)\Big) \ (|\lambda| < 1/\varepsilon).$$

Concentrating now on the case $\delta(\lambda) \neq 0$, we have that the linear system (6.2.9) has unique solution $\vec{x} = [\delta(\lambda)]^{-1}\underline{A}_\lambda \vec{y}_\lambda$, which we write component-wise as

$$(6.2.15) \qquad x_\mu = [\delta(\lambda)]^{-1}\sum_{\nu=1}^{n} \alpha_{\mu\nu}(\lambda)\langle\ [I+\lambda G_\lambda]g\ ,\ b_\mu{}^\bullet\ \rangle\ (1 \leq \mu \leq n)$$

and substitute these entries into equation (6.2.10) to obtain thereby

$$(6.2.16) \qquad f(re^{i\phi}) = (g+\lambda G_\lambda g)(re^{i\phi}) + \lambda[\delta(\lambda)]^{-1} \times$$

$$\Big(\sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(a_\mu + \lambda G_\lambda a_\mu) \otimes (b_\mu{}^\bullet + \lambda G_\lambda b_\mu{}^\bullet)\Big)(g)(re^{i\phi}).$$

This immediately follows after we transfer $I+\lambda G_\lambda$ in $\langle\ (I+\lambda G_\lambda)g\ ,\ b_\mu{}^\bullet\ \rangle$ to the right of the comma by means of transposition - i.e.

$$\langle\ g+\lambda G_\lambda g\ ,\ b_\mu{}^\bullet\ \rangle = \langle\ g\ ,\ (I+\lambda G_\lambda)^T b_\mu{}^\bullet\ \rangle = \langle\ g\ ,\ b_\mu{}^\bullet + \lambda G_\lambda{}^T b_\mu{}^\bullet\ \rangle\ (1 \leq \nu \leq n)\ \text{-}$$

and apply properties (5.5.48) of tensor product operators illucidated in the previous chapter. Owing to equation (6.2.16), we expect that the Fredholm Resolvent of $K \in \mathfrak{K}_2$ in terms of the Radon Split as given in Definition 6.2.1 - i.e. $K = Q + \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet$ - to turn out to be

$$(6.2.17) \quad H_\lambda = H_\lambda(K) = G_\lambda + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(a_\mu + \lambda G_\lambda a_\mu) \otimes (b_\mu{}^\bullet + \lambda G_\lambda{}^T b_\mu{}^\bullet)$$

$$\text{with } G_\lambda = \sum_{m=0}^{\infty} \lambda^m Q^{m+1}\ (|\lambda| < 1/\varepsilon),\ \text{provided } \delta(\lambda) \neq 0.$$

It satisfies the Fredholm Resolvent equations, as can readily be seen from the following arguments. We rewrite the relationship (6.2.12) between the matrix $\underline{I} - \lambda\underline{k}_\lambda$

and its classical adjoint $\underline{A}_\lambda$ as

(6.2.18) $\lambda \underline{A}_\lambda \underline{k}_\lambda = \underline{A}_\lambda - \delta(\lambda)\underline{I} = \lambda \underline{k}_\lambda \underline{A}_\lambda$ and calculate as follows:

$$\lambda H_\lambda K = \lambda\Big\{G_\lambda + [\delta(\lambda)]^{-1} \sum_{\mu,\eta=1}^{n} \alpha_{\mu\eta}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big) \otimes \Big([I+\lambda {G_\lambda}^T]b_\eta{}^\bullet\Big)\Big\}\times$$

$$\Big\{Q+\sum_{\nu=1}^{n} a_\nu \otimes b_\nu{}^\bullet\Big\} = \lambda G_\lambda Q + [\delta(\lambda)]^{-1} \sum_{\mu,\eta=1}^{n} \alpha_{\mu\eta}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes\Big(Q^T[I+\lambda {G_\lambda}^T]b_\eta{}^\bullet\Big)+$$

$$\sum_{\nu=1}^{n}(\lambda G_\lambda a_\nu)\otimes b_\nu{}^\bullet + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\Big\{\sum_{\eta=1}^{n}\lambda\alpha_{\mu\eta}(\lambda)\langle\, a_\nu\,,\ [I+\lambda {G_\lambda}^T]b_\eta{}^\bullet\,\rangle\Big\}\times$$

$$\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes b_\eta{}^\bullet = \lambda G_\lambda Q + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes\Big([\lambda {G_\lambda}^T]b_\nu{}^\bullet\Big)+$$

$$\sum_{\nu=1}^{n}(\lambda G_\lambda a_\nu)\otimes b_\nu{}^\bullet + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\Big\{\sum_{\eta=1}^{n}\lambda\alpha_{\mu\eta}(\lambda)k_{\eta\nu}(\lambda)\Big\}\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes b_\nu{}^\bullet =$$

$$G_\lambda - Q + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes(\lambda {G_\lambda}^T b_\nu{}^\bullet) + \sum_{\nu=1}^{n}(\lambda G_\lambda a_\nu)\otimes b_\nu{}^\bullet +$$

$$[\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\Big\{\alpha_{\mu\nu}(\lambda)-\delta(\lambda)\delta_{\mu\nu}\Big\}\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes b_\nu{}^\bullet =$$

$$G_\lambda - Q + [\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes(\lambda {G_\lambda}^T b_\nu{}^\bullet) + \sum_{\nu=1}^{n}(\lambda G_\lambda a_\nu)\otimes b_\mu{}^\bullet +$$

$$[\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes b_\nu{}^\bullet - \sum_{\nu=1}^{n}\Big([I+\lambda G_\lambda]a_\mu\Big)\otimes b_\mu{}^\bullet = G_\lambda +$$

$$[\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(a_\mu+\lambda G_\lambda a_\mu)\otimes(b_\mu{}^\bullet + \lambda {G_\lambda}^T b_\mu{}^\bullet) - \Big\{Q+\sum_{\nu=1}^{n}a_\nu\otimes b_\nu{}^\bullet\Big\} = H_\lambda - K.$$

We used herein in the second equality the contra-variance of transposition T - i.e.

$$Q^T[I+\lambda {G_\lambda}^T]b_\eta{}^\bullet = Q^T[I+\lambda G_\lambda]^T b_\eta{}^\bullet = [(I+\lambda G_\lambda)Q]^T b_\eta{}^\bullet = [I+\lambda G_\lambda]^T b_\mu{}^\bullet = {G_\lambda}^T b_\mu{}^\bullet,$$

and $\langle\, a_\nu\,,\ [I+\lambda {G_\lambda}^T]b_\mu{}^\bullet\,\rangle = \langle\, a_\nu\,,\ [I+\lambda G_\lambda]^T b_\mu{}^\bullet\,\rangle = \langle\, [I+\lambda G_\lambda]a_\mu\,,\ b_\mu{}^\bullet\,\rangle = k_{\mu\nu}(\lambda)$

as result of equation (6.2.8), and in the third equality we availed ourselves of equation (6.2.18). On the other hand,

$$\lambda K H_\lambda = \lambda\Big\{Q + \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet\Big\}\Big\{G_\lambda + [\delta(\lambda)]^{-1} \sum_{\eta,\nu=1}^{n} \alpha_{\eta,\nu}(\lambda)\times$$

$$\Big([I+\lambda G_\lambda]a_\eta\Big) \otimes \Big([I+\lambda G_\lambda{}^T]b_\mu{}^\bullet\Big)\Big\} = \lambda Q G_\lambda + \sum_{\mu=1}^{n} a_\mu \otimes \Big(\lambda G_\lambda{}^T b_\mu{}^\bullet\Big) + [\delta(\lambda)]^{-1}\times$$

$$[\delta(\lambda)]^{-1} \sum_{\eta,\nu=1}^{n} \alpha_{\eta\nu}(\lambda)\Big(\lambda Q[I+\lambda G_\lambda]a_\eta\Big) \otimes \Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big) +$$

$$[\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \Big\{\sum_{\eta=1}^{n} \lambda\langle\, [I+\lambda G_\lambda]a_\eta \;,\; b_\mu{}^\bullet\,\rangle \alpha_{\eta\nu}(\lambda)\Big\} a_\mu \otimes \Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big) =$$

$$\lambda Q G_\lambda + \sum_{\mu=1}^{n} a_\mu \otimes \Big(\lambda G_\lambda{}^T b_\mu{}^\bullet\Big) + [\delta(\lambda)]^{-1} \sum_{\eta,\nu=1}^{n} \alpha_{\eta\nu}(\lambda)(\lambda G_\lambda a_\eta) \otimes \Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big) +$$

$$[\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \Big\{\sum_{\eta=1}^{n} \lambda k_{\mu\eta}(\lambda)\alpha_{\eta\nu}(\lambda)\Big\} a_\mu \otimes \Big([I+\lambda G_\lambda{}^T]b_\mu{}^\bullet\Big) = G_\lambda - Q +$$

$$\sum_{\mu=1}^{n} a_\mu \otimes \Big(\lambda G_\lambda{}^T b_\mu{}^\bullet\Big) + [\delta(\lambda)]^{-1} \sum_{\eta,\nu=1}^{n} \alpha_{\eta\nu}(\lambda)(\lambda G_\lambda a_\eta) \otimes \Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big) +$$

$$[\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \Big\{\alpha_{\mu\nu} - \delta(\lambda)\delta_{\mu,\nu}\Big\} a_\mu \otimes \Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big) = G_\lambda - Q +$$

$$\sum_{\mu=1}^{n} a_\mu \otimes (\lambda G_\lambda{}^T b_\mu{}^\bullet) + [\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(\lambda G_\lambda a_\mu) \otimes \Big([I+\lambda G_\lambda{}^T]b_\mu{}^\bullet\Big) +$$

$$[\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda) a_\mu \otimes \Big([I+G_\lambda{}^T]b_\nu{}^\bullet\Big) - \sum_{\mu=1}^{n} a_\mu \otimes (b_\mu{}^\bullet + \lambda G_\lambda{}^T b_\mu{}^\bullet) = G_\lambda +$$

$$[\delta(\lambda)]^{-1} \sum_{\mu\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(a_\mu + \lambda G_\lambda a_\mu) \otimes (b_\nu{}^\bullet + \lambda G_\lambda{}^T b_\nu{}^\bullet) - \Big\{Q + \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet\Big\} = H_\lambda - K,$$

and thus the Fredholm Resolvent Equations (4.1.8) are satisfied, if $\delta(\lambda) \neq 0$. $H_\lambda = H_\lambda(K)$ has ϕ-parameter family of $\mathcal{L}_2$ kernels

$$H_\lambda(r, r', \phi) = H_\lambda(K; r, r', \phi) = G_\lambda(Q; r, r', \phi) + \tag{6.2.19}$$

$$[\delta(\lambda)]^{-1}\sum_{\mu,\nu=1}^{n}\alpha_{\mu\nu}(\lambda)(a_\mu+\lambda G_\lambda a_\mu)(re^{i\phi})(b_\mu{}^\bullet+\lambda G_\lambda{}^T b_\mu{}^\bullet)(r'e^{i\phi})$$

$$(r,r'>0;\ 0\le\phi\le\pi;\ \delta(\lambda)\neq 0,\ \text{and } |\lambda|<1/\varepsilon)$$

with inverse Mellin-Transform representation

$$(6.2.20)\quad H_\lambda(K;r,r',\phi)=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathbf{H}_\lambda(K;t,t')(re^{i\phi})^{-it-1/2}(r'e^{i\phi})^{it'-1/2}dt'dt$$

$$(r,r'>0;\ 0\le\phi\le\pi),\ \text{where } \mathbf{H}_\lambda(K;t,t')=\mathbf{G}_\lambda(Q;t,t')+[\delta(\lambda)]^{-1}\sum_{\mu\nu=1}^{n}\alpha_{\mu\nu}(\lambda)\times$$

$$(\mathbf{a}_\mu+\lambda\mathbf{G}_\lambda\mathbf{a}_\mu)(t)(\mathbf{b}_\mu{}^\bullet+\lambda\mathbf{G}_\lambda{}^T\mathbf{b}_\mu{}^\bullet)(-t')\ ((t,t')\in\mathbb{R}^2,\ \delta(\lambda)\neq 0,\ \text{and } |\lambda|<1/\varepsilon).$$

Further, it is very clear from these results, that the set of regular values of $K\in\mathfrak{K}_2$ is an open subset of $\mathbb{C}$, because if $\delta(\lambda_0)\neq 0$ for $|\lambda_0|<1/\varepsilon$, then the condition $\delta(\lambda)\neq 0$ may be continued into a neighborhood of λ_0 contained within the open disc $\{\lambda\in\mathbb{C}:|\lambda|<1/\varepsilon\}$ on account of the holomorphy of $\delta(\lambda)$ in $\{\lambda\in\mathbb{C}:|\lambda|<1/\varepsilon\}$.

6.3. Radon Split of the Transpose of $K\in\mathfrak{K}_2$.

Let us look at the transpose K^T of $K\in\mathfrak{K}_2$ under the Radon Split $K=Q+P$ with $P\in\mathfrak{K}_2$ of finite rank and $|||Q|||_{s(2)}<\varepsilon$. This induces also a Radon Split of $K^T\in\mathfrak{K}_2$, namely $K^T=Q^T+P^T$ with $|||Q^T|||_{s(2)}<\varepsilon$, owing to $Q^T(r,r',\phi)=Q(r',r,\phi)$ and thus $|||Q^T|||_{s(2)}=|||Q|||_{s(2)}$, because

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\phi(t-t')}|\mathbf{Q}^T(t,t')|^2dtdt'=|||(Q^T)_\phi|||^2=$$

$$|||Q_\phi|||^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{2\phi(t-t')}|\mathbf{Q}(t,t')|^2dtdt'\ (0\le\phi\le\pi).$$

The transposed radial integral equation

$$(6.3.21)\qquad u(re^{i\phi})=v(re^{i\phi})+\lambda\int_0^\infty K^T(r,r',\phi)u(r'e^{i\phi})e^{i\phi}dr'\ (re^{i\phi}\in\Pi_+)$$

with given $v \in \mathfrak{H}_2$ and $u \in \mathfrak{H}_2$ sought after, reduces in terms of the Radon Split $K^T = Q^T + P^T$ to $u = v + \lambda Q^T u + \lambda P^T u$ or $(I - \lambda Q^T)u = v + \lambda P^T u$, which equation we operate on from the left by $I + \lambda {G_\lambda}^T = (I - \lambda Q^T)^{-1}$ and thereby obtain

$$(6.3.22) \qquad u = [I + \lambda {G_\lambda}^T]v + [I + \lambda {G_\lambda}^T]P^T u, \text{ where } P^T = \sum_{\mu=1}^{n} {b_\mu}^\bullet \otimes a_\mu \text{ or}$$

$$(6.3.23) \quad u(z) = \Big([I + \lambda {G_\lambda}^T]v\Big)(z) + \lambda \sum_{\nu=1}^{n} \langle\, a_\nu \ , \ u \,\rangle \Big([I + \lambda {G_\lambda}^T]{b_\mu}^\bullet\Big)(z) \ (z \in \Pi_+).$$

Calculating the bilinear products

$$\langle\, a_\mu \ , \ u = [I + \lambda {G_\lambda}^T]v + \lambda \sum_{\nu=1}^{n} \langle\, a_\nu \ , \ u \,\rangle [I + \lambda {G_\lambda}^T]{b_\nu}^\bullet \,\rangle \ (1 \le \mu \le n)$$

under the observation that

$$\langle\, a_\mu \ , \ [I + \lambda {G_\lambda}^T]{b_\mu}^\bullet \,\rangle = \langle\, [I + \lambda G_\lambda]^T a_\mu \ , \ {b_\mu}^\bullet \,\rangle = k_{\nu\mu}(\lambda) \ (1 \le \mu \le n),$$

we readily see that the radial integral equation (6.3.22) is solvable, if and only if the linear system of equation

$$(6.3.24) \qquad [\underline{I} - \lambda {\underline{k}_\lambda}^T]\vec{x} = \begin{pmatrix} \langle\, a_1 \ , \ [I + \lambda {G_\lambda}^T]v \,\rangle \\ \langle\, a_2 \ , \ [I + \lambda {G_\lambda}^T]v \,\rangle \\ \vdots \\ \langle\, a_n \ , \ [I + \lambda {G_\lambda}^T]v \,\rangle \end{pmatrix} = \begin{pmatrix} \langle\, [I + \lambda G_\lambda]a_1 \ , \ v \,\rangle \\ \langle\, [I + \lambda G_\lambda]a_2 \ , \ v \,\rangle \\ \vdots \\ \langle\, [I + \lambda G_\lambda]a_n \ , \ v \,\rangle \end{pmatrix}$$

$$\text{is solvable for } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^{(n \times 1)},$$

and the solution of the transposed radial integral equation is given by

$$(6.3.25) \qquad u(re^{i\phi}) = \Big([I+\lambda G_\lambda{}^T]v\Big)(re^{i\phi}) + \lambda\sum_{\nu=1}^{n} x_\nu\Big([I+\lambda G_\lambda{}^T]b_\nu{}^\bullet\Big)(re^{i\phi})$$

$$(re^{i\phi} \in \Pi_+).$$

Since $rank(\underline{I}-\lambda\underline{k}_\lambda) = rank([\underline{I}-\lambda\underline{k}_\lambda]^T) = rank(\underline{I}-\lambda\underline{k}_\lambda{}^T)$, we have that K and K^T have the same characteristic values and on account of

$$N(I-\lambda K^T) = \Big\{\lambda\sum_{\nu=1}^{n} x_\nu\Big([I+\lambda G_\lambda{}^T]b_\mu{}^\bullet\Big)(re^{i\phi}) : [\underline{I}-\lambda\underline{k}_\lambda{}^T]\vec{x} = \vec{0}\Big\}, \text{ we have that}$$

$$(6.3.26) \qquad dim\Big(N(I-\lambda K^T)\Big) = dim\Big(N(I-\lambda K^T)\Big).$$

From linear algebra, we know that the range $\mathcal{R}(\underline{I}-\lambda\underline{k}_\lambda)$ has $N\Big((\underline{I}-\lambda\underline{k}_\lambda)^T)\Big) = N(\underline{I}-\lambda\underline{k}_\lambda{}^T)$ as its annihilator and therefore, the radial integral equation (1.0.5) has a solution, if and only if

$$\lambda\sum_{\nu=1}^{n}\langle\, [I+\lambda G_\lambda]g\ ,\ b_\mu{}^\bullet\,\rangle z_\nu = 0 \text{ for all } \vec{z}\in N(\underline{I}-\lambda\underline{k}_\lambda{}^T) \text{ or}$$

$$\lambda\sum_{\nu=1}^{n} z_\nu\langle\, g\ ,\ [I+\lambda G_\lambda]^T b_\mu{}^\bullet\,\rangle = 0, \text{ even better } \langle\, g\ ,\ \lambda\sum_{\nu=1}^{n} z_\nu[I+\lambda G_\lambda{}^T]b_\mu{}^\bullet\,\rangle = 0.$$

Hence, $f = g + \lambda K f$ is solvable in $\mathfrak{H}_2$ for unknown f, if and only if

$$(6.3.27) \qquad \langle\, g\ ,\ N(I-\lambda K^T)\,\rangle = 0.$$

Further, due to the self-evident property of $K^{TT} = (K^T)^T = K$ and the fact that transposition preserves the Radon Split, as indicated in the sentence immediately preceding equation (6.3.21), we have that the transposed radial integral equation

$u = v + \lambda K^T u$ ($v \in \mathfrak{H}_2$ given, and $u \in \mathfrak{H}_2$ sought after) is solvable, if and only if $\langle\, v\ ,\ N(I - \lambda(K^T)^T)\,\rangle = 0$, which is the same as saying, if and only if

$$\langle\, N(I - \lambda K)\ , v\,\rangle = 0. \tag{6.3.28}$$

For the conditions (6.3.27) and (6.3.28) to be valid, it must be assumed that our $\varepsilon > 0$ for the Radon Split is such that $\varepsilon|\lambda| < 1$, in order that

$G_\lambda = \sum_{n=0}^{n} \lambda^n Q^{n+1}$ and $G_\lambda{}^T = \sum_{n=0}^{n} \lambda^n (Q^T)^{n+1}$ stand for operators belonging to $\mathfrak{K}_2$.

6.4. Summary and Conclusion.

We therefore summarize the results accumulated in this chapter in the following four theorems.

THEOREM 6.4.1. *If $K \in \mathfrak{K}_2$, then the radial integral equation $f = g + \lambda K f$ ($f = g + \lambda K^T f$) is solvable if and only if*

$$\langle\, g\ ,\ N(I - \lambda K^T)\,\rangle = 0\ \Big(\langle\, N(I - \lambda K)\ ,\ g\,\rangle = 0\Big).$$

THEOREM 6.4.2. *IF $K \in \mathfrak{K}_2$, then K and its transpose K^T have the same characteristic values, which characteristic values cannot accumulate in the infinite complex plane $\mathbb{C}$; moreover,*

$$dim\Big(N(I - \lambda K)\Big) = dim\Big(N(I - \lambda K^T)\Big) \text{ for all } \lambda \in \mathbb{C}.$$

THEOREM 6.4.3. *If λ is a regular value of $K \in \mathfrak{K}_2$, then any Radon Split*

$$K = Q + P\ \Big(P = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^{\bullet} \in \mathfrak{H}_2 \otimes \mathfrak{H}_2,\ |||Q|||_{s(2)} < \varepsilon\Big)$$

of K yields the Fredholm Resolvents

$$H_\lambda(K) = G_\lambda + [\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(a_\mu + \lambda G_\lambda a_\mu) \otimes (b_\nu{}^\bullet + \lambda G_\lambda{}^T b_\nu{}^\bullet) \in \mathfrak{K}_2 \text{ and}$$

$$H_\lambda(K^T) = G_\lambda{}^T + [\delta(\lambda)]^{-1} \sum_{\mu,\nu=1}^{n} \alpha_{\mu\nu}(\lambda)(b_\nu{}^\bullet + \lambda G_\lambda b_\nu{}^\bullet) \otimes (a_\mu + \lambda G_\lambda a_\mu) \in \mathfrak{K}_2$$

of K and K^T respectively, where the holomorphic functions $\delta(\lambda)$ and $\alpha_{\mu\nu}(\lambda)(1 \le \mu, \nu \le n)$ of variable λ in the open disc $\{\lambda \in \mathbb{C} : |\lambda| < 1/\varepsilon\}$ are defined in terms of the matrix $\underline{k}_\lambda$ and $\underline{A}_\lambda$ as they appear in the relations (6.2.8) , (6.2.11) and (6.2.12).

THEOREM 6.4.4. *Given the Radon Split $K = Q + P$ ($P = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{H}_2 \otimes \mathfrak{H}_2$, $|||Q|||_{s(2)} < \varepsilon$) of $K \in \mathfrak{K}_2$, then the radial integral equations $f = g + \lambda K f$ and $f = g + \lambda K^T f$, for the regular value λ of K with $\varepsilon|\lambda| < 1$, have the unique solution in terms of the Fredholm Resolvents given by*

$$f(z) = (g + \lambda G_\lambda g)(z) + \lambda[\delta(\lambda)]^{-1} \times \tag{6.4.29}$$

$$\sum_{\mu,\nu=1}^{n} \alpha_{\mu,\nu}(\lambda)\langle\, g \ ,\ b_\nu{}^\bullet + \lambda G_\lambda{}^T b_\nu{}^\bullet\, \rangle (a_\mu + \lambda G_\lambda a_\mu)(z) \text{ and}$$

$$f(z) = (g + \lambda G_\lambda{}^T g)(z) + \lambda[\delta(\lambda)]^{-1} \times \tag{6.4.30}$$

$$\sum_{\mu,\nu=1}^{n} \alpha_{\mu,\nu}(\lambda)\langle\, a_\mu + \lambda G_\lambda a_\mu \ ,\ g\, \rangle (b_\nu{}^\bullet + \lambda G_\lambda{}^T b_\nu{}^\bullet)(z) \text{ for all } z \in \Pi_+ \text{ respectively.}$$

What about the Fredholm Resolvent of the adjoint $K^\bullet$ of $K \in \mathfrak{K}_2$ in the Hilbert space $\langle\, \mathfrak{H}_2 \mid \mathfrak{H}_2 \,\rangle$? Due to Theorem 5.5.3 and relation (5.5.75) for $H_\lambda(K^\bullet)$, it is not simple to construct a Radon Split $K = Q + P$ with $P = \sum_{\mu=1}^{n} a_\mu \otimes b_\mu{}^\bullet \in \mathfrak{H}_2 \otimes \mathfrak{H}_2$ and $|||Q|||_{s(2)} < \varepsilon$ so that

$$K^* = Q^* + P^*, \text{ where } P^* = \sum_{\mu=1}^{n} b_\mu \otimes a_\mu{}^\bullet \in \mathfrak{H}_2 \otimes \mathfrak{H}_2,$$

on account of the counter-example described in the paragraph prior to Theorem 5.5.3, namely the operator $a \otimes b \in \mathfrak{K}_2$ of rank 1 with adjoint $(a \otimes b)^* \notin \mathfrak{K}_2$. The investigation of the Radon Split so that

$$K^* = Q^* + P^* \in \mathfrak{K}_2 \text{ with } P^* = \sum_{\mu=1}^{n} b_\mu \otimes b_\mu{}^\bullet \in \mathfrak{K}_2 \text{ and } |||Q^*|||_{s(2)} < 1/\varepsilon$$

requires an extensive and very detailed study of the interplay between the Hilbert spaces $\langle\ \mathfrak{H}_2 \mid \mathfrak{H}_2\ \rangle$ and $\langle\ \boldsymbol{\mathfrak{H}_2} \mid \boldsymbol{\mathfrak{H}_2}\ \rangle$ and the dual system $\langle\ \boldsymbol{\mathfrak{H}_2}\ ,\ \mathfrak{H}_2\ \rangle$, which is beyond the scope of this M. Sc. thesis.

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